Chapter 2
Advance Approximate Analytical Solutions of the Contact Problem for an Inhomogeneous Layer

Sergei M. Aizikovich, Polina A. Lapina, and Sergei S. Volkov

Abstract The paper considers contact problems on the shear of the surface of an elastic layers that are inhomogeneous with depth and lie on a non-deformable and elastic base. Contact problems are reduced to solving integral equations. An analysis of the proximity of the kernel transforms of integral equations depending on the parameters of the problem is carried out. The solutions of integral equations are constructed by asymptotic methods, including the Wiener-Hopf method and the bilateral asymptotic method. The closeness of contact stresses is investigated. It is obtained, that when the value of the shear modulus at the lower boundary of the layer exceeds the value of the shear modulus at the layer surface by more than e^2 times, the divergence of contact stresses for layers on a non-deformable and elastic base does not exceed 5 percent.

Key words: Contact problem · Inhomogeneous material · Exponential shear modulus · Layer on the non-deformable base · Layer on the elastic base

2.1 Introduction

Inhomogeneous coatings of various structures have various applications and are of interest to researchers. The choice of adequate mathematical models for the calculation of inhomogeneous materials remains an important issue in modern mechanics [1]-[9].

The paper analyzes the equivalence of solutions for two models of a layer inhomogeneous with depth. The solution of the problems is carried out on the example of contact problems on the pure shear of the surface of an elastic layers inhomogeneous with depth on a non-deformable and elastic base. The exponential law of

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shear modulus change was chosen for the analysis, that is also considered in various studies [10]-[18]. The choice of the exponential law made it possible to obtain the kernels of the integral equations in an analytical form. Models of layers on a non-deformable and elastic base are close in a certain range of problem parameters. In this range, the proximity of solutions to contact problems is studied. The problems were solved approximately analytically using asymptotic methods using the simplest approximations of the kernel transforms of integral equations.

2.2 Statement of the Problem of a Shear of the Surface of an Inhomogeneous Layer

Let us consider the contact problem of pure shear of the surface of an inhomogeneous layer lying on a non-deformable or an elastic base by a strip punch with a flat base. The equation of the theory of elasticity in the case of antiplane deformation without taking into account the forces of friction has the form

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0, \tag{2.1}$$

where stresses $\sigma_{zx}(x,y), \sigma_{zy}(x,y)$ and deformation $w(x,y)$ are related by the relations

$$\sigma_{zx} = \mu(y)\frac{\partial w}{\partial x}, \quad \sigma_{zy} = \mu(y)\frac{\partial w}{\partial y}. \tag{2.2}$$

where $\mu(y)$ is shear modulus of a layer inhomogeneous by the vertical coordinate.

In the case of a layer lying on a non-deformable base (Fig. 2.1a), the boundary conditions of the contact problem have the form

$$w(x,0) = \varepsilon \quad |x| \le a, \qquad w(x,H) = 0 \quad |x| < \infty \tag{2.3}$$

$$\sigma_{yz}(x,0) = \begin{cases} -\varphi(x) & |x| \le a \\ 0 & a < |x| < \infty \end{cases} \tag{2.4}$$

where ε is the value, to which the punch is shifted, a is the half-width of the contact area, $\varphi(x)$ are contact stresses under the punch to be determined, H is the layer thickness.

A layer lying on an elastic base (Fig. 2.1b) is represented by an inhomogeneous half-space, the shear modulus of which is given as a piecewise function

$$\begin{cases} \mu(y) & 0 \le y \le H \\ \mu_\infty & H < y < \infty \end{cases} \tag{2.5}$$

where μ_∞ is elastic base shear modulus. In this case, the boundary conditions of the contact problem consist of conditions (2.3), (2.4) on the layer surface, to which the

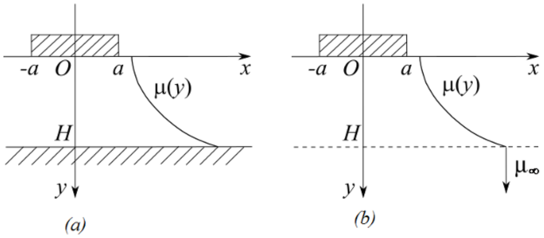

Fig. 2.1 Layer models on a non-deformable (a) and elastic (b) base.

condition of decreasing solution at infinity is added

$$\lim_{\sqrt{x^2+y^2}\to\infty}\left\{w, \frac{\partial w}{\partial x}\right\} = 0. \tag{2.6}$$

2.3 Integral Equations of Contact Problems

The paper considers the case when the shear modulus of an inhomogeneous layer changes according to the exponential law

$$\mu(y) = \mu_0 e^{2dy}, \quad 0 \le y < \infty, \tag{2.7}$$

where μ_0 is the value of the shear modulus on the upper surface of the inhomogeneous layer, d is the value that characterizes the rate of change in the shear modulus with depth.

In this case, in problem (b), the shear modulus of the elastic base is $\mu_\infty = \mu_0 e^{2dH}$, and the shear modulus of the elastic base is e^{2dH} times larger than the shear modulus of the layer on the surface ($\mu_\infty/\mu_0 = e^{2dH}$). For the exponential law of the shear modulus change, it is possible to construct solutions to contact problems in an analytical form. Using the integral Fourier transform, the solutions of problems are reduced to solving integral equations for unknown contact stresses.

For the problem (a), the integral equation in dimensionless form has the form

$$\int_{-1}^{1} \varphi(\xi)\mathrm{d}\xi \int_{-\infty}^{\infty} K(\alpha)e^{i\alpha\frac{\xi-x}{\lambda}}\mathrm{d}\alpha = 2\pi a^{-1}\varepsilon\mu_0, \quad |x| \le 1, \quad \lambda = \frac{H}{a}, \tag{2.8}$$

$$K(\alpha) = \frac{1}{d_0 + V\coth V}, \quad V = \sqrt{d_0^2+\alpha^2}, \quad d_0 = Hd \tag{2.9}$$

Integral equation (2.8) is an integral equation of the Fourier convolution type of the first kind with a difference kernel with respect to unknown contact stresses $\varphi(\xi)$. The transform $K(\alpha)$ has the following asymptotic properties

$$K(\alpha) = |\alpha|^{-1} + O(\alpha^{-2}) \text{ at } |\alpha| \to \infty, \quad K(\alpha) = K(0) + O(\alpha^2) \text{ at } |\alpha| \to 0, \tag{2.10}$$

When constructing analytical solutions of integral equation (2.8), asymptotic methods with respect to the dimensionless parameter λ are used in a similar way to [15]-[17] for the case of an inhomogeneous half-space. For small values of the parameter $\lambda \in (0, \lambda_0)$ the solution of the integral equation is constructed in the framework of the Wiener-Hopf method [19]. The solution is written as the zero term of the Neumann series with further use of the simplest approximation of the kernel transform $K(\alpha)$ in the following form

$$K(\alpha) = \frac{1}{\sqrt{A^2+\alpha^2}} \frac{B^2+\alpha^2}{C^2+\alpha^2}, \quad K(0) = \frac{1}{A} \frac{B^2}{C^2} \tag{2.11}$$

For large values of the parameter $\lambda \in (\lambda_0, \infty)$, as in [20], the solution of the integral equation is written as a double functional series in powers of the parameter λ.

In the case of problem (b), the integral equation for unknown contact stresses according to [18] is written in the same form as (2.8), (2.9), taking into account the notation

$$K(\alpha) = \frac{1}{|\alpha|} L(\alpha) \tag{2.12}$$

The function $L(\alpha)$ is obtained analytically and has the form

$$L(\alpha) = \frac{(|\alpha| - d_0)\sinh V + V \cosh V}{(|\alpha| + d_0)\sinh V + V \cosh V}, \quad V = \sqrt{d_0^2 + \alpha^2}, \quad d_0 = Hd \tag{2.13}$$

The function $L(\alpha)$ given by expression (2.13) has the following asymptotic properties

$$L(\alpha) = 1 + O(|\alpha|^{-1}) \text{ at } |\alpha| \to \infty, \quad L(\alpha) = L(0) + O(|\alpha|) \text{ at } |\alpha| \to 0. \tag{2.14}$$

When solving the integral equation of problem (b), the bilateral asymptotic method [21]-[24] is used. The kernel transform is approximated by the following product

$$L(\alpha) = L_\Pi^N(\alpha) \equiv \prod_{i=1}^{N} \frac{\alpha^2 + A_i^2}{\alpha^2 + B_i^2}; (B_i - B_k)(A_i - A_k) \neq 0 \text{ where } i \neq k \tag{2.15}$$

Based on the proposed approximation, an approximate analytical solution of the integral equation is constructed, which is effective over the entire range of values of the dimensionless geometric parameter λ.

2.4 Numerical Analysis

Figures 2.2-2.4 show the kernel transforms of integral equations for two models of a layer that is inhomogeneous with depth at different values of the parameter $d_0 = 2, 1, 0.5$, which corresponds to cases where the shear modulus at the lower boundary of the layer (the shear modulus of the elastic base in problem b) is e^4, e^2, e times greater than the shear modulus at the layer surface respectively. For model (a), the graphs show functions $|\alpha|K(\alpha)$, for model (b), the graphs show functions $L(\alpha)$. As the α grows, the graphs $|\alpha|K(\alpha)$ for the model (a) and the graphs $L(\alpha)$ for the model (b) begin to coincide and tend to the value 1 at $\alpha \to \infty$. We consider $d_0 > 0$, which means that the shear modulus increases with depth. At $d_0 \to +0$, the inhomogeneous layer transforms into a homogeneous one. In this case, as the parameter d_0 decreases, models (a) and (b) cease to be similar, which is confirmed by the graphs of the transforms in Figs. 2.2-2.4 .

For values of the shear modulus parameter d_0, at which the relative difference between the kernel transforms $|\alpha|K(\alpha)$ and $L(\alpha)$ of the integral equations of problem (a) and problem (b) is small, the contact stresses are constructed. Note that thin layers are of interest from the point of view of applications. Figures 2.5 and 2.6 show the relative contact stresses $a\varepsilon^{-1}\mu_0^{-1}P^{-1}\varphi(\xi)$ for different values of the parameters d_0 and λ. Here P is the force acting on the punch. In the case of the model (a), the solution is constructed by the Wiener-Hopf method, which corresponds to the case of small values of the dimensionless parameter $\lambda \in (0, \lambda_0)$ (thin layers). In the case of the model (b), the solution is constructed by the bilateral asymptotic method.

Numerical results confirm the coincidence of contact stresses for the two layer models (on a non-deformable and elastic base) in the case when the problem parameters are such that the models are physically close to each other. Thus, at $d_0 \geq 1$, when the value of the shear modulus at the lower boundary of the layer exceeds the value of the shear modulus at the upper boundary of the layer by more than a e^2 times, the contact stresses constructed for the case of a layer on a non-deformable base are close to the contact stresses constructed for the case of a layer on an elastic base, and the error between solutions does not exceed 5 percent.

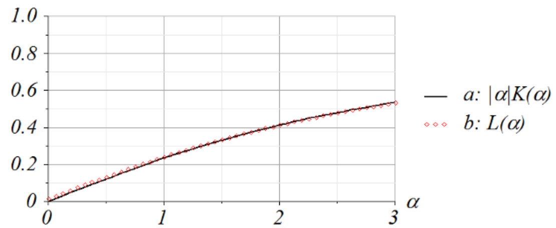

Fig. 2.2 $|\alpha|K(\alpha)$ for model (a) and $L(\alpha)$ for model (b) at $d_0 = 2, \mu_\infty/\mu_0 = e^4$, the solid black line - problem (a), red dots - problem (b).

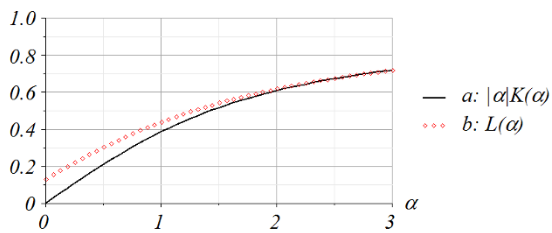

Fig. 2.3 $|\alpha|K(\alpha)$ for model (a) and $L(\alpha)$ for model (b) at $d_0 = 1, \mu_\infty/\mu_0 = e^2$, the solid black line - problem (a), red dots - problem (b).

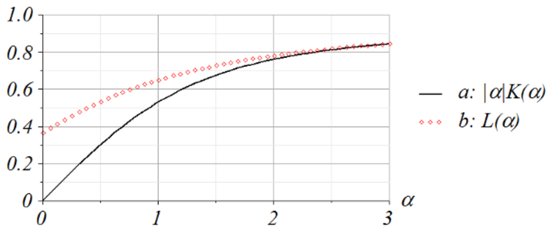

Fig. 2.4 $|\alpha|K(\alpha)$ for model (a) and $L(\alpha)$ for model (b) at $d_0 = 0.5, \mu_\infty/\mu_0 = e$, the solid black line - problem (a), red dots - problem (b).

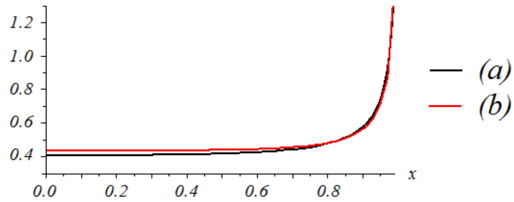

Fig. 2.5 Relative contact stresses $a\varepsilon^{-1}\mu_0^{-1}P^{-1}\varphi(\xi)$ at $d_0 = 1, \lambda = 1$, the solid black line - problem (a), the solid red line - problem (b).

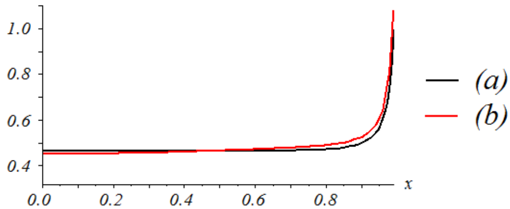

Fig. 2.6 Relative contact stresses $a\varepsilon^{-1}\mu_0^{-1}P^{-1}\varphi(\xi)$ at $d_0 = 2, \lambda = 0.5$, the solid black line - problem (a), the solid red line - problem (b).

2.5 Closure

A comparative analysis of the proximity of solutions and the equivalence of two contact problems on the shear of the surface of an elastic layers inhomogeneous with depth on a non-deformable and elastic base is carried out. For the values of the problem parameter, when the value of the shear modulus at the lower boundary of the layer exceeds the value of the shear modulus at the layer surface by more than e^2 times, the error between solutions does not exceed 5 percent. In the further studies the solutions are planned to be used for the interpretation of the experimental data, including shear of the coatings [25]-[27], porous materials [28] and scaffold walls [29] in the course of in situ tests.

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