Chapter 7
Keeping Theorizing in Touch with Practice: Practical Rationality as a Middle Range Theory of Mathematics Teaching

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Abstract This chapter characterizes the practical rationality of mathematics teaching as a middle range theory, a theory that is developed through the practice of research. We argue that a middle range theory of teaching permits theorizing that keeps in touch with practice, the exploration of complementarities and mutual sharpening of competing constructs, and the pursuit of an agenda of scientific research on mathematics teaching. We illustrate how empirical research on practical rationality has enabled not only the progressive characterization of phenomena hypothesized by the basic concepts of the theory (e.g., what are the norms of instructional situations) or the uncovering of relationships among those concepts (e.g., complementarities and tensions among contractual and situational norms) but also the drawing of relationships with other constructs (e.g., teachers' beliefs and knowledge). We use this example to argue that progress in theorizing teaching can benefit from a middle-range theory, to illustrate in what way subject-specificity and subject-genericity can complement each other in theorizing, and to speculate on what the field needs from different theorizations to advance toward better understanding of the practice of teaching.

Keywords Practical rationality · Instructional norms · Professional obligations · Middle-range theory · Didactical contract

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1 Teaching as an Object of Study and Our Position as Researchers of Mathematics Teaching

Theorizing is a crucial activity among researchers' efforts to understand the world. The identification of the chunk of the world on which to focus our understanding efforts, the means we use to endeavor in such understanding efforts, and the content and form of such understanding are all tasks that call for the involvement of theory. Because education researchers are part of the social world they seek to understand, theorizing assists those who study education phenomena in the struggle against an *illusion of transparency* caused by the immediacy and practical validity of the knowledge of the world that enables us, as participants, to live in it (Bourdieu et al., 1991).

The practice of teaching is one chunk of the world that can use theorizing, if anything because the existence of the social role of teacher makes it all too easy to think one knows what teaching is. Older and well developed fields of scholarship such as psychology, sociology, and economics have been considering education for decades, reducing education to objects and methods of study from their disciplines. In theorizing the practice of teaching, however, we claim that this practice deserves to be constructed into an object of study, to have its own special gaze or regard, one that draws from other disciplines but is not reducible to them. Thus, to the question that Hill (this volume) imagines David Cohen asking, why would you want to develop a theory of teaching (hereafter, Cohen's question), our answer is, simply, to understand the practice. Vieluf and Klieme (Chap. 3, this volume) ascribe such a goal to practice theory applied to teaching; in our case, the notion that ours is a practice theory of mathematics teaching is an important modifier that, as we show below, connects more specifically to Cohen's notion of instruction.

Our research program pursues a basic or fundamental understanding of the practice of teaching through theorizing of a particular kind. In Chazan et al. (2016) we described a fundamental approach to the study of teaching in contrast to an instrumental approach. An instrumental approach to the study of teaching would be interested in teaching as a variable that can be manipulated in order to optimize some of its outcomes (e.g., meaningful learning, student achievement, equitable opportunity, etc.). Research on what kind of teaching produces desired outcomes (e.g., Hiebert & Grouws, 2007), or instrumental research on teaching, is important and necessary for improving education (see Hiebert & Stigler, Chap. 2, this volume). However, instrumental research does not necessarily construct teaching as an object of study and does not provide a basic understanding of the sort of practice that teaching is. A fundamental approach, in contrast, sees teaching as a phenomenon in the social world that exists in response to societal and institutional conditions of existence just as much as an expression of the will and technical knowledge of its actors and enablers. A cornerstone of our theorizing about teaching is the commitment to understanding the teaching practice that exists as a result of those conditions. At the same time, we also seek a way of theorizing that allows the practice we aim to understand to speak back to our theorizing and keep it grounded.

One resource we have in this regard is our own identities and experiences. We identify as former teachers of secondary and college level mathematics courses.[1] In both of our cases, our transition to becoming mathematics education scholars resulted from our commitment to understanding the practice we were engaged in, an understanding that could use the resources of the academy—including its time, community, and stringent criteria for intellectual work. Indeed, for Chazan, this transition included an extended opportunity to engage in scholarly inquiry into teaching by teaching (Chazan, 2000), what Ball (2000) calls first-person research on teaching.

Thus, we find ourselves in the position of aspiring to study a field of practice of which we have intimate knowledge having been its agents. At the same time, in our study of that practice, we aspire to the goals of science, to describe, explain, and predict. We seek to use those resources to produce accounts of the field of mathematics teaching that, like those of Simon and Tzur (1999), explicate the teacher's perspective from the researchers' perspective. In doing so, there are two traps into which we must not fall. On the one hand, as articulated earlier, we must not fall prey to the illusion of transparency and assume that our experiences as teachers are best described as we experienced them when we taught. On the other hand, we must not assume, either, that the external descriptions of teaching that we are now able to craft as observers obviate the need to consider the experiences of practitioners.

Put another way, we must apply to ourselves Bourdieu's (1990) simultaneous critiques of structuralism and phenomenology. Theorization of the social world requires a critique of the objectivizing dispositions of researchers who may propose structures in the social world partly because their social position allows them to extricate themselves from practice. Theorization of the social world also requires a critique of the subjectivizing dispositions of participants who may promote the epistemological status of their lived experience without consideration of the social conditions and constraints that made such experience and reflection possible. We apply those requirements to ourselves as former mathematics teachers-become-social researchers. We bring to our theorizing both personal experience as mathematics teachers living the tension between the compulsion of sociotechnical norms and the sometimes frustrated and sometimes successful motives of individual agency and our present ability to contemplate that reality as outsiders not immediately engaged in it. Furthermore, that ability is supported by the resources of our present positions, including the relative intellectual autonomy and abundant scholarship available to tenured university faculty in the United States.

We are therefore disposed to see and propose structures to which we can now see ourselves having been adapting when we were teachers; at the same time, we cannot shed the sense of the agency and responsibility we felt we had as teachers and for the study of which other constructs (e.g., teacher beliefs) and measures have been

[1] It has become common in education research for scholars to state their positionality, particularly with regard to their race, gender, and social class and how those situate them in relation to the communities they address in their writing. We adopt that practice in a slightly different manner to disclose our connection to the practice we seek to study.

developed. We see the work of teaching as including making decisions in spaces where there are normative expectations, as well as publicly justifiable alternatives. In describing our theorization in terms compatible with those of Simon and Tzur (1999), we use both our personal experience as decision agents and the conceptual and methodological tools social research has for objectifying the world to explicate the teacher's perspective from the researchers' perspective.

Hence, our theorizing efforts are, as von Glasersfeld (1991) would have it, adapted to fit our experiential world rather than to discover an objective reality. Along those lines, theorizing teaching is akin to an observer's modeling of their experience observing teachers' actions. This modeling includes ongoing empirical research and responsive theorizing moves. On the one hand, empirical research on provisional versions of a theory may generate perturbations to those initial versions. On the other hand, theorizing may respond by adapting the theory to neutralize the perturbations generated by empirical research. Put another way, through empirical research, practice can speak back to theory and enable theory to respond. The potential result of such a dialectic is theorizing that is in closer contact with the practice it theorizes. Thus, empirical research can play a crucial role in the development of a theory.

This chapter illustrates that dialectic: In particular, it illustrates how reliance on empirical work to support and constrain the production of theory is a crucial element in constructing a theory of teaching practice that accounts for the perspective of the practitioner.

2 Practical Rationality, Theorization, and Middle Range Theories

Our contribution to this book on theories of teaching makes use of our research aimed at the development of a theory of, what we call, the *practical rationality of mathematics teaching*. By that name we allude to the basis upon which the practice of mathematics teaching can be understood as rational or sensible. We have explained practical rationality elsewhere (e.g., Chazan et al., 2016; Herbst & Chazan, 2003, 2011, 2012), so this chapter does not do that. Rather, this chapter takes practical rationality as a case of a particular kind of theory (middle range theory; Merton, 1967) and shows examples of what theorizing looks like in that kind of theory. The examples we present serve to argue for the development of a middle range theory of teaching as the way to mitigate the illusion of transparency.

In characterizing theories of the middle range, Merton (1967) was distancing himself from specific hypotheses and grand theories—with the former amenable to be tested empirically and the latter being large sets of ideal constructs designed speculatively to be used to read the world. We are aware that aspects of our theorizing represent strong commitments we have and that might be spun into grand theories. For example, we are committed to understanding teaching as an outcome or a result of complex processes, rather than reducing it to being a voluntary expression of individual teachers; to give up this commitment would represent a change of focus. Other aspects of our theorizing are more responsive to empirical work. The specific concepts that flesh out our basic commitments have not been defined apriori of empirical research operations but rather in relationship with empirical research operations. Also, constructs proposed and empirical research results obtained outside practical rationality (e.g., in theories of mathematical knowledge for teaching) can be engaged to inform, challenge, or complement such theoretical development. Along those lines we consider practical rationality to be an example of what Merton (1967) called a *theory of the middle range*, a theory that is developed through the practice of research.

In saying that practical rationality is a middle range theory, we take distance from grand theorizing. However, we are less interested in classifying practical rationality among theories than in demonstrating how the use of an initial set of commitments and a perspective to steer empirical research on mathematics teaching support theorizing that keeps in touch with practice. The latter includes, in particular, reconciling empirical facts that may be couched in different uses of language, seeking to understand relationships with other theoretical constructs, and organizing them in larger systems of ideas and questions that could guide researchers toward the understanding of general constructs. In our interpretation, the name *practical rationality of mathematics teaching* neither points to a well outlined system of abstractions made from speculation nor does it identify a specific assertion as amenable to being tested empirically. Rather, as the name of a middle range theory, *practical rationality* designates a shell within which we are developing empirical research that seeks to enable theorizing as a means of understanding.

In this chapter, we reflect on a number of aspects of the continued development of practical rationality that illustrate the mutually reinforcing relationships among theorizing teaching, practitioners' tacit knowledge of teaching practice, and empirical research. We argue that a middle range theory of teaching permits theorizing that keeps in touch with practice, the exploration of complementarities and mutual sharpening of competing constructs, and the pursuit of an agenda of scientific research on mathematics teaching (Shavelson & Towne, 2002). There is a parallel between our approach and what Cai and colleagues (Chap. 8, this volume) propose when they describe theories of teaching as including two dynamic processes of *theory for teaching* and *teaching for theory*, in that both their proposal and ours make room for practitioners' knowledge in the development of theory. The difference is in the intent; while Cai et al. (Chap. 8, this volume) and to some extent also Schoenfeld (Chap. 6, this volume) and Hiebert and Stigler (Chap. 2, this volume) assume that the development of theories of teaching seeks to guide the practice of teaching, our intent is more proximal, to understand the practice of teaching in order to further guide research on teaching. Along those lines, coming back to Cohen's question, our goal to develop a theory of mathematics teaching has been to enable research on mathematics teaching to attend to the mathematical specificity of the work of teaching which can be noticed by teachers.

3 Practical Rationality as a Scientific Effort to Study the Work of Mathematics Teaching

In this section, we provide just the theoretical material needed to later describe the empirical work in which our theoretical ideas have been tested and from which the theory has been receiving feedback to pursue theorizing. In later sections we exemplify how this empirical work has supported three different kinds of theoretical developments within practical rationality and the building of connections with two other theoretical perspectives.

3.1 Focusing on Institutionalized Mathematics Teaching

Our work theorizing the practical rationality of mathematics teaching was stimulated not only by our goal to understand the work of mathematics teaching but also by the challenge in Shavelson and Towne's (2002) call for scientific research in education (Herbst & Chazan, 2011). Seeking to avoid interpreting scientific education research solely as evaluation of education interventions, our image of what it means to do scientific, fundamental research in education is tightly connected to Merton's (1967) description of middle range theories and to an interplay of theorizing and testing of theory as means to construct a scholarly understanding of the phenomenon of mathematics teaching. Over time, as a result of both the identification of the constructs that articulate practical rationality and the understanding of relationships among those constructs and the more individual-centered constructs others have proposed, we have come to conjecture that the work of teaching involves decisions and actions that can be explained in terms of a combination of factors. Figure 7.1, below, shows this and also provides a basis for understanding how our

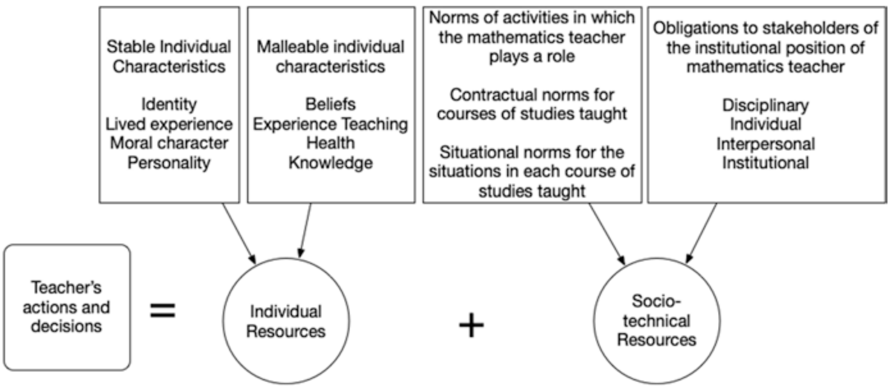

Fig. 7.1 Practical rationality's account of decision making

approach can answer the questions posed by Charalambous and Praetorius to the authors of chapters in this volume.

Our efforts to develop a theory of the practical rationality of mathematics teaching draw from Cohen's (2011) definition of teaching in which he takes distance from both teaching as an accomplishment (i.e., teaching is that which produces learning; see also Scheffler, 1960) and as an occupation (i.e., teaching is what teachers do). Cohen (2011) constructs teaching as an object of study, defining it as the work that teachers do which is deliberately oriented to—even if not effective in—producing learning.

Our theorizing effort focuses on teaching practice in the context of mathematics instruction, while remaining aware that teaching practice responds to other demands. In order to maintain attention to the work a teacher does to support the learning of mathematics, while recognizing the legitimacy of other work teachers do which might be oriented to students' learning of other things (e.g., students' self-concept, social values, other disciplines), we elaborate on the definition of teaching Cohen proposed. Our elaboration of Cohen's (2011) definition of teaching takes advantage of Cohen et al.'s (2003) definition of instruction as a system of interactions among teacher, students, and content in environments, often referred to as the instructional triangle. According to Cohen et al. (2003), instruction is a complex activity in which teachers play a role; in instruction, teaching involves what a teacher does with content and what the teacher does with students in environments. These environments are sociocultural as well as institutional. Applying this definition of instruction to mathematics as the content of instruction in educational institutions allows us to propose a distinction between the overall work of teaching and the work of teaching in mathematics instruction. The work of teaching involves a teacher in many roles in a range of activities that can be oriented to students' learning of something (not necessarily disciplinary content); mathematics instruction is one of those activities. Mathematics teaching is the work a teacher does which is deliberately oriented to students' learning of the mathematical content at stake in instruction. This focused distinction of mathematics teaching from the whole of teaching relies in particular on the institutionally sanctioned content of studies.

Our attention to mathematics teaching is a commitment not only to a focus but also to a gaze or perspective. As noted above, we attend to the work of teaching in mathematics instruction by articulating a gaze that is mathematically specific–that attempts to see the mathematical specificity of mathematics teaching as this is noticeable (even if not always noticed) by teachers. We commit to articulating how mathematics is needed as a resource in the effort of describing, explaining, and predicting the work of mathematics teaching. More succinctly, we seek a subject-specific theory of mathematics teaching. We do not expect that every topic of content taught needs to be part of the theoretical language used to describe the work itself, but we do expect that the theoretical language to describe mathematics teaching will be mathematical in some way and that the discovery of how mathematics needs to be involved in the development of such theoretical language will be shaped by the expectation that such language should show value in the ways that are usually expected of scientific theories—enabling description, explanation, and prediction.

We are also committed to acknowledging the role that institutions like the department, the school, and larger educational systems play in influencing mathematics teaching. Our experiences as teachers of mathematics, where we taught the same material to different groups of students and coordinated work with other teachers of the same material, make us especially aware that an account of the practice of mathematics teaching that explicates the perspective of the teacher from the researcher's perspective needs to be more general than an account of the teaching of a particular group of students and more specific than an account of the teaching of mathematics in general. The institutionalized existence of mathematics instruction provides the course of study as a more or less stable unit for such accounting; courses of study have standard durations (e.g., a semester, a year), a target student population, and a relatively stable share of the curriculum in terms of scope and sequence. Thus, we start from the assumption that the practical rationality of teaching different courses of mathematical studies may have similarities and differences. A natural direction for research is to find out more about those similarities and differences. What ideas are useful to create accounts of the work of teaching across courses of mathematical study, and what distinctions are needed in order to improve explanation and prediction of what a teacher would do in those courses?

We apply Cohen et al.'s (2003) definition of instruction to model instruction in courses of mathematical study within the educational institutions that provide environments for such instruction. This allows us to restrict the content of instruction to that which is institutionally sanctioned as content for a course of mathematical study. For us, mathematics instruction concerns the interactions among teacher, students, and the mathematics content at stake in a course of study, or the knowledge designated to be learnt, in an educational institution (Herbst & Chazan, 2020; see also Chevallard, 1991). Therefore, building on Cohen (2011), when we talk about the work of teaching in mathematics instruction, we limit it to the work that teachers do which is deliberately oriented to—even if not effective in—producing the learning of the mathematical content at stake in a course of study. This definition allows us to describe the work of teaching (within mathematics classrooms) as involving tension between playing the role of teacher in mathematics instruction (i.e., managing transactions of content) and playing the role of teacher in other activities that (legitimately) depart from mathematics instruction (e.g., talking to the class about appropriate use of language or supporting a student's self concept). Such tensions create complexities that teachers must manage, as Hill and Lampert (this volume) remind us.

The distinction between instruction (as a focused activity) and other activities of teachers and the notion that all these activities legitimately compete for the time and energy of the teacher is not meant to discount the possible synergies among those activities (e.g., building students' self-concept might support students' doing the mathematical work related to the knowledge at stake). It does not mean we discount aspects of the teacher's work which are tangentially related to mathematical content; instead, the distinction provides us with analytic power. The distinction helps us describe variability in the work teachers do and build models of decision making in mathematics teaching that attend both to matters that are specific to the knowledge at stake and to matters that are more general about the institutionally sanctioned work of teachers. In turn, this helps us prevent too early a subsumption of subject-specific variability under larger, more general concepts. In examining the practical rationality of mathematics teaching, we center our efforts on the work the teacher does in mathematics instruction, the work the teacher does which is oriented toward students' learning of whatever mathematics is at stake at the moment and define this work to be the management of transactions of content with students.

3.2 Instructional Exchanges, Instructional Situations, and Instructional Norms

An important goal of our research within practical rationality has been to identify and confirm the existence of instructional norms, in particular norms that regulate what teachers are expected to do in instruction. In seeking to identify norms we have looked at them as reference points or benchmarks around which individual differences in practice distribute, even if individuals are not consciously aware of these norms. Because we seek a subject-specific language of description of teaching practice, we have been attentive to the role that mathematics plays in identifying those norms.

The notions of *instructional exchange* and *instructional situation* are the building blocks of a subject-specific theory of mathematics teaching in instruction, including their genealogical relationships with the more widely used notions of *task* (Doyle, 1983; Stein et al., 1996) and *didactical contract* (Brousseau, 1997). Later in this chapter, we describe how empirical research on instructional situations operationalizing associated ideas such as *norm*, *breach*, and *repair* supported creating local, descriptive, and testable models of teaching specific subject matter.

In other writing we have explained how the work of teaching in instruction involves the management of transactions of content with students and that these transactions include enabling and confirming instructional exchanges between two manifestations of content: content as instructional goals at stake and content as the specific mathematical work (the enacted mathematical tasks) to do or be done on behalf of the former (see also Chazan et al., 2016; Herbst & Chazan, 2012, 2020). Whereas the notion of didactical contract (Brousseau, 1997) has inspired our identification of some general norms for those instructional exchanges (e.g., the teacher has the right and is expected to pose problems to students; students are expected to show their work on problems), we have been more interested in norms that regulate specific, recurrent instructional exchanges. The notion of *instructional situation* (Herbst, 2006) designates local contracts for recurring instructional exchanges (e.g., solving equations, doing proofs; see Chazan & Lueke, 2009; Herbst et al., 2009) that frame expectations about specific mathematical work. Our research on instructional norms has, therefore, aimed at identifying the norms of instructional situations (also known as situational norms).

Our reading of the notion of didactical contract and offering of the notion of instructional situation help us operationalize three key commitments of our perspective on theorizing teaching. First, the norms we aim to identify and organize are subject-specific constructs in the sense that they make use of mathematics in the description, explanation, and prediction of teaching. Second, these norms are practice-specific in the sense that they find mathematical specificity in the practice of teaching mathematics rather than import this specificity from outside with categories brought over from the discipline of mathematics, even if our familiarity with the discipline helps us identify those norms. Third, these norms are instruction specific in that they account for the work of teaching in instruction, defined as the management of transactions between content as instructional goals and content as mathematical work inside educational institutions.

One can note something of a tension between commitments to subject-specificity and practice-specificity, which highlights why we have started this chapter with a discussion of our position as secondary-mathematics-teachers-become-education--researchers. In principle, that tension may be resolved subjectively. Both of us can commit to enforce self discipline in avoiding reductions of the practice of teaching mathematics to either a generic practice of teaching that brackets the content or to a mere application of mathematics that brackets the activity of teaching. Yet the management of that tension is also aided by the cognate theoretical constructs that in some cases we build on (e.g., contract, task, norm) and in some cases we differentiate from (e.g., activity structure) which require us to look at mathematics teaching practice from the stance of a detached observer. In what follows, we speak at length about instructional norms as one of the sociotechnical factors that help account for the work of teaching; they are sociotechnical in the sense that they describe ways in which humans handle knowledge in organizations.[2] The other set of sociotechnical factors we allude to includes the professional obligations of mathematics teaching (elaborated at length in Chazan et al., 2016) which include an obligation to the discipline of mathematics, to students as individuals, to the societal values and needs at stake in classroom interaction, and to the institutions of schooling.

The management of that tension is also aided by the expectations around scientific research in education (Shavelson & Towne, 2002). The extent to which the notions of didactical contract and instructional situation can be grounded empirically and evince some degree of intersubjectivity is precisely the purpose of the empirical research we have done and a crucial resource to build a theory of the middle-range.

3.3 *Coordinating Individual and Socio-technical Factors to Understand Teacher Decision Making*

The socio-technical factors alluded to above and discussed below in Sect. 7.4 are one set of elements we bring to understanding the work of teaching. The metaphorical equation in Fig. 7.1 shows how the constructs of practical rationality explain the

[2] A useful example to anchor the meaning of instructional norm is the statement that when solving equations in one variable teachers expect students to manipulate algebraically both sides of the equal sign.

actions or decisions a teacher makes (or will make) as dependent on two sets of factors. On the one hand, there are individual factors that the actions or decisions of a teacher can be seen as expressing. Knowledge, beliefs, and experience teaching are malleable examples of these individual factors that may change over time; other individual factors are more stable examples of those individual factors such as lived experience (e.g., as a member of a social group) or personality (Goe, 2007, calls the first group qualifications and the second characteristics). On the other hand, there are socio-technical factors that describe the context in which the individual is operating and that enable or constrain the actions of the teacher. The institutional position, within educational institutions of a society, is one source of description of the context; stakeholders of those institutions obligate the teacher as a professional. These obligations are identified generically by the four obligations named above, but the extent to which teachers are beholden to each of those obligations may vary (e.g., by school level, by culture). Within an educational institution, there are several activities that a teacher engages in to respond to some of those obligations (e.g., stewarding prosocial behavior is part of what American teachers are expected to do in high schools). Instruction is a particularly central one of those activities, and practical rationality seeks to account for the decisions and actions a teacher makes when engaging in this activity. To support accounts of how the activity of instruction impinges on the actions a teacher takes when engaging in that activity, practical rationality models that activity in terms of systems of norms—expectations on how teacher and students are to manage the content of studies. These norms include the contractual norms of a course of studies, and within a course of studies, the situational norms associated with the instructional situations in that course of studies. The socio-technical resources available to account for teacher action and decision making vary both by the mathematics being taught (different courses of study may include different instructional situations and hence different norms) and the institutional contexts within which mathematics is being taught (different institutions in different cultures may obligate teachers differently to their various stakeholders).

In our empirical work, we have been especially interested in explaining what decisions teachers make in lessons, particularly regarding how they present problems to students and how they respond to students' contributions. In that context, the metaphorical equation of Fig. 7.1 would describe the decision of what move to make as dependent not only on individual teacher resources (e.g., their resources, orientations, and goals, as described by Schoenfeld, Chap. 6, this volume) but also on the norms of the course of study in which the lesson is taught and the professional obligations of the teacher. The norms of the didactical contract may constrain, for example, what kind of problem might be posed and how the problem might be posed. The professional obligations associated with the role of teacher in school can serve to justify any departures from norms incurred by the decision to pose that problem. The instructional situations available in that course of studies may serve as resources for the teacher to frame, and therefore enable, the students' work on the problem, and the norms of the situation may condition how the teacher responds to what students produce. Those demands are socio-technical in the sense that they concern social as well as technical (especially mathematical but also psychological and legal[3]) expectations of how teachers do their work. Like in other settings, it is clear that individual agents might be able to act in ways that deviate from norms or that fail to abide by obligations, likely with the assistance of personal resources, and that actions that deviate from those expectations will require special justification, while actions that fulfill expectations may go without comment.

Both the metaphorical equation in Fig. 7.1 and its use to explain the decisions a teacher makes illustrate how practical rationality handles various types of specificity in describing teaching. The institutionalized nature of some mathematics teaching in schools, for example, as contributing to the societal function of schooling, in contrast with other kinds of mathematics teaching (e.g., in camps, clubs, or at home), is represented by the professional obligations. Those who teach outside of schooling institutions may be subject to obligations to other or fewer stakeholders. In particular, society as the source of the interpersonal obligation[4] may act as a stakeholder of mathematics teaching in different ways in different countries, promoting the classroom cultivation of different social values in different societies, and within a given society, different school organizations (e.g., primary or secondary schools, universities) may obligate teachers differently. Within a kind of organization (e.g., secondary schools), the didactical contract may have different norms in different courses of study and within a course of study, different instructional situations may create different expectations (e.g., the norms a teacher recognizes for how they have to outfit a diagram when including it in a geometry problem may depend on whether the diagram will be used in a proof, a calculation, or a construction).

3.4 One Reason to Create a Middle-Range Theory of Mathematics Teaching

In developing practical rationality, we were mindful that we wanted to affirm the role of mathematics in the description of the work of mathematics teaching in such a way that this role would persist when data was aggregated to construct measures and test hypotheses. Instructional situations are a key element of the theory in that they afford ways of focusing on teacher decision making about mathematics and ways of finding commonalities across different instances of mathematics teaching. Toward this goal, an important contribution of our work has been the identification and empirical demonstration of the norms of instructional situations (or situational

[3] The technical part of sociotechnical alludes to all the disciplinary bases of professional practice. The mathematical basis of professional practice is salient for mathematics teachers, but insofar as legitimating how teachers are supposed to attend to the needs of children or conduct themselves within the confines of the workplace, other technical bases are relevant (e.g., psychology and the law).

[4] Chazan et al. (2016) name the interpersonal obligation to describe how society obligates teachers to steward social values and needs (e.g., social equity, work ethic, civic and prosocial behavior, etc.).

norms). These are subject-specific statements, in the sense that the statements of the norms use mathematics to describe the work a teacher or their students are expected to do. They are also empirically verifiable statements, in the sense that we can demonstrate at scale that teachers reliably recognize the differences stated by those norms across instances of teaching work that have other surface similarities and differences (Boileau, 2021).

Going back to the statement of our positionality, as mathematics-teachers-become-researchers, we are disposed to see and propose structures to which we can now see ourselves having been adapting when we were teachers. At the same time, we cannot shed the agency and responsibility we perceive we had as teachers, which included making consistent mathematical considerations across instances not only in consciously drawing on the resources we had available but also in tacitly adapting to the instructional contexts in which we were working.

Our positionality, in turn, enables us to see the work of teaching as the making of decisions in spaces where there are normative expectations as well as justifiable alternatives. Situational norms are represented in observer statements that describe what, as practitioners, we may have experienced as compelling demands without having mental representations of them but that now, with the support of scholarly uses of intellectual assets like norm, experimental design, instrumentation, psychometrics, Systemic Functional Linguistics, and so on, we can turn into measurable constructs (namely, we can define, detect, and measure practitioners' recognition of a norm). In doing that, we expect that such knowledge will help explain what actions practitioners take in the field, without presupposing that the practitioners themselves are, or need to be, explicitly aware of the norm statements we make to describe the regularities of those actions.

Thus, at its core, practical rationality suggests that we can do scientific research that builds a mathematically specific theory of mathematics teaching. But while the possibility of such an account is apparent, the necessity of such an account may seem compelling only for researchers who are disposed to using the resources of social research to model the perspective of the practitioner in accounting for the practice of mathematics teaching. This is one obvious way in which practical rationality is value laden, like any theory. In our view, the representation of the mathematics teacher's perspective on mathematics teaching articulated from the researchers' perspective using the theoretical resources represented in Fig. 7.1 is a compelling way to construct mathematics teaching as an object of study, a way to represent the work of mathematics teaching that overcomes the illusion of transparency. The value of such an endeavor, for us, lies in the possibility of creating a basis for research, instructional improvement, and teacher advocacy that is rooted in what a mathematics teacher can perceive and appreciate, given the conditions and constraints in which they work. Those eventual ends (instructional improvement, teacher advocacy) resemble those proposed in other chapters (e.g., Cai et al., Chap. 8, this volume; Hiebert & Stigler, Chap. 2, this volume). However, along the lines that practical rationality is not a grand theory but rather a theory of the middle-range, the ideas sketched out above regarding norms, obligations, and the way they may complement personal resources in accounting for actions in teaching only provide language to scope a terrain of work, they do not flesh out the theory. Rather, the theory is built through the practice of research scoped by those ideas. To exemplify that, we now show how the theoretical ideas of instructional situation and norm have inspired empirical research and how this empirical research has begotten insights that expand the theory of practical rationality.

4 Empirical Research on Situational Norms in Instruction

Having introduced key elements of practical rationality, we now illustrate its character as a theory of the middle range by demonstrating how empirical research has supported its growth. Because these examples illustrate how empirical research has supported the development of our understandings of instructional norms, in this section we elaborate on the notion of instructional norm and describe how we have studied the recognition of norms empirically.

By norm we mean the statement, made by an observer, of what participants in a social setting behave *as if* they held as appropriate and expected to do when they relate to each other and to the stuff they handle (including mathematical stuff). In making that definition, our use of *statement* aims to take stock of the critique of objectivism formulated by Bourdieu (1990; Taylor, 1993). This notion of norm emphasizes the role of the observer in stating what may not exist as a rule statement in the participants' social experience and may not even be stateable as a rule for participants because participants never experience the need to make such a statement. In this sense, the norm statements an observer makes may be tacit knowledge (of the *collective tacit knowledge* type; Collins, 2010). Such statements reveal themselves as knowledge to an observer because participants acquainted with the practice act in ways that others do not.

The following considerations of the definition of norm are important in order to understand the type of empirical research on norms we have engaged in. Since the word *norm* is charged with a variety of meanings, we underscore that, in our usage, norms are observer constructs and contrast with two other usages which have valid but limited associations. On the one hand, the word *norm* tends to be associated with what is correct and with prescription. The association here is valid in the sense that an observer who states a norm, states that participants behave as if doing what the norm states is correct and as if that is what participants think they are expected to do. However, the association is not valid in the sense that in stating the norm the observer is not rendering their own judgment as to the appropriateness of those actions or of the expectations recognized by participants. The observer's statement is descriptive of what appears to be a prescription (albeit, often tacit) for participants.

On the other hand, the word norm tends to be associated with frequency and with a distribution of observations. The association here is valid in the sense that an observer should expect actions that take place at moments when a given norm would be activated to form a distribution. However, it is not the case that these actions have to be exact instances of behaviors described in a norm statement for an observer to record an instance that might count toward documenting the actual hold of the norm in practice. As Garfinkel (1967) and others have noted (Mehan & Wood, 1975), participants often use repair strategies when their actions depart from the norm, thereby signaling that a norm is being breached. In order to use a frequentist interpretation of norm, an observational approach to research on norms should attend not only to the presence of compliant actions but also to the presence of repair strategies.

Although we have used video records of instructional practice and analysis of those video records to support the statement of norms through abductive reasoning (Dimmel & Herbst, 2018; Herbst et al., 2009), our empirical work on instructional norms does not define norms as descriptions of what the majority of people do. Rather, norms represent socially shared expectations of what people ought to do. The gathering of empirical evidence that could enable us to claim that these norms describe what participants experience as expectations has required some innovation.

4.1 Virtual Breaching Experiments: Designing Studies of Instructional Norms

Building on Garfinkel's ethnomethodology (Garfinkel, 1967; Mehan & Wood, 1975), we developed a methodology of virtual breaching experiments (Herbst & Chazan, 2015) that consisted of engaging practitioners with representations of practice in which hypothesized norms of practice had been complied with or breached. We attended to the descriptions of and reactions to the represented practice that practitioners offered. In such discourse and evaluations, we found evidence toward confirming the hypotheses made.

Our virtual breaching experiments first used video records (Nachlieli et al., 2009) and animations of classroom scenarios (Chazan & Herbst, 2012; Chazan et al., 2012; Herbst et al., 2011) with focus groups of teachers. Then the virtual breaching experiments used online scenario-based questionnaires responded to by individual teachers, where scenarios of practice were represented using storyboards of cartoon characters (Buchbinder et al., 2019; Dimmel & Herbst, 2017, 2018; Herbst et al., 2018). The decision to engage empirically with norms in these ways aligns with the goal of maintaining the subject-specificity of the norms on which we focus. The statements of the norms of instruction refer to elements of the instructional situation that regulates a type of instructional exchange, hence they use mathematically-specific as well as practice-specific language. The decision to engage empirically with norms in this way has also allowed us to control for surface content variations (e.g., topics, task statements) that would inevitably have to be dealt with if using a frequentist notion of norm and an observational research approach.

Our empirical work has been oriented to establishing the viability of norms, namely the extent to which a norm statement fits (as opposed to matches; see von Glasersfeld, 1991) the practice that it describes, as this practice is attested to by practitioners. Because these norms may be tacit knowledge from the perspective of practitioners, we could not trust the efficacy of traditional surveys that might pose general statements of the norm and ask practitioners to rate whether they agreed that those statements described actions they considered normative. This critique of traditional surveys follows the goal to overcome the traps of objectification and subjectification noted above—neither the reflected experience of practitioners nor the explicit language of researchers are adept to study the practical rationality of mathematics teaching.

Virtual breaching experiments confronted practitioners with multimodal representations of instances of the practice (initially video records or video animations, later storyboards with cartoon characters) in which the participant expected the norm to apply, but the norm had been breached. These experiments produced artifacts in which we could observe the participants' reactions to those representations. How these reactions were mediated mattered in how the viability of a norm was assessed. We used three kinds of artifacts: (1) group conversations in which verbal reactions and commentary to videos or animations were offered; (2) individual written reactions to storyboards in response to prompts to describe what they saw in an episode or to say more about a rating they provided; (3) individual ratings of the appropriateness that the participant attributed to actions represented in storyboards. In all cases, these breaching experiments were virtual because they confronted participants not with actual events where a norm was breached but with representations of those events. The nature of the data collected required us to distinguish different operational constructs that bridged the general notion of a situational norm to the particulars of the data we collected.

Our earlier work with virtual breaching experiments was done having groups of teachers of a course of study engage with video records or animations. Kosko and Herbst (2012) exemplified how we drew from Halliday's systemic functional linguistics (SFL; Halliday & Matthiessen, 2004), especially from what SFL calls the *modality* system of language, for linguistic indicators that participants were repairing on the breach of a norm. Modal verbs (e.g., should) and adverbs (e.g., always) were used as possible indicators of what Lemke (1998) called attitudinal meanings. When these modality resources were associated with actions of the teacher (or student) in events where a norm had been breached, we took that as possible evidence of participants' recognition of the breach of a norm. While this data modeling allowed for some quantitative analysis (Herbst & Kosko, 2014a), there were limitations, both in the data model and in the sample size, that threatened the construct and internal validity of any claims that norms were viable descriptions of participants' expectations of practice. Specifically, our data reduction model attended to a limited set of lexical items in turns of speech within a group conversation. Consequently, we could not consider the sample as composed of individual teachers but as composed of interactions among teachers in a single group. Group sessions could be parsed into smaller intervals demarcated by participants' own ways of organizing interaction and then intervals of conversation could be inspected for evidence of repair of the breach of a norm (Herbst et al., 2011), but this method lacked any systematic search for counterfactuals. Some advances in linguistics and our own move to developing multimedia surveys using storyboards with cartoon characters allowed for us to improve the methods used.

As we moved to analyzing data that included written descriptions of storyboards (paragraphs authored by a person in response to an online storyboard of instruction), we were able to ascribe orientation toward a norm to an individual based on what they wrote and collect such responses in larger numbers. Martin and White's (2005) appraisal theory, a contribution to a systemic functional analysis of discourse, was key for us to grasp the discursive—as opposed to merely lexico-grammatical—nature of the linguistic realization of repair of a breach and move beyond modality as indicator of attitudinal meanings. Two different empirical manifestations of the norm became useful to consider. On the one hand, participants' descriptions of what they saw happening in a scenario had the chance to include discursive elements that alluded to the aspects of the norm that had been breached. They could also use discursive resources to indicate their attitudes toward those events. Coders could reduce that data accordingly, distinguishing individuals' recognition of the norm in their responses to scenarios. On the other hand, participants were asked to rate the appropriateness of the teaching they had observed. We were able to create similar scenarios that did not stage breaches of a norm and asked the participants the same questions, which helped provide a baseline against which to measure the effects of breaches of norms. These online questionnaires were eventually used with a nationally distributed sample of high school mathematics teachers (Boileau, 2021; Herbst et al., 2018).

5 How the Analysis of Research Data Contributed to Theory Development

In this section we discuss how the analysis of empirical data collected to examine viability of norms contributed to the theorization of practical rationality. We present three cases. In the first, the study of a norm in algebra led to our better understanding of how norms of the global didactical contract and norms of the instructional situation interact. In the second, the study of a norm in geometry led to revisions and elaboration of the norm itself. In the third, the study of repairs of a norm led to proposing the professional obligations as a new construct.

5.1 Becoming Aware of Tensions Between Situational and Contractual Norms

Our study of teachers' recognition of the norms in solving equations in algebra focused on what norm teachers recognize for responding to students' solving of equations. As in other cases, teachers were offered opportunities to react to scenarios that either breached or did not breach the norm, and we coded the participants' descriptions of scenarios for evidence of recognition (or lack thereof) of norms when norms were (vs. were not) breached. The coding of participants' descriptions required us to attend to four basic contingencies: whether actions described by the norm are present or absent in the representation provided to the participant and whether the actions described by the norm are present or absent in the participant's description.

Some norms further complicated matters in terms of participants' repair of what was expected of the teacher. These were norms that we came to call *tactical*, inasmuch as they described what the teacher was expected to do in response to possible student actions.[5] In representing the work to be described, both the student actions calling for the teacher's work and the work of the teacher need to be included, and doing this required attending to more than two theoretically distinct possibilities: whether the response from the teacher could be normative or not and whether the student work to which the teacher was responding could be normative or not. In coding the descriptions of those events, coders needed to be attentive to whether and how participants described the events that called for the teacher's intervention, as well as whether and how they described the teacher's intervention. An early example of how empirical work led us to advance the theory comes from the examination of participants' responses to the way a teacher responded to students' use of alternative solution methods in solving equations in one variable.

Our work on the solving of equations has focused on the solving of linear equations in one variable, where a *canonical method* for solving equations has developed as the teaching of algebra has become a part of institutionalized schooling (Buchbinder et al., 2015). This canonical method involves manipulating the expressions on both sides of the equal sign in a set order: gathering linear terms on one side and constants on the other first, operating on those separately, eventually dividing the constant by the coefficient of the linear term. In our modeling of this instructional situation (Chazan & Lueke, 2009), the instructional exchange involves students submitting work that uses the canonical method to solve linear equations and the teacher's judging students to have learned to solve equations. Building on a variety of calls to reform the teaching of the solving of linear equations (Star & Seifert, 2006; Yerushalmy & Gilead, 1997) as a way to breach our model of the situation, we represented student work that offered other mathematically correct, symbolic solutions that nevertheless did not follow the canonical method. These included, for example, dividing an equation through by a common factor or simplifying the equation by an implicit change of variable (e.g., treating $x + 1$ as a variable).

Our survey instrument of norms in the instructional situation of solving equations explored the hypothesis that participants would consider appropriate for a teacher to discourage solutions that did not use the canonical method, requesting a more usual solution regardless of whether the answer found was correct. By

[5] Tactical norms are circumstance-dependent norms, while strategic norms are goal-dependent norms.

contrast, the depicted teacher's response to canonical solutions was to accept the solution and to move on to a new problem. Our hypothesis was that, when asked to judge the appropriateness of the teacher's response to the student's work, teachers would find it more appropriate to dwell on students' use of the canonical method. Yet not all responses to the non-canonical solutions fit our hypothesis. In some cases, the teacher's shifting of the class's attention away from alternative solutions and towards the canonical method were evaluated as somewhat inappropriate (Buchbinder et al., 2019). Our qualitative analysis of the open-ended responses suggested that participants were taken aback by the teacher's lack of acknowledgment of the correctness of the student's response. For example:

> The teacher was too dismissive, not acknowledging Blue's correct answer.

> 'Usual way' says to the student that they did something wrong, when in fact their math was correct. Suggesting or hinting that a student did something wrong mathematically is wrong and will cause students to shut down.

These critiques were of how the depicted teacher had spoken to the student but did not suggest that the depicted teacher had missed an important opportunity to engage the class in justification of methods used for solving equations. In other words, the teacher was not doing something wrong in terms of the requirements of the instructional situation. There was something else they were violating. These comments were not directly targeted at the negative reaction to the student's method, which they might agree did not use the expected method. The comments were instead targeted at the teacher's lack of acknowledgment that the students had produced a (correct) answer to the problem. In terms of instrument design, this observation was developmental for us at the pilot study stage and suggested that before responding to what method the student had used, the depicted teacher should thank the student for their contribution. However, the observation raised a more important theoretical point, which had to do with the interaction between contractual and situational norms.

The expectation that the teacher should respond to students' contributions has been documented in the literature as part of the default pattern of interaction in classroom recitations, whereby the teacher is expected to evaluate what students say in responses to questions (Mehan, 1979). Over the years, professional development on classroom discourse has sought to provide teachers with resources to respond which provide better alternatives to Evaluation (e.g., Milewski & Strickland, 2020). Therefore, it is reasonable to assume that didactical contracts may still lay on teachers the expectation to respond to students' contributions, although the expectation that such response be an evaluation of the student's contribution may be more variable. These observations about norms of the didactical contract present an interesting backdrop against which to set observations about the instructional situation of solving equations. Inasmuch as the initiation by the teacher, posing an equation to be solved, may frame students' work in the context of this instructional situation, there are expectations on the student as well as on the teacher. Some of those expectations are contractual (e.g., students have to do work and offer it for scrutiny), and some are specific to the situation (e.g., in solving equations, the students have to use the canonical method). Our participants' expectation that the teacher would acknowledge students appropriately for volunteering work in response to a problem is an example of a norm of the didactical contract. This is made even more important when, as in the storyboard used in this case, the work students had done resulted in the correct value for the unknown in the equation, albeit through non-normative means. The framing of the work as solving equations also activated the expectation that teachers would use cases of solving equations to provide students with practice in a method of solving equations, but the specific equations and the specific values of the unknowns in those equations were not expected to have intrinsic value. Our participants therefore put us before an interesting theoretical problem that seems to have some generality within the theory.

The problem is in general one of how contractual and situational norms interact. Our stumbling upon it revealed that we had made an assumption about the relationship between contract and instructional situation, and that assumption should be questioned. In proposing instructional situations as local contracts for recurrent instructional exchanges, we seemed to have assumed that the relationship between situational norms and contractual norms is one of inclusion, namely that every situational norm is a perhaps more specific instance of a contractual norm and that compliance with a situational norm would imply compliance with a contractual norm. In the example being used here, providing corrective feedback on the lack of use of the canonical method for solving equations seemed like a teacher's compliance with the contractual expectation to evaluate students' work.

The data from the participants in the algebra survey not only brought that assumption to question but also reinforced two important points at the base of the theory itself. The first point is one that Brousseau (1997) made, that the relationship between the teacher, student, and content needs to be maintained against all odds. A breach of contract rarely ushers in a state of anomie; instead, it calls for a negotiation of a new contract, even if this negotiation is reduced to the teacher's statement of a new rule or making a new allowance. This seems to be highly visible in how our participants expected the teacher to respond, facing the fact that students had voluntarily offered contributions that, though correct, were not the preferred ones. Their responses, facing a teacher who had actually complied with the expectation to discourage students' dispreferred even if correct and effortful responses, was to propose more sympathetic reactions, acknowledging the students' responses. Our participants seemed aware that the hypothetical teacher's interactions with the students would continue and the teacher would need to procure students' participation in the future. This required the teacher to acknowledge the students' responses. As a result, revisions of the items included first some gratitude from the teacher for having done work, followed by feedback on the way the student had solved the equation. However, the second point suggested that the assumption of alignment between situational and contractual norms is itself questionable; this suggested the need to revisit the theory.

The second point is that an instructional situation involves students in work with some specific content that instantiates not only the content at stake but also, possibly, other valuable mathematical properties. Although the equations presented put at stake knowledge of solving equations and had been assigned as opportunities for students to practice the canonical method of manipulating both sides of the equal sign (Chazan & Herbst, 2012), they also put in play objects like specific expressions, numbers, and operations and their specific relationships (e.g., numbers might have common factors, expressions might have common factors). The work with these specific objects may therefore elicit valuable knowledge from students and, as Herbst and Chazan (2020) note, knowledge that may only come up in the context of work assigned for the sake of opportunity to learn something else. In the case of these students' non-canonical solutions of equations, not only had they found the correct number for the unknown but they had also used properties (e.g., factoring numbers or expressions) that could be valuable within the contract at large. Thus, a teacher that enforced the norm of the situation by noting the dispreferred nature of the students' work might be seen as breaching a contractual norm by not allocating value to what the students had done, which might also be contractually valuable. More generally, this data suggested to us that while norms of a situation may be related to norms of the contract along the lines of specific (situational) to general (contractual), compliance with a situational norm may still involve a breach of the didactical contract and call for negotiation. Recommendations for teachers to use problems that allow for several solution approaches are instructional circumstances where such conflicts may occur regularly for teachers. The elements of the theory, specifically the notion that situational and contractual norms may oppose each other, seem like a useful analytic tool to describe those experienced conflicts.

5.2 Developing a More Precise Formulation of a Situational Norm: The Diagrammatic Register

In the second example of how theorization benefited from empirical work, we briefly recount a story told by Herbst et al. (2013) about the development of a more precise formulation of a situational norm as a result of difficulties instrumenting the study of its less precise version. From the analysis of teachers' responses to animations, Weiss and Herbst (2007) had proposed the norm that *proof problems in high school geometry are presented in a diagrammatic register*[6]—by which they referred to a difference with how geometric theorems and their proofs are presented in the discipline (e.g. Hilbert, 1902). In the discipline, geometric theorems and their proofs rely on an interaction between two registers: a conceptual register, in which theorems state general properties of concepts and a generic register, in which generic objects that represent those concepts are selected in order to be used in the proofs. While theorems in high school geometry are often also stated in conceptual terms, proof problems are often not; rather, they are stated in terms of particular objects

[6]The use of the word register in this context is connected to that of Duval (2006). We have not worked out its compatibility with the SFL notion of register (see also Morgan, 2006).

(Otten et al., 2014). Moreover, these objects are usually not generic but diagrammatic, in the sense that they avail themselves of properties not only by what is explicitly predicated of them but also by how they are presented in a diagram (Laborde, 2005). We formulated the hypothesis that in assigning proof problems to students, a teacher is expected to present those problems using a diagrammatic register.

With that general conceptualization in mind, we created storyboards in which a teacher assigned proof problems but breached the diagrammatic register norm. At the time, our attempts to operationalize what a breach of the diagrammatic register norm could be were only guided by a general sense of what the diagrammatic register was. We thought, for example, that not including a diagram or referring to a given diagram using the names of the concepts involved in the problem would constitute breaches of the norm. Yet we did not have a precise statement of the diagrammatic register norm. The instrument we created included a set of five storyboards representing breaches of that sense of the diagrammatic register norm, each of which required the participant to describe what they saw happening and to rate the appropriateness of the way the teacher had presented the problem. When we looked at the pilot data results, we noticed very low internal consistency among the appropriateness ratings for those items. This low internal consistency prompted us to ponder whether we really had clearly identified the properties of the diagrammatic register. We thus attempted to spell out what "complying with the diagrammatic register norm" could mean in terms of simple clauses and arrived at six of those that, while expectably related in practice, could be separated for analysis. In order to study them empirically, we designed a different type of instrument with questions that asked participants to choose between two ways of presenting a proof problem and where in each choice only one of the hypothesized properties of the diagrammatic register was the source of the difference between the problem presentations. As a result of analyzing responses to that instrument, we arrived at a third specification of the norm that maintained the five properties of the diagrammatic register that could be confirmed.

Our current conceptualization of the diagrammatic register norm includes five assertions about how proof problems are presented: (1) a diagram is included to represent the figure alluded by the proof problem, (2) the diagram represents with relative accuracy the properties that are true about the figure alluded by the proof problem, (3) the diagram has labels for the points referred to in the proof problem and for others which are useful for the proof, (4) the statement of the proof problem refers to geometric objects using the labels in the diagram, and (5) the statement of the proof problem asserts properties about congruence, parallelism, and perpendicularity while it does not state explicitly (but relies on the diagram to communicate) properties of incidence, collinearity, and separation. A sixth assertion, that the diagram includes diacritical markings to represent properties of congruence, parallelism, and perpendicularity given in the problem statement, was not confirmed to be normative. This identification supported our design of new scenarios for implicit norm recognition that resulted in a set of items with better internal consistency. Eventually these revised items enabled us to show that teachers are more likely to react to the breach of the diagrammatic register norm (recognizing the breach of any of the five components) than to scenarios in which all components of the diagrammatic register norm have been complied with (Herbst et al., 2016).

This second example shows how instrument development and empirical research on norms has been useful, not only to confirm aspects of the theory but also to refine the theory by helping us arrive at a more specific statement of the norm. The second example also shows that situational norms can be tacit and subject specific. The diagrammatic register norm illustrates how participants may reliably recognize aspects of the norm when they are breached even if these are not explicit to them when they construct their practice. The diagrammatic register norm also illustrates the subject-specific nature of the norm—it describes the acts of teaching in terms of actions on geometric diagrams and ways of referring to and reading them. These expectations on how a teacher has to present a proof problem do not easily or validly generalize to considerations of communication modality or literacy but rather require attention to geometry and proof, while supporting some generalization across geometric figures and across the properties of those figures being proved. The diagrammatic register norm is, therefore, not only an example of how research on practical rationality supports the development of the theory of practical rationality but also of how this theory of teaching pursues subject specific statements of norms made by an observer to describe teachers' acting as if they were following them. Furthermore, the example shows that this subject-specificity of the theory in describing teaching is not reducible to combining generic pedagogical moves with specific mathematical topics. Rather, our subject specific approach requires a disposition to generalize across similar instructional exchanges.

5.3 After Detecting Breaches of Norms: Justifying Actions and the Professional Obligations

The third example illustrates how empirical research on situational norms led to our proposal of new elements of the theory. The technique of virtual breaching experiments (Herbst & Chazan, 2015) has been used to confirm that our proposed norms of instructional situations fit with the reactions from teachers to representations of practice (Boileau, 2021; Dimmel, 2015). But the study of how participants responded to breaches of norms also provided more concrete insights into the rationality of teaching. Brousseau's argument that the didactical relationship between teacher and student needs to be maintained at all costs can also serve to understand what may happen if a task is originally framed in the context of an instructional situation but its norms are breached. Consider, as an example, an episode we recorded on video, where a student was doing a proof at the board and after making a statement could not come up with a justification for it. Instead of insisting that he justified the statement, inviting another student to it, or providing the justification himself, the teacher encouraged the student to move along with the proof, making the next statement, while leaving the justification blank with the idea that they would come back later to the missing justification (Herbst & Chazan, 2003). When we used this episode in virtual breaching experiments with teacher focus groups, we noticed that participants not only indicated discomfort or pointed to what the teacher had done as being unexpected, they also provided justifications or rationalizations either for what had been done or for what they thought could have been done instead (Nachlieli et al., 2009).

We started documenting these rationales when introducing the general idea of practical rationality (Herbst & Chazan, 2003) with the intention of mapping the competing commitments and dispositions that often justify different decisions in teaching. We expected then that, while individual teachers might differ in what they decide to do, the grounds they use to justify what they decide to do in front of colleagues might have some commonalities. We then named those commonalities *the professional obligations of mathematics teaching* (Herbst & Chazan, 2012) and identified four: disciplinary, individual, interpersonal, and institutional. Subsequently, Chazan et al. (2016) elaborated on the conceptualization of the obligations as sources of public justification for teachers even though those sources may not obligate individual teachers, or groups of teachers across institutions or cultures, in the same way. Chazan et al. (2016) elaborated theoretically on how the same obligation could relate different dispositions (or commitments). For example, a disposition to challenge individual students intellectually and a disposition to care for students' emotional wellbeing might justify alternative decisions, but the common obligation to individual students could serve as the grounds upon which to compare and critique those alternative decisions, and perhaps also find a compromise.

In the intervening years, the concept of professional obligations has been used to investigate sources of justification for instructional decisions that deviate from the norm (Bieda et al., 2015; Chazan et al., 2012). We have also developed instruments that could detect participants' recognition of the different obligations. The PROSE (Professional Obligations Scenario Evaluation; Herbst et al., 2014; Herbst & Ko, 2018) instrument is made of items in which a scenario is provided wherein a teacher is seen departing from a contractual norm in a way that we might consider attends to a professional obligation, and respondents are asked to indicate their degree of agreement with a statement that says the teacher should have stuck to the instructional goal. This instrument has been used both with high school mathematics teachers and university instructors, and we have found not only that it is possible to measure recognition of the obligations using it but also that these items may be used to compare recognition of obligations across instructors of different levels of schooling (Ko et al., 2021).

6 Theorizing by Connecting Practical Rationality Constructs to Those from Other Theories

An important goal of practical rationality is to explain the work of teaching, especially the decisions that teachers make and the actions they take in instruction. The concepts of instructional situation, norm, and obligation that have been developed through our research account for the socio-technical characteristics of the work of teaching, providing tools for understanding structures that form the context of instruction.

In characterizing practical rationality as a middle range theory we indicated two aspirations. One is to develop the constructs of practical rationality in relationship with research operations, as illustrated in Sect. 7.5. The other is to accommodate relationships with constructs from other theories. Section 7.5 provided three examples of the first aspiration, this section now turns to the relationships between constructs we have developed and existing constructs.

The individual resources teachers bring with themselves to the work of teaching have been a focus of research on mathematics teaching for decades, especially through programs of research that focused on teachers' beliefs and knowledge (see the review by Herbst & Chazan, 2017). Theory that explains teaching as an expression of teachers' individual characteristics and resources (e.g., Schoenfeld, Chap. 6, this volume) has been in the mainstream of research on mathematics teaching since the mid 1980s and has provided important constructs and measures. But these individual-centered approaches have shown limits, theoretically, in failing to sufficiently account for how various interpretations of context affect what individuals believe, know, and do. More limited are their practical implications; individual-centered accounts of teaching can lead to descriptions that highlight deficits in individual teachers and support policies for instructional improvement that rely only on improving individuals by developing in them the proper beliefs, knowledge, or skills.

Rather than ignoring individual-based explanations, we have been interested in investigating how individual-based explanations of the work of teaching could be connected to explanations that use the constructs of practical rationality to describe the socio-technical context of the work of teaching. By attending both to the individual resources teachers bring to the work and to the ways in which those resources adapt to the socio-technical characteristics of the work itself, we aim to craft better descriptions of teacher decision making.

6.1 Connecting Mathematical Knowledge for Teaching to Practical Rationality

The interest in explaining the work of teaching as requiring professional knowledge has been a mainstream trend in research on mathematics teaching and teacher education in the last three decades. Highlights have been the conceptualizations of mathematical knowledge for teaching (MKT; Ball et al., 2008) and scales to measure it (Hill et al., 2004). Our work has sought to investigate relationships between the construct known as MKT (mathematical knowledge for teaching; Ball et al., 2008) and practical rationality.

This rapprochement started with a modest theoretical reconciliation aimed at developing a measurement instrument; we adopted the domain definitions and heuristics for item development from Hill et al. (2004) to create items that measured knowledge at stake in the U.S. high school geometry course of studies, which resulted in the MKT-G test (Herbst & Kosko, 2014b). The use of this instrument permitted us to observe significant associations between experience teaching geometry and MKT-G scores that could not be accounted for by experience teaching secondary mathematics in general. Furthermore, Herbst and Kosko's (2014b) examination of single item responses led to the conjecture of a relationship between instructional situations and teacher knowledge. The effects of experience teaching geometry were especially noticeable in MKT-G items that were contextualized in instructional situations that recur in geometry courses (e.g., geometric calculation), whereas items contextualized in novel tasks were equally difficult for teachers with different experience teaching geometry. In an effort to better connect MKT with instructional situations, Ko (2019) was able to show that it is possible to create psychometrically distinguishable scales to measure the mathematical knowledge for teaching needed in different instructional situations (including geometric calculation and doing proofs).

6.2 *Connecting Teachers' Beliefs to Practical Rationality*

As another example of how research on practical rationality has looked for ways to reconcile the constructs we developed with those that sought to account for the work of teaching using teacher beliefs. The relationship between beliefs and practice has been a persistent theme in mathematics education research on teaching since the 1980s (Leder et al., 2003). Some researchers have inferred beliefs from practice, while others have used the inconsistency between beliefs and practice as a source for questioning the conceptualization of teacher beliefs (Philipp, 2007). The theme is also present in Schoenfeld's ROG theory (Chap. 6, this volume).

Shultz (2020, 2022) explored relationships between university instructors' recognition of professional obligations, beliefs they hold about teaching and learning, and their use of particular instructional practices. She used our PROSE instrument for college instructors along with Clark et al.'s (2014) beliefs questionnaire and her own INQUIRE instrument which gathers instructors' self-reported use of inquiry-oriented instruction practices. Her findings show the potential for obligations to explain why inquiry-supporting beliefs espoused by instructors might not be reflected in their reported use of inquiry-oriented practices. For example, the practice of having students make presentations was less present than expected based solely on student-centered beliefs (e.g., that students should be allowed to struggle), but a moderate negative correlation with recognition of the disciplinary obligation helped explain it—instructors with high recognition of the disciplinary obligation would gravitate less to having students give presentations, regardless of their beliefs that students need to struggle (Shultz, 2020).

We bring this short example here to show how, as expected from middle range theories, practical rationality is capable of assimilating constructs developed outside of this theoretical perspective (e.g., beliefs, inquiry oriented instruction) and offer a possible solution to pre-existing theoretical problems (namely, the inconsistency between beliefs and practices could be reconciled by accounting for a measure of recognition of the disciplinary obligations).This study has helped support a basic proposition of practical rationality, whereby the decisions that teachers make are explained in relation to a combination of individual factors (the knowledge or beliefs individual instructors have) and social factors, including ones associated with the role of the teacher in instruction and ones associated with the position of the teacher in an educational institution.

6.3 The Uses of Practical Rationality

Beyond its scientific contribution to the understanding of mathematics teaching practice, practical rationality has much to offer to the work of researching the connection between instruction and learning, as well as professionalizing the practice of mathematics teaching and improving this practice. The parsing of instrumental research on teaching proposed by Hiebert and Stigler (Chap. 2, this volume) between theories that describe how teaching produces student learning opportunities and theories that describe how those learning opportunities produce student learning allows us to locate practical rationality as providing an instance of the first group of theories. The research agenda of practical rationality can serve to explain the learning opportunities afforded by intact lessons and identify grain sizes for local instructional theories (e.g., instructional exchanges) and variables that can be manipulated (e.g., norms) to investigate the viability of generating conceivable opportunities to learn (similar to "teaching for theory" in Cai et al., Chap. 8, this volume). Basic research characterizing instructional contracts and situations and their norms across courses of study, school levels, and cultures is an important prerequisite for that kind of improvement-oriented research.

Hiebert et al. (2002) have argued for the need for a professional knowledge base for teaching and highlighted the role of lessons in that knowledge base (see also Cai et al., Chap. 8, this volume). The concepts of practical rationality are useful to conceive of those lessons in terms of choices made from systems of possibilities, where those possibilities are borne of personal and socio-technical resources. Hiebert and Stigler (2017) have recommended that improvement efforts shift from being focused on improving teachers to improving teaching. Practical rationality can support a focus on improving teaching by improving the teaching of lessons, as the theory provides means for mapping the choices available for teachers as they manage a lesson. The notions of instructional situation and professional obligations can be articulated to form local instructional theories (Gravemeijer, 2004; cf. Cai et al., Chap. 8, this volume, notion of "teaching for theory") in a mathematical course of study and for specific conceptual development. The choices offered by the theory may become usable for teachers in the form of conditional rules for their management of a lesson (e.g., whether to frame work using one or another instructional situation) and may also be inscribed in artifacts (worksheets, diagrams, software applications) that the teacher can choose to use to support their work. The subject-specificity of the theory is essential not only for identifying the pertinent choices a teacher can make when teaching the lesson but also to orient the choice of lessons that might be useful to work on as contexts for improving teaching. It is worth stressing that reform notions like engaging students in productive struggle or in cognitively demanding tasks are observer-centered notions. For them to be operational for teachers, they need to be anchored in practice—the norms of instructional situations are such anchors. Both research questions (e.g., how does engaging students in productive struggle vary in teaching difficulty or in the qualities of the student learning opportunities created across the instructional situations of a course of studies) and improvement questions (e.g., what does it take to enable teachers to engage students in productive struggle across the instructional situations of a course of study?) are feasible to ask using the concepts of practical rationality.

The involvement of practical rationality in designing the improvement of teaching, however, requires better understanding the relationships between teaching practice and students' learning from teaching (Hiebert & Grouws, 2007; Hiebert & Stigler, Chap. 2, this volume). A possible direction ahead includes reconciling our account of practical rationality with theories of student learning from teaching and of teacher learning in teaching. Given specific goals for students' learning, practical rationality concepts, such as the notion of instructional exchanges, may support the creation of content-specific infrastructure to support teachers in managing such exchanges (Olsher et al. 2016) and chart what teachers may need to learn from teaching practice in order to enable such student learning. The development of new instructional situations, their expansion via breaches and repairs of norms, and their complementation with existing instructional situations are basic, general ways of thinking about how teaching can produce students' learning opportunities. The cost to that operation is, however, the turning of the theory into explicit teacher knowledge, which raises the questions of whether, how, and when teachers can (and should) be expected to hold on to and make productive use of a theory that represents practice as intellectually and morally complex.

At the same time that it offers means to work on the improvement of teaching, practical rationality also provides intellectual resources to build a professional discourse of advocacy for mathematics teachers. Too often policymakers make individual teachers responsible for enacting reforms. The concepts of norms and obligations are useful to account for what enables and constrains teaching; they could also be helpful in developing a public discourse about teaching that focuses less on burdening or shaming teachers and more on advocating for adjusting the systems in which teachers work.

7 Conclusion: Addressing the Questions Posed

In their invitation to write a chapter for this book, Charalambous and Praetorius asked us to address questions about the nature of theorizing about teaching. Stepping back from the particular empirical studies that have helped develop constructs of practical rationality, against the background of this chapter, here are responses to five of their questions:

- *What is a theory (of teaching)?*

We use practical rationality as a theory of teaching as a resource for our response. We have described practical rationality as a middle-range theory of teaching oriented to the fundamental, scientific aims of describing, explaining, and predicting mathematics teaching as a phenomenon that results from a combination of expression of individual resources and adaptation to socio-technical context. Unlike other authors in this volume, we did not assume that the theory should guide teaching on how to achieve particular kinds of learning; rather, along the lines of what Vieluf and Klieme (Chap. 3, this volume) call practice theory, practical rationality provides intellectual resources to understand mathematics teaching practice from a perspective that reconciles structural and agentic perspectives. Such a theory of teaching is a growing organization of constructs, assumptions, and empirical statements that seeks to describe the natural variability in the work of teaching, explain how differences observed in that variability are related to other phenomena, and predict changes in aspects of that variability as a result of natural or provoked changes in the related phenomena. Whereas much of that definition could describe theories in general, what makes this a theory of teaching is its fundamental aims; it takes the work of teaching as the object of study and makes its purpose to explain the variability of teaching, as potentially caused by other phenomena. It is middle-range in that it grows through the work of empirical research, and it is fundamental because it seeks to provide the means to understand all teaching rather than to specify a desirable kind of teaching (what we would call a prescriptive theory). As noted above, however, an application of practical rationality to the improvement of teaching can lead to the use of practical rationality concepts in the design of local instructional theories that might have more of a prescriptive orientation.

In appealing to socio-technical resources (norms and obligations) as sources of explanation, practical rationality proposes rational (not causal) explanations for variability observed in the work of teaching. The specific mechanisms that link those socio-technical resources to individuals' actions need to be discovered as they might hold keys for ways in which the work of teaching could be improved. Thus, a path for growth in this theory of teaching involves reconciling our theory of the rationality of practice, which pays attention to the public justification of actions in teaching, with theories of teacher thinking and decision making that account for the cognitive, neurological, or socioemotional mechanisms that explain causally how practitioners make decisions in teaching (e.g., Kaplan & Garner, 2018; Schoenfeld, 2010; Sherin et al., 2011).

- *What should a theory of teaching contain and why?*

A theory of teaching should be a theory of the practice in which teachers engage as opposed to a theory of the individuals who do the practice, though it may articulate with ways of describing the individual resources people bring to teaching. It should aim to describe, explain, and predict this practice. As far as description, it should include resources for representing the practice of teaching that permit one to draw similarities across some instances of the practice while also sustaining differences across some other instances of the practice, both within and across the practices of individual teachers. It should contain some technical language and other semiotic tokens whose definitions are provided, some technical uses of language whose definitions are sought through research, and nontechnical uses of language that support reading and writing without calling attention to themselves.

As far as explanation, a theory of teaching should provide the means to express relationships that connect instances of practice, not only in terms of similarity or difference but also more generally in terms of how categories of instances of practice form larger systems of practice such as lessons, units, courses, and programs of study. A theory of teaching should identify some sources or dimensions of complexity as ones that will not be reduced but whose texture is to be dissected and understood. A theory should contain connections among constructs of the theory and other phenomena, both possible causes and possible consequences.

As far as prediction, a theory of teaching should contain connections among constructs of the theory and sources of empirical evidence or measures of those constructs. It should contain empirically falsifiable propositions and experimentally falsifiable explanations. It should articulate how the interplay of theorization and empirical research enables theorists to manage critically the objectifying and subjectifying tendencies of social research.

At the same time, descriptions and predictions should at least be expressible in ways that practitioners can adjudicate their face validity, but we do not expect that practitioners will come to adopt the language of educational theorists. This raises the question of whether our field might develop a semiotic infrastructure that goes beyond language and permits researchers and teachers to transact practice without having to rely solely on words (see Herbst et al., in preparation). Such possibilities suggest the need for mathematics educators to continue to elaborate theoretically the notion of representations of practice (Herbst et al., 2016).

- *Can such a theory accommodate differences across subject matters and student populations taught? If so, how? If not, why?*

From our experience developing practical rationality, we can answer this question both in the affirmative and the negative. Some of the procedures for developing theory and some of the constructs of the theory can be applicable across subjects and student populations, while others may need to be specialized for different subjects. The question itself is interesting, also, inasmuch as it ignores other sources of possible difference across teaching practice such as cultures or institutions that are important to investigate as well. At some level of theorization, a theory of teaching practice could take all those layperson sources of difference and elaborate them theoretically.

Our own work studying teaching across high school algebra and geometry shows that the constructs of practical rationality are useful across courses of studies, which suggests that while the specific instructional situations and their norms may not translate from subject matter to subject matter, the notion that there are instructional situations that frame instructional exchanges and that such situations are regulated by norms may be useful across subject-matters. At the very least, we believe that practical rationality can be used to study mathematics teaching in different courses of study in educational institutions, as long as there are institutional mechanisms for identifying what knowledge is at stake in instruction. In particular, we believe practical rationality can account for the work of teaching mathematics at all levels of compulsory schooling, as well as in university mathematics courses.

We think it is possible to posit equivalent constructs regulating the teaching of other fields of knowledge. The instructional situations we have identified (e.g., solving equations, doing proofs) are specific to mathematics, but the notion of instructional situation could be applied in other fields of knowledge (e.g., physics or history). In school subjects such as social studies or science, there may be a need to stipulate more than one disciplinary obligation to account for the various sources of epistemological vigilance of each of those school subjects. Indeed, it is a compelling theoretical question for us to investigate the purchase that these ideas have in helping understand similarities and differences in the teaching of mathematics and other school subjects—not only subjects associated with academic disciplines like physics or biology but also very different subjects, such as the performance or visual arts. What could a comparative study look like that aimed at understanding the teaching of different disciplines (e.g., history or painting) in regard to how instructional transactions are managed? Documenting the range of applicability of instructional exchanges as the cornerstone of a theory of teaching practice seems more interesting to us, however, than finding a general theory of teaching.

- *Do we already have a theory/theories of teaching? If so, which are they?*

There are multiple kinds of theories of teaching. Some theories describe the work of teaching. Herbst & Chazan (2017) reviewed how different theories rely on different conceptualizations of teaching, behavioral, cognitive, social interactionist, sociocultural, and more. Practical rationality aspires to explore complementarities and contrasts with all of those. There also are descriptions of teaching that attempt to prescribe what teaching should look like in order for it to achieve some desired ends. While not often called theories, expressions like ambitious instruction, complex instruction, direct instruction, equitable practice, inquiry-oriented instruction, student-centered instruction, and others have been used to designate some aspirational kinds of teaching that can have the force of prescriptive theory. Insofar as practical rationality is a fundamental theory of teaching, its goals are to describe, explain, and predict all kinds of teaching, not to prescribe a particular kind of teaching. However, the concepts of practical rationality can be used to study the implementation of more prescriptive approaches to teaching. In particular, these concepts can be used to explain and predict what aspects of reform in teaching may be more or less viable, illustrating the value of a fundamental theory of teaching and enabling a discourse of teaching advocacy to complement the discourse of teaching imperatives often present in reforms and policy documents.

- *In the future, in what ways might it be possible, if at all, to create a (more comprehensive) theory of teaching?*

Some commitments have played important roles in our development that might not be as fundamental for other theorists, but we have made them explicit in this chapter: (1) the commitment to understanding the teaching of mathematics and to use mathematical resources to describe its teaching, (2) the recognition of teaching as complex systemic work describable by modeling the perspective of the practitioner but irreducible to the characteristics or the lived experience of the actors, (3) the fundamental research orientation to describe and explain all kinds of teaching and predict the outcomes of improvement efforts, (4) the commitment to avoid voluntarism and deficit-thinking in improvement design, and (5) the embracing of social science methods including the provisional acceptance of some amount of reduction are all commitments we have embraced.

For our field to make progress toward a theory of teaching, we need theorists to make explicit the commitments on which they build. We need to develop instruments that can gather information on constructs from different theories so that we can use them to develop a better understanding of how competing constructs are related and have a publicly accessible source of data that many people can contribute to steward and mine. We need to pre-register experiments that will allow different theories to compete to explain or predict the outcomes of these experiments. Framing all that, we need a scientific consensus, not only on the need to articulate commitments but also on shared rules of engagement (e.g., to recognize our scholarly practice also as complex and demanding us to hold on to the tensions among sets of competing values such as ecumenism and consistency, complexity and parsimony, and so on) in order to make such progress.

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