Chapter 3
The Restricted Dual of an Algebra

As we already have seen in the previous chapters, if C is a coalgebra, then the dual space $C^* = \mathrm{L}(C, \mathbb{F})$ is an algebra with the convolution product

$$\mu_{C^*} = \Delta^*|_{C^* \otimes C^*}.$$

However, the categorial duality between algebras and coalgebras does not allow us to conclude that the dual space of an algebra is a coalgebra with respect to the dual structural maps. The reason is that for a vector space V, the inclusion $V^* \otimes V^* \subset (V \otimes V)^*$ is strict if V is infinite dimensional. This means that, the dual vector space A^* of an algebra is a coalgebra with respect to the dual structural maps only if $\mu^*(A^*) \subset A^* \otimes A^*$. This motivates the definition of the restricted dual of an algebra.

Definition 3.1 The *restricted (or finite) dual* A^o of an algebra A is the vector subspace of A^* given by the inverse image of the tensor square of the dual vector space A^* by the dual of the product of A, i.e.

$$A^o := (\mu^*)^{-1}(A^* \otimes A^*). \tag{3.1}$$

3.1 The Restricted Dual and Finite Dimensional Representations

In this section, elements of the restricted dual A^o are characterised in terms of finite dimensional representations of A and A^o is shown to be a coalgebra with respect to the dual structural maps, that is $\mu^*(A^o) \subset A^o \otimes A^o$.

When A is finite dimensional, one always has the equality $A^o = A^*$. When A is infinite dimensional, A^o is a subspace of A^* which can be both the whole space

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$A^o = A^*$ or the trivial subspace $A^o = 0$. The result of this section implies that $A^o = 0$ in the case when A does not admit any finite dimensional representations.

In order to characterise elements of A^o, we consider the matrix elements of finite dimensional representations of A.

To begin with, let $\rho\colon A \to \mathbb{F}$ be an algebra morphism which corresponds to a one-dimensional representation. This means that ρ is a linear form with a specific behaviour with respect to the algebra structure of A, namely

$$\langle \rho, xy\rangle = \langle \rho, x\rangle\langle \rho, y\rangle$$

for any $x, y \in A$, and $\langle \rho, 1\rangle = 1$. Let us rewrite $\langle \rho, xy\rangle$ as follows:

$$\langle \rho, xy\rangle = \langle \rho, \mu(x \otimes y)\rangle = \langle \mu^*\rho, x \otimes y\rangle. \tag{3.2}$$

By writing also

$$\langle \rho, x\rangle\langle \rho, y\rangle = \langle \rho \otimes \rho, x \otimes y\rangle, \tag{3.3}$$

we see that $\mu^*\rho = \rho \otimes \rho$, which means that ρ, considered as a linear form on A, is contained in the restricted dual of A.

Assume now, more generally, that V is an n-dimensional (left) A-module, i.e. that we have an algebra morphism $\lambda\colon A \to \mathrm{End}(V)$. Let us choose a linear basis $\{v_i\}_{i\in\underline{n}} \subset V$ with $\underline{n} = \{0, 1, \ldots, n-1\}$, and for any $x \in A$ and $i \in \underline{n}$, consider the vector $(\lambda x)v_i$. As any other vector in V, it is a linear combination of the basis vectors where the coefficients are linear functions of x:

$$(\lambda x)v_i = \sum_{j\in\underline{n}} v_j\langle \lambda_{j,i}, x\rangle, \tag{3.4}$$

where the elements $\lambda_{j,i} \in A^*$ are called *matrix coefficients* of the representation λ with respect to the basis $\{v_i\}_{i\in\underline{n}}$. Writing

$$(\lambda(xy))v_i = \sum_{j\in\underline{n}} v_j\langle \lambda_{j,i}, xy\rangle = \sum_{j\in\underline{n}} v_j\langle \mu^*\lambda_{j,i}, x \otimes y\rangle \tag{3.5}$$

and

$$(\lambda x)(\lambda y)v_i = \sum_{k\in\underline{n}}(\lambda x)v_k\langle \lambda_{k,i}, y\rangle = \sum_{j,k\in\underline{n}} v_j\langle \lambda_{j,k}, x\rangle\langle \lambda_{k,i}, y\rangle$$

$$= \sum_{j,k\in\underline{n}} v_j\langle \lambda_{j,k} \otimes \lambda_{k,i}, x \otimes y\rangle,$$

and using the equality $\lambda(ab) = (\lambda a)(\lambda b)$, we conclude that

$$\mu^* \lambda_{j,i} = \sum_{k \in \underline{n}} \lambda_{j,k} \otimes \lambda_{k,i}, \quad \forall i, j \in \underline{n}, \tag{3.6}$$

i.e. $\{\lambda_{j,i}\}_{i,j \in \underline{n}} \subset A^o$ and $\mu^*(\{\lambda_{j,i}\}_{i,j \in \underline{n}}) \subset A^o \otimes A^o$.

Remark 3.1 The matrix coefficients $\{\lambda_{j,i}\}_{i,j \in \underline{n}}$ generate a finite dimensional subcoalgebra of A^o which is an isomorphic image of the matrix coalgebra from Example 1.13.

Theorem 3.1 *The restricted dual A^o of any algebra A is the linear span of the matrix coefficients of all finite dimensional representations of A.*

Proof Taking into account the preceding consideration, it suffices to show that, for any non zero element f of A^o, there exists a finite dimensional (left) A-module V_f such that f is a linear combination of the matrix coefficients of this representation (with respect to some basis).

The dual space A^* is a left A-module corresponding to the dual right multiplications $R_x^* \in \operatorname{End}(A^*)$, where $x \in A$ and $R_x \in \operatorname{End}(A)$ is defined by $R_x y = yx$. Indeed, for any $x, y, z \in A$ and $\alpha \in \mathbb{F}$, we verify the linearity

$$R_{x+\alpha y} z = z(x + \alpha y) = zx + \alpha zy = R_x z + \alpha R_y z = (R_x + \alpha R_y) z$$
$$\Rightarrow R_{x+\alpha y} = R_x + \alpha R_y \Rightarrow R_{x+\alpha y}^* = R_x^* + \alpha R_y^*$$

and it is easily checked that

$$R_x^* R_y^* = (R_y R_x)^* = R_{xy}^*, \quad R_1^* = (\mathrm{id}_A)^* = \mathrm{id}_{A^*}. \tag{3.7}$$

Let $V_f := R_A^* f \subset A^*$ be the orbit of f with respect to this action of A on A^*. The linear dependence of R_x^* on x implies that the set V_f is a vector subspace of A^*, and the map $\lambda \colon A \to \operatorname{End}(V_f)$ defined by $\lambda x = R_x^*|_{V_f}$ is an algebra morphism.

The condition $f \in A^o$ implies that

$$\mu^* f = \sum_{i \in \underline{n}} g_i \otimes h_i \tag{3.8}$$

for some $n \in \mathbb{Z}_{>0}$ and $g, h \in (A^*)^{\underline{n}}$. The calculation

$$\langle R_x^* f, y \rangle = \langle f, yx \rangle = \langle \mu^* f, y \otimes x \rangle = \sum_{i \in \underline{n}} \langle g_i, y \rangle \langle h_i, x \rangle = \left\langle \sum_{i \in \underline{n}} g_i \langle h_i, x \rangle, y \right\rangle \tag{3.9}$$

shows that for any $x \in A$, the element $R_x^* f$ finds itself in the linear span of the elements $\{g_i\}_{i \in \underline{n}}$:

$$R_x^* f = \sum_{i \in \underline{n}} g_i \langle h_i, x \rangle. \tag{3.10}$$

Thus, $m := \dim(V_f) \leq n < \infty$.

Let $\{v_i\}_{i \in \underline{m}}$ be a linear basis of V_f with $\underline{m} = \{0, 1, \dots, m-1\}$. Then, for any $x \in A$, we have

$$R_x^* f = \sum_{i \in \underline{m}} v_i \langle w_i, x \rangle \tag{3.11}$$

for some $w \in (A^*)^{\underline{m}}$. In particular,

$$f = R_1^* f = \sum_{i \in \underline{m}} v_i \langle w_i, 1 \rangle. \tag{3.12}$$

Let $z \in A^{\underline{m}}$ be such that

$$v_i = R_{z_i}^* f, \quad \forall i \in \underline{m}. \tag{3.13}$$

We have

$$(\lambda x) v_i = (\lambda x) R_{z_i}^* f = R_{x z_i}^* f = \sum_{j \in \underline{m}} v_j \langle w_j, x z_i \rangle = \sum_{j \in \underline{m}} v_j \langle R_{z_i}^* w_j, x \rangle. \tag{3.14}$$

Thus, the matrix coefficients $\{\lambda_{i,j}\}_{i,j \in \underline{m}}$ of the representation λ, corresponding to the basis $\{v_i\}_{i \in \underline{m}}$, are given by

$$\lambda_{i,j} = R_{z_j}^* w_i, \quad \forall i, j \in \underline{m}. \tag{3.15}$$

Let us show that f is a linear combination of $\lambda_{i,j}$'s.

By using (3.11), for any $x \in A$, we write

$$\langle f, x \rangle = \langle R_x^* f, 1 \rangle = \sum_{i \in \underline{m}} \langle v_i, 1 \rangle \langle w_i, x \rangle = \left\langle \sum_{i \in \underline{m}} \langle v_i, 1 \rangle w_i, x \right\rangle \tag{3.16}$$

which means that

$$f = \sum_{i \in \underline{m}} \langle v_i, 1 \rangle w_i. \tag{3.17}$$

By applying $R^*_{z_j}$ to both sides of this decomposition, we obtain

$$v_j = R^*_{z_j} f = \sum_{i \in \underline{m}} \langle v_i, 1 \rangle R^*_{z_j} w_i = \sum_{i \in \underline{m}} \langle v_i, 1 \rangle \lambda_{i,j}. \tag{3.18}$$

Finally, by substituting this into (3.12), we obtain

$$f = \sum_{i,j \in \underline{m}} \langle v_i, 1 \rangle \langle w_j, 1 \rangle \lambda_{i,j}. \tag{3.19}$$

Corollary 3.1 *For any algebra A, one has the inclusion $\mu^*(A^o) \subset A^o \otimes A^o$.*

This follows immediately from (3.6).

Exercise 3.1 For any algebra A let $\iota_A : A^o \to A^*$ be the canonical inclusion map. Let $f : A \to B$ be an algebra morphism. Show that

1. there exists a unique coalgebra morphism $f^o : B^o \to A^o$ such that

$$f^* \iota_B = \iota_A f^o;$$

2. $(\mathrm{id}_A)^o = \mathrm{id}_{A^o}$;
3. $(fg)^o = g^o f^o$ for any algebra morphism $g : Z \to A$;

Remark 3.2 The parts (2) and (3) of Exercise 3.1 reflect the functorial nature of the restricted dual which directly follows from the functorial nature of the duality correspondence for vector spaces. The restricted dual is, in fact, a contravariant functor from the category $\mathbf{Alg}_{\mathbb{F}}$ of $\mathbb{F}$-algebras to the category $\mathbf{Coalg}_{\mathbb{F}}$ of $\mathbb{F}$-coalgebras. One can also show that there exists a natural equivalence

$$\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(A, C^*) \simeq \mathrm{Hom}_{\mathbf{Coalg}_{\mathbb{F}}}(C, A^o), \quad \forall (A, C) \in \mathbf{Alg}_{\mathbb{F}} \times \mathbf{Cog}_{\mathbb{F}}. \tag{3.20}$$

Exercise 3.2 Let $f : A \to B$ be a surjective morphism of algebras. Show that $f^o : B^o \to A^o$ is an injective morphism of coalgebras.

3.1.1 An Algebra with Trivial Restricted Dual

Theorem 3.1 implies that, if an algebra A does not admit finite dimensional representations, then its restricted dual is trivial, i.e. $A^o = 0$. For example, consider the Heisenberg subalgebra A_{Heis} of $\mathrm{End}(\mathbb{C}[z])$ generated by the multiplication and differentiation operators x and ∂ defined by

$$x(p(z)) = zp(z), \quad \partial(p(z)) = \frac{\mathrm{d}p(z)}{\mathrm{d}z}, \quad \forall p(z) \in \mathbb{C}[z]. \tag{3.21}$$

They satisfy the commutation relation

$$\partial x - x\partial = \mathrm{id}_{\mathbb{C}[z]} \,. \tag{3.22}$$

The Heisenberg algebra does not admit finite dimensional representations. Indeed, assume that there is an algebra homomorphism $\lambda\colon A_{\mathrm{Heis}} \to \mathrm{End}(V)$, where $n := \dim(V) \in \mathbb{Z}_{>0}$. By taking the trace of the identity

$$(\lambda\partial)(\lambda x) - (\lambda x)(\lambda\partial) = \mathrm{id}_V , \tag{3.23}$$

and using the cyclic property of the trace, we obtain the equality $0 = n > 0$ which is a contradiction. Thus, $(A_{\mathrm{Heis}})^o = 0$.

3.1.2 An Infinite Dimensional Algebra A with $A^o = A^*$

Let V be an infinite dimensional vector space. Define an algebra A_V which, as a vector space, is the direct sum $\mathbb{F} \oplus V$ and the product

$$\mu((\alpha, v) \otimes (\beta, w)) = (\alpha, v)(\beta, w) = (\alpha\beta, \alpha w + \beta v) \tag{3.24}$$

Let $p \in A_V^*$ be the linear form defined by

$$\langle p, (\alpha, v)\rangle = \alpha. \tag{3.25}$$

For any $f \in A_V^*$, we have

$$\begin{aligned}
&\langle \mu^* f, (\alpha, v) \otimes (\beta, w)\rangle = \langle f, (\alpha\beta, \alpha w + \beta v)\rangle \\
&= \langle f, (1, 0)\rangle\alpha\beta + \langle f, (0, \alpha w + \beta v)\rangle = \langle f, (1, 0)\rangle\alpha\beta + \alpha\langle f, (0, w)\rangle + \beta\langle f, (0, v)\rangle \\
&= -\langle f, (1, 0)\rangle\alpha\beta + \alpha\langle f, (\beta, w)\rangle + \beta\langle f, (\alpha, v)\rangle \\
&= -\langle f, (1, 0)\rangle\langle p \otimes p, (\alpha, v) \otimes (\beta, w)\rangle \\
&+ \langle p \otimes f, (\alpha, v) \otimes (\beta, w)\rangle + \langle f \otimes p, (\alpha, v) \otimes (\beta, w)\rangle \\
&= \langle p \otimes f + f \otimes p - \langle f, (1, 0)\rangle p \otimes p, (\alpha, v) \otimes (\beta, w)\rangle.
\end{aligned} \tag{3.26}$$

Thus, $f \in A_V^o$ with

$$\mu^* f = p \otimes f + f \otimes p - \langle f, (1, 0)\rangle p \otimes p. \tag{3.27}$$

3.2 The Restricted Dual of the Tensor Product of Two Algebras

Lemma 3.1 *For any algebras A and B, the canonical embedding*

$$\alpha_{A,B} \colon A^o \otimes B^o \hookrightarrow (A \otimes B)^o \tag{3.28}$$

is a coalgebra isomorphism such that, for any pair of algebra morphisms $f \colon A \to U$ and $g \colon B \to V$, one has the equality

$$(f \otimes g)^o \alpha_{U,V} = \alpha_{A,B}(f^o \otimes g^o). \tag{3.29}$$

Proof

(1) Let A and B be algebras. Define the canonical algebra inclusions

$$\imath \colon A \hookrightarrow A \otimes B, \quad \jmath \colon B \hookrightarrow A \otimes B,$$
$$\imath x = x \otimes 1_B, \quad \jmath y = 1_A \otimes y, \quad \forall (x, y) \in A \times B. \tag{3.30}$$

Denoting $\alpha := \alpha_{A,B}$, let us show that the map

$$\beta := (\imath^o \otimes \jmath^o)\Delta_{(A\otimes B)^o} \colon (A \otimes B)^o \to A^o \otimes B^o \tag{3.31}$$

is the inverse of α.

For any $(\varphi, x, y) \in (A \otimes B)^o \times A \times B$, denoting $\Delta := \Delta_{(A\otimes B)^o}$, we have

$$\langle \alpha\beta\varphi, x \otimes y\rangle = \langle \beta\varphi, x \otimes y\rangle = \langle \Delta\varphi, \imath x \otimes \jmath y\rangle = \langle \varphi, (\imath x)(\jmath y)\rangle = \langle \varphi, x \otimes y\rangle \tag{3.32}$$

implying that β is a right inverse of α, and, for any $(f, g, x, y) \in A^o \times B^o \times A \times B$, we also have

$$\langle \beta\alpha(f \otimes g), x \otimes y\rangle = \langle \beta(f \otimes g), x \otimes y\rangle = \langle \Delta(f \otimes g), \imath x \otimes \jmath y\rangle$$
$$= \langle f \otimes g, (\imath x)(\jmath y)\rangle = \langle f \otimes g, x \otimes y\rangle \tag{3.33}$$

implying that β is a left inverse of α.

(2) In order to show that $\alpha_{A,B}$ is a morphism of coalgebras, it suffices to show that

$$\Delta_{(A\otimes B)^o}\alpha_{A,B} = (\alpha_{A,B} \otimes \alpha_{A,B})\Delta_{A^o\otimes B^o} \tag{3.34}$$

and

$$\epsilon_{(A\otimes B)^o}\alpha_{A,B} = \epsilon_{A^o} \otimes \epsilon_{B^o}. \tag{3.35}$$

Indeed, for any $(\varphi, \psi) \in A^o \times B^o$ and $(x, y, u, v) \in A^2 \times B^2$, we have

$$\begin{aligned}&\langle \Delta_{(A\otimes B)^o}\alpha_{A,B}(\varphi\otimes\psi), x\otimes u\otimes y\otimes v\rangle = \langle \varphi\otimes\psi, xy\otimes uv\rangle = \langle\varphi, xy\rangle\langle\psi, uv\rangle \\ &= \langle \Delta_{A^o}\varphi, x\otimes y\rangle\langle\Delta_{B^o}\psi, u\otimes v\rangle = \langle(\Delta_{A^o}\varphi)\otimes(\Delta_{B^o}\psi), x\otimes y\otimes u\otimes v\rangle \\ &= \langle\Delta_{A^o\otimes B^o}(\varphi\otimes\psi), x\otimes u\otimes y\otimes v\rangle = \langle(\alpha_{A,B}\otimes\alpha_{A,B})\Delta_{A^o\otimes B^o}(\varphi\otimes\psi), x\otimes u\otimes y\otimes v\rangle\end{aligned}$$

and

$$\begin{aligned}\langle\epsilon_{(A\otimes B)^o}\alpha_{A,B}, \varphi\otimes\psi\rangle &= \langle\varphi\otimes\psi, \eta_{A\otimes B}1\rangle \\ &= \langle\varphi\otimes\psi, \eta_A 1\otimes\eta_B 1\rangle = \langle\varphi, \eta_A 1\rangle\langle\psi, \eta_B 1\rangle \\ &= \langle\epsilon_{A^o}, \varphi\rangle\langle\epsilon_{B^o}, \psi\rangle = \langle\epsilon_{A^o}\otimes\epsilon_{B^o}, \varphi\otimes\psi\rangle.\end{aligned}$$

(3) Let $f: A \to U$ and $g: B \to V$ be algebra morphisms. For any quadruple $(\varphi, \psi, x, y) \in U^o \times V^o \times A \times B$, we have

$$\begin{aligned}&\langle(f\otimes g)^o\alpha_{U,V}(\varphi\otimes\psi), x\otimes y\rangle = \langle\varphi\otimes\psi, fx\otimes gy\rangle = \langle\varphi, fx\rangle\langle\psi, gy\rangle \\ &= \langle f^o\varphi, x\rangle\langle g^o\psi, y\rangle = \langle f^o\varphi\otimes g^o\psi, x\otimes y\rangle = \langle\alpha_{A,B}(f^o\otimes g^o)(\varphi\otimes\psi), x\otimes y\rangle.\end{aligned}$$

3.3 The Restricted Dual of a Hopf Algebra

The restricted dual H^o of a Hopf algebra H is defined as the restricted dual of the underlying algebra. In this subsection we show that the Hopf algebra operations of H imply that the restricted dual is itself a Hopf algebra.

Exercise 3.3 Let $f: X \to U$ and $g: Y \to V$ be two linear maps between vector spaces. Show that

$$(f\otimes g)^*|_{U^*\otimes V^*} = f^*|_{U^*}\otimes g^*|_{V^*}.$$

Proposition 3.1 *For any Hopf algebra $H = (H, \mu, \eta, \Delta, \epsilon, S)$, the restricted dual H^o is a Hopf algebra with respect to the dual structural maps*

$$\mu_{H^o} = \Delta^*|_{H^o\otimes H^o}, \quad \eta_{H^o} = \epsilon^*: 1\mapsto\epsilon, \quad \Delta_{H^o} = \mu^*|_{H^o}, \quad \epsilon_{H^o} = \eta^o = \eta^*|_{H^o}, \\ S_{H^o} = S^o = S^*|_{H^o}.$$

Proof By the functorial nature of the restricted dual, the vector space H^o is a coalgebra with the coproduct $\mu^*|_{H^o}$ and the counit η^o, and the algebra morphisms $\epsilon\colon H \to \mathbb{F}$ and $\Delta\colon H \to H \otimes H$ induce coalgebra morphisms $\epsilon^o\colon \mathbb{F} \to H^o$ and $\Delta^o\colon (H \otimes H)^o \to H^o$. By Lemma 3.1, the canonical inclusion

$$\alpha_{H,H}\colon H^o \otimes H^o \hookrightarrow (H \otimes H)^o$$

is an isomorphism of coalgebras and the composed map

$$\Delta^o \alpha_{H,H}\colon H^o \otimes H^o \to H^o$$

coincides with the restriction $\Delta^*|_{H^o \otimes H^o}$. This means that the triple

$$(H^o, \Delta^o \alpha_{H,H}, \epsilon^o)$$

is an algebra as a subalgebra of the convolution algebra H^*. Thus, the tuple

$$(H^o, \Delta^o \alpha_{H,H}, \epsilon^o, \mu^*|_{H^o}, \eta^o)$$

is a bialgebra.

Finally, we verify that S^o is the inverse of id_{H^o} in the convolution algebra $\mathrm{End}(H^o)$. By functoriality of the dual of a vector space, we have the equality

$$\epsilon^* \eta^* = \Delta^*(S \otimes \mathrm{id}_H)^* \mu^*\colon H^* \to H^* \tag{3.36}$$

which implies that

$$\begin{aligned}\eta_{H^o} \epsilon_{H^o} &= \Delta^*(S \otimes \mathrm{id}_H)^* \mu^*|_{H^o} = \Delta^*(S \otimes \mathrm{id}_H)^*|_{H^o \otimes H^o} \Delta_{H^o} \\ &= \Delta^*|_{H^o \otimes H^o}(S^o \otimes \mathrm{id}_{H^o}) \Delta_{H^o} = \mu_{H^o}(S^o \otimes \mathrm{id}_{H^o}) \Delta_{H^o}\end{aligned}$$

where, in the third equality, we used Exercise 3.3. The second relation is verified similarly. □