Chapter 6
On the Coercivity of Strain Energy Functions in Generalized Models of 6-Parameter Shells

Mircea Bîrsan and Patrizio Neff

Abstract In this paper we consider geometrically nonlinear 6-parameter shell models. We establish some existence proofs by the direct methods of the calculus of variations. In contrast to more classical approaches, we also investigate models up to order h^5 in the shell thickness, where the form of the equations is determined by a dimensional descent from a three-dimensional Cosserat model.

Key words: Geometrically nonlinear shells · 6-parameter shells · Cosserat models · Strain energy density · Coercivity · Existence of minimizers

6.1 Introduction

The theory of shells is a vast subject. It is useful to distinguish three types of models: the shells of Kirchhoff–Love–Koiter type (with normality assumption, see e.g. [1]-[6]), the 5-parameter shells of Reissner–Mindlin type (allowing for transverse shear, see e.g. [7]-[9]), and the 6-parameter shells (in addition allowing for in-plane drill [10]-[12]). In the linearized setting, all these models are well-posed: one can prove existence and uniqueness of solutions (see, e.g., [13]-[16]).

However, in the geometrically nonlinear setting astounding differences appear. Indeed, as will be shown in a forthcoming paper, the Koiter and the Reissner–Mindlin

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models (based on quadratic strain and curvature energies) are both not well-posed in the sense that global minimizers do not exist. This may come as a surprise since they are often used by engineers. On the contrary, the general 6-parameter (Cosserat, micropolar) shell model is well-posed. This has been shown for the first time in [17] for the flat shell problem (see also [18, 19]) and in general in [20], and here we will expand on our knowledge of that. Also from an engineering point of view, 6-parameter shells have certain advantages, e.g. the imposition of boundary conditions is transparent, and these shells can easily be coupled with beam elements. Therefore, we strongly advocate the use of 6-parameter shell models, although it means that a numerical code implementing such models must be able to handle the rotation map.

Outline of the paper. In Sect. 6.2 we present briefly the governing equations of 6-parameter shells, including the differential geometry of the reference midsurface, the strain measures and the general stress-strain relations. In Sect. 6.3 we introduce the isotropic Cosserat 6-parameter shell model of order $O(h^3)$ and show that the strain energy function is coercive under certain conditions on the constitutive coefficients. Then, we apply the direct methods of the calculus of variations to prove the existence of minimizers to the variational problem. In Sect. 6.4 we consider the higher order Cosserat 6-parameter shell model and show the coercivity of the areal strain energy density. Also, we use this coercivity property to prove existence results for the minimization problem associated to equilibrium of Cosserat (6-parameter) shells of order $O(h^5)$.

Summary of notations. We present first some useful notations which will be used throughout this paper. The Latin indices $i, j, k, ...$ range over the set $\{1,2,3\}$, while the Greek indices $\alpha, \beta, \gamma, ...$ range over the set $\{1,2\}$. The Einstein summation convention over repeated indices is used. A subscript comma preceding an index i (or α) designates partial differentiation with respect to the variable x_i (or x_α, respectively), e.g. $f_{,i} = \frac{\partial f}{\partial x_i}$. We denote by δ_i^j the Kronecker symbol and employ the direct tensor notation. Thus, $\otimes$ designates the dyadic product, while $\mathrm{axl}(\boldsymbol{W})$ stands for the axial vector of any skew-symmetric tensor $\boldsymbol{W}$. For any second order tensor $\boldsymbol{X}$, let $\mathrm{tr}(\boldsymbol{X})$ designate the trace, $\mathrm{sym}(\boldsymbol{X})$ the symmetric part, and $\mathrm{skew}(\boldsymbol{X})$ the skew-symmetric part of $\boldsymbol{X}$. The scalar product between any second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{tr}(\boldsymbol{A}^T \boldsymbol{B})$. For any vector $\boldsymbol{v}$ and second order tensor $\boldsymbol{A}$ we write also $\boldsymbol{v}\boldsymbol{A} = \boldsymbol{A}^T \boldsymbol{v}$.

6.2 General 6-Parameter Elastic Shells. Governing Equations

Let us present first the geometry and kinematics of general 6-parameter shells. Denote by $\mathcal{S}_c$ the deformed (current) configuration of the shell and by $\mathcal{S}_\xi$ its reference configuration. The midsurface of the reference configuration is denoted by ω_ξ. This surface is determined by the position vector $\boldsymbol{y}_0(x_1, x_2)$, where $\boldsymbol{y}_0 : \omega \subset \mathbb{R}^2 \to \omega_\xi \subset \mathbb{R}^3$ is a parametric representation. The curvilinear coordinates (x_1, x_2) are convected coordinates on the surface ω_ξ.

Concerning the differential geometry of the midsurface ω_ξ, we introduce the covariant base vectors $\boldsymbol{a}_\alpha$ and the contravariant base vectors $\boldsymbol{a}^\alpha$ by

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{y}_0}{\partial x_\alpha}, \qquad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta \qquad (\alpha,\beta = 1,2). \tag{6.1}$$

Moreover, we introduce the vectors

$$\boldsymbol{a}_3 = \boldsymbol{a}^3 = \boldsymbol{n}_0, \qquad \text{where} \qquad \boldsymbol{n}_0 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|} \tag{6.2}$$

is the unit normal vector to the surface ω_ξ. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the first fundamental tensor and the second fundamental tensor of the midsuface ω_ξ, respectively, i.e.

$$\begin{aligned} \boldsymbol{a} &= \mathrm{Grad}_s\, \boldsymbol{y}_0 = \boldsymbol{a}_\alpha \otimes \boldsymbol{a}^\alpha = a_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta = a^{\alpha\beta}\boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta, \\ \boldsymbol{b} &= -\mathrm{Grad}_s\, \boldsymbol{n}_0 = -\boldsymbol{n}_{0,\alpha} \otimes \boldsymbol{a}^\alpha = b_{\alpha\beta}\,\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta = b^\alpha_\beta\, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}^\beta, \end{aligned} \tag{6.3}$$

where Grad_s is the surface gradient operator defined by $\mathrm{Grad}_s\, \boldsymbol{f} = \boldsymbol{f}_{,\alpha} \otimes \boldsymbol{a}^\alpha$ for any $\boldsymbol{f}$. Also, we employ the surface divergence operator given by $\mathrm{Div}_s\, \boldsymbol{T} = \boldsymbol{T}_{,\alpha}\, \boldsymbol{a}^\alpha$ for any second order tensor $\boldsymbol{T}$. The so-called alternator tensor $\boldsymbol{c}$ of the surface is

$$\boldsymbol{c} = \frac{1}{\sqrt{a}}\, \epsilon_{\alpha\beta}\, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta = \sqrt{a}\, \epsilon_{\alpha\beta}\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta, \tag{6.4}$$

where

$$a = \det\big(a_{\alpha\beta}\big)_{2\times 2} > 0$$

and $\epsilon_{\alpha\beta}$ is the two-dimensional alternator

$$\epsilon_{12} = -\epsilon_{21} = 1\,,\ \epsilon_{11} = \epsilon_{22} = 0.$$

Let

$$H = \frac{1}{2}\mathrm{tr}\, \boldsymbol{b} = \frac{1}{2} b^\alpha_\alpha$$

be the mean curvature and

$$K = \det \boldsymbol{b} = \det\big(b^\alpha_\beta\big)_{2\times 2}$$

be the Gauß curvature of the surface ω_ξ. Then, the relation of Cayley-Hamilton type holds

$$\boldsymbol{b}^2 - 2H\boldsymbol{b} + K\boldsymbol{a} = \boldsymbol{0}. \tag{6.5}$$

The last relation is equivalent to $\boldsymbol{b}(2H\boldsymbol{a} - \boldsymbol{b}) = K\boldsymbol{a}$. Hence, we introduce the tensor

$$\boldsymbol{b}^* = 2H\boldsymbol{a} - \boldsymbol{b}, \tag{6.6}$$

which can be regarded as the cofactor of $\boldsymbol{b}$ in the tangent plane. We also denote by κ_1, κ_2 the principal curvatures of the reference midsurface and we assume as usual that $|\kappa_\alpha h| < 1$ $(\alpha = 1,2)$.

To describe its deformation, we refer the shell to a Cartesian coordinate frame $Ox_1x_2x_3$ with orthonormal base vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. The reference configuration is characterized by the position vector $\boldsymbol{y}_0$ and the initial microrotation tensor $\boldsymbol{Q}_0$ as

$$\begin{aligned} &\boldsymbol{y}_0 : \omega \subset \mathbb{R}^2 \to \omega_\xi \subset \mathbb{R}^3, \qquad && \boldsymbol{y}_0 = \boldsymbol{y}_0(x_1,x_2), \\ &\boldsymbol{Q}_0 : \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3), \qquad && \boldsymbol{Q}_0 = \boldsymbol{d}_i^0(x_1,x_2) \otimes \boldsymbol{e}_i\,. \end{aligned} \tag{6.7}$$

The parameter domain ω is a bounded open domain with Lipschitz boundary $\partial\omega$ in the Ox_1x_2 plane. The reference directors $\{\boldsymbol{d}_1^0, \boldsymbol{d}_2^0, \boldsymbol{d}_3^0\}$ are orthonormal and the third director $\boldsymbol{d}_3^0$ is chosen to coincide with the unit normal in the reference configuration, i.e. $\boldsymbol{d}_3^0 = \boldsymbol{n}_0$. The shell deformation is characterized by the deformation function $\boldsymbol{m}$ and the microrotation tensor $\boldsymbol{Q}_e$ given by

$$\begin{aligned} &\boldsymbol{m} : \omega \to \omega_c\,, \qquad && \boldsymbol{m} = \boldsymbol{m}(x_1,x_2), \\ &\boldsymbol{Q}_e : \omega \to \mathrm{SO}(3), \qquad && \boldsymbol{Q}_e = \boldsymbol{Q}_e(x_1,x_2) = \boldsymbol{d}_i \otimes \boldsymbol{d}_i^0\,. \end{aligned} \tag{6.8}$$

Here, ω_c is the deformed midsurface and $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ is the orthonormal triad of directors in the deformed configuration.

We introduce the strain measures of 6-parameter shells as follows [10, 11, 21]: the shell strain tensor is

$$\boldsymbol{E}^e = \boldsymbol{Q}_e^T \mathrm{Grad}_s \boldsymbol{m} - \boldsymbol{a} \tag{6.9}$$

and the shell bending-curvature tensor is

$$\boldsymbol{K}^e = \mathrm{axl}\big(\boldsymbol{Q}_e^T \boldsymbol{Q}_{e,\alpha}\big) \otimes \boldsymbol{a}^\alpha. \tag{6.10}$$

The local equilibrium equations for 6-parameter shells have the following form (see, e.g. [21, 22])

$$\mathrm{Div}_s \boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \qquad \mathrm{Div}_s \boldsymbol{M} + \mathrm{axl}\big(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T\big) + \boldsymbol{l} = \boldsymbol{0}, \tag{6.11}$$

where $\boldsymbol{N}$ is the internal surface stress tensor and $\boldsymbol{M}$ the internal surface couple stress tensor (of the first Piola-Kirchhoff type). The tensor

$$\boldsymbol{F} = \mathrm{Grad}_s \boldsymbol{m} \,=\, \boldsymbol{m}_{,\alpha} \otimes \boldsymbol{a}^\alpha$$

is the shell deformation gradient, while the vectors $\boldsymbol{f}$ and $\boldsymbol{l}$ are the external body forces and body couples, respectively. We consider the following boundary conditions of mixed type prescribed on the boundary curve $\partial\omega_\xi$ [23, 24, 20]

$$
\begin{aligned}
\boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{N}^*,\ \boldsymbol{M}\boldsymbol{\nu} = \boldsymbol{M}^* \ \text{ along } \partial\omega_f\,, \\
\boldsymbol{m} = \boldsymbol{m}^*, \quad \boldsymbol{Q}_e = \boldsymbol{Q}^* \quad \text{along } \partial\omega_d\,,
\end{aligned}
\tag{6.12}
$$

where $\partial\omega_f \cup \partial\omega_d = \partial\omega_\xi$ is a disjoint partition of the boundary curve $\partial\omega_\xi$. Here, $\boldsymbol{N}^*$ and $\boldsymbol{M}^*$ are the external boundary force and couple vectors respectively, applied along the deformed boundary curve, but measured per unit length of $\partial\omega_f$. The vector $\boldsymbol{\nu}$ is the outer unit normal to the boundary curve $\partial\omega_\xi$, lying in the tangent plane.

Let the areal strain energy density for 6-parameter shells be given as a function of the strain measures in the form

$$
\mathcal{W}_{\text{shell}} = \mathcal{W}_{\text{shell}}(\boldsymbol{E}^e, \boldsymbol{K}^e). \tag{6.13}
$$

Under hyperelasticity assumptions, the stress and couple stress tensors are expressed by the following constitutive relations

$$
\boldsymbol{Q}_e^T \boldsymbol{N} = \frac{\partial \mathcal{W}_{\text{shell}}}{\partial \boldsymbol{E}^e}\,, \qquad \boldsymbol{Q}_e^T \boldsymbol{M} = \frac{\partial \mathcal{W}_{\text{shell}}}{\partial \boldsymbol{K}^e}\,. \tag{6.14}
$$

To obtain the explicit form of the stress-strain relations we need the specific expression of the strain energy function $\mathcal{W}_{\text{shell}}$. In [21], Eremeyev and Pietraszkiewicz have presented the general form of a quadratic energy density for isotropic shells, but the constitutive coefficients are not determined in terms of three-dimensional material constants. For instance, the following simplified expression of the energy density is proposed [21]

$$
\begin{aligned}
2\widehat{\mathcal{W}}_{\text{shell}}(\boldsymbol{E}^e, \boldsymbol{K}^e) = {} & \alpha_1 \big[\text{tr}(\boldsymbol{a}\boldsymbol{E}^e)\big]^2 + \alpha_2 \text{tr}\big[(\boldsymbol{a}\boldsymbol{E}^e)^2\big] + \alpha_3 \|\boldsymbol{a}\boldsymbol{E}^e\|^2 + \alpha_4 \|\boldsymbol{n}_0 \boldsymbol{E}^e\|^2 \\
& + \beta_1 \big[\text{tr}(\boldsymbol{a}\boldsymbol{K}^e)\big]^2 + \beta_2 \text{tr}\big[(\boldsymbol{a}\boldsymbol{K}^e)^2\big] + \beta_3 \|\boldsymbol{a}\boldsymbol{K}^e\|^2 + \beta_4 \|\boldsymbol{n}_0 \boldsymbol{K}^e\|^2,
\end{aligned}
$$

or equivalently, in view of the relation $\text{tr}(\boldsymbol{X}^2) = \|\text{sym}(\boldsymbol{X})\|^2 - \|\text{skew}(\boldsymbol{X})\|^2$,

$$
\begin{aligned}
2\widehat{\mathcal{W}}_{\text{shell}}(\boldsymbol{E}^e, \boldsymbol{K}^e) = {} & (\alpha_2 + \alpha_3)\|\text{sym}(\boldsymbol{a}\boldsymbol{E}^e)\|^2 + (\alpha_3 - \alpha_2)\|\text{skew}(\boldsymbol{a}\boldsymbol{E}^e)\|^2 \\
& + \alpha_1 \big[\text{tr}(\boldsymbol{a}\boldsymbol{E}^e)\big]^2 + \alpha_4 \|\boldsymbol{n}_0 \boldsymbol{E}^e\|^2 + (\beta_2 + \beta_3)\|\text{sym}(\boldsymbol{a}\boldsymbol{K}^e)\|^2 \\
& + (\beta_3 - \beta_2)\|\text{skew}(\boldsymbol{a}\boldsymbol{K}^e)\|^2 + \beta_1 \big[\text{tr}(\boldsymbol{a}\boldsymbol{K}^e)\big]^2 + \beta_4 \|\boldsymbol{n}_0 \boldsymbol{K}^e\|^2,
\end{aligned}
\tag{6.15}
$$

where α_k and β_k $(k = 1,2,3,4)$ are constant constitutive coefficients.

Let us introduce the surface deviator operator dev_s defined in [25] by

$$
\text{dev}_s \boldsymbol{T} = \boldsymbol{T} - \frac{1}{2}(\text{tr}\,\boldsymbol{T})\boldsymbol{a}
$$

for any $\boldsymbol{T}$. Then, we can decompose any tensor of the type $\boldsymbol{X} = X_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ as a direct sum (orthogonal decomposition) in the form

$$
\boldsymbol{X} = \text{dev}_s(\text{sym}\,\boldsymbol{X}) + \text{skew}\,\boldsymbol{X} + \frac{1}{2}(\text{tr}\,\boldsymbol{X})\boldsymbol{a}. \tag{6.16}
$$

Hence, it follows that

$$\|\mathrm{sym}\,\boldsymbol{X}\|^2 = \|\mathrm{dev}_s(\mathrm{sym}\,\boldsymbol{X})\|^2 + \frac{1}{2}(\mathrm{tr}\,\boldsymbol{X})^2 \tag{6.17}$$

and relation (6.15) can be written in the form

$$\begin{aligned} 2\widehat{W}_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) = {} & (\alpha_2+\alpha_3)\|\mathrm{dev}_s\mathrm{sym}(\boldsymbol{a}\boldsymbol{E}^e)\|^2 + (\alpha_3-\alpha_2)\|\mathrm{skew}(\boldsymbol{a}\boldsymbol{E}^e)\|^2 \\ & + \Big(\alpha_1 + \frac{\alpha_2+\alpha_3}{2}\Big)\big[\mathrm{tr}(\boldsymbol{a}\boldsymbol{E}^e)\big]^2 + \alpha_4\|\boldsymbol{n}_0\boldsymbol{E}^e\|^2 \\ & + (\beta_2+\beta_3)\|\mathrm{dev}_s\mathrm{sym}(\boldsymbol{a}\boldsymbol{K}^e)\|^2 + (\beta_3-\beta_2)\|\mathrm{skew}(\boldsymbol{a}\boldsymbol{K}^e)\|^2 \\ & + \Big(\beta_1 + \frac{\beta_2+\beta_3}{2}\Big)\big[\mathrm{tr}(\boldsymbol{a}\boldsymbol{K}^e)\big]^2 + \beta_4\|\boldsymbol{n}_0\boldsymbol{K}^e\|^2. \end{aligned} \tag{6.18}$$

From the last relation we see that the strain energy function $\widehat{W}_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$ is coercive in terms of $\boldsymbol{E}^e$ and $\boldsymbol{K}^e$ provided that the coefficients verify the conditions

$$\begin{aligned} & 2\alpha_1+\alpha_2+\alpha_3>0, \qquad \alpha_2+\alpha_3>0, \qquad \alpha_3-\alpha_2>0, \qquad \alpha_4>0, \\ & 2\beta_1+\beta_2+\beta_3>0, \qquad \beta_2+\beta_3>0, \qquad \beta_3-\beta_2>0, \qquad \beta_4>0. \end{aligned} \tag{6.19}$$

Indeed, if we denote by

$$c_1 = \min\{2\alpha_1+\alpha_2+\alpha_3\,,\,\alpha_2+\alpha_3\,,\,\alpha_3-\alpha_2\,,\,\alpha_4\} > 0$$

and

$$c_2 = \min\{2\beta_1+\beta_2+\beta_3\,,\,\beta_2+\beta_3\,,\,\beta_3-\beta_2\,,\,\beta_4\} > 0,$$

then from (6.16) and (6.18) we obtain

$$\begin{aligned} 2\widehat{W}_{\mathrm{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \geq {} & c_1\Big(\|\mathrm{dev}_s\mathrm{sym}(\boldsymbol{a}\boldsymbol{E}^e)\|^2 + \|\mathrm{skew}(\boldsymbol{a}\boldsymbol{E}^e)\|^2 + \frac{1}{2}\big[\mathrm{tr}(\boldsymbol{a}\boldsymbol{E}^e)\big]^2 \\ & + \|\boldsymbol{n}_0\boldsymbol{E}^e\|^2\Big) + c_2\Big(\|\mathrm{dev}_s\mathrm{sym}(\boldsymbol{a}\boldsymbol{K}^e)\|^2 + \|\mathrm{skew}(\boldsymbol{a}\boldsymbol{K}^e)\|^2 \\ & + \frac{1}{2}\big[\mathrm{tr}(\boldsymbol{a}\boldsymbol{K}^e)\big]^2 + \|\boldsymbol{n}_0\boldsymbol{K}^e\|^2\Big) \\ \geq {} & c_1\Big(\|\boldsymbol{a}\boldsymbol{E}^e\|^2 + \|\boldsymbol{n}_0\boldsymbol{E}^e\|^2\Big) + c_2\Big(\|\boldsymbol{a}\boldsymbol{K}^e\|^2 + \|\boldsymbol{n}_0\boldsymbol{K}^e\|^2\Big) \\ = {} & c_1\|\boldsymbol{E}^e\|^2 + c_2\|\boldsymbol{K}^e\|^2, \end{aligned} \tag{6.20}$$

i.e., the strain energy function is coercive. Remark that the same conditions (6.19) have been imposed in [15] to establish existence results for linear 6-parameter shells, see also [26, 27].

The energy function $\widehat{W}_{\mathrm{shell}}$ has been employed to solve shell problems in [11, 28]. To this aim, the following values of the coefficients α_k and β_k have been chosen for isotropic shells made of a material with Poisson ratio ν and Young modulus E:

$$
\begin{aligned}
&\alpha_1 = C\,\nu, \qquad \alpha_2 = 0, \qquad \alpha_3 = C\,(1-\nu), \qquad \alpha_4 = \alpha_s\, C\,(1-\nu),\\
&\beta_1 = D\,\nu, \qquad \beta_2 = 0, \qquad \beta_3 = D\,(1-\nu), \qquad \beta_4 = \alpha_t\, D\,(1-\nu).
\end{aligned}
\tag{6.21}
$$

Here, we denote by $C = E\,h/(1-\nu^2)$ the stretching (membrane) stiffness of the shell, $D = E\,h^3/12(1-\nu^2)$ is the bending stiffness, h is the thickness, and $\alpha_s = 5/6$, $\alpha_t = 7/10$ are two shear correction factors. Inserting (6.21) into (6.15) and using the relations

$$
\begin{aligned}
&C\,\frac{1+\nu}{2} = h\,\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}, \qquad C(1-\nu) = 2\mu h,\\
&D\,\frac{1+\nu}{2} = \frac{h^3}{12}\,\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}, \qquad D(1-\nu) = \frac{\mu h^3}{6},
\end{aligned}
\tag{6.22}
$$

where λ and μ are the Lamé constants of the isotropic elastic material, we see that the specific form of the strain energy density for 6-parameter shells commonly used in the literature is

$$
\begin{aligned}
\widehat{\mathcal{W}}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) = {}& h\Big[\mu\,\|\boldsymbol{a}\boldsymbol{E}^e\|^2 + \frac{\lambda\mu}{\lambda+2\mu}\big[\mathrm{tr}(\boldsymbol{a}\boldsymbol{E}^e)\big]^2 + \mu\,\alpha_s\,\|\boldsymbol{n}_0\boldsymbol{E}^e\|^2\Big]\\
&+\frac{h^3}{12}\Big[\mu\,\|\boldsymbol{a}\boldsymbol{K}^e\|^2 + \frac{\lambda\mu}{\lambda+2\mu}\big[\mathrm{tr}(\boldsymbol{a}\boldsymbol{K}^e)\big]^2 + \mu\,\alpha_t\,\|\boldsymbol{n}_0\boldsymbol{K}^e\|^2\Big].
\end{aligned}
\tag{6.23}
$$

In view of inequalities (6.19) and relations (6.21) and (6.22), we deduce that strain energy density (6.23) is coercive, provided that the Lamé constants satisfy the conditions

$$
\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{6.24}
$$

These conditions are usually assumed to hold for isotropic elastic materials.

In the next sections we consider two generalized models of 6-parameter shells made of Cosserat material, in which the constitutive relations are more elaborate. For these models we investigate the coercivity of the strain energy functions.

6.3 The Order h^3 Model of 6-Parameter Shells made of Cosserat Material

Starting from an isotropic three-dimensional Cosserat parent model, we have performed in [29, 30] a dimensional reduction and have obtained a generalized 6-parameter shell model of higher order, which has been investigated mathematically in [31]. Then, using a different method suggested by the classical shell theory [5, 6], we have derived in [22] a related 6-parameter Cosserat shell model of order $O(h^3)$, which will be analysed in details in this section.

Thus, the explicit form of the areal strain energy density $\mathcal{W}_{\text{shell}}$ has been obtained in [22] as a quadratic function of $(\boldsymbol{E}^e,\boldsymbol{K}^e)$, in which the coefficients are expressed in terms of the three-dimensional material constants and depend also on the curvature of the reference midsurface. Let λ, μ, and μ_c denote the Lamé constants and the Cosserat couple modulus, respectively, of the three-dimensional Cosserat material. We introduce the bilinear form $W_{\text{Coss}}(\cdot,\cdot)$ defined for any tensors $\boldsymbol{X} = X_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$, $\boldsymbol{Y} = Y_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ by

$$\begin{aligned} W_{\text{Coss}}(\boldsymbol{X},\boldsymbol{Y}) = \mu\langle \text{sym}(\boldsymbol{aX}), \text{sym}(\boldsymbol{aY})\rangle + \mu_c\langle \text{skew}(\boldsymbol{aX}), \text{skew}(\boldsymbol{aY})\rangle \\ + \frac{\lambda\mu}{\lambda+2\mu}\,\text{tr}(\boldsymbol{aX})\,\text{tr}(\boldsymbol{aY}) + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,(\boldsymbol{n}_0\boldsymbol{X})\cdot(\boldsymbol{n}_0\boldsymbol{Y}), \end{aligned} \tag{6.25}$$

as well as the associated quadratic form

$$\begin{aligned} W_{\text{Coss}}(\boldsymbol{X}) = \mu\,\|\text{sym}(\boldsymbol{aX})\|^2 + \mu_c\|\text{skew}(\boldsymbol{aX})\|^2 + \frac{\lambda\mu}{\lambda+2\mu}\big[\text{tr}(\boldsymbol{aX})\big]^2 \\ + \frac{2\mu\,\mu_c}{\mu+\mu_c}\|\boldsymbol{n}_0\boldsymbol{X}\|^2. \end{aligned} \tag{6.26}$$

Moreover, let $W_{\text{curv}}(\cdot,\cdot)$ be the bilinear form defined by

$$\begin{aligned} W_{\text{curv}}(\boldsymbol{X},\boldsymbol{Y}) = \mu\,L_c^2\Big[\; b_1\langle \text{sym}\boldsymbol{X}, \text{sym}\boldsymbol{Y}\rangle + b_2\langle \text{skew}\boldsymbol{X}, \text{skew}\boldsymbol{Y}\rangle \\ + \big(b_3 - \frac{b_1}{3}\big)(\text{tr}\boldsymbol{X})(\text{tr}\boldsymbol{Y})\Big] \end{aligned} \tag{6.27}$$

and the associated quadratic form

$$W_{\text{curv}}(\boldsymbol{X}) = \mu\,L_c^2\Big[\,b_1\,\|\,\text{sym}\,\boldsymbol{X}\|^2 + b_2\,\|\,\text{skew}\,\boldsymbol{X}\|^2 + \big(b_3 - \frac{b_1}{3}\big)\big(\text{tr}\,\boldsymbol{X}\big)^2\Big], \tag{6.28}$$

where the coefficients b_1, b_2, b_3 are dimensionless constitutive coefficients and the parameter $L_c > 0$ introduces an internal length (characteristic for the Cosserat material, see details in [17]-[19], [32]).

With these notations, the explicit form of the shell strain energy density for the order h^3 model is given by (see [22, Eq. (68)])

$$\begin{aligned} \mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) = \Big(h - K\,\frac{h^3}{12}\Big)\Big[W_{\text{Coss}}\big(\boldsymbol{E}^e\big) + W_{\text{curv}}\big(\boldsymbol{K}^e\big)\Big] \\ + \frac{h^3}{12}\Big[W_{\text{Coss}}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) - 2W_{\text{Coss}}\big(\boldsymbol{E}^e, \boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big) + W_{\text{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big)\Big], \end{aligned} \tag{6.29}$$

Notice that this strain energy function satisfies the invariance properties required by the local symmetry group of isotropic 6-parameter shells, which have been established by Eremeyev and Pietraszkiewicz in [21, Sect. 9].

6.3.1 Coercivity Results for the Model of Order $O(h^3)$

With a view toward proving the coercivity of the strain energy function, let us verify first that the quadratic forms $W_{\text{Coss}}(\boldsymbol{X})$ and $W_{\text{curv}}(\boldsymbol{X})$, which appear in (6.29), are coercive. Indeed, using (6.17), (6.26) and (6.28) we can write the equivalent forms

$$\begin{aligned} W_{\text{Coss}}(\boldsymbol{X}) &= \mu\,\|\text{dev}_{\text{s}}\text{sym}(\boldsymbol{a}\boldsymbol{X})\|^2 + \mu_c\|\text{skew}(\boldsymbol{a}\boldsymbol{X})\|^2 + \frac{\mu(3\lambda+2\mu)}{2(\lambda+2\mu)}\big(\text{tr}\boldsymbol{X}\big)^2 \\ &\quad + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,\|\boldsymbol{n}_0\boldsymbol{X}\|^2, \\ W_{\text{curv}}(\boldsymbol{X}) &= \mu L_c^2\Big(b_1\|\text{dev}_{\text{s}}\text{sym}(\boldsymbol{a}\boldsymbol{X})\|^2 + b_2\|\text{skew}(\boldsymbol{a}\boldsymbol{X})\|^2 + \big(b_3 + \frac{b_1}{6}\big)\big(\text{tr}\boldsymbol{X}\big)^2 \\ &\quad + \frac{b_1+b_2}{2}\,\|\boldsymbol{n}_0\boldsymbol{X}\|^2\Big). \end{aligned} \tag{6.30}$$

Then, under the usual assumptions on material constants (6.24) from classical elasticity, together with the conditions $\mu_c > 0$ and $b_i > 0$ for the Cosserat material, we see that quadratic forms (6.30) are coercive, since it holds

$$W_{\text{Coss}}(\boldsymbol{X}) \geq \bar{C}_1\,\|\boldsymbol{X}\|^2 \qquad \text{and} \qquad W_{\text{curv}}(\boldsymbol{X}) \geq \bar{C}_2\,\|\boldsymbol{X}\|^2, \tag{6.31}$$

where the positive constants are

$$\bar{C}_1 = \min\Big\{\mu\,,\mu_c,\ \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}\,,\ \frac{2\mu\,\mu_c}{\mu+\mu_c}\Big\} > 0$$

and

$$\bar{C}_2 = \mu L_c^2\min\Big\{b_1, b_2, 2b_3 + \frac{b_1}{3}\,,\ \frac{b_1+b_2}{2}\Big\} > 0.$$

For the sake of brevity and for later convenience, let us denote by $\boldsymbol{\Phi}^e$ the mixed bending tensor

$$\boldsymbol{\Phi}^e = \boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e. \tag{6.32}$$

Then, using (6.5) and (6.32) we can decompose the strain energy density (6.29) as follows

$$\mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) = \mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) + \mathcal{W}^{(3)}_{\text{bend,curv}}(\boldsymbol{K}^e), \tag{6.33}$$

where the membrane-bending part is given by

$$\begin{aligned} \mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) &= \Big(h + K\,\frac{h^3}{12}\Big)W_{\text{Coss}}\big(\boldsymbol{E}^e\big) + \frac{h^3}{12}\,W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big) \\ &\quad - 2\,\frac{h^3}{12}\,W_{\text{Coss}}\big(\boldsymbol{E}^e,\boldsymbol{\Phi}^e\boldsymbol{b}^*\big). \end{aligned} \tag{6.34}$$

and the bending-curvature part is

$$\mathcal{W}^{(3)}_{\text{bend,curv}}(\boldsymbol{K}^e) = \Big(h - K\,\frac{h^3}{12}\Big)W_{\text{curv}}\big(\boldsymbol{K}^e\big) + \frac{h^3}{12}\,W_{\text{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big). \tag{6.35}$$

Firstly, the bending-curvature part $\mathcal{W}^{(3)}_{\text{bend,curv}}$ of the energy density is obviously coercive, since for $|\kappa_\alpha h| < 1$ we have

$$|K|\,h^2 = |\kappa_1 h|\cdot|\kappa_2 h| < 1\,, \tag{6.36}$$

so we can write using $(6.31)_2$

$$\mathcal{W}^{(3)}_{\text{bend,curv}}(\boldsymbol{K}^e) \geq \Big(h - K\frac{h^3}{12}\Big)W_{\text{curv}}\big(\boldsymbol{K}^e\big) \geq \Big(h - \frac{h}{12}\Big)W_{\text{curv}}\big(\boldsymbol{K}^e\big) \geq \frac{11}{12}h\bar{C}_2\|\boldsymbol{K}^e\|^2. \tag{6.37}$$

Secondly, for the membrane-bending part of the energy we establish the following result.

Lemma 6.1. *Assume that the constitutive coefficients satisfy the conditions*

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \text{and} \qquad \mu_c > 0. \tag{6.38}$$

Let κ be the maximum of the absolute value of the principal curvatures $|\kappa_\alpha|$ on the midsurface ω_ξ, i.e. $\kappa = \max_{\omega_\xi}\{|\kappa_1|,|\kappa_2|\}$. Assume that the product κh satisfies

$$\kappa h \,<\, \min\left\{\,\frac{1}{2}\,,\,\left(\frac{47}{32}\cdot\frac{\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{\lambda+\mu}\right)^{1/2},\left(\frac{47}{8}\cdot\frac{\min\{\mu,\mu_c\}}{\mu+\mu_c}\right)^{1/2}\right\}. \tag{6.39}$$

Then, there exists a positive constant $C_1 > 0$ such that

$$\mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \,\geq\, C_1\,\|\boldsymbol{E}^e\|^2\,. \tag{6.40}$$

Proof. We notice first that the radicands in (6.39) are always positive, in view of the conditions (6.38).

In order to estimate the coupling term in energy function (6.34) we employ a similar technique as in the classical shell theory (see, e.g. [6]). Using the Cauchy-Schwarz inequality for the scalar product $W_{\text{Coss}}\big(\cdot,\cdot\big)$ we can write

$$\big|W_{\text{Coss}}\big(\boldsymbol{E}^e,\boldsymbol{\Phi}^e\boldsymbol{b}^*\big)\big| \leq \sqrt{W_{\text{Coss}}\big(\boldsymbol{E}^e\big)}\,\sqrt{W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{b}^*\big)}\,. \tag{6.41}$$

To express the last term, we use the spectral representation of the tensor $\boldsymbol{b}$ in the form

$$\boldsymbol{b} \,=\, \kappa_1\,\boldsymbol{u}_1\otimes\boldsymbol{u}_1 + \kappa_2\,\boldsymbol{u}_2\otimes\boldsymbol{u}_2\,, \tag{6.42}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the orthonormal principal vectors. Then, we have

$$\boldsymbol{a} = \boldsymbol{u}_1\otimes\boldsymbol{u}_1 + \boldsymbol{u}_2\otimes\boldsymbol{u}_2$$

and the cofactor $\boldsymbol{b}^*$ given by (6.6) can be written as

$$\begin{aligned} \boldsymbol{b}^* &= H\boldsymbol{a} + (H\boldsymbol{a} - \boldsymbol{b}) \\ &= H\boldsymbol{a} + \Big[\frac{\kappa_1 + \kappa_2}{2} (\boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \boldsymbol{u}_2 \otimes \boldsymbol{u}_2) - (\kappa_1 \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \kappa_2 \boldsymbol{u}_2 \otimes \boldsymbol{u}_2) \Big] \\ &= H\boldsymbol{a} + \frac{\kappa_1 - \kappa_2}{2} (\boldsymbol{u}_2 \otimes \boldsymbol{u}_2 - \boldsymbol{u}_1 \otimes \boldsymbol{u}_1). \end{aligned} \tag{6.43}$$

Thus, we obtain

$$\boldsymbol{\Phi}^e \boldsymbol{b}^* = H \boldsymbol{\Phi}^e + \frac{\kappa_1 - \kappa_2}{2} \boldsymbol{\Phi}^e \boldsymbol{\delta} \tag{6.44}$$

where

$$\boldsymbol{\delta} := \boldsymbol{u}_2 \otimes \boldsymbol{u}_2 - \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 .$$

We notice that

$$\Big(\frac{\kappa_1 - \kappa_2}{2}\Big)^2 = \Big(\frac{\kappa_1 + \kappa_2}{2}\Big)^2 - \kappa_1 \kappa_2 = H^2 - K,$$

so we can estimate

$$\begin{aligned} W_{\text{Coss}}\big(\boldsymbol{\Phi}^e \boldsymbol{b}^*\big) &= W_{\text{Coss}}\Big(H\boldsymbol{\Phi}^e + \frac{\kappa_1 - \kappa_2}{2} \boldsymbol{\Phi}^e \boldsymbol{\delta}\Big) \\ &\leq 2\big[H^2 W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big) + (H^2 - K) W_{\text{Coss}}\big(\boldsymbol{\Phi}^e \boldsymbol{\delta}\big)\big]. \end{aligned} \tag{6.45}$$

With the notation

$$\kappa = \max_{\omega_\xi} \{|\kappa_1|, |\kappa_2|\}$$

we can write

$$\begin{gathered} H^2 = \Big(\frac{\kappa_1 + \kappa_2}{2}\Big)^2 \leq \frac{\kappa_1^2 + \kappa_2^2}{2} \leq \kappa^2 \\ \text{and} \\ H^2 - K = \Big(\frac{\kappa_1 - \kappa_2}{2}\Big)^2 \leq \frac{\kappa_1^2 + \kappa_2^2}{2} \leq \kappa^2 . \end{gathered} \tag{6.46}$$

Hence, relation (6.45) yields

$$W_{\text{Coss}}\big(\boldsymbol{\Phi}^e \boldsymbol{b}^*\big) \leq 2\kappa^2 \big[W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big) + W_{\text{Coss}}\big(\boldsymbol{\Phi}^e \boldsymbol{\delta}\big)\big]. \tag{6.47}$$

To estimate the sum in the last brackets we decompose

$$\begin{aligned} W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big) &= W_{\text{Coss}}\big(\boldsymbol{\Phi}^e (\boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \boldsymbol{u}_2 \otimes \boldsymbol{u}_2)\big) \\ &= W_{\text{Coss}}\big(\boldsymbol{\Phi}^e \boldsymbol{u}_1 \otimes \boldsymbol{u}_1\big) + W_{\text{Coss}}\big(\boldsymbol{\Phi}^e \boldsymbol{u}_2 \otimes \boldsymbol{u}_2\big) \\ &\quad + 2\, W_{\text{Coss}}\big(\boldsymbol{\Phi}^e \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 ,\, \boldsymbol{\Phi}^e \boldsymbol{u}_2 \otimes \boldsymbol{u}_2\big) \end{aligned} \tag{6.48}$$

and

$$
\begin{aligned}
W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{\delta}\big) &= W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e(\boldsymbol{u}_2\otimes\boldsymbol{u}_2-\boldsymbol{u}_1\otimes\boldsymbol{u}_1)\big)\\
&= W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2\big)+W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1\big)\\
&\quad -2\,W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1\,,\,\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2\big),
\end{aligned}
\tag{6.49}
$$

so we get by addition

$$
W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\big)+W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{\delta}\big) = 2\,W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1\big)+2\,W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2\big). \tag{6.50}
$$

Then, inequality (6.47) becomes

$$
W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{b}^*\big) \le 4\,\kappa^2\Big[W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1\big)+W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2\big)\Big]. \tag{6.51}
$$

Inserting this in (6.41), we obtain for membrane-bending energy (6.34) the inequality

$$
\begin{aligned}
\mathcal{W}^{(3)}_{\mathrm{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \ge{}& \Big(h+K\,\frac{h^3}{12}\Big)W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\big)+\frac{h^3}{12}\,W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\big)\\
&-4\,\kappa\,\frac{h^3}{12}\sqrt{W_{\mathrm{Coss}}\big(\boldsymbol{E}^e\big)}\cdot\sqrt{W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1\big)+W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2\big)}\,.
\end{aligned}
\tag{6.52}
$$

To show that the right-hand side is positive, let us decompose the tensor $\boldsymbol{\Phi}^e$ using the orthonormal basis $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{n}_0\}$: denoting its components by $\varphi_{i\alpha}$, we have

$$
\boldsymbol{\Phi}^e = \varphi_{\alpha\beta}\,\boldsymbol{u}_\alpha\otimes\boldsymbol{u}_\beta+\varphi_{3\beta}\,\boldsymbol{n}_0\otimes\boldsymbol{u}_\beta\,.
$$

Then,

$$
\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1 = \varphi_{\alpha1}\,\boldsymbol{u}_\alpha\otimes\boldsymbol{u}_1+\varphi_{31}\,\boldsymbol{n}_0\otimes\boldsymbol{u}_1
$$

and

$$
\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2 = \varphi_{\alpha2}\,\boldsymbol{u}_\alpha\otimes\boldsymbol{u}_2+\varphi_{32}\,\boldsymbol{n}_0\otimes\boldsymbol{u}_2\,,
$$

and using definitions (6.25), (6.26) we compute directly

$$
\begin{aligned}
W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1\big) &= \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\varphi_{11}^2+\frac{\mu+\mu_c}{2}\,\varphi_{21}^2+\frac{2\mu\mu_c}{\mu+\mu_c}\,\varphi_{31}^2\,,\\
W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2\big) &= \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\varphi_{22}^2+\frac{\mu+\mu_c}{2}\,\varphi_{12}^2+\frac{2\mu\mu_c}{\mu+\mu_c}\,\varphi_{32}^2\,,\\
W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{u}_1\otimes\boldsymbol{u}_1\,,\,\boldsymbol{\Phi}^e\boldsymbol{u}_2\otimes\boldsymbol{u}_2\big) &= \frac{\lambda\mu}{\lambda+2\mu}\,\varphi_{11}\,\varphi_{22}+\frac{\mu-\mu_c}{2}\,\varphi_{12}\,\varphi_{21}\,.
\end{aligned}
\tag{6.53}
$$

Substituting (6.53) in (6.48) we obtain

$$
\begin{aligned}
W_{\mathrm{Coss}}\big(\boldsymbol{\Phi}^e\big) ={}& \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2)+\frac{2\lambda\mu}{\lambda+2\mu}\,\varphi_{11}\,\varphi_{22}+\frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2)\\
&+(\mu-\mu_c)\,\varphi_{12}\,\varphi_{21}+\frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2).
\end{aligned}
\tag{6.54}
$$

Since

$$|\kappa_\alpha h| < \frac{1}{2}$$

we have

$$|K|\,h^2 = |\kappa_1 h|\cdot|\kappa_2 h| < \frac{1}{4} \qquad \text{and} \qquad |H|\,h \le \frac{1}{2}\left(|\kappa_1 h| + |\kappa_2 h|\right) < \frac{1}{2}\,, \tag{6.55}$$

so we can write the inequality

$$h + K\,\frac{h^3}{12} > h - \frac{h}{48} = \frac{47}{48}\,h = \frac{\delta}{48}\,h + \frac{47-\delta}{48}\,h \tag{6.56}$$

for any $\delta > 0$. The choice of δ will be specified later in (6.61). Using relations (6.53), (6.54) and (6.56) we can transform inequality (6.52) as follows

$$\begin{aligned}
\mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \ge{}& \frac{\delta h}{48} W_{\text{Coss}}(\boldsymbol{E}^e) + h\Bigg[\frac{47-\delta}{48} W_{\text{Coss}}(\boldsymbol{E}^e) - \frac{h^2\kappa}{3}\sqrt{W_{\text{Coss}}(\boldsymbol{E}^e)} \\
&\times\left(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2) + \frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2) + \frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\right)^{1/2} \\
&+\frac{h^2}{12}\Big(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2) + \frac{2\lambda\mu}{\lambda+2\mu}\,\varphi_{11}\,\varphi_{22} + \frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2) \\
&\qquad +(\mu-\mu_c)\,\varphi_{12}\,\varphi_{21} + \frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\Big)\Bigg].
\end{aligned} \tag{6.57}$$

To prove that the expression in square brackets in (6.57) is positive, we regard it as a quadratic function in

$$\sqrt{W_{\text{Coss}}(\boldsymbol{E}^e)}$$

and show that its discriminant Δ is negative. Indeed, the discriminant is

$$\begin{aligned}
\Delta ={}& \frac{h^4\kappa^2}{9}\left(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2) + \frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2) + \frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\right) \\
&-\frac{47-\delta}{12}\,\frac{h^2}{12}\Big(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(\varphi_{11}^2+\varphi_{22}^2) + \frac{2\lambda\mu}{\lambda+2\mu}\,\varphi_{11}\,\varphi_{22} + \frac{\mu+\mu_c}{2}(\varphi_{12}^2+\varphi_{21}^2) \\
&\qquad +(\mu-\mu_c)\,\varphi_{12}\,\varphi_{21} + \frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\Big).
\end{aligned}$$

Then, we can combine the terms and write

$$
\begin{aligned}
\frac{9}{h^2}\,\Delta = & \Big[\Big(\kappa^2 h^2 - \frac{47-\delta}{16}\Big)\,\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2) - \frac{47-\delta}{16}\,\frac{2\lambda\mu}{\lambda+2\mu}\,\varphi_{11}\,\varphi_{22}\Big]\\
& + \Big[\Big(\kappa^2 h^2 - \frac{47-\delta}{16}\Big)\,\frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2) - \frac{47-\delta}{16}\,(\mu-\mu_c)\,\varphi_{12}\,\varphi_{21}\Big]\\
& + \Big(\kappa^2 h^2 - \frac{47-\delta}{16}\Big)\,\frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2),
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
\frac{9}{h^2}\,\Delta = & \Big(\kappa^2 h^2 - \frac{47-\delta}{16}\Big)\Big[\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\Big(\varphi_{11}^2+\varphi_{22}^2 - \frac{47-\delta}{16\,\kappa^2 h^2-(47-\delta)}\\
& \times\frac{\lambda}{\lambda+\mu}\,\varphi_{11}\,\varphi_{22}\Big) + \frac{\mu+\mu_c}{2}\Big(\varphi_{12}^2+\varphi_{21}^2 - \frac{47-\delta}{16\,\kappa^2 h^2-(47-\delta)}\,\frac{\mu-\mu_c}{\mu+\mu_c}\,2\varphi_{12}\,\varphi_{21}\Big)\\
& + \frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\Big].
\end{aligned}
\tag{6.58}
$$

Notice that it holds

$$
\frac{\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{2(\lambda+\mu)} = 1-\frac{|\lambda|}{2(\lambda+\mu)} \quad\text{and}\quad \frac{2\min\{\mu,\mu_c\}}{\mu+\mu_c} = 1-\frac{|\mu-\mu_c|}{\mu+\mu_c}\,. \tag{6.59}
$$

Then, inequality (6.39) can be written equivalently

$$
\kappa\,h < \min\Big\{\,\frac{1}{2}\,,\ \frac{\sqrt{47}}{4}\Big(1-\frac{|\lambda|}{2(\lambda+\mu)}\Big)^{1/2},\ \frac{\sqrt{47}}{4}\Big(1-\frac{|\mu-\mu_c|}{\mu+\mu_c}\Big)^{1/2}\Big\}\,. \tag{6.60}
$$

From the last relation we deduce

$$
47-16\kappa^2 h^2\Big(1-\frac{|\lambda|}{2(\lambda+\mu)}\Big)^{-1} > 0 \qquad\text{and}\qquad 47-16\kappa^2 h^2\Big(1-\frac{|\mu-\mu_c|}{\mu+\mu_c}\Big)^{-1} > 0.
$$

Hence, we can choose the positive constant $\delta > 0$ such that

$$
\delta < \min\Big\{43\,,\,47-16\,\kappa^2 h^2\Big(1-\frac{|\lambda|}{2(\lambda+\mu)}\Big)^{-1},\,47-16\,\kappa^2 h^2\Big(1-\frac{|\mu-\mu_c|}{\mu+\mu_c}\Big)^{-1}\Big\}. \tag{6.61}
$$

From $\delta < 43$ and $\kappa h < \frac{1}{2}$ it follows that

$$
\kappa^2 h^2 - \frac{47-\delta}{16} < \kappa^2 h^2 - \frac{4}{16} < 0. \tag{6.62}
$$

Further, from (6.61) we also derive that

$$\Big|\frac{47-\delta}{16\kappa^2h^2-(47-\delta)}\cdot\frac{\lambda}{2(\lambda+\mu)}\Big|<1 \qquad \text{and} \qquad \Big|\frac{47-\delta}{16\kappa^2h^2-(47-\delta)}\cdot\frac{\mu-\mu_c}{\mu+\mu_c}\Big|<1. \tag{6.63}$$

Using inequalities (6.62) and (6.63) in (6.58) we deduce that

$$\frac{9}{h^2}\,\Delta<\Big(\kappa^2h^2-\frac{47-\delta}{16}\Big)\Big[\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\big(|\varphi_{11}|-|\varphi_{22}|\big)^2+\frac{\mu+\mu_c}{2}\,\big(|\varphi_{12}|-|\varphi_{21}|\big)^2$$
$$+\frac{2\mu\mu_c}{\mu+\mu_c}\,\big(\varphi_{31}^2+\varphi_{32}^2\big)\Big]\le 0. \tag{6.64}$$

Thus, the discriminant Δ is negative, so the expression in square brackets in (6.57) is positive. Hence, relations (6.57) and $(6.31)_1$ infer

$$\mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e)\;\ge\;\frac{\delta}{48}\,h\,W_{\text{Coss}}\big(\boldsymbol{E}^e\big)\;\ge\;C_1\,\|\boldsymbol{E}^e\|^2,$$

for some positive constant $C_1>0$. The proof is complete.

Let us present now another auxiliary result (a variant of Lemma 6.1), in which inequality (6.39) is replaced by an alternative assumption. Thus, we prove the following coercivity result which is complementary to the previous lemma.

Lemma 6.2. *Assume that the constitutive coefficients fulfil the conditions* (6.38) *and that the product* $\kappa h<1$ *is small enough such that*

$$\kappa h<\min\left\{\left(\frac{12\,\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{8(\lambda+\mu)+\min\{\lambda+2\mu\,,3\lambda+2\mu\}}\right)^{1/2},\left(\frac{12\,\min\{\mu,\mu_c\}}{2(\mu+\mu_c)+\min\{\mu,\mu_c\}}\right)^{1/2}\right\}. \tag{6.65}$$

Then, the membrane-bending energy $\mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$ *satisfies the inequality* (6.40) *for some positive constant* C_1.

Proof. We begin the proof in the same way as in the case of Lemma 6.1 and establish relations (6.41)-(6.54). If we insert (6.53) and (6.54) into Eq. (6.52) and use the relation

$$1+K\,\frac{h^2}{12}\;\ge\;1-\frac{\kappa^2h^2}{12}\;=\;\epsilon+\Big(1-\frac{\kappa^2h^2}{12}-\epsilon\Big)\qquad\text{for any }\;\epsilon>0, \tag{6.66}$$

then we obtain the inequality

$$
\begin{aligned}
\mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \geq\; & h\,\epsilon\, W_{\text{Coss}}(\boldsymbol{E}^e) + h\Bigg[\Big(1-\frac{\kappa^2h^2}{12}-\epsilon\Big)W_{\text{Coss}}(\boldsymbol{E}^e) \\
& -\frac{h^2\kappa}{3}\sqrt{W_{\text{Coss}}(\boldsymbol{E}^e)}\,\bigg(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2) + \frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2) \\
& +\frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\bigg)^{1/2} + \frac{h^2}{12}\bigg(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2)+\frac{2\lambda\mu}{\lambda+2\mu}\,\varphi_{11}\,\varphi_{22} \\
& +\frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2)+(\mu-\mu_c)\,\varphi_{12}\,\varphi_{21}+\frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\bigg)\Bigg].
\end{aligned}
\tag{6.67}
$$

We regard the expression in square brackets in (6.67) as a quadratic function in $\sqrt{W_{\text{Coss}}(\boldsymbol{E}^e)}$ and denote by D its discriminant. To show that the discriminant D is negative, we compute

$$
\begin{aligned}
D =\; & \frac{\kappa^2h^4}{9}\bigg(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,(\varphi_{11}^2+\varphi_{22}^2)+\frac{\mu+\mu_c}{2}\,(\varphi_{12}^2+\varphi_{21}^2)+\frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\bigg) \\
& -\frac{h^2}{3}\Big(1-\frac{\kappa^2h^2}{12}-\epsilon\Big)\bigg(\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}(\varphi_{11}^2+\varphi_{22}^2)+\frac{2\lambda\mu}{\lambda+2\mu}\,\varphi_{11}\,\varphi_{22} \\
& \qquad +\frac{\mu+\mu_c}{2}(\varphi_{12}^2+\varphi_{21}^2)+(\mu-\mu_c)\,\varphi_{12}\,\varphi_{21}+\frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\bigg),
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
\frac{9}{h^2}\,D =\; & \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\bigg[\Big(\kappa^2h^2-\big(3-\frac{\kappa^2h^2}{4}-3\epsilon\big)\Big)(\varphi_{11}^2+\varphi_{22}^2)-\big(3-\frac{\kappa^2h^2}{4}-3\epsilon\big) \\
& \times\frac{\lambda}{\lambda+\mu}\,\varphi_{11}\,\varphi_{22}\bigg] + \frac{\mu+\mu_c}{2}\bigg[\Big(\kappa^2h^2-\big(3-\frac{\kappa^2h^2}{4}-3\epsilon\big)\Big)(\varphi_{12}^2+\varphi_{21}^2) \\
& -\big(3-\frac{\kappa^2h^2}{4}-3\epsilon\big)\frac{\mu-\mu_c}{\mu+\mu_c}2\varphi_{12}\varphi_{21}\bigg]+\frac{2\mu\mu_c}{\mu+\mu_c}\Big(\kappa^2h^2-\big(3-\frac{\kappa^2h^2}{4}-3\epsilon\big)\Big)(\varphi_{31}^2+\varphi_{32}^2).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\frac{3}{h^2}\,D =&-\Big(1-\frac{5}{12}\kappa^2h^2-\epsilon\Big)\bigg\{\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\bigg[(\varphi_{11}^2+\varphi_{22}^2)+\frac{1-\frac{1}{12}\kappa^2h^2-\epsilon}{1-\frac{5}{12}\kappa^2h^2-\epsilon}\\
&\times\frac{\lambda}{\lambda+\mu}\,\varphi_{11}\varphi_{22}\bigg]+\frac{\mu+\mu_c}{2}\bigg[(\varphi_{12}^2+\varphi_{21}^2)+\frac{1-\frac{1}{12}\kappa^2h^2-\epsilon}{1-\frac{5}{12}\kappa^2h^2-\epsilon}\cdot\frac{\mu-\mu_c}{\mu+\mu_c}\,2\varphi_{12}\,\varphi_{21}\bigg]\\
&+\frac{2\mu\mu_c}{\mu+\mu_c}\,(\varphi_{31}^2+\varphi_{32}^2)\bigg\}.
\end{aligned}
\tag{6.68}
$$

Now, we want to choose a positive number ϵ such that the following inequalities are fulfilled

$$
\begin{aligned}
&1-\frac{5}{12}\kappa^2h^2-\epsilon>0 \qquad \text{and} \qquad \frac{1-\frac{1}{12}\kappa^2h^2-\epsilon}{1-\frac{5}{12}\kappa^2h^2-\epsilon}\cdot\frac{|\lambda|}{\lambda+\mu}<1\\
&\text{and} \qquad \frac{1-\frac{1}{12}\kappa^2h^2-\epsilon}{1-\frac{5}{12}\kappa^2h^2-\epsilon}\cdot\frac{|\mu-\mu_c|}{\mu+\mu_c}<1\,.
\end{aligned}
\tag{6.69}
$$

If we solve inequations (6.69) with respect to ϵ, then we find, respectively

$$
\begin{aligned}
&\epsilon<1-\frac{5}{12}\kappa^2h^2 \qquad \text{and} \qquad \epsilon<1-\frac{\kappa^2h^2}{3}\bigg[\frac{1}{4}+\Big(1-\frac{|\lambda|}{2(\lambda+\mu)}\Big)^{-1}\bigg]\\
&\text{and} \qquad \epsilon<1-\frac{\kappa^2h^2}{3}\bigg[\frac{1}{4}+\Big(1-\frac{|\mu-\mu_c|}{\mu+\mu_c}\Big)^{-1}\bigg].
\end{aligned}
\tag{6.70}
$$

Further, using relations (6.59) we can write

$$
\begin{aligned}
&\frac{1}{4}+\Big(1-\frac{|\lambda|}{2(\lambda+\mu)}\Big)^{-1}=\frac{8(\lambda+\mu)+\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{4\min\{\lambda+2\mu\,,3\lambda+2\mu\}} \qquad \text{and}\\
&\frac{1}{4}+\Big(1-\frac{|\mu-\mu_c|}{\mu+\mu_c}\Big)^{-1}=\frac{2(\mu+\mu_c)+\min\{\mu\,,\mu_c\}}{4\min\{\mu\,,\mu_c\}}\;.
\end{aligned}
\tag{6.71}
$$

Substituting (6.71) into (6.70) we find the equivalent form

$$
\begin{aligned}
&\epsilon<1-\frac{5}{12}\kappa^2h^2 \qquad \text{and} \qquad \epsilon<1-\kappa^2h^2\cdot\frac{8(\lambda+\mu)+\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{12\min\{\lambda+2\mu\,,3\lambda+2\mu\}}\\
&\text{and} \qquad \epsilon<1-\kappa^2h^2\cdot\frac{2(\mu+\mu_c)+\min\{\mu\,,\mu_c\}}{12\min\{\mu\,,\mu_c\}}\;.
\end{aligned}
\tag{6.72}
$$

By virtue of assumptions (6.65), we see that the right-hand sides of inequalities (6.72) are all positive, so there exists a constant $\epsilon>0$ having properties (6.72), i.e., ϵ satisfies inequalities (6.69). Then, using (6.69) in Eq. (6.68) we derive

$$\frac{3}{h^2}\,D < -\Big(1-\frac{5}{12}\kappa^2h^2-\epsilon\Big)\bigg[\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\big(|\varphi_{11}|-|\varphi_{22}|\big)^2+\frac{\mu+\mu_c}{2}\,\big(|\varphi_{12}|-|\varphi_{21}|\big)^2 + \frac{2\mu\mu_c}{\mu+\mu_c}\,\big(\varphi_{31}^2+\varphi_{32}^2\big)\bigg] \leq 0.$$

Thus, the discriminant D is negative and, hence, the expression in square brackets in relation (6.67) is always positive. Consequently,inequations (6.67) and $(6.31)_1$ imply that there exists a constant $C_1 > 0$ such that

$$\mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \,\geq\, h\,\epsilon\, W_{\text{Coss}}\big(\boldsymbol{E}^e\big) \,\geq\, C_1\,\|\boldsymbol{E}^e\|^2,$$

which completes the proof.

We are now able to prove the coercivity of the strain energy density $\mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$. We obtain the following result as a consequence of the above lemmas.

Theorem 6.1. *Assume that the constitutive coefficients fulfil the inequalities*

$$\mu>0,\qquad 3\lambda+2\mu>0,\qquad \mu_c>0,\qquad \textit{and}\qquad b_i>0, \tag{6.73}$$

and that κh satisfies at least one of the conditions (6.39) *and* (6.65). *Then, the areal strain energy density $\mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$ given by* (6.33)-(6.35) *is coercive, i.e. there exist some positive constants $C_1>0$, $C_2>0$ such that*

$$\mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \,\geq\, C_1\,\|\boldsymbol{E}^e\|^2+C_2\,\|\boldsymbol{K}^e\|^2. \tag{6.74}$$

Proof. In view of the hypotheses of the theorem, we can apply Lemma 6.1 or Lemma 6.2. In both case, inequality (6.40) holds true. Then, taking into account relations (6.37) and (6.40) we obtain

$$\mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)=\mathcal{W}^{(3)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e)+\mathcal{W}^{(3)}_{\text{bend,curv}}(\boldsymbol{K}^e)\geq C_1\|\boldsymbol{E}^e\|^2+C_2\|\boldsymbol{K}^e\|^2$$

for some positive constants C_1, C_2. The proof is complete.

Remark 6.1. We mention that Theorem 6.1 is the main original result in this work. It improves a similar coercivity result established for a related Cosserat shell model of order $O(h^3)$. This related Cosserat shell model has been derived and investigated in [29, 30, 31] and the comparison with the present model has been presented in [22, Sect. 5.2]. More precisely, Theorem 6.1 improves the result in [31, Proposition 4.1] in two ways: Firstly, the conditions on constitutive coefficients given by (6.73) are less restrictive, since in [31, Proposition 4.1] the inequalities $\mu>0$ and $2\lambda+\mu>0$ are assumed, which are more restrictive. Secondly, the conditions on the product κh given by (6.39) and (6.65) are more convenient to check, since they involve only the constitutive coefficients. In contrast, the corresponding conditions (i) and (ii) listed in [31, Proposition 4.1] involve also the smallest and largest eigenvalues of the quadratic forms $W_{\text{curv}}(\boldsymbol{X})$, $W_{\text{shell}}(\boldsymbol{X})$ and, hence, they are more difficult to check.

Remark 6.2. Notice that hypothesis (6.65) in Lemma 6.2 can be written in the equivalent form

$$\kappa h < \min\left\{1\,,\,2\sqrt{3}\left(\frac{2(\lambda+\mu)-|\lambda|}{10(\lambda+\mu)-|\lambda|}\right)^{1/2},\,2\sqrt{3}\left(\frac{\mu+\mu_c-|\mu-\mu_c|}{5(\mu+\mu_c)-|\mu-\mu_c|}\right)^{1/2}\right\}. \tag{6.75}$$

Indeed, by a straightforward calculation we verify that

$$\begin{aligned}
&\frac{1}{4}+\left(1-\frac{|\lambda|}{2(\lambda+\mu)}\right)^{-1} = \frac{1}{4}\cdot\frac{10(\lambda+\mu)-|\lambda|}{2(\lambda+\mu)-|\lambda|} \qquad \text{and}\\
&\frac{1}{4}+\left(1-\frac{|\mu-\mu_c|}{\mu+\mu_c}\right)^{-1} = \frac{1}{4}\cdot\frac{5(\mu+\mu_c)-|\mu-\mu_c|}{\mu+\mu_c-|\mu-\mu_c|}\,.
\end{aligned} \tag{6.76}$$

Then, from relations (6.71) and (6.76) we deduce

$$\begin{aligned}
&\frac{8(\lambda+\mu)+\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{\min\{\lambda+2\mu\,,3\lambda+2\mu\}} = \frac{10(\lambda+\mu)-|\lambda|}{2(\lambda+\mu)-|\lambda|} \qquad \text{and}\\
&\frac{2(\mu+\mu_c)+\min\{\mu\,,\mu_c\}}{\min\{\mu\,,\mu_c\}} = \frac{5(\mu+\mu_c)-|\mu-\mu_c|}{\mu+\mu_c-|\mu-\mu_c|}\,.
\end{aligned} \tag{6.77}$$

Hence, conditions (6.65) and (6.75) are indeed equivalent.

Remark 6.3. In the case $\lambda \geq 0$ the hypotheses of Theorem 6.1 (namely relations (6.39) and (6.65)) simplify. Indeed, for $\lambda \geq 0$ we get

$$\left(\frac{47}{32}\cdot\frac{\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{\lambda+\mu}\right)^{1/2} = \left(\frac{47}{32}\cdot\frac{\lambda+2\mu}{\lambda+\mu}\right)^{1/2} > 1$$

and condition (6.39) in the statements of Lemma 6.1 and Theorem 6.1 reduces to the simpler form

$$\kappa h < \min\left\{\frac{1}{2}\,,\left(\frac{47}{8}\cdot\frac{\min\{\mu,\mu_c\}}{\mu+\mu_c}\right)^{1/2}\right\}. \tag{6.78}$$

Also, if $\lambda \geq 0$ the inequalities (6.38) imply

$$\frac{12\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{8(\lambda+\mu)+\min\{\lambda+2\mu\,,3\lambda+2\mu\}} = \frac{12(\lambda+2\mu)}{8(\lambda+\mu)+\lambda+2\mu} > 1.$$

Hence, in the case $\lambda \geq 0$ condition (6.65) can be reduced to

$$\kappa h < \min\left\{1\,,\left(\frac{12\min\{\mu,\mu_c\}}{2(\mu+\mu_c)+\min\{\mu,\mu_c\}}\right)^{1/2}\right\}. \tag{6.79}$$

Remark 6.4. In the case $\lambda < 0$ condition (6.39) from Lemma 6.1 takes the following form

$$\kappa h < \min\left\{ \frac{1}{2}\, ,\, \left(\frac{47}{32} \cdot \frac{3\lambda + 2\mu}{\lambda + \mu} \right)^{1/2} ,\, \left(\frac{47}{8} \cdot \frac{\min\{\mu, \mu_c\}}{\mu + \mu_c} \right)^{1/2} \right\}, \tag{6.80}$$

while condition (6.65) from Lemma 6.2 reduces to

$$\kappa h < \min\left\{ 1\, ,\, 2\sqrt{3} \left(\frac{3\lambda + 2\mu}{11\lambda + 10\mu} \right)^{1/2} ,\left(\frac{12 \min\{\mu, \mu_c\}}{2(\mu+\mu_c) + \min\{\mu, \mu_c\}} \right)^{1/2} \right\}. \tag{6.81}$$

6.3.2 Existence of Minimizers

Let us write the variational formulation for equilibrium of Cosserat shells. To this aim, we consider the usual Lebesgue and Sobolev spaces for vectors and tensors

$$\begin{aligned} L^p(\omega, \mathbb{R}^3) &= \{ \boldsymbol{v} = v_i \boldsymbol{e}_i \,|\, v_i \in L^p(\omega) \}, \\ L^p(\omega, \mathbb{R}^{3\times 3}) &= \{ \boldsymbol{T} = T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \,|\, T_{ij} \in L^p(\omega) \} \qquad (p \geq 1), \\ H^1(\omega, \mathbb{R}^3) &= \{ \boldsymbol{v} = v_i \boldsymbol{e}_i \,|\, v_i \in H^1(\omega) \}, \\ H^1(\omega, \mathbb{R}^{3\times 3}) &= \{ \boldsymbol{T} = T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \,|\, T_{ij} \in H^1(\omega) \}, \end{aligned} \tag{6.82}$$

We also introduce the subsets

$$L^p\big(\omega, \mathrm{SO}(3)\big) = \big\{ \boldsymbol{Q} \in L^p(\omega, \mathbb{R}^{3\times 3}) \,|\, \boldsymbol{Q}(x_1, x_2) \in \mathrm{SO}(3) \ \text{ for a.e. } (x_1, x_2) \in \omega \big\}$$

by abuse of notation, with the induced strong topology of $L^p(\omega, \mathbb{R}^{3\times 3})$, as well as

$$H^1\big(\omega, \mathrm{SO}(3)\big) = \big\{ \boldsymbol{Q} \in H^1(\omega, \mathbb{R}^{3\times 3}) \,|\, \boldsymbol{Q}(x_1, x_2) \in \mathrm{SO}(3) \ \text{ for a.e. } (x_1, x_2) \in \omega \big\}$$

with the induced strong and weak topologies of $H^1(\omega, \mathbb{R}^{3\times 3})$.

We assume that the boundary data in (6.12) satisfy the regularity $\boldsymbol{m}^* \in H^1(\omega, \mathbb{R}^3)$ and $\boldsymbol{Q}^* \in H^1\big(\omega, \mathrm{SO}(3)\big)$, and we define the set of admissible pairs $(\boldsymbol{m}, \boldsymbol{Q}_e)$ by

$$\mathcal{A} = \big\{ (\boldsymbol{m}, \boldsymbol{Q}_e) \in H^1(\omega, \mathbb{R}^3) \times H^1\big(\omega, \mathrm{SO}(3)\big) \,\big|\, \boldsymbol{m}\big|_{\partial\omega_d} = \boldsymbol{m}^*, \boldsymbol{Q}_e\big|_{\partial\omega_d} = \boldsymbol{Q}^* \big\}, \tag{6.83}$$

where the boundary conditions hold in the sense of traces. For boundary-value problem (6.11), (6.12) we assume the existence of the potential $\Lambda(\boldsymbol{m}, \boldsymbol{Q}_e)$ of external surface loads $\boldsymbol{f}$, $\boldsymbol{l}$ and boundary loads $\boldsymbol{N}^*$, $\boldsymbol{M}^*$ (cf. [23]), such that the total energy functional can be written as

$$
\begin{aligned}
\mathcal{E}^{(3)}(\boldsymbol{m},\boldsymbol{Q}_e) &= \iint_{\omega_\xi} \mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)\,\mathrm{d}a - \Lambda(\boldsymbol{m},\boldsymbol{Q}_e) \\
&= \iint_{\omega} \mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)\,a(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 - \Lambda(\boldsymbol{m},\boldsymbol{Q}_e),
\end{aligned}
\tag{6.84}
$$

where the tensors $\boldsymbol{E}^e,\boldsymbol{K}^e$ are given in terms of $\boldsymbol{m},\boldsymbol{Q}_e$ by Eqs. (6.9), (6.10). Here, the external loading potential has the form

$$
\Lambda(\boldsymbol{m},\boldsymbol{Q}_e) = \iint_{\omega_\xi} \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}a + \Pi_{\omega_\xi}(\boldsymbol{Q}_e) + \int_{\partial\omega_f} \boldsymbol{N}^*\cdot\boldsymbol{u}\,\mathrm{d}s + \Pi_{\partial\omega_f}(\boldsymbol{Q}_e), \tag{6.85}
$$

where $\boldsymbol{u} := \boldsymbol{m} - \boldsymbol{y}_0$ is the displacement vector and we assume that $\boldsymbol{f} \in L^2(\omega,\mathbb{R}^3)$ and $\boldsymbol{N}^* \in L^2(\partial\omega_f,\mathbb{R}^3)$. The potential $\Pi_{\omega_\xi} : L^2\big(\omega,\mathrm{SO}(3)\big) \to \mathbb{R}$ of the external surface couples $\boldsymbol{l}$ and the potential $\Pi_{\partial\omega_f} : L^2\big(\partial\omega_f,\mathrm{SO}(3)\big) \to \mathbb{R}$ of the external boundary couples $\boldsymbol{M}^*$ are assumed to be continuous and bounded operators.

We can prove now the existence of minimizers for the shell model of order $O(h^3)$ following closely the initial idea presented in [17].

Theorem 6.2. *Consider the minimization problem for the equilibrium of Cosserat 6-parameter elastic shells:*

$$
\text{minimize} \quad \mathcal{E}^{(3)}(\boldsymbol{m},\boldsymbol{Q}_e) \quad \text{w.r.t.} \quad (\boldsymbol{m},\boldsymbol{Q}_e) \in \mathcal{A}, \tag{6.86}
$$

where the total energy functional $\mathcal{E}^{(3)}$ *is given by* (6.84) *and the admissible set* $\mathcal{A}$ *is defined by* (6.83). *Assume that the constitutive coefficients satisfy inequalities* (6.73) *and that* κh *is small enough such that at least one of conditions* (6.39) *and* (6.65) *holds. Moreover, the external loads and boundary data are assumed to satisfy the regularity conditions*

$$
\boldsymbol{f} \in L^2(\omega,\mathbb{R}^3), \quad \boldsymbol{N}^* \in L^2(\partial\omega_f,\mathbb{R}^3), \quad \boldsymbol{m}^* \in H^1(\omega,\mathbb{R}^3), \quad \boldsymbol{Q}^* \in \boldsymbol{H}^1(\omega,SO(3)), \tag{6.87}
$$

while the reference configuration of the shell fulfils the regularity conditions

$$
\boldsymbol{y}_0 \in H^2(\omega,\mathbb{R}^3), \quad \boldsymbol{Q}_0 \in H^1(\omega,\mathrm{SO}(3)), \quad \boldsymbol{a}_\alpha \in L^\infty(\omega,\mathbb{R}^3), \quad a(x_1,x_2) \geq a_0 > 0, \tag{6.88}
$$

where a_0 *is a positive constant. Then, minimization problem* (6.86) *admits at least one minimizing solution pair* $(\hat{\boldsymbol{m}},\hat{\boldsymbol{Q}}_e)$ *in the admissible set* $\mathcal{A}$.

Proof. To prove this assertion we employ the general existence result established in [20] for 6-parameter shells. We can verify that the hypotheses of Theorem 6 from [20] are satisfied, i.e. that the strain energy density $\mathcal{W}^{(3)}_{\text{shell}}$ is a coercive and uniformly convex quadratic function of $(\boldsymbol{E}^e,\boldsymbol{K}^e)$.

Indeed, in view of Theorem 6.1 the function $\mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)$ is coercive. Since $\mathcal{W}^{(3)}_{\text{shell}}$ is a quadratic form in terms of $(\boldsymbol{E}^e,\boldsymbol{K}^e)$, which is also positive definite, we see that it is also convex. Thus, all the hypotheses of Theorem 6 from [20] are fulfilled. Applying this general result, we can derive the existence of a minimizing pair $(\hat{\boldsymbol{m}},\hat{\boldsymbol{Q}}_e)\in\mathcal{A}$. In what follows, we present only the main steps of the existence proof and refer to [20, Theorem 6] for further details.

We estimate first the external loading potential and show that

$$|\Lambda(\boldsymbol{m},\boldsymbol{Q}_e)|\leq\; c_1\left(\|\boldsymbol{m}\|_{H^1(\omega)}+1\right)\qquad\text{for any}\quad(\boldsymbol{m},\boldsymbol{Q}_e)\in\mathcal{A},\tag{6.89}$$

for some positive constant c_1. Then, using coercivity relation (6.74) we deduce that there exist some constants $c_2>0$ and c_3, c_4 such that

$$\mathcal{E}^{(3)}(\boldsymbol{m},\boldsymbol{Q}_e)\geq c_2\,\|\nabla\boldsymbol{m}\|^2_{L^2(\omega)}-c_3\|\,\boldsymbol{m}\,\|_{H^1(\omega)}-c_4\,.\tag{6.90}$$

Here, we denote by $\|\cdot\|_{L^2(\omega)}$ and $\|\cdot\|_{H^1(\omega)}$ the norms in the Lebesgue and Sobolev spaces, respectively. Using the Poincaré–inequality we deduce from (6.90) that

$$\mathcal{E}^{(3)}(\boldsymbol{m},\boldsymbol{Q}_e)\geq c_5\,\|\boldsymbol{m}-\boldsymbol{m}^*\|^2_{H^1(\omega)}-c_6\|\boldsymbol{m}-\boldsymbol{m}^*\|_{H^1(\omega)}+c_7\quad\text{for any}\;\;(\boldsymbol{m},\boldsymbol{Q}_e)\in\mathcal{A},\tag{6.91}$$

where $c_5>0$ and c_6, c_7 are some constants. Hence, the functional $\mathcal{E}^{(3)}(\boldsymbol{m},\boldsymbol{Q}_e)$ is bounded from below over $\mathcal{A}$. Therefore, there exists an infimizing sequence $(\boldsymbol{m}_n,\boldsymbol{Q}_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\mathcal{E}^{(3)}(\boldsymbol{m}_n,\boldsymbol{Q}_n)=\inf\left\{\mathcal{E}^{(3)}(\boldsymbol{m},\boldsymbol{Q}_e)\,\middle|\,(\boldsymbol{m},\boldsymbol{Q}_e)\in\mathcal{A}\right\}.\tag{6.92}$$

For this infimizing sequence we show that $(\boldsymbol{m}_n)$ is bounded in $H^1(\omega,\mathbb{R}^3)$ and $(\boldsymbol{Q}_n)$ is bounded in $H^1(\omega,\mathbb{R}^{3\times3})$. Then, there exist some subsequences (not relabeled) and the limit pair $(\hat{\boldsymbol{m}},\hat{\boldsymbol{Q}}_e)\in\mathcal{A}$ such that the following weak and strong convergences hold

$$\begin{aligned}&\boldsymbol{m}_n\rightharpoonup\hat{\boldsymbol{m}}\quad\text{in }H^1(\omega,\mathbb{R}^3)\qquad\text{and}\qquad\boldsymbol{m}_n\to\hat{\boldsymbol{m}}\quad\text{in }L^2(\omega,\mathbb{R}^3),\\&\boldsymbol{Q}_n\rightharpoonup\hat{\boldsymbol{Q}}_e\quad\text{in }H^1(\omega,\mathbb{R}^{3\times3})\qquad\text{and}\qquad\boldsymbol{Q}_n\to\hat{\boldsymbol{Q}}_e\quad\text{in }L^2(\omega,\mathbb{R}^{3\times3}).\end{aligned}\tag{6.93}$$

Since the pairs $(\boldsymbol{m}_n,\boldsymbol{Q}_n)$ and $(\hat{\boldsymbol{m}},\hat{\boldsymbol{Q}}_e)$ are elements of the admissible set $\mathcal{A}$, we can construct the corresponding shell strain measures $(\boldsymbol{E}^e_n,\boldsymbol{K}^e_n)$ and $(\hat{\boldsymbol{E}}^e,\hat{\boldsymbol{K}}^e)$, respectively, using definitions (6.9), (6.10). Then, we can extract some subsequences (not relabeled) such that we have the following weak convergences

$$\boldsymbol{E}^e_n\rightharpoonup\hat{\boldsymbol{E}}^e\quad\text{in }L^2(\omega,\mathbb{R}^{3\times3})\qquad\text{and}\qquad\boldsymbol{K}^e_n\rightharpoonup\hat{\boldsymbol{K}}^e\quad\text{in }L^2(\omega,\mathbb{R}^{3\times3}).\tag{6.94}$$

We use now the convexity of the energy density function $\mathcal{W}^{(3)}_{\text{shell}}$ and obtain

$$\iint\limits_{\omega}\mathcal{W}^{(3)}_{\text{shell}}(\hat{\boldsymbol{E}}^e,\hat{\boldsymbol{K}}^e)\,a\,\mathrm{d}x_1\mathrm{d}x_2\;\leq\;\liminf_{n\to\infty}\iint\limits_{\omega}\mathcal{W}^{(3)}_{\text{shell}}(\boldsymbol{E}^e_n,\boldsymbol{K}^e_n)\,a\,\mathrm{d}x_1\mathrm{d}x_2\,.\tag{6.95}$$

Finally, by virtue of (6.84) and (6.95) we get

$$\mathcal{E}^{(3)}(\hat{\boldsymbol{m}},\hat{\boldsymbol{Q}}_e) \leq \liminf_{n\to\infty} \mathcal{E}^{(3)}(\boldsymbol{m}_n,\boldsymbol{Q}_n), \tag{6.96}$$

so $(\hat{\boldsymbol{m}},\hat{\boldsymbol{Q}}_e)$ is a minimizing solution pair of minimization problem (6.86).

6.4 The Higher Order Model of Cosserat 6-Parameter Shells

The first Cosserat 6-parameter shell model of order $O(h^5)$ has been established in [29, 30, 31] by a dimensional descent from the three-dimensional nonlinear Cosserat elasticity. Then, using an alternative derivation procedure suggested by the classical shell theory [5, 6], we have derived in [33] a refined higher order Cosserat shell model, in which we have optimized some terms.

Let us investigate in this section the 6-parameter shell model of order $O(h^5)$ derived in [33] for shells made of Cosserat material. Thus, the following areal strain energy density has been obtained (see [33, Eq. (119)])

$$\begin{aligned}\mathcal{W}^{(5)}_{\rm shell}(\boldsymbol{E}^e,\boldsymbol{K}^e) = {} & \Big(h - K\frac{h^3}{12}\Big)\Big[W_{\rm Coss}\big(\boldsymbol{E}^e\big) + W_{\rm curv}\big(\boldsymbol{K}^e\big)\Big] - \frac{h^3}{6}W_{\rm Coss}\big(\boldsymbol{E}^e,\boldsymbol{c}\boldsymbol{K}^e\boldsymbol{b}^*\big)\\ & + \Big(\frac{h^3}{12} - K\,\frac{h^5}{80}\Big)\Big[W_{\rm Coss}\big(\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e\big) + W_{\rm curv}\big(\boldsymbol{K}^e\boldsymbol{b}\big)\Big]\\ & + \frac{h^5}{80}\Big[W_{\rm Coss}\big((\boldsymbol{E}^e\boldsymbol{b} + \boldsymbol{c}\boldsymbol{K}^e)\boldsymbol{b}\big) + W_{\rm curv}\big(\boldsymbol{K}^e\boldsymbol{b}^2\big)\Big].\end{aligned} \tag{6.97}$$

If we employ relation (6.5) and notation (6.32), we can decompose the above energy density in the form

$$\mathcal{W}^{(5)}_{\rm shell}(\boldsymbol{E}^e,\boldsymbol{K}^e) = \mathcal{W}^{(5)}_{\rm memb,bend}(\boldsymbol{E}^e,\boldsymbol{K}^e) + \mathcal{W}^{(5)}_{\rm bend,curv}(\boldsymbol{K}^e) \qquad \text{with} \tag{6.98}$$

$$\begin{aligned}\mathcal{W}^{(5)}_{\rm memb,bend}(\boldsymbol{E}^e,\boldsymbol{K}^e) = {} & \Big(h + K\,\frac{h^3}{12}\Big)W_{\rm Coss}\big(\boldsymbol{E}^e\big) - 2\,\frac{h^3}{12}\,W_{\rm Coss}\big(\boldsymbol{E}^e,\boldsymbol{\Phi}^e\boldsymbol{b}^*\big)\\ & + \Big(\frac{h^3}{12} - K\,\frac{h^5}{80}\Big)W_{\rm Coss}\big(\boldsymbol{\Phi}^e\big) + \frac{h^5}{80}W_{\rm Coss}\big(\boldsymbol{\Phi}^e\boldsymbol{b}\big),\end{aligned} \tag{6.99}$$

$$\begin{aligned}\mathcal{W}^{(5)}_{\rm bend,curv}(\boldsymbol{K}^e) = {} & \Big(h - K\,\frac{h^3}{12}\Big)W_{\rm curv}\big(\boldsymbol{K}^e\big) + \Big(\frac{h^3}{12} - K\,\frac{h^5}{80}\Big)W_{\rm curv}\big(\boldsymbol{K}^e\boldsymbol{b}\big)\\ & + \frac{h^5}{80}\,W_{\rm curv}\big(\boldsymbol{K}^e\boldsymbol{b}^2\big).\end{aligned} \tag{6.100}$$

Let us consider first the bending-curvature part $\mathcal{W}^{(5)}_{\rm bend,curv}(\boldsymbol{K}^e)$ and prove its coercivity. Since $|\kappa_\alpha h| < 1$, we have $|Kh^2| < 1$ (cf. (6.36)) and, hence,

$$h - K\frac{h^3}{12} > h - \frac{h}{12} = \frac{11}{12}\,h \qquad \text{and} \qquad \frac{h^3}{12} - K\frac{h^5}{80} > \frac{h^3}{12} - \frac{h^3}{80} = \frac{17}{240}\,h^3. \tag{6.101}$$

Then, Eqs. (6.100), (6.101) and $(6.31)_2$ yield

$$\begin{aligned}\mathcal{W}^{(5)}_{\text{bend,curv}}(\boldsymbol{K}^e) &\geq \frac{11}{12}\, h\, W_{\text{curv}}\big(\boldsymbol{K}^e\big) + \frac{17}{240}\, h^3 W_{\text{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}\big) + \frac{h^5}{80}\, W_{\text{curv}}\big(\boldsymbol{K}^e\boldsymbol{b}^2\big)\\ &\geq \frac{11}{12}\, h\, W_{\text{curv}}\big(\boldsymbol{K}^e\big) \;\geq\; C_3\, \|\boldsymbol{K}^e\|^2,\end{aligned} \tag{6.102}$$

for some positive constant C_3 . Thus, the energy function $\mathcal{W}^{(5)}_{\text{bend,curv}}(\boldsymbol{K}^e)$ is coercive.

We turn our attention now to membrane-bending part (6.99) and establish an auxiliary result.

Lemma 6.3. *Assume that the condition*

$$|\kappa_\alpha h| < \frac{1}{2}$$

holds and the constitutive coefficients satisfy inequalities (6.38). *Then, there exist some positive constants* $C_4 > 0$, $C_5 > 0$ *such that the membrane-bending energy density* (6.99) *satisfies inequality*

$$\mathcal{W}^{(5)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \;\geq\; C_4\, \|\boldsymbol{E}^e\|^2 + C_5\, \|\boldsymbol{\Phi}^e\|^2\,. \tag{6.103}$$

Proof. We can put the membrane-bending energy density (6.99) in the form

$$\begin{aligned}\mathcal{W}^{(5)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) &= \Big(h + K\,\frac{h^3}{12}\Big) W_{\text{Coss}}\big(\boldsymbol{E}^e\big) - 2\,\frac{h^3}{12}\, W_{\text{Coss}}\big(\boldsymbol{E}^e, \boldsymbol{\Phi}^e\boldsymbol{b}^*\big)\\ &\quad + \Big(\frac{h^3}{12} + (4H^2 - K)\frac{h^5}{80}\Big) W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big) + \frac{h^5}{80} W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{b}^*\big) - 4H\frac{h^5}{80} W_{\text{Coss}}\big(\boldsymbol{\Phi}^e, \boldsymbol{\Phi}^e\boldsymbol{b}^*\big)\\ &= \Big(\frac{11}{36}\, h + K\,\frac{h^3}{12}\Big) W_{\text{Coss}}\big(\boldsymbol{E}^e\big) + h\Big[\frac{25}{36}\, W_{\text{Coss}}\big(\boldsymbol{E}^e\big) - 2\,\frac{h^2}{12}\, W_{\text{Coss}}\big(\boldsymbol{E}^e, \boldsymbol{\Phi}^e\boldsymbol{b}^*\big)\\ &\quad + \frac{4}{5}\,\frac{h^4}{80}\, W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{b}^*\big)\Big] + \Big(\frac{h^3}{12} - (16H^2 + K)\frac{h^5}{80}\Big) W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big)\\ &\quad + \frac{h^5}{80}\Big[20H^2 W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big) - 4H W_{\text{Coss}}\big(\boldsymbol{\Phi}^e,\, \boldsymbol{\Phi}^e\boldsymbol{b}^*\big) + \frac{1}{5}\, W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\boldsymbol{b}^*\big)\Big]\\ &= \frac{h}{12}\Big(\frac{11}{3} + K h^2\Big) W_{\text{Coss}}\big(\boldsymbol{E}^e\big) + h\, W_{\text{Coss}}\Big(\frac{5}{6}\,\boldsymbol{E}^e - \frac{h^2}{10}\,\boldsymbol{\Phi}^e\boldsymbol{b}^*\Big)\\ &\quad + \Big(\frac{h^3}{12} - H^2\,\frac{h^5}{5} - K\,\frac{h^5}{80}\Big) W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big) + \frac{h^5}{16}\, W_{\text{Coss}}\big(2H\boldsymbol{\Phi}^e - \frac{1}{5}\,\boldsymbol{\Phi}^e\boldsymbol{b}^*\big).\end{aligned} \tag{6.104}$$

In view of

$$|\kappa_\alpha h| < \frac{1}{2}$$

we have

$$|K|\, h^2 = |\kappa_1 h| \cdot |\kappa_2 h| < \frac{1}{4} \qquad \text{and} \qquad |H|\, h \;\leq\; \frac{1}{2}\,\big(|\kappa_1 h| + |\kappa_2 h|\big) < \frac{1}{2}\,, \tag{6.105}$$

so it holds

$$\frac{11}{3}+Kh^2 > \frac{11}{3}-\frac{1}{4}=\frac{41}{12}$$
and
$$\frac{h^3}{12}-H^2\frac{h^5}{5}-K\frac{h^5}{80} > h^3\Big(\frac{1}{12}-\frac{1}{4}\cdot\frac{1}{5}-\frac{1}{4}\cdot\frac{1}{80}\Big)=\frac{29}{960}h^3. \tag{6.106}$$

Using inequalities (6.106) and $(6.31)_1$, from relation (6.104) we get

$$\begin{aligned}\mathcal{W}^{(5)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e) &\geq \frac{41}{144}\,h\,W_{\text{Coss}}\big(\boldsymbol{E}^e\big)+\frac{29}{960}\,h^3W_{\text{Coss}}\big(\boldsymbol{\Phi}^e\big)\\ &\geq C_4\,\|\boldsymbol{E}^e\|^2+C_5\,\|\boldsymbol{\Phi}^e\|^2,\end{aligned} \tag{6.107}$$

for some positive constants C_4, C_5. The lemma is proved.

We are now able to prove the coercivity of the shell strain energy function for the higher order model.

Theorem 6.3. *Assume that*
$$|\kappa_\alpha h| < \frac{1}{2}$$
and the constitutive coefficients satisfy inequalities (6.73). *Then, the areal strain energy density* $\mathcal{W}^{(5)}_{\text{shell}}$ *given by* (6.98)-(6.100) *is coercive, i.e. there exist some positive constants* $C_3 > 0$, $C_4 > 0$ *such that*

$$\mathcal{W}^{(5)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e) \geq C_4\,\|\boldsymbol{E}^e\|^2+C_3\,\|\boldsymbol{K}^e\|^2. \tag{6.108}$$

Proof. By virtue of the relation (6.102) and the Lemma 6.3 we deduce that

$$\mathcal{W}^{(5)}_{\text{memb,bend}}(\boldsymbol{E}^e,\boldsymbol{K}^e)+\mathcal{W}^{(5)}_{\text{bend,curv}}(\boldsymbol{K}^e) \geq C_4\,\|\boldsymbol{E}^e\|^2+C_5\,\|\boldsymbol{\Phi}^e\|^2+C_3\,\|\boldsymbol{K}^e\|^2,$$

so coercivity inequality (6.108) holds true.

In a similar way as in Subsect. 6.3.2 we can prove the existence of minimizers for the Cosserat 6-parameter shell model of order $O(h^5)$. In this case, the total energy functional is expressed by

$$\mathcal{E}^{(5)}(\boldsymbol{m},\boldsymbol{Q}_e)=\iint\limits_{\omega}\mathcal{W}^{(5)}_{\text{shell}}(\boldsymbol{E}^e,\boldsymbol{K}^e)\,a(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2-\Lambda(\boldsymbol{m},\boldsymbol{Q}_e), \tag{6.109}$$

where the areal strain energy density $\mathcal{W}^{(5)}_{\text{shell}}$ is given by (6.98)-(6.100) and the external loading potential Λ has the form (6.85).

Theorem 6.4. *Consider the minimization problem for the equilibrium of Cosserat 6-parameter elastic shells:*

$$\text{minimize}\quad \mathcal{E}^{(5)}(\boldsymbol{m},\boldsymbol{Q}_e)\quad \text{w.r.t.}\quad (\boldsymbol{m},\boldsymbol{Q}_e)\in\mathcal{A}, \tag{6.110}$$

where the total energy functional $\mathcal{E}^{(5)}$ *is given by* (6.109) *and the admissible set* $\mathcal{A}$ *is defined by* (6.83). *Assume that* $|\kappa_\alpha h| < \frac{1}{2}$ *and the constitutive coefficients satisfy the inequalities* (6.73). *Also, the reference configuration of the shell is assumed to satisfy regularity conditions* (6.88)*, while the external loads and boundary data fulfil regularity conditions* (6.87). *Then, the minimization problem* (6.110) *admits at least one minimizing solution pair* $(\hat{\boldsymbol{m}}, \hat{\boldsymbol{Q}}_e)$ *in the admissible set* $\mathcal{A}$.

Proof. We proceed similarly as in the proof of Theorem 6.2. Since all the hypotheses of Theorem 6.3 are satisfied, we deduce that the strain energy density $\mathcal{W}^{(5)}_{\text{shell}}(\boldsymbol{E}^e, \boldsymbol{K}^e)$ is coercive. Further, from relations (6.98)-(6.100) we see that this function is a quadratic form in $(\boldsymbol{E}^e, \boldsymbol{K}^e)$, which is positive definite. Then, $\mathcal{W}^{(5)}_{\text{shell}}(\boldsymbol{E}^e, \boldsymbol{K}^e)$ is also convex. We can apply the general existence theorem in [20], since all its hypotheses are fulfilled. From [20, Theorem 6] we obtain the existence of minimizers to our minimization problem (6.110) for the Cosserat 6-parameter shell model of order $O(h^5)$. We refer to [20] for the details of the proof.

Remark 6.5. We mention that Theorem 6.4 is similar to the existence result presented previously in [31, Theorem 3.3] for a related Cosserat shell model including terms up to order $O(h^5)$. This related Cosserat shell model has been investigated in [30, 31, 34] using the matrix formulation. A detailed comparison between the two Cosserat approaches to 6-parameter shells (which employ either matrix formulation or tensorial notation) can be found in [33, Sect. 5.3].

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