Cai Tianxin

A Brief History of Mathematics

A Promenade through the Civilizations of Our World

Birkhäuser

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The book of time

like numbers are divided into odd and even,
a book has its front and back,
and time is cut into day and night.

but when I turn the pages over,
there is no obvious difference
between the first and last line.

this is the uniqueness of twilight,
and also of dawn – the profound
and subtle nature of the book of time.

Lake Geneva, 2007[1]

[1] Tr. Robert Berold and Cai Tianxin. Published in *Every Cloud has its Own Name: Selected Poems and Essays of Cai Tianxin*, 1 Plus Books, San Francisco (2017).

Preface

In the summer of 2011 the world received news of a shocking event in Norway. A solitary political extremist detonated a bomb in a government center in Oslo before proceeding to a youth camp taking place on the island of Utoya not far away and carrying out an attack that left more than 80 people dead. Norway is among the most peaceful, comfortable, and rich nations in the world today, home of the great nineteenth-century mathematical genius Niels Henrik Abel. The first Fields medal was awarded in Oslo in 1936, and the Nobel Peace Prize has its home there, as does the Abel Prize in mathematics. It was difficult and painful to imagine such a tragic incident taking place there.

Abel was the first Norwegian to achieve international fame for his achievements. Although he died of tuberculosis in 1829 in obscurity at the age of 26, he was soon recognized as one of the greatest mathematicians of the nineteenth century or for that matter in human history. His life and accomplishments seem to have inspired the talents of fellow Norwegians. The playwright Henrik Ibsen was born one year before his death, and not long after there appeared also the composer Edvard Grieg, the artist Edvard Munch, and the explorer Roald Amundsen. I could not help but think about all of this when I heard about the shootings. Perhaps Abel's early death, the long expatriation of Ibsen, and the naked dread in Munch's famous painting *The Scream* speak also to some sense of tragedy in that northern country.

In every book related to mathematics, Abel's name always appears first in the index. This book also discusses his life at length, as well as his fellow Norwegian, the mathematician Sophus Lie (1842–1899), after whom are named two of the most important topics in contemporary mathematics—Lie groups and Lie algebras. In 1872, the German mathematician Felix Klein (1849–1925) initiated his so called *Erlangen* program in an effort to use group theory to unify all of geometry and even all of mathematics by way of ideas developed by Lie.

This book does not have occasion to introduce another great Norwegian mathematician, Atle Selberg (1917–2007), who passed away in 2007 and with whom the author had the opportunity to meet and discuss number theory. Selberg was awarded a Fields Medal in 1950 for his many accomplishments, including an elementary proof of the prime number theorem. In perhaps another coincidence, one of the last figures to make an appearance in this book is the Austrian philosopher Ludwig Wittgenstein (1889–1951) who formed a deep attachment during his life to the remote Norwegian landscape. During his time at Cambridge University he built a cabin in a rural area in western Norway to which he would retreat to think, sometimes staying for as long as a year. It was there that he began work on his great posthumous work *Philosophical Investigations*.

It must be clear already from the passages in this preface that this is a book that does not want to leave out any great mathematician or mathematical line of thought or to pass up any opportunity to investigate the interminglings of mathematics with all the other modes of culture and human activity. The contents and style of writing are intended precisely to suit this end. Such a book as far as I know has no blueprint. Probably its most similar precursor is *Mathematics in Western Culture* by Morris Kline (1908–1992). The appearance, however, of the words *western* and *culture* in the latter title point toward a substantial difference. I feel we have no choice here but to take into consideration the history of humanity across the globe. The domain of mathematics is beyond the scope of any single culture. The British mathematician and philosopher Alfred North Whitehead (1861–1947) observed that modern science was born in Europe, but its home is the whole world.

From the perspective of composition, although there are many possibilities, in the main the most important consideration was whether or not to take the history of mathematics itself as an organizing principle. Although Kline had based his work on chronology (and the same is true of his other masterpiece, *Mathematical Thought from Ancient to Modern Times*), he devotes a topic in each chapter to the relationship between mathematics and culture. It is evident that Kline is both proficient in mathematics and possesses a deep familiarity with western culture since Ancient Greece. I felt it would be difficult to surpass him in this regard.

However, it is immediately apparent in reading his works that Kline had in mind in the course of composition an audience consisting of experts in mathematics or the history of western culture. I have in mind for this book a much broader readership. I hope to reach readers who have studied only elementary mathematics, maybe a little calculus, and readers who perhaps do not really know much about the history of mathematics and its relationship to human culture. The role that mathematics has played in the development of civilization is not widely enough understood, especially with respect to the origins of modern mathematics, and modern culture and art.

It seems to me that mathematics, science, and the various humanities all stand as equals as representatives of the human mind and the process of intellectual development. In any given period they are all intermixed with one another and exhibit similar processes and characteristics. I have therefore organized this book chronologically, moving freely from one region of the earth to another in any given period. In this way it is possible to find points of commonality and distinction between the various approaches. For example, the mathematics of ancient Egypt and Babylon was driven primarily by the need for survival, whereas mathematics in Greece was closely connected with philosophy. The motivation for mathematics in China was driven largely by questions of calendar reform. In India mathematics sprang from religion; in Persia and the Arab world, mathematics was inseparable from astronomy.

The Renaissance in Europe is an indisputable milestone in the history of human civilization. During this period, the development of geometry was inspired by artistic considerations. In the seventeenth century the invention of the calculus emerged in the wake of the scientific and industrial revolutions to meet its newly discovered needs. Mathematicians in the time of the French Revolution concerned themselves with mechanics and innovation in civil and military engineering. During the first half of the nineteenth century, mathematics and art leapt forward into the modern era almost simultaneously. These revolutions were marked by the appearance of noncommutative algebra and non-Euclidean geometry on the one hand, and the poetry of Edgar Allan Poe (1809–1849) and Charles Baudelaire (1821–1867) on the other. Since the beginning of the twentieth century, abstraction has become the common law of mathematics and the humanities.

In mathematics there appeared the new disciplines of set theory and the axiomatic method; in art there was the abstract movement, action painting, and various other developments towards the abstract. The reunion inaugurated by these developments between mathematics and philosophy has produced modern logic and facilitated the remarkable innovations of Wittgenstein and Gödel. Perhaps what is most surprising in all of this is that mathematics has not proved indulgent in its tendency toward abstraction, but rather has become only more useful, with new applications to theoretical physics, biology, economics, computer science, chaos theory. This conforms splendidly with the state of things at every stage of its historical development. Nevertheless, the future of the mathematical horizon remains unclear.

The definitive feature of this book therefore is a comparative analysis and interpretation of modern mathematics and modern civilization. This is the fruit of many years of writing and active research in mathematics. With respect to ancient mathematics I have chosen to focus especially on those mathematical topics that retain a vital interest from a modern perspective. For example, the discussion of Egyptian mathematics places special emphasis on the topic of Egyptian fractions, a topic which provides number theorists with deep and challenging problems even in the twenty-first century. As another example, it seems to have been the Babylonians who first discovered the Pythagorean theorem, and began to investigate the problem of Pythagorean triples, a beautiful achievement that was recapitulated throughout ancient history, for example, in Greece more than a thousand years later. But it also has deep connections to one of the most modern and challenging problems of the twentieth century, the proof of Fermat's Last Theorem.

Another characteristic feature of this book is the sections are mostly named after colorful and important figures in the history of mathematics and culture; I have written them in the hope that they are easy to understand, to enjoy, and to remember. I have also included more than a hundred carefully selected illustrations, some of which I photographed myself, in order to bring to life the various topics in mathematics, science, art, literature, and education within. I hope that anyone who reads this book can experience through it a closer connection to a subject that is sometimes considered forbidding and relentlessly abstract, and reflect more deeply upon the relationships between mathematics and the other creative disciplines, perhaps even on the nature of human life and civilization.

My Chinese readers will recognize this book as a revision of my textbook *Mathematics and Human Civilization*, published in 2012 by the *Commercial Press* in Beijing. After the copyright expired, I revised the entire book according to suggestions offered by the CITIC Publishing House, and updated many of the illustrations. In 2017 we renamed the book *A Brief History of Mathematics*, which is more in line with the original intention of the book. Since the book focuses on the history of mathematics and its relationship to human civilization, it has proved advantageous to avoid many of the intricate complexities of modern mathematics and I have endeavored instead to offer the reader a variety of different angles from which to approach the subject. I have since been gratified to see versions published in Korean, Russian, and Taiwanese.

I would like to bring this preface to a close with a poem. I wrote it in the summer of 2005, when I traveled with four graduate students to participate in an international seminar on number theory and cryptography at the University of the Philippines in Manila, the country where Magellan died and which has been too easily overlooked in a history of mathematics and culture. There appear in the poem some geometric phrases, lines, arcs, circles, surfaces, and topological deformations, transformed of course into a language suitable to poetry. The poem at first glance seems to be operating within the abstruse territories of mathematics, but its aim is toward the pure emotions of life.

Skipping

Each bright and clean rice stalk
is covered entirely in silver moonlight
and then woven into the rope.

Like a silver chain on the ankle
the circle around the circle
also covered in moonlight

The tip of the eyebrow, the temple
and the scald mark on the arm
All oscillate through the rope.[2]

West Brook, Hangzhou, China Tianxin Cai
Late Spring, 2023

[2] Tr. Robert Berold and Cai Tianxin

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About the Author

Cai Tianxin was born in Taizhou, in southeastern China in 1963, and was admitted to Shandong University at the age of 15. He received his doctorate in science in 1987, and currently serves as an outstanding professor at the School of Mathematics at Zhejiang University. He is also a poet, writer, and photographer, who has published more than 40 literary and scholarly works, translated variously into more than 20 languages, including seven works translated into English. In 2007, he was invited to be a resident writer at Château de Lavigny in Switzerland; he served as judge in a Haiku competition in Tokyo in 2008, was a guest at the Arabic Capital of Culture in Baghdad, Iraq, in 2014, participated in the Frankfurt Book Fair in 2010 and the International Writing Program in Iowa, USA, in 2018. In total, he has participated in more than 30 international festivals of poetry and literature. Since 2010, he has served as the founding editorial board member of the quarterly magazine *Mathematical Culture* based in Hong Kong.

Since the start of the new millennium, Cai has given hundreds of public lectures around the world, including at Princeton University, London School of Economics, the Australian National University, the National University of Singapore, Pontifical Catholic University of Peru, and University of Nairobi. Poetry readings and photography exhibitions dedicated to his work have been held in New York, Paris, Los Angeles, Chicago, San Francisco, Houston, St. Louis, Cambridge, Kharkiv, and other cities around the world. He first saw a train on his journey to college, and now his footprints have spread across the provinces and cities and to more than one hundred countries.

In his capacity as a professor of mathematics, Cai proposed a class of Diophantine equations that was hailed as a "truly original contribution" by the British mathematician and recipient of the Fields Medal Alan Baker. In his capacity as a writer and a poet, Cai was awarded the Naji Naaman Poetry prize (Beirut) in 2013, and the Kathak Literary Award (Dakar) in 2019. In 2022, he was selected as an influential writer on Dangdang (the Chinese analogue to Amazon.com).

Chapter 1
The Middle East, or the Beginning

It must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number 2: the degree of abstraction involved is far from easy.

Bertrand Russell

The Origins of Mathematics

The Beginnings of Counting

There is a story in *The Odyssey* by the Greek poet Homer: after the itinerant hero Odysseus had struck out the eye of the cyclops Polyphemus, the unfortunate creature lived out his days in isolation tending to his sheep from his mountain cave. Each morning as the sheep went out to graze, the blind giant would take out a single stone from a pile and cast it to one side. When they returned to the cave in the evening, he would toss the stones back into the pile one by one. Only after he had thrown back every stone that he had gathered in the morning he could rest assured that all of the sheep had made their way home to rest. In this story we can catch a glimpse of the birth of mathematics from the counting of sheep. Just as poetry has its origins in prayers for a bountiful harvest, these two most basic inventions of humanity are the product of our needs (Fig. 1.1).

Of course the earliest figures in the story of mathematics are lost amid the shrouds of prehistory, and we can only speculate as their inventions only by way of the inevitable concomitant developments in the history of human civilization. Many millions of years ago, our ancestors in caves must already have hit upon the concept of number, adding or taking away individual objects from a collection say of foodstuffs to distinguish between few and many. Indeed, even many animals possess this ability. With the slow passage of time, humanity came to develop some conception of the basic numbers $1, 2, 3, \ldots$. In this way the tribal leaders came to know the number of their purview, and the shepherds the number of their sheep.

Counting and basic arithmetic also sprang up before the dawn of recorded history. The hunter, for example, at some point discovered that two arrows and three arrows together make five arrows. Just as different peoples around the world have all

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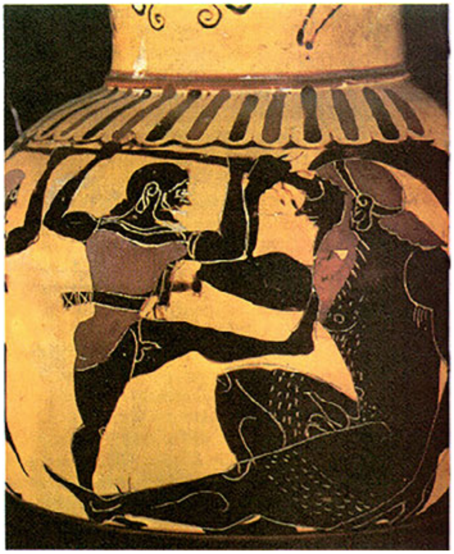

Fig. 1.1 Pottery scene: Odysseus blinds the cyclops

alike similar sounding words to *mama* and *papa* to designate the most fundamental familial relations, so too it must have happened that the earliest form of counting was carried out everywhere in the past as it is by children today, by means of the fingers. Some time later there appeared three standard approaches to the basic problem of enumeration: (1) by pebbles or small wooden rods, (2) by knots, and (3) by notches recorded in bricks of mud, blocks of wood, stone, the bark of trees, and the bones of animals. It was possible by way of these methods not only to keep track of larger quantities of things but also to create a more permanent record of their number and to add together or compare to one another two separate tallies.

Naturally counting proved useful in war and in hunting as well. Some peoples recorded the number of their defeated enemies by means of scalps, and others the number of their slaughtered prey by means of teeth. Among certain peoples it was custom among the women to adorn their necks with rings representing age in years. In more recent history, it used to be custom among British bartenders to tally up the drinks of the customers with marks on a slate, while in Spain the bartenders would toss little pebbles into the customer's hat. Perhaps there are some cultural observations to be drawn from all these differences.

As language emerged and developed, there came with it a variety of linguistic means for the expression of numbers. Initially it was commonplace for number words and signs to come with predetermined attachments to specific objects and categories, as, for example, in English one says a team of horses, a yoke of oxen, a span of mules, a pair of shoes, and so one. In Chinese the variety of measure words is even greater, and a great many of them have survived to the present day.

Eventually however humanity managed to abstract the concept of a number from its specific contexts. The British philosopher and mathematician Bertrand Russell (1872–1970) describes this scope of this achievement as follows: "It must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number 2: the degree of abstraction involved is far from easy." The invention of mathematics proper must have come a little later still, when people managed the conceptual leap from such propositions as two chickens and three chickens makes five chickens, two arrows and three arrows makes five arrows, etc. to the identity $2 + 3 = 5$.

Number Bases

As people had occasion to communicate more extensively, it became necessary to make systematic the notations of enumeration. The same basic solution seems to sprung up independently around the world among various peoples: take some fixed ensemble of symbols $1, 2, \ldots, b$ and use them in combination with one another to represent arbitrarily large numbers. The number b in this system is called its base.

There is some evidence that 2, 3 and 4 have all been used as number bases at one point or another. For example, the aboriginal people of Queensland in northern Australia reckon their numbers as 1, 2, and 2 and 1, 2, and 2, ... Among certain groups in Africa, the first six numbers are reckoned as *a*, *oa*, *ua*, *oa-oa*, *oa-oa-a*, and *oa-oa-oa*. Both systems are essentially binary, a way of counting that has since proved useful in computer science. Examples of number systems in base 3 and 4 have been found in parts of South America.

Because people possess five fingers on each hand and five toes on each foot, it is not difficult to imagine that number systems in base five would turn up sooner or later, and indeed it seems that some people in South America count as 1, 2, 3, 4, hand, hand and 1, and so on. The German lunar calendar took five as its base until as late as 1880. In 1937 the bones of a young wolf were excavated in Moravia in Czechoslovakia with tallies arranged in groups of five. The Yukaghir people in Siberia, who live off the Lena river in the coldest part of the inhabited earth, use a system that resembles a combination of base five and decimal (Fig. 1.2).

The number 12 has also been a common choice of base. The American mathematician H.W. Eves (1911–2004) has argued that this may be because twelve is a highly divisible number, with six divisors. Or it may be because there are 12 lunar months in a year. In any case, examples are plentiful: there are 12 inches to the foot and 12 pence to the shilling. There are also the words dozen and gross (meaning a dozen dozens). As late as the 1980s, it was common for scales in Chinese villages to include measurements in both decimal and hexadecimal (with base 16).

Twenty seems also to have occurred here and there throughout history as a number base, which again suggests counting by means of the fingers and toes. The most famous example of mathematics in base 20 was among the Mayan people, who used it to construct their calendar as recorded in three codices which survived the Spanish Inquisition in the new world: the Dresden codex, the Madrid codex, and the Paris codex, named for the cities in which they each eventually settled. The Dresden codex is generally considered to be the most important of the three and

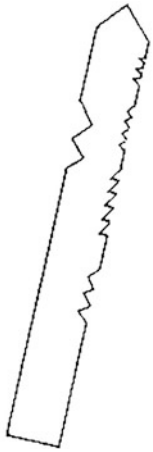

Fig. 1.2 British chipped wood talley

contains the richest treasury of mathematical content. Its final page also warns of the coming of the end of the world, that is, the famous "2012 doomsday prophecy," and even describes the scene, in which a flood triggered by crocodiles brings about the destruction of the whole world. The Dresden codex was probably copied from stone engravings of the twelfth century. The bulk of the codex is concerned with astronomical tables and events, climate forecasting, and the record of religious ceremonies and details. The codex was purchased for the Royal Library at Dresden from a private owner in Vienna in 1739. The original suffered serious water damage during the bombing of Dresden in 1945 (Fig. 1.3).

The French language contains some interesting vestiges of base 20 enumeration: the French word for 80, for example, is *quatre-vingt* (four twenties), and the French word for 90 is *quatre-vingt-dix* (four twenties and ten), reminiscent of the archaic English word *score*. Similar remnants persist in other languages of disparate geographic origin, including Danish, Welsh, Gaelic, and Chinese, which has the word "廿." Surprisingly, not all of these places are in temperate zones. Finally there is the prevalence of base 60, which seems to have originated among the ancient Sumerian and Babylonian people and remains in use today for the measure of time and angles.

One way or another, however, humanity seems eventually to have settled almost everywhere on decimal notation. The hieroglyphic numbers of Ancient Egypt, the oracle bones and counting rods in Ancient China, the attic numerals of Ancient Greece, and the Brahmi numeral system of Ancient India all adopt decimal notation. Somehow ten seems to be the most natural number base of the human mind, just as two is the most natural number base for digital systems. The Greek philosopher Aristotle naturally speculated that this is a basic fact of anatomy. This situation is reflected in the language we use as well. In English the word digits stood originally for the fingers and toes and was adopted only later to for the numbers one through nine. Traditional and modern cultures alike preserve through the present-day rich representation systems for small numbers by means of gestures and configurations

Fig. 1.3 Dresden codex original, containing the infamous Mayan 2012 prophecy

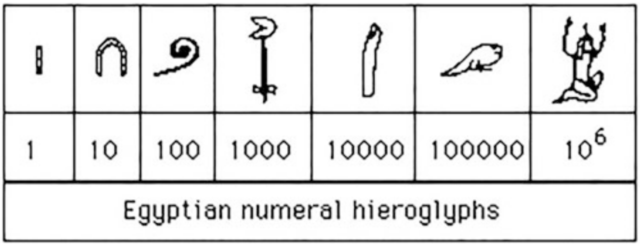

Fig. 1.4 Ancient Egyptian hieroglyphic numerals

of the hand. In modern China it is still possible to work out what region or province a person comes from by the gestures he or she makes while casting fingers for numbers in a popular drinking game (Fig. 1.4).

Arabic Numerals

According to the archaeological evidence, the use of tally marks in enumeration first appeared about 30,000 years ago. It was many millenia again before written numbers and number systems appeared, around 3000 BCE. Corresponding perhaps to a direct representation of the fingers, the earliest written number systems seem to have comprised vertical or horizontal tallies in groups of 1, 2, 3, or 4. Vertical strokes appeared in hieroglyphics, in Attic numbers, Mayan numerals, and the vertical counting chips of Ancient China. Horizontal strokes turn up among the oracle bones and horizontal counting chips in China and the Brahmi numerals in India (excluding the numeral 4).

It is interesting to note that the above systems in which the first four were represented by horizontal or vertical strokes without exception all made the transition to decimal notation in the same way. The other two famous number systems of antiquity, namely, the sexagesimal system of ancient Babylon and the vigesimal system of the ancient Mayans, made use of a notation consisting of little isosceles triangles and circles. As for the numbers greater than four, each of the tally-based enumerations handled them differently. The number ten, for example, was represented in the Egyptian system by the symbol $\cup$, resembling the symbol for the union of two sets in modern set theory. The Greeks used the symbol Δ, the fourth letter of their alphabet. In China, the number ten was represented by an ensemble of four vertical strokes and one horizontal (Fig. 1.5).

The system of Arabic numerals refers to the decimal writing system represented by the digits $0, 1, 2, 3, \cdots, 9$ and their combinations. For example, in the number 911, the rightmost 1 stands for 1, the middle 1 for 10, and the 9 for 9 times 100. Among the thousands of languages extant in the world today, these ten Arabic numerals are the only universal symbols, far more commonplace than even the Latin alphabet. It is conceivable that if there were no Arabic numeral system, international communication in science and technology, culture, economics, military affairs, and sport would be much more difficult, even impossible.

This system is sometimes also referred to as the system of Hindu-Arabic numerals, since they were invented originally in India and subsequently transmitted to Europe by way of the Arabic world. The transition to the modern system in India was complete by the twelfth century, and all earlier forms of numerals disappeared from view until the development of modern archaeology facilitated their rediscovery on cave walls and stone pillars in India. It is believed today that those records of an earlier age were all made between the years 250 BCE and 200 CE. It is worth mentioning that no notion of the number zero occurs in any of the early systems. In the year 825, the Persian mathematician Al-Khwarizmi wrote a book *On the Use of Indian Numerals* in Arabic, which presented a complete treatment of the Hindu numeral system including zero. The modern English word *zero* came into the language from Arabic by transliteration.

Arabic numerals entered Europe by way of Northern Africa and Spain during a period of great Arabic flourishing and prosperity. A pivotal figure in this story was the Italian mathematician Leonardo of Piso, more commonly known as Fibonacci. In his youth, Fibonacci was able to receive instruction from a Spanish Muslim mathematician and to travel to Northern Africa. When he returned to Italy around 1202, he published an influential book called the *Liber Abacia (Book of Calculation)*. This book is an incomparable milestone in the history of mathematics. In addition to inaugurating the transmission of Hindu-Arabic numerals from the Muslim world to greater Europe, it also contributed a number of topics of profound

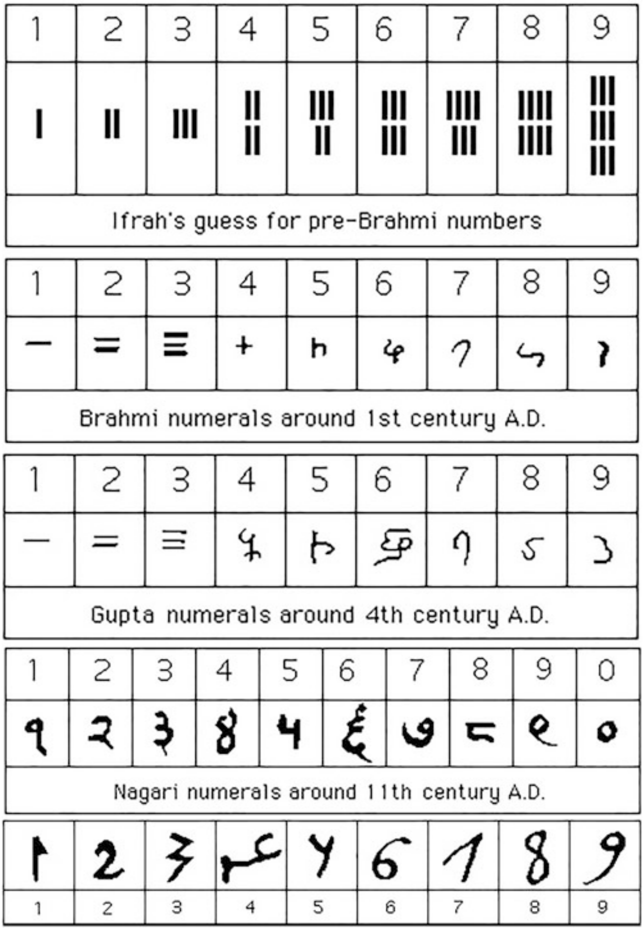

Fig. 1.5 The evolution of Hindu-Arabic numerals

importance for the future direction of mathematical research during the Italian Renaissance and indeed throughout the world. It was also during the same century that the Venetian traveller Marco Polo (1254–1324) completed the first European expeditions to the far East. At that time Constantinople (now Istanbul in Turkey) was troubled by continuous strife and conflict. Marco Polo was able to bypass the Mediterranean by way of North Africa and the Middle East, travelling in a direction exactly opposite to the transmission of Arabic numerals.

Shape and Geometry

The invention of number systems facilitated the development of written mathematics, arithmetic, and computation. The operations of addition, subtraction, multiplication, division, and even the rather advanced concept of prime numbers seem to have sprung up independently among various ancient civilizations. The subsequent unification of notation spurred mathematicians onto deeper and deeper discoveries and inventions. Similarly the science of geometry emerged at first from the basic human intuition for shape. People across the world must have recognized from the beginning that there is something fundamentally different between the form of the moon and the tall trees. It is easy to imagine that geometry was the product of observations of this sort in the natural world.

The earliest notion of a straight line seems to have been derived from the experience of a tightly tied rope. In fact the Greek word *hypotenuse* simply means *to stretch*. It is clear how the word for *arms* came to stand for the two sides of a right triangle. In this way, the concept of a triangle seems to have originated in our experience of our bodies. Coincidentally, it was exactly the same in China, where the words for *hook* and *thigh* were adopted to represent, respectively, the shorter and longer sides of a right-angled triangle. This is why the Chinese word for the Phythagorean theorem translates directly into English as *hook-thigh-theorem*. Ancient pottery pieces discovered at the Banpo archaeological site in Xi'an, China, show patterns of congruent triangles, each side connected by eight small holes separated by equal intervals. Tomb murals in Thebes, the ancient capital of Egypt, also include geometric figures made up of straight lines, triangles, and curvilinear arcs. Presumably the other basic geometric structures, circles, squares, and rectangles were also the byproducts of everyday experience.

The Ancient Greek historian Herodotus (ca. 480–425 BCE) described geometry as the gift of the Nile. As early as the fourteenth century BCE, the pharaohs in Egypt were already carrying out large-scale projects in land surveying and taking a census, such that every citizen was allotted a portion of land equal in area to every other. When the Nile flooded in spring, anyone whose land was affected was to report the loss to the pharaoh, who would send a specialist to measure the extent of the loss and calculate a corresponding decrease in its taxation. This process marked the birth of geometry proper in the ancient world. The word *geometry* itself means *the measurement of the earth* in Greek, and the specialists in its practical application were known as *rope stretchers* (Fig. 1.6).

Demand for precise measurements of land also led to the development of geometry in Babylon. The characteristic feature of Babylonian geometry was its arithmetic nature. As early as 1600 BCE, the Babylonians were comfortable with calculations about rectangles, right triangles, isosceles triangles, and the areas of various trapezoids. In India, the development of geometry was connected with religion and architecture. The *Sulbasutras*, written sometime between the eighth century BCE and the second century CE, record a method using ropes to solve geometric problems and facilitate the construction of temples and altars. In China,

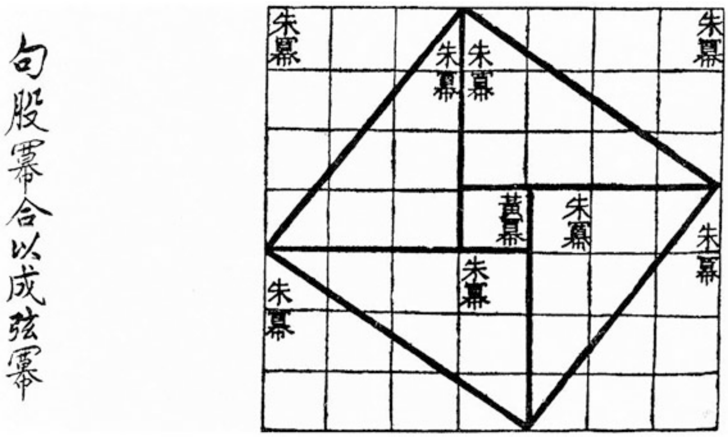

Fig. 1.6 Diagram from the Zhoubi Suanjing, containing the smallest Pythagorean triple (3,4,5)

the primary motivation for the development of geometry came from astronomical observations. The text *Zhoubi Suanjing* written around the first century BCE detailed various geometrical approaches to astronomy.

Civilization on the Nile River

A Peculiar Terrain

According to the European conception of geography, the Near East, or Middle East, designates the eastern coast of the Mediterranean and also includes the Asiatic part of Turkey and North Africa, that is, the Mediterranean and its nearby vicinity from the Black Sea to the Sea of Gibraltar. The Near East is both the cradle of human civilization and the birthplace of western civilization. The American historian of mathematics Morris Kline observed:

> While the more restless abandoned this birthplace to roam the plains of Europe, their kinsman remained behind to found civilizations and cultures. Many centuries later the wise men of the East had to assume the task of educating their still untutored relatives.

Egypt lies at the southeast corner of the Mediterranean, the intersection of the Middle East and North Africa. To the west and to the south is the Sahara, the largest desert in the world. The east and the north are for the most part closed in by the waters of the Red Sea and the Mediterranean. The only passage by land from out of Egypt is through the Sinai Peninsula, an area of only 60,000 square kilometers sandwiched between the Gulf of Aqaba and Gulf of Suez on its east and west sides, respectively. This peninsula is moreover covered mostly by desert and high mountains. It is therefore only a narrow passage that connects Egypt to

Fig. 1.7 Map of Egypt

Israel. Although Roman leaders such as Julius Caesar were able to invade Egypt via this route, such an occurrence in ancient times would have been almost impossible. Egypt was therefore able to maintain its stability across a long period (Fig. 1.7).

In addition to its protective geographic barrier, Egypt also boasts the clear waters of the Nile river, the longest river in the world; this river runs through the entire territory of Egypt from south to north until at last it flows into the Mediterranean Sea, forming in its path a long and fertile valley. This has been called the largest oasis in the world, flanked as it is by the vast Sahara to the west and Arabia to the east. In fact, the original Greek meaning of the word *nile* is valley or river valley. It is precisely because of these two special geographic features that the ancient Egyptian civilization, home of hieroglyphics and massive pyramids, could last for some 3000 years (FIg. 1.8).

The hieroglyphics were developed before the year 3000 BCE. These were a completely pictorial system of writing, later split into the hieratic or priestly and demotic or popular scripts in order to facilitate easier use. Around the third century CE, concomitant to the rise of Christianity, the ancient religions of Egypt

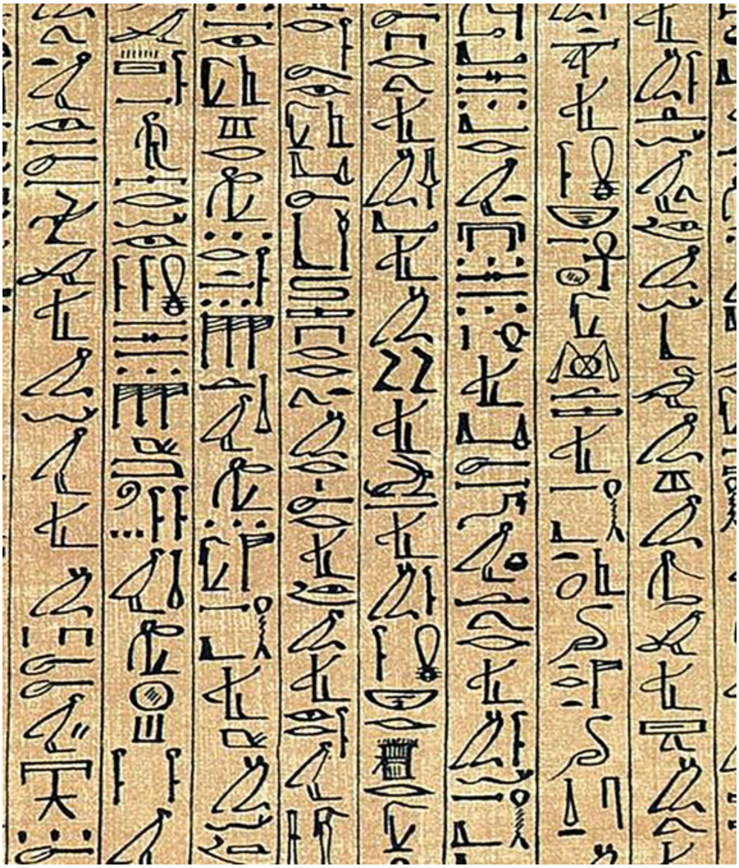

Fig. 1.8 Hieroglyphs on papyrus

faded away, and with them the hieroglyphs too died out. The last occurrence of hieroglyphs among extant artifacts is an inscription from the year 394 CE. Christian Egyptians switched instead to a somewhat modified Greek alphabet and, later, to Arabic, after the Muslim invasion in the seventh century. The hieroglyphs became an enigma.

In 1799, French soldiers accompanying Napoleon on his campaign in Egypt discovered a stone monument with an area no larger than a single square meter in Rosetta, an ancient port city not far from Alexandria. The stone was engraved with three copies of a single inscription, one in hieroglyphics, one in the demotic script, and one in Ancient Greek. Building upon earlier work by the British doctor and physicist Thomas Young (1773–1829), the French historian and linguist Jean-François Champollion (1790–1832) finally completed the interpretation of all three inscriptions. This opened the door for modern people to begin reading hieroglyphics and the demotic literature and to gain thereby a better understanding of Egyptian civilization, including Egyptian mathematics. The stone tablet has since come to be known as the famous Rosetta Stone; it has been on display on the British Museum in London since 1802.

The Rhind Papyrus

If ever you have the chance to visit Cairo, in addition to visiting the pyramids and the museums, taking a boat along the Nile, and taking in the tradition belly dancing, your friend or guide will also take you to the marketplace and workshops where papyrus is made and sold (often these two activities are combined into one). The papyrus plant is a wetland sedge that grows in the Nile delta. After it is picked, the pith at the center of its stem is cut into long narrow strips, pressed together, and dried out to form a smooth and thin writing surface. The ancient Egyptians did their writing on such paper, and it was later adopted as well by both the Greeks and the Romans. It was only in the third century CE that papyrus was replaced by parchment paper (produced in what is today Turkey), which is cheaper and able to be written upon on two sides rather than one. Parchment was in use up until the eighth century.

What we call papyrus books consist of texts written and bound in papyrus. The knowledge we have today of mathematics in ancient Egypt derives in the main from two papyrus books. The first of these is the Rhind Mathematical Papyrus, named after the Scottish lawyer and antiquarian Alexander Henry Rhind (1833–1863); it too is housed today in the British Museum in London. The other is the Moscow Mathematical Papyrus, purchased in Thebes by the Russian Egyptologist Vladimir Golenischev (1856–1947); today it is in the Pushkin State Museum of Fine Arts in Moscow. The Rhind papyrus is sometimes also referred to as the Ahmes papyrus, in honor of the scribe who copied out this text in about 1650 BCE. This makes Ahmes the first person in human history to make a name for himself for his contributions to mathematics. The whole volume is 525 cm long and 33 cm wide, with a few fragments missing from the middle; these missing fragments can be seen today in the Brooklyn Museum in New York City (Fig. 1.9).

Both the Rhind papyrus and the Moscow papyrus are written in the hieratic script, and both are very old. In his prefatory remarks, Ahmes states that the text had been in circulation for at least two centuries before he made his copy of it; for the Moscow papyrus, the modern research consensus is that it was written around 1890 BCE. These two books can therefore safely be regarded as the oldest works of written mathematics. In terms of content, they are essentially just compilations of various types of mathematical problem. The main text of the Rhind papyrus consists of 85 questions, and the Moscow papyrus 25. Most of these problems are drawn from real-life applications, for example, the composition of bread and the composition of beer, the feed ratio of cattle and poultry, and grain storage. These problems were compiled by the authors as demonstrative examples (Fig. 1.10).

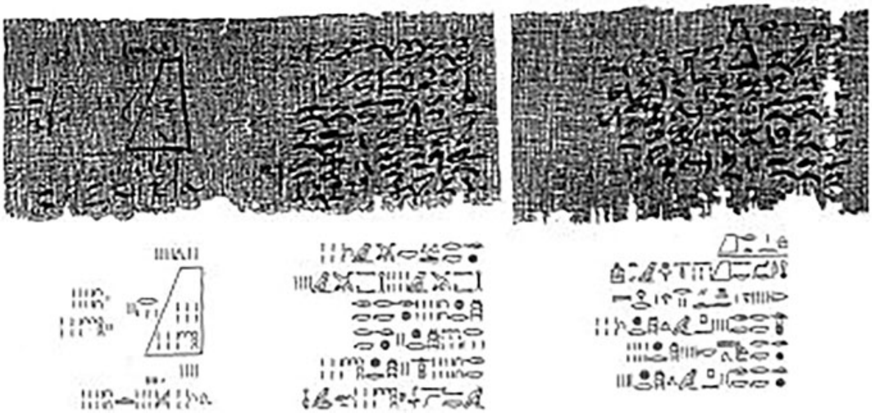

Fig. 1.9 Detail from the Moscow papyrus

Fig. 1.10 Scottish antique dealer Alexander Henry Rhind

We consider next the achievements of the Egyptians in geometry, the mathematical gift of the Nile. In one ancient land contract, we have a formula for the area of an arbitrary quadrilateral: if a, b, c, and d represent opposite sides of the quadrilateral and S its area, then this formula is

$$S = \frac{(a+b)(c+d)}{4}.$$

This is a very bold but rough attempt at a general formula; only in the special case where the given quadrilateral is rectangular is it correct. There is also a calculation for the area of a circle: in the 50th question in the Rhind papyrus, a circle with a diameter given by 9 is said to have an area equal to a square with sidelength 8. From this we can conclude that the Egyptian approximation of π (if they had any such concept) is

$$(8 \times \frac{2}{9})^2 \approx 3.1605.$$

In light of these rather imprecise approximations, it is perhaps surprising that the Egyptians achieved a very high level of accuracy in the determination of solid volumes, which problem they considered in connection with the storage of foodstuffs. For example, they knew already that the volume of a cylinder is the product of its base and its height. For the volume of a square frustum with height h and top and base a and b, respectively, Question 14 of the Moscow papyrus gives the formula

$$V = \frac{h}{3}(a^2 + ab + b^2).$$

This conclusion is correct, a remarkable achievement. The American historian of mathematics Eric Temple Bell called this the greatest pyramid.

Egyptian Fractions

In the Stone Age, the only numbers of which people had need were the integers; but advancing into the Bronze Age, the concepts of fraction and sign came into play. From the papyrus books, we observe an interesting and important feature of the Egyptian treatment of fractions: that is, they favored the use of unit fractions (fractions of the form $\frac{1}{n}$ with n an integer), and they would express any rational number smaller than 1 as a sum of unit fractions. For example,

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{15},$$
$$\frac{7}{29} = \frac{1}{6} + \frac{1}{24} + \frac{1}{58} + \frac{1}{87} + \frac{1}{232}.$$

It is not clear today why the Egyptians maintained such a preference for unit fractions, especially in light of the fact that the familiar four basic arithmetic operations are quite tedious to carry out in terms of such fractions. In fact, it is precisely because of this that later generations of mathematicians identified an important mathematical problem as the problem of Egyptian fractions; this is the most substantial mathematical problem associated with the Rhind papyrus. The problem of Egyptian fractions belongs to a branch of theory known as indefinite equations (or Diophantine equations, after Diophantus, the last great mathematician of ancient Greece). This problem concerns the solutions in positive integers to equations of the form

$$\frac{4}{n} = \frac{1}{x_1} + \cdots + \frac{1}{x_k}.$$

In fact, Egyptian fractions have given rise to a great number of mathematical problems, many of which have not yet been resolved, and this topic continues to this day to generate new questions. It is no exaggeration to say that in any given year, they are the research topic of any number of master's theses and doctoral dissertations; professional mathematicians also continue to carry out research into this issue. We present here a few examples. In 1948, the Hungarian mathematician Paul Erdős (1913–1996), who shared the 1983/1984 Wolf Prize in Mathematics with Shiing-Shen Chern, conjectured together with the German-American mathematician and assistant to Einstein Ernst G. Strauss (1922–1983) that the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has solutions in integers for all $n > 1$.

It is not hard to see that it is only necessary to verify the conjecture when $n = p$ is a prime number. The American-born British mathematician Louis J. Mordell (1888–1972), who was awarded a Fields Medal for his proof of one of his own conjectures, obtained a partial proof. He showed the result holds for all $n > 1$ except for $n \equiv 1$, 11^2, 13^2, 17^2, 19^2, or $23^2 \pmod{840}$, where $a \equiv b \pmod{m}$ (pronounces a is congruent to b modulo m) means that m divides $a - b$. For $n \equiv 2 \pmod{3}$, we can also produce an explicit solution

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{\frac{n-2}{3} + 1} + \frac{1}{n\left(\frac{n-2}{3} + 1\right)}.$$

The conjecture has also been verified by inspection for all $n < 10^{14}$.

It is a natural extension of this problem to consider also the equation

$$\frac{5}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Fig. 1.11 Egyptian license plates feature two different numeral systems; photograph by the author, in Cairo

In 1956, the Polish mathematician Wacław Sierpiński (1882–1969) proposed as a conjecture that this equation too has solutions for all $n > 1$. It has since been verified that this conjecture holds for all $n < 10^9$ and all $n > 1$ not of the form $n = 1 + 278,460k$ for some $k \geq 1$ (Fig. 1.11).

Apart from these partial results, it seems that a complete resolution of the two problems just mentioned is still quite out of reach. I have presented them here in some detail only to emphasize that the mathematics of the ancient Egyptians is not so simple as we might think; also I hope that this example shows how in mathematics it often happens that studying some classic and apparently simple problems leads to new insight all the way through to the present day and modern civilization. Another good example of this phenomenon is Fermat's Last Theorem; this theorem was first proposed by a French mathematician in the seventeenth century as he was reading the mathematical works of a third-century Greek. Here we see the truth behind the saying from Ezra Pound, the twentieth-century American poet and leader of the modernist movement in poetry, that the most ancient is also the most contemporary.

Between the Rivers

Babylonia

The waters of the Nile flow gently through Egypt from the capital city Cairo all the way to where it enters the sea; in contrast, the Tigris river that flows through Baghdad and its companion the Euphrates river are turbulent and rough. The people who lived in between these rivers too suffered through many wars and hardships. This region is known by its ancient Greek name Mesopotamia, meaning simply between the rivers, situated in the main in modern-day Iraq. According to surviving records, Mesopotamia was invaded and occupied across its history by more than a

Fig. 1.12 Sumerian cylinder seals; photograph by the author, in Baghdad

dozen foreign nations. Nevertheless, the people of this region maintained a highly unified culture; the economy developed rapidly during times of peace, and it was always a necessary destination along the routes of travelling merchant caravans. Three great civilizations emerged in Mesopotamia: the Sumerian, Babylonian, and Neo-Babylonian civilizations. All of them were characterized by the use of a special kind of writing called cuneiform, which undoubtedly played an important role in the cultural unity of the Mesopotamian peoples (Fig. 1.12).

Babylonia was located in the southeastern part of Mesopotamia, containing modern Baghdad, with the Persian Gulf to its south. The capital city of this territory during its greatest ages was the city of Babylon, for which reason the whole area is sometimes also referred to as Babylon. Like the Egyptians, the Babylonians lived on the banks of the river, where the land was fertile and easy to immigrate, and gave birth to a splendid civilization. In addition to cuneiform writing, the Babylonians produced the earliest known legal codes, established flourishing city-states, and developed a wide array of technological inventions, including pottery wheels, sailboats, and ploughs. They were also capable and pioneering architects, as attested in legend to by the famous Hanging Gardens of Babylon and the Tower of Babel. The editors of the Encyclopedia Brittanica have cited the influence of Babylonian literature, music, and architectural style on all of western civilization.

With respect to counting, too, the Babylonians were creative and original. They used a sexagesimal system (base 60), written in an interesting notation consisting only of a vertical downward wedge and a horizontal leftward wedge, the permutations and combinations of which were sufficient to represent every natural number. The familiar division of the day into 24 hours, with 60 minutes to the hour and 60 seconds to the minute, is also due to the Babylonians; this division subsequently spread across the globe and has been in use now for more than 4000 years.

Their writing system was different from that of the Egyptians, who preferred to write on papyrus. The Mesopotamians instead used pointed reed pipes to carve their wedge-shaped characters on wet clay, which they dried in the sun or by direct application of heat. They clay texts composed in this way were more durable than papyrus, and the cursive script holds its form easily. To date, more than 500,000 texts have been unearthed, and these form the main documents and research objects through which we have studied the Babylonian civilization. On the other hand, the modern interpretation of Babylonian cuneiform writing was accomplished later than that of Egyptian hieroglyphics. It was completed around the mid-nineteenth century, via the writings on a rock relief in a stone cliff on Mount Behistun, near the city of Kermanshah in western Iran, not far from its border with Iraq.

Similarly to the Rosetta Stone, the Behistun inscriptions consist of a single text engraved in three different cuneiform languages: Babylonian, Old Persian, and Elamite, all three of which later were forgotten together. The first person to decipher the inscriptions was a British officer named Sir Henry Rawlinson (1810–1895). In 1835, at the age of 23, he and other officers of the British East India Company army were sent to Persia to assist and reorganize the troops of the Shah of Iran. He became interested in Persian inscriptions and local monuments, and he used his knowledge of the Persian language to transcribe and eventually decipher the Old Persian text. With this completed, it became possible to decipher and read completely the entirety of the inscriptions.

The text relates the rise to the throne of Darius I, commonly referred to as Darius the Great, the most famous king of the Persian empire; Darius ascended to the throne by overthrowing its legitimate heir and putting down a long succession of rebellions. It is related by Herodotus that Darius died after learning of the defeat of his army in the famous battle at Marathon, during the first Persian invasion of Greece.

In any case, even after the Babylonian written language had successfully been deciphered, it was not until the 1930s and 1940s that the first breakthroughs were achieved in the interpretation of the mathematical contents of the surviving clay documents.

The Clay Tablets

Among the 500,000 clay tablet works so far discovered, more than 300 are mathematical texts, and our understanding today of Babylonian mathematical achievement is based on these sources. As we have already seen, the Babylonians used a sexagesimal system of cuneiform numerals, represented by repeated short lines or circles, and divided the hours and minutes into 60 units. Unlike the Egyptians, the Babylonians used a positional numeral system, the first such system in recorded history. This is a remarkable achievement. Later, they even extended this notation

Fig. 1.13 Cuneiform writing on clay tablets

to fractions other than the integers. In particular, they were not restricted in their notion of fractions to unit fractions, as were the Egyptians (FIg. 1.13).

The Babylonians were also more sophisticated in their arithmetic than the Egyptians. They developed many effective algorithms, for example, for the computation of square roots as follows: let a_1 be a first approximation to $\sqrt{a}$, let $b_1 = \frac{a}{a_1}$, and set $a_2 = \frac{a_1+b_1}{2}$; the continuing, $b_2 = \frac{a}{a_2}$, $a_3 = \frac{a_2+b_2}{2}$, and so on. The numbers a_n will oscillate around the true value of $\sqrt{a}$, approaching every closer to it with each iteration. In one clay tablet (No. 7289) in the collection of Yale University in the United States, the square root of two is calculated by this method as

$$\sqrt{2} \approx 1 + \frac{24}{60} + \frac{50}{60^2} + \frac{10}{60^3} = 1.41421296\ldots,$$

an approximation not at all far off from the true value

$$\sqrt{2} = 1.41421356\ldots.$$

The Babylonians also achieved impressive results in algebra. The Egyptians were able to solve on linear equations and the simplest quadratic equations of the form

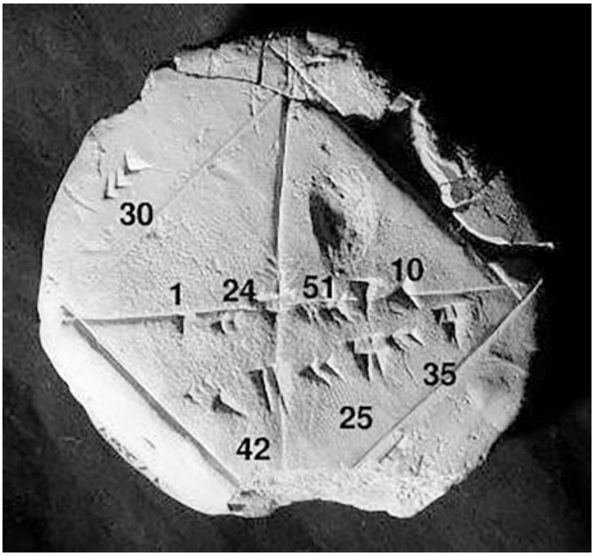

Fig. 1.14 Ancient Babylonian approximation for $\sqrt{2}$, approximate to five decimal places

$ax^2 = b$. By contrast, another clay tablet in the Yale collection gives a process equivalent to the general formula for quadratic equations of the form $x^2 - px - q = 0$

$$x = \sqrt{\left(\frac{p}{2}\right)^2 + q} + \frac{p}{2},$$

as well as correct solutions to the two remaining cases of monic quadratic polynomials with at least one negative coefficient; a quadratic polynomial with all positive coefficients, of course, does not have a positive root (Fig. 1.14).

This result, which we might well call the *Babylonian formula*, can be considered a precursor to the general relationship between the roots and coefficients of a polynomials obtained in the sixteenth century by the French mathematician François Viète, which is valid in the quadratic case for the more general equation $ax^2 + bx + c = 0$.

As for cubic equations, the Babylonians were not able to work out a general solution, even for special cases such as $x^3 = a$ and $x^3 + x^2 = a$; nevertheless, they compiled the solutions to these special cases in tables, the former being simply the table of cube roots.

When it comes to geometry, however, the Babylonians did not surpass the Egyptians. Their estimate for the area of a quadrilateral, for example, is consistent with the Egyptian approximation, which we have seen already to be very rough. For the area of a circle, they seem to have generally assumed its value to be three times the square of the radius, corresponding in modern notation to an approximation $\pi \approx 3$; the Egyptians did better. There is some evidence that the Babylonians knew how to use to concept of similar figures to determine the length of a line segment, and they had a formula for the volume of a square frustum similar to the formula in the Moscow papyrus praised by E.T. Bell as the greatest pyramid.

Plimpton 322

There are some problems in the clay tablet texts that suggest that the Babylonians sustained a theoretical interest in mathematics separate from its practical applications, which does not seem to have been true of the Egyptians. A good example of this is the clay tablet known as Plimpton 322. The origins of this tablet are uncertain, but the publisher George Arthur Plimpton bought it from an archaeological dealer in around 1922 and subsequently bequeathed it along with the rest of his collection to Columbia University in New York; the number 322 refers to its index in this collection. In fact, it is possible that this tablet is part of a larger document, since the left side is broken and contains traces of glue, suggesting that a defective portion was left behind when the tablet was unearthed (Fig. 1.15).

The remaining plate is small, with length and width only 12.7 cm and 8.8 cm, respectively. The text written on it is in Babylonian, which puts the date of its composition as late as 1600 BCE. The contents consist only of a table containing 4 columns and 15 rows, a total of 60 sexagesimal numbers. Because of this, it was at first assumed to be a simple bit of accounting, and for a long time, it was not taken seriously. It was only in 1945 that Otto E. Neugebauer (1899–1990), at that

Fig. 1.15 Plimpton 322

time editor of the journal *Mathematical Reviews* under the auspices of the American Mathematical Society, recognized the number theoretical significance of Plimpton 322. Since then, Plimpton 322 has attracted substantial interest.

Neugebauer noticed that table in Plimpton 322 is related to Pythagorean triplets. A Pythagorean triplet consists of three positive integers a, b, c, satisfying the equation

$$a^2 + b^2 = c^2.$$

The smallest three such numbers are $a = 3$, $b = 4$, and $c = 5$. In ancient China, such numbers were also known as right triangle numbers; from the geometrical perspective, such numbers represent the lengths of the three sides of a right triangle with integer-valued sidelengths. Neugebauer had observed that the corresponding numbers in the second and third columns of the table give the lengths of the hypotenuse c and leg b of a right triangle. There were also four deviations from this rule, which Neugebauer attributed to a clerical error and corrected.

As an example, the middle columns of rows 1, 5, and 11 of the table (in sexagesimal) are $(1, 59; 2, 49)$, $(1, 5; 1, 37)$, and $(45; 1, 15)$, respectively. Converting this into decimal gives $(119, 169)$, $(65, 97)$, $(45, 75)$, which correspond to sidelengths of right triangles with third side given by 120, 72, and 60, respectively, all integers. After filling in the missing length, Neugebauer found that the number in the first column is obtained as $s = (a/c)^2$ (the number in the final column is simply the row index); in words, s is the square of the secant of the angle opposite side b. If we write B for the angle opposite B, we have

$$s = \csc^2 B.$$

Therefore this column actually gives a table of squared secants for angles from 31° to 45° at 1° increments.

The Pythagorean triplets (a, b, c) in the table can be determined from the parametrization $a = 2uv$, $b = u^2 - v^2$, $c = u^2 + v^2$ where u and v are relatively prime with one even and the other odd. It is a mystery, however, how the Babylonians calculated these numbers, whether by this method or another.

Since the mathematical reputation of the Babylonians today rests so much upon this discovery by Neugebauer, I would like to introduce in a bit more detail this figure in the history of mathematics. Otto E. Neugebauer was born in Innsbruck, Austria, in the last year of the nineteenth century. His parents died when he was still young, and he was raised by his uncle. During World War I, in order to avoid his graduation examinations, Neugebauer enlisted in the Austrian Army as an artilleryman, and he spent the end of the war in an Italian prisoner-of-war camp alongside his countryman the philosopher Ludwig Wittgenstein. After the end of the war, he enrolled in several different universities to study mathematics and physics, before he undertook a study of the history of mathematics at Göttingen University. After graduating, he eventually emigrated to America and took American citizenship. He became a professor at Brown University, where he remained for many years, until he moved to the Institute for Advanced Study in Princeton. He was proficient in both ancient Egyptian and Babylonian and founded journals in both Germany and the United States.

Conclusion

In addition to their mathematical achievements detailed above, the Egyptians and the Babylonians also made extensive use of practical mathematics in their daily lives. They recorded accounts, promissory notes, debts, sales receipts, mortgage contracts, outstanding payments, and the distribution of profits in papyrus or in clay, respectively. They made use of algebra and arithmetic in their commercial transactions and of geometric formulas in the calculation of area, the reckoning of their canals, and the storage of their foodstuffs in circular or conic silos. And of course in architecture, the great Egyptian pyramids, the fabled Hanging Gardens of Babylon, and the Tower of Babel, all attest to mathematical wisdom and skill.

Prior to the use of mathematics and astronomy to calculate calendars and facilitate navigation, humankind possessed for countless eras an instinctive relation of fear and curiosity toward nature. They observed and considered the movement sun and the stars and the moon year after year. The Egyptians knew already that there are 365 days in the year and understood and mastered the niceties of seasonal change. Through observations of the position and angle of the sun, they could predict the flooding of the Nile; from the position and direction of the stars, they could determine the orientation of a ship in the Mediterranean or Red Sea. The Babylonians could not only predict the positions of the planets from day to day but also determine the time of the number to within an accuracy of a few minutes.

In another direction, the connection in both Babylonia and Egypt between mathematics and painting, architecture, religion, and the mysteries of the natural world was not as significant than the application of mathematics to commerce and agriculture. The priests of Babylon and Egypt may well have mastered general mathematical principles, but they kept this knowledge secret and communicated it only verbally, perhaps in order to increase the sense of awe among the common people toward the ruling class. From this perspective, especially in comparison with a civilization without a ruling priestly caste, conditions seem to have been in some ways unfavorable for the development and dissemination of advanced mathematics (Fig. 1.16).

Of course, the numbers and the nature of number play a role in the history of religious mysticism and its expression. It is generally believed that numerology and the mystical or even magical attitude toward number was a Babylonian innovation, later transmitted to the Hebrew people. For example, the number seven was first emphasized by the Babylonians as a point of harmony between the power of the gods and the complexity of nature. In the hands of the Hebrews, seven became the number of the days in a week, and it of course plays a fundamental role in their

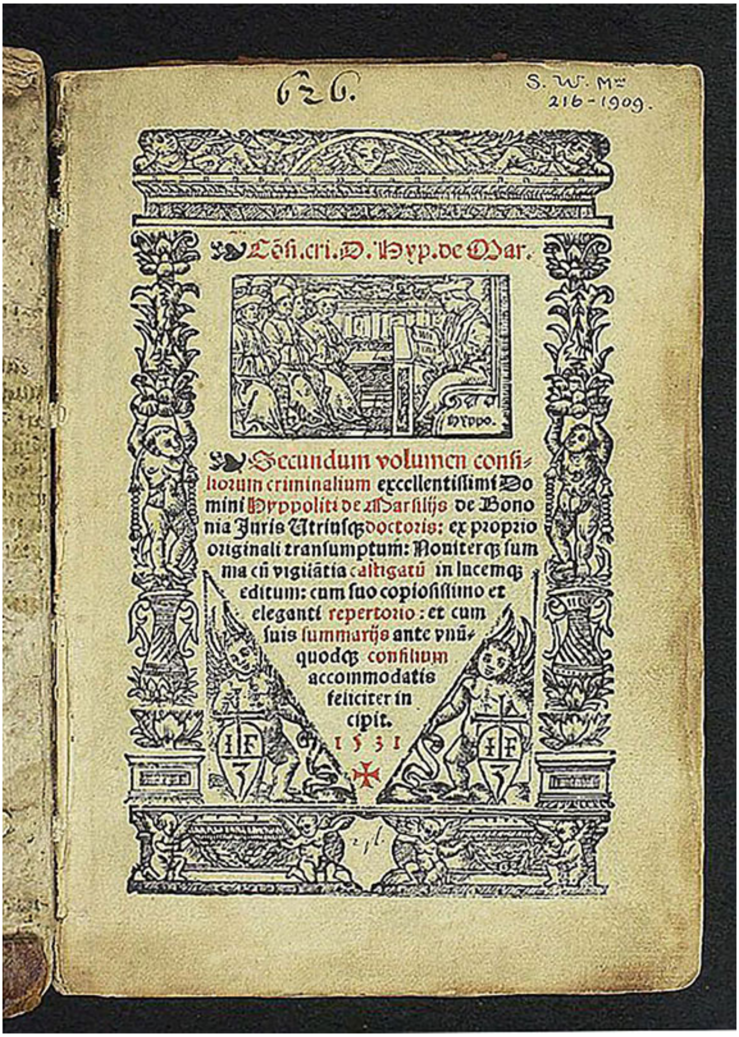

Fig. 1.16 Latin Bible from 1531

sacred text, which states that God created the world and its inhabitants in 6 days, and on the seventh day he rested.

There are also some outstanding mysteries. For example, why did the Babylonians divide the circle in 360 degrees? This seems to have been an innovation from the last century BCE, and not a straightforward artifact of the use of sexagesimal and concomitant to the division of hours and minutes into units of 60. In the second century CE, the influential Greek astronomer Ptolemy (ca. 100 CE–ca. 170 CE) accepted and adopted the Babylonian division of the circle, and it has remained in use ever since. Another the Egyptians used their knowledge of geometry and astronomy to build their temples such that on the longest day of the year sunlight would enter into the temple and illuminate the gods on the altar. The orientation of the pyramid with respect to the space and the eastward facing sphinx seem also to have been intentional.

It seems that the endless needs and interests of human beings, and our inborn tendency to meditate upon the sky, give birth to the potential for mathematical inspiration. Nature itself seems to contain somehow the laws of mathematics, or rather the laws of nature take on the form of mathematics. Plato is said to have claimed that God is always doing geometry, and Jacobi later commented on this with the revision that God is always doing arithmetic. They both seem to have meant that the creation of world itself is a mathematical phenomenon. In this way we can consider that mathematics not only derives from the needs of human survival but also brings about a return into this world.

Unfortunately, both Egypt and Babylon suffered constant invasion by external forces throughout history, and there have been frequent changes in culture and civilization in the Middle East in the subsequent millenia. In particular, in the middle of the seventh century CE, Arabic rule reoriented the languages and religious customs throughout the region. Later, both of these territories can be said to have gone through a rocky transition to modernity, and neither was able to quickly reach a high level of development and economic productivity, although Iraq is home to the second largest concentration of oil reserves in the world. Since the advent of the twenty-first century, there has been back to back the Iraq war and the Jasmine Revolution. In any case, it follows that the mathematical and cultural development in a region at one point in history serves as no guarantee of the continuous and effective development of its economy and society.

Chapter 2
The Sages of Ancient Greece

Whatever we Greeks receive, we improve and perfect it.

Plato

The Birth of Mathematicians

The Greek Arena

Sometime around the seventh century BCE, Greek civilization emerged in modern-day southern Italy, Greece, and Asia Minor (the western part of Turkey in Asia). This civilization was different in many respects from the Egyptian and Babylonian civilizations discussed in the previous chapter. The British writer H.G. Wells (1866–1946) observed that Egypt and Babylonia developed over a long and slow period, starting as primitive agricultural societies and eventually building up around a culture of temples and the priestly caste; the nomadic Greeks, on the other hand, came into their territory from abroad, and the peoples of the land they occupied were already developed in the ways of agriculture, seafaring, the formation of city-states, and even writing. So the Greeks did not so much grow up with their civilization but rather tore down another and set up their own atop its ruins. It is perhaps for this reason that much later the Greeks were able to accept conquest by the Macedonians who readily assimilated their invaders into their own culture.

Bertrand Russell remarks in his discussion of the Egyptian and Babylonian civilizations that the free pursuit of intellectual activity was restrained by their religious preoccupations. The Egyptian religion was obsessed with the afterlife, and the greatest monuments of Egypt, its pyramids, were tombs. The Babylonian interest in religion was rather in service of prosperity in this world; they recorded with care the movements of the stars and performed various rituals and divinations for this purpose. In Greece, however, there were no figures equivalent to a prophet or a high priest and no concept of a singular and omniscient god presiding over all creation. The Greeks, perhaps in light of their nomadic origins, had a pioneering spirit. They were unwilling to bind themselves too closely to tradition and preferred instead

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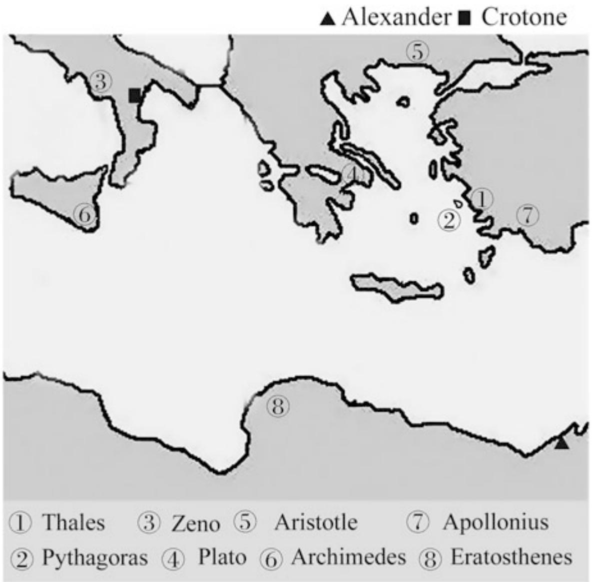


Fig. 2.1 Birthplaces of ancient Greek mathematicians, as drawn by the author

to come into contact with new ideas and learn from them. As one example, the Greeks quietly changed the hieroglyphics they had originally used to adopt instead the Phoenician phonetic alphabet (Fig. 2.1).

Another significant factor in the development of the culture of ancient Greece is its geography. Any tourist who has ever visited can attest that the land is rugged and uneven, partitioned by barren mountains, and that land transportation is by no means convenient. There are no smooth rivers or river systems and only a few fertile plains. When it was not possible to sustain its population, the people of Greece would cross the sea and establish new colonies. Greek towns pepper the landscape from Sicily and southern Italy to the shores of the Black Sea. The level of emigration necessitated the development of regular nautical routes connecting the ports of the Eastern Mediterranean with the Black Sea in order to facilitate trade and visits home from abroad. Indeed, this phenomenon continues through to the present day, with a dense network of routes between Athens and the islands of the Aegean Sea. Following the early migration of many of the Cretans to Asia Minor on account of an earthquake, the Greeks came more and more into contact with the East.

Originally, Greece was relatively close to the civilizations of the two river valleys and much susceptible to their cultural influence. When the many Greek merchants and scholars who travelled to Egypt and Babylon made their return, they brought with them new mathematical knowledge, and in the atmosphere of rational enquiry peculiar to the Greek city-states, these empirical arithmetical and geometric rules were raised up into a system of logical structure and mathematical demonstration. People often put forward the questions, "why are the two base angles of an isosceles triangle equal?" and "why does the diameter of a circle divide it in half?" The American historian of mathematics Howard Eves (1911–2004) has pointed out that in the empirical tradition of the ancient East, the question *how* can be answered with confidence, but the more scientific question *why* is a source of more trepidation.

We close this section with a discussion of the city-states and political characteristics of ancient Greece. Unlike the vast and long unified eastern civilizations of the ancient world, the Greek city-states existed always in a state of stark separation from one another. This of course was a product of geography: the mountains and the sea scattered the people across distant coasts. As for the social structure of ancient Greece, there were in the main two distinct classes the nobility and the commoners (in some regions the aboriginal people were also present as farmers, craftsmen, or slaves). These two classes were not, however, completely isolated from one another, and they followed the same ruler into war, the ruler being simply the head of one of the noble families. The society so organized produced an environment ripe for democracy and rationalism.

All of these various factors set the Greeks up to play an important role on the stage of world civilization.

The First Proofs

History never lacks for coincidence and rhyme. The mathematicians and philosophers of ancient Greece poured out in a multitude, just as much later the writers and artists in Italy during the Renaissance period. If we consider 1266, 1 year after the great poet Dante Alighieri (1265–1321) was born in Florence, that city produced also Giotto (1266–1377), the most outstanding painter of his time. Italians generally date the start of the greatest period in the history of art by his life. The art historian Sir Ernst Gombridge (1909–2001) has argued that prior to Giotto, artists were viewed in the same light as a carpenter or tailor, and they even signed their works only infrequently; since Giotto, art history became the history of artists (Fig. 2.2).

The mathematicians made their debut quite a bit earlier. The first mathematician to make his name famous to future generations was Thales of Miletus (ca. 625–ca. 547 BCE), some eighteen centuries before Giotto. Thales was born in Miletus in Asia Minor (now near the mouth of the Maeander river in the Aydin province of southwestern Turkey). At that time it was the largest eastern city in Greece. Most of the residents in the region at that time were members of the Ionian tribe who had some time early spread out across the area; for this reason the whole territory was also referred to as Ionia. In the city of Miletus, the mercantile class had replaced the aristocracy as political leaders, and the intellectual air was more free and open, producing a wealth of notable figures in literature and what we would now call philosophy of science. Inherited legend has it that the poet Homer (ca. ninth century–ca. eighth century BCE) and the historian Herodotus were both native to Ionia (Fig. 2.3).

The life of Thales is related to us mainly by way of later philosophers. In his early years, he traveled on business to both Egypt and Babylon, where he quickly encountered and mastered their astronomy and mathematics. In addition to these

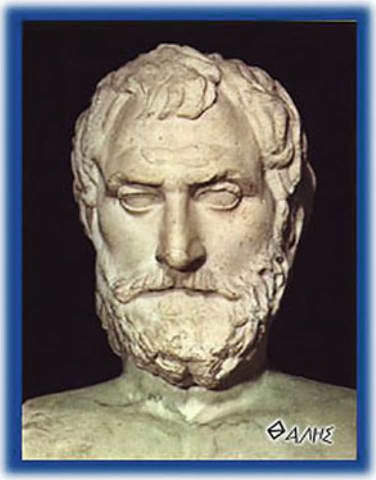


Fig. 2.2 Bust of Thales

Fig. 2.3 Remains of a columned Ionic Stoa at Miletus

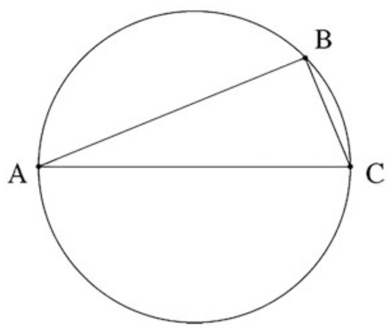
fields, he also carried out investigations in physics, engineering, and philosophy. Aristotle tells an amusing story: Thales predicted a plentiful olive harvest in a certain year on the basis of his agricultural knowledge and the meteorological data, so he put down a substantial deposit on all the olive presses in the area at a very low price. When his prediction proved true, he was able to let out again the presses for considerable profit, and he became wealthy. All this was not so as to become rich, but rather to refute some ridicule he had received: “If you’re so smart, why aren’t you rich?”

Plato recounts another amusing anecdote: at one time, Thales was looking up at the sky when he accidentally fell into a ditch. A beautiful woman passing by laughed and asked him how could he hope to know the sky when he could not even see what is at his feet? Thales made no response to this, but he was troubled to respond to another question put to him by Solon, archon of Athens. Solon had come to Miletus to visit Thales and one day asked him why he had never married (indeed, Thales may be the first of many renowned scholars who lived along his entire life). At the time, Thales made no answer. Some days later, Solon received news that his son had died in Athens, and he became distraught with grief. Thales came to him with a smile and, after telling him that the news was false, explained that the reason he had never married and had children was his fear of the pain of losing a loved one (Fig. 2.4).

The first historian of mathematics, Eudemus (ca. fourth century BCE), wrote that Thales first introduced the study of geometry from Egypt into Greece; moreover, he himself discovered many propositions and instructed his students to research the basic principles from which other propositions can be derived. According to legend, he measured the height of the pyramids in Egypt from their shadows by comparing human height to human shadows. One of Plato’s disciples alleged in writing that several propositions in plane geometry were first proved by Thales, namely, that the diameter of a circle divides the circle into two equal parts, that the base angles of an isosceles triangle are equal to one another, that the opposite angles formed by two intersecting lines are equal to one another, and that, if two triangles have two angles and the side between them equal, respectively, then the triangles are congruent. The most interesting result attributed to Thales is that the angle formed on the circumference of a circle by a triangle with its diameter as base is a right angle; this result is now known as Thales’s theorem.

But much more important than any single result, Thales introduced the concept of a proof of a mathematical proposition, the idea that a given proposition should be demonstrated by logical argumentation on the basis of axioms and more

Fig. 2.4 Thales’s Theorem



basic propositions the truth of which has already been established. This marks an extraordinary leap in the history of mathematics and set the precedent for mathematical demonstration. There is no primary source to prove that all of these achievements were due to Thales; rather, his contributions were passed down to the future as oral history, identifying him as the first mathematician proper and the founder of demonstrative geometry. The theorem named after him is also the first theorem in history to be named after a mathematician.

Thales achieved great things outside the discipline of mathematics as well. He developed a cosmology with water as the essence of all things, following the cycle of water evaporating into steam in the sunlight to form clouds and falling to the earth again as rain. His cosmology proved false, of course (he also believed the earth to be a disk floating in water), but he had dared to gaze into nature and develop his own system of thought. For this reason, he is recognized as one of the founding figures in Greek philosophy. In physics, Thales is also said to have first discovered the phenomenon of static electricity generated in amber by friction. Herodotus, himself known as the father of history, claimed that Thales had accurately predicted a solar eclipse, and Eudemus states that the division of the seasons according to the vernal equinox, the summer solstice, the autumnal equinox, and the winter solstice is not an equal division.

Pythagoras

Following after the leadership of Thales, Miletus produced two more significant philosophers, Anaximander (ca. 610 BCE–ca. 545 BCE) and Anaximenes (ca. 588 BCE–ca. 526 BCE), and the writer Hecataeus (ca. 550 BCE–ca. 476 BCE), who wrote the first travelogue, in a simple and beautiful prose style, and also produced pioneering work in geography and ethnography. Anaximander believed that the world is not composed in its essence of water, or any of the other elements, but rather a certain special basic substance whose form is not known to us. He thought also that the earth is a freely floating cylinder at the center of an infinite universe. He developed a methodology of *reductio ad absurdum* and applied it to conclude that humankind had evolved from marine life, with a period of time spent maturing within the bodies of fish. Anaximenes had yet another theory; he held air to be the primordial element, with all other matter formed from its various condensations and evacuations (Fig. 2.5).

Meanwhile in the Aegean Sea, just a stone's throw from Miletus, there is a small island known as Samos. The inhabitants of this island were of a more conservative bent than their neighbors on the mainland, and they practiced a kind of loose Orphism, without a strict dogma, frequently gathering into the fold people of similar beliefs. It may have been here that philosophy first came to be associated with a way of life, and the forerunner of this new philosopher was called Pythagoras of Samos (ca. 580 BCE–500 BCE). He left Samos as an adult and travelled to Miletus to study, but Thales turned him away on account of his advanced age, suggesting that

Fig. 2.5 Monument to Pythagoras at Samos

he go instead to Anaximander. Pythagoras quickly came to realize that philosophy in Miletus was a practical undertaking, contrary to his own preference for meditation and worldly detachment. Pythagoras held that humankind is naturally partitioned into three categories: the lowest of the three engage in sales and purchases, members of the middle category are suitable for participation in competitions such as the Olympic games, and the highest category contains the spectators of the world, the scholars or philosophers, as they came to be called.

Pythagoras abandoned Miletus and travelled alone to Egypt, where he lived for 10 years, and studied Egyptian mathematics. Later, he was taken captive by Persians in Egypt and transported to Babylon, where he spent another 5 years and mastered the more advanced mathematics that was known there. Including the duration of his journey home by boat, Pythagoras spent a total of 19 years abroad, longer than Faxian (337–322) of the Eastern Jin dynasty in China and Xuanzang (602–664) of the Tang dynasty spent in India studying the Buddhist teachings.

However, Samos was still too conservative to accommodate his way of thinking, and Pythagoras was obliged to travel across the ocean again to Croton, in southern Italy, where he settled down, married, had children, and recruited disciples. This was the birth of what is known as the Pythagorean School; although this was a very secretive society, subject to strict discipline, it was not influenced by existing religious dogmas. Rather, they created a scientific (but mainly mathematical) tradition that has retained its influence for some two millenia. Indeed, the words *philosophy* (φιλοσοφία) and *mathematics* (μάθημα) are said to have been coined by Pythagoras himself. The former means *love of wisdom*, and the latter *knowledge that can be learned*.

The mathematical achievements of the Pythagorean school include the Pythagorean theorem; the introduction of special categories of natural numbers such as perfect numbers, amicable numbers, triangular numbers, and Pythagorean triplets; the construction of the regular polyhedra; the irrationality of $\sqrt{2}$; the identification of the golden ratio; and so on. Some of these topics have not yet been resolved completely (e.g., the perfect numbers and amicable numbers); others appear routinely in all aspects of daily life, and still others have generated deep and modern flowers at the heart of mathematics, as, for example, the Pythagorean theorem can be said to have generated Fermat's Last Theorem. The Pythagorean School also emphasized harmony, order, and moderation, believing these to be fundamental goods, and at the same time valued the expression of form, proportion, and number.

In Greece, the birthplace of Homer, poetry was at the foundations of all education, and many families of means, including that of the chief magistrate of Samos, hired poets to serve as tutors to their children; in some cases poetry schools were also established. Pythagoras is alleged to have described the theorem that bears his name in verse:

The square of the hypotenuse,
If I have made no errors,
Is equal to the sum of squares
Of the remaining two sides.

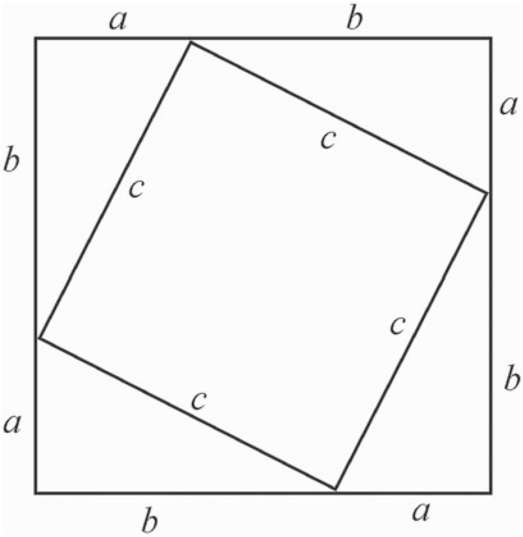
Pythagoras is also attributed with the first proof of this theorem, which had been discovered already by the Babylonians and the Chinese. He is said to have hugged his wife, who was mute, and shouted, "I finally found it!" He discovered also that the sum of the three interior angles of a triangle is equal to the sum of two right angles, and that the plane can be tiled with regular triangles, regular quadrilaterals, or regular hexagons. A result due to the later theory of tessellations states that it is impossible to tile the plane with any other regular polygons (Fig. 2.6).

As for the proof, one widespread theory is that he used a method of subdivision. If we let a, b, and c denote the two legs and the hypotenuse of a right triangle, respectively, as in the figure, we construct a square with sidelength $a+b$. This square is subdivided into five areas: a square with sidelength given by the hypotenuse c and four triangles congruent with the original right triangle. Computing the area in two ways and simplifying give

$$a^2 + b^2 = c^2.$$

When it comes to the natural numbers, the most interesting contributions due to Pythagoras were the definitions of perfect numbers and amicable numbers. A perfect number is a number equal to the sum of its proper divisors; for example, 6 and 28

Fig. 2.6 Proof of the Pythagorean theorem



are perfect numbers, since

$$6 = 1 + 2 + 3,$$
$$28 = 1 + 2 + 4 + 7 + 14.$$

Both of these numbers played an important role in ancient mysticism: in the Bible, it is written that God created the world in 6 days, with the seventh day set aside for rest; and the geocentric theory of the ancient Greeks held that the time required for the moon to revolve around the earth is 28 days. It was also believed as late as Copernicus that there are only six planets in the solar system. It is necessary to point out that there remain many unresolved questions concerning the perfect numbers. For example, people have discovered so far only 51 even perfect numbers, and not even a single odd perfect numbers, but nobody has been able to prove whether or not there are infinitely many perfect numbers or whether an odd perfect number exists.

It is not difficult to prove that the decimal representation of every perfect number terminates in either a 6 or an 8. The ancient Greeks believed that these two final digits alternate among successive perfect numbers, but this turned out not to be true. Nevertheless, the author of this book has noticed that among the first 50 perfect numbers, those ending with 6 and those ending with 8 occur 19 and 31 times, respectively, and the ratio 19 : 31 between them is very close to the golden ratio 0.618.... The numbers 19 and 31 are also the height and width in meters of the Parthenon in Athens. The golden ratio is another concept associated with Pythagoras, but he did not think to make any connection between it and the perfect numbers.

The amicable numbers consist of pairs of natural numbers such that each is equal to the sum of the proper divisors of the other, for example, the numbers 220 and

284 form an amicable pair. Among later generations, the amicable numbers were frequently associated with magic and astrology, and various mystical properties were attributed to them. The number 220 also makes an appearance in the Bible, as the number of sheep that Esau gave to his brother Jacob as an expression of his love. Nevertheless, it was more than 2000 years before the second pair of amicable numbers was discovered. These are 17926 and 18416, discovered by the French mathematician Pierre de Fermat; his contemporary and fellow countryman René Descartes discovered the third pair. Using modern computers and modern mathematical techniques, more than 100 million pairs of amicable numbers have since been found. The smallest pair after the first (1184 and 1210), however, was discovered by a 16-year-old Italian boy named B. Nicolò I. Paganini in the nineteenth century (Fig. 2.7).

But the greatest achievement that can be attributed to Pythagoras is the influence his thought continued to exert upon later generations. In the Middle Ages, he was regarded as the originator of the *quadrivium* of academic disciplines: arithmetic, geometry, music, and astronomy. During the Renaissance, his views on harmonious proportions and the golden ratio become important in aesthetics. At the start of the sixteenth century, Copernicus claimed the inheritance of the Pythagorean philosophical tradition for his proposed heliocentric model of the solar system, and Galileo, who discovered the law of freely falling bodies, was likewise considered to be a Pythagorean. In the seventeenth century, Leibniz, who created calculus, thought of himself as the last of the Pythagoreans.

Pythagoras in particular considered music to be the most effective source of purification in human life. He discovered the relationship between harmonious musical intervals and the ratios of the lengths of strings used to produce them. For example, if the length of a tuned string is halved, the note it produces will sound

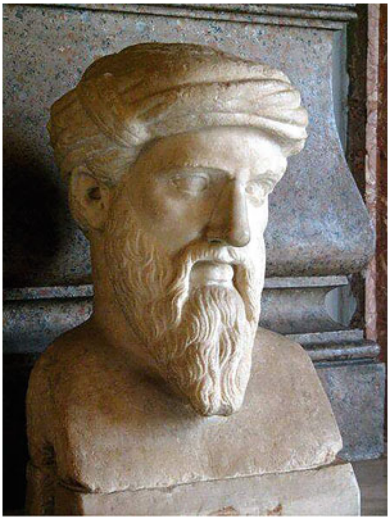


Fig. 2.7 Bust of Pythagoras, now in the Capitoline Museums in Rome

an octave higher than the original; if it is shorted by a ratio of two thirds, the ratio is a perfect fourth; and so on. The concepts of tuned strings and harmony played an important role in Greek philosophy. Harmony means balance, the adjustment and union of opposing forces, just as the proper tuning of an interval requires its adjustment upward and downward in musical space. Russell argued that such concepts of the Golden Mean in ethics and moral philosophy also bear the imprint of these musical discoveries.

Musical considerations also led Pythagoras to the conclusion that all things are number, the most important tenet of his philosophy, in stark contrast with the philosophies of the three philosophers of Miletus. Pythagoras held that whoever can master the structure of numbers can master also the world. Prior to this, mathematical interest arose mainly out of practical needs, as, for example, the Egyptians required a certain minimum of mathematics to measure their land and build their pyramids. But for Pythagoras, however, Herodotus tells us that the purpose of mathematics was simply in order to explore the world. We see this also in the Pythagorean nomenclature, *philosophy* and *mathematics*. Or, as another example, the original meaning of the word *calculation* is *the manipulation of stones* (Fig. 2.8).

Pythagoras referred to numbers as the language of the gods: whereas most of the material of human life and the human world is fleeting and ephemeral, passing

Fig. 2.8 Ruins of the Pythagorean Academy; photograph by the author, in Crotone

through for only a brief moment and then dying away again, only numbers and the gods are deathless and eternal. The rise of the digital age in the modern world seems to confirm some aspects of this Pythagorean hypothesis. On the other hand, it is perhaps regrettable that while numbers today lie at the heart of so much of the material world, the sacred or aesthetic aspect of nature remains underemphasized.

The Platonic Academy

Zeno's Tortoise

The Pythagorean School allied itself politically with the aristocratic system, and so it suffered a blow with the rise of Greek democracy, and gradually its influence collapsed. Pythagoras himself was forced to flee from Croton, and he was killed not long afterward. After the Greek victory in the first Greco-Persian war, the political, economic, and cultural center of Greece shifted to Athens, especially during the era of Pericles (ca. 495–ca. 429 BCE), who made significant contributions to the formation of Athenian political institutions and social development, including the construction of the Acropolis in 447 BCE (Fig. 2.9).

Greek mathematics and philosophy also prospered throughout this period, and many competing schools of thought sprang up. The first to gain prominence was the Eleatic school, which was established by Parmenides (ca. 515 BCE–ca. 445 BCE),

Fig. 2.9 Bust of Pericles

who had been a disciple of one of the latter day Pythagoreans. Parmenides lived in Elea in southern Italy (a bit more than a hundred kilometers southeast of modern-day Naples), and his views were most vigorously propounded by his student Zeno (ca. 490 BCE–ca. 425 BCE). These two, master and pupil, were the most accomplished thinkers of pre-Socratic Greece.

Parmenides was one of the few Greek philosophers to express his thought in verse; his only surviving work is a poem fragment whose title is not known for certain, but which is generally referred to as *On Nature*. Its first part is called "the way of truth" which includes a logical argument that attracted considerably the interest of later philosophers. Parmenides believed that the diversity of experience and its shifting forms are but manifestations of timeless, changeless, and uniform existence, encapsulated by the slogan, "everything is one." He argued also that whatever cannot be thought cannot exist, and whatever exists necessarily can be thought. This is in contrast to his philosophical predecessor Heraclitus (ca. 540 BCE–ca. 470 BCE), who put forward a paradoxical philosophy of the unity of opposites, being and nonbeing. Parmenides also introduced to philosophy the method of rational proof as the basis of judgment and is considered for this reason to be the founder of metaphysics. It is interesting to note that each of these three philosophers, Pythagoras, Heraclitus, and Parmenides, were considered to be overseas Ionians.

In his dialogue *Parmenides*, Plato describes in an ambiguous tenor a visit paid by Parmenides and his student Zeno to Athens. He wrote that "[Parmenides] was, at the time of his visit, about 65 years old, very white with age, but well favoured. Zeno was nearly 40 years of age, tall and fair to look upon; in the days of his youth he was reported to have been beloved by Parmenides.[1]" Later Greek scholars speculated that this visit was a fictitious invention, but accurately and reliably represented the philosophical views and arguments of the two visiting philosophers. Zeno in particular defended the difficult ontological theories of his teacher by way of a peculiar *reductio ad absurdum*, claiming that the view that things are many entails more ridiculous conclusions than the hypothesis that everything is one.

This method is the origin of the famous *Zeno's Paradoxes*. Taking as his starting point the two hypotheses that everything is many and motion is possible, Zeno developed in total some 40 different paradoxes, of which unfortunately only 8 have survived. The most famous are the paradoxes of motion, which are discussed in Aristotle's *Physics* and elsewhere. Even these famous paradoxes were not entirely appreciated by subsequent generations, who viewed and criticized them only as interesting fallacies and diversions. It was only in the second half of the nineteenth century that scholars reevaluated them and realized their close connection with such mathematical concepts as continuity and infinitude.

We introduce in turn the four paradoxes of motion presented by Zeno. The quotations below are taken directly from Aristotle's *Physics*.[2]

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[1] Tr. Benjamin Jowett

[2] Tr. R.P Hardie and R.K. Gaye

(1) **Motion Between Two Points:** "The first asserts the non-existence of motion on the ground that which is in locomotion must arrive at the half-way stage before it arrives at the goal."
(2) **Achilles and the Tortoise:** Achilles, the soldier at the center of *The Iliad* by Homer and renowned for his swift speed as a runner, can never catch up to the tortoise with a head start, because in the time he catches up to its initial position, the tortoise will have advanced some distance.
(3) **The Flying Arrow:** "…the flying arrow is at rest, which result follows from the assumption that time is composed of moments …."
(4) **The Stadium:** Space and time cannot be made up of indivisible units; suppose that there are three rows A, B, and C of stationary bodies of equal size situated with the bodies in A situated at the left of the racetrack of a stadium, the bodies in C situated at its right, and the bodies in B situated at the center. As the bodies in A move to the right, and the bodies in C move to the left, each moving a single unit of distance in a single unit of time, then the bodies in A cross two of the bodies in C in each unit of time, implying the existence of a still smaller unit of time.

The first two of these paradoxes take aim at the view that things are infinitely divisible, and the latter two consider the notion of indivisible units and infinitesimals. A full clarification of these paradoxes requires concepts from modern mathematics, in particular the concepts of a limit, continuity, and infinite sets, all of which were still many centuries away, and so even careful thinkers such as Aristotle struggled to give an explanation. Aristotle did take note of the fact that Zeno proceeded in his arguments by taking as his starting point the position of his opponent and deriving from it some contradiction. For this reason, Aristotle referred to Zeno as the inventor of dialectic. And of course, all this vigorous disputation was possible only in light of the freedom of speech and open academic atmosphere in Greece that gave free reign to its thinkers to seek out truth.

Zeno grew up in the countryside and sustained a lifelong passion for athletics. Perhaps he put forward his paradoxes purely in the spirit of curiosity and play, and not to upset the sensitivities of urban intellectuals. In any case, they mark him out as standing against Pythagorean thought, since the latter attributed all things to number. E.T. Bell remarks that Zeno discussed, "...in non-mathematical language, ... the sort of difficulties that early grapplers with continuity and infinity encountered." Today, more than 2400 years later, it is clear that Zeno's name will never be absent from any history of mathematics or philosophy. The German idealist philosopher G.W.F. Hegel discusses Zeno in his *Lectures on the History of Philosophy* and concludes that he investigated the nature of motion objectively and dialectically. He reiterates and extends the praise originally due to Aristotle: "Zeno had the very important character of being the originator of the true objective dialectic.[3]"

[3] Tr. E.S. Haldane

Plato's Academy

We turn now to Plato (427 BCE–347 BCE), one of the three great philosophers of ancient Greece, alongside his teacher Socrates (460 BCE–399 BCE) and student Aristotle (384 BCE–322 BCE). All three of them are connected deeply with Athens: Socrates and Plato were both born there, and Aristotle spent time in Athens as both a student and a teacher. Socrates left behind no written works and founded no institutions, and we have learned what we know of him mainly through Plato and another of his students named Xenophon (440 BCE–354 BCE), who was a military general, but also spent his time composing histories and essays. Socrates did not make much in the way of contributions to mathematics, but both of his disciples just mentioned observed that he made major contributions to logic, specifically the principles of induction and generalization.

Socrates exerted an inestimable influence on Plato, in spite of the great difference between their stations in life: the latter was born into a prominent family, while the former had a carver and a midwife for parents. Socrates was notably unbeautiful in appearance, but he exercise throughout his life an amazing restraint in worldly things, and he was observed on many occasions to fall into a kind of contemplative reverie midspeech. He rarely drank alcohol, but whenever he did so, it would happen that one of his fellow drinkers would end up drunk and under the table while Socrates himself remained sober. His martyrly death by hemlock, having been accused and convicted of corrupting the youth of Athens and turning them away from the gods, made a deep impression on Plato and convinced him to turn away from political life and devote himself to philosophy. He later referred to Socrates in writing as the bravest, wisest, and most just person he had ever known (Fig. 2.10).

Following the death of Socrates, Plato left Athens and spent 10 or 12 years wandering, travelling in succession to Asia Minor, Egypt, Cyrene (now in Libya), southern Italy, and Sicily. Along his way, he came into contact with several mathematicians and undertook a personal study of mathematics. After his return to Athens, Plato founded an academy, similar in spirit to a modern private university (the word academia derives from the Athenian hero Akademos and refers now to the cultural accumulation of knowledge, as in the academies of science). There were classrooms, dining rooms, auditoriums, gardens, and dormitories. Plato served as the head of school, and he and his assistants lectured on various topics. Apart from several visits to Sicily as a guest lecturer, Plato spent the last 40 years of his life in his academy, and the academy itself continued to exist as an institution for a miraculous nine centuries (Fig. 2.11).

As a philosopher, Plato made a most profound impression on European philosophy and the development of all western culture and society. He composed 36 philosophical texts in his lifetime, mostly in the form of dialogues. The content for the most part concerns political and moral questions but also touches upon metaphysics, epistemology, theology, and cosmology. In his famous dialogue *The Republic*, Plato argued that all people, men and women alike, should share the opportunity to display their talents and enter into the rulership class. In *Symposium*,

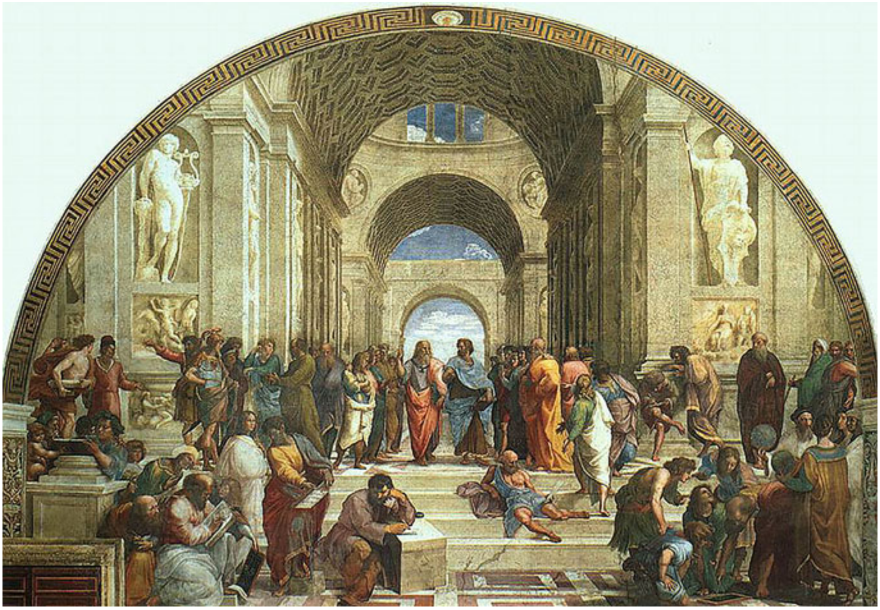


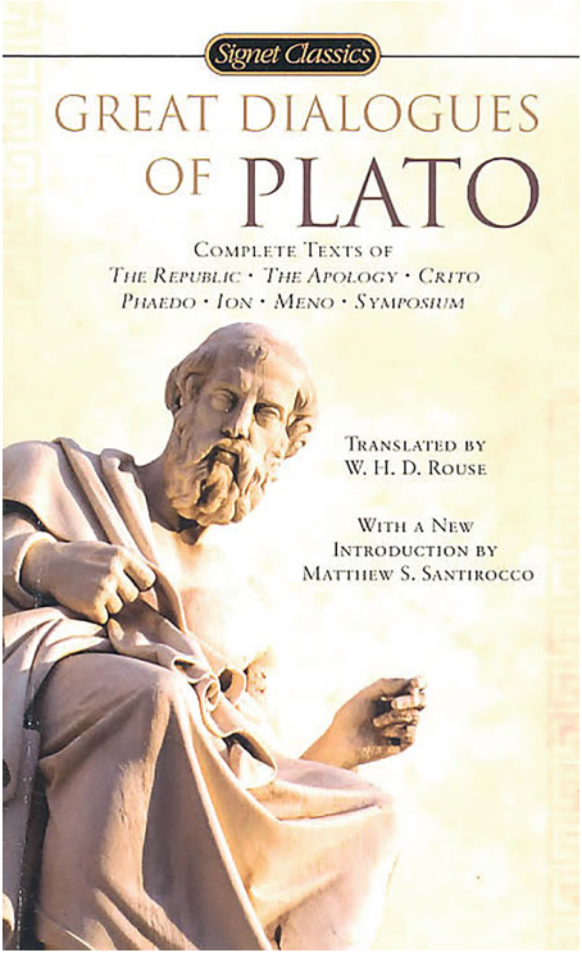
Fig. 2.10 Raphael's famous *School of Athens*; Plato and Aristotle are at the center; Pythagoras, Zeno, and Euclid appear among the other figures

the topic is *eros*, or erotic love: "... if, has as already been admitted, love is of the everlasting possession of the good, all men will necessarily desire immortality together with the good—wherefore love is of immortality." In common language, the dialogue concludes that to love a beautiful person is to pursue immortality by way of the beauty of the body and the production of descendants (Fig. 2.12).

Although Plato himself did not make any deep new contributions to mathematics (some people attribute to him the distinction between analysis and reduction to absurdity), his school was the center of mathematical activity in his time, and the most significant mathematical achievements of the period were obtained by his followers. These include the irrationality of square roots and n-th roots of integers other than n-th powers and concomitant the resolution of the mathematical crisis caused by the discovery of irrational numbers, the construction of the regular octahedron and regular icosahedron, the invention of conic sections (developed in pursuit of a solution to the problem of doubling the cube,[4]) the method of

[4] The problem of doubling the cube, known also as the Delian problem, was one of three major problems in compass and straightedge geometry of ancient Greece. The other two were to square the circle, or more precisely to construct a square having the same area as a given circle, and develop a general construction for the trisection of any given angle. In each case, the solution was required to use constructions that could be completed using only a straightedge and a compass. It was only in the nineteenth century that mathematicians were able to show that all three of these problems are without solution.

Fig. 2.11 2008 English edition of Plato's dialogues



exhaustion, and so on. Even the later great mathematician Euclid spent some time learning geometry at the Academy in his youth. Plato's Academy became known as the great incubator of mathematicians.

Investigations into the philosophy of mathematics also start from Plato. He argued that the proper object of mathematical research is eternal relations and ideas in the abstract world of forms, and not the volatile objects of everyday reality. He not only distinguished mathematical objects from the objects approximating them in our world but also from the geometric figures and specific constructions used to illustrate them in demonstrations. For example, the form *triangle* denotes a unique concept, although there are many triangles, all different from one another in their details, and also many imperfect representations of these triangles, that is to say, concrete objects of triangular shape. In this way Plato advanced the process of abstraction in mathematics that had been initiated by Pythagoras.

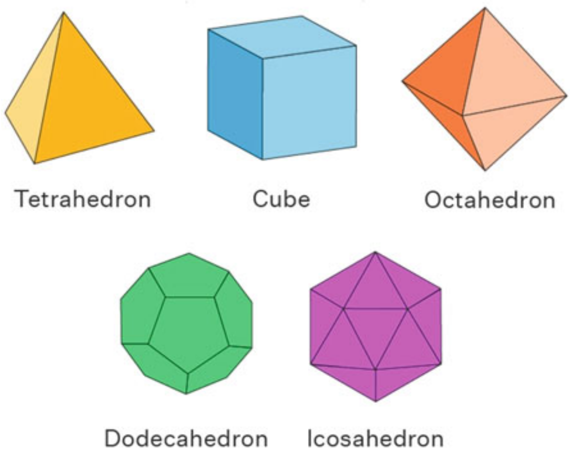


Fig. 2.12 The Platonic solids

Among the many works of Plato, the most influential by far has been *The Republic*. This dialogue consists of ten books. In addition to its political contents, this book contains at its core an outline of his metaphysics and philosophy of science, and the sixth book discusses mathematical assumptions and proof. He writes

> I suppose you know that the men who work in geometry, calculation, and the like treat as known the odd and the even, the figures, three forms of angles, and other things akin to these in each kind of inquiry. These things they make hypotheses and don't think it worthwhile to give any further account of them to themselves or others, as though they were clear to all. Beginning from them, they go ahead with their exposition of what remains and end consistently at the object toward which their investigation was directed.[5]

It can be inferred from this the role that deductive reasoning had already begun to play at the Academy. It was also Plato who insisted on the restriction in mathematical figure drawing to the use of straightedge and compass, which restriction played an important role in the axiomatic development of Euclidean geometry.

As for geometry, it is well known that Plato admired this subject first and foremost among all forms of knowledge, and he gave it a place of prominence in the 10 years of education in the exact sciences for which he advocated. He described god as a great geometer, and he himself explained in a systematic way the diagrams and characteristics of the only five regular polyhedrons, which in later generations came to be called the Platonic solids. A tradition originating in the sixth century holds that the words "let no one ignorant of geometry enter" were engraved at the entrance to the Academy. In any case, it is clear that Plato considered mathematics to be an essential piece of search for human ideals. In his posthumous book *The Laws*,

[5] Tr. Allan Bloom

he went so far as to describe those who neglect the significance of mathematics as pigs.

Aristotle

In the year 347 BCE, Plato became uncomfortable at the wedding banquet of a friend and retreated to a quiet corner of the house, where he died peacefully at the age of 80. Although there is no record of this fact, it is likely that one of the attendees at his funeral was a student by the name of Aristotle (384 BCE–322 BCE), who had studied personally under Plato and continued to work with him for some 20 years after he had been sent to the Academy by his guardian at the age of 17. Aristotle is without question the most exceptional student produced by the Academy, and he went on to become the greatest philosopher and scientist in the ancient history of the world. He had an influence on the nature and content of western culture to which no other thinker can be compared.

Aristotle was born on the peninsula of Chalkidice in the northern part of Greece, at that time under the rule of Macedonia, and now the tourist center of all of northern Greece. His father served as royal physician to the King of Macedonia, and it was perhaps due to the influence of his father that Aristotle developed an interest in biology and the empirical sciences; but later, his time with Plato cultivated in him an obsession with philosophical reasoning. After Plato died, Aristotle took to wandering, just as Plato had taken to wandering after the death of Socrates. Along with a classmate and companion, he spent 3 years in Assos in Asia Minor, before he traveled to Mytilene on the island of Lesbos, where he set up a research center and set about a program of biological investigations (Fig. 2.13).

At the age of 42, Aristotle was invited by King Philip II of Macedon to resettle in the capital Pella and serve as tutor to the 13-year-old prince Alexander. Aristotle based his tutelage after the models of the Homeric epics in the hopes that the young prince could come to embody the highest ideals of Greek civilization. After several years, Aristotle returned to his hometown, where he stayed until the year 335 BCE, when Alexander came into the throne. He then returned to Athens and established his own school there, known as the Lycaeum. He devoted himself to teaching and administration at the Lycaeum for some 12 years, in addition to his continued efforts as a researcher and writer. He was known to prefer delivering his lectures on walks through the gardens, from which habit the Peripatetic school of philosophy that was associated with the Lycaeum got its name.

Both the Lycaeum and Plato's Academy were situated on the outskirts of Athens. In contrast with Plato's interest in mathematics, Aristotle preferred biology and history. All the same, he had spent some 20 years in the Academy and could not but inherit some of its mathematical way of thinking. He discussed certain mathematical definitions in more detail than had Plato and carried out a detailed investigation into the basic principles of mathematical reasoning. He introduced a distinction between

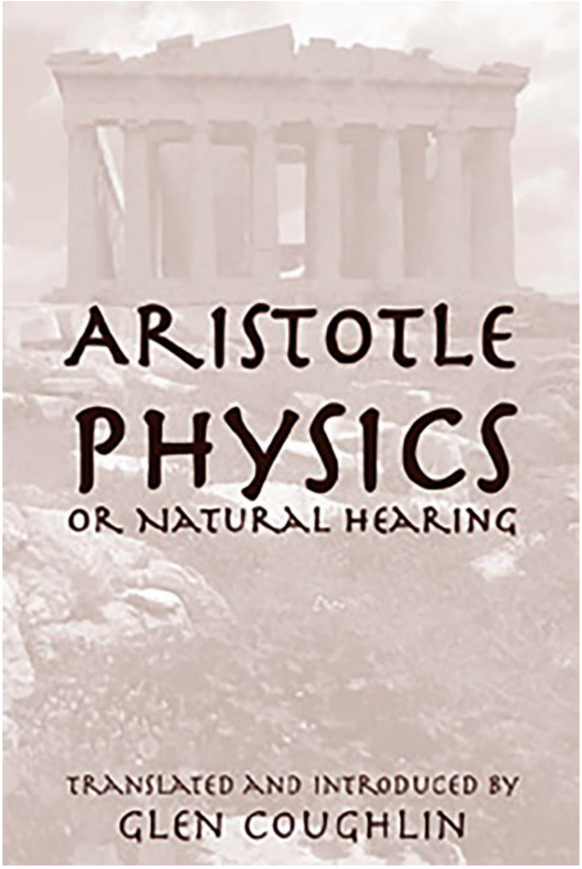


Fig. 2.13 2004 English edition of Aristotle's *Physics*

axioms and postulates, with axioms defined to be the truths common to all sciences and postulates the original principles peculiar to a particular scientific discipline.

His most important contribution to the field of mathematics was his systematization and standardization of mathematical reasoning. The most basic principles are the law of contradiction, which states that a proposition cannot be both true and false, and the law of the excluded middle, which states that every proposition is either true or false. These two principles have long been the centerpiece of mathematical proof. His greatest contribution to philosophy was the invention of formal logic, most notably the logical structure known as the syllogism, a central pillar of his many achievements. The formal logic developed by Aristotle was held to be the standard of reasoning and deductive thought by subsequent generations. At the time of its invention, it laid the methodological foundations for Euclidean geometry, an indisputable apex of the golden age of Greek mathematics (Fig. 2.14).

Aristotle was also the originator of statistics, until recently a discipline altogether independent from mathematics. His voluminous but mostly lost compilations of city-state constitutions included a survey and comparative analysis of the social and

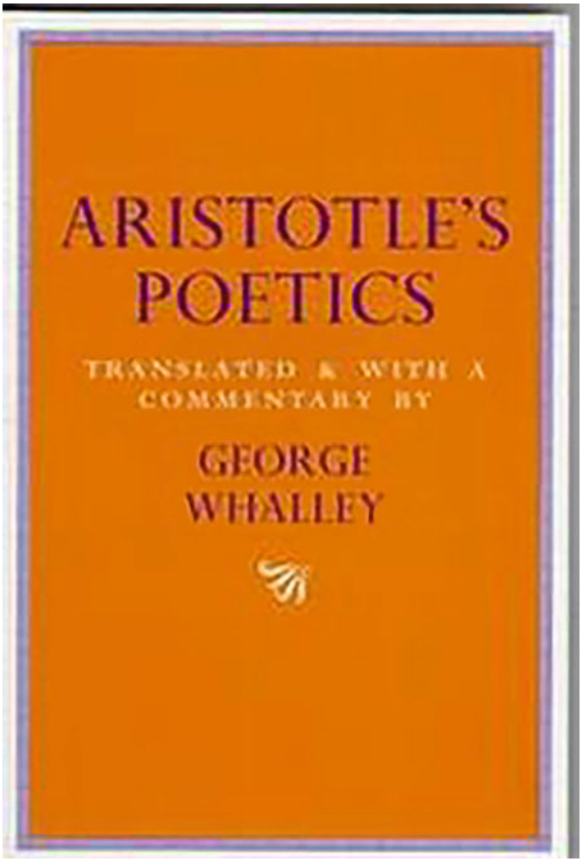


Fig. 2.14 1997 English edition of Aristotle's *Poetics*

economic conditions of various city-states, encompassing their histories, administrations, sciences, art, populations, resources, and wealth. This manner of analysis prevailed for more than 2000 years, until it was replaced by a new methodology in the middle of the seventeenth century and quickly developed into modern statistics, which word still contains the etymological traces of its origins in statecraft.

Finally, a few words concerning the *Poetics*. This treatise instructs its readers not only in the composition of poetry, but also painting and acting. A thin pamphlet in comparison with Euclid's *Elements*, which appeared not long afterward, these two works involve the imitation of three-dimensional space, although the former concerns the imitation of images and the latter a kind of abstract imitation. They are the two greatest representatives of literary and artistic theory on the one hand and mathematical theory on the other in the ancient world.

The Alexandrian School

Euclid's Elements

There are two notable figures named Euclid in the history of Greek thought, one a philosopher, and the other a mathematician. The philosopher, Euclid of Megara (ca. 435 BCE–ca. 365 BCE), was yet another disciple of Socrates, several years older than Plato. He was born in Megara, to the west of Athens, and established the Megarian school of philosophy, also referred to sometimes as the Eristic school. Euclid and the Megarian philosophers were deeply influenced by both Socrates and the Eleatics, unifying the Socratic notion of the good with the preeminence of unity inherited from Parmenides. In short, the good and the one are the same, and there is nothing else that exists. They were also skilled in disputation, most notably Eubulides of Miletus, who studied under Euclid and became famous for seven logical paradoxes, the most famous of which is the liar's paradox: "What I am saying right now is a lie." On the basis of these paradoxes, Eubulides emphasized the inherent contradiction of things, the problematic distinction between qualitative and quantitative change, and promoted the development of logic (Fig. 2.15).

The mathematician Euclid, by contrast, was born much later and left behind no details or even clues concerning his life. We cannot even say today what continent he was born on, whether Europe, Asia, or Africa, or the years of his birth and

Fig. 2.15 Statue of Euclid

death. This is a frustrating elision in the history of mathematics. It is known only that he studied for a time at the Academy in Athens and sometime around the year 300 BCE was hired to teach mathematics in Alexandria in Egypt. He left behind a book entitled *Elements*, which became an essential mathematical textbook for more than 2000 years and indeed contains the main content of even modern elementary mathematical pedagogy. In light of the importance of mathematics for human reason, he is considered to be the most influential mathematician in all of world history.

First, a few words about the city of Alexandria. Following the Peloponnesian War, Greece suffered a period of political disunity. The Macedonians to the north took advantage of this weakness and not long afterward captured Athens. When young Alexander inherited the throne, although possessed by deep reverence toward Greek civilization, he turned his ambitions to world conquest. Wherever his army was victorious, he would look to good locations for the establishment of new cities, and after he took Egypt, he set up in 332 BCE a city on the Mediterranean bearing his own name more than 200 kilometers northwest of Cairo. He not only invited the best architects of the time to contribute to it but also personally supervised the planning, construction, and the management of immigration.

Nine years later, Alexander returned from an expedition in India and died suddenly of illness in Babylon at the age of 32. His massive empire was divided into three, although united under the banner of Greek culture. When Ptolemy the Great became ruler of Egypt, he made Alexandria its capital. In order to attract the greatest minds to the city, he established the Mouseion, a research institution comparable in size and scope to modern research universities. At its center was a large library, the famous Library of Alexandria, said to have contained more than 600,000 papyrus scrolls. From that time on, Alexandria was the spiritual and cultural capital of Greece for nearly a thousand years. As late as the nineteenth and twentieth centuries, Constantine P. Cavafy (1863–1933), the most celebrated modern Greek poet, chose to spend most of his life in Alexandria.

Euclid came to Alexandria during this period of flourishing, and presumable the *Elements* was written around the same time. Almost all of the theorems of geometry and number theory presented in the book were known before his time, generally alongside the proofs he gives for them, but Euclid organized and systematically laid out in logical order these preexisting materials and set them up atop an appropriate selection of axioms and postulates. This is by no means an easy task, but rather requires tremendous judgment and insight. He was careful also in the arrangement of the theorems to ensure that each new theorem was logically consistent with its precursors. Euclid was recognized as the foremost authority on geometry in ancient Greece, and the *Elements* quickly replaced every earlier textbook.

The nature of the *Elements* presumably exhibits some of the influence on its author of his time at the Platonic Academy. Plato had emphasized the abstract nature of reality and the importance of mathematics for philosophical thought. His influence led some mathematicians, including Euclid, to divorce their mathematical activities from practical considerations. In what became the most important mathematical textbook of the ancient world, perhaps even of all time, Euclid began his treatise with a statement of its definitions, axioms, and postulates. He defined a point to be that which has no part, a line (which we would now more commonly call a curve) to be a breadthless length, a straight line to be a line which lies evenly with the points on itself, and so on. The text is divided into 13 books: Books 1 through 6 discuss plane geometry, Books 7 through 9 discuss number theory, Book 10 is about irrational numbers, and Books 11 through 13 present solid geometry. In total there are 465 propositions, built atop 5 axioms and 5 postulates. In particular, attempts to prove or replace Euclid's fifth postulate led to the birth of non-Euclidean geometry, as we will discuss in detail in Chap. 7.

I would like to introduce specifically the sections of the book dealing with number theory. Many of the number theoretic propositions still appear today in textbooks on elementary number theory. Chapter 7 introduces a method for determining the greatest common divisor of two or more positive integers, known today as the Euclidean algorithm, and uses it to determine when two numbers are relatively prime. Proposition 14 in Book 9 is equivalent to the fundamental theorem of arithmetic, which states that any positive integer larger than 1 can be uniquely decomposed as a product of prime powers. Proposition 20 in the same book proves the infinitude of prime numbers, and the proof itself is frequently presented as a model example of mathematical proof. All of these results are indispensable to modern number theory and appear in any elementary treatment of the subject. Proposition 36 presents a sufficient condition for an even number to be a perfect number, contributing to a problem originating with Pythagoras and still not completely resolved at present.

Finally, I would like to relate two famous anecdotes concerning Euclid, both of which appear in annotations to the *Elements* by later Greek commentators. It is said that Ptolemy the Great found the book too difficult to understand and asked Euclid if there was not some easier way to master its contents, to which Euclid replied, "there is no royal road to geometry!" On another occasion, one of his students asked Euclid what could be gained from the study of geometry. Euclid made no direct answer, but rather sent a servant to bring the student a coin with the words "give him a coin, since he must profit from what he learns."

Since Gutenberg's invention of the printing press in fifteenth-century Germany, Euclid's *Elements* has appeared in thousands of editions around the world. It is considered to have made a major contribution to the development of modern science, and its logical structure and rigorous deductive reasoning continue to fascinate even contemporary thinkers. The first Latin version of Euclid's *Elements* was translated from Arabic, since versions in its original language had vanished at that time from the world following the successful burning of the Library of Alexandria first by the Roman army and later by Christian extremists. It was first translated into Chinese by the Italian missionary Matteo Ricci (1552–1610), in collaboration with Xu Guangqi (1562–1633) in the seventeenth century, but they translated only the first six books. It was two and half centuries before the British missionary Alexander Wylie (1815–1887) and Li Shanlan (1810–1882) prepared a complete translation.

Archimedes

On the strength of Euclid's reputation, the Mouseion at Alexandria became famous for mathematics and attracted young talents from all walks of life. The most famous of these was Archimedes (287 BCE–212 BCE). Biographical details concerning Archimedes are more reliable than for many mathematicians of the period due to the careful efforts of Roman historians. He was born in Syracuse in southeastern Sicily, the son of an astronomer. In his youth, Archimedes studied under the disciples of Euclid in Egypt, and he remained collegiate with the Alexandrian scholars even after his return to Alexander. Indeed, the letters between them form the main record of his academic achievements. For this reason, he is numbered among the members of the Alexandrian school (Fig. 2.16).

Archimedes wrote prolifically, mostly in the form of brief manuscripts rather than long treatises, perhaps the first person in the history of mathematics to adopt a format that anticipates the modern journal article. The contents of these works include mathematics, mechanics, and astronomy. The geometric works are *Measurement of a Circle*, *Quadrature of the Parabola*, *On Spirals*, *On the Sphere and the Cylinder*, *On Conoids and Spheroids*, and *On the Equilibrium of Planes*. The works dealing with mechanics are *On Floating Bodies* and *The Method of Mechanical Theorems*. There is also a sort of astronomical divertimento entitled *The Sand Reckoner*, written for a young prince who later inherited the throne and continued to look kindly upon Archimedes, an apparently apocryphal text entitled *Book of Lemmas* that survived only in Arabic, and a mathematical poem of 44

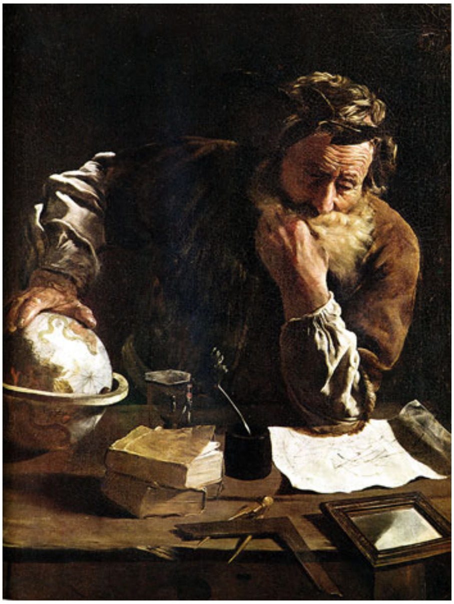
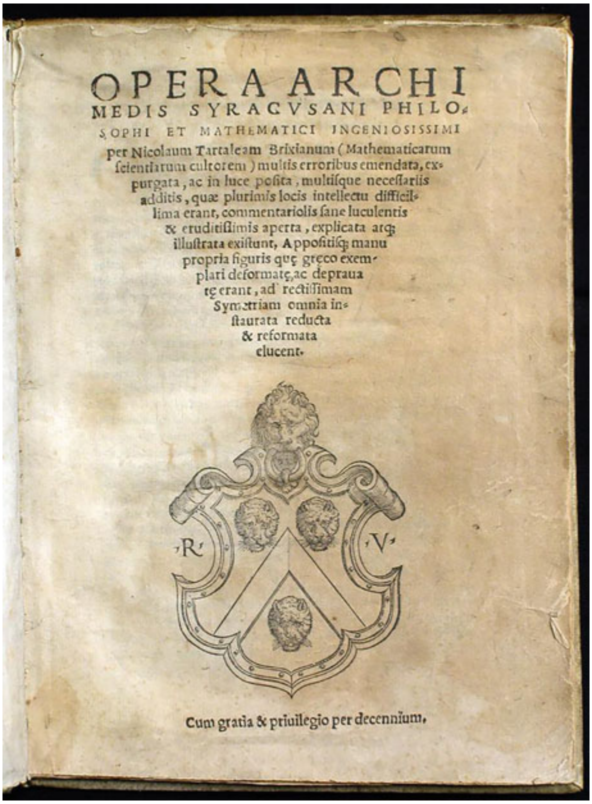


Fig. 2.16 Portrait of Archimedes (1620)

lines entitled *The Cattle Problem* and addressed in a subtitle to the Alexandrian mathematician Eratosthenes and his colleagues.

As a geometer, Archimedes was most adept in the calculation of areas, volumes, and related problems, in which he far surpassed the results in Euclid. As one example of his work, he evaluated the circumference of a circle using the method of exhaustion: by increasing the number of sides of regular polygons inscribing and circumscribing a given circle, he was able to achieve better and better approximations with the increase in the number of sides of the polygons. In particular, he was able to conclude from polygons with 96 sides that the number π lies greater than $\frac{223}{71}$ and smaller than $\frac{22}{7}$, the most precise approximation in ancient history. By the same method, he was able to show that the surface area of a sphere is four times the area of a great circle lying on it and therefore calculate precisely the surface areas of given spheres (Fig. 2.17).

But while the method of exhaustion is a powerful tool for proving results already known or conjectured to be true, it is ill-suited to the discovery of new results. For this purpose, Archimedes developed a method involving equilibrium and the law of the lever; this method makes use of infinitesimals and anticipates the notion of



OPERA ARCHI
MEDIS SYRACVSANI PHILO-
SOPHI ET MATHEMATICI INGENIOSISSIMI
per Nicolaum Tartaleam Brixianum (Mathematicarum
scientiarum cultorem) multis erroribus emendata, ex-
purgata, ac in luce posita, multisque necessariis
additis, quae plurimis locis intellectu difficil-
lima erant, commentariolis sane luculentis
& eruditissimis aperta, explicata atq;
illustrata existunt, Appositisq; manu
propria figuris quę greco exem-
plari deformatę, ac deprava-
tę erant, ad rectissimam
Symetriam omnia in-
staurata reducta
& reformata
elucent.

R V

Cum gratia & priuilegio per decennium.

Fig. 2.17 1534 edition of the writings of Archimedes limits and the differential method of modern integral calculus. Archimedes was able to discover using this method the formula

$$V = \frac{4}{3}\pi r^3$$

for the volume V of a sphere in terms of its radius r. He also provided a rigorous proof of this result using the method of exhaustion. By the same method, Archimedes was able to determine that the ratio between the area of enclosed between a parabolic arc and a straight line and a triangle with equal base and height is $4 : 3$. The discovery of this proposition is a corroboration of the proportionality of Pythagorean numbers.

Archimedes was more of an applied mathematician than Euclid seems to have been, and there are many stories and legends concerning his practical achievements. The Roman architect Vitruvius composed a work in ten volumes called *De architectura* dedicated to the preservation of classical traditions in the architecture of temples and public buildings. The ninth volume of this work relates an anecdote that has remained famous through the ages. The King of Syracuse had ordered as a display of gratitude and largesse for a golden crown to be fashioned and placed in a certain temple. After the completion of this work, rumor began to spread that a certain portion of gold had been pilfered by the goldsmith and replaced with silver. The king invited Archimedes to appraise it, and as he was thinking about the problem during his daily bath, Archimedes noticed that as he lowered his body into the water, more and more of it flowed over the sides. He realized that this was the solution of the problem, that the specific gravity and composition of a solid can be measured by the water it displaces. Allegedly he jumped out of the baths and immediately ran home without even remembering to get dressed, shouting along the way “Eureka!”, meaning “I’ve got it!”

Less trivially, Archimedes discovered the basic principle of fluid mechanics, or the law of floating bodies, after much careful thought and experimentation: the weight of an object in a fluid medium is equal to the weight of the water displaced by it. According to Pappus, the last great geometer of ancient Greece, Archimedes once also declared “if you give me a lever and a place to stand, I can move the world.” As evidence for this claim, he is supposed to have devised a set of pulleys that enabled the king to move with his own strength a three-masted galleon. The king was suitably impressed and replied that from that time on it would be necessary to believe anything that Archimedes said. Even in modern times, the huge ships that pass through the Panama Canal or the Suez Canal are pulled on their way by means of tracked pulleys.

Archimedes was able to make such bold and confident claims because he had mastered already the principle of leverage. He used this knowledge and his expertise to defend his city and finally died for his country. During the third and second centuries BCE, commercial and territorial conflicts between Carthage,[6] neighboring Syracuse, and the Romans had bubbled over into wars, known as the Punic Wars. The second of these wars saw Syracuse dragged into the fray as Carthaginian allies. In 214 BCE, the Romans surrounded Syracuse.

According to legend, the citizens of Syracuse first employed cranes invented by Archimedes to lift the ships near the shore or the walls of the city and slam them down again with great force and powerful machinery to rain boulders down on the retreating invaders like torrential rains. An especially extravagant embellishment has it that Archimedes used a huge mirror to focus the rays of the sun upon the Roman ships and light them ablaze. A more credible account is that they set the ships on fire with burning projectiles. The Romans changed strategy eventually and mounted a long siege until finally Syracuse fell due to exhaustion and scarcity of food and ammunition. Archimedes is said to have been drawing up geometric figures in the sand when the Romans finally took the city, and a reckless soldier put him to death on the spot. The death of Archimedes marked the beginning of the decline of Greek mathematics and the splendid culture of Greece. The Romans that replaced them established in their place a philistine rule.

Other Mathematicians

Just at the time that the Romans captured Syracuse, another representative of the Alexandrian school of mathematics named Apollonius of Perga (ca. 262 BCE–ca. 190 BCE) was on the verge of completing his life's work. Apollonius was born in Pamphylia in southern Asia Minor, not far from the island of Rhodes. Like Archimedes, Apollonius studied mathematics at Alexandria in his youth. His greatest achievement was the treatise *Conics*, in which appeared for the first time in a mathematical book the parabola, hyperbola, and ellipse with which we are all familiar today (Fig. 2.18).

Apollonius defined the conic sections as follows. Given any circle and a point lying outside the plane of the circle, draw any straight line connecting the point to a point on the circumference of the circle. The movement of the line along the circumference generates the surface of a cone with two branches. A section of the cone is the curve produced by its intersection with a plane. If the plane does not intersect the circular base, then the curve is an ellipse; if it does intersect the base, but does not run parallel to any of the generating lines, then the section is a hyperbola; and if it intersects the base and also runs parallel to one of the generating lines, then the section is a parabola. Apollonius also defined and studied the diameters, tangents, centers, asymptotes, foci, etc. of the conic sections.

[6] An ancient country established by Phoenicians and centered about what is now Tunisia in North Africa. During its greatest period, its territory stretched from Sicily in the east, to Morocco and Spain in the west.

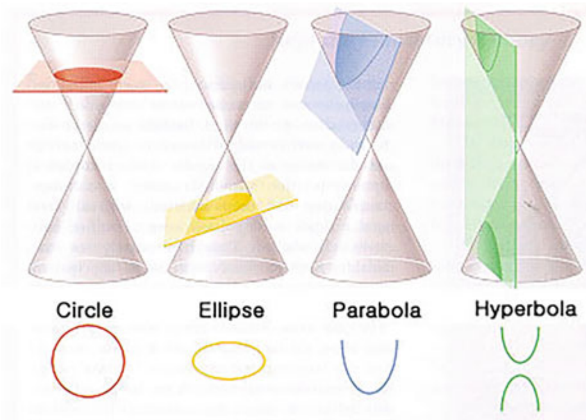


Fig. 2.18 Geometric construction of the conic sections

Apollonius used the methods of pure geometry to obtain some of the major results of analytic geometry, introduced nearly two millenia later, a remarkable achievement. His theory of conic sections marks perhaps the high point of Greek deductive geometry, and later generations referred to him alongside Euclid and Archimedes as the three great mathematicians of early Alexandria. They are the main figures of the golden age of Greek mathematics. Subsequently, after the expansion of the Roman empire, academic research in Athens and other cities quickly withered away. Nevertheless, the inertial influence of Greek civilization and the lax attitude of the Romans toward freethinking in remote Alexandria meant that there were still mathematicians producing substantial results even in later Alexandria.

For the most part, this second school of Alexandrian mathematicians did not contribute much to geometry. The only result worth mention is Heron's formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

for the area Δ of a triangle in terms of its three sides a, b, c and its perimeter $2s = a + b + c$. It came to light later that even this formula had been discovered earlier by Archimedes, although it does not appear in any of his extant works.

More significant was the establishment of trigonometry. Work in this area was presented in an influential astronomical text entitled the *Almagest* and written by a mathematician, geographer, and astronomer with the same name Ptolemy as the earlier pharaohs of Ptolemaic Egypt. This book propounded a detailed geocentric model of the solar system and became a classic of western astronomy throughout the Middle Ages. Its author is considered to have been the greatest astronomer of ancient Greece. Ptolemy also gave a fast an efficient approximation method for π

and obtained the value (in sexagisemal) 3; 8, 30 $= \frac{377}{120} = 3.141666\ldots$. Finally, there is Ptolemy's theorem, which states:

> if a quadrilateral is inscribable in a circle then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.

An important feature of the later period of Alexandrian mathematics was that it broke from the geometric tradition of earlier times and treated arithmetic and algebra as independent subjects. The arithmetic of the Greeks is what today we call number theory, although some authors retain the archaic usage of arithmetic; for example, the preeminent journal of number theory in Poland is *Acta Arithmetica*. After Euclid, the most important work in number theory was *Arithmetica*, by Diophantus (ca. 246–330). The title is in Latin, translated from an Arabic translation of the original, which is lost. This book is famous for its treatment of indeterminate equations, also called Diophantine equations. These are algebraic equations with integer coefficients and solutions required to be integers. Usually, the number of unknowns is more than the number of equations.

The best known problem in this book is Problem 8 in Volume 2 which requires the statement of a known square number as a sum of two square numbers. In the seventeenth century, the French mathematician Pierre de Fermat added a note to this margin of his copy of *Arithmetica* next to this problem, and this note became Fermat's Last Theorem, which attracted the interest of the entire world some time later. Diophantus, who is generally believed to have lived around 250 CE, also makes an appearance in Greek anthology of number games and puzzles from the first year of the sixth century, in the form of a poem or epitaph:

> Here lies Diophantus,' the wonder behold.
> Through art algebraic, the stone tells how old:
> 'God gave him his boyhood one-sixth of his life,
> One twelfth more as youth while whiskers grew rife;
> And then yet one-seventh ere marriage begun;
> In five years there came a bouncing new son.
> Alas, the dear child of master and sage
> After attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life.

This riddle is equivalent to solving the equation

$$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4,$$

which has solution $x = 84$, implying that Diophantus died when he was 84 years old.

Pappus, whom we have already mentioned, lived sometime around the year 320, at which time the Chinese mathematician Liu Hui was alive already. Like Diophantus, Pappus left behind a single well-known work, the *Synagoge*, or *Collection*, which is regarded as a kind of requiem for Greek mathematics. The most notable conclusion in it is that a circle is the largest plane area that can be contained by a closed curve of fixed circumference. This is an example of an extreme value problem and can be considered a part of relatively advanced mathematics. This book also recounts four attempts to solve the problem of doubling the cube, the first of which was due to Eratosthenes (ca. 276–ca. 194 BCE). Eratosthenes was born in Cyrene in modern-day Libya and studied in Alexandria. He earned a reputation as a second Plato, but he was a much more versatile character, a poet, philosopher, historian, astronomer, and pentathlon athlete.

In number theory, there is the so-called Sieve of Eratosthenes associated with him, an algorithm for finding all prime numbers up to any given size, and the original method for the construction of prime tables. Even as late as the twentieth century, research into Goldbach's conjecture that every even number larger than two can be expressed as the sum of two primes relied primarily on variations of this method. Eratosthenes was also the first person to achieve a reasonably accurate calculation of the circumference of the earth, a considerably better result than his colleague in Alexandria Archimedes was able to achieve. In the practical sphere, Eratosthenes took the lead in marking out the five climatic zones of the earth, a division still in use today. He also analyzed and compared the waters of the Mediterranean, belonging to the Atlantic water system, and the Red Sea, belonging to the Indian Ocean water system. He concluded that the two were connected, paving the way for the Portuguese explorer Vasco de Gama to reach India by water at the end of the fifteenth century (Fig. 2.19).

Nevertheless, the map of the world drawn by Eratosthenes, which was supposedly the first of its kind in human history, revealed that the known world of the Greeks was still very limited (in the figure, the Arabian Gulf is the Red Sea, and the Erythraean Sea is the Indian Ocean, missing of course the eastern part of Asia,

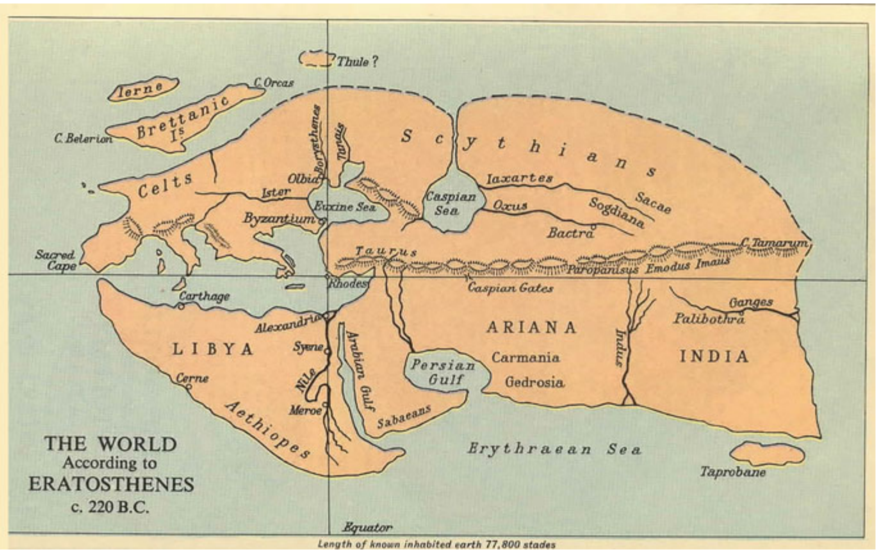


Fig. 2.19 Map of the world by Eratosthenes, ca. 220 BCE

the Americas, Australia, and Antarctica). This suggests that their achievements in mathematics and art, which marked the high point of human achievement in the classical era, were still piecemeal and incomplete, planting seeds for the emergence and growth of modernism in the first half of the nineteenth century, which saw mathematical expression as non-Euclidean geometry and noncommutative algebra.

Conclusion

It is easy to see from the discussion above that Greek mathematics was characterized by two outstanding features: its abstraction and deductive spirit on the one hand and on the other its connection with philosophy. As Morris Kline put it, the mathematical knowledge accumulated by the Egyptians and the Babylonians was like a castle in the sky, or a house made of sand, that crumbles and collapses under the slightest touch, while the Greeks built up impregnable and eternal mathematical palaces. And just as music lovers appreciate music purely for its combination of structure, harmony, and melody, the Greeks appreciated beauty as order, completeness, consistency, and clarity. Plato wrote, “Whatever we Greeks receive, we improve and perfect it.”

Plato truly loved geometry, and Aristotle was not willing to separate mathematics from aesthetics. He felt that the principles of order and symmetry that are so important to the appreciation of beauty are not difficult to find in mathematics. In fact, the ancient Greeks considered the sphere to be the most beautiful of all shapes and therefore that it was sacred and good. They also admired the circle. It was not unnoticed that the planets in their heavenly trajectories seem to move along a circle, while in the sublunary world below, things tend to move in straight lines. It was precisely because mathematics held some aesthetic appeal for the Greeks that they insisted on exploring mathematical theorems and principles beyond the remit of nature.

The Greeks were also philosophers by nature, possessed a deep and sincere love for reason, sport, and spiritual activity, a significant distinction from other civilizations of the time. The period from Thales of Miletus in the sixth century BCE to the death of Plato in 337 BCE was the honeymoon of mathematics and philosophy, and the two disciplines frequently coexisted even in a single person. One of the distinguishing features of Greek philosophy is that it takes the entire universe as its object of study; it is all-encompassing. There is no doubt connected with the infancy of the development of mathematics at that time. The mathematicians could only discuss simple geometry and arithmetic and could not do anything with movement and change, leading to such problems as Zeno’s paradoxes. They had no choice but to take on the role of philosophers to interpret the world.

As the Greek city-states fell under the rule of Macedonia after 338 BCE, the center of Greek mathematics shifted from Athens to Alexandria in the southern Mediterranean, and the honeymoon period of mathematics and philosophy came to an end. In spite of this, the highest crystallization of logical deduction in the

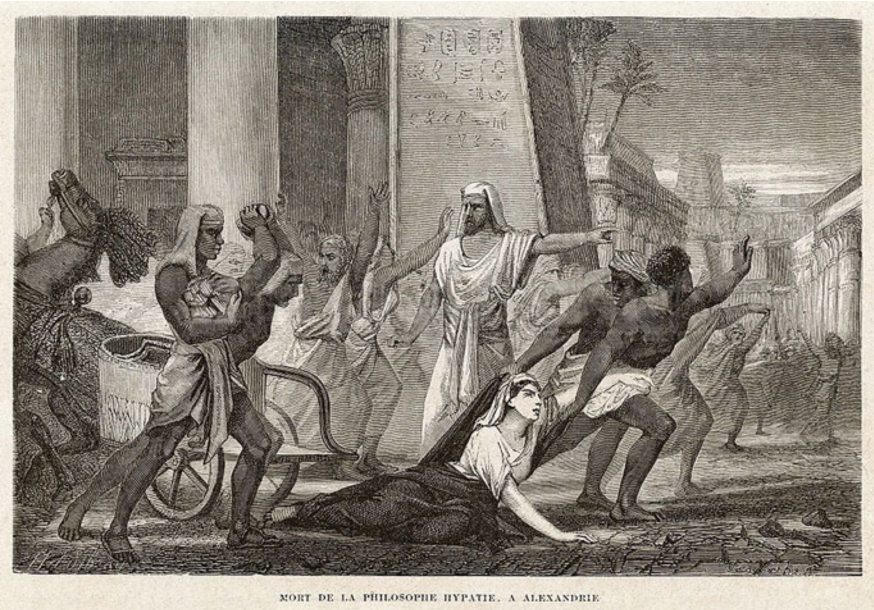


Fig. 2.20 Death of the philosopher Hypatia, Alexandria (1866)

ancient world emerged in its afterglow, Euclid's *Elements*. The significance of this book was not only the many wonderful theorems it contains but rather the spirit of rational deduction that it inculcates. Generation after generation of European thinkers learned impeccable reasoning from this book. No doubt its influence is also felt in the enduring democratic institutions and rich judicial systems of western society (Fig. 2.20).

It must be pointed out that the leisurely pursuit of rational enquiry in Greece was facilitated by the many native and foreign slaves who were responsible for farming the land, harvesting crops, and engaging in the menial tasks of the city-state. Such a lifestyle cannot last in the absence of material abundance. Eventually, the pragmatic Romans took the place of the aesthetic Greeks, as much later the spirit of material progress in the United States gained prominence over the more idealist temperament of Europe. In the year 415 CE, the first known female mathematician in the history of mathematics Hypatia (ca. 370–415) was slaughtered by a Christian mob in her home city Alexandria, and the inevitable final decline of Greek civilization became undeniable.

Hypatia's father provided the most authoritative ancient commentary on Euclid's *Elements*, and Hypatia herself contributed annotations to the *Arithmetica* by Diophantus and the *Conics* by Apollonius. She was also the leader of the Alexandrian Neoplatonists, and she was said to have attracted a large number of admirers on account of her beauty, kindness, and extraordinary talents. Unfortunately, her commentaries have all been lost, and we do not even have any record of her philosophical writings. The only remaining documents connected with Hypatia are letters written to her by her students, asking her about the construction of the astolabe and the water clock.

Following the retreat of the Greek civilization, in both the ancient Roman period and the long Middle Ages, mathematics and philosophy drifted apart. We shall see however in the next two chapters that this created an opening for several ancient eastern countries to take the stage of our world history. It was not until the sixteenth century that a reunification began. In the words of Russell:

> Of much greater importance in the thinking of the Italian humanists was the renewed emphasis on the mathematical tradition of Pythagoras and Plato. The numeric structure of the world was once again emphasized, thus displacing the Aristotelian tradition that had overshadowed it.

In the seventeenth century, the birth of calculus brought mathematics and philosophy closer to one another again, but by that time the philosophers had narrowed the scope of their research to determining only how it is that people understand the world.

Chapter 3
The Chinese Middle Ages

The carpenter's square is not square, compasses cannot make circles; The shadow of the flying bird never moves.

Hui Shi (as recorded in the Zhuangzi)

Prologue

The Pre-Qin Era

Just at the time that the civilizations of Egypt and Babylon were developing in the borderlands of the three continents of Asia, Africa, and Europe, another very different civilization was emerging in the far east and spreading out along the river basins of the Yellow River and the Yangtze River: the Chinese civilization. Scholars generally believe today that in ancient times migration between the Tarim Basin in modern-day Xinjiang and the Euphrates riverlands was impossible on account of a forbidding series of mountain ranges, harsh deserts, and the ferocity of the nomadic tribes of the region. Sometime between the years 2700 BCE and 2300 BCE, the Five Emperors of legend emerged in ancient northern China and after them a series of dynasties one after another.[1] Although the bamboo boards which were traditionally used for the inscription of Chinese characters are not so durable as clay tablets or papyrus books, nevertheless the science historian Joseph Needham has pointed out that a great wealth of ancient texts have survived intact in China due to the diligent record keeping of the Chinese people (Fig. 3.1).

Like both Babylon and Egypt, China in ancient times had grasped already the mathematical seeds of number and shape. Although the Shang oracle bones have been only incompletely deciphered, they have been found to contain a complete decimal system; strict calculations and counting appeared at the latest in the spring

[1] The announcement in 2007 of the discovery of ancient city wall relics at the Liangzhu Ancient City in Zhejiang Province suggests that the Xia Dynasty was not the first dynasty in the history of Chinese civilization.

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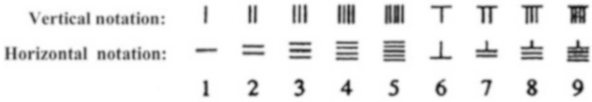


Fig. 3.1 Arithmetical notation of ancient China

and autumn period and the Warring States period. This notational system consisted of vertical and horizontal counting rods, representing even and odd digits, with a blank space where zero would go. The first century BCE Chinese historian Sima Qian (ca. 145 BCE–ca. 90 BCE) writes in his *Records of the Grand Historian, Annals of the Xia Dynasty* (史记 · 夏本纪); "[Xia Yu] surveyed the nine mountains, with a water level and chalk line in his left hand, a compass and carpenter's square in his right. . . .[2]" This can be regarded as an early application of geometry.

It is perhaps more noteworthy that, just at the time when the Athenian school in Greece was overflowing with discourse on philosophy and theoretical mathematics, the Warring States period in China (475 BCE–221 BCE) too was teeming with all manner of scholars, belonging to what has been called the Hundred Schools of Thought. This was the time in world history when philosophers were springing up across the globe, sometimes called the axial age, a term coined by the German philosopher Karl Jaspers (1883–1969). Among the works of this period, the *Mojing* (墨经) is representative of the Mohist philosophy of logic and rational thought; in it appear certain laws for formal logic, and built atop them a series of abstract mathematical definitions, even involving the concept of infinity. The logicians of the Ming school (or *Mingjia*, 名家), known for their eloquence, expressed a deeper understanding of the infinite. The landmark book *Zhuangzi* (庄子) of the Taoist philosophical tradition records the proposition of the representative of the Ming school Hui Shi: "The largest thing has nothing beyond it; it is called the One of largeness. The smallest thing has nothing within it; it is called the One of smallness." The largest thing here indicates the infinite universe; the smallest thing can be considered equivalent to the atoms of Democritus.

Hui Shi (ca. 370 BCE–ca. 310 BCE) was a philosopher of the Song state under the Zhou Dynasty in modern-day Henan Province, and his reputation in his time was second only to Confucius and Mozi. He served for 15 years as chief minister of the Wei state and advocated for the unification with the Qi and Chu states against the Qin with considerable political success. Hui Shi and his contemporary Zhuang Zhou, author of the *Zhuangzi*, were at once friends and rivals. The *Debate on the Joy of Fish* between the two of them is among the famous dialogues in Chinese philosophy. After the death of Hui Shi, Zhuang Zhou is said to have remarked that there was no one left to talk to anymore. Hui Shi and the Ming school with which

[2] Tr. Tsai-fa Cheng, Zongli Lu, William H. Nienhauser, Jr., Robert Reynolds

he was associated are remembered for many wonderful and paradoxical statements involving mathematical concepts:

> The carpenter's square is not square, compasses cannot make circles;
> The shadow of the flying bird never moves;
> No matter how swift the barbed arrow, there are times when it is neither moving nor at rest;
> Take a pole one foot long, cut away half of it every day, and at the end of ten thousand generations, there will still be some left;[3]

and so on. It is easy to see the resemblance to the paradoxes invented by Zeno in Greece about a century earlier. The immediate successor to Hui Shi was Gongsun Long (325 BCE–250 BCE), who was famous for the aphorism "white horses are not horses." This paradox is generally interpreted as pointing toward the distinction between the general and the particular, but inevitably it has also given rise to accusations of shallow sophistry.

Regrettably, the Ming school and Moist philosophical traditions were exceptions among pre-Qin thought. The more socially influential works in the Confucian, Taoist, and Legalist traditions paid little heed to mathematics and abstract topics, but rather were focused exclusively on the successful governance of the state and the world, social ethics, and the sound cultivation of body and mind, markedly at odds with the austere rationalism of ancient Greek philosophy. After Qin Shi Huang, the first emperor of Qin, unified China, he put a decisive stop to the contention of the Hundred Schools of Thought and burned the history books and folk collections of various states. By the time of Emperor Wu of the Han dynasty (around 140 BCE), the only female emperor in the history of China, Confucianism had monopolized the intellectual landscape, and the mathematical disputations of the Ming school and the Moists had no opportunity for further development. On the other hand, due to a long period of social stability and increased exposure to the outside world, the economy had blossomed to an unprecedented level of prosperity, driving the development of mathematics along practical and computational lines, with greater success.

Zhoubi Suanjing

In the year 47 BCE, the Library of Alexandria was partially burned by the Roman army under the command of Julius Caesar in the course of military operations intended to assist his lover Cleopatra in the seizure of Egyptian power. Cleopatra was the second daughter of Ptolemy XII Auletes, and she ruled alongside her two younger brothers Ptolemy XIII and Ptolemy XIV and her son with Caesar, Caesarion. At this time, China was under the rule of the Western Han dynasty and experiencing its first period of ascendance in mathematical achievement. It is generally believed the greatest masterpiece of classical Chinese mathematics, the *Nine Chapters on the Mathematical Art*, was written during this era (around the

[3] Tr. Burton Watson

Fig. 3.2 The earliest known mathematical work in China, the *Book of Numbers and Computations*

first century BCE). The oldest Chinese mathematical classic, the *Zhoubi Suanjing*,[4] presumably came a bit earlier (Fig. 3.2).

It is worth mentioning here, however, that although Needham agrees that the mathematical level of *Nine Chapters on the Mathematical Art* is more advanced than that of the *Zhoubi Suanjing*, nevertheless the earliest date we can assign to the latter according to the archaeological evidence is in fact two centuries later than the former. This lacuna is a source of some disappointment to archaeologists and historians of mathematics. Needham himself remarks in his landmark *Science and Civilization in China* that some of the results in the *Zhoubi Suanjing* are so early that it seems impossible not to believe that its composition dates back to the Warring States period.

In addition to the uncertain provenance of the *Zhoubi Suanjing*, its author is also completely unknown, a situation very different from the fate of Euclid's *Elements* in Greece. There are two most interesting mathematical results in this book. One of these is the Gougu theorem, as the Pythagorean theorem concerning right triangles is known in China. This was derived earlier than Pythagoras, but there is no detailed proof of this result like that of Proposition 47 Book 1 of the *Elements*. Rather this proposition is recorded in the form of a dialogue between the Duke of Zhou and his

[4] In fact, an earlier Western Han text written across 190 bamboo strips and entitled *Book of Numbers and Computations* (算数书), was unearthed in a tomb in Zhangxiangshan in Hubei Province in 1984. This text, which consists of a collection of problems, is now the earliest known Chinese mathematical text.

astronomer and mathematician Shang Gao in the early years of the Western Zhou dynasty (eleventh century BCE). This marks these two out as the earliest characters in the history of Chinese mathematics.

The Duke of Zhou, whose personal name was Dan (旦), was the fourth son of King Wen of Zhou and the younger brother of King Wu. After King Wu died and left the kingdom to his son, the Duke of Zhou became regent and oversaw the administration of the kingdom, provoking revolts, which he successfully put down, before dutifully acquiescing to a peaceful transfer of power when King Cheng came of age after 7 years had passed. As regent, the Duke of Zhou is also credited with formalizing the legal and ritual basis of the feudal system of ancient China, the foundations atop which the Zhou dynasty endured for a further 800 years. Confucius revered him as a model of the ideal.

Returning to the *Zhoubi Suanjing*, Shang Gao answers the question posed to him by the Duke of Zhou with the remarks:

> . . . a base of three in breadth, the altitude makes four, and the diameter is five diagonally.

This is a special case of the Pythagorean theorem, which for this reason is also referred to as the Shang Gao theorem in China. Shang Gao also outlined a proof of the theorem. Its other name in Chinese uses the characters 勾 (*gou*) and 股 (*gu*), meaning *hook* and *thigh* (or *thigh bone*), respectively, but which were understood in ancient Chinese to refer to the shorter and longer sides about the right angle of a right triangle, that is, its base and altitude (Fig. 3.3).

The *Zhoubi Suanjing* also records a dialogue between the two figures Chen Zi and Rong Fang who are presumed to be later descendants of the Duke of Zhou (sixth and seventh centuries BCE) which includes the general form of the Pythagorean theorem:

> Take the point beneath the sun as the base, and the height of the sun as altitude, square both the base and the altitude and add them, and take the square root to get the oblique distance to the sun.

It is easy to see that this rule was obtained as part of the study of astronomical measurements. Another important mathematical result contained in the *Zhoubi Suanjing* is solar height formula, which was widely used in early astronomy and

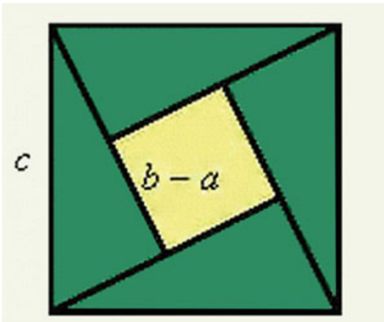


Fig. 3.3 Graphical proof of the Pythagorean theorem by Zhao Shuang

calendrical calculation. For a long time, it was not known how this formula came about, until 1975 when the contemporary Chinese mathematician Wu Wenjun (1919–2017) restored its proof.

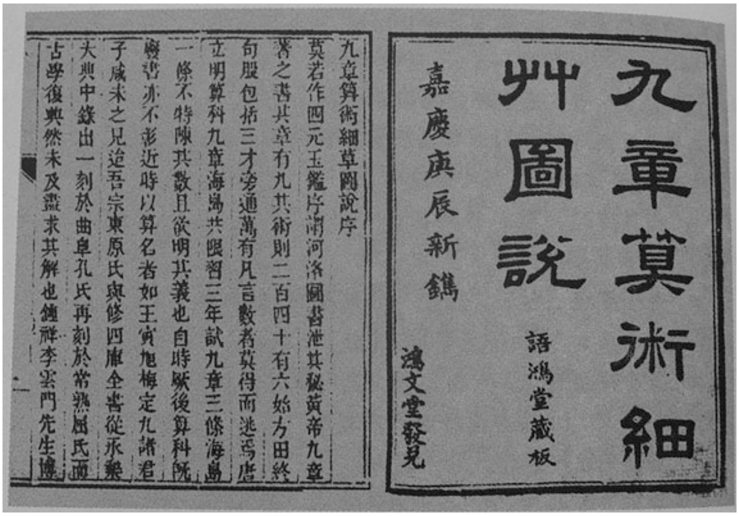
In addition, there appear also the use of fractions, a discussion of multiplication, and a method for finding greatest common denominators, indicating that the concept of square roots was already in use. It is also worth mentioning that the dialogue in the *Zhoubi Suanjing* touches also upon the rules and regulations of the three mythological figures Yu the Great, who was said to have controlled the waters, Fu Xi, and Nüwa. The discussion reveals an early familiarity with surveying methodology and applied mathematics. There are also sporadic bits of geometry, arising as questions of measurement. Needham argues that this seems to indicate that the Chinese people have exhibited arithmetical and mercantile acumen since a very early date. On the other hand, there does not seem to have been much interest in abstract geometry made up of general theorems and propositions atop axiomatic foundations, without specific numerical motivation.

It is gratifying, however, that the Eastern Wu mathematician Zhao Shuang, a third-century commentator on the *Zhoubi Suanjing*, independently proved the Pythagorean theorem in a very beautiful way, by a method of complementary areas. Let the lengths of the two sides about the right angle of a right triangle be a and b as in the figure, with $b > a$. Then the square with hypotenuse c as its sidelength can be decomposed into five areas consisting of a square with sidelength $b - a$ and four triangles congruent to the original right triangle. After some simplification, this gives again $a^2 + b^2 = c^2$. This is similar in favor to the proof we have encountered already in our discussion of Pythagoras above, but whereas that proof is attributed to him only by way of later speculation, the proof presented by Zhao Shuang is authoritatively documented, and moreover he included with his annotations a very beautiful diagram.

Nine Chapters on the Mathematical Art

Unlike the *Zhoubi Suanjing*, somewhat more is known about the authorship and year of composition of the classic *Nine Chapters on the Mathematical Art*. This book was almost certainly developed from the *Nine Arithematical Arts*, one of six compulsory courses (the six arts) taught to the sons of Western Zhou nobles; later it was compiled and supplemented by two mathematicians during the Western Han dynasty, under the leadership of Zhang Cang, a famous politician and thinker who had personally contributed to the formulation of laws, measures, and weights as prime minister under Emperor Wen of Han. In general, the *Nine Chapters on the Mathematical Art* seems to be the product of a continual process of synthesis and revision lasting from the pre-Qin era through to the middle of the Western Han dynasty (Fig. 3.4).

The book takes the form of a problem set, containing 246 problems divided across its 9 chapters, which are as follows: (1) *Fangtian* (方田) – Bounding Fields,



九章算術細艸圖說

嘉慶庚辰新鐫

語鴻堂藏板

鴻文堂發兌

九章算術細草圖說序

莫若作四元玉鑑序謂河洛圖書泄其秘黃帝九章著之書其章有九其術則二百四十有六始方田終句股包括三才旁通萬有凡言數者莫得而逃焉唐立明算科九章海島共限習三年試九章三條海島一條不特陳其數且欲明其義也自時厥後算科既廢習亦不彰近時以算名者如王寅旭梅定九諸君子咸未之見迨吾宗東原氏與修四庫全書從永樂大典中錄出一刻於曲阜孔氏再刻於常熟屈氏而古學復興然未及盡求其解也鍾祥李雲門先生博

Fig. 3.4 The Nine Chapters on the Mathematical Art, Qing dynasty engraving

(2) *Sumi* (粟米) – Millet and Rice, (3) *Cuifen* (衰分) – Proportional Distribution, (4) *Shaoguang* (少廣) – Dimension Reduction, (5) *Shanggong* (商功) – Figure Construction, (6) *Junshu* (均輸) – Equitable Taxation, (7) *Yingbuzu* (盈不足) – Excess and Deficit, (8) *Fangcheng* (方程) – Equations, and (9) *Gougu* (勾股) – Right Triangles. It can be seen from the chapter titles alone that the primary focus of the book is calculation and mathematical applications. The only materials related to geometry concern primarily the calculation of areas and volumes.

The chapters entitled *Millet and Rice*, *Proportional Distribution*, and *Equitable Taxation* deal with proportions of numbers in a way that contrast sharply with the geometric theory of proportions developed by the Greeks via line segments. The topic of *Proportional Distribution* is concerned with distribution of wealth and commodities according to fixed proportional rates, *Equitable Taxation* addresses more advanced problems of proportion, and *Millet and Rice* concerns the solution to even distribution of the burden of grain transportation.

The most academically valuable arithmetic problem in the book is the method of excess and deficit, which concerns the solution of equations using the principle known later as the rule of false position. Consider an equation $f(x) = 0$, and suppose the two values $f(x_1) = y_1$ and $f(x_2) = -y_2$ are known. Then the root is given by

$$x = \frac{x_1 y_2 + x_2 y_1}{y_1 + y_2} = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_1) - f(x_2)}.$$

If f is linear, then this solution is exact, whereas for nonlinear f, it provides only an approximation. From the modern perspective, this technique is equivalent to the method of linear interpolation.

In the thirteenth century, the Italian mathematician Fibonacci included in his treatise the *Liber Abaci* a chapter devoted to the method of excess and deficit, which he called the *Method Elchataym*, transliterating an Arabic word which has been conjectured to refer to the archaic designation *Khitan* or *Cathai* for China, although it also translates directly as *the two errors*. It is all the same by no means inconceivable that this method was spread to Arabic countries through Central Asia by way of the Silk Road and later transmitted to the western world via Arabic sources.

The *Nine Chapters on the Mathematical Art* presents more substantial results in the field of algebra. In the chapter *Fangcheng* dealing with equations, there appear already solutions to linear systems of equations, for example,

$$\begin{cases} x + 2y + 3z &= 26 \\ 2x + 3y + z &= 34 \\ 3x + 2y + z &= 39 \end{cases}.$$

Such systems are presented without the use of any symbol for unknown or indeterminate quantities. Rather, the coefficients and constants are presented as an array or matrix, as in

$$\begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \\ \hline \end{array}.$$

Then by a method referred to as *multiply and directly divide*, this system is transformed so that there are zeros everywhere except along the antidiagonal:

$$\begin{array}{|ccc|} \hline 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \\ 11 & 17 & 37 \\ \hline \end{array},$$

from which the solution can be obtained. This method is equivalent to that known in western countries as Gaussian elimination, and this art of equation solving is considered a jewel in the history of Chinese mathematics (Fig. 3.5).

There are two more very notable features of the *Nine Chapters on the Mathematical Art*. The first is the inclusion of both positive and negative numbers and the rules for the addition and subtraction of both. The other concerns the root extraction method, about which it is stated, "if the root extraction method continues without end, then it is impossible to extract the root." The former shows that Chinese mathematicians were comfortable with negative quantities very early on, in contrast with Indian mathematicians, who introduced negative numbers in the

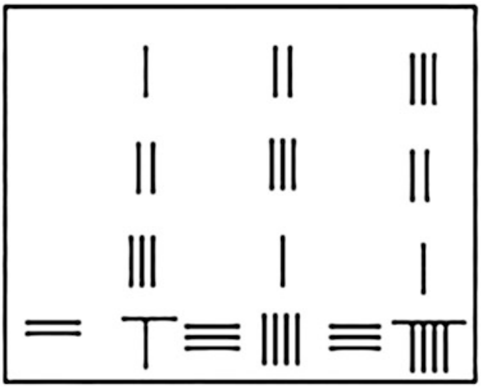


Fig. 3.5 Representation of simultaneous equations by counting-rods

seventh century, and western mathematicians, who accepted them only much later. The latter shows that Chinese mathematicians were aware already of the existence of irrational numbers, although they did not pay it serious heed on account of its inclusion as a curiosity in the process of solving equations. The Greeks, who prized rigorous deduction above all things, took more notice; they were not ones to easily abandon an opportunity worth pursuing.

It is in the treatment of geometrical problems in the *Nine Chapters on the Mathematical Art* that the deficiencies of ancient Chinese mathematicians become apparent. For example, the formula for the approximation of the area of a circle in the *Bounding Fields* chapter makes use of the approximate value $\pi \approx 3$, identical to the value used by the ancient Babylonians. The formula given for the volume of a sphere is only half the exact value obtained in Greece by Archimedes, and incorporating into this formula the very imprecise approximation for π, the error is even worse. On the other hand, there are basically correct formulas for the areas and volumes of linear geometric figures. One way to summarize the situation is that the *Nine Chapters on the Mathematical Art* arithmetizes or algebraizes geometric problems, just as Euclid's *Elements* geometrizes algebraic problems. Unfortunately, no derivation is given for the algorithmic treatment of geometric problems in the text, so it can be considered really only as a practical geometrical toolkit.

From Circle Divisions to the Method of Four Unknowns

Liu Hui's π Algorithm

In the year 391 CE, after years of conflict both within the Christian church and between the local church and the Holy See in Rome, Emperor Theodosius I, who abolished the Olympic games and divided Rome in two, sanctioned or at least failed to prevent the destruction of the Temple of Serapis at Alexandria, and with it the treasures and Greek manuscripts Cleopatra had earlier ordered to be rescued from the old Library. In China at that time, the Eastern Han dynasty, which had produced Cai Lun, who had improved the science of papermaking, and Zhang Heng,[5] a remarkable scientist and polymath, had already split apart, and the Sui dynasty had not yet risen to power. This was the turbulent period of the Wei, Jin, and Southern and Northern dynasties. After a long period in which Confucianism was the dominant trend in intellectual life, this period saw a newfound spirit of speculative thought, producing the Wei-Jin philosophy and the Seven Sages of the Bamboo Grove, remembered still today.

The Wei-Jin style refers to the habits and demeanor of the leading figures of the period; it has sometimes also been called Wei-Jin romanticism. The central principles of this style were its admiration for nature, detachment, directness, and magnanimity. Its adherents admired refined eloquence, enjoyed alcohol, and cared little for worldly affairs, preferring instead an aesthetic seclusion. The Wei-Jin thinkers referred to the seminal texts the *I Ching* (or *Book of Changes*), the *Zhuangzi*, and the *Laozi* as *The Three Xuan* (三玄, meaning *three profound studies*), and *qingtan* (（清谈,*idle conversation*) or *xuantan* (玄谈, *profound conversation*) came to refer to the doctrine of pure conversation in metaphysics and philosophy. At the end of the Wei dynasty and the beginning of the Jin dynasty, the representatives of the Wei-Jin school were the Seven Sages of the Bamboo Grove, a collective of scholars, writers, and musicians headed by the poets Ruan Ji and Ji Kang. In later times, the Wei-Jin style became a popular aesthetic ideal for the demeanor and self-expression of the scholar-official (Fig. 3.6).

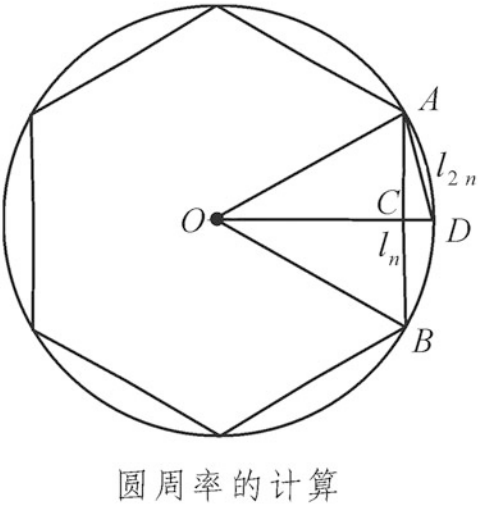
In the atmosphere of this social and humanistic environment, Chinese mathematics also experienced a new flourishing. Several academic works appeared in the form of commentaries on the *Zhoubi Suanjing* or the *Nine Chapters on the Mathematical Art*, in particular providing proofs for some of the important conclusions in these books. One of the pioneers of this practice was Zhao Shuang (from the Eastern Wu state of the Three Kingdoms period), whom we have already encountered, and its most accomplished practitioner was Liu Hui. Like Zhao Shuang, we do not know the dates of his birth or death, only that he lived sometime in the third century and that he wrote his *Notes on the Nine Chapters on the Mathematical Art* in the year 263, before the collapse of the Wei and Wu states. It is difficult to determine whether Zhao Shuang or Liu Hui was the earlier mathematician; both are recognized as the earliest Chinese mathematicians to have made major individual contributions to mathematics (Fig. 3.7).

Liu Hui was able to verify and justify various geometrical calculations in the *Nine Chapters on the Mathematical Art* by a method of subdivision and complementary areas identical to the method used by Zhao Shuang in his proof of the Pythagorean theorem, inaugurating a standard of logical proof for mathematical propositions in

[5] Zhang Heng (78–139) is famous among other things for having invented the first seismoscope. He is also said to have given the value $\frac{730}{232} = 3.1466$ as an approximation for π. If true, this approximation predates Liu Hui, but unfortunately none of his mathematical works have survived. He was also known for his talents as a painter and a writer.

Fig. 3.6 Liu Hui, a mathematician of the Wei and Jin period

Fig. 3.7 Calculation of π



ancient Chinese mathematics. Liu Hui also noticed an important limitation of this method: it cannot be extended to three-dimensional figures, since it is not possible in three dimensions as it is in two to transform any figure into another of identical volume by a process of planar cuts and rearrangements. In order to circumvent this obstacle to the determination of volume formulas, Liu Hui resorted to infinitesimal methods, just as Archimedes had. He used two such methods, a method of limits and a method of indivisibles, and he determined in this way that the formula for the volume of a sphere in the *Nine Chapters on the Mathematical Art* was incorrect.

In more detail, Liu Hui considered two cylinders inscribed in a cube with perpendicular axes, such that their intersection just touches the sphere inscribed in the cube. He called this figure a *box-lid* (牟合方盖) and determined that the ratio of the volume of the sphere to the volume of the box-lid should be as $\frac{\pi}{4}$. His innovations in this argument come very close to Cavalieri's principle, developed many centuries later by an Italian mathematician who played an important role in the development of integral calculus. Liu Hui did not however take the further step of calculating its general form and was not able to determine the volume of his box-lid or correspondingly the volume of the sphere. On the other hand, his methods paved the way for Zu Chongzhi and his son Zu Geng to complete this work some two centuries later.

In addition to his annotations to *Nine Chapters on the Mathematical Art*, Liu Hui added as a tenth chapter to this book an essay of his own composition, later published separately as *The Sea Island Mathematical Manual* (海岛算经). In this book, Liu Hui develops his *double-difference-algorithm* (重差术), an important tool in ancient astronomy, and *The Sea Island Mathematical Manual* became a classic in the field of surveying. But Liu Hui's most famous and valuable work is the technique of circle division he introduces in his commentary on the first chapter of the *Nine Chapters on the Mathematical Art* for the determination of the circumference and area of a circle and an approximation algorithm for π. The basic idea is to approximate a circle by an inscribed regular polygon. Liu Hui writes:

> If the division is fine, then the deficit is less, and if the process of division is continued and continued until the point of indivisibility, then it will become as one with the circle without any deficit whatsoever.

He also noticed that the sidelength l_{2n} of a regular $2n$-gon can be easily obtained from the sidelength l_n of a regular n-gon by a double application of the Pythagorean theorem. In the figure, if the radius of the circle is r, then

$$\begin{aligned} l_{2n} &= AD = \sqrt{AC^2 + CD^2} \\ &= \sqrt{\left(\frac{1}{2}l_n\right)^2 + \left(r - \sqrt{r^2 - \left(\frac{1}{2}l_n\right)^2}\right)^2}. \end{aligned}$$

If we put $r = 1$ and start from the regular hexagon and double the number of sides five times, we obtain from the regular 192-gon ($192 = 6 \times 2^5$) an approximation

$$\pi \approx \frac{157}{50} = 3.14,$$

which Liu Hui argued was a fine enough approximation for practical purposes. This is all basically consistent with the results and methods employed by Archimedes in the third century BCE, except that Archimedes made use of both inscribed and circumscribed polygons and was able therefore to obtain the same value from polygons with only $96 = 6 \times 2^4$ sides. In a note that cannot definitely be attributed to Liu Hui, this computation is carried out as far as a polygon with $3072 = 6 \times 2^9$ sides to obtain the approximation

$$\pi \approx \frac{3927}{1250} = 3.1416.$$

In light of his extraordinary achievements in mathematics, Emperor Huizong of Song honored him as a noble man of Zi (淄乡男) in the year 1109. Since this honorific was customarily designated after the hometown of its recipient at that time, we can infer from this that Liu Hui was from either Linzi or Zibo in Shandong Province. As the birthplace of Confucius and Confucianism in the time of the Qi and Lu states in the spring and autumn period, the academic atmosphere of this region was refined from throughout the Han dynasty up to the Wei and Jin period, a rich cultural environment in which Liu Hui would have had exposure to extensive scholarly debate and history. It can be seen in his writings that he was indeed familiar with a wide range of earlier thought and worked from within the position of freedom from ideology in his time. This no doubt contributed to his ability to achieve such remarkable results in mathematics.

Three years after Liu Hui completed his annotations to *Nine Chapters on the Mathematical Art*, China experienced its second reunification (the first being the establishment of the Qin dynasty), when Sima Yan, a general of the Wei state, established the Jin dynasty (Western Jin) as its first emperor, Emperor Wu of Jin. Increased economic development and interregional exchange during this period stimulated the emergence of geography as an intellectual discipline, culminating in the works of the cartographer Pei Xiu, who proposed six principles for cartography, including consistent scaling, and standards for orientation and distance, setting down the theoretical framework for the future of Chinese cartography. New customs and habits also sprang up during this period, including the consumption of tea, and several new tools were invented in order to save labor, including the wheelbarrow and the water mill. In the year 283, the Daoist naturalist and alchemist Ge Hong was born.

The northern regions however still suffered under constant threat of foreign invasion. In the year 317, the Jin family was forced to relocate to the south of the Yangtze River and set up the capital of their empire in Jiankang (now Nanjing). This became the Eastern Jin dynasty, which lasted for just over a century, during which time the north split up into 16 small countries. Subsequently the Jin dynasty in the south was destroyed, and four military figures in succession took power by force and changed the name of the regime: first the Liu Song dynasty and then in order the Southern Qi dynasty, the Liang dynasty, and the Chen dynasty, collectively referred to as the Southern dynasties. This period lasted about 170 years, with the capital still at Jiankang throughout. In the year 429, 10 years into the Liu Song dynasty, Zu Chongzhi was born into an erudite and respected family of calendarists in the

Fig. 3.8 Zu Chongzhi, a mathematician of the Liu Song and Southern Qi dynasties

capital city. Although his professional achievements consist of minor official posts in Zhenjiang (Southern Xuzhou), Suzhou, and other places, his central achievement was in mathematics, for which he earned a place in history as the first mathematician in China to be listed in the official dynastic histories (FIg. 3.8).

In the *Book of Sui*, the official history of the Sui dynasty, Zu Chongzhi is credited with the lower and upper bounds

$$3.1415926 < \pi < 3.1415927$$

for the value of π, which is accurate to the seventh decimal place. This is his most important mathematical achievement, and this level of accuracy was not surpassed until the year 1424, when the Persian mathematician Jamshīd al-Kāshī obtained an approximation valid up to the 17th decimal digit. Consensus opinion is that Zu Chongzhi achieved this approximation via Liu Hui's method of circle division, a feat of incredible perseverance: by this method, it is necessary to carry out the computation up to a polygon with 24576 sides to arrive at the data above.

In the same book, there appears another result due to Zu Chonghi's calculations with π: the fraction approximations $\pi \approx \frac{22}{7}$ and $\pi \approx \frac{355}{113}$. The former is consistent with approximations by Archimedes and valid to two decimal places; the latter is accurate to six decimal places. In modern mathematics these fractions appear as the first few convergents in the continued fraction presentation of π

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \ldots.$$

The first term is of course the approximation used by the Babylonians and in the *Nine Chapters on the Mathematical Art*; we can call it the ancient approximation. The second and fourth terms are known as the approximate ratio (约率) and close ratio (密率) in China due to Zu Chongzhi. The latter is the best rational approximation for π with numerator and denominator not exceeding 1000.

In 1913, the Japanese mathematician and historian of mathematics Yoshio Mikami advocated in his influential book *The Development of Mathematics in China and Japan* that the rational approximation $\frac{355}{113}$ for π be designated as Zu's ratio in honor of Zu Chongzhi. In Europe, this result was not recapitulated until 1573, when it was rediscovered by the German mathematician Valentinus Otho. Unfortunately, we do not know even to this day how Zu Chongzhi arrived at this approximation. It cannot be reached directly by the method of circle division, and there is no evidence that ancient Chinese mathematicians had any concept of or practical experience with continued fractions. Some historians have speculated that he used a fractional interpolation method known as *harmonization of the divisor of the day* (调日法) developed by He Chengtian, a mathematician and calendarist contemporary to Zu Chongzhi.

Briefly stated, this method consists of updating lower and upper rational approximations $\frac{a}{b}$ and $\frac{c}{d}$, respectively, to a better approximation $\frac{ma+nc}{mb+nd}$ by a suitable selection of weights m and n. If you take $m = 1, n = 9$ with known upper and lower approximations $\frac{157}{50}$ and $\frac{22}{7}$ or, alternatively $m = 1, n = 16$ with known upper and lower approximations $\frac{3}{1}$ and $\frac{22}{7}$, in either case you obtain the close ratio $\frac{355}{113}$. We can speculate that after Zu Chongzhi obtained his rational approximations by this method, he used the method of circle division to verify their validity, much as Archimedes verified his results obtained by arguments from equilibrium by proofs using the method of exhaustion.

Like Liu Hui, another mathematical achievement of Zu Chongzhi is in the calculation of the volume of a sphere. This result appeared in a chapter on calendrics in a political work entitled *Song Shu* and was mostly likely also included in his mathematical treatise *Methods for Interpolation* or *Zhui Shu* (缀术), which has unfortunately been lost since the Song dynasty. Intriguingly, the Tang dynasty mathematician Li Chunfeng referred to this result in yet another annotation of the *Nine Chapters on the Mathematical Art* as Zu Geng's cube root extraction technique. Zu Geng was Zu Chongzhi's son and also an accomplished mathematician. Modern historians generally attribute the derivation in China of the correct formula for the volume of a sphere to the Zu family, father and son together (Fig. 3.9).

According to Li Chunfeng's description, they calculated the volume of Liu Hui's box-lid as follows. Take first a cube with sidelength given by the radius r of a circle. Fix one vertex as the center of a circle with radius r, and remove the cross-section of the cube cut out by this circle. Carrying out this process both horizontally and vertically produces a truncated cube obtained as the intersection of two cylinders with perpendicular axes. In total, the cube is subdivided into four volumes: the intersection of two cylinders is one (the interior, covered by $\frac{1}{8}$ of the box-lid), and there are three exterior volumes. The key to the problem was the calculation

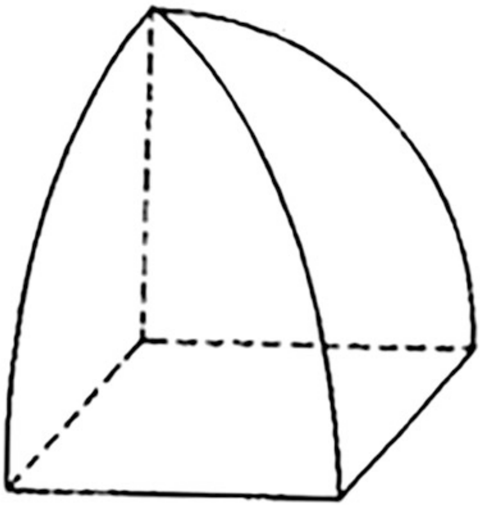


Fig. 3.9 One eighth of the box-lid

of the volume of the outer three components. Zu Geng found that the sum of the cross-sectional area of these three parts at any given height is equal to that of an inverted square cone with volume equal to $\frac{1}{3}$ the volume of the cube. It follows that the volume of the inner component is $\frac{2}{3}$ the volume of the cube, and therefore the volume of the box-lid is given by $\frac{16}{3}r^2$. Finally, from Liu Hui's calculation that the ratio of the volume of the sphere to the volume of the box-lid is $\frac{4}{\pi}$, we get Archimedes's formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3.$$

The contemporary Chinese historian of mathematics Li Wenlin has observed:

> The work of Liu Hui and the father and son Zu Chongzhi and Zu Geng is very profound. It reflects the tendency towards disputation and rigor that appeared in Chinese classical mathematics throughout the Wei, Jin, and Southern and Northern dynasties, and marks the culmination of this tendency. But what is puzzling is that this tendency came to a very abrupt end with the end of this period.

The text *Zhui Shu* in which Zu Chongzhi compiled his mathematical results was listed alongside the *Nine Chapters on the Mathematical Art* as an official textbook in both the Sui and Tang dynasties, and the School of Mathematics at the Imperial Academy (Guozijian, 国子监) included it as required reading with a recommended period of study lasting as long as 4 years. The influence of this book spread even as far as Korea and Japan, but it disappeared completely after the tenth century.

The Sun Zi-Qin Jiushao Theorem

In the year 639, Arabic forces invaded Egypt on a large scale. At this time, the Romans had long since withdrawn, and Egypt was under the administrative control of Byzantium. After 3 years of fighting, the Byzantine army was forced to withdraw. The last few scraps of the former academic treasure trove that was Alexandria were burned, and ancient Greek civilization came to its decisive end. After that, Cairo came into being, and the Egyptian people took up the Arabic language and embraced the Muslim religion. At the same time in China, the Tang dynasty was seeing its golden age under the rule of Emperor Taizong (Li Shimin). This was the most prosperous era in the history of feudal China, a period of continuous territorial expansion. The capital city Chang'an, known today as Xi'an, was a gathering place for merchants and luminaries from various countries, and China was in frequent contact with western regions and other lands (Fig. 3.10).

Although the Tang dynasty did not produce any mathematicians comparable in achievement to those of the previous Wei, Jin, Southern, and Northern dynasties, or the later Song and Yuan dynasties, nevertheless this period saw substantial achievements in the establishment of systematic mathematical education and the compilation of earlier mathematical classics. The Tang dynasty extended the "School of Computation" initiated during the Northern and Sui dynasties and established *Doctor of Arithmetic*[6] as an official title. Mathematics was also added during this time to the imperial examinations, and anyone who could successfully pass the mathematical examination would be awarded an official title, although this title was the lowest ranking among all official titles and it was abolished in the late Tang dynasty. But in general, the predominant strains in the intellectual atmosphere of the Tang dynasty were humanistic, without much concern for science and technology, somewhat similar in favor to the Italian renaissance. The most significant mathematical event of the Tang dynasty, which lasted for nearly 300 years, was the compilation and publication of the *Ten Computational Canons* by Li Chunfeng under the rule of Emperor Gaozong (Li Zhi) (Fig. 3.11).

Li Chunfeng (602–670) was known also for his astronomical work and the composition of a remarkable fortunetelling book entitled *Massage-Chart Prophecies* (推背图). In his *Yisizhan*, one of the earliest monographs on meteorology in world history, Li Chunfeng classified wind strength into 8 levels, or rather 10 if no wind and a light breeze are included, a system that was echoed in 1805 when a British hydrographer introduced a scale from 0 to 12 for wind speed that remains in use today.

In addition to the *Zhoubi Suanjing*, the *Nine Chapters on the Mathematical Art*, the *Sea Island Mathematical Manual*, and the *Zhui Shu*, there are three more books in the *Ten Computational Canons* worth mentioning. These are the *Sunzi Suanjing*

[6] This was not the earliest title to designate a specialist in a single art. The first was *Doctor of Law*, established during the Western Jin dynasty, and after that *Doctor of Medicine* was added under the Northern Wei dynasty.

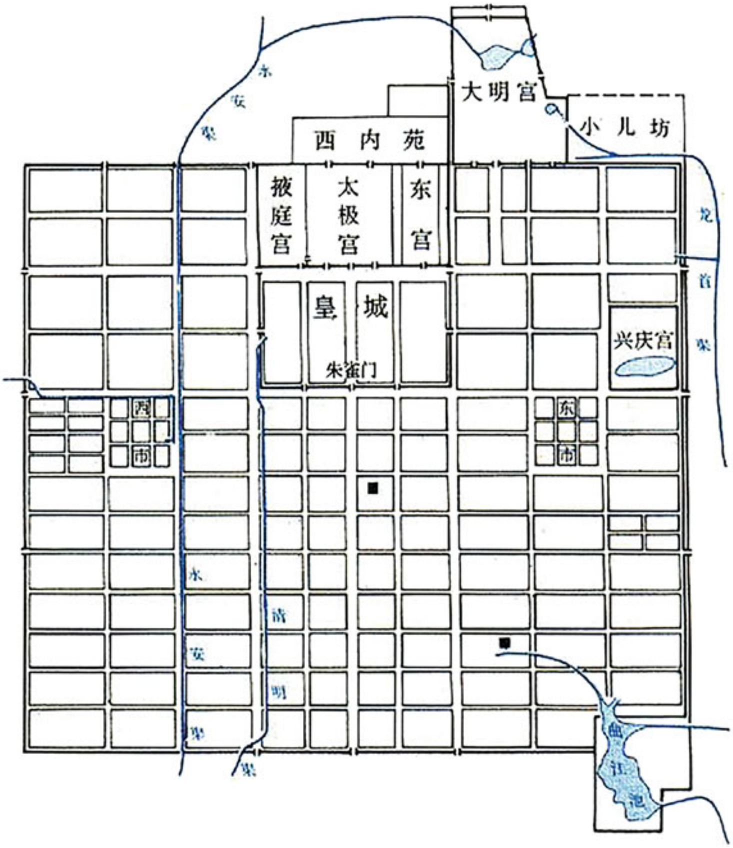


Fig. 3.10 Plan of Chang-an city during the Tang dynasty, featuring rectangles arranged in squares

(孙子算经), or *The Mathematical Classic of Sun Zi*; the *Zhang Qiujian Suanjing* (张丘建算经), or *The Mathematical Classic of Zhang Qiujian*; and the *Jigu Suanjing* (缉古算经), or *The Continuation of Ancient Mathematics Classic*. Each of these books raises some very valuable question to pass down to the world (Fig. 3.12).

The author of the *Sunzi Suanjing* is not known today, although presumably his surname was Sun; this book is generally believed to have been written sometime in the fourth century. The best known feature of the *Sunzi Suanjing* is the problem of the unknown number, which is stated as:

> Now there are unknown number of things; if we count by threes there is a remainder of two, if we count by fives a remainder of three, and if we count by sevens a remainder of two. What is the number?

Fig. 3.11 Chunfeng temple; photograph by the author in Langzhong, Sichuan province

This is equivalent to the system of congruences

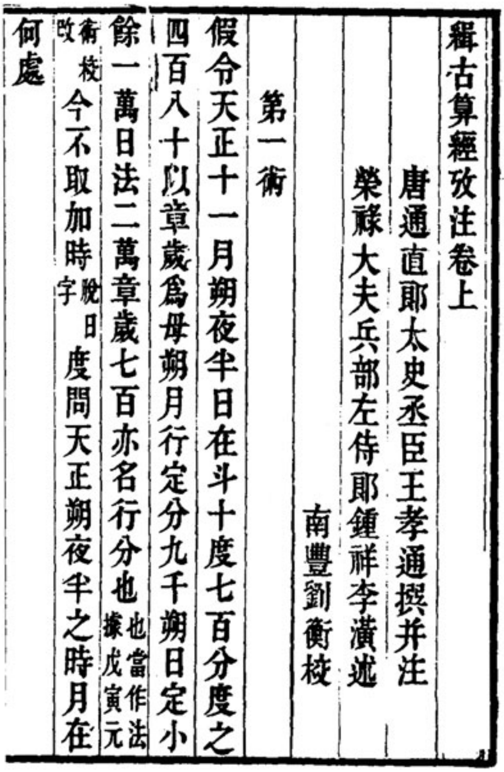
$$\begin{cases} n & \equiv 2 \pmod 3 \\ n & \equiv 3 \pmod 5 \\ n & \equiv 2 \pmod 7 \end{cases}.$$

The given answer is $n = 23$, which is the smallest positive integer simultaneously satisfying these three congruences. The book also discusses the method for solving this problem, where the remainders 2, 3, and 2 can be replaced by any numbers, a special case of the Chinese remainder theorem. For this reason this theorem is also known as Sun Zi's theorem, although a fully general method was not given until the Song dynasty when Qin Jiushao presented it. In the eighth century, the Tang dynasty monk and astronomer Yi Xing (673–727) used this result to formulate the calendar.

The *Zhang Qiujian Suanjing* was written in the fifth century, and its author was a native of the Northern Wei dynasty. The highlight of this book is its last topic, generally known as the *Hundred Fowls Problem*. The problem statement is as follows:

> Now one cock is worth 5 qian, one hen 3 qian and 3 chicks 1 qian. It is required to buy 100 fowls with 100 qian. In each case, find the number of cocks, hens and chicks bought.[7]

[7] Tr. Lam Lay Yong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假令天正十一月朔夜半日在斗十度七百分度之四百八十以章歲爲母朔月行定分九千朔日定小餘一萬日法二萬章歲七百亦名行分也也當作法據戊寅元改衡校今不取加時脫日字度問天正朔夜半之時月在何處

Fig. 3.12 Qing dynasty edition of the *Jigu Suanjing*

If we label the number of cocks, hens, and chicks x, y, z, respectively, then in modern notation, this problem asks for solutions in positive integers of the system of indefinite equations

$$\begin{cases} x + y + z & = 100 \\ 5x + 3y + \frac{z}{3} & = 100 \end{cases}.$$

Zhang Qiujian gives all of the three possible solutions with each of x, y, z nonzero:

$$\begin{cases} x = 4, \ y = 18, \ z = 78 \\ x = 8, \ y = 11, \ z = 81 \\ x = 12, \ y = 4, \ z = 84 \end{cases}.$$

These can be obtained by transforming the two linear equations in three variables into equations for y and z in terms of a parameter $x = 4t$ and solving for positive values of y. In modern times, we know that a linear equation in several variables can give rise to general solutions. But issues along these lines were not explored

Fig. 3.13 Statue of the Tang dynasty monk and mathematician Yi Xing; photograph by the author in Xi'an

until much later, when Fibonacci investigated similar problems in thirteenth-century Italy, as did Jamshīd al-Kāshī in fifteenth-century Iran. Unlike Sun Zi, whose work was extended by Qin Jiushao, Zhang Qiujian did not follow up his computational achievement with any efforts toward a more general result, and nobody seems to have looked more deeply into it (Fig. 3.13).

The *Jigu Suanjing* is the most recent of the books in the *Ten Computational Canons*, written in the seventh century by Wang Xiaotong, a Doctor of Mathematics of the early Tang dynasty and probably the most accomplished among all the mathematicians to hold this title. This book is yet again a collection of practical problems, but people found it very difficult at the time. Most of the problems concern astronomical calendry, civil engineering, warehouse and storage cellar sizes, and Pythagorean problems, and most require biquadratic or higher-order polynomials. The books lists 28 equations of the form

$$x^3 + px^2 + qx = c$$

in positive coefficients and provides annotations detailing the origins of each coefficient. The author supplies the positive rational roots, but no general solution method. Nevertheless, this is the oldest document in the history of world mathematics concerning the numerical solution of cubic equations and their applications.

It is worth a mention that the oldest surviving paper book in the world, the Chinese edition of the Indian Buddhist classic *The Diamond Sutra*, was printed during the Tang dynasty, in the year 868. A copy of this book was found among the *Dunhuang* (敦煌) manuscripts in 1900 and purchased by the British archaeologist

Sir Marc Aurel Stein (1862–1943). It was displayed at one point at the British Museum in London and is now housed at the British Library. In any case, the much earlier *Ten Computational Canons* certainly has not survived in any original edition. When the Italian missionary Matteo Ricci spent time in China much later during the Ming dynasty, China had extremely large volumes of books in circulation available at very low prices.

In spite of the economic and cultural prosperity of the Tang dynasty, in later periods after the end of the ninth century, many semiautonomous governments of hereditary rule began to spring up around the borderlands, and the bureaucratic central government was no longer able to restrain them. Increasing levels of taxation and the participation of these chieftans in the suppression of the Huang Chao peasant uprising expanded their power significantly, and by the year 907, the Tang dynasty had come to an end, and China was once again a state divided. This was the beginning of the Five Dynasties period, which saw the quick succession of five separate dynasties in the span of only half a century: the Later Liang, Later Tang, Later Jin, Later Han, and Later Zhou dynasties. The capital was moved to Kaifeng or Luoyang, two cities nearby to one another in the heart of Henan Province. The aftermath of all this unrest caused the loss of many classics, including Zu Chongzhi's *Zhui Shu*. During this time there separately appeared also ten small countries in the south, including the Southern Tang kingdom with its capital at Jinling, another name for Nanjing. The last ruler Li Yu of Southern Tang became a great lyric poet following the destruction of his country.

But "the empire long divided must unite, long united must divide,[8]" as goes the famous opening line of the *Romance of the Three Kingdoms* by Luo Guanzhong. In the year 960, a soldier from Henan Province named Zhao Kuazngyin took power at the urging of his soldiers and became the first emperor Emperor Taizu of the Song dynasty in a bloodless coup, after which he "dissolved the military power over a glass of wine" and released many of his generals into retirement to return to their hometowns with a general prohibition against looting and violence. Following this reunification, there were developments in Chinese society that were altogether conducive to cultural and scientific undertakings. A special form of prose poetry known as *Songci* (宋词) brought literary culture to its highest peak since the Tang dynasty. Commerce and craft saw a period of great prosperity and produced a flurry of technological advancements, including three of the four great inventions of ancient China: printing, gunpowder, and the compass. All this injected a new vitality into the cultivation of mathematics. In particular, the invention of movable type printing technology facilitated the convenient preservation and dissemination of mathematical texts. The first known mathematical book to be printed was Liu Hui's *Sea Island Mathematical Manual*.

Needham remarks in his *Science and Civilization in China* that Sun Zi's result is not of sufficient generality to quite be considered a theorem; but he also points out that the four greatest mathematicians in the history of ancient Chinese

[8] Tr. Moss Roberts

Fig. 3.14 Statue of Qin Jiushao; photograph by the author in Nanjing

mathematics appeared in the (Southern) Song dynasty, around the thirteenth century, coincidentally the last days of the European Middle Ages. These were Yang Hui, Qin Jiushao, Li Ye, and Zhu Shijie, known as the four great masters of the Song and Yuan dynasties. In addition to these four mathematicians, there were also two significant mathematicians of the Northern Song dynasty: Shen Kuo and Jia Xian. Of the six of them, Qin Jiushao is the most legendary and best-known; he is perhaps the most accomplished mathematician of ancient China (Fig. 3.14).

Qin Jiushao (1202 or 1208–1261) is known to us on the basis of a relatively short academic career. His ancestors came from what is now Fan County, in Henan Province, though sometimes this territory has also fallen under the administrative control of Shandong Province, and Qin Jiushao himself was born in Anyue, in Sichuan. His hometown was a tumultuous place for many years, and he and his family spent part of his youth living in the capital city Lin'an. As an adult he left Sichuan again, passed the imperial examinations, and served as an administrator in Hubei, Anhui, Jiangsu, Fujian, and other places. During his tenure in Nanjing, his mother passed away, and Qin Jiushao left his post to return to Huzhou in Zhejian province. It was during a period of 3 years in mourning in Huzhou that he took up seriously the study of mathematics and wrote his treatise *Mathematical Treatise in Nine Sections* (数书九章), a work completely surpassing its predecessor the *Nine Chapters on the Mathematical Art*.

The two most important achievements in the *Mathematical Treatise in Nine Sections* are the "positive and negative evolution method" and the "Da Yan Shu" (大衍总数术). The positive and negative evolution method, known also as

Qin Jiushao's algorithm, is an algorithm for the numerical solution of algebraic equations of any degree, that is, polynomial equations of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

with positive or negative coefficients. Ordinarily, the solution of such an equation requires an iterative method in which the value of the polynomial is repeatedly evaluated, with each evaluation in turn requiring $\frac{n(n+1)}{2}$ multiplications and n additions, but Qin Jiushao converts the problem into a system of n linear equations, requiring only n multiplications and n additions to solve. Even through to the present day, Qin Jiushao's method has important applications in the age of computer algorithms (Fig. 3.15).

The Da Yan Shu is a mathematically precise generalization and statement of Sun Zi's theorem. In modern notation and terminology, suppose $m_1, \ldots, m_k$ are pairwise relatively prime integers larger than 1. Then for any integers $a_1, \ldots, a_k$, the system

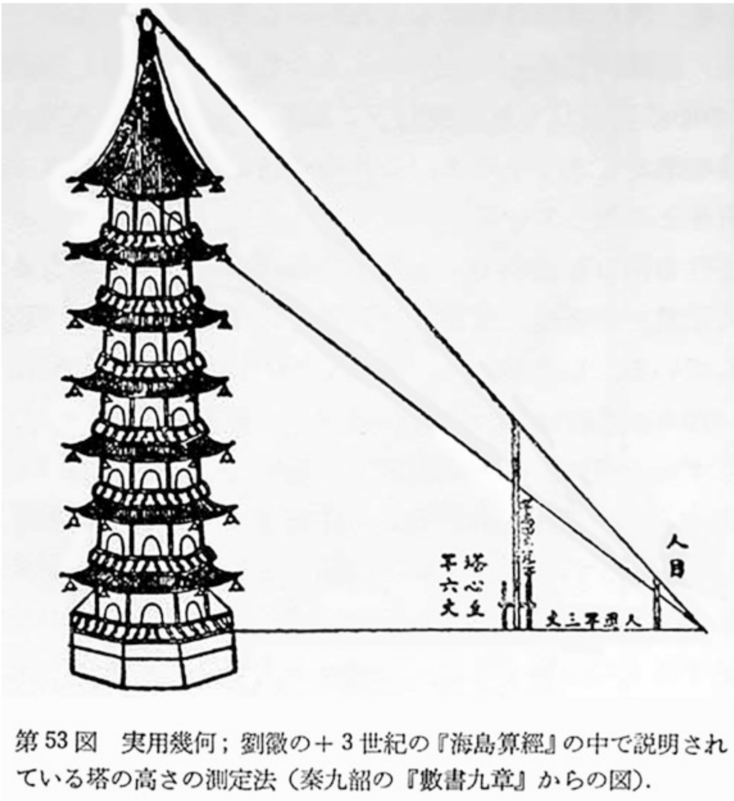

Fig. 3.15 Illustration from a Japanese edition of the *Mathematical Treatise in Nine Sections*

of simultaneous congruences

$$\begin{cases} x & \equiv a_1 \pmod{m_1} \\ & \vdots \\ x & \equiv a_k \pmod{m_k} \end{cases}$$

has a unique solution modulo $m_1 \cdots m_k$. Qiu Jinshao further discusses the detailed method for determining this solution, for which purpose he introduces the linear congruence

$$ax \equiv 1 \pmod{m}$$

where a and m are relatively prime integers. He makes use of the algorithm known in modern elementary number theory as the division algorithm, or the Euclidean algorithm, which he calls in particular the Da Yan Qiu Yi Shu. His method is completely correct and rigorous, with important modern applications in cryptography, in particular the RSA key algorithm.

Sun Zi's theorem is the most perfect and beautiful result in the history of ancient Chinese mathematics. It appears in every modern textbook on number theory, and in western textbooks it is known as the Chinese remainder theorem, perhaps due to a general parcity of well-known results with origins in China. The author of this book feels it should rather be known as the Sun Zi-Qin Jiushao Theorem, or simply the Qin Jiushao Theorem, and refers to it in this way in his own textbook on number theory, *A Modern Introduction to Classical Number Theory* (经典数论的现代导引). Like other ancient Chinese mathematicians, who rarely entered into theoretical abstractions and viewed mathematics primarily from the perspective of applications to calendry, engineering, taxation, and military purposes, Qin Jiushao did not provide a proof of his theorem, although his solution falls really only a single step short of a proof. He did however consider the case in which the moduli are not pairwise relatively prime and provided a computational method to reduce this to the relatively prime case.

In Europe, questions of divisibility and congruence were studied systematically by Euler in the eighteenth century and Gauss in the nineteenth, and they obtained results identical to Qin Jiushao's theorem, including rigorous proofs. After the British missionary and sinologist Alexander Wiley published his *Jottings on the science of the Chinese* in 1853, European academic circles became aware of the pioneering work of Chinese mathematicians and Qin Jiushao in particular in this area, and the Chinese remainder theorem took the name by which it is known today. This result has a generalization in the field of modern algebra, and its scope today extends to branches of mathematics other than number theory. The German historian of mathematics Moritz Cantor referred to Qin Jiushao as the luckiest genius, and the Belgian-born American chemist and historian of science George Sarton wrote that he was "...one of the greatest mathematicians of his race, of his time, and indeed of all times."

Fig. 3.16 Shen Kuo, a naturalist of the Northern Song dynasty

Other Mathematicians

Traveling backward now some 170 years, we arrive at Shen Kuo (1031–1095), who was born in Qiantang (modern-day Hangzhou) and wrote one of the wonderful works in the history of Chinese science, entitled *Dream Pool Essays* (or *Mengxi Bitan*, 梦溪笔谈) in 1086. In his later years, Shenkuo settled on the outskirts of modern-day Zhenjiang, in Jiangsu Province, and purchased there a lavish garden which he named Dream Brook Estate, perhaps in honor of the Dongtiao river flowing through his hometown (Fig. 3.16).

Shen Kuo was a successful candidate in the *Jinshi* (进士) system of imperial examination; he participated in the reforms initiated by the writer Wang Anshi (1021–1086) and came also into contact with the poet Su Shi (1037–1101). Later, he was sent as an envoy to the Khitan Liao Dynasty, and upon his return he served as a member of the Hanlin Imperial Academy as an imperial secretary, with outstanding political achievements. In the course of any and all of his travels, in addition to fulfilling his official obligations, Shen Kuo would diligently record whatever materials of scientific or technological significance he encountered (Fig. 3.17).

He can also be regarded as the greatest naturalist of ancient China, and the *Dream Pool Essays* includes a survey of all the known natural and social sciences of his time. As an example, it was Shen Kuo who identified and measured the inconsistency in the length of the days throughout the year, with the summer solstice as the longest day and the winter solstice the shortest, and he introduced a bold calendar reform consisting of 12 solar months with longer months of 31 days and shorter months of 30 days. In physics, he performed experiments with concave mirror imaging and sound resonance. In geography and geology, Shen Kuo successfully explained the origins of strange landforms as due to the intrusion of

Fig. 3.17 Tomb of Shen Kuo; photograph by the author, in Yuhang, Hangzhou

flowing water, inferred the evolution of geological features from the presence of fossils, and so on.

We turn now to the mathematical achievements recorded in Shen Kuo's writings. In geometry, Shen Kuo rose to the challenge of measuring the lengths of circular arcs and developed a technique for substituted straight lengths for curved ones, later the basis for spherical trigonometry in China. In algebra, he gave a formula for the sum of squares of consecutive adjacent integers as part of the solution to the problem of finding the number of wine barrels that fit in a shape like the frustum of a square pyramid. This is the first example in Chinese mathematics of a sum of higher-order arithmetic series. As a mathematician, Shen Kuo was introspective and considered the essence of mathematics to lie in simplicity. He observed that everything has its own fixed shape, and every shape has its own true number, a mathematical philosophy not far removed from the perspective of Pythagoras.

In contrast, very little is known about the life of Jia Xian (ca. 1010–ca. 1070), a mathematician who was contemporary to Shen Kuo. He wrote a book entitled *The Detailed Solutions of the Yellow Emperor to the Nine Chapters on the Mathematical Art*, which has since been lost. Fortunately, the main content of this book appeared in excerpts some 200 years later in the book *Xiangjie Jiuzhang Suanfa* (祥解九章算法, *A Detailed Analysis of the Nine Chapters on the Mathematical Art*, 1261) by the Southern Song mathematician Yang Hui. This book records Jia Xian's method for the expansion of higher-order binomials according to a source map, which is simply a table of the coefficients in the expansion of $(x + a)^n$ for $0 \leq n \leq 6$:

$$\begin{array}{lccccccccccccc}
n = 0 & & & & & & & 1 & & & & & & \\
n = 1 & & & & & & 1 & & 1 & & & & & \\
n = 2 & & & & & 1 & & 2 & & 1 & & & & \\
n = 3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n = 4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n = 5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n = 6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

This triangle is of course known as Pascal's triangle in western countries, after a French mathematician who discussed it more than 600 years later. In China, it is known as the Jia Xian triangle or the Yang Hui triangle. Jia Xian used this table to compute square roots and achieved unexpected results in this direction, known as the additive-multiplicative method.

As early as the Five Dynasties period, there existed in the northeastern dynasties and in Mongolia a dynasty known as the Liao dynasty or the Khitan Empire, under the rule of the Khitan people and established just at the tail end of the Tang dynasty. At the start of the Song dynasty, Emperor Taizong personally led or sent troops to attack the Liao dynasty, but he quickly found himself on the defensive, and in the end the Song dynasty was compelled to pay a tribute to the Liao and set up a precedent for the regular delivery of property. We have seen in the previous section that Shen Kuo once acted as an envoy to the Liao dynasty. During the same period, there was also a group tribe living in the Heilongjiang river basin in the northeastern part of China, later known as Mongolia, called the Jurchen people (女真), renowned for their skill at horseback riding. The Jurchen people had suffered as vassals of the Khitan rulers of the Liao dynasty, and when the winds of fortune shifted in their favor, they established the Jin dynasty and sent troops to bring about the destruction of the Liao dynasty. They went on to attack the heart of the Northern Song dynasty in Bianjing (Kaifeng), and they captured the father and son Emperors Huizong and Qinzong. The youngest brother of Qinzong took rule as Emperor Gaozong of Song and moved the capital to Hangzhou (at that time called Lin'an) in 1127. This was the beginning of the Southern Song dynasty (Fig. 3.18).

Although the northern threat was ever present, the people of the Southern Song dynasty lived happily through a time of even greater prosperity and cultural development. The mathematician Yang Hui, like Shen Kuo before him, was from the capital city Lin'an. Although we do not have the dates of his birth or death, it is known that Yang Hui lived in the thirteenth century; served as a local official in Taizhou, Suzhou, and elsewhere; and studied mathematics in his spare time. In the space of 15 years spanning 1261 to 1275, Yang Hui completed five substantial mathematical works, including that *Xiangjie Jiuzhang Suanfa* discussed above. His writing is simple and profound, and he developed such a reputation as a mathematician and mathematics educator that people would ask his advice wherever he went.

Fig. 3.18 Korean edition of the Yang Hui Suanfa (1433)

Following up upon Jia Xian's additive-multiplicative method, Yang Hui presented an example of its use to solve quartic equations. This is a highly mechanical computation that can be applied to solve polynomial equations of any degree, essentially identical to Horner's method, developed in 1819 a widely used thereafter in the western world. Yang Hui also used his method of multiplicative piles to calculate the volume of a square prism, and in order to facilitate a fast algorithmic implementation, he introduced for the first time in China the concept of prime numbers, presenting all of the 16 prime numbers between 200 and 300. His research into this topic was of course less sophisticated than what is in Euclid, both in scope and rigor (Fig. 3.19).

His most interesting mathematical contribution however was in the study of magic squares, which at that time were known as vertical-horizontal figures (纵横图). Magic squares first appeared in the *Classic of Changes* or *I Ching* (易经), the oldest classical text in Chinese culture, with provenance stretching as far back as the eleventh century BCE. In this book there appear two cosmological diagrams of numbers called the *Yellow River Map* (*He Tu*, 河图) and *Inscription of the River Luo* (*Luo Shu*, 洛书). According to the legend, Emperor Yu who controlled the waters (other legends say it was Fuxi) appeared on the banks of the Yellow River riding a dragon horse sometime around the year 2200 BCE during a time of deluge and flooding, and there emerged from the waters a magical turtle with the Luo Shu pattern on its shell. The *Yellow River Map* is a figure consisting of five elements arranged in a cross, with two numbers corresponding to each element, one even, one odd, and at the center the number five. The *Luo Shu* is as follows, represented

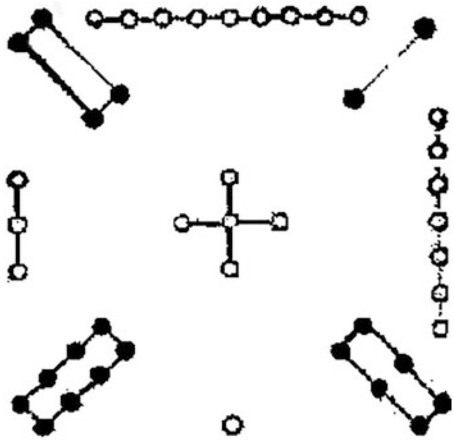

Fig. 3.19 The *Luo Shu* magic square

here in Arabic numerals:

4	9	2
3	5	7
8	1	6

.

The sum along any vertical, horizontal, or diagonal line is in every case 15.

Prior to the thirteenth century, Chinese mathematicians viewed such systems as mere number games, perhaps shrouded in a certain aura of mystery, but not to be taken seriously. Yang Hui however devoted considerable effort to the nature of magic squares and discovered that such systems are governed by rules and regularity. In particular, he used the summation formula for arithmetic series to cleverly construct magic squares of orders three and four. For magic squares of orders five and higher, he gave only examples without indicating any method, but his examples for orders five, six, and even ten were all correct, showing that he had mastered the rules of their composition. Yang Hui called his magic square of order ten, with row and column sums given by the number 505, the hundred numbers figure. He also invented and investigated magic circles. As seen in the diagram, the sum of the eight numbers on any of the four circles or four diameters is 138, except one of each given by 140. It seems likely that he was inspired in this research by the *Luo Shu* (Fig. 3.20).

There were at the same time other mathematicians in Persia, Arabia, and India carrying out research into magic squares. In Europe, magic squares came under scrutiny much later, but there is one especially famous example in the engraving *Melencolia I* by the German painter and printmaker Albrecht Dürer, which we shall discuss later. It is not difficult to see that any magic square remains a magic square if subject to rotation or reflection about an axis. Without counting more than once the eight squares that are equivalent to one another under these operations, there

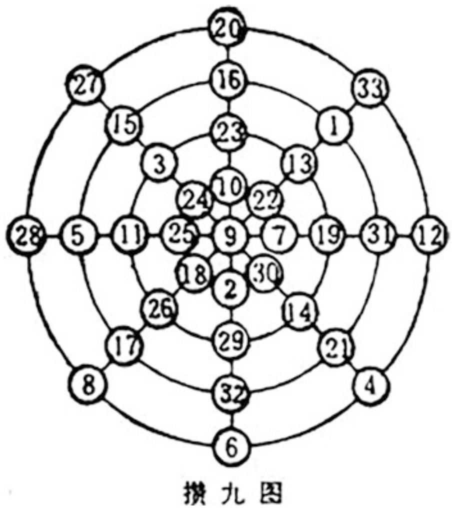

Fig. 3.20 Yang Hui's circular magic square

is only a single magic square of order three, while there are 880 magic squares of order four, and 275, 305, 224 of order five.

Yang Hui like Qin Jiushao spent his life and career based in the south; we consider next two other great mathematicians of the Song dynasty, Li Ye and Zhu Shijie, both of whom were based in the north. Li Ye (1192–1279) was born in Daxing (now the outskirts of Beijing), under the rule of the Jin dynasty. His name at birth was Li Zhi, but since it was later noticed that this is the same as the name of Emperor Gaozong of the Tang dynasty, he changed his name by the removal of a single stroke (so that instead his name was the same as that of one of the four great female poets of the Tang dynasty). Li Ye's father was a respected local official and erudite scholar, and Li Ye was influenced from childhood to value knowledge more than wealth. He took an early interest in literature, history, and mathematics, and he was admitted to the imperial academy where he earned praise for his intellectual talents. After the Mongol invasion under Ögedei Khan, he did not go to Shaanxi as planned but rather took up an administrative post in Henan.

In the year 1232, the Mongols invaded the Central Plains. Li Ye, who was 40 years old at the time, took up civilian attire and began a long and arduous journey into exile. Two years later, the Jin dynasty came to end. Li Ye did not however escape to the Southern Song dynastic territory, but rather remained in the north under the Mongolian rule of the Yuan dynasty. He had his reasons: the Jin dynasty and the Southern Song dynasty had always been at odds, and Kublai Khan, who established the Yuan dynasty, extended his courtesy to the intellectuals of the Jin dynasty and even to Li Ye personally, whom he had summoned on three occasions to provide scientific counsel. On one of these occasions, Li Ye persuaded Kublai Khan to reduce the severity of his penal measures and put an end to his conquest.

This was the turning point in his life, and Li Ye embarked upon an academic career lasting nearly half a century (he 3 years longer even than Diophantus). He returned to his hometown in Hebei and spent his final years teaching near Fenglong Mountain in the southwestern suburbs of modern Shijiazhuang. He wrote books and various essays recording his thoughts on all manner of topic.

The book of which Li Ye was most proud was his *Sea mirror of circle measurements* (测圆海镜, 1248), which laid the foundations for the Tian Yuan Shu system of algebraic notation for polynomial equations. In the *Nine Chapters on the Mathematical Art*, quadratic equations occur only in narrative form, and there was no notion of indeterminate quantities. In the Tang dynasty, although mathematicians had begun to work with cubic equations, these were presented geometrically, requiring skill and cleverness, and not suitable to easy generalization. For a long time afterward, algebra was tied to geometric thinking, prohibiting nonpositive constant terms, and avoiding polynomials of degree higher than three. It was only during the time of the Northern Song dynasty that Jia Xian and others were able to find positive roots for equations of higher degree (Fig. 3.21).

More complex problems, however, generated an urgent need for a more general method for handling polynomials of arbitrarily large degree, and the Tian Yuan Shu system met this need. Li Ye recognized that it was necessary to abandon geometric thinking altogether and establish universal procedures that do not rely on the specific

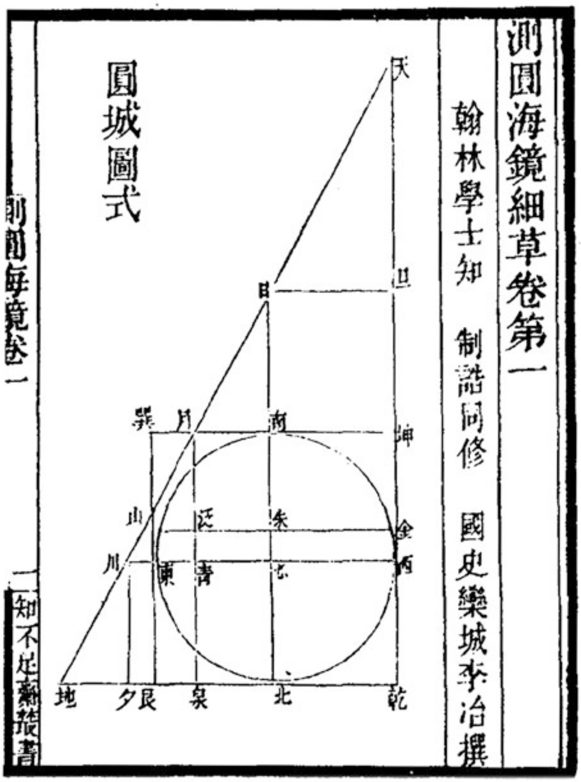

Fig. 3.21 Illustration from *Sea mirror of circle measurements* by Li Ye

Fig. 3.22 Li Ye introduced the use of slashes through numbers to indicate negative quantities

details of the problem to be solved. The Tian Yuan (or *heavenly variable*) played the same role as symbols x, y, etc. in modern algebra: "let the heavenly element be such and such" in place of "let x be such and such." The word *yuan* (元) was to be placed adjacent to the coefficient of the term in the first degree, with coefficients of all the terms arranged vertically with the degrees of the terms increasing from top to bottom. Moreover, its meaning was purely algebraic, and there was no requirement that the square term represents an area or the cubic term a volume. The constant term could be either positive or negative. With this system, it became trivial to represent polynomials of any degree, a challenge that had troubled Chinese mathematicians for more than a thousand years (Fig. 3.22).

Li Ye also used the symbol ◯in place of the empty space previously in use in decimal notation. The *Mathematical Treatise in Nine Sections*, which had come out 1 year earlier in the south, adopts the same notation, and the number zero quickly gained popularity throughout China. Finally, Li Ye introduced a notation for negative numbers (a slash drawn through the numeral), filling out a very simple and practical system of decimal notation. These two notational innovations appeared in China two and four centuries earlier than in Europe, respectively. At this point, Chinese algebra was in a semisymbolic state: there were still no operational symbols or relational symbols such as an equal sign. It seems that Li Ye was of a philosophical bent and believed that for all their infinite mystery, numbers can be simply understood (Fig. 3.23).

In the same year that Li Ye died, the Southern Song dynasty fell to the Yuan dynasty. Before this, there had been very little intellectual exchange, mathematical or otherwise, between the north and the south. Zhu Shijie (1249–1314) was the last of the four great masters of the Song and Yuan dynasties, and he was born late enough to enjoy the best mathematical offerings of both north and south. Since Zhu Shijie never embarked upon any official career, we do not know his family history. Whatever information we have about his life is drawn from prefatory material to his two books *Introduction to Computational Studies* (*Suanxue Qimeng*, 算学启蒙, 1299) and *Jade Mirror of the Four Unknowns* (四元玉鉴, 1303). Zhu Shijie like Li Ye was born near modern Beijing, but at that time the Jin dynasty had already been destroyed by the Yuan dynasty, and Beijing (or Yanjing) had become an important political and cultural center.

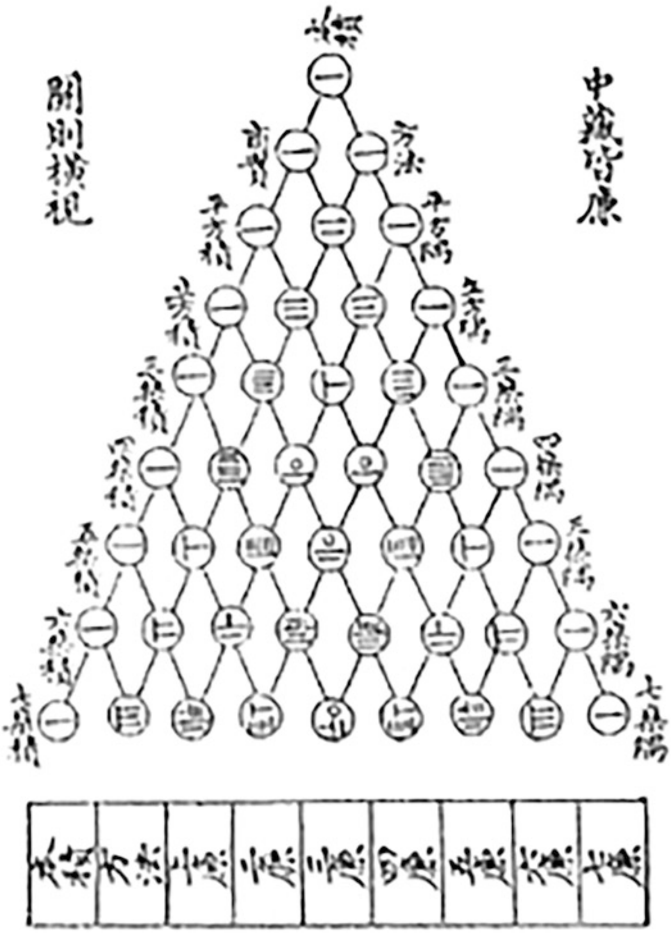

Fig. 3.23 Korean reprint of the *Introduction to Computational Studies* by Zhu Shijie

After more than 20 years of travel and study, Zhu Shijie settled in Yangzhou, where he published the two books just mentioned. The *Introduction to Computational Studies* begins from the four basic arithmetical operations and discusses all the important mathematical achievements of the time, including the extraction of higher-order roots, the Tian Yuan Shu system, achieving a very thorough synthesis of extant materials to serve as an excellent pedagogical text for the development of mathematics. Perhaps due to the influence of the practical and mercantile use of mathematics in the Southern Song dynasty, Zhu Shijie includes in a frontispiece the nine-nine multiplication song, the nine-nine division song, and other such formulas to entice a broader readership.

According to historical records, the Jiajing Emperor of the Ming dynasty (1507–1566) studied from the *Introduction to Computational Studies* and discussed it with his ministers, but this book was lost in China by the end of the Ming dynasty. Fortunately, it had spread to Korea and Japan shortly after its publication, where it was frequently annotated and exerted a special influence on Japanese *wasan* mathematics. It was not until the reign of the Daoguang Emperor of the Qing Dynasty (1839) that this book was republished in Yangzhou, its birthplace, on the basis of a Korean version.

In comparison with the populist aims of the *Introduction to Computational Studies*, the *Jade Mirror of the Four Unknowns* is a crystallization of years of personal research. Its most important contribution is an extension of the Tian Yuan Shu system to systems of indeterminate equations in two, three, or four variables, the four unknowns of the title.

In the method of four unknowns, the constant term appears in the center, and the indeterminate quantities which today we would write as x, y, z, w are labelled as the heavenly element on the bottom, the earthly element on the left, the human element on the right, and the material element on top. For example, the equation

$$x + 2y + 3z + 4w + 5xy + 6zw = A$$

would be written as

$$\begin{array}{|ccc|} & 4 & 6 \\ 2 & A & 3 \\ 5 & 1 & \end{array}.$$

In addition to developing this notation for indeterminate equations in four variables, Zhu Shijie also invented the elimination method for reducing the number of unknowns in a system of polynomial equations to a single variable. In Europe, it was not until the nineteenth century that Sylvester, Cayley, and others carried out a more comprehensive analysis using matrix methods. Zhu Shijie also presents a detailed treatment of the summation higher-order arithmetic series and continues the work of Shen Kuo and Yang Hui with more complex calculations of triangular piles. Finally, he anticipates the interpolation formulas later rediscovered by Isaac Newton in 1676 (Fig. 3.24).

Sarton praised the *Jade Mirror of the Four Unknowns* as the most important work of Chinese mathematics and one of the most outstanding mathematical works of the middle ages. George Sarton (1884–1956) is remembered today as the father of the history of science in recognition of his role as the founder of this discipline. He was proficient in 14 languages, including Chinese and Arabic, and taught the Chinese linguist Zhao Yuanren (1892–1982) during his time at Harvard University. The George Sarton Medal is the most prestigious prize given by the History of Science Society, and its recipients include Sarton himself in 1955, Joseph Needham in 1968; Thomas S. Kuhn, author of the influential book *The Structure of Scientific Revolutions*, in 1982; and Richard Westfall, author of a biography of Newton, in 1985.

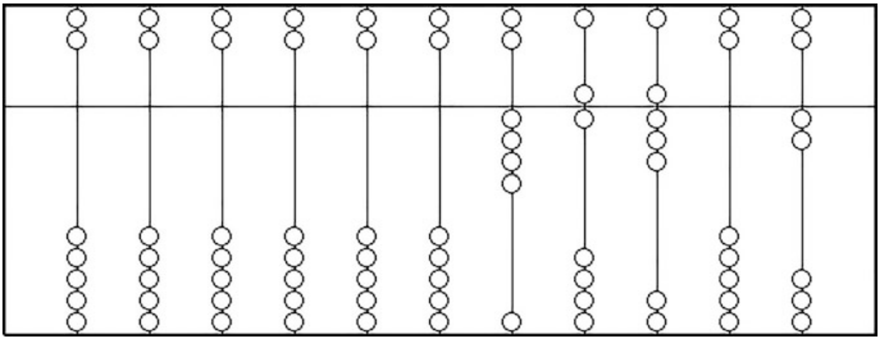

Fig. 3.24 The abacus might not have been invented in China but enjoyed its widest use there

Conclusion

After the *Jade Mirror of the Four Unknowns*, the Yuan dynasty produced no further mathematical works of note. In the Ming dynasty, agricultural, commercial, and industrial development continued apace, and Western classics such as Euclid's *Elements* were introduced in China; the rigid ideology of neo-Confucianism, the selection of scholars by overly standardized criteria, harsh penalties including imprisonment for impolitic speech and writing, all conspired to stifle the free creation of open thought. The mathematical level of the Ming dynasty fell far short of that of the Song and Yuan dynasties, and mathematicians could no longer understand the additive-multiplicative method, the Tian Yuan Shu, and the method of four unknowns. The mathematical works of the Han, Tang, Song, and Yuan dynasties not only went out of print, but many of them were even lost. It was not until the late Qing dynasty that Li Shanlan emerged as a new pioneer and propagator of modern science. He also introduced translations for many mathematical terms, which remain in use today. But by that time, Chinese mathematics had fallen far behind the mathematics of the west, and Li Shanlan alone could not catch up (Fig. 3.25).

I would like to also say a few words here about Japanese mathematics, which was influenced deeply by Chinese culture. While Chinese mathematics stagnated in the late Ming and early Qing dynasties, the mathematical prodigy Seki Takakazu (1642–1708) was born in Edo (now Tokyo). He was just a few months older than Newton and has since been recognized as the founder of Japanese mathematics. His foster father had been a samurai, and he himself served as a samurai under the shōgun before he became involved with a surveying project. Takakazu improved upon Zhu Shijie's Tian Yuan Shu system and established a theory of determinants both earlier and in a more extensive than what Leibniz achieved. He is also credited with early contributions to calculus, but due to the humility of the samurai tradition and the secrecy between competing schools of the time, these cannot be attributed to him with certainty. The body of work produced by Takakazu and his successors

Fig. 3.25 Qing dynasty mathematician Li Shanlan

Fig. 3.26 Seki Takakazu, the mathematical sage of Japan

who formed the dominant school in Japanese mathematics during the Edo period is the most substantial body of *wasan* mathematics, and he is remembered today as the mathematical sage of Japan (Fig. 3.26).

Looking over the history of Chinese mathematics through the Middle Ages, most mathematicians pursued attractive research programs only after achieving a certain degree of renown in the composition of formulaic essays. There were no institutions for group research or large-scale data centers like the Library of Alexandria or the Academy in Greece, and as a result it was difficult to devote professional efforts entirely to research. In the Song dynasty, for example, when mathematics developed rapidly, most of the significant mathematicians were minor officials who focused their attention primarily on issues important to the daily lives of ordinary people and technicians. They could not attend to theoretical work, and their writings came mainly in the form of annotations of classics.

Nevertheless, in comparison with the mathematical development of other ancient peoples, such as the Egyptians, the Babylonians, the Indians, the Arabic peoples, and even the Europeans of the middle ages, the Chinese people have much to be proud of. In terms of rigor and systematic abstraction, Greek mathematics as represented by Euclid's *Elements* represents an absolute peak, but in the field of algebra, it cannot be said that the Chinese mathematicians were inferior, and in some ways they may have achieved even better results. The biggest defect of Chinese mathematics is that there never developed in ancient China the notion of rigorous verification and proof, and mathematics for its own sake was a very rare phenomenon (one prominent example is the difference between ruler drawing and Euclidean diagrams). The situation is like that of the literary luminaries who chase after fame; altogether it is a kind of utilitarianism.

This attitude of course has firm social roots: it is natural that scholars work first toward the solution of problems required by the ruling classes. In ancient China, mathematics came to prominence mainly by way of its relation to the calendar. After Zhao Shuang proved the Pythagorean theorem, his first application of it was to find the roots of quadratic equations that came up in calendry. Zu Chongzhi obtained very fine rational approximations for π, which were used to calculate the leap year cycle. Qin Jiushao's Da Yan Shu, or the Chinese remainder theorem, was used mainly to calculate the years of the superior epoch, from which were determined certain astronomical constants such as tropical years and synodic moons (Fig. 3.27).

Fig. 3.27 The new Daogu Bridge; photograph by the author, in Hangzhou

In ancient China, whenever the harvest were bad for several years running and the population gave way to famine, the rulers would begin to worry about rebellions and peasant uprisings. Certainly one good excuse was to lay the blame at the feet of an insufficiently accurate calendar. At such times, the imperial court would issue an edict calling upon the scholars to undertake calendar reform, and the result of all this is that the greatest mathematical minds of ancient times were always drawn back again and again into ancient calculations. There were few opportunities and little courage to strike out for new mathematical worlds. But in modern times, contemporary Chinese mathematicians such as Wu Wenjun have taken inspiration from ancient Chinese algorithmic ideas. He developed an algorithmic method for solving multivariate polynomial equations with powerful applications for mechanical theorem proving in elementary geometry.

Finally, a few stories are connecting ancient Chinese mathematicians with modern China, and especially the city of Hangzhou, home of Chen Jiangong (1893–1971) and Su Buqing (1902–2003), two Zhejiang natives who earned doctoral degrees in mathematics in Japan. They established the Chen-Su School of Mathematics at Zhejiang University. Among the ancient mathematicians we have discussed in this chapter, two of them were born in Hangzhou: Shen Kuo and Yang Hui. Qin Jiushao, whose courtesy name was Daohu, also remarked that he lived in Hangzhou for some years with his family when he was young. There is a stone bridge near the Xixi Campus of Zhejiang University called Daogu Bridge; tradition has it that this bridge was initiated, designed, and built by Qin Jiushao. It was built across the Xixi River and originally called Xixi Bridge; its change of name was proposed by Zhu Shijie.

In his later years and after his death, two literary rivals wrote articles alleging that Qin Jiushao was corrupt and immoral, severely damaging his reputation. His name and the name of his bridge seem to be flickering out of sight, and it was not until the Qing dynasty that some sympathetic admirers defended him and denounced this slander. Sadly, in the twenty-first century, a municipal project caused the bridge to be destroyed and the river filled in so that the only remnant of it to remain was the Daogu Bridge Bus Station. In 2012, at the author's suggestion, the city authorities agreed to name a new bridge Daogu Bridge in honor of Qin Jiushao and about a hundred meters away from its original location.

In comparison with Zu Chongzhi's approximations of π and formula for the volume of a sphere, Qin Jiushao's algorithm and the Chinese remainder theorem are the more substantial achievements. But stories related to π are more easily digested by the public and more in line with the heroic imagination of the Chinese people.

Chapter 4
India and Arabia

One might write a history of India coming down to four hundred years ago and hardly mention the sea.

H.G. Wells

In the Rubáiyát we read that the history of the universe is a spectacle that God conceives, stages, and watches

Jorge Luis Borges

From the Indus River to the Ganges

The Indo-European Past

About 4000 years ago, at the time when the Egyptians, Babylonians, and Chinese were developing their river valley civilizations after their separate fashions, a nomadic people speaking a language that has since been classified as Indo-European made a long journey from Central Asia across the Gangdisi Mountains in the Transhimalaya system and settled in what is now northern India. These people later referred to themselves as Aryans, a word derived from Sanskrit and meaning *noble* or *landowner*. Some of them also travelled westward and became the ancestors of the Iranian people and some European peoples. It was believed in the middle nineteenth century and early twentieth century that the most purely Aryan people are found in Germany and the Nordic countries, a fallacy that saw widespread use by Hitler and his followers in the 1930s and 1940s to justify their theory of the noble race (Fig. 4.1).

Prior to the arrival of the Aryans, there were already indigenous people living in India, known as the Dravidian peoples. The history of these peoples can be traced back at least a further thousand years, when they are believed to have spread across the Indus River basins from western Pakistan. In modern times, about a quarter of the population of India still speaks languages belonging to the Dravidian language family, and four such languages, including Tamil and Telugu, are among the many official languages of India. Unfortunately the hieroglyphs used by the early

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T. Cai, *A Brief History of Mathematics*,
https://doi.org/10.1007/978-3-031-26841-0_4

Fig. 4.1 Bathing pilgrimage festival at Allahabad

Dravidians are as difficult to decipher as the oracle bone inscriptions of ancient China, and very little is known today about the Indus valley civilization, including the state of their mathematical knowledge.

After they had gained a foothold in northwestern India, the Aryan people continued to advance eastward, crossing the Indo-Gangetic Plain and eventually reaching the region known today as the state of Bihar, with a population of more than 100 million and twice the population density of Japan. They conquered the Dravidian people and made the northern regions into the cultural core of India. The major religions of India, including the predecessor Vedicism of Hinduism, Jainism, Buddhism, and much later Sikhism, were all born here, and the Aryan influence gradually spread throughout India. Sometime in the first millennium after their arrival, written and spoken Sanskrit were developed, as well as the Vedic religion, the oldest documented religion in India. It could be said that the Vedic religion and the Sanskrit language are the roots of the culture of ancient India.

The historical Vedic religion was a kind of polytheism with a heavy emphasis on ritual, in particular the ritual worship of various male deities associated with the sky and other natural phenomena, very different from the later tradition of Hinduism. The Vedic rituals involved animal sacrifice and the consumption of a sacred drink called soma, derived from a plant the identity of which is now unknown. This drink, which was extracted from the stems, pressed through wool, and mixed with water or milk, was cherished by adherents of the Vedic religion for its effects as a stimulant and even a hallucinogen. As for the purposes of the sacrifices, they sprang from the belief that the gods would repay them with material offerings such as plentiful livestock, good fortune, health, longevity, and male descendants. However the abundance of cumbersome rituals and exacting precepts may have contributed to the gradual decline of Vedicism.

The Vedic religion takes its name from its holy texts, the *Vedas*, transmitted and written down over a period of about a thousand years lasting from the fifteenth century to the fifth century BCE. The original meaning of the word *veda* (वेदः) in the Vedic Sanskrit in which they were written is *knowledge* or *light*. These books present sacrificial speeches in prose and poetry, as well as commentary, exegesis, and various philosophical ruminations. These texts also introduce the division of Indian society into four castes: Brahmins (priests), Kshatriyas (warriors), Vaishyas (skilled traders, merchants), and Shudras (unskilled workers or slaves), a division which persisted through later Hinduism (Fig. 4.2).

The exegetic materials appear primarily in the *Brahmanas*, the *Aranyakas*, and the *Upanishads*. The *Brahmanas* contain explanations of ritual procedure, the *Aranyakas* discuss the meaning of ritual sacrifice and the worship of heaven, and the *Upanishads* discuss the relation between individual souls and cosmic reality, the obliteration of ignorance, and concomitant the liberation from attachments to

Fig. 4.2 Cover page of a Chinese edition of the *Upanishads*

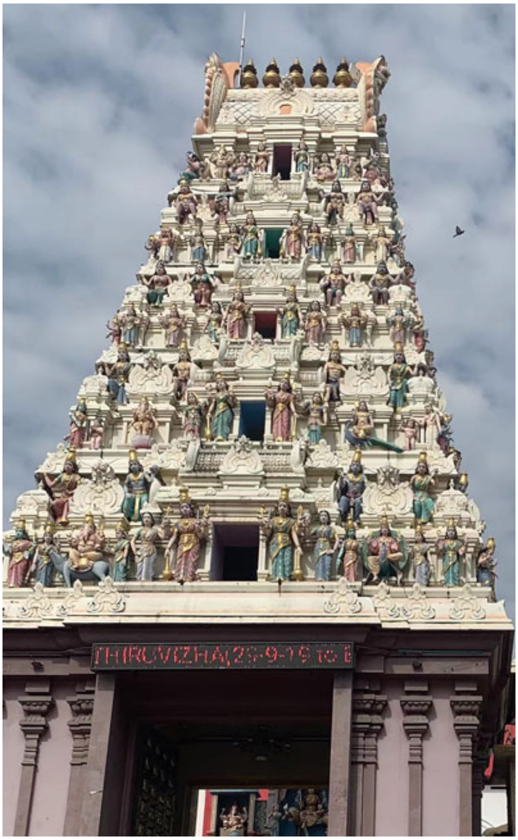

Fig. 4.3 A typical Hindu temple, with the form of an isosceles trapezoid; photograph by the author, in Malaysia

the material world, secular temptations, and physical ego. These texts comprise the sacred *heard text* (*shruti*) tradition; there is also a tradition of *remembered text* (*smriti*), with the famous *Bhagavad Gita* as the most representative example. Among the doctrines of this book, it is remembered for the teaching that tranquility is achieved through yogic practice (Fig. 4.3).

The *Vedas* were originally transmitted orally and later recorded on palm leaves or bark. Although many of the original texts have been lost, it is fortunate from the perspective of the history of mathematics that the *Shulba Sutras* are among those that have survived. The word *shulba* means *string*, *cord*, or *rope*, and the *Shulba Sutras* discuss the design and measurement of altars and temples. This is the earliest mathematical literature in the history of India, excluding a few fragments of mathematical notation appearing on coins and inscriptions. The mathematical content of the *Shulba Sutras* includes some geometric figures and algebraic calculations related to altar design, including an application of the Pythagorean theorem and a general treatment of the diagonal of a rectangular figure, some discussion of similar plane figures, and some basic instructions for the construction of geometric diagrams.

The most essential features of the *Shulba Sutras* are the method of measurement by drawn ropes and basic area calculations.

The Shulba Sutras and Buddhism

The *Shulba Sutras* were written sometime between the eighth century BCE and second century CE, no later than the two classical Indian epics *Mahābhārata* and *Rāmāyaṇa*. Four major *Shulba Sutras* have survived, each named individually after its author or the school of thought represented by its author. These books contain detailed instructions for the construction of fire altars, including in particular the appropriate shapes and sizes. The three most commonly used shapes are square, circle, and semicircle, but an important regulation is that the altar should have a fixed area regardless of its shape. For these reason, ancient Indian mathematicians had to have learned or already known how to construct a circle with the same area as a given square or twice the area of a given square. Another common shape is the isosceles trapezoid, and even more exotic geometric figures appear, all constrained to a given area. This raises a host of new and interesting problems in plane geometry.

The design of the altar according to a prescribed shape requires some basic geometric knowledge and results, for example, the Pythagorean theorem. The Indian mathematicians give a unique statement of this theorem:

> The areas produced separately by the lengths of the breadths of a rectangle taken together equal the area produced by its diagonal.

This is very obviously different in character from the solar height presentation in the *Zhoubi Suanjing*. More generally, the Indian mathematics of this period is a scattered assemblage of approximate laws for the calculation of areas and volumes, expressed in words rather than in notation. A few of these laws included deductive proofs, while most of them were purely empirical.

As an example, the construction of a circular area with twice the area of a given square (say, for a semicircular altar) requires an approximation of π. The *Shulba Sutras* record the following approximate value, which we present in an equivalent formulation with unit fractions:

$$\pi \approx 4\left(1 - \frac{1}{8} + \frac{1}{8 \cdot 29} - \frac{1}{8 \cdot 29 \cdot 6} - \frac{1}{8 \cdot 29 \cdot 6 \cdot 8}\right) \approx 3.0883.$$

The approximations $\pi \approx 3.004$ and $\pi \approx 4 \times \left(\frac{8}{9}\right)^2 \approx 3.16049$ seem also to have been in use. Similarly, the construction of a square altar with area 2 requires a value for $\sqrt{2}$. The *Shulba Sutras* give

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \approx 1.414215686,$$

Fig. 4.4 Statue of a reclining Buddha in the Ajanta Caves

accurate to the fifth decimal place. It is interesting that both of these approximations can be recognized as unit fractions, suggesting perhaps that such fractions also held significance for Indian mathematicians as they did for Egyptian mathematicians, whether by coincidence or as the product of cultural exchange (Fig. 4.4).

The founding promulgator of Jainism in India, Mahavira, was born in about 599 BCE Bihar, not far from where the founder Gautama Buddha of Buddhism was born some 36 years later. The parallels between the lives of these two pivotal figures are manifold. Both grew up in an environment of prosperity and material abundance, which later they renounced along with all family and property at about the age of 30, when both left their homes to lead perambulatory lives in pursuit of truth. One minor difference is Gautama Buddha is said to have left behind a wife and infant son in course of his renunciation and Mahavira is said to have had a daughter instead. In any case, Buddhism and Jainism arose in India at almost exactly the same time and probably due to a common cause: opposition to the bureaucratic fuss of Vedicism and the elitism of the Brahman caste in the caste system.

The original Sanskrit word *jain* derives from the base *ji* (जि) , meaning *victor* or *to conquer*. Jainism is a religion without any central creative deity; rather both time and the universe are held to be eternal and endless, with all things divided into the categories of souls and nonsouls. The foundational texts of Jainism cover a wide range of intellectual territory. In addition to expounding and clarifying the basic doctrine, they also include important contributions to literature, drama, art, architecture, and other topics, as well as some basic principles and conclusions in mathematics and astronomy. Many Jain texts were written between the fifth century BCE and second century CE in the vernacular Prakrit language; the word *prakrit* means *natural*, distinguished it from the classical *sanskrit* language, meaning *refined*. These texts include some mathematical results, including approximations for the circumference of a circle $C = \sqrt{10}d$, for an arc length $s = \sqrt{a^2 + 6h^2}$, and other approximation formulas of a similar flavor (Fig. 4.5).

In contrast, Buddhist doctrine holds that all things are impermanent and transient, subject to constant change both within and without the human experience. Such things as the area of an altar are impossible to state with fixed specificity. Buddhism

Fig. 4.5 The pattern on the altar

accepted all people without regard for caste and does not recognize any essential differences between different people. Buddhism is more of a philosophical concept than a precise religious doctrine than either Jainism or Hinduism, especially as it is practiced in India. The concept of time in Buddhism is also very unique and somewhat of a mathematical character. For example, perhaps because India has three seasons (rainy season, hot season, and dry season), the Buddhist scriptures also divide the days and the nights each into three parts: early day, midday, and late day and similarly early night, midnight, and late night. As for the years, 100 years is one lifespan, 500 years is an evolution, 1000 years is a turn, and 12,000 years is an epoch.

More interesting is the subdivision of time. The shortest period of time is the single moment (kṣāṇa), the time between 2 thoughts, and 60 of these make up the time of a finger snap. From this we get also the saying “the sixty-three moments of youth is a snap of a finger.” However, the true duration of a moment cannot be known except to the Buddha. It is as in the following poem:

> We see the fullness of the moon and understand that time is always moving
> We understand the birth and death of our minds, we know the shortness of time.

At the same time that Buddhism and Jainism emerged in India in the sixth century BCE, the concepts of the reincarnation of the soul, karma, and the transcendence of reincarnation through meditation became widespread amongst the Vedics. This became what is now known as the Hindu religion.

Since that time, this reform religion involving almost every aspect of life has gradually come to dominate almost the entire Indian continent, and its philosophy and moral code have even spread into the beliefs, customs, and social religious systems of many ethnic groups in South Asia, including the Nepalese and Sri Lankans. This is in contrast with Buddhism, which developed a broad influence in Southeast Asia, but in India comprised mainly a philosophical system, and with Jainism, the sphere of influence of which remained restricted primarily to some western and northern Indian states. Around this time, mathematics broke free of its connection with religion and developed independently instead as a powerful astronomical tool (Fig. 4.6).

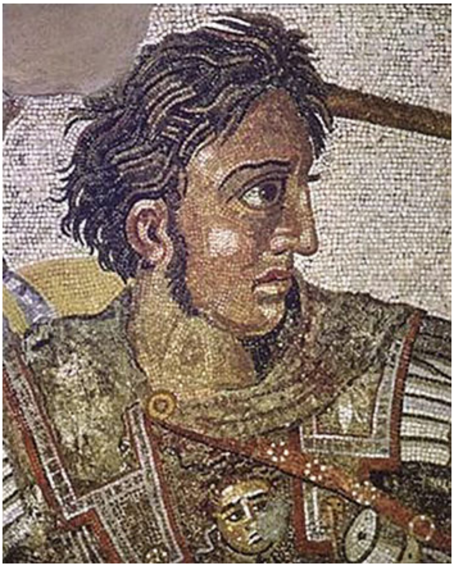

Fig. 4.6 Alexander the Great, whose expeditions bridged the East and the West

The Number Zero and Hindu Numerals

By the middle of the fifth century BCE, the people of the Magadha region, encompassing Bihar, had expanded the boundaries of their territory to include the entire Indo-Gangetic Plain. This laid the foundations for a period of prosperity under the Maurya Empire (322 BCE–185 BCE), which reach its peak under the rule of the emperor Ashoka the Great, often remembered as the greatest monarch in the history of India. He devoted his life and his rule to the promotion and dissemination of Buddhism and served a role in that religion similar to that of Paul the apostle in Christianity, facilitating its eventual spread around the world. His grandfather Chandragupta Maurya had been the founder of the Maurya Empire, who expelled the forces left behind by Alexander the Great during his campaign in India and around the same time or slightly later conquered and unified northern India. This established the first empire in Indian history. This campaign by Alexander was in itself a remarkable expedition, forming a bridge so to speak between Greece in the west and India in the east. They reached the southern shores of the Caspian Sea and continued to march eastward and laid the foundations for the modern cities of Kandahar and Herāt in Afghanistan, before heading north into Samarkand in Central Asia. He did not occupy this territory but rather used it to send his troops southward again through the crevices of the Hindu Kush mountain range and enter India via the Khyber Pass east of Kabul. His original intent was to continue further and further to the east, crossing the desert to the Ganges river basin. Years of fighting had exhausted his soldiers, however, and in 325 BCE Alexander withdrew with his troops from the Indus valley and returned to Persia. He left behind an army and a governor in Punjab, later driven away by Chandragupta Maurya.

Despite its failure, this short campaign forged indelible traces, including the facilitation of future trade and exchange between India and Greece. It is believed that by Roman times Alexandrian merchants had established many settlements throughout South India and even built a temple to Augustus in Muziris. These settlements were usually guarded by two teams of Roman troops, and at this time, the Roman emperor also sent envoys to South India. Moreover, this encounter with Greek civilization certainly had an influence in India on the development of mathematics and the other sciences. A fifth-century Indian astronomer observed that although the Greeks were impure from the perspective of Indian religious belief, nevertheless they ought to be revered for their training in the sciences, in which direction they surpass all other peoples.

In the summer of 1881, in the small village of Bakhshali (near Peshawan in modern Pakistan, but a part of India for most of its history), a local peasant unearthed a birch bark manuscript containing written text. This is the Bakhshali document, which records a rich compilation of Jain mathematics, including treatments of fractions, squares, sequences, proportions, problems involving revenue, expenditure, and the calculation of profits, series summations, algebraic equations, and so forth. This manuscript makes use of a sign for negative numbers, identical visually to the modern addition sign $+$, but placed to the right of a numeral rather than to its left. And even more consequentially, this manuscript includes numerals written in complete decimal notation, with a solid dot to represent the number zero.

This solid dot evolved over time into a circle and finally into the modern symbol 0. The earliest verified evidence of the modern notation appears in an engraving at a temple in Gwalior date to the year 876. The city of Gwalior is in northern India, in its most densely populated state of Madhya Pradesh, adjacent to Bihar in the Ganges river basin. This engraving is on a stele in a garden and supposed to indicate the number of wreaths or corollas to be supplied daily to local temples. Although the two occurrences of the numeral 0 are small, they are clear and unmistakable (Fig. 4.7).

The use of a circle to indicate zero and more broadly the introduction of the concept of zero as an independent number are a great contribution to mathematics from India. The use of zero as a placeholder in decimal notation had predecessors in early Babylonian cuneiform and in Chinese arithmetic prior to the Song and Yuan dynasties, where it was indicated by an empty space, and with a specific symbol in later Babylonian texts and in Mayan mathematics (which made use of a shell or eye symbol), but it was not regarded as in India as an independent number with a role to play in calculations. Perhaps there is some connection here with religious culture: in Indian religions the idea that the universe was born from nothing was widespread. The Indian mathematicians were also comfortable with the distinction between positive numbers to represent property and negative numbers to represent debts.

The numerals engraved at the Gwalior temple are also closer in form to the so-called Arabic numerals used throughout the world than the numbers used in the Arabic world. After the eighth century CE, Indian digits and the zero numeral were spread successively first to the Arabic-speaking world and later to Europe. After these notations were accepted and formalized in Europe, having been introduced in the influential book *Liber Abaci* by the thirteenth-century Italian mathematician Fibonacci, they have played an invaluable role in the progress of modern science. Since that time, the history of mathematics in India is also the history of some of the leading mathematicians in the world.

From North India to South India

Aryabhata

In the year 476 CE, on the south bank of the Ganges not far from the historical city Pataliputra, known today as the capital city Patna of Bihar, Aryabhata, the earliest Indian mathematician known by name, was born. This city obtained its modern name when the Afghans invaded and rebuilt it in the sixteenth century. Gautama Buddha also taught here in his later years, and this city was the capital of the two most powerful empires in Indian history: the Maurya Empire and the Gupta Empire

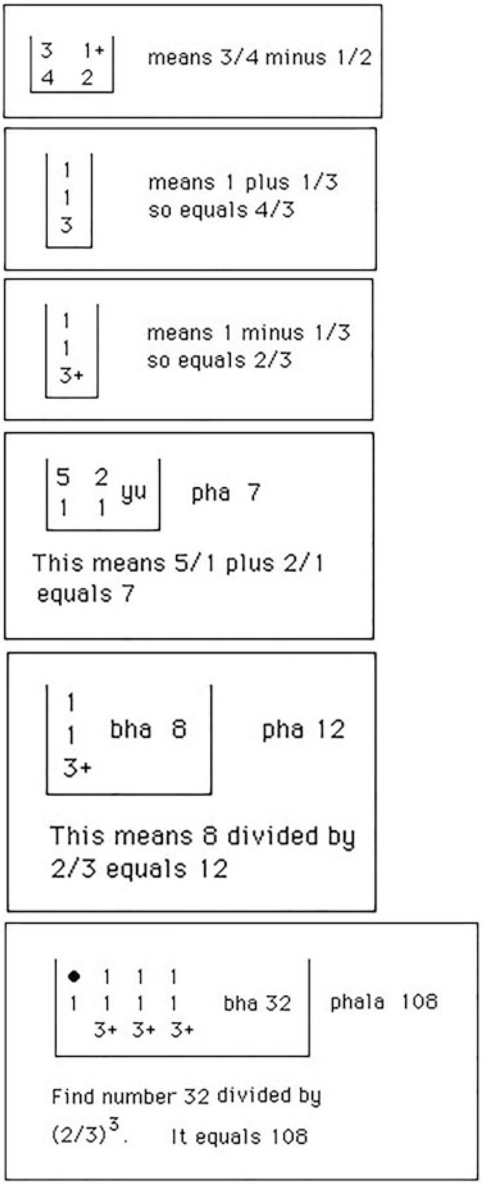

Fig. 4.7 Arithmetic problems from the Bakhshali document

Fig. 4.8 Statue of Aryabhata in Pune

(ca. 320–540), the first empire to unify India during the Middle Ages. The territory of the Gupta Empire included most of the northern, central, and western parts of modern India. It was during this period that decimal notation, Hindu art, and the great Sanskrit epic *Abhijnanashakuntalam* or *The Sign of Shakuntala* and its author the poet Kālidāsa (ca. fifth century) were all born. The Eastern Jin dynasty Buddhist monk Faxian (法显), who travelled by foot from China to India, also came to this city to study during his travels (Fig. 4.8).

By the time of Aryabhata's birth, the capital of the Gupta Empire had moved further west, and the city of Pataliputra had begun to decline from the height of its glory. Nevertheless it was still a significant academic center: indeed, the influential Tang dynasty Buddhist monk Xuanzang (玄奘) visited in around the year 631. As with later Indian mathematicians, Aryabhata studied mathematics mainly for its uses in astronomy and astrology. His two representative works are the *Aryabhatiya* and a book of computations the *Arya-siddhanta*, which has since been lost. The *Aryhabhata* is primarily an astronomical compendium, but it contains significant mathematical content, including arithmetic, a discussion of the measurement of time, spheres, and so on. Around the year 800, this book was translated into Latin and made its way into Europe. Its influence in Indian intellectual history has been huge, especially in South India, and it has been the subject of many commentaries and annotations by subsequent mathematicians.

Aryabhata gives expressions for the sums of squares and cubes of the first n consecutive positive integers:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

He was also the first person in India to obtain the valued $\pi \approx 3.1416$, but it is not known by what method he arrived at this approximation, which can be compared with ancient approximations in China. There is some speculation that he calculated the perimeter of an inscribed regular polygon with 384 sides. In trigonometry, Aryabhata is remembered for his sine tables. Earlier such tables had been compiled by Ptolemy in Greece, but Ptolemy used different units of length for circular arcs and the straight radius, an unwieldy system. Aryabhata assumed instead a common measure of length for straight and curved lines and completed his table on this basis from 0° to 90° in 3°45 increments. Aryabhata used the word *ardha-jya* meaning half-chord for *sine*. This was later simplified to *jya* (the bowstring of a hunter), which became *jaib*, meaning *pocket* or *fold*, in Arabic and eventually by way of Arabic *sinus* (meaning *cove* or *bay*) in Latin transliteration. This is the origin of English word *sine*.

In arithmetical work, Aryabhata frequently makes use of trial solutions and working backward from the given conditions. As an example, he states a certain problem as:

> Oh beautiful girl with smiling eyes, can you tell me please, what number multiplied by 3 and then adding again $\frac{3}{4}$ of this product, dividing by 7, subtracting $\frac{1}{3}$ of the result and multiplying it with itself, then subtracting from this 52, taking the square root and adding to it 8, and finally dividing by 10, gives 2?

According to the method of working backward, the solution is found by starting from the solution 2 and inverting the operations as follows:

$$((2 \times 10) - 8)^2 + 52 = 196,$$

$$\sqrt{196} = 14,$$

$$14 \times \frac{3}{2} \times 7 \times \frac{4/7}{3} = 28,$$

which is the required number. We see in this example also that Indian mathematics were prone to express their mathematical work in the language of poetry.

Aryabhata's most significant contribution is the solution of the first-order indefinite equation

$$ax + by = c.$$

He uses a method he calls *kuṭṭaka*, meaning *pulverization*. For example, assume $a > b > 0$ and $c = \gcd(a, b)$ is the greatest common divisor of a, b. Then we find integers $q_1, r_1, \ldots, q_{n+1}, r_n$ such that

$$\begin{aligned} a &= bq_1 + r_1 (0 \le r_1 < b), \\ b &= r_1 q_2 + r_2 (0 \le r_2 < r_1), \\ &\vdots \\ r_{n-2} &= r_{n-1} q_n + r_n (0 \le r_n < r_{n-1}), \\ r_{n-1} &= r_n q_{n+1}. \end{aligned}$$

By iteration, the number $c = \gcd(a, b) = r_n$ can be expressed as a linear combination of a and b, providing integer solutions x and y to the indeterminate equation given above.

In fact, this method was also used later by Qin Jiushao in his *Da Yan Shu* and the positive and negative evolution method, with a rudimentary version appearing earlier in the *Nine Chapters on the Mathematical Art*. In western countries, this method is known as the extended Euclidean algorithm, but the version given by the Greek is incomplete; even the last master of number theory among the Greeks, Diophantus, considered only positive integer solutions to such equations. This restriction was lifted by Aryabhata and his followers.

Aryabhata also made important contributions to astronomy. He used mathematical methods to calculate the circumference of the earth, the movement of the ascending and descending nodes of the ecliptic and lunar path, the latest points and points of slowest motion of certain stars, and even proposed methods for the accurate prediction of solar and lunar eclipses. He also promoted the notion of the rotation of the earth, but this idea was not recognized and carried forward by subsequent generations. In the year 1975, India launched its first man-made satellite, named *Aryabhata*, in honor of his achievements and significance in the history of Indian science.

Brahmagupta

After Aryabhata, it was more than a century for the appearance of another great mathematician: Brahmagupta (ca. 598–ca. 668). It is interesting to note that in fact there do not seem to have been any important mathematicians anywhere in the world during this century. Brahmagupta may have had ancestors from the region of the Sindh province in modern Pakistan, home to its capital and largest city Karachi, but Brahmagupta was probably born in Bhillamāla in Gurjaradesa and spent the better part of his life in Ujjain, a city in the southwestern part of the Indian state of Madhya Pradesh. Alongside its neighboring state of Bihar, these two states formed

Fig. 4.9 Brahmagupta, deep in calculation

the political, cultural, and scientific hub of ancient India, a role similar to that of the Guanzhong Basin and the Central Plains in China (Fig. 4.9).

Although Ujjain was never the capital of a unified empire in India (and indeed India had been divided since the end of the Gupta Empire), it is regarded as one of seven holy cities in India. The Tropic of Cancer passes through the northern suburbs of the city, as does the first meridian set down by Indian geographers. After the decline of Pataliputra, Ujjain was the second city to occupy the center of ancient Indian mathematics and astronomy. It was also the birthplace of the poet and dramaturge Kālidāsa, often considered the greatest writer in the history of India. Since these two cities are separated by nearly a thousand kilometers, with Ujjain closer to Mumbai than Pataliputra, this represented a shift in the center of Indian intellectual activity toward the southwest. It is also believed that Ashoka spent time in Ujjain as a governor before his succession to the throne. Brahmagupta served as the head of the astronomical observatory in Ujjain, one of the oldest and most prestigious observatories in the world prior to the invention of the telescope.

Brahmagupta left behind two important astronomical works: the *Brāhmasphuṭasi ddhānta* or *Correctly Established Doctrine of Brahma*, composed in the year 628, and the *Khaṇḍakhādyaka* (meaning *a morsel*), written around 665 and published after the death of its author. The latter work includes a new sine table, calculated using a different method than that of Aryabhata: the quadratic interpolation method. The *Brāhmasphuṭasiddhānta* contains more extensive mathematical content. This book was written entirely in verse, divided into 24 chapters, with 2 of them devoted entirely to mathematics. These are *Lectures on Arithmetic* and *Lectures on Indefinite Equations*. The former discusses triangles, quadrilaterals, the quadratic formula, and the arithmetical properties and operations related to zero and negative numbers. The latter introduces the study of first-order and second-order indefinite equations. The remaining chapters discuss astronomical research but call upon a rich foundation of mathematical knowledge.

For example, Brahmagupta gives the following rules for calculations involving zero:

> A negative number minus zero is negative, a positive number minus zero is positive; zero minus zero is zero ... the product of zero with a positive number or with a negative number or with zero is zero ... zero divided by zero is zero, and a positive or a negative number divided by zero is a ratio with zero as the divisor.

This is the earliest record of the problem of division by zero in the Indian literature, although the conclusion is different from the modern understanding of the situation. The idea of operating with zero as with any other number survived in the works of later Indian mathematicians.

Brahmagupta also elucidated the concept and notation for negative numbers and provided rules for their treatment in computations:

> The sum of a positive number and a negative is the difference of their absolute values or zero if they are equal; the product of a positive number and a negative number is negative, and the product of a two positive numbers or two negative numbers is positive.

This is the first consideration of its kind in the history of the world.

The most important mathematical achievement due to Brahmagupta is his solution of the indefinite equation

$$nx^2 + 1 = y^2$$

where n is a nonsquare integer. Brahmagupta was the first mathematician to consider such equations; they were later misattributed by the eighteenth-century Swiss mathematician Leonhard Euler to a British mathematician from the previous century named John Pell, and they are known today collectively as Pell's equation. In fact, the first mathematician to discuss this equation in Europe was Fermat, and it was Lagrange who resolved it. Brahmagupta found solutions in the special case $n = 92$ (and other special cases) using a method he called *samasa* and which is known today as the *method of compositions* or an application of *Brahmagupta's identity*. His approach was clever and powerful and deserves its special place in the history of mathematics.

Brahmagupta also gave a general formula for a root of a quadratic equation in one unknown, although he neglected to account for the second root of such equations. He provided a formula for the area of a quadrilateral with sidelengths a, b, c, d

$$s = \sqrt{(p-a)(p-b)(p-c)(p-d)},$$

where $p = \frac{1}{2}(a + b + c + d)$. This result is only valid for a quadrilateral inscribed in a circle, as Brahmagupta himself pointed out, but it is still a very impressive result. Finally, he also gave a beautiful proof of the Pythagorean theorem, using the proportional relationship between the sidelengths of two adjacent triangles.

Mahāvīra

Brahmagupta was a sophisticated mathematician, but unfortunately very little information about his thought, life, and work is known to us. He wrote that just as the stars are eclipsed by the rays of the sun, so too are the works of learned scholars eclipsed by those who can put forward problems in algebra and even more so by those who can solve them. Presumably the academic atmosphere during his lifetime was very refined; there was also a movement in the study of history known as the Ujjain school. But for four centuries after the death of Brahmagupta, no further mathematicians of note appeared in Ujjain, perhaps due to political turmoil and dynastic change. Rather during this time, two mathematicians of genius emerged in the relatively remote state of Karnataka in southwestern India: Mahāvīra and Bhāskara.

Although the area within the borders of India is only about 3 million square kilometers, and the distance between its eastern and western borders is greater than between its northern and southern borders, nevertheless the distinct character of South India has long played a deep role in the self-conception of the Indian people. The towering Deccan Plateau of South India and the two mountains at its northern edge, along with the Narmada river, form a natural defense against incursion by the northern imperial powers (the name of the Deccan Plateau derives from a Sanskrit word meaning *south*). And indeed, many attempts at conquest by the north were met with fierce resistance by the south. As a result, the dietary habits of the Indo-Aryan peoples never spread to the south, the armies of Alexander never set foot there, the various invasions of the Mongols and Muslim armies all fell short, and even in later times the influence of France and Britain was more muted in southern India (Fig. 4.10).

Very little is known about the history of South India prior to the time of Ashoka, but it is clear that even in spite of factionalism, its culture was no less advanced or deep than that of the north with respect to religion, philosophy, moral customs, artistic expression, and material development. Several large states or dynasties competed for dominance throughout history, but none of them succeeded in unifying the entire southern territories of India. Every such dynasty maintained a developed maritime trade with Southeast Asia and established variously a capital city as a cultural center, characterized by extensive temple architecture.

Among these several dynasties, one was the Rashtrakuta dynasty, who ruled the Deccan Plateau and a strip of nearby land from about the year 755 to 975. The originators of this dynasty may have been Dravidian farmers, and their empire

Fig. 4.10 Birthplaces of Indian mathematicians

became so great that a Muslim traveller and commentator of the time referred to its ruler as one of the four great emperors of the world, the other three being the Caliph, the emperor of Byzantium, and the emperor of China. The Trinidad and Tobago-born British writer V.S. Naipaul (1932–2018) observed that there remains a relic of the capital of the kingdom of Vijayanagar some 200 miles from Bangalore. This city was among the most resplendent in the world by the fourteenth century.

The Jain mathematician Mahāvīra, who shared a name with one of the founding figures of that religion, was born during the peak of the Rashtrakuta dynasty, probably in the city of Mysore. Mysore was the second largest city in the state of Karnataka (the word Karnataka probably originates from a word meaning *highlands*), located on the southwestern coast of India, between the two famous cities Bangalore and Kolkata. Bangalore is the capital city of Karnataka, now widely regarded as the Silicon Valley of India and home to the National Institute of Mathematics. Kolkata is a famous port city, the city where the Chinese explorer Zheng He died and to which the Portuguese explorer Vasco da Gama arrived much later via the first voyage to India by sea from Europe, around the Cape of Good Hope.

Not much is known about the life of Mahāvīra: as an adult, he lived in the court of the Rashtrakuta dynasty, where he served as court mathematician. Around the year 850, he wrote a book entitled *Gaṇitasārasañgraha* or *Compendium on the Gist of Mathematics*, later widely used as a textbook throughout South India. This book was translated into English in 1912 and published in Madras. This is the first book in the history of India with the form of a modern mathematical textbook, and it contains already glimpse of the topics and structure of current mathematical textbooks. It is especially remarkable that the *Gaṇitasārasañgrah* really is a treatise on pure mathematics, with hardly any mention of astronomical matters. The book is divided into nine chapters, and its most valuable research results include further discussion of zero, quadratic equations, calculations with interest rates, the properties of integers, and topics in what is now known as combinatorics.

Mahāvīra observed that multiplication of a number by zero yields zero and subtracting zero from any number does not reduce its value. He further noticed that division by a number is equivalent to multiplication by its reciprocal and even put forward that division by zero yields an infinitely large value. On the other hand, he asserted that negative numbers have no square roots, contrary to the modern theory of complex numbers. It is interesting to note that in the same way as the Chinese mathematicians who were fascinated by magic squares, Mahāvīra devoted considerable energy to the study of a numerical game he called *garland numbers*. These are pairs of numbers whose product is a palindromic number

$$14287143 \times 7 = 100010001,$$

$$12345679 \times 9 = 111111111,$$

$$27994681 \times 441 = 12345654321.$$

This terminology of garland number survives in Chinese poetry to indicate palindromic numbers. There is also a variation on palindromic numbers known as Scherezade numbers, after the storyteller in *One Thousand and One Nights*. Palindromic numbers also frequently occur as powers, for example, $121 = 11^2$, $343 = 7^3$, and $14641 = 11^4$, but nobody has ever found a palindromic number occurring as a fifth power.

Earlier classic of Jain mathematics had already introduced some simple problems in permutations and combinations. In the course of summarizing these topics, Mahāvīra introduced for the first time the formula for binomial coefficients with which we are familiar today

$$\binom{n}{r} = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{r \times (r-1) \times \cdots \times 1},$$

where $1 \leq r \leq n$. This was two centuries before the birth of Jia Xian in China.

Mahāvīra improved upon the *kuṭṭaka* method for indeterminate equations of Aryabhata and also carried out extensive research into Egyptian fractions. He determined that the number 1 can be expressed as a sum of any fixed number n

unit fractions and found ways to express arbitrary fractions as sums of unit fractions subject to various constraints. He also studied in detail the solutions to certain higher-order indeterminate equations and construction problems in plane geometry and gave approximation formulas for the perimeter of an ellipse and area of a circular arc, in agreement with results that appear in the Chinese classic *Nine Chapters on the Mathematical Art*.

Bhāskara II

We turn finally to the greatest mathematician and astronomer of ancient and medieval India: Bhāskara II, the second of two mathematicians named Bhāskara in the history of India. The first of them lived in the seventh century, and Bhāskara II in the twelfth century. He was born in 1114, probably in Bidar in the western part of the Deccan Plateau in South India. This city is located today along the railway connecting Mumbai and Hyderabad (host city of the 2010 International Congress of Mathematicians) and like the hometown of Mahāvīra a part of the Karnataka state. His father was on orthodox Hindu Deshastha Brahmin who authored a popular book on astrology. As an adult, Bhāskara worked at the astronomical observatory at Ujjain and later became its director, a worthy successor to Brahmagupta.

By the twelfth century, Indian mathematics had already accumulated a considerable corpus of research results, which Bhāskara assimilated and built upon to achieve deeper results than any of his predecessors. His literary talents were also formidable, and a poetic flavor permeates his works. His two most significant works are entitled *Līlāvatī* (named after his daughter) and *Bījagaṇita* (*Algebra*), two volumes of a larger work entitled *Siddhānta-Śiromani* (*Crown of Treatises*) (Fig. 4.11).

The *Bījagaṇita* includes discussion of positive and negative numbers, linear equations, low-order indeterminate equations in integer coefficients, and so on. There appear also two beautiful proofs of the Pythagorean theorem, one of which is very similar to the method of Zhao Shuang, the other of which was only rediscovered in the seventeenth century, by the British mathematician John Wallis. With reference

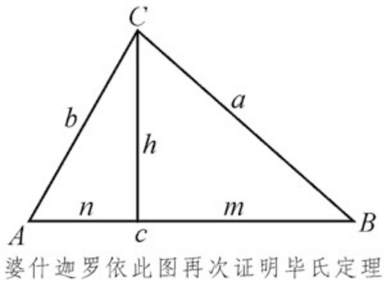

Fig. 4.11 Diagram of the proof of the Pythagorean theorem by Bhaskara

to the figure, we can obtain from the properties of similar triangles that

$$\frac{c}{a} = \frac{a}{m},$$
$$\frac{c}{b} = \frac{b}{n},$$

from which it follows that $cm = a^2$, $cn = b^2$; adding these two identities,

$$a^2 + b^2 = c(m + n) = c^2.$$

Bhāskara also discusses formally a crude version of the concept of mathematical infinity. He writes:

> If we divide a number by zero the result is a quotient with divisor zero, for example three divided by zero is the quotient $3/0$. Such a quotient is called infinity. Just as during the period of cosmic dissolution beings merge into the creator and during the period of creation beings emerge out of him, but the creator himself remains unaffected, likewise nothing happens to the number infinity when any other number is added to it or subtracted from it; it remains unchanged.

The content of *Līlāvatī*, which opens with a salutary Hindu prayer, is more extensive. There is also an interesting legend surrounding the authorship of this book. According to the legend, Bhāskara had concluded through his studies into the horoscope of his daughter Līlāvatī after whom the book was named that she would remain unmarried and childless throughout her life, unless her wedding were held at a precise time on a certain auspicious day. He placed a cup with a small hole in a vessel of water such that the cup would sink at precisely the correct moment for the wedding and warned his daughter not to approach or disturb it. But her curiosity led her to investigate the device, and in the process, a pearl from her bridal headdress dropped into the cup, upsetting it and causing the specified moment to pass by unnoticed. After the marriage, the unfortunate Līlāvatī lost her husband, and Bhāskara promised his devastated daughter to teach her arithmetic and write a book in her honor that would preserve her name to history.[1]

Bhāskara's main mathematical contributions include the use of abbreviated words and symbols to represent unknown quantities and operations; a thorough mastery of trigonometric identities, including the now familiar sum and difference rules; and a more comprehensive discussion of negative quantities, which he referred to as *losses* or *deficits* and indicated by way of a small dot over the numeral. He observed that the square of a positive or a negative number is always positive and that positive numbers have two square roots, one positive and one negative. He also held that negative numbers are not square numbers and have no square roots.

[1] Since the 2010 meeting of the International Congress of Mathematicians in Hyderabad, this institution now offers once every 4 years a *Leelavati Award* for work in the dissemination and popularization of mathematics.

More significantly, Bhāskara and other Indian mathematicians made free use of irrational numbers and did not distinguish them from rational numbers in arithmetic computations. This is in contrast with the Greeks, who had a worked out theory of incommensurable qualities but did not admit irrational numbers into the pantheon of numbers proper.

As heir to the mathematical tradition of Brahmagupta, Bhāskara studied carefully and understood deeply the complete works of his predecessor and made improvements to some of his results, in particular with respect to Pells' equation $nx^2 + 1 = y^2$. He was also productive and successful as an astronomer, which discipline he always pursued from the perspective of a mathematician, introducing innovations in spherical trigonometry, cosmography, astronomical instruments, and so on. Perhaps most astonishing is his use of a technique of instantaneities essentially identical with what would centuries later be the differential calculus to study the motion of the planets. It is said that some generations after his lifetime, a stone stele was discovered in Pataliputra with records of the donation of a sum of money on August 9th, 1207, by local dignitaries to a local educational institution to sponsor the study of his writings. At that time he had already passed away more than 20 years earlier, around the year 1185 (Fig. 4.12).

Remarkably, another very advanced mathematician and astronomer appeared in Kerala at the southern tip of India more than two centuries later. This was Mādhava of Sangamagrāma (ca. 1340–ca. 1425). He was the leading representative of the Kerala school of mathematicians, and the results attributed to him include power series expansions for trigonometric functions and their derivations. We will say something about this work in connection with Leibniz in the final section of Chapter

Fig. 4.12 Srinivasa Ramanujan, Indian mathematician of genius

Fig. 4.13 Indian mathematician Kanakanahalli Ramachandra; photograph by the author, in Bangalore

5. Indeed it is necessary to point out that at the time that Mādhava was born, two pioneers of the European Renaissance, the poet Dante Alighieri and Giotto, the father of European painting, had already died.

We close this section with a discussion of a few mathematicians born in India in the nineteenth and early twentieth centuries, at which time India was under the control of the British Empire as a colony. In addition to the British writers William Thackeray, Eric Blair (better known by his pen name George Orwell), and Rudyard Kipling, two British mathematicians were also born in India during this period. The first of these was the mathematical logician Augustus De Morgan, born in Madurai in the state of Tamil Nadu in South India in 1806, and about a century later, the algebraic topologist J.H.C. Whitehead was born in Chennai, at that time known as Madras, in the same state. De Morgan was responsible for a significant revolution in mathematical logic, freeing it from the restrictive laws passed down by Aristotle, and is recognized today as one of the founders of modern mathematical logic. Whitehead made significant contributions to the development of homotopy theory, one of the fundamental tools of algebraic topology, and also gave the first precise definition of differentiable manifolds (Fig. 4.13).

It was also in this same state of Tamil Nadu that one of the great geniuses of Indian mathematics was born: Srinivasa Ramanujan (1887–1920). Ramanujan was an almost entirely self-taught prodigy who made remarkable contributions to number theory, especially in the theory of integer partitions, but also in the fields of elliptic functions, hypergeometric functions, and divergent series. Alongside the polymath and poet Rabindranath Tagore (1861–1941), these two figures are the pride of modern Indian intellectual history. The inspiration provided by Ramanujan in particular spurred India to achieve great progress in mathematics and the natural sciences in the second half of the twentieth century. The Indian mathematician Kanakanahalli Ramachandra (1933–2011) formed an Indian school of number theory, establishing him as perhaps the true successor of Ramanujan. In physics, the University of Madras has produced two Nobel Laureates: CV Raman (1888–1970) and Subrahmanyan Chandrasekhar (1910–1995). The latter of these two was 10 years old in the year that Ramanujan died.

Sacred Land

The Arabian Empire

We turn now from India to the Arabian Peninsula, where the prosperity of the Arabian Empire contributes one of the most exciting episodes in the history of humanity. The starting point of this narrative is of course the legendary life of the prophet Muhammad. Muhammad ibn Abdullah was born around the year 570 in Mecca in the southwestern Arabian Peninsula. In contrast with the founding figures of Jainism and Buddhism, who were raised in opulence, Muhammad was orphaned in childhood, which, although his grandfather was an important tribal leader, cut him off from his inheritance. At that time Mecca was also a remote place, far removed from every commercial, cultural, and artistic centers, and Muhammad grew up under difficult conditions. When he was 25 years old, Muhammad married the businesswoman widow of a successful businessman, and his economic situation improved considerably, but it was not until the age of 40 that the events that have cemented his place in history took place (Fig. 4.14).

Muhammad experienced a revelation that there is one almighty god Allah with dominion over the world, who had chosen Muhammad as prophet to preach his faith. This was the birth of the religion of Islam. The word *islām* means *submission* in Arabic, and the believers in this religion are called Muslim, meaning *those who submit*. Islam teaches that the end of the world will see the resurrection of the dead and the judgment of all people according to their actions. Muslims have an

Fig. 4.14 Mosque geometry

obligation to relieve poverty and the suffering of others, the accumulation of wealth or mistreatment of the poor are considered social ills that will receive the harsh judgment of later generations. It is also emphasized in Islam that all believers are brothers and should live together in a close group. One saying has it that Allah is closer to his followers than the blood vessels in their necks.

In the year 622, Muhammad had gathered around him about 70 followers, and persistent persecution caused them to migrate from Mecca to Medina, about 200 kilometers to the north. This marked a turning point in the story of Islam and led to a rapid accumulation of new followers. The Bedouin living at that time on the Arabian Peninsula were nomadic Arabic-speaking tribes who were known for their bravery and martial talents, but they had long been divided and had never united in rivalry against the tribes living on the arable land further north along the peninsula. Muhammad brought them together through religious and institutional maneuvers such as intermarriage and began an unprecedented period of conquest, including a campaign that he himself led that pushed as far as the edge of Syria (Fig. 4.15).

In the 10 years following Muhammad's death in the year 632, the army he had assembled, led by two heirs to the caliphate, both of whom were father-in-law to Muhammad, defeated the Sassanid army of Persia; occupied Mesopotamia, Syria, and Palestine; and seized Egypt by way of Byzantium, landing the final blow to the Alexandrian civilization. Around the year 650, a holy text known as the Quran

Fig. 4.15 Cover page of the Quran

was published, based on the words of Muhammad and his followers. This book is considered by Muslims to be a revelation directly from God, written in the language of Allah, and forms the basic text of the Islam religion, one of the four valid sources of law in Usuli jurisprudence, alongside the teachings and normative examples of Muhammad known as the *sunnah* and composed of oral accounts called *hadith*, the consensus of scholars (referred to as *ijmā*ʿ), and the guidance of the rational faculty of the soul (*ʻaql*).

The Muslim campaign continued its rapid expansion: in the year 711, they swept North Africa and proceeded toward the Atlantic Ocean; they crossed the Strait of Gibraltar to the north and occupied Spain. During this period, China was still experiencing the peace and prosperity of the Tang Dynasty, the poet Li Bai was still in his youth, and his younger friend and fellow poet Du Fu was still in his mother's womb. From the perspective of mathematical history, it had been half a century since the death of Brahmagupta, and there were not at this time any significant mathematical figures alive in either the east or the west. It was not inconceivable that the Muslim army would conquer the entirety of Christian Europe. But in the year 732, the Muslim invaders had reached the Aquitane region in what is now central France, where they were defeated in the Battle of Tours.

The Muslim army had expanded their territory to a vast area from India in the east to the Atlantic Ocean to the west, encompassing areas north of the Caspian Sea stretching into Central Asia. This was the Umayyad Caliphate, probably the largest empire in human history to this date, committed throughout its expansion to the dissemination of Islam. But internal power struggles caused the caliphate to split into two powers, one with its capital at Córdoba in Spain and the other with its capital in Damascus, in Syria. The latter became the Abbasid Caliphate, founded by descendants of Muhammad's uncle Abbas ibn Abdul-Muttalib, and its center of power switched gradually eastward to Baghdad in Iraq. Here the Abbasids built a city unparalleled in the world, a center of science, culture, philosophy, and invention during the period later known as the Golden Age of Islam. The Abbasid Caliphate became the longest and most renowned dynastic power in the history of Islam.

The House of Wisdom in Baghdad

Baghdad is located along the Tigris River, at the point where the distance between the Tigris and the Euphrates is smallest, and surrounded by a flat alluvial plain. The word *baghdad* is most likely a Middle Persian compound meaning *a gift from God*, and this city quickly began to prosper after it was established as the capital of the Abbasid Caliphate by its second caliph, Al-Mansur in 762. Palaces and various buildings sprang up from the ground between the circular city walls, and the city reached the peak of its economic and academic prosperity by the late eighth century and the first half of the ninth century, under the governance of Al-Mahdi and his successors Harun al-Rashid and Al-Ma'mun. At this time, Baghdad was one of the richest cities in the world (another was Chang'an in China) (Fig. 4.16).

Fig. 4.16 1237 illustration of the House of Wisdom

The ninth century in world history begins with the appearance on the world stage of two emperors with a dominant role in international affairs: Charlemagne, King of the Franks, and Harun al-Rashid. Charlemagne was the grandson of Charles Martel, who had defeated the Umayyad invasion of Aquitaine at the Battle of Tours, and Charlemagne was crowned Emperor of the Romans by Pope Leo III on Christmas Day in the year 800. Harun al-Rashid exerted an even greater influence on the course of history, and these two leaders in the east and the west, respectively, established a diplomatic alliance and even a personal friendship, characterized by frequent exchanges of valuable gifts. Charlemagne hope that Harun could aid him in his efforts against the Byzantine efforts, and Harun in turn hoped for Charlemagne's assistance in his struggle against the surviving Umayyad dynasty in Spain.

Both legend and historical evidence confirm that the period under Harun al-Rashid was the most glorious time in the history of Baghdad. In less than half a century, the city had grown from its humble beginnings as a deserted village to a cosmopolitan center of extraordinary wealth. Only Constantinople in Byzantium could compete with it for splendor at the time. Harun was a typical Muslim monarch, whose magnetic generosity attracted poets, musicians, singers, dancers, and trainers for hounds and cock-fighting, in a word all manner of remarkable people, to the capital. He was immortalized as an extravagant, even profligate, caliph in *The Thousand and One Nights*.

About the year 771, 9 years after the establishment of Baghdad, an Indian traveller brought with him two scientific papers to the capital. One of these was an astronomical treatise, which Al-Mansur had translated into Arabic. This

Fig. 4.17 Arabic translation of Greek writings

translator became the first astronomer of the Islamic world. Previously, the nomadic inhabitants of the Arabian Peninsula had sustained a deep interest in the position of the stars, but they had not conducted any systematic scientific research into this phenomenon. But the rise of Islam provided a need for careful astronomical calculations due to the requirement to pray facing the direction of Mecca five times daily. This obligation to prayer (*Salah*) is one of the five pillars of Islam; the others are the declaration of faith (*Shahada*), almsgiving (*Zakat*), fasting (*Sawm*), and pilgrimage (*Hajj*) (Fig. 4.17).

The second paper was a mathematical essay by Brahmagupta. The Lebanese-American historian Philip K. Hitti has observed that the numerals known to Europeans as Arabic numerals, and known in Arabic as Indian numerals, were first introduced to the Muslim world by this paper. Apart from this however, the cultural exports of India were very few, and eventually the Greeks came to have the most substantial influence on Arabic thought of this period, especially after the conquest of Syria and Egypt. They began to actively seek out Greek works, including Euclid's *Elements*, Ptolemy's *Almagest*, and the dialogues of Plato, all of which were translated one after another into Arabic.

Here we point out that also Chinese papermaking technology had not long earlier been introduced to the Arabic world at this time, four centuries before its introduction to Europe by way of the Middle East and North Africa (bypassing the Mediterranean, like Hindu-Arabic numerals). Indeed, there was a paper mill in the city of Baghdad already. The Chinese had kept the art of papermaking a deeply held secret for many centuries after improvements were made to this craft in the second century by Cai Lun of the Eastern Han dynasty. But in 751, a Tang dynasty army was defeated by Muslim forces in the Battle of Talas in Kazakhstan, and a group of paper workers were taken prisoner to Samarqand, where they were compelled to divulge their knowledge.

The Greek influence reached its peak after Harun al-Rashid was succeeded by his son al-Ma'mun, who became obsessed with rational inquiry. He is said to have been visited by Aristotle in a dream, who assured him that there was no substantial conflict between reason and then teachings of Islam. In 830, al-Ma'mun ordered the construction of the *Bayt al-Ḥikmah* or *The House of Wisdom* in Baghdad. This was a joint institution with integrated functions as a library, science academy, and translation center. This was undoubtedly the most important academic institution to appear since the establishment of the Library of Alexandria in the third century BCE, and it quickly became the center of the academic world, a place of extensive research activity in philosophy, medicine, zoology, botany, astronomy, mathematics, mechanical technology, architecture, Islamic theology, Arabic grammar, and more.

The Algebra of al-Khwarizmi

In the latter half of the long and effective era of translations under the Abbasid caliphate, Baghdad became home to an age of scientific originality. The most important figure in this story is the mathematician and astronomer Muḥammad ibn Mūsā al-Khwārizmī (780–850). He was born more than a century after the death of Brahmagupta and before the birth of Mahāvīra. Relatively little is known about his life, although it is generally believed on the basis of his name that he was born in the Khwarazm region of Greater Iran, where the Amu Darya river flows into the Aral Sea, not far from the city of Khiva in modern Uzbekistan. Another theory has it that al-Khwarizmi was born on the outskirts of Baghdad but descended from people of the Khwarazm region (Fig. 4.18).

One of his epithets also implies that al-Khwarizmi had among his ancestors adherents of the ancient Zoroastrian religion. Zoroastrianism, also known as

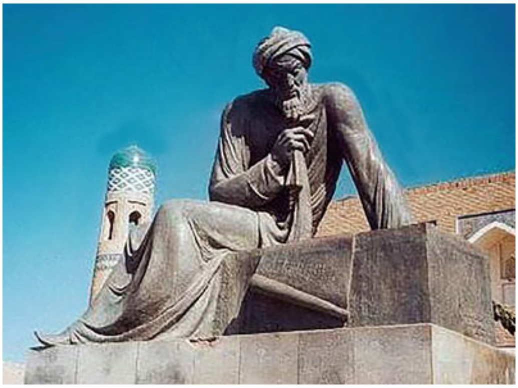

Fig. 4.18 Statue of al-Khwarizmi in Samarkand

Mazdayasna, is one of the oldest continuously practiced religions in the world, with a history stretching back even at that time more than two and a half millennia. Its tenets include a form of elementalism with special prominence given to the role of fire, a theology of cosmic dualism, and its opposition to abstinence, celibacy, fasting, and other such forms of ascetic self-denial. The founding prophet Zoroaster (or Zarathustra) of Zoroastrianism probably lived about the same time as the early Jainist preacher Mahavira, perhaps about 30 years earlier, somewhere in the northern parts of what is today Iran. Since his death, the religion he founded became at several times the state religion of the Persian Empire.

For these reasons, it can be inferred that al-Khwarizmi had Persian ancestors; even if this is not the case, for example, if he was rather of Central Asian descent, his spiritual life was essentially linked to Persia, a nation with a long and deep cultural tradition, and he was proficient in Arabic. Al-Khwarizmi spent the years of his early education in his hometown, before travelling to the ancient city of Merv in Central Asia to continue his studies. He also visited Afghanistan, India, and other places in pursuit of learning. He quickly became a well-known and respected scholar, and al-Ma'mun, at that time governor of Khorāsān in the eastern Iranian Plateau, summoned him back to Merv. Later, after al-Ma'mun became caliph of the Abbasid Caliphate, he hired al-Khwarizmi to work in the capital Baghdad, where al-Khwarizmi eventually became one of the lead researchers at the House of Wisdom. After the death of al-Ma'mun, his successor continued to employ al-Khwarizmi, who remained in Baghdad until his own death. This was a time of tremendous political stability, economic development, and cultural and scientific prosperity for the Islamic empire (Fig. 4.19).

Al-Khwarizmi left behind two influential mathematical treatises: *Algebra* and *On the Calculation with Hindu Numerals*. The full title of the book referred to colloquially as *Algebra* was *The Compendious Book on Calculation by Completion*

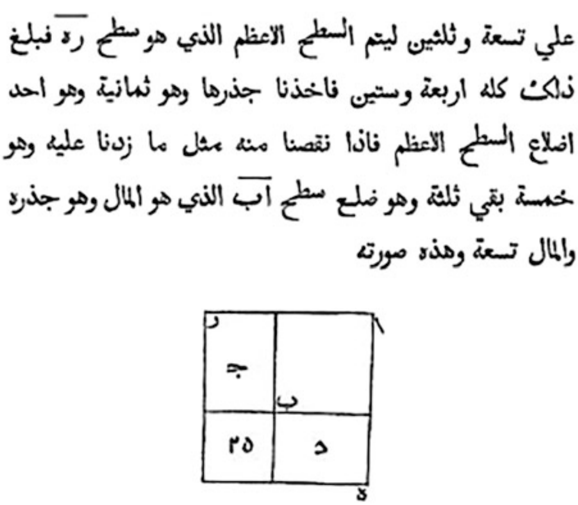
علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

Fig. 4.19 Manuscript of the *Algebra* of al-Khwarizmi *and Balancing*, and the modern word *algebra* derives from the term *al-jabr* contained within it, meaning *restoration* and designating the process of adding a number to both sides of an equation to cancel or consolidate terms. This book was translated into Latin in the twelfth century during a period that saw a burst of translation activity and exerted an inestimable influence on the subsequent development of European sciences. It would not be inaccurate to say that algebra was an Arabic invention, as perhaps geometry was an Egyptian invention; perhaps algebra was even something like the grammar through which the people of the Arabian Peninsula understood the workings of numbers.

Al-Khwarizmi completed *Algebra* in about the year 820. The problems he discusses are not more difficult than those in Diophantus or Brahmagupta, but since he concerns himself with general rather than particular solutions, his perspective is much closer to that of modern elementary algebra than anything that appears in the ancient Greek or Indian mathematical literature and constitutes a remarkable achievement in the history of mathematics. The book discusses the algebraic treatment of linear equations and provides a general algebraic solution to the quadratic equation. More importantly, it also introduces generalized algebraic tools, such as shifting terms from one side of an equation to the other and merging like terms, paving the way for algebra to develop as the science of equation solving. It is hardly surprising that al-Khwarizmi's book became a standard textbook in Europe for several centuries, an unusual situation for a scientific work from the east.

Whereas Brahmagupta provided only a single solution for quadratic equations in one variable, al-Khwarizmi gives both. He was perhaps the first mathematician in world history to observe explicitly that quadratic equations have two roots. On the other hand, although he was aware that such equations can have negative roots, he did not admit either negative roots or zero roots as solutions to quadratic equations. He also pointed out that if the discriminant of a quadratic equation is negative, then there are no (real) roots (here we introduce modern terminology to describe this observation). After having provided solutions to various typical equations, al-Khwarizmi provides also geometric proofs for his results, a practice which shows the obvious fingerprints of Euclid and the Greeks. Therefore it is reasonable to say that unlike other mathematicians of the Arabian Peninsula, al-Khwarizmi was influenced by the two civilizations of Greece and India, which of course also reflects their relative geographic positions.

On the Calculation with Hindu Numerals was another immensely important book in the history of mathematics, since it provides a systematic introduction to Hindu numerals and decimal notation, both of which had previously been described to scientists in Baghdad by Indian visitors, but which had not yet attracted widespread attention. Like *Algebra*, this book was translated into Latin and widely disseminated in the twelfth century. An early Latin edition is housed today in the University of Cambridge library. Subsequently, the Indian system of numeral notation gradually came to replace the alphabetic system employed by the Greeks, and the system of Roman numerals, until eventually it became the universal number system across the globe. On account of the history of its adoption, such numerals are generally known today as Arabic numerals or Hindu-Arabic numerals. It is also worth mentioning that the original title of the Latin translations of *On the Calculation with Hindu Numerals* was *Algoritmi de numero Indorum*, where *Algoritmi* was the Romanization of the name al-Khwarazmi. It is from this that the modern term *algorithm* in mathematics and computer science derives.

Al-Khwarizmi also made contributions to geometry, in particular to the measurement of area. He classified triangles and quadrilaterals and gave, respectively, the formulas for the calculation of their areas. He also gave an approximate formula for the area of a circle

$$A = \left(1 - \frac{1}{7} - \frac{1}{2} \times \frac{1}{7}\right) d^2,$$

where d is the diameter of the given circle, corresponding to the value $\pi \approx 3.14$. We see also here that the Arabic mathematics like the Indians adopted the Egyptian preference for unit fractions. Al-Khwarizmi also gave an area formula for a circular segment, considering separately the cases where the segment is larger or smaller than the semicircle, respectively.

In astronomy, al-Khwarizmi compiled trigonometric and astronomical tables to calculate the positions of the stars and lunar and solar eclipses and wrote numerous books on astrolabes, the sine quadrant, sundials, and the calendar. In astronomy in particular, al-Khwarizmi's work inspired a remarkable successor: al-Battani (ca. 858–929), who was born in Syria and discovered for the first time that the radius between the earth and the sun varies throughout the year and determined that the apogee of the sun (the point when it is furthest from the earth) produces an annular solar eclipse. Al-Battani replaced the geometric method in astronomy with trigonometry, introduced the use of the sine function in calculations, and corrected some of errors in the works of Ptolemy, including improved calculations for the orbits of the sun and certain planets. His astronomical works were also translated into Latin in the twelfth century and became the best known works of their kind in medieval Europe.

The intellectual achievements of al-Khwarizmi were not limited to mathematics and astronomy. He also wrote the first works of academic history in Arabic and promoted thereby the development of history as a research subject. He also participated in an important project of the period, the creation of a world map, which was in high demand for its military and commercial utility (the people of the Arabian Peninsula have proved to be especially savvy businesspeople). This led to the composition of a book entitled *Kitâb Sûrat al-Ard*, or *Book of the Image of the Earth*, the first geographical monograph in medieval Muslim history, which describes the important settlements, mountains, rivers, lakes, seas, and islands of the known world at that time, accompanied by four maps.

The Scholars of Persia

Omar Khayyam

Although the mathematics and science of the Arabian Peninsula in the medieval period exhibited mainly the influence of Greece and India, the main influence on the culture of this civilization was undoubtedly Persia, in a relationship similar to that of Macedonia under the sway of the Greek civilization, the fruit of which was the great polymath Aristotle. The people of the Arabian Peninsula were known not only for their decisiveness and bravery but also for their excellent organizational and management skills, and an attitude of tolerance and generosity, but in rational philosophy they lagged behind the Persians. In the end, there were two main indigenous features of this civilization: the rise of Islam, which became the state religion, and the Arabic language, which was preserved as the official national language. In other respects, the Persian influence was everywhere: in Baghdad, Persian titles, Persian wine, Persian romance, Persian songs, and more gradually became more and more fashionable (Fig. 4.20).

Fig. 4.20 Mausoleum of Omar Khayyam, Nishapur

According to legends, it was al-Mansur himself who first came to revere and emulate the Persians, and his subjects naturally followed his lead. He also initiated a Persian revival under his government, including a heavy dependence on Persian diplomats. Al-Mansur appointed the first Persian vizier Yahya ibn Khalid, and the two became close to the point that their wives nursed each of the other's children, who were born around the same time, and Jafar bin Yayha, son of Yayha ibn Khalid, served as a tutor to Harun al-Rashid. Eventually however there was a falling out between these two families, according to story, because Jafar bin Yayha violated the terms of what was intended to be a purely formal marriage with Harun's beloved sister Abbāsa, and the two of them had conceived a child. In any case, the Barmakid family to which Yayha belonged fell out of favor, Jafar was beheaded, and many of the remaining family members were imprisoned.

Following the death of al-Ma'mun, the Abbasid Caliphate went into decline, and many smaller autonomous dynastic powers begin to spring up at the periphery of Baghdad. The political situation became increasingly turbulent, competing religious factions proliferated, the empire gradually fractured, and what central power remained fell to the military over the course of multiple troop riots and finally a slave uprising known as the Zanj rebellion. All this provided an opening for the Turks and the Persians to direct again the point of their blades at the heart of Baghdad.

In spite of all this, noteworthy intellectual activity continued in Baghdad. In the tenth century, the Persian mathematician al-Karaji (953–1029) carried out research into the binomial coefficients (later than the Indians, but earlier than Jia Xian) and the algebra of exponents, and made contributions to the theories of linear equations and mathematical induction. In 1065, the first official institution of higher learning in the Muslim world, the Al-Nizamiyya, was established in Baghdad, but it failed to attract the brightest young talents of the period, such as Omar Khayyam, perhaps the greatest mind of the medieval Islamic world.

Omar Khayyam (1048–1131) was born in 1048 in Nishapur, an ancient city in the Khorasan region of northeastern Iran. The name *khayyam* means *tent-maker*, which suggests that his father or more distant forebears were engaged in the practice of tent-making. Perhaps for this reason, his childhood was spent roaming freely with his father, first within the confines of his hometown, later in the province of Bukhara, home to a famous library, and the small city of Balkh in northern Afghanistan, where he studied, before finally he moved to Samarkand, the oldest city in Central Asia, where he gained political favor. Here he began to compose his substantial mathematical works, under the patronage of the governor and chief judge of the city (Fig. 4.21).

Already in Euclid's *Elements*, there appear geometric solutions to quadratic equations of the form $x^2 + ax = b^2$, where one of the solutions is given by

$$\sqrt{\left(\frac{a}{2}\right)^2 + b^2} - \frac{a}{2}.$$

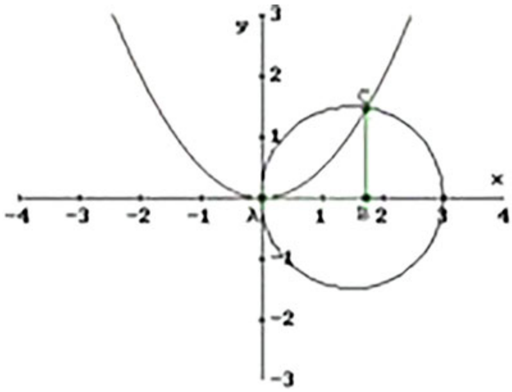

Fig. 4.21 Graph used by Omar Khayyam to find solutions to cubic equations

This can be proven by using right triangles and the Pythagorean theorem: construct a right triangle with sides of length $\frac{a}{2}$ and b about the right angle. Then after removing a length $\frac{a}{2}$ from the hypotenuse, what remains will be the solution as given. But the solution of cubic equations is obviously more complicated. Omar Khayyam considers 14 types of cubic equations in his research and determined their roots by considering the intersections of two conic sections.

Consider as an example the cubic equation $x^3 + ax = b$. This can be rewritten as $x^3 + c^2x = c^2h$, which equation Omar Khayyam considered as determined by the abscissa x of the intersection C of the parabola $x^2 = cy$ and the semicircle $y^2 = x(h - x)$ (see the figure), since the variable y can be eliminated from the latter two equations to recover the original cubic equation. As a result, Omar Khayyam obtained solutions to cubic equations by way of conic sections and paved the way for the study of polynomial equations of higher degree. He presented his results in an important treatise *Maqāla fi l-jabr wa l-muqābala* (*On proofs for problems concerning Algebra*). He almost contributed a historically important attempt at a proof of Euclid's fifth postulate, the parallel postulate.

In the eleventh century, a Turko-Persian empire called the Seljuk Empire swept into power and gained control over a vast area, stretching from western Anatolia and the Levant in the west to Hindu Kush in the east and from Central Asia in the north to the Persian Gulf in the south, also under the banner of Islam. Omar Khayyam was commissioned by the Sultan Malik-Shah I of the Seljuk Empire to travel to the Isfahan to preside over the establishment of a new observatory and the reform of the Persian calendar. This became his main source of livelihood, while mathematical research remained an important recreational pursuit. In his calendar work, he proposed the addition of 8 leap days every 33 years to the basic 365 days of the normal year. This reduces the difference from the actual year to an error of 19.37 seconds, or 1 day every 4460 years, more accurate even than the Gregorian calendar in use around the world today. A change of leadership however had the unfortunate result that his reforms were not implemented (Fig. 4.22).

Omar Khayyam spent most of his life in Isfahan. It can be said that he experienced in his lifetime each of the three dominant strands of his period: Islam, the Suljuk

Fig. 4.22 Poem of four lines by Omar Khayyam, with illustration

court, and Persian culture. But all this turmoil and an eccentric personality made a solitary and somewhat unsettled life, and he recorded his occasionally untimely thoughts in the form of Persian poems of four lines, a style popular in Khorasan at the time. Probably he could not have imagined that some eight centuries later in 1859 an Englishman named Edward FitzGerald would translate and publish a selection of these poems under the title *Rubáiyát of Omar Khayyám* with the result that Omar Khayyam became known the world over for his poetry, while his mathematical achievements drifted under its shadow. In one of these poems (No. 57 in the numbering of the *Rubáiyát*), he laments the failure of his calendar reform:

> Ah, but my Computations, People say,
> Reduced the Year to better reckoning?–Nay
> 'Twas only striking from the Calendar
> Unborn To-morrow, and dead Yesterday.[2]

The ancient Iranian people referred to themselves as *Aryans* and probably belonged to the same group of Indo-European speaking Central Asian nomads who travelled toward Europe sometime between the years 2000 and 1000 BCE. Indeed, the modern Persian name of Iran means *the land of the Aryans*, and likely these were the same people who earlier had migrated into India, where some remained and intermarried with the aboriginal Dravidian peoples. The name Persia itself derives from *Parsa*, the name of the people from whom Cyrus the Great of the Achaemenid

[2] Tr. Edward FitzGerald

dynasty emerged. These people gave their name to the region where they settled as *Fars* (or *Pars*), today a province of Iran containing the central city of Shiraz, known as the city of poets and flowers.

This was the birthplace of modern Persia. Cyrus the Great was born in the sixth century BCE and started as a local leader in his native region, before eventually managing to defeat the Babylonians and many other powers and establishing a great empire between India and the Mediterranean. After the death of Cyrus, one of his sons, and the son Darius of one of his ministers, continued to expand the territories of the empire to encompass Egypt, which is when legend has it that Pythagoras was taken captive in Babylon. The cuneiform inscriptions at Mount Behistun in western Iran that we have already encountered relate how Darius came to the throne. It seems also that after the decline of Plato's Academy in Greece, many Greek scholars travelled to Persia and contributed to the development of its civilization.

Nasir al-Din al-Tusi

About 70 years after the death of Omar Khayyam, during which time both the Italian mathematician Fibonacci and the Chinese mathematician Li Ye were born, the great Persian polymath Nasir al-Din al-Tusi (1201–1274) was born in the city of Tus, also in Khorasan. At this time Tus was the intellectual center of the Arabian Peninsula, and Harun al-Rashid also died there. Nasir-al-Din Tusi's father was a scholar of jurisprudence, who encouraged his son to take seriously his studies and saw himself to his elementary education before his early death, while an uncle in the same city taught Nasir-al-Din al-Tusi logic and philosophy. He also took up algebra and geometry during this time. In his later youth, Nasir-al-Din al-Tusi moved to Nishapur, home of Omar Khayyam, to study medicine and mathematics with disciples of the legendary Persian philosopher and scientist Ibn Sina (ca. 980–1037), more famous in western countries by his Latin name Avicenna. Gradually Nasir-al-Din al-Tusi made a name for himself as a thinker (Fig. 4.23).

At this time, the armies of Genghis Khan were sweeping westward, and the vestiges of the Islamic empire were crumbling. In the absence of any stable academic environment, Nasir-al-Din al-Tusi was invited to move from stronghold to stronghold at the behest of the Nizari Ismaili state, and it was in this context that he composed several important works of mathematics and philosophy. Finally in 1256 the grandson Hulagu Khan of Genghis Khan (the brother of Mongke Khan and Kublai Khan) conquered northern Persia and occupied the stronghold Maymun-Diz where Nasir-al-Din al-Tusi had settled. He was captured, but managed to earn the respect of Hulagu Khan, who appointed him as scientific advisor to the Mongols. Two years later, Nasir-al-Din al-Tusi served under Hulagu Khan in a brutal and bloody expedition against Iraq that signaled the final end of the Abbasid Caliphate.

After the death of Mongke Khan, Kublai Khan succeeded to the throne, and Hulagu Khan was made king of the Ilkhanate territory and charged with the subjugation of Persia and any remaining Muslim states in southwestern Asia, which

Fig. 4.23 Iranian commemorative stamp in honor of Nasir al-Din al-Tusi

he accomplished at the head of a massive Mongol army, establishing a capital in the city of Tabriz in northwestern Iran adjacent to Azerbaijan. With the approval and funding of Hulagu Khan, Nasir-al-Din al-Tusi had established the Maragheh observatory in this region, and he set about recruiting talented scholars, writing and carrying out research, and commissioning the production of advanced instruments of observation, so that the Maragheh observatory became an important center of academic activity. Nasir-al-Din al-Tusi has been twice featured on commemorative stamps in Azerbaijan on account of his importance to the region.

In 1274, Nasir-al-Din al-Tusi paid a visit to Baghdad, where he succumbed to an illness and was buried in the suburbs. Hulagu Khan had already died at this time, having conquered an area including all of Persia and of course Baghdad in particular. By the time his grandson came into power, the Ilkhanate extended from the Amu Darya river in the east to the Mediterranean Sea in the west, from the Caucasus in the north to the Indian Ocean in the south (Fig. 4.24).

Nasir-al-Din al-Tusi was a diligent writer throughout his entire life, leaving behind a trove of treatises and letters in Arabic, as well as a small number of philosophical texts in Persian. He was also reputed to be familiar with the Greek language, and Turkish makes an appearance in some of his writings. He touched upon every aspect of the Islamic intellectual world of the time, in particular mathematics, astronomy, logic, philosophy, ethics, and theology. These works are not only classics of Islamic scholarship but also influenced deeply the awakening of

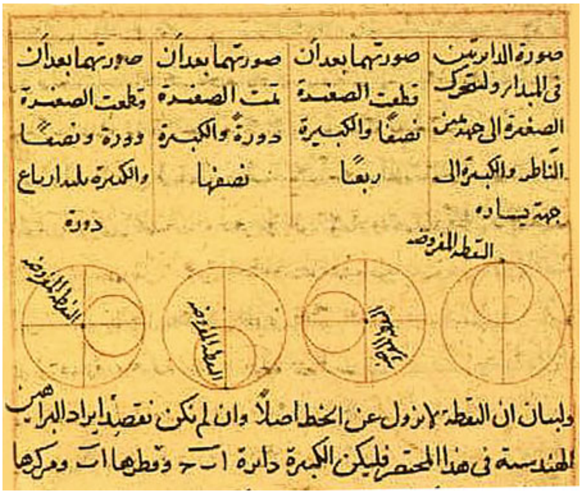

Fig. 4.24 A manuscript of Nasir al-Din al-Tusi

European science. Some of his astronomical instruments may also have made their way to China where they served as reference for the astronomers there.

Three of his books in particular are noteworthy mathematical treatises. In a collection of arithmetical writings entitled *Jami' al-hisab bi-' l-Takht wa- 'l-turab*, Nasir-al-Din al-Tusi presents familiar results from Omar Khayyam and extends the research of numbers more deeply in the direction of irrational numbers and other fields. The numerals are Hindu throughout and include some discussion of Pascal's triangle and methods for finding fourth roots and other higher roots of numbers. This seems to be the earliest extant work on this topic. It is also fascinating that Nasir-al-Din al-Tusi discovered the important number theoretic result that the sum of squares of two odd numbers cannot be square, which is usually proven by way of the theory of congruences (Fig. 4.25).

A more substantial work is contained across the *Al-Risala al-shafiya 'an al-shak fi al-khutut mutawaziyya* (*Treatise healing the doubt about the parallel lines*) and the *Tahrir al-Usul al-Handasiya li-Uqlidis* (*Exposition of Euclid's Elements*), comprising two revisions and annotations of the *Elements* and containing a more detailed consideration of the parallel postulate. Nasir-al-Din al-Tusi argued that the parallel postulate ought not to be a postulate but rather something than can be proved on the basis of the four other fundamental postulates of Euclidean geometry. He followed in this regard the methodology of Omar Khayyam: if $ABCD$ is a quadrilateral, with segments DA and CB equal in length and perpendicular to the side AB, then the angles at C and D are equal. If these two angles are acute, Nasir-al-Din al-Tusi showed that it follows that the sum of the interior angles of a triangle is less than two right angles, which is the basic starting point of Lobachevskian geometry, although Nasir-al-Din al-Tusi did not pursue this line of thought any further.

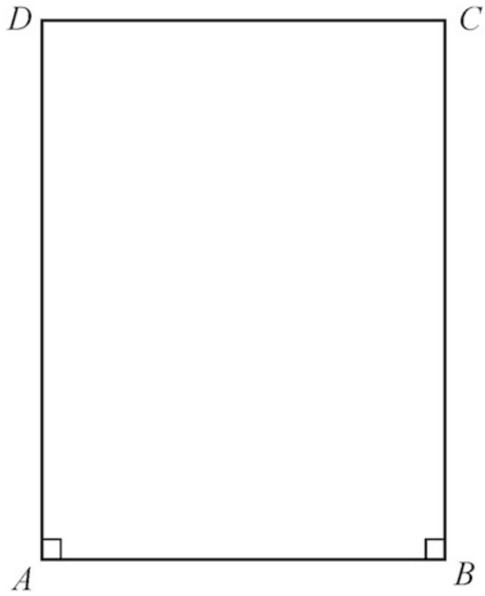

Fig. 4.25 Quadrilateral used by Nasir al-Din al-Tusi in connection with the parallel postulate

The most important mathematical work of Nasir-al-Din al-Tusi is his *Treatise on the Quadrilateral*, the first mathematical monograph devoted solely to trigonometry in the history of mathematics – prior to this, trigonometric results appeared only in astronomical texts, as a computational tool, and it was only after the work of Nasir-al-Din al-Tusi that trigonometry developed as an independent branch of pure mathematics. This book contains also the first statement of the sine law for plane triangles: if A, B, C are three angles in a triangle, and a, b, c the lengths of the sides opposite to them, respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Nasir-al-Din al-Tusi made equally outstanding contributions to astronomy, which we will not go into in more detail here. His two sons seem to have also worked at the Maragheh observatory, as well as a Chinese astronomer whose name and origins cannot be identified. The *History of Yuan* (元史) lists *seven western instruments* of Arabic design, some of which are very similar to those of Nasir-al-Din al-Tusi. Much later in the eighteenth century, several observatories built by Indians in Delhi and other places imitated the structure and appearance of the observatory established by Nasir-al-Din al-Tusi at Maragegh.

Jamshīd al-Kāshī

It is a historical fact concerning the broad appeal of Islam that although the territory captured by the various Muslim powers may be lost in time, the people inhabiting almost always will have long converted to Islam. Iran, or Persia, is a typical example. After the Arabic conquest in 640 and the advent of Islam after a series of debilitating wars with the Byzantine Empire, the rule over this territory changed hands again and again, as it was occupied by various powers. But the influence of Islam has remained a constant and is reflected through to today in the national emblem and national flag of Iran. The former contains four crescents, a sword in the shape of a water lily, and a stylized script, symbolizing, respectively, Islam, power, and the Quran. The latter is green, white, and red, with that Takbir ("*Allah is great*") written 11 times each in the Kufic script along the edges of the green and red bands.

We turn now to the last great mathematician and astronomer of the ancient Arabic world and indeed of the ancient east: Jamshīd al-Kāshī, whose death in 1429 marks the end of an era. There is no record of the year of his birth, but probably it was in around 1380. Rather the earliest record of his existence dates from June 2nd, 1406, when he observed a lunar eclipse from his hometown Kashan, in the eastern foothills of the central mountain range of Iran, today along the railway line between the old capital Isfahan and the modern capital Tehran. Although al-Kāshī seems to have come from a humble background, nevertheless like Omar Khayyam and Nasir-al-Din al-Tusi before him, his talents were early recognized and appreciated by the political elite.

At the end of the fourteenth century, a descendent of Genghis Khan named Timur (or Timūr Gurkānī), who had been crippled for life during a failed raid in his youth, established the Timurid Empire, with its capital at Samarkand. Timur belonged to the Turco-Mongol tradition and a believer in Islam, and he carried out a fierce and unstoppable campaign across lands stretching from Russia to India and the Mediterranean in pursuit of the restoration of the Mongol empire. He did not return to Samarkand until he had secured tribute from the Sultan of Egypt and the Byzantine emperor. Although Timur himself was illiterate, he enjoyed the company of learned scholars with whom he could play chess and discuss various questions of history, Islamic theology, and applied science (Fig. 4.26).

In 1405, as he was preparing to embark upon a new campaign against China, where the Yuan dynasty had already come to an end, Timur died of an illness. His grandson Ulugh Beg had no appetite for martial affairs, but rather developed an obsession with the sciences and astronomy in particular; he himself discovered through his own observations of calculation errors in the works of Ptolemy. He also wrote poetry, studied history and the Quran, and became a powerful patron and protector of art and science. He established early on an institute for science and theology in Samarkand and made plans to build an observatory as well. Samarkand quickly became the most important academic center in the eastern world at the time (Fig. 4.27).

The academic career of al-Kāshī was closely linked with Ulugh Beg. Although he was trained as a doctor, his passion was for mathematics and astronomy, and he

Fig. 4.26 Uzbekistani commemorative stamp in honor of Ulugh Beg

Fig. 4.27 Ancient city gates of Samarkand

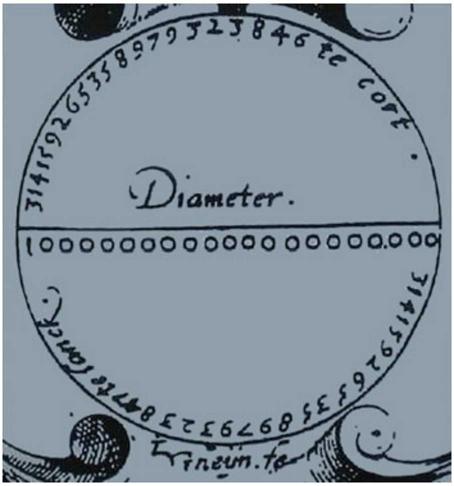

Fig. 4.28 Illustration of the value of π obtained by Al-Kāshī

found stable patronage at Samarkand in the court of Ulugh Beg after a long period of poverty and hesitation. He participated in the construction of the observatory and the installation of its various instruments and became its first director after its completion. In his astronomical works, such as *Sullam al-sama'* (*Ladder to the Sky*), al-Kāshī discussed the distance and size of the stars and other celestial bodies and introduced the armillary sphere and other astronomical instruments, some of which were his own inventions. He also participated like almost every other ancient scholar in calendar reform.

In a letter to his father, al-Kāshī praised in the highest terms the knowledge, organizational skills, and mathematical talents of Ulugh Beg and emphasized especially the spirit of academic freedom in the court at that time, which he considered a necessary prerequisite for scientific progress. Ulugh Beg seems to have been deeply sympathetic to the scientists working under him and was very willing to tolerate in al-Kāshī his lack of refined court etiquette and somewhat unconventional habits. In the preface of a calendar book bearing his name as its title, Ulugh Beg praised al-Kāshī as an outstanding scientist and one of the greatest scholars in the world, well versed in the ancient scientific sources and capable of solving the most difficult problems (Fig. 4.28).

Al-Kāshī contributed two landmark mathematical achievements: the first is his approximation of π, and the second is his approximation of $\sin 1°$. Throughout ancient times, research into the calculation of π reflected to a certain extent the mathematical development of a civilization, just as today the calculation of large prime numbers stands as a benchmark for the computer power available to a corporation or even a country. In 1424, Al-Kāshī set a new record for accuracy in the calculation of π, 962 years after Zu Chongzhi had established the previous record of accuracy up to the seventh decimal place. Al-Kāshī obtained

$$\pi \approx 3.14159265358979325,$$

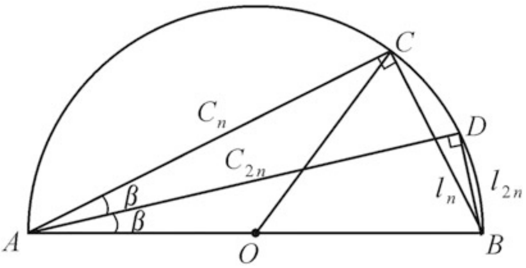

Fig. 4.29 Calculation of π by Al-Kāshī

accurate to 17 decimal places, by determining the perimeter of a polygon with 3×2^{28} sides. This record held until 1596, when Dutch mathematician Ludolph von Ceulen used inscribed and circumscribed polygons of 60×2^{33} sides to obtain an accuracy of 20 decimal places (Fig. 4.29).

We present here the method used by Al-Kāshī in his calculation. As shown in the figure, suppose $AB = d = 2r$ is the diameter of the circle and l_n (respectively, l_{2n}) is the length of one side of the regular polygon with n (respectively, $2n$) sides inscribed in it. Then the other two sides about the right angle have a recurrence relationship given by

$$c_{2n} = d \cos \beta = d\sqrt{\frac{1 + \cos 2\beta}{2}} = \sqrt{r(2r + c_n)}.$$

Then by the Pythagorean theorem,

$$l_n = \sqrt{(2r)^2 - c_n^2}.$$

A similar calculation gives the length of one side of the circumscribed regular polygon, and taking the arithmetic average of the two provides an approximation for the circumference of the circle from which the value of π can be obtained. In comparison with Liu Hui's method of circle division, Al-Kāshī is able to double the sides of the relevant polygon by calculating a single root, using the half-angle formula for cosines.

Conclusion

Bhāskara II died in Ujjain in about 1185, and afterward scientific activity in India went into gradual decline and mathematical progress more or less ceased entirely. In 1206, the long-lasting Delhi Sultanate was established, and India came into the Muslim sphere. About a century later, parts of the south became independent, and there began a protracted struggle for power. In contrast with India, mathematics in Persia both rose later and declined later. But shortly after the assassination of Ulugh

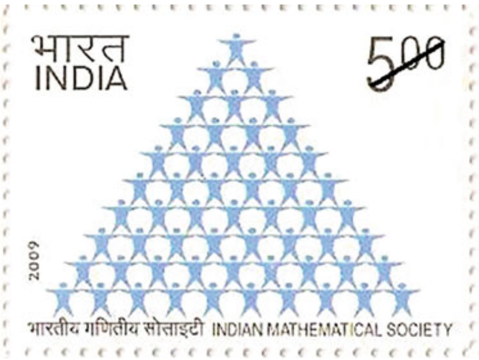

Fig. 4.30 Commemorative stamp of the Indian Mathematical Society

Beg, allegedly orchestrated by his own son, the Safavid dynasty, which was still essentially martial and internally constrained in nature, took power, and the glorious age of mathematics in Persia and indeed the entire Arabic world came to an end. But at precisely this time, the European Renaissance lit a new fire starting in the Apennine Mountains (Fig. 4.30).

As in Egypt, the mathematical minds of India were almost all clerical figures or otherwise belonged to a higher caste. This is in contrast with Greece, with the doors of mathematics in spirit at least were open to all. Another point of contrast is the Indian mathematicians, with the exception of Mahāvīra, were almost all astronomers by profession, whereas the Greeks viewed mathematics from the start as an independent discipline, worthy of study in its right (mathematics for the sake of mathematics). Third, the Indians expressed their mathematical thought in poetic language; the works could be mysterious or mystical, although of course they also introduced the zero numeral, and the results were mainly empirical, without derivations and proofs. The Greeks preferred a logical and even austere presentation and required proofs for every result. The astronomical bent of the Indian mathematicians produced however some splendid results. For example, Indian astronomers recognized that when the moon is half-full, the positions of the sun, the moon, and the earth form the vertices of a right triangle and were able to use this fact and their knowledge of the sine function to conclude that the distance between the moon and the earth is one-fortieth the distance between the sun and the earth.

The Persians were more accomplished in geometry than the Indians, though not so much as the Greeks, with a natural peak given by the geometric solution of cubic equations by Omar Khayyam. Like the Indian mathematics, the Arabic mathematicians for the most part considered themselves to be astronomers, and this emphasis on astronomy facilitated substantial contributions in trigonometry. The four mathematicians mentioned previously all carried out excellent work in astronomy, and the names of many stars even today use Latinate forms of Arabic words, for example, Aldebaran in Taurus, Vega in Lyra, Betelgeuse in Orion, Megrez in the Big Dipper asterism of Ursa Major, and Algol in Perseus. Arabic mathematicians also made substantial contributions to algebra, and many of the questions that appear later in Fibonacci's *Liber Abaci* are taken from al-Khwarizmi's *Algebra*.

The emphasis on astronomy under Islamic rule stemmed in part from the requirement to pray five times a day while facing toward Mecca, a challenging problem of coordination across a vast empire. To this end, they spared no expense in the construction of observatories and the recruitment of talented mathematical minds to staff them. The main work of these scholars included improvements to astronomical data, the construction of observatory sites, and the development of the science of optics. The Arabic interest in mathematics can be said to have arisen primarily through the practical demands of astronomy, astrology, and optics. But they were also excellent businesspeople and so had need to calculate distributions, inheritance, dividends, and so on. This led to the emphasis on algebra and especially calculation.

From the perspective of the history of mathematics, the Arabic mathematicians also served as a conduit through which the mathematical writings of India and ancient Greece were transmitted to Europe during a period of intense interest in translation. Translations of various mathematical works, including Euclid's *Elements*, survived in good condition in the House of Wisdom in Baghdad, long after the original texts had been lost or burnt. Later, this book was translated into Latin by European scholars, primarily in the western part of the Islamic empire centered in Toledo, the former capital of Spain. But as in the Indian and Chinese civilizations of the Middle Ages, Arabic mathematicians emphasized mostly practical results and did not inaugurate any new theoretical peaks or sustained development.

We now compare a bit the different philosophical attitudes of the Greeks and the thinkers to the east. The twentieth-century French philosopher Jacques Maritain (1882–1974) argued that the Indian philosophers viewed wisdom as a form of liberation, salvation, or divine wisdom and concomitantly their metaphysics never took the form of pure speculation in the practical sciences associated with the Greeks, who regarded wisdom rather as the target of rational human inquiry. This begins in the lower realm of earthly things, the tangible reality of change and movement, and the diversity of existence. It is somewhat paradoxical to note that the divine perspective of Indian philosophy contributed to simple and practical mathematical requirements, while for the earthly considerations of the Greeks and eventually all of Europe, mathematics developed its independent existence through the perfection of logical deduction.

It is worth a mention in closing that since the start of the twenty-first century, two mathematicians of Indian descent and two mathematicians of Persian or Iranian descent have earned Fields Medals. These are Manjul Bhargava in 2014 and Akshay Venkatesh in 2018 and Caucher Birkar in 2018 and Maryam Mirzakhani in 2014; Mirzakhani remains the first and only female recipient of the Fields Medal. Other developing countries have contributed several Fields Medalists in this century: Chi-Shen Tao, better known as Terence Tao, of Chinese descent in 2006, Ngô Bảo Châu of Vietnamese descent in 2010, and Artur Avila from Brazil in 2014.

Chapter 5
From the Renaissance to the Birth of Calculus

I would wish that the painter could be as learned as possible in the liberal arts, but first and foremost I would wish that he know geometry.

Leon Battista Alberti

Here is buried Isaac Newton, Knight, who by a strength of mind almost divine, and mathematical principles peculiarly his own, explored the course and figures of the planets, the paths of comets, the tides of the sea…

Inscription at Newton's monument

The Renaissance in Europe

Medieval Europe

During the period when the ancient civilizations of China, India, and Arabia in the east were making new contributions in mathematics, Europe was in the midst of its long Dark Age, a term first used by the Italian poet and scholar Petrarch (1304–1374), often considered the father of the Renaissance. The start of this period is marked by the collapse of Roman civilization in the fifth century, but there is no universal agreement as to its end, which could be considered to belong to the fourteenth, fifteenth, or even sixteenth century, with the start of the European Renaissance. The Dark Ages, which lasted for a thousand years, was later called the Middle Ages by Italian humanists in order to highlight their own works and ideals and to mark out the echoes of classical Greece and Rome formed by their era in contrast with the intervening centuries (Fig. 5.1).

Prior to the Middle Ages, the European territories outside of Greece and Rome had not done much to leave behind any deep marks on the history of human civilization, and since later there was no sign of intellectual revival in Greece, such terms as the Dark Age and Middle Ages alike, with the exception of the epidemic of the Black Plague, were mainly technical terms of academic humanism, limited in scope to Italy. In fact, even along the Apennines, the situation of mathematics

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T. Cai, *A Brief History of Mathematics*,
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Fig. 5.1 Likeness of Pope Sylvester II on a French stamp

during these times was not so bleak. Pope Sylvester II (ca. 945–1003) in particular admired and endorsed mathematics, and his election to the papacy was not unrelated to his mathematical facility, establishing him as something of a legend in the history of mathematics.

This pope, originally known as Gerbert of Aurillac, was born in central France and spent 3 years in Spain in his youth, where he studied the quadrivium at a monastery north of Barcelona, where the level of mathematics was high as a result of the legacy of Muslim Spain. When later he visited Rome, where he met the pope and the emperor, who were impressed by his mathematical knowledge, the latter hired him as a tutor for the young prince. With the further support of the subsequent emperor, Gerbert was elected eventually to the papacy and took the name Sylvester II. He is also said to have constructed an abacus and an armillary sphere, reintroducing them to Europe, and to have invented the first mechanical clock. In his mathematical work *De geometria*, he solved an open problem of the period: given the hypotenuse and area of a right triangle, determine the length of its remaining two sides (Fig. 5.2).

The period of Pope Sylvester II corresponds more or less with the era of translation in the history of science, when the classic works of Greek mathematics and science began to reappear in Western Europe, having been for centuries preserved primarily in the Islamic world long after they had disappeared from Alexandria and other centers of Greek academic activity. Whereas the translation of these works from Greek into Arabic had taken place mainly in the House of

Fig. 5.2 Toledo, Spanish capital city after the fall of the Roman empire; photograph by the author

Wisdom in Baghdad, the route from Arabic to Latin was more varied, taking place in the ancient Spanish city Toledo (which flooded with European scholars after the Christian defeat of the Muslims) and Sicily (which had been under Muslim rule for a period) and involving also diplomats in Baghdad and Constantinople.

The works translated into Latin included not only Euclid's *Elements*, Ptolemy's *Almagest*, *Measurement of a Circle* by Archimedes, *Conics* by Apollonius of Perga, and other Greek classics but also the more recent gems from the Islamic world, such as al-Khwarizmi's *Algebra*. All this took place mostly in the twelfth century, as the center of economic power in this part of the world shifted gradually from the eastern Mediterranean to the west. The primary mover of this change came from developments in agriculture, when the cultivation of pulses provided for the first time in history a guaranteed source of protein, leading to a population explosion that became one of the factors contributing to the disintegration of the old feudal structure.

By the thirteenth century, an endless proliferation of different social organizations emerged in Italy, including various guilds, associations, civic councils, churches, and so on, all of them desperate for some measure of autonomy. The idea of representation in the determination of important laws developed and spread

Fig. 5.3 Portrait of Fibonacci as a court mathematician

until finally a political assembly was formed whose members had authority to make binding decisions on behalf of all citizens participating in their election. In art, the classic models of Gothic architecture and sculpture sprang into being, and in terms of cultural and intellectual life, the methodology of scholastic philosophy took place of prominence. The representative figure of this trend was St. Thomas Aquinas (ca. 1225–1274), a Christian philosopher who was born in Sicily and took tremendous inspiration from the works of Aristotle (the French philosopher Jacques Maritain mentioned in the previous chapter was an important modern Thomist). For the first time, longstanding conservative beliefs came up against scientific rationalism (Fig. 5.3).

Fibonacci's Rabbits

In this relatively open and humanistic political atmosphere, mathematics did not lag far behind. The most outstanding mathematician of the European Middle Ages Fibonacci (ca. 1170–ca. 1250) was born during this time, a bit later than Bhaskara II in India and a bit earlier than Li Ye in China. Fibonacci, known in his time as Leonardo Bonacci or Leonardo of Pisa, was born in Pisa. His father was a merchant and customs official who brought his young son with him to Bugia (now Algeria), where Fibonacci was exposed to Islamic mathematics and learned to use Hindu-Arabic numerals. He subsequently visited Egypt, Syria, Byzantium, and Sicily, acquainting himself with the calculation of both the East and the Middle East. Not

Fig. 5.4 Graphical representation of the Fibonacci numbers

long after he returned to Pisa, he wrote and published his masterpiece the *Liber Abaci* (*The Book of Calculation*). The title of this book suggests some connection with the abacus, but this is misleading: actually, it is in reference to sand table calculations without the use of an abacus. The original 1202 manuscript is not known to exist; rather, the work survives in a copy from 1227, dedicated to one of the scientific advisors of the Holy Roman Emperor Frederick II (1194–1250).

The first section of the *Liber Abaci* introduces the basic arithmetic of numbers, with calculations in sexagesimal; a noteworthy innovation is the introduction also of the horizontal bar demarcating the numerator and denominator of a fraction, which notation is still in use today. The second section consists of word problems related to commerce, including the *Hundred Fowls Problem* from China. This problem, first posed by Zhang Qiujian, seems to have spread to the Arabic world. The third section contains miscellaneous problems and mathematical oddities, including a problem concerning rabbits that has proved significant. This problem asks how many rabbits can be bred in 1 year, starting from a single pair, under the stipulations that each pair of rabbits begins to breed at the age of 2 months and can produce thereafter a new pair of rabbits each month (Fig. 5.4).

Subsequent generations have referred to the sequence of numbers determined by the rabbit problem as the *Fibonacci sequence*:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots .$$

This sequence can also be described by the recurrence relation

$$\begin{cases} F_1 = F_2 = 1 \\ F_n = F_{n-1} + F_{n-2} \text{ (for } n \geq 3) \end{cases},$$

one of the first recurrence relations to appear in mathematics. There is also a remarkable explicit expression for the terms of this sequence involving the irrational number $\sqrt{5}$, that is:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

The Fibonacci sequence has many interesting properties and important applications. For example, as n grows larger and larger approaching infinity ($n \longrightarrow \infty$),

$$\frac{F_{n+1}}{F_n} \longrightarrow \frac{1+\sqrt{5}}{2} \approx 1.618.$$

This number is related to the golden ratio identified as a ratio of line segments by Pythagoras in the early years of the history of mathematics. In addition to tendrils stretching into many areas of mathematics, the Fibonacci sequence has also turned up in applied problems related to the reproduction of bees, the petals of certain flowers, and aesthetics.

Around the year 1220, Fibonacci was summoned by Frederick II, who was visiting Pisa at the time. His scientific advisors posed to Fibonacci a series of mathematical problems, which Fibonacci answered one by one. One of these problems was to find the roots of the cubic equation $x^3+2x^2+10x = 20$. Fibonacci used an approximation method to give the answer in sexagesimal, accurate up to nine digits after the decimal point. Afterward, Fibonacci maintained a long correspondence with the emperor and his court, where mathematics was held in high regard (according to some accounts, he was rather hired by the emperor to serve at the palace and became the first court mathematician in European history). Frederick II seems to have had almost limitless energy and served simultaneously as the King of Sicily, the King of Germany, and later the Holy Roman Emperor and the King of Jerusalem.

Fibonacci devoted his second substantial work, *The Book of Squares* (*Liber Quadratorum*), to Frederick II. In this book, Fibonacci presents the profound proposition that x^2+y^2 and x^2-y^2 cannot both be perfect squares simultaneously. This book is perhaps the first monograph ever devoted to a specific class of problems in number theory and established Fibonacci as the significant number theorist between the times of Diophantus and Fermat. Considering the legacy of Fibonacci, he not only played a pioneering role in the revival of European mathematics but also served as an important bridge in the transfer of mathematics from east to west. Gerolamo Cardano, the finest Italian mathematician of the sixteenth century, remarked: "We can conclude that all the knowledge we have of mathematics outside of Greece is due to the appearance of Fibonacci."

Judging from the surviving likenesses of Fibonacci, he had a charm similar to that of his compatriot the painter Raphael, who lived three centuries later, and he seems to have regarded himself as a kind of wanderer. The name Leonardo of Pisa by which he is also known places him in the company of Leonardo da Vinci, the painter of the *Mona Lisa*. In the year 1963, a group of American mathematicians inspired by the rabbit problem established the Fibonacci Association and began to publish *Fibonacci Quarterly* in the United States, dedicated to mathematical research papers related to the Fibonacci sequence. Since 1984, the Fibonacci Association has also hosted biannually an *International Conference on Fibonacci Numbers and Their Applications* around the world. The development of such a rich universe of research from a simple model of rabbit reproduction is another miraculous legend in the history of mathematics.

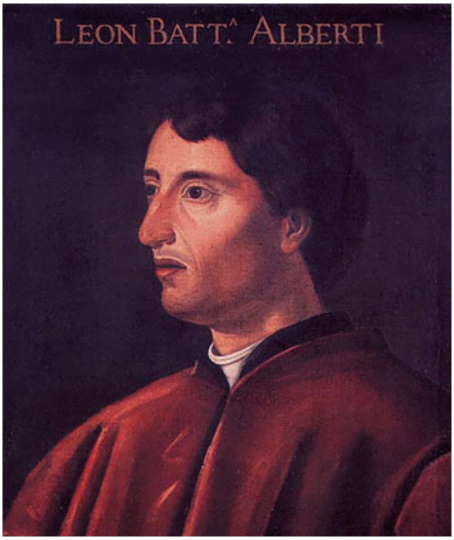


Fig. 5.5 Alberti the humanist

Alberti's Perspective Method

After the collapse of the old feudal structure in Europe, there followed an astonishing sequence of events that taken together signified the birth of a new era governed by a totally new mental outlook: the strengthening of the Italian city-states; the rise of the monarchies in Spain, France, and England; the development of secular education; the discovery of new maritime routes and the New World; the radical proposal of a heliocentric solar system by Nicolaus Copernicus;[1] the invention and application of movable type printing; and so on. This new era recalled and took inspiration from the scholarship, wisdom, and values of the classical world; for this reason, it was called the Renaissance.

The Italian thinkers of the Renaissance period embraced a humanistic ideal with man at the center of the universe and capable of unlimited development. It followed naturally for many such thinkers that is incumbent on humankind to pursue the total acquisition of knowledge and the development of abilities, that is, the refinement of skill and capability not only in every intellectual field but also in physical training, social activity, literature, and art. Such polymathic individuals are often referred to today as Renaissance men in honor of this ideal. The archetypal Renaissance man was Leon Battista Alberti (1404–1472), a humanist, artist, writer, mathematician, and thinker, who also excelled in horsemanship and martial arts (Fig. 5.5).

[1] Copernicus was studying at the University of Kraków, a medium-sized city in Poland that was home to the two Nobel Laureates Czesław Miłosz and Wisława Szymborska at the same time in the early twenty-first century, at the time when Columbus reached the New World.

Alberti was born in Genoa, the illegitimate son of a wealthy Florentine banker, who taught him mathematics in his youth. He took to writing early on, composing Latin comedies, and later obtained a doctorate in law, took holy orders, and served the papal court. Alberti used his knowledge of geometry to determine for the first time in history precise laws for the representation of a three-dimensional scene on a flat wooden block or the surface of a wall. This had an immediate effect on Italian painting and relief-making and facilitated the production of an accurate, rich, and geometrically correct perspective style. Alberti wrote, "a man can do anything if he but wills it," and "I would wish that the painter could be as learned as possible in the liberal arts, but first and foremost I would wish that he know geometry.[2]"

Prior to Alberti, Florence had already produced the great architect Filippo Brunelleschi (1377–1446), responsible for the dome over the most famous cathedral to this day in this art capital. According to one saying, he loved mathematics since childhood and took up painting only in order to engage with geometry; of course, this was not enough to achieve mastery in mathematics, and he turned later to engineering and architecture, but nonetheless he was the first person to study perspective, and Alberti's own interest in perspective came about because of his connection to such predecessors. The basic principles of perspective that Alberti introduced can be described as follows:

> Place upright a glass screen between the eyes of the observer and the scene, and imagine rays of light emitted from one eye to every point of the scene, creating a cross-section as they pass through the screen. This cross-section should present to the eye the same impression as the scene itself, so that the problem of realistic painting is precisely the production of this cross-section of glass on the canvas. Alberti noticed that if two such glass screens are placed between the eyes and the scene at different locations, the results will be different, and similarly if the eyes look through the same screen from two different positions, the cross-section in glass again will be in each case different.

In every case, Alberti asked what is the mathematical relationship between any two parallel scenes; this question is the starting point of projective geometry (Fig. 5.6).

Alberti also discovered that in a realistic representation of the scene, parallel lines (except those parallel to the glass screen or to the plane of the image) must intersect in a certain point. This point is called the vanishing point, and its discovery was a turning point in the history of painting. In the past, it was rare to paint so accurately, but subsequent generations of painters mostly followed this principle, although of course the vanishing point itself need not appear in the painting. The origin or rather cause of the existence of the vanishing point is as follows: any two parallel lines in the real scene form two intersecting planes containing the observation point, and the point where the line of intersection of these two planes meets the glass screen forms the vanishing point. It is precisely because of his work with perspective and the vanishing point that Alberti became the most important art theorist of the Renaissance (Fig.,5.7).

[2] *On Painting*, Tr. Rocco Sinisgalli

Fig. 5.6 Dome of the Florence Cathedral, designed by Brunelleschi

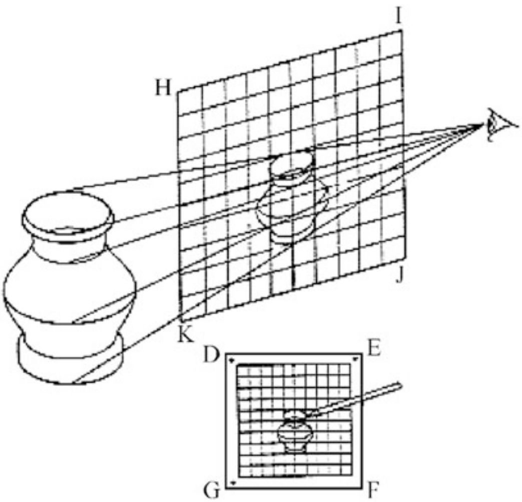


Fig. 5.7 Alberti's perspective method

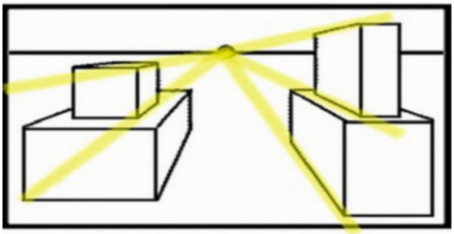


Fig. 5.8 Alberti's vanishing point

Throughout his entire body of work, Alberti always maintained the outlook of civic humanism that flourished in Florence at the time. For example, he wrote the first Italian grammar, arguing that Italian was as regular as Latin and equally suited to literary composition; he also wrote a pioneering treatise in cryptography, in which appear the first polyalphabetic cipher and the first table of letter frequencies. His final work, written several years before his death, was a dialogue entitled *De iciarchia* (*On Ruling the Household*), in which he praises human accomplishment and public service as virtues, fully in line with the spirit of humanism in pursuit of public welfare. According to the biographer Giorgio Vasari (1511–1574), Alberti died quietly and contented (Fig. 5.8).

Da Vinci and Dürer

When Alberti was approaching 50 years of age, the most glorious figure of the Renaissance period was born in a village called Vinci on the outskirts of Florence: Leonardo da Vinci (1452–1519). He was born out of wedlock to a peasant woman who later married a craftsman and a successful Florentine notary and landlord, who also married shortly afterward. His first of several wives was unable to bear children, however, and Leonardo's father took custody of him early on and provided for his elementary education in reading, writing, and arithmetic. He became a studio boy in his adolescence and took up painting as an apprentice and after the age of 30 turned his attention to advanced geometry and arithmetic. His two famous works *The Last Supper* and the *Mona Lisa* were painting in his middle age and old age, respectively (Fig. 5.9).

The artistic achievements of Leonardo da Vinci are common knowledge and need no further introduction here; his name even inspired a suspense novel in the twenty-first century that became a worldwide bestseller. He believed deeply that the foundation of painting was the accurate reproduction of the original impression, which can be achieved only through rigorous adherence to the mathematics of perspective, which he referred to as the steering wheel and guiding principle of painting. It was probably in response to this attitude that the twentieth-century French avant-garde painter Marcel Duchamp made his *L.H.O.O.Q.*, consisting of a cheap postcard reproduction of the *Mona Lisa* on top of which the artist drew a

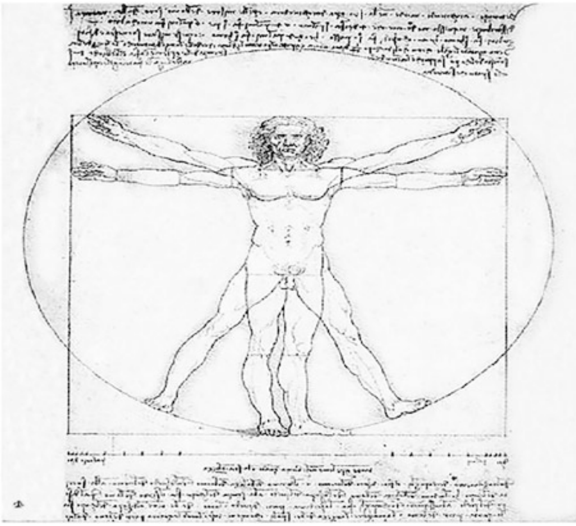


Fig. 5.9 Leonardo da Vinci's famous Vitruvian Man, drawn around 1487

Fig. 5.10 Statue of Leonardo da Vinci in Amboise, France

moustache and beard in pencil. In geometry, Leonardo's main achievement was the determination of the position of the center of gravity of the tetrahedron, given as a quarter of the distance along the line to the opposite vertex from the center of gravity of the base triangle. On the other hand, he made an error in his similar determination of the center of gravity of the isosceles trapezoid, providing two methods, only one of which was correct (Fig. 5.10).

Leonardo also achieved outstanding results beyond the scope of art and mathematics. His observations of the celestial bodies led him to secretly record in his notebooks that "il sole no si muove" or "the sun does not move" earlier than Copernicus, in contradiction with the doctrine of the Bible that God created the sun and the moon and made them to travel around the earth. The flight of birds inspired him to investigate air resistance and sketch the first designs for a flying

Fig. 5.11 *Self Portrait*, Albrecht Dürer (1498)

machine. Some dynamicists believe today that if Leonardo had access to a light fuel source at the time, he could have made it to the heavens. He also personally dissected more than 30 corpses in his research in the mysteries of human anatomy and life. All of these various researches were abandoned midway, but contributed to the development of his observational powers and accuracy in painting (Fig. 5.11).

Also in the fifteenth century, another versatile artist and Renaissance figure appeared in Nuremburg in Bavaria, Germany, in Northern Europe. This was Albrecht Dürer (1471–1528), born 1 year before the death of Alberti, whose humanistic ideals lent to his art its characteristic air of knowledge and rationality. Dürer spent about 20 years of his life travelling and living in Holland, Switzerland, Italy, and other places. He also maintained some connection to his fellow religious reformer Martin Luther (1483–1546), a few years his junior, and the various figures surrounding him. He produced creative work in a very broad range of fields, including oil painting, printmaking, woodcutting, illustration, and so on, and it is obvious from his work that Dürer was well versed in the perspective method introduced by Alberti (Fig. 5.12).

Among all Renaissance artists, Dürer is generally considered to be the one with the greatest knowledge of mathematics. His *Four Books on Measurement* (or *Instructions for Measuring with Compass and Ruler*) deals mainly with geometry and touches also on linear perspective. Among its innovations is a treatment of the projections on to the plane of curves in space and the introduction of the epicycloid, the curve traced by the trajectory of a fixed point on the circumference of a rolling circle. Even more impressive, Dürer considered the orthogonal projections of curves or figures onto two or three mutually perpendicular planes, a very advanced topic that was not developed further until the eighteenth century, when the French

Fig. 5.12 *Melencolia I*, Albrecht Dürer (1514)

mathematician Gaspard Monge created the field of descriptive geometry, earning a place for his own name in the history of mathematics.

In his large 1514 engraving *Melencolia I*, Dürer depicts in the foreground a winged young woman sitting in contemplative manner with her head resting in her left hand. In the background, there is a fourth-order magic square:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

.

In this magic square, every row, column, and diagonal sums to 34, as do the terms of the five second-order submatrices in the four corners and the center, and even four of the third-order submatrices and the unique fourth-order submatrix, as well as other such arrays.

Comparing this to the example cited in the works of Yang Hui of the Southern Song dynasty in China, the only difference is in the order of the rows. The presence of this magic square undoubtedly contributes to the enigmatic melancholia of the engraving; it is interesting also to note that the middle two numbers of the final row serve to date the picture: 1514. This year saw the death of his mother, and he may have made this engraving as an expression of his grief. The magic square contained within it, however, is not altogether perfect. There is one inscribed at the entrance to Parshvanatha temple in Khajuraho, India, in the twelfth century which is more satisfactory in some respects although less so in others. In particular, the terms of all nine of its second-order submatrices also sum to 34.

Speaking broadly with respect to painting, it is generally the case that colors are more expressive of emotion, while line is more expressive of reason. In line with the German reputation for rational thought, German painters have proved excellent in their use of line. Certainly this holds for Dürer. His precise line drawings directly reveal the subtlety of his observation and the complexity of his conception, and the combination of his cerebral approach with his ardent ideals produces a unique effect. In addition to the visual arts and mathematics, Dürer worked in art theory and scientific writing, including works on draftsmanship, human proportions, and architectural engineering, featuring his own illustrations.

The Invention of Calculus

The Awakening of New Mathematics

Although the artists of the Renaissance offered novel insights into mathematics, the revival of mathematics, and indeed the rise of modern mathematics, did not take place until the sixteenth century. The first new advances in mathematics began with algebra: for example, trigonometry had been separated from astronomy, the study of perspective gave rise to projective geometry, and the invention of logarithms facilitated easier computation, but the main breakthrough was in the solution of cubic and quartic equations and the development of symbolic algebra. After the *Algebra* of al-Khwarizmi was translated into Latin, it was widely circulated and used as a textbook throughout Europe. At this time, people considered the solution of cubic and quartic equations to be a problem as difficult as the three unsolved geometric problems of Ancient Greece. But at the turn of the century, two Italian mathematicians were born who managed to settle this issue completely: Tartaglia and Cardano (Fig. 5.13).

Tartaglia (1499–1557), whose name at birth was Niccolò Fontana, was born in Brescia, not far from Milan, to a dispatch rider father who was murdered several years later. In a further misfortune, Tartaglia's jaw and palate were sliced with a saber by an invading French soldier in 1512, leaving him with a speech impediment that earned him the nickname by which he is remembered today (*tartaglia* means *stammerer*). As an adult, Tartaglia made his living teaching mathematics; he made a name for himself with his claim that he could solve any cubic equation lacking either the linear or quadratic term, that is, equations of the form $x^3+mx^2 = n$ or $x^3+mx =$

Fig. 5.13 Gerolamo Cardano, prominent physician, lawyer, and politician

n with $m, n > 0$. A professor at the University of Bologna doubted this claim and sent a student to challenge Tartaglia, which challenge Tartaglia readily met, since his opponent could handle only those cubic equations lacking the quadratic term.

In 1539, a mathematics enthusiast and medical practitioner in Milan named Gerolamo Cardano (1501–1576) invited Tartaglia to stay at his home as a guest for 3 days to discuss mathematics. After sharing together a full and satisfying meal, Cardano cajoled Tartaglia into revealing his solution to the cubic equation with the promise never to publish it. Tartaglia presented his solution encoded in a poem of 25 lines. Several years later, Cardano encountered the same solution in an unpublished work and determined that his promise was no longer binding. Tartaglia was shocked to see his solution published by Cardano in his book *Ars Magna* to quite some fanfare, and a bitter enmity developed between the two mathematicians.

Tartaglia's solution, which we present here in modernized exposition, was as follows. In light of the identity

$$a^3 - b^3 = (a - b)^3 + 3ab(a - b),$$

choose appropriate a and b such that

$$\begin{cases} 3ab = m, \\ a^3 - b^3 = n \end{cases}.$$

Then $a-b$ is a solution of the equation $x^3+mx=n$, and it is not difficult to further solve for a and b to obtain them as

$$\sqrt[3]{\pm\frac{n}{2}+\sqrt{\left(\frac{n}{2}\right)^2+\left(\frac{m}{3}\right)^3}}.$$

This solution is what is known as Cardano's equation, although Cardano was careful to give credit for it to Tartaglia. Cardano also considered the case $m<0$ and gave in this case also the complete solution. As for cubic equations lacking a linear term, they can always be transformed into equations of this type by a change of variables.

Even more impressively, the *Ars Magna* also gave a general solution for the quartic equation, which was also not due to Cardano, but rather to his former servant and eventual student Lodovico Ferrari (1522–1565). Ferrari had begun his career at the age of 15 as a house servant to Cardano, known at that time mainly as a doctor. Cardano quickly recognized his intelligence and began to teach him mathematics. And indeed, Ferrari quickly discovered a way to convert quartic equations into cubic ones and became as a result the first mathematician to successfully solve the quartic equation. He also represented Cardano in a second mathematical challenge against Tartaglia, this time in Milan; on this occasion, Tartaglia did not emerge victorious (Fig. 5.14).

After making a name for himself while still in his teens, Ferrari quickly obtained a prestigious teaching post in Rome, from which he retired at the age of 42 to move back to his hometown and serve as a professor of mathematics at the University of Bologna. Unfortunately, he died not long after at the young age of 43 of arsenic poisoning, given to him according to legend by a widowed and greedy sister. The question of polynomials of degree five and higher was not resolved until the Norwegian mathematician Niels Henrik Abel proved their insolvability by radicals in the nineteenth century; from this, it is evident that the achievements and stories associated with these Italian mathematicians circulated among their mathematical colleagues and successors for a long stretch of time.

From the discussion above, we can conclude that although Tartaglia and Ferrari were more adept at discovering clever solutions to specific problems, Cardano played a more important and unifying role in this story. In this respect, he was a kind of Euclidean character for this period in the history of mathematics. Another such character emerged in France in the sixteenth century: François Viète (1540–1603), more commonly known by the Latinized form of his name, Franciscus Vieta. Vieta is credited with the creation of the first symbolic algebra, with which he was able to make substantial contributions to the theory of equations. Middle school mathematics textbooks today include a special case of Vieta's formula, which relates the two roots x_1 and x_2 of the quadratic polynomial ax^2+bx+c to its coefficients:

$$x_1+x_2=-\frac{b}{a}, \text{ and } x_1x_2=\frac{c}{a}.$$

Fig. 5.14 François Viète, lawyer and politician

Vieta was a lawyer and politician by profession. During the wars between France and Spain, Vieta used his mathematical talent to uncover the key to a Spanish cipher. It was during a period of political frustration that he devoted himself to mathematical research and he developed the ideas of using letters to represent algebraic terms while reading the writings of Diophantus. Although most of the symbols he himself used have since been replaced, Vieta is still recognized as the father of symbolic algebra. In particular, Vieta used consonants to represent known constants, and vowels to represent unknown terms, with the symbol $\sim$ indicating a negative quantity. Considering more generally the standard notation in modern mathematics texts, the addition and subtraction signs $+$ and $-$ and notation for powers were introduced in the fifteenth century, the equals and greater-than signs $=$ and $>$ were introduced in the sixteenth century, and the less-than sign $<$, the radical $\sqrt{\ }$, the multiplication and division signs $\times$ and $\div$, the use of the bottom of the alphabet (a, b, c, …) for knowns and the top of the alphabet (…, x, y, z) for unknowns, and exponential notation were all introduced in the seventeenth century (Fig. 5.15).

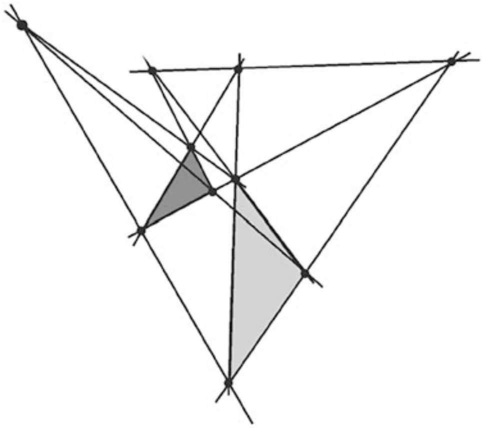


Fig. 5.15 Desargues's theorem

Analytic Geometry

After the advent of the seventeenth century, various mathematical theories and branch began to spring up like bamboo shoots after a rain. It is impossible for us to analyze here these developments in comprehensive detail, and inevitably, we will have to leave out even some of the more important mathematicians. We press on all the same and turn first to the French mathematician Girard Desargues (1591–1661). It was Desargues who answered the outstanding questions about perspective left over from Alberti and established the discipline of projective geometry. Indeed, Desargues is considered as the founder of this branch of mathematics. Desargues was originally a soldier and later earned his living as an engineer and architect. He was a regular participant in the mathematical salons organized by the priest and polymath Marin Mersenne, where he won the respect of such young mathematicians as Descartes, Pascal, and others (Fig. 5.16).

One of the fundamental contributions that Desargues made to projective geometry is the concept of the point at infinity, unifying the classes of parallel lines and intersecting lines in the plane by allowing parallel lines to intersect in the point at infinity; this point of view would later prove very fruitful for the development of non-Euclidean geometry. It follows that in projective geometry, every pair of lines lying in the same plane eventually intersect, which is the starting point on which the theory is built. An additional innovation is that Desargues concerned himself only with the interrelationships between geometric figures without any reference to measurement, also a novel and forward-looking idea in geometry. Finally, there is also Desargues' theorem, which states that if the lines formed by three corresponding pairs of vertices of two triangles all intersect, respectively, then the three sets of intersection points so obtained are each individually colinear. From the point of view of the painter, this theorem can be stated as follows: if two triangles can be seen in perspective from a single external point (which turns out to be just at two different sections of the cone), the points of intersection of the extended corresponding edges are colinear, provided none of the edge pairs are parallel.

Fig. 5.16 Fashion show with designs based on Desargues's theorem

More generally, geometric research in the seventeenth century broke out along two main strands. The path taken by Desargues can be described as a continuation of the tradition of synthetic geometry, but under conditions of a broader generality. The second path proved ultimately to be the more brilliant and influential; this was to introduce the use of algebraic tools to the study of geometry, specifically, the discipline of analytic geometry established by Descartes.

At its essence, the contrast between modern mathematics and ancient mathematics is that modern mathematics is concerned with variables, whereas ancient mathematics was concerned with constants. The development of capitalist production following upon the Renaissance created new demands on science and technology: the widespread use of machinery, for example, necessitated the study of mechanical motion; the development of a maritime industry driven by trade created a demand for more accurate and convenient methods for the determination of the positions of ships, leading people to study the laws of motion governing the celestial bodies; and the improvement of weapons technology stimulated research into problems of ballistics. All of these various topics and questions indicate that the study of movement and change had become the central research topic in the natural sciences and mathematics (Fig. 5.17).

The first milestone in the mathematics of variables was the invention of analytic geometry. The basic idea of analytic geometry is to introduce coordinates to the

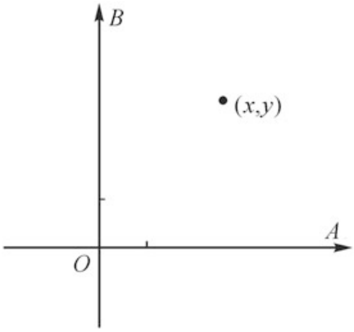


Fig. 5.17 Cartesian coordinates

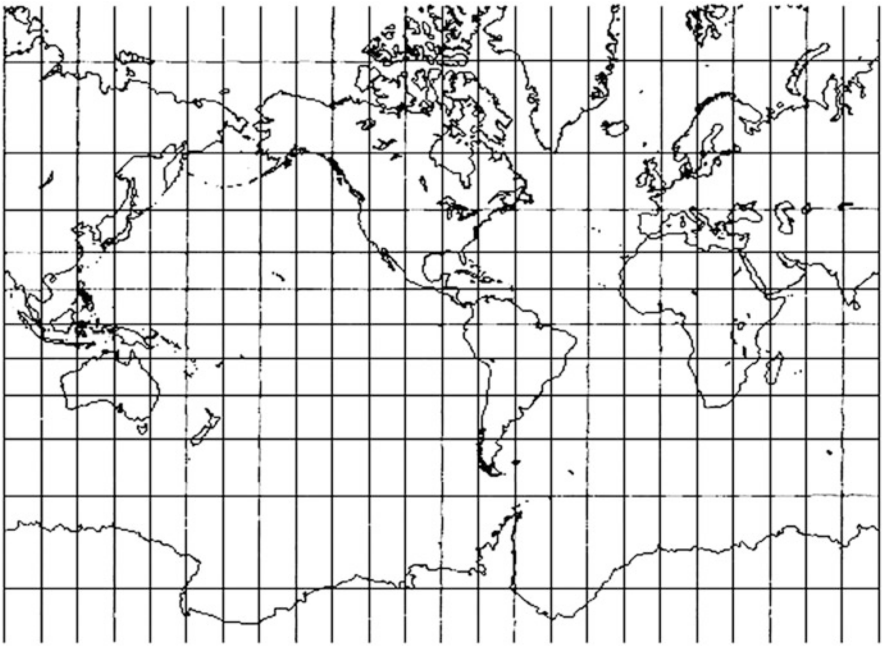


Fig. 5.18 Map of the world by Mercator

plane; for this reason, analytic geometry is known also by the name coordinate geometry. The coordinates are determined by a coordinate system as follows: fix any two intersection straight lines A and B in the plane and designate their point of intersection O as the *origin* of the system. The two lines A and B are referred to as the coordinate axes, and the coordinate system is established by fixing unit coordinates along the two axes. In this way, every ordered pair (x, y) of real numbers corresponds to a unique point in the coordinate plane, and vice versa (Fig. 5.18).

With the tools of analytic geometry in hand, it becomes possible to associate the solution set of any algebraic equation of the form

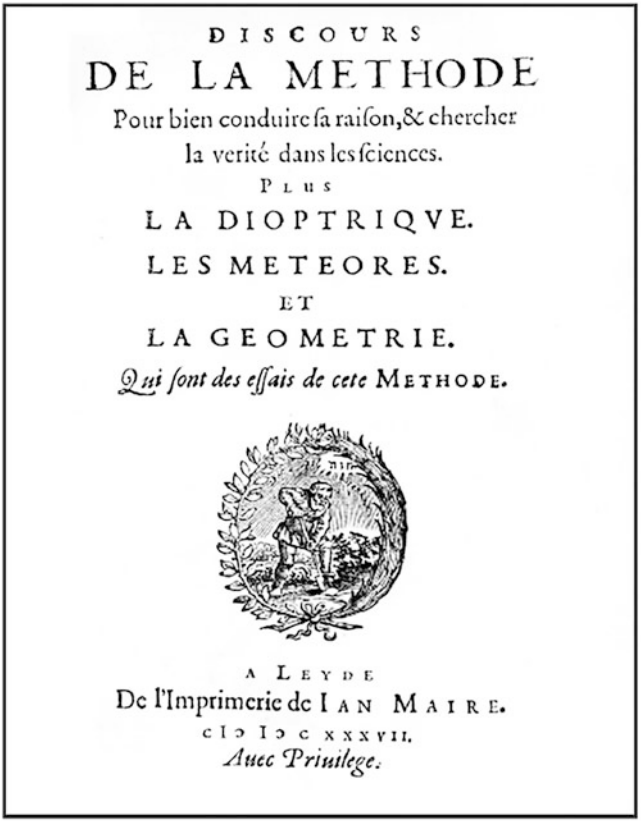
$$f(x, y) = 0$$

with a curve in the plane. This amounts to a reduction of problems in geometry to algebraic problems, so that new geometric results can be obtained through the study of problems in algebra. In the other direction, this association produces a natural geometric interpretation of algebraic problems.

There were several precursors to this innovation throughout history: the fourteenth-century French mathematician Nicole Oresme (ca. 1320–1382) borrowed from geography the terms *longitude* and *latitude* to describe his geometric figures (he was also the inventor of the + symbol in mathematics), and in the sixteenth century, the Flemish geographer Gerardus Mercator (1512–1594) used orthogonal longitudinal and latitudinal lines to draw the first atlas in history. He was also the first to use the term *atlas*. He was deeply versed in the mathematics and physics of his time and applied his knowledge freely in his work; he was in addition an excellent engraver and calligrapher. But in any case, neither of these two forerunners took the further step of establishing a direct association between numbers and geometric figures. Rather the credit for the invention of analytic geometric belongs properly to two later French mathematics, Descartes and Fermat.

It is necessary to point out that Descartes and Fermat alike both took as their starting point the general consideration of oblique coordinate systems, with the system of rectangular coordinates with axes A and B perpendicular to one another, say as horizontal and vertical, considered only as a kind of special case. They also both discussed the further possibility of a coordinate system in three dimensions. It has since become customary to refer to the coordinate system as Cartesian coordinates, or to the plane equipped with a coordinate system as the Cartesian plane, although this should not be taken to mean that Descartes achieved earlier or more brilliant results in this domain than Fermat. The main difference between the two is that Descartes considered his invention to mark a sharp break from Greek tradition and in particular emphasized the power of algebraic methods, while Fermat regarded his work as a straightforward restoration of the mathematics of Apollonius. But Fermat was also decidedly more explicit in his emphasis on the use of equations to define trajectories and curves. He gives directly the modern forms for the equations of many curves, including straight lines, circles, ellipses, parabolas, and hyperbolas (Fig. 5.19).

Although Descartes and Fermat arrived at analytic geometry by different routes and for different purposes, nevertheless, they became caught up in a priority dispute. Descartes had published his results in analytic geometry, in 1637, under the title *Geometry* (*La Géométrie*), an appendix to his *Discourse on Method* (*Discours de la méthode*), a broad philosophical treatise. Fermat never published his work, but he had discovered the basic principles of coordinate geometry as early as 1629. This was published only after his death in 1665, along with many other of his mathematical discoveries. Perhaps because they were both French, this dispute


DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher
la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
CIↃ IↃ C XXXVII.
Auec Priuilege.

Fig. 5.19 Title page of *Discourse on Method* by Descartes

never bubbled over into troubling proportions, but each had their supporters: Pascal took up with Fermat, and Desargues with Descartes (Fig. 5.20).

This was not the only coordinate system invented in this period. In 1671, 2 years after the publication of Fermat's work in coordinate geometry, Isaac Newton in Britain invented his own system of coordinates, known today as polar coordinates. In modern terminology, polar coordinates are determined by a fixed point O in the plane and a half-line A extended in any direction from O. Then any point B in the plane is determined by the distance r between the points O and B, and the angle θ formed by the intersection of lines OA and OB. The elements of the ordered pair (r, θ) are called the polar coordinates of the point B. As everybody has learned in middle school, some geometric figures lend themselves to simpler expression in polar coordinates than in Cartesian coordinates, for example, Archimedean spirals, catenary curves, cardioids, three- and four-leafed rose curves, and so on.

Fig. 5.20 Descartes

The Pioneers of Calculus

The invention of analytic geometry not only enabled the application of algebraic methods to geometric problems but also introduced variables into mathematics, paving the way for the creation of calculus; but an even more crucial development was the establishment of the concept of a function.

In the year 1642, 5 years after the publication of Descartes' *Geometry*, Isaac Newton (1642–1727) was born in a hamlet in the country of Lincolnshire in England. It was also the same year in which Galileo died. Newton was a posthumous child, born 3 months after the death of his father, and did exhibit in childhood the signs of a prodigy. He did however develop a love for extracurricular reading and in middle school picked up the habit of compiling notebooks, which he referred to as his *waste book*. This habit, which some generations later was also practiced by Gauss, would prove very important, and later he brought his notebooks with him to Cambridge University, where he used them for notes on mathematics and mechanics, including his work on calculus and the theory of gravity.

At around the age of 22, Newton began to include in his notes a record of his work on calculus, in which he always used the word *fluent* to denote a relationship between variables. It was the German mathematician Leibniz who first used the word *function* to designate to any quantity that changes according to the change in position of a point on a curve. The familiar notation $f(x)$ to represent a function in the variable x was only introduced in 1734, by the Swiss mathematician Leonhard

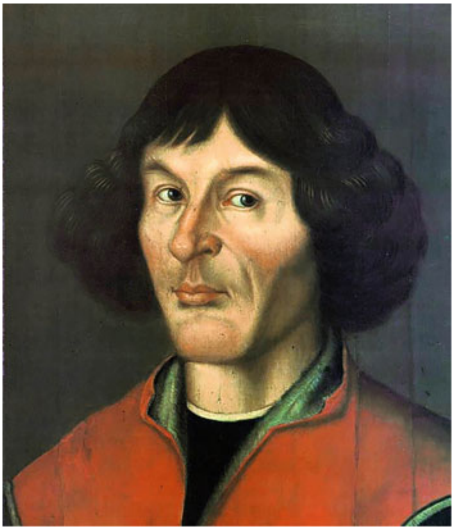


Fig. 5.21 Polish astronomer Copernicus

Euler, when functions had long already been the centerpiece of the conceptual machinery of calculus (Fig. 5.21).

In fact, the basic ideas of calculus, and in particular integral calculus, can be traced back to ancient times. As we have discussed already, the calculation of areas and volumes has been a topic of interest to mathematicians since ancient times, and there appear many examples of the use of infinitesimal arguments to compute the areas, volumes, or arc lengths of various special figures in the mathematical writings of Ancient Greece, China, and India. These include the work of Archimedes in Greece and Zu Chongzhi and his son in China on the calculation of the volume of a sphere. The example of Zeno's paradox also introduces the idea of the infinite division of an ordinary constant. As for differential calculus, Archimedes and Apollonius discussed, respectively, the tangent lines to spirals and conic sections, although only individually or statically. But calculus in its modern form was introduced mainly in order to solve the scientific problems of the seventeenth century (Fig. 5.22).

The first half of the seventeenth century in Europe saw successive major advances in the fields of astronomy and mechanics. First, a Dutch lensmaker invented the telescope in 1608, and when the Italian scientist Galileo Galilei (1546–1642) heard of its invention, he quickly built a powerful telescope of his own and used it to discover many hitherto unknown secrets of the solar system. In particular, his observations confirmed the validity of the heliocentric model of the solar system first proposed in modern times in the fifteenth century by Polish astronomer Nicolaus Copernicus (1473–1543). This remarkable achievement however brought upon Galileo a series of disasters, including interrogation and persecution by the church and leading eventually to blindness and despondency at the end of his life. Simultaneously, the German astronomer Johannes Kepler (1571–1630), 7 years

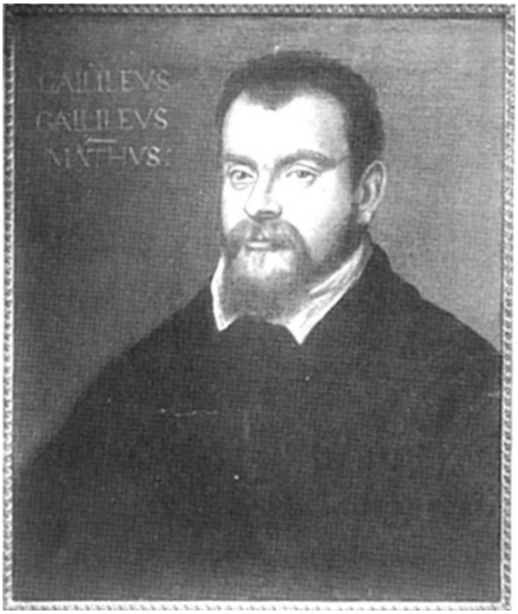


Fig. 5.22 Italian physicist Galileo

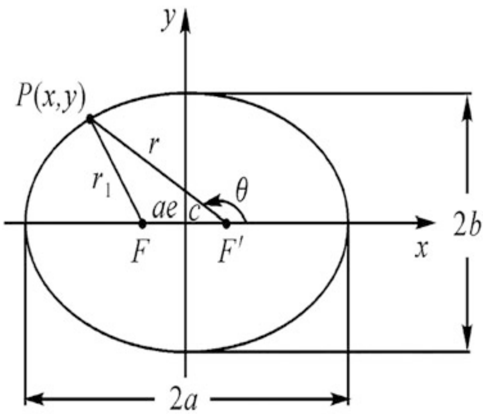


Fig. 5.23 Kepler determined that the orbits of the planets are elliptical

Galileo's junior, was in the process of obtaining a more precise mathematical argument for the heliocentric model on the basis of data collected by his predecessor and employer, the Danish astronomer Tycho Brahe (1546–1601) (Fig. 5.23).

Copernicus and Tycho Brahe however both believed that the orbits of the planets were circular (and Galileo too said nothing against this theory). It was Kepler who first stated as his first law of planetary motion, which states that:

> the orbit of every planet is an ellipse, with the sun at one of its two foci.

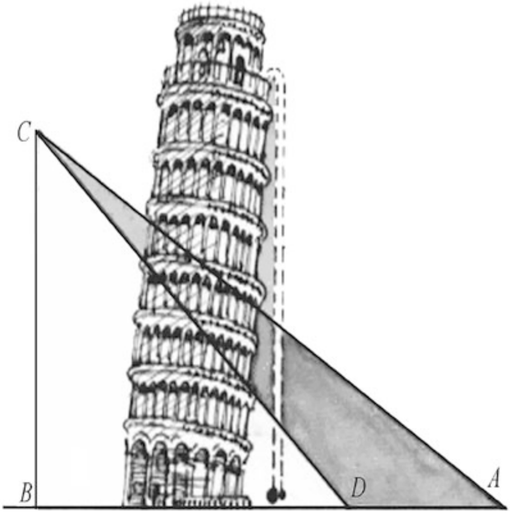


Fig. 5.24 The Leaning Tower of Pisa and associated with it a famous experiment on freely falling bodies

His second and third laws of planetary motion demonstrate even more thoroughly his mathematical ability, which was probably greater even than that of Galileo. The second states that:

> the line segment joining a planet and the sun sweeps out an equal area in equal intervals of time,

and the third that:

> the square of the orbital period of a planet is proportional to the cube of the length of the semi-major axis of its orbit.

This is not to say however that the achievements of Galileo lagged behind those of Kepler. In the first half of his life, that is, in the late sixteenth century, Galileo introduced the law of free fall $s = \frac{1}{2}gt^2$ and the law of inertia or Galilean relativity; he was also a great pioneer in the use of experimental methods in science (Fig. 5.24).

Neither Stuttgart, not far from which Kepler was born, nor Prague, where he later lived, was at the center of European civilization at the time, and his work did not receive as much attention as it deserved. On the other hand, he also avoided the religious persecutions suffered by Galileo. Still, his life was not altogether a happy one: he was a premature and sickly baby and the child of an unhappy marriage, and he himself later suffered through two disastrous marriages and a series of family troubles. He was comforted in his difficulties by his belief in the mathematical harmony of the heavens as revealed to him in mathematics and astronomy, a doctrine showing the distance influences of Pythagoras and Plato, and it was this conviction that set him in pursuit of the laws of planetary motion. In another story from his life, Kepler apparently was at one time deeply dissatisfied with the rough calculation for the volume of a wine barrel employed by the merchant who sold it to him and took it upon himself to discover a method for determining

Fig. 5.25 Statue of Tycho Brahe and Johannes Kepler in Prague

precisely the volume contained in a surface of revolution, a generalization of the spherical volume formula discovered by Archimedes (Fig. 5.25).

Kepler discovered the first two of his three laws of planetary motion in the year 1609, but it took him another 10 years to produce the third. The main obstacle was the complexity of the date left behind by Tycho Brahe and the computational challenges posed by it, in particular the continuous need to multiply together very large numbers. In 1614, the Scottish landowner and mathematician John Napier (1550–1617) invented the logarithm, which simplified the calculations involved in multiplication and division to addition and subtraction. But the practical use of logarithms only became possible 2 years later, when the British mathematician Henry Briggs (1561–1630) paid a visit to Napier in Scotland and encouraged him to reformulate his logarithms in base ten and to compile the first comprehensive logarithm tables. News of this innovation reached Kepler and played a critical role in the development of his third law of planetary motion.

The method that Kepler employed was precisely the method of infinitesimal elements of integral calculus; in modern terminology, this is to take the sum of infinitely many infinitely small elements to determine the area contained in a curve or the volume contained in a surface of revolution. A contemporary Italian mathematician Bonaventura Cavalieri (1598–1647), who was a disciple of Galileo but more committed to pure mathematics, devoted his life to the study of indivisible elements, another precursor to infinitesimal calculus, according to which lines, surfaces, and solids, respectively, are considered as composed of infinitely many surfaces, lines, and planes. Using this method, Cavalieri was able to calculate the definite integral of the power function x^n subject to the constraint that n is a positive integer. The British mathematician John Wallis (1616–1703) considered the more general power function $x^{p/q}$, but only managed to resolve the case $p = 1$. John Wallis was the most direct predecessor of Isaac Newton in terms at least of chronology.

In another direction, tracing backward the roots of differential calculus, we cite also the works of three different predecessors: Descartes, Fermat, and Isaac Barrow (1630–1677), who was Newton's teacher. Descartes and Barrow had endeavored to calculate the tangent line to a generic curve in the plane, using, respectively, an algebraic method known as the circle method or the method of normals and a geometric method making use of the so-called differential triangle. Fermat meanwhile used the nascent methods of differential calculus to determine the extreme values of a function, except for a difference in sign. He realized in fact that it was also possible to obtain tangent lines with this method but mentioned it only in passing in a letter to Mersenne, accompanied by the remark that he would discuss it on another occasion. All things considered, Fermat came closest of the various mathematicians discussed above to success, but it remained to Newton and Leibniz to complete the work.

Newton and Leibniz

As we have seen in the previous section, the seventeenth century brought with it a host of new scientific problems that were closely related to the development of calculus. For example, tangent lines to a curve can be used to determine not only the direction of motion of a moving body at a given moment but also the angle of refraction formed between a ray of light entering a lens and the normal line of the lens; the extreme values of a function can be used to determine the launch angle of a projectile such that it achieves its maximum range and also to determine the closest and furthest distances between a planet and the sun. There was also the basic problem of dynamics: given the distance travelled by a moving body as a position of time, to calculate its velocity and acceleration at any moment. It was above all this uncomplicated problem and its inverse that prompted Newton to the creation of the calculus (Fig. 5.26).

Newton established his formulation of calculus using what he called the method of fluxions, which concept started brewing in his thoughts during his time as a student at Cambridge and burst forth in full maturity during 2 years he spent in his hometown in Lincolnshire during the years of the plague. According to his own accounting, Newton invented the fluxion calculus (differential calculus) in

Fig. 5.26 Newton's apple tree; photograph by the author, Cambridge

November of 1665 and the inverse fluxion calculus (integral calculus) in May of the following year. It follows that Newton, in contrast with all his colleagues who were working toward the calculus earlier, considered and resolved the two topics of differential calculus and integral calculus as inverse operations, as did his contemporary competitor Leibniz. It is interesting to note that Newton indicates in his *Waste Book* that although he had studied under Isaac Barrow at Cambridge, he had been more deeply influenced by the work of John Wallis, who taught at Oxford, and that of Descartes; rather, it was Leibniz who absorbed the teachings of Barrow, during his time in Paris. Barrow himself also proved a the fundamental theorem of calculus in geometric formulation several years later, in his 1670 treatise the *Lectiones Geometricae.*

In the year 1669, upon his return to Cambridge, Newton distributed to his colleagues a mathematical work entitled *De analysi per aequationes numero terminorum infinitas*[3] (*On Analysis by Equations with an Infinite Number of Terms*), having previously made public some similar considerations from a kinematic

[3] At that time, Latin was the universal language of academia, and Newton composed his major works in Latin.

perspective. In this paper, Newton considered a curve y such that the area beneath it is given by the equation

$$z = ax^n$$

where n is an integer or rational number. An infinitesimal increment in x is written as o, and the area enclosed by the x-axis, the y-axis, the curve, and the ordinate at $x + o$ is represented by $z + oy$, where oy is the incremental area:

$$z + oy = a(x + o)^n.$$

Making use of his own generalization of the binomial theorem, the right-hand side of this equation is written as an infinite series; subtracting it from the previous equation, dividing each side of the equation by o, and omitting any terms in which the factor o still occurs give

$$y = nax^{n-1}.$$

In the language of modern mathematics, the rate of change of the area under the curve at any point x is the value of y at x. Conversely, if the curve $y = nax^{n-1}$ is given, then the area underneath it is given by $z = ax^n$. This is the basic prototype of the differential and integral calculus. Two years later, Newton presented a fuller account in a book entitled *Method of Fluxions*; in his terminology, a variable was called a *fluent*, and its rate of change the *fluxion*, from which he derived the name for his method (Fig. 5.27).

In the same period, Newton put his calculus of fluxions and inverse fluxions to work in the calculation of tangent lines, curvatures, inflection points, arc lengths, the force of gravity, centers of mass, and so on. But like Fermat, he was reluctant to publish his results: the first of the two treatises discussed in the previous paragraph was published only inn 1711, after much urging by his colleagues, and the *Method of Fluxions* only in 1736, after his death. In his landmark work *Philosophiæ Naturalis Principia Mathematica*, which was published earlier in 1687, Newton cloaked his calculus in geometric costume, and its significance was not fully recognized straight away. This book nevertheless quickly earned its reputation as the greatest scientific work of modern times, on the basis of the establishment of the law of universal gravitation and a strict mathematical derivation of Kepler's three laws of planetary motion; this was easily enough to ensure its immortality (Fig. 5.28).

In contrast with Newton, Gottfried Wilhelm Leibniz (1646–1716) published his results in calculus earlier, in 1684 and 1686, although he arrived at their invention later; this sparked a protracted and bitter debate over priority. Leibniz also proceeded from a geometric rather than kinematic point of view. Specifically, he took his first inspiration from a paper by Pascal on circles that he read in 1673: consider as in the figure a characteristic small triangle with hypotenuse parallel to the tangent line

Fig. 5.27 Statue of Isaac Newton at Trinity College Chapel; photograph by the author, Cambridge

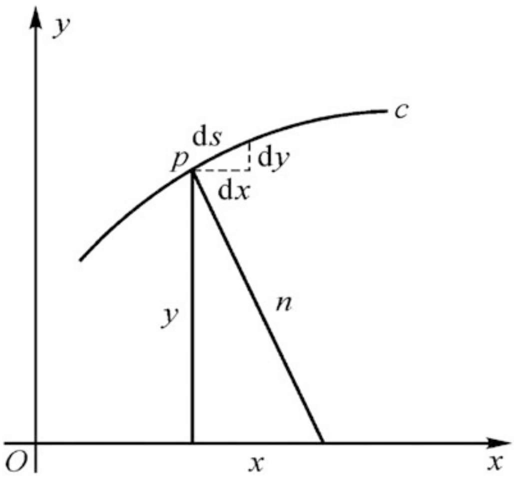


Fig. 5.28 Leibniz formulation of the principles of differential calculus

at any point P on a curve C; then from the proportional relationship between the sidelengths of similar triangles,

$$\frac{ds}{n} = \frac{dx}{y},$$

where n represents the normal line to the curve C at P. Taking the sum,

$$\int yds = \int ndx.$$

This result however was expressed rather vaguely, in words rather than in mathematical notation, and it was 4 years later that Leibniz explicitly stated the fundamental theorem of calculus in a manuscript.

On the other hand, as early as 1666, Leibniz had considered in a published paper entitled *De Arte Combinatoria* (*On the Combinatorial Art*) the first-order and second-order differences of the square sequence

$0, 1, 4, 9, 16, 25, 36, \ldots,$

which are

$1, 3, 5, 7, 9, 11, \ldots$

and

$2, 2, 2, 2, 2, 2, \ldots$

respectively. He noticed that the original sequence is obtained by taking successive sums of the first terms in the sequence of first-order differences, indicating the inverse relationship between summation and difference. It was this that led him to the relationship in calculus between differentiation and integration. In the notation of the Cartesian coordinate system, he wrote the ordinates of an infinite sequence of points on a curve as y and the corresponding abscissas as x. If the ordinates are given in terms of x and the sequence of differences between any consecutive values of y is considered, Leibniz was thrilled to discover that the derivative is simply a kind of difference and the integral a sum.

From this observation, although his progress was not altogether smooth, Leibniz gradually arrived from the notion of the discrete difference to consider the increments of any arbitrary function. In 1675, he introduced the important symbol $\int$ to represent the integral, and in the following year, he obtained the derivative and integral formulas for the power function. As for the fundamental theorem of calculus, it can be stated in modern terminology as follows: in order to find the area under a curve whose ordinate is y, it is only necessary to find a curve with ordinate z such that the slope $\frac{dz}{dx}$ of its tangent line is given by the rule $\frac{dz}{dx} = y$. If the interval

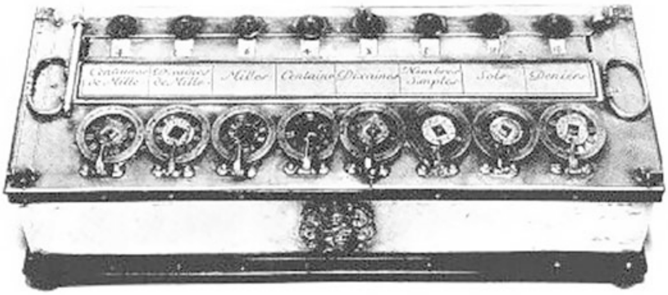


Fig. 5.29 Pascal's calculator

under consideration is $[a, b]$, subtract the area on the interval $[0, a]$ from the area on the interval $[0, b]$ to obtain

$$\int_a^b y dx = z(b) - z(a).$$

This is also known as the Newton-Leibniz formula.

It is interesting and unusual that Leibniz developed his enthusiasm for mathematics initially for reasons of political ambition. At that time, Germany was in a divided state of separate feudal rule, not unlike the situation in the Spring and Autumn and Warring States periods in China more than two millennia prior. During one summer, Leibniz met the former chief minister of the Elector of Mainz.[4] Although he had at that time been dismissed from his post, the erudite former chief minister retained his connection to the Elector of Mainz, to whom he recommended the learned and entertaining young Leibniz for a position as an assistant.

France had become a major power center in Europe by that time, the peak of the rule of the Sun King Louis XIV, and was prone to attack its neighbors to the north at any time. As an assistant to the legal advisor of the Elector, Leibniz proposed a brilliant strategy to distract the French king with the prospect of conquering Egypt. Leibniz was sent at the age of 26 as a diplomatic to Paris where he spent 4 years. Although Descartes, Pascal, and Fermat had already passed away by that time, Leibniz came into contact during his time in Paris with the Dutch mathematician Christiaan Huygens (1629–1695), the inventor of the pendulum clock and the wave theory of light (Fig. 5.29).

Leibniz soon realized the limitations of his mathematical education in technologically regressive Germany, and he applied himself with humility and diligence to his studies under the careful guidance of Huygens. Due to his persistence and talent, and the still incompletely developed mathematical foundations of that era, Leibniz had already made major mathematical discoveries by the time he left Paris,

[4] Historically, the Archbishops of Mainz were the most important of the electors of the Holy Roman Emperor; it was also in Mainz that Gutenberg invented his movable type printing press.

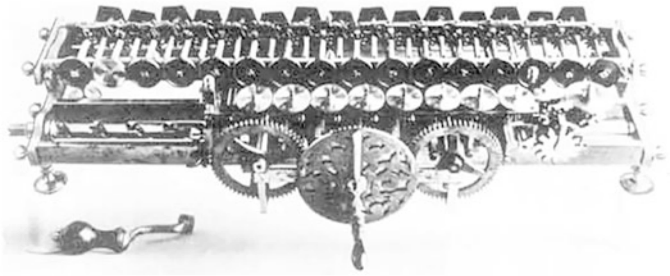


Fig. 5.30 Leibniz's calculating machine, capable of multiplication

although his original plans for intrigue involving France and Egypt had already been shelved. During this time, he first introduced the binary system and subsequently made improvements to Pascal's mechanical calculator, inventing the first calculating machine capable of multiplication, division, and the operation of squaring a number. Of course, his most important contribution was his work with infinitesimals, that is, the invention of the calculus (Fig. 5.30).

This was indeed an epochal contribution to the history of science, and it was precisely because of this innovation that mathematics came to play an outsize role in the natural sciences and social life. It also created a space for thousands of careers in mathematics in subsequent generations, not unlike the role played by the invention of the computer in the twentieth century. In addition, Leibniz also created the elegant theory of determinants and extended the binomial theorem to any number of variables with a beautifully symmetric formulation. Perhaps the most aesthetic of his results for the layman is the infinite series expression for π that he discovered during a visit to London in 1673 at the age of 27:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

In fact, this formula had been independently discovered and forgotten at least twice earlier: by the Scottish mathematician and astronomer James Gregory (1638–1675) and in South India in the fourteenth century by Madhava of Sangamagrama. Madhava derived it from the power series expansion of the arctangent function and made use of his work to calculate π to 13 decimal places. In 1424, 1 year before the death of Madhava, the Persian mathematician Jamshid al-Kashi had used an ancient technique to approximate π to 17 decimal places. Madhava and his followers also obtained power series expansions for the sine and cosine functions, as well as the Taylor series expansions of various functions. Their work was recorded by an astronomer of the Kerala school in the 1530 in a treatise entitled *Yuktibhasa*, meaning *Rationale* in the Malayalam language. This treatise consists of seven chapters, the last of which includes the results just mentioned. It was first published as a book in modern India in 1948, and a critical edition was published with English

Fig. 5.31 Newton's rival, Leibniz

Fig. 5.32 Leibniz's grave; photograph by the author, in Hanover

alongside the original by Springer in 2008. For these reasons, the formula above is sometimes referred to as the Madhava-Gregory-Leibniz formula (Fig. 5.31).

Not long after Leibniz returned from Paris, his patron passed away; his repeated applications to the French Academy of Science as a foreign honorary member were rejected, and he was forced to earn his living as a tutor. In October of 1676, at the age of 30, Leibniz accepted the invitation from the Duke of Brunswick to travel north to Hanover to serve as a legal advisor and librarian. Leibniz continued however to devote himself to mathematics, philosophy, and science with remarkable results and became an honored guest in many of the royal houses of Europe (Fig. 5.32).

I would like to close this section with a discussion of mathematical inheritance, in a sense broader than that of the relationship between mentor and student; rather, I mean something closer to intellectual telepathy or synchronicity. Just as later Euler carefully learned from that mathematical legacy of Fermat, Leibniz developed a particular affinity for the work of Pascal. His original inspiration for the invention of the calculus came from his familiarity with the characteristic triangle invented by Pascal, and his mechanical calculator was also an improvement upon one of the latter's inventions. Pascal is also famous for his work with the binomial coefficients, known familiarly as Pascal's triangle, and Leibniz extended its scope to expansions in any number of variables. In philosophy and the humanities more generally, Leibniz also walked a path paved by the footsteps of Pascal, and the two were alike even in never having married.

Conclusion

Starting from the twelfth century, the Europeans learned from China by way of the Arabian Peninsula the art of papermaking from hemp and cotton as a replacement for parchment and papyrus, and in the middle of the fifteenth century, Johannes Gutenberg (ca. 1400–1468) invented his movable type printing press. In short order, a large number of works on mathematics and astronomy appeared in print. As we have discussed in the previous chapter, the scholarly works of Ancient Greece were translated into Latin by way of the Arabic translations in which they had survived and in this way reappeared in Europe. In 1482, the first Latin edition of Euclid's *Elements* was published in Venice. During this time, the compass and gunpowder were also introduced to Europe from China, the former facilitating voyages across the seas and the latter changing the nature of warfare and the structure and design of military fortifications. In particular, the study of ballistics became important.

As Greek texts began to proliferate across Europe, certain concepts associated in the popular imagination with Ancient Greece also experienced a revival, especially in Italy, including an emphasis on the exploration of nature, admiration for and dependence upon reason, enjoyment of the material world, the pursuit of physical and intellectual perfection, desire and freedom of expression, and so forth. Artists were the first to embody these principles through their love of nature and commitment to the Greek doctrine that mathematics is the essence of nature. They learned their mathematics through practice, in particular geometry, and this led to the rise of such Renaissance figures as Alberti and Leonardo da Vinci. Alberti also contributed directly to the birth of projective geometry as a branch of mathematics through his interest in perspective.

The natural sciences were also increasingly dominated throughout this period by deductive reasoning, which led them to become more mathematical in nature and to an increase in the importance of mathematical terminology, methods, and results. The integration of mathematics and the sciences also fostered an acceleration in their development. From Galileo through to Descartes, the prominent thinkers of the age all believed the world is composed of matter in motion and that the purpose of science is to reveal the mathematical laws governing the motion of moving bodies. The finest examples of this movement are the law of universal gravitation and the three laws of motion, all of them due to Newton.

Fig. 5.33 French mathematician Blaise Pascal

The most important invention in mathematics since the appearance of the Euclidean geometry was the calculus, and as such, it emerged from the background of a rich social tapestry. Most directly, the calculus was designed expressly to handle the major scientific problems of the seventeenth century, in physics, astronomy, optics, and military science alike. But also it met the needs for internal development in pure mathematics, posed by such problems as the determination of the tangent lines to given curves. And its path was paved by the advent of analytic geometry, which introduced the notion of a variable into mathematics and allowed for the quantitative representations of change and motion (Fig. 5.33).

The history of great mathematics is also the history of great mathematicians, and the seventeenth century in particular has been dubbed the century of genius by the British philosopher Alfred North Whitehead. It is no exaggeration to say that the seventeenth century played a crucial role in the developmental history of human civilization, and this was in no small part due to the expansion in the scope and depth of mathematics exemplified by the birth of analytic geometry and calculus. It was also during this time that philosophy and mathematics reentwined in the works of such great thinkers as Descartes, Pascal, and Leibniz, after a long period of separation since the decline of Ancient Greece. All in all, a glorious chapter in the book of history (Fig. 5.34).

I have not yet discussed the upbringings of the two French mathematicians René Descartes (1596–1650) and Blaise Pascal (1623–1662). They were both born in the provinces (as was Fermat), lost their mothers in childhood, and were known to be frail as children; moreover, both of them had fathers who provided for them a good education, and they both came to an interest in mathematics spontaneously. At the age of 12, Pascal discovered on his own without any relevant training the theorem in geometry that the three interior angles of a triangle sum to two right angles; it

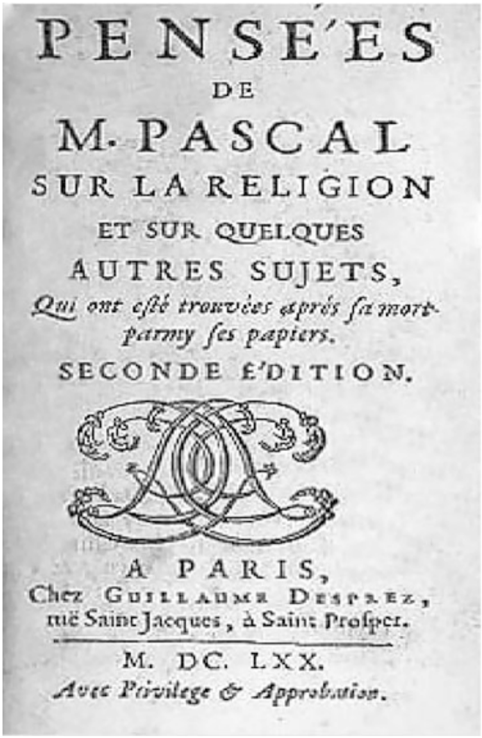


Fig. 5.34 *Pensées* by Pascal

was only later that his father, who was himself an amateur mathematician, began to give Pascal lessons in Euclidean geometry. Descartes for his part developed an interest in mathematics when he saw the solutions to mathematical problems written on the barracks blackboard of the military camp during his time as a soldier in the Netherlands.

Despite their considerable achievements in the discovery of new results in mathematics and science, Descartes and Pascal both resisted the honors associated with their work and preferred to direct their scientific interests toward the spiritual world. Descartes composed the important philosophical texts *Discourse on Method*, *The World*, *Meditations on First Philosophy*, and *Principles of Philosophy*. Pascal left behind his *Lettres provinciales* and *Pensées*. Of the two of them, Descartes was the more committed to abstract metaphysics, even perhaps to the point of indulgence; this was probably in response to the trial and conviction of Galileo, punished as one might suppose for having grounded his doctrine too much in reality. This made for remarkable philosophy, but somewhat less successful science. Pascal, who led a lonely life of deep but terrifying piety, composed aphoristic works of intense feeling and spirituality, marking a fascinating chapter in the history of French literature, indeed of world literature.

In philosophy, Descartes is regarded as the liberator of philosophical thinking from the shackles of the scholastic tradition, and later generations have referred to him as the father of modern philosophy. He is famous for his philosophy of dualism, promoting a stark division between the mind and the body; this has been often encapsulated by reference to his famous dictum *cogito ergo sum*, or *I think, therefore I am*, one of the most powerful and well-known propositions in the history of philosophy. This was in contrast to the philosophy of the Greeks, including Pythagoras, who tended to believe that all the phenomena of the world were composed of a single substance. Pascal was a thinker more grounded in human reality: he understood early and all too well the limitations of human faculties, our frailties and faults. His work in mathematics contributed an awe bordering on terror for the concepts of the infinitely large and infinitely small, and his mathematical discoveries as a result were also confined to a limited space.

It was worth also discussing here a bit the relationship of Pascal's triangle to mathematical induction. We have seen already that various interesting properties of the triangle of binomial coefficients were known to Chinese, Indian, and Persian mathematicians many centuries before the life of Pascal. But it was Pascal who first made use of mathematical induction to give rigorous proofs that (e.g.) the sum of the k-th and $k + 1$-th elements in the n-th row is equal to the $k + 1$-th element of the $n + 1$-th row. In fact, this is perhaps the first explicit and clear formulation and use of mathematical induction in the history of mathematics, although its prototype can be traced back to the proof in Euclid's *Elements* that there exist infinitely many prime numbers. Since that time, mathematical induction has become a basic tool in the arsenal of mathematics, used to prove all manner of propositions about infinite sets of numbers, and in particular the positive integers. It provides an effective mean to prove infinite results from finite hypotheses. The name *mathematical induction* was coined in the nineteenth century by the British mathematician and philosopher Augustus de Morgan.

Descartes and Pascal are both giants in this history of human thought, in both the sciences and the humanities. It is probably in no small part due to their influence that mathematics became such an integral part of the traditional intellectual culture of the French people, perhaps indeed its most excellent aspect. French mathematics has prospered and proliferated since the seventeenth century, with great masters emerging one after another. In typical French fashion, their mathematical geniuses have accepted the honors accrued to them without ever having viewed mathematics as a mere stepping stone. Since the establishment of the Fields Medal in 1936, 11 French mathematicians have been awarded this highest honor, second only in number to the 13 Fields Medalists of the United States.

It was also because of his encounter with French mathematics and the intellectual atmosphere of France that Leibniz turned to mathematics during his stay in Paris and eventually developed as a thinker to such an extent that Bertrand Russell later said of him that "Leibniz was one of the supreme intellects of all time." In addition to having invented the calculus simultaneously to Newton, Leibniz propounded an influential philosophy which he referred to as *monadology*. The central tenet of this philosophy was that the universe is composed of infinitely many *windowless monads*, each resembling the soul to varying degrees. These monads are the ultimate, inextensible, spiritual essence at the foundation of all things. In particular, this implies that humans differ from animals and indeed living things from inanimate objects only as a matter of degree, and indeed, Leibniz pointed out in support of this that many of our thoughts and behaviors occur only at the trigger of subconscious impulses, which doctrine brings closer together than had ever been previously suggested the spheres of human and animal behavior. He derived also from this philosophy his belief that all things are interconnected and in particular that any singular entity is inseparable from its connection with every other entity in existence.

Chapter 6
The Age of Analysis and the French Revolution

Everything our minds can comprehend is interrelated.

Leonhard Euler

The Age of Analysis

The King of the Amateurs

Considering the art of the Renaissance, it is not difficult to arrive at the conclusion that painting as the representative of the spatial arts is intimately tied to geometry, just as in Ancient Greece, the Pythagoreans recognized that algebra or arithmetic is closely related to music, the representative of the temporal arts. It is interesting to note in this light that the great masters of modern music in Europe did not appear until late in the seventeenth century, with the appearance of such figures as Antonio Vivaldi (1678–1741), Johann Sebastian Bach (1685–1750) in Germany, and George Frideric Handel (1685–1759), who was also born in Germany, but spent most of his life in England. They arrived on the scene much later than the master painters and sculptors of the Renaissance. Perhaps this is related to the fact that prior to the invention of calculus, geometry occupied the undisputed place of prominence in mathematics, with Euclidean geometry as its core (Fig. 6.1).

From antiquity, most mathematicians in Europe referred to themselves as geometers; this is exemplified by the most famous epigrams associated with ancient mathematics, Euclid's remark that "there is no royal road to geometry," and the inscription "let no one ignorant of geometry enter here" at the entrance to Plato's Academy. Much later, Pascal refers to geometers in the broad sense in his melancholic aphorism from the *Pensées*, "all geometers would be intuitive, if only they had clear sight, ... and all intuitive minds would become geometers, if only they could direct their sight to the unfamiliar principles of geometry."

The establishment of the Cartesian coordinate system provided a bridge linking the study of geometry to the use of algebraic methods, and the impression of algebra as a subordinate discipline in mathematics also changed. All the same, the primary

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Fig. 6.1 Pierre de Fermat

focus of research in algebra at the time revolved still around problems relation to solving equations, and it would have to wait until the nineteenth century for truly revolutionary changes in algebra to appear, as for that matter in geometry. Rather the first branch of mathematics to experience true breakthroughs was number theory, the most ancient topic in mathematics, concerned with the properties of the natural numbers or the integers and their interrelationships, a topic which could be described as frequently stepping out from the garden of algebra. This was due mainly to the private interest and efforts of an unassuming amateur named Pierre de Fermat (1607–1665), a civilian official in the small town of Toulouse in the south of France (Fig. 6.2).

At his provincial remove far from the capital city Paris, Fermat spent his days occupied in judicial affairs and devoted the evenings and the holidays almost exclusively to his passion for mathematics and its study. Partially, this was on account of opposition in France at the time to private social activity among its councilors, in light of the fact that friends and acquaintances might someday find themselves before the court. This forced isolation from the upper echelons of Toulouse society to which he might otherwise naturally have belonged enabled Fermat to focus on his research hobby; he spent nearly all his nights engrossed in mathematics, and he was drawn especially to problems in number theory. He proposed a wealth of propositions and conjectures, many of which have kept mathematicians busy through the centuries since.

There are not so many complete conclusions associated with Fermat for which he himself provided proofs; among these, the most famous are as follows: every

Fig. 6.2 Andrew Wiles, who proved Fermat's last theorem

odd prime number can be expressed as a difference of two square numbers in one and only one way, and every odd prime number of the form $4n + 1$ denotes the hypotenuse of a right triangle with integer sides in exactly one way when not raised to any power, in two ways when squared, in three ways when cubes, and so on, as, for example,

$$5^2 = 3^2 + 4^2,$$
$$25^2 = 15^2 + 20^2 = 7^2 + 24^2,$$
$$125^2 = 75^2 + 100^2 = 35^2 + 120^2 = 44^2 + 117^2.$$

More often, Fermat would present his results either in his correspondences or by way of a mathematical challenge simply with a statement of the conclusion, without any proof. These include the following: the area of a right triangle with integer side lengths can never be a square number, and every natural number can be written as a sum of four or fewer square numbers. There is a famous generalization of this latter conclusion known as Waring's problem. Research on Waring's problem attracted international attention to the autodidact Chinese mathematician Hua Luogeng (1910–1985), who was partially paralyzed by typhoid fever in his youth and made contributions to mathematics in the fields of analytic number theory, algebra, the theory of functions of several complex variables, numerical analysis, and others.

The two propositions just mentioned were only proved later by the French mathematician Joseph-Louis Lagrange; the Swiss mathematician Leonhard Euler also devoted considerable energy to the resolution of various questions left unresolved by Fermat, and it is for this reason that we have delayed a discussion of Fermat until the beginning of this chapter, since both Lagrange and Euler were mathematicians of the eighteenth century. In fact, throughout his long career, Euler carried out deep and meticulous research into almost problem posed by Fermat. As one example, Fermat famously conjectured that for every nonnegative inter n, the number

$$F_n = 2^{2^n} + 1$$

is prime; such prime numbers are called Fermat primes. Fermat himself verified this conjecture for the cases $0 \leq n \leq 4$. But Euler discovered that F_5 is not prime and even identified a prime factor 641 of F_5. Since that time, no new Fermat primes have been found.

For another example, Fermat had proposed in 1740 in a correspondence with a friend that following divisibility result: if p is a prime number, and a is any integer relatively prime to p (i.e., the greatest common factor of a and p is 1, which means simply that a is not a multiple of p when p is prime), then $a^{p-1} - 1$ is divisible by p. Nearly a century later, Euler not only proved this proposition but also generalized it considerably to the case where p in the proposition is replaced by any positive integer. For this generalization, he introduced what has since come to be called the Euler totient function $\phi(n)$, which counts the number of positive integers not exceeding n that are relatively prime to n. So $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = \phi(6) = 2$ (because, e.g., 1 and 5 are the only two positive integers not exceeding 6 that are relatively prime to it), $\phi(5) = 4$, and so on. Euler's generalization states that for any two relatively prime positive integers n and a, $a^{\phi(n)} - 1$ is divisible by n.

The special case and its generalization just discussed are known as Fermat's little theorem and Euler's theorem, respectively. It is somewhat astonishing that Euler's theorem has sprouted important applications in modern society several centuries later: it plays an important role in the RSA public key cryptosystem developed in 1977 and widely used today for secure data transmission. But in contrast with Fermat's little theorem, Euler could make no dents in the conjecture and eventual theorem that came to be known as Fermat's last theorem, first written down by Fermat in 1637. Fermat's last theorem states that there are no solutions x, y, z in positive integers for the equation

$$x^n + y^n = z^n$$

whenever $n \geq 3$. Of course when $n = 2$, there are infinitely many solutions; these are precisely the Pythagorean triples, which can be easily and completely characterized. Fermat himself proved that there are no solutions when $n = 4$, and Euler resolved the case $n = 3$ (which is more difficult than the case $n = 4$). But a fully general proof remained completely out of reach.

For more than three centuries after it was first written down, this conjecture continued to attract innumerable bright and intelligent mathematicians to make their own contributions to it, until finally it was proved toward the end of the twentieth century by the British mathematician Andrew Wiles (1953-), working at Princeton in the United States. This news made the front page of *The New York Times*, alongside a portrait of Fermat. In fact, what Wiles, with assistance from his student

Richard Taylor (1962-), proved is actually something known at the time as the Taniyama-Shimura conjecture, proposed in 1957 by two Japanese mathematicians and now referred to as the modularity theorem. More precisely, Wiles and Taylor proved a special case of the conjecture as applied to semistable elliptic curves, which was sufficient to prove Fermat's last theorem as a corollary. The modularity theorem elucidates the relationship between elliptic curves and modular forms; the former are geometric objects with profound arithmetic properties, and the latter are highly periodic functions derived from the field of analysis.

In addition to the two Japanese mathematicians just mentioned, many mathematicians have made important contributions to general mathematics along the road toward a proof of Fermat's last theorem. Particularly worthy of mention is the German mathematician Ernst Kummer (1810–1893), who introduced the theory of ideal numbers and thereby established the discipline of algebraic number theory, a development that is probably more important than Fermat's last theorem itself. His extended family also included the composer Felix Mendelssohn and the mathematician Peter Gustav Lejeune Dirichlet.

Finally, there is a famous story concerning the origins of Fermat's last theorem: Fermat wrote his conjecture in the margins to his Latin copy of the book *Arithmetica* by the Ancient Greek mathematician Diophantus. Following it, the mischievous recluse scribbled an additional remark: "I have discovered a truly marvelous proof of this, which this margin is too narrow to contain."

Fermat also carried out important research outside the scope of number theory. In optics, there is Fermat's principle, which states that the path taken by a ray of light between two points is that which can be travelled in the least amount of time, whether a straight line or bent due to refraction. A corollary is that light travels in straight lines through a vacuum. Returning to mathematics, Fermat also discovered the basic principles of analytic geometry independent of Descartes, and his methods for finding the maxima and minima of curves established him as a founder of differential calculus. And in his correspondence with Pascal, the two mathematicians inaugurated probability as mathematical subject. They were interesting in particular in a gambling problem: suppose A and B are two gamblers with a comparable level of skills playing a game in which A needs to earn at least two points in a round to win, while B needs at least three; what is the probability of victory for each?

Fermat analyzed the situation in a table as follows, using the lowercase letters a and b to indicate a point earned by A or B, respectively, and taking into account that every game is completed in at most four rounds:

aaaa	aaab	abba	bbab
baaa	baba	abab	babb
abaa	bbaa	aabb	abbb
aaba	baab	bbba	bbbb

From this, the solution can be read off directly: the probability of victory for A is $\frac{11}{16}$, and for B, it is $\frac{5}{16}$.

It is necessary here also to include some discussion of statistics, which appeared later than probability and consists mainly of the collection of data, the use of probability theory for the construction of models, quantitative analysis, consolidation of results, and ultimately inference and prediction; all of this makes statistics an invaluable tool for research and decision-making, in areas as diverse as physics and the social sciences, the humanities, and business and government. Major applications in particular are in insurance, epidemiology, census making, and public polling. In the modern zoology of disciplines, statistics has separated from mathematics and established itself like computer science as an independent field of research derived from mathematical origins.

We mentioned in the first chapter that statistics had its earliest development in the work of Aristotle, but this did not yet include its maturation as an independent discipline. Modern statistics, like the theory of probability, grew from not altogether reputable origins, the latter from the study of gambling and the former from the analysis of death. In 1666, the Great Fire of London swept through the city destroying such notable buildings as St. Paul's Cathedral and possibly helping to bring an end to the plague years. One of its victims was a local haberdasher named John Graunt (1620–1674), who was bankrupted by the devastation, who had made a study of 130 years worth of death records in London. He used survival rates at ages 6 and 76 to extrapolate the proportion of the population that had lived to other ages and determine their life expectancies. A similar study was carried out in 1693 by the British astronomer Edmond Halley (1656–1742), who conducted a statistical survey of the mortality rate in the German city of Breslau (now known as Wrocław and part of Poland).

We close this section with some further remarks on Fermat's last theorem, which has been likened to a goose that lays golden eggs. When Wiles announced that he had conquered this problem, the mathematical community was at once overjoyed but also concerned that there would be no more such problems that would stimulate so fruitfully the development of number theory. But within a few years, the *abc* conjecture emerged as an important candidate for its replacement. The *abc* conjecture is an inequality relating the two fundamental integer operations of addition and multiplication. We introduce first a bit of notation: if n is a natural number, define its radical $\text{rad}(n)$ as the product of its distinct prime factors. For example, $\text{rad}(12) = 6$, since the distinct prime factors of 12 are 2 and 3. The *abc* conjecture, which was proposed in 1985 independently by the French mathematician Joseph Oesterlé (1954-) and the British mathematician David Masser (1948-), states, in its weaker form, that if a, b, and c are relatively prime integers such that $a + b = c$, then

$$c \leq (\text{rad}(abc))^2 .$$

The resolution of the *abc* conjecture or its weaker version could lead to the solution of a number of important and outstanding problems in number theory. It is also easy to derive directly some well-known theorems and conjectures as corollaries to the *abc* conjecture, including four results that earned Fields Medals for their proofs, one of these being Fermat's last theorem. Taking this as an example, suppose $n \geq 3$ and $x^n + y^n = z^n$. Then with $a = x^n$, $b = y^n$, and $c = z^n$, the weak form of the *abc* conjecture states that

$$z^n \leq (\mathrm{rad}(x^n y^n z^n))^2 < (xyz)^2 < z^6.$$

This limits the possible values of n to $n = 3$, 4, or 5, and these cases can be handled by purely elementary methods.

The Further Development of Calculus

For the Western European powers at the center of the recent scientific developments, the transition from the seventeenth to the eighteenth century was a relatively smooth period, but the northern regions experience some turbulence and change during this time. In the year 1700, Tsar Peter the Great of Russia adopted the Julian calendar, with January 1st as the first day of the new year, and at the same time began the undertaking of various reforms of a military nature. That summer, only a week after the conclusion of a 30-year truce agreement with Turkey, Russia, with Poland and Denmark as allies, launched the Great Northern War against Sweden. Denmark withdrew from the effort not long afterward, however, when King Charles XII of Sweden, who was fond incidentally of mathematics, painting, and architecture, led his troops to Copenhagen. In Germany at this time, the Royal Prussian Academy of Sciences was established in Berlin, with Leibniz as its first president.

The rapid development of calculus shortly after its invention was facilitated precisely by the peace and prosperity of this era. Its applications also spread wide and quickly, resulting in many new branches of mathematics, collected together under the umbrella term *analysis* as an ensemble of distinct concepts and methods. The eighteenth century became known in mathematics as the era of analysis, an important period of transition from ancient to modern mathematics. Intriguingly, just as analysis presented a synthesis of geometry and algebra, there also appeared a new synthesis in the arts between spatial art and temporal art. The characteristic form of synthetic art is theater, and eventually film, which comprises both a spatial component alike to the visual arts such as painting and sculpture and a temporal component, for which the classical analogues are poetry and music. It was after the Renaissance that European theater began its rapid development (Fig. 6.3).

In France, the golden age of drama was the seventeenth century, in which time the great dramaturges Pierre Corneille (1606–1684), Molière (born Jean-Baptiste Poquelin, 1622–1673), and Jean Racine (1639–1699) all lived and worked. Much as English Elizabethan drama, and most notably Shakespeare, was heavily influenced by the Italian Renaissance (see, e.g., *The Merchant of Venice*, *Romeo and Juliet*, *The Tempest*, and so on, all of which were set in the Apennines), modern French drama

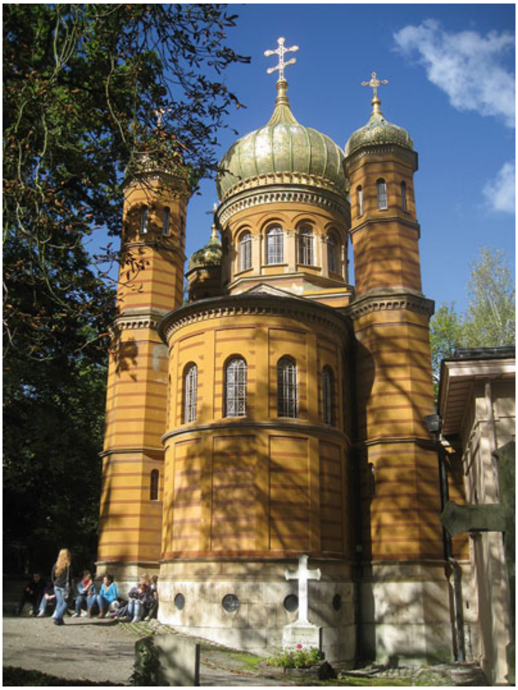


Fig. 6.3 The Russian Orthodox Chapel of Weimar, situated next to the Weimarer Fürstengruft, which houses the coffins of Goethe and Schiller; photograph by the author

drew upon Spanish dramas; for example, the protagonist of *Le Cid* by Corneille was a Spanish national hero. In Germany, drama sprang to life in the eighteenth century, with the emergence of such figures as Gotthold Lessing (1729–1781), Johann Wolfgang von Goethe (1749–1832), and Friedrich Schiller (1759–1805).

Returning to the development of calculus, the mathematicians of the eighteenth century were presented with a full plate of new problems and even disciplines left over in germinal form in the original works of Newton and Leibniz. But before these developments could be seen through to their completion, it was necessary to carry out the perfection and expansion of calculus itself, and the first task at hand was to achieve a full understanding of elementary functions. An example of the issues involved is the logarithmic function, which had originated as a description of the termwise relationship between the geometric and arithmetic series and was later recognized as the integral of the rational function $\frac{1}{1+x}$; at the same time, this function also serves as the inverse of the exponential functional, a particularly simple characterization.

In the period after Newton, the main results in British mathematics were in the study of power series expansions. A particularly important result is due to Brook Taylor (1685–1731), known today as the Taylor series:

$$f(x+h) = f(x) + hf^{(1)}(x) + \frac{h^2}{2!}f^{(2)}(x) + \cdots,$$

which makes it possible to expand any function as a power series and quickly proved to be a powerful tool for the development of calculus, to the extent that the French mathematician Lagrange even later referred to it as the basic principle of differential calculus.

On the other hand, Taylor's proof of this result was by no means rigorous, and he did not even take into account the question of the convergence or divergence of this series. These shortcomings in his work can perhaps be overlooked in light of his additional talents as a painter, which inspired him to compile a comprehensive treatment of perspective in his 1715 essay *Linear Perspective*, in which he introduced for the first time the term *vanishing point*, and provided the first full explanation of the geometry of multipoint perspective. An important special case of the Taylor series is the Maclaurin series, corresponding to the evaluation at $x = 0$, familiar today to any high school student.

Colin Maclaurin (1698–1746) was 13 years younger than Taylor and arrived at his result later, but it is his name that has been attached to it ever since. Partially, this is because Taylor was not well known during his lifetime, but Maclaurin was also a savvy academician, an early promoter of Newton's method of fluxions, who was admitted as a member of the Royal Society at the age of 21. After the deaths of these two mathematicians, British mathematics suffered a long period of decline. One cause of this was a conservative and nationalistic mentality among British mathematicians of the period inspired by the priority dispute over the invention of calculus. They were loath to acknowledge let alone overcome the weaknesses of the fluxion formulation associated with Newton during a time when their continental counterparts were taking full advantage of the symbolic and conceptual clarity of the calculus as developed by Leibniz to achieve fast and fruitful results.

Consider Switzerland, for example. This small, landlocked country in Central Europe was home to several of the most important mathematicians of the eighteenth century. These included Johann Bernoulli, the first to provide a formal definition for the concept of a function and who also introduced various integration techniques such as substitution of variables and integration by parts, and then his student at the University of Basel, Leonhard Euler (1707–1783), arguably the greatest mathematician of the century, who carried out meticulous research into every corner touched by calculus (Fig. 6.4).

Euler proceeded from the loose notion of a function as consisting of an analytical expression of a certain form involving a variable and constants; this was enough to encompass polynomials, power series, exponential and logarithmic functions, trigonometric functions, and even multivariate functions. Euler also separated the algebraic operations involved in the definition of a function into two categories: rational operations, involving only the four basic arithmetic operations, and irrational operations, involving, for example, square roots (Fig. 6.5).

Euler gave the definition of some important functions in terms of limits, for example, the logarithmic function, which he defined for $x > 0$ by

$$\log x = \lim_{n\to\infty} n\left(x^{1/n} - 1\right),$$

Fig. 6.4 The tomb of Euler; photograph by the author, in St. Petersburg

Fig. 6.5 Leonhard Euler

and along with this the exponential function defined as

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n .$$

The symbol e is generally regarded now as a tribute to Euler, although he does not seem to have introduced it for this constant with any special meaning in mind.

The Euler family included many generations of craftsmen, originally based on the shore of Lake Constance at the border between Switzerland and Germany. Toward the end of the seventeenth century, they had made their way down along the Rhine River to Basel, where Euler was born in 1707. He graduated from the University of

Fig. 6.6 Euler's formula, relating five of the most important mathematical constants

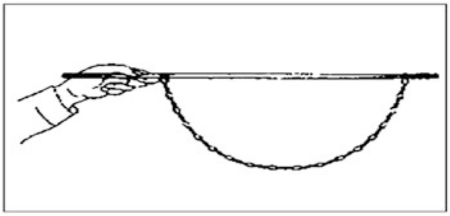


Fig. 6.7 Bernoulli's catenary

Basel at the age of 15 and earned a master's degree a year later. He also came to the attention of the Bernoulli family early on, and they became important mentors and friends. In 1727, he entered the Paris Academy prize competition for the first time; he would go on to win this prize a total of 12 times (Fig. 6.6).

When he was 20, Euler moved to Russia after having failed to earn a physics professorship at his alma mater. He obtained a position with the Imperial Russian Academy of Sciences in Saint Petersburg and succeeded Daniel Bernoulli as a professor of mathematics in 1733. Although he never returned to his home country, Euler retained his Swiss nationality his entire life. He spent 25 years in Berlin at the Prussian Academy of Sciences. The remainder of his life took place in Saint Petersburg. Euler was a remarkably prolific mathematician and father to 13 children, only 5 of whom survived to adulthood. He made seminal contributions to number theory, analysis, geometry, topology, graph theory, and mechanics (Fig. 6.7).

Euler further introduced distinctions between explicit and implicit functions, single-valued and multiple-valued functions, and algebraic and transcendental functions and provided a definition for continuous functions equivalent to the

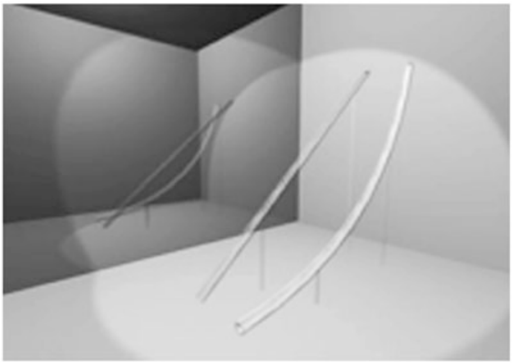


Fig. 6.8 The path of fastest descent between two points is not a straight line

modern notion of an analytic function. He considered the power series expansions of various functions and made the assertion that any function can be expanded in a power series, which is not strictly correct from a modern perspective. His work touched deeply upon physics, astronomy, architecture, and navigation. Euler was a remarkably productive mathematician, who also found time to raise a large family of many children. He famously remarked: "everything our minds can comprehend is interrelated."

The Influence of Calculus

At the same time that the calculus was undergoing internally a continuous development, rigorization, and refinement, and the concept of functions was to become more and more deep, the scope of calculus was also expanding widely and rapidly in its application to other fields, leading to the formation of some new branches of mathematics. One of the most notable developments was that mathematics and mechanics grew more closely related to one another than they had ever been. Most of the Western mathematicians of the period also carried out work in mechanics,[1] much as in ancient times in the east most mathematicians were also astronomers. These emerging disciplines included among them the study of ordinary and partial differential equations, the calculus of variations, differential geometry, and the theory of algebraic equations. Moreover, the influence of the calculus extended beyond simply mathematics and the natural sciences and penetrated even into the humanities and the social sciences (Fig. 6.8).

The theory of ordinary differential equations sprang up directly concomitantly with the growth of calculus. Starting at the end of the seventeenth century, practical problems related to cycloid motion, the theory of elasticity, and celestial mechanics

[1] Later, in the twentieth century, many colleges and universities in China established departments of mathematics and mechanics.

Fig. 6.9 Mathematician and Enlightenment thinker Jean le Rond d'Alembert

produced a series of equations involving differentials, laying down a challenge at the feet of the mathematicians. The most famous of these was the catenary problem, which asks for the equation of the curve formed by an idealized flexible but inelastic cable hanging between two fixed points in a uniform gravitational field. The problem was first posed explicitly as a challenge by Jacob Bernoulli, the brother of Johann Bernoulli, and given its name by Leibniz. Johann Bernoulli derived the equation

$$y = c \cosh \frac{x}{c}$$

for the catenary curve, where c is a constant determined by the weight per unit rope length and cosh is the hyperbolic cosine function.

Subsequently, the theory of ordinary differential equations developed from first-order equations, to higher-order equations with constant coefficients, and then on to higher-order equations with variable coefficients. Finally, this topic was perfected by the two great mathematicians Euler and Lagrange. Euler also established the important distinction between the particular and general solutions of an ordinary differential equation (Fig. 6.9).

Partial differential equations appeared later and were first studied in 1747 by the French mathematician and polymath of the Age of Enlightenment Jean le Rond d'Alembert (1717–1783), who published a paper on the mechanics of string vibrations containing within it the concept of the partial derivative. D'Alembert had been abandoned by his parents as an infant and was later adopted by the wife of a glazier. His name was taken from the patron saint of the church on the steps of which he was found, in keeping with the custom of the time. His knowledge of mathematics was almost entirely self-taught. Later, Euler provided a particular solution involving the sine and cosine functions under the assumption that the initial condition is sinusoidal. Motivated by applications to musical aesthetics and instrument design, Euler and Lagrange both also studied the vibration of tympanic membranes and the wave equation generated by the propagation of sound.

Another important contribution to the development of partial differential equations came from the French mathematician Pierre-Simon Laplace (1749–1827), who introduced the so-called Laplace equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Here, V refers to a potential function, and for this reason, this equation is sometimes also called the potential equation. Potential theory provided a solution to a problem much clamored about in mechanics: the determination of the gravitational force between two objects. If the mass of the objects is negligible in comparison with the distance between them, then the partial derivative of V is the gravitational component between them, determined by Newton's formula for universal gravitation.

In contrast, the genesis of the calculus of variations was more dramatic, and its eventual applications were extremely broad, including both the study of soap bubbles and the theory of relativity, geodesics and minimal surfaces, and isoperimetric problems (the determination of maximal areas enclosed by a curve of a fixed perimeter). The original problem for which this discipline was invented however was a simple one: the identification of the line of fastest descent. This problem is as follows: given two points that do not lie in the same plane horizontally or along the same line vertically, determine the curve between them along which a particle travels in the least time subject only to the action of gravity. After Johann Bernoulli publicly posed this problem in 1696, it attracted the great mathematical minds from around Europe, including Newton, Leibniz, and Johann's brother Jacob Bernoulli. At its core, the problem boiled down to the identification of a pole of a certain special function. Among the various correct solutions that appeared, Newton submitted a solution anonymously, but Johann Bernoulli quickly discerned the identity of its author, famously remarking that he could be recognized "as the lion from its claw."

Through the joint effort of many mathematicians in the establishment of the above various offshoots of calculus, the broad mathematical discipline of *analysis* was born. This became one of the three major areas of modern mathematics, alongside algebra and geometry, and its fruits in this time were the most numerous of the three. Even today, greater weight is placed on mathematical analysis than algebra or geometry as the foundation of mathematical education at the undergraduate level. Calculus also exerted a profound influence on the study of algebra and geometry, starting with the birth of differential geometry. But in the eighteenth century, this was limited to a discussion of geometrical properties in the region near a point or local differential geometry as we would say today; we discuss this in some detail below.

Due to the rise of calculus and its connection with the other natural sciences, this period aroused the enthusiasm of remarkable and careful thinkers and inspired in them confidence in the power of rational thought and the application of mathematical methods to physics and even the normative sciences. There was much faith that this success could be extended to the totality of knowledge. Descartes, for example, believed early on that all problems can be reduced to mathematical problems, all mathematical problems can be reduced to algebraic problems, and all algebraic problems can be reduced to basic equation solving. It could be said that he regarded mathematical reasoning as the only reliable method of thought and sought to reconstruct all knowledge atop these sure foundations.

Leibniz went further even than the ambitious goals outlined by Descartes in his attempt to create a framework for universal logical calculation and universal conceptual language that would render the solution of all human problems trivial. Mathematics was not only the starting point for his program but also its beating heart. Among his other proposals, he suggested that the human mind can be factored into basic and distinct parts, just as the number 24 can be written as a product of its prime factors 2 and 3. Although neither Leibniz nor his successors could ever see this program through to completion, the development of mathematical logic in the second half of the nineteenth century and the twentieth century was based on his idea of a purely formal language, and for this reason, he has sometimes been celebrated as the father of modern logic.

The birth of calculus and turn toward faith in mathematics had an even more direct and obvious influence on religion, which at that time played a central role in both spiritual and secular life. Although Newton attributed to God the power to create the universe, he limited his role in daily life, and Leibniz further depreciated his influence. Although Leibniz too acknowledged his role in creation, he believed that God was constrained to proceed according to established mathematical order. The increased emphasis on reason in this period also contributed to a decrease in devotion to religion, although was not necessarily an outcome intended by mathematicians and scientists of the time. Just as Plato described God as a geometer, Newton believed him to be a capable physicist and mathematician (Fig. 6.10).

In the eighteenth century, the further development of calculus introduced further changes to the spiritual and intellectual landscape. The pioneer and spiritual leader of the French Enlightenment François-Marie Arouet (1694–1778), better known as Voltaire, was a stalwart advocate of Newtonian mathematics and physics and simultaneously a leading proponent of the emerging philosophy of Deism, a theological system in which reason and nature were equated to one another that quickly gained popularity among the intellectuals of the period. In the United States, its adherents included Thomas Jefferson and Benjamin Franklin, the former of whom did much to encourage the instruction of advanced mathematics. In fact, none of the first seven presidents of the United States, including its first president George Washington, identified themselves as Christian. Among the disciples of Deism, nature was God, and Newton's *Principia* is its bible. With philosophy and theology as its accomplices, calculus has exerted a remarkably broad-reaching influence on just about every sphere of human activity, including economics, law, literature, and aesthetics (Fig. 6.11).

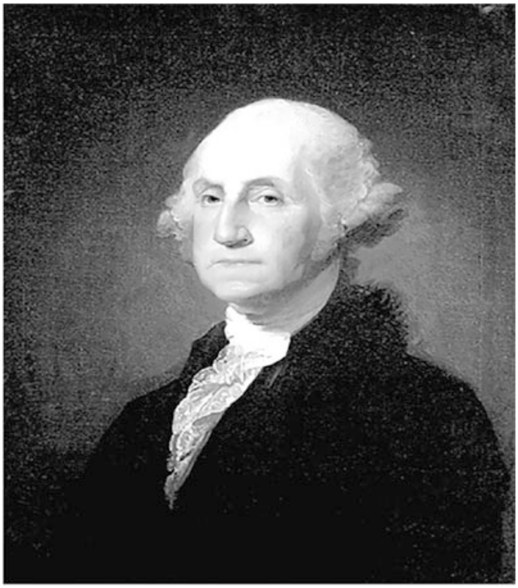


Fig. 6.10 George Washington, during his time as public land surveyor

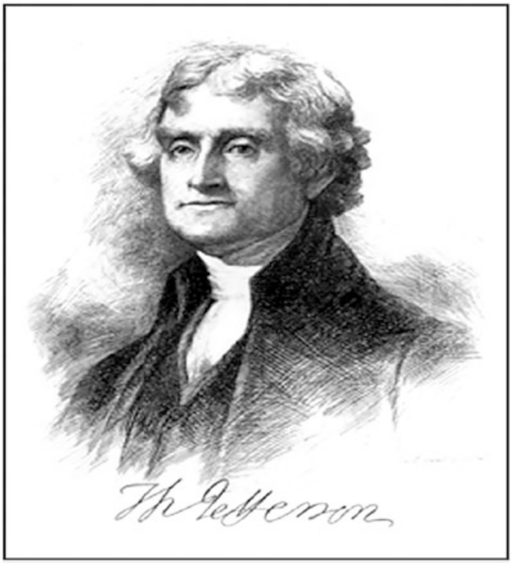


Fig. 6.11 Thomas Jefferson, coauthor of the *Declaration of Independence*

The Bernoulli Family

We have mentioned several times already in the preceding sections the outstanding contributions of the Bernoulli brothers Johann and Jacob, and their Swiss compatriot Euler, in the development and application of the calculus. We discuss now in detail this most famous mathematical family in history, a family seemingly destined for calculus. The Bernoulli family was originally based in Antwerp in Belgium, at that time part of the Spanish Netherlands, where they were practicing Huguenots, a protestant sect which suffered persecution by the Catholic church, like the Calvinists and the Puritans. As a result, Jakob Bernoulli fled his hometown in 1583, taking refuge first in Frankfurt in Germany and eventually settling in Basel in Switzerland, where he married into a prominent local family and established a career as a well-connected merchant of medicinal herbs.

More than a century later, the first of many mathematicians in this family was born; this was Jacob Bernoulli (1654–1705), who mastered the new discipline of calculus as formulated by Leibniz through diligent self-study and later served as a professor of mathematics at the University of Basel. Initially intended for a life as a man of the cloth, Jacob initially studied theology and entered the ministry, but he became obsessed with mathematics over the objections of his father and eventually rejected his church appointments. In 1690, he was the first to introduce the term *integral* into the mathematical lexicon, and in the following year, he studied catenary curves and applied the fruits of his research to bridge design. His other important research areas included the theory of permutations and combinations, the law of large numbers in probability, the Bernoulli numbers derived from the sums of integer powers, and the calculus of variations, discussed already above (Fig. 6.12).

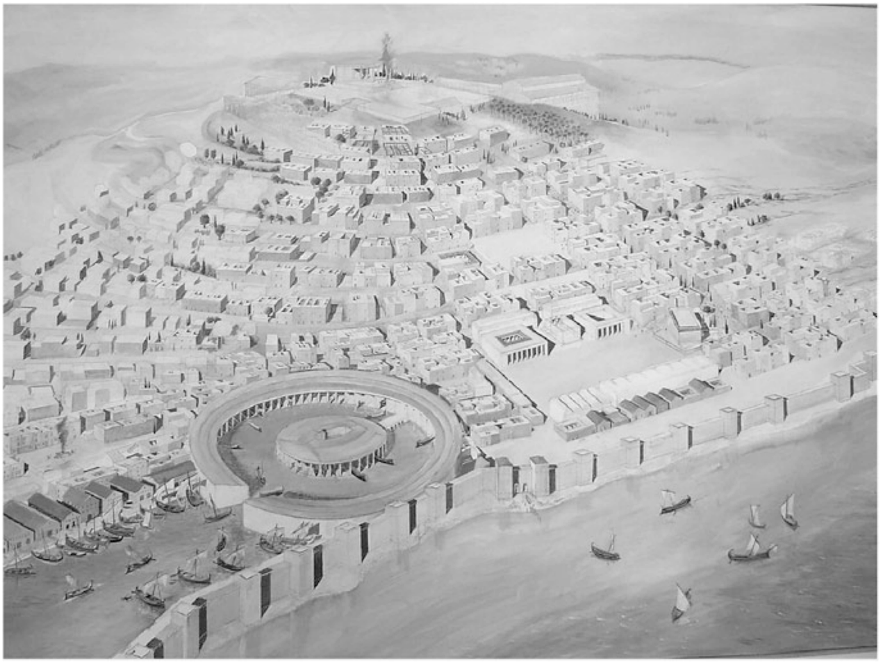


Fig. 6.12 Map of the ancient city of Carthage; photograph by the author, in Tunisia

The basic idea of the calculus of variations makes some interesting and beautiful appearances in ancient literature: according to Greek legend, the founder Queen Dido of Carthage cleverly offered a high price to purchase for the establishment of her new city all the land that could be enclosed in a single leather hide. She then proceeded to cut the hide into a very thin continuous strip, long enough to enclose all the territory she needed for the city. In another version of this story, Dido fled her home city of Tyre upon discovering that her cruel brother Pygmalion had orchestrated the murder of her husband; she made her way to the coast of Africa and purchased a plot of land for the establishment in Carthage, the territory demarcated in such a way as to match the size of a ditch dug in a single day. A very moving and tragic love story involving Dido and Aeneas, the legendary founder of Rome, as well as Dido's sister Anna, appears in the *Aeneid* by the Roman poet Virgil (70 BCE–19 BCE) and in the *Heroides* by Ovid (43 BCE–14 CE).

Returning to Jacob Bernoulli, the Bernoulli numbers B_n named in his honor play an invaluable role in number theory. These numbers can be defined recursively as

$$B_0 = 1,\ B_1 = \frac{1}{2},\ B_n = \sum_{k=0}^{n} \binom{n}{k} B_k\ (n \geq 2),$$

where the numbers $\binom{n}{k}$ are the usual binomial coefficients. From this, it is obvious that every B_n is a rational number, and these numbers exhibit some remarkable properties. For example, it is easy to prove that $B_n = 0$ whenever $n \geq 3$ is an odd number; and for odd prime numbers p, the special case of Fermat's last theorem with exponent given by p can be directly resolved by way of the number B_{p-3}. The Bernoulli polynomials, which also play an important role in number theory, as well as in the theory of functions, are also defined in terms of the Bernoulli numbers. Upon his death, Jacob Bernoulli requested that his gravestone be engraved with a logarithmic spiral and the motto *Eadem mutata resurgo* (*Although changed, I rise again the same*), but instead it was engraved with an Archimedean spiral.

The mathematical contributions of his younger brother Johann Bernoulli (1667–1748) were no less significant; some of them have been discussed already above. Johann first studied medicine and earned a doctorate in Basel for a thesis on muscle contraction. Later, like Jacob over the objections of his father, he studied mathematics with his brother and went on to become a professor of mathematics at the University of Groningen in the Netherlands. He returned to Basel only many years later, shortly after his brother had succumbed to tuberculosis.

The best-known mathematical discovery associated with Johann Bernoulli is his method for determining the limit of a fraction of functions as both numerator and denominator tend to zero, a familiar favorite of calculus students. This rule states that if two functions $f(x)$ and $g(x)$ both admit derivatives $f'(x)$ and $g'(x)$ in the neighborhood of a certain point a with

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

Fig. 6.13 Daniel Bernoulli: the second generation of Bernoulli family mathematicians

and $g'(x) \neq 0$ for all $x \neq a$ near a, then if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, we can calculate

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This rule was included in the first systematic textbook on infinitesimal calculus, written by a French former student of Johann Bernoulli's named Guillaume de l'Hôpital, and it has ever since been referred to as l'Hôpital's rule. Johann also used calculus to tackle the problem of fastest descent, known as the brachistochrone problem, and to determine the lengths and enclosed areas of related tautochrone curves (Fig. 6.13).

The brothers Johann and Jacob Bernoulli were academic colleagues and both were friends to Leibniz, but between them, there developed frequent academic rivalry. Johann was known for a quick temper and jealous disposition. In spite of this, he seems to have been an impressive teacher: not only did he nurture such impressive students as l'Hôpital, but he also trained his three sons for lives as mathematicians, although he encouraged both the eldest Niklaus and the second eldest Daniel to pursue careers other than mathematics; the former studied law and the latter medicine, but they both eventually took posts as professors of mathematics at the newly founded Saint Petersburg Academy. It was these two who introduced their close friend Euler to Russia, where he spent the better part of his life. The youngest of the three, Johann II, succeeded his father as professor of mathematics at the University of Basel, after an earlier stint as a professor of rhetoric. The legacy did not end with this generation: the two sons Johann and Jakob of Johann II also found their way to mathematics, after some detours (Fig. 6.14).

In general, the second and third generations of Bernoulli mathematicians did not achieve the same heights as the first, with the notable exception of Daniel Bernoulli (1700–1782), who at several points throughout his life could be said to rival his

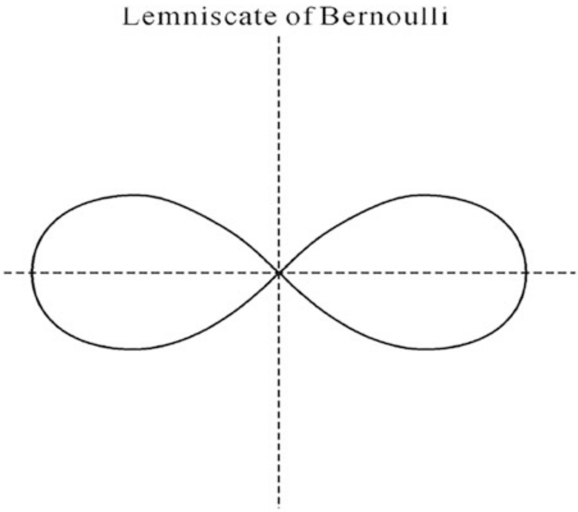


Fig. 6.14 The lemniscate of Bernoulli

friend and contemporary, the great Euler, with whom he shared several of the ten prizes he was awarded by the French Academy of Science. Upon his return to Basel from Saint Petersburg, Daniel successively served as a professor of medicine, metaphysics, and natural philosophy while continuing to make contributions in a number of different fields within mathematics, including calculus, differential equations, and the theory of probability.

The most famous result due to Daniel Bernoulli is Bernoulli's principle, a result in fluid dynamics that has direct applications in modern aircraft design. This theorem states that the total energy of a moving fluid (gas or liquid) remains constant; this includes its kinetic energy and dynamic pressure, potential energy due to gravity and static pressure, and internal energy. As an example, a fluid flowing horizontally experiences no change in its gravitational potential energy, and from this, it follows that its static pressure decreases with an increase in the speed of its flow. This principle provides the theoretical basis for many problems in engineering, notably in the design of aircraft wings: since the airflow along the curved upper surface of the wing is faster than along its lower surface, Bernoulli's principle implies that the pressure along the lower surface is greater than along the upper surface, thereby generating lift.

In the 1990s, the Bernoulli Society for Mathematical Statistics and Probability in the Netherlands introduced the *Bernoulli Journal* in commemoration of the mathematical contributions of the Bernoulli family. This is now the second mathematical journal we have encountered to derive its name from a significant mathematician or mathematical family, after the *Fibonacci Quarterly*.

The French Revolution

Napoleon Bonaparte

In the year 1769, the situation of the two French mathematicians Laplace and Lagrange was as follows: Lagrange was 31 years old and serving as the director of mathematics at the Prussian Academy of Sciences in Berlin; Laplace, 11 years his junior, was employed as a professor of mathematics at the École Militaire. At this time, their future student and friend Napoleon Bonaparte was born in Ajaccio, capital of the Mediterranean island of Corsica. Only a year earlier, this island had belonged to the Republic of Genoa in the Apennine Peninsula. If its transfer to France had been delayed for even a few more years, Napoleon might have found himself as an adult fighting instead for the territorial defense and expansion of Italy, or a part of the underground resistance against France, as indeed his father had been. In fact, his paternal ancestors the Bonapartes descended from a family of minor nobles in Tuscany, whose capital city Florence had been the central city of the Italian Renaissance (Fig. 6.15).

The Corsican resistance against France quickly collapsed, however, and Napoleon's father was obliged to submit to French rule and serve in his capacity as an attorney for the new regime, eventually becoming the representative of Corsica to the court of Louis XVI. All this paved the way for young Napoleon, at the age of 9, to move to the French mainland and enroll briefly in a religious school

Fig. 6.15 Napoleon, the amateur geometer

Fig. 6.16 *La Marseillaise*, the national anthem of France, was born during the French Revolution

before obtaining a scholarship to a military academy at Brienne-le-Château, and eventually, many transfers later, to graduate from the École Militaire in Paris. It was while Napoleon was at the École Militaire that his father died, compelling him to complete his course in a single year. He exhibited some talent for mathematics during his studies and was examined for graduation at the age of 16 by none other than Laplace (Fig. 6.16).

After he graduated from the military academy, Napoleon became a second lieutenant in an artillery regiment. During this time, he carried out an extensive reading of military treatises. Not long afterward, he returned for 2 years to Corsica, and it seems that he sustained a strong affection for his homeland, to which he returned again several times in later years, and which had things gone differently he might still have helped to achieve its independence. But as the French Revolution, Napoleon was more and more attracted to Paris; he was a loyal student of Voltaire and Rousseau and fervently believed that political change was necessary for France. All the same, when the revolution actually arrived and the Parisians stormed the Bastille, a fortress that served as a symbol of the tyranny of the king, on July 14, 1789 (later the national holiday of France), Napoleon was in the provinces.

The French Revolution not only brought an end to the ancient regime in France but also marked a change in the entire political climate of Europe. Although historians disagree about the precise causes of this revolution, there are five generally agreed-upon explanations: (1) France had the largest population of any European nation at the time and could no longer adequately support it; (2) the rise of a wealthy bourgeoisie class and its exclusion from political power, a divide more extreme than that which existed in other nations of the period; (3) a deepening understanding among the peasants of the situation and with it their inability to tolerate a feudal system that subjected them to fraud and exploitation; (4) the appearance of radical philosophers promoting political and social change and the widespread circulation of their works; and (5) the depletion of the national treasury due to French participation in the American Revolution.

There is no doubt however that Napoleon's march into Paris required many blessings from the goddess of fate. In January of 1793, King Louis XVI was executed by guillotine for high treason. This was a period of intense crisis within France and even throughout Europe, as the revolutionaries had earlier declared war against the counterrevolutionary forces in various European countries. The following winter, Napoleon led the artillery of the republican forces in the port city of Toulon to defeat the royalist British navy assembled at the behest of the Baron d'Imbert and force them to evacuate. This battle brought him fame and recognition and earned him a promotion to brigadier general. Another year passed, in which time the royalists carried out their White Terror campaign and an attempt to seize power in Paris. Napoleon crushed their efforts; by the age of 26, this young Corsican soldier was widely recognized as the savior and hero of the French Revolution.

Also in the year 1795, the old University of Paris and the Academy of Sciences in Paris were abolished by the National Assembly in the name of egalitarianism, replaced by the École Polytechnique and the University of France, the latter of which absorbed the Paris Academy of Sciences as one of its three branches, as well as a new normal academy in Paris founded the year earlier and later rebranded in 1808 as the École Normale Supérieure. Although these two schools were originally conceived as training schools for engineers and teachers, respectively, both placed a high importance on mathematics, perhaps not unrelated to the fact that the Marquis de Condorcet, whose was involved in the establishment of the new state education, was himself a mathematician. He brought in the most respected mathematicians in France at the time: Lagrange, Laplace, Legendre, and Gaspard Monge, who later also served as the first Director of the École Polytechnique.

It was a few more years however before Napoleon became the First Consul of the Republic; during this time, he led military campaigns both northward and southward, leaving behind him his footprints in Italy, Malta, and Egypt, where he commanded more victories than losses. It was after he returned to France that he could truly be said to have consolidated beneath him the military and political power, not unlike the return of Caesar from Egypt to Rome. On the last Christmas of the eighteenth century, France established a new national constitution in which Napoleon was designated the first consul for a period of 10 years, with basically unlimited powers, including the appointment of ministers, generals, civil officers, magistrates, and legislators. Afterward, Napoleon included his mathematician friends among the high-ranking officials.

Although his rise to power was facilitated by the French Revolution, Napoleon himself was a man of tremendous ambition, and his belief in the sovereignty of the people and the virtue of free legislative debate quickly proved illusory.

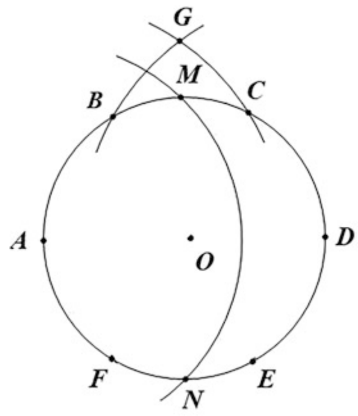


Fig. 6.17 Answer to a question raised by Napoleon

Rather Napoleon took up the mantle of the philosopher king of pure reason and intellect, with mathematics and jurisprudence as his advisors. The war effort was still unresolved however, and his campaign of territorial expansion had just begun. In his capacity as First Consul, Napoleon considered the management of the army as in need of most careful attention in order to consolidate power and achieve his imperial ends. As a result, the École Polytechnique was militarized and charged with the training of artillery officers and engineers, and its professors were encouraged to turn their attention to mechanics, the development of artillery shells and other weaponry, and to maintain close contact with the consulate (Fig. 6.17).

The mathematical talent nourished in his early years and his continued contact with eminent mathematicians encouraged Napoleon to propose a question in geometry: using only a compass but no straightedge, how to divide a circle into four equal parts? This problem was solved by the Italian mathematician Lorenzo Mascheroni (1750–1800), who had been trapped in Paris by the war. Mascheroni also wrote a book entitled *Geometria del Compasso*, dedicated to Napoleon, in which he proved that any geometrical construction than can be accomplished by compass and straightedge can also be accomplished by compass alone, that is, that the straightedge of classical Euclidean geometry is superfluous. It was discovered by later generations that this result had in fact already been proven in an obscure book by the Danish mathematician Jørgen Mohr (1640–1697) (Fig. 6.18).

The specific method for the division of the circle into four parts is as follows: let A be any point on the given circle O, and with one bisector at A, set up a total of six bisectors at A, B, C, D, E, and F, dividing the circle into six equal parts as shown in the figure. Construct two circles with centers A and D and radius AC or BD, intersecting in the point G. Construct another circle with A as its center and with radius OG, meeting the circle O in points M and N. Then the points A, M, D, and N divide the circle into four equal arcs. Indeed, according to the Pythagorean theorem, $AG^2 = AC^2 = (2r)^2 - r^2 = 3r^2$, and therefore, $AM^2 = OG^2 = AG^2 - r^2 = 2r^2$, $AM = \sqrt{2}r$, so AO and MO are perpendicular.

Fig. 6.18 Marquis de Condorcet, a revolutionary and a mathematician

The Lofty Pyramid

We turn now to Joseph-Louis Lagrange (1736–1813), considered alongside Euler as one of the two greatest mathematicians of the eighteenth century. As for which of the two was in fact the greater, this has been a topic of much debate and not immune to the preference in mathematical interests of the supporters of one or the other. Lagrange was born in Turin, a famous city in northwestern Italy, known today as the home of Fiat and the Juventus Football Club. Its close proximity to France had meant that Turin was for a time occupied by France, during the sixteenth century, and by the time that Lagrange was born, it was the capital of the Kingdom of Sardinia. Its status did not afterward change until the nineteenth century, when Turin was at the political and ideological center of the struggle for Italian unification, to the extent that it was even briefly the capital of the newly independent Kingdom of Italy.

Lagrange was of mixed French and Italian ancestry. His great-grandfather had been a captain in the French cavalry, who settled in Turin and married into a prominent local family after having served under the king of Sardinia, a Mediterranean island that is today a part of Italy. His father briefly had charge of the king's military chest and served as Treasurer of the Office of Public Works and Fortifications in Turin, but all the same he failed to effectively manage his family property, and Lagrange, who was the firstborn of 11 children, received only a small inheritance. Later he regarded this as the luckiest thing that could have happened to him, reasoning that a large fortune might have cut him off from his fate as a mathematician.

In his school years, Lagrange was drawn first to classical literature and not much inspired by his encounters with the geometric works of Euclid and Archimedes. Later he stumbled by accident across a popular work written by Edmond Halley (1656–1742), a friend of Newton responsible for the discovery of Halley's Comet, in which the topic of calculus was introduced and exalted. Lagrange became fascinated with this new subject and quickly mastered through self-study the full body of knowledge in analysis of his era. At the age of either 19 or 16 (accounts vary), Lagrange was appointed Sostituto del Maestro di Matematica (assistant professor of mathematics) at the Royal Military Academy of the Theory and Practice of Artillery and embarked upon one of the most glorious careers in the history of mathematics. By the age of 25, Lagrange was already regarded as one of the greatest mathematicians in the world (Fig. 6.19).

Unlike any earlier mathematician, Lagrange was an analyst right from the start of the career, further evidence that analysis had already become the most popular branch of mathematics in that period. This preference achieved its full realization in his *Mécanique analytique* (*Analytical Mechanics*), which Lagrange first conceived at the age of 19, although its publication in Paris did not appear until he was already

Fig. 6.19 Lagrange, a descendant of France and Italy

52, by which time he had largely lost interest in mathematics. In the preface to this work, Lagrange writes:

> No diagrams will be found in this work. The methods that I explain require neither geometrical, nor mechanical, constructions or reasoning, but only algebraical operations in accordance with regular and uniform procedure.

All the same, his framework for mechanics is novel in its appeal to the geometry of four dimensions: three coordinates representing spatial position and a fourth coordinate representing time. According to this conception, the mechanics of a moving point is determined entirely by its geometrical description.

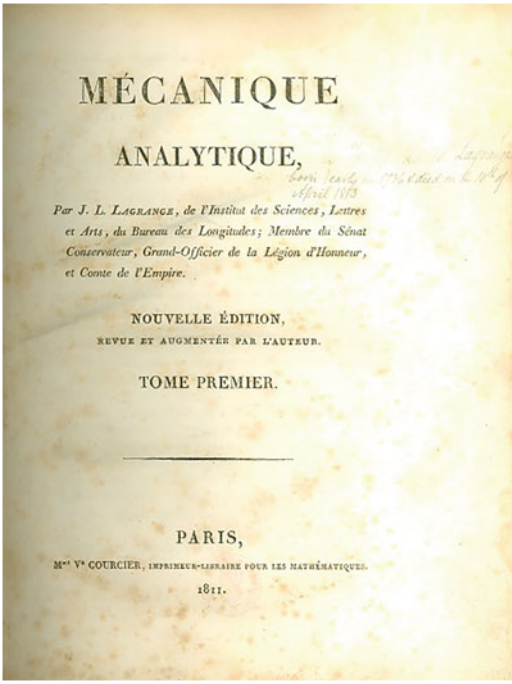
Lagrange also introduced the notations $f'(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$, etc., for the derivatives of a function $f(x)$ with which we are familiar today, and discovered an early version of the mean value theorem, sometimes referred to as Lagrange's mean value theorem. This theorem states that if a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one point ζ in the interval $a < \zeta < b$ satisfying

$$f'(\zeta) = \frac{f(b) - f(a)}{b - a}.$$

In addition, Lagrange developed approximation methods for determining the real roots of polynomial equations using continued fractions and investigated the question of the representation of arbitrary functions by power series.

In his *Analytical Mechanics*, which the Irish mathematician William Rowan Hamilton referred to in the nineteenth century as "a scientific poem," Lagrange reduced the general equations of solid and fluid dynamics to a single principle from which he derived the general equations of dynamical systems, including what have since come to be called the Lagrange equations. This work also includes some of his best known results concerning differential equations, partial differential equations, and the calculus of variations. Its importance to general mechanics is as great as the importance of Newton's law of universal gravitation for celestial mechanics. Not to say, however, that Lagrange paid no heed to the celestial bodies; in fact, he solved the problem of the moon's libration, that is, why is it that the moon presents the same face to the earth at almost all times. His analytical approach to problem-solving in mechanics marked a departure from the classical Greek tradition, and even from the study of mechanics by Newton and his immediate successors, which still made use of geometry and figures (Fig. 6.20).

From the start of his career, Lagrange received generous praise and support from his potential rival Euler, almost 30 years his senior, and the intellectual selflessness of their relationship has become one of the pivotal stories in the history of mathematics. Like Euler, Lagrange applied himself primarily to analysis and its applications but also indulged his curiosity with countless investigations into number theoretic questions: we have seen already that he resolved two important conjectures left over by Fermat. There is also Lagrange's theorem in modular arithmetic, which states that if some prime number p does not divide every


MÉCANIQUE

ANALYTIQUE,

Par J. L. Lagrange, de l'Institut des Sciences, Lettres et Arts, du Bureau des Longitudes; Membre du Sénat Conservateur, Grand-Officier de la Légion d'Honneur, et Comte de l'Empire.

NOUVELLE ÉDITION,
REVUE ET AUGMENTÉE PAR L'AUTEUR.

TOME PREMIER.

PARIS,
Mme Ve COURCIER, IMPRIMEUR-LIBRAIRE POUR LES MATHÉMATIQUES.
1811.

Fig. 6.20 First volume of *Mécanique analytique*, by Lagrange (1811)

coefficient of a polynomial $f(x)$ of degree n, then the congruence $f(x) \equiv 0 \pmod{p}$ has at most n distinct solutions modulo p. But the most famous theorem associated with the name Lagrange is Lagrange's theorem in group theory, which states that the order of any subgroup of a finite group G is a factor of the order of G.

In light of his achievements, Lagrange obtained funding from the king of Sardinia to travel to Paris and London, but he fell ill during his time in Paris and returned early to Turin upon his recovery. Not long afterward, he travelled again, this time to Berlin at the invitation of King Frederick of Prussia, and he remained there for 11 years until the death of the king, at which time France did not miss a second opportunity to invite him to Paris at the behest of King Louis XVI. This was in the year 1787, and Lagrange had already turned his attention mainly to the humanities, medicine, and botany. He became close with the king and his queen Marie Antoinette, who looked after him with care and did her best to soothe his bouts of depression (Fig. 6.21).

Two years later, the French Revolution reached its climax in Paris, and this seems to have penetrated through the intellectual lethargy into which Lagrange had sunk and inspired him to become active in mathematics. He wrote several academic works and textbooks and declined an invitation to return to Berlin, surviving through the reign of terror by virtue of his silence and discretion; his friend the chemist Antoine Lavoisier (1743–1794) was not so lucky and died under the guillotine. When the École Normale Supérieure was established, Lavoisier was appointed as a professor, and later, he also became the first professor at the École Polytechnique, teaching mathematicians to young military engineers in the service of Napoleon; among

Fig. 6.21 *The Death of Marat*, by Jacques-Louis David (1793)

them was the future mathematician Augustin-Louis Cauchy. Napoleon, who turned his attention to internal affairs in the intermission between two campaigns, paid frequent visits to Lagrange to discuss mathematics and philosophy and honored him by making him a Senator and a Count of the Empire. This towering emperor who had invaded Egypt described Lagrange as "the lofty pyramid of the mathematical sciences."

The French Newton

In his later years, Lagrange referred to Newton with a measure of envy, remarking that although he was no doubt a particularly gifted man, also he was the luckiest of scientists, since there history admits but one opportunity to explain the universe. In this sense, Pierre-Simon Laplace (1749–1827) could be said to have been less fortunate than Lagrange; he too came too late to achieve the monumental revolutions of Newton, and his career spread evenly across the eighteenth and nineteenth centuries, the former dominated by the shadow of Euler and Lagrange and the latter by that of Gauss; he is automatically disqualified therefore for such titles as the greatest mathematician of this or that century. All the same, his was a brilliant life marked by a tremendous intellect, diligence, and his association with his student Napoleon (Fig. 6.22).

Laplace was born to farmer parents in Beaumont-en-Auge in Calvados, Lower Normandy, not far from the English Channel, site of the Allied invasion of Western Europe during World War II. He exhibited considerable talent as a student at the

Fig. 6.22 Pierre-Simon Laplace, the "French Newton"

Fig. 6.23 Laplace metro station, Paris; photograph by the author

village school, including special eloquence in theological debates, attracting the attention of his wealthy neighbors and securing for him a place as a day student at a local military school. It may have been on account of his prodigious memory rather than his mathematical ability, but in any case, he secured a letter of recommendation from an influential figure to travel for the first time to Paris at the age of 18 to advance his fortune (Fig. 6.23).

This letter nearly proved his undoing. It was addressed to d'Alembert, a famous mathematician and co-editor of the *Encyclopédie*, who did not as a rule pay much attention to letters of introduction submitted to him and turned the young man away. Returning dejected to his residences, Laplace wrote overnight a new introduction containing a treatment of the principles of mechanics and possibly the solution of a problem posed to him in passing by d'Alembert, and it was this letter rather that caught the attention of d'Alembert, who wrote back after having read it and invited Laplace for an immediate audience, remarking that he should not have brought any letter of recommendation in the first place as he had introduced himself so much more capably. Just a few days later, at the recommendation of d'Alembert, Laplace was awarded a teaching position at the École Militaire, where he came into contact with his future student Napoleon.

Laplace devoted less energy to pure mathematics and achieved in it fewer results than Lagrange, preferring instead to carry out his research toward applications in astronomy. Among the results associated with him, there is the Laplace expansion for the calculation of the determinant of a matrix; in its most general form, this states that the determinant can be obtained as an expansion along any arbitrarily selected k rows or columns of the matrix by taking a weighted sum of the products of determinants of various submatrices and their complements determined by the particular choice of k rows or columns. There is also the Laplace transform for differential equations; this transform replaces a suitable function $F(t)$ with another type of function $f(p)$ via the improper integral

$$f(p) = \int_0^\infty e^{-pt} F(t)dt.$$

But the work for which Laplace is best known is of course his treatise *Celestial Mechanics* (*Traité de mécanique céleste*) in five volumes which earned for him this nickname as the French Newton. Starting from the age of 24, Laplace had carried out research into the application of the Newtonian theory of gravity to the solar system as a whole and sought to answer why it is that the orbit of Saturn is expanding while that of Jupiter is shrinking. He proved that the mutual action of two planets could only ever produce small changes to their eccentricities and inclinations. He showed also that the acceleration of the moon is related to the eccentricity of the orbit of the earth, providing a theoretical solution to the last anomaly in the dynamical observations of the solar system. His name is inseparably linked with the nebular theory for the formation and evolution of planetary systems throughout the universe, and another testimony to his achievements is the Laplace equation for potential energy that we have introduced already in our discussion of the influence of calculus.

The respective and comparative work of Laplace and Lagrange, two giants in the history of science, is a topic much discussed among their successors. The nineteenth-century French mathematician Siméon Denis Poisson observed a profound difference between the thought and working methods of the two in everything they did, from pure mathematics to studying the libration of the moon.

"Lagrange," Poisson observed, "often appeared to see in the questions he treated only mathematics, of which the questions were the occasion – hence the high value he put upon elegance and generality. Laplace saw in mathematics principally a tool, which he modified ingeniously to fit every special problem as it arose."

There were also stark differences between them in terms of their personal attributes. Joseph Fourier remarked of Lagrange:

> By his whole life he proved, in the moderation of his desires, his immovable attachment to the general interests of humanity, by the noble simplicity of his manners and the elevation of his character, and finally by the accuracy and depth of his scientific works.

In contrast, Laplace developed a reputation among mathematicians as a political actor and a snob. The historian of mathematics E.T. Bell summarizes his character by his greed for titles, casual political flexibility, and an intense desire to gain the respect of the public at the center of its ever-shifting attention.

But Laplace was not without his candid and sincere side. His dying words were reported by Fourier to have been, "What we know is not much. What we do not know is immense." Napoleon also found much fault with his administrative work as Minister of the Interior on account of his fastidiousness and tendency to look for the subtlest nuances in all things; he even quipped that Laplace brought the spirit of the infinitesimal with him into his efforts as an administrator. All the same, Napoleon heaped upon him many honors, making him a senator, a Count of the Empire, appointing him to the Bureau of Longitudes, and awarding him the Legion of Honour. In spite of this, Laplace signed the decree to banish Napoleon and continued to flourish under the Bourbon Restoration, during which time he was awarded the further title of Marquis de Laplace and a seat in the Chamber of Peers; he served also during this time as chairmen of the committee for the reorganization of the École Polytechnique.

In the eighteenth century and early nineteenth century, French mathematicians spoke of the *three Ls* of mathematics: Lagrange, Laplace, and Legendre. Legendre spent his entire life in Paris; he became a professor at the École Militaire at the age of 23. Later in 1795, he became a professor at the École Normale Supérieure. His outstanding work on elliptic integration provided fundamental analytic tools for mathematical physics, and along with Gauss, he introduced the least squares method, proposed the prime number theorem as a conjecture, and proved the law of quadratic reciprocity in number theory. There is also the Legendre symbol in number theory, which appears in every introductory course in that topic. His book *Éléments de géométrie* replaced Euclid's *Elements* as the basic geometry textbook in European and American universities.

The Emperor's Friend

There is a widely circulated legend about Laplace that he had presented Napoleon with a copy of the *Celestial Mechanics* after the latter had become emperor and received from him the pointed question how is it that he had written a work on the system of the world without any mention of its author. Laplace replied, “I had no need of that hypothesis.” This sentence recalls to mind the response of Euclid to Ptolemy I, “There is no royal road to geometry.” In fact, Laplace may have had in mind with this omission a contrast the Newtonian tradition, since Newton did in fact make reference to God in his works, and Laplace considered his celestial mechanics to accommodate a wider scope than the solar system as conceived by Newton.

Both Laplace and Lagrange enjoyed a relationship with Napoleon that fit the classical conception of the relation between scientist and enlightened monarch; in particular, the distinction between monarch and subject was well delineated. Not so with Monge. Gaspard Monge (1746–1818) was 3 years older than Laplace and somewhat less talented in mathematics, but his personal experiences and open personality led him to establish a close friendship with the young Napoleon. As a result, during the Bourbon Restoration, Monge was not showered in glory as Laplace was but rather he became a wanted man and went into hiding, widely regarded as a close confidante of the Corsican emperor, as indeed he was: Napoleon had said of him that Monge loved him as a man loves a mistress (Fig. 6.24).

Monge was born in Beaune, a small town in Côte-d’Or in central France, belonging to the Burgundy region, famous for its wine, and located to the southwest of Dijon. Today, it is a stopping point along the high-speed rail line between Monte Carlo and Paris. His father was a hawker and knife sharpener who placed great emphasis on the education of his son, and as a result, his son took naturally to a leadership position in everything from sports to crafts. When he was 14 years old, Monge designed a fire truck without reference to any preexisting diagrams, relying only on his own perseverance and dexterity, and presented his construction with geometrical precision. Two years later, he drew up a detailed map of his hometown

Fig. 6.24 Gaspard Monge, among the few who dared to contradict Napoleon

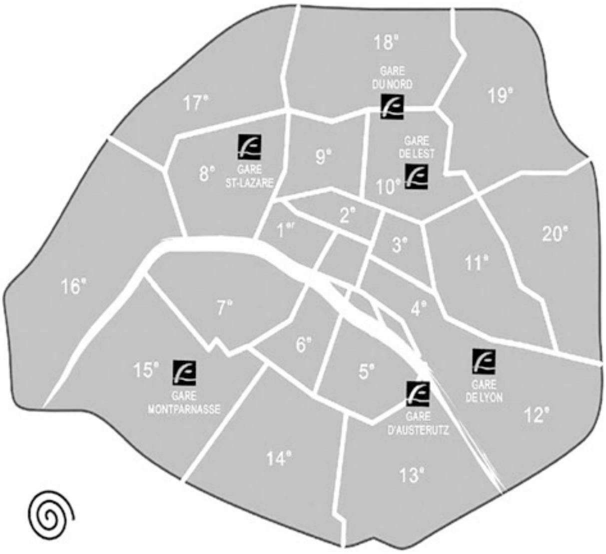


Fig. 6.25 Zoning map of Paris, exhibiting an Archimedean spiral

on a large scale and was recommended for a teaching position in physics at the Collège de la Trinité at Lyon (Fig. 6.25).

On one occasion, Monge encountered in the course of his return home from Lyon an officer of engineers who had seen his map and recommended him for a position at the École Royale du Génie at Mézières, capital of the Champagne-Ardenne region in northern France. This city is only 14 kilometers from the Belgian border, and nearby Charleville was the birthplace of the great poet Rimbaud more than a century later. Monge worked during this time as draftsman, responsible for measurements and drawing, and took advantage of this experience to create a new form of geometry, now known as descriptive geometry, which involves the representation of three-dimensional objects in the two-dimensional plane according to specific rules. He also worked as a teacher, and one of his students Lazare Carnot (1753–1823) later enjoyed a fruitful career as a geometer and participated in the French Revolution.

In 1768, when he was 22 years old, Monge began to teach mathematics at the École Royale du Génie at Mézières, and a few years later, he was appointed there as a professor of mathematics and physics. He left the city only in 1783 when he travelled to Paris and took a post as an examiner of naval candidates. Before moving to Paris, he married a young widow renowned for her beauty and devotion. He was surrounded in Paris by powerful figures and became enmeshed in the petty struggles of the city elite; inevitably, he was drawn into the French Revolution when it broke out. He was compelled to serve for a time as the Minister of the Marine after the formation of an executive council by the new Legislative Assembly. When the École Polytechnique was established in Paris in 1795, Monge was heavily involved in its founding and served there afterward as a professor of descriptive geometry. The

Fig. 6.26 The tombs at the Panthéon in Paris, where Lagrange, Monge, Carnot, and Condorcet are buried; photograph by the author

birth of this school and the École Normale Supérieure in the same year marked the beginning of a glorious period in the history of French mathematics (Fig. 6.26).

The next year, Monge received a letter from the young Corsican who had already ascended quite some way along his rise to power. This letter recalled first the cordial welcome that Napoleon had received as an unknown artillery officer from Monge in his capacity as an examiner for the navy, remarking with gratitude that he had now already risen to the rank of general of the army and was on an expedition to Italy. As an expression of his appreciation, Napoleon appointed Monge as a commissioner to select various paintings, sculptures, and other works of art for return to Paris. Fortunately, Monge succumbed to his conscience after carrying out his task to a suitable degree of completion counselled moderation rather than strip Italy completely of its masterpieces. Subsequently, Monge and Napoleon began a long and close friendship; it has been remarked that after Napoleon had become emperor, Monge was along among his friends who dared to contradict him or speak plain truths in his hearing.

Napoleon however was not at this time entirely occupied by domestic affairs, and in 1798, he led an expedition to Egypt, and Monge accompanied him as a member of the Legion of Culture, alongside Fourier, inventor of the well-known Fourier series expansion of functions. Along the voyage to the Mediterranean, Napoleon seems to have summoned Monge and others to his flagship each morning for a discussion on the same major topic, for example, the age of the earth, the possibility of its destruction by fire or flood, the existence of any other habitable planets, and so on. Upon their arrival to Cairo, Monge helped to establish the Institut d'Égypte, after the model of the Institut de France.

Finally, we turn to the contributions of Gaspard Monge to mathematics. In addition to the creation of descriptive geometry, Monge is also remembered as the father of differential geometry, a form of geometry that makes use of the tools of calculus to study curves, surfaces, and their various extensions and applications. Monge greatly advanced the theory of curves and surfaces in space, a topic characterized by its close connection with differential equations; various properties of curves and surfaces can be represented in terms of differential equations, which is also why this branch of mathematics is called differential geometry. As one example, Monge obtained the general representation of a class of surfaces known as developable surfaces and showed that with the exception of cylindrical surfaces perpendicular to the xy plane, such surfaces always satisfy the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = 0.$$

Monge served for a time as director of the École Polytechnique and continued to occasionally deliver lectures to its students. During one such lecture, he discovered an ingenious theorem in geometry concerning the properties of the tetrahedron. Recall that a tetrahedron is a solid with four faces and six edges, each of which meets every other edge in a point except for one, called its opposite edge. Monge's tetrahedron theorem states that the six planes passing through the midpoints of each edge and perpendicular to its opposite edge all meet in a point, now known as the Monge point. We close this section with the remark any readers who have the opportunity to visit Paris can find in that city a Rue Monge, a Place Monge, and a Café Monge.

While Monge was serving at the École Polytechnique, there was a student there named Jean-Victor Poncelet (1788–1867), who went on to become the proper founder of modern projective geometry and serve as the director of his alma mater. Poncelet was born in Metz in eastern France, an illegitimate child, later legitimated. At the age of 24, he participated in an expedition to Moscow under Napoleon as an engineer lieutenant. He was captured and turned his attention to mathematical problems during his time in a prison camp along the Volga River, using charcoal intended for heating to scribble on the walls. During this time, he wrote his most influential work, the *Traité des propriétés projectives des figures*, which presented his central projection treatment of conic sections, now the starting point for projective geometry of three dimensions. This paved the way for his career as a mathematician. Starting a year after his death, the French Academy of Sciences began to offer the Poncelet Prize for mechanics, applied mathematics, and the advancement of science.

Conclusion

The development of mathematics has proven throughout its history to need occasional nutrition from external sources; among these, physics has been consistently the most fruitful, and of course, it is also physics that has benefited most from the contributions of mathematics. Problems in physics have given much impetus to mathematics, especially in analysis, which has been closely linked with mechanics right from the birth of the calculus, and perhaps since the late nineteenth century also in geometry. This was the source of Lagrange's great masterpiece, the *Mécanique analytique*. But Lagrange himself perhaps loved number theory most of all the mathematical disciplines, and he was very proud of his proof that every positive integer can be represented as a sum of four or fewer squares. Another impetus for mathematics came from the demands for military and technological innovation ignited by the French Revolution. Since that time, the revolving door linking the development of mathematics and its applications has never closed.

It is necessary to observe that during the time after Newton and Leibniz had completed their work but before the appearance of Lagrange, the greatest mathematical minds in Europe were all concentrated in the small mountain country of Switzerland, at that time with a still relatively underdeveloped economy, culture, and scientific atmosphere: these of course were Leonhard Euler and the various members of the Bernoulli family, all of them from the same small city of Basel. The first-generation Johann and Jacob of the Bernoullis served as teachers to Euler in their capacities as professors at the University of Basel. After he graduated from this university, Euler spent most of his life in two distant and exotic cities, Berlin and Saint Petersburg. After his death, Euler appeared on the 10 Swiss francs banknote; alongside Newton on the 1 Pound Sterling banknote in Britain and Niels Henrik Abel on the 500 Norwegian Kroner banknote, he was one of three major mathematicians to appear on European currency still in circulation today. We mention also that Euler made his European debut in a prize competition organized by the Paris Academy of Sciences, which prize he won 12 times.

The École Polytechnique played an important historical role as the start of a new type of university; it also provided reliable employment for mathematicians, especially applied mathematicians. Lagrange and Monge were the first mathematical luminaries to appear among the professorship of this institution, and young students competed fiercely for admission with the goal of entry into service as an officer or engineer. The most significant member of this next generation was Augustin-Louis Cauchy, responsible for deep and humanistic achievements, although in later years some shallowness of mind or conceit caused him to ignore his younger colleagues, in particular Abel. The tradition of this institution later spread across the globe, with notable examples after its prototype being the Massachusetts Institute of Technology and California Institute of Technology in the United States, Tsinghua University in China, and the Indian Institute of Technology, with seven independent campuses located in different cities across the country.

Prior to Cauchy, there were two other great mathematicians to pass through the École Polytechnique. These were Joseph Fourier (1768–1830) and Siméon Denis Poisson (1781–1840). The greatest work by Fourier was his *The Analytic Theory of Heat*, which later James Clerk Maxwell is said to have called a great mathematical poem. In this book, he proved the important result that any function (today this claim requires some further qualification) can be expanded as a series of sine functions in multiples of the variable. Such expansions, known as Fourier series expansions, are important in the theory of boundary-constrained partial differential equations and also contributed an extension of the scope of the concept of a function. Poisson was the son of a former soldier and district president and became the first person to study integration along paths in the complex plane. His name comes up frequently in university mathematics: there is the Poisson integral, Poisson's equation in potential theory, Poisson's ratio in the theory of elasticity, the Poisson distribution and Poisson law in probability, the Poisson bracket in the theory of differential equations, and so on (Fig. 6.27).

Fourier famously remarked: "The deep study of nature is the most fruitful source of mathematical discoveries." There are also various interesting rumors surrounding Fourier and Poisson. Among them, it is said that during his time as Prefect in Egypt, he adopted the habit of wearing thick layers of clothes in the hot desert as part of his research into thermodynamics, and this aggravated his heart condition; when he died in Paris at the age of 63, he was alleged to have been as hot as if he had just been boiled. Poisson was looked after by a caretaker in his childhood. One day his father came for a visit and discovered the caretaker in absentia and his son hanging

Fig. 6.27 French mathematician Joseph Fourier

Fig. 6.28 Tomb of Fourier, at Père Lachaise Cemetery, Paris

in a cloth bag from a stud in the wall. The caretaker later offered the explanation that this was to prevent the child from catching an illness from the floor. Perhaps there is some connection between this event and his later years devoted to the study of pendulums (Fig. 6.28).

It is not unreasonable to say that the number of great mathematicians to emerge in the eighteenth century was greater than in any previous period, including even the seventeenth century, which was not lacking for geniuses. On the other hand, no singular giant of Renaissance proportions appeared among them, and the increasingly pragmatic turn of the times led to a separation between mathematics and philosophy. For this reason, the eighteenth century has sometimes been referred to as the century of invention. As a point of fact, there was not a single mathematician-philosopher of note during this time, and both Euler and Lagrange came to feel in their later years that the supply of mathematical ideas was beginning to run out. They could not have anticipated that this would merely mark a new turning point in the story of the development of mathematics (Fig. 6.29).

On the other hand, the astonishing new achievements in mathematics and through its applications and with it the elevated light in which mathematics came to be regarded shook longstanding systems of philosophical and religious thought. For intellectuals of the period, a devout piety and religious was increasingly impossible, and philosophers took this as an opportunity to inquire more deeply into the foundations of truth and its discovery. The German philosopher Immanuel Kant (1724–1804) wrote extensively on the subject, and he took as an example the Euclidean axiom that a straight line is the shortest distance between two points

Fig. 6.29 The philosopher Immanuel Kant; like Fermat, he remained in his small hometown (Königsberg) his entire life

to argue that truth cannot be obtained from experience alone but rather requires comprehensive rational judgment.

Another example from Kant was his introduction of the antinomies; these comprise inherent contradictions between two propositions each established in accordance with generally recognized principles. Antinomy is a fundamental concept in his philosophy, in particular in his *Critique of Pure Reason*, in which Kant presented with proofs four sets of antinomies in the form of thesis and antithesis. Among them, two are mathematical in nature and take the form of mathematical paradoxes:

Third Antinomy
Thesis: Causality as determined by the laws of nature is not sufficient to derive one and all of the appearances of the world; there must also be another form of causality in the form of spontaneity.
Antithesis: There is no such thing as spontaneity, and everything in the world takes place solely according to the laws of nature.
Fourth Antinomy
Thesis: There exists in the world either as part of it or as its cause some being that is absolutely necessary.
Antithesis: There exists no absolutely necessary being in the world, nor does one exist outside of it as its cause.

The main pillar of the philosophical system established by Kant, at least with respect to mathematical truths then, is that mathematical truths contain both Euclidean geometry and paradox.

Chapter 7
Modern Mathematics, Modern Art

Out of nothing I have created a strange new universe.

János Bolyai

Whatever mud you give me, I can turn it into gold.

Charles Baudelaire

The Rebirth of Algebra

Toward a Rigorous Treatment of Analysis

In mathematics and in the arts, the first half of the nineteenth century marks a crucial turning point in the long march to modernity. In poetry, the works of Edgar Allen Poe and Charles Baudelaire announced the appearance of an entire host of new stylistic and thematic concerns. In mathematics, the development of non-Euclidean geometry and noncommutative algebra shook the foundations of the established order. Some two millennia in which *Poetics* of Aristotle and *Elements* by Euclid had served as standard bearers for their respective arts were coming to an end. Nevertheless, analysis remained the most active area of mathematical research during this time, and the mathematicians of the period introduced and developed substantial refinements with respect to rigor and clarity in the foundations of analysis, although perhaps without the revolutionary flavor of contemporary discoveries in geometry and algebra (Fig. 7.1).

Among the many talented analysts in France in the nineteenth century, the most prominent was Augustin-Louis Cauchy (1789–1857). Cauchy was born in Paris in the summer of 1789, about a month after the citizens of the city had stormed the Bastille and initiated the French Revolution. His father had served as a magistrate prior to the revolution. Subsequently, after the tumult had died down, he was able to rise quickly through the bureaucratic ranks and eventually became the Secretary-General of the newly formed Senate under Napoleon. In this capacity, he worked directly under Laplace and came into frequent contact with Lagrange, so that his son had opportunity from an early age to come into contact with two of the greatest

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Fig. 7.1 French mathematician Augustin-Louis Cauchy

mathematicians of the time. One story has it that Lagrange happened to observe Cauchy performing some calculations on scratch paper in his father's office one day and remarked in passing, "This child! One day he will surpass us all." But in any case, Lagrange also advised his father, on account of the sensitive constitution of his son, not to permit Cauchy to devote himself wholly to mathematics until he had completed his basic education.

Starting in his childhood, Cauchy studied literature and classical languages with great success before eventually deciding to pursue a career in engineering. When he was 16, Cauchy sat the entrance examination at the prestigious École Polytechnique in Paris and was awarded admission. He completed his studies there 2 years later and transferred to the École des Ponts et Chausées (School for Roads and Bridges) to study civil engineering. Upon graduation, Cauchy accepted a post in Cherbourg, a harbor city in northwest France along the English channel where Napoleon intended to develop a naval base. Throughout this time, however, Cauchy sustained an intense interest in pure mathematics and devoted a considerable amount of his spare time to its study. Eventually, he decided to return to Paris, ostensibly due to illness, but also in order to better facilitate his mathematical research. Both Lagrange and Laplace welcomed his returned and encouraged his mathematical activity. At the age of 27, Cauchy accepted a contract as a professor of mathematics and mechanics at the École Polytechnique and subsequently replaced the mathematician Gaspard Monge as a member of the French Academy of Science, following the exile of Napoleon. Apart from some years spent abroad after having refused to take an oath to a new king at odds with his political sensitivities, Cauchy lived out the remainder of his life in peace and contributed tremendously to the development of mathematics in his time.

During his time at the École Polytechnique, Cauchy used his own analytic results to prepare an influential course of lectures with the objective of formulating and presenting a rigorous treatment of various topics in analysis, including variables, functions, limits, continuity, derivatives, differentials, and the other basic concepts of calculus. For example, Cauchy was the first to state the definition of the derivative as the limit of the expression

$$\frac{\Delta x}{\Delta y} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

as Δx becomes arbitrarily close to zero. Cauchy was also the first to use the notation $dy = f'(x)dy$ for the differential of a function. He provided rigorous definitions for the limiting processes of infinite sequences and series and established the important Cauchy criterion for the convergence of a sequence of real numbers. Cauchy explained the conditions of this criterion as follows:

> The necessary and sufficient condition for the convergence of a sequence x_n is this: for any $\epsilon > 0$ there must exist some positive integer N such that $|x_n - x_m| < \epsilon$ whenever both $m > N$ and $n > N$.

Another important result is the so-called Cauchy mean value theorem, which generalizes the Lagrange mean value theorem discussed in the previous chapter and by means of which Cauchy was able to provide a new proof of the fundamental theorem of calculus, which says that if $f(x)$ is a continuous function defined on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(x)dx,$$

defined at every point in $[a, b]$, satisfies the relation $F'(x) = f(x)$.

The definitions and arguments given by Cauchy already exhibit the flavor of modern analysis, and represent a crucial turning point in its transition towards rigor. One story has it that Laplace, already advanced in years at the time, was in the audience when Cauchy presented his results on convergent series at the Academy of Science. At the conclusion of the lecture, Laplace rushed home and took up his copy of *Celestial Mechanics* from off the bookshelf so that he could use the new methods introduced by Cauchy to confirm the validity of various series invoked in its arguments. Only after he had checked them all was Laplace able to relax (Fig. 7.2).

Nevertheless, notwithstanding a considerable increase in rigor as compared to its predecessors, there remained various substantial logical gaps in Cauchy's work. In particular, Cauchy made free use of such concepts as "infinite approach," "sufficiently large," and similar intuitive formulations. Moreover, his use of limits in the proof of the existence of integrals of continuous functions and elsewhere relies

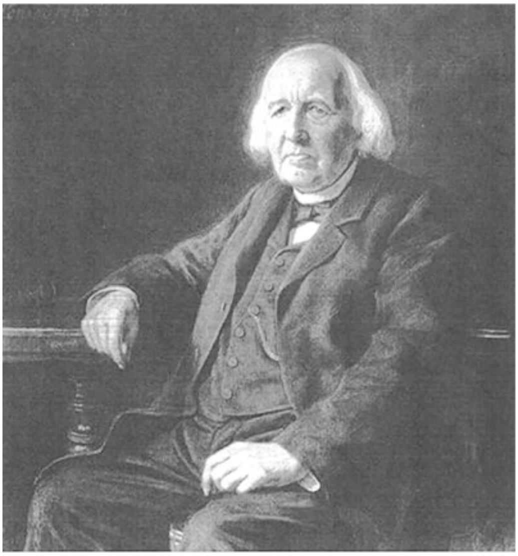


Fig. 7.2 German mathematician Karl Weierstrass

implicitly on the completeness of the system of real numbers, a concept that at the time had not yet been introduced to mathematics with any clarity.

The next major developments in the history of nineteenth-century analysis came from Germany. In particular, the analytic baton was taken up by a German teacher of middle school mathematics named Karl Weierstrass (1815–1897). In the same year that Napoleon suffered defeat at Waterloo, Weierstrass was born in Westphalia, a province in the western part of Germany. After a misguided career choice in his youth, Weierstrass lost some years to the study of law, finance, and economics. At the age of 26, he returned to the region of his birth and spent 15 years in obscurity, teaching mathematics, physics, botany, and gymnastics at various middle schools in the region. Only in 1857, in the same year that Cauchy died, and at the age of 42, was Weierstrass able to obtain a post as an assistant professor at the Technical Institute of Berlin. Despite a close friendship with the Russian mathematician Sofya Kovalevskaya (1850–1891) that seems to have exceeded the usual bounds of their relationship as student and teacher, Weierstrass, as well as his three siblings, never married (Fig. 7.3).

Speaking of Sofya Kovalevskaya, she too was a legendary figure in the history of modern mathematics. She was born in Moscow. Her father was a Lieutenant General in the Russian Army and her mother a descendant of German immigrants. At the time, it was forbidden in Russia for women to study abroad, and so, not unlike the young Brahman Gandhi in India around the same time, Kovalevskaya elected to enter into a contrived marriage with Vladimir Kovalevsky, a young paleontology student, in order to obtain permission to travel to Germany to continue her studies. They settled first in Heidelberg, where Kovalevskaya was able to attend courses in mathematics and physics under professors including Herman von Helmholtz (1821–1894), a natural scientist of wide-ranging interests who had made

Fig. 7.3 Russian mathematician Sofya Kovalevskaya

important contributions to the theory of conservation of energy and many other topics. Afterward, Kovalevskaya relocated to Berlin, where she embarked upon a course of private lessons in mathematics under Weierstrass.

In 1874, Kovalevskaya became the first woman to have been awarded a doctoral degree by a European university, on the basis of a thesis presented to the University of Göttingen on the topic of partial differential equations. Her paper included results remembered today as the Cauchy-Kovalevskaya theorem, on the existence and uniqueness of solutions for certain analytic partial differential equations. Weierstrass served as her advisor and helped her to obtain exemption from the usual requirement for oral examination. In 1888, Kovalevskaya received the prestigious *Prix Bordin* by the French Academy of Science for work on the rotation of rigid bodies around a fixed point, and in the same year, she became the first woman to serve as a Corresponding Member of the Russian Academy of Sciences, on the recommendation of the mathematician Pafnuty Chebyshev (1821–1894) and many others. A posthumous memoir entitled *A Russian Childhood* (1893) depicted details from her life growing up in mid-nineteenth-century Russia and enjoyed immediate success.

We return now to the contributions of Weierstrass to the development of analysis. At that time, the system of real numbers was still incompletely understood, and this misunderstanding contributed to the commonly accepted but mistaken belief that every continuous function is necessarily differentiable. Weierstrass provided the first counterexample to this claim. Specifically, he was able to show that the function

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$$

where a is a positive odd integer, b is any real number with $0 < b < 1$, and a, b jointly satisfy $ab > 1 + \frac{3\pi}{2}$ is everywhere continuous and nowhere differentiable. This came as a serious shock to the mathematical community.

In addition to this discovery, Weierstrass introduced to mathematics the language of δ—ϵ arguments that remains to this day to the most basic gadget in the analytic toolkit. By way of such arguments, Weierstrass was able to make rigorous the notion of an infinitesimal limit inherited from Cauchy and also to formulate a rigorous construction of the real number system: Weierstrass first defined the rational numbers in terms of the integers and proceeded to define the real numbers in terms of infinite sets of rationals. On the basis of this construction, Weierstrass was able to clarify the concepts of limits, continuity, and other such basic elements of calculus. His innovations have led subsequent generations of mathematics to sometimes refer to Weierstrass as the father of modern analysis.

In the generation after Weierstrass, his investigations into the logical structure of the real number system were carried further by his fellow countrymen Richard Dedekind (1831–1916) and Georg Cantor (1845–1918). The former presented a new construction of the real numbers as partitions of the set of rationals and the latter as limits of certain classes of sequences of rational numbers. Both Dedekind and Cantor succeeded in proving the completeness of the set of real numbers according to their respective constructions. Georg Cantor is also responsible for the invention of set theory, which today occupies a position of fundamental importance at the heart of modern mathematics. He was born in Saint Petersburg, son of a Danish father and Russian mother, and subsequently emigrated with his family to Germany at the age of 11, where he eventually had the opportunity to study under Weierstrass. Although he and Dedekind (who, incidentally, had been Gauss's last student) developed competing formulations of a broad array of mathematical concepts and constructions, the two men sustained a lifelong correspondence characterized by mutual respect and encouragement (Fig. 7.4).

Abel and Galois

In 1821, the same year that Napoleon died at Saint Helena, a young Norwegian student named Niels Henrik Abel (1802–1829) enrolled at the Royal Frederick University. A mere 3 years later, Abel published at his own expense a remarkable article entitled *Mémoire sur les équations algébraiques òu on démontre l'impossibilité de la résolution de l'équation générale du cinqiuème degré* (Memoire on algebraic equations, in which the impossibility of solving the general equation of fifth degree is proven). In this paper, Abel proved the following result: it is impossible to produce a general equation involving only radicals and basic arithmetical operations to determine the roots of all polynomials in any fixed degree larger than four as a function of the coefficients. The historical and mathematical significance of this result was tremendous. The simplest case of polynomials in degree two had already been worked out many centuries earlier by Arabic mathematicians of the Middle

Fig. 7.4 Niels Henrik Abel, genius of Norwegian mathematics

Ages via a variant of the modern quadratic formula. During the Renaissance, Italian mathematicians had discovered similar solutions for polynomials in degrees three and four. And then for more than two centuries, mathematicians endeavored in vain to find an equation for polynomials of degree greater than four. Abel proved that no such equation exists.

Abel was born in the village of Nedstrand, on the southwestern coast of the Nedstrand peninsula in Norway, the second child of a local pastor. Although Norway is counted among the wealthiest countries in Europe today, at the time, its economy was still relatively undeveloped, and at the time, the country had yet to produce any scientists of note. Abel was to be the first. In his youth, Abel had the good fortune to come into contact with an excellent mathematics teacher at the Cathedral School where he was attending lessons who recognized his talents and encouraged him to read advanced works in the subject by Euler, Lagrange, Gauss, and others. Abel took an interest in the problem of fifth-degree polynomials and believed at first that he had managed to discover a solution. However, there were no mathematicians in Norway at the time capable of confirming or disconfirming his result. Abel sent a manuscript containing his solution to a well-respected professor of mathematics in Copenhagen, who could not find any faults with it, but who asked Abel to produce a concrete numerical example demonstrating his method. As he was preparing his reply, Abel himself discovered a mistake in his argument, which led him eventually to his result on the nonexistence of a general solution by radicals for polynomials in degree greater than four.

After he had made a name for himself in mathematics, Abel decided to apply to the government for a stipend to study abroad in France and Germany, which was awarded only on the condition that he first devote 2 years to the study of

French and German. At the age of 23, shortly after the completion of his university studies, Abel finally obtained permission to travel and made his way to Berlin, where he befriended the publisher August Leopold Crelle, who was preparing to publish a new mathematical journal entitled *Journal für die reine und angewandte Mathematik* (Journal for Pure and Applied Mathematics), to which Abel contributed seven articles in the first year of its distribution, including an article on his results in the theory of polynomials. This was one of the first journals published outside the remit of any academy and is one of the oldest mathematical journals still in publication today. At the same time, Abel came to realize that various mathematical results he had sent to Gauss and other prominent mathematicians had not yet made any impression and made a roundabout way around Göttingen en route to Paris. In Paris, Abel submitted what he believed to be his most important and impressive result to the French Academy of Sciences, where it was to be reviewed by Cauchy. His work was almost completely unknown in France at the time, however, and Cauchy, like Gauss before him, put the work aside and neglected to review it.

After 2 years abroad, Abel returned to Norway. He had exhausted his funds and contracted pulmonary tuberculosis in the course of his travels. For the remainder of his short life, Abel supported himself via a single loan and engagements as a private tutor. In the meantime, a small ensemble of sympathetic European mathematicians began to recognize the value of his work, which contained a great many deep and significant results in addition to the celebrated theorem on polynomial solutions. Abel also worked on the theory of elliptic integrals and in the process discovered a class of functions called elliptic functions. Subsequently, he was able to establish the double periodicity of such functions. The theory of elliptic functions, which was independently worked out by the German mathematician Carl Jacobi (1804–1851) around the same time, ranks among the brightest gems of nineteenth-century complex analysis. In the spring of 1829, Abel was finally awarded a position as a professor of mathematics at the University of Berlin, on the recommendation of his friend August Crelle. Sadly, Abel died of tuberculosis 2 days before the news arrived.

After Abel's death, the mathematical community gradually came to recognize the significance of his work. Today, he is remembered as among the greatest nineteenth-century mathematicians and a true pioneer in the modernization of mathematical thought. In Norway, his life and achievements have been commemorated on postage stamps, on banknotes, and on coins. The generation that followed saw a great flowering of artistic and intellectual talent in Norway. First, there was the great playwright Henrik Ibsen (1828–1906), born in the year before Abel died. Then in short succession appeared the composer Edvard Grieg (1843–1907), the painter Edvard Munch (1863–1944), and the explorer Roald Amundsen (1872–1928), who led the first expedition to the South Pole, via sled.

In the course of his work on the quintic polynomial, Abel considered various special cases of equations with solutions in radicals, considerations which amounted to the introduction into mathematics of what came to be known in modern algebra as an algebraic number field. Gauss had proved already at the end of the previous century that every polynomial with complex coefficients has a complex root, a result

Fig. 7.5 Évariste Galois, the "Rimbaud of mathematics"

generally referred to as the fundamental theorem of algebra, and it was natural in light of Abel's work for mathematicians to ask what sort of equations were sufficient to express such roots. In particular, what exactly are the conditions according to which a given polynomial equation has solutions in radicals? This question was settled conclusively by yet another mathematician with a short and tragic biography: Évariste Galois (1811–1832). In the course of 2 years following Abel's death, Galois was able to establish the necessary and sufficient conditions for the solvability of polynomial equations by radicals (Fig. 7.5).

The main insight that Galois pursued was to investigate the complete collection of roots of a given polynomial and in particular their behavior under permutations. As an example, suppose x_1, x_2, x_3, and x_4 are the four roots of some fourth-degree polynomial. Exchanging the first two of these roots corresponds to a certain permutation on four letters, expressed mathematically by the array

$$P = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_3 & x_4 \end{pmatrix}.$$

Given two such permutations, define their product as the new permutation obtained by first applying one and then the other. Then if one considers the set of all possible permutations on some fixed number of letters, this product rule satisfies three important properties: (1) it is associative: $(P_1 P_2) P_3 = P_1 (P_2 P_3)$ for all choices of permutations P_1, P_2, P_3, (2) there exists an identity element P_0 such that $PP_0 = P_0P = P$ for every choice of permutation P given by the permutation that leaves every letter unchanged, and (3) every permutation is invertible; that is, there is some other permutation that undoes the action of any given permutation, such that the composition of the two is the identity permutation.

In modern terminology, a set of elements satisfying the above conditions is called a group (and if the group operation is also commutative, then it is called an abelian group). The group of permutations on the roots of a degree n polynomial is written S_n. Galois designated certain subgroups having desirable properties as normal subgroups and investigated the largest normal subgroup of a given group. According to Lagrange's theorem, the order of a finite group must be a multiple of the order of any of its subgroups, where the order of a group or subgroup means the number of its elements. The miracle of Galois theory is that a polynomial of degree n is solvable in radicals if and only if the order of each factor group in the largest subnormal series (the final subnormal group is always the trivial group) of the permutation group S_n is prime. Such groups are called solvable groups. When $n = 3$, the largest normal subgroup series has three terms, and the orders $2, 3$ of the nontrivial factor groups are prime. When $n = 4$, the largest normal subgroup series has five terms, and again, the orders $2, 3, 2, 2$ of the nontrivial factor groups are prime. But when $n \geq 5$, the largest normal subgroup series has *three* terms, with nontrivial orders n and $n!/2$. The latter number is composite.

Galois was born in 1811 in a small town at the southern outskirts of Paris. His father was a republican and an active participant in the French Revolution who eventually became mayor of the town. Galois enjoyed in his childhood a rich education under the tutelage of his mother. When Galois was 18 years old, his father committed suicide in the course of the fallout of a local political dispute. A few days later, Galois tried and failed for the second time to obtain admission to the École Polytechnique. He matriculated instead to the less prestigious École Normale and was expelled the following year on the basis of his political activities. In the period that followed, he was also arrested several times and eventually imprisoned for 6 months. Not long after his release, Galois participated in a duel for reasons that remain murky but which seem to have been connected somehow to a tumultuous love affair. In any case, Galois was shot in the abdomen and died the following day and was buried in a common grave whose exact location is unknown. He was 20 years old.

Like Abel, Galois had the good fortune to come into contact with a gifted mathematics teacher in his high school years, who introduced to him the wonders of the mathematical universe. Galois abandoned his mathematics textbook and began to read in its place advanced contemporary works by Lagrange, Euler, Gauss, and Cauchy. He seems to have hit upon the idea of a group not long afterward. Unlike Abel, Cauchy lived in Paris and attended some of its most prestigious schools. He ought to have been able to avoid the neglect suffered by Abel. Unfortunately, the various papers and results that he submitted to the Academy of Sciences were either ignored or otherwise lost. On the night before the duel that took his life, Galois became convinced of his impending death and spent the night composing a letter to which he attached several of his mathematical manuscripts. It is this letter that forms the beautiful legacy that Galois gave to the mathematical future. In total, Galois published a single brief article in his lifetime, and the works assembled after his death comprise a mere 60 pages.

Today, the work of Galois is recognized as the first chapter in the story of modern algebra. He brought to its complete resolution the problem of solvability by radicals with which mathematicians had been engaged for more than 300 years. But more importantly, Galois introduced to mathematics the concept of a group and thereby initiated a profound revolution in the contents and methods of algebra. In the subsequent development of mathematics and the natural sciences, groups have come to play an increasingly significant role in topics as diverse as the structure of crystals, the theory of fundamental particles, and quantum mechanics. The rapid rise in prominence of group theory in relation to the other sciences is illustrated by the following remarks. At the very start of the twentieth century, in the course of curriculum discussions among Princeton physicists and mathematicians, it was proposed that group theory could be omitted, since no use for it had yet been found in physics. But within the space of the following 20 years, no fewer than three monographs were published on the new science of quantum mechanics in terms of group theory. Meanwhile, the work of Abel and Galois and others prompted algebraists to shift their attention away from questions about the solutions to specific equations and turn instead toward development and innovation inside mathematics.

Another mathematician of the same generation worthy of particular mention is Joseph Liouville (1809–1882), born 2 years before Galois. Liouville entered the École Polytechnique at the age of 16 and subsequently earned a position there as an assistant professor. He is remembered for pioneering work on rational approximations of algebraic numbers and the theory of transcendental numbers. Algebraic and transcendental numbers are defined as follows: a complex number is called an algebraic number if it is the root of a nonzero polynomial in one variable with rational complex numbers; a complex number that is not an algebraic number is called a transcendental number. The concept of a transcendental number had first been introduced by Euler in his seminal text *Introduction to Analysis of the Infinite* (1748), and Liouville provided the first proof that such numbers actually exist almost a century later in 1844. In particular, Liouville used power series methods to obtain infinitely many transcendental numbers known today as Liouville numbers, the most famous of which is the Liouville constant

$$\sum_{n=1}^{\infty} \frac{1}{10^{n}} = 0.1100010000\cdots.$$

In 1873, the French mathematician Charles Hermite (1822–1901) proved that Euler's number $e = 2.7182818\cdots$ is a transcendental number, and in 1882, the German mathematician Ferdinand von Lindemann (1852–1939) proved that the number π is also transcendental. Lindemann is a lesser known figure at the center of the story of modern German mathematics: he completed his doctoral dissertation under the supervision of the great mathematician Felix Klein and went on to supervise the doctoral work of such notable figures as David Hilbert and Hermann Minkowski, who obtained their degrees in the same year as one another.

There remain to this day many open questions in the theory of transcendental numbers. For example, nobody has been able to prove that the sum $\pi + e$ is transcendental, or even irrational, although it is widely assumed to be. Similarly, it is unknown whether or not the Euler-Mascheroni constant

$$\gamma = \lim_{N\to\infty} (\sum_{n=1}^{N} \frac{1}{n} - \log N) = 0.5772156649\cdots$$

is irrational and, if so, whether it is algebraic or transcendental.

The Quaternions of William Rowan Hamilton

After Galois had introduced the concept of a group, the next major development in the history of modern algebra was the discovery of the system of quaternions. Although the impact of the quaternions was not as deep or far reaching as the group theory of Galois or the theory of Abelian elliptic functions developed by Abel, nevertheless, they represent a revolutionary moment in the history of algebra as the first number system to come equipped with a noncommutative multiplication operation. They represent also the first contribution to this chapter of the history of mathematics from the United Kingdom, which in the nineteenth century included Ireland. In fact, for several generations following the death of Newton, the mathematical stage in Europe had been predominantly occupied by mathematicians from France and Germany. To the delight of the English-speaking world, this changed in the nineteenth century via the work of William Rowan Hamilton (1805–1865), who introduced the quaternions.

Hamilton was born in Dublin in 1805. His father worked as a lawyer, and his mother was a woman of substantial learning. In any case, Hamilton was sent at an early age to a somewhat distant village to live with his uncle, a local priest. This uncle had a background in linguistics, and Hamilton developed under his care a great facility for languages as well. By the age of 13, Hamilton had acquired already as many languages as his age, including Latin, Hebrew, Arabic, Persian, Sanskrit, Hindustani, Bengali, Syriac, and Malay. He was preparing at the time to take up the study of Chinese, but the death of his parents and other events in his personal life caused him to turn his attention away from languages and focus instead on mathematics.

Hamilton appears to have been almost entirely self-taught as a mathematician. He was able to quickly master analytic geometry and analysis and went on to read the *Principia Mathematica* by Newton and the *Celestial Mechanics* by Laplace. Hamilton discovered a mathematical error in the latter text, which attracted early attention to his abilities. The following year, and despite having received almost

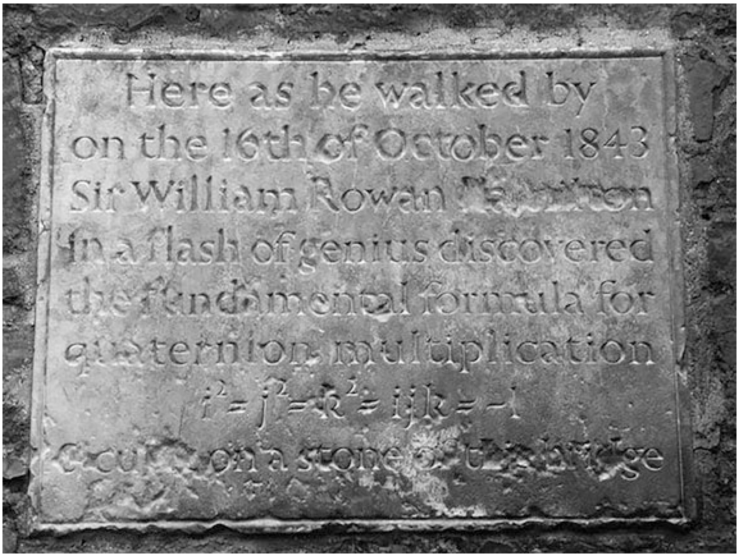


Fig. 7.6 Broome Bridge in Dublin, where Hamilton wrote down the fundamental formula for quaternions

no formal training, Hamilton was awarded admission to Trinity College in Dublin, having obtained first marks in his entrance examinations. By the time he ought to have graduated, Hamilton had already invented a new system of geometrical optics. In fact, he never completed his degree as he was instead appointed directly to the Andrews Professorship of Astronomy at the University of Dublin. He was not yet 22 years old. Although Hamilton shares with Abel and Galois a biography characterized by early brilliance and accomplishment, his fate in later life was more happy than theirs. In particular, Hamilton enjoyed many honors and recognitions throughout his life: at the age of 30, he was knighted, and 2 years later, he was elected to the president's chair of the Royal Irish Academy (Fig. 7.6).

Although he made a name for himself early on for his contributions in physics and astronomy, Hamilton viewed pure mathematics as the field to which he was most fully devoted. Nevertheless, recognition and achievement in the domain of pure mathematics came a bit more slowly than in the natural sciences. Hamilton devoted considerable energy in particular to the problem of extending the system of complex numbers, which led eventually to his discovery of the quaternions. In the early years of the nineteenth century, Gauss and other mathematicians had worked out a geometrical interpretation of the system of complex numbers: every complex number $a + bi$ was taken to represent what we now call a vector in the plane. This point of view was particularly fruitful from the perspective of physics, in which discipline a great many of the most fundamental properties take the form of vectors, including force, velocity, and acceleration. Such directed quantities obey the so-called parallelogram law of addition, which is identical in two dimensions to the addition law for complex numbers.

The particular advantage in interpreting vectors as complex numbers is that it facilitates the study of the behavior by proper purely algebraic means, without recourse to geometric figures. On the other hand, this method suffers from serious limitations: if, for example, several forces acting on some body in space fail to lie all in a single plane, there is simply no way to represent them separately by complex numbers. Rather some three-dimensional analogue of the system of complex numbers is required in this circumstance. If vectors in space are represented by Cartesian coordinates of the form (x, y, z), the problem is to determine what law the operations of addition and multiplication should take such that the system so obtained shares certain desired properties with the system of complex numbers. This is the problem to which Hamilton devoted so much of his attention.

In 1837, Hamilton published a paper in which he proposed that the use of the addition sign in the notation $a + bi$ for complex numbers was a matter of historical accident rather than mathematical necessity. He observed that it was possible to define complex numbers rather as ordered pairs (a, b) of real numbers equipped with operations of addition and multiplication defined by the laws

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \times (c, d) = (ac - bd, ad + bc)$$

Hamilton proved that the set of such ordered pairs is closed under the two operations so defined and that both operations are associative and commutative. His goal was to generalize the ordered pairs involved in this definition to arbitrary arrays of numbers retaining the basic properties of real and complex numbers. In this he succeeded only partially: after much effort, Hamilton discovered a generalization to arrays with four components rather than with three. Moreover, it was necessary in this case to abandon the commutative law for multiplication that had been taken for granted in arithmetic since antiquity. He called his new system the quaternions.

The generic quaternions has the form $a+bi+cj+dk$, where a, b, c, and d are real numbers and i, j, k satisfy the relations $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. These relations ensure that any two quaternions can be multiplied by one another according to the usual distributive law of arithmetic to obtain a new quaternion. For example, if $p = 1+2i+3j+4k, q = 4+3i+2j+k$, then

$$pq = -12 + 6i + 24j + 12k, qp = -12 + 16i + 4j + 22k.$$

Although $pq \neq qp$, the multiplication operation for quaternions does obey the associative law, as Hamilton himself verified. When Hamilton discovered this result in 1843, he opened wide a door to many possibilities for the future of algebra. From that time on, mathematicians enjoyed a newfound freedom to introduce new and exotic number systems.

It is necessary to point out that although the introduction of the system of quaternions was a significant moment in the history of mathematics, in many ways, the quaternions were not well suited to use in physics. Subsequent mathematicians and mathematical physicists found it more convenient to separate the first term in the definition of a quaternion from the remaining three, viewed as a single vector, and to redefine the operations on i, j, and k in terms of what we recognize today in modern vector analysis as the dot product and cross product of vectors. At the same time, the German mathematician and polymath Hermann Grassmann (1809–1877) discovered in 1844 a much more general system for working with arbitrary arrays of numbers. The abstract notion of linear algebra that Grassmann introduced subsequently became an object of considerable independent research and one of the cornerstones of modern mathematics.

The final years of Hamilton's life have a certain tragedy about them. Although Hamilton believed that quaternions held the secrets to the universe and ought to play a role in the development of nineteenth-century science analogous to the discoveries of Newton in the seventeenth century, in fact, the system of quaternions departed the scene not long after they were invented and was regarded for some time afterward as a mathematical curiosity. Nevertheless, the system of quaternions did enjoy a period of popularity during Hamilton's lifetime across the Atlantic in America, and the United States National Academy of Sciences included his name at the top of the list among their first round of foreign associates.

The history of matrix algebras, developed around the same time, is rather different. The English mathematician Arthur Cayley (1821–1895) invented the concept of a matrix in 1857, 8 years before Hamilton died, and defined the laws of their arithmetic in such a way as to identify matrix operations with compositions of linear transformations. He discovered that the addition of matrices is both associative and commutative and that their multiplication is associative and distributes across addition. On the other hand, the multiplication of matrices, like that of the quaternions, is noncommutative. For example, one has

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\neq\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

It is impossible to overstate the significance of the theory matrices and linear algebra, and in fact, it is primarily in connection with matrices that Hamilton's name lives in on mathematics. Specifically, there is the Cayley-Hamilton theorem:

> if A is an $n \times n$ matrix with entries drawn from a commutative ring, say the real or complex numbers, and if $f(\lambda) = \det(\lambda I_n - A)$ is the characteristic polynomial of A, then $f(A) = 0$ (the zero matrix).

In the year 1925, the physicist Max Born (1882–1970) discovered in collaboration with Werner Heisenberg (1901–1976) that the algebra of matrices was especially well suited to serve as a mathematical framework for the newly emerging theory of quantum mechanics. In this way, it came to be appreciated that the arithmetic of certain physical quantities is noncommutative. One consequence of the new formalism is the famous uncertainty principle. It is worth mentioning in passing that the name matrix was coined by the English mathematician James

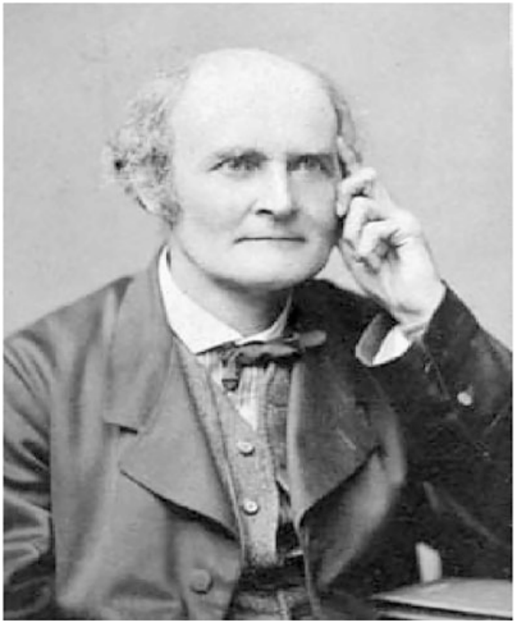


Fig. 7.7 British mathematician Arthur Cayley

Joseph Sylvester (1814–1897), who collaborated with Cayley on the theory of algebraic invariants, which played a role in the creation of both quantum mechanics and the theory of relativity. Sylvester studied in the United States for a period of a time and became an early pioneer in the development of mathematics in the New World (Fig. 7.7).

Cayley's father worked in Saint Petersburg as a merchant. Although Cayley was born in London, he spent his earliest years in Russia. His father intended for him to go into business, but Cayley exhibited a talent for mathematics from an early age, and one of his schoolteachers encouraged his father to allow Cayley to enroll at the University of Cambridge. These three mathematicians, Hamilton, Cayley, and Sylvester, inaugurated the first renaissance of pure mathematics in Britain since Newton. In addition to his achievements in mathematics, Cayley worked for some years as a lawyer, during which time he nevertheless maintained an impressive standard of mathematical research. There are some parallels to be drawn here with life of the great American poet Wallace Stevens (1879–1955), who served for much of his adult life as an executive at an insurance company. Cayley also took a keen interest in the education of women and played a role in bringing to an end the prohibition of their admission to Cambridge University (Fig. 7.8).

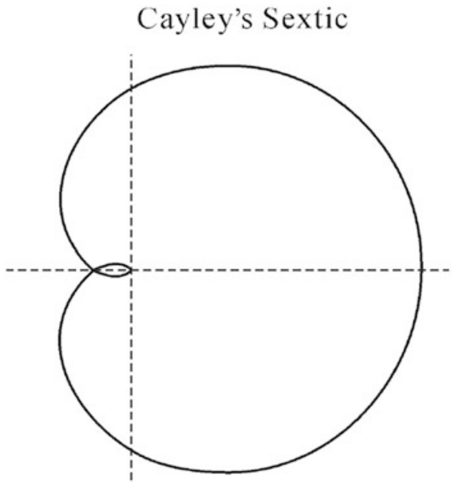


Fig. 7.8 Cayley's cardioid curve

A Revolution in Geometry

A Scandal in Elementary Geometry

During the same period that algebra was undergoing such tremendous change and modernization, a great revolution was also taking place in the ancient and venerable mathematical discipline of geometry. On the other hand, due to its rich history and close connection to human thought, the new developments in geometry were more difficult to accept. The story begins in Ancient Greece, where Euclid established the rigorous and logical standards of mathematical practice that would govern all its future development. For some two millennia afterward, the sanctity of his mathematical methodology and results was unassailable. So, for example, even as Newton was revolutionizing mathematics and physics with the invention of calculus, he followed the example of his teacher Isaac Barrow, who had praised Euclid extensively, and presented his new ideas in the old language of Euclidean geometry.

Although the analytic geometry of Descartes had proved an important update to the methods of geometrical research, it had not introduced any changes to the basic contents and presuppositions of Euclidean geometry, and Descartes himself was always careful to rework results obtained from his graphical methods into proofs in the Euclidean style. The philosophers of the period, including Thomas Hobbes (1588–1679), John Locke 1632–1704, Leibniz, and later G.W.F. Hegel (1770–1831), also each agreed from their various perspectives and analyses that the validity of Euclidean geometry constituted a clear and necessary truth. The most famous argument for this position was given by Immanuel Kant (1724–1804) in his influential treatise, *Critique of Pure Reason*. Kant argued that the nature of human

Fig. 7.9 David Hume, the agnostic

sense perception is mediated by spatial intuition and that spatial intuition necessarily imposes the structure of Euclidean geometry upon the world (Fig. 7.9).

Kant developed his arguments in response to the radical ideas of the Scottish philosopher David Hume (1711–1776). In the year 1739, Hume had published a remarkable work in which he denied that we can know with certainty any set of laws governing the behavior of the universe. In particular, since in his account scientific knowledge can come about only through direct experience, Hume denied the exactitude of Euclidean geometry as a description of reality.

In fact, the system of Euclidean geometry was not without certain deep fault lines from a purely mathematical perspective. Since its inception, there had been a question about its foundations that had perpetually puzzled mathematicians, namely, the problem of the parallel postulate. The parallel postulate occurs fifth and last among the postulates that Euclid sets out at the beginning of his *Elements of Geometry*, and its formulation is strikingly different in character from the four that precede it. It reads as follows:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

It is a convoluted and finicky statement, entirely lacking the simplicity and clarity of the other postulates. Moreover, many mathematicians felt that it has more the flavor of a theorem than an axiom, that there ought therefore to be some way to derive it from the others. This state of affairs prompted d'Alembert to refer to the parallel postulate in 1767 as the scandal of elementary geometry (Fig. 7.10).

Mathematicians endeavored to clear up this scandal along two main avenues of pursuit. In the first case, many mathematicians really did search for a way to prove it from simpler hypotheses, including the mathematicians Omar Khayyam and Nasir al-Din al-Tusi mentioned in the fourth chapter. Other mathematicians sought to

Fig. 7.10 Scottish mathematician John Playfair

settle the issue by reformulating the fifth postulate in some equivalent but simpler and more palatable version. The version that appears in most modern textbooks is associated with the eighteenth-century Scottish mathematician and physicist John Playfair (1748–1819). It reads as follows:

> In a plane, given a line and a point not on it, exactly one line parallel to the given line can be drawn through the point.

It is necessary to point out however that this version of the fifth postulate in fact predates Playfair by many centuries. The Greek astronomer Ptolemy had presented perhaps the first attempt at a proof of the parallel postulate in the second century, and several centuries later, the philosopher Proclus (410–485) observed in his commentary on *Elements* that Ptolemy had implicitly assumed Playfair's axiom in his proof. This shows that it was not Playfair who first proposed the axiom that bears his name.

In the centuries after Proclus wrote his commentary, the text of *Elements* largely disappeared for a time from Europe and survived only in Arabic. A few Latin translations appeared during this time as well, but it was really only after the end of the medieval period that the story of the parallel postulate picks up again in Europe. In the time of Newton, Wallis made a thorough inquiry into the issue and proposed various proofs, but each of his proofs was found ultimately to rest on some axiom equivalent to the parallel postulate or otherwise to involve some other mistaken inference. Afterward, the only new and interesting developments prior to the eighteenth century were contributed by three relatively unknown mathematicians.

In fact, the methods pursued by these three mathematicians were not essentially different than those of Omar Khayyam and Nasir al-Din al-Tusi. They all considered a quadrilateral, say $ABCD$ in which the angles $\angle A = \angle B$ are assumed to be right angles, and endeavored by the method of contradiction to rule out case by case the possibility that the angles $\angle C = \angle D$ are acute or obtuse. But none of them were able to make any progress on the obtuse angle case. Slowly, the idea began to take hold in Europe that it might not in fact be possible to prove the parallel postulate on the basis of the other postulates. Notably, various mathematicians in Italy, Switzerland, and Germany tiptoed close to the precipice of non-Euclidean geometry, but each of them recoiled in turn from the prospect of such an apparently absurd proposition.

The Arrival of Non-Euclidean Geometry

The several mathematicians just mentioned were close to a revolution in mathematics that would prove as substantial as the invention of analytic geometry by Descartes and Pascal or the invention of calculus by Newton and Leibniz. The decisive step however remained for a later generation to take, when, in a remarkable turn of events, three mathematicians hailing from three different countries, and working completely independently from one another, established once and for all the foundations of non-Euclidean geometry. These were Gauss in Germany, János Bolyai (1802–1860) in Hungary, and Nikolai Lobachevsky (1792–1856) in Russia. This of course is the same Gauss whose formidable name has already come up in a variety of contexts. The remaining two mathematicians were both newcomers to the mathematical scene when they completed the work for which they are primarily remembered today.

All three of them began their investigations by way of Playfair's axiom. There were three cases to resolve: given a line in the plane and some point not lying on it, it is possible to assume either that there exist several lines passing through the given point and parallel to the given line, or one line, or no lines. In fact, these three possibilities correspond one to one to presence of either right, acute, or obtuse angles in the quadrilaterals discussed above. Each of the three believed that the acute angle case ought to yield a consistent geometry, although they did not give a formal proof of its consistency. Rather, they provided proofs of various geometric and trigonometric results on the assumption of the acute angle hypothesis. In this way, a new geometry was born (Fig. 7.11).

Here, we provide a simple example of the new results. Consider any arbitrary quadratic curve, for example, an ellipse, and the area it encloses. Such a curve can be taken to represent a geometric space satisfying the hypotheses of Lobachevsky. The line connecting any two points A and B on the curve (points at infinity) defines a straight line in the space. Then for any third point P lying inside the ellipse or on its perimeter, but not in the line joining AB, the two lines joining AP and BP intersect the ellipse in points C and D, respectively. Then according to the theorem of Desargues discussed in the fifth chapter, the two lines APC and BPD formed with the points at infinity are both parallel to the line AB and pass through the points P (Fig. 7.12).

In this example, it is not actually the case that the first four Euclidean postulates are completely satisfied. This defect however is readily emended. It is necessary only to replace the straight lines in the example with curves selected so as to satisfy the postulates and meeting the ellipse at right angles. In this way, it really is possible

Fig. 7.11 Likeness of Lobachevsky on a Soviet postage stamp

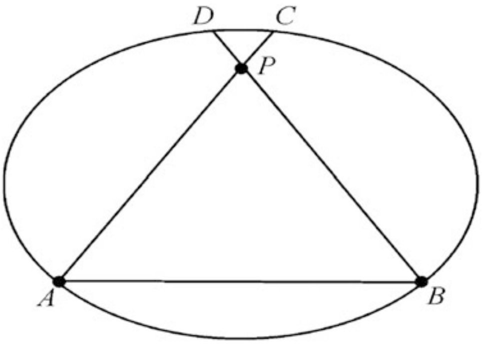


Fig. 7.12 Geometric representation of Lobachevskian space

to represent an arbitrary number of straight lines passing through a given point and parallel to a given line.

It was Gauss who coined the term non-Euclidean geometry, so that every system of geometry retained some association with the name of its ancient progenitor. Gauss himself however was reluctant to make his results public and shared them instead only privately with a few close colleagues, recognizing perhaps that they brought him into conflict with the philosophical ideas of Kant, which were hugely popular at the time. Gauss himself said only that he feared the clamor of the Boeotians if he were to openly express his radical ideas. This created the opportunity for two young mathematicians of the next generation to earn a place for themselves in the history of mathematics as the inventors of non-Euclidean geometry and to earn this supreme honor for their motherlands (Fig. 7.13).

János Bolyai belonged to the same generation as Abel. He was born in a small town in Transylvania, today a county in Romania, although at the time it had belonged to Hungary for more than eight centuries. His father Farkas Bolyai had

Fig. 7.13 Likeness of János Bolyai on a Hungarian postage stamp

studied mathematics at Göttingen, where he became lifelong friends with Gauss, before eventually returning to Transylvania and taking up a teaching position for mathematics and sciences, which he held for the remainder of his life. He instructed his son in mathematics, who mastered calculus and the elements of analytical mechanics by the age of 13. At 16, Bolyai enrolled at the Imperial and Royal Military Academy in Vienna. After graduating, Bolyai entered military service, but retained an active interest in mathematics and in particular the foundations of geometry (Fig. 7.14).

Although Bolyai's father always encouraged him in his mathematical research, he resolutely opposed any investigations into the parallel postulate, with which he too had become obsessed, writing: "You must not attempt this approach to parallels. I know this way to the very end. I have traversed this bottomless night, which extinguished all light and joy in my life. I entreat you, leave the science of parallels alone...." But Bolyai could not be dissuaded, and in 1823, he took advantage of a winter vacation spent at home to present his father with a treatise on the theory of parallel lines. Some 6 years later, Bolyai's work appeared as an appendix to a textbook written by his father. This appendix eventually made its way to Gauss, who was impressed by its genius, but replied only after a long time with the claim that he had in fact hit upon the same theory some 30 years earlier.

In any case, the appendix did not make much impression on the mathematical community at the time. In the years that followed, Bolyai suffered a debilitating accident and retired from the military. He returned home, where he suffered misfortune after misfortune—poverty and, like his father, a disastrous relationship and marriage. When he learned that the Russian mathematician Lobachevsky had published results in the theory of parallel lines somewhat earlier than his own treatise had appeared in print, Bolyai abandoned mathematics altogether

Fig. 7.14 Home of Farkas Bolyai in Göttingen; photograph by the author

and devoted his final years to private efforts in the composition of literary and philosophical works. Bolyai died forgotten and dejected after a protracted illness. Some 30 years later, the Hungarian government restored his grave and erected a statue in his honor. Today, the International János Bolyai prize is still awarded in his honor every 5 years by the Hungarian Academy of Sciences. Its notable recipients include the mathematicians Henri Poincaré, David Hilbert, and Albert Einstein.

We turn now to Lobachevsky, the first mathematician to publish any results in non-Euclidean geometry. He was born 10 years earlier than Bolyai in the city of Nizhny Novgorod, some 400 kilometers to the east of Moscow. His father worked as clerk in a land surveying office and died when Lobachevsky was still a child. His mother, who was diligent and open minded, traveled with her three children to Kazan, where Lobachevsky attended Kazan Gymnasium. Four years later, Lobachevsky received a scholarship at the age of 14 to attend Kazan University. Later Kazan University would be recognized as one of the most prestigious educational institutions in Russia, alongside the universities of Moscow and Saint Petersburg, but at the time, it was still new and unknown. Lobachevsky remained in Kazan for the rest of his life.

Like so many mathematicians discussed already in this chapter, Lobachevsky had the good fortune to encounter an exceptional mathematics teacher during his schoolboy years, who exerted a profound influence on his development. He began to read primary texts in mathematics and began to exhibit a remarkable talent. Notwithstanding a headstrong personality that led to occasional violations of school discipline, Lobachevsky excelled in his studies, and his teachers gave him their support. After he completed his master's degree in physics and mathematics, Lobachevsky stayed on as a lecturer at Kazan University. Eventually, he was promoted to a full professor and occupied various additional positions of honor on the basis of his administrative abilities and mathematical achievements. Just at the time that Leo Tolstoy enrolled as a student in the Department of Oriental Languages at Kazan University, Lobachevsky had ascended through the ranks to the position of rector. Later, Vladimir Lenin enrolled in the Law Department of the same university.

Although Lobachevsky enjoyed a brilliant academic career, the significance of his work on non-Euclidean geometry was not widely appreciated at first. The state of the sciences in Russia at the time was still relatively undeveloped, and Lobachevsky struggled to find an audience for his ideas. In 1823, Lobachevsky completed his first major work, entitled *Geometriya*, one part of which contained the germ his conception of non-Euclidean geometry. Three years later, he presented an expanded account of his ideas at the Kazan department of physics and mathematics where it was regarded as fantastical and absurd and attracted no further attention; the manuscript was subsequently lost. After another 3 years, he succeeded in publishing an article on the subject called *A Concise Outline of the Foundations of Geometry* in the university journal *Kazan Messenger*, news of which slowly made its way to mathematicians in Western Europe.

In any case, a new universe of geometry was born and eventually came to be called Lobachevskian geometry, in spite of the contemporary contributions of Gauss and Bolyai to its development. Lobachevsky himself referred to his new system of geometry as imaginary geometry, while Bolyai used the name absolute geometry for his. The influence of non-Euclidean geometry was slow to spread, as people viewed the entire subject with considerable suspicion at first. But after Gauss died and the contents of notebooks pertaining to non-Euclidean geometry were made public, the imprimatur of his tremendous status and fame forced the issue, and the widespread feeling that the laws of geometry were single and immutable began to waver.

We turn at last to the so-called prince of mathematics Gauss himself. Johann Carl Friedrich Gauss was born in 1777 in Brunswick in Northern Germany, the only child of poor farmers. He was a child prodigy, and legend has it that he discovered a mistake in his father's account books at the age of 5. According to another story, when Gauss was 9, one of his teachers invited the class to add up all the numbers between 1 and 100 to pass some time in peace, and Gauss arrived at the correct answer almost immediately by working out an explicit expression for such sums. In any case, his extraordinary talent attracted the attention and subsequent financial support of the Duke of Brunswick. Gauss was able to attend the University of Göttingen, where he eventually became a professor of astronomy and director of the observatory (Fig. 7.15).

Gauss exhibited also a talent for languages, and he hesitated at first in deciding between linguistics and mathematics before eventually devoting himself wholeheartedly into mathematical research. His breakthrough came at the age of 19, when Gauss managed to use number theoretical methods to solve an outstanding problem with a 2000-year history: the construction of a regular polygon with 17 sides using only a straightedge and compass as permitted by the rules of classical geometry. Afterward, Gauss maintained an astonishing pace of discovery and invention for some 50 years. In 1801, when he was only 24 years old, Gauss

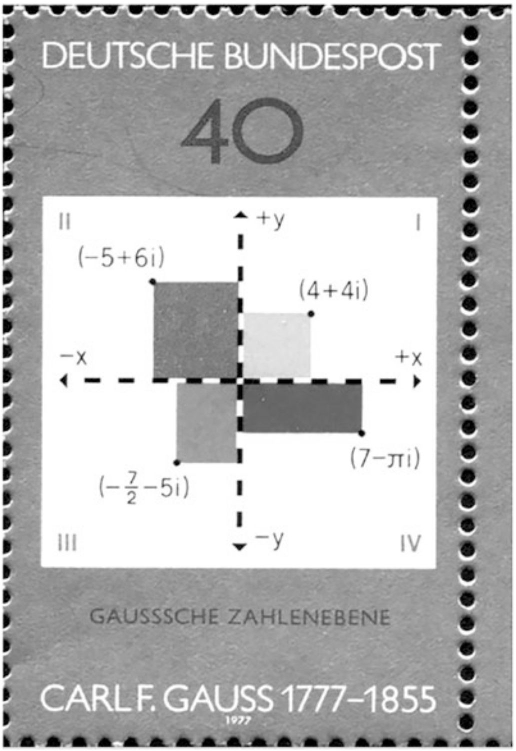


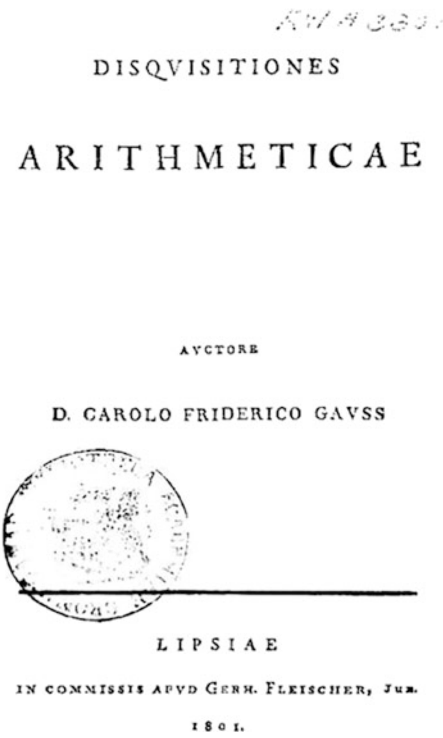
Fig. 7.15 Postage stamp featuring the Gaussian integers

published his *Disquisitiones Arithmeticae*, which ushered in the modern era in the history of number theory. In it, Gauss discussed among other things the problem of regular polygon constructions, introduced the theory of modular arithmetic and the congruence symbol, and presented two proofs of the beautiful law of quadratic reciprocity (Fig. 7.16).

In his later years, Gauss contributed continuously to just about every branch of mathematical research. He was also one of the most accomplished physicists and astronomers of his generation. Nevertheless, number theory remained without doubt his most beloved subject. He referred to number theory as the queen of mathematics and said of it that "the enchanting charms of this sublime science reveal themselves in all their beauty only to those who have the courage to go deeply into it." Perhaps this preference for the purity and elegance of number theory was another reason that Gauss was reluctant to make public his results in non-Euclidean geometry.

Riemannian Geometry

After the introduction of non-Euclidean geometry, it still remained to verify formally its internal consistency and sort out its actual mathematical significance.


DISQVISITIONES

ARITHMETICAE

AVCTORE

D. CAROLO FRIDERICO GAVSS

LIPSIAE

IN COMMISSIS APVD GERH. FLEISCHER, IUN.

1801.

Fig. 7.16 Title page of *Disquisitiones Arithmeticae* by Gauss

Lobachevsky devoted considerable effort to this endeavor, but he was unable to settle the issue successfully. But 2 years before Lobachevsky died, a younger mathematician was able to set out an extraordinarily general system of geometry that expanded upon the work that Lobachevsky and others had initiated. This was Bernhard Riemann (1826–1866), one of the greatest mathematicians in the history of Germany and the world, and the system of geometry he developed included both Euclidean geometry and the new geometry of Lobachevsky as special cases. Before Riemann, mathematicians assumed that the obtuse angle hypothesis mentioned in the preceding section and the hypothesis that straight lines can be extended indefinitely in space contradict one another and for this reason neglected to investigate the former hypothesis (Fig. 7.17).

Riemann took as his starting point a subtle logical distinction between the mathematical ideas of the infinite on the one hand and the arbitrarily large on the other. He denied that the requirement that a line admit arbitrary extension is equivalent to the requirement that it have infinite length. Rather, such a line is determined by an absence of terminal points or other such bounds. On the basis of this distinction, Riemann was able to show that it is possible to establish also a consistent system of geometry under the obtuse angle hypothesis, known today as Riemannian geometry or elliptical geometry. As an example, the great circles on the surface of a sphere can be regarded as the straight lines. It is easy to see that in this geometry any two straight lines necessarily intersect in two points.

Fig. 7.17 Bernhard Riemann, the greatest of Gauss's students

Riemann carried out his research on foundations established by Gauss in the topic of intrinsic differentiable geometry on curved surfaces, a particularly fertile source of mathematical invention in the nineteenth century. In the original form given to it by French mathematician Gaspard Monge, differential geometry treats curved surfaces as embedded in ordinary Euclidean space. Gauss however presented a new and radical perspective in his treatise *General Investigation of Curved Surfaces*, completed in 1828. Gauss proposed to consider curved surfaces as spaces in their own right without any reference to some ambient space. He discovered that many of their properties, including the distance between points, the measurement of angles, and the curvature itself, could be determined from within the space by purely intrinsic considerations. Incidentally, the Chinese mathematician Shiing-Shen Chern (1911–2004) made fundamental contributions to the later development of this subject and is sometimes referred to for this reason as the father of modern differential geometry. Chern proved a generalization to higher-dimensional Riemannian manifolds of the Gauss-Bonnet theorem relating the Euler characteristic of a two-dimensional surface to its curvature. His student Shing-Tung Yau (b. 1949) further proved a famous conjecture due to Eugenio Calabi (b. 1923) concerning the existence of Riemannian metrics of a certain form for complex manifolds subject to certain conditions, a result of deep significance for superstring theory in modern physics and which constituted one piece of a rich body of research that earned Yau a Fields Medal in 1983 (Fig. 7.18).

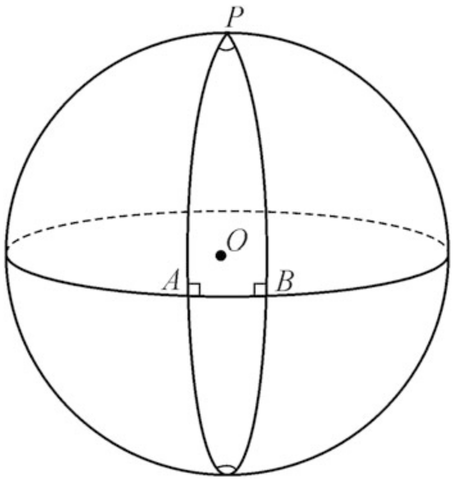


Fig. 7.18 Geometrical representation of Riemannian geometry

In 1854, Riemann delivered in Göttingen a lecture entitled *On the hypotheses which underlie Geometry* in order to qualify to teach as a *privatdozent*. The topic had been selected specifically for him by his advisor Gauss from a list of three that Riemann had originally proposed. In this lecture, Riemann introduced for the first time the concept of an n-dimensional space for arbitrary n, which he called a manifold, and extended the work that his teacher Gauss had initiated into the properties of curvature to such higher-dimensional spaces. Riemann defined a point in a manifold as an ordered n-tuple of numbers or coordinates. He generalized the notions of distance, length, and angle to arbitrary manifolds. Riemann took advantage of these generalizations to define the curvature of a submanifold at a given point within it in terms of properties contained in the definition of the manifold itself. He was particularly interested in the properties of spaces with constant curvature, determined by the property that the curvatures at any two points in the space are identical.

In the case of three-dimensional spaces, there are three possibilities to consider:

the curvature is either a positive constant, or a negative constant, or zero.

Riemann observed that the latter two cases correspond, respectively, to the geometries of Lobachevsky and Euclid. The first case corresponds to the new geometry he himself had invented, in which there exist no straight lines parallel to any given line through a given point not contained in it. In light of this, it is fair to say that Riemann was the first mathematician to really grasp the generality and significance of non-Euclidean geometry.

There remained however a technical problem—it was still necessary to prove formally the consistency of non-Euclidean geometry and the independence of the parallel postulate from the other Euclidean axioms. The matter was soon settled however by mathematicians working independently in Italy, England, Germany, and France. They each adopted the same basic methodology: specifically, they constructed concrete models of abstract non-Euclidean spaces embedded within familiar Euclidean geometry. From this, it follows that any inconsistency in non-Euclidean geometry entails a corresponding inconsistency in Euclidean geometry itself. That is, if Euclidean geometry is consistent, so too is non-Euclidean geometry. They closed the book entirely on questions of the legitimacy of non-Euclidean geometry.

Many of theorems of Euclidean geometry carry over directly into the new geometries discovered by Riemann and Lobachevsky; for example, certain simple congruence criteria for triangles are valid in every system. On the other hand, certain properties of non-Euclidean geometry fly completely in the face of common human experience and intuition. In Riemannian geometry, for example, there are infinitely many lines perpendicular to a given line at a given point, and it is possible for two straight lines to define the border of a closed plane region. Moreover, there exist no parallel lines in Riemannian geometry, and the indefinite extension of a straight line produces a line of finite length. All of this is easy to see by analogy with the surface of a sphere: the curve formed by the shortest path connecting two points is the arc of a great circle passing through the two points, with the center of the sphere as its center.

According to this analogy, a triangle on the surface of a sphere is defined as the region set out by three great circles. It is not difficult to discover that the interior angles of such a triangle add up to more than the 180 degrees of ordinary Euclidean geometry. In fact, it is possible for two sides of a triangle to both be perpendicular to the third. It is an interesting fact that in Lobachevskian geometry, precisely the opposite is true: the interior angles of any triangle add up to less than 180 degrees. Moreover, the deficiency increases as the area of the triangle increases in the case of Lobachevskian geometry and decreases in the case of Riemannian geometry. A few more final quirks of non-Euclidean are worth mentioning: in Lobachevskian geometry, similar triangles are always congruent, and parallel lines are separated by a distance that becomes smaller and smaller in one direction and larger and larger in the other (Fig. 7.19).

Riemann was born in 1826 in a small village in the Kingdom of Hanover. His father was a poor Lutheran pastor; his mother was the daughter of a court magistrate. She died due to financial hardship and malnutrition while he was still young. Riemann began his education under the direction of his father and took a particular interest the history of Poland and the hardships endured by its people, which inspired in him a deeply felt sense of compassion. He also became obsessed with arithmetic and used to invent problems in mathematics for his brothers and sisters. When he was 14, Riemann moved to Hanover to live with his grandmother and attend the lyceum. His teachers were amazed by his mathematical ability, and the principal allowed him to borrow some books on the subject. He quickly mastered Legendre's monumental treatment of number theory and Euler's calculus texts. Nevertheless, he planned at the time to pursue a career as a missionary.

Riemann enrolled at the University of Göttingen when he was 19 in order to study theology and philosophy, but his attention was diverted almost immediately by lectures in mathematics delivered by Gauss, and he quickly changed his major to

Fig. 7.19 Riemann's home in Göttingen; photograph by the author

mathematics. His father approved of the change, and Riemann eventually transferred to the University of Berlin, where he had the opportunity to study under Carl Jacobi and Peter Gustav Lejeune Dirichlet (1805–1859). Riemann stayed in Berlin for 2 years to study mechanics, algebra, number theory, and analysis before he returned to Göttingen and completed his degree. He was 23 by that time and stayed on at Göttingen to pursue doctoral research with Gauss as advisor. He would quickly prove to be Gauss's greatest student, and his doctoral thesis on the theory of functions of a complex variable earned him his teacher's highest praise.

In 1859, following the death of Dirichlet, Riemann became the head of mathematics at Göttingen. Following his promotion, he married, and he and his wife had a daughter. Not long afterward, he contracted pleurisy and tuberculosis and died at the age of 40 in the course of a journey to Italy in Selasca on Lake Maggiore, where he was buried. In his short life, Riemann made deep and radical contributions to many areas of mathematics which exerted a profound influence on the subsequent directions of geometry and analysis. His bold ideas about space have since found a place among the mathematical cornerstones of modern theoretical physics and paved the way for the development of relativity theory in the twentieth century. Of the many mathematical statements and concepts named after Riemann, the most famous and challenging is the famous Riemann hypothesis.

The Riemann hypothesis concerns the distribution of zeros of the so-called Riemann zeta function, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for complex numbers s such that the series converges, and by analytic continuation for all other complex numbers. It is not too difficult to prove that the Riemann zeta function satisfies $\zeta(s) = 0$ for negative even integers $s = -2, -4, -6, \ldots$. These are called its trivial zeros. Riemann conjectured in 1859 that the nontrivial zeros all lie on the line $x = \frac{1}{2}$ in the complex plane. The implications of this conjecture extend like a thread across all of number theory and the theory of functions. Already Euler had discovered identities that relate the Riemann zeta function to prime numbers. Today, it is universally regarded as the deepest and most challenging open problem in mathematics, and it seems that no one believes its resolution is yet close at hand. According to mathematical folklore, someone once asked the German mathematician David Hilbert what his first question would be if he went to sleep and woke up 500 years in the future. He replied, "has the Riemann hypothesis been proven?".

A New Era of Art

Edgar Allan Poe

In the first month of 1809, when Hamilton was still 3 years old, learning to read English and carry out a little basic arithmetic, and preparing to embark upon his studies in Latin, Greek, and Hebrew, when Abel and Bolyai were each 6 years old and already showing the first signs of their prodigious mathematical talents, the American poet Edgar Allen Poe was born in Boston. At that time, America was still a young nation of immigrants and had not yet produced any mathematicians, but already several poets of note had been born there, including Ralph Waldo Emerson (1803–1882) and Henry Wadsworth Longfellow (1807–1882). Poe's parents were both actors. His father was prone to drinking and gambling and abandoned the family not long after Poe was born. His mother died the following year. Poe was an orphan by the age of 3. He was taken in and raised by a childless couple living in Virginia, from whom he received the name Allan. In this respect, his life bears some early similarities to that of Hamilton, who was also raised far from home from an early age (Fig. 7.20).

The Allan family moved to Britain when Poe was 6, and he stayed with them there and attended school for 4 years before they all returned to Virginia. The Allans argued frequently, and Poe was a melancholic child. His performance in school was

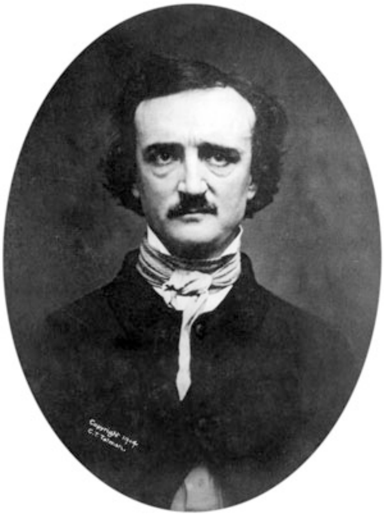


Fig. 7.20 Edgar Allan Poe, father of modernist literature

not altogether bad, although he fell in love with the mother of a classmate, to whom in 1831 dedicated the poem *To Helen*:

Helen, thy beauty is to me
Like those Nicean barks of yore,
That gently, o'er a perfum'd sea,
The weary way-worn wanderer bore
To his own native shore.

Although this poem is not representative of the best of his mature style, it explores already an ideal of feminine beauty to which Poe would return again and again. He would later write in his essay *The Philosophy of Composition*:

> ... I asked myself —'Of all melancholy topics what, according to the universal understanding of mankind, is the most melancholy?' Death, was the obvious reply. 'And when,' I said, 'is this most melancholy of topics most poetical?' From what I have already explained at some length the answer here also is obvious —'When it most closely allies itself to Beauty: the death then of a beautiful woman is unquestionably the most poetical topic in the world' ...

Although this melancholic way of thinking would perhaps not be out of place among the words of a twentieth-century film director, in the early years of the nineteenth century, they expressed a radical view of art. Consider as a point of comparison the transcendentalism of Emerson and Longfellow, born only a few years earlier than Poe. The former also admitted a certain negativity into his art, but it was always a positive negativity. His guiding principle was faithfulness to oneself. The latter was a sensational narrative poet whose name was given to a bridge spanning the Charles River in Boston. His poetry expresses a certain yearning for traditional beauty and myth, and he has sometimes since been criticized as derivative and moralistic (Fig. 7.21).

Fig. 7.21 Illustration for The Raven by Édouard Manet

Poe himself felt little love for the ideas of Emerson and the transcendentalist school, who promoted a fundamentally positive and stoic vision of humanity and nature. He also had no patience for extended narrative forms and indeed expressed in writing his opinion that a poem ought never to exceed 50 lines in length. Finally, Poe steadfastly denied that it is the business of poetry to cultivate moral sentiment or preserve received legend. He believed rather that art necessarily justifies its own independent existence. When Poe was 36, he published *The Raven* and became famous almost immediately. In this poem, Poe gave free and full expression to his symbolist aesthetic conception. In this poem, the mournful and repeated imagery of a raven, a statue, a chamber door gives voice to that melancholy beauty associated in Poe's poetical philosophy with the death of a beautiful woman. Unfortunately, his reputation was subsequently tarnished by public disputes with Longfellow and other poets and a perhaps somewhat less than upstanding lifestyle that took its toll on his reputation and health.

When he was 17, Poe had fallen in love with a young woman named Sarah Elmira Royster, and his feelings were finally reciprocated. After he had enrolled at the University of Virginia, her father intervened, however, and she eventually married another man. Poe spent less than a year at the University of Virginia before abandoning his studies, perhaps due to his disappointment in love or perhaps simply on account of irregular behavior and lifestyle. His time at the school is memorialized through the present day in the maintenance of his dormitory room No. 13 on the

West Range. Subsequently, he served for a period of time in the army and eventually matriculated to West Point academy. He again failed to complete his studies. Many years later, when Poe was 39, he briefly resumed a relationship with Royster, whose husband had died, although they never married. The following year, Poe collapsed delirious in the streets of Baltimore. He died in a hospital nearby the same night.

After his death, Poe's poems, short stories, and literary commentaries exerted a tremendous influence in France. The poets Baudelaire and Mallarmé among others held him in the greatest esteem. His life resembles in this respect the similarly tragic lives of Abel, Bolyai, and Galois in mathematics. The differences are also interesting. Poetry after all is invented, while mathematics is discovered. According to the poets, it is necessary to tear down the works of the past in order to build up new works in their place. According to the mathematicians on the other hand, each generation adds a new story to the vast edifice built up by past generations. This perhaps is why so many poets have doubled as literary critics, whereas mathematicians are more than anything else apprehensive of great clashes of personality and priority disputes.

Baudelaire

In the spring of 1821, a year after Poe had returned with his foster family to America from England, the poet Charles Baudelaire was born in Paris. His father, who was already 62 at the time, had been born to a family of means in a small village and received an excellent education. He cultivated a lifelong love of literature and the arts, and worked variously as a teacher and a private tutor at the house of a duke, and eventually held an administrative post in the Senate after the French Revolution, where he could have encountered Laplace and Lagrange and Louis François Cauchy (Fig. 7.22).

Baudelaire's mother was born in London to French emigrés. Her parents died when she was young, and she returned to Paris with her relatives at the age of 21.

Fig. 7.22 Portrait of Baudelaire as a young man, by Gustave Courbet

Five years later, she married Joseph François Baudelaire, 34 years her senior. But he died when their son was 6, and she remarried a year later. According to an analysis by the French philosopher and author Jean-Paul Sartre (1905–1980), this was a critical moment in the life of Baudelaire, whose life and work were spurred by a deep-seated desire to recover the fervent love of his mother that he believed himself to have enjoyed before her husband died and she remarried. His new stepfather was a military officer whose brilliantly successful career ensured for Baudelaire a prestigious education; Baudelaire was a melancholic, lonely, and rebellious child.

When he was 15, Baudelaire began to read seriously the notable French poets and literary critics of the time, including Victor Hugo (1802–1885), Charles Augustin Sainte-Beuve (1804–1869), and Théophile Gautier (1811–1872), who coined the slogan *l'art pour l'art*. Baudelaire began to teach himself the craft of poetry from these texts, but he was not so critical of his predecessors as Poe. The next year, he was awarded special recognition at school for his examination results in Latin poetry. When he was 19, Baudelaire took his first mistress, a prostitute named Sara, who inspired several poems, and began to lead an increasingly dissolute life. His stepfather arranged for Baudelaire to travel by sea to India in an effort to reform him. In the summer of 1841, as China was suffering through the First Opium War, Baudelaire embarked in Bordeaux on the *Paquebot des Mers du Sud*, headed for Calcutta.

After it had rounded the Cape of Good Hope at the southern extremity of Africa, the ship bypassed Madagascar and the Mozambique Straight and proceeded directly to Mauritius. Baudelaire does not seem to have caught the spirit of travel and viewed the journey rather as an exile. In his famous poem *The Albatross* from *Les Fleurs du mal*, Baudelaire sets up a metaphorical comparison between solitary travel at sea and the sense of displacement the poet feels among the people of the world. The last stanza reads as follows:

> Le Poète est semblable au prince des nuées
> Qui hante la tempête et se rit de l'archer;
> Exilé sur le sol au milieu des huées,
> Ses ailes de géant l'empêchent de marcher.
>
> The Poet is like this monarch of the clouds
> riding the storm above the marksman's range;
> exiled on the ground, hooted and jeered,
> he cannot walk because of his great wings.[1]

Baudelaire wrote *The Albatross* when he was 20 years old in course of his travels.

The *Paquebot des Mers du Sud* remained in Port Louis, the capital of Mauritius, for 3 weeks and then set sail for the nearby island of Réunion, where Baudelaire hesitated for 26 days before he decided to abandon his voyage and made arrangements to board a different ship en route to Paris. His complete indifference to travel may seem strange from the perspective of the modern backpacker youth, but Baudelaire

[1] Tr. Richard Howard

was determined by this point to become a poet and impatient to return to his home country and make his beginning.

Two months after his return to Paris, Baudelaire came into the sizable inheritance left to him by the death of his father many years earlier. He returned immediately to his old lifestyle and squandered most of it within the course of several years, at which point his family arranged to have what remained placed in trust, providing him with a monthly allowance of 200 francs. When he was 27, Baudelaire encountered the writings of Edgar Allan Poe for the first time, just 1 year before Poe himself died. For the next 17 years, Baudelaire was an active advocate for Poe in France and provided translations of his poems and short stories. The influence of Poe on Baudelaire can be seen in his third essay on Poe, *New Notes on Edgar Poe*:

> For him, imagination is queen of faculties; but by this word he understands something greater than that which is understood by the average reader ... Imagination is an almost divine faculty which perceives immediately and without philosophical methods the inner and secret relations of things, the correspondences and analogies.[2]

In 1857, as Riemann was completing work in the theory of complex variables that would eventually give rise to the discipline of topology, Baudelaire published a collection of poems called *Les Fleurs du mal (Flowers of Evil)*. Within 20 days of its appearance, the celebrated author Gustave Flaubert (1821–1880), born in the same year as Baudelaire, and who had published *Madame Bovary* just the year before, sent him a letter expressing his deepest admiration. The French government was less impressed, and both author and publisher were prosecuted for *outrage aux bonnes mœurs* (insult to public decency). Baudelaire was fined 300 francs, and six of the poems contained in *Les Fleurs du mal* were suppressed in France until 1949. Naturally, the scandal catapulted Baudelaire to notoriety and fame. Today, the influence of *Les Fleurs du mal* is undisputed, and Baudelaire is recognized as a great pioneer in the then yet nascent symbolist and modernist movements in European arts and letters.

In his preface to the revised edition of *Les Fleurs du mal*, Baudelaire wrote "What is poetry? What is its aim? What is the difference between the beautiful and good? What is the beauty of evil?". He elaborated, "It seemed pleasing to me, a task altogether more agreeable than difficult, to extract beauty from evil." Naturally, Baudelaire is not referring here to formal beauty but rather to a kind of inner or mystic beauty. Baudelaire inaugurated with his ideas a new conception of poetry. He used whatever artistic means were best suited to bring the world within to the surface, without reference to established formal requirements. His poems simultaneously invent and express new and altogether modern forms of anxiety and melancholy.

---

[2] Tr. Lois Boe Hyslop and Francis E. Hyslop, Jr.

We can see in the following lines from *Le Vin des Chiffonniers* (*The Rag-Picker's Wine*) how Baudelaire extracted new imagery and materials from contemporary life:

> In the muddy maze of some old neighborhood,
> Often, where the street lamp gleams like blood,
> As the wind whips the flame, rattles the glass,
> Where human beings ferment in a stormy mass,
>
> One sees a ragpicker knocking against the walls,
> Paying no heed to the spies of the cops, his thralls,
> But stumbling like a poet lost in dreams;
> He pours his heart out in stupendous schemes.[3]

There is a certain common universality in these lines that suggest a horizon of new possibilities for poetry. We can see the fruit of this style, for example, in some lines from *Morning at the Window* by T.S. Eliot (1888–1965), who was deeply influenced by Baudelaire:

> The brown waves of fog toss up to me
> Twisted faces from the bottom of the street.

Baudelaire adhered to a certain maxim: "Whatever mud you give me, I can turn it into gold." It is interesting to compare this with a comment Bolyai made as he was beginning to realize the extent of his discoveries in non-Euclidean geometry: "Out of nothing I have created a strange new universe." Along these lines, the critic Sainte-Beuve, to whom *Les Fleurs du mal* was dedicated, suggested to Baudelaire the following defense in the face of its prosecution:

> In the domain of poetry everything has already been taken. Lamartine took the heavens, Hugo took the earth—no, more than the earth, Laprade took the forests, Musset took passion and its dazzling orgy, still others took the hearth and the rustic life ... and so what remained to Baudelaire?

There are many similarities to draw between birth of modernism in poetry and the revolutions in mathematics in the nineteenth century. The innovations of Gauss, Bolyai, Lobachevsky, and Riemann in non-Euclidean geometry in particular upended a tradition and system of mathematical thinking that had stood for two millennia. And just as the future history of mathematics and physics was permanently altered by these innovations, the poetry of Baudelaire had a lasting influence on subsequent generations of poets and artists, notably the poets Stéphane Mallarmé (1841–1898), Paul Verlaine (1844–1896), and Arthur Rimbaud (1854–1891); the Belgian painters Gustave Moreau (1826–1898), who taught Matisse and Rouault, and Felicien Rops (1833–1898); and the sculptor Auguste Rodin (1840–1917). Perhaps the most famous of his acolytes in the English-speaking world was T.S. Eliot, who received the Nobel Prize for Literature in 1948 and was voted Britain's favorite poet in a survey conducted by the BBC in 2009 (Fig. 7.23).

[3] Tr. C.F. Macintyre

Fig. 7.23 The tomb of Baudelaire; photograph by the author, Paris

From Imitation to Wit

Prior to the birth of the modern art, all creative work was inseparably tied to imitation. Aristotle had said that imitation is the origin of art and supported his claim with the argument that humanity naturally takes pleasure in imitation. Even such objects as normally cause discomfort to the viewer, a corpse, for example, can become a source of pleasure when they are reproduced by artistic imitation. He links this pleasure to the human drive for knowledge. In considering an imitation, the viewer compares it to the original and experiences recognition and realization at once. Although there were many revolutions throughout the history of art in technique and style, the basic instinct for imitation was ever present (Fig. 7.24).

For example, the most basic problem in painting has always been the representation of objects in space on the flat surface of the canvas. Already in the earliest fresco paintings of Ancient Egypt depicting scenes of hunting and fishing, there appear some efforts at perspective and projective effects. In the early part of the fifteenth century, the discovery of the method of the vanishing point was a critical point in art history. Afterward, the techniques of linear and spatial perspective dominated four centuries of European art. Through the end of the nineteenth century, artists were accustomed to use darkness to represent shadow, bending trees and flowing hair to represent wind, and unstable contortions to represent the movement of the body. Even the early impressionist painters who introduced subtle confusions of nature and transformations of color retained a basically representative or imitative attitude toward art.

As for the subject matter itself, the classicists naturally inclined toward the ancient, while the romantics preferred medieval or oriental themes. In literature, the

Fig. 7.24 English poet T.S. Eliot

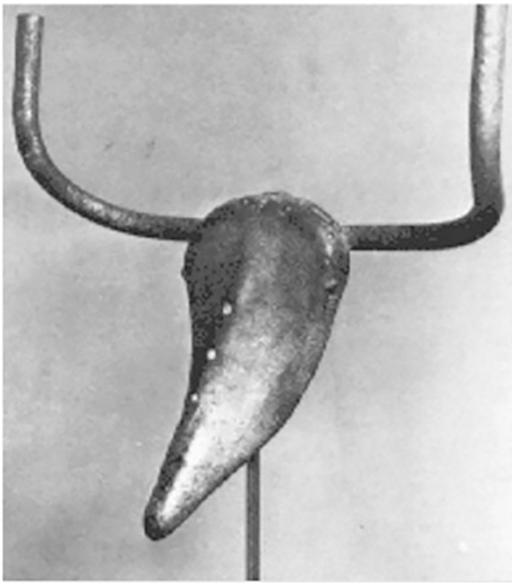


Fig. 7.25 *Bull's Head*, Picasso (1942)

realists and fantasists alike of course modelled their materials on the experiences of human life. In his criticism of *Emma* by Jane Austen (1775–1887), the Scottish writer Sir Walter Scott (1771–1832) wrote that the art which really and truly imitates nature itself is not that which presents the reader with splendid scenes of an imaginary world but rather the daily realities of life itself (Fig. 7.25).

Fig. 7.26 Commemorative stamp featuring the French poet Guillaume Apollinaire

But imitation is not without limitations. Pascal in his *Pensées* observes that two faces that resemble each other make us laugh by their juxtaposition, though neither of them makes us laugh on its own. This suggests the rudimentary nature of imitation as a creative force. Moreover, the apprehension of beauty calls necessarily upon new and novel presentations. Modern artists began to feel that there was something vulgar in the direct representation of ordinary experience. This compelled them to find methods more resourceful than imitation—the first such new resources is wit. As the French poet Guillaume Apollinaire (1880–1918) put it: "When men resolved to imitate walking, he invented the wheel, which in no way resembles a leg (Fig. 7.26)."

Wit is the products of human intelligence at an already advanced stage of development. It consists of a certain unexpected accuracy, and its invention lies in the capacity for quick and clever association between ideas and things. The Spanish philosopher George Santayana (1863–1952) writes in *The Sense of Beauty*: "It is characteristic of wit to penetrate into hidden depths of things, to pick out there some telling circumstance or relation, by noting which the whole object appears in a new and clearer light." The charm of wit lies exactly in this—it is an experience of things obtained only through thought. Wit produces in an intelligent mind the feeling of a riddle, a sparkle, a lightness. The American philosopher and writer Susanne Langer (1895–1982) has observed that whenever an emotion is expressed indirectly, it signifies a height of artistic expression (Fig. 7.27).

Some examples are as follows: in 1943, Pablo Picasso (1881–1973) welded together the seat and handlebars of an abandoned bicycle and turned the contraption on its side to create a sculpture of a bull's head. When I was young, I remember seeing a painting by Chagall (1887–1985) in which the hip of a girl and the body of a violin were fused together. Finally, there is *The Promenades of Euclid* by the Belgian Surrealist artist René Magritte, which depicts a city view through a window, with a straight line subjected to strong perspective and deformation such that it resembles the conic shape of the adjacent tower (Fig. 7.28).

Fig. 7.27 *The Birthday*, Marc Chagall (1915)

Fig. 7.28 *The Promenades of Euclid*, Magritte (1955)

Conclusion

Just as the development from classical art to modern art followed the progression established in poetry, the revolutions in science were prefigured by innovations in mathematics, in particular the developments in geometry in the nineteenth century. In both cases, there is a passage from imitation to invention, from representation to abstraction. This shared evolution is no doubt the product of the development of human thought in accordance with natural law. In any case, it is easy to imagine the tremendous difficulties inherent to innovation. Non-Euclidean geometry, for example, was met with the same initial incredulity as the heliocentric revolution of Copernicus, the law of universal gravitation first proposed by Newton, or Darwin's theory of evolution and similarly instigated revolutions in science, religion, and philosophy.

Fig. 7.29 Carl Friedrich Gauss, "Prince of Mathematics"

Aristotle regarded imitation as the paradigmatic form of art. In the other sphere of human creativity activity, so has natural science and in particular mathematics always been regarded as the standard of truth. Ancient mathematics in the Western world occupied a cultural position as sacred and unimpeachable as religion, and Euclid was its patron saint.

The universality and validity of Euclidean geometry remained sufficiently beyond doubt for so long that the German philosopher Kant, who died in 1804, set up the vast machinery of his rich and esoteric philosophy atop its foundations. But by 1830 there had appeared among the rarefied airs of quantitative reasoning and spatial forms a fog of new and mutually contradictory geometries, each of them with equal claim to the legitimacy of internal consistency (Fig. 7.29).

In fact, the basic ideas of non-Euclidean geometry had been within eyesight for thousands of years already, just as the source materials of modern poetry had always been available. But even the greatest mathematicians had never considered to take the geometry on the surface of a sphere as a geometrical system in its own right. Many of them endeavored rather to prove the parallel postulate by way of quadrilateral arguments and never noticed that even the planet is a model for a different kind of geometry. Such is the power of custom and inertia when it comes to human thought. It is no wonder that Gauss was reluctant to reveal his discoveries in non-Euclidean geometry. In any case, his prudence paved the way for the younger generation.

The age of Euclidean geometry came at last to an end, and its end posed a serious challenge to the cherished idea of absolute truth. Similarly, the works of Edgar Allan Poe and Baudelaire heralded the end of the romantic era in poetry. And in each case, the dissolution of certainty was accompanied by a concomitant gain in freedom. The mathematician Georg Cantor expressed the new attitude as follows: “The essence of mathematics lies in its freedom.” The situation of mathematics prior to 1830 can be compared to that of an artist with a deep love for pure beauty confined to the task of producing a stream of magazine covers. Its change in status and attitude is no doubt primarily the product of the discovery of non-Euclidean geometry, one of the great achievements of human ingenuity. The innovations in geometry and algebra in the nineteenth century are not at all like the earlier revolutions brought about by the invention of the calculus. They were not developed to satisfy the demands of science or socioeconomic development but rather by the internal development of pure mathematics itself.

As for utility, the situation is roughly captured by the following schematic description: the ordinary circumstances of daily life are best described by Euclidean geometry; at the scale of the cosmos at one end, and the scale of fundamental particles at the other, Lobachevskian geometry has proved most useful; for questions related to navigation, aviation, and the representation of the earth, Riemann geometry is most suitable. But of course there are many subtleties involved in the relationships between geometry and physics, and there remain to this day open questions concerning the geometry of physical space. Moreover, it is possible to describe identical physical situations with respect to different geometrical systems via suitable modifications in the physical hypotheses. In any case, the development of non-Euclidean geometry necessitates a stricter division than had ever been necessary before between pure mathematics and the natural sciences, in the same way that the natural sciences had split off from philosophy, and before that philosophy from religion. Mathematicians were now free to explore the vast universe of possible systems of geometry, algebra, etc. with only consistency as prerequisite. And the discoveries of the mathematicians in this new and fertile territory promised to prompt new ideas and possibilities in the natural sciences in turn. We shall see in the next chapter that Einstein struck upon general theory of relativity by way of non-Euclidean geometry.

I would like to close this chapter with an anecdote. In early 1830, 2 years after Lobachevsky had published his paper on non-Euclidean geometry in distant Kazan, the Cambridge mathematician George Peacock (1791–1858) published his *A Treatise on Algebra*, in which he sought to place algebra on solid logical foundations analogous to the axiomatic structure of Euclid’s *Elements*. He stipulated five basic algebraic principles: the commutative laws for addition and multiplication, the associative laws for addition and multiplication, and the distributive law relating multiplication to addition. These five laws comprise a generalization of the arithmetic of integers. But just as Peacock and his colleagues were preparing to publicize and extend his results, Hamilton and Grassmann introduced the theory of quaternions. Peacock’s theories disappeared somewhat from view, and Peacock left his job at Cambridge in 1839 to become the Dean of Ely Cathedral in Cambridgeshire.

Chapter 8
Abstraction: Mathematics Since the Twentieth Century

Number is the ultimate abstract expression of all forms of art.

Wassily Kandinsky

Philosophy must be of some use and we must take it seriously

Frank Ramsey

The Road to Abstraction

Set Theory and Axiomatic Systems

The revolutions in mathematics in the nineteenth century paved the way for rapid development and unprecedented expansion in mathematics in the twentieth. Modern mathematics no longer comprises only geometry, algebra, and analysis. Rather, mathematics today is a vast web of interconnected and evolving disciplines and concepts, characterized not only by rigorous logic but also by high abstraction and wide applicability. This indicates the basic division of modern mathematical research into pure mathematics and applied mathematics. The latter classification has expanded in recent decades to include computer science, the importance of which in the modern world goes without saying: from the perspective of employment opportunities alone, it has already exceeded every other branch of mathematics (Fig. 8.1).

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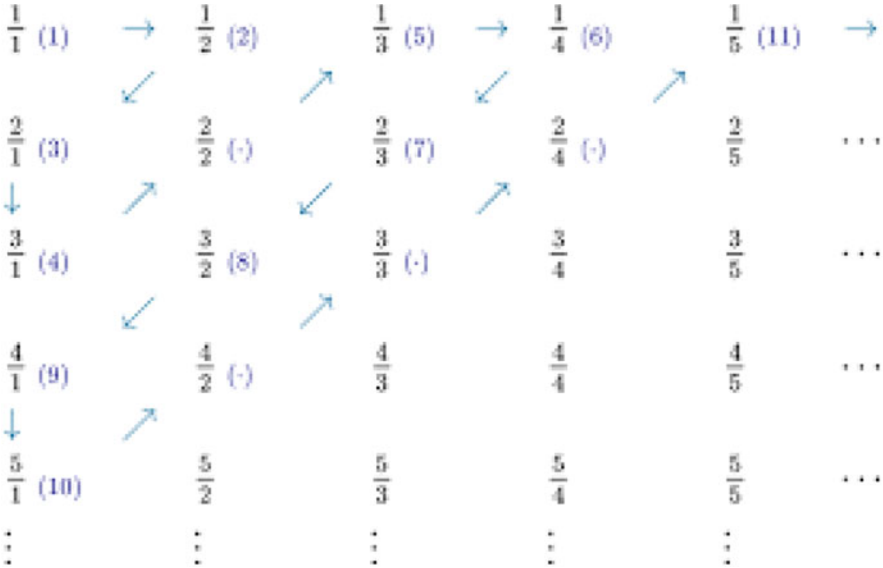
T. Cai, *A Brief History of Mathematics*,
https://doi.org/10.1007/978-3-031-26841-0_8

Fig. 8.1 Georg Cantor, founder of set theory

The modernization of pure mathematics was driven primarily by two innovations: the invention of set theory and the introduction of axiomatic methods. Set theory was created in the nineteenth century by Georg Cantor. Its invention was initially ill received by the mathematics community, notably Kronecker, but eventually achieved widespread success. Sets were originally conceived as collections of numbers or points, but the definition of a set quickly expanded to include collections of arbitrary elements, for example, sets of functions, sets of shapes satisfying a given property, and so forth. Today, it is the universal language of mathematics in which the basic concepts of mathematics, say integrals, functions, and spaces of various kinds, are all expressed. The introduction of set theory has also had a profound influence on the machinery of mathematical logic and motivated the debate between mathematical intuitionism and formalism, which is the subject of the present chapter.

Georg Cantor was born in 1845 in Saint Petersburg into a family of second-generation German emigrants. His father was a businessman with connections in Hamburg, London, and even New York. When Cantor was 11 years old, his father became ill, and the family returned to Germany. He completed his secondary education in Amsterdam and attended universities in both Zurich, Switzerland, and Berlin. He had a talent for painting which was a source of considerable pride for his family, but settled eventually upon a career in mathematics.

As Cantor saw it, a set consists of any abstract collection of well-distinguished objects. He introduced the notion of the cardinality of a set in order to compare the sizes of different sets, whether finite or infinite. His definition relies on the notion of a one-to-one correspondence between sets, which is illustrated by a surprising and beautiful demonstration: Cantor discovered and proved that it is possible to set up a one-to-one correspondence between the rational numbers and the natural numbers. The proof is encapsulated by the following diagram.



Such infinite sets that can be put into a one-to-one correspondence in this way with the natural numbers are called countable. Infinite sets that cannot be put into any one-to-one correspondence with the natural numbers are called uncountable. Cantor proved that the set of real numbers is uncountable.

Moreover, Cantor was able to use set theoretical arguments to provide a simple nonconstructive proof for the existence of transcendental numbers: since it is not difficult to see that the set of algebraic numbers, which includes as a subset the set of rational numbers, is countable. Since every real number is either algebraic or transcendental, and the set of real numbers is uncountable, it follows that the majority of real numbers must be transcendental. The study of transcendental numbers became a deep and active area of research in twentieth-century mathematics.

The philosophical assumptions and implications at the heart of Cantor's research were not uncontroversial. In particular, the successful and influential mathematician Leopold Kronecker opposed the introduction of actual infinities into mathematics. Kronecker was head of mathematics at the University of Berlin and a successful businessman, and his vigorous public opposition to Cantor may have prevented Cantor from ever obtaining a post there, and Cantor spent the entirety of his career at the less prestigious University of Halle.

Cantor borrowed from Hebrew the notation $\aleph_0$ (aleph null) to stand for the cardinality of the natural numbers and showed that it is possible to construct an increasing sequence $\aleph_0 < \aleph_1 < \aleph_2 < \cdots$ of transfinite cardinalities. Since the cardinality of the real numbers is strictly larger than the cardinality $\aleph_0$ of the natural numbers, Cantor proposed a natural conjecture, referred to today as the continuum hypothesis: there exists no cardinal number lying strictly between the two. When David Hilbert presented his famous list of open problems at the turn of the twentieth century at the International Congress of Mathematicians in Paris in 1900, the problem of the continuum hypothesis was first among them (a problem related to transcendental numbers was seventh).

Cantor corrected a serious defect in the foundations of mathematics that had persisted since the time of Zeno in Ancient Greece. The philosopher Bertrand Russell discusses the historical significance of his work in his *Mathematics and the Metaphysicians*, published in 1901:

> Zeno was concerned, as a matter of fact, with three problems, each presented by motion, but each more abstract than motion, and capable of a purely arithmetical treatment. These are the problems of the infinitesimal, the infinite, and continuity . . . From him to our own day, the finest intellects of each generation in turn attacked the problems, but achieved, broadly speaking, nothing. In our own time, however, three men—Weierstrass, Dedekind, and Cantor—have not merely advanced the three problems, but have completely solved them. The solutions, for those acquainted with mathematics, are so clear as to leave no longer the slightest doubt or difficulty. This achievement is probably the greatest of which our age has to boast . . . Of the three problems, that of the infinitesimal was solved by Weierstrass; the solution of the other two was begun by Dedekind, and definitively accomplished by Cantor.

Unfortunately, Cantor's Promethean efforts and many personal insecurities and misfortunes led to his own mental breakdown at the age of 40, and he spent much of his later life in and out of sanatoriums, in one of which he died some many years later (Fig. 8.2).

The story of axiomatization in mathematics also begins in Ancient Greece, with Euclid and his *Elements of Geometry*. In it, he introduced the five axioms discussed at length in the previous chapter. His system however was incomplete and imperfect. The mathematician David Hilbert introduced a new system of axioms for geometry

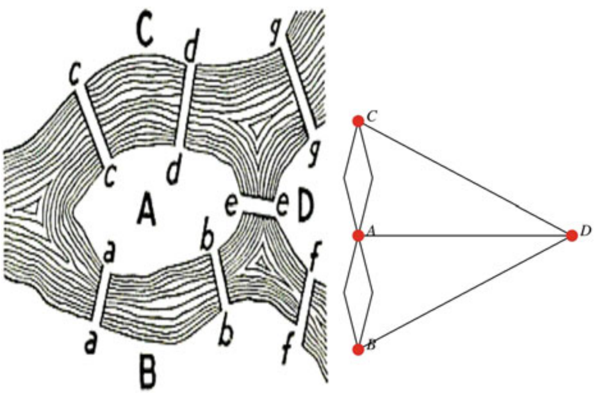


Fig. 8.2 A commemorative stamp issued by the Democratic Republic of the Congo featuring David Hilbert

in order to clear up its ambiguities. He is reported to have described the objective of his axiomatic system with the words: “One must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs.”

In Euclid, points, lines, and planes have descriptive definitions in terms of their spatial properties. Hilbert endeavored to replace these descriptive definitions with purely formal definitions. Points, lines, and planes become purely abstract objects with no specific content, and the axioms define formal relations between them. Hilbert established three legitimacy requirements for an axiomatic system: consistency, independence, and completeness. Of course, axiomatization at this stage was only a methodological question and does not possess as rich a content as set theory. Nevertheless, Hilbert provided with his method a rigorous foundation for geometry, and since then, the method of axiomatization has gradually seeped into other branches of mathematics and become a powerful tool for refining mathematics and a specific topic of mathematical research in its own right.

David Hilbert was born in 1862 in the outskirts of Königsberg, a Prussian city that today is part of Russia and known as Kaliningrad. Probably the most famous resident in the history of Königsberg was Kant, who spent his entire life there. The city is also associated with a famous problem in mathematics. There are seven bridges across the river Pregel running through it, some of them connecting the mainland to one or the other of two large islands at its center, one of them joining the two islands to one another (Fig. 8.3).

The problem was to find a walk through the city that would cross each of the bridges once and only once, and it was resolved by Euler in the eighteenth century, who proved that no such walk exists. This seemingly simple mathematical problem eventually gave rise to the modern theory of topology. Another mathematically famous resident of Königsberg was Christian Goldbach (1690–1764), responsible for a famous eponymous open conjecture in mathematics, that every even integer larger than 2 admits a presentation as a sum of two primes. Perhaps the greatest progress toward the resolution of this problem was provided by the Chinese mathematician Chen Jingrun, who proved in 1966 that every sufficiently large even

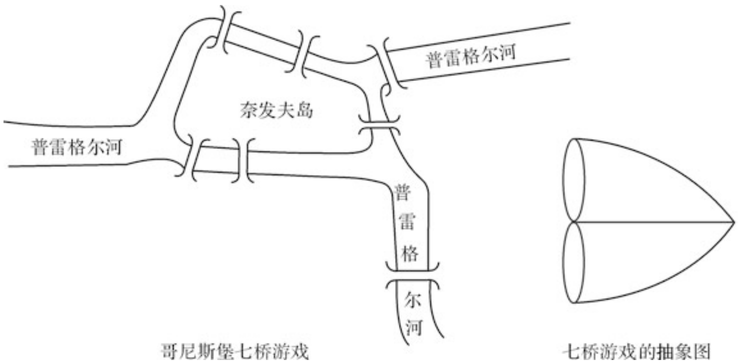


Fig. 8.3 Abstract illustration of the Seven Bridges of Königsberg problem

number can be written as a sum of either two primes or the sum a prime and the product of two primes. In 2013, Zhang Yitang (1955–), another mathematician born and raised in China, made a breakthrough in the study of the twin prime conjecture, which states that there exist infinitely many pairs of prime numbers with a difference of two, such as, for example, 5 and 7, 11 and 13. His result was subsequently improved by a new method created by the British mathematician James Maynard (1987–), who was awarded the Fields Medal in 2022.

During Hilbert's lifetime, the Königsberg mathematician who played the largest role in his mathematical career was his colleague Hermann Minkowski (1864–1909) who was born 2 years after Hilbert in the Russian town of Aleksotas, now part of Kaunas in Lithuania, and moved with his family when he was 8 years old to Königsberg, where they lived across the river from Hilbert. This talented mathematician earned the prestigious Mathematics Prize of the French Academy of Sciences when he was 18 years old for a manuscript on the theory of quadratic forms. His brother Oskar Minkowski (1858–1931) was also a successful medical researcher, who discovered the relationship between the pancreas and diabetes, which led to the discovery of insulin as a treatment of the disease.

Hilbert's talent was in no way outshone by the remarkable talent of Minkowski, but rather he was impelled to hone and accumulate his skills and quietly endeavor to build for himself an even more solid foundation. The two of them developed a remarkable friendship that spanned more than a quarter century until Minkowski's sudden death due to appendicitis in 1909. Hilbert lived to see his eighties and became one of the most accomplished and respected elder statesman of mathematics in his time. The famous list of open questions and research projects that he introduced at the turn of the century remain to this day an influential guidepost for the entire discipline.

We say a bit here about Hilbert's ninth problem, which was partially resolved by the work of the Austrian mathematician Emil Artin (1898–1962) and the Japanese mathematician Teiji Takagi (1875–1960) with the creation of class field theory. Takagi pursued his doctorate at the University of Göttingen under the supervision of Hilbert and later returned to his country where he trained a generation of outstanding Japanese mathematicians: indeed, following the end of World War II, Japan produced three Fields Medalists, the first of them being Kunihiko Kodaira (1915–1997).

The Abstraction of Mathematics

Set theory and the axiomatic method became the paradigms for mathematical abstraction in the twentieth century, even more so after they were integrated into a singular foundational approach to all of modern mathematics. Eventually, four central disciplines emerged: real analysis, functional analysis, topology, and modern (or abstract) algebra. It is interesting to note that all the mathematicians mentioned in the previous section in connection with this development hailed from Germany, a country which has always nurtured a talent for the abstract, whether in art, music, or the humanities and social sciences.

The introduction of set theory brought about a revolution in integral calculus which led to development of the modern theory of functions of a real variable. The rigorous treatment of analysis in the nineteenth century had forced into the light a variety of pathological functions such as the Weierstrass's function, discussed in the previous chapter. Another example is Dirichlet function, named after another of Gauss's students, who discovered it:

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number} \\ 0 \text{ otherwise} \end{cases} .$$

This function has the interesting property of being discontinuous at every real number. Such examples forced mathematicians to study a more general class of functions than that which had typically been admitted into calculus (Fig. 8.4).

The first significant success in this d irection was achieved by the French mathematician Henri Lebesgue (1875–1941). He adopted a set theoretical approach to invent a new mathematical discipline called measure theory. In measure theory, certain familiar geometrical concepts including length and area are generalized and made abstract by the introduction of a measure on a given space. Similarly, Lebesgue extended the integral of classical calculus by defining the Lebesgue integral. On the basis of these foundations, it is possible to recover the fundamental theorem of calculus relating the differential operation and the integral operation and the other familiar theorems in calculus due to Leibniz and Newton. The contributions of Lebesgue became the building blocks of modern real analysis. However, his work received a hostile reception from classical analysts, and he struggled to find consistent work for a period of time after its publication. Its importance is recognized today by the division in analysis between classical analysis

Fig. 8.4 Henri Lebesgue, father of modern analysis

and modern analysis, the latter of which refers to any topic in analysis which makes use of his innovations.

Another deep development in analysis in the twentieth century was the development of modern functional analysis. The word functional was coined by Jacques Hadamard (1865–1963) to describe a function whose argument is another function. We have had occasion already to discuss examples of such functions in our treatment of the calculus of variations. The list of mathematicians who contributed important results in functional analysis is a long one. Hilbert, for example, studied the space of square-summable sequences: sequences $(a_1, a_2, \ldots, a_n, \ldots)$ of real numbers subject to the requirement that the series $\sum_{n=0}^{\infty} a_n^2$ converges. He defined the notion of an inner product on such a space, as well as its various operations, and provided in this way the first example of an infinite-dimensional vector space. This space is referred to today as a Hilbert space.

Ten years later, the Polish mathematician Stefan Banach (1892–1945) presented a more general class of vector spaces, the so-called Banach spaces. He replaced the inner product of Hilbert with a real valued function called a norm by means of which it is possible to provide general definitions of the length of a vector, the convergence of a sequence of vectors, and so on. The study of general Banach spaces marked a considerable expansion and abstraction in the scope of functional analysis as a discipline. Around the same time, considerable progress was made toward a more abstract and general concept of a function. We present here only an example of this work: the so-called Dirac delta function $\delta(x)$, invented by the British physicist Paul Dirac (1902–1984)[1] and defined by the properties

$$\delta(x) = 0 \text{ for all } x \neq 0, \text{ and } \int_{-\infty}^{+\infty} \delta(x)dx = 1.$$

Of course, there exists no function in the classical sense satisfying these properties, but the Dirac delta function proved extremely useful for physics, and eventually, a mathematical formalism was discovered to handle such cases. Today, functional analysis is among the areas of mathematics that has proved most useful to physics and the other sciences, in particular engineering technology (Fig. 8.5).

At the same time that set theoretical methods were facilitating revolutions in real analysis and functional analysis, the axiomatic method was also extending its reach into every area of mathematics. The most significant developments were in abstract algebra. Ever since Galois had first introduced the group concept into mathematics, mathematicians had expanded the class of groups to include finite groups, discrete groups, infinite groups, and continuous groups. A host of other

[1] In 1928, Dirac introduced the theory of relativity into quantum mechanics and established the relativistic version of the Schrödinger equation, known as the Dirac equation. That year, he and Schroödinger both won the Nobel Prize in Physics.

Fig. 8.5 Emmy Noether, a founding figure in abstract algebra

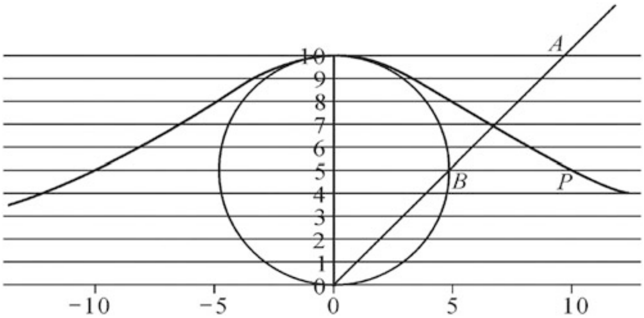


Fig. 8.6 The witch of Agnesi curve

algebraic systems also appear, including rings, fields, lattices, ideals, etc. The focus of algebraic research began to shift toward abstract algebraic structures; such a structure consists of a set equipped with some number of finitary operations subject to a list of prescribed axioms (Fig. 8.6).

It is generally believed that the first mathematician to formally set down the idea of modern abstract algebra was the German mathematician Emmy Noether (1882–1935) in her 1921 paper *Idealtheorie in Ringbereichen (Theory of Ideals in Ring Domains)*. She was one of the finest mathematicians of her age or any age and contributed to the axiomatic treatment of the general theory of ideals and noncommutative algebra. At the time of her death, she was memorialized as the greatest woman mathematician of all time, having surpassed in accomplishment the mathematicians Hypatia (c. 350–415) of Ancient Gaetana Agnesi (1718–1799) of Italy, Sophie Germain (1776–1831) of France, and Kovalevskaya of Russia.

Sex discrimination prevented her for many years from obtaining a regular post at Göttingen despite the fervent recommendations of David Hilbert, and she often worked for no pay. After the rise of Hitler the Nazi party, she was removed from her position and eventually moved to America where she spent her final years lecturing at Bryn Mawr College.

In addition to abstract algebra, probability theory also benefited from axiomatization. The main work in this area was carried out by the Soviet mathematician Andrey Kolmogorov (1903–1987). Kolmogorov graduated from Moscow State University in 1925 and immediately began to carry out research at the same institution. Four years later, he published his *General Theory of Measure and Probability Theory*, in which he proposed six axioms as a foundation for probability. He also contributed to the practical development of probability theory through his work on continuous-time Markov process. Leaving probability aside, Kolmogorov also carried out important work in functional analysis, topology, the theory of turbulence, information theory, dynamic systems, and classical mechanics.

In 1980, Kolmogorov shared the Wolf Prize in Mathematics with the French mathematician Henri Cartan (1904–2008). Two years earlier, his student Israel Gelfand (1913–2009) had received the first ever Wolf Prize in Mathematics for his work on functional analysis, group theory, and representation theory; Gelfand shared this award with the German mathematician Carl Ludwig Siegel (1896–1981). Israel Gelfand was born into a poor Jewish family in the Odessa Oblast (province) of Ukraine, where he was expelled from high school, according to his own account for political reasons related to his father's status as a mill owner. At the age of 17, he and his father made his way to Moscow to live with some distant relatives. Two years later, without having received a high school diploma or university degree, Gelfand begin postgraduate studies at Moscow State University under the supervision of Kolmogorov. His doctoral dissertation introduced the theory of normed rings; he also proved an important theorem concerning the space of maximal ideals in rings of continuous functions and established the general spectral theory of C^*-algebras.

We turn finally to topology. The great German-born American mathematician Hermann Weyl (1885–1955) famously said, "In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics." This indicates something of the great importance of these two disciplines. The premodern origins of topology however appear much earlier than those of abstract algebra, and its motivating examples are more immediately accessible. These include the problem of the bridges of Königsberg (1736), the four-color problem for maps (1852), and the famous Möbius strip (1858). The basic objects of interest in topology are abstractions of geometric shapes subject to continuous processes – two topological structures are considered to be equivalent to one another if one can be obtained from the other by an invertible continuous transformation (intuitively, transformations that can be achieved by stretching or distorting, but without introducing any cuts or joins). The word topology seems to have first been coined by a student of Gauss in 1847. In Greek, it means the study of position.

Modern topology is subdivided into point-set topology, also called general topology, and algebraic topology. In point-set topology, the basic structure is that of a set equipped with a collection of distinguished subsets referred to as open sets or neighborhoods. The entire ensemble is known as a topological space. In this way, it is possible to give abstract definitions for various properties of interest to mathematicians, including continuity, connectedness, and dimension, and also some more specialized concepts such as compactness and separability. The theory has some interesting and surprising applications. For example, it follows from the famous fixed point theorem of topology that at any given time there is always some point on the surface of the earth at which there is no wind (like the eye of a hurricane) and that there is some point on the surface of the earth from which every direction points southward, specifically the North Pole. The fixed point theorem states: every continuous map from an n-dimensional object (satisfying certain conditions) to itself has a fixed point.

Algebraic topology was founded by the French mathematician Henri Poincaré (‘854–1912). Just as a wall is made up of bricks, Poincaré began by partitioning geometric spaces into finitely many little regions. He defined in terms of these regions the topological concepts of higher-dimensional manifolds, homeomorphisms, and homology. Subsequent mathematicians also developed such related concepts as homotopy and homology. This procures a translation of topological problems into the domain of abstract algebra. One of the earliest results in what is now referred to as algebraic topology was first discovered by Descartes in 1635 and independently rediscovered by Euler in 1752. This is the famous Descartes-Euler polyhedral formula which says that for any simply connected convex polyhedron, the sum of the number of vertices and the number of faces minus the number of edges is always equal to 2. Another famous result in algebraic topology is the Poincaré conjecture, which states that every simply connected closed 3-manifold is homeomorphic to the 3-sphere. Poincaré first proposed his conjecture in 1904, and it was proved by the Russian mathematician Grigori Perelman (b. 1966) in 2006 (Fig. 8.7).

Henri Poincaré was born in Nancy, Meurthe-et-Moselle, in 1854, the same year in which Riemann developed his theory of non-Euclidean geometry. He exhibited a prodigious intelligence from an early age, although he became seriously ill with

Fig. 8.7 French mathematician Henri Poincaré

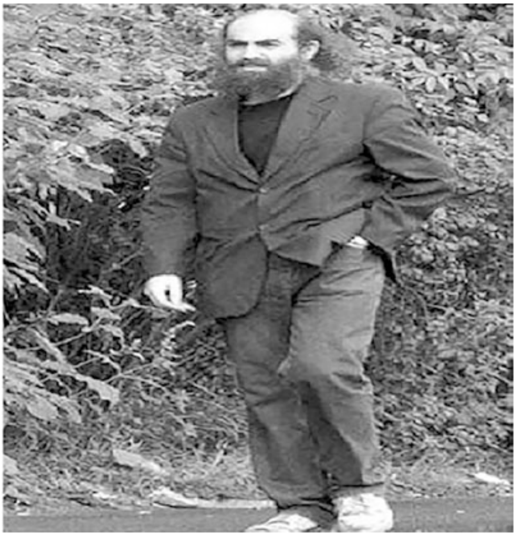


Fig. 8.8 Grigori Perelman, who proved Poincaré's conjecture

diphtheria when he was 5 and sometimes had trouble expressing his thoughts fluently for a period afterward. Nevertheless, he enjoyed all manner of games and dancing as a child and developed a reputation as a remarkably quick and attentive reader. In school, he excelled in all his subjects and especially in written composition. His interest in mathematics flowered somewhat late, probably when he was about 15, but his talent quickly revealed itself. He enrolled at the École Polytechnique when he was 19 (Fig. 8.8).

Poincaré never stayed too long in one area of research – one of his contemporaries described him as more a conqueror than a colonizer. To some extent, he planted his flag in every discipline in mathematics, and several disciplines outside it, but his most important contributions were certainly in topology. Research into the Poincaré conjecture and its generalizations and eventual proof produced three separate Fields Medalists at intervals separated by 20 years: first in 1966 and then again in 1986 and 2006 (Fig. 8.9).

Poincaré was also an exceptional popularizer of mathematics. His popular works were translated into many languages and read with interest by people from all walks of life with an influence not unlike that of *A Brief History of Time* by Stephen Hawking (1942–2018) in the present day. Finally, Poincaré sustained an active interest in philosophy throughout his life and published three influential works on the philosophy of science: *Science and Hypothesis*, *The Value of Science*, and *Science and Method*. He famously argued for the position of conventionalism in physics, which holds that the laws that govern physics and physical space are subject to competing equivalent formulations and that the choice of one or another particular system of formulations is a question of convention and convenience. At the same time, he was opposed to the use of infinite sets in mathematics and believed instead that the most basic concept in mathematics is the concept of the natural numbers.

Fig. 8.9 *Les Demoiselles d'Avignon*, Picasso (1907)

In this respect, he was one of the earliest proponents of intuitionism. In connection with this belief, Poincaré always emphasized the role of creativity in mathematics and its relation to the arts. He wrote in *The Value of Science* that "it is only through science and art that civilization is of value."

At a time when people were still actively debating the legitimacy of non-Euclidean geometry, Poincaré presented in powerful intuitive guides to the geometry of space in four dimensions. In *Science and Hypothesis*, he argues "consider a purely visual impression, due to an image formed on the back of the retina. A cursory analysis shows us this image as continuous, but as possessing only two dimensions. However, sight enables us to appreciate distance, and therefore to perceive a third dimension." Just as information in three spatial dimensions can be translated onto the two dimensions of the retina, it is possible to imagine that the three dimensions of physical space are projections onto a surface in four-dimensional space not unlike the artistic choice of perspective on a canvas. This argument had a profound influence on Pablo Picasso, who was inspired by it to begin his experiments in cubism with the painting *Les Demoiselles d'Avignon* in 1907.

Science and Hypothesis also had a profound effect on another member of Picasso's circle, the Paris actuary Maurice Princet (1875–1973), who is generally credited with introducing its ideas to the cubists who lived and met at the *Bateau-Lavoir* building in the Montmartre district. The writer and critic Guillaume

Fig. 8.10 *Self Portrait 1875*, Cézanne

Apollinaire (1880–1918), who moved in the same circles and invented the term cubism, observes in his book *The Cubist Painters* (1913) that "geometry, the science of space, its dimensions and relations, has always determined the norms and rules of painting." He likened the idea of a fourth spatial dimension to the "immensity of space eternalizing itself in all directions at any given moment," a great metaphor containing the seeds of an entirely new art. He further pointed out that "geometric figure is as essential to painting, and geometry is as important to the plastic arts, as grammar is to writing." We can perhaps regard Cubism as a second great encounter between painting and geometry after the Renaissance (Fig. 8.10).

Abstraction in Art

The word "abstract" as a noun occurs frequently at the beginning of mathematical and other scientific papers, just beneath the title, author, and institution, where it has the meaning of "summary." In this section, we discuss its more usual descriptive meaning in the context of art and mathematics (Fig. 8.11).

Just as the introduction of set theory and the tendency toward abstraction in mathematics in the early part of the twentieth century was not met without a certain amount of resistance and controversy, the abstract movement in art has also been cause for significant dispute. Ever since Aristotle, the ultimate aim of painting and sculpture had always been the imitation of nature.

Fig. 8.11 *The Card Players*, Cézanne (1893)

Fig. 8.12 *La Nuit étoilée (Starry Night)*, van Gogh (1889)

It was only in the mid-nineteenth century that artists began to view their project differently and regard painting as an end in itself without reference to verisimilitude. Over time, a new style emerged: specific forms increasingly were exaggerated and deformed and transformed for expressive effect. The pioneer of this new style was Paul Cézanne (1839–1906). Cézanne took inspiration from his own idiosyncratic optical theories according to which the eyes perceive a scene continuously in time and from a variety of perspectives. His innovative ideas concerning nature, people, and painting are all on display in his paintings of mountains, rivers, and still life compositions in his native Provence. For Cézanne, abstraction was a tool for restoring to painting its natural beauty and independence (Fig. 8.12).

Cézanne is known as the father of modern art, and his guidance initiated a great wave of modernism in art in the late nineteenth and early twentieth century. His

Fig. 8.13 *Landscape at Murnau*, Kandinsky (1908)

immediate heirs were the Fauvists, represented by Henri Matisse (1869–1964), and the Cubists, represented by Picasso. All of these artists however retained in their work some connection to the representation of natural forms. Their work cannot yet be called abstract art, but rather only abstracted art, or perhaps half-abstract. The word abstract here is merely descriptive and does not have the status of a proper noun, as in "abstract art" and the mathematical term "abstract algebra." Rather the phrase abstract art in its fullest sense refers to works with no identifiable subject matter (Fig. 8.13).

The first truly abstract artist was probably the Russian painter Wassily Kandinsky (1866–1944). Since the eighteenth century, Russia under Peter the Great and Catherine II had engaged in large-scale patronage in the arts and sciences. Beneficiaries of this patronage in mathematics alone included Euler and the Bernoulli brothers. Russians at that time travelled often to France, Italy, Germany, and other countries, and by the nineteenth century, Russian literature, music, drama, and ballet had all developed to an extraordinarily high degree of refinement.

It was in this context that Kandinsky was born in Moscow in the same year that Riemann died in Germany and only a few months before Baudelaire died in Paris. His father was a tea merchant from Siberia, and his grandmother a princess of Chinese Mongolian descent. His mother was a Moscow local. When he was still young, Kandinsky travelled with his parents to Italy. After his parents divorced, he lived with an aunt in Odessa on the shores of the Black Sea in modern Ukraine and completed his education there. He took up piano and cello and began to teach himself painting (Fig. 8.14).

When he was 20, Kandinsky enrolled at the University of Moscow to study law and economics and eventually obtained a degree equivalent to a modern doctorate. He maintained a strong interest in painting however and was especially influenced by the colorful folk art he experienced as part of an ethnographic research expedition

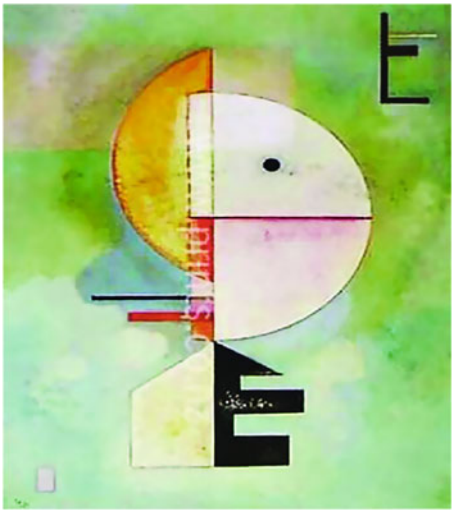


Fig. 8.14 Abstract painting by Kandinsky

Fig. 8.15 *On the Spiritual in Art*, German edition (1912)

to the Vologda region north of Moscow. In 1896, when he was already 30 years old, Kandinsky decided once and for all to become a painter. He abandoned a promising teaching career and took the train for Germany, where he studied privately at first and later enrolled as an art student at Munich Academy. Among his classmates was a young Swiss artist named Paul Klee (1879–1940) who later became one of the great painters of the early twentieth century alongside Kandinsky (Fig. 8.15).

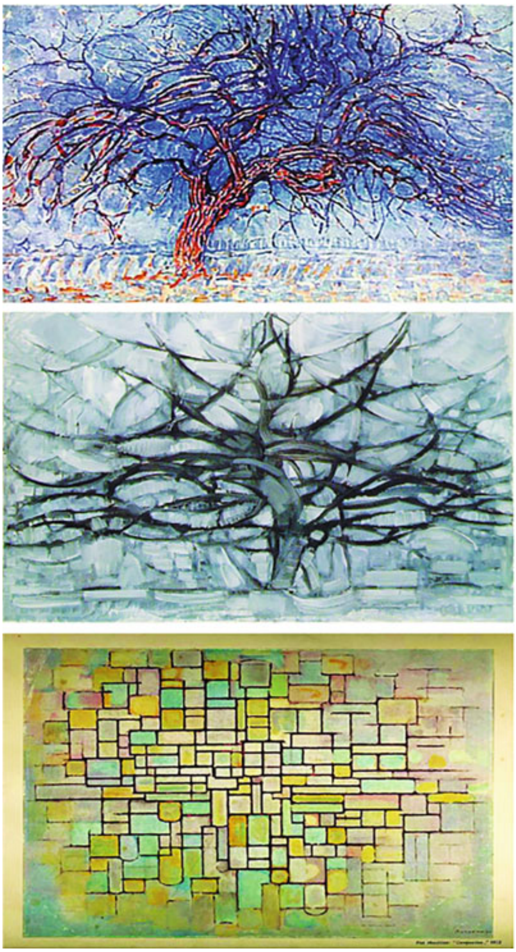


Fig. 8.16 Progression from representation to abstraction: *Flowering Trees* by Mondrian

It was during his time in Munich that Kandinsky began to develop his mature ideas about nonobjective and nonrepresentational art. After a period of exploration, he struck upon his purpose in art: the creation of decisive spiritual and emotional reaction by way of line and color, space and movement, without reference to the representation of natural objects. In his tract *Concerning the Spiritual in Art*, Kandinsky discusses his first encounter with the impressionist paintings of Édouard Manet (1832–1883) and the attraction he felt toward an art in which the material reality of its objects was deemphasized. Revolutionary advances in the natural sciences in his lifetime further corroded his commit to the world of direct sense perception (Fig. 8.16).

Kandinsky endeavored in his art to give spiritual expression to mystic inner experience independent from external reality on the one hand and technical refinement on the other. He believed that the harmony of color and form must always take as

Fig. 8.17 Painting by Kazimir Malevich

its primary objective the task of reaching the human soul. In middle age, Kandinsky wrote an autobiography in which he described his experience of colors:

> The colors which made the greatest impression on me were bright green, white, magenta, black, yellow. Even now I have memories of them from when I was three years old. I noticed them again and again in a variety of shapes and objects, and over time the objects became less clear in my eyes than the colors themselves.

In his later years, Kandinsky began to develop a more geometric style of abstraction built in circles and in triangles. His ideas are reflected in the titles of some of his works: *Concentric Circles*; *A Center*; *Yellow, Red and Blue*; and *Sounds*. In another important treatise, *Point and Line to Plane*, Kandinsky analyzed the specific emotional effect of formal elements in painting, claiming, for example, that a horizontal line has a coldness to it, while a vertical line is hot. In any case, his works are characterized by an immediately recognizable feeling for color and form that suggest the new horizons of expression facilitated in art by the turn toward the abstract, in much the same way that non-Euclidean geometry had conjured up a broader space of possibilities in mathematics (Fig. 8.17).

After Kandinsky, the prominent representatives of abstraction in art have included the Russian painter Kazimir Malevich (1879–1935), the Dutch paint Piet Mondrian (1872–1944), and the American painter Jackson Pollock (1912–1956). Malevich brought geometric abstract to its ultimate and simplest form of expression, for example, in such *Black Square*. Both Malevich and his contemporary Mondrian had also deeply influenced by the Cubist movement (Fig. 8.18).

Pollock, inspired by the Surrealists, worked in a very different style, sometimes called action painting, which involved subconscious and bodily techniques such as the dripping and pouring of paint onto the surface of the canvas or even the hood of

Fig. 8.18 Action painting by Jackson Pollock

a car. The success that he and his fellow traveller Willem de Kooning (1904–1997) enjoyed (de Kooning was born in Holland and came to America as a stowaway) suggests the shift in the center of gravity of the art world from Europe to America in the second half of the twentieth century.

Applications of Mathematics

Theoretical Physics

At the beginning of this chapter, we mentioned that research in modern mathematics split into two major directions, pure mathematics and applied mathematics. The previous section introduced briefly the four main branches of modern abstract mathematics; the interactions between these branches also contributed to the birth of further branches, such as algebraic geometry, differential topology, and so on. Given the limitations space and scope of this book, we will not discuss these in any further detail. Instead, we turn now to the penetration of mathematics into the other intellectual crystallizations of human civilization, that is, the sciences, starting with physics. The eighteenth century had been the golden age for the synthesis of mathematics with classical mechanics, and in the nineteenth century, the greatest mathematical applications to physics occurred in the theory of electricity and magnetism, and its best representative was James Clerk Maxwell (1831–1879), associated with the mathematical physics school at Cambridge University. Maxwell established a complete system of electromagnetic theory consisting of four concise partial differential equations. He seems to have first developed a more complicated formulation, but started over on the basis of his belief that the mathematics representing the physical world should be beautiful (Fig. 8.19).

Fig. 8.19 Maxwell at Cambridge

Maxwell joined a long line of Scottish thinkers and inventors; indeed, this small country has contributed the largest number of inventors relative to its population of any in the world. Prior to Maxwell, there was James Watt (1736–1819), who contributed one of the early practical steam engines, and afterward, there appeared also Alexander Graham Bell (1847–1922), inventor of the telephone; John Macleod (1876–1935), a coauthor in the discovery and isolation of insulin; Alexander Fleming (1881–1955), who discovered penicillin; and John Logie Baird (1888–1946), who contributed to the invention of television and demonstrated the first true working television in London in 1927. Scotland was also home to Adam Smith (1723–1790), who presented the first complete and systematic theory of economics. The central concept of his masterpiece *The Wealth of Nations* is that the apparent chaos of the free market consists in fact of the workings of a self-regulating mechanism that tends as if automatically to the production of those products that are most desired and needed by society (Fig. 8.20).

After the advent of the twentieth century, mathematics has occupied the center of such disciplines in theoretical physics as relativity, quantum mechanics, and elementary particle theory. In 1908, the German mathematician Hermann Minkowski (1864–1909) proposed his four-dimensional spacetime model $\mathbb{R}^{3,1}$ equipped with the metric relation

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

where c is the speed of light. This provided the most suitable mathematical model for the special theory of relativity introduced only a few years early in 1905 by Albert Einstein (1879–1955); this model is now referred to as Minkowski space. Incidentally, Minkowski had been among the teachers of Einstein, although he was unimpressed by the mathematical ability of his early student.

Afterward, Einstein sought to expand his theory to account for the gravitational field; he achieved a basic outline of his new theory by summer of 1912, but he lacked sufficiently sophisticated mathematical tools to develop it completely. But

Fig. 8.20 Einstein's mathematics teacher, Hermann Minkowski

during this time, he reacquainted with an old classmate in Zurich who had since become a professor of mathematics, who introduced him to Riemannian geometry and more generally to differential geometry, which Einstein referred to as tensor calculus. After more than 3 years of hard work, in a paper completed on November 25, 1915, Einstein derived the gravitational field equations

$$R_{\mu\nu} = kT_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the stress-energy tensor, R is the scalar curvature, $g_{\mu\nu}$ is the metric tensor, and k is a constant related to the gravitational constant and the speed of light. With these equations in hand, Einstein remarked that the logical construction of general relativity was now complete.

Although Einstein had completed his derivation of the general theory of relativity in 1915, his work was published only the next year. It is fascinating that at almost the exact same time, the German mathematician David Hilbert obtained the same gravitational field equations from along a different line of thought. Hilbert took an axiomatic approach based on the theory of invariants for continuous groups developed by Emmy Noether. He submitted this paper to the Göttingen Academy of Sciences on November 20, 1915; it was published 5 days earlier than Einstein's paper.

On the basis of his theory of general relativity, Einstein predicted the existence of gravitational waves and black holes, which were confirmed experimentally in 2017 and 2019, respectively; more precisely, in 2017, scientists directly detected gravitational waves produced by a collision of binary neutron stars, and in 2019, the first photograph of a black hole was produced. These remarkable achievements were the result of a collaboration between many scientists from many different countries. Another consequence of general relativity is that spacetime taken as a whole is not uniform; it is uniform only across tiny regions. Mathematically, this nonuniformity can be expressed via the Riemannian metric

$$ds^2 = \sum_{\mu,\ \nu=1}^{2} g_{\mu\nu} dx_\mu dx_\nu.$$

The mathematical description of general relativity revealed for the first time the practical significance of non-Euclidean geometry and stands as one of the greatest achievements of applied mathematics in history. This perhaps does not quite place its realization on a level with the establishment by Newton of the law of universal gravitation, since Newton unlike Einstein also developed the entire mathematical basis for his new mechanics (Fig. 8.21).

In contrast with the theory of relativity, the development of quantum mechanics is not associated with the name of any single physicist but rather with an ensemble of scientists working around the same time. The pioneers were Max Planck (1855–1947), Einstein, and Niels Bohr (1855–1962) and subsequently Erwin Schrödinger (1887–1961), Werner Heisenberg (1901–1976), and Paul Dirac (1902–1984); they established formulations of quantum mechanics in terms of wave mechanics, matrix mechanics, and operator theory, respectively. The integration of these various theo-

Fig. 8.21 Einstein's home; photograph by the author, Bern

ries into a unified system called for new mathematical theories. Hilbert introduced analytical tools such as integral equations for this purpose, and John von Neumann (1903–1957) further extended what is known as the theory of Hilbert spaces to solve the eigenvalue problem in quantum mechanics. He also finally extended the spectral theory introduced by Hilbert to address the situation of unbounded operators that frequently arise in quantum mechanics. This laid the rigorous mathematical foundations for the discipline.

In the second half of the twentieth century, there were further developments in theoretical physics that required applications from the abstract branches of pure mathematics; two well-known examples are gauge theory and superstring theory. In 1954, the Chinese physicist Yang Chen-Ning (1922-), who shared a Nobel Prize in 1957 with another Chinese physicist Tsung-Dao Lee (1926-), and the American physicist Robert Mills (1927–1999) introduced Yang-Mills theory, which proposes gauge invariance as the common feature of the four fundamental forces of nature (electromagnetic force, gravitational force, and the strong and weak forces), bringing back into the spotlight the theory of gauge fields which by that time had already been long established. They attempted to achieve through this theory a unification of the interactions between known forces. Mathematicians quickly observed that the necessary mathematical tools were already available in the form of the fiber bundles of differential geometry. The Yang-Mills equations were recognized as a set of partial differential equations, and research into these equations has promoted the further development of mathematics. Another bridge between pure mathematics and theoretical physics by way of Yang-Mills theory came from the Atiyah-Singer index theorem, proved in 1963 and subsequently determined to have important applications in Yang-Mills theory. The research areas involved in this topic include analysis, topology, algebraic geometry, partial differential equations, functions of several complex variables, and other core disciplines in pure mathematics, a remarkable instance of the unity of modern mathematics.

Superstring theory, and string theory more generally, emerged in the 1980s. This theory views the elementary particles as a kind of stretch one-dimensional stringlike massless forms, about 10^{-33} centimeters in length (i.e., on the order of the Planck length), in place of the dimensionless points in spacetime that feature in other theories. This theory takes aim at a unified mathematical description of gravitation, quantum mechanics, and elementary particle interactions and has become one of the most active areas of collaboration between mathematicians and physicists. In particular, the mathematics involved includes differential topology, algebraic geometry, differential geometry, group theory, infinite-dimensional algebra, complex analysis, the moduli spaces of Riemann surfaces, and so on; countless physicists and mathematicians have now associated themselves with this research.

Biology and Economics

Outside of physics, mathematics has also played an important role in other disciplines in the natural sciences and social sciences. For reasons of space, we limit our discussion here to a treatment of mathematics in biology and mathematical economics as representative examples. Modern biology is a younger discipline than physics, which took off in earnest only after the invention of the microscope in the seventeenth century, but alongside physics these are the two most important disciplines within natural science. The introduction of mathematical methods to research in biology was also relatively slow to get off the ground, and the story begins at the start of the twentieth century, when the versatile British mathematician Karl Pearson (1857–1936) began to apply statistics to the study of problems in genetics and the theory of evolution. In 1901, he founded the journal *Biometrika*, the first journal in the discipline of biomathematics.

In 1926, Italian mathematician Vito Volterra (1860–1940) proposed the system of differential equations

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = cxy - dy \end{cases}$$

as a successful model of the dynamics of fish populations in the Mediterranean Sea. Here, x represents the number of small fish eaten as prey and y the number of large carnivorous fish. These equations, known also as the Lotka-Volterra equation, set a precedent for the use of differential equations in biological modelling (Fig. 8.22).

In 1953, 2 years after Hartline and Ratliff introduced their model, the American biochemist James Watson (1928-) and the British biophysicist Francis Crick (1916–2004) discovered the double helix structure of DNA (deoxyribonucleic acid); this not only marked the birth of molecular biology as a discipline but also introduced abstract topology as a tool in biology. Since the double helix strands exhibit winding

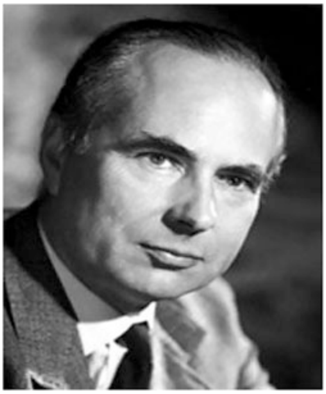


Fig. 8.22 Biologist Sir Andrew Huxley, grandson of the physiologist Thomas Henry Huxley and brother to novelist Aldous Huxley

Fig. 8.23 Watson and Crick display their DNA model

and kinking under the gaze of the electron microscope, a sub-branch of algebraic topology known as knot theory came into play, fulfilling a prediction made by Gauss more than a century earlier. In 1984, the New Zealand mathematician Vaughan Jones (1952–2020) established the Jones polynomial as an invariant of an ordered knot, which has proved useful to biologists for the classification of knots observed in the structure of DNA. Jones himself received the Fields Medal in 1990 for his work (Fig. 8.23).

Watson and Crick were awarded the Nobel Prize in Physiology or Medicine in 1962, and the significance of their discovery has still not been fully unraveled, and I would like to say here a bit more about it. We contrast the scope of various disciplines: physics and classical mechanics takes as its object primarily the macroscopic world, and the importance of the internal structure of atoms is seen also at the level of the large via the tremendous energy of nuclear fusion and fission; the objects of biology such as cells and genes on the other hand are mainly microscopic. Darwin's theory of evolution can be compared to Galileo's law of free fall insofar they express the external life, motion, and development of things. On the other hand, Newton's law of universal gravitation introduced the internal laws and causes governing the motions of objects, even the universe. The corresponding achievement to this in biology is precisely the discovery of the double helix structure of DNA, which reveals the internal mysteries of life. Watson and Crick announced this monumental result at the Eagle Pub in Cambridge, where they were frequent patrons alongside their various colleagues.

We discuss finally another pair of recipients of the Nobel Prize in Physiology or Medicine; in 1979, it was awarded to the South African-born American physicist Allan M. Cormack (1924–1998) and the British electrical engineer Sir Godfrey N. Hounsfield (1919–2004), both of them nonspecialists in biology. While he was working part-time in the radiology department at a hospital in Cape Town alongside

Fig. 8.24 Thomas Nash, mathematician and protagonist of the film *A Beautiful Mind*

his regular job as a physics lecturer, Cormack became interested in X-ray imaging of human soft tissue and tissue layers of different densities. After he began teaching in the United States, he established the mathematical basis for computerized scanning, specifically a formula for determining the amount of X-ray absorption in different human tissues. This formula was rooted in integral geometry and lay the theoretical foundations for digital tomography, which prompted Hounsfield to invent the first computerized tomography scanner (CT scanner), which achieved profound success in clinical trials (Fig. 8.24).

Leaving biology aside, we turn next to mathematical economics. This discipline was introduced by the Hungarian mathematician John von Neumann, who coauthored a book entitled *Theory of Games and Economic Behavior* in 1944, in which he proposed a mathematical model of competition and its application to problems in economics. A full half-century later, the American mathematician John Nash (1928–2015) and the German economist Reinhard Selten (1930–2016) shared the Nobel Prize in Economics for achievements in game theory. Nash was the subject of the successful film *A Beautiful Mind*, and he developed the concept of Nash equilibrium as an attempt to explain the dynamics of conflict and action between competitors. In the last year of his life, Nash was awarded the highest honor in mathematics, the Abel Prize, for his contributions to the theory of nonlinear partial differential equations.

Two relatively simple further contributions came from the Soviet mathematician and economist Leonid Kantorovich (1912–1986), who created the discipline of linear programming, and the Dutch-American mathematician Tjalling Koopmans (1910–1985), who studied in particular the relationship between inputs and outputs in production. They shared the Nobel Prize in Economics in 1975 for their contributions to the theory of optimal resource allocation. More profound mathematics also began to appear in economic applications as well: the French-born American economist Gerard Debreu (1921–2004) and the American economist Kenneth Arrow (1921–2017) introduced tools from topology into economics, in particular the theories of convex sets and fixed points. Following upon their research on equilibrium price theory, others added additional abstract mathematical concepts to the toolkit, including differential topology, algebraic topology, the theory of dynamical systems, and global analysis. These two also received Nobel Prizes in Economics, but many years apart from one another: Arrow in 1972 and Debreu in 1983.

Since the 1970s, stochastic analysis has emerged as a fundamental tool in economics. In particular, the American economist Fischer Black (1938–1995) and Canadian-American economist Myron Scholes (1941-) developed the Black-Scholes model, which reduces options pricing in the stock exchange to the solution of a stochastic differential equation to obtain the Black-Scholes formula, an option pricing formula that is consistent with real market behavior. Previously, investors struggled to precisely determine the value of future options, but with the introduction of this formula and its inclusion in the risk premium in the price of the stock, the complexity risk of investing in stocks was diminished significantly. Following upon their work, the American economist Robert C. Merton (1944-) removed many of the restrictions on this model, expanding the scope of its application to other areas of financial activity, such as residential mortgages. The Nobel Prize in Economics was jointly awarded to Merton and Scholes in 1997 for this work.

However, the development of the world economy in the twenty-first century has been significantly affected by the subprime mortgage crisis in the United States and the global financial crisis precipitated by it in 2008. In particular, people had become reluctant to apply for bank loans as they would under normal circumstances in such circumstances as poor credit conditions. As a result, many leading lending institutions began to issue loans under looser credit requirements but with higher interest. Such subprime loans involve a greater risk of default, mainly due to the derivative products based on them. The relevant departments were generally reluctant to take on risk on their own and instead sold package deals to investment banks or even insurance or hedging institutions. The derivative products became invisible and intangible, and their prices and packaging schemes were in accessible to estimation by ordinary human judgment; all of this required and encouraged the development of a new branch of mathematics, which became financial mathematics or quantitative finance.

The pricing process of derivatives involves two especially important parameters, the discount rate and the default probability. The former is a stochastic differential equation, and the latter is given as a Poisson probability density function. The global financial crisis made it clear that these and other methods related to pricing and estimation were in need of refinement. In the 1990s, the Chinese mathematician Peng Shige (1947-) and French mathematician Étienne Pardoux (1947-), who were born in the same year, collaborated to develop the theory of backward stochastic differential equations, which has become an important tool for the study of pricing of financial products. In the eighteenth century, Jacob Bernoulli had remarked that anyone who carries out research physics without understanding mathematics is actually investigating without sense. In the twenty-first century, this has also proven to hold for the financial and banking industries. Citibank, based in New York City, has claimed that some 70% of their business depends on mathematics, emphasizing that they could not survive without this dependence on mathematics.

Finally, we return to the linear programming theory of Kantorovich to remark that it was one of the earliest mature research branches of operations research, which is the study of analytic methods based in mathematics and logic for decision-making and organization management in order to obtain optimal results. This was born as a scientific discipline in the flames of World War II, alongside the applied mathematical disciplines of cybernetics and information theory, founded by the American mathematicians Norbert Wiener (1894–1964) and Claude Shannon (1916–2004), respectively. Both Wiener and Shannon were professors at MIT until their retirements and served as influential public figures. Wiener had received his doctorate from Harvard at the age of 18 and later published two autobiographies, *Ex-Prodigy: My Childhood and Youth* and *I am a Mathematician*. Shannon is widely considered the preeminent founding figure of the age of digital communication.

As formulated by Wiener, cybernetics takes as the object of its study laws of control and communication that govern both machines and living things, and the maintenance in such a dynamic system of stability or equilibrium under changing environmental conditions. He coined the name *cybernetics* for his new research program, borrowing from the Greek word κυβερνητική, meaning *governance* and derived in turn from the word for *navigation* or *steering*. Plato used this word often in his writings to describe the art of managing and governing human affairs. Information theory refers to the use of mathematical statistics to study the measurement, transmission, and transformation of information. It is important to point out however that information in this context has a specialized meaning and refers to a specific hierarchy order or degree of non-randomness that can be measured and quantified as precisely as mass, energy, and other such physical quantities.

Computers and Chaos Theory

As a definition, the word *computer* refers to any automated electronic device capable of storing and processing data according to programmatic instructions and returning the results of its operations as output. Throughout the history of computing, the most important figures contributing to its innovations have almost all been mathematicians. In China, computer science majors were for the most enrolled in mathematics departments through to the end of the 1970s, just as in the past, say in the time of Kant, mathematics was considered a part of philosophy departments. Today, most universities have one or two schools dedicated to computer science. It has long been a human desire to replace manual computation with automated machines; perhaps the best early example is the abacus, which may not have first been used by the Chinese people, but enjoyed the widest use for the longest period of time in China. In a book published in 1371, during the Ming dynasty, there appear illustrations showing the ten-speed abacus. In fact, its invention was much earlier. Later, the mathematician Cheng Dawei (1533–1606) published his *Suanfa Tongzong* (算法統宗, *General Source of Computation Methods*), in which he detailed the system and use methods of the abacus, marking its technological maturity. This book spread to Korea and Japan, where the abacus also gained widespread popularity.

The first to propose a mechanical calculating machine was the German scientist Wilhelm Schickard (1592–1635), who described his idea in a letter to Johannes Kepler. The first working mechanical calculator, capable of addition and subtraction, was invented by Pascal in 1642, and 30 years later, Leibniz created a calculator further capable of multiplication, division, and root extraction. A key step in the transition toward modern computing was achieved by the English mathematician Charles Babbage (1791–1871) who had the bold insight to make the arithmetic operations of his device programmable. In number theory, there is also a congruence relation related to the binomial coefficients named after Babbage. The Analytical Engine that Babbage proposed in 1837 as a successor to his earlier Difference Engine was divided into a storage component and a processing component, as well as a special mechanism for the operation of its programming. He envisaged for it the possibility of various arithmetical operations according to the instructions given in zeros and ones on punch cards; this was the prototype for the modern electronic computer (Fig. 8.25).

In a tragic turn, Babbage devoted the remainder of his life and most of his property to the promulgation of his ideas and inventions, to the extent that eventually he was compelled to turn his resignation as a Lucasian professor at Cambridge, but few people could understand his thinking. He seems to have had only three true supporters: his son, Major General Henry Prevost Babbage (1824–1918), who continued the struggle to promote the Difference Engine and Analytical Engine even after the death of his father; Luigi Menabrea (1809–1896), a professor of mechanics and construction at the University of Turin who later became Prime Minister of Italy; and Ada Lovelace (1815–1852), daughter of the poet Lord Byron. Ada was the only daughter of Byron and his wife, who separated a month after her birth. She compiled calculation programs for various functions and can therefore be regarded as the first modern programmer. Due however to the limitations of the times, there were huge technical obstacles to the implementation of the Analytical Engine, and the ingenious and forward-looking idea that Babbage dreamed up to control digital computers by general-purpose programs would not be realized for more than a century. From the beginning of the twentieth century, the rapid development of science and technology brought with it a mountain of new problems for data analysis. In particular, the computing needs of the military during World War II brought urgency to the requirement for improved computing speed. The first steps were the replacement of mechanical gears with electrical components. In 1944, the American physicist and mathematician Howard Aiken (1900–1973), working at Harvard University, designed and manufactured the first practical general-purpose

Fig. 8.25 Charles Babbage on a British postage stamp

programmable computer, which occupied a space of 170 square meters. The first of these made only partial use of electronic components, but he quickly followed up with another computer containing entirely electronic components, specifically relays. Meanwhile, at the University of Pennsylvania, computers were produced using vacuum tubes in place of relays. The first programmable, electronic, general-purpose, digital computer was the ENIAC (Electronic Numerical Integrator and Computer), produced the following year in 1945, a thousand times faster than the computer made by Aiken (Fig. 8.26).

In 1947, von Neumann arrived at the idea of replacing the external programs used by the ENIAC with internally stored programs. Computers made after this model operate according to stored instructions, and the programs can be modified by making changes to these instructions. A year earlier, von Neumann had coauthored a paper proposing a comprehensive structure for parallel programming and stored-program computers, which ideas had a profound impact on the design of later digital computers. John von Neumann was born in Budapest, Hungary, and became an extraordinarily prolific and versatile thinker; he made remarkable contributions to mathematics, physics, economics, meteorology, explosion theory, and computing. He is said to have met the designer of ENIAC while they were both waiting at a station for the train to arrive. The latter caught his attention and asked him to explain some technical problems related to computing (Fig. 8.27).

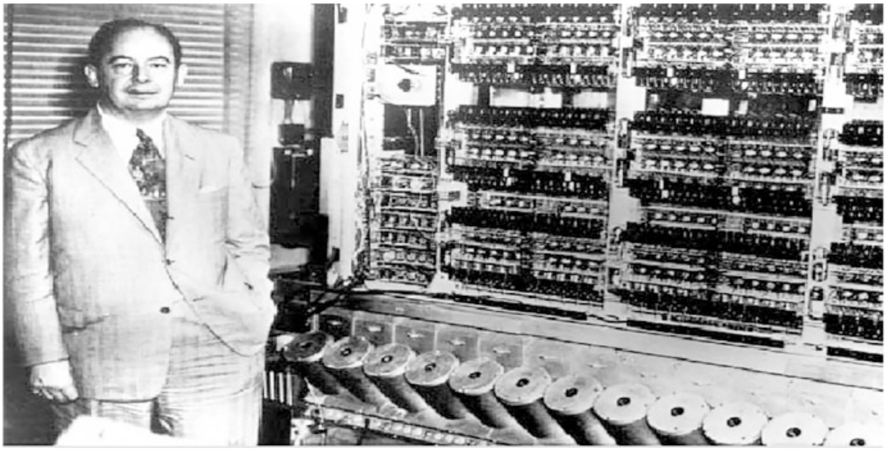


Fig. 8.26 John von Neumann with his computer

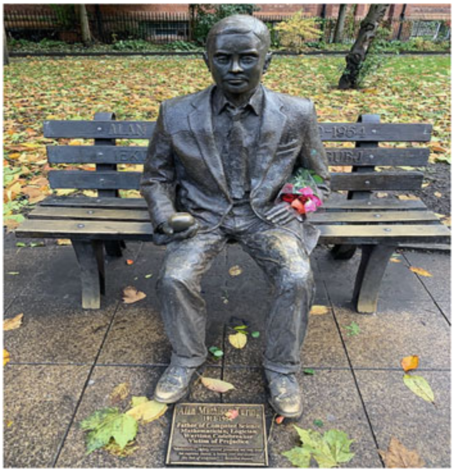


Fig. 8.27 Bronze statue of Alan Turing; photograph by the author, in Manchester

Another outstanding contribution to the concept and development of computer design came from the British mathematician Alan Turing (1912–1954). In order to solve theoretical problems in mathematical logic, in particular compatibility and the problem of mechanical determination of solvability or computability in mathematics, Turing introduced the concept of an abstract automatic machine (now referred to as a universal machine), an idealized model of the computer from which they have not departed to this day. This model comprises:

- Input and output (infinite memory tape divided into cells and a machine head capable of reading and writing)

- Memory (a table)
- Central processing unit (or control mechanism)

Turing also investigated the question of an artificial thinking machine, making him an early pioneer in the field of artificial intelligence. He proposed a famous standard for machine intelligence known as the Turing test, which requires that at least 30% of a team of human interrogators could not correctly identify the test subject as human or machine. His life had a tragic end, unfortunately: he had been persecuted and eventually prosecuted for his sexuality and died of poisoning after having eat an apple laced with cyanide shortly before his 42nd birthday. In 1966, Intel Corporation established the Turing Award, to this day the highest distinction in computer science. His death seems also to have inspired the logo of Apple Inc., founded in 1976 and famous around the world today for its computers and iPhones. The logo suggests also that only imperfection can lead to progress and the pursuit of perfection. Since 2019, Turing has appeared on the $50 banknote.

An interesting influence on Turing during his time at Cambridge was the mathematician G.H. Hardy (1877–1947), who was a natural leader in the mathematics department at Cambridge who is credited with establishing the Cambridge school of number theory. Hardy was obsessed with the Riemann conjecture and proved that there are infinitely many zeros along the critical line (the line in the complex plane with real part equal to $\frac{1}{2}$). Turing wrote the last research paper of his life on the Riemann hypothesis, in which he proposed a numerical method for verifying it and its implementation on an early computer. He seems to have believed that the Riemann hypothesis is false and hoped to find a nontrivial zero off the critical line through his method. Of course, he did not succeed; perhaps if he had, it would have furnished him with some encouragement and prevented him from succumbing to despair at the end of his life.

Through four successive generations of digital computers, from tubes and transistors to integrated circuits and eventually very massive integrated circuits, binary switches have remained a constant, and this will not change even if someday electronic computers are replaced, for example, by quantum computers (a recently developing kind of physical device that uses the laws of quantum mechanics to perform mathematical and logical operations at high speed and store and process quantum information; this discipline is called quantum computing). This is a natural extension of the system of symbolic logic developed by the British mathematician George Boole (1815–1864) in the nineteenth century. Boole completed work dreamed of by Leibniz two centuries earlier, the creation of ideographic symbols standard for simple or atomic concepts and their combination into complex ideas. He was born into a poor family, the son of a cobbler, and his knowledge of mathematics came mainly through self-study, eventually enabling him to earn a post as professor of mathematics at Queen's College, Cork, in Ireland and an election as a Fellow of the Royal Society. His life was cut short at the age of 49 by pneumonia brought on by a walk in heavy rain. Earlier in the same year, his youngest daughter Ethel Lilian Voynich (*née* Boole, 1864–1960), who went on to write the novel *The Gadfly*, was born.

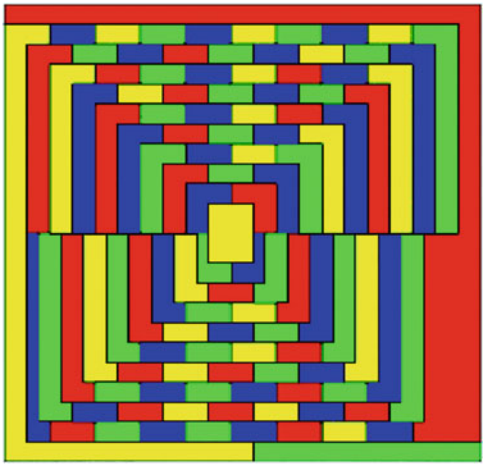


Fig. 8.28 Illustration of the four-color theorem of maps

As a shining example of the applicability of abstract mathematics, the computer has also become a powerful tool for mathematical research and even a source of new problems for mathematical inquiry, leading to the birth of the new branch of mathematics, computational mathematics. This branch is concerned with the design and improvement of various numerical methods, as well as problems of error analysis, convergence, and stability related to these calculations. von Neumann appears again here as an important early founder of this research area. He introduced a new method for numerical calculation known as the Monte Carlo method and led a team of researchers to use ENIAC to accomplish numerical weather prediction for the first time. The centerpiece of this effort was the solution of the relevant hydrodynamical equations. In the 1960s, the Chinese mathematician Feng Kang (1920–1993) created another method for numerical analysis known as the finite element method, independent of simultaneous research efforts in the Western world. The finite element method has found applications in the calculations involved in aviation, the study of electromagnetic fields, and the design of bridges (Fig. 8.28).

In the fall of 1976, two mathematicians at the University of Illinois named Kenneth Appel (1932–2013) and Wolfgang Haken (1928–) proved with the aid of computers a result known as the four-color theorem for maps, a problem with a history stretching back more than a century, perhaps the most inspiring example of the use of computers to solve a big problem in mathematics. The four-color theorem was proposed as a conjecture in 1852 by the British mathematician Francis Guthrie (1831–1899), who had just earned a double bachelor's degree at University College London. As part of his research, he undertook to color a map of the counties of England and noticed that four colors were sufficient to complete the task such that no two neighboring counties shared the same color. But neither he nor his younger brother, at that time still a student, could prove that this is always sufficient, and his well-established teachers, De Morgan and Hamilton, were also defeated by the problem. Arthur Cayley heard of the problem and presented it as a report to the London Mathematical Society, and it became a famous problem in mathematics.

Since that time, computers have become a powerful tool for the study of pure mathematics. Perhaps the most outstanding example of this is the discovery of solitons and chaos theory, the two core problems of nonlinear dynamics, which can be described as the two beautiful flowers of mathematical physics. The history of solitons predates the formulation of the four-color theorem. In 1834, the British engineer John Scott Russell (1808–1882) followed the water waves caused by sudden stops of ships in the canal on horseback and observed that they mostly maintain their shape and speed in the course of their propagation. He reproduced this effect in a water tank and named such waves of translation; today, they are referred to as solitons or solitary waves. More than a century later, mathematicians discovered that two solitons remain solitons upon collision, which explains the etymology of their name. Such waves occur in large numbers in optical fiber communication, activity at the Great Red Spot on Jupiter, nerve impulse conduction, and other fields. Chaos theory is another powerful tool for the description of irregular phenomena in nature and considered one of the major revolutions in modern physics following relativity and quantum mechanics.

The rapid development of computer science has not only been inseparable from mathematical logic but also promoted the transformation and even creation of other related branches of mathematics. A characteristic example of the former comes from combinatorics, while a field of the latter type is fuzzy logic. The origins of combinatorics can be traced by to the ancient Chinese legend of the *Luo Shu*. The term *combinatorics* was first proposed by Leibniz in his *Dissertation on the Art of Combinations* (*Dissertatio de arte combinatoria*). Over time, mathematicians resolved some substantial problems in this field, such as the Seven Bridges of Königsberg problem (which gave birth to graph theory, the main branch of combinatorial mathematics), the 36 officers problem, Kirkman's schoolgirl problem, and the problem of Hamiltonian cycles. But since the second half of the twentieth century, problems of computer system design and information storage and recovery have injected the study of combinatorics with a new and powerful impetus.

In contrast with the long history of combinatorics, fuzzy mathematics is a truly young discipline: it was introduced only in 1965. Fuzzy mathematics is established as an alternative to classical set theory, in which every set is defined as composed of its elements, and membership in the set is a clear and binary proposition, for example, given by the characteristic function

$$\mu_{\mathcal{A}}(x) = \begin{cases} 1 \text{ if } x \in \mathcal{A} \\ 0 \text{ if } x \notin \mathcal{A} \end{cases} .$$

In fuzzy logic, the characteristic function is replaced by a membership function satisfying $0 \leq \mu_{\mathcal{A}}(x) \leq 1$. In this case, $\mathcal{A}$ is called a fuzzy set, and $\mu_{\mathcal{A}}(x)$ the degree of membership of x in $\mathcal{A}$. The values $\mu_{\mathcal{A}}(x) = 1$ and $\mu_{\mathcal{A}}(x) = 0$ of classical theory correspond to 100% and 0% membership in $\mathcal{A}$, but such situations

Fig. 8.29 Lee Sedol does battle against AlphaGo in 2016

as $\mu_{\mathcal{A}}(x) = 0.2$ corresponding to 20% membership in $\mathcal{A}$ or $\mu_{\mathcal{A}}(x) = 0.8$ corresponding to 80% inclusion in $\mathcal{A}$ have no place in classical set theory.

Fuzzy mathematics was created in a paper by the mathematician Lotfi A. Zadeh (1921–2017), born in Azerbaijan but later based in Iran and eventually the United States. Since human thought encompasses both precise and fuzzy aspects, fuzzy mathematics has played an important role in the simulation process of artificial intelligences and related aspects of modern computer design. As a branch of mathematics, however, fuzzy mathematics is not yet fully mature (Fig. 8.29).

We now discuss artificial intelligence in more detail. The name and concept of artificial intelligence was first formally proposed at a research seminar hosted at the Dartmouth Institute in 1956. Its main practical goal is enable machines to carry out complex tasks that ordinarily require human intelligence, including language and image recognition and processing, robotics, and so forth, which involve tools from machine learning, computer vision, and other recent fields. The mathematical foundations of machine learning include statistics, information theory, and cybernetics, and the mathematical tools involved in computer vision also include projective geometry, matrix and tensor algebra, and model estimation. Artificial intelligence was considered alongside space technology and energy technology as one of the three most cutting-edge technological areas of the twentieth century, starting especially in the 1970s, and developments in artificial intelligence were rapid and plentiful in the past half-century, as were its applications in various fields with outstanding results. In the twenty-first century, artificial intelligence remains at the forefront, but the other two most cutting-edge technologies of our times are probably genetic engineering and nanoscience.

Artificial intelligence does not exhibit the same contours as human intelligence, but machines can think as a human does and may eventually surpass general human intelligence. One striking example of this was the 1997 defeat of the Azerbaijan-born Russian chess master Garry Kasparov (1963-) by the Deep Blue chess supercomputer developed by IBM. In 2016 and 2017, AlphaGo, developed by the subsidiary DeepMind Technologies of Google, also defeated two world champions of Go, Lee Sedol (1983-) of South Korea and Ke Jie (1997-) of China. Advances in this area have benefited from the development of cloud computing, big data, neural network technology, and the progression of Moore's law. At present, artificial intelligence has already surpassed human thought in terms of mechanical or logical reasoning, but achievements in cognitive emotion and decision-making remain very limited. Experts believe that artificial intelligence remains for the time being a mathematical problem and has not yet reach a stage of sufficiently advanced development to require ethical discussions as, for example, is the case for cloning technology.

We consider next cloud computing and big data. The cloud is a metaphor for the internet, and cloud computing refers to shared computing across a large number of servers distributed through the cloud. The user sends instructions to the service provider through his or her personal computer, and the service provider returns the result to the user via a calculation that can be compared to a nuclear explosion of computing activity. Since the era of cloud computing, big data has received more and more attention as a mode of thought. The explosion of data and its analysis have replaced the traditional cognitive tools of experience and intuition with an influence on decision-making in business, economics, and beyond. In 2013, the Austrian researcher Viktor Mayer-Schönberger (1966-) and the editor Kenneth Cukier (1968-) of *The Economist* published a book entitled *Big Data: A. Revolution That Will Transform How We Live, Work and Think* that has proved a pioneering work in the development of big data. The authors pointed out as their title suggests that big data and the resulting storm of information associated with it are transforming every aspect of our lives, thought, and work. Mayer-Schönberger believes that the core feature of big data is its predictive power, which suggests three subversive conceptual shifts: first, everything is data, and not random sampling; second, big data provides general direction rather than precise guidance; and third, correlation takes precedence over causality. The latter is equivalent to replacing the question *why?* with the question *what?*, which recalls also the traditional mode of thought of the Chinese people (Fig. 8.30).

As we have seen, every leap forward in computer technology has been inseparable from the work of mathematicians, but at the same time, advances in computing have promoted new directions in mathematical research. We introduce here a final example of a wonderful interaction between computer science and geometry. In the twentieth century, there occurred two great developments in geometry: in the first half of the century, the study of finite-dimensional spaces was extended to infinite-dimensional spaces, and in the second half of the century, integer-dimensional spaces were expanded to fractional-dimensional spaces. The latter refers to fractal geometry, which provides mathematical foundations for the emerging scientific discipline of chaos theory. The geometry of fractals was established through a study of self-similarity carried out by a Polish-born Lithuanian mathematician with

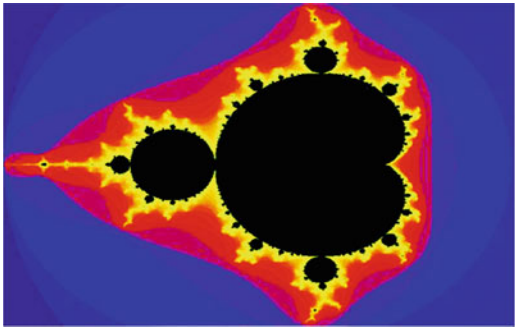


Fig. 8.30 The Mandelbrot set

dual French and American nationality named Benoit Mandelbrot (1924–2010). The new features uncovered by this geometry include spots, pits, broken, twisted, and winding and kinking spaces, which feature a kind of dimensionality not necessarily measured in integers.

In 1967, Mandelbrot published *How Long is the Coast of Britain?*. He had consulted the encyclopedias of Spain and Portugal, and Belgium and the Netherlands, and found that the estimates of these neighboring counties of their shared borders differed by up to 20%. It turns out that the length of a coastline or national border depends on the length of the scale used to measure it; for example, an observer attempting to estimate the length of a coastline from aboard a satellite will arrive at a smaller number than surveyor working directly on its bays and beaches. The latter in turn will provide a smaller number than say an erudite snail crawling across its every pebble.

Common sense suggests that while each of these successive estimates is larger than the last, they should converge toward a certain value that represents the true length of the coastline. But Mandelbrot proved that this is not so, and in fact every coastline is in a certain sense infinite, as its bays and peninsulas give way to smaller and smaller sub-bays and sub-peninsulas. This is a kind of self-similarity, a special type of symmetry with respect to scale that is associated with recursion and patterns within patterns. It is not a new concept, and in fact, it has ancient roots in Western culture. As early as the seventeenth century, Leibniz had imagined that a single drop of water includes within itself an entire variegated universe. Later, the English poet and painter William Blake (1757–1827) wrote in his *Auguries of Innocence*:

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour.

Mandelbrot considered the simple function $f(z) = z^2 + c$ where z is a complex variable and c an arbitrary complex parameter. Starting from an initial point x_0 and iterating this function generates a set of points $x_1, x_2, x_3, \ldots$ where $x_{n+1} = f(x_n)$. In 1980, Mandelbrot noticed that for some values of the parameter c, the values x_n would fall into a cyclical repetition or at least remain bounded in value, while for

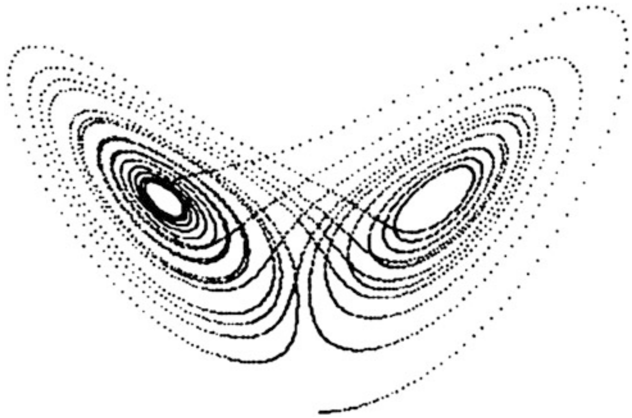


Fig. 8.31 The Lorenz attractor and chaos butterfly

other values of c, the values of x_n would explode without bound. Parameters of the former kind are called attractors, and the latter type chaotic; the set of all attractors in the complex plane is known now as the Mandelbrot set (Fig. 8.31).

Since the complex iterative process requires a huge number of calculations even for relatively simple equations (or dynamical systems), research into fractal geometry and chaos theory can only be carried out with the aid of high-speed computers. The visuals associated with this subject have proved popular as book illustrations and even wall calendars, but the practical applications are many: fractal geometry and chaos theory have been used to describe and explore many irregularities in nature, such as the shape of coastlines, atmospheric movements, ocean turbulence, wildlife, and even the fluctuations of stocks and funds.

In its aesthetics, this new geometry also brings the hard sciences in line with the particularities of modern taste, in particular the return to wild, uncivilized, and natural forms that became popular with postmodern artists since the 1970s. Mandelbrot expressed the view that satisfying art should not be fixed to any specific scale, or rather that it should contain attractive elements in every dimension. As an antithesis to the boxy skyscraper, he points to the Palais des Beaux-Arts in Paris, with its sculptures and gargoyles, horns and jambs, and swirls of arches and cornices with gutter dents, all of which present some pleasing detail to an observer situation at any distance away. As you approach it, the construction itself changes, revealing new structural elements.

Mathematics and Logic

Russell's Paradox

Since the twentieth century, the turn toward abstraction in mathematics has not only brought it into closer alignment with science and art but has also facilitated a resurgence in dialogue between mathematics and philosophy, for the third time considering the earlier periods of their sympathetic harmony, first in Ancient Greece and later in seventeenth-century Europe. It is perhaps no coincidence that mathematics has also struggled through three periods of crisis, corresponding to these moments in history. The first was the discovery of irrational numbers or incommensurable quantities in Ancient Greece, in contradiction with the doctrine that all numbers are represented by integers or ratios of integers. The second occurred in the seventeenth century, when calculus ran up against serious theoretical obstacles, and in particular the question whether an infinitesimal or vanishing quantity was identical with zero or in fact has some nonzero value. The problem is apparent: if it is zero, how can it appear as a divisor?; but if it is not zero, how is it permissible to eliminate terms involving infinitesimal quantities?

Recall that it was the Pythagoreans who first discovered that the diagonal of a square with unit sides is neither an integer nor can it be written as a ratio of integers. This triggered the first crisis, and one legend has it that the response was so severe that a disciple of Pythagoras named Hippasus who is credited either with revealing the existence of irrational numbers was thrown overboard into the Mediterranean Sea to drown for his offenses. In a strange coincidence, the birthplace Metapontum of Hippasus was also the site of the murder of Pythagoras. In any case, the crisis was resolved some two centuries later by Eudoxus, who introduced a geometrical formulation of incommensurable quantities. According to Eudoxus, two line segments are said to be commensurable if there is some third segment that can simultaneously measure each of them and otherwise incommensurable. For the sides and diagonal of a square, there is no such third line segment, and they are therefore incommensurable with one another. But as long as the existence of incommensurable quantities is admitted in geometry, the crisis is dissolved.

More than two millennia later, the birth of calculus introduced the second crisis of basic theoretical contradictions, sowing chaos within the foundations of mathematics. This crisis involved the definition of infinitesimal quantities, among the most basic concepts involved in calculus. In the course of very typical derivations, Newton would introduce the infinitesimal as a denominator by which to divide a quantity or expression; afterward, he would treat the infinitesimal as though it were zero and eliminate any terms still containing infinitesimal terms once the division is carried through. Although their application to mechanics and geometry allowed for no doubt that the formulas obtained by this process were correct, the process itself is logically self-contradictory, and this problem was not clarified until the first half of the nineteenth century, when Cauchy developed his

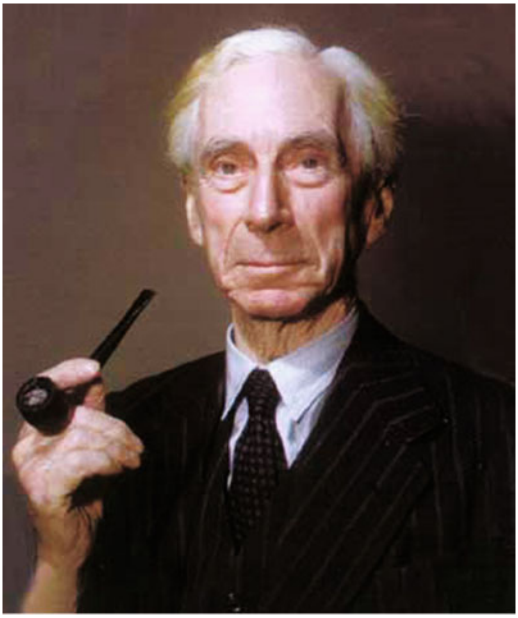


Fig. 8.32 The versatile Bertrand Russell

theory of limits. Cauchy treated the infinitesimal as an arbitrarily small but positive quantity quantified in such a way as to permit it to behave as a vanishing variable.

After the advances in analytical rigor at the end the nineteenth century and in particular its crowning achievement in the birth of set theory, mathematicians believed that it should be possible to eliminate all crises and even the possibility of crisis from the foundations of mathematics once and for all. In 1900, Henri Poincaré even declared to the International Congress of Mathematicians in Paris that complete rigor had at last been achieved. But a new paradox in set theory, which seemed the simplest and most clear of theories, provoked a new debate concerning the foundations of mathematics and triggered its third crisis. In order to resolve this crisis, mathematicians turned to a deeper consideration of the basis of mathematics and undertook the development of mathematical logic, another important trend in pure mathematics in the twentieth century (Fig. 8.32).

A key figure in this story is Bertrand Russell (1872–1970), who was born in 1872 into an aristocratic family in England. His grandfather had twice served as Prime Minister of the United Kingdom. Russell lost both his parents by the age of 3, and the strict puritanical bent of his subsequent education made him suspicious of religion as early as the age of 11. Rather, he began to consider the world always through a skeptical eye, inclined to consider how much we know and do not know and with what degrees of certainty and uncertainty. Starting around the onset of puberty, loneliness and despair began to take hold in his thoughts, and Russell struggled with suicidal thoughts. In the end, it was an obsession with mathematics than enable to him to break free of his darker impulses, and at the age of 18, he was admitted to Cambridge University after having spent the entirety of his previous

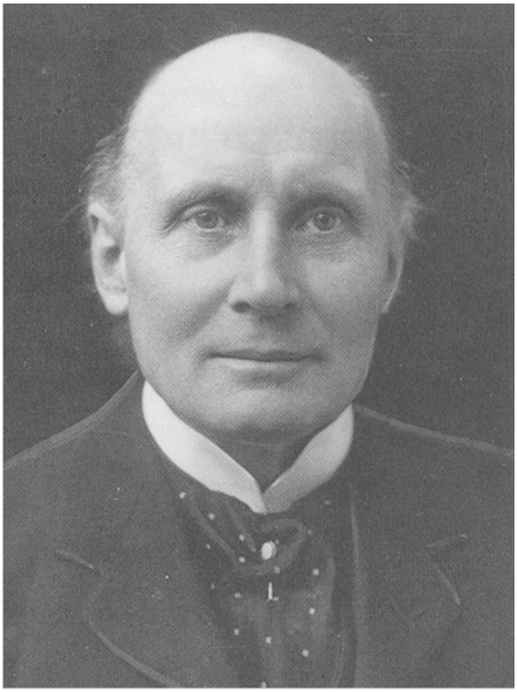


Fig. 8.33 Russell's teacher, Alfred North Whitehead

schooling at home. He continued to search for perfect and definite goals for his mathematical ambitions, but during his final year became attracted to the writings of Hegel and turned to philosophy (Fig. 8.33).

It seems obvious that the most natural area of research for Russell was in mathematical logic and philosophy of mathematics, which had been established not long earlier as a unified discipline by the German philosopher and mathematician Gottlob Frege (1848–1925). Fortunately, Cambridge University offered both fertile grounds and admirable colleagues for this pursuit. These included Alfred North Whitehead (1861–1947), a teacher and a friend; George Edward Moore (1873–1958), 1 year Russell's junior; and later his brilliant student Ludwig Wittgenstein (1889–1951). Russell was proficient early on in mathematics and a passionate believer in the basic correctness of the scientific worldview, and on this basis, he identified for himself three major goals as a philosopher. The first was to reduce the vanity and pretense to which human cognition is by nature subject to an absolute minimum and express himself as simply as possible, the second was to establish a link between logic and mathematics, and the third was to find a path of inference from language to the world it describes. These three goals were each of them eventually achieved with more or less success by Russell and his colleagues, setting the stage for analytic philosophy.

A significant factor in the wide reach of Russell's influence was also his natural ability as a popularizer. His philosophical prose is clear and beautiful, and many philosophers have been first drawn to the subject by way of his popular works, *Introduction to Western Philosophy*, *Wisdom of the West: A Historical Survey of Western Philosophy*, and even the somewhat more specialist work *Human Knowledge: Its Scope and Limits*. Russell was also prone to venture beyond the ivory tower in his writings and touch upon social, political, and moral issues, never shirking from addressing sensitive issues with passion. He was twice imprisoned, fined, and at one point dismissed from his position at Trinity College, Cambridge, for his controversial views and activities as a conscientious objector. Nevertheless, he was awarded the Nobel Prize in Literature in 1950. Later recipients of this award have also included writers with a background in mathematics: the Russian novelist Aleksandr Solzhenitsyn (1918–2008), who won the Nobel Prize in Literature in 1970, and the South African-Australian writer J.M. Coetzee (1940-) who won it in 2003 both studied mathematics as undergraduate students.

The paradox in set theory known as Russell's paradox goes like this: consider the menagerie of sets as divided into two categories. The first kind consists of sets that do not contain themselves as elements; most ordinary sets are like this. The second kind consists of sets $\mathcal{A}$ satisfying $\mathcal{A} \in \mathcal{A}$. An example of a set of this kind would be the set of all sets, if such a thing exists. It is obvious that every set $\mathcal{A}$ belongs to one of these kinds. Let $\mathcal{M}$ be the set of all sets of the first kind, that is, the set containing every set that does not contain itself. Then the natural question is, does $\mathcal{M}$ belong to the first kind or the second kind? Suppose it belongs to the first kind; then $\mathcal{M}$ does not contain itself, and it follows then from the definition of $\mathcal{M}$ that $\mathcal{M} \in \mathcal{M}$, a contradiction. But suppose instead that it belongs to the second kind. Then $\mathcal{M} \in \mathcal{M}$, from which it follows again by the definition of $\mathcal{M}$ that $\mathcal{M}$ is not an element of $\mathcal{M}$, another contradiction (Fig. 8.34).

Fig. 8.34 The village barber challenges the mathematicians

In 1919, Russell presented a colloquial version of this paradox known as the barber paradox:

> Consider a village barber who shaves all those and only those who do not shave themselves. Does this barber shave himself?

In both the formal and informal case, it is apparent that the construction leads to an unresolvable contradiction, and this pointed to a flaw in the very foundations of set theory as it had been formulated up to that point. Recall that the second crisis in mathematics, the crisis of calculus, had been resolved through the development of the theory of limits. But the theory of limits was in turn based on set theory. Therefore, the appearance of Russell's paradox in set theory formed an even deeper crisis for the foundations of mathematics.

In order to remove this paradox, mathematical logicians began to work toward an axiomatization of set theory. The first attempt was made by the German mathematician Ernst Zermelo (1873–1953), who put forward seven axioms that support a set theory free from paradoxes. This system was further refined by the German-born Israeli mathematician Abraham Fraenkel (1891–1965), resulting in **ZF** set theory, which remains the most widely used axiomatic foundation for mathematics in use today (commonly with the somewhat controversial axiom of choice append to it to form **ZFC** set theory). This eased the severity of the mathematical crisis, although nobody can prove that this system itself is consistent, and indeed it follows from Gödel's second incompleteness theorem that it cannot prove its consistency; few mathematicians however suspect that there are hidden inconsistencies lurking with ZFC, but there are nevertheless mysteries still to be unraveled in the foundations of mathematics. One particularly noteworthy example: the American mathematician Paul Cohen (1934–2007) proved in 1963 that the continuum hypothesis cannot be proved within the Zermelo-Fraenkel system, which taken in conjunction with an earlier result due to Kurt Gödel shows that in fact it is independent of the Zermelo-Fraenkel axioms, a resolution of sorts to Hilbert's first problem, and perhaps that most complete resolution of it that can be expected. Cohen received a Fields Medal in 1966 (Fig. 8.35).

Further efforts to find a logical solution to the paradox of sets led to the formation of three major philosophies of mathematics. The first is logicism, represented by Frege and Russell. The second was called intuitionism, introduced by the Dutch mathematician L.E.J. Brouwer (1881–1966), and the third was formalism, represented by Hilbert. The formation and activity of these competing schools of thoughts elevated the question of the foundations of mathematics to an unprecedented height. Although these efforts failed to achieve a completely satisfactory resolution to the situation, they contributed substantially to the formation and development of the program of mathematical logic first initiated by Leibniz. Due to space limitations, we present only a few of the arguments associated with each school below.

The first position is logicism, as promoted by Russell and his school. According to logicism, mathematics is simply an extension of logic, and there is no need to introduce any special axioms to demarcate the two. Rather, all of mathematics can be written in language of logic, mathematical concepts are simply a certain family of logical concepts, and mathematical theorems can be derived entirely

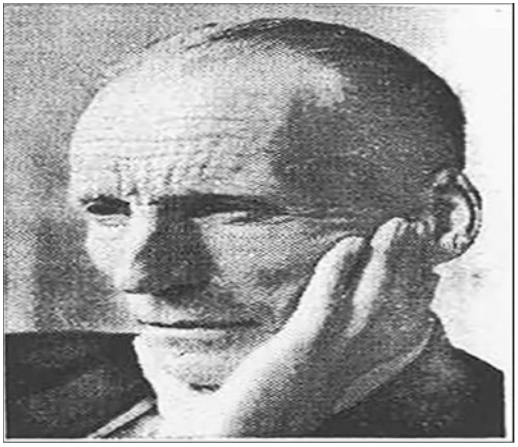


Fig. 8.35 L.E.J. Brouwer, one of the founders of topology, who introduced and proved the fixed-point theorem

from the axioms of logical and logical rules of deduction. As for the development of logic itself, the proceedings are entirely axiomatic. For the reconstruction of mathematics, the logicists first defined the theory of propositional functions and classes; proceeded to the construction of cardinal and ordinal numbers, and in particular the natural numbers; and on this basis established the real and complex number systems, functions, and analysis; the contents of geometry can also be fully reproduced atop these foundations. In this way, mathematics became the mathematics of philosophers, with no special content of its own, only a special form of logical thought.

Intuitionism stands in direct contrast with logicism and holds that mathematics exists independent of logic in the mental activity of humans. The essence of intuitionism is its insistence on purely constructive approaches to mathematical objects. Brouwer in particular held that the proof that this or that mathematical object exists is valid if and only if it as accompanied by a construction or proof of construction that can be carried out in finitely many steps. In set theory, for example, the intuitionists admit only the existence of finite constructible sets, in this way easily avoiding the paradoxes associated with infinite sets such as the set of all sets. One striking consequence of this perspective is that it necessitates the denial of the so-called law of the excluded middle, which states that either every proposition is true or its negation is true. It is also necessary to throw out the general theory of irrational numbers, and even the well-ordering principle of the natural numbers, which states that every subset of the natural numbers, including of course infinite subsets, has a smallest element.

Hilbert replied: "Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists[2]." As part of his criticism against intuitionism, Hilbert brought out

[2] Tr. Jean van Heijenoort

his long incubating Hilbert program for the foundations of mathematics, referred to later as formalism. The main idea is that the basic objects of mathematical thinking are mathematical symbols themselves, rather than any meaning attached to them say as geometrical or physical objects, and therefore that all of mathematics can and should be reduced to the laws governing the use of symbols in formulas, without any reference to their interpretation. Formalism absorbed some ideas from intuitionism, but retains the law of the excluded middle, and permits the fundamental transfinite axiom that goes a certain way toward proving the consistency of the theory of natural numbers, with some restrictions. Any hopes for a more complete realization of this program, however, were dashed by the work of a young logician named Kurt Gödel, as we will discuss in more detail below.

Wittgenstein

But before we discuss Gódel’s incompleteness theorems, we turn to one of Bertrand Russell’s most brilliant students and collaborators, Ludwig Wittgenstein (1889–1951), who elevated in his works the abstract discipline of logic to the heights of pure philosophy. Wittgenstein was born in Vienna in 1889 in a wealthy Jewish entrepreneurial family, the youngest of eight children. He was educated at home until the age of 14 and only afterward underwent formal schooling with some hardship. After a study of engineering in Berlin, Wittgenstein enrolled at Victoria University of Manchester in 1908 to pursue a doctorate. His focus was on aeronautical projects and patented the design of a propeller jet with small engines in 1911. All this fostered in him an interest in applied mathematics. His preference soon turned toward pure mathematics, and he became eager to understand more deeply the foundations of mathematics and eventually mathematical philosophy (Fig. 8.36).

In 1912, the 23-year-old engineering student made his way to Cambridge, where he spent five semesters at Trinity College and quickly caught the attention of the philosophers Russell and Moore, both of whom regarded him as an intellectual equal. The outbreak of World War I however led Wittgenstein to volunteer in the Austrian army as an artilleryman on the eastern front; he wound up in Turkey,

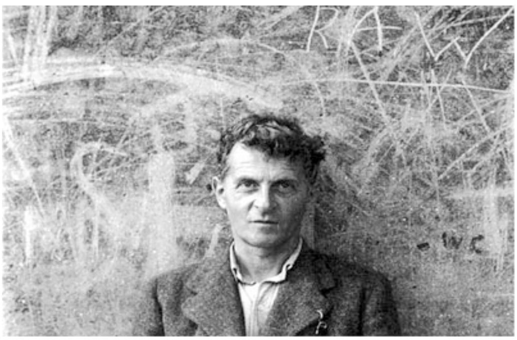


Fig. 8.36 Philosopher Ludwig Wittgenstein

where he was captured by Italian soldiers in winter of 1918. He lost contact with his connections at Cambridge, and Russell wrote in his *Introduction to Mathematical Philosophy*, published the following year, that it was not clear whether or not he was even still alive.

But in the same year, Wittgenstein wrote a letter to Russell from the prisoner-of-war camp where he was being held. He had read his former teacher's book while in prison and believed that he had answered successfully several of the questions raised within it. Both teacher and student hoped to meet as soon as possible after his release for a discussion of philosophy. By this time, however, Wittgenstein was destitute, having been persuaded by the writings of the great Russian author Leo Tolstoy to renounce his wealth and leave his considerable inheritance divided among his siblings under the condition that they not leave it in trust to him. Russell resorted to the sale of some of his furniture left behind in Cambridge in order to cover his travel expenses, and the two were able to meet at last in Amsterdam.

Wittgenstein is rare even among philosophers of genius for having developed two brilliant and highly original systems of thought at two completely different periods in his life, the two of them also very different from one another. The first of these is represented by his classic, the *Tractatus Logico-Philosophicus*, published in 1921, and the second by his *Philosophical Investigations*, published posthumously in 1953. Both of these works exhibit a refined and bold style of writing and thinking and exerted a profound influence on the course of subsequent philosophy. Apart from a short essay entitled *Some Remarks on Logical Form*, the *Tractatus Logico-Philosophicus* was the only work published by its author during his lifetime (Fig. 8.37).

This short book is an undisputed philosophical masterpiece, constructed from out of its central premise that philosophy in the final analysis is nothing other than the study of language. The central question of the book is *how is it that language can be language?*, prompted by a thoroughly familiar fact with which every living person is thoroughly familiar, but which surprised and amazed Wittgenstein: a person can understand a sentence that he or she has never heard before. He explains this fact as follows: a sentence or a proposition that describes something creates a picture of the world being a certain way. Propositions have a certain meaning, and the world has a certain state, and these are phenomena of the same kind. Wittgenstein argued that all proposition schemes and all possible states of the world are committed to the same logical form, which is simultaneously a form of representation and a form of reality.

The nature of this logical form itself however cannot be discussed; rather, it is meaningless in a very literal sense of this word. Wittgenstein makes this claim by way of a very famous analogy:

> My propositions serve as elucidations in the following way: anyone who understands me eventually recognizes them as nonsensical, when he has used them— as steps— to climb beyond them. (He must, so to speak, throw away the ladder after he has climbed up it.)[3]

[3] Tr. David Pears and Brian McGuinness

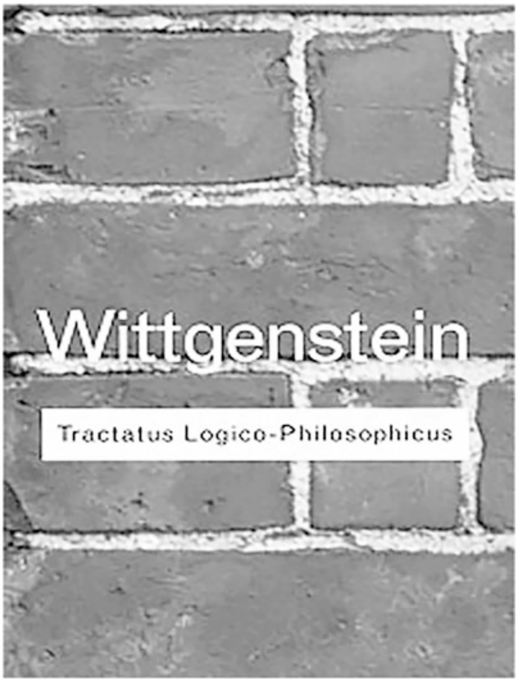


Fig. 8.37 The *Tractatus Logico-Philosophicus*

There are certain things that simply cannot be spoken in language: the necessary existence of the simple elements of reality, the existence of the self of thought and will, and the existence of absolute values. These inexplicable things cannot even be imagined, because the limits of language are identical with the limits of thought. The last sentence of the book is associated with its author as a kind of motto: *whereof we cannot speak, thereof we must be silent*.

Language had become the central topic in philosophy starting from the work of Gottlob Frege, mentioned above as the founder of modern philosophy of mathematics and who introduced the important distinction in language between *sense* and *reference*. Wittgenstein admired Frege deeply and visited him at the University of Jena in 1911 to show him some work on philosophy of mathematics and logic. In fact, he hoped to study under Frege, who recommended instead that he attend the University of Cambridge to learn from Russell. Wittgenstein later credited these two figures, Frege and Russell, as the source of his best ideas in philosophy. Frege was also an important influence upon the works of Russell and also Edmund Husserl; the former once communicated his deep admiration in a letter. Frege himself famously remarked, "Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician."

Wittgenstein believed sincerely that philosophy is not a merely theory or body of doctrine, but rather an activity whose goal is to clarify the propositions of

Fig. 8.38 Wittgenstein's tomb; photograph by the author, Cambridge

natural science and expose the emptiness of metaphysics.[4] Since he believed that his work in this direction was completed in the *Tractatus Logico-Philosophicus*, he disappeared from philosophy after its publication and spent the next several years work as a primary school teacher in mountain villages in southern Austria, having also previously built for himself an isolated log cabin in the remote Norwegian countryside. Eventually, he returned however to England and submitted the *Tractatus Logico-Philosophicus* as his doctoral dissertation to Cambridge University. Naturally, he earned his degree, and he was elected shortly afterward as a Fellow at Trinity College (Fig. 8.38).

Wittgenstein remained there as a lecturer for a further 6 years, during which time he became increasingly dissatisfied with the contents of the *Tractatus Logico-Philosophicus*. He began to dictate some new and original developments in his thought to two of his students. He paid a visit to the Soviet Union and considered settling there before spending a year in his cabin in Norway. He made his way again to Cambridge and succeeded the chair in philosophy vacated by Moore. After World War II broke out, he became disgusted with professional philosophy and worked instead a volunteer at Guy's Hospital in London and then as a laboratory assistant at the Royal Victoria Infirmary in Newcastle upon Tyne. It was during this time that he began the writing of *Philosophical Investigations*. After the conclusion of the war, he returned to Cambridge as a professor for a further 2 years before resigning finally and making his way to Ireland, where he spent 2 years finishing the book.

As for the *Philosophical Investigations*, although it is not so inextricably devoted to logic as the *Tractatus Logico-Philosophicus*, all the same it retains some connection with mathematics. In this masterpiece, Wittgenstein abandoned the idea of a unified nature underlying the endless varieties of language. He compares language to games, observing that there is no property common to all games, only a certain *family resemblance*. When we consider all the various activities that make

[4] cf. *Tractatus Logico-Philosophicus* **4.112**

up games, there emerges a complex web of overlaps and intersecting similarities, sometimes broad, sometimes in specific details.

In the course of his elucidation of this argument, Wittgenstein introduces as examples several integer sequences, since in his view numbers also constitute such a family of resemblances. His question is: what does it mean to grasp a mathematical pattern? One example is as follows. Suppose one person sees another write down the numbers

$$1, 5, 11, 19, 29, \ldots,$$

concluding with the notorious phrase, "and so on." Of course, there are various ways to continue the sequence, and the observer endeavors to write down various formulas to describe it, for example, $a_n = n^2 + n - 1$. Or even without identifying this formula, he or she recognizes that the first number is $1^2 + 0$, the second number $2^2 + 1$, and the third $3^3 + 2$ and therefore obtains the next number as $6^2 + 5 = 41$ or notices instead that the differences between pairs of successive numbers make up the arithmetic sequence

$$4, 6, 8, 10, \ldots$$

and on this basis concludes that the next number should be $29 + 12 = 41$. In any case, it requires little effort to continue.

His point is that it is not necessary to derive an explicit formula to have successfully grasped the pattern governing the sequence. On the other hand, it is imaginable that the viewer equipped with the formula may experience a comprehension of the sequence that extends no further than the contents of the formula, unaccompanied by any intuitive epiphany or other special experience. The lesson of it all is that a pattern is not the same thing as a straitjacket; at all times, we are free to accept or reject the dictates of the pattern. He also insisted that the outcome of the mathematical process is not predetermined: although we follow a procedure that seems clear to us, we cannot predict exactly where it will lead.

Gödel's Theorems

At the end of the last century, the American magazine *Time* published its list of the hundred most influential people of the previous hundred years, one-fifth of which consisted of leading scientists and technological and academic figures. Among these 20, one of them was a philosopher and the other a mathematician. The philosopher was Wittgenstein, and the mathematician was Kurt Gödel, to whom we turn now. In fact, these two have much in common: both occupied an intellectual position at the intersection of mathematics and philosophy, and both were Austrian but wrote in English as a second language. But one made his way to England and Cambridge University to pass the latter part of his life, and the other to the United States and

Fig. 8.39 Kurt Gödel

Princeton University. And of course, neither of them remained Austrian citizens by the time of his death (Fig. 8.39).

In 1906, Gödel was born in Brünn in Austria-Hungary, known now as Brno in the Czech Republic. It was in a monastery in this city that the nineteenth-century Austrian geneticist Gregor Mendel (1822–1844) discovered the principles of genetics, and it was also home to the Czech composer Leoš Janáček (1854–1928). As for the broader Moravia region, both the father of psychoanalysis Sigmund Freud (1856–1939) and the father of phenomenology Edmund Husserl (1859–1938) were born there. Husserl had a background in mathematics and earned his doctorate from the University of Vienna for a thesis entitled *Contributions to the Calculus of Variations*. Gödel also ended up at the University of Vienna, after spending his youth entirely in his hometown, and he studied theoretical physics there before developing a keen interest in mathematics and philosophy and teaching himself mathematics to a more advanced level (Fig. 8.40).

By his third year at university, Gödel was entirely preoccupied with mathematics, and his library card for this period showed that in particular he read a number of works devoted to number theory. He also began to participate in some of the proceedings of the famous Vienna Circle, having been introduced to him by his mathematics teacher. The Vienna Circle comprised an assortment of philosophers, mathematicians, and scientists who met to discuss primarily the linguistic nature and methodology of science; this group came to occupy an important position in the history of twentieth-century philosophy. At the age of 23, Gödel was the youngest of 14 members to attach his name to the manifesto of the Vienna Circle, *Wissenschaftliche Weltauffassung: Der Wiener Kreis* or *The Scientific Conception of the World: The Vienna Circle*. The following year, he completed his doctorate on the

Fig. 8.40 Gödel and Einstein

basis of a remarkable thesis *On the Completeness of the Logical Calculus*. Not long afterward, he obtained his world-shatter first and second incompleteness theorems.

In January of 1931, when he had not yet reached the age of 25, Gödel published his *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I* or *On Formally Undecidable Propositions of Principia Mathematica and Related Systems I* in the *Monthly Journal of Mathematics and Physics* of Vienna. Within a few years, it was already considered among the most monumental milestones in the history of mathematics. The results of this paper are of their nature first and foremost negative results, overturning the belief among mathematicians of every stripe that mathematics as a whole could be subject to axiomatization and eradicating any hope of proving the internal consistency of mathematics as envisioned by Hilbert. But at the same time, this negative result eventually led to an epochal change in basic mathematics research, introducing a fundamental distinction between the concepts *true* and *provable* and also introducing analytic logic to the basic toolkit of mathematical thought.

Gödel's first incompleteness theorem states:

> Any consistent formal system F that is strong enough to carry out the basic arithmetic of numbers contains statements S such that both S and its negation are both not provable within F.

In brief, any consistent axiomatization of the natural number system is incomplete. It follows immediately that no formal system completely describes all of mathematical theory. A few years later, the American mathematician Alonzo Church (1903–1995) proved an even stronger result along the same lines: (Church's theorem) given any consistent formal system strong enough to contain the natural number system, there is no algorithmic decision process to determine whether a given arbitrary proposition is or is not a theorem of the system.

On the basis of his first incompleteness theorem, Gödel also proved the second incompleteness theorem:

> If F is a consistent formal system strong enough to contain the natural number system, then the consistency of F cannot be proven within F.

In other words, among the propositions of the system that are true but unprovable, within it occurs the proposition that the system itself is consistent. This put a full stop to the hopes of Hilbert and his program. It appeared now that the internal consistency of classical mathematics cannot be obtained except by way of sophisticate principles of reasoning that are subject to questions of consistency no less worthy of suspicion as the question of the consistency of classical mathematics itself.

Taken together, the two incompleteness theorems show that basic mathematics is as a whole out of the reach of axiomatization and furthermore that it is impossible to guarantee that mathematics harbors no hidden inconsistencies. These are strict limitations of the axiomatic approach and suggest that the procedure of mathematical proof cannot and does not conform to the procedure of formal axiomatization. Taken in a positive light, they suggest also that the role of human intuition and insight in mathematics cannot be fully formalized. In formal systems, it is possible to mechanically reproduce the provable content, but this is guaranteed not to exhaust the full spectrum of true statements within the system. Or in other words, all provable statements are true in the system, but not all true statements are provable within it (Fig. 8.41).

Gödel's two incompleteness theorems are indisputably among the most important theorems in the history of mathematics; we do not prove it here, since the proof is more technical than the general tenor of this book. It is worth mentioning however that the concept of a recursive function that appears in the proof was proposed to Gödel in a letter from a friend, who died suddenly and unexpectedly 3 months after writing it. After the appearance of the incompleteness theorem, recursive functions

Fig. 8.41 Gödel's tomb; photograph by the author, Princeton

became widely known and used and eventually formed the basic starting point for the theory of algorithms. It was also this idea that led Turing to develop his idea of Turing machines and universal Turing machines, another foundational moment in the history of the electronic computer. Since that time, the controversy surrounding paradoxes and mathematical foundations has settled a bit, and concerns about such questions do not much intrude upon the daily work of ordinary mathematics; they did however contribute to a resurgence of interest and energy in mathematical logic, leading to a flurry of development within this discipline.

Conclusion

In modern times, the natural progression toward increased division of labor has led to an extension of the period of time dedicated to studies among aspiring scholars in various fields, and the content of their studies has become more complex and abstract. This is the case not only in mathematics but in every area of human civilization. In poetry, it is no longer possible to compose clear and simple poems such as *Climbing Stork Tower* by Wang Zhihuan (688–742); in mathematics, such easily derived low hanging fruit as Fermat's little theorem seems to have been exhausted. Simultaneously, in mathematics, in the natural sciences, and in the arts and humanities, there have also been great changes in aesthetic preferences and conceptions, and complexity, abstraction, and depth have become completely standard measures of judgment (Fig. 8.42).

This is not to say that abstraction has not relegated pure mathematics to the back shelf; if anything, its application is wider today than ever before, further confirming that the process of abstraction in mathematics is altogether in line with the developments and changes in social trends more broadly. With the birth of calculus, mathematics had emerged as a powerful tool in the course of the scientific and technological revolutions of the seventeenth and eighteenth centuries, with mechanical motion as the main protagonist. After 1860, the new stars of the tech-

Fig. 8.42 Notre-Dame du Haut Chapel by Le Corbusier (1953), in Ronchamp, France

Fig. 8.43 The Guggenheim Museum in New York City, by Frank Lloyd Wright (1959). Photograph by the author

nological revolution appeared: generators, motors, and electronic communications. Finally, since the 1940s, electronic computers, atomic energy technology, space technology, the automation of production, and communications technology have all been inseparably linked to mathematics. The branches of mathematics called upon by newer fields of science such as relativity, quantum mechanics, superstring theory, molecular biology, mathematical economics, and chaos theory in particular are esoteric, abstract, and modern (Fig. 8.43).

With the progression of science and technology and the increasingly complex developmental needs of human society, new mathematical theories and disciplines are constantly appearing. Here, we present two examples: catastrophe theory and wavelet analysis. Catastrophe theory was introduced in 1972 by the French topologist and Fields Medalist René Thom (1923–2002) in his book *Structural Stability and Morphogenesis*; its object of study is the methodology and classification of system control variables subject to sudden massive shifts in behavior. As a mathematical discipline, it is a branch of geometry, and the behavior and trajectories of its variables occur as curves or surfaces. An example of its application is the arch bridge, which deforms at first more or less uniformly under pressure until the load reaches a certain critical point, after which the shape of the bridge undergoes an instantaneous change and it collapses. Concepts from catastrophe theory were later used by sociologists to study such phenomena as gang warfare.

Turning next to wavelet theory, it has sometimes been referred to as the microscope of mathematics, and it represents a milestone in the development of harmonic analysis. Around the year 1975, the French geophysicist Jean Morlet (1931–2007) invented the word *wavelet* to describe functions he was using to study signal processing problems for oil prospecting. Wavelet analysis or wavelet transform refers to the use of wavelike oscillations with finite length and fast decay to represent signals. As with the Fourier transform, these oscillations can be written as a sum of sinusoidal functions, but wavelets are local with respect to both time and frequency, whereas the Fourier transform in general is local only with respect to frequency. The computational complexity of the wavelet transform is also small: it is of $O(N)$, in comparison with the time $O(N \log N)$ required for the fast Fourier transform. In addition to signal analysis, wavelet analysis has been used for military intelligence, computer classification and recognition problems, music and language synthesis, mechanical fault diagnosis, data processing for seismology, and so on. In medical imaging in particular, the wavelet transform allows for fast imaging times and improved resolution in B-scan ultrasonography, CT, and MRI.

The mainstream of mathematics in the twentieth century can be described as structural mathematics, promoted and developed by a major school of French mathematicians known pseudonymously as Nicolas Bourbaki. The research objects of structural mathematics are not the classical objects of numbers and shapes in any traditional sense, and mathematics is no longer split up into the clean disciplines of algebra, geometry, and analysis, but rather organized according to the occurrences within it of equivalent structures. For example, linear algebra and elementary geometry are isomorphic to one another in the sense that it is possible to carry out a complete translation of statements between the two, and in this sense, they are considered simultaneously. The mathematician and historian of mathematics André Weil (1906–1998), who was a major figure in the Bourbaki school and a recipient of the Wolf Prize in Mathematics, was close with the cultural anthropologist Claude Lévi-Strauss (1908–2009), who borrowed structuralist ideas to study the mythologies of various cultures. He identified various *isomorphic correspondences* between them, a striking example of the influence of the new mathematics on linguistics and anthropology. This inaugurated a new trend in French philosophy in the 1960s known as structuralism. Its most famous adherents were Jacques Lacan (1901–1981), Roland Barthes (1915–1980), Louis Althusser (1918–1990), and Michel Foucault (1926–1984), who used structuralist ideas to investigate psychoanalysis, literature, Marxism, and socio-historical topics, respectively. Jacques Derrida (1930–2004) introduced his influential theory of deconstruction as a critique of linguistic structuralism.

Looking now to the future, the major question facing mathematics is whether or not it can achieve some kind of unification. This has long been a preoccupation among mathematicians: as early as 1872, in the second year of German reunification, the young German mathematician Felix Klein (1849–1925) published his famous *Erlangen program*, an attempt to unify modern geometry and mathematics from the perspective of group theory. The Erlangen program developed from collaborations with the Norwegian mathematician Sophus Lie (1842–1899), inventor of Lie groups and Lie algebras, and took its name from the university at which Klein was employed at the time, now known as the University Erlangen-Nürnberg, in Bavaria. Lie groups also played a deep role for the Bourbaki school, who regarded them as a synthesis of group theory and topology. The group theoretical perspective has since become commonplace in every area of mathematics, but the full achievement of the goals set forth by the Erlangen program has remained out of reach.

Nearly a century later, the Canadian mathematician Robert Langlands (1936-) set up the banner of his *Langlands program*. In a 1967 letter to Weil, and then in

Fig. 8.44 The Beijing CCTV Headquarters, by Rem Koolhaas and Ole Scheeren (2007)

1970, Langlands proposed a series of conjectures entailing a web of relationships intertwining the Galois groups of number theory, automorphisms in analysis, and representation theory in algebra. Langlands was awarded the Abel Prize in 2018. Meanwhile, André Weil, whose sister was the famous philosopher Simone Weil (1909–1943), proposed in 1948 an analogue to the Riemann hypothesis in algebraic geometry, which was later proved by the Belgian mathematician Pierre Deligne (1944-), using methods pioneered by his uniquely brilliant mentor and doctoral advisor, the stateless mathematician Alexander Grothendieck (1928–2014). Both Grothendieck and Deligne received Fields Medals, in 1966 and 1978, respectively (Fig. 8.44).

On the other hand, although there has emerged since the nineteenth century a trend toward the interpenetration and integration of disparate subjects in mathematics, which has led to the formation of new disciplines, at present, mathematics as a whole is still a highly differentiated domain, characterized in modern times by abstraction and generalization, but also intense specialization. A very considerable portion of new mathematics is necessarily divorced from the natural world and scientific applications, perhaps a troubling phenomenon. It is reasonable to ask then if abstraction or structuralism can provide a framework for mathematical unification. Certainly it is possible, but it seems also likely that mathematics cannot become unified in a context of isolation within itself.

There is an analogy to be drawn with art, where collage has gradually become a central technique and in some cases even the predominant conception of art. Modern philosophers have also embraced collage as a kind of ideal myth. In the past, collage was considered primarily as an artistic technique involving the random combination of unrelated pictures, words, sounds, and so on in order to produce a special effect. Today, it seems that the range of this word should be expanded to include the combination of disparate ideas. In this sense, collage has played a role in modern

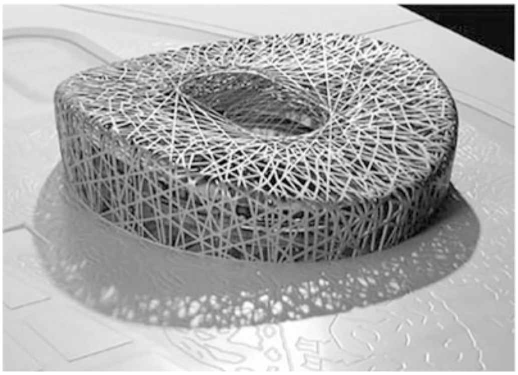


Fig. 8.45 The Beijing National Stadium, or Bird's Nest, by Jacques Herzog and Pierre de Meuron (2008)

mathematics, even in the nature of modern civilizations. For example, many of the new interdisciplinary topics in mathematics could be considered as an instance of collage. To some extent, collage and abstraction are identical phenomenon, except that the use of one word is more common in the art world and the other in mathematics (Fig. 8.45).

For reasons of space, we have only considered the medium of painting, but abstraction has also occurred in other forms of art. Architecture, for example, has undergone tremendous changes with respect to content, form, and decoration. In his classic *De architectura*, the Roman architect Vitruvius held up the three words *strength*, *utility*, and *beauty* as the cornerstone of architecture, and these three words became the basic criteria for quality of buildings or architectural plans. In the Renaissance, Alberti subdivided the category of *beauty* into *the beautiful* and *the decorative*, where *the beautiful* is defined by harmonious proportion and *the decorative* consists of mere auxiliary splendor. Since the twentieth century, architects have rejected the dismissal of ornament as auxiliary splendor and treated it rather as an indispensable and ubiquitous aesthetic component, not unlike collage for painting. Geometric figures, both classical and modern, have played a particularly important role here.

Like music, painting, architecture, and the other arts, mathematics is without borders and suffers little from the limitations of language barriers. It has been an essential part of human civilization, and it seems not unreasonable to speculate that if there exists any alien civilization, mathematics has played just an important role there as it has here. Indeed, if extraterrestrial intelligences exist, it seems very possible that they can understand mathematics and may even be proficient in it, and many have suggested that mathematics is the most suitable arena for the first attempts at communication. As early as 1820, Gauss proposed to use a graphical proof of the Pythagorean theorem cut into the vast Siberian forest as a signal to space indicating the presence of human civilization. Some 20 years later, the Austrian astronomer Joseph Johann von Littrow (1781–1840) proposed instead to fill a large circular canal dug out in the Sahara desert with burning kerosene for the same purpose.

In any case, they both agreed that signals containing such prominently mathematical imagery should attract the attention of any intelligent alien life, although neither of these ideas was ever put into practice. Carl Devito, a mathematician at the University of Arizona, has argued that accurate communication with a civilization from another planet must start from an exchange of scientific information, with the first step being the establishment of units of measurement. In recent years, he has collaborated with a linguist in an attempt to construct a language derived from universal scientific concepts. For example, differences in the chemical composition of the atmosphere or the energy output of a planet may facilitate communication. The basic idea is that both civilizations should have arrived at mathematical methods and computations, discovered chemical elements and the periodic table, and carried out quantitative studies of the states of matter.

But of course there remain many difficulties and obstacles in the way of communication with an alien civilization even in the case of contact. Perhaps they have derived their laws of motion along very different mathematical lines and arrived at formulations very different from the ones with which we are familiar. The mathematical basis for our study of motion is calculus; indeed, calculus is the basis for many fields of science. Should this also be true of an alien civilization? Or as another example, will the natural starting point in geometry for a distant civilization be Euclidean as it was for ours or some non-Euclidean geometry? Their physics may be so different from ours that they would not recognize the theory of our solar system introduced by Copernicus or our picture of the universe. And afterward, there is the equally challenging question: how to present other aspects of human civilization in terms of mathematics. It is exactly this question, which still stands in need of much intercultural research and further discussion, that this book has endeavored to explore.

Appendix A
A Mathematical Chronology

3000 BCE Hieroglyphic numbers appear in Egypt

2400–1600 BCE Hexadecimal numbers and arithmetic appear in Babylonian texts; the Pythagorean theorem is already known

1850–1650 BCE Decimal arithmetic in use in Egyptian papyrus books

1400–1100 BCE Decimal digits in use on oracle bones in China; by the eleventh century, the Duke of Zhou and Shang Gao are familiar already with the 3–4–5 Pythagorean triple

Ca. 600 BCE Thales introduces the demonstration of propositions in Greece; in China, Rong Fang and Chen Zi are aware of the Pythagorean theorem

Ca. 540 BCE The Pythagoreans prove the Pythagorean theorem; discovery of the incommensurability of $\sqrt{2}$

Ca. 500 BCE The *Shulba Sutras* in India provide an accurate value for $\sqrt{2}$; the Pythagorean theorem is already known in India

Ca. 460 BCE Greek philosophers introduce the three classical problems of geometric construction

Ca. 450 BCE Zeno and Eleatic philosophers in Greece propose Zeno's paradoxes

Ca. 380 BCE Plato establishes his Academy in Athens and advocates the cultivation of logical thinking through the study of geometry

Ca. 335 BCE Eudemus of Rhodes writes *History of Geometry*; this establishes him as the first historian of mathematics

Ca. 300 BCE Euclid writes his geometrical masterpieces, *Elements*; this marks the appearance of deductive mathematics with axiomatic foundations

287–212 BCE Archimedes of Syracuse in Greece obtains the formula for the volume of the sphere and an approximation for π, using methods that anticipate modern calculus

230 BCE Eratosthenes invents a sieve method, known today as the sieve of Eratosthenes, to determine a table of prime numbers

225 BCE Apollonius of Perga writes his treatise, *Conics*

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Ca. 150 BCE The earliest mathematical text in China, the *Book on Numbers and Computation*, is written; afterward, there appear also the *Zhoubi Suanjing* and the *Nine Chapters on the Mathematical Art*

Ca. 150 CE In Greece, Ptolemy writes the astronomical treatise *Almagest*, which introduces developments in the field of trigonometry

Ca. 250 Diophantus in Greece writes *Arithmetica*, in which he introduces indefinite equations, the concept of unknown quantities, and equations containing symbols for known and unknown quantities

Ca. 370 Hypatia is born in Alexandria, the first female mathematician known to history

462 Zu Chongzhi of China calculates an approximation for π accurate up to seven decimal places, given by the ration $\frac{355}{113}$

820 Al-Khwarizmi writes the treatise *Algebra*, from which the modern word for this subject was derived after its introduction to Europe in the twelfth century

850 Mahavira in India writes his *Compendium on the Gist of Mathematics*, in which the familiar formula for the binomial coefficients appears for the first time

Ca. 870 Decimal numerals appear in India, including the zero numeral; later, these spread to Arabia and become the Hindu-Arabic numerals in use today

1100 Omar Khayyam in Iran uses the intersection of a circle and a parabola to solve geometrically a cubic polynomial equation

1150 Bhaskara II recognizes negative numbers and admits the existence of irrational numbers

1202 Fibonacci of Italy writes *Liber Abaci*, in which appears a famous rabbit problem from which derives the Fibonacci sequence

1247 Qin Jiushao in China writes *Mathematical Treatise in Nine Sections*, containing the Da Yan Shu method and Qin Jiushao's algorithm

1482 The first Latin translation of Euclid's *Elements* is published

1545 Cardano of Italy writes *Ars Magna*, containing the general solutions of cubic and quartic polynomials

1572 Bombelli of Italy writes *L'Algebra*, introducing the rudiments of the theory of imaginary numbers

1591 Vieta of France identifies relations between the roots of a polynomial and its coefficients and inaugurates the use of modern symbolic algebra

1614 Napier of Scotland invents logarithms

1629 Dutch mathematician Albert Girard proposes the fundamental theorem of algebra

1637 In France, Descartes invents analytic geometry, and Fermat proposes Fermat's last theorem

1642 Pascal invents in France the first mechanical calculator in the world, capable of addition and subtraction

1657 Huygens of the Netherlands introduces the concept of mathematical expectation, building upon earlier discussions of probability in communications between Pascal and Fermat

1665 Newton in England and Leibniz in Germany invent the calculus, the former using a geometric method of fluents and fluxions and the latter algebraically; Leibniz publishes first

1666 Leibniz writes *De Arte Combinatoria*, the founding document of mathematical logic

1680 Seki Takakazu of Japan develops *wasan* mathematics and develops the theory of determinants

1736 Euler of Switzerland solves the problem of the seven bridges of Königsberg and establishes the disciplines of graph theory and geometric topology

1777 Buffon in France introduces Buffon's needle problem, promoting further development in probability theory

1799 Monge of France creates descriptive geometry

1801 In Germany, Gauss writes *Disquisitiones Arithmeticae*, laying the foundations of modern number theory

1802 Jean-Étienne Montucla and Jérôme Lalande of France publish *Histoire des Mathématiques* in four volumes; this is the earliest work to systematically discuss the history of mathematics

1810 The first specialized academic journal of mathematics appears in France, the *Annales de Mathématiques Pures et Appliquées*, published by Joseph Diez Gergonne

1812 The first mathematical society is established at Cambridge, the Analytical Society of Cambridge

1824 Norwegian mathematician Abel proves that there is no general solution in radicals for polynomials in degree five or higher

1829 Russian mathematician Lobachevsky publishes the first treatise on non-Euclidean geometry, *On the principles of geometry*

1832 In France, Galois resolves completely the question of the solvability by radicals of polynomial equations; in the process, he invents group theory

1843 Irish mathematician Hamilton discovers the quaternions and proposes for the first time the concept of noncommutative algebra

1851 Riemann introduces the Riemann hypothesis; 3 years later, he introduces Riemannian geometry

1864 The first dedicated research society in mathematics is established in Moscow, the Moscow Mathematical Society

1868 Italian mathematician Beltrami proposes the pseudosphere as a model of hyperbolic geometry

1871 German mathematician Cantor establishes set theory and introduces for the first time the concept of infinite sets

1872 German mathematician Klein publishes his *Erlangen program* in an attempt to unify geometry on the basis of group theory and projective geometry

1889 Italian mathematician Peano introduces the Peano axioms as an axiomatic foundation for the system of natural numbers

1897 The first International Congress of Mathematicians is held in Zurich, Switzerland

1898 Mathematical statistics is founded by the English mathematician Pearson

1899 German mathematician Hilbert writes *The Foundations of Geometry*, a pioneering work in the axiomatic method in mathematics
1900 Hilbert presents his famous 23 problems in mathematics to the International Congress of Mathematicians in Paris
1903 English mathematician and philosopher Russell introduces Russell's paradox in set theory, causing the third great crisis in mathematics
1904 Poincaré proposes the Poincaré conjecture
1907 German mathematician Minkowski proposes his four-dimensional spacetime model, which provided the most appropriate mathematical setting for special relativity
1910 Hilbert introduces Hilbert spaces, advancing geometry from the study of finite-dimensional spaces to include infinite-dimensional spaces
1931 Austrian mathematician Gödel presents his two incompleteness theorems concerning formal systems of mathematics
1933 In the Soviet Union, Kolmogorov establishes an axiom system for probability theory
1936 The Oslo International Congress of Mathematicians awards the first Fields Medal
1938 The pseudonymous collective Bourbaki publishes *Éléments de mathématique*
1944 Hungarian-American mathematician von Neumann introduces game theory
1948 In the United States, Wiener publishes *Cybernetics: Or Control and Communication in the Animal and the Machine*
1949 The first electronic computer with stored programs is designed and manufactured at Cambridge University: the EDSAC (Electronic Delay Storage Automatic Calculator)
1967 Canadian mathematician Langlands introduces the Langlands program, an attempt at a grand unified theory of mathematics linking number theory, algebraic geometry, and the representation theory of groups
1976 Appel and Haken in the United States use computer assistance to prove the four-color theorem for maps
1977 French mathematician Mandelbrot introduces fractal geometry, expanding the range of dimensionality in geometry to include rational numbers
1978 The Wolf Prize in Mathematics is awarded for the first time
1995 British mathematician Wiles proves Fermat's last theorem
2003 The Abel Prize for outstanding contributions to mathematics is awarded for the first time
2006 The mathematics community accepts after a long process of verification the proof of the Poincaré conjecture by Russian mathematician Perelman

Appendix B
The Origin of Some Common Mathematical Symbols

Symbol name	**Symbol**	Earliest usage	Year
Fraction symbol	–	Fibonacci (Italy)	1202
Addition sign	$+$	Johannes Widmann (Germany)	1489
Subtraction sign	$-$	Widmann (Germany)	1489
Parentheses	( )	Christopher Clavius (Germany)	Ca. mid-sixteenth century
Equals sign	$=$	Robert Recorde (England)	1557
Multiplication sign	$\times$	William Oughtred (England)	1618
Inequality symbol	$\neq$	Thomas Harriot (England)	1631
Square root sign	$\sqrt{\ }$	Descartes (France)	1637
Known and unknown quantities	a, b, c, x, y, z	Descartes (France)	1637
Percentage sign	%	Anonymous	Ca. 1650
Infinity sign	∞	John Wallis (England)	1655
Division sign	$\div$	Johann Rahn (Switzerland)	1659
Integration sign	$\int$	Leibniz (Germany)	1675
Circle constant	π	William Jones (England)	1706
Summation sign	$\sum$	Euler (Switzerland)	1755
Congruence symbol	$\equiv$	Gauss (Germany)	1801
Product sign	$\prod$	Gauss (Germany)	1812
Absolute value sign	$\lvert\ \rvert$	Karl Weierstrass (Germany)	1841
Set membership	$\in$	Peano (Italy)	1889

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Bibliography

1. M. Kline, *Mathematical Thought from Ancient to Modern Times* (Oxford University Press, 1990)
2. M. Kline, *Mathematics in Western Culture* (Oxford University Press, 1964)
3. M. Kline, *Mathematics: The Loss of Certainty* (Oxford University Press, 1982)
4. W.C. Dampier, *A History of Science and its Relations with Philosophy & Religion* (Cambridge University Press, 1929)
5. A.N. Whitehead, *Science and the Modern World* (Free Press, New York, USA, 1997)
6. E.T. Bell, *Men of Mathematics: The Lives and Achievements of the Great Mathematicians from Zeno to Poincaré* (Simon & Schuster, New York, USA 1937)
7. H. Eves, *An Introduction to the History of Mathematics* (Holt, Rinehart and Winston, New York, USA, 1969)
8. H. Eves, *Great Moments in Mathematics* (Mathematical Association of America, Washington D.C., USA, 1983)
9. W. Wu (editor-in-chief), *Famous Mathematicians of the World* (世界著名数学家传记) (China Science Publishing and Media (北京科学出版社), Beijing, China, 1995)
10. W. Li, *An Introduction to the History of Mathematics* (数学史概论) (Higher Education Press, (北京：高等教育出版社), Beijing, China 2000)
11. Z. Hu, M. Deng, *Bright Prospects for Mathematics* (大有可为的数学) (Hebei Education Press, (石家庄：河北教育出版社), Hebei, China 2006)
12. A. Qu, *The Chinese Calendar and Mathematics* (中国历法与数学) (China Science Publishing and Media, (北京科学出版社), Beijing, China, 2005)
13. J. Needham, C. Ronan, *The Shorter Science and Civilization in China* (Cambridge University Press, 1980–1994)
14. H.G. Wells, *The Outline of History: Being a Plain History of Life and Mankind* (Garden City Books, New York, USA, 1961)
15. *Encyclopaedia Britannica* (Chinese edition), Encyclopedia of China Publishing House (中国大百科全书出版社), Beijing, China, (1999)
16. P.K. Hitti, *History of the Arabs* (Palgrave Macmillan, London, United Kingdom, 2002)
17. *The Bhagavad Vita* (Tr. Eknath Easwaran) (Nilgiri Press, California, USA, 2007)
18. G.E. You, Q. Wang et al., *History of Chinese Literature* (中国文学史), People's Literature Publishing House (北京：人民文学出版社), Beijing, China (1982).
19. Plato, *The Republic* (Tr. Allan Bloom) (Basic Books, New York, USA, 2016)
20. Aristotle, *Poetics* (Tr. Malcolm Heath) (Penguin Classics, London, United Kingdom, 1997)
21. B. Pascal, *Pensées* (Tr. A.J. Krailsheimer) (Penguin Classics, London, United Kingdom, 1995)

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
T. Cai, *A Brief History of Mathematics*,
https://doi.org/10.1007/978-3-031-26841-0

22. I. Kant, *Critique of Pure Reason* (Tr. Paul Guyer and Allen W. Wood), Cambridge University Press: The Cambridge Edition of the Works of Immanuel Kant (1999)
23. B. Russell, *A History of Western Philosophy* (Simon & Schuster/Touchstone, New York, USA, 1967)
24. B. Russell, *Wisdom of the West* (Crescent Books, New York, USA 1959)
25. A.J. Ayer, *Philosophy in the Twentieth Century* (Weidenfeld and Nicolson, London, United Kingdom, 1982)
26. H. Wang, *Reflections on Kurt Gödel* (The MIT Press: A Bradford Book, 1990)
27. E. Gombrich, *The Story of Art* (Phaidon, New York, USA, 1950)
28. H.H. Arnason, *History of Modern Art* (Pearson College Division, New York, USA, 2003)
29. F. Cajori, *A History of Physics* (Maven Books, Kolkata, India, 2019)
30. J. Gleick, *Chaos: Making a New Science* (Penguin Books, London, United Kingdom, 2008)
31. J. Tabak, *Probability and Statistics: The Science of Uncertainty* (Checkmark Books, New York, USA, 2005)
32. J. Maritain, *Science and Wisdom* (Scribner's, New York, USA, 1954)
33. P. Gorman, *Pythagoras: A Life* (Routledge, Kegan & Paul, Oxfordshire, United Kingdom, 1978)
34. S. Singh, *Fermat's Last Theorem* (HarperCollins, New York, USA, 2002)
35. C. Tord, *Carl Friedrich Gauss: A Biography* (The MIT Press, 1970)
36. C. Reid, *Hilbert* (Copernicus Publications, Göttingen, Germany, 1996)
37. A.I. Miller, *Einstein, Picasso: Space, Time, and the Beauty that Causes Havoc* (Basic Books, New York, USA, 2001)
38. C. Baudelaire, *The Flowers of Evil* (Tr. Richard Howard) (David R. Godin Publisher, Massachusetts, USA, 2022)
39. J.-P. Sartre, *Baudelaire: Critical Study* (Tr. Martin Turnell) (New Directions, New York, USA, 1950)
40. W. Kandinsky, *Concerning the Spiritual in Art* (Tr. M.T.H. Sadler) (Dover Publications, New York, USA, 1977)
41. W. Kandinsky, *Reminiscences*, printed in Kandinsky: *Complete Writings on Art* (Da Capa Press, Massachusetts, USA, 1994)
42. V.S. Naipaul, *India: A Wounded Civilization* (André Deutsch Limited, London, United Kingdom, 1977)
43. P. Jones, *Introduction to Fifty American Poets* (Heinemann Educational Books, New Hampshire, USA, 1980)
44. Y.N. Harari, *Sapiens: A Brief History of Humankind* (Harper, New York, USA, 2015)
45. T. Cai, *Mathematics and Arts* (数学与玫瑰) (The Commercial Press, Beijing, (商务印书馆), Nanjing, China, 2012)
46. T. Cai, *Mathematical Legends* (数学传奇) (The Commercial Press, (商务印书馆), Beijing, China, 2016)
47. T. Cai, *The Story of Mathematics*, （数学的故事) (CITIC Press Group, (中信出版社), Beijing, China, 2018)
48. T. Cai, *England, A Country Without Tigers: Notes on Study Abroad at Cambridge* (英国，没有老虎的国家: 剑桥游学记) (CITIC Press Group, (中信出版社), Beijing, China, 2011)
49. T. Cai, *Germany, Wisdom of Unknown Origins: Notes on Study Abroad at Göttingen* (德国：来历不明的才智：哥廷根游学记)，(Zhonghua Book Company, (北京：中华书局), Beijing, China, 2015)
50. World Atlas (DK Publishing, London, United Kingdom, 2003)
51. J. Muir, *Of Men and Numbers: The Story of the Great Mathematicians* (Dover Press, New York, 1996)
52. W. Scharlau, H. Opolka, *From Fermat to Minkowski: Lectures on the Theory of Numbers and Its Historical Development* (Tr. Walter K. Bühler and Gary Cornell), Springer: Undergraduate Texts in Mathematics, New York, USA, (1985)
53. R.K. Guy, *Unsolved Problems in Number Theory* (Springer: Problems Books in Mathematics, New York, USA, 2004)

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T. Cai, *A Brief History of Mathematics*,
https://doi.org/10.1007/978-3-031-26841-0