Perfectness of G-generalized Join of Graphs

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Abstract. In this paper, we prove that the G-generalized join of complete or totally disconnected graphs is perfect if and only if G is perfect. As a result, we deduce some results proved in (Saeid et al. Rocky Mountain J. Math. 48(3) (2018), 729–751) and (Nilesh et al. arXiv (2022), arXiv:2205.04916). We also characterize rings, posets and reduced semigroups whose zero-divisor graphs and ideal based zero-divisor graphs are perfect. As a consequence, we characterize distributive lattices with 0, reduced semirings and boolean rings whose zero divisor graphs are perfect, which are proved in (Patil et al. in Discrete Math. 340: 740–745, 2017).

Keywords: Perfect graphs · G-generalized join of graphs · Zero-Divisor Graphs

1 Introduction

All the graphs considered in the paper are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph. For $v \in V(G)$ and $S \subseteq V(G)$, let $N_G(v)$ denote the open neighborhood of v in G and $\langle S \rangle$ denote the subgraph induced by S. Let $\overline{G}$ denote the complement of a graph G. A *proper k-coloring* of a graph G is a function from $V(G)$ into a set of k colors such that no two adjacent vertices receive the same color. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the least positive integer k such that there exists a proper k-coloring of G. A *clique* in a graph G is a complete subgraph of G. The *clique number* of G is the largest size of a clique in G and it is denoted by $\omega(G)$. Let G be a graph with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $H_1, H_2, \ldots, H_n$ be pairwise disjoint graphs. The *G-generalized join* graph, denoted by $G[H_1, H_2, \ldots, H_n]$, of $H_1, H_2, \ldots, H_n$ is the graph obtained by replacing each vertex u_i of G by H_i and joining each vertex of H_i to each vertex of H_j by an edge if u_i is adjacent to u_j in G. If $H_i \cong H$, for $1 \leq i \leq n$, then $G[H_1, H_2, \ldots, H_n]$ becomes the standard lexicographic product $G[H]$.

For a graph G, we define a relation $\sim_G$ on $V(G)$ as follows: For any $x, y \in V(G)$, define $x \sim_G y$ if and only if $N_G(x) = N_G(y)$. Clearly, $\sim_G$ is an equivalence relation on $V(G)$. Let $[x]$ be the equivalence class which contains x and S be the set of all equivalence classes of this relation $\sim_G$. Based on this equivalence classes we define the reduced graph G_r of a graph G as follows. The *reduced graph* G_r of G (defined

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in [13]) is the graph with vertex set $V(G_r) = S$ and two distinct vertices $[x]$ and $[y]$ are adjacent in G_r if and only if x and y are adjacent in G.

Note that if $V(G_r) = \{[x_1], [x_2], \ldots, [x_k]\}$, then G is the G_r-generalized join of $\langle [x_1] \rangle, \langle [x_2] \rangle, \ldots, \langle [x_k] \rangle$, that is, $G = G_r[\langle [x_1] \rangle, \langle [x_2] \rangle, \ldots, \langle [x_k] \rangle]$ and each $[x_i]$ is an independent subset of G (that is, $\langle [x_i] \rangle$ has no edge). Clearly, G_r is isomorphic to an induced subgraph of G. It is easy to observe the following observation.

Observation 1. If G_r is the reduced graph of G with $\omega(G_r) = \chi(G_r)$, then $\chi(G) = \omega(G_r)$.

Let G be a graph with $\omega(G) = k$, and let $\Delta_k(G)$ be the set of all the vertices of a graph G which lie in some clique of size k of G. A connected graph G is called a *generalized complete k-partite* graph (see [13]) if the vertex set $V(G)$ of G is a disjoint union of A and H satisfying the following conditions:

(1) $A = \Delta_k(G)$ and the subgraph induced by A is a complete k-partite graph with parts, say, $A_i, i = 1, 2, \ldots, k$.
(2) For any $h \in H$ and $i \in \{1, 2, \ldots, k\}$, h is adjacent to some vertex of A_i if and only if h is adjacent to any vertex of A_i.
Set $W(h) = \{1 \leq i \leq k \mid N(h) \cap A_i \neq \emptyset\}$ for any $h \in H$.
(3) For any $h_1, h_2 \in H, h_1$ is adjacent to h_2 if and only if $W(h_1) \cup W(h_2) = \{1, 2, \ldots, k\}$.

A graph G is called a *compact* graph (see [13]) if G contains no isolated vertices and for each pair x, y of non-adjacent vertices of G, there is a vertex z in G with $N(x) \cup N(y) \subseteq N(z)$. A graph G is said to be *k-compact* if it is compact and $\omega(G) = k$.

Throughout this paper, rings are finite non-zero commutative rings with unity. Let R be a ring. A non-zero element x of R is said to be a *zero-divisor* if there exists a non-zero element y of R such that $xy = 0$. A non-zero element u of R is *unit* in R if there exists v in R such that $uv = 1$. For $x \in R$, the *annihilator* of x is the set $Ann(x) = \{y \in R \mid xy = 0\}$. A ring R is said to be *local* if it has unique maximal ideal M. The *nilradical* of a ring R is the set $J = \{x \in R : x^t = 0, \text{ for some positive integer } t\}$. The *index of nilpotency* of J is the least positive integer m for which $J^m = \{0\}$, where $J^m = JJ \ldots J$ (m-times). A ring R is said to be *reduced* if $J = \{0\}$. A ring is said to be *indecomposable* if it can not be written as a direct product of two rings. Let $\mathbb{Z}_n$ be the ring of integer modulo n.

For any ring R, in [6], Beck associated a simple graph with R whose vertices are the elements of R and any two distinct vertices x and y are adjacent if and only if $xy = 0$ in R. Beck conjectured that (see [6]) the chromatic number and clique number of this graph are the same and this was disproved by Anderson and Naseer in [2] (also, see [10]). It can be observed that for the graph associated with the ring, the vertex 0 is adjacent to every other vertex. Anderson and Livingston in [5] slightly modified the definition of the graph associated with a ring by considering the zero-divisors as the vertices and any two distinct vertices x and y are adjacent if and only if $xy = 0$ in R. They called this *zero-divisor graph* of the ring R and it is denoted by $\Gamma(R)$. Zero-divisor graphs have been extensively studied in the past. This can be seen in [1,3,4,11,20].

The following definitions and results can be found in [4,20]. For $x, y \in R$, define $x \sim_R y$ if and only if $Ann(x) = Ann(y)$. It is proved in [4] that the relation $\sim_R$ is an equivalence relation on R. For $x \in R$, let $D_x = \{r \in R \mid x \sim_R r\}$ be the equivalence class of x. Let $R_E = \{D_{x_1}, D_{x_2}, \ldots, D_{x_k}\}$ be the set of all equivalence classes of the relation $\sim_R$. The *compressed zero-divisor graph* $\Gamma_E(R)$ of R (defined in [20]) is a simple graph with vertex set $R_E \backslash \{D_0, D_1\}$ and two distinct vertices D_x and D_y are adjacent if and only if $xy = 0$. The following result can be found in [18].

Theorem 1 [18]. *If R is a ring, then*

(i) $\Gamma(R) \cong \Gamma_E(R)[\langle D_{x_1}\rangle, \langle D_{x_2}\rangle, \ldots, \langle D_{x_{k-2}}\rangle]$, *where* $D_{x_i} \neq D_0, D_1$, *for* $1 \leq i \leq k-2$,
(ii) $\langle D_{x_i}\rangle$ *is complete if and only if* $x_i^2 = 0$, *and*
(iii) $\langle D_{x_i}\rangle$ *is totally disconnected (that is,* $\langle D_{x_i}\rangle$ *has no edge) if and only if* $x_i^2 \neq 0$.

The following result is proved in [3].

Theorem 2 [3]. *If R is a non-zero reduced ring, then there exists a positive integer k such that the compressed zero-divisor graph* $\Gamma_E(R) \cong \Gamma(\mathbb{Z}_2^k)$, *where* $\mathbb{Z}_2^k = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ *(k-times).*

In [9], Hala and Jukl introduced the concept of the zero-divisor graph of a poset. Let $(P, \leq)$ be a finite poset with the least element 0. For any $a, b \in P$, denote $L(a, b) = \{c \in P \mid c \leq a \text{ and } c \leq b\}$. A non-zero element $a \in P$ is said to be a *zero-divisor* if $L(a, b) = \{0\}$ for some $0 \neq b \in P$. We say a non-zero element $a \in P$ is an *atom (primitive)* if for any $0 \neq b \in P$, $b \leq a$ implies $a = b$. The *zero-divisor graph* $\Gamma(P)$ *of a poset* P is a graph whose vertex set $V(\Gamma(P))$ consists of the zero-divisors of P, in which a is adjacent to b if and only if $L(a, b) = \{0\}$. It is shown in [9] that for any poset P, the clique number and the chromatic number of $\Gamma(P)$ are the same.

By a *semigroup,* we mean a finite commutative semigroup with the zero element 0. A semigroup S is said to be *reduced* if for any $a \in S$ and any positive integer n, $a^n = 0$ implies $a = 0$. A semigroup S is said to be *idempotent* (it is a so-called semilattice, see [13]) if for each $a \in S$, $a^2 = a$.

We define a *zero-divisor graph of a semigroup* in a similar manner in the definition of zero-divisor graph of a ring.

Let $R = \mathbb{Z}_2^k$. Clearly, it is a Boolean ring and it becomes a poset by defining $a \leq b$ iff $ab = a$ for any $a, b \in R$. Note that, the zero-divisor graphs of R as a ring (or a semigroup) and as a poset coincide. Let H be a subgraph of $\Gamma(\mathbb{Z}_2^k)$. We say that H is *minimal* (see [13]) if H is an induced subgraph of $\Gamma(\mathbb{Z}_2^k)$ which contains all the atoms of the poset $\mathbb{Z}_2^k$, and we say H is *minimal closed* (see [13]) if H is minimal and $V(H) \cup \{0\}$ is a sub-semigroup of $\mathbb{Z}_2^k$. The following results can be found in [13].

Theorem 3 [13]. *Let G be a simple graph with $\omega(G) = k$. Then the following statements are equivalent:*

(i) G is the zero-divisor graph of a poset.
(ii) G is a k-compact graph.
(iii) G is a generalized complete k-partite graph.
(iv) The reduced graph G_r of G is isomorphic to a minimal subgraph of $\Gamma(\mathbb{Z}_2^k)$.

Theorem 4 [13]. *Let G be a simple graph with $\omega(G) = k$. Then the following statements are equivalent:*

(i) G is the zero-divisor graph of a reduced semigroup with 0.
(ii) G is a generalized complete k-partite graph such that for any non-adjacent vertices $a, b \in V(G)$, there is a vertex $c \in V(G)$ with $W(c) = W(a) \cup W(b)$.
(iii) The reduced graph G_r of G is isomorphic to a minimal closed subgraph of $\Gamma(\mathbb{Z}_2^k)$.
(iv) G is the zero-divisor graph of a semilattice (or equivalently, idempotent semigroup) with 0.

A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. The following result was proved by Lovasz, see [12].

Theorem 5 [12]. *The complement of every perfect graph is perfect.*

In [7], Berge conjectured the following and it was proved by Chudnovsky et al., see [8].

Theorem 6 (Strong Perfect Graph Theorem [8]). *A graph G is perfect if and only if it does not contain an induced subgraph which is either an odd cycle of length at least 5 or the complement of such a cycle.*

The paper mainly deals with the results on perfect graph using the Strong Perfect Graph Theorem. As a result, we deduced many known results in the literature. This is precisely as follows.

In Sect. 2, we prove that the G-generalized join of complete graphs and totally disconnected graphs is perfect if and only if G is perfect. As a consequence, we deduce the results proved in [14] and [17] and prove that the lexicographic product of a perfect graph and a complete graph and the lexicographic product of a perfect graph and a complement of a complete graph are perfect.

In Sect. 3, we characterize rings, posets and reduced semigroups whose zero-divisor graphs and ideal based zero-divisors are perfect. As a result, we characterize distributive lattices with 0, reduced semirings and boolean rings whose zero divisor graphs are perfect, which are proved in [15]. Further, we completely characterize rings the ideal based zero-divisor graph of the ring $\mathbb{Z}_n$ is perfect.

2 When a G-generalized Join of Complete and Totally Disconnected Graphs is Perfect

In this section, we prove the following result on perfect graphs.

Theorem 7. *If G is a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $H_1, H_2, \ldots, H_n$ are graphs such that each H_i is either complete or a totally disconnected graph, then G is perfect if and only if $G[H_1, H_2, \ldots, H_n]$ is perfect.*

Proof. Let $G' = G[H_1, H_2, \ldots, H_n]$. It is enough to prove if G is perfect, then G' is perfect. Suppose G' is not perfect, then by Theorem 6, G' contains either an odd cycle of length at least 5 as an induced subgraph or the complement of an odd cycle of length at least 5 as an induced subgraph.

Case 1. G' contains an odd cycle C_{2k+1} as an induced subgraph, where $k \geq 2$.

Let $V(C_{2k+1}) = \{x_0, x_1, \ldots, x_{2k}\}$ such that x_i is adjacent to x_{i+1} (where the addition in subscript is taken modulo $2k+1$) and x_i is not adjacent to x_j, where $j \neq i-1, i+1$. Suppose there exists $1 \leq t \leq n$ such that $|V(C_{2k+1}) \cap V(H_t)| \geq 2$.

First, if there exists $0 \leq i \leq 2k$ such that $x_i, x_{i+1} \in V(H_t)$. Then H_t is complete and hence $x_{i-1} \notin V(H_t)$ (otherwise, C_{2k+1} would not be induced in G'). Thus there exists $1 \leq s \leq n$ with $s \neq t$ such that $x_{i-1} \in V(H_s)$ and hence x_{i-1} is adjacent to x_{i+1}, which is a contradiction.

Next, if there exist $0 \leq i, j \leq 2n$ such that $j \neq i-1, i, i+1$ and $x_i, x_j \in V(H_t)$. Then H_t has no edge in G' and $x_{i+1}, x_{i-1} \notin V(H_t)$. Suppose if $j \neq i+2$, then there exists $1 \leq s \leq n$ such that $s \neq t$ and $x_{i+1} \in V(H_s)$ and hence x_j is adjacent to x_{i+1}, (because of $x_i x_{i+1} \in E(C_{2k+1})$) which is a contradiction. Therefore, if $j = i+2$, then there exists $1 \leq s \leq n$ such that $s \neq t$ and $x_{i-1} \in V(H_s)$ and therefore x_{i-1} is adjacent to x_j, which is again a contradiction.

Hence $|V(C_{2k+1}) \cap V(H_i)| = 1$, for $0 \leq i \leq 2k$ which implies that G contains an odd cycle of length at least 5 as an induced subgraph, which is a contradiction.

Case 2. G' contains a complement of an odd cycle of length at least 5 as an induced subgraph.

Let $\overline{C_{2k+1}}$ be the complement of the odd cycle C_{2k+1} as an induced subgraph of G', where $k \geq 2$ with $V(\overline{C_{2k+1}}) = \{x_0, x_1, \ldots, x_{2k}\}$ such that x_i is not adjacent to x_j for $j = i-1, i+1$ and x_i is adjacent to x_j, for $j \neq i-1, i, i+1$ (where the addition in subscripts is taken modulo $2k+1$). Suppose there exists $1 \leq t \leq n$ such that $|V(C_{2k+1}) \cap V(H_t)| \geq 2$.

First, if there exists $0 \leq i \leq 2k$ such that $x_i, x_{i+1} \in V(H_t)$. Then H_t has no edge and $x_{i-1} \notin H_t$ and hence there exists $1 \leq s \leq n$ with $s \neq t$ such that $x_{i-1} \in V(H_s)$. But x_{i+1} is adjacent to x_{i-1} and hence x_i is adjacent to x_{i-1}, which is a contradiction.

Next, if there exist $0 \leq i, j \leq 2n$ such that $j \neq i-1, i, i+1$ and $x_i, x_j \in V(H_t)$. Then H_t is complete and $x_{i-1}, x_{i+1} \notin V(H_t)$. Suppose if $j \neq i+2$, then there exists $1 \leq s \leq n$ such that $s \neq t$ and $x_{i+1} \in V(H_s)$. But x_j is adjacent to x_{i+1} and therefore x_i is adjacent to x_{i+1}, which is impossible. Hence, if $j = i+2$, then there exists $1 \leq s \leq n$ such that $s \neq t$ and $x_{i-1} \in V(H_s)$ and therefore x_{i-1} is adjacent x_i, which is again a contradiction.

Thus $|V(\overline{C_{2k+1}}) \cap V(H_i)| = 1$, for $0 \leq i \leq 2k$, which implies that G contains a complement of an odd cycle of length at least 5 as an induced, which is a contradiction.

The following corollary is an immediate consequence of Theorem 7.

Corollary 1. *If G is perfect and n is a positive integer, then $G[K_n]$ and $G[K_n^c]$ are perfect.*

Proof. As $G[K_n] \cong G[K_n, K_n, \ldots, K_n]$ and $G[K_n^c] \cong G[K_n^c, K_n^c, \ldots, K_n^c]$, the result follows from Theorem 7.

The following result proved in [17] is deduced from Theorem 7.

Corollary 2 (Corollary 3.2, [17]). *A graph G is perfect if and only if it's reduced graph G_r is perfect.*

The following relation is defined on a graph G in [14]. For $x, y \in V(G)$, define $x \approx y$ if and only if either $x = y$ or $xy \in E(G)$ and $N(x)\backslash\{y\} = N(y)\backslash\{x\}$. Clearly, it is an equivalence relation. Let $[x]$ be the equivalence class of x, and $S = \{[x_1], [x_2], \ldots, [x_r]\}$ be the set of all equivalence classes of the relation $\approx$. Based on these equivalence classes of the relation $\approx$, we defined (This can be seen in [14]) the graph G_{red} with vertex set $V(G_{red}) = S$ and two distinct vertices $[x]$ and $[y]$ are adjacent in G_{red} if and only if x and y are adjacent in G. Clearly, for any graph G, $G = G_{red}[\langle[x_1]\rangle, \langle[x_2]\rangle, \ldots, \langle[x_r]\rangle]$ and $\langle[x_i]\rangle$ is complete, for $1 \leq i \leq r$.

By Theorem 7, we deduce the following result proved in [14].

Corollary 3 (Theorem 4.4, [14]). *A graph is perfect if and only if G_{red} is perfect.*

3 Perfect Zero-Divisor Graph of a Ring

In this section, we ask the following interesting question. When does the zero-divisor graph of a ring R perfect? To answer this question, we provide a necessary and sufficient condition for which the zero-divisor graph of a ring is perfect.

Theorem 8. *If R is a ring, then $\Gamma(R)$ is perfect if and only if its compressed zero-divisor graph $\Gamma_E(R)$ of R is perfect.*

Proof. The result follows from Theorems 1 and 7.

Let $R_1, R_2, \ldots, R_k$ be rings. For $x_j \in R_1 \times R_2 \times \ldots \times R_k$, there exists a unique $x_j(i) \in R_i$, for $1 \leq i \leq k$, such that $x_j = (x_j(1), x_j(2), \ldots, x_j(k))$.

Note that there are several rings satisfying Beck's conjecture; see [2,4,6,9,10,20]. One of them is a finite reduced ring. Using Observation 1, we give a shorter proof of this result as follows.

Corollary 4 [6,20]. *If R is a non-zero reduced ring, then $\chi(\Gamma(R)) = \omega(\Gamma(R))$.*

Proof. By Observation 1 and Theorem 2, it is enough to prove $\omega(\Gamma(\mathbb{Z}_2^k)) = \chi(\Gamma(\mathbb{Z}_2^k))$. Clearly $\{e_i \mid 1 \leq i \leq k\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, induces a clique. Color first e_i by i, for $1 \leq i \leq k$.

For any $x = (x(1), x(2), \ldots, x(k)) \in V(\Gamma(\mathbb{Z}_2^k))\backslash\{e_i \mid 1 \leq i \leq k\}$, there exists a least j with $1 \leq j \leq k$, such that $x(i) = 0$ for $1 \leq i \leq j-1$ and $x(j) = 1$. Color x by j, then the resulting coloring is a proper k-coloring of $\Gamma(\mathbb{Z}_2^k)$.

The following result gives a necessary condition for a product of rings whose zero-divisor graphs are perfect.

Theorem 9. *Let $R = R_1 \times R_2 \times \ldots \times R_k$, where R_i's are indecomposable rings. If $\Gamma(R)$ is perfect, then $k \leq 4$.*

Proof. Suppose $k \geq 5$. Then the set of vertices $\{(1,1,0,0,0,0,\ldots,0),(0,0,1,1,0,0,\ldots,0),\ (1,0,0,0,1,0,\ldots,0),(0,1,0,1,0,0,\ldots,0),(0,0,1,0,1,0,\ldots,0)\}$ forms an induced cycle of length 5. By Theorem 6, we get a contradiction.

Next, let us prove the following result.

Theorem 10. *If $R = \mathbb{Z}_2^4$ $(= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, then $\Gamma(R)$ is perfect.*

Proof. Suppose $\Gamma(R)$ is not perfect. Then, by Theorem 6, we consider the following cases.

Case 1. $\Gamma(R)$ contains an odd cycle of length at least 5 as an induced subgraph.

Let C_{2r+1} be an induced cycle in $\Gamma(R)$ of length $2r+1$ with the vertex set $\{x_0, x_1, \ldots, x_{2r}\}$, where $r \geq 2$. If exactly one co-ordinate of x_i is non-zero, for $0 \leq i \leq 2r$, then $2r+1 \leq 4$, a contradiction. Therefore there exists an x_i containing at least two non-zero co-ordinates. WLOG, $x_i = (1, 1, x_i(3), x_i(4))$, for some i, $0 \leq i \leq 2r$. Then the 1^{st} two coordinates of x_{i-1}, x_{i+1} are zeros, that is, $x_{i-1}(1) = x_{i-1}(2) = x_{i+1}(1) = x_{i+1}(2) = 0$. Since x_{i-1} and x_{i+1} are not adjacent, either the third coordinate or forth coordinate of x_{i-1} and x_{i+1} are non-zero. WLOG, $x_{i-1}(3) = x_{i+1}(3) = 1$. If $x_{i-1}(4) = 1$, then $x_{i+1}(4) = 0$, as $x_{i-1} \neq x_{i+1}$ and hence $x_{i-1} = (0,0,1,1)$ and $x_{i+1} = (0,0,1,0)$. Since x_{i-2} is adjacent to x_{i-1}, we have $x_{i-2} = (x_{i-2}(1), x_{i-2}(2), 0, 0)$. Thus x_{i-2} is adjacent to x_{i+1}, which is a contradiction. Hence $x_{i-1}(4) = 0$, which implies that $x_{i+1}(4) = 1$ and thus $x_{i+1} = (0,0,1,1)$ and $x_{i-1} = (0,0,1,0)$. Since x_{i+2} is adjacent x_{i+1}, we have $x_{i+2} = (x_{i+2}(1), x_{i+2}(2), 0, 0)$ and hence x_{i+2} is adjacent to x_{i-1}, which is a contradiction.

Case 2. $\Gamma(R)$ contains the complement of an odd cycle of length at least 5 as an induced subgraph.

Let $\overline{C_{2r+1}}$ be an induced subgraph of $\Gamma(R)$ with vertex set $\{x_0, x_1, \ldots, x_{2r}\}$, where $r \geq 2$. If no x_i contains exactly two coordinates that are non-zeros, then there exists j, $1 \leq j \leq k$ such that x_j contains exactly three that coordinates that are non-zero (otherwise $2r+1 \leq 4$), which is impossible. Thus there exists i, $1 \leq i \leq k$ such that x_i contains exactly two coordinates that are non-zeros. WLOG, $x_i = (1, 1, x_i(3), x_i(4))$. Since x_i is adjacent to $2r-2$ vertices in $\overline{C_{2r+1}}$, namely $x_{i+2}, x_{i+3}, \ldots, x_{i+2r-1}$ (where the addition in subscripts taken modulo $2r+1$), we have the 1^{st} two coordinates of $x_{i+2}, x_{i+3}, \ldots, x_{i+2r-1}$ are zero's and hence $x_{i+2}, x_{i+3}, \ldots, x_{i+2r-1} \in \{(0,0,1,1),(0,0,1,0),(0,0,0,1)\}$. Thus $2r-2 \leq 3$, which implies $2r+1 \leq 6$. As it is an odd number and $r \geq 2$, we have $2r+1 = 5$. Therefore $\overline{C_5} \cong C_5$. By Case 1, which is impossible.

The following result in [14] is a consequence of Theorems 9 and 10.

Corollary 5 [14]. *If $R = \mathbb{Z}_2^k$, then $\Gamma(R)$ is perfect if and only if $k \le 4$.*

Proof. By Theorems 9 and 10, it is enough to prove that $\Gamma(R)$ is perfect if $k \le 3$. In this case we have $|V(\Gamma(R))| \le 6$, and hence $\Gamma(R)$ does not contain a cycle of length 5 as an induced subgraph of $\Gamma(R)$ and, thus the result follows.

It is well-known that any finite non-zero reduced commutative ring R is isomorphic to a finite direct product of finite fields, say $\mathbb{F}_{p_1^{\alpha_1}}, \mathbb{F}_{p_2^{\alpha_2}}, \ldots, \mathbb{F}_{p_\ell^{\alpha_\ell}}$, where p_i's are prime numbers and α_i's are positive integers, that is $R \cong \mathbb{F}_{p_1^{\alpha_1}} \times \mathbb{F}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{F}_{p_\ell^{\alpha_\ell}}$.

By Theorem 2, the compressed zero-divisor graph of a reduced ring R is isomorphic to the zero-divisor graph of $\mathbb{Z}_2^k$, for some $k \ge 1$, that is $\Gamma_E(R) \cong \Gamma(\mathbb{Z}_2^k)$. So, the following result is a consequence of Theorem 9 and Corollary 5.

Theorem 11. *If $R \cong \mathbb{F}_{p_1^{\alpha_1}} \times \mathbb{F}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{F}_{p_\ell^{\alpha_\ell}}$ is a non-zero reduced ring, where $\mathbb{F}_{p_i^{\alpha_i}}$'s are finite fields, then $\Gamma(R)$ is perfect if and only if $\ell \le 4$.*

Proof. The first part is clear from Theorem 9. For the second part, let us assume that $\ell \le 4$. Then $\omega(\Gamma(R)) \le 4$. By the above discussion, $\Gamma_E(R) \cong \Gamma(\mathbb{Z}_2^k)$ for some $k \ge 1$. Suppose $k \ge 5$, then $\Gamma(\mathbb{Z}_2^k)$ contains a clique $\langle \{e_i : 1 \le i \le k\} \rangle$ of size at least 5 (where e_i's are defined in Corollary 4) and hence $\omega(\Gamma(R)) \ge 5$, which is impossible. Thus $k \le 4$ and therefore, by Corollary 5 $\Gamma(\mathbb{Z}_2^k)$ is perfect, and hence $\Gamma(R)$ is perfect by Theorem 8.

The following result in [15] is an immediate consequence of Corollary 5, because every finite Boolean ring R is isomorphic to $\mathbb{Z}_2^k$, for some $k \ge 1$.

Corollary 6 [15]. *Let R be a finite Boolean ring. Then the following are equivalent,*

(1) $\Gamma(R)$ is perfect.
(2) $\Gamma(R)$ does not contain K_5 as a subgraph.
(3) $|R| \le 2^4$.

3.1 Perfect Ideal Based Zero-Divisor Graph of Rings

In this subsection, we characterize rings whose ideal based zero-divisor graphs are perfect. In particular, under what values of n, the ideal based zero divisor graph of the ring $\mathbb{Z}_n$ of integers modulo n is perfect.

The following observation is observed in [16] and [21].

(i) If I is an ideal of R and $x_1 + I, x_2 + I, \ldots, x_k + I$ are the distinct cosets of I, which are zero-divisors of the quotient ring $\frac{R}{I}$, then $\Gamma_I(R)$ is a $\Gamma(\frac{R}{I})$-generalized join of $\langle x_1 + I \rangle, \langle x_2 + I \rangle, \ldots, \langle x_k + I \rangle$, that is,

$$\Gamma_I(R) = \Gamma\Big(\frac{R}{I}\Big)\big[\langle x_1 + I \rangle, \langle x_2 + I \rangle, \ldots, \langle x_k + I \rangle\big],$$

(ii) $\langle x_i + I \rangle$ is a complete subgraph of $\Gamma_I(R)$ if and only if $x_i^2 \in I$,

(iii) $\langle x_i + I \rangle$ is a totally disconnected subgraph of $\Gamma_I(R)$ if and only if $x_i^2 \notin I$.

Hence, by Theorems 7 and 8, we have

Theorem 12. *Let I be an ideal of R, then the following are equivalent,*

(i) $\Gamma_I(R)$ is perfect;
(ii) $\Gamma(\frac{R}{I})$ is perfect;
(iii) $\Gamma_E(\frac{R}{I})$ is perfect.

We recall the following result proved in [19].

Theorem 13 [19]. *The zero divisor graph $\Gamma(\mathbb{Z}_n)$ of a ring $\mathbb{Z}_n$ is perfect if and only if $n = p^a, p^a q^b, p^a qr$, or $pqrs$, where p, q, r and s are distinct primes and a and b are positive integers.*

It is well known that if I is an ideal of $\mathbb{Z}_n$ generated by m, then $\frac{\mathbb{Z}_n}{I} \cong \mathbb{Z}_m$. So, we have

Corollary 7. *If I is an ideal of $\mathbb{Z}_n$ generated by m, then $\Gamma_I(\mathbb{Z}_n)$ is perfect if and only if $m = p^a, p^a q^b, p^a qr$, or $pqrs$, where p, q, r and s are distinct primes and a and b are positive integers.*

Proof. By Theorems 12 and 13, $\Gamma_I(\mathbb{Z}_n)$ is perfect if and only if $\Gamma(\mathbb{Z}_m)$ is perfect if and only if $m = p^a, p^a q^b, p^a qr$, or $pqrs$.

3.2 Zero-Divisor Graph of Rings, Reduced Semigroups and Posets

In [13], it is shown that the chromatic number is equal to the clique number of zero-divisor graphs of poset, reduced semiring with 0 and reduced semigroup with 0. So it is interesting to consider the following problem.

Problem. Characterize the posets, reduced rings and reduced semigroups whose zero-divisor graphs are perfect.

Now we characterize posets whose zero-divisor graphs are perfect using Theorem 3.

Theorem 14. *Let G be a zero-divisor graph of a poset with 0 and $\omega(G) = k$. Then the following are equivalent,*

(i) G is perfect.
(ii) The reduced graph G_r of G is perfect.
(iii) The reduced graph H_r of H (where H is in the Definition of generalized complete k-partite graph) is perfect.

Proof. (i) $\Leftrightarrow$ (ii) It follows from Corollary 2.
(ii) $\Rightarrow$ (iii) It follows from the definition of perfect.

(iii) $\Rightarrow$ (ii) Suppose G_r is not perfect graph, then by the Theorem 6, G_r contains an odd cycle of length at least 5 as an induced subgraph or the complement of an odd cycle of length at least 5 as an induced subgraph.

Let $e_1, e_2, \ldots, e_k$ be the atoms of G.

Case 1. G_r contains an odd cycle of length at least 5 as an induced subgraph.

Let C_{2s+1} be an odd cycle of G_r as an induced subgraph with vertex set $V(C_{2s+1}) = \{a_0, a_1 \ldots, a_{2s}\}$, where $s \geq 2$. Then $V(C_{2s+1})$ is not a subset of $V(H_r)$. As the atoms forms a clique, we have $|V(C_{2s+1}) \cap \{e_1, e_2, \ldots, e_k\}| \leq 2$. First if $|V(C_{2s+1}) \cap \{e_1, e_2, \ldots, e_k\}| = 2$, then there exist $i, j \in \{1, 2, \ldots, k\}$ and $\ell \in \{0, 1, 2, \ldots, 2s\}$ such that $a_\ell = e_i$ and $a_{\ell+1} = e_j$. Since $a_{\ell+2}$ and $a_{\ell+3}$ are not adjacent to $a_\ell = e_i$, we have $i \notin W(a_{\ell+2}) \cup W(a_{\ell+3})$ and hence $W(a_{\ell+2}) \cup W(a_{\ell+3}) \neq \{1, 2, \ldots, k\}$, which is a contradiction to the definition of generalized complete k-partite graph. Next if $|V(C_{2s+1}) \cap \{e_1, e_2, \ldots, e_k\}| = 1$, then we get a contradiction in a similar way as above. Thus $V(C_{2s+1}) \cap \{e_1, e_2, \ldots, e_k\} = \emptyset$ and hence C_{2s+1} is an induced odd cycle of H_r, which is a contradiction.

Case 2. G_r contains the complement of an odd cycle of length at least 5 as an induced subgraph.

Let $\overline{C_{2s+1}}$ be the complement of the odd cycle C_{2s+1} in G_r with vertex set $V(\overline{C_{2s+1}}) = \{a_0, a_1, \ldots, a_{2s}\}$, where $s \geq 2$. If $V(\overline{C_{2s+1}}) \cap \{e_1, e_2, \ldots, e_k\} \neq \emptyset$, then there exists $i \in \{1, 2, \ldots, k\}$ such that $e_i = a_\ell$, for some $\ell \in \{0, 1, 2, \ldots, 2s\}$. Then $a_{\ell-1}, a_{\ell+1} \notin \{e_1, e_2, \ldots, e_k\}$ and they are not adjacent to e_i and hence $i \notin W(a_{\ell-1}) \cup W(a_{\ell+1})$, which is impossible. Thus, $V(\overline{C_{2s+1}}) \cap \{e_1, e_2, \ldots, e_k\} = \emptyset$ and therefore $\overline{C_{2s+1}}$ lies in H_r, which is a contradiction.

Next, we present equivalent conditions for a zero-divisor graph of a reduced semigroup to be perfect using Theorem 4.

Theorem 15. *Let G be a zero-divisor graph of a reduced semigroup with $\omega(G) = k$. Then the following are equivalent,*

(i) G is perfect.
(ii) The reduced graph G_r of G is perfect.
(iii) The reduced graph H_r of H (where H is given in the definition of generalized complete k-partite graph) is perfect.

Proof. The proof is similar to that of Theorem 14.

A lattice $L = (L, \wedge, \vee)$ with 0 is *distributive* if for $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. As every lattice is a poset, we have the following result proved in [15].

Corollary 8 [15]. *Let L be a distributive lattice with 0. Then the following are equivalent,*

(i) $\Gamma(L)$ is perfect.
(ii) $\Gamma(L)$ contains no induced cycle of length 5.
(iii) $\omega(\Gamma(L)) \leq 4$, (equivalently, the number of atoms of $\Gamma(L)$ is at most 4).

Proof. (i) ⇒ (ii) It is trivial from the definition of perfect graph.
(ii) ⇒ (iii) If $\langle\{a_1, a_2, \ldots, a_s\}\rangle$ is a clique in $\Gamma(L)$, where $s \geq 5$, then the subgraph induced by $\{a_1 \vee a_2, a_3 \vee a_4, a_1 \vee a_5, a_2 \vee a_3, a_4 \vee a_5\}$ is an induced cycle of length 5 (as L is distributive) which is a contradiction.
(iii) ⇒ (i) Suppose $\Gamma(L)$ is not perfect. Then by Theorem 14, the reduced subgraph H_r of H, defined in Theorem 14, is not perfect. By Theorem 6, H_r contains an odd cycle of length at least 5 as an induced subgraph or its complement of an odd cycle of length at least 5 as an induced subgraph. If H_r contains an induced odd cycle C_{2s+1} with vertex set $V(C_{2s+1}) = \{a_1, a_2, \ldots, a_{2s+1}\}$, where $s \geq 2$. Then $a_i \wedge a_{i+1} = 0$, for $1 \leq i \leq 2s$, $a_{2s+1} \wedge a_1 = 0$ and $a_i \wedge a_j \neq 0$, for $j \neq i-1, i, i+1$ and hence the subgraph induced by $\{a_1 \wedge a_3, a_1 \wedge a_4, a_2 \wedge a_4, a_2 \wedge a_5, a_3 \wedge a_{2s+1}\}$ is a clique in $\Gamma(L)$ of size 5, which is a contradiction. Similarly if H_r contains the complement $\overline{C_{2s+1}}$ of an induced odd cycle C_{2s+1} with vertex set $V(\overline{C_{2s+1}}) = \{a_1, a_2, \ldots, a_{2s+1}\}$, where $s \geq 2$, then the subgraph induced by $\{a_1 \wedge a_2, a_2 \wedge a_3, a_3 \wedge a_4, a_4 \wedge a_5, a_5 \wedge a_1\}$ is a clique in $\Gamma(L)$ of size 5, which is again a contradiction.

As every semiring is a semigroup and by Theorem 15, we have the following result proved in [15].

Corollary 9 [15]. *Let R be a reduced semiring with 0. Then the following are equivalent,*

(i) $\Gamma(R)$ *is perfect.*
(ii) $\Gamma(R)$ *contains no induced cycle of length 5.*
(iii) $\omega(\Gamma(R)) \leq 4$, *(equivalently, the number of atoms of* $\Gamma(R)$ *is at most 4).*

Proof. The proof is similar to that of Corollary 8 by replacing $\vee$ and $\wedge$ by addition and multiplication, respectively.

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