Fuzzy Free Logic with Dual Domain Semantics

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Abstract. In this work we propose a fuzzy free logic, that is a generalized system which can give proper interpretation to sentences containing vagueness of non-referring singular terms. This logic is an amalgamation of classical positive free logic system with predicate rational Pavelka logic. The semantics given to the fuzzy free logic is based on dual-domain semantics. A graded similarity measure is introduced in the system which allows comparing two objects based on some properties and assign a degree of similarity. Soundness of the proposed system is proved.

Keywords: Free logic · Positive free logic · Dual-domain semantics · Rational Pavelka logic · Graded similarity measure · Empty-names

1 Introduction

Most of our daily communications use ordinary language and a good part of our thinking is done in it. In our daily lives, we often talk about sentences with referentless names, i.e., empty names, like, Dark matter, Santa Claus, Vulcan, Pegasus, Byomkesh Bakshi etc. These names may come from fictional stories (like story of Christmas, Roman Mythology, Greek Mythology, stories of Satyanweshi), from metaphysics ('nothingness') or from scientific discussions about existence of unobserved hypothetical entities (like dark matter). Also, in the ordinary language, we often use words whose meanings are vague, e.g., 'tall', 'beautiful' etc.

Though the referentless names are unavoidable in ordinary discourse, classical first order logic is not well-equipped to reason with such names. Because, if we attempt to assign arbitrary truth values to such sentences by assuming some kind of interpretation of the empty names in the domain of discourse, i.e., both existent and non-existent objects are placed in a single domain without any distinction, then some counter-intuitive results, like "Pegasus exists", "Round square exists", "something exists that does not exist" etc. can be derived. Furthermore, some logical rules/principles, i.e., principle of Existential Generalization (EG) and principle of Universal Instantiation (UI) are invalidated [8,9]. Also classical logic is inadequate to deal with graded valuation arising from vague terms.

So, to cope with the problem of empty names, free logic emerges [10,12], where all classical principles are valid in case of sentences containing empty names. But free

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logic is unable to deal with vague properties and fuzzy reasoning. To deal with vague properties of objects and for approximate reasoning, fuzzy logic is used, which allows graded valuation according to the degree to which an object satisfies a particular property. But fuzzy logic also has its drawbacks. A sentence, like, "A Pegasus is more similar to a horse than a Hyppogriff", which involves non-referring names and the fuzzy notion of similarity, cannot simultaneously be dealt within fuzzy predicate logic.

In this paper, we propose a new fuzzy logical system that can simultaneously deal with vague properties and also empty names. The system is named as **Fuzzy Free Logic with Dual Domain Semantics** (FFDS). The proposed system is new in the sense that it amalgamates dual domain positive free logic with predicate rational Pavelka logic [5], thus giving a many-valued variation of free logic. Also in the system we introduce a graded similarity measure that broadens the scope to comparing different objects having different degrees of similarity. Not only that, the similarity measure is defined in such a way so that we can express different degrees of similarity between two objects based on different sets of properties. In this respect the proposed system is more generalized than systems containing only strict identity, because strict identity cannot capture different grades of similarity. The similarity relation plays significant role in fuzzy approximate reasoning [14] and also to build information systems [6]. Also the proposed system extends the scope of dealing with sentences that are used in ordinary language.

To the best of our knowledge this type of amalgamation of fuzzy logic and free logic has not been studied yet. In [2] a basic outline of a fuzzy free logic was proposed. But their approach was quite different from the system presented here. Firstly, in [2] the authors allowed truth-value gaps in the semantics; while here the proposed semantics is based on total valuation. Secondly, no detailed mathematical analysis and axiomatization was presented in [2]. Thirdly, we have incorporated a notion of graded similarity in our system.

2 Fuzzy Free Logic with Dual Domain Semantics (FFDS)

In this work, we propose a fuzzy free logic, which is an amalgamation of predicate rational Pavelka logic ($RPL\forall$) and positive free logic with dual domain semantics. This logic resolves the problem of empty names of fuzzy predicate logic by dispensing with the tacit existential assumption of singular terms by means of an existence predicate $E!$. Simultaneously, the proposed system is capable of dealing with vague properties and graded truth values of sentences.

There are three major semantics for free logic [12], namely positive [7, 10], negative [11] and neutral [3]. Positive semantics allows some empty-termed atomic formulas, not of the form '$E!t$' (where, t is a term), to be true. On the other hand, negative free logic semantics require all empty-termed atomic formulas to be false. In neutral semantics all empty-termed atomic formulas, not of the form '$E!t$', are truthvalueless. Here, in positive free logic the sentence $t = t$ is true, even if t is empty; whereas in negative free logic the sentence $t = t$ is false if t is empty.

The semantics of the proposed system is based on the dual domain semantics of positive free logic. Dual domain positive semantics, uses two domains of interpretation [13], namely *inner* and *outer* domain; where, the inner domain captures the class of existing objects and the outer domain includes non-existent objects. Hence, in a dual-domain semantics the interpretation function is total and empty names are not non-referring as they refer to non-existent objects from the outer domain. The quantifiers range over the inner domain only. The domain of interpretation can be given as:

$$D = D_i \cup D_o,$$

where, D_i and D_o are *inner* and *outer domain* respectively, and D_i and D_o are disjoint sets.

Single and dual-domain semantics differ in the ways of assigning truth values to atomic formulas containing empty names. In dual-domain semantics the extension of a predicate ranges over the union of inner and outer domain, and truth value of any atomic formula is evaluated in the Tarskian fashion; i.e., an atomic formula is true if and only if the ordered n-tuple of objects referred to by its singular terms belongs to the extension of the predicate. Such evaluation is not possible in case of single domain semantics as empty names have no reference; hence truth values of such formulas are fixed either to be true or to be false depending on the semantics, which may lead to counter-intuitive results. For instance, the atomic proposition "*Santa Claus is the President of United States*" is true in single-domain positive free semantics and "*Harry Potter is a fictitious character*" is false in negative free semantics. This issue is resolved by considering two domains of interpretation.

Predicate rational Pavelka logic ($RPL\forall$) is a conservative extension of Łukasiewicz's infinite valued logic [5], which extends Łukasiewicz logic with truth constants $\bar{r}$ for each rational $r \in [0,1]$, where $\bar{r}$ is treated as an atomic formula whose truth value is r under each evaluation. This allows to derive partially true conclusions from partially true statements.

2.1 The Role of Similarity

One essential feature of the proposed system is that, here, along with strict identity ($=$) a graded similarity measure is introduced, so that two objects can be compared based on how much they have in common.

Strict identity follows Leibnizian principle, which in a modified form, can be stated as [1]:

$$x = y \text{ iff } x \text{ has } p \text{ implies } y \text{ has } p \text{ and conversely;}$$

where, p represents a property belonging to a specified collection of properties or attributes. Say, the specified collection of properties is denoted by A.

However, graded similarity offers a broader scope for comparing two objects, which can't be captured by strict identity. This is illustrated with the help of following two cases.

Firstly, in case of vague properties, e.g., tall, beautiful etc., the concern is not only whether an object has those properties but also to what extents the properties are present in the object. Hence for any object $x \in D$ and any vague property $p \in A$, it is not enough, only to specify that whether x is p or not, but to what extent x has the property of being p is to be specified. For this, $RPL\forall$ is appropriate to assign a graded valuation from $[0,1]$

to the statement "x is p". In such a scenario, two objects, x and y, may both have some property p, but with different degrees. Thus x and y are not identical, but have certain similarity with respect to the vague property p. The same scenario can arise when a set of vague properties is considered instead of a single property p. This aspect cannot be handled using strict identity.

Secondly, even if we are not considering vague properties, the notion of graded similarity may come into the picture [6]. Instead of the strict identity between two objects (or entities), finding some common properties becomes important in reasoning with ordinary discourse. Bivalent identity does not capture these aspects. Suppose two objects $x, y \in D$ have certain (crisp) properties $p_1, .., p_n \in A$. Whereas, x and y mismatch over the rest of the properties in A. Then clearly x and y are not identical; but they are *similar*. But this similarity is not fuzzy, two objects can either be similar or not.

In the proposed logic, a graded similarity measure is introduced along with the concept of identity of classical free logic, that can deal with both of the aforementioned cases.

In order to define an appropriate similarity measure it must be kept in mind that similarity between two objects (or entities) should be judged with respect to many parameters, not just one. For instance, a horse is similar to a Pegasus with respect to their *appearance*; Santa Claus is similar to Pegasus because both are *non-existent*; Hippogriff is similar to Harry Potter since both are characters from *same fantasy story*, whereas they have very different appearance. Hence, a single similarity measure is not capable of dealing with these various *parameters of comparison*, i.e., sets of properties, based on which similarity of two objects are to be evaluated.

2.2 Representation of Properties and Similarity

Any unary predicate stands for a property or attribute of the corresponding argument. For instance, if $Intelligent$ is considered to be a unary predicate then, $Intelligent(x)$ depicts a property of x. In case of binary predicates there are two arguments. Thus a binary predicate depicts a property of one of its arguments relative to the other one. For instance, in case of the binary predicate *Greater_than*; $Greater\_than(x, x_1)$ says $x\ is\ greater\ than\ x_1$; i.e. the property of x being greater than x_1. This binary predicate can be considered to be an attribute for one of its arguments if the other argument is not a free variable. If the second argument is some constant, say 2, then $Greater\_than(x, 2)$, having only one free variable, corresponds to the attribute *being greater than 2*. The position of the free variable is important in case of non-symmetric non-unary predicates. $Greater\_than(x, 2)$ does not depict the same property as $Greater\_than(2, x)$. Same holds true for higher arity predicates, where only one argument is kept for the free variable.

In a general way it can be said that any well formed formula with a single free variable can be associated with a property. For instance, the property of *being tall and blonde* can be represented with $Tall(x) \wedge Blonde(x)$. The property of *being tall or not blonde* can be depicted as $Tall(x) \vee \neg Blonde(x)$.

Hence, corresponding to each property, depicted by any wff with a single free variable, a similarity measure can be induced over any two objects.

2.3 Language of $FFDS$

In the language, L_F, of the proposed system, there are: a list of variables $(x, x_1, x_2, x_3, ..)$, a list of individual constants $(c, c_1, c_2, ...)$, a list of predicate symbols, P_i^n, where, n stands for the arity and i for identifier, logical connectives $\neg$, &, $\underline{\vee}$, $\rightarrow$, $\wedge$, $\vee$, the equality predicate $=$, truth constants $\bar{r}$ for each $r \in [0,1]$, universal and existential quantifiers $\forall$, $\exists$, an *existence* predicate $E!$, a special binary predicate symbol $Sim\_meas_{\varphi,s}$. Here φ and s are parameters.

The corresponding truth functions of t-norms & and $\wedge$ are taken to be Lukaseiwicz t-norm ($\odot$) and *min* respectively; truth functions of t-conorms $\underline{\vee}$ and $\vee$ are taken to be Lukaseiwicz t-conorm ($\oplus$) and *max* respectively; truth function of implication ($\rightarrow$) is taken to be Lukaseiwicz residuated implicator ($\Rightarrow$) [4]. The *existence* predicate $E!$ is taken from free logic.

In the language L_F, a *term* is a variable or an individual constant. For any n-ary predicate P_i^n, $P_i^n(t_1,..,t_n)$ is an atomic formula, where $t_1,..t_n$ are terms. For any wff φ and terms s, t, t_1, $Sim\_meas_{\varphi,s}(t,t_1)$ and $E!s$ are atomic formulas. For each rational $r \in [0,1]$, $\bar{r}$ is a formula. If φ, ψ, χ are formulas, x is a variable and s is a term, then $\neg\varphi$, $\varphi \rightarrow \psi$, $\varphi \& \psi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \underline{\vee} \psi$, $(\forall x)\psi$, $(\exists x)\varphi$ are formulas. Each formula that results from atomic formulas and using the deduction rule, are also formulas in the system.

The similarity measure between two terms t and t_1 with respect to a wff φ and a term s, denoted as $Sim\_meas_{\varphi,s}(t,t_1)$, is used to represent the similarity between two terms t and t_1 with respect to the property depicted by the wff φ, where the term s serves to denote the arguments to be substituted to capture the property depicted by φ.

For instance, consider φ is $IQ(t_1) \wedge Pass\_Exm(t_1) \wedge Good\_univ(t_2) \wedge Admit(t_1,t_2)$, which considers a student's IQ, whether he/she passed the exam and whether he/she has taken admission to a good university. Then $Sim\_Meas_{\varphi,t_1}(s,t)$ stands for the similarity of two students based on φ, where, s and t are to be substituted in place of t_1.

2.4 Semantics of FFDS

The semantics of the system, $FFDS$, is based on dual-domain positive semantics and semantics of $RPL\forall$. In the dual domains, inner domain represents the domain of existent entities and the outer domain represents the domain of non-existent entities. So, in the semantics, the interpretation function is total.

The model structure is specified as follows;

$$M = \langle D_i, D_o, \langle d_c \rangle_c, \langle r_P \rangle_P \rangle.$$

The truth values range over $[0,1]$ and the domain of interpretation $D_i \bigcup D_o$ is non-empty.

In the model structure,

- D_i, D_o are two disjoint sets, called *inner* and *outer* domain respectively.
- For each object constant c, d_c is an element of $D_i \bigcup D_o$.
- For an n-ary predicate P_i^n, $r_{P_i^n} : (D_i \bigcup D_o)^n \rightarrow [0,1]$ associates to each tuple $(d_{c_1}, ..., d_{c_n}) \in (D_i \bigcup D_o)^n$ the membership value of $(d_{c_1}, ..., d_{c_n})$ to the fuzzy relation P_i^n, which is $r_{P_i^n}(d_{c_1}, ..., d_{c_n}) \in [0,1]$.

Definition 1. *For the language L_F and model structure M for L_F, an M-evaluation of variables is a mapping v, that assigns an element from $D_i \bigcup D_o$ to each object variable x. Let, v, v' be two assignments, then $v \equiv_x v'$ means $v(x_1) = v'(x_1)$ for each variable x_1 distinct from x.*

The interpretation of a term by M, v, is given as; $||x||_{M,v} = v(x)$ and $||c||_{M,v} = d_c$. The truth value of a sentence φ, denoted as $||\varphi||_{M,v}$, is obtained under the following conditions:

1. $||P_i^n(t_1,...,t_n)||_{M,v} = r_{P_i^n}(||t_1||_{M,v},...,||t_n||_{M,v})$;
2. $||\varphi \rightarrow \psi||_{M,v} = ||\varphi||_{M,v} \Rightarrow ||\psi||_{M,v}$;
3. (a) $||\varphi \& \psi||_{M,v} = ||\varphi||_{M,v} \odot ||\psi||_{M,v}$;
   (b) $||\varphi \veebar \psi||_{M,v} = ||\varphi||_{M,v} \oplus ||\psi||_{M,v}$;
   (c) $||\varphi \wedge \psi||_{M,v} = min(||\varphi||_{M,v}, ||\psi||_{M,v})$;
   (d) $||\varphi \vee \psi||_{M,v} = max(||\varphi||_{M,v}, ||\psi||_{M,v})$;
   (e) $||\neg\varphi||_{M,v} = 1 - ||\varphi||_{M,v}$
4. $||\bar{r}||_{M,v} = r, r \in \mathbb{Q} \cap [0,1]$;
5. $||\forall x\varphi||_{M,v} = inf\{||\varphi||_{M,v'} \mid v \equiv_x v' \text{ and } v'(x) \in D_i\}$;
6. $||\exists x\varphi||_{M,v} = sup\{||\varphi||_{M,v'} \mid v \equiv_x v' \text{ and } v'(x) \in D_i\}$;
7. $||t = t_1||_{M,v} = \begin{cases} 1, & ||t||_{M,v} = ||t_1||_{M,v} \\ 0, & \text{otherwise} \end{cases}$
8. $||E!t||_{M,v} = \begin{cases} 1, & ||t||_{M,v} \in D_i \\ 0, & \text{otherwise} \end{cases}$;
9. $\left|\left|Sim\_meas_{\varphi,s}(t,t_1)\right|\right|_{M,v} = 1 - \left| \, ||\varphi(t//s)||_{M,v} - ||\varphi(t_1//s)||_{M,v} \right|$
   where, $\varphi(t//s)$ means replacing zero or more (but not necessarily all) occurrences of s in φ with t. Also a restriction is imposed that, in $\varphi(t//s)$ and $\varphi(t_1//s)$, t and t_1 are substituted in the same occurrences of s.

It can be seen that in the proposed semantics the valuation is *total*, because each and every well formed formula is given some valuation and no truth-value gap is allowed. Here, one thing is to be noted that predicates' extensions range over both the inner and outer domain. Semantic condition 1 ensures that the truth values of atomic sentences with empty names, that are not of the form $E!t$, are being assigned in the Tarskian fashion, rather that being rigidly true or false. This would result in positive dual-domain semantics if the valuation would have been restricted to $\{0,1\}$ and also this clearly points out the distinction of the proposed system from negative and neutral semantics. The valuation can be constructed in such a way to assign 0 to the sentence "*Santa Claus is the President of the Unites States*" and to assign 1 to the sentence "*Harry Potter is a fictitious character*"; whereas negative semantics would assign false to both of them and neutral semantics would assign truth-value gap to both of them.

In condition 9, the similarity between two terms t and t_1 with respect to any property depicted by φ is measured by the extents to which t and t_1 satisfies φ. If both $||\varphi(t//s)||_{M,v}$ and $||\varphi(t_1//s)||_{M,v}$ are close to each other then both t and t_1 are similar with respect to φ and $\left|\left|Sim\_Meas_{\varphi,s}(t,t_1)\right|\right|_{M,v}$ is close to 1. On the other hand if t and

t_1 are dissimilar with respect to φ, then one of $\|\varphi(t//s)\|_{M,v}$ and $\|\varphi(t_1//s)\|_{M,v}$ will be close to 1 and the other one is close to 0; thus resulting $\left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v}$ to be close to 0. The similarity defined here, is reflexive, symmetric and transitive.

Lemma 1: $\|\varphi(t_1//s)\|_{M,v} \geq \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} \odot \|\varphi(t//s)\|_{M,v}$.

Proof: $R.H.S. = \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} \odot \|\varphi(t//s)\|_{M,v}$

$$= \left(1 - \left| \|\varphi(t//s)\|_{M,v} - \|\varphi(t_1//s)\|_{M,v} \right| \right) \odot \|\varphi(t//s)\|_{M,v}$$
$$= max\left\{0, 1 - \left| \|\varphi(t//s)\|_{M,v} - \|\varphi(t_1//s)\|_{M,v} \right| + \|\varphi(t//s)\|_{M,v} - 1\right\}$$
$$= max\left\{0, \|\varphi(t//s)\|_{M,v} - \left| \|\varphi(t//s)\|_{M,v} - \|\varphi(t_1//s)\|_{M,v} \right| \right\}$$

Case 1: If $\|\varphi(t//s)\|_{M,v} \leq \|\varphi(t_1//s)\|_{M,v}$

$$R.H.S. = max\left\{0, 2\,\|\varphi(t//s)\|_{M,v} - \|\varphi(t_1//s)\|_{M,v}\right\}$$
$$= max\left\{0, 2\left(\|\varphi(t//s)\|_{M,v} - \|\varphi(t_1//s)\|_{M,v} \right) + \|\varphi(t_1//s)\|_{M,v}\right\} \leq \|\varphi(t_1//s)\|_{M,v}$$

Case 2: If $\|\varphi(t_1//s)\|_{M,v} \leq \|\varphi(t//s)\|_{M,v}$

$$R.H.S. = max\left\{0, \|\varphi(t//s)\|_{M,v} - \|\varphi(t//s)\|_{M,v} + \|\varphi(t_1//s)\|_{M,v}\right\}$$
$$= max\left\{0, \|\varphi(t_1//s)\|_{M,v}\right\} = \|\varphi(t_1//s)\|_{M,v}$$

2.5 Axiom Schema of FFDS

The axiomatization of the proposed system is developed by converging free logic with predicate rational Pavelka logic ($RPL\forall$), with introduction of a similarity measure.

Let φ, ψ, χ be well formed formulas, x is a variable and $s, t, t_1, t_2, ...t_{i-1}$ be terms, then the axioms are as follows:

1. $\varphi \to (\psi \to \varphi)$;
2. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;
3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$;
4. $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$;
5. $(\varphi \to \forall x\varphi)$, x is not free in φ (Blanket Axiom);
6. $(\forall x(\varphi \to \psi)) \to (\forall x\varphi \to \forall x\psi)$ (Dist);
7. $\forall x\varphi \to (E!t \to \varphi(t/x))$ (Restricted form of the principle of Specification);
8. $t = t$

9. $Sim\_Meas_{\varphi,s}(t,t_1) \rightarrow (\varphi(t//s) \rightarrow \varphi(t_1//s))$;
   where, in $\varphi(t//s)$ and $\varphi(t_1//s)$, t and t_1 are substituted in the same occurrences of s.
10. $\forall x E!x$;
11. (a) $\overline{r} \rightarrow \overline{k} \equiv \overline{r \Rightarrow k}$
    (b) $\neg\overline{r} \equiv \overline{1-r}$, where r, k are rationals from $[0,1]$.

In axiom 7, $\varphi(t/x)$ stands for replacing each free occurrence of the variable x in $\varphi(x)$ with t. This principle states that, if a term t refers to any existing object in the domain of discourse, then only it can be used to replace the occurrences of the variable x in the formula φ. Axiom 7 is the restricted form of principle of Specification, which is the key behind the success of free logic for dealing with empty names.

In axiom 9, $\varphi(t//s)$ stands for replacing zero or more (but not necessarily all) occurrences of s in φ by t. In the axiom, $Sim\_Meas_{\varphi,s}(t,t_1)$ replaces the identity $(t = t_1)$ occurring in the relevant axiom in classical free logic. This depicts that the t can substitute t_1, not only when they are *identical*; but also when they both possess certain set of properties *relevant to* the wff φ. For instance, if the consequent is concerned about a property such as *beautiful*, then the substitutivity of t by t_1 would depend upon their similarity based on their beauty only. This relaxes the substitutivity condition.

The axiom 10, stipulates that the quantifiers range over all objects that satisfy $E!$.

Axiom 11 corresponds to the "bookkeeping axioms" for truth constants.

2.6 Deduction Rules

The deduction rule of $FFDS$ is modus ponens (MP) (from φ, $\varphi \rightarrow \psi$ infer ψ). There is another derived deduction rule in $RPL\forall$ [4] as stated below;

$$\frac{(\varphi, r) \qquad (\varphi \rightarrow \psi, k)}{(\psi, r \odot k)}$$

Here, $r, k, r \odot k$ are rational elements from $[0,1]$. The pair of the form (φ, r) is a graded formula such that φ is a formula and is a shorthand representation of $\overline{r} \longrightarrow \varphi$ and '$\odot$' is Łukasiewicz t-norm.

3 Soundness of the System

Now we prove the soundness of the system $FFDS$. The truth degree and provability degree are defined following $RPL\forall$.

Definition 2: *For a theory T (a closed set of formulas) in $FFDS$ and a formula φ;*

1. *The truth degree of φ over T is $\|\varphi\|_T = inf\{\|\varphi\|_M \,|M$ is a model structure of $T\}$, where, $\|\varphi\|_M = inf\{\|\varphi\|_{M,v}\,|v$ is M-evaluation$\}$*
2. *The provability degree of φ over T is $|\varphi|_T = sup\{r|T \vdash (\varphi, r)\}$.*

Theorem 1: For each theory, T in the system and each formula, φ, we have,

$$|\varphi|_T \leq \|\varphi\|_T$$

Proof: It immediately follows from if $T \vdash \psi$ then $\|\psi\|_{M,v} = 1$, for each model of φ [4]. The soundness of deduction rules are proved in [4]. Thus, for proving soundness of the system we will have to prove that for any axiom α, $\|\alpha\|_{M,v} = 1$. Axioms 1 to 4 and axiom 8 and 11 are taken directly from $RPL\forall$; hence for them the proof is already well-established [[4], Ch 5]. For the rest of the axioms, proofs are presented here.

Axiom 5: $(\varphi \rightarrow \forall x\varphi)$ [x is not free in φ]

$$\begin{aligned}
\|\varphi \rightarrow \forall x\varphi\|_{M,v} &= \|\varphi\|_{M,v} \Rightarrow \|\forall x\varphi\|_{M,v}\,;\\
&= min(1 - \|\varphi\|_{M,v} + \|\forall x\varphi\|_{M,v}\,, 1);\\
&= min(1 - \|\varphi\|_{M,v} + inf\{\|\varphi\|_{M,v'}\,|\,v \equiv_x v' \text{ and } v'(x) \in D_i\}, 1);\\
&= min(1 - \|\varphi\|_{M,v} + \|\varphi\|_{M,v}\,, 1); \quad [\text{since, } x \text{ is not free in } \varphi]\\
&= 1.
\end{aligned}$$

Axiom 6: $(\forall x(\varphi \rightarrow \psi)) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$
To prove, $\|(\forall x(\varphi \rightarrow \psi)) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)\|_{M,v} = 1$
we need to show;

$$\begin{aligned}
&\|\forall x\varphi \rightarrow \forall x\psi\|_{M,v} \geq \|\forall x(\varphi \rightarrow \psi)\|_{M,v}\\
\text{or, } &(\|\forall x\varphi\|_{M,v} \Rightarrow \|\forall x\psi\|_{M,v}) \geq \|\forall x(\varphi \rightarrow \psi)\|_{M,v}\,;\\
\text{or, } &min(1 - \|\forall x\varphi\|_{M,v} + \|\forall x\psi\|_{M,v}\,, 1)\\
&\qquad \geq inf\{\|\varphi \rightarrow \psi\|_{M,v'}\,|\,v \equiv_x v' \text{ and } v'(x) \in D_i\};\\
\text{or, } &min(1 - inf\{\|\varphi\|_{M,v'}\,|\,v \equiv_x v' \text{ and } v'(x) \in D_i\}\\
&\qquad + inf\{\|\psi\|_{M,v'}\,|\,v \equiv_x v' \text{ and } v'(x) \in D_i\}, 1)\\
&\qquad \geq inf\{\|\varphi \rightarrow \psi\|_{M,v'}\,|\,v \equiv_x v' \text{ and } v'(x) \in D_i\}
\end{aligned}$$

Now suppose, for some constants c_1, c_2, with $d_{c_1}, d_{c_2} \in D_i$;

$$\begin{aligned}
inf\{\|\varphi\|_{M,v'}\,|\,v \equiv_x v' \text{ and } v'(x) \in D_i\} &= \|\varphi\|_{M,v(x|c_1)}\\
inf\{\|\psi\|_{M,v'}\,|\,v \equiv_x v' \text{ and } v'(x) \in D_i\} &= \|\psi\|_{M,v(x|c_2)}
\end{aligned}$$

where, $v(x|c)$ is same as v, except at the variable x, which is assigned the value d_c, i.e.,

$$v(x|c)(x_1) = \begin{cases} v(x_1), & x_1 \neq x \\ d_c, & x_1 = x \end{cases}$$

So,
$$\begin{aligned}
& min(1 - inf\{\|\varphi\|_{M,v'} \,|\, v \equiv_x v' \text{ and } v'(x) \in D_i\} \\
& \qquad + inf\{\|\psi\|_{M,v'} \,|\, v \equiv_x v' \text{ and } v'(x) \in D_i\}, 1) \\
&= min(1 - \|\varphi\|_{M,v(x|c_1)} + \|\psi\|_{M,v(x|c_2)}, 1) \\
&\geq min(1 - \|\varphi\|_{M,v(x|c_2)} + \|\psi\|_{M,v(x|c_2)}, 1) \\
& \qquad\qquad [Since, \|\varphi\|_{M,v(x|c_2)} \geq \|\varphi\|_{M,v(x|c_1)}] \\
&= \|\varphi \to \psi\|_{M,v(x|c_2)} \\
&\geq inf\{\|\varphi \to \psi\|_{M,v'} \,|\, v \equiv_x v' \text{ and } v'(x) \in D_i\}
\end{aligned}$$

Hence, $\|(\forall x(\varphi \to \psi)) \to (\forall x\varphi \to \forall x\psi)\|_{M,v} = 1$

<u>Axiom 7:</u>

$\forall x\varphi \to (E!t \to \varphi(t/x))$

It is to be proved;

$$\begin{aligned}
& \|\forall x\varphi \to (E!t \to \varphi(t/x))\|_{M,v} = 1; \\
\text{or, } & (\|\forall x\varphi\|_{M,v} \Rightarrow \|E!t \to \varphi(t/x)\|_{M,v}) = 1; \\
\text{or, } & \|E!t \to \varphi(t/x)\|_{M,v} \geq \|\forall x\varphi\|_{M,v}; \\
\text{or, } & (\|E!t\|_{M,v} \Rightarrow \|\varphi(t/x)\|_{M,v}) \geq \|\forall x\varphi\|_{M,v}; \\
\text{or, } & min(1,\ 1 - \|E!t\|_{M,v} + \|\varphi(t/x)\|_{M,v}) \\
& \qquad \geq inf\{\|\varphi\|_{M,v'} \,|\, v \equiv_x v' \text{ and } v'(x) \in D_i\}
\end{aligned}$$

Now consider the following two cases:

Case-1:

Let, $\|E!t\|_{M,v} = 1$, i.e., $\|t\|_{M,v} \in D_i$

Then,

$$\begin{aligned}
L.H.S &= min(1,\ 1 - \|E!t\|_{M,v} + \|\varphi(t/x)\|_{M,v}) \\
&= min(1,\ 1 - 1 + \|\varphi(t/x)\|_{M,v}) \\
&= min(1, \|\varphi(t/x)\|_{M,v}) \\
&= \|\varphi(t/x)\|_{M,v} \\
&\geq inf\{\|\varphi\|_{M,v'} \,|\, v \equiv_x v' \text{ and } v'(x) \in D_i\} = R.H.S
\end{aligned}$$

Case-2: Let, $\|E!t\|_{M,v} = 0$, i.e., $\|t\|_{M,v} \in D_o$;

Then,

$$\begin{aligned}
L.H.S &= min(1,\ 1 - \|E!t\|_{M,v} + \|\varphi(t/x)\|_{M,v}) \\
&= min(1, 1 - 0 + \|\varphi(t/x)\|_{M,v}) \\
&= min(1, 1 + \|\varphi(t/x)\|_{M,v}) \\
&= 1 \\
&\geq inf\{\|\varphi\|_{M,v'} \,|\, v \equiv_x v' \text{ and } v'(x) \in D_i\} \\
&= R.H.S
\end{aligned}$$

Hence, it is proved that,

$$\|E!t \rightarrow \varphi(t/x)\|_{M,v} \geq \|\forall x\varphi\|_{M,v}$$
$$\text{and,}\quad \|\forall x\varphi \rightarrow (E!t \rightarrow \varphi(t/x))\|_{M,v} = 1$$

Axiom 9:

$Sim\_Meas_{\varphi,s}(t,t_1) \rightarrow (\varphi(t//s) \rightarrow \varphi(t_1//s))$ It is to show that,

$$\left\|Sim\_Meas_{\varphi,s}(t,t_1) \rightarrow (\varphi(t//s) \rightarrow \varphi(t_1//s))\right\|_{M,v} = 1$$
$$\text{or, } \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} \Rightarrow \left(\|\varphi(t//s)\|_{M,v} \Rightarrow \|\varphi(t_1//s)\|_{M,v}\right) = 1$$
$$\text{or, } \left(\|\varphi(t//s)\|_{M,v} \Rightarrow \|\varphi(t_1//s)\|_{M,v}\right) \geq \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v}$$
$$\text{or, } min\left(1, 1 - \|\varphi(t//s)\|_{M,v} + \|\varphi(t_1//s)\|_{M,v}\right) \geq \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v}$$

Case-1:

Suppose, $\|\varphi(t//s)\|_{M,v} \leq \|\varphi(t_1//s)\|_{M,v}$. So,

$$\begin{aligned} L.H.S &= min(1 - \|\varphi(t//s)\|_{M,v} + \|\varphi(t_1//s)\|_{M,v}, 1) \\ &= 1 \\ &\geq \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} = R.H.S \end{aligned}$$

Case-2:

Suppose, $\|\varphi(t//s)\|_{M,v} > \|\varphi(t_1//s)\|_{M,v}$

Then, we need to show that,

$$\begin{aligned} & min(1, 1 - \|\varphi(t//s)\|_{M,v} + \|\varphi(t_1//s)\|_{M,v}) \\ & = 1 - \|\varphi(t//s)\|_{M,v} + \|\varphi(t_1//s)\|_{M,v} \\ & \geq \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} \end{aligned}$$

Now,

$$\begin{aligned} & 1 - \|\varphi(t//s)\|_{M,v} + \|\varphi(t_1//s)\|_{M,v} \\ & \geq 1 - \|\varphi(t//s)\|_{M,v} + (\left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} \odot \|\varphi(t//s)\|_{M,v}) \\ & \qquad\qquad [\text{replacing } \|\varphi(t_1//s)\|_{M,v} \text{ using Lemma 1.}] \\ & = 1 - \|\varphi(t//s)\|_{M,v} + max(0, \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} + \|\varphi(t//s)\|_{M,v} - 1) \\ & = max(0, \|\varphi(t//s)\|_{M,v} + \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} - 1) - (\|\varphi(t//s)\|_{M,v} - 1) \\ & = max(0, \|\varphi(t//s)\|_{M,v} + \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} - 1) \\ & \qquad - (\|\varphi(t//s)\|_{M,v} + \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} - 1) \\ & \qquad\qquad + \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} \\ & \geq \left\|Sim\_Meas_{\varphi,s}(t,t_1)\right\|_{M,v} [\text{Since, } max(0,a) - a \geq 0]. \end{aligned}$$

Hence, it is proved that,

$$\left\|Sim\_Meas_{\varphi,s}(t,t_1) \to (\varphi(t//s) \to \varphi(t_1//s))\right\|_{M,v} = 1.$$

Axiom 10: $\forall x E!x$

$$\begin{aligned}\|\forall x E!x\|_{M,v} &= inf\{\|E!x\|_{m,v'} \,|\, v \equiv_x v' \text{ and } v'(x) \in D_i\};\\ &= inf\{\|E!c\| \,|\, d_c \in D_i\};\\ &= 1.\end{aligned}$$

This completes the proof of soundness of the $FFDS$ system.

4 Conclusion

In this paper a fuzzy version of positive free logic is proposed that is a unification of positive free logic with dual-domain semantics and predicate rational Pavelka logic. Having in it the properties of both the systems, the resulting system, namely $FFDS$, can resolve the problems that arise in classical logic and fuzzy logic due to inclusion of referentless singular terms corresponding to non-existent objects and also supports graded valuation which can handle vague concepts. This makes the system more appealing for reasoning in ordinary discourse of which non-referring empty names and vagueness are inseparable parts.

Another very crucial improvement, that makes the system more general and flexible is adding the notion of a graded similarity measure along with strict identity. This allows comparing two different objects based on their common properties and assign a degree of similarity, which can't be captured by identity. This notion of similarity is an important characteristic of logic for information systems. This system has accommodated more than one similarity measure so that similarity between any two objects can be judged from different perspectives. The system $FFDS$ is sound. The investigation of completeness is aimed to be our future work. This type of fuzzy free logic based on Gödel logic and product logic, instead of $RPL\forall$, can be further studied.

So the new system $FFDS$ filters out the limitations of positive free logic as well as fuzzy logic. In our daily lives this system can be used for dealing with reasoning in ordinary discourse. This system presents a vast scope of applications.

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