Chapter 3
The Direct Approach for Plates Considering Hygrothermal Loading and Residual Kinetics

Marcus Aßmus, Zia Javanbakht, and Holm Altenbach

Abstract The direct approach for plates is based on Cosserat continuum theory, or rather on the Cosserat surface. The same principles as used in classical continuum mechanics are applied whereby a restriction to two-dimensional body manifolds is imposed. At the same time independent rotational degrees of freedom are introduced. Pavel Andreevich Zhilin then proposed a reduced treatment that is widely used in engineering, especially in composite mechanics. However, the approach still lacks important extensions to include significant quantities influencing the mechanical behaviour. These quantities result from moisture exposure, temperature changes and initial tense in the material. We here delineate extensions based on physical justifications while discussing three-dimensional causation and two-dimensional impact. A resulting set of equations for the combined loading is derived and discussed.

Key words: Generalized plate theory · Hygroscopic impact · Thermal effects · Residual kinetics

3.1 Introduction

We consider a rectangular plate of length L_1, width L_2 and constant thickness H with arbitrary supports at four edges, cf. Fig. 3.1. The plate is subjected to moisture exposure and temperature change. Within the plate, fields exist that result from

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residual quantities, namely residual forces and moments. Such influences are to be treated in the context of the direct approach for homogeneous thin-walled structures. A first attempt was successfully utilized in [1] where a three-layered plate was considered with reduced influence quantities. We here sharpen our reflections and reduce ourselves to a single layer.

The direct approach for plates is strongly associated with the ideas of Zhilin, cf. [2], even if other scientists have had similar thoughts before, cf. Cosserat & Cosserat [3] and Ericksen & Truesdell [4]. We here reduce our considerations to an initially plane (undeformed) surface on which in analogy to the procedure at classical three-dimensional continua, kinematic and kinetic measures are defined and constitutive relations are introduced. The resulting theory is an analogy to a generalized plate theory in accordance with the works of Kirchhoff [5], Reissner [6], and Mindlin [7]. The generalization is meant in the spirit of [8], i.e., where in-plane, out-of-plane and transverse shear state of a plate may superimpose.

We are considering a monolithic conventional material with isotropic and purely elastic material behaviour. Thereby, we restrict ourselves to the small deformation setting. We furthermore consider

- the thermo-static case ($\Delta\dot{\theta} = 0 \wedge \dot{\square} = 0 \; \forall \, \square \in \{v_\alpha, w, \varphi_\alpha\}$),
- the hygro-static case ($\Delta\dot{\mu} = 0 \wedge \dot{\square} = 0 \; \forall \, \square \in \{v_\alpha, w, \varphi_\alpha\}$), and what we call
- the residual-static case ($\dot{\square}^0_{\alpha\beta} = 0 \forall \square \in \{N, L\} \wedge \dot{\square} = 0 \; \forall \, \square \in \{q^0_\alpha, v_\alpha, w, \varphi_\alpha\}$)

for the entire plate at any time. Herein, θ, μ, $N^0_{\alpha\beta}$, $L^0_{\alpha\beta}$, q^0_α, v_α, w, φ_α are the degrees of freedom, which we will specify in more detail in the further course. Their consideration results in a description for two-dimensional body manifolds which is similar to the description of classical three-dimensional Cauchy continuum in a hygrothermal environment with residual stresses. With this analogy we formulate a generalized plate theory for hygro-thermo-elastostatics with preload.

Notation: Throughout present treatise, a direct tensor notation is used. First- and second-order tensors are denoted by lowercase and uppercase bold letters, e.g., $\boldsymbol{a}$ and $\underset{\sim}{\boldsymbol{A}}$, respectively. Fourth-order tensors are designated by uppercase calligraphic letters, e.g. $\mathcal{A}$. Sets groups and spaces are written in blackboard bold letters, e.g. $\mathbb{S}\mathrm{ym}$. If index notation is required, we will make use of implicit summation over indices 1 to 3 in case of latin letters, and over indices 1 to 2 in case of greek letters. Furthermore, $\cdot$, $:$, $\otimes$, and $\times$ are the scalar product, the double scalar product, the dyadic product, and the cross product. We additionally introduce the three-dimensional $\boldsymbol{\nabla}_3 = \boldsymbol{e}_i \, {}^{\partial\square}/_{\partial x_i}$ and the two dimensional Hamiltonian $\boldsymbol{\nabla}_2 = \boldsymbol{e}_\alpha \, {}^{\partial\square}/_{\partial x_\alpha}$. With its help the operations divergence $\boldsymbol{\nabla}\cdot\square$ and gradient $\boldsymbol{\nabla}\square$ of a tensor $\square$ can be specified compactly. Also, the symmetric part of a gradient $\boldsymbol{\nabla}^{\mathrm{sym}}\square = {}^1/_2[\boldsymbol{\nabla}\square + \boldsymbol{\nabla}^\top\square]$ can be given. For detailed penetrations of these operations we refer to [9] or [10].

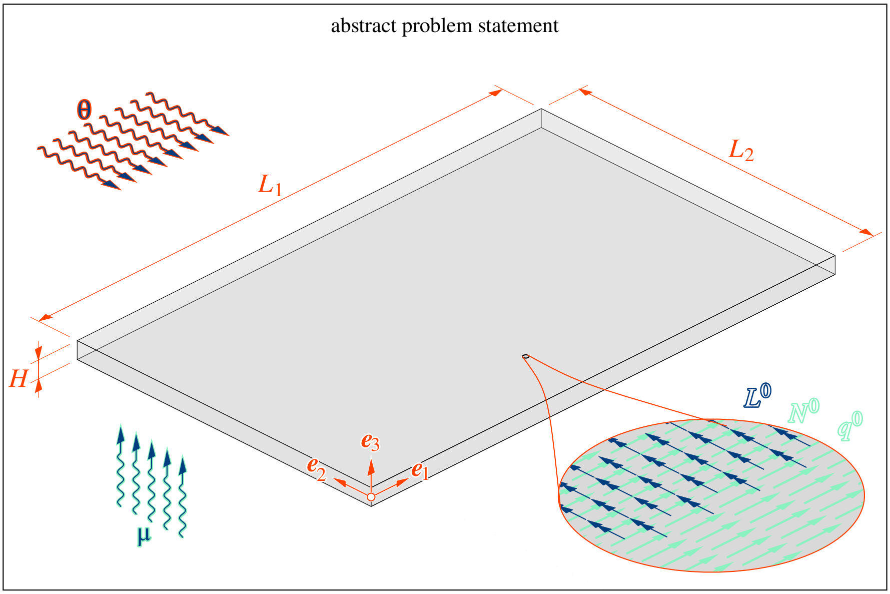

Fig. 3.1: A plate exposed to moisture (μ) and temperature (θ) as well as as kinetic quantities (N^0, L^0, q^0) which are already present in the interior of the unloaded state.

3.2 Frame of Reference

In plate theory it is generally accepted that the plate can be represented by a deformable plane. Such considerations can traced back to Germain [11], Lagrange [12], Navier [13], Poisson [14], and of course Kirchhoff [5], among others (for further discussion see [15]). Hence, all contemplations are reduced to a two-dimensional problem due to the slenderness of the body considered. For this reason, one selects a surface $\mathfrak{S}$ related to planar dimensions. It is reasonable to choose the mid surface due to material and geometrical symmetries in transverse direction. We reduce our concern to coplanarity of all material points of this surface, i.e. initially plane surfaces. For present treatise we furthermore assume that each material point is an infinitesimal rigid body with five kinematic degrees of freedom, cf. Fig. 3.2 at the right-hand side. As will be described in more detail later, these are three translations and two rotations. Thus, we introduce a plate theory as proposed by Zhilin [2]. We directly enter the topic by introducing the degrees of freedom. In present case these are three translations $\boldsymbol{a}$ and two rotations $\boldsymbol{\psi}$, which components are visualized in Fig. 3.2 on the ride-hand-side.

$$\boldsymbol{a} = \boldsymbol{v} + w\boldsymbol{n} \qquad \text{with } \boldsymbol{v} = v_\alpha \boldsymbol{e}_\alpha, \tag{3.1}$$

$$\boldsymbol{\psi} = -\varphi_2 \boldsymbol{e}_1 + \varphi_1 \boldsymbol{e}_2. \tag{3.2}$$

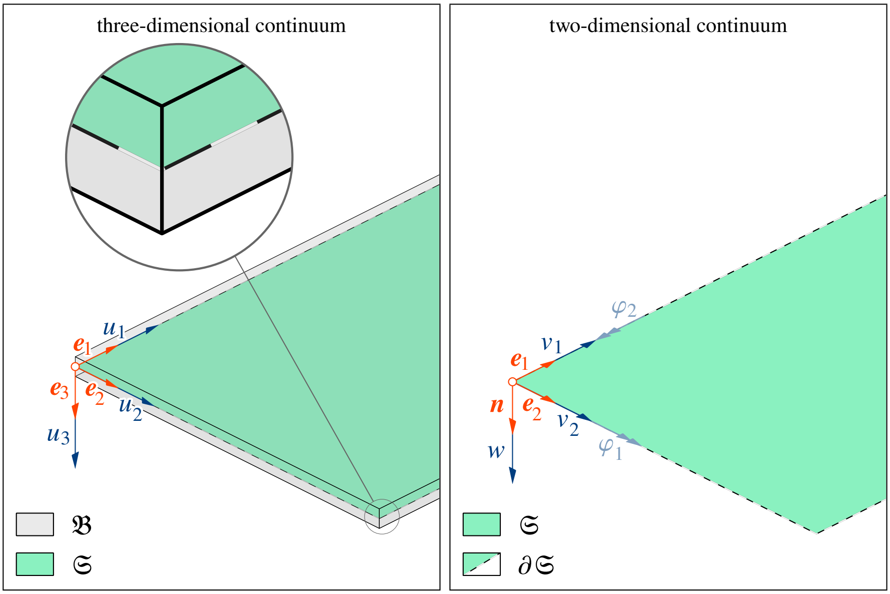

Fig. 3.2: Mid surface $\mathfrak{S}$ in context of a three-dimensional body $\mathfrak{B}$ with degrees of freedom assigned.

Herein, our tripod is still a set of orthogonal unit vectors with longitudinal directions $\boldsymbol{e}_\alpha$ and one direction perpendicular to the plane spanned by the longitudinal ones, called $\boldsymbol{n}$. However, rotations around the surface normal remain unconsidered since they have no physical justification due to the fact that the in-plane shear stiffness of classical engineering structures is much larger that the bending stiffness, cf. [2]. However, instead of employing the rotational vector $\boldsymbol{\psi}$, we introduce the vector $\boldsymbol{\varphi} = \varphi_\alpha \boldsymbol{e}_\alpha$, which is related via $\boldsymbol{\varphi} = \boldsymbol{\psi} \times \boldsymbol{n}$.

Typical deformation measures are the membrane strains $\underset{\sim}{\boldsymbol{G}}$, the curvature changes $\underset{\sim}{\boldsymbol{K}}$, and the transverse shear strains $\boldsymbol{g}$. In a geometrically linear framework, we can define these measures as follows.

$$\underset{\sim}{\boldsymbol{G}} = \nabla^{\mathrm{sym}} \boldsymbol{v} = G_{\gamma\delta} \boldsymbol{e}_\gamma \otimes \boldsymbol{e}_\delta, \tag{3.3}$$

$$\underset{\sim}{\boldsymbol{K}} = \nabla^{\mathrm{sym}} \boldsymbol{\varphi} = K_{\gamma\delta} \boldsymbol{e}_\gamma \otimes \boldsymbol{e}_\delta, \tag{3.4}$$

$$\boldsymbol{g} = \nabla w + \boldsymbol{\varphi} = g_\beta \boldsymbol{e}_\beta. \tag{3.5}$$

The decoupling of the constitutive equations is reasonable due to

- coplanarity of all material points of the surface considered,
- geometrical symmetry in thickness direction,
- material symmetry in thickness direction, and
- isotropic material behaviour.

We can now introduce a decoupled, which means the in-plane, out-of-plan and transverse shear behaviour not influencing each other, strain energy function of the deformable plane surface in terms of a Taylor series limited to quadratic terms where positive definiteness holds true.

$$W(\underset{\sim}{\boldsymbol{G}},\underset{\sim}{\boldsymbol{K}},\boldsymbol{g}) = \frac{1}{2}\left(\underset{\sim}{\boldsymbol{G}}:\mathcal{A}:\underset{\sim}{\boldsymbol{G}}+\underset{\sim}{\boldsymbol{K}}:\mathcal{D}:\underset{\sim}{\boldsymbol{K}}+\boldsymbol{g}\cdot\underset{\sim}{\boldsymbol{Z}}\cdot\boldsymbol{g}\right) \tag{3.6}$$

Therein, we have introduced the constitutive tensors, whereby $\mathcal{A} = A_{\alpha\beta\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{e}_\gamma\otimes\boldsymbol{e}_\delta$ is the fourth-order in-plane stiffness tensor, $\mathcal{D} = D_{\alpha\beta\gamma\delta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta\otimes\boldsymbol{e}_\gamma\otimes\boldsymbol{e}_\delta$ is the fourth-order out-of-plane stiffness tensor, and $\underset{\sim}{\boldsymbol{Z}} = Z_{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta$ is the second-order transverse shear stiffness tensor. They depend on the material and geometrical properties. In the isotropic case, they depend on two material parameters only and the thickness of the plate. The kinetic measures arising from this potential.

$$\frac{\partial W}{\partial\underset{\sim}{\boldsymbol{G}}} = \mathcal{A}:\underset{\sim}{\boldsymbol{G}}, \qquad \frac{\partial W}{\partial\underset{\sim}{\boldsymbol{K}}} = \mathcal{D}:\underset{\sim}{\boldsymbol{K}}, \qquad \frac{\partial W}{\partial\boldsymbol{g}} = \underset{\sim}{\boldsymbol{Z}}\cdot\boldsymbol{g}. \tag{3.7}$$

Hence we have found dual quantities, i.e. the in-plane forces $\underset{\sim}{\boldsymbol{N}}$, the moments $\underset{\sim}{\boldsymbol{L}}$, and the transverse shear forces $\boldsymbol{q}$.

$$\frac{\partial W}{\partial\underset{\sim}{\boldsymbol{G}}} = \underset{\sim}{\boldsymbol{N}} = N_{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta, \tag{3.8}$$

$$\frac{\partial W}{\partial\underset{\sim}{\boldsymbol{K}}} = \underset{\sim}{\boldsymbol{L}} = L_{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta, \tag{3.9}$$

$$\frac{\partial W}{\partial\boldsymbol{g}} = \boldsymbol{q} = Q_\alpha\boldsymbol{e}_\alpha. \tag{3.10}$$

The constitutive equations are then given as linear mappings.

$$\underset{\sim}{\boldsymbol{N}} = \mathcal{A}:\underset{\sim}{\boldsymbol{G}}, \tag{3.11}$$

$$\underset{\sim}{\boldsymbol{L}} = \mathcal{D}:\underset{\sim}{\boldsymbol{K}}, \tag{3.12}$$

$$\boldsymbol{q} = \underset{\sim}{\boldsymbol{Z}}\cdot\boldsymbol{g}. \tag{3.13}$$

The isotropic stiffness tensors can be given in following form [16].

$$\mathcal{A} = 2Bh\ \mathcal{P}_1^{\mathfrak{S}} + 2Gh\ \mathcal{P}_2^{\mathfrak{S}}, \tag{3.14}$$

$$\mathcal{D} = 2B\frac{h^3}{12}\mathcal{P}_1^{\mathfrak{S}} + 2G\frac{h^3}{12}\mathcal{P}_2^{\mathfrak{S}}, \tag{3.15}$$

$$\underset{\sim}{\boldsymbol{Z}} = \kappa Gh\ \underset{\sim}{\boldsymbol{P}}. \tag{3.16}$$

Herein we have introduced the compression modulus of the surface $B = {}^{Y}\!/_{2(1-\nu)}$ [16] with Young's modulus Y, Poisson's ratio ν and a correction factor $0 < \kappa \leq 1$ accounting for transverse shear, cf. [6, 7] and the discussion in [17]. The fourth-order isotropic projectors are given as follows.

$$\mathcal{P}_1^{\mathfrak{S}} = \frac{1}{2}\underset{\sim}{\boldsymbol{P}} \otimes \underset{\sim}{\boldsymbol{P}}, \qquad \mathcal{P}_2^{\mathfrak{S}} = \mathcal{P}^{\text{sym}} - \mathcal{P}_1^{\mathfrak{S}}. \tag{3.17}$$

Herein, $2\mathcal{P}^{\text{sym}} = \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta \otimes (\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta + \boldsymbol{e}_\beta \otimes \boldsymbol{e}_\alpha)$ is the symmetric part of the fourth-order identity of the surface.

Finally, the equilibrium equations of the two-dimensional planar continuum are written as

$$\nabla \cdot \underset{\sim}{\boldsymbol{N}} + \boldsymbol{s} = \mathbf{o}, \tag{3.18a}$$

$$\nabla \cdot \boldsymbol{q} + p = 0, \tag{3.18b}$$

$$\nabla \cdot \underset{\sim}{\boldsymbol{L}} - \boldsymbol{q} + \boldsymbol{m} \times \boldsymbol{n} = \mathbf{o}. \tag{3.18c}$$

where $\underset{\sim}{\boldsymbol{N}}, \underset{\sim}{\boldsymbol{L}}$ are the tensors of in-plane forces and out-of-plane moments, respectively, $\boldsymbol{q}$ is the vector of transverse shear forces, $\boldsymbol{m}$ is the surface moment vector (excluding drilling moments in line with the assumptions of a five degree of freedom continuum). The vector of surface forces $\boldsymbol{t}$ is split in two parts: $\boldsymbol{s}$ and $\boldsymbol{p}$, where the first one is the vector of tangent surface forces and the second one is the vector transverse surface loads.

$$\boldsymbol{t} = \boldsymbol{s} + \boldsymbol{p} \quad \text{with} \quad \boldsymbol{p} = p\boldsymbol{n} \quad \text{and} \quad \boldsymbol{s} = -s_\alpha \boldsymbol{e}_\alpha \tag{3.19}$$

It is worth mentioning that any general body forces are applied as surface loads to the planar structure ($\mathfrak{S}$):

$$\boldsymbol{t} := \int_{-\frac{H}{2}}^{+\frac{H}{2}} \rho \boldsymbol{b} \, \mathrm{d}X_3 = \boldsymbol{s} + \boldsymbol{p}. \tag{3.20}$$

where $\boldsymbol{t}$ is the general surface load (tangent and transverse) and ρ is the mass density of the body $\mathfrak{B}$. In a similar fashion, the moment of the body loads along directions 1 and 2 are applied as a surface moment vector (per unit area):

$$\boldsymbol{m} := -\int_{-\frac{H}{2}}^{+\frac{H}{2}} \rho X_3 \underset{\sim}{\boldsymbol{P}} \cdot \boldsymbol{b} \, \mathrm{d}X_3. \tag{3.21}$$

By excluding any mixed boundary conditions on the surface $\mathfrak{S}$, only independent Drichlet (on $\partial\mathfrak{S}_\text{D}$) and Neumann boundary conditions (on $\partial\mathfrak{S}_\text{N}$) are considered over the boundary of the domain $\partial\mathfrak{S}$:

$$\partial\mathfrak{S}_\text{D} \cup \partial\mathfrak{S}_\text{N} = \partial\mathfrak{S}, \qquad \partial\mathfrak{S}_\text{D} \cap \partial\mathfrak{S}_\text{N} = \varnothing. \tag{3.22}$$

where $\partial\mathfrak{S}_\text{D}$ and $\partial\mathfrak{S}_\text{N}$ are non-intersecting non-trivial subsets of the boundary $\partial\mathfrak{S}$ over which an outward normal direction vector $\boldsymbol{\upsilon}$ is defined, i.e., $\boldsymbol{\upsilon} \cdot \boldsymbol{n} = 0$. In addition, the arc-length variable ℓ parametrises the distance along the contour $\partial\mathfrak{S}$.

Consequently, the Drichlet boundary conditions consist of prescribed kinematic values for in-plane displacements $\boldsymbol{v}_\star(\ell)$, deflections $w_\star(\ell)$, and rotations $\boldsymbol{\varphi}_\star(\ell)$:

$$\text{on } \partial\mathfrak{S}_\mathrm{D}: \qquad \boldsymbol{v} \overset{!}{=} \boldsymbol{v}_\star(\ell), \tag{3.23a}$$

$$w \overset{!}{=} w_\star(\ell), \tag{3.23b}$$

$$\boldsymbol{\varphi} \overset{!}{=} \boldsymbol{\varphi}_\star(\ell), \tag{3.23c}$$

In addition, the Neumann boundary conditions consists of prescribed static values (distributed per unit length) for in-plane forces $\boldsymbol{s}_\star(\ell)$, transverse forces $\boldsymbol{p}_\star(\ell)$, and in-plane moments $\boldsymbol{m}_\star(\ell)$:

$$\text{on } \partial\mathfrak{S}_\mathrm{N}: \qquad \boldsymbol{\nu}\cdot\underset{\sim}{\boldsymbol{N}} \overset{!}{=} \boldsymbol{s}_\star(\ell), \tag{3.24a}$$

$$\boldsymbol{\nu}\cdot\boldsymbol{q} \overset{!}{=} p_\star(\ell), \tag{3.24b}$$

$$\boldsymbol{\nu}\cdot\underset{\sim}{\boldsymbol{L}} \overset{!}{=} \boldsymbol{m}_\star(\ell). \tag{3.24c}$$

3.3 Thermal Effects and Hygroscopic Impact

Environmental thermal cycles result in expansion/contraction of materials with non-zero coefficients of thermal expansion ($\alpha^\theta \neq 0$). A homogeneous temperature change in an unconstraint isotropic material results in a volumetric change, i.e., it nullifies the thermal curvature change tensor $\underset{\sim}{\boldsymbol{K}}^\theta = \underset{\sim}{\boldsymbol{0}}$ and thermal transverse shear vector $\boldsymbol{g}^\theta = \mathbf{o}$. Thus, the following Duhamel-Neumann form [18, 19, 20] can be written if only normal in-plane thermal strains are considered:

$$\underset{\sim}{\boldsymbol{N}} = \mathcal{A} : (\underset{\sim}{\boldsymbol{G}} - \underset{\sim}{\boldsymbol{G}}^\theta), \tag{3.25a}$$

$$\underset{\sim}{\boldsymbol{L}} = \mathcal{D} : \underset{\sim}{\boldsymbol{K}}, \tag{3.25b}$$

$$\boldsymbol{q} = \underset{\sim}{\boldsymbol{Z}}\cdot\boldsymbol{g}, \tag{3.25c}$$

where $\underset{\sim}{\boldsymbol{G}}^\theta$ is the tensor of thermal in-plane strains:

$$\underset{\sim}{\boldsymbol{G}}^\theta = \alpha^\theta \Delta\theta \underset{\sim}{\boldsymbol{P}}. \tag{3.26}$$

Polymeric materials can absorb moisture and undergo swelling. Such a volumetric change is similar to what happens during thermal expansion in the sense that both are considered to be dilatoric eigenstrains ($\boldsymbol{q}^\mu = \mathbf{o}$). By assuming a uniform through-thickness moisture absorption (Δm), an isotropic plane will experience no curvature change ($\underset{\sim}{\boldsymbol{K}}^{\mu} = \boldsymbol{0}$) but only planar eigenstrains:

$$\underset{\sim}{\boldsymbol{N}} = \mathcal{A} : (\underset{\sim}{\boldsymbol{G}} - \underset{\sim}{\boldsymbol{G}}^{\mu}), \tag{3.27a}$$

$$\underset{\sim}{\boldsymbol{L}} = \mathcal{D} : \underset{\sim}{\boldsymbol{K}}, \tag{3.27b}$$

$$\boldsymbol{q} = \underset{\sim}{\boldsymbol{Z}} \cdot \boldsymbol{g}, \tag{3.27c}$$

where $\underset{\sim}{\boldsymbol{G}}^{\mu}$ includes the in-plane moisture strains:

$$\underset{\sim}{\boldsymbol{G}}^{\mu} = \alpha^{\mu} \Delta m \underset{\sim}{\boldsymbol{P}}, \tag{3.28}$$

where α^{μ} is the coefficient of moisture expansion, and Δm is the change in the moisture mass (absorption/desorption) per unit mass of the plate.
It is noteworthy that the mechanical properties of the material are deemed independent of the hygrothermal effects. Moreover, this smeared representation does not include individual constituents (e.g., fibres within a matrix) and their mesoscopic structures. More importantly, the hygroscopic component does not entail any residual stresses due to violating the compatibility condition—hence, it is a stress-free eigenstrain, see [21, 22].

3.4 Residual Kinetics

Based on the concept of residual stresses, we also introduce residual kinetic quantities, i.e. in-plane forces, moments, and transverse shear forces. Such quantities may be present even in the absence of any external loading, thermal gradients etc. These quantities are often associated with plastic deformations, inhomogeneous microstructural transformations or hygrothermal effects.

In the context of present approach, a general set of incompatible strains results in residual fields for in-plane forces $\underset{\sim}{\boldsymbol{N}}^0$, moments $\underset{\sim}{\boldsymbol{L}}^0$, and transverse shear forces $\boldsymbol{q}^0$. In analogy to the problems of three-dimensional continua, these quantities can only be measured by the available non-destructive and destructive methods where the deformations corresponding to the residual stresses are measured. In order not to lapse into details, we refer to [23] or [24] and only disclose the transfer of measured residual Cauchy stresses $\underset{\sim}{\boldsymbol{T}}^0 = T^0_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ to the quantities introduced here. In this sense, the following residual forces and moments are introduced:

$$\underset{\sim}{\boldsymbol{N}}^0 = \int\limits_{-\frac{H}{2}}^{+\frac{H}{2}} \underset{\sim}{\boldsymbol{P}} \cdot \underset{\sim}{\boldsymbol{T}}^0 \cdot \underset{\sim}{\boldsymbol{P}} \, \mathrm{d}X_3, \tag{3.29a}$$

$$\underset{\sim}{\boldsymbol{L}}^0 = \int\limits_{-\frac{H}{2}}^{+\frac{H}{2}} X_3 \underset{\sim}{\boldsymbol{P}} \cdot \underset{\sim}{\boldsymbol{T}}^0 \cdot \underset{\sim}{\boldsymbol{P}} \, \mathrm{d}X_3, \tag{3.29b}$$

$$q^0 = \int_{-\frac{H}{2}}^{+\frac{H}{2}} \underset{\sim}{P} \cdot \underset{\sim}{T}^0 \cdot n \, \mathrm{d}X_3. \tag{3.29c}$$

The strain energy function now should be extended:

$$W(\underset{\sim}{G}, \underset{\sim}{K}, g) = \frac{1}{2} \left(\underset{\sim}{G} : \mathcal{A} : \underset{\sim}{G} + \underset{\sim}{K} : \mathcal{D} : \underset{\sim}{K} + g \cdot \underset{\sim}{Z} \cdot g \right) + \underset{\sim}{N}^0 : \underset{\sim}{G} + \underset{\sim}{L}^0 : \underset{\sim}{K} + q^0 \cdot g \tag{3.30}$$

Considering the constitutive equations originally introduced, we can now extend this set as follows.

$$\underset{\sim}{N} = \mathcal{A} : \underset{\sim}{G} + \underset{\sim}{N}^0 \tag{3.31a}$$

$$\underset{\sim}{L} = \mathcal{D} : \underset{\sim}{K} + \underset{\sim}{L}^0 \tag{3.31b}$$

$$q = \underset{\sim}{Z} \cdot g + q^0 \tag{3.31c}$$

3.5 Conclusion

In present treatise, the direct approach was adopted to set up the field equations of a generalized plate theory under various influences. More specifically, thermal and hygroscopic effects along with residual quantities are considered. When all the quantities taken into account are combined, the Duhamel-Neumann analogy takes subsequent form.

$$\underset{\sim}{N} = \mathcal{A} : (\underset{\sim}{G} - \underset{\sim}{G}^{\theta} - \underset{\sim}{G}^{\mu}) + \underset{\sim}{N}^0, \tag{3.32a}$$

$$\underset{\sim}{L} = \mathcal{D} : (\underset{\sim}{K}) + \underset{\sim}{L}^0, \tag{3.32b}$$

$$q = \underset{\sim}{Z} \cdot (g) + q^0. \tag{3.32c}$$

Obviously, hygroscopic and thermal loads effects the in-plane force tensor solely, at least in present case. Contrary, residual quantities effect all three states. The field equations of Eq. (3.18) my then be rearranged with the reduced forces and moments of Eqs. (3.32) to set up the linear hygro-thermo-static equilibrium with residual quantities. A concise visual representation can be found in Fig. 3.3.

To be honest, present execution concerns the simplest case of a two-dimensional theory where the reference surface halves the plate thickness. In absence of such geometrical symmetry, quantities like thermal moments etc. also occur then. Future work should take these quantities into account. Moreover, we are here limited to purely static considerations. In continuation, fluctuations, at least of temperature and moisture, should find entrance in the direct approach.

However, potential applications of present ansatz can be found in the analysis of composite structures as presented for example in [25].

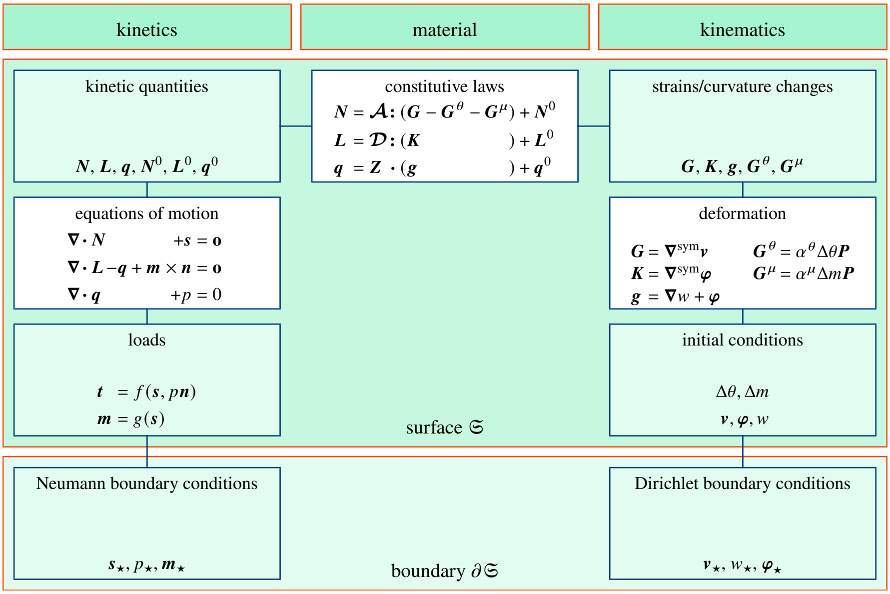

Fig. 3.3: Tonti diagram [26] for the composed plate problem of present treatise.

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