Chapter 1
Groups and Hopf Algebras

The main goal of this chapter is

- to give the definition of a Hopf algebra and to motivate it on the basis of the notion of a group which is of fundamental importance in mathematics;
- to introduce the graphical notation of string diagrams;
- to introduce the algebraic structures closely related to Hopf algebras, namely the notions of algebra, module, coalgebra, comodule, convolution algebra, and bialgebra;
- to establish few basic properties of Hopf algebras.

Section 1.1 in this chapter is the most abstract one where we briefly discuss the notions of a monoidal category, a braided monoidal category and a symmetric monoidal category. We do so for at least three reasons. First, those notions are used in the last Chap. 6 where knot invariants are defined in the general context of monoidal categories. The second reason is that, we motivate the definition of a Hopf algebra by the definition of a group (reformulated by using the structural maps), and these two definitions differ only by the underlying symmetric monoidal categories: vector spaces with tensor product in the case of Hopf algebras and sets with Cartesian product in the case of groups. The third reason is that many general constructions and statements in the book can be expressed in the language of string diagrams, and the latter make sense also in the context of arbitrary symmetric monoidal categories. In principle, with the exception of the last Sect. 1.1.3, where the graphical notation of string diagrams is introduced, Sect. 1.1 is optional for five Chaps. 1–5 where we mainly work only in the framework of multilinear algebra, that is the symmetric monoidal category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$ with the tensor product $\otimes_{\mathbb{F}}$ as the monoidal product. Thus, one can start the reading right from Sect. 1.1.3 and return back to Sect. 1.1 only before reading Chap. 6 dedicated to applications in knot theory.

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1.1 Monoidal Categories

In this section we give few basic definitions that concern monoidal or tensor categories, without elaborating details but providing few concrete examples. It is assumed that the reader is familiar with definitions of a category, a functor, a natural transformation, and a commutative diagram.

Since our main example of a monoidal category is the category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over a fixed base field $\mathbb{F}$ equipped with the tensor product $\otimes_{\mathbb{F}}$ as a monoidal product, on first reading, this section can be viewed as a summary of general properties of the category $\mathbf{Vect}_{\mathbb{F}}$. A more systematic and detailed presentation of monoidal categories can be found in Chapter 1 of the book [42] and chapter XI of the book [19].

1.1.1 Monoidal Categories

For any category $\mathcal{C}$, let $\mathcal{C} \times \mathcal{C}$ be the cartesian square of $\mathcal{C}$, which is a category whose objects are ordered pairs of objects of $\mathcal{C}$, morphisms are ordered pairs of morphisms of $\mathcal{C}$, and the composition is component-wise composition in $\mathcal{C}$.

A category $\mathcal{C}$ is called *monoidal* if it is equipped with a functor

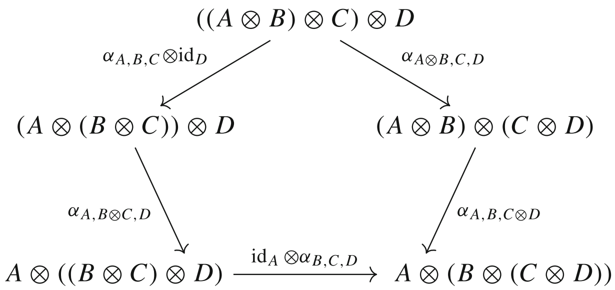
$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad (A, B) \mapsto A \otimes B, \quad (f, g) \mapsto f \otimes g, \tag{1.1}$$

called the *tensor* or *monoidal product*, an object I called the *unit* or *identity object*, and natural isomorphisms

$$\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad A, B, C \in \operatorname{Ob} \mathcal{C}, \tag{1.2}$$

$$\lambda_A \colon I \otimes A \to A, \quad \rho_A \colon A \otimes I \to A, \quad A \in \operatorname{Ob} \mathcal{C}, \tag{1.3}$$

respectively called *associator*, *left unitor* and *right unitor*, which satisfy two families of *coherence conditions* corresponding to commutative diagrams



(1.4)

and

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array} \tag{1.5}$$

respectively called the *pentagon* and the *triangle diagrams*, and the equality

$$\lambda_I = \rho_I : I \otimes I \to I. \tag{1.6}$$

A monoidal category is called *strict* if the natural isomorphisms α, λ, ρ are identities. It is known that any monoidal category is equivalent to a strict monoidal category, see for the proof, for example, [19].

Example 1.1 The category **Set** of sets is a monoidal category with the Cartesian product as the monoidal product and any one-element set, say $\underline{1} = \{0\}$, as the unit object. This is a prototypical example of a monoidal category. □

Example 1.2 The category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over a base field $\mathbb{F}$ with $\mathbb{F}$-linear maps as morphisms is a monoidal category with the tensor product $\otimes_{\mathbb{F}}$ as the monoidal product and the base field $\mathbb{F}$, viewed as a vector space of dimension one, as the unit object. This is the principal monoidal category we will be working with in this book. □

Just for the sake of clarity, to make the abstract definition less abstract, below we give some more less intuitive examples of monoidal categories, although they will not be used in any way in the following.

Before giving the next example, let us recall the definition of the direct sum of a family of vector spaces.

Definition 1.1 Let $\{V_i\}_{i\in I}$ be a family of vector spaces over a fixed base field. The *direct sum* of this family is the vector space $V := \bigoplus_{i\in I} V_i$ of all maps from the index set I to the set-theoretical union of all the vector spaces in the family,

$$x : I \to \cup_{i\in I} V_i, \quad i \mapsto x_i, \tag{1.7}$$

that satisfy the condition $x_i \in V_i$ for all $i \in I$, and x_i is the zero vector of V_i for all but finitely many i's.

For each index $i \in I$, there are two canonical linear maps associated to the direct sum $V = \bigoplus_{i\in I} V_i$ of a family vector spaces indexed by I. These are the projection map

$$p_i : V \to V_i, \quad x \mapsto x_i, \tag{1.8}$$

and the inclusion map

$$q_i\colon V_i \to V, \quad v \mapsto x, \quad x_j = \begin{cases} v & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

The projection and inclusion maps satisfy the following relations:

$$p_i q_j = \begin{cases} \mathrm{id}_{V_i} & \text{if } i = j; \\ 0 & \text{otherwise,} \end{cases} \quad \sum_{i \in I} q_i p_i = \mathrm{id}_V, \tag{1.10}$$

where the sum always truncates to a finite sum when applied to an element of V.

Example 1.3 The monoidal category $\mathbf{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded $\mathbb{F}$-vector spaces is a subcategory of $\mathbf{Vect}_{\mathbb{F}}$ defined as follows.

An object V of $\mathbf{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ is the direct sum of a $\mathbb{Z}$-indexed family of $\mathbb{F}$-vector spaces $\{V_n\}_{n\in\mathbb{Z}}$, while a morphism $f\colon V \to W$ is a linear map such that $f(V_n) \subset W_n$ for any $n \in \mathbb{Z}$. The tensor product $V \otimes_{\mathbb{F}} W$ of two $\mathbb{Z}$-graded vector spaces is the direct sum of the family

$$(V \otimes_{\mathbb{F}} W)_n = \bigoplus_{k\in\mathbb{Z}} (V_k \otimes_{\mathbb{F}} W_{n-k}), \quad n \in \mathbb{Z}. \tag{1.11}$$

The unit object I is the direct sum of the family

$$I_n = \begin{cases} \mathbb{F} & \text{if } n = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{1.12}$$

The tensor product $f \otimes_{\mathbb{F}} g$ of two morphisms $f\colon X \to U$ and $g\colon Y \to V$ is the usual tensor product of linear maps. □

Example 1.4 Define a category $\mathbf{Mat}(\mathbb{F})$ of matrices over a field $\mathbb{F}$ where the objects are elements of the set of non-negative integers $\omega = \mathbb{Z}_{\geq 0}$ and a morphism $f : m \to n$ is a (set-theoretical) map $f\colon \underline{m} \times \underline{n} \to \mathbb{F}$. The composition of $f\colon l \to m$ and $g\colon m \to n$ is the morphism $g \circ f : l \to n$ defined by

$$(g \circ f)_{i,j} = \sum_{k\in\underline{m}} f_{i,k} g_{k,j}, \quad \forall (i, j) \in \underline{l} \times \underline{n}.$$

This is a strict monoidal category where $m \otimes n = m + n$, the unit object being 0. As $\underline{0} = \emptyset$, for any object $n \in \omega$, there is only one morphism from 0 to n and from n to 0.

For two morphisms $f : m \to n$ and $g : k \to l$, their tensor (monoidal) product is the morphism $f \otimes g : m + k \to n + l$ defined by

$$(f \otimes g)_{i,j} = \begin{cases} f_{i,j} & \text{if } (i, j) \in \underline{m} \times \underline{n}; \\ g_{i-m,j-n} & \text{if } (i - m, j - n) \in \underline{k} \times \underline{l}; \\ 0 & \text{otherwise.} \end{cases}$$

A morphism $f : m \to n$ in the category $\mathbf{Mat}(\mathbb{F})$ can also be viewed as a m-by-n matrix

$$M_f = (f_{i,j})_{i \in \underline{m}, j \in \underline{n}}.$$

With this interpretation, the matrix associated to the composition $g \circ f$ is given by the matrix product

$$M_{g \circ f} = M_f M_g,$$

while the tensor product of two morphisms $f \otimes g$ is represented by the block matrix

$$M_{f \otimes g} = \begin{pmatrix} M_f & 0 \\ 0 & M_g \end{pmatrix}.$$

Example 1.5 Let G be a group and $H \subset G$ a normal subgroup. Denote by

$$C(H) := \{g \in G \mid gh = hg \ \forall h \in H\}$$

the commutant of H in G (which is also a normal subgroup of G). We define a category $\mathcal{G}_{H,G}$ where objects are elements of the quotient group G/H and a morphism $f : xH \to yH$ is a pair $(u, v) \in yH \times xH$ modulo the equivalence relation $(u, v) \sim (uh, vh)$, $h \in H$, such that $uv^{-1} \in C(H)$. Notice that the latter element does not change under the equivalence relation. The composition of morphisms $f = (u, v) : xH \to yH$ and $g = (p, q) : yH \to zH$ is given by the formula

$$g \circ f = (p, q) \circ (u, v) = (pq^{-1}u, v) : xH \to zH.$$

This formula is explicitly compatible with the equivalence relation and thus is well defined. Associativity of the composition is verified straightforwardly

$$\begin{aligned}(f \circ g) \circ h &= ((p, q) \circ (s, t)) \circ (u, v) = (pq^{-1}s, t) \circ (u, v) = (pq^{-1}st^{-1}u, v) \\ &= (p, q) \circ (st^{-1}u, v) = (p, q) \circ ((s, t) \circ (u, v)) = f \circ (g \circ h).\end{aligned}$$

The identity morphism id_{xH} is represented by (x, x).

This is a strict monoidal category where $xH \otimes yH = xyH$, the unit object being $eH = H$, and for two morphisms $f = (u, v)$ and $g = (p, q)$ their tensor product is defined by the component-wise multiplication $f \otimes g = (up, vq)$, and the condition $uv^{-1} \in C(H)$ for a morphism (u, v) ensures compatibility of the tensor product with the composition

$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d).$$

Example 1.6 A special case of the previous example corresponds to a group G with the trivial subgroup $H = \{e\}$. In this case, the category $\mathcal{G}_{H,G}$ is given by G as the set objects and, for any pair of objects $x, y \in G$, there is exactly one morphism $(y, x)\colon x \to y$. The group multiplication of G gives the monoidal structure of the category. □

1.1.2 Braided Monoidal Categories

For any category $\mathcal{C}$, the *exchange* functor

$$\varsigma\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C} \tag{1.13}$$

is defined by exchanging the components, that is

$$\varsigma(A, B) = (B, A), \quad \forall (A, B) \in \mathrm{Ob}(\mathcal{C} \times \mathcal{C}), \tag{1.14}$$

for objects and

$$\varsigma(f, g) = (g, f), \quad (f, g)\colon (A, B) \to (C, D), \tag{1.15}$$

for arrows (morphisms).

Let $\mathcal{C}$ be now a monoidal category. We have two functors

$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad (A, B) \mapsto A \otimes B, \quad (f, g) \mapsto f \otimes g \tag{1.16}$$

and

$$\otimes^{\mathrm{op}} = \otimes \circ \varsigma\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad (A, B) \mapsto B \otimes A, \quad (f, g) \mapsto g \otimes f. \tag{1.17}$$

A *braiding* in a monoidal category $\mathcal{C}$ is a natural isomorphism

$$\beta\colon \otimes \to \otimes^{\mathrm{op}} \tag{1.18}$$

such that the following diagrams are commutative:

$$
\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{\beta_{A,B\otimes C}} & (B \otimes C) \otimes A & \xrightarrow{\alpha_{B,A,C}} & B \otimes (C \otimes A) \\
\uparrow{\scriptstyle \alpha_{A,B,C}} & & & & {\scriptstyle \mathrm{id}_B \otimes \beta_{A,C}}\uparrow \\
(A \otimes B) \otimes C & \xrightarrow{\beta_{A,B}\otimes \mathrm{id}_C} & (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C)
\end{array}
\tag{1.19}
$$

and

$$
\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\beta_{A\otimes B,C}} & C \otimes (A \otimes B) & \xrightarrow{\alpha^{-1}_{C,A,B}} & (C \otimes A) \otimes B \\
\uparrow{\scriptstyle \alpha^{-1}_{A,B,C}} & & & & {\scriptstyle \beta_{A,C}\otimes \mathrm{id}_B}\uparrow \\
A \otimes (B \otimes C) & \xrightarrow{\mathrm{id}_A \otimes \beta_{B,C}} & A \otimes (C \otimes B) & \xrightarrow{\alpha^{-1}_{A,C,B}} & (A \otimes C) \otimes B
\end{array}
\tag{1.20}
$$

A *braided monoidal category* is a monoidal category with a braiding.

In view of applications in knot theory, braided monoidal categories constitute a very important class of monoidal categories, and the quantum double construction for Hopf algebras of Chap. 5 implicitly gives rise to braided monoidal categories, see, for example, [19, 21, 41].

A *symmetric monoidal category* is a braided monoidal category where the braiding satisfies the conditions

$$\beta^{-1}_{A,B} = \beta_{B,A}, \quad \forall A, B \in \mathrm{Ob}\,\mathcal{C}. \tag{1.21}$$

In this case, the braiding is called *symmetry* and denoted as σ.

Example 1.7 The category **Set** of sets, see Example 1.1, is a symmetric monoidal category where the symmetry is given by the exchange maps

$$\sigma_{X,Y} \colon X \times Y \to Y \times X, \quad (x, y) \mapsto (y, x). \tag{1.22}$$

As we will see in Sect. 1.2, any group can be interpreted as an object of this symmetric monoidal category. □

Example 1.8 The category $\mathbf{Vect}_{\mathbb{F}}$ of $\mathbb{F}$-vector spaces, see Example 1.2, is a symmetric monoidal category where the symmetry is given by the exchange maps extended by linearity

$$\sigma_{V,W} \colon V \otimes_{\mathbb{F}} W \to W \otimes_{\mathbb{F}} V, \quad x \otimes_{\mathbb{F}} y \mapsto y \otimes_{\mathbb{F}} x. \tag{1.23}$$

Hopf algebras are objects of this category, and the symmetry enters in their definition in one of the defining properties, see Definition 1.6 of Sect. 1.4. □

Example 1.9 For any $q \in \mathbb{F}_{\neq 0}$, the category $\mathbf{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded $\mathbb{F}$-vector spaces, see Example 1.3, is a braided monoidal category with the braiding

$$\beta_{U,V} : U \otimes V \to V \otimes U \tag{1.24}$$

defined by

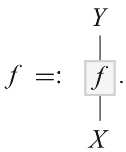
$$\beta_{U,V}(x \otimes_{\mathbb{F}} y) = q^{mn} y \otimes_{\mathbb{F}} x, \quad x \in U_m, \quad y \in V_n, \quad m, n \in \mathbb{Z}. \tag{1.25}$$

It is a symmetric monoidal category if $q^2 = 1$. □

1.1.3 *The Graphical Notation of String Diagrams*

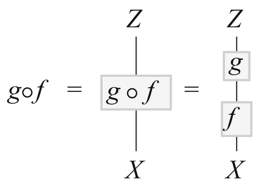
Throughout this book, we will find it convenient sometimes to use the graphical notation of *string diagrams*.

Let $\mathcal{C}$ be a category. To any morphism $f : X \to Y$ in $\mathcal{C}$, we associate a graphical picture



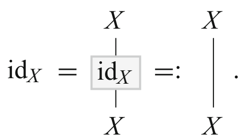
(1.26)

If $f : X \to Y$ and $g : Y \to Z$ are two composable morphisms, then their composition is described by the vertical concatenation of graphs



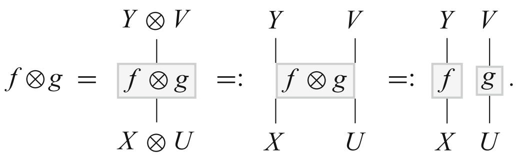
(1.27)

In particular, for the identity morphism id_X it is natural to use just a line



(1.28)

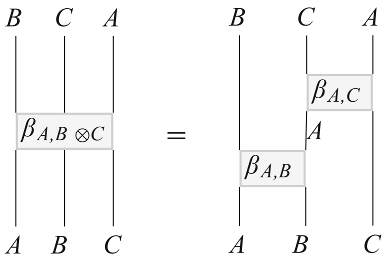
The string diagrams are especially useful in the case when $\mathcal{C}$ is a strict monoidal category, because the tensor (monoidal) product can be drawn by the horizontal juxtaposition. Namely, for two morphisms $f: X \to Y$ and $g: U \to V$, their tensor product $f \otimes g: X \otimes U \to Y \otimes V$ is drawn as follows:



(1.29)

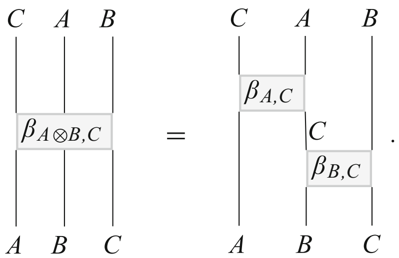
By taking into account the distinguished role of the identity object I, it is natural to associate to it the empty graph.

In this notation, for example, the commutative diagram (1.19) for a braiding, in the context of a strict monoidal category, corresponds to the following diagrammatic equality



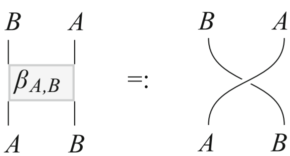
(1.30)

and the graphical equality corresponding to the commutative diagram (1.20)



(1.31)

These relations become intuitively natural and almost tautological, if one uses a notation for a braiding borrowed from knot diagrams



(1.32)

which, in the case of a symmetric monoidal category, can be further simplified by removing the indication of under-passing strands.

In the case of non-strict monoidal categories, the graphical calculus becomes less convenient because of the non-associativity of the tensor product. In this case, the horizontal juxtaposition is not enough so that one should provide an extra structure, for example, the relative distance between the vertical lines.

More systematic and detailed explanation of the graphical notation of string diagrams can be found in Chapter 2 of the book [42].

1.2 Groups in Terms of Structural Maps

At first glance, the formal definition of a Hopf algebra (to be given later in Sect. 1.4) looks neither simple nor intuitively motivated. For this reason, we start by reviewing the notion of a group which we reformulate by using the structural maps as the basic entities. Such a reformulation will make the definition of a Hopf algebra very natural, at least from the viewpoint of group theory.

Recall that a *group* is a set G where, for any two elements $g, h \in G$, there corresponds a unique element gh called the *product* of g and h, a distinguished element e called the *identity element*, and, for any element g, there corresponds a unique element g^{-1} called the *inverse element* such that the following axioms are satisfied:

$$\text{associativity}: \quad (fg)h = f(gh), \quad \forall f, g, h \in G, \tag{1.33}$$

$$\text{unitality}: \quad eg = ge = g, \quad \forall g \in G, \tag{1.34}$$

$$\text{invertibility}: \quad gg^{-1} = g^{-1}g = e, \quad \forall g \in G. \tag{1.35}$$

We formalize the definition of a group by introducing three *structural maps*:

$$\text{product} \quad \mu\colon G \times G \to G, \quad (g, h) \mapsto gh, \tag{1.36}$$

$$\text{unit} \quad \eta\colon \underline{1} \to G, \quad 0 \mapsto e, \tag{1.37}$$

$$\text{inverse} \quad S\colon G \to G, \quad g \mapsto g^{-1}. \tag{1.38}$$

By rewriting

$$(fg)h = \mu(fg, h) = \mu(\mu(f, g), h) = \mu((\mu \times \mathrm{id})(f, g, h))$$
$$= \mu \circ (\mu \times \mathrm{id})(f, g, h) \tag{1.39}$$

and

$$f(gh) = \mu(f, gh) = \mu(f, \mu(g, h)) = \mu((\text{id} \times \mu)(f, g, h)) \\ = \mu \circ (\text{id} \times \mu)(f, g, h), \qquad (1.40)$$

we conclude that the associativity axiom is equivalent to the following equality for the product map

$$\mu \circ (\mu \times \text{id}) = \mu \circ (\text{id} \times \mu) \qquad (1.41)$$

which corresponds to the commutative diagram

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \text{id}} & G \times G \\ {\scriptstyle \text{id} \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\mu} & G \end{array} . \qquad (1.42)$$

Before going further with the unitality axiom, let us agree on the following convention.

Let $\underline{1} := \{0\}$ be the set consisting of one element denoted by 0. For any set X, the (cartesian) product sets $\underline{1} \times X$ and $X \times \underline{1}$ are identified with X through the obvious canonical bijections $(0, x) \mapsto x$ and $(x, 0) \mapsto x$ which allows us to have natural identifications $(0, x) = (x, 0) = x$. With this convention, for the unitality axiom, we have

$$e \cdot g = \mu(\eta(0), g) = \mu \circ (\eta \times \text{id})(0, g) = \mu \circ (\eta \times \text{id})(g) \qquad (1.43)$$

and

$$g \cdot e = \mu(g, \eta(0)) = \mu \circ (\text{id} \times \eta)(g, 0) = \mu \circ (\text{id} \times \eta)(g) \qquad (1.44)$$

so that the unitality axiom can be stated as the following equations for the structural maps

$$\mu \circ (\eta \times \text{id}) = \text{id} = \mu \circ (\text{id} \times \eta) \qquad (1.45)$$

which correspond to the commutative diagram

$$\begin{array}{ccccc} \underline{1} \times G & = & G & = & G \times \underline{1} \\ {\scriptstyle \eta \times \text{id}}\downarrow & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id} \times \eta} \\ G \times G & \xrightarrow{\mu} & G & \xleftarrow{\mu} & G \times G \end{array} . \qquad (1.46)$$

In order to describe the invertibility axiom in terms of equations for the structural maps, we need to use two other maps which are canonically defined for any set X. These are the *diagonal* or *coproduct* map

$$\Delta \colon X \to X \times X, \quad x \mapsto (x, x), \quad \forall x \in X, \tag{1.47}$$

and the *counit* map

$$\epsilon \colon X \to \underline{1}, \quad x \mapsto 0, \quad \forall x \in X. \tag{1.48}$$

These names come from the fact that they are similar to the product and the unit maps of a group in the following sense.

Definition 1.2 A commutative diagram Γ is called a *categorial* or *diagrammatic dual* of another commutative diagram Γ', if Γ can be obtained from Γ' by reversing all arrows and relabelling the objects.

The diagonal map satisfies the equality

$$(\mathrm{id} \times \Delta) \circ \Delta = (\Delta \times \mathrm{id}) \circ \Delta \tag{1.49}$$

corresponding to the commutative diagram

$$\begin{array}{ccc} X \times X \times X & \xleftarrow{\Delta \times \mathrm{id}} & X \times X \\ {\scriptstyle \mathrm{id} \times \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ X \times X & \xleftarrow{\quad \Delta \quad} & X \end{array} \tag{1.50}$$

which is the categorial dual of the commutative diagram (1.42) corresponding to equality (1.41). For this reason, equality (1.49) is called the *coassociativity* property.

The counit map enters the *counitality* equalities

$$(\epsilon \times \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \times \epsilon) \circ \Delta. \tag{1.51}$$

corresponding to the commutative diagram

$$\begin{array}{ccccc} \underline{1} \times X & = & X & = & X \times \underline{1} \\ {\scriptstyle \epsilon \times \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id} \times \epsilon} \\ X \times X & \xleftarrow{\Delta} & X & \xrightarrow{\Delta} & X \times X \end{array} \tag{1.52}$$

which is the categorial dual of the commutative diagram (1.46) corresponding to equalities (1.45) expressing the unitality axiom.

In order to express the invertibility axiom in terms of equations for structural maps, we write

$$gg^{-1} = \mu(g, g^{-1}) = \mu \circ (\mathrm{id} \times S)(g, g) = \mu \circ (\mathrm{id} \times S) \circ \Delta(g), \tag{1.53}$$

$$g^{-1}g = \mu(g^{-1}, g) = \mu \circ (S \times \mathrm{id})(g, g) = \mu \circ (S \times \mathrm{id}) \circ \Delta(g), \tag{1.54}$$

and

$$e = \eta(0) = \eta(\epsilon(g)) = \eta \circ \epsilon(g). \tag{1.55}$$

Thus, the invertibility axiom is equivalent to the equations

$$\mu \circ (\mathrm{id} \times S) \circ \Delta = \eta \circ \epsilon = \mu \circ (S \times \mathrm{id}) \circ \Delta \tag{1.56}$$

corresponding to the commutative diagram

$$\begin{array}{ccccc} G \times G & & \xrightarrow{\mathrm{id} \times S} & & G \times G \\ \Delta \uparrow & & & & \downarrow \mu \\ G & \xrightarrow{\epsilon} & \underline{1} & \xrightarrow{\eta} & G \\ \Delta \downarrow & & & & \uparrow \mu \\ G \times G & & \xrightarrow{S \times \mathrm{id}} & & G \times G \end{array} \tag{1.57}$$

which is the categorial dual of itself.

Finally, by using the canonical exchange map

$$\sigma = \sigma_{G,G} \colon G \times G \to G \times G, \quad (x, y) \mapsto (y, x), \tag{1.58}$$

we remark that the product and the coproduct satisfy the *compatibility* equality

$$(\mu \times \mu) \circ (\mathrm{id} \times \sigma \times \mathrm{id}) \circ (\Delta \times \Delta) = \Delta \circ \mu \colon G \times G \to G \times G \tag{1.59}$$

corresponding to the commutative diagram

$$\begin{array}{ccccc} G \times G \times G \times G & & \xrightarrow{\mathrm{id} \times \sigma \times \mathrm{id}} & & G \times G \times G \times G \\ \Delta \times \Delta \uparrow & & & & \downarrow \mu \times \mu \\ G \times G & \xrightarrow{\mu} & G & \xrightarrow{\Delta} & G \times G \end{array} \tag{1.60}$$

which is also the categorial dual of itself. Moreover, identity (1.59) holds even if G is replaced by any set X and the product μ by any binary operation $f : X \times X \to X$. Indeed, for any $(x, y) \in X \times X$, we have

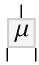
$$\begin{aligned}&(f \times f) \circ (\mathrm{id} \times \sigma \times \mathrm{id}) \circ (\Delta \times \Delta)(x, y)\\ &= (f \times f) \circ (\mathrm{id} \times \sigma \times \mathrm{id})(x, x, y, y) = (f \times f)(x, y, x, y) = (f(x, y), f(x, y))\\ &= \Delta(f(x, y)) = \Delta \circ f(x, y). \qquad (1.61)\end{aligned}$$

As the matter of fact, this calculation reflects an elementary general property of the cartesian symmetric monoidal category of sets which will be described in Sect. 1.3.

1.2.1 The Structural Maps of a Group in Graphical Notation

We are ready now to use the graphical notation of string diagrams introduced in Sect. 1.1.3 to rewrite the definition of a group. The monoidal category we are working in is the symmetric monoidal category **Set** of sets with the tensor product specified by the Cartesian product of sets, see Example 1.7.

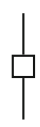
Let us introduce the following graphical notation for the structural maps of a group (all lines correspond to the underlying set of the group G and the singleton $\underline{1}$ carries no line):

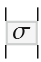
product μ =:  =:  (1.62)

coproduct Δ =:  =:  (1.63)

unit η =:  =:  (1.64)

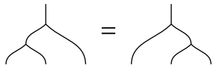
counit ϵ =:  =:  (1.65)

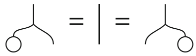
inverse or antipode S =:  =:  (1.66)

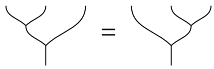
exchange or symmetry σ =:  =: . (1.67)

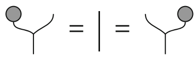
For the inverse map in (1.66) we also put the term “antipode” in anticipation of its counterpart in the case of Hopf algebras, see Definition 1.6 of Sect. 1.4.

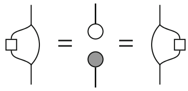
Recall from Sect. 1.1.3 that the composition of maps corresponds to vertical concatenation of the corresponding graphical objects, while the Cartesian product corresponds to horizontal juxtaposition. With this notation, the structural equations of a group take the following form:

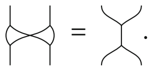
associativity:  (1.68)

unitality:  (1.69)

coassociativity:  (1.70)

counitality:  (1.71)

invertibility:  (1.72)

compatibility:  (1.73)

Remark 1.1 A motivational idea behind the definition of a Hopf algebra is to think of these diagrams in the context of other symmetric monoidal categories. The corresponding realizations are called *group objects*. In particular, as we will see later in Sect. 1.4, a *Hopf algebra* can be identified as a group object in the symmetric monoidal category of vector spaces with the tensor product as the monoidal product, see Example 1.8.

1.3 Monoids and Comonoids

Given an algebraic notion, for example a group, it is often useful and instructive to consider other structures obtained from the initial one by dropping some of the defining properties/axioms. In the definition of a group, if we remove the inverse map together with the invertibility axiom, then we obtain the notion of a monoid.

Definition 1.3 A *monoid* is a set M together with two maps

$$\mu\colon M \times M \to M, \quad \eta : \underline{1} \to M, \tag{1.74}$$

respectively called *product* and *unit*, which satisfy the associativity axiom (1.41) and the unitality axiom (1.45).

Exercise 1.1 (Uniqueness of Inverses) An element $x \in M$ of a monoid M is called *invertible* if there exists an element $y \in M$, called *inverse* of x, such that $\mu(x, y) = \mu(y, x) = \eta(0)$. Show that any invertible element x admits a unique inverse.

Definition 1.4 Let $M = (M, \mu_M, \eta_M)$ and $N = (N, \mu_N, \eta_N)$ be two monoids. A map $f : M \to N$ is called *morphism of monoids* if it commutes with the structural maps in the sense of the relations

$$f \circ \mu_M = \mu_N \circ (f \times f), \quad f \circ \eta_M = \eta_N. \tag{1.75}$$

The notion of a *comonoid* is obtained by taking the categorial dual of the notion of a monoid, i.e. by reversing all arrows in the definition of a monoid in terms of commutative diagrams, see Definition 1.2.

Definition 1.5 A *comonoid* is a set C provided with two maps

$$\Delta : C \to C \times C, \quad \epsilon : C \to \underline{1}, \tag{1.76}$$

called *coproduct* and *counit*, which satisfy the coassociativity axiom (1.49) and the counitality axiom (1.51).

Exercise 1.2 Give a definition of a morphism of comonoids.

The following proposition shows that the notion of a comonoid is not particularly meaningful in a set-theoretic context.

Proposition 1.1 *Any set admits a unique comonoid structure and any map between two sets is a morphism of comonoids.*

Proof Notice that the counit map of any comonoid, being a map to the singleton, is uniquely fixed, and it is thus uniquely defined also for any set. Moreover, it is easily seen that any set X is a comonoid with the diagonal map as the coproduct and the map to the singleton as the counit. The formal proof is identical to the case of groups, see Eqs. (1.49)–(1.52). Let us show that there are no other comonoids.

Let $C = (C, \Delta, \epsilon)$ be a comonoid. Then the coproduct Δ corresponds to two maps $\alpha, \beta : C \to C$ defined by

$$\Delta(x) = (\alpha(x), \beta(x)). \tag{1.77}$$

Substituting this into the counitality axiom (1.51), we obtain

$$\begin{aligned}(\epsilon \times \mathrm{id}) \circ \Delta(x) &= (\epsilon \times \mathrm{id})(\alpha(x), \beta(x)) \\ &= (\epsilon(\alpha(x)), \beta(x)) = (0, \beta(x)) = \beta(x) = x,\end{aligned} \tag{1.78}$$

and

$$\begin{aligned}(\mathrm{id} \times \epsilon) \circ \Delta(x) &= (\mathrm{id} \times \epsilon)(\alpha(x), \beta(x)) \\ &= (\alpha(x), \epsilon(\beta(x))) = (\alpha(x), 0) = \alpha(x) = x \end{aligned} \tag{1.79}$$

where we use the convention on the equality for the canonical identifications

$$X \times \underline{1} \simeq X \simeq \underline{1} \times X.$$

Thus, the coproduct is necessarily the diagonal map $\Delta(x) = (x, x)$.

Finally, any map between two sets $f : X \to Y$ enters the obvious commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\Delta} & & {\scriptstyle\Delta}\downarrow \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle\epsilon}\searrow \quad \swarrow{\scriptstyle\epsilon} & \\ & 1 & \end{array} \tag{1.80}$$

which mean that f is a morphism of comonoids. □

1.4 Hopf Algebras

In this section, we introduce the central object of this book, a Hopf algebra. The definition that follows is motivated by the notion of a group which we reformulated in Sect. 1.2 in terms of structural maps. As the notions of a monoid or/and a comonoid are the results of dropping some of the structural maps and axioms from the definition of a group, in the subsequent sections of this chapter, we also introduce the analogous notions of an algebra (Sect. 1.6) and a coalgebra (Sect. 1.7) by dropping the corresponding structural maps and axioms from the definition of a Hopf algebra. There is also a notion of a bialgebra (Sect. 1.10), which in the set-theoretical context, corresponds to a monoid, but in the context of vector spaces, it is not true that every algebra (or coalgebra) is a bialgebra, though any Hopf algebra is a bialgebra. The reason for this difference comes from the fact that any set is canonically a comonoid in a unique way, as we have seen in the previous section, see Proposition 1.1, while a given vector space can admit many structures of a coalgebra, and, for a given algebra, there could be different possibilities, for example, non of available coalgebra structures on the underlying vector space can be compatible with the algebra structure, etc.

In what follows, we let $\mathbb{F}$ denote a field, write $\otimes$ instead of $\otimes_{\mathbb{F}}$ and omit the composition symbol in the case of linear maps.

Definition 1.6 A *Hopf algebra* (over a field $\mathbb{F}$) is a $\mathbb{F}$-vector space H of strictly positive dimension together with the following five linear maps:

$$\text{product} \quad \mu\colon H \otimes H \to H, \tag{1.81}$$

$$\text{coproduct} \quad \Delta\colon H \to H \otimes H, \tag{1.82}$$

$$\text{unit} \quad \eta\colon \mathbb{F} \to H, \tag{1.83}$$

$$\text{counit} \quad \epsilon\colon H \to \mathbb{F}, \tag{1.84}$$

$$\text{antipode} \quad S\colon H \to H, \tag{1.85}$$

which satisfy the following equations (axioms):

$$\text{associativity}: \quad \mu(\mu \otimes \mathrm{id}_H) = \mu(\mathrm{id}_H \otimes \mu), \tag{1.86}$$

$$\text{unitality}: \quad \mu(\eta \otimes \mathrm{id}_H) = \mathrm{id}_H = \mu(\mathrm{id}_H \otimes \eta), \tag{1.87}$$

$$\text{coassociativity}: \quad (\mathrm{id}_H \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_H)\Delta, \tag{1.88}$$

$$\text{counitality}: \quad (\epsilon \otimes \mathrm{id}_H)\Delta = \mathrm{id}_H = (\mathrm{id}_H \otimes \epsilon)\Delta, \tag{1.89}$$

$$\text{invertibility}: \quad \mu(\mathrm{id}_H \otimes S)\Delta = \eta\epsilon = \mu(S \otimes \mathrm{id}_H)\Delta, \tag{1.90}$$

$$\text{compatibility}: \quad (\mu \otimes \mu)(\mathrm{id}_H \otimes \sigma \otimes \mathrm{id}_H)(\Delta \otimes \Delta) = \Delta\mu, \tag{1.91}$$

where the symmetry map $\sigma = \sigma_{H,H}\colon H \otimes H \to H \otimes H$ acts by $\sigma(x \otimes y) = y \otimes x$.

Remark 1.2 The list of axioms (1.86)–(1.91) exactly corresponds to the list of graphical relations (1.68)–(1.73), and we will often use the same graphical notation in this new context with the replacements:

$$\begin{aligned} \text{sets} &\mapsto \text{vector spaces} \\ \text{set theoretical maps} &\mapsto \text{linear maps} \\ \text{the singleton } \underline{1} = \{0\} &\mapsto \text{the base field } \mathbb{F} \text{ (a 1-dimensional vector space)} \\ \text{the cartesian product} &\mapsto \text{the tensor product.} \end{aligned}$$

Definition 1.7 Let

$$H = (H, \mu_H, \eta_H, \Delta_H, \epsilon_H, S_H) \text{ and } L = (L, \mu_L, \eta_L, \Delta_L, \epsilon_L, S_L)$$

be two Hopf algebras. A linear map $f\colon H \to L$ is called a *morphism of Hopf algebras* or a *Hopf algebra morphism* if it commutes with all the structural maps, that is the following diagrams are commutative:

$$\begin{array}{ccc} H \otimes H & \xrightarrow{f \otimes f} & L \otimes L \\ \downarrow{\scriptstyle \mu_H} & & {\scriptstyle \mu_L}\downarrow \\ H & \xrightarrow{f} & L \end{array} , \qquad \begin{array}{ccc} H & \xrightarrow{f} & L \\ {\scriptstyle \eta_H}\nwarrow & & \nearrow{\scriptstyle \eta_L} \\ & \mathbb{F} & \end{array} , \tag{1.92}$$

$$\begin{array}{ccc} H \otimes H & \xrightarrow{f \otimes f} & L \otimes L \\ {\scriptstyle \Delta_H}\uparrow & & \uparrow{\scriptstyle \Delta_L} \\ H & \xrightarrow{f} & L \end{array} , \qquad \begin{array}{ccc} H & \xrightarrow{f} & L \\ {\scriptstyle \epsilon_H}\searrow & & \swarrow{\scriptstyle \epsilon_L} \\ & \mathbb{F} & \end{array} , \tag{1.93}$$

$$\begin{array}{ccc} H & \xrightarrow{f} & L \\ {\scriptstyle S_H}\downarrow & & \downarrow{\scriptstyle S_L} \\ H & \xrightarrow{f} & L \end{array} . \tag{1.94}$$

These commutative diagrams correspond to the following equalities between linear maps

$$\mu_L(f \otimes f) = f\mu_H, \quad f\eta_H = \eta_L, \tag{1.95}$$

$$(f \otimes f)\Delta_H = \Delta_L f, \quad \epsilon_L f = \epsilon_H, \tag{1.96}$$

$$S_L f = f S_H. \tag{1.97}$$

Remark 1.3 In any Hopf algebra, we have the inequality $\eta \neq 0$ as otherwise the Hopf algebra would be zero-dimensional.

1.5 Group Algebras as Hopf Algebras

In this section we consider a class of examples of Hopf algebras coming from groups. Given the fact that we have motivated the definition of a Hopf algebra by considering the definition of a group, it is not very surprising that the two notions are related.

Definition 1.8 For any set X, we denote by $\delta_{a,b}$ the *Kronecker delta* function which is the characteristic function $\chi_{\Delta(X)}\colon X \times X \to \{0, 1\}$ of the diagonal $\Delta(X)$ in $X \times X$, i.e.

$$\delta_{a,b} = \chi_{\Delta(X)}(a, b) = \begin{cases} 1 \text{ if } a = b, \\ 0 \text{ if } a \neq b. \end{cases} \tag{1.98}$$

Definition 1.9 Let X be a set. The vector space of all maps $f\colon X \to \mathbb{F}$ of finite support, that is a map that takes all but finitely many values zero, is called the *vector space freely generated by* X, and it is denoted as $\mathbb{F}[X]$. A natural linear basis in $\mathbb{F}[X]$ is given by the set of single element characteristic functions $\{\chi_a\}_{a\in X}$ defined by

$$\chi_a(b) = \delta_{a,b}, \quad \forall (a, b) \in X^2. \tag{1.99}$$

Remark 1.4 The vector space $\mathbb{F}[X]$ can also be described as the direct sum of a family of 1-dimensional vector spaces $\mathbb{F}$ indexed by the set X:

$$\mathbb{F}[X] = \bigoplus_{x\in X} \mathbb{F}, \tag{1.100}$$

see Definition 1.1.

For any group G, let $\mathbb{F}[G]$ be the vector space freely generated by the set G. We define the product

$$\mu\colon \mathbb{F}[G] \otimes \mathbb{F}[G] \to \mathbb{F}[G], \quad (\mu(f \otimes g))(a) = \sum_{b\in G} f(b)g(b^{-1}a), \quad \forall a \in G, \tag{1.101}$$

the unit

$$\eta\colon \mathbb{F} \to \mathbb{F}[G], \quad (\eta 1)(a) = \delta_{e,a}, \quad \forall a \in G, \tag{1.102}$$

the coproduct

$$\Delta\colon \mathbb{F}[G] \to \mathbb{F}[G]\otimes\mathbb{F}[G], \quad (\Delta f)(a, b) = \delta_{a,b} f(a), \quad \forall (a, b) \in G^2, \tag{1.103}$$

the counit

$$\epsilon f = \sum_{a\in G} f(a), \tag{1.104}$$

and the antipode

$$(Sf)(a) = f(a^{-1}), \quad \forall a \in G. \tag{1.105}$$

Remark that in the definition of the product (1.101), the sum is finite due to the fact that the functions f and g are finitely supported. For the same reason, the function $\mu(f \otimes g)$ is also finitely supported. Indeed if $\mathrm{Supp}_f \subset G$ is the support of $f \in \mathbb{F}[G]$, then we have the inclusion

$$\mathrm{Supp}_{\mu(f\otimes g)} \subset \bigcup_{a\in \mathrm{Supp}_f} a\, \mathrm{Supp}_g \,. \tag{1.106}$$

With respect to the natural basis of single element characteristic functions $\{\chi_a\}_{a\in G}$, the structural maps take the form

$$\mu(\chi_a \otimes \chi_b) = \chi_{ab}, \quad \Delta\chi_a = \chi_a \otimes \chi_a, \tag{1.107}$$

$$\eta 1 = \chi_e, \quad \epsilon\chi_a = 1, \quad S\chi_a = \chi_{a^{-1}}. \tag{1.108}$$

Exercise 1.3 Show that the data $(\mathbb{F}[G], \mu, \eta, \Delta, \epsilon, S)$ satisfy the Hopf algebra axioms.

1.6 Algebras

Here we introduce the notion of an algebra by dropping some of the data in the definition of a Hopf algebra, namely we leave only the product and the unit as structural maps and impose on them the axioms of associativity and unitality corresponding to diagrammatic equations (1.68) and (1.69). An algebra is a monoidal object in the monoidal category of vector spaces with the tensor product as the monoidal product.

Definition 1.10 An *algebra* over a field $\mathbb{F}$ or $\mathbb{F}$-*algebra* is a triple (A, μ, η) consisting of a $\mathbb{F}$-vector space A, a linear map $\mu\colon A \otimes A \to A$ called *product*, and a linear map $\eta\colon \mathbb{F} \to A$ called *unit* such that

$$\mu(\mu \otimes \mathrm{id}_A) = \mu(\mathrm{id}_A \otimes \mu) \tag{1.109}$$

and

$$\mu(\eta \otimes \mathrm{id}_A) = \mu(\mathrm{id}_A \otimes \eta) = \mathrm{id}_A. \tag{1.110}$$

Example 1.10 As Eqs. (1.109) and (1.110) coincide respectively with Eqs. (1.86) and (1.87), any Hopf algebra is an algebra, if we keep the product and the unit and forget about all other structural maps. □

Example 1.11 Let V be a vector space. Then, the vector space $\mathrm{End}(V)$ of all endomorphisms of V is an algebra with the product

$$\mu(f \otimes g) = fg, \quad \forall (f, g) \in (\mathrm{End}(V))^2, \tag{1.111}$$

and the unit

$$\eta 1 = \mathrm{id}_V \, . \tag{1.112}$$

In particular, the base field $\mathbb{F} \simeq \mathrm{End}(\mathbb{F})$ is an algebra. □

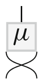
Definition 1.11 Let $A = (A, \mu_A, \eta_A)$ and $B = (B, \mu_B, \eta_B)$ be two algebras. A linear map $f : A \to B$ is called a *morphism of algebras* or an *algebra morphism* if it commutes with the structural maps in the sense of the equations

$$f \mu_A = \mu_B (f \otimes f) \tag{1.113}$$

and

$$f \eta_A = \eta_B . \tag{1.114}$$

Definition 1.12 The *opposite product* of an algebra $A := (A, \mu, \eta)$ is the linear map μ^{op} obtained by composing the product with the exchange map,

$$\mu^{\mathrm{op}} := \mu \sigma_{A,A} =$$  $$: A \otimes A \to A, \quad x \otimes y \mapsto \mu(y \otimes x). \tag{1.115}$$

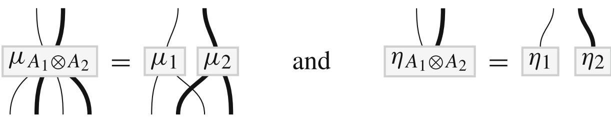
The algebra A is called *commutative* if the opposite product coincides with the product, $\mu^{\mathrm{op}} = \mu$.

Exercise 1.4 Show that if $A := (A, \mu, \eta)$ is an algebra then $A^{\mathrm{op}} := (A, \mu^{\mathrm{op}}, \eta)$ is also an algebra.

Definition 1.13 Let $A_1 = (A_1, \mu_1, \eta_1)$ and $A_2 = (A_2, \mu_2, \eta_2)$ be two algebras. The *tensor product* of A_1 and A_2 is the algebra

$$(A_1 \otimes A_2, (\mu_1 \otimes \mu_2)(\mathrm{id}_{A_1} \otimes \sigma_{A_2,A_1} \otimes \mathrm{id}_{A_2}), \eta_1 \otimes \eta_2) \tag{1.116}$$

or graphically



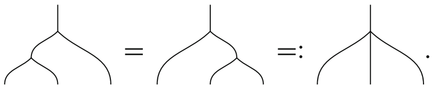
(1.117)

where the thin lines correspond to A_1 and thick lines to A_2, and we implicitly identify $\mathbb{F} \otimes \mathbb{F}$ with $\mathbb{F}$.

Exercise 1.5 Let $A = (A, \mu, \eta)$ be an algebra. Show that the unit $\eta \colon \mathbb{F} \to A$ is always a morphism of algebras, while the product $\mu \colon A \otimes A \to A$ is a morphism of algebras if and only if A is commutative.

1.6.1 Iterated Products

Let $A = (A, \mu, \eta)$ be an $\mathbb{F}$-algebra. In calculations, it is the common practice to write just xy instead of $\mu(x \otimes y)$. In particular, as the associativity axiom (1.109) implies that $(xy)z = x(yz)$, one can just write xyz without any ambiguity. Graphically, this means that we can use multivalent vertices:


(1.118)

This can be formalised by introducing the set of iterated products

$$\{\mu^{(m)} \colon A^{\otimes m} \to A\}_{m \in \omega} \tag{1.119}$$

defined recursively as follows:

$$\mu^{(m)} := \mu(\mu^{(m-1)} \otimes \mathrm{id}_A), \quad \mu^{(0)} := \eta, \tag{1.120}$$

so that, in particular, we have

$$\mu^{(1)} = \mathrm{id}_A, \quad \mu^{(2)} = \mu. \tag{1.121}$$

The n-th iterated product $\mu^{(n)}$ graphically can be represented by any binary tree with n inputs and one output, because the associativity of the product allows to ensure that any such tree gives one and the same linear map which we denote by a multivalent vertex.

Exercise 1.6 Prove that

$$\mu^{(k_1+\cdots+k_m)} = \mu^{(m)}\left(\mu^{(k_1)} \otimes \cdots \otimes \mu^{(k_m)}\right), \quad \forall (k_1, \ldots, k_m) \in \omega^m. \tag{1.122}$$

Exercise 1.7 Let $f \colon A \to B$ be an algebra morphism. Prove that

$$\mu_B^{(m)} f^{\otimes m} = f \mu_A^{(m)}, \quad \forall m \in \omega. \tag{1.123}$$

1.6.2 Modules

In the context of vector spaces, the notion of a module over an algebra corresponds to an M-set in the set-theoretical context, that is a set on which a monoid M acts.

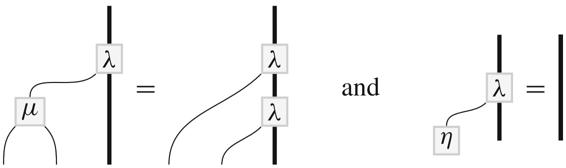
Definition 1.14 Let $A = (A, \mu, \eta)$ be an algebra over a field $\mathbb{F}$. A *left module* over A (or simply a *left A-module*) is a $\mathbb{F}$-vector space V together with a linear map

$$\lambda \colon A \otimes V \to V \tag{1.124}$$

such that the diagrams

$$\begin{array}{ccc} A \otimes A \otimes V & \xrightarrow{\mu \otimes \mathrm{id}_V} & A \otimes V \\ {\scriptstyle \mathrm{id}_A \otimes \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ A \otimes V & \xrightarrow{\lambda} & V \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{F} \otimes V & = & V \\ {\scriptstyle \eta \otimes \mathrm{id}_V}\searrow & & \nearrow{\scriptstyle \lambda} \\ & A \otimes V & \end{array} \tag{1.125}$$

are commutative. In terms of our graphical notation, the commutative diagrams (1.125) correspond to the equations



(1.126)

where the thick lines correspond to V and thin lines to A.

Remark 1.5 For two vector spaces X and Y, let $\mathrm{L}(X, Y)$ be the set of all linear maps from X to Y. The natural bijection between two sets of linear maps

$$\mathrm{L}(A \otimes V, V) \simeq \mathrm{L}(A, \mathrm{End}(V)) \tag{1.127}$$

descents to a natural bijection between the sets of left A-module structures on V and algebra morphisms from A to $\mathrm{End}(V)$. For this reason, a left A-module structure on a vector space V is often called *representation* of A in V.

Exercise 1.8 Give a definition of a right module over an algebra A.

1.7 Coalgebras

The notion of a coalgebra is the categorial dual of that of an algebra in the sense that the commutative diagrams expressing the defining properties of an algebra and coalgebra are related through the categorial duality, see Definition 1.2. The definition of a coalgebra is obtained by dropping all the structural maps in the definition of a Hopf algebra, apart from the coproduct and the counit, and by keeping the axioms of coassociativity and counitality. These axioms correspond to two diagrammatic equations (1.70) and (1.71).

Definition 1.15 A *coalgebra* over a field $\mathbb{F}$ or a $\mathbb{F}$-*coalgebra* is a triple (C, Δ, ϵ) consisting of a $\mathbb{F}$-vector space C, a linear map $\Delta\colon C \to C \otimes C$ called *coproduct*, and a linear map $\epsilon\colon C \to \mathbb{F}$ called *counit* such that

$$(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta \tag{1.128}$$

and

$$(\epsilon \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \epsilon)\Delta = \mathrm{id}_C. \tag{1.129}$$

Example 1.12 As Eqs. (1.128) and (1.129) coincide respectively with Eqs. (1.88) and (1.89), any Hopf algebra is a coalgebra, if we keep the coproduct and the counit and forget about all other structural maps. □

Example 1.13 For a finite non-empty set I, let $\mathbb{F}[I^2]$ be the $\mathbb{F}$-vector space freely generated by the set $I^2 = I \times I$. Then, $\mathbb{F}[I^2]$ is a coalgebra, if, for the natural linear basis $\{\chi_{(i,j)}\}_{(i,j)\in I^2}$ of $\mathbb{F}[I^2]$, we define a coproduct

$$\Delta\chi_{(i,j)} = \sum_{k\in I} \chi_{(i,k)} \otimes \chi_{(k,j)} \tag{1.130}$$

and a counit

$$\epsilon\chi_{(i,j)} = \delta_{i,j}. \tag{1.131}$$

Through the duality relation between algebras and coalgebras to be discussed later, this coalgebra is closely related to the endomorphism algebra $\mathrm{End}(V)$, see Example 1.11, associated to a vector space of dimension given by the cardinality $|I|$ of the set I. This algebra, in its turn, through a choice of a basis in V, becomes the algebra of square matrices of size $|I|$. For this reason, this coalgebra is called *matrix coalgebra*. □

Example 1.14 The vector space $\mathbb{F}[\mathbb{Z}_{>0}]$ freely generated by the set of strictly positive integers $\mathbb{Z}_{>0}$ is a coalgebra with the coproduct

$$\Delta\chi_m = \sum_{a\in\mathrm{Div}(m)} \chi_a \otimes \chi_{m/a}, \tag{1.132}$$

where $\mathrm{Div}(m)$ is the set of all (positive) divisors of m, and the counit

$$\epsilon\chi_m = \delta_{1,m}. \tag{1.133}$$

This coalgebra will be called *Dirichlet coalgebra* because of its role in analytic number theory, see Example 1.17 in the next Sect. 1.8. □

Exercise 1.9 Give a definition of a morphism of coalgebras.

Definition 1.16 The *opposite coproduct* in a coalgebra $C := (C, \Delta, \epsilon)$ is the linear map Δ^{op} obtained by composing the coproduct with the exchange map

$$\Delta^{\mathrm{op}} := \sigma\Delta =$$  . (1.134)

The coalgebra C is called *cocommutative* if the opposite coproduct coincides with the coproduct, $\Delta^{\mathrm{op}} = \Delta$.

Exercise 1.10 Show that if $C = (C, \Delta, \epsilon)$ is a coalgebra, then $C^{\mathrm{cop}} := (C, \Delta^{\mathrm{op}}, \epsilon)$ is also a coalgebra.

The following definition is motivated by the behavior of the canonical basis elements of group (Hopf) algebras under the coproduct, see relations (1.107).

Definition 1.17 A non zero element g of a coalgebra is called *grouplike* if $\Delta g = g \otimes g$.

Exercise 1.11 Show that any set of grouplike elements of a coalgebra is linearly independent.

Exercise 1.12 Show that the matrix coalgebra of Example 1.13 contains a grouplike element only if it is 1-dimensional.

The following definition introduces the notion of an element of a coalgebra which can be viewed as simplest among non grouplike elements.

Definition 1.18 A non zero element x of a coalgebra is called *primitive* if

$$\Delta x = g \otimes x + x \otimes h$$

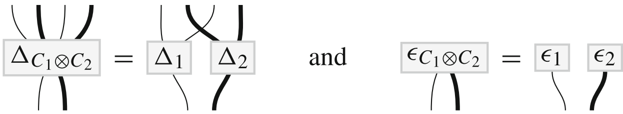
where g, h are grouplike elements.

Exercise 1.13 Find grouplike and primitive elements in the Dirichlet coalgebra of Example 1.14.

Definition 1.19 Let $C_1 = (C_1, \Delta_1, \epsilon_1)$ and $C_2 = (C_2, \Delta_2, \epsilon_2)$ be two coalgebras. The *tensor product* of C_1 and C_2 is the coalgebra

$$(C_1 \otimes C_2, (\mathrm{id}_{C_1} \otimes \sigma_{C_1,C_2} \otimes \mathrm{id}_{C_2})(\Delta_1 \otimes \Delta_2), \epsilon_1 \otimes \epsilon_2) \tag{1.135}$$

or graphically

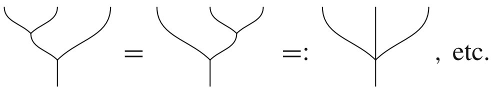


(1.136)

where the thin lines correspond to C_1 and thick lines to C_2.

1.7.1 Iterated Coproducts

Similarly to the case of algebras, due to the coassociativity property, it is convenient to use multivalent vertices in graphical representation of iterated coproducts:



(1.137)

Elements of the infinite set of all iterated coproducts

$$\{\Delta^{(m)} : C \to C^{\otimes m}\}_{m\in\omega} \tag{1.138}$$

are defined recursively

$$\Delta^{(m)} := (\Delta^{(m-1)} \otimes \mathrm{id}_C)\Delta, \quad \Delta^{(0)} = \epsilon, \tag{1.139}$$

so that, in particular, we have

$$\Delta^{(1)} = \mathrm{id}_C, \quad \Delta^{(2)} = \Delta. \tag{1.140}$$

1.7.2 Sweedler's Sigma Notation for the Iterated Coproducts

Originally introduced in the book [39], Sweedler's *sigma notation* allows to write formally the coproduct of an element of a coalgebra in the form

$$\Delta x = \sum_{(x)} x_{(1)} \otimes x_{(2)} \tag{1.141}$$

where the meaning of the sum is that it is a finite sum of the form

$$\Delta x = \sum_{i=1}^{n} a_i \otimes b_i \tag{1.142}$$

where the number n and the elements a_i, b_i with $1 \leq i \leq n$ are determined non uniquely by x. The sigma notation allows to avoid mentioning the number n and the associated elements all together thus simplifying writing. More generally, one can use a similar notation also for iterated coproducts

$$\Delta^{(m)} x = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(m)}, \quad \forall m \geq 2. \tag{1.143}$$

In this notation, for example, the equality $\Delta^{(3)}(x) = ((\Delta \otimes \mathrm{id}_C) \circ \Delta)(x)$ takes the form

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = \sum_{(x)} \sum_{(x_{(1)})} x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)}. \tag{1.144}$$

For examples and exercices on using the sigma notation, see the book [39].

1.7.3 The Fundamental Theorem of Coalgebras

Despite the fact that coalgebras are categorially dual objects to algebras, there is an important difference between them. Namely, there is no a conterpart for algebras of the following theorem.

Theorem 1.1 (The Fundamental Theorem of Coalgebras) *Let $C = (C, \Delta, \epsilon)$ be a coalgebra and $x \in C$. Then, there exists a finite dimensional sub-coalgebra $X \subset C$ containing x.*

Proof As the case $x = 0$ is trivial, we assume that $x \neq 0$.

Let $\{\alpha_i\}_{i\in I}$ and $\{\beta_j\}_{j\in J}$ be two non empty finite sets of linearly independent elements of C such that

$$\Delta^{(3)}x = \sum_{(i,j)\in I\times J} \alpha_i \otimes x_{i,j} \otimes \beta_j \tag{1.145}$$

and let $X \subset C$ be the vector subspace generated by the elements $\{x_{i,j}\}_{(i,j)\in I\times J}$. We have $\dim(X) \leq |I||J| < \infty$ and

$$x = (\epsilon \otimes \mathrm{id}_C \otimes \epsilon)\Delta^{(3)}x = \sum_{(i,j)\in I\times J} \epsilon(\alpha_i)\epsilon(\beta_j)x_{i,j} \in X. \tag{1.146}$$

Let us show that X is a sub-coalgebra of C, that is $\Delta(X) \subset X \otimes X$.

We have the equalities

$$\Delta^{(4)}x = (\Delta \otimes \mathrm{id}_{C^{\otimes 2}})\Delta^{(3)}x = \sum_{(k,j)\in I\times J} (\Delta\alpha_k) \otimes x_{k,j} \otimes \beta_j \tag{1.147}$$

$$\Delta^{(4)}x = (\mathrm{id}_C \otimes \Delta \otimes \mathrm{id}_C)\Delta^{(3)}x = \sum_{(i,j)\in I\times J} \alpha_i \otimes (\Delta x_{i,j}) \otimes \beta_j \tag{1.148}$$

$$\Delta^{(4)}x = (\mathrm{id}_{C^{\otimes 2}} \otimes \Delta)\Delta^{(3)}x = \sum_{(i,l)\in I\times J} \alpha_i \otimes x_{i,l} \otimes (\Delta\beta_l). \tag{1.149}$$

Comparing the right hand sides of (1.147) and (1.148) and using the linear independence of the family $\{\beta_j\}_{j\in J}$, we obtain the equalities

$$\sum_{k\in I}(\Delta\alpha_k) \otimes x_{k,j} = \sum_{i\in I} \alpha_i \otimes (\Delta x_{i,j}), \quad \forall j \in J, \tag{1.150}$$

which, in their turn, due to the linear independence of the family $\{\alpha_i\}_{i\in I}$, imply that

$$\Delta\alpha_k = \sum_{i\in I} \alpha_i \otimes \alpha_{i,k}, \quad \forall k \in I, \tag{1.151}$$

for some elements $\{\alpha_{i,k}\}_{i,k\in I} \subset C$ and

$$\Delta x_{i,j} = \sum_{k\in I} \alpha_{i,k} \otimes x_{k,j} \in C \otimes X, \quad \forall (i,j) \in I \times J. \tag{1.152}$$

By a similar reasoning, comparing the right hand sides of (1.149) and (1.148), we obtain

$$\Delta\beta_l = \sum_{j\in I} \beta_{l,j} \otimes \beta_j, \quad \forall l \in J, \tag{1.153}$$

for some elements $\{\beta_{l,j}\}_{l,j\in J} \subset C$ and

$$\Delta x_{i,j} = \sum_{l\in J} x_{i,l} \otimes \beta_{l,j} \in X \otimes C, \quad \forall (i,j) \in I \times J. \tag{1.154}$$

Finally, putting together (1.152) and (1.154), we conclude that

$$\Delta x_{i,j} \in X \otimes X, \quad \forall (i,j) \in I \times J \quad \Rightarrow \quad \Delta(X) \subset X \otimes X. \tag{1.155}$$

□

The fundamental theorem of coalgebras allows to reduce many questions about general coalgebras to questions about finite-dimensional coalgebras. Notice also that the category of finite dimensional coalgebras is equivalent to the category of finite dimensional algebras in the sense that the dual vector space of a finite dimensional algebra is canonically a finite dimensional coalgebra and vice versa.

1.7.4 Comodules

The notion of a comodule over a coalgebra is the categorial dual to that of a module over an algebra in the sense of Definition 1.2.

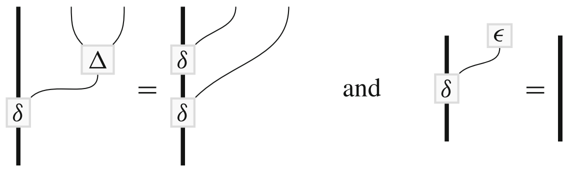
Definition 1.20 Let $C = (C, \Delta, \epsilon)$ be a coalgebra over a field $\mathbb{F}$. A *right comodule* over C (or simply a *right* C*-comodule*) is a $\mathbb{F}$-vector space V together with a linear map

$$\delta \colon V \to V \otimes C \tag{1.156}$$

such that the diagrams

$$\begin{array}{ccc} V\otimes C\otimes C & \xleftarrow{\mathrm{id}_V\otimes\Delta} & V\otimes C \\ \uparrow{\scriptstyle \delta\otimes\mathrm{id}_C} & & {\scriptstyle\delta}\uparrow \\ V\otimes C & \xleftarrow{\ \ \delta\ \ } & V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V\otimes\mathbb{F} & = & V \\ {\scriptstyle \mathrm{id}_V\otimes\epsilon}\nwarrow & & \swarrow{\scriptstyle\delta} \\ & V\otimes C & \end{array} \tag{1.157}$$

are commutative which, in the graphical notation, correspond to the equations



(1.158)

where the thick lines correspond to V and thin lines to C.

Exercise 1.14 Give a definition of a left comodule over a coalgebra C.

Example 1.15 An obvious example of a C-comodule (both right and left ones) is the coalgebra C itself with $\delta = \Delta$. □

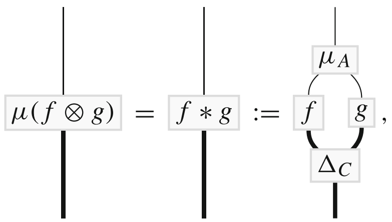
1.8 Convolution Algebras

The dual vector space of any coalgebra is canonically an algebra called the convolution algebra of a coalgebra. This is a special case of a more general convolution algebra associated to an algebra and a coalgebra.

Proposition-Definition 1.1 *Let A be an algebra and C a coalgebra. Then, the vector space* $\mathrm{L}(C, A)$ *of linear maps from C to A is an algebra, called* convolution algebra*, with the product* $\mu\colon \mathrm{L}(C, A) \otimes \mathrm{L}(C, A) \to \mathrm{L}(C, A)$ *defined by*

$$\mu(f \otimes g) =: f * g := \mu_A(f \otimes g)\Delta_C \tag{1.159}$$

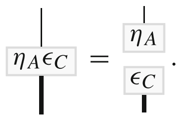
or diagrammatically



(1.160)

where the thick lines correspond to C and thin lines to A, and the unit $\eta\colon \mathbb{F} \to \mathrm{L}(C, A)$ *is defined by*

$$\eta 1 = \eta_A \epsilon_C =$$



(1.161)

Proof We verify the associativity property

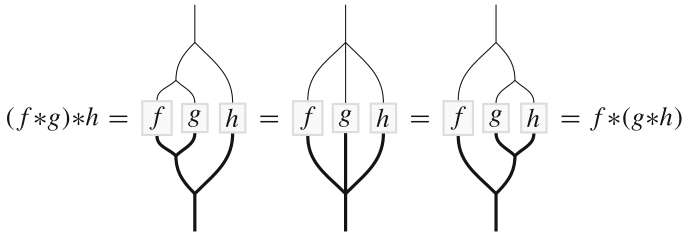
$$\begin{aligned}(f * g) * h &= \mu_A((f * g) \otimes h)\Delta_C = \mu_A((\mu_A(f \otimes g)\Delta_C) \otimes h)\Delta_C \\ &= \mu_A(\mu_A \otimes \mathrm{id}_A)(f \otimes g \otimes h)(\Delta_C \otimes \mathrm{id}_C)\Delta_C = \mu_A^{(3)}(f \otimes g \otimes h)\Delta_C^{(3)} \\ &= \mu_A(\mathrm{id}_A \otimes \mu_A)(f \otimes g \otimes h)(\mathrm{id}_C \otimes \Delta_C)\Delta_C = \mu_A(f \otimes (\mu_A(g \otimes h)\Delta_C))\Delta_C \\ &= \mu_A(f \otimes (g * h))\Delta_C = f * (g * h) \end{aligned} \tag{1.162}$$

and the unitality property

$$\begin{aligned} f * (\eta 1) &= \mu_A(f \otimes (\eta 1))\Delta_C = \mu_A(f \otimes (\eta_A \epsilon_C))\Delta_C \\ &= \mu_A(\mathrm{id}_A \otimes \eta_A) f (\mathrm{id}_C \otimes \epsilon_C)\Delta_C = \mathrm{id}_A \, f \, \mathrm{id}_C = f \end{aligned} \tag{1.163}$$

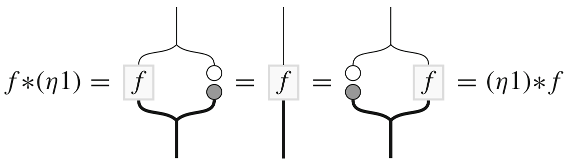
and similarly for the product $(\eta 1) * f$. □

Remark 1.6 In order to illustrate the effectiveness of the graphical calculus of string diagrams in this context, here is the diagrammatic proof of the associativity of the convolution product (cf. (1.162)):



(1.164)

and the unitality property of the convolution product (cf. (1.163)):



where we are using the simplified notation for the structural maps of the algebra A and the coalgebra C.

As the base field $\mathbb{F}$ is canonically an algebra, see Example 1.11, a particular case of the convolution algebra $L(C, A)$ with $A = \mathbb{F}$ corresponds to an algebra structure on the dual vector space $C^* = L(C, \mathbb{F})$ of a coalgebra C given by the product

$$\langle f * g, x\rangle = \langle f \otimes g, \Delta_C x\rangle = \sum_{(x)} \langle f, x_{(1)}\rangle \langle g, x_{(2)}\rangle, \tag{1.165}$$

where $\langle\cdot,\cdot\rangle : C^* \times C \to \mathbb{F}$ is the evaluation map of a linear form on a vector, and the unit element $\eta 1 = \epsilon_C \in C^*$. This algebra is called the *convolution algebra of a coalgebra*.

Example 1.16 The convolution algebra of the matrix coalgebra from Example 1.13 is isomorphic to the algebra of n-by-n matrices where $n = |I|$ is the cardinality of the set I. It is also identified with the endomorphism algebra $\mathrm{End}(\mathbb{F}^n)$. □

Example 1.17 The convolution algebra of the Dirichlet coalgebra (see Example 1.14) is known as the *Dirichlet convolution algebra*. Its subalgebra of arithmetic functions plays an important role in analytic number theory, where the corresponding convolution product is called Dirichlet product or Dirichlet convolution, see, for example, Chapter 2 of the book [2]. □

Exercise 1.15 An element of the Dirichlet convolution algebra $f \in (\mathbb{F}[\mathbb{Z}_{>0}])^*$ is called *multiplicative* if $\langle f, \chi_{ab}\rangle = \langle f, \chi_a\rangle\langle f, \chi_b\rangle$ for all mutually prime pairs of positive integers $a, b \in \mathbb{Z}_{>0}$ and $\langle f, \chi_1\rangle = 1$. Show that if $f, g \in (\mathbb{F}[\mathbb{Z}_{>0}])^*$ are multiplicative, then their convolution product $f * g$ is also a multiplicative element.

1.9 Some Properties of Hopf Algebras

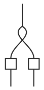
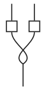
For a Hopf algebra H, the invertibility axiom (1.90) is nothing else but the condition that the antipode is the inverse of the identity map id_H in the convolution algebra $\mathrm{End}(H)$.

By the uniqueness of inverses, this means that a Hopf algebra cannot admit more than one antipode. Indeed, assuming that $\tilde{S}$ is another element of $\mathrm{End}(H)$ satisfying the invertibility axiom, we write the associativity condition for the triple of elements $(\tilde{S}, \mathrm{id}_H, S)$ in the convolution algebra $\mathrm{End}(H)$:

$$(\tilde{S} * \mathrm{id}_H) * S = \tilde{S} * (\mathrm{id}_H * S) \Leftrightarrow (\eta_{\mathrm{End}(H)} 1) * S = \tilde{S} * (\eta_{\mathrm{End}(H)} 1) \Leftrightarrow S = \tilde{S}. \tag{1.166}$$

Remark 1.7 Definition 1.6 of a Hopf algebra differs from the standard definition(s) in the literature. Specifically, in Definition 1.6 we do not assume that the counit (respectively the unit) is an algebra (respectively a coalgebra) morphism. Below, we derive these properties from the axioms listed in Definition 1.6. These derivations are based on the interpretations of the product μ and the coproduct Δ as invertible elements of the convolution algebras $L(H \otimes H, H)$ and $L(H, H \otimes H)$, respectively.

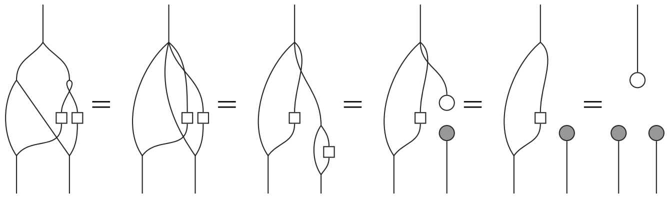
Lemma 1.1 *In any Hopf algebra* $H = (H, \mu, \eta, \Delta, \epsilon, S)$*, the product* μ *(respectively the coproduct* Δ*) is an invertible element of the convolution algebra* $\mathrm{L}(H \otimes H, H)$ *(respectively* $\mathrm{L}(H, H \otimes H)$*) with the inverse*

$$\bar{\mu} := \mu^{\mathrm{op}}(S \otimes S) =$$  $$\Big(\textit{respectively } \bar{\Delta} := (S \otimes S)\Delta^{\mathrm{op}} =$$  $$\Big). \quad (1.167)$$

Here the opposite product and the opposite coproduct are defined by

$$\mu^{\mathrm{op}} := \mu\sigma_{H,H}, \quad \Delta^{\mathrm{op}} := \sigma_{H,H}\Delta. \quad (1.168)$$

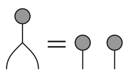
Proof Here is a graphical proof of the fact that $\bar{\mu}$ is a right convolution inverse of μ:

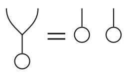
$$\mu * \bar{\mu} =$$  $$(1.169)$$

where, in the second equality, we convert the three trivalent vertices corresponding to the product into a multivalent vertex corresponding to an iterated product; in the third equality, by using associativity of the product and properties of the symmetry, we "pulled out" appropriately chosen trivalent vertex from the multivalent vertex, and in the last three equalities, we use twice the invertibility axiom and once the unitality axiom.

The rest of the proof goes along the same type of graphical calculations. □

Proposition 1.2 *In any Hopf algebra, the counit (respectively unit) is a morphism of algebras (respectively coalgebras). This means that*

$$\epsilon\mu = \epsilon \otimes \epsilon \quad \Leftrightarrow$$  $$, \quad (1.170)$$

$$\Delta\eta = \eta \otimes \eta \quad \Leftrightarrow$$  $$, \quad (1.171)$$

$$\epsilon\eta = \mathrm{id}_{\mathbb{F}} \quad \Leftrightarrow$$  $$= 1 \quad \Leftrightarrow \quad \epsilon\eta 1 = 1. \quad (1.172)$$

Proof The compatibility, the unitality and the counitality axioms imply that

$$(\eta\epsilon\mu) * \mu = \mu$$

in the convolution algebra $\mathrm{L}(H \otimes H, H)$. As μ is an invertible element, we conclude that

$$\eta\epsilon\mu = \eta_{\mathrm{L}(H\otimes H,H)}1 = \eta(\epsilon\otimes\epsilon) \quad \Rightarrow \quad \epsilon\mu = \epsilon\otimes\epsilon \quad \Rightarrow \quad \epsilon\eta = \mathrm{id}_{\mathbb{F}}\,. \tag{1.173}$$

By the duality symmetry, the compatibility, the unitality and the counitality axioms imply that

$$(\Delta\eta\epsilon) * \Delta = \Delta$$

in the convolution algebra $\mathrm{L}(H, H \otimes H)$. As Δ is an invertible element, we conclude that

$$\Delta\eta\epsilon = \eta_{\mathrm{L}(H,H\otimes H)}1 = (\eta\otimes\eta)\epsilon \quad \Rightarrow \quad \Delta\eta = \eta\otimes\eta. \tag{1.174}$$

□

Exercise 1.16 Show that if $H := (H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra, then

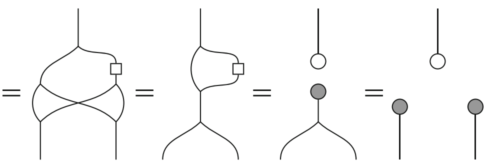
$$H^{\mathrm{op,cop}} := (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \epsilon, S) \tag{1.175}$$

is also a Hopf algebra.

Proposition 1.3 *In any Hopf algebra H, the antipode is a Hopf algebra morphism from H to $H^{\mathrm{op,cop}}$.*

Proof By Lemma 1.1, the convolution inverse of μ is the map $\bar{\mu}$ defined in (1.167). On the other hand, the composition $S\mu$ is also the convolutional inverse of μ as shows the following diagrammatic calculation:

$\mu*(S\mu) =$



(1.176)

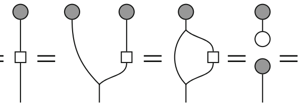
and likewise for the product $(S\mu) * \mu$. Thus, by uniqueness of inverses, we have the equality

$$S\mu = \mu^{\mathrm{op}}(S\otimes S) \tag{1.177}$$

and likewise

$$\Delta S = (S \otimes S)\Delta^{\mathrm{op}}. \tag{1.178}$$

To finish the proof, we check that

$\epsilon S =$  $= \epsilon$ (1.179)

and similarly

$$S\eta = \eta. \tag{1.180}$$

□

1.10 Bialgebras

Bialgebras, like Hopf algebras, are categorially self-dual algebraic objects (in the sense of Definition 1.2) that carry compatible structures of an algebra and a coalgebra but without assuming the existence of the antipode.

Definition 1.21 A *bialgebra* is a tuple $(B, \mu, \eta, \Delta, \epsilon)$, where (B, μ, η) is an algebra, (B, Δ, ϵ) is a coalgebra, and the linear maps Δ and ϵ are algebra morphisms (or, equivalently, μ and η are coalgebra morphisms).

Exercise 1.17 Give a definition of a bialgebra morphism.

Remark 1.8 By forgetting the antipode, any Hopf algebra becomes a bialgebra if one keeps the property that the counit is a morphism of algebras. A bialgebra B originates in this way from a Hopf algebra if and only if the identity map id_B is invertible in the convolution algebra $\mathrm{End}(B)$ of endomorphisms of B.

Example 1.18 Let M be a monoid, i.e. a set with associative product and the unit element $e \in M$, see Definition 1.3. The monoid bialgebra is the vector space $\mathbb{F}[M]$ freely generated by the set M, where the structure maps are given in terms of the linear basis of characteristic functions of points $\{\chi_a\}_{a \in M}$ by the formulae

$$\mu(\chi_a \otimes \chi_b) = \chi_{ab}, \quad \eta 1 = \chi_e, \tag{1.181}$$

$$\Delta\chi_a = \chi_a \otimes \chi_a, \quad \epsilon\chi_a = 1. \tag{1.182}$$

These relations coincide with the relations (1.107) for group algebras, so that verification of the axioms follow the same line of reasoning as in the case of group algebras. □

Exercise 1.18 Show that a monoid bialgebra $\mathbb{F}[M]$ admits the structure of a Hopf algebra if and only if M is a group.