Chapter 19
Multistability of Convective Flows in a Porous Enclosure

Vasily Govorukhin, Mezhlum Sumbatyan, and Vyacheslav Tsybulin

Abstract We discuss multistability for the problem of convection in a porous medium. Nontrivial phenomena of extreme multistability arise in the mathematical model of the Darcy convection with a linear temperature profile on the boundary. It manifests in the appearance of one-parameter families of steady states. An illustrative example concerns the anisotropic convection problem. We review here some issues of transition from extreme to standard multistability. Then we describe numerical and algorithmic aspects of the extreme multistability and provide necessary references.

Key words: Convection · Porous medium · Darcy-Boussinesq equations · Anisotropy · Multistability · Cosymmetry

19.1 Introduction

Investigation of the convective motions of fluids in porous media is an important subject of interest to many researchers [1]-[4]. Geophysics and some branches of industry require new models, to describe various observed phenomena [5, 6]. Nonlinearity of the corresponding mathematical models typically leads to the multistability – a phenomenon of coexistence of different solutions.

Often, multistability is a consequence of some symmetry group in a system. Existence of a one–parameter family of solutions (extreme multistability) may be caused by a continuous symmetry group (translational or rotational invariance). Another reason for this phenomena is cosymmetry, which was introduced by Yudovich, to explain the existence of one–parameter family of steady states in the problem with no continuous symmetry [7]-[9]. Such an extreme multistability was found in the

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H. Altenbach et al. (eds.), *Sixty Shades of Generalized Continua*,
Advanced Structured Materials 170, https://doi.org/10.1007/978-3-031-26186-2_19

planar Darcy convection problem [7], [10]-[12]. The cosymmetry is a vector field which is orthogonal to a given vector field in a Hilbert space H. For example, cosymmetry of a differential equation $\dot{u} = \Phi(u)$, $u \in R^n$ is a nontrivial vector field $L(u)$, such that $(\Phi(u), L(u)) = 0$ at each point of R^n. In [7] there are formulated the conditions for existence of the family of equilibria in the system with cosymmetry. Assume u_0 to be a solution of $\Phi(u_0) = 0$, and $J = \nabla\Phi$ is the Jacobian matrix. Let the zero be a simple eigenvalue of the operator $J(u_0)$, and ϕ_0 is the basis vector in $\ker J(u_0)$. Then the given differential equation has the one–parameter family of solutions: $u_\alpha = u_0 + \alpha\phi_0 + O(\alpha^2)$, for a sufficiently small θ. This family of the equilibria cannot be a result of any symmetry group [8]. Moreover, the stability spectrum of the equilibria u_α depends on parameter α. The cosymmetry differs from the symmetry case, where the spectrum is identical for all equilibria. The nontrivial cosymmetry indicates that the system has a hidden free parameter, i.e. it is underdetermined.

An analytic study of the one-parameter cosymmetric family of steady-state regimes is possible only for those values of parameters which are close to a bifurcation in its appearance [7, 8]. For other situations only numerical approaches can be used as a rule. Some special numerical methods, to study cosymmetric dynamical systems, were developed in [13, 14] and continued in [14]-[25].

Yudovich proposed a selective function for studying the situation when the cosymmetry does not takes place and the continuous families of steady states break down or disappear [9]. Two scenarios are possible for a family of equilibria: the disinteg,ration of the family to a finite number of equilibria (ordinary multistability) or the appearance of slow periodic motions. The destruction of the family of steady states was investigated in [10, 19, 26, 27]. There is observed a memory effect about broken cosymmetry: relaxation oscillations and slow dynamics in a vicinity of the disappeared family of equilibria [19, 28].

Let us present a short survey of some results in this field, where authors' papers play a key role. We outline the numerical approaches for investigation both cosymmetric and close to cosymmetric problems. This permits the study of scenarios of bifurcations for some families [29, 30]. A spectral-difference method was derived for the convection in the rectangle [20]. The convection of a multi-component fluid is studied in [31]. A discretization in cylindrical coordinates is applied in [25, 32, 33]. A three–dimensional problem is studied in [24, 34, 35]. An anisotropy in a gravitational convection for rectangular enclosure is analyzed in [15, 26, 36, 37]. A selection of states under extreme multistability is considered in [17, 19, 38].

We consider the gravitational convection in a porous medium saturated by the heat-conducting fluid. The enclosure is heated from below. We are interested in an appearance of convective flows when the quiescent state (mechanical equilibrium) loses its stability. The resulting system is cosymmetric under some additional conditions for the parameters of the problem. It means that the continuous family of stationary regimes branches off from the quiescent state. After discretization with preserving the cosymmetry we obtain a system of ordinary differential equations. To compute the extreme multistability, a special continuation procedure is developed.

19.2 Mathematical Formulation of the Problem

Let us consider the problem of heating a rectangular container

$$\Omega = [0,a] \times [0,b],$$

see Fig. 19.1. Let us set the impermeability conditions on the boundary $\partial\Omega$, as well as a linear temperature profile. The gravity acts in the direction opposite to coordinate y:

$$\begin{aligned} u(0,y,t) &= u(a,y,t) = 0, \\ v(x,0,t) &= v(x,b,t) = 0, \\ T_*(y) &= T_0 + A - \frac{y}{b}A, \end{aligned}$$

where u, v are the horizontal and vertical velocities respectively, T_0 and $T_0 + A$ correspond to the temperature at the upper ($y = b$) and lower ($y = 0$) boundaries. The following system describes anisotropic convection in the porous medium:

$$0 = \frac{\partial p}{\partial x} + \mu(M_{11}u + M_{12}v), \tag{19.1}$$

$$0 = \frac{\partial p}{\partial y} + \mu(M_{21}u + M_{22}v) - \rho_0\beta(T - \hat{T})g, \tag{19.2}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \tag{19.3}$$

$$\begin{aligned} (\rho c_p)_m \frac{\partial T}{\partial t} &= -(\rho c_p)_f \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial x}\left(\Lambda_{11}\frac{\partial T}{\partial x} + \Lambda_{12}\frac{\partial T}{\partial y} \right) \\ &+ \frac{\partial}{\partial y}\left(\Lambda_{21}\frac{\partial T}{\partial x} + \Lambda_{22}\frac{\partial T}{\partial y} \right). \end{aligned} \tag{19.4}$$

Here t is time, x, y are the Cartesian coordinates, p is the pressure, T is the temperature, μ is the fluid viscosity, ρ is the density, g is the gravity acceleration, Λ_{ij} and M_{ij} are the components of tensors of the temperature conduction and the inverse permeability,

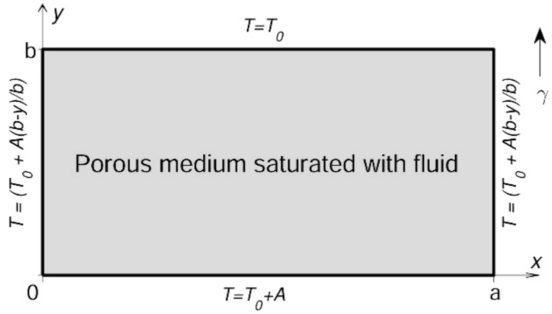

Fig. 19.1 Flow region

correspondingly. Then, ρ_0 is the density at the reference temperature T_0, the parameter $(\rho c_p)_m$ characterizes the product of the density by the specific heat at constant pressure averaged for the fluid and the porous medium, and $(\rho c_p)_f$ is an analogous characteristics for the fluid; β is the thermal expansion coefficient.

We introduce the deviation of the temperature θ as

$$T(x,y,t) = T_*(y) + \theta(x,y,t)$$

and dimensionless variables and parameters, as follows:

$$t = t_*\tilde{t}, \quad x = h_*\tilde{x}, \quad y = h_*\tilde{y}, \quad u = v_*\tilde{u}, \quad v = v_*\tilde{v}, \quad \theta = \tau_*\tilde{\theta},$$
$$\hat{P} = P_*\tilde{P}, \quad \hat{P} = p - \int_0^y \rho_0 \beta g (T_* - \hat{T}) dy, \quad M_{ij} = M_{22}\mu_{ij}, \quad d_{ij} = \frac{1}{\Lambda_{22}} \Lambda_{ij}.$$

The physical characteristics of length h_*, velocity v_*, pressure P_*, time t_*, and temperature τ_* are, respectively:

$$h_* = b, \quad v_* = \frac{\Lambda_{22}}{(\rho c_p)_f b}, \quad P_* = \frac{\mu \Lambda_{22} M_{22}}{(\rho c_p)_f} \quad \tau_* = A t_* = \frac{b^2 (\rho c_p)_m}{\Lambda_{22}} \tag{19.5}$$

The Rayleigh number λ is defined by the formula

$$\lambda = \frac{(\rho c_p)_f A b \rho_0 \beta g}{M_{22} \mu \Lambda_{22}}\ . \tag{19.6}$$

As a result, for dimensionless quantities (the “tilde” sign above the variables is omitted), the following system is obtained:

$$\begin{aligned} \frac{\partial p}{\partial x} + \mu_{11} u + \mu_{12} v \qquad &= 0, \\ \frac{\partial p}{\partial y} + \mu_{21} u + \mu_{22} v - \lambda \theta &= 0. \end{aligned} \tag{19.7}$$

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \qquad &= 0, \\ \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} - v &= L_D \theta, \end{aligned} \tag{19.8}$$

Here operator L_D contains the dimensionless thermal conductivity coefficients d_{ij}, being given by the formula

$$L_D = \frac{\partial}{\partial x}\left(d_{11}\frac{\partial}{\partial x} + d_{12}\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial y}\left(d_{21}\frac{\partial}{\partial x} + d_{22}\frac{\partial}{\partial y}\right). \tag{19.9}$$

The resulting system of equations for velocity, pressure, and temperature deviation can be written in the vector form, as follows:

$$\nabla p + \boldsymbol{M} \cdot \boldsymbol{V} + \lambda\theta\boldsymbol{\gamma} = 0, \tag{19.10}$$

with

$$\nabla \cdot \boldsymbol{V} = 0$$

and

$$\dot{\theta} = L_D\theta - \boldsymbol{V} \cdot \boldsymbol{\gamma} - (\boldsymbol{V} \cdot \nabla)\theta. \tag{19.11}$$

Here dot-sign denotes differentiation with respect to time t, $\boldsymbol{M}$ represents the tensor of dimensionless inverse permeability coefficients μ_{ij}, $\boldsymbol{\gamma} = (0, -1)$ is a unit vector which determines the direction of the gravity.

These equations are supplemented with the boundary conditions of impermeability and homogeneity of the temperature deviation at the boundary $\partial\Omega$

$$\begin{aligned} u(0,y,t) &= u(a,y,t) = 0, \\ v(x,0,t) &= v(x,b,t) = 0, \end{aligned} \tag{19.12}$$

$$\theta(x,0,t) = \theta(0,y,t) = \theta(a,y,t) = \theta(x,b,t) = 0. \tag{19.13}$$

To analyze the two-dimensional problem, it is convenient to introduce the stream function ψ by $u = \psi_y, v = -\psi_x$, to come to a system for two unknowns functions: stream function ψ and temperature θ (temperature deviation). By applying differentiation to first equation (19.7) with respect to y, and to second one – with respect to x, and then taking their difference, one obtains the following equation:

$$0 = \mu_{11}\psi_{yy} - (\mu_{12} + \mu_{21})\psi_{xy} + \mu_{22}\psi_{xx} + \lambda\theta_x, \tag{19.14}$$

Besides, Eq. (8) is reduced to:

$$\dot{\theta} = L_D\theta - \psi_x - J(\psi,\theta), \quad J = \theta_x\psi_y - \theta_y\psi_x \tag{19.15}$$

The boundary conditions follow from (19.12):

$$\psi(x,0,t) = \psi(0,y,t) = \psi(a,y,t) = \psi(x,b,t) = 0. \tag{19.16}$$

The system for the isotropic porous medium follows from equations (19.14)-(19.16), (19.9) with

$$\mu_{ii} = d_{ii} = 1, (i = 1,2)$$

and

$$\mu_{ij} = d_{ij} = 0, (i \neq j).$$

It is shown in [7] that in this case the problem has a cosymmetry, and there may arise a family of stationary convective motions.

The following properties of the Jacobian $J(\psi,\theta)$ hold

$$\int_\Omega J(\psi,\theta)\psi dxdy = 0, \quad \int_\Omega J(\psi,\theta)\theta dxdy = 0. \tag{19.17}$$

To discretize the equations (19.14)-(19.15) properly, we must preserve the discrete analogues of the integral identities (19.17). The numerical schemes constructed appear to be conservative.

The system (19.14)-(19.15) is invariant under transformation

$$R_{xy} : \{x, y, \psi, \theta\} \mapsto \{a - x, b - y, \psi, -\theta\}.$$

If $\mu_{12} = \mu_{21} = d_{12} = d_{21} = 0$, then there is also an invariance

$$R_x : \{x, y, \psi, \theta\} \mapsto \{a - x, y, -\psi, \theta\}.$$

To analyze convective motions, system (19.9), (19.14)-(19.16) must be supplemented with the initial temperature distribution

$$\theta(x, y, 0) = \theta_0(x, y). \tag{19.18}$$

The initial condition for the stream function is not required, and the corresponding distribution for $t = 0$ is found from problem (19.14), (19.16) applied to the given function $\theta_0(x, y)$. The resulting initial-boundary problem describes the non–stationary gravitational anisotropic convection of a heat-conducting fluid in a porous rectangle heated from below.

The formulated problem demonstrates nontrivial effects of thermal and mass transfer. In the isotropic case, the onset of the convection is accompanied with an extreme multistability. Such phenomena make clear that there appears a one-parameter continuum family of the steady convective regimes. It means that there exist a hidden parameter in the system. The reason for this is the cosymmetry, the conditions for its existence are established in [7].

In the anisotropic case, the cosymmetry is the vector function

$$L = (\theta, -\psi)$$

and this exists if the following conditions are fulfilled

$$\mu_{11} = d_{22}, \quad \mu_{22} = d_{11}, \quad \mu_{21} = -d_{21}, \quad \mu_{12} = -d_{12}, \tag{19.19}$$

see [36]. In primitive variables the cosymmetry is given by the vector $(\theta_y, -\theta_x, 0, \psi)$.

If the conditions for parameters (19.19) are violated, then vector-function

$$L = (d_{22}\theta, -\mu_{11}\psi)$$

is not a cosymmetry. As a result, only the finite number of convective regimes (steady or time-dependent) exist, instead of the one-parametric family. To analyze this case, Yudovich developed the selective function techniques [9]. In the considered problem, the selective function has the form

$$S(s) = \int_\Omega \psi(s)\theta(s)(d_{22}\mu_{22} - \mu_{11}d_{11})dxdy. \tag{19.20}$$

Here s is a hidden parameter, $\psi(s)$ and $\theta(s)$ are the members of a family. Thus, for the analysis of destruction, one needs to calculate solutions for any s.

19.3 Numerical Methods and Extreme Multistability

Numerical analysis of cosymmetric and near-cosymmetric systems results in specific computational problems. First, the numerical method should preserve the fundamental properties of the problem. This includes preserving cosymmetry through discretization of the partial differential equations. Violation of the cosymmetry by discretization can lead to a wrong behavior, such as destruction of the family of steady states. Another challenging problem arises, due to an inability to apply standard methods to study cosymmetric problems because of its strong degeneration [39, 40].

Based on the Galerkin method, the study of the continuum family of stationary solutions in isotropic convection problems in porous media has been started in [16, 17]. To analyze dynamics in a wide range of parameters, some special methods for calculation of convective steady states and continuation along a hidden parameter are developed in [18, 41, 42]. Subsequently, finite-difference and spectral-difference approaches preserving cosymmetry were derived in [21]-[23].

With all above mentioned variants of discretization, one obtains a system of ordinary differential equations in the form

$$\dot{\Theta} = \Phi(\Theta) + \delta K(\Theta) \tag{19.21}$$

The right-hand side of this equation contains both cosymmetric (Φ) and noncosymmetric (δK) parts. Vector $\Theta(t)$ is a discrete approximation of $\theta(x, y, t)$. Cosymmetry $\hat{L}$ for Φ follows from the approximation of L. So, the case $\delta = 0$ corresponds to a cosymmetric finite-dimensional problem. Further, we define a discrete version of the selective function as $\hat{S} = (K, \hat{L})$.

19.3.1 Spectral Global Galerkin Method

Let us briefly describe the Galerkin method for analysis of problem (19.14)–(19.15), see [17, 42]. Functions ψ and θ are approximated by the series

$$\psi = \sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{i,j}(t)\phi_{i,j}(x, y), \quad \theta = \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_{i,j}(t)\phi_{i,j}(x, y), \tag{19.22}$$

with

$$\phi_{i,j}(x, y) = \frac{2}{\sqrt{ab}} \sin\left(i\frac{\pi x}{a}\right) \sin\left(j\frac{\pi y}{b}\right)$$

Substitution of (19.22) to (19.14)–(19.15) and projection operations lead to a system of ODEs of the order $N = n^2$ in form (19.21). Here

$$\Theta = (\theta_{11}, \theta_{12}, \ldots, \theta_{1n}, \theta_{21}, \ldots, \theta_{nn})$$

Similarly, one obtains an approximating system for the selective function (19.20). It is easy to prove that approximation (19.22) preserves the cosymmetry of the isotropic problem.

19.3.2 Cosymmetry Preserving Finite-Difference Approximations

The approximation of the problem in primitive variables (19.9)-(19.11) is carried out based on the staggered grids. The implementation for the isotropic case is presented in [22]. Firstly, we introduce the mesh on coordinates x and y:

$$0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = a, 0 = y_0 < y_1 < \ldots < y_m < y_{m+1} = b.$$

The nodes (x_i, y_j) form the main grid on which the temperature θ_i^j is determined:

$$\omega_\theta = \{ (x_i, y_j), \quad i = 0, \cdots, n+1, \;\; j = 0, \cdots, m+1 \}.$$

We place auxiliary nodes in the middle of the intervals formed by neighboring nodes of the main grid

$$x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1}), \quad y_{j+1/2} = \frac{1}{2}(y_j + y_{j+1}).$$

As a result, three additional grids are obtained, offset relative to the main one:

$$\begin{aligned}
\omega_u &= \{ (x_i, y_{j+1/2}) \, , \; i = 0, \cdots, n+1, \;\; j = 0, \cdots, m \}, \\
\omega_v &= \{ (x_{i+1/2}, y_j) \, , \; i = 0, \cdots, n, \;\; j = 0, \cdots, m+1 \}, \\
\omega_p &= \{ (x_{i+1/2}, y_{j+1/2}) \, , \; i = 0, \cdots, n, \;\; j = 0, \cdots, m \}.
\end{aligned}$$

The horizontal and vertical components of the velocity vector are calculated at grids ω_u and ω_v, and the pressure at node ω_p is calculated. The following notation is used for grid variables:

$$\begin{aligned}
u_i^{j+1/2} &= u(x_i, y_{j+1/2}), \quad v_{i+1/2}^j = u(x_{i+1/2}, y_j), \\
p_{i+1/2}^{j+1/2} &= u(x_{i+1/2}, y_{j+1/2}), \quad \theta_i^j = u(x_i, y_j).
\end{aligned}$$

To approximate (19.9)-(19.11), discrete analogues of differential operators are used. First-order difference operators are introduced on the two-point stencil

$$(\delta_x f)_{i+1/2}^{j+1/2} = \frac{f_{i+1}^{j+1/2} - f_i^{j+1/2}}{x_{i+1} - x_i}, \tag{19.23}$$

$$(\delta_y f)_{i+1/2}^{j+1/2} = \frac{f_{i+1/2}^{j+1} - f_{i+1/2}^{j}}{y_{j+1} - y_j}. \tag{19.24}$$

and operators for calculating the weighted average on the interval

$$(\delta_0^x f)_{i+1/2}^{j+1/2} = \frac{(x_{i+1} - x_{i+1/2}) f_{i+1}^{j+1/2} + (x_{i+1/2} - x_i) f_i^{j+1/2}}{x_{i+1} - x_i}, \tag{19.25}$$

$$(\delta_0^y f)_{i+1/2}^{j+1/2} = \frac{(y_{j+1} - y_{j+1/2}) f_{i+1/2}^{j+1} + (y_{j+1/2} - y_j) f_{i+1/2}^{j}}{y_{j+1} - y_j}. \tag{19.26}$$

Formulas (19.23)-(19.26) are valid for integer and half-integer values of indices i and j. With the help of the introduced operators, the difference relations on the three-point templates are determined:

$$(D_x f)_i^j = \left(\delta_x \delta_0^x f\right)_i^j, \quad (D_y f)_i^j = \left(\delta_y \delta_0^y f\right)_i^j.$$

Using the introduced operators, differential operator L_D (19.9) is approximated as follows:

$$L_D^h = \delta_x d_{11} \delta_x + d_x d_{12} d_y + d_y d_{21} d_x + \delta_y d_{22} \delta_y, \tag{19.27}$$

since

$$d_x d_y = D_x D_y$$

and the operators d_x, d_y and D_x, D_y are commutative. It should be noted that for the calculation of terms with d_{11} and d_{22}, respective operators on three-dot templates are used, and the use of operators $d_x d_x$ and $d_y d_y$ leads to the loss of cosymmetry for the grid analogue of the problem.

Next, first-order difference derivatives and operators for calculating the average on a rectangular template are introduced:

$$d_0 = \delta_0^x \delta_0^y \equiv \delta_0^y \delta_0^x, \quad d_x = \delta_x \delta_0^y \equiv \delta_0^y \delta_x, \quad d_y = \delta_y \delta_0^x \equiv \delta_0^x \delta_y. \tag{19.28}$$

Due to the non-uniformity of the grid, the operators on the rectangles are different for half-integer and integer index values, see [23].

To approximate the Jacobian J, a linear combination

$$(1-\alpha) J_D + \alpha J_d$$

of two discrete analogs of J is used

$$J_D = D_x(\theta D_y \psi) - D_y(\theta D_x \psi), \quad J_d = d_x(d_0 \theta d_y \psi) - d_y(d_0 \theta d_x \psi). \tag{19.29}$$

The value of the parameter $\alpha = 1/3$ makes it possible to satisfy the cosymmetry.

To approximate the problem (19.14)–(19.15), we use the approach [21, 23]. There are introduced the uniform grids

$$x_i = ih, i = 0, 1, \ldots, n+1, h = a/(n+1), y_j = jg, j = 0, 1, \ldots, m+1, g = b/(m+1),$$

with respective values of the stream functions ψ_{ji} and the temperatures θ_{ji} at gridpoints (x_i, y_j).

The resulting system of ordinary differential equations can be written as:

$$\dot{\Theta} = A\Theta + B\Psi - F(\Theta, \Psi), \quad \Psi = \lambda A^{-1} B\Theta \tag{19.30}$$

where the block-three-diagonal matrix A corresponds to the approximation of the Laplacian, and matrix B corresponds to the differential operator of the first order. The nonlinear vector-function $F(\Theta, \Psi)$ comes from a finite-difference approximation of the Jacobian. The problem at hand can also be reduced to (19.21).

19.3.3 Continuation on the Hidden Parameter Method

The continuation method is based on the cosymmetric implicit function theorem, see [43]. The first version of this method is presented in [14]. The one-parameter family of steady state regimes corresponds to the curve of equilibria $\hat{\Theta}(s)$ (here s is a hidden parameter) of approximating system of ordinary differential equations (19.21). This equilibrium curve may be continued with respect to parameter s from the point $\hat{\Theta}_0$, solving the following Cauchy problem

$$\frac{d\hat{\Theta}(s)}{ds} = \phi(s), \qquad \hat{\Theta}(0) = \hat{\Theta}_0 \tag{19.31}$$

Here $\phi(s)$ is a vector from the kernel of the linearized operator $\Phi'\hat{\Theta}(s)$.

Let us briefly describe the algorithm step by step, as follows:

1. Find any point $\hat{\Theta}_0$ on the curve by the modified Newton method.
2. Calculate the kernel of Jacobi matrix $\Phi'\hat{\Theta}_0$ (SVD method), check whether this is not degenerated (otherwise all next calculations are impossible) and choose the direction of the continuation along the curve.
3. Do one step of the Runge-Kutta method for the problem (19.31), where respective right-hand-side is an eigenvector corresponding to a trivial eigenvalue of $\Phi'\hat{\Theta}(s)$.
4. Accuracy checking. If the accuracy of calculations is not satisfactory, correct the solution.
5. Calculation of the selection function $S(s)$, stability analysis, verification of the exit conditions, then go to step 3 to find next point on the equilibria curve.

The described algorithm was applied to analyze a number of problems of filtration convection with various functions $K(\Theta)$.

19.4 Multistability

When conditions (19.19) are satisfied, the extreme multistability occurs simultaneously with convection. This results in the formation of a family of stationary states, see Fig. 19.2. Firstly, all stationary regimes are stable, but differ by the flow structure and heat distribution. At the same time, selection mechanisms, i.e. realizations of certain convective states, are very complicted, see [42]. As the Rayleigh parameter increases, the family of stationary flows becomes more complex: the structure of streamlines, the stability spectrum, etc. A further increase in heating leads to the appearance of instability on the family, see the right side of Fig. 19.2. Unstable states can form arcs, see [16]-[18], [20].

In the presence of heat sources inside and at the boundary, the fluid penetrates through the boundaries, with other influences, that leads to violation of the cosymmetry. In such cases the extreme multistability is lost [9, 19, 38]. Here, different scenarios for the transformation of convective movements are possible. The most typical is a multistability, which is expressed in the coexistence of a finite set of convective regimes. The selective function technique [9] allows one to determine the number of remaining regimes. An example of the destruction of the family is the inclusion of internal heat sources [19]. In Fig. 19.2 the signs on the central curves mark the regimes which are implemented for various options, if placing heat sources inside the region. With so doing, bold signs correspond to stable states.

The number of regimes which remain, when the cosymmetry is broken, can be different. It depends on the type of perturbation and the Rayleigh number. Figure 19.3 shows the results of exploring selective function (19.20) when the conditions (19.19) are violated. The analysis shows that there remain eight different convective regimes in this case. Thus, the presented examples show the transition from extreme to ordinary multistability.

Acknowledgements This research was carried out in the Laboratory of Computational Mechanics of Southern Federal University, with a financial support of the Government of the Russian Federation (Contract no. 075-15-2019-1928).

Besides, the authors are deeply grateful to Professor Victor Eremeyev for many years of scientific friendship: positive examples, good advice, kind help, as well as for creating an atmosphere of goodwill.

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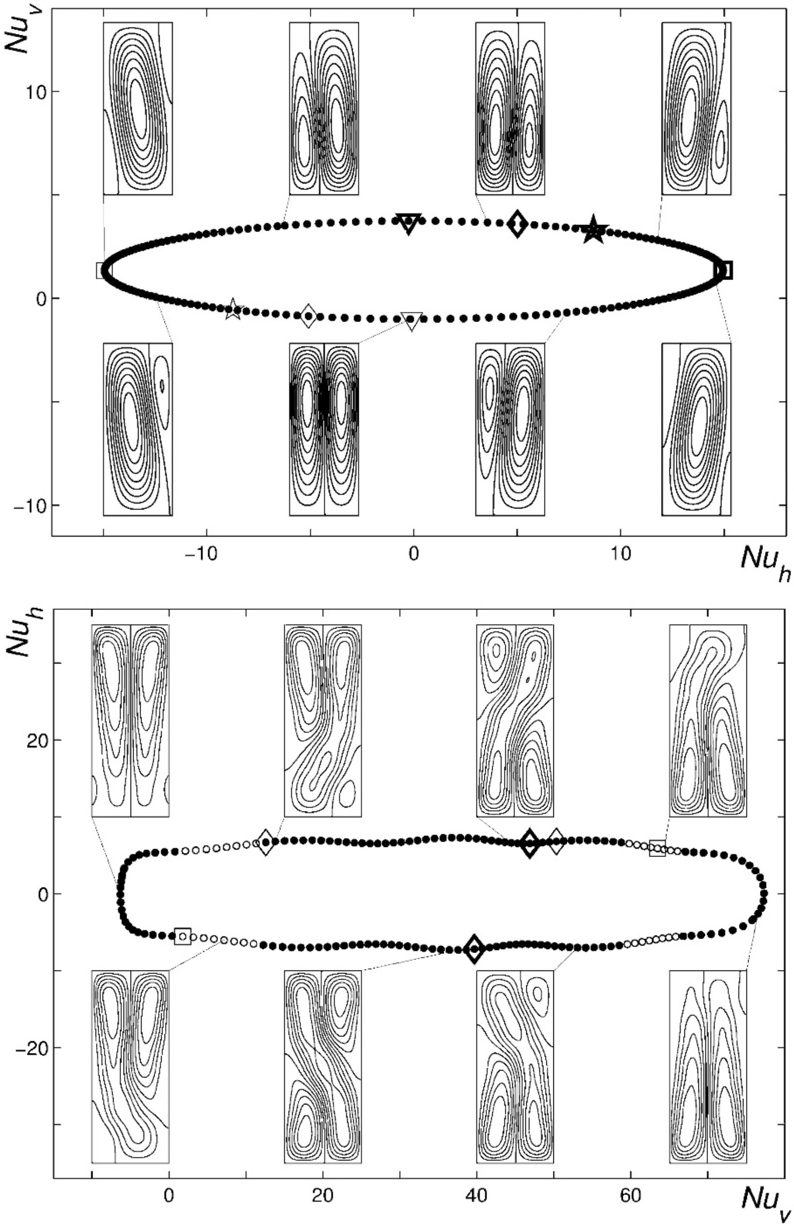

Fig. 19.2: Families of steady states and streamlines of convective flows: at top – after branching off from a quiescent state, at bottom – close to loss of stability on the family. Filled circle marks stable state, and unstable one is depicted by empty circle. .

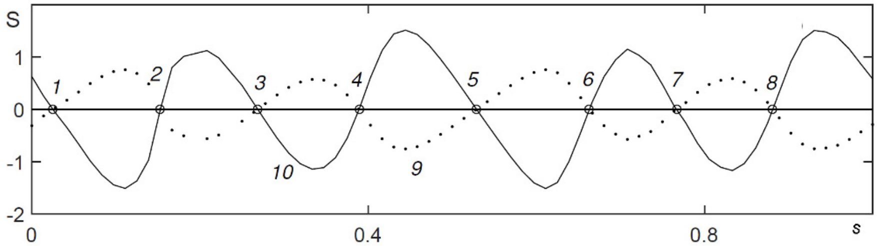

Fig. 19.3: Selective function S: $\mu_{11} = d_{22} = 1.2$ and $\mu_{22} = 1.0$; $\lambda = 270$, (1–8) are the zeros of the selective function, $d_{11} = 0.95$ and 1.1 (curves 9 and 10, respectively). .

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