Rectilinear Voronoi Games with a Simple Rectilinear Obstacle in Plane

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Abstract. We study two player single round rectilinear Voronoi games in the plane for a finite set of clients where service paths are obstructed by a rectilinear polygon. The players wish to maximize the net number of their clients where a client is served by the nearest facility of players in $\mathbb{L}_1$ metric. We prove the tight bounds for the payoffs of both the players for the class of games with simple, convex and orthogonal convex polygons. We also generalize the results for $\mathbb{L}_\infty$ metric in the plane.

1 Introduction

Motivation. A *Voronoi game* is a *competitive facility location problem* where the goal is to maximize a service in the Voronoi cells of the players. Rectilinear versions of such problems naturally arise in several applications that deal with rectilinear paths, such as those related to city maps, electronic circuits, raster graphics, warehousing, architecture, civil engineering, network flows, etc. The motivation for the problem comes from real-life situations where an impassable zone restricts every player in a competitive facility location problem. We prove lower and upper bounds on the payoffs of such games in $\mathbb{L}_1$ and $\mathbb{L}_\infty$.

Previous Results. The concept of Voronoi games was introduced by Ahn et al. [1] for line segments and circles. Several variants of Voronoi games are available in the literature [2–15]. Ahn et al. [1], Cheong et al. [10] and, Fekete and Meijer [13] studied the versions where they tried to maximize the Voronoi cells themselves. Briefly, they proved that Bob is guaranteed at least half of the total payoff for their versions of Voronoi games. They also game suitable strategies for both the players. Durr and Thang [12], Teramoto et al. [15], Bandyopadhyay et al. [2] and Sun et al. [14] studied intractability of Voronoi games in graphs. Banik et al. [3] described a discrete vesion of the problem for line segments. Banik et al. [4,5] and, later, Berg et al. [9] solved the single round where two players can place a fixed number of facilities in a single round. Banik et al. [8] introduced Voronoi games in the interior of simple polygons and devised polynomial time optimal strategies for both Alice and Bob.

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Banik et al. [6,7] described the version of the problem that we study in this paper. Banik et al. [6] and Das et al. [11] extended, generalized and improved the solutions of the problem. They also proved several tight lower and upper bounds on the payoffs of a similar nature as in this paper.

New Results. In this paper, we study the rectilinear Voronoi games with rectilinear polygonal obstacles for players similar to the games mentioned in [6,7,11]. A notable difference is that we restrict the players outside of a fixed polygon. We formally describe the rectilinear Voronoi game in Sect. 2.

We prove that the optimal payoff of Alice $\geq \lceil n/3 \rceil$ and $\leq n/2$ and that the optimal payoff of Bob $\geq n/2$ and $\leq \lfloor 2n/3 \rfloor$, where the net number of served clients is n. These bounds are tight. We also prove that these bounds hold irrespective of whether we fix the class of polygonal obstacles as simple polygons, convex polygons, or orthogonal convex polygons in contrast with the results of [6,7,11]. We then generalize these results for $\mathbb{L}_\infty$ metric.

Organization. We present some preliminary definitions, concepts and observations in Sect. 2. In Sect. 3, we prove the bounds for rectilinear Voronoi games for simple, convex and orthogonal convex polygonal obstacles. We show in Sect. 4 that the same bounds hold for extensions to $\mathbb{L}_\infty$.

2 Preliminaries

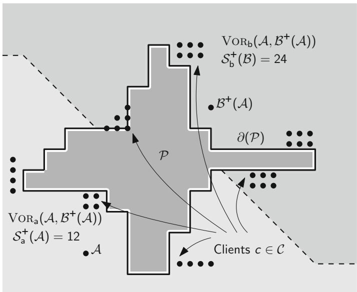

Fig. 1. A two player single round rectilinear Voronoi game with an orthogonal convex polygon obstacle in $\mathbb{L}_1$ metric in $\mathbb{R}^2$. Alice is at $\mathcal{A}$ and Bob wins with a payoff of 24 by playing at $\mathcal{B}^+(\mathcal{A})$.

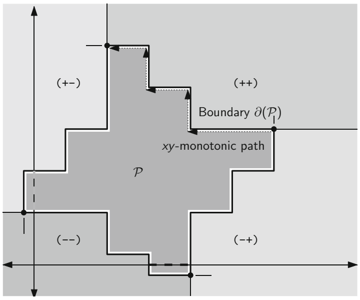

Fig. 2. An illustration for orthogonal convex polygon with explanation of some notation for quadrants.

We present some definitions, notations and conventions first. A *Voronoi game* is a competitive game in which players compete to serve a set of clients by placing their facilities. A facility serves the clients in its Voronoi cell and shares the clients on its Voronoi cell boundary. The *payoff of each player* is determined by the net number of clients they serve. The *two players single round rectilinear Voronoi game with a polygon obstacle* $\mathcal{P}$, denoted by $\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})$, is a single round Voronoi game played between two players, conveniently named Alice and Bob. They place a single facility each in a region containing a *finite set of point* *clients* $\mathcal{C} \subset \mathbb{R}^2$ with the *open simple polygonal obstacle* $\mathcal{P}$ in the $\mathbb{L}_1$ plane. Alice places her facility first, followed by Bob. The facility locations of Alice and Bob are denoted by $\mathcal{A} \in \mathbb{R}^2$ and $\mathcal{B} \in \mathbb{R}^2$, respectively. Effectively, $\mathcal{A} \in \mathbb{R}^2 \setminus \mathcal{P}$ and $\mathcal{B} \in \mathbb{R}^2 \setminus \mathcal{P}$, since $\mathcal{A} \in \mathcal{P}$ and $\mathcal{B} \in \mathcal{P}$ fetch exactly zero payoffs for Alice and Bob respectively. The *distance* from a facility f to a client c, denoted by $d^{\mathcal{P}}_{\mathbb{L}_1}(f, c)$ is measured as the $\mathbb{L}_1$-length of any shortest path from f to c that avoids interior of $\mathcal{P}$. The *payoffs of Alice and Bob*, denoted by $\mathcal{S}_{\mathrm{a}}(\mathcal{A}, \mathcal{B})$ and $\mathcal{S}_{\mathrm{b}}(\mathcal{A}, \mathcal{B})$, respectively, are the net count of clients they serve. See Fig. 1 for an example. We note that neither the shortest paths nor the best locations for Alice and Bob have to be unique. Moreover, we allow overlapping of clients and facilities, and in some cases, we also permit degenerate simple polygonal obstacles. Two problems arise naturally for these Voronoi games that we describe subsequently.

Problem 1. Let Alice and Bob play a two player single round rectilinear Voronoi game with a polygonal obstacle $\mathcal{P}$. What is an optimal location of Alice that maximizes her minimum payoff? What is an optimal location of Bob that maximizes his payoff for a fixed Alice's facility location? ∎

An optimal location of Bob for a fixed location $\mathcal{A}$ for Alice's facility is denoted by $\mathcal{B}^+(\mathcal{A})$ and the optimal payoff $\mathcal{S}^+_{\mathrm{b}}(\mathcal{A})$. The corresponding payoff of Alice is denoted by $\mathcal{S}^+_{\mathrm{a}}(\mathcal{A})$. Then,

$$\mathcal{S}^+_{\mathrm{a}}(\mathcal{A}) = \min_{\mathcal{B} \in \mathbb{R}^2} \mathcal{S}_{\mathrm{a}}(\mathcal{A}, \mathcal{B}) = \mathcal{S}_{\mathrm{a}}(\mathcal{A}, \mathcal{B}^+(\mathcal{A}))$$

$$\mathcal{S}^+_{\mathrm{b}}(\mathcal{A}) = \max_{\mathcal{B} \in \mathbb{R}^2} \mathcal{S}_{\mathrm{b}}(\mathcal{A}, \mathcal{B}) = \mathcal{S}_{\mathrm{b}}(\mathcal{A}, \mathcal{B}^+(\mathcal{A}))$$

We can compute $\mathcal{B}^+(\mathcal{A})$ by solving any one of the above equations. An optimal locations of Alice and Bob are denoted by $\mathcal{A}^*$ and $\mathcal{B}^*$, respectively, and the optimal payoffs by $\mathcal{S}^*_{\mathrm{a}}$ and $\mathcal{S}^*_{\mathrm{b}}$, respectively. $\mathcal{B}^* = \mathcal{B}^+(\mathcal{A}^*)$. Then,

$$\mathcal{S}^*_{\mathrm{a}} = \max_{\mathcal{A} \in \mathbb{R}^2} \min_{\mathcal{B} \in \mathbb{R}^2} \mathcal{S}_{\mathrm{a}}(\mathcal{A}, \mathcal{B}) = \mathcal{S}^+_{\mathrm{a}}(\mathcal{A}^*) = \mathcal{S}_{\mathrm{a}}(\mathcal{A}^*, \mathcal{B}^*)$$

$$\mathcal{S}^*_{\mathrm{b}} = \min_{\mathcal{A} \in \mathbb{R}^2} \max_{\mathcal{B} \in \mathbb{R}^2} \mathcal{S}_{\mathrm{b}}(\mathcal{A}, \mathcal{B}) = \mathcal{S}^+_{\mathrm{b}}(\mathcal{A}^*) = \mathcal{S}_{\mathrm{b}}(\mathcal{A}^*, \mathcal{B}^*)$$

We can compute $\mathcal{A}^*$ and then $\mathcal{B}^*$ by solving for $\min_{\mathcal{A} \in \mathbb{R}^2} \max_{\mathcal{B} \in \mathbb{R}^2} \mathcal{S}_{\mathrm{b}}(\mathcal{A}, \mathcal{B}) = \max_{\mathcal{B} \in \mathbb{R}^2} \mathcal{S}_{\mathrm{b}}(\mathcal{A}^*, \mathcal{B}) = \mathcal{S}_{\mathrm{b}}(\mathcal{A}^*, \mathcal{B}^*)$.

Next, we propose the problem of determining the upper and lower bounds of Alice's and Bob's payoffs.

Problem 2. Let Alice and Bob play a two player single round rectilinear Voronoi game with polygonal obstacle $\mathcal{P}$. What are the upper and lower bounds on the payoffs of Alice and Bob? ∎

The lower and upper bounds of Alice's payoffs are mathematically determined by expressions $\min_{\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})} \mathcal{S}^*_{\mathrm{a}}$ and $\max_{\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})} \mathcal{S}^*_{\mathrm{a}}$ respectively. Similarly, the lower and upper bounds of Bob's payoffs are $\min_{\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})} \mathcal{S}^*_{\mathrm{b}}$ and $\max_{\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})} \mathcal{S}^*_{\mathrm{b}}$ mathematically.

We can prove that the Voronoi game is a constant sum game. Hence

Theorem 1. $\mathcal{S}_{\mathrm{a}}(\mathcal{A},\mathcal{B}) + \mathcal{S}_{\mathrm{b}}(\mathcal{A},\mathcal{B}) = \mathcal{S}^{+}_{\mathrm{a}}(\mathcal{A}) + \mathcal{S}^{+}_{\mathrm{b}}(\mathcal{A}) = \mathcal{S}^{*}_{\mathrm{a}} + \mathcal{S}^{*}_{\mathrm{b}} = |\,\mathcal{C} \setminus \mathcal{P}\,|$

Proof. We note that $\mathcal{P}$ is open, and any client in the (strict) interior of $\mathcal{P}$ is not served. Other clients are either fully served by Alice or Bob or equally shared by them. Thus the net total of the payoffs in any Voronoi game is always equal to $|\,\mathcal{C} \setminus \mathcal{P}\,|$. □

We study the class of the rectilinear Voronoi games when the obstacles are simple, convex or orthogonal convex polygons. We can also extend the Voronoi games and related problems described above to $\mathbb{L}_\infty$ metric. Instead of orthogonal convex polygons, we look at polygons that are oblique orthogonal convex polygons described later in Sect. 4.

We represent a simple polygon, and likewise, a simple polygonal region, $\mathcal{P}$ by its boundary $\partial(\mathcal{P})$ and assume that the polygon contains its open interiors. The boundary $\partial(\mathcal{P})$ is assumed to be represented by a non-crossing counter-clockwise sequence of edges such that the interior of $\mathcal{P}$ is on the left. The simple polygons are open and bounded. Likewise, both the interiors and exteriors are open. Though simply connected, they may possibly be degenerate. An *orthogonal convex polygon* is an open rectilinear polygon such that every horizontal and vertical line intersects the polygon no more than once in an interval. See Fig. 2 for an example. Let S be any finite or infinite bounded set of points. An *orthogonal convex hull* of S, possibly non-unique, is a minimal open orthogonal convex polygon that contains S and is denoted by $\mathrm{OCH{\scriptsize ULL}}(S)$. The *smallest containing box* of S is denoted by $\mathrm{B{\scriptsize OX}}(S)$, i.e., $\mathrm{B{\scriptsize OX}}(S) = \{\,(q_{\mathrm{x}}, q_{\mathrm{y}}) \mid x_{\min}(S) < q_{\mathrm{x}} < x_{\max}(S),\ y_{\min}(S) < q_{\mathrm{y}} < y_{\max}(S)\,\}$ where $x_{\min}(S)$, $x_{\max}(S)$, $y_{\min}(S)$ and $y_{\max}(S)$ are respectively the left, right, bottom and top extremes of the set S.

We implicitly use the Voronoi regions of Alice and Bob in the discussion. The Voronoi regions of Alice and Bob are denoted by $\mathrm{V{\scriptsize OR}}_{\mathrm{a}}(\mathcal{A},\mathcal{B})$ and $\mathrm{V{\scriptsize OR}}_{\mathrm{b}}(\mathcal{A},\mathcal{B})$ respectively for their facility locations $\mathcal{A}$ and $\mathcal{B}$ respectively.

$$\mathrm{V{\scriptsize OR}}_{\mathrm{a}}(\mathcal{A},\mathcal{B}) = \{\, p \in \mathbb{R}^2 \setminus \mathcal{P} \mid d^{\mathcal{P}}_{\mathbb{L}_1}(\mathcal{A}, p) \le d^{\mathcal{P}}_{\mathbb{L}_1}(\mathcal{B}, p)\,\},$$

$$\mathrm{V{\scriptsize OR}}_{\mathrm{b}}(\mathcal{A},\mathcal{B}) = \{\, p \in \mathbb{R}^2 \setminus \mathcal{P} \mid d^{\mathcal{P}}_{\mathbb{L}_1}(\mathcal{A}, p) \ge d^{\mathcal{P}}_{\mathbb{L}_1}(\mathcal{B}, p)\,\}.$$

The clients $c \in \mathcal{C}$ for which $d^{\mathcal{P}}_{\mathbb{L}_1}(\mathcal{A}, c) = d^{\mathcal{P}}_{\mathbb{L}_1}(\mathcal{B}, c)$ are shared equally between Alice and Bob and contribute $1/2$ to each of the payoffs.

3 Bounds for Rectilinear Voronoi Games with Polygonal Obstacles

3.1 Unrestricted

Let S be a finite set of points in the polygonal region R. We define xy-*median* of S in R to be a point c_{m}, such that, any open horizontal and vertical chord of R that avoids c_{m} contain $\le \lfloor |S|/2 \rfloor$ points of S on the other side of c_{m}. We can

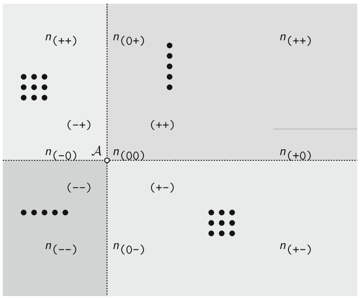

Fig. 3. Lemma 2: The lower bound of Alice's payoff for the unrestricted rectilinear Voronoi game in plane.

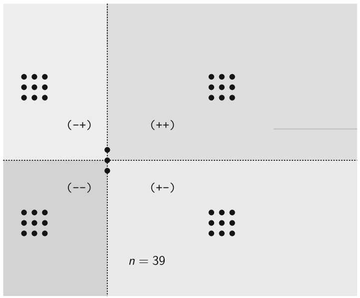

Fig. 4. Theorem 3: An unrestricted rectilinear Voronoi game proving tight bounds for non-overlapping clients.

argue that it is always possible to compute xy-*median* for any set of clients S in any bounded or unbounded simply connected region R. See Fig. 5.

Let $\mathcal{G}_{\mathbb{L}_1}(\mathcal{C})$ be *an unrestricted rectilinear Voronoi game* with a finite set of clients $\mathcal{C}$ in plane, where the service paths are not restricted by any obstacle. Let $|\mathcal{C}|$ be n. We show that in an unrestricted rectilinear Voronoi game, Alice and Bob have an optimal strategy so that the other player does not have an advantage in their payoff. This is similar to the original result of [1].

Lemma 2. *$\mathcal{S}_{\mathrm{a}}^* \geq {}^n\!/2$ and $\mathcal{S}_{\mathrm{b}}^* \geq {}^n\!/2$.*

Proof. To prove Alice's bound, we put Alice's facility at the xy-median of $\mathcal{C}$. Then Alice is guaranteed a payoff of ${}^n\!/2$. See Fig. 3 for the sketch of the proof. In the figure, $\lceil {}^n\!/2 \rceil - (n_{(0+)} + n_{(0-)} + n_{(00)}) \leq (n_{(++)} + n_{(+-)} + n_{(+0)}) < \lceil {}^n\!/2 \rceil$, etc. where $n_{(++)}$, etc., denotes the number of clients in the (++) quadrant, etc. We can show that $\mathcal{S}_{\mathrm{a}}(\mathcal{A}, \mathcal{B}) > {}^n\!/2$ by formulating it is an integer linear program while optimizing for the max-min payoff. To prove Bob's bound, we put Bob's facility overlapping Alice's facility. Bob is guaranteed a payoff of ${}^n\!/2$ there. □

Moreover, as an immediate consequence of Lemma 2, we can show that the lower and upper bounds of $\mathcal{S}_{\mathrm{a}}^* = \mathcal{S}_{\mathrm{b}}^* = {}^n\!/2$ are indeed same, invariably constant, and hence, tight for unrestricted case.

Theorem 3. *Let $\mathcal{G}_{\mathbb{L}_1}(\mathcal{C})$ be an unrestricted rectilinear Voronoi game in $\mathbb{R}^2$ with n clients. Then $\mathcal{S}_{\mathrm{a}}^* = \mathcal{S}_{\mathrm{b}}^* = {}^n\!/2$.*

Proof. We put Alice at an xy-median of $\mathcal{C}$. Bob is forced to place his facility at the same location to maximize his payoff. See Fig. 4 for an example unrestricted rectilinear Voronoi game. □

The implicit technique employed above is used several times later with some tight modifications for regions with obstacles.

3.2 Simple Polygon Obstacle

Let $\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})$ be a Voronoi game in $\mathbb{R}^2$ with a simple polygonal obstacle $\mathcal{P}$ and clients $\mathcal{C}$. Let $n = |\mathcal{C} \setminus \mathcal{P}|$. We note that Bob has a simple strategy to ensure a payoff of at least $n/2$.

Lemma 4. *Let $\mathcal{A}$ be fixed. Then $\mathcal{S}_{\mathrm{b}}^{+}(\mathcal{A}) \geq {}^{n}/_{2}$ for any simple polygon $\mathcal{P}$.*

Proof. We fix $\mathcal{B}^{+}(\mathcal{A})$ overlapping $\mathcal{A}$. Due to the space limitations, we omit important (and technical) details, as the formal proof requires several more definitions and claims. We request the interested reader to see the full version of this paper. □

The strategy for Alice to ensure at least a minimum payoff is non-trivial. We show below a facility location where she can get a payoff of $\lceil {}^{n}/_{3} \rceil$. With the aid of numerous figures, we sketch the main idea in our proof. We omit important (and technical) details, as the formal proof requires several more definitions and claims. It is impossible to provide all the necessary details within the page limit of the conference submission.

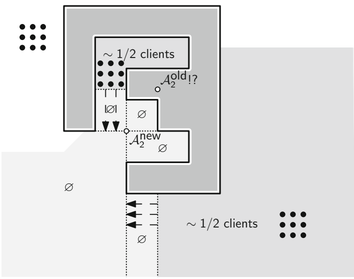

Fig. 5. The median horizontal and vertical chords for xy-median do not intersect in R for a set of points. We show the existence of another valid location for xy-median.

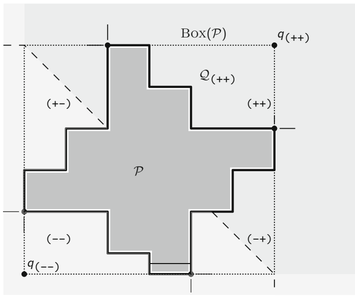

Fig. 6. The $\mathcal{Q}_{(++)}$ extended quadrant with respect to an orthogonal convex polygon $\mathcal{P}$. Naturally, $\mathcal{Q}_{(++)} \cup \mathcal{Q}_{(--)} = \mathbb{R}^2 \setminus \mathcal{P}$. Hence, $\mathcal{C} \setminus \mathcal{P} \subset \mathcal{Q}_{(++)} \cup \mathcal{Q}_{(--)}$.

We define the *extended quadrants relative to* $\mathcal{P}$ for the purpose of proofs below. Let us consider the Voronoi diagram with obstacle $\mathcal{P}$ in $\mathbb{L}_1$ of the two points $q_{(++)} = (x_{\max}(\mathcal{P}), y_{\max}(\mathcal{P}))$ and $q_{(--)} = (x_{\min}(\mathcal{P}), y_{\min}(\mathcal{P}))$. The closed (++) extended quadrant, denoted by $\mathcal{Q}_{(++)}$, is the set of points in the Voronoi cell of $q_{(++)}$, i.e., $\{\, p \in \mathbb{R}^2 \setminus \mathcal{P} \mid d_{\mathbb{L}_1}^{\mathcal{P}}(q_{(++)}, p) \leq d_{\mathbb{L}_1}^{\mathcal{P}}(q_{(--)}, p)\}$. Likewise, we define $\mathcal{Q}_{(-+)}$, $\mathcal{Q}_{(--)}$ and $\mathcal{Q}_{(+-)}$ closed extended quadrants. See Fig. 6 for an illustration of the extended quadrant $\mathcal{Q}_{(++)}$.

Before proving a lower bound of $\lceil {}^{n}/_{3} \rceil$, we show first that $\lceil {}^{n}/_{4} \rceil$ is a weaker lower bound for $\mathcal{S}_{\mathrm{a}}^{*}$.

Lemma 5. $\mathcal{S}_a^* \geq \lceil n/4 \rceil$ *for any orthogonal convex polygon* $\mathcal{P}$.

Proof. We observe that at least one of any pair of diametrically opposite extended quadrants of $\mathcal{P}$, for example one of $\mathcal{Q}_{(++)}$ or $\mathcal{Q}_{(--)}$, will contain $\geq \lceil \frac{n}{2} \rceil$ clients because they cover $\mathbb{R}^2 \setminus \mathcal{P}$ and hence $\mathcal{C} \setminus \mathcal{P}$. We can show that $\mathcal{A}$ on a xy-median of the extended quadrant of the four that contains the most clients will get $\geq \lceil n/4 \rceil$ payoff. See Fig. 7. □

Later, in Theorem 10, we show that $\lceil n/3 \rceil$ is the tight bound of $\mathcal{S}_a^*$ for even orthogonal convex polygonal obstacles.

Lemma 6. $\mathcal{S}_a^* \geq \lceil n/3 \rceil$ *for any simple polygon* $\mathcal{P}$.

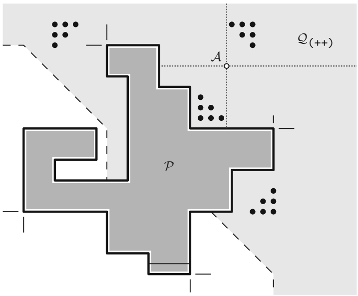

Fig. 7. A weak lower bound for Alice's payoff for rectilinear Voronoi game with a simple orthogonal polygonal obstacle. $\mathcal{Q}_{(++)}$ quadrant contains $\geq \lceil \frac{n}{2} \rceil$ clients. Alice gets at least half of the clients in $\mathcal{Q}_{(++)}$.

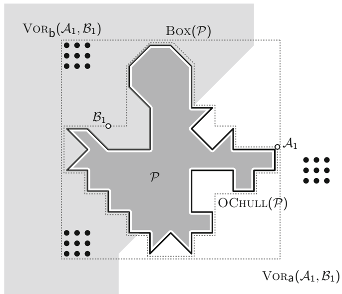

Fig. 8. Lower bound for Alice's payoff for rectilinear Voronoi game with a simple polygonal obstacle. Possible candidate location $\mathcal{A}_1$ in Lemma 6.

Proof. We consider two possible candidates for Alice's facility location for our claim. One of these will guarantee a payoff of $\lceil n/3 \rceil$. The first candidate location, denoted by $\mathcal{A}_1$, is a rightmost point of $\partial(\mathcal{P})$. See Fig. 8. If $\mathcal{S}_a^+(\mathcal{A}_1) \geq \lceil n/3 \rceil$ then the proof is complete.

Otherwise, let $\mathcal{B}^+(\mathcal{A}_1) = \mathcal{B}_1$. We compute $\mathcal{V} = \text{VOR}_b(\mathcal{A}_1, \mathcal{B}_1)$. Naturally $\mathcal{C} \cap \mathcal{V}$ contains $> \lfloor 2n/3 \rfloor$, since, $\mathcal{S}_a^+(\mathcal{A}_1) < \lceil n/3 \rceil \implies \mathcal{S}_b^+(\mathcal{A}_1) > n - \lceil n/3 \rceil$. We fix $\mathcal{A}_2$ on the xy-median of the clients in $\mathcal{V}$, i.e., $\mathcal{C} \cap \mathcal{V}$. We give a proof sketch below that $\mathcal{S}_a^+(\mathcal{A}_2) \geq \lceil n/3 \rceil$. Let $\mathcal{A}_2 = (a_x, a_y)$.

The following subcases arise.

Case 1. $\mathcal{A}_2 \in \mathbb{R}^2 \setminus \text{BOX}(\mathcal{P})$.

If $\mathcal{A}_2$ is, without loss of generality, such that $a_x \geq x_{\min}(\mathcal{P})$ and $a_y \geq y_{\max}(\mathcal{P})$, then clearly there are at least $\lceil n/3 \rceil$ clients above and $\lceil n/3 \rceil$ clients right of $\mathcal{A}_2$. If $\mathcal{A}_2$ is, without loss of generality, such that $a_x \geq x_{\min}(\mathcal{P})$ and $y_{\min}(\mathcal{P}) < a_y > y_{\max}(\mathcal{P})$, then we can show that $\mathcal{B}_2$ either serves only shared clients of

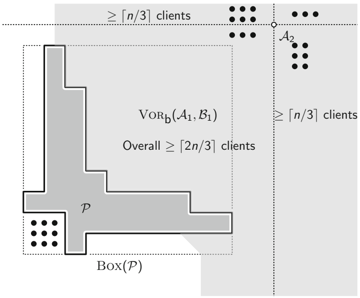

Fig. 9. Lemma 6: Case 1. (a) Both the median lines for $\mathcal{A}_2$ are unbounded.

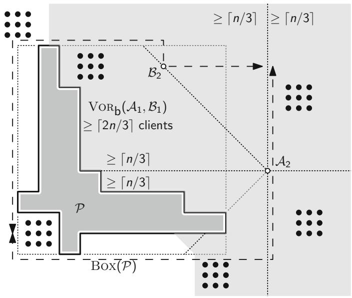

Fig. 10. Lemma 6: Case 1. (b) One of the median lines for $\mathcal{A}_2$ is semi-bounded and the other one is unbounded.

$\mathcal{C} \cap \mathcal{V}$ in any diametrically opposite quadrants relative to $\mathcal{A}_2$ or serves clients of $\mathcal{C} \cap \mathcal{V}$ in only one of the any diametrically opposite quadrants relative to $\mathcal{A}_2$. So, in either situation, $\mathcal{A}_2$ will get a payoff of at least $\lceil n/3 \rceil$. See Figs. 9 and 10.

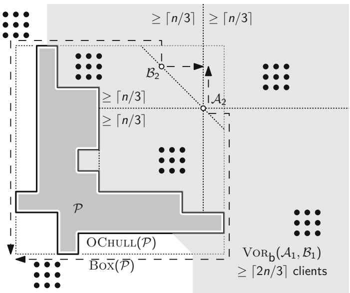

Fig. 11. Lemma 6: Case 2. (a) Both $\mathcal{A}_2$ and $\mathcal{B}_2$ in the same quadrant with respect to $\mathcal{P}$.

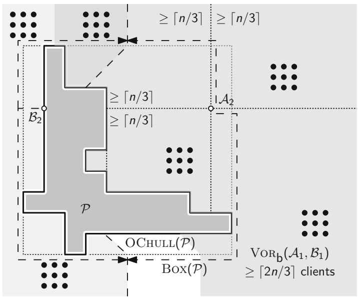

Fig. 12. Lemma 6: Case 2. (b) $\mathcal{A}_2$ and $\mathcal{B}_2$ in different quadrants with respect to $\mathcal{P}$.

Case 2. $\mathcal{A}_2 \in \textsc{Box}(\mathcal{P}) \setminus \textsc{OHull}(\mathcal{P})$.
If $\mathcal{A}_2$ is, without loss of generality, in $\mathcal{Q}_{(++)}$, then we can show that $\mathcal{B}_2$ either serves only shared clients of $\mathcal{C} \cap \mathcal{V}$ in any diametrically opposite quadrants relative to $\mathcal{A}_2$ or serves clients of $\mathcal{C} \cap \mathcal{V}$ in only one of the diametrically opposite quadrants relative to $\mathcal{A}_2$. Thus, $\mathcal{A}_2$ is guaranteed a payoff of at least $\lceil n/3 \rceil$. See Figs. 11 and 12.

Case 3. $\mathcal{A}_2 \in \textsc{OHull}(\mathcal{P}) \setminus \mathcal{P}$.
If $\mathcal{A}_2$ is sufficiently deep in a pocket such that the horizontal and vertical chords passing through $\mathcal{A}_2$ are also in the same pocket and, without loss of generality, the opening to the exterior is towards (++) quadrant with respect to $\mathcal{A}_2$ then, we can argue that since there are $\lceil n/3 \rceil$ clients in $\mathcal{C} \cap \mathcal{V}$ below the

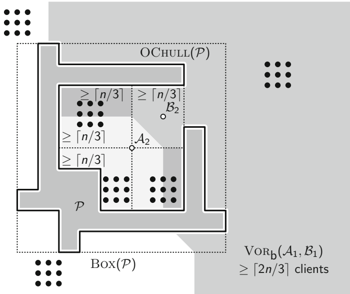

Fig. 13. Lemma 6: Case 3. (a) Both the median chords for $\mathcal{A}_2$ are bounded.

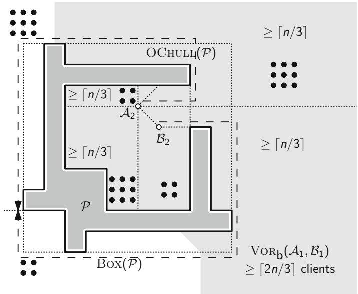

Fig. 14. Lemma 6: Case 3. (b) One of the median chord for $\mathcal{A}_2$ is bounded and the other semi-bounded.

horizontal chord and left of the vertical chord, Alice gets a payoff of at least $\lceil n/3 \rceil$. If $\mathcal{A}_2$ is shallow in a pocket, then too, we can show that $\mathcal{B}_2$ either serves only shared clients of $\mathcal{C} \cap \mathcal{V}$ in any diametrically opposite quadrants relative to $\mathcal{A}_2$ or serves clients of $\mathcal{C} \cap \mathcal{V}$ in only one of the diametrically opposite quadrants relative to $\mathcal{A}_2$. Thus $\mathcal{A}_2$ ensures a payoff of at least $\lceil n/3 \rceil$. See Figs. 13 and 14.

Case 4. $\mathcal{A}_2 \in \mathcal{P}$

This case does not arise.

Consequently, in all the cases, $\mathcal{S}_a^+(\mathcal{A}_2) \geq \lceil n/3 \rceil$. □

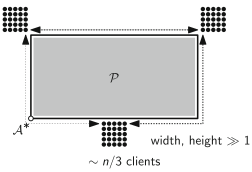

Fig. 15. The lower bound for Alice's payoff for rectilinear Voronoi game with a simple polygonal obstacle is tight.

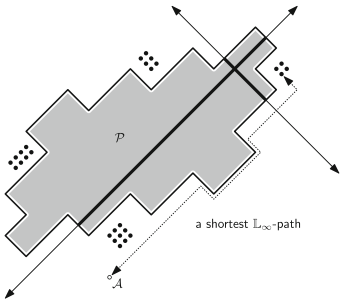

Fig. 16. An oblique orthogonal convex polygon $\mathcal{P}$.

Theorem 7. *Let $\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C}, \mathcal{P})$ be a Voronoi game in $\mathbb{R}^2$ with a simple polygonal obstacle $\mathcal{P}$ and n clients $\mathcal{C}$. Then $\lceil n/3 \rceil \leq \mathcal{S}_a^* \leq n/2$ and $n/2 \leq \mathcal{S}_b^* \leq \lfloor 2n/3 \rfloor$. The bounds are tight.*

Proof. The bounds are consequences of Lemma 4 and Lemma 6. For tightness, we construct two Voronoi games as follows. We fix $\mathcal{P}$ as a rectangular region with three sets of about $n/3$ nearly overlapping clients totaling n clustered at three equidistant locations. We can show that $\mathcal{S}^{+}_{\mathrm{a}}(\mathcal{A}) = \lceil n/3 \rceil$. See Fig. 15 for the construction. For the tightness of the upper bound of $\mathcal{S}^{+}_{\mathrm{a}}$, we construct a Voronoi game as before with a single cluster of n overlapping clients. The polygonal obstacle does not matter. □

3.3 Convex Polygon Obstacle

In [6], Alice was guaranteed a share of payoff for convex polygon case compared to the general case. In [11], both Alice and Bob were guaranteed a share of payoff for convex polygon case. However, unlike [6,11], in rectilinear Voronoi games with obstacles, there is no such advantage for either Alice or Bob, if we specialize to the class of convex polygonal obstacles. The proofs, on the other hand, are simplified. Also, we note that the convex polygons are special cases of orthogonal convex polygons though the opposite is not true. Thus we have the following theorem.

Theorem 8. *Let $\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})$ be a Voronoi game in $\mathbb{R}^2$ with a convex polygonal obstacle $\mathcal{P}$ and n clients $\mathcal{C}$. Then $\lceil n/3 \rceil \leq \mathcal{S}^{*}_{\mathrm{a}} \leq n/2$ and $n/2 \leq \mathcal{S}^{*}_{\mathrm{b}} \leq \lfloor 2n/3 \rfloor$. The bounds are tight.*

Proof. The proof is similar to that of Theorem 7 though much simplified because of the convexity of the polygonal obstacle. The tightness's follow from the same constructions. □

3.4 Orthogonal Simple Polygon Obstacle

Next, we consider the class of orthogonal simple polygonal obstacles, a subclass of simple polygons. The class of such polygons includes degenerate polygons though the paths should not cross the boundary edges. Again, we can conclusively show that the bounds are the same and tight. Hence,

Theorem 9. *Let $\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})$ be a Voronoi game in $\mathbb{R}^2$ with a orthogonal simple polygonal obstacle $\mathcal{P}$ and n clients $\mathcal{C}$. Then $\lceil n/3 \rceil \leq \mathcal{S}^{*}_{\mathrm{a}} \leq n/2$ and $n/2 \leq \mathcal{S}^{*}_{\mathrm{b}} \leq \lfloor 2n/3 \rfloor$. The bounds are tight.*

3.5 Orthogonal Convex Polygon Obstacle

Lastly, we show the bounds for the class of orthogonal convex polygonal obstacles. As mentioned earlier, since the convex polygons are special cases of orthogonal convex polygons, hence the bounds here are valid for the earlier section too. Again this is unlike [11].

Theorem 10. *Let $\mathcal{G}_{\mathrm{P},\mathbb{L}_1}(\mathcal{C},\mathcal{P})$ be a Voronoi game in $\mathbb{R}^2$ with a orthogonal simple polygonal obstacle $\mathcal{P}$ and n clients $\mathcal{C}$. Then $\lceil n/3 \rceil \leq \mathcal{S}^{*}_{\mathrm{a}} \leq n/2$ and $n/2 \leq \mathcal{S}^{*}_{\mathrm{b}} \leq \lfloor 2n/3 \rfloor$. The bounds are tight.*

Proof. Case 3 of the proof of Lemma 6 does not arise because the orthogonal convex polygonal obstacle will not have any pockets. Also, for the tightness, we had deliberately constructed the Voronoi game so that the obstacle is at the same time simple, convex, orthogonal simple and orthogonal convex. Hence the same example game proves the tightness of each of these classes of obstacles. □

4 Bounds for $\mathbb{L}_\infty$ Metric in Plane

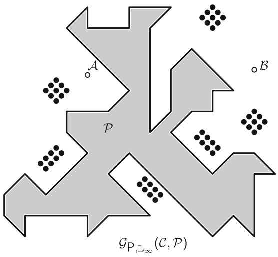

Fig. 17. A Voronoi game $\mathcal{G}_{\mathrm{P},\mathbb{L}_\infty}(\mathcal{C},\mathcal{P})$ in $\mathbb{L}_\infty$ metric with the polygonal obstacle $\mathcal{P}$ in plane.

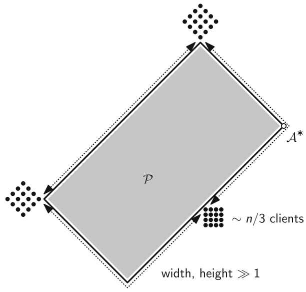

Fig. 18. Tightness of the lower bound for Alice's payoff for the Voronoi game $\mathcal{G}_{\mathrm{P},\mathbb{L}_\infty}(\mathcal{C},\mathcal{P})$ in $\mathbb{L}_\infty$ metric with the convex polygonal obstacle $\mathcal{P}$ in plane.

Let $\mathcal{G}_{\mathrm{P},\mathbb{L}_\infty}(\mathcal{C},\mathcal{P})$ be a Voronoi game in $\mathbb{L}_\infty$ metric with the polygonal obstacle $\mathcal{P}$ in plane with a set of clients $\mathcal{C}$. We note that the $\mathbb{L}_\infty$ metric is very similar to $\mathbb{L}_1$. Though it is not apparent, we can easily extend the bounds to these Voronoi games by modifying our proofs. Most of the arguments are valid if we choose an oblique pair of reference axes, i.e., if we choose the lines $x = y$ and $x + y = 0$ as the x-axis and the y-axis, respectively. Moreover, an *oblique orthogonal convex polygon* $\mathcal{P}$, which is an extension of orthogonal convex polygons, has a property that any lines parallel to the above two oblique axes will intersect the polygon $\mathcal{P}$ in at most one interval. See Figs. 16 and 17 for examples of an oblique orthogonal convex polygon and a Voronoi game $\mathcal{G}_{\mathrm{P},\mathbb{L}_\infty}(\mathcal{C},\mathcal{P})$ respectively (Fig. 18).

Theorem 11. *Let $\mathcal{G}_{\mathrm{P},\mathbb{L}_\infty}(\mathcal{C},\mathcal{P})$ be a Voronoi game in $\mathbb{R}^2$ with a simple polygonal obstacle and n clients. Then $\lceil n/3 \rceil \le \mathcal{S}_{\mathrm{a}}^* \le n/2$ and $n/2 \le \mathcal{S}_{\mathrm{b}}^* \le \lfloor 2n/3 \rfloor$.*

Also, there exist Voronoi games with a convex polygonal obstacle and n clients such that $\mathcal{S}_{\mathrm{a}}^* = \lceil n/3 \rceil$ and $\mathcal{S}_{\mathrm{b}}^* = n/2$.

Corollary 12. $\lceil n/3 \rceil \le \mathcal{S}_{\mathrm{a}}^* \le n/2$ *and* $n/2 \le \mathcal{S}_{\mathrm{b}}^* \le \lfloor 2n/3 \rfloor$ *for subclasses of Voronoi games in $\mathbb{L}_\infty$ with convex polygonal obstacles, oblique orthogonal polygonal obstacles and oblique orthogonal convex polygonal obstacles. The bounds are tight.*

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