On the Cauchy Problem for the Nonlinear Wave Equation with Damping and Potential

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Abstract In this note, we study the Cauchy problem for the nonlinear wave equation with damping and potential terms. The aim of this study is to generalize the result in Georgiev et al. (J. Differ. Equ. 267(5):3271–3288, 2019) into two directions. One is to relax the condition which characterizes the behavior of the coefficient of the damping term at spatial infinity as in (6). The other is to treat the slowly decreasing initial data. The decaying rate of the data affects the global behavior of the solutions even if the nonlinear exponent lies in the super-critical regime (see Theorem 5 below).

1 Introduction

This paper is concerned with the Cauchy problem for the nonlinear wave equation with damping and potential:

$$\begin{cases} (\partial_t^2 + 2w(r)\partial_t - \Delta + V(r))U = |U|^p & \text{in } (0,T)\times\mathbb{R}^3, \\ U(0,x) = \varepsilon f_0(r), \quad (\partial_t U)(0,x) = \varepsilon f_1(r) & \text{for } x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $r = |x|$ and $p > 1$. In the earlier work [8], the coefficients of damping and potential terms are supposed to satisfy the relation:

$$V(r) = -w'(r) + w(r)^2 \quad \text{for } r > 0, \tag{2}$$

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where

$$w(r) = 1/r \quad \text{for } r \geq 1.$$

Keeping such a relation between the coefficients of damping and potential terms, we relax the assumption on the initial data at spatial infinity. Actually, we obtain upper bound of the lifespan for slowly decreasing initial data in Theorems 1 and 5 below. Moreover, we are able to broaden the choice of the damping coefficient, essentially, as $w(r) = \mu/(2r)$ for $\mu \geq 0$ and $r \geq 1$. The number μ affects on the shift of the critical exponent of the Strauss type, as we shall see below.

Before going into further details, we recall some known results. The case without any damping term, i.e. the case when $w = V = 0$, has been intensively studied for few decades (see [4, 6, 9, 11, 14, 17, 20], or references in [5]) and in this case there is a critical nonlinear exponent known as Strauss critical exponent that separates the global existence and blow-up of the small data solutions. This critical exponent $p_0(n)$ is given by the positive root of

$$\gamma(p, n) := 2 + (n + 1)p - (n - 1)p^2 = 0.$$

For the semilinear wave equation with potential

$$(\partial_t^2 - \Delta + V(x))U = |U|^p \quad \text{in } (0, T) \times \mathbb{R}^3,$$

one can find blow up result in [18] or global existence part in [7].

In the case where the coefficient of the damping term is a function of time variable, D'Abbicco et al. [3] derived the critical exponent for the Cauchy problem to

$$\left(\partial_t^2 + \frac{2}{1+t}\partial_t - \Delta\right) U = |U|^p \quad \text{in } (0, T) \times \mathbb{R}^3, \tag{3}$$

by assuming the radial symmetry. Indeed, they proved that the problem admits a global solution for sufficiently small initial data if $p > p_0(5)$, and that the solution blows up in finite time if $1 < p < p_0(5)$. This result can be interpreted as an effect of the damping term in (3) that shifts the critical exponent for small data solutions from $p_0(3)$ to $p_0(5)$. The assumption about the radial symmetry posed in [3] was removed by Ikeda and Sobajima [10] for the blow-up part (actually, they treated more general damping term $\mu(1 + t)^{-1}\partial_t u$ with $\mu > 0$), and by Kato and Sakuraba [12] and Lai [16] for the existence part, independently.

In the next section, we formulate our problem and describe the statements to the problem.

2 Formulation of the Problem and Results

Since we are interested in spherically symmetric solutions to the problem (1), we set

$$u(t,r) = rU(t, r\omega) \quad \text{with } r = |x|,\ \omega = x/|x|.$$

Then, by the relation (2) we obtain

$$\begin{cases} (\partial_t - \partial_r + w(r))(\partial_t + \partial_r + w(r))u = |u|^p/r^{p-1} & \text{in } (0,T)\times(0,\infty), \\ u(0,r) = \varepsilon\varphi(r), \quad (\partial_t u)(0,r) = \varepsilon\psi(r) & \text{for } r > 0, \\ u(t,0) = 0 \quad \text{for } t \in (0,T), \end{cases} \tag{4}$$

where $\varphi(r) = rf_0(r)$ and $\psi(r) = rf_1(r)$.

In order to express the solution of (4), we set $W(r) = \int_0^r w(\tau)d\tau$ for $r \geq 0$ and define

$$E_-(t,r,y) = e^{-W(r)}e^{2W(2^{-1}(y-t+r))}e^{-W(y)} \quad \text{for } t, r \geq 0,\ y \geq t-r. \tag{5}$$

We suppose that $w(r)$ is a function in $C([0,\infty)) \cap C^1(0,\infty)$ satisfying

$$w(r) = \frac{\mu}{2r} + \tilde{w}(r), \quad |\tilde{w}(r)| \lesssim r^{-1-\delta} \quad \text{for } r \geq r_0 \tag{6}$$

with some positive number r_0, $\mu \geq 0$, and $\delta > 0$. This assumption implies

$$e^{W(r)} \sim \langle r\rangle^{\mu/2}, \quad r > 0.$$

Then the definition (5) of E_- implies

$$E_-(t,r,y) \sim \frac{\langle r-t+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}}. \tag{7}$$

Following the argument in [8], we see that the problem (4) can be written in the integral form

$$u(t,r) = \varepsilon u_L(t,r) + \frac{1}{2}\iint_{\Delta_-(t,r)} E_-(t-\sigma,r,y)\frac{|u(\sigma,y)|^p}{y^{p-1}}dyd\sigma \tag{8}$$

for $t > 0$, $r > 0$, where we have set

$$\Delta_-(t,r) = \{(\sigma,y) \in (0,\infty)\times(0,\infty);\ |t-r| < \sigma + y < t+r,\ \sigma - y < t-r\}.$$

Besides, we put

$$u_L(t,r) = \frac{1}{2}\int_{|t-r|}^{t+r} E_-(t,r,y)\left(\psi(y)+\varphi'(y)+w(y)\varphi(y)\right)dy \qquad (9)$$
$$+\chi(r-t)E_-(t,r,r-t)\varphi(r-t),$$

where $\chi(s)=1$ for $s\geq 0$, and $\chi(s)=0$ for $s<0$.

Then, the blow-up result in [8] where the case of $\mu = 2$ is handled can be extended as follows.

Theorem 1 *Suppose that* (6) *holds. Let* φ, $\psi \in C([0,\infty))$ *satisfy*

$$\varphi(r)\equiv 0, \quad \psi(r)\geq 0, \quad \psi(r)\not\equiv 0 \quad \text{for } r\geq 0. \qquad (10)$$

If $1<p\leq p_0(3+\mu)$, *then*

$$T(\varepsilon)\leq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & \text{if} \quad p=p_0(3+\mu), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & \text{if} \quad 1<p<p_0(3+\mu). \end{cases}$$

Here $T(\varepsilon)$ *denotes the lifespan of the problem* (4).

On the other hand, when $p>p_0(3+\mu)$, we expect that the solution exists globally. Actually, when the initial data decays rapid enough, one can show the following result analogously to [8]. But the pointwise estimate (12) is improved in the region away from the light cone, due to the factor $\langle t+r\rangle^{-1}$.

Theorem 2 *Suppose that* (6) *holds. Assume* $p>p_0(3+\mu)$ *and* $\kappa\geq(\mu/2+1)p-1$. *Let* $\varphi\in C^1([0,\infty))$, $\psi\in C([0,\infty))$ *satisfy*

$$|\varphi(r)|\leq r\langle r\rangle^{-\kappa}, \quad |\varphi'(r)|+|\psi(r)|\leq r\langle r\rangle^{-\kappa-1} \ \text{for } r\geq 0. \qquad (11)$$

Then there exists $\varepsilon_0>0$ *so that the corresponding integral Eq.* (8) *to the problem* (4) *has a unique global solution satisfying*

$$|u(t,r)|\lesssim \varepsilon r\,\langle r\rangle^{-\mu/2}\langle t+r\rangle^{-1}\langle t-r\rangle^{-\eta}, \quad \eta:=(\mu/2+1)(p-1)-1 \qquad (12)$$

for $t>0$, $r>0$ *and any* $\varepsilon\in(0,\varepsilon_0]$.

This theorem leads us to one natural question, that is, what will happen when the initial data decays more slowly. In view of the work of Asakura [1], the self-similarity comes into play (see also [2, 13, 15, 19]). Namely, the global behavior would be different between the cases $\kappa\geq 2/(p-1)$ and $\kappa<2/(p-1)$. Indeed, we are able to show the global existence result in the former case.

Theorem 3 *Let $\kappa > \mu/2$. Suppose that* (6) *holds. Assume $p > p_0(3+\mu)$ and $\kappa \geq 2/(p-1)$. Let $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$ satisfy* (11)*. Then there exists $\varepsilon_0 > 0$ so that the integral Eq.* (8) *has a unique global solution for $\varepsilon \in (0, \varepsilon_0]$.*

The proof of Theorem 3 is based on the contraction mapping principle in a suitable weighted L^∞-space, similarly to the proof of Theorem 2. But we need to replace the weight function according to the size of κ as

$$w(r,t) = \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases} \langle t+r\rangle^{-(\kappa-\mu/2)} & (\mu/2 < \kappa < \mu/2+1), \\ \langle t+r\rangle^{-1}\left(1+\log\dfrac{1+t+r}{1+|t-r|}\right) & (\kappa = \mu/2+1), \\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-(\kappa-\mu/2-1)} & (\mu/2+1 < \kappa \leq (\mu/2+1)p-1). \end{cases}$$

for $t > 0$, $r > 0$. Note that $w(r,t)$ coincides with the upper bound appeared in (12) when $\kappa = (\mu/2+1)p-1$.

When either $p > p_0(3+\mu)$ and $\kappa < 2/(p-1)$ or $1 < p \leq p_0(3+\mu)$, we obtain the following lower bounds of the lifespan.

Theorem 4 *Let $\kappa > \mu/2$ and set $\kappa_1 := \mu/2+1+1/p$. Suppose that* (6) *holds. Let $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$ satisfy (11). Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$*

$$T(\varepsilon) \geq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & (p = p_0(3+\mu) \text{ and } \kappa > \kappa_1), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & (1 < p < p_0(3+\mu) \text{ and } \kappa > \kappa_1), \\ \exp(C\varepsilon^{-(p-1)}) & (p = 1+2/\kappa = p_0(3+\mu)), \\ Cb(\varepsilon) & (1 < p < 1+2/\kappa \text{ and } \kappa = \kappa_1), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1+2/\kappa \text{ and } \kappa < \kappa_1). \end{cases}$$

Here $b(\varepsilon)$ is defined by

$$\varepsilon^{p(p-1)} b^{\gamma(p,3+\mu)/2}(\log(1+b))^{p-1} = 1.$$

In order to prove Theorem 4, we reformulate the integral Eq. (8) to the following one:

$$v(t,r) = \frac{1}{2}\iint_{\Delta_-(t,r)} E_-(t-\sigma, r, y)\frac{|\varepsilon u_L(\sigma,y)+v(\sigma,y)|^p}{y^{p-1}}dyd\sigma \tag{13}$$

for $t > 0$, $r > 0$, by introducing the new unknown function $v = u - \varepsilon u_L$, as in the proof of Theorem 2.3 in [14]. It is rather easy to treat the integral Eq. (13) than the original one. Indeed, the solution v can be presumably assumed to satisfy the essentially same upper bound as in (12), although the solution u_L to the homogeneous equation does not satisfy such an estimate if the size of κ is small. Moreover, since u_L exists globally in time, the maximal existence time of the solution u of (8) is the same as that of the solution v of (13), so that the desired conclusion follows from the study of (13).

To conclude the optimality of those lower bounds in Theorem 4 with respect to ε, the upper bounds given in Theorem 1 are not enough for the last three cases. However, the following result enable us to conclude the optimality in these cases.

Theorem 5 *Suppose that* (6) *holds. Let* φ, $\psi \in C([0,\infty))$ *satisfy*

$$\varphi(r) \equiv 0, \quad \psi(r) \geq (1+r)^{-\kappa} \quad \text{for } r \geq 0 \tag{14}$$

for some $0 < \kappa \leq \kappa_1$. *Then there exist* $C > 0$, $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-(p-1)}) & (p = 1 + 2/\kappa = p_0(3+\mu)), \\ Cb(\varepsilon) & (1 < p < 1 + 2/\kappa \text{ and } \kappa = \kappa_1), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1 + 2/\kappa \text{ and } \kappa < \kappa_1). \end{cases}$$

Thanks to the assumption (14), if the solution of (8) exists globally in time, then we can prove that for any (t, r) satisfying $0 < t \leq 2r$ and $t - r \geq b$ with a positive number b, and for any natural number n, the following type of lower bound of the solution:

$$u(t,r) \geq \frac{(t-r)^{\mu/2+1}}{r^{\mu/2}(t-r-b)^{2/(p-1)}} \exp(p^n \log J(t,r)), \tag{15}$$

$$J(t,r) = \varepsilon E\,(t-r-b)^{2/(p-1)+\mu/2+1}(t-r)^{-\kappa-(\mu/2)-1} \tag{16}$$

holds, when $1 < p < 1 + 2/\kappa$, for instance. Here E is a positive constant independent of t, r, n, and ε. By choosing (t, r) far away from the origin on the line $t = 2r$ so that $\log J(t, r)$ is strictly positive, we find that the value of $u(t, r)$ becomes unbounded as $n \to \infty$. This gives a contradiction together with the upper bound of the lifespan.

This paper is organized as follows. We shall prove only Theorems 1 and 2 in this note, because the proofs of other theorems are rather technical and will appear elsewhere. In the Sect. 3, we give preliminary facts. The Sect. 4 is devoted to the proof of a blow-up result given in Theorem 1. In the Sect. 5, we derive a priori upper bounds and complete the proof of Theorem 2.

3 Preliminaries

In this section we prepare a couple of lemmas which will be used in the proofs of Theorems 1 and 2. For the proofs of Lemmas 1 and 2, see [14], Lemma 2.2 and Lemma 2.3.

Lemma 1 *Let* $0 < a < b$ *and* $\mu, \nu \geq 0$. *Then there exists* $C = C(\mu, \nu) > 0$ *such that*

$$\int_a^b \frac{(\rho - a)^\nu}{\rho^\mu} d\rho \geq \frac{C}{a^{\mu-\nu-1}} \left(1 - \frac{a}{b}\right)^{\nu+1}.$$

Lemma 2 *Let* $C_1, C_2 > 0$, $\alpha, \beta \geq 0$, $\theta \leq 1$, $\varepsilon \in (0, 1]$, *and* $p > 1$. *Suppose that* $f(y)$ *satisfies*

$$f(y) \geq C_1 \varepsilon^\alpha, \quad f(y) \geq C_2 \varepsilon^\beta \int_1^y \left(1 - \frac{\eta}{y}\right) \frac{f(\eta)^p}{\eta^\theta} d\eta, \quad y \geq 1.$$

Then, $f(y)$ *blows up in a finite time* $T_*(\varepsilon)$. *Moreover, there exists a constant* $C^* = C^*(C_1, C_2, p, \theta) > 0$ *such that*

$$T_*(\varepsilon) \leq \begin{cases} \exp(C^* \varepsilon^{-\{(p-1)\alpha+\beta\}}) & \text{if } \theta = 1, \\ C^* \varepsilon^{-\{(p-1)\alpha+\beta\}/(1-\theta)} & \text{if } \theta < 1. \end{cases}$$

Lemma 3 *Let* $0 \leq a \leq b$ *and* $k \in \mathbb{R}$. *Then we have*

$$\int_a^b \langle x \rangle^{-k} dx \lesssim (b - a) \times \begin{cases} \langle b \rangle^{-k} & (k < 1), \\ \langle b \rangle^{-1} \langle a \rangle^{-k+1} & (k > 1), \\ \langle b \rangle^{-1} \Psi(a, b) & (k = 1). \end{cases} \tag{17}$$

Here, for $0 \leq a \leq b$, *we put*

$$\Psi(a, b) := 2 + \log \frac{1 + b}{1 + a}. \tag{18}$$

Proof

(i) When $k > 1$, we have

$$\int_a^b \langle x \rangle^{-k} dx \lesssim \frac{1}{k-1} \left\{ \frac{1}{(1+a)^{k-1}} - \frac{1}{(1+b)^{k-1}} \right\}$$
$$\lesssim \frac{1}{(1+a)^{k-1}} \left\{ 1 - \left(\frac{1+a}{1+b} \right)^{k-1} \right\}.$$

Note that

$$1 - s^l \leq \max\{1, l\}(1 - s) \quad \text{for} \quad l \geq 0, \ 0 \leq s \leq 1. \tag{19}$$

Hence we obtain (17) for $k > 1$.

(ii) When $k < 1$, we have in the similar manner

$$\begin{aligned}\int_a^b \langle x\rangle^{-k} dx &\lesssim \frac{1}{1-k}\left\{\frac{1}{(1+b)^{k-1}} - \frac{1}{(1+a)^{k-1}}\right\}\\ &\lesssim \frac{1}{(1+b)^{k-1}}\left\{1-\left(\frac{1+a}{1+b}\right)^{1-k}\right\}\\ &\lesssim (b-a)\langle b\rangle^{-k}.\end{aligned}$$

(iii) When $k = 1$, It follows that

$$\int_a^b \langle x\rangle^{-1} dx \lesssim \log\left(\frac{1+b}{1+a}\right). \tag{20}$$

If $a \geq b/2$, since $\log(1+s) \leq s$ $(s \geq 0)$, we find that

$$\int_a^b \langle x\rangle^{-1} dx \lesssim \log\left(1+\frac{b-a}{1+a}\right) \lesssim \frac{b-a}{1+a} \lesssim (b-a)\langle b\rangle^{-1}.$$

If $a \leq b/2$ and $b \geq 1$, we find that $b - a \geq b/2$. Hence we have from (20)

$$\int_a^b \langle x\rangle^{-1} dx \lesssim \frac{b-a}{b}\log\left(\frac{1+b}{1+a}\right) \lesssim (b-a)\langle b\rangle^{-1}\log\left(\frac{1+b}{1+a}\right).$$

If $0 < b \leq 1$, we obtain

$$\int_a^b \langle x\rangle^{-1} dx \lesssim b-a \sim (b-a)\langle b\rangle^{-1}.$$

Therefore we get (17). This completes the proof. □

Lemma 4 *Let* $k_1, k_2, k_3 \geq 0$ *and* $\alpha \geq 0$. *Then we have*

$$\int_{-\alpha}^{\alpha} \langle\alpha+\beta\rangle^{-k_1-k_2}\langle\beta\rangle^{-k_1-k_3} d\beta \lesssim \langle\alpha\rangle^{-k_1} \times \begin{cases}\langle\alpha\rangle^{1-(k_1+k_2+k_3)} & (k_1+k_2+k_3<1),\\ 1 & (k_1+k_2+k_3>1),\\ \log(2+\alpha) & (k_1+k_2+k_3=1),\end{cases}$$

Proof First of all, we prove for $a, b \geq 0$ and $\alpha \geq 0$

$$\int_{-\alpha}^{\alpha} \langle\alpha+\beta\rangle^{-a}\langle\beta\rangle^{-b} d\beta \lesssim \begin{cases}\langle\alpha\rangle^{1-(a+b)} & (a+b<1),\\ 1 & (a+b>1),\\ \log(2+\alpha) & (a+b=1).\end{cases} \tag{21}$$

We note that for $-\alpha < \beta < -\alpha/2$, we have $|\beta| > \alpha+\beta$ and that for $-\alpha/2 < \beta < \alpha$, we have $|\beta| < \alpha + \beta$. Then we see that the β-integral is bounded by the sum of

$$\int_{-\alpha}^{-\alpha/2} \langle \alpha + \beta \rangle^{-(a+b)} d\beta \le \int_{-\alpha}^{0} \langle \alpha + \beta \rangle^{-(a+b)} d\beta,$$
$$\int_{-\alpha/2}^{\alpha} \langle \beta \rangle^{-(a+b)} d\beta \le 2 \int_{0}^{\alpha} \langle \beta \rangle^{-(a+b)} d\beta.$$

Then we get (21) by a direct computation.

We now divide the β-integral into I_1 and I_2:

$$I_1 := \int_{-\alpha}^{-\alpha/2} \langle \alpha + \beta \rangle^{-k_1-k_2} \langle \beta \rangle^{-k_1-k_3} d\beta,$$
$$I_2 := \int_{-\alpha/2}^{\alpha} \langle \alpha + \beta \rangle^{-k_1-k_2} \langle \beta \rangle^{-k_1-k_3} d\beta.$$

Then we get from (21)

$$I_1 \lesssim \langle \alpha/2 \rangle^{-k_1} \int_{-\alpha}^{\alpha} \langle \alpha + \beta \rangle^{-k_1-k_2} \langle \beta \rangle^{-k_3} d\beta$$
$$\lesssim \langle \alpha \rangle^{-k_1} \times \begin{cases} \langle \alpha \rangle^{1-(k_1+k_2+k_3)} & (k_1+k_2+k_3 < 1), \\ 1 & (k_1+k_2+k_3 > 1), \\ \log(2+\alpha) & (k_1+k_2+k_3 = 1). \end{cases}$$

As to I_1, we have

$$I_2 \lesssim \langle \alpha/2 \rangle^{-k_1} \int_{-\alpha}^{\alpha} \langle \alpha + \beta \rangle^{-k_2} \langle \beta \rangle^{-k_1-k_3} d\beta,$$

which implies the desired estimate by (21). This completes the proof. □

4 Proof of Theorem 1

Let u denote the solution of the problem (4) in what follows. When $\varphi \equiv 0$, it follows from (8), (9) and (7) that

$$u(t,r) \gtrsim \varepsilon u_L(t,r) + \widetilde{I_-}(|u|^p/y^{p-1})(t,r), \tag{22}$$
$$u_L(t,r) \gtrsim \widetilde{J_-}(\psi)(t,r) \tag{23}$$

holds for $t, r > 0$, where we put

$$\widetilde{I_-}(F)(t,r) = \iint_{\Delta_-(t,r)} \frac{\langle -t+\sigma+r+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} F(\sigma,y)dyd\sigma, \tag{24}$$

$$\widetilde{J_-}(\psi)(t,r) = \int_{|t-r|}^{t+r} \frac{\langle r-t+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \psi(y)dy. \tag{25}$$

Our first step is to obtain basic lower bounds of the solution to the problem (4). By (10), we may assume that ψ is strictly positive in an interval $[a, b]$.

Lemma 5 *We assume* (10) *holds. Then we have*

$$u_L(t,r) \gtrsim \frac{c_0}{\langle r\rangle^{\mu/2}}, \quad c_0 := \min_{a\le r\le b} \psi(r) \tag{26}$$

for

$$t < r < t+a, \ t+r > b. \tag{27}$$

Moreover, if u is the solution to (4)*, then we have*

$$u(t,r) \gtrsim \frac{\varepsilon^p}{\langle r\rangle^{\mu/2}(t-r)^{\eta}}, \quad \eta = (\mu/2+1)(p-1)-1 \tag{28}$$

for $0 < t < 2r$ and $t - r > b$.

Proof First, we show (26). Let (t, r) satisfy (27). Then, from (23) we have

$$\begin{aligned} u_L(t,r) &\gtrsim \int_{r-t}^{t+r} \frac{\langle r-t+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \psi(y)dy \\ &\gtrsim c_0 \int_a^b \frac{1}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} dy, \end{aligned}$$

which implies (26).

Next we show (28). Let $0 < t < 2r$ and $t - r > b$. If we set

$$\widetilde{\Sigma}(t,r) = \{(\sigma,y) \in (0,\infty)\times(0,\infty);\ 0 \le y-\sigma \le a,\ t-r < \sigma+y < t+r\},$$

then $\widetilde{\Sigma}(t,r) \subset \Delta_-(t,r)$. In addition, we see from (22) and (26) that $u(\sigma, y) \gtrsim \varepsilon\langle y\rangle^{-\mu/2}$ for $(\sigma, y) \in \widetilde{\Sigma}(t,r)$. Therefore, from (22) we get

$$u(t,r) \gtrsim \varepsilon^p \iint_{\widetilde{\Sigma}(t,r)} \frac{\langle -t+\sigma+r+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \frac{1}{\langle y\rangle^{(\mu/2+1)p-1}} dyd\sigma.$$

Now, introducing the coordinates $\alpha = \sigma + y$, $\beta = \sigma - y$, we obtain

$$u(t,r) \gtrsim \varepsilon^p \int_{t-r}^{t+r} d\alpha \int_{-a}^{0} \frac{\langle \alpha - t + r\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle \alpha - \beta\rangle^{(\mu/2+1)p+\mu/2-1}} d\beta$$
$$\gtrsim \varepsilon^p \int_{t-r}^{t+r} \frac{\langle \alpha - t + r\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle \alpha\rangle^{(\mu/2+1)p+\mu/2-1}} d\alpha.$$

Since $t < 2r$, we have $t + r > 3(t - r)$, so that

$$\langle r\rangle^{\mu/2} u(t,r) \gtrsim \varepsilon^p \int_{t-r}^{3(t-r)} \frac{(\alpha - t + r)^{\mu}}{\langle \alpha\rangle^{(\mu/2+1)p+\mu/2-1}} d\alpha$$
$$\gtrsim \varepsilon^p \langle t - r\rangle^{-((\mu/2+1)p+\mu/2-1)} \int_{t-r}^{3(t-r)} (\alpha - t + r)^{\mu} d\alpha$$
$$\gtrsim \varepsilon^p (t - r)^{-\eta}$$

for $t - r > b$. This completes the proof. □

For $\rho > 0$, we introduce the following quantity:

$$\langle u\rangle(\rho) = \inf\{\langle y\rangle^{\mu/2}(\sigma - y)^{\eta}|u(\sigma, y)|;\ (\sigma, y) \in \Sigma(\rho)\},$$

where we set

$$\Sigma(\rho) = \{(\sigma, y);\ 0 \le \sigma \le 2y,\ \sigma - y \ge \rho\}.$$

For simplicity, we assume $0 < b \le 1$. Then, (28) yields

$$\langle u\rangle(y) \ge C_1 \varepsilon^p \quad \text{for } y \ge 1. \tag{29}$$

Let $\xi \ge 1$ and $(t, r) \in \Sigma(\xi)$, so that $t - r \ge 1$. For $\rho > 0$ we set

$$\tilde{\Sigma}(\rho, t - r) = \{(\sigma, y);\ y \ge t - r,\ \sigma + y \le 3(t - r),\ \sigma - y \ge \rho\}.$$

It is easy to see that $\tilde{\Sigma}(\rho, t - r) \subset \Delta_-(t, r)$ for any $\eta > 0$ and $(t, r) \in \Sigma(\xi)$ and that $(\sigma, y) \in \tilde{\Sigma}(1, t - r)$ implies $(\sigma, y) \in \Sigma(\sigma - y)$. Therefore, from (22) we have

$$u(t,r)$$
$$\gtrsim \iint_{\tilde{\Sigma}(1,t-r)} \frac{(-t + \sigma + r + y)^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \frac{[\langle u\rangle(\sigma - y)]^p}{\langle y\rangle^{(\mu/2+1)p-1}(\sigma - y)^{p\eta}} dy\, d\sigma$$
$$\gtrsim \frac{(t - r)^{\mu}}{\langle r\rangle^{\mu/2}} \iint_{\tilde{\Sigma}(1,t-r)} \frac{[\langle u\rangle(\sigma - y)]^p}{\langle y\rangle^{(\mu/2+1)p+\mu/2-1}(\sigma - y)^{p\eta}} dy\, d\sigma,$$

because $-t+\sigma+r+y = -t+r+(\sigma-y)+2y \geq 1+(t-r)$ for $(\sigma, y) \in \tilde{\Sigma}(1, t-r)$. Changing the variables by $\beta = \sigma - y$, $z = y$, we have

$$\begin{aligned} u(t,r) &\gtrsim \frac{(t-r)^{\mu}}{\langle r\rangle^{\mu/2}} \int_1^{t-r} d\beta \int_{t-r}^{(3(t-r)-\beta)/2} \frac{[\langle u\rangle(\beta)]^p}{\langle z\rangle^{(\mu/2+1)p+(\mu/2-1)}\beta^{p\eta}} dz \\ &\gtrsim \frac{1}{\langle r\rangle^{\mu/2}(t-r)^{(\mu/2+1)(p-1)}} \int_1^{t-r} \frac{t-r-\beta}{2} \frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}} d\beta \\ &\gtrsim \frac{1}{\langle r\rangle^{\mu/2}(t-r)^{(\mu/2+1)p-(\mu/2+2)}} \int_1^{t-r} \left(1 - \frac{\beta}{t-r}\right) \frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}} d\beta. \end{aligned}$$

Since the function

$$y \mapsto \int_1^y \left(1 - \frac{\beta}{y}\right) \frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}} d\beta$$

is non-decreasing, for any $(t, r) \in \Sigma(\xi)$, we have

$$\langle r\rangle^{\mu/2}(t-r)^{(\mu/2+1)p-(\mu/2+2)} u(t,r) \geq C_2 \int_1^{\xi} \left(1 - \frac{\beta}{\xi}\right) \frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}} d\beta.$$

Thus, recalling $\eta = (\mu/2+1)p - (\mu/2+2)$ from (28), we obtain

$$\langle u\rangle(\xi) \geq C_2 \int_1^{\xi} \left(1 - \frac{\beta}{\xi}\right) \frac{[\langle u\rangle(\beta)]^p}{\eta^{p\eta}} d\beta, \quad \xi \geq 1 \tag{30}$$

Proof of Theorem 1 By (29) and (30), we can apply Lemma 2 as $\alpha = p$, $\beta = 0$, and $\theta = p\eta$. Since $1 < p \leq p_0(3+\mu)$ implies $\theta \leq 1$, the maximal existence time $T_*(\varepsilon)$ of $\langle u\rangle(\xi)$ satisfies the following estimates:

$$T_*(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & \text{if } \theta = 1, \\ C\varepsilon^{-p(p-1)/(1-\theta)} & \text{if } \theta < 1. \end{cases}$$

Since $\theta = 1$ and $\theta < 1$ correspond to $p = p_0(3+\mu)$ and $1 < p < p_0(3+\mu)$, respectively, we obtain the desired conclusion. □

5 Proof of Theorem 2

Our first step is to obtain the following estimates for the homogeneous part of the solution to the problem (8).

Lemma 6 *Assume that* (6) *holds and* $\varphi \in C^1([0,\infty))$*,* $\psi \in C^0([0,\infty))$ *satisfy* (11)*, so that*

$$|\varphi(r)| \lesssim r\,\langle r\rangle^{-\kappa}, \quad |\psi(r) + \varphi'(r) + w(r)\varphi(r)| \lesssim \langle r\rangle^{-\kappa} \text{ for } r \geq 0 \tag{31}$$

holds with some positive constant κ*. We put*

$$\nu := \kappa - \mu/2 - 1. \tag{32}$$

Then we have

$$\begin{aligned}&\left|\int_{|t-r|}^{t+r} E_-(t,r,y)\big(\psi(y) + \varphi'(y) + w(y)\varphi(y)\big)\,dy\right| \\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases} \langle t+r\rangle^{-\kappa+\mu/2} & (\nu < 0), \\ \langle t+r\rangle^{-1}\Psi(|t-r|, t+r) & (\nu = 0), \\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu > 0) \end{cases}\end{aligned} \tag{33}$$

for $t > 0$*,* $r > 0$*, where* $\Psi(a,b)$ *is defined in* (18)*. Moreover, for* $0 < t \leq r$ *we have*

$$|E_-(t,r,r-t)\varphi(r-t)| \lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases} \langle t+r\rangle^{-\kappa+\mu/2} & (\nu < 0), \\ \langle t+r\rangle^{-1}\Psi(r-t, t+r) & (\nu = 0), \\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu > 0). \end{cases} \tag{34}$$

Proof We begin with the proof of (33). In the following, let $t > 0$, $r > 0$. Since $0 \leq r - t + y \leq 2y$ for $y \geq |t-r|$, from (7) we have

$$|E_-(t,r,y)| \lesssim \langle y\rangle^{\mu/2}/\langle r\rangle^{\mu/2} \quad \text{for } y \geq |t-r|. \tag{35}$$

Therefore, by using the assumptions on the data, the left hand side of (33) is estimated by

$$\langle r\rangle^{-\mu/2}\int_{|r-t|}^{t+r}\langle y\rangle^{\mu/2}\left|\psi(y) + \varphi'(y) + w(y)\varphi(y)\right|dy \lesssim \langle r\rangle^{-\mu/2}\int_{|r-t|}^{t+r}\langle y\rangle^{-\kappa+\mu/2}dy.$$

From (32) and Lemma 3, the last integral is estimated as follows:

$$\int_{|r-t|}^{t+r}\langle y\rangle^{-\kappa+\mu/2}dy \lesssim r \times \begin{cases} \langle t+r\rangle^{-\kappa+\mu/2} & (\nu < 0), \\ \langle t+r\rangle^{-1}\Psi(|t-r|, t+r) & (\nu = 0), \\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu > 0). \end{cases}$$

Therefore we obtain (33).

Next we prove (34), by assuming $0 < t \le r$. From (31) and (35) we have

$$|E_-(t,r,y)\varphi(y)| \lesssim \frac{y}{\langle r\rangle^{\mu/2}\langle y\rangle^{\kappa-\mu/2}} \quad \text{for } y \ge |t-r|. \tag{36}$$

Let $r \ge 1$. It follows from (32) and (36) that

$$\begin{aligned}|E_-(t,r,r-t)\varphi(r-t)| &\lesssim \frac{r-t}{\langle r\rangle^{\mu/2}\langle r-t\rangle^{\kappa-\mu/2}}\\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2+1}\langle r-t\rangle^{\kappa-\mu/2-1}}\\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases}\langle r+t\rangle^{-\kappa+\mu/2} & (\nu \le 0),\\ \langle r+t\rangle^{-1}\langle r-t\rangle^{-\nu} & (\nu > 0).\end{cases}\end{aligned}$$

Let $0 < r \le 1$. We obtain from (36)

$$\begin{aligned}|E_-(t,r,r-t)\varphi(r-t)| &\lesssim \frac{r}{\langle r\rangle^{\mu/2}\langle r-t\rangle^{\kappa-\mu/2}}\\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases}\langle r+t\rangle^{-\kappa+\mu/2} & (\nu \le 0),\\ \langle r+t\rangle^{-1}\langle r-t\rangle^{-\kappa+\mu/2} & (\nu > 0).\end{cases}\end{aligned}$$

Hence, we obtain the desired estimate (34). This completes the proof. □

For $t > 0$, $r > 0$, it follows from (9) and Lemma 6 that

$$|u_L(t,r)| \le \tilde{C}_0 r\,\langle r\rangle^{-\mu/2} \times \begin{cases}\langle t+r\rangle^{-\kappa+\mu/2} & (\nu < 0),\\ \langle t+r\rangle^{-1}\Psi(|t-r|,t+r) & (\nu = 0),\\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu > 0)\end{cases} \tag{37}$$

with some positive constant $\tilde{C}_0$, provided (6) and (11) hold.

Our next step is to consider the integral operator appeared in (8):

$$I_-(F)(t,r) := \frac{1}{2}\iint_{\Delta_-(t,r)} E_-(t-\sigma,r,y)F(\sigma,y)dyd\sigma.$$

For $(\sigma,y) \in \Delta_-(t,r)$ we have $y \ge |t-r-\sigma|$, so that (35) yields

$$E_-(t-\sigma,r,y) \lesssim \langle r\rangle^{-\mu/2}\langle y\rangle^{\mu/2} \quad \text{for} \quad (\sigma,y) \in \Delta_-(t,r).$$

Hence we get

$$|I_-(F)(t,r)| \lesssim \langle r\rangle^{-\mu/2}\iint_{\Delta_-(t,r)} \langle y\rangle^{\mu/2}|F(\sigma,y)|dyd\sigma. \tag{38}$$

In order to derive an apriori estimate, we introduce the following weighted L^∞-norm:

$$\|u\|_1 = \sup_{(t,r)\in[0,\infty)\times[0,\infty)} \{w_1(t,r)^{-1}|u(t,r)|\}, \tag{39}$$

where we put

$$w_1(t,r) := r\,\langle r\rangle^{-\mu/2}\langle t+r\rangle^{-1}\langle t-r\rangle^{-\eta}. \tag{40}$$

Here we choose

$$\eta = (\mu/2+1)(p-1)-1$$

as in (28), so that $\eta > 1/p$, by the assumption $p > p_0(3+\mu)$.

Lemma 7 *Let $\eta > 0$ be as above. Then, there exists a positive constant $\widetilde{C}_1$ such that*

$$\|I_-(F)\|_1 \le \widetilde{C}_1 \|u\|_1^p \tag{41}$$

with $F(t,r) = |u(t,r)|^p/r^{p-1}$.

Proof From (39) and (40), we obtain

$$\langle r\rangle^{\mu p/2}\langle t+r\rangle^p\langle t-r\rangle^{\eta p}|F(t,r)| \le r\|u\|_1^p.$$

It follows from (38) that

$$|I_-(F)(t,r)| \lesssim \langle r\rangle^{-\mu/2}\|u\|_1^p I(t,r),$$

where we put

$$I(t,r) := \iint_{\Delta_-(t,r)} \frac{y}{\langle y\rangle^{\mu(p-1)/2}\langle \sigma+y\rangle^p\langle \sigma-y\rangle^{\eta p}} dy d\sigma. \tag{42}$$

In order to show (41), it is enough to prove

$$I(t,r) \lesssim \frac{r}{\langle t+r\rangle\langle t-r\rangle^\eta}.$$

To evaluate the integral (42), we pass to the coordinates

$$\alpha = \sigma + y, \quad \beta = y - \sigma$$

and deduce

$$I(t,r) \lesssim \int_{|r-t|}^{t-r} \int_{r-t}^{\alpha} \frac{1}{\langle\alpha\rangle^{p}\langle\alpha+\beta\rangle^{\mu(p-1)/2-1}\langle\beta\rangle^{\eta p}} d\beta d\alpha. \tag{43}$$

First, suppose $r \geq t$. Then we get

$$I(t,r) \lesssim \int_{r-t}^{t-r} \frac{d\alpha}{\langle\alpha\rangle^{\eta+1}} \int_{r-t}^{\alpha} \frac{1}{\langle\beta\rangle^{\eta p}} d\beta.$$

Since $p\eta > 1$, we have from Lemma 3

$$I(t,r) \lesssim \int_{r-t}^{t+r} \frac{1}{\langle\alpha\rangle^{\eta+1}} d\alpha \lesssim \frac{r}{\langle t+r\rangle\langle t-r\rangle^{\eta}}.$$

Next, suppose $r < t$. Since $p\eta > 1$, we have from (43), Lemma 3, and Lemma 4 with $k_1 = \eta$, $k_2 = 0$, and $k_3 = \eta(p-1)$

$$\begin{aligned} I(t,r) &\lesssim \int_{t-r}^{t+r} \frac{d\alpha}{\langle\alpha\rangle} \int_{-\alpha}^{\alpha} \frac{1}{\langle\alpha+\beta\rangle^{\eta}\langle\beta\rangle^{\eta p}} d\beta \\ &\lesssim \int_{t-r}^{t+r} \frac{1}{\langle\alpha\rangle^{\eta+1}} d\alpha \lesssim \frac{r}{\langle t+r\rangle\langle t-r\rangle^{\eta}}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2 Let X be the linear space defined by

$$X = \{u(t,r) \in C([0,\infty) \times [0,\infty))\ ;\ \|u\|_1 < \infty\}.$$

We can verify easily that X is complete with respect the norm $\|\cdot\|_1$. We define the sequence of functions $\{u_n\}$ by

$$u_0 = \varepsilon u_L, \quad u_{n+1} = \varepsilon u_L + I_-(|u_n|^p/r^{p-1}) \quad (n = 0, 1, 2, \cdots).$$

Since $\kappa \geq (\mu/2+1)p - 1$ and $\nu = \kappa - \mu/2 - 1$, we have $\nu \geq \eta$. Therefore, it follows from (37), (39) and (40) that $\|u_0\|_1 \leq \varepsilon\widetilde{C}_0$. Hence $u_0 \in X$.

Now, by choosing ε is sufficiently small, we find from Lemma 7 that $\{u_n\} \in X$ for all n. Moreover, we can prove that $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that u_n converges uniformly to u as $n \to \infty$. Clearly, u satisfies (8). This completes the proof. □

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