Chapter 17
Geometrically Nonlinear Cosserat Elasticity with Chiral Effects Based upon Granular Micromechanics

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Abstract In this short contribution, we exemplify how the Cosserat kinematics, introduced into classical continuum mechanics at the turn of the XX^{th} century, arises when we consider the granular microstructure inherent in all matter. The discussed discrete-continuum identification, that follows Piola's *ansatz*, is utilized to develop kinematic measures in the framework of finite deformations. These kinematic measures are then utilized to express the internal deformation (strain) energy in terms of both grain-scale and continuum kinematic measures. As a result, we have identified the macroscopic elastic constants of geometrically nonlinear Cosserat continuum model in terms of the grain scale parameters.

Key words: Cosserat media · Granular micromechanics · Microstructured solids · Generalized continua · Finite deformations

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17.1 Introduction

All materials are characterized by hierarchies of microstructures, which at some spatial scale, can be described as composed of discrete sub-bodies or structural elements [1]-[8]. The relative geometrical arrangements and inter-connectivity (interphases or interfaces) of these discrete elements (grains) forms the microstructure of the matter irrespective of the spatial scale [9]-[15]. Furthermore, the geometrical/mechanical attributes of the interconnections informs the micromechanics of the grain motions [16]-[19]. The symbiotic effects produced through interaction of the microstructure and micromechanics can be termed as micro-mechano-morphology [20]-[25]. The mathematical description of the material deformation by accounting for the micro-mechano-morphological effects requires introduction of additional kinematical descriptors in continuum models [26]-[32]. At the turn of the XX$^{\text{th}}$ century, the Cosserat brothers developed the continuum model of deformable bodies by introducing the rotational kinematic degree-of-freedom associated with every continuum material point [33]-[35]. The Cosserat model and its refinements/modifications, known as the micro-polar continuum theory, have come to play an essential role in explaining certain size-dependent phenomena exhibited by the so-called "microstructured" solids [36]-[43], including materials with granular microstructures [35]. In this sense, the Cosserat or the micro-polar models are considered an important step in the endeavor to capture in greater details the effect of the microstructure in continuum models of material deformation [20].

In this paper, we derive a geometrically nonlinear Cosserat continuum theory on the basis of granular micromechanics by linking the grain-scale deformation to the continuum kinematic measures [44]-[47]. In this regard, we first employ Piola's *ansatz* to make the discrete-continuum identification [26],[48]-[53] such that the objective relative motion between grains can be related to the continuum strain measures. Moreover, we can then define the elastic strain energy at the grain-scale utilizing the objective relative motion between grains, thus laying the foundation for the micro-macro identification of the elastic constants for a hemitropic Cosserat media exhibiting chirality.

17.2 Discrete and Continuous Models for Granular Systems

17.2.1 Identification via Piola's Ansatz

In the discrete model, the reference configuration of the considered set of n grains is given by their position

$$\{X_1, X_2, \ldots, X_n\} \in \left(E^2\right)^n,$$

where E^2 is the Euclidean two dimensional space and reference angular arrangement,

$$\{\mathfrak{R}_1 = I, \mathfrak{R}_2 = I, \ldots, \mathfrak{R}_n = I\} \in \left(\mathrm{Orth}^+\right)^n,$$

where Orth^+ is the space of orthogonal tensor with positive determinant (thus, the space of rotation). For the sake of simplicity, the reference angular arrangement is assumed to be constant in space and equal to the identity tensor I.

They displace, respectively, with the following displacement functions

$$u_1(X_1) = \chi_1(X_1) - X_1, u_2(X_2) = \chi_2(X_2) - X_2, \ldots, u_n(X_n) = \chi_n(X_n) - X_n,$$

and rotate as

$$\{R_1, R_2, \ldots, R_n\} \in \left(\mathrm{Orth}^+\right)^n,$$

where $\chi_i(X_i)$ is the placement of the i-th grain eventually depending on time t and where Orth^+ is the space of orthogonal tensor with positive determinant (thus, the space of rotation).

In the continuum model, we have a continuous body $\mathfrak{B}$ which, in the reference configuration, is constituted by infinite particles having position X, i.e. $X \in \mathcal{B}$. Each particle has displacement $u(X) = \chi(X) - X$, where $\chi(X)$ is the placement function of the continuous body $\mathfrak{B}$ and rotation $R(X)$.

In the continuum-discrete models identification, the following Piola's ansatz will be assumed

$$\begin{array}{llll} u(X_1) = u_1(X_1), & u(X_2) = u_2(X_2), & \ldots & u(X_n) = u_n(X_n), \\ R(X_1) = R_1(X_1), & R(X_2) = R_2(X_2), & \ldots & R(X_n) = R_n(X_n), \end{array}$$

which means that the displacements and rotation of the n grains correspond to the displacement $u(X)$ and rotation $R(X)$ of the continuous body $\mathfrak{B}$ evaluated at the points X_i with $i = 1, \ldots, n$ where the grains are located in the reference configuration.

17.2.2 Relative Intergranular Displacement and Related Continuum Deformation Measures

Let us assume that the distance between the particles at X_n and X_p is L and the unit vector $\hat{c}$ is defined as follows,

$$X_n - X_p = \hat{c}L. \tag{17.1}$$

In the reference configuration, therefore, the vector attached to the position X_n and pointing to the position X_p is $\hat{c}L$ and is given in (17.1).

In the current configuration, the positions of the two particles at X_n and X_p are, respectively, $\chi(X_n)$ and $\chi(X_p)$. The vector in (17.1) in the current figuration yields

$$\chi(X_n) - \chi(X_p). \tag{17.2}$$

The difference between the vectors in (17.1) and (17.2) is called the relative displacement δ_{np} of the two grains n and p,

$$\delta^{np} = \chi(X_n) - \chi(X_p) - (X_n - X_p) = u_n(X_n) - u_p(X_p). \tag{17.3}$$

In order to define an objective relative displacement (i.e. a relative displacement that is a measure of the contribution of the $n-p$ pair to the deformation of the granular assembly), we consider both the deformation gradient

$$F = \mathcal{F}(X_p) = \nabla\chi, \tag{17.4}$$

as the gradient of the placement function χ and the rotation

$$R = \mathcal{R}(X_p). \tag{17.5}$$

Thus, we define two objective relative displacements, i.e. the objective macro relative displacement u^{np} and the objective micro-macro relative displacement γ^{np} as follows,

$$u^{np} = \mathcal{F}^T(X_p)\left[\chi(X_n) - \chi(X_p)\right] - (X_n - X_p), \tag{17.6}$$

$$\gamma^{np} = \mathcal{R}^T(X_p)\left[\chi(X_n) - \chi(X_p)\right] - (X_n - X_p). \tag{17.7}$$

In the current configuration, the rotations of the two particles at X_n and X_p are, respectively, $\mathcal{R}(X_n)$ and $\mathcal{R}(X_p)$. Thus, we define an objective relative tensor,

$$m^{np} = \mathcal{R}^T(X_p)\mathcal{R}(X_n) - I \tag{17.8}$$

that is called the intergranular micro deformation. Let us now assume that the two grains n and p are neighboring grains. Let us restrict the present model to the case they place and rotate similarly in the present configuration, and therefore the following Taylor's series expansions are possible and yield

$$\chi(X_n) \cong \chi(X_p) + (\nabla\chi)_{X_p}(X_n - X_p) = \chi(X_p) + \mathcal{F}(X_p)(X_n - X_p). \tag{17.9}$$

$$R(X_n) \cong \mathcal{R}(X_p) + (\nabla\mathcal{R})_{X_p}(X_n - X_p). \tag{17.10}$$

It is notable that this simplification may loose applicability for materials that possess strong variations and discontinuities of stiffnesses or have highly disordered geometries, which could be deliberately designed as in pantographic metamaterials [11, 21, 49], [54]-[68] or in granular (meta)materials [69]. In these cases, Taylor expansion is not representative and additional kinematic descriptors may be introduced to accurately describe the response as in Cosserat or micromorphic media [20, 36].

The Green–Saint-Venant tensor G,

$$G = \frac{1}{2}\left(F^T F - I\right), \tag{17.11}$$

is defined in terms of the deformation gradient F as well as a relative micro-macro Green–Saint-Venant tensor Υ

$$\Upsilon = \frac{1}{2}\left(R^T F - I\right). \tag{17.12}$$

The last four equations (17.9-17.11-17.10-17.12) involve third order tensors and it is more convenient to show them in index notation

$$\chi_i^n = \chi_i^p + F_{ij}^p \hat{c}_j L, \qquad G_{ij}^p = \frac{1}{2}\left(F_{ai}^p F_{aj}^p - \delta_{ij}\right),$$
$$R_{aj}^n = R_{aj}^p + R_{aj,h}^p \hat{c}_h L, \qquad \Upsilon_{ij}^p = \frac{1}{2}\left(R_{ai}^p F_{aj}^p - \delta_{ij}\right), \tag{17.13}$$

where superscripts n or p denote that the functions are evaluated, respectively, at the points X_n or X_p. Thus, the objective relative displacements in (17.6) and (17.7) and tensor (17.8) are in index form,

$$u_i^{np} = F_{ai}^p \left(\chi_a^n - \chi_a^p\right) - \left(X_i^n - X_i^p\right),$$
$$\gamma_i^{np} = R_{ai}^p \left(\chi_a^n - \chi_a^p\right) - \left(X_i^n - X_i^p\right),$$
$$m_{ij}^{np} = R_{ai}^p R_{aj}^n - \delta_{ij},$$

that, from (17.1) and $(17.13)_1$ yields

$$u_i^{np} = F_{ai}^p \left(F_{aj}^p \hat{c}_j L\right) - \hat{c}_i L = F_{ai}^p F_{aj}^p \hat{c}_j L - \delta_{ij}\hat{c}_j L = 2G_{ij}^p \hat{c}_j L, \tag{17.14}$$

$$\gamma_i^{np} = R_{ai}^p \left(F_{aj}^p \hat{c}_j L\right) - \hat{c}_i L = \left[R_{ai}^p F_{aj}^p - \delta_{ij}\right] \hat{c}_j L = 2\Upsilon_{ij}^p \hat{c}_j L, \tag{17.15}$$

$$m_{ij}^{np} = R_{ai}^p R_{aj}^n - \delta_{ij} = R_{ai}^p \left(R_{aj}^p + R_{aj,h}^p \hat{c}_h L\right) - \delta_{ij} = R_{ai}^p R_{aj,h}^p \hat{c}_h L, \tag{17.16}$$

where δ_{ij} is the Kronecker symbol. In absolute notation (17.14), (17.15) and (17.16) are

$$u^{np} = 2LG\hat{c}, \qquad \gamma^{np} = 2L\Upsilon\hat{c}, \qquad m^{np} = LR^T\left(\nabla R\right)\hat{c}, \tag{17.17}$$

where all the superscripts p have been omitted because all the quantities are locally evaluated at $X = X_p$.

17.2.3 On the Objective (Macro and Micro-macro) Displacement Vectors

The projections of the objective relative displacements on the unit vector $\hat{c}$, defined in (17.1), is the so called normal displacement u_η, and normal micro relative displacement that are defined,

$$u_\eta = \frac{1}{2} u^{np} \cdot \hat{c}, \qquad \gamma_\eta = \frac{1}{2}\gamma^{np} \cdot \hat{c}. \tag{17.18}$$

Insertion of (17.14) and (17.15) into (17.18) provides

$$u_\eta = LG_{ij}\hat{c}_i\hat{c}_j, \qquad \gamma_\eta = L\Upsilon_{ij}\hat{c}_i\hat{c}_j. \tag{17.19}$$

Their squares are,

$$u_\eta^2 = L^2 \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b G_{ij} G_{ab}, \qquad \gamma_\eta^2 = L^2 \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \Upsilon_{ij} \Upsilon_{ab}. \tag{17.20}$$

The objective relative displacement vectors u^{np} and γ^{np} projected orthogonally to the unit vector $\hat{c}$, defined in (17.1), are the so called tangent displacements u_τ and γ_τ that are vectors and defined as follows,

$$u_\tau = u^{np} - (u^{np} \cdot \hat{c})\,\hat{c}, \qquad \gamma_\tau = \gamma^{np} - (\gamma^{np} \cdot \hat{c})\,\hat{c}. \tag{17.21}$$

The only objective quantities derived from u_τ, γ_τ and $\hat{c}$ are the squares of both u_τ and γ_τ and their scalar product, i.e.

$$u_\tau^2 = [u^{np} - (u^{np} \cdot \hat{c})\,\hat{c}] \cdot [u^{np} - (u^{np} \cdot \hat{c})\,\hat{c}] = u^{np} \cdot u^{np} - (u^{np} \cdot \hat{c})^2, \tag{17.22}$$

$$\gamma_\tau^2 = [\gamma^{np} - (\gamma^{np} \cdot \hat{c})\,\hat{c}] \cdot [\gamma^{np} - (\gamma^{np} \cdot \hat{c})\,\hat{c}] = \gamma^{np} \cdot \gamma^{np} - (\gamma^{np} \cdot \hat{c})^2, \tag{17.23}$$

$$\begin{aligned} u_\tau \cdot \gamma_\tau &= [u^{np} - (u^{np} \cdot \hat{c})\,\hat{c}] \cdot [\gamma^{np} - (\gamma^{np} \cdot \hat{c})\,\hat{c}] \\ &= u^{np} \cdot \gamma^{np} - (u^{np} \cdot \hat{c})\,(\gamma^{np} \cdot \hat{c}), \end{aligned} \tag{17.24}$$

or, in index notation and taking (17.20) into account,

$$u_\tau^2 = \left(2G_{ij}\hat{c}_j L\right)(2G_{ik}\hat{c}_k L) - 4u_\eta^2, \tag{17.25}$$

$$\gamma_\tau^2 = \left(2\Upsilon_{ij}\hat{c}_j L\right)(2\Upsilon_{ik}\hat{c}_k L) - 4\gamma_\eta^2, \tag{17.26}$$

$$u_\tau \cdot \gamma_\tau = \left(2G_{ij}\hat{c}_j L\right)(2\Upsilon_{ik}\hat{c}_k L) - 4u_\eta \gamma_\eta, \tag{17.27}$$

or in a more compact form,

$$u_\tau^2 = 4L^2 G_{ij} G_{ab} \left(\delta_{ia}\hat{c}_j\hat{c}_b - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right), \tag{17.28}$$

$$\gamma_\tau^2 = 4L^2 \Upsilon_{ij} \Upsilon_{ab} \left(\delta_{ia}\hat{c}_j\hat{c}_b - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right), \tag{17.29}$$

$$u_\tau \cdot \gamma_\tau = 4L^2 G_{ij} \Upsilon_{ab} \left(\delta_{ia}\hat{c}_j\hat{c}_b - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right). \tag{17.30}$$

17.2.4 On the Objective Tensor

In the 2D case, the rotation tensor R is assumed to be a rotation of an angle φ around the third axis orthogonal to the 2D plane. Besides, the unit vector $\hat{c}$ is parameterized by the anticlockwise angle θ with respect to the horizontal axis within the 2D plane and the displacement vector u has only two components, i.e. $u_3 = 0$. Finally, the deformation gradient F is an in-plane tensor. In formulas we have,

$$R = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}, \quad F = \begin{pmatrix} 1+u_{1,1} & u_{1,2} & 0 \\ u_{2,1} & 1+u_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{17.31}$$

The gradient of the rotation tensor is a third order tensor but its projection along the unit vector $\hat{c}$ is a second order tensor with the following components

$$[(\nabla R)\,\hat{c}]_{ij} = (\nabla R)_{ijh}\,\hat{c}_h = R_{ij,h}\hat{c}_h. \tag{17.32}$$

Thus, evaluating component by component the second order tensor defined in (17.32) we have,

$$\begin{aligned}
[(\nabla R)\,\hat{c}]_{11} &= R_{11,1}\hat{c}_1 + R_{11,2}\hat{c}_2 = -\sin\varphi\,(\varphi_{,1})\cos\theta - \sin\varphi\,(\varphi_{,2})\sin\theta,\\
[(\nabla R)\,\hat{c}]_{12} &= R_{12,1}\hat{c}_1 + R_{12,2}\hat{c}_2 = \cos\varphi\,(\varphi_{,1})\cos\theta + \cos\varphi\,(\varphi_{,2})\sin\theta,\\
[(\nabla R)\,\hat{c}]_{21} &= R_{21,1}\hat{c}_1 + R_{21,2}\hat{c}_2 = -\cos\varphi\,(\varphi_{,1})\cos\theta - \cos\varphi\,(\varphi_{,2})\sin\theta,\\
[(\nabla R)\,\hat{c}]_{22} &= R_{22,1}\hat{c}_1 + R_{22,2}\hat{c}_2 = -\sin\varphi\,(\varphi_{,1})\cos\theta - \sin\varphi\,(\varphi_{,2})\sin\theta,
\end{aligned}$$

or in a matrix compact form,

$$(\nabla R)\,\hat{c} = \begin{pmatrix} -\sin\varphi\,(\varphi_{,1}\cos\theta + \varphi_{,2}\sin\theta) & \cos\varphi\,(\varphi_{,1}\cos\theta + \varphi_{,2}\sin\theta) & 0 \\ -\cos\varphi\,(\varphi_{,1}\cos\theta + \varphi_{,2}\sin\theta) & -\sin\varphi\,(\varphi_{,1}\cos\theta + \varphi_{,2}\sin\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that can be reduced as follows,

$$(\nabla R)\,\hat{c} = \nabla\varphi\cdot\hat{c}\begin{pmatrix} -\sin\varphi & \cos\varphi & 0 \\ -\cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Keeping this in mind, the objective relative tensor in (17.16) is easily represented from $(17.31)_1$ as follows,

$$\begin{aligned}
m^{np} = LR^T\,(\nabla R)\,\hat{c} &= L\nabla\varphi\cdot\hat{c}\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\sin\varphi & \cos\varphi & 0 \\ -\cos\varphi & -\sin\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}\\
&= \kappa\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{17.33}$$

where,

$$\kappa = L\nabla\varphi\cdot\hat{c}. \tag{17.34}$$

The representation (17.33) of the objective relative tensor in terms of a single scalar is worth noting.

17.2.5 The Objective Scalar Deformation Measures

Finally, we can list the scalar quantities of the relative deformation measures of the pairs (17.6), (17.7) and (17.8),

$$u_\eta, \gamma_\eta, u_\tau^2, \gamma_\tau^2, \kappa, u_\tau \cdot \gamma_\tau. \tag{17.35}$$

from (17.18), (17.28), (17.29), (17.30) and (17.34).

17.3 Elastic Energy Function

The elastic energy function U^{np} for a given couple of particles, say the couple $n-p$ considered in Section 17.2.2, is assumed to be, in the discrete and (because of the Piola's ansatz) in the continuum models, a quadratic form of normal and tangent parts components of the objective relative displacements listed in (17.35),

$$\begin{aligned} U^{np} = & \frac{1}{2} k_\eta \left(1 - D_\eta\right) u_\eta^2 + \frac{1}{2} k_{\gamma\eta} \left(1 - D_{\gamma\eta}\right) \gamma_\eta^2 + \frac{1}{2} k_\tau \left(1 - D_\tau\right) u_\tau^2 + \\ & + \frac{1}{2} k_{\gamma\tau} \left(1 - D_{\gamma\tau}\right) \gamma_\tau^2 + \frac{1}{2} k_\kappa \left(1 - D_\kappa\right) \kappa^2 + \\ & + k_{u\tau\gamma\tau} u_\tau \cdot \gamma_\tau + k_{u\eta\gamma\eta} u_\eta \gamma_\eta + k_{u\eta\kappa} u_\eta \kappa + k_{\gamma\eta\kappa} \gamma_\eta \kappa, \end{aligned} \tag{17.36}$$

where the coefficients of this quadratic form are the stiffness of the grain pair elastic interaction, that can be divided into the coefficients of the quadratic terms,

$$k_\eta, k_{\gamma\eta}, k_\tau, k_{\gamma\tau}, k_\kappa, \tag{17.37}$$

and that of the coupling terms,

$$k_{u\tau\gamma\tau}, k_{u\eta\gamma\eta}, k_{u\eta\kappa}, k_{\gamma\eta\kappa}. \tag{17.38}$$

For the sake of simplicity, we have assumed that damage affects only the stiffness listed in (17.37) with the following, and respectively, damage variables

$$D_\eta, D_{\gamma\eta}, D_\tau, D_{\gamma\tau}, D_\kappa.$$

The total elastic energy density, for a given position X, is the some of that of every pairs or, in the continuum approximation, the integral over all the orientation

$$
\begin{aligned}
U = &\int_{\mathcal{S}^1} \frac{1}{2} k_\eta \left(1 - D_\eta\right) \left(L^2 \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b G_{ij} G_{ab}\right) \\
&+ \int_{\mathcal{S}^1} \frac{1}{2} k_{\gamma\eta} \left(1 - D_{\gamma\eta}\right) L^2 \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b \Upsilon_{ij} \Upsilon_{ab} \\
&+ \int_{\mathcal{S}^1} \frac{1}{2} k_\tau \left(1 - D_\tau\right) \left(4 L^2 G_{ij} G_{ab} \left(\delta_{ia} \hat{c}_j \hat{c}_b - \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right)\right) \\
&+ \int_{\mathcal{S}^1} \frac{1}{2} k_{\gamma\tau} \left(1 - D_{\gamma\tau}\right) 4 L^2 \Upsilon_{ij} \Upsilon_{ab} \left(\delta_{ia} \hat{c}_j \hat{c}_b - \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right) \\
&+ \int_{\mathcal{S}^1} \frac{1}{2} k_\kappa \left(1 - D_\kappa\right) L^2 \varphi_{,i} \varphi_{,j} \hat{c}_i \hat{c}_j \\
&+ \int_{\mathcal{S}^1} k_{u\tau\gamma\tau} 4 L^2 G_{ij} \Upsilon_{ab} \left(\delta_{ia} \hat{c}_j \hat{c}_b - \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right) \\
&+ \int_{\mathcal{S}^1} k_{u\eta\gamma\eta} 2 L \hat{c}_i \hat{c}_j G_{ij} 2 L \Upsilon_{ab} \hat{c}_a \hat{c}_b \\
&+ \int_{\mathcal{S}^1} k_{u\eta\kappa} 2 L \hat{c}_i \hat{c}_j \hat{c}_h G_{ij} \varphi_{,h} \\
&+ \int_{\mathcal{S}^1} k_{\gamma\eta\kappa} L \Upsilon_{ij} \hat{c}_i \hat{c}_j \varphi_{,h} \hat{c}_h
\end{aligned}
\tag{17.39}
$$

where insertion of (17.18), (17.28), (17.29), (17.30) and (17.34) have been used and the integral over the unit circle is intended as follows,

$$
\int_{\mathcal{S}^1} (\bullet) = \int_0^{2\pi} (\bullet)\, d\theta .
$$

Grouping the terms of (17.39), we have

$$
\begin{aligned}
U = &\int_{\mathcal{S}^1} \frac{1}{2} L^2 \left[k_\eta \left(1 - D_\eta\right) \left(\hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right)\right] G_{ij} G_{ab} \\
&+ \int_{\mathcal{S}^1} \frac{1}{2} L^2 \left[k_\tau \left(1 - D_\tau\right) \left(\left(\delta_{ia} \hat{c}_j \hat{c}_b + \delta_{ib} \hat{c}_j \hat{c}_a + \delta_{ja} \hat{c}_i \hat{c}_b + \delta_{jb} \hat{c}_i \hat{c}_a\right)\right.\right. \\
&\left.\left. -4\, \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right)\right] G_{ij} G_{ab} \\
&+ \int_{\mathcal{S}^1} \frac{1}{2} L^2 \left[k_{\gamma\eta} \left(1 - D_{\gamma\eta}\right) \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right. \\
&\left. + k_{\gamma\tau} \left(1 - D_{\gamma\tau}\right) 4 \left(\delta_{ia} \hat{c}_j \hat{c}_b - \hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right)\right] \Upsilon_{ij} \Upsilon_{ab}
\end{aligned}
$$

$$+\int_{\mathcal{S}^1} \frac{1}{2}L^2 k_\kappa \left(1-D_\kappa\right) \hat{c}_i\hat{c}_j\varphi_{,i}\varphi_{,j}$$

$$+\int_{\mathcal{S}^1} L^2 4\left[k_{u\tau\gamma\tau}\left(\frac{1}{2}\left(\delta_{ia}\hat{c}_j\hat{c}_b+\delta_{ja}\hat{c}_i\hat{c}_b\right)-\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right)+k_{u\eta\gamma\eta}\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right]G_{ij}\Upsilon_{ab}$$

$$+\int_{\mathcal{S}^1} k_{u\eta\kappa}2L\hat{c}_i\hat{c}_j\hat{c}_hG_{ij}\varphi_{,h}$$

$$+\int_{\mathcal{S}^1} k_{\gamma\eta\kappa}L\hat{c}_i\hat{c}_a\hat{c}_b\Upsilon_{ab}\varphi_{,i}$$

or, in a compact form, it yields

$$U=\frac{1}{2}\mathbb{C}_{ijab}G_{ij}G_{ab}+\frac{1}{2}\mathbb{C}^{\gamma}{}_{ijab}\Upsilon_{ij}\Upsilon_{ab}+\frac{1}{2}\mathbb{K}_{ij}\varphi_{,i}\varphi_{,j}$$
$$+\mathbb{K}^{u\gamma}_{ijab}G_{ij}\Upsilon_{ab}+\mathbb{K}^{u\kappa}_{ijh}G_{ij}\varphi_{,h}+\mathbb{K}^{\kappa\gamma}_{iab}\varphi_{,i}\Upsilon_{ab}, \tag{17.40}$$

where the elastic stiffness $\mathbb{C}$, $\mathbb{C}^\gamma$, $\mathbb{K}$, $\mathbb{K}^{u\gamma}$, $\mathbb{K}^{u\kappa}$, and $\mathbb{K}^{\kappa\gamma}$ are identified in (17.40) as follows, with the symmetrization induced by the symmetry of the strain tensor G,

$$\mathbb{C}_{ijab}=L^2\int_{\mathcal{S}^1} k_\eta\left(1-D_\eta\right)\left(\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right)$$
$$+4L^2\int_{\mathcal{S}^1} k_\tau(1-D_\tau)\left(\frac{1}{4}(\delta_{ia}\hat{c}_j\hat{c}_b+\delta_{ib}\hat{c}_j\hat{c}_a+\delta_{ja}\hat{c}_i\hat{c}_b+\delta_{jb}\hat{c}_i\hat{c}_a)-\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) \tag{17.41}$$

$$\mathbb{C}^{\gamma}{}_{ijab}=L^2\int_{\mathcal{S}^1} k_{\gamma\eta}\left(1-D_{\gamma\eta}\right)\left(\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right)$$
$$+4L^2\int_{\mathcal{S}^1} k_{\gamma\tau}\left(1-D_{\gamma\tau}\right)\left(\delta_{ia}\hat{c}_j\hat{c}_b-\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) \tag{17.42}$$

$$\mathbb{K}_{ij}=L^2\int_{\mathcal{S}^1} k_\kappa\left(1-D_\kappa\right)\hat{c}_i\hat{c}_j \tag{17.43}$$

$$\mathbb{K}^{u\gamma}_{ijab}=4L^2\int_{\mathcal{S}^1}\left(k_{u\eta\gamma\eta}\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right)$$
$$+4L^2\int_{\mathcal{S}^1}\left\{k_{u\tau\gamma\tau}\left[\frac{1}{2}\left(\delta_{ia}\hat{c}_j\hat{c}_b+\delta_{ja}\hat{c}_i\hat{c}_b\right)-\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right]\right\} \tag{17.44}$$

$$\mathbb{K}^{u\kappa}_{ijh}=2L\int_{\mathcal{S}^1}\left(k_{u\eta\kappa}\hat{c}_i\hat{c}_j\hat{c}_h\right) \tag{17.45}$$

$$\mathbb{K}^{\kappa\gamma}_{iab} = L \int_{\mathcal{S}^1} \left(k_{\gamma\eta\kappa} \hat{c}_i \hat{c}_a \hat{c}_b \right) \tag{17.46}$$

17.4 Identification of the Undamaged Isotropic Case

17.4.1 Characterization of the Undamaged Isotropic Case

In the non-damaged isotropic case, we have both

$$D_\eta = D_\tau = D_{\gamma\eta} = D_{\gamma\tau} = D_\kappa = 0,$$

and constant distribution of stiffnesses with respect to the orientation

$$\begin{aligned}
&k_\eta = \widetilde{k}_\eta(\theta) = \frac{\bar{k}_\eta}{2\pi}, \qquad k_\tau = \widetilde{k}_\tau(\theta) = \frac{\bar{k}_\tau}{2\pi},\\
&k_{\gamma\eta} = \widetilde{k}_{\gamma\eta}(\theta) = \frac{\bar{k}_{\gamma\eta}}{2\pi}, \qquad k_{\gamma\tau} = \widetilde{k}_{\gamma\tau}(\theta) = \frac{\bar{k}_{\gamma\tau}}{2\pi},\\
&k_\kappa = \widetilde{k}_\kappa(\theta) = \frac{\bar{k}_\kappa}{2\pi},\\
&k_{u\eta\gamma\eta} = \widetilde{k}_{u\eta\gamma\eta}(\theta) = \frac{\bar{k}_{u\eta\gamma\eta}}{2\pi}, \qquad k_{u\tau\gamma\tau} = \widetilde{k}_{u\tau\gamma\tau}(\theta) = \frac{\bar{k}_{u\tau\gamma\tau}}{2\pi},\\
&k_{u\eta\kappa} = \widetilde{k}_{u\eta\kappa}(\theta) = \frac{\bar{k}_{u\eta\kappa}}{2\pi}, \qquad k_{\gamma\eta\kappa} = \widetilde{k}_{\gamma\eta\kappa}(\theta) = \frac{\bar{k}_{\gamma\eta\kappa}}{2\pi}
\end{aligned}$$

where $\bar{k}_\eta$, $\bar{k}_\tau$ $\bar{k}_{\gamma\eta}$, $\bar{k}_{\gamma\tau}$ $\bar{k}_\kappa$, $\bar{k}_{u\eta\gamma\eta}$, $\bar{k}_{u\tau\gamma\tau}$, $\bar{k}_{u\eta\kappa}$ and $\bar{k}_{\gamma\eta\kappa}$ are the integrated stiffness over the set of possible orientations, that are defined in the general anisotropic case as follows,

$$\begin{aligned}
&\bar{k}_\eta = \int_0^{2\pi} \widetilde{k}_\eta(\theta)\, d\theta, \qquad \bar{k}_\tau = \int_0^{2\pi} \widetilde{k}_\tau(\theta)\, d\theta,\\
&\bar{k}_{\gamma\eta} = \int_0^{2\pi} \widetilde{k}_{\gamma\eta}(\theta)\, d\theta, \qquad \bar{k}_{\gamma\tau} = \int_0^{2\pi} \widetilde{k}_{\gamma\tau}(\theta)\, d\theta,\\
&\bar{k}_\kappa = \int_0^{2\pi} \widetilde{k}_\kappa(\theta)\, d\theta,\\
&\bar{k}_{u\eta\gamma\eta} = \int_0^{2\pi} \widetilde{k}_{u\eta\gamma\eta}(\theta)\, d\theta, \quad \bar{k}_{u\tau\gamma\tau} = \int_0^{2\pi} \widetilde{k}_{u\tau\gamma\tau}(\theta)\, d\theta,
\end{aligned}$$

$$\bar{k}_{u\eta\kappa} = \int_0^{2\pi} \widetilde{k}_{u\eta\kappa}(\theta)\, d\theta, \quad \bar{k}_{\gamma\eta\kappa} = \int_0^{2\pi} \widetilde{k}_{\gamma\eta\kappa}(\theta)\, d\theta.$$

With these hypotheses, stiffness tensors (17.41), (17.42), (17.43), (17.44), and (17.46) reduce to the following simplified form

$$\begin{aligned}\mathbb{C}_{ijab} &= 2L^2 \frac{\bar{k}_\eta}{\pi} \int_{\mathcal{S}^1} \left(\hat{c}_i \hat{c}_j \hat{c}_a \hat{c}_b\right) \\ &\quad + 2L^2 \frac{\bar{k}_\tau}{\pi} \int_{\mathcal{S}^1} \left[\frac{1}{4}\left(\delta_{ia}\hat{c}_j\hat{c}_b + \delta_{ib}\hat{c}_j\hat{c}_a + \delta_{ja}\hat{c}_i\hat{c}_b + \delta_{jb}\hat{c}_i\hat{c}_a\right) - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right]\end{aligned} \tag{17.47}$$

$$\mathbb{C}^\gamma{}_{ijab} = 2L^2 \frac{\bar{k}_{\gamma\eta}}{\pi} \int_{\mathcal{S}^1} \left(\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) + 2L^2 \frac{\bar{k}_{\gamma\tau}}{\pi} \int_{\mathcal{S}^1} \left(\delta_{ia}\hat{c}_j\hat{c}_b - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) \tag{17.48}$$

$$\mathbb{K}_{ij} = \frac{\bar{k}_\kappa}{2\pi} L^2 \int_{\mathcal{S}^1} \left(\hat{c}_i\hat{c}_j\right) \tag{17.49}$$

$$\begin{aligned}\mathbb{K}^{u\gamma}_{ijab} &= 2L^2 \frac{\bar{k}_{u\eta\gamma\eta}}{\pi} \int_{\mathcal{S}^1} \left(\hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right) \\ &\quad + 2L^2 \frac{\bar{k}_{u\tau\gamma\tau}}{\pi} \int_{\mathcal{S}^1} \left[\left(\frac{1}{2}\left(\delta_{ia}\hat{c}_j\hat{c}_b + \delta_{ja}\hat{c}_i\hat{c}_b\right) - \hat{c}_i\hat{c}_j\hat{c}_a\hat{c}_b\right)\right]\end{aligned} \tag{17.50}$$

$$\mathbb{K}^{u\kappa}_{ijh} = 0 \tag{17.51}$$

$$\mathbb{K}^{\kappa\gamma}_{iab} = 0. \tag{17.52}$$

It is worth noting that the coupling tensor

$$\mathbb{K}^{u\gamma}_{ij[ab]} = L^2 \frac{\bar{k}_{u\tau\gamma\tau}}{\pi} \int_{\mathcal{S}^1} \left[\frac{1}{2}\left(\delta_{ia}\hat{c}_j\hat{c}_b + \delta_{ja}\hat{c}_i\hat{c}_b\right) - \frac{1}{2}\left(\delta_{ib}\hat{c}_j\hat{c}_a + \delta_{jb}\hat{c}_i\hat{c}_a\right)\right]$$

is null for the isotropic case,

$$\mathbb{K}^{u\gamma}_{ij[ab]} = L^2 \frac{\bar{k}_{u\tau\gamma\tau}}{2\pi} \left[\pi\delta_{ia}\delta_{jb} + \pi\delta_{ib}\delta_{ja} - \pi\delta_{ib}\delta_{ja} - \pi\delta_{ia}\delta_{jb}\right] = 0.$$

17.4.2 Macroscopic Isotropic Stiffness Matrices

The standard isotropic representation of 2nd and 4th order stiffness tensors, for the 2D case, is given by following expressions (17.53), (17.55), (17.57), and (17.59),

$$\mathbb{K}_{ij} = \kappa\delta_{ij} \tag{17.53}$$

$$\kappa = \mathbb{K}_{11} \tag{17.54}$$

$$\mathbb{C}_{ijab} = \mu\delta_{ia}\delta_{jb} + \mu\delta_{ib}\delta_{ja} + \lambda\delta_{ij}\delta_{ab} \tag{17.55}$$

$$\mathbb{C}_{1212} = \mu, \qquad \mathbb{C}_{1122} = \lambda, \tag{17.56}$$

$$\mathbb{C}^{\gamma}_{ijab} = \mu^{\gamma}_1\delta_{ia}\delta_{jb} + \mu^{\gamma}_2\delta_{ib}\delta_{ja} + \lambda^{\gamma}\delta_{ij}\delta_{ab} \tag{17.57}$$

$$\mathbb{C}^{\gamma}_{1212} = \mu^{\gamma}_1, \quad \mathbb{C}^{\gamma}_{1221} = \mu^{\gamma}_2, \quad \mathbb{C}^{\gamma}_{1122} = \lambda^{\gamma}, \tag{17.58}$$

$$\mathbb{K}^{u\gamma}_{ijab} = \mu^{u\gamma}_1\delta_{ia}\delta_{jb} + \mu^{u\gamma}_2\delta_{ib}\delta_{ja} + \lambda^{u\gamma}\delta_{ij}\delta_{ab} \tag{17.59}$$

$$\mathbb{K}^{u\gamma}_{1212} = \mu^{u\gamma}_1, \quad \mathbb{K}^{u\gamma}_{1221} = \mu^{u\gamma}_2, \quad \mathbb{K}^{u\gamma}_{1122} = \lambda^{u\gamma} \tag{17.60}$$

where identifications of Lamè coefficients in terms of the stiffness matrix components have also been expressed in (17.54), (17.56), (17.58) and (17.60).

17.4.3 Identification of the Macroscopic Isotropic Stiffness Matrices

In the 2D case, a standard representation of the unit vector $\hat{c}$ is

$$\hat{c}_1 = \cos\theta, \ \hat{c}_2 = \sin\theta, \tag{17.61}$$

where θ is the anti-clockwise angle from the first unit vector $\hat{e}_1$ of the frame of reference to $\hat{c}$. Besides, trivial analytical results are as follows

$$\int_0^{2\pi}\left(\sin^2\theta\right)d\theta = \int_0^{2\pi}\left(\cos^2\theta\right)d\theta = \pi, \tag{17.62}$$

$$\int_0^{2\pi}\left(\sin^2\theta\cos^2\theta\right)d\theta = \frac{1}{4}\pi, \tag{17.63}$$

$$\int_0^{2\pi}\sin^4\theta d\theta = \int_0^{2\pi}\cos^4\theta d\theta = \frac{3}{4}\pi. \tag{17.64}$$

From (17.54), (17.49) and (17.62)

$$\kappa = \mathbb{K}_{11} = \frac{\bar{k}_\kappa}{2\pi}L^2\int_0^{2\pi}\left(\cos^2\theta\right) = \frac{\bar{k}_\kappa}{2}L^2. \tag{17.65}$$

From $(17.56)_1$, (17.47) and (17.63)

$$\mu = \mathbb{C}_{1212} = 2L^2 \frac{\bar{k}_\eta}{\pi} \int_0^{2\pi} \left(\sin^2\theta \cos^2\theta\right) + 2L^2 \frac{\bar{k}_\tau}{\pi} \int_0^{2\pi} \left[\frac{1}{4} - \sin^2\theta\cos^2\theta\right] = \frac{1}{2} L^2 \left(\bar{k}_\eta + \bar{k}_\tau\right). \quad (17.66)$$

From $(17.56)_2$, (17.47) and (17.63)

$$\lambda = \mathbb{C}_{1122} = 2L^2 \frac{\bar{k}_\eta}{\pi} \int_0^{2\pi} \left(\sin^2\theta \cos^2\theta\right) + 2L^2 \frac{\bar{k}_\tau}{\pi} \int_0^{2\pi} \left[-\sin^2\theta\cos^2\theta\right] = \frac{1}{2} L^2 \left(\bar{k}_\eta - \bar{k}_\tau\right). \quad (17.67)$$

From $(17.58)_1$, (17.57), (17.64) and (17.63)

$$\mu_1^\gamma = \mathbb{C}_{1212}^\gamma = \mathbb{C}_{2121}^\gamma = 2L^2 \frac{\bar{k}_{\gamma\eta}}{\pi} \int_0^{2\pi} \left(\sin^2\theta \cos^2\theta\right) + 2L^2 \frac{\bar{k}_{\gamma\tau}}{\pi} \int_0^{2\pi} \left[\sin^2\theta - \sin^2\theta\cos^2\theta\right] = \frac{1}{2} L^2 \left(\bar{k}_{\gamma\eta} + 3\bar{k}_{\gamma\tau}\right). \quad (17.68)$$

From $(17.58)_2$, (17.57) and (17.63)

$$\mu_2^\gamma = \mathbb{C}_{1221}^\gamma = 2L^2 \frac{\bar{k}_{\gamma\eta}}{\pi} \int_0^{2\pi} \left(\sin^2\theta \cos^2\theta\right) + 2L^2 \frac{\bar{k}_{\gamma\tau}}{\pi} \int_0^{2\pi} \left[-\sin^2\theta\cos^2\theta\right] = \frac{1}{2} L^2 \left(\bar{k}_{\gamma\eta} - \bar{k}_{\gamma\tau}\right). \quad (17.69)$$

From $(17.58)_3$, (17.57) and (17.63)

$$\lambda^\gamma = \mathbb{C}_{1122}^\gamma = 2L^2 \frac{\bar{k}_{\gamma\eta}}{\pi} \int_0^{2\pi} \left(\sin^2\theta \cos^2\theta\right)$$

$$+2L^2\frac{\bar{k}_{\gamma\tau}}{\pi}\int_0^{2\pi}\left[-\sin^2\theta\cos^2\theta\right] = \frac{1}{2}L^2\left(\bar{k}_{\gamma\eta}-\bar{k}_{\gamma\tau}\right). \quad (17.70)$$

From $(17.60)_1$, (17.50), (17.64) and (17.63)

$$\mu_1^{u\gamma} = \mathbb{K}_{1212}^{u\gamma} = 2L^2\frac{\bar{k}_{u\eta\gamma\eta}}{\pi}\int_0^{2\pi}\left(\sin^2\theta\cos^2\theta\right)+$$
$$+2L^2\frac{\bar{k}_{u\tau\gamma\tau}}{\pi}\int_0^{2\pi}\left[\sin^2\theta-\sin^2\theta\cos^2\theta\right] = \frac{1}{2}L^2\left(\bar{k}_{u\eta\gamma\eta}+3\bar{k}_{u\tau\gamma\tau}\right). \quad (17.71)$$

From $(17.60)_2$, (17.50), (17.64) and (17.63)

$$\mu_2^{u\gamma} = \mathbb{K}_{1221}^{u\gamma} = 2L^2\frac{\bar{k}_{u\eta\gamma\eta}}{\pi}\int_0^{2\pi}\left(\sin^2\theta\cos^2\theta\right)+$$
$$+2L^2\frac{\bar{k}_{u\tau\gamma}}{\pi}\int_0^{2\pi}\left[-\sin^2\theta\cos^2\theta\right] = \frac{1}{2}L^2\left(\bar{k}_{u\eta\gamma\eta}-\bar{k}_{u\tau\gamma\tau}\right). \quad (17.72)$$

From $(17.60)_2$, (17.50), (17.64) and (17.63)

$$\lambda^{u\gamma} = \mathbb{K}_{1122}^{u\gamma} = 2L^2\frac{\bar{k}_{u\eta\gamma\eta}}{\pi}\int_0^{2\pi}\left(\sin^2\theta\cos^2\theta\right)+$$
$$2L^2\frac{\bar{k}_{u\tau\gamma\tau}}{\pi}\int_0^{2\pi}\left[-\sin^2\theta\cos^2\theta\right] = \frac{1}{2}L^2\left(\bar{k}_{u\eta\gamma\eta}-\bar{k}_{u\tau\gamma\tau}\right). \quad (17.73)$$

17.5 Conclusion

The primary result of the presented work is the deduction of a set of elastic material constants for a isotropic Cosserat media derived by exploiting micro-macro identification of granular micro-mechano-morphology. To this end, we have utilized Piola's *ansatz* to make the identification of a discrete granular system with a continuum body to develop relationships between grain-scale and continuum kinematic measures in the framework of finite deformations. The internal deformation (strain) energy is then expressed in terms of these kinematic measures leading to the mentioned Cosserat model.

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