Reconstruction from Boundary Measurements: Complex Conductivities

Ivan Pombo

Abstract In this paper we show that following Nachman's method we can still reconstruct complex conductivities in $C^{1,1}$ from its Dirichlet-to-Neumann map in three and higher dimensions. For such, we analyze all of his results and pinpoint what really needs to be shown for complex conductivities. Moreover, we show the existence of non-exceptional points for low frequency and $C^{1,1}$-domains. As far as we are aware, this is the first reconstruction procedure for complex conductivities, even though the proof follows easily by extending some of the theorems obtained by Nachman to the complex case.

1 Introduction

In Electrical Impedance Tomography (EIT) we determine the interior impedance inside a bounded domain Ω by applying alternating electrical currents and measuring the corresponding voltages at the boundary $\partial\Omega$, or vice-versa. Impedance is the inverse of admittance which is defined through $\gamma = \sigma + i\omega\epsilon$, where ω is the angular frequency, $\sigma,\ \epsilon$ are the electrical conductivity and permittivity of materials inside Ω, respectively.

Our working assumptions are

$$\gamma \in C^{1,1}(\bar{\Omega}) \text{ and isotropic}, \quad \sigma \geq c > 0, \quad \epsilon \geq 0, \quad \omega \in \mathbb{R}^+, \tag{1}$$

$$\Omega \text{ is a bounded domain with } C^{1,1} \text{ boundary in } \mathbb{R}^n,\ n \geq 3 \tag{2}$$

In applications, most data acquisition systems and respective algorithms focus on computing the conductivity σ. However, in certain applications it is highly valuable to also obtain permittivity from boundary measurements. It brings extra

I. Pombo (✉)
Universidade de Aveiro Department of Mathematics, Aveiro, Portugal
e-mail: ivanpombo@ua.pt

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knowledge to the table and allows us to distinguish more clinical conditions than is possible with the conductivity alone. An example is the ability to distinguish between pneumothorax and hyperinflation. Both scenarios correspond to regions of low resistivity, which implies high conductivity, but the pneumothorax has zero permittivity while the hyperinflation corresponds to low yet positive permittivity. Other possible application is in multi-frequency EIT since the properties σ and ϵ vary with the applied angular frequency ω, while in the real case the frequency is somewhat discarded.

Mathematically, the direct problem concerns the unique determination of the electrical potential $u \in H^1(\Omega)$ given a voltage $f \in H^{1/2}(\partial\Omega)$ set at the boundary, modelled by

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, & \text{in } \Omega \\ u|_{\partial\Omega} = f \end{cases} \tag{3}$$

Uniqueness in $H^1(\Omega)$ holds from the assumption $\operatorname{Re} \gamma > 0$, which implies by the weak formulation that 0 is not a Dirichlet eigenvalue of the operator $\nabla \cdot (\gamma \nabla u)$ in Ω.

Formally, from each voltage $f \in H^{1/2}(\partial\Omega)$ and each corresponding electrical potential $u \in H^1(\Omega)$ we can determine the electrical current measured at the boundary given by $\gamma \frac{\partial u}{\partial \nu}$. In essence, we can define for $\gamma \in L^\infty(\Omega)$ the Dirichlet-to-Neumann map $\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}$, which holds weakly by

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega), \\ f \quad &\mapsto \quad \langle \Lambda_\gamma f, g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx, \end{aligned} \tag{4}$$

where $v \in H^1(\Omega)$ has trace $g \in H^{1/2}(\partial\Omega)$.

In 1980 A.P. Calderón [11] was the first to pose the mathematical problem whether the conductivity $\sigma \in L^\infty(\Omega)$ can be uniquely determined by boundary measurements, Λ_γ, and if so how to reconstruct it. He showed that the linearized problem at constant conductivities has a unique solution. In mathematical literature, this is designated as Calderón's problem or inverse conductivity problem. In medical imaging the problem is know by Electrical Impedance Tomography (EIT).

After the initial work of Calderón there were many extensions to global uniqueness results. In [34], Sylvester and Uhlmann used ideas of scattering theory, namely the exponential growing solutions of Faddeev [15] to obtain global uniqueness in dimensions $n \geq 3$ for smooth conductivities. Using this foundations the uniqueness for lesser regular conductivities was further generalized for dimensions $n \geq 3$ in the works of [1, 7, 8, 12, 13, 19, 26, 29, 32]. Currently, the best know result is due to Haberman [18] for conductivities $\gamma \in W^{1,3}(\Omega)$. The reconstruction procedure for $n \geq 3$ was obtained in both [26] and [30] independently. As far as we are aware, there seems to be no literature concerning reconstruction for conductivities with less than two derivatives.

In two dimensions the problem seems to be of a different nature and tools of complex analysis were used to establish uniqueness. Nachman [27] obtained uniqueness and a reconstruction method for conductivities with two derivatives. The uniqueness result was soon extend for once-differentiable conductivities in [9] and a corresponding reconstruction method was obtained in [22]. In 2006, Astala and Päivärinta [2] gave a positive answer Calderón's problem for $\sigma \in L^\infty(\Omega)$, $\sigma \geq c > 0$, by providing the uniqueness proof through the reconstruction process.

The first extension to admittances, and here forward also designated by complex-conductivities, was in two-dimensions in [16]. Francini extended the work of Brown and Uhlmann [9] in two-dimensions by proving uniqueness for small angular frequencies and $\gamma \in W^{2,\infty}$. Afterwards, Bukgheim influential paper [10] proved the general result in two-dimensions for complex-conductivities in $W^{2,\infty}$. He reduced the (3) to a Schrödinger equation and shows uniqueness through the stationary phase method (based on is work many extensions followed [3, 5, 31]). Recently, by mixing techniques of [9] and [10], Lakshtanov et al. obtained in [24] uniqueness for Lipschitz complex-conductivities in $\mathbb{R}^2$. In [33], the author followed up their work to show that it is possible to reconstruct complex-conductivity with a jump at least in a certain set of points.

As far as we are aware, in dimensions higher than 2 there is no explicit literature for complex-conductivities. As stated in [6], it is possible to obtain uniqueness for twice differentiable complex-conductivities by the approach obtained in [34] and [29]. Furthermore, there is reference of a theoretical work for direct reconstruction method in the case of complex conductivities, since most works restrict themselves to the real scenario.

However, in [20] Nachman's reconstruction method is used to find complex conductivities from boundary measurements. This is a promising result that requires a theoretical background to support it, even if for some researchers it seems folklore.

Hence, in this paper, we show that Nachman's reconstruction method holds for complex conductivities. To be precise, the main result of this paper is the following:

Theorem 1 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $\gamma_1, \gamma_2 \in C^{1,1}(\Omega)$ be complex-valued conductivities, such that Re $\gamma_j \geq c > 0$ for $j = 1, 2$. Further, let Λ_1, Λ_2 be their corresponding Dirichlet-to-Neumann maps.*

If $\Lambda_1 = \Lambda_2$, then $\gamma_1 = \gamma_2$ in Ω.

Under careful examination of [26], we highlight here that the only requirement for the reconstruction method to hold concerns the uniqueness of boundary value problems with complex coefficient for $f \in H^{3/2}(\partial\Omega)$. For convenience of the reader, we present here the most essential results of Nachman's magnificent work, taking a sequential tour through the pieces needed to make this work. Hence, in essence this paper works as a review of Nachman's procedure highlighting the requirements for it to work for complex-conductivities.

Furthermore, following the work of [14] we show that the complex-conductivity can also be obtained from low-frequency asymptotics through the exponential growing solutions.

2 Uniqueness of Schrödinger Inverse Problem

The recurring idea on Calderón's problem is to convert our equation into one that has the coefficient in the lowest order terms. Here, we transform into the Schrödinger equation with complex-potential. We start by following the uniqueness result presented in [29], under the assumption of complex-valued potentials in $L^\infty(\bar{\Omega})$. In their work, there is no mention and need of the potential to be real, therefore we present their proof in its entirety.

Let $u \in H^1(\Omega)$ be the unique solution of (3) with trace $f \in H^{1/2}(\partial\Omega)$ at the boundary. Then the substitution $u = \gamma^{-1/2}w$ yields with $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$,

$$\begin{cases} -\Delta w + qw = 0, \text{ in } \Omega, \\ w|_{\partial\Omega} = \gamma^{1/2} f. \end{cases} \tag{5}$$

Notice that if $\gamma \in C^{1,1}(\Omega)$ and $\sigma \geq c > 0$ then $\gamma^{1/2}$ is well-defined and twice weakly differentiable. Therefore, q is well-defined and in $L^\infty(\Omega)$. As previously stated, the assumptions on γ lead to 0 not being a Dirichlet eigenvalue of $\nabla \cdot (\gamma \nabla u)$. The relation above implies a bijection between solutions of the (3) and of (5). Therefore, 0 is also not a Dirichlet eigenvalue of the Schrödinger problem.

In general, if 0 is not a Dirichlet eigenvalue of the Schrödinger operator in Ω, then the Dirichlet-to-Neumann map, Λ_q, is well-defined from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ and formally is given by

$$\Lambda_q f = \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega}$$

for w being the unique solution of $(-\Delta + q)w = 0$, in Ω and $w|_{\partial\Omega} = f$. Hence, here the corresponding inverse problem is to determine q from the boundary measurements Λ_q uniquely.

In this manner, we can cast our focus into the Schrödinger equation. First, we can extend q to zero outside the domain and study solutions of

$$-\Delta w + qw = 0, \quad \text{in } \mathbb{R}^n \tag{6}$$

which behave like

$$w = e^{ix\cdot\zeta}\left(1 + \psi(x, \zeta)\right), \text{ for } \zeta \in \mathbb{C}^n, \ \zeta \cdot \zeta = 0.$$

In Calderón's paper [11] he already uses the family of exponential harmonic functions, $e^{ix\cdot\zeta}$ in its proof, but it was Sylvester and Uhlmann [34] that first used this type of solutions to dispense the requirement of σ be close to a constant. Substituting into (6), it follows that ψ must satisfy

$$-\Delta\psi - 2i\zeta\cdot\nabla\psi + q\psi = -q \tag{7}$$

From scattering theory, we inherited the Faddeev-Green's function (see [15]) which takes a principal role in the study of the above Eq. (7). For $\zeta \in \mathbb{C}^n$ with $\zeta\cdot\zeta = 0$ it is given by $G_\zeta(x) = e^{ix\cdot\zeta}g_\zeta(x)$, where g_ζ is the fundamental solution of operator $(-\Delta - 2i\zeta\cdot\nabla)$ and is defined as:

$$g_\zeta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\zeta\cdot\xi}\,d\xi, \tag{8}$$

Recall, that as a fundamental solution G_ζ differs from the classical one G_0 by an harmonic function H_ζ.

From this, solutions of (6) with the desired asymptotics can be obtained by solving the integral equation

$$w(x,\zeta) = e^{ix\cdot\zeta} - \int G_\zeta(x-y)q(y)w(y,\zeta)\,dy \tag{9}$$

with ψ solving

$$\psi + g_\zeta * (q\psi) = -g_\zeta * q. \tag{10}$$

The study of these integral equations follows by a weighted L^2 estimate for g_ζ obtained in [34], which guarantees unique solvability of (9) for $|\zeta|$ large, even for complex conductivities. This estimate is one of the most important elements in scattering works, since it allows the existence and uniqueness of solutions and already puts into light their behavior in terms of ζ.

Let $\langle x\rangle = (1+|x|^2)^{1/2}$. We define the weighted L^2-space for $\delta\in\mathbb{R}$ as

$$L^2_\delta(\mathbb{R}^n) := \left\{ f : \|f\|_\delta := \|\langle x\rangle^\delta f\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$

Then the convolution operators with g_ζ and G_ζ satisfy the following estimates

Proposition 1 *For all $\zeta\in\mathbb{C}^n$ with $\zeta\cdot\zeta = 0$ and $|\zeta| \geq a$ the operator of convolution with g_ζ satisfies*

$$\left\|g_\zeta * f\right\|_{\delta-1} \leq \frac{c(\delta,a)}{|\zeta|}\|f\|_\delta, \quad \textit{for}\ \ 0<\delta<1 \tag{11}$$

Moreover, let $H^2_\delta(\Omega) := \left\{f : D^\alpha f \in L^2_{-\delta}(\mathbb{R}^n), 0 \le |\alpha| \le 2\right\}$ *be the weighted Sobolev space with norm*

$$\|f\|_{2,\delta} = \left(\sum_{|\alpha|\le 2} \|D^\alpha f\|^2_\delta\right)^{1/2}.$$

Then, for any $\zeta \in \mathbb{C}^n$ *with* $\zeta \cdot \zeta = 0$ *it holds for* $\delta \in (1/2, 1)$ *that*

$$\|g_\zeta * w\|_{2,-\delta} \le c(\delta, \zeta)\|w\|_{2,\delta}.$$

Furthermore, under the definition

$$\mathbf{G}_\zeta w(x) = \int_\Omega G_\zeta(x-y)w(y)\,dy$$

it holds that

$$\|\mathbf{G}_\zeta w\|_{H^2(\Omega)} \le c(\zeta, \Omega)\|w\|_{L^2(\Omega)}.$$

Proof The first estimate can be found in [34, Corollary 2.2], while the rest is in [26, Lemma 2.11]. □

For the uniqueness proof our interest resides in studying the exponential growing solutions given through Eq. (9)

Corollary 1 *Let* $0 < \delta < 1$ *and* $q \in L^\infty(\Omega)$ *be complex-valued and extended to zero outside* Ω*. Then there exists an* $R > 0$ *such that for all* $\zeta \in \mathbb{C}^n$ *with* $\zeta \cdot \zeta = 0$ *and* $|\zeta| > R$ *the integral equation (9) is uniquely solvable with* $e^{-ix\cdot\zeta}w(x,\zeta) - 1 \in L^2_{\delta-1}(\mathbb{R}^n)$*. Furthermore, it holds*

$$\|e^{-ix\cdot\zeta}w(x,\zeta) - 1\|_{\delta-1} \le \frac{\tilde{c}(R,\delta)}{|\zeta|}\|q\|_\delta. \tag{12}$$

Proof Let $M_q\phi = q\phi$, i.e., the operator of multiplication with q. We show that for $q \in L^\infty(\mathbb{R}^n)$ with compact support, $M_q : L^2_{\delta-1}(\mathbb{R}^n) \to L^2_\delta(\mathbb{R}^n)$ is a bounded operator.

Let $f \in L^2_{\delta-1}(\mathbb{R}^n)$. Then

$$\begin{aligned}\|M_q f\|_\delta &= \left[\int_{\mathbb{R}^n} (1+|x|^2)^\delta |q(x)f(x)|^2\,dx\right]^{1/2}\\ &= \left[\int_{\mathbb{R}^n} \left(1+|x|^2\right)|q(x)|^2\left(1+|x|^2\right)^{\delta-1}|f(x)|^2\,dx\right]^{1/2}\\ &\le \|\langle x\rangle q\|_\infty \|f\|_{\delta-1}.\end{aligned}$$

We define the operator $A_\zeta = C_\zeta M_q$, where C_ζ is the convolution with g_ζ, that is

$$A_\zeta f(x) = \int_{\mathbb{R}^n} g_\zeta(x-y)q(y)f(y)\,dy = C_\zeta M_q f \tag{13}$$

By Proposition 1 for $|\zeta| \geq R$ we obtain

$$\|A_\zeta f\|_{\delta-1} = \|C_\zeta M_q f\|_{\delta-1} \leq \frac{c(\delta, R)}{|\zeta|}\|M_q f\|_\delta \leq \frac{c(\delta, R)}{|\zeta|}\,\|\langle x\rangle q\|_\infty\,\|f\|_{\delta-1}$$

Therefore, A_ζ is bounded in $L^2_{\delta-1}(\mathbb{R}^n)$. Furthermore, if we consider

$$|\zeta| > R := c(\delta, R)\,\|\langle x\rangle q\|_\infty$$

then A_ζ is a contraction and $I + A_\zeta$ is invertible.

Since $q \in L^\infty$ and as compact support then it is in L^2_δ and therefore the right-hand side of (10) is in $L^2_{\delta-1}$. Hence, the unique solution to (7) is given by

$$\psi(x,\zeta) = -\left[I + A_\zeta\right]^{-1}\left(g_\zeta * q\right).$$

From here, we already know that

$$w = e^{ix\cdot\zeta}\left(1 - \left[I + A_\zeta\right]^{-1}\left(g_\zeta * q\right)\right)$$

solves the integral equation (9). Furthermore, the estimate (12) easily follows from $[I + A_\zeta]^{-1}$ being bounded in $L^2_{\delta-1}$, Proposition 1 and $g_\zeta * q \in L^2_{\delta-1}$.

Now, let us suppose that there exist two solutions w_1, w_2 of (9) such that

$$\phi_j = e^{-ix\cdot\zeta}w_j - 1 \in L^2_{\delta-1}.$$

Then, their difference is also in $L^2_{\delta-1}$ and both fulfill the equation

$$[I + A_\zeta]\phi_j = -g_\zeta * q.$$

This implies $[I + A_\zeta]\left(e^{-ix\cdot\zeta}(w_1 - w_2)\right) = 0$ and thus $w_1 \equiv w_2$ by the invertibility of $I + A_\zeta$ in $L^2_{\delta-1}$. Hence, uniqueness of the integral equation (9) for exponential growing solutions follows. □

We designate the values ζ for which this solution does not exist or is not unique as exceptional points.

Definition 1 Let $q \in L^\infty(\Omega)$ complex-valued and extended to zero outside Ω.

Let $\zeta \in \mathcal{V} := \{\zeta \in \mathbb{C}^n \setminus \{0\}\zeta \cdot \zeta = 0\}$. Then we call $\zeta \in \mathcal{V}$ an exceptional point for q if there is no unique exponential growing solution of $(-\Delta + q)w = 0$ in $\mathbb{R}^n$, that is, there is no unique solution of the type

$$w(x,\zeta) := e^{ix\cdot\zeta}\,(1+\mu(x,\zeta))\,,\ \text{with}\ \mu \in L^2_{\delta-1}(\mathbb{R}^n),\ 0<\delta<1.$$

The uniqueness proof of this section given by Nachman et al. [29] and Nachman's reconstruction method [26], only require large non-exceptional points ζ. However, in this sense the reconstruction process is very unstable. Hence, one of our desires is to mimic the theory in two-dimensions, where we are able to reconstruct γ from small values of non-exceptional points ζ by the D-bar method. We will see in a further section how under circumstances one can still apply a version of it.

Now a first step in the uniqueness proof is to show that the exponential growing solutions outside Ω are uniquely identified by the Dirichlet-to-Neumann map.

Lemma 1 *Let $q_1, q_2 \in L^\infty(\Omega)$ and extended to zero outside Ω, such that 0 is not a Dirichlet eigenvalue of $-\Delta + q_j$ in Ω for $j = 1, 2$. Further, let $\zeta \in \mathcal{V}$ a non-exceptional point for q_1, q_2. Suppose that $\Lambda_{q_1} = \Lambda_{q_2}$ and w_1, w_2 are the unique solutions of $(-\Delta + q_j)w_j = 0$ in $\mathbb{R}^n$ of the form $e^{ix\cdot\zeta}\left(1+\mu_j\right)$. Then*

$$w_1 = w_2, \quad \text{in } \mathbb{R}^n \setminus \Omega.$$

Proof Let $v \in H^1(\Omega)$ be the unique solution of

$$-\Delta v + q_2\, v = 0, \quad \text{in } \Omega$$
$$v|_{\partial\Omega} = w_1|_{\partial\Omega}\,.$$

Then we define

$$h = \begin{cases} v, & \text{in } \Omega \\ w_1, & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Since, $\Lambda_{q_1} = \Lambda_{q_2}$ it holds that $\Lambda_{q_1}\, w_1|_{\partial\Omega} = \Lambda_{q_2}\, w_1|_{\partial\Omega}$ and thus $\frac{\partial w_1}{\partial \nu} = \frac{\partial v}{\partial \nu}$. This implies that h is continuous over $\partial\Omega$, as well as, $\frac{\partial h}{\partial \nu}$. Therefore, h solves $-\Delta h + q_2\, h = 0$ in $\mathbb{R}^n$ and has the appropriate asymptotics since w_1 has them. By the uniqueness theorem it follows that $h = w_2$ and thus $w_1 = w_2$ in $\mathbb{R}^n \setminus \Omega$. □

Now the uniqueness theorem obtained in [29] follows also for complex-conductivities directly.

Theorem 2 *Let $q_1, q_2 \in L^\infty(\Omega)$ extended to zero outside Ω. Suppose that 0 is not a Dirichlet eigenvalue of $-\Delta + q_j$, $j = 1, 2$ on Ω. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$.*

Proof Let $k \in \mathbb{R}^n$ be fixed and for $m, s \in \mathbb{R}^n$ we set

$$\zeta = \frac{1}{2}\left((k+s)+im\right)\ \text{and}\ \tilde{\zeta} = \frac{1}{2}\left((k-s)-im\right)$$

with $k \cdot s = k \cdot m = s \cdot m = 0$ and $|k|^2 + |s|^2 = |m|^2$. The $\zeta,\ \tilde{\zeta}$ are in $\mathbb{C}^n$ and fulfill the condition $\zeta \cdot \zeta = 0$. Hence, taking $s,\ m$ large enough we obtain solutions w_j of the integral equation (9) for their respective potentials for $\tilde{\zeta}$. By Green's identity it holds that

$$\int_\Omega e^{ix\cdot\zeta} q_j(x) w_j(x)\, dx = \int_\Omega e^{ix\cdot\zeta} \Delta w_j(x) - w_j \Delta e^{ix\cdot\zeta}\, dx$$

$$= \int_{\partial\Omega} e^{ix\cdot\zeta} \frac{\partial w_j}{\partial \nu} - w_j\, (\nu \cdot i\zeta)\, e^{ix\cdot\zeta}\, d\sigma(x).$$

By hypothesis and the previous lemma $\Lambda_{q_1} = \Lambda_{q_2}$ and thus $w_1|_{\partial\Omega} = w_2|_{\partial\Omega}$. Therefore, it also holds that

$$\left.\frac{\partial w_1}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial w_2}{\partial \nu}\right|_{\partial\Omega}$$

as w_j solve the interior problem $(-\Delta + q_j)w_j = 0$. Hence, the right-hand side of the integral above is equal for both q_j and assuming the asymptotics of w_j w.r.t. $\tilde{\zeta}$ it follows

$$\int_\Omega e^{ix\cdot\zeta}\, (q_1 w_1 - q_2 w_2)\; dx = 0$$

which is equivalent to

$$\int_\Omega e^{ix\cdot(\zeta+\tilde{\zeta})}\, (q_1 - q_2)\; dx = \int_\Omega e^{ix\cdot(\zeta+\tilde{\zeta})}\, (q_1\psi_1 - q_2\psi_2)\; dx$$

Using $\zeta + \tilde{\zeta} = k$ and taking modulus we obtain by Cauchy-Schwarz inequality and Corollary 1

$$\left|\int_\Omega e^{ix\cdot k}\, (q_1 - q_2)\; dx\right| \le \sum_{j=1}^{2} \int_\Omega \left|q_j\psi_j\right| \le \sum_{j=1}^{2} \|q_j\|_{1-\delta} \|\psi_j\|_{\delta-1}$$

$$\le \sum_{j=1}^{2} \frac{C}{|\tilde{\zeta}|} \|q_j\|_{1-\delta} \|q_j\|_{\delta}.$$

Since $\tilde{\zeta}$ was arbitrarily depending on s, we can take the limit as $|s| \to \infty$. This implies that the left-hand side equals to zero for each fixed $k \in \mathbb{R}^n$. Given that the proof holds for all k we have

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2)\, dx = 0, \quad \forall k \in \mathbb{R}^n$$

Therefore, by Fourier inversion theorem we obtain $q_1 = q_2$ in Ω. □

Note We do not require more assumptions for q being complex; second the following uniqueness proof only works for $n \geq 3$ due to the required choice of $\zeta, \tilde{\zeta}$.

Hence, uniqueness is extended for complex-potentials in $L^\infty(\Omega)$ with 0 not a Dirichlet eigenvalue of $(-\Delta + q)$, directly from the work of [29]. To extend uniqueness for admittivities $\gamma \in C^{1,1}(\bar{\Omega})$ it is still necessary to establish a relation between Λ_γ and Λ_q. We will present this in a later section. Now, we proceed to explain how Nachman's reconstruction method equally holds for complex-conductivities.

3 Preliminaries for Reconstruction

In the following section, we present the necessary results to follow Nachman's approach [26]. The two main pinpoints in our extension to complex-conductivities are the Lemma 4 and Proposition 3.

The first result concerns an estimate for the single layer operator which will help us prove that there are no-exceptional points close to zero for complex-conductivities. The estimate was obtained in [14].

The second result concerns the uniqueness of the interior Schrödinger problem. In essence, this is were Nachman's work needs to be extended, since is version of this proposition was proven by estimates for real-coefficients. Even though, the machinery to prove a complex-coefficient version does not require anything novel it seems essential to provide a clear statement into why Nachman's method still works.

Analogously to the classical single and double layer potentials we define the respective operators for G_ζ. The single layer operator is defined as

$$S_\zeta f(x) = \int_{\partial\Omega} G_\zeta(x-y) f(y)\, ds(y)$$

and the double layer as

$$D_\zeta f(x) = \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial \nu}(x-y) f(y)\, ds(y).$$

Moreover, taking the trace of double layer potential it holds

$$B_\zeta f(x) := \text{ p.v.} \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial \nu}(x-y) f(y)\, ds(y), \text{ for } x \in \partial\Omega.$$

Since the singularity of G_ζ for x near y is the same as G_0, it is locally integrable on $\partial\Omega$ and the trace of S_ζ is still "itself".

We state here the properties that Nachman established and are essential for the later proofs.

Proposition 2 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$.*

(i) For $0 \leq s \leq 1$

$$\left\| S_\zeta f \right\|_{H^{s+1}(\partial\Omega)} \leq c(\zeta, s) \| f \|_{H^s(\partial\Omega)}. \tag{14}$$

(ii) For $0 \leq s \leq \frac{3}{2}$ we have that B_ζ is bounded in $H^s(\partial\Omega)$.

Let ρ_0 be a number large enough so that $\bar{\Omega} \subset \{x : |x| < \rho_0\}$. For any $\rho > \rho_0$ we define $\Omega'_\rho = \{x : x \notin \bar{\Omega},\ |x| < \rho\}$.

Lemma 2 *If $f \in H^{1/2}(\partial\Omega)$, the function $\phi = S_\zeta f$ has the following properties*

(i) $\Delta\phi = 0$ in $\mathbb{R}^n \setminus \partial\Omega$.
(ii) $\phi \in H^2(\Omega)$ and $\phi \in H^2(\Omega'_\rho)$ for any $\rho > \rho_0$.
(iii) ϕ satisfies an analogue to the Sommerfeld radiation condition. For almost every x it holds

$$\lim_{\rho\to\infty} \int_{|y|=\rho} \left[G_\zeta(x-y) \frac{\partial\phi}{\partial\nu(y)} - \phi(y) \frac{\partial G_\zeta}{\partial\nu(y)}(x-y) \right] ds(y) = 0. \tag{15}$$

In fact, for $\rho > \rho_0$ the above identity holds for $|x| < \rho$ even without taking the limit.

(iv) Let $B_\zeta^\dagger$ denote the operator on the boundary

$$B_\zeta^\dagger f(x) = \text{ p.v.} \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial\nu(x)}(x-y) f(y)\, ds(y). \tag{16}$$

It follows that the (nontangential) limits $\partial\phi/\partial\nu_+$, $\partial\phi/\partial\nu_-$ of the normal derivative of ϕ as the boundary is approached from the outside and inside Ω, respectively, are given by

$$\frac{\partial\phi}{\partial\nu_\pm} = \mp\frac{1}{2} f(x) + B_\zeta^\dagger f(x), \quad \textit{for almost every } x \in \partial\Omega. \tag{17}$$

(v) The boundary values ϕ_+, ϕ_- of ϕ from outside and inside of Ω, respectively, are identical as elements of $H^{3/2}(\partial\Omega)$ and agree with the trace of the single layer potential $S_\zeta f$.

Lemma 3 *If $f \in H^{3/2}(\partial\Omega)$ the function $\psi = D_\zeta f$ defined in $\mathbb{R}^n \setminus \partial\Omega$ has the properties (i), (ii) and (iii) of the Lemma 2.*

Moreover, the non-tangential limits ψ_+, ψ_- of ψ as we approach the boundary from outside and inside of Ω, respectively, exist and satisfy

$$\psi_\pm(x) = \pm\frac{1}{2} f(x) + B_\zeta f(x), \textit{ for almost every } x \in \partial\Omega. \tag{18}$$

Lemma 4 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. The Faddeev fundamental solution G_ζ can be given through the decomposition*

$$G_\zeta(x) = G_0(x) + H_\zeta(x),$$

where G_0 is the classical fundamental solution and H_ζ is an harmonic function.

Moreover, the single and double layer operators have a similar decomposition and, for our own convenience, we present here the case for the single layer. For $f \in H^{1/2}(\partial\Omega)$ we have

$$S_\zeta f(x) = S_0 f(x) + \int_{\partial\Omega} H_\zeta(x-y) f(y)\, ds(y) =: S_0 f(x) + \mathcal{H}_\zeta f(x).$$

Further, it holds

$$\|\mathcal{H}_\zeta\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{3/2}(\partial\Omega))} \leq C|\zeta|^{n-2},$$

where the constant C only depends on the domain.

Proof See [14] for further details. □

Now we provide the proof for uniqueness of the interior problem with a complex-potential. This is the only "new" and required statement to bring forth the proof of Nachman's reconstruction to complex-admittivities.

Proposition 3 *Let Ω be a bounded $C^{1,1}-$domain in $\mathbb{R}^n$, $n \geq 3$. Suppose that $q \in L^\infty(\bar{\Omega})$ is complex-valued and that 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$ in Ω. Then for every $f \in H^{3/2}(\partial\Omega)$ there is a unique $w \in H^2(\Omega)$ such that*

$$\begin{cases} (-\Delta + q)\, w = 0 \text{ in } \Omega \\ w|_{\partial\Omega} = f. \end{cases} \tag{19}$$

The solution operator is defined by $P_q f := w$ and has the mapping property

$$P_q : H^{3/2}(\partial\Omega) \to H^2(\Omega).$$

Moreover, the Dirichlet-to-Neumann map operator has the mapping property

$$\Lambda_q : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega).$$

Proof The proof follows by studying first the Laplacian and showing that multiplication by q is a compact operator from $H^2(\Omega)$ to $L^2(\Omega)$.

Thus, let

$$P_0 : H^2(\Omega) \to L^2(\Omega) \times H^{3/2}(\partial\Omega), \quad u \mapsto (-\Delta u, \operatorname{tr} u).$$

By the definition of $H^2(\Omega)$ and the trace properties on this space and $C^{1,1}-$domains the operator P_0 is linear and bounded. By Theorem 9.15. of [17] and under our conditions on the domain, there always exists a unique solution in $H^2(\Omega)$ of

$$\begin{cases} -\Delta u = f, \\ u|_{\partial\Omega} = g. \end{cases}$$

Therefore, the operator P_0 is bijective and invertible. In particular is Fredholm of index zero.

Analogously, we define the operator

$$P_q : H^2(\Omega) \to L^2(\partial\Omega) \times H^{3/2}(\partial\Omega), \quad u \mapsto ([-\Delta + q]u,\ \operatorname{tr} u)\,.$$

Then, the difference of the operators $P_q - P_0$ maps u to $(qu,\ 0)$ between the same spaces. Since, the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it immediately follows that multiplication by $q \in L^\infty(\Omega)$ is a compact operator. Hence, by definition $P_q - P_0$ is a compact and since $P_q = P_0 + (P_q - P_0)$ is the sum of a Fredholm operator of index zero and a compact operator, it still is Fredholm of index zero. Thus, to show invertibility we prove that $\ker P_q = \{0\}$. Let $w \in \ker P_q$. By definition this implies w is a solution in $H^2(\Omega)$ of

$$\begin{cases} -\Delta w + qw = 0, \\ w|_{\partial\Omega} = 0, \end{cases}$$

but due to the assumption of 0 not being a Dirichlet eigenvalue of $(-\Delta + q)$ in Ω it follows that $w \equiv 0$. □

Our main assumption is that $\gamma \in C^{1,1}(\bar{\Omega})$, thus it is in $H^2(\Omega)$. Therefore, for potentials q given by the complex-conductivity the following statement holds.

Corollary 2 *Let Ω be a bounded $C^{1,1}-$domain in $\mathbb{R}^n$, $n \geq 3$. For $\gamma \in C^{1,1}(\bar{\Omega})$ such that Re $\gamma \geq c > 0$.*

Then $q \in L^\infty(\Omega)$ given by $q = \Delta(\gamma^{1/2})/\gamma^{1/2}$ is well-defined and 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$. Then the unique solution $w \in H^2(\Omega)$ of

$$\begin{cases} -\Delta w + qw = 0 \\ w|_{\partial\Omega} = \gamma^{1/2} \end{cases} \tag{20}$$

is $w \equiv \gamma^{1/2}$.

This corollary brings to light that if we know the boundary values of the conductivity and the potential q in Ω, we can find γ by solving the above boundary value problem.

4 Boundary Integral Equation

The properties of the previous section allows us to establish a one-to-one correspondence between the solution of a boundary integral equation and of the following exterior problem

$$
\begin{aligned}
&(i)\ \Delta\psi = 0,\ \text{ in } \Omega' := \mathbb{R}^n \setminus \bar{\Omega}, \\
&(ii)\ \psi \in H^2(\Omega'_\rho),\ \text{ for any } \rho > \rho_0, \\
&(iii)\ \psi(x,\zeta) - e^{ix\cdot\zeta} \text{ satisfies } (15), \\
&(iv)\ \frac{\partial\psi}{\partial\nu_+} = \Lambda_q\psi \text{ on } \partial\Omega.
\end{aligned}
\tag{21}
$$

In this section, we assume that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$ and $q \in L^\infty(\Omega)$ is a complex-potential for which 0 is not a Dirichlet eigenvalue. Further, most proofs follow directly from Nachman's work [26], but we provide them here for convenience of the reader. We will highlight the new pieces needed to put the puzzle together.

Lemma 5 *Let $\zeta \in \mathcal{V}$.*

(a) Suppose ψ solves the exterior problem (21). Then its trace $f_\zeta = \psi_+ = \psi|_{\partial\Omega}$ solves the boundary integral equation

$$f_\zeta = e^{ix\cdot\zeta} - \left[S_\zeta\Lambda_q - B_\zeta - \frac{1}{2}I\right] f_\zeta. \tag{22}$$

(b) Conversely, suppose $f_\zeta \in H^{3/2}(\partial\Omega)$ solves (22). Then the function $\psi(x,\zeta)$ defined for $x \in \Omega'$ by

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \left(S_\zeta\Lambda_q - D_\zeta\right) f_\zeta(x) \tag{23}$$

solves the above exterior problem under all conditions. Furthermore, $\psi|_{\partial\Omega} = f_\zeta$.

Proof

(a) Assume ψ solves (21). We apply Green's identity to G_ζ and ψ in Ω'_ρ, $\rho > \rho_0$. It holds

$$
\begin{aligned}
&\left(\int_{|y|=\rho} - \int_{\partial\Omega}\right)\left[G_\zeta(x-y)\frac{\partial\psi}{\partial\nu_+} - \psi_+(y,\zeta)\frac{\partial G_\zeta}{\partial\nu_+(y)}(x-y)\right] ds(y) \\
&= \int_{\Omega'_\rho}\left[G_\zeta(x-y)\Delta\psi(y,\zeta) - \psi(y,\zeta)\Delta_y G_\zeta(x-y)\right] dy.
\end{aligned}
\tag{24}
$$

Since, ψ is harmonic on Ω'_ρ and G_ζ is the fundamental solution of $-\Delta$ we obtain for arbitrary $x \in \Omega'_\rho$

$$\begin{aligned}\psi(x,\zeta) = &\int_{|y|=\rho}\left[G_\zeta(x-y)\frac{\partial\left(\psi - e^{iy\cdot\zeta}\right)}{\partial\nu}\right.\\ &\left. - \left(\psi - e^{iy\cdot\zeta}\right)\frac{\partial G_\zeta}{\partial\nu_+(y)}(x-y)\right] ds(y) \qquad (25)\\ &+ \int_{|y|=\rho}\left[G_\zeta(x-y)\frac{\partial e^{iy\cdot\zeta}}{\partial\nu} - e^{iy\cdot\zeta}\frac{\partial G_\zeta}{\partial\nu(y)}(x-y)\right] ds(y)\\ &- \int_{\partial\Omega}\left[G_\zeta(x-y)\frac{\partial\psi}{\partial\nu_+} ds(y) - \int_{\partial\Omega}\psi_+(y,\zeta)\frac{\partial G_\zeta}{\partial\nu(y)}(x-y)\right] ds(y)\end{aligned}$$

By hypothesis (21-iii), the first integral vanishes. The function $e^{iy\cdot\zeta}$ is harmonic and a re-application of Green's identity to the second integral on $|y| < \rho$ equals $e^{ix\cdot\zeta}$. Finally, due to (21-iv) the last integral is $\left[S_\zeta\Lambda_q - D_\zeta\right]\psi$. Thus, the function ψ fulfills for $x \in \Omega'$ the identity

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \left[S_\zeta\Lambda_q - D_\zeta\right] f_\zeta.$$

Taking the non-tangential limit to the boundary from the outside we obtain by Lemmas 2 and 3

$$f_\zeta(x) = e^{ix\cdot\zeta} - \left[S_\zeta\Lambda_q - B_\zeta - \frac{1}{2}I\right] f_\zeta(x).$$

(b) Conversely, suppose $f_\zeta \in H^{3/2}(\partial\Omega)$ solves the boundary integral equation (22). Define a function ψ in Ω' by

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \left[S_\zeta\Lambda_q - D_\zeta\right] f_\zeta(x). \qquad (26)$$

We show that this ψ solves the exterior problem (21) from properties of the single and double layer (Lemmas 2 and 3).

It is immediate to see that ψ fulfills the property (i) of (21), since for $\zeta\cdot\zeta = 0$ the exponential $e^{ix\cdot\zeta}$ is harmonic, and $S_\zeta\Lambda_q f_\zeta$, $D_\zeta f_\zeta$ are harmonic in Ω' by the above mentioned lemmas. Moreover, it holds that $S_\zeta\Lambda_q f_\zeta,\ D_\zeta f_\zeta \in H^2(\Omega'_\rho),\ \rho > \rho_0$ and further the identity (15) also holds. Hence, the property (ii) and (iii) of the exterior problem follow.

To show the last property, we approach the boundary $\partial\Omega$ non-tangentially from the outside and we obtain, as in part (a),

$$\psi|_{\partial\Omega} = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I\right] f_\zeta.$$

By virtue of f_ζ fulfilling the boundary integral equation the right-hand side equals f_ζ and therefore $\psi|_{\partial\Omega} = f_\zeta$. From this and the first three properties of (21), that we already showed ψ fulfills, we can obtain analogously to part (a)

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - S_\zeta\left(\frac{\partial\psi}{\partial\nu_+}\right) + D_\zeta f_\zeta, \qquad \text{for } x \in \Omega'. \tag{27}$$

Subtracting both formulations of ψ, (27) and (26), the following equality holds throughout Ω'

$$S_\zeta\left[\Lambda_q f_\zeta - \frac{\partial\psi}{\partial\nu_+}\right] = 0. \tag{28}$$

By taking traces from the outside, it actually holds on the boundary $\partial\Omega$. We are reminded that $S_\zeta\left[\Lambda_q f_\zeta - \frac{\partial\psi}{\partial\nu_+}\right]$ is harmonic in $\mathbb{R}^n \setminus \partial\Omega$ and since the trace is 0 on $\partial\Omega$ uniqueness of the interior problem for $q \equiv 0$ implies that the equality (28) holds everywhere. Then, its normal derivatives will be zero and subtracting them on $\partial\Omega$ with the help of (17) we obtain

$$\left[\Lambda_q - \partial\psi/\partial\nu_+\right] = \frac{\partial S_\zeta\left[\Lambda_q - \partial\psi/\partial\nu_+\right]}{\partial\nu_-} - \frac{\partial S_\zeta\left[\Lambda_q - \partial\psi/\partial\nu_+\right]}{\partial\nu_+} = 0. \tag{29}$$

Thus the last property of the exterior problem follows. □

Furthermore, we are able to obtain a relation between the exterior problem and the solutions of integral equation (9).

Lemma 6 *Let $\zeta \in \mathcal{V}$.*

(a) Suppose $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is a solution of

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\mathbb{R}^n} G_\zeta(x-y)q(y)\psi(y,\zeta).$$

Then the restriction of ψ to Ω' solves the exterior problem (21) and fulfills the respective properties (i)–(iv).

(b) Conversely, if ψ solves the exterior problem (21), there is a unique solution $\tilde{\psi} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ of the integral equation (9), such that $\tilde{\psi} = \psi$ in Ω'.

Proof

(a) From the Proposition 1 it follows $\psi \in H^2_{\text{loc}}(\mathbb{R}^n)$, which immediately implies property (ii) of the exterior problem. Moreover, in $\mathbb{R}^n$ it holds $(-\Delta + q)\psi = 0$, thus due to $q \equiv 0$ on Ω' the property (i) holds, i.e., $-\Delta\psi = 0$ in Ω'.

Applying Green identity on $|y| < \rho$ yields

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial \psi}{\partial \nu(y)} - \psi(y,\zeta)\frac{\partial G_\zeta}{\partial \nu(y)}(x-y) \right] ds(y)$$
$$= \int_{|y|<\rho} G_\zeta(x-y)q(y)\psi(y,\zeta)\, dy + \psi(x,\zeta), \text{ for a.e. } x \text{ with } |x| < \rho.$$

Now, we can choose ρ large in order to contain the supp of q. Since ψ solves integral equation this means that the right-hand side equals $e^{ix\cdot\zeta}$. Moreover, we already showed that

$$e^{ix\cdot\zeta} = \int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial e^{iy\cdot\zeta}}{\partial \nu(y)} - e^{iy\cdot\zeta}\frac{\partial G_\zeta}{\partial \nu(y)}(x-y) \right] ds(y).$$

Then passing the exponential to the right-hand side, we obtain

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial \left(\psi - e^{iy\cdot\zeta}\right)}{\partial \nu(y)} - \left(\psi(y,\zeta) - e^{iy\cdot\zeta}\right)\frac{\partial G_\zeta}{\partial \nu(y)}(x-y) \right]$$
$$\times\, ds(y) = 0$$

for all $\rho > \rho_0$ Thus property iii) follows by taking the limit as $\rho \to \infty$.

Immediately, we can see that $\Lambda_q\psi_- = \frac{\partial\psi}{\partial\nu_-}$ and since $\psi \in H^2$ in a two-sided neighborhood of $\partial\Omega$ it holds that $\psi_- = \psi_+$ and $\frac{\partial\psi}{\partial\nu_-} = \frac{\partial\psi}{\partial\nu_+}$. This leads to ψ fulfilling the iv) property. Therefore, the restriction of ψ to Ω' solves the exterior problem (21).

(b) Suppose ψ defined in Ω' solves the exterior problem (21). Set $\tilde\psi$ by $\tilde\psi = P_q\psi_+$ in Ω and $\tilde\psi = \psi$ in Ω'. Then on $\partial\Omega$,

$$\tilde\psi_- = (P_q\psi_+) = \psi_+ = \tilde\psi_+$$

and

$$\frac{\partial\tilde\psi}{\partial\nu_-} = \Lambda_q\psi_+ = \frac{\partial\psi}{\partial\nu_+} = \frac{\partial\tilde\psi}{\partial\nu_+}$$

due to (iv). Thus $\tilde{\psi}$ solves $(-\Delta + q)\tilde{\psi} = 0$ on $\mathbb{R}^n$. Applying Green's formula in $|y| < \rho$ yields

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial \psi}{\partial \nu(y)} - \psi(y,\zeta)\frac{\partial G_\zeta}{\partial \nu(y)}(x-y) \right] ds(y)$$
$$= \int_{|y|<\rho} G_\zeta(x-y)q(y)\tilde{\psi}(y,\zeta)\, dy + \tilde{\psi}(x,\zeta)$$

for almost every x with $|x| < \rho$. Thus by letting $\rho \to \infty$ the radiation condition (iii) implies that the left-hand side is $e^{ix\cdot\zeta}$. Hence $\tilde{\psi}$ verifies the desired integral equation in $\mathbb{R}^n$.

To finalize we prove that this extension is unique. Suppose that we have two extensions $\tilde{\psi}^1, \tilde{\psi}^2 \in L^2_{\text{loc}}(\mathbb{R}^n)$ of ψ which agree in Ω' and solve the integral equation everywhere. As in part (a), we see that $\tilde{\psi}^1, \tilde{\psi}^2 \in H^2_{\text{loc}}(\mathbb{R}^n)$ and $(-\Delta + q)\tilde{\psi}^j = 0$ in $\mathbb{R}^n$ for $j = 1, 2$. Hence, they are in H^2 on a two-sided neighborhood of $\partial\Omega$. This implies that $\tilde{\psi}^j_+ = \tilde{\psi}^j_-$, for $j = 1, 2$, which promptly leads to $\tilde{\psi}^1_- = \tilde{\psi}^2_-$ since they agree on Ω'. Now, from the uniqueness of the interior problem it follows that $\tilde{\psi}^1 = \tilde{\psi}^2$. □

Remark The two previous lemmas establish that a solution of the boundary integral equation is equivalent to a exponential growing solution of the Schrödinger equation in $\mathbb{R}^n$. The interesting remark is that there was no explicit requirement of ζ being large. Hence, by showing that the boundary integral equation is uniquely solvable for small values of ζ we guarantee the existence of exponential growing solutions for these ζ.

Moreover, on all the proves above there is no explicit difference of q being real or complex.

Keeping this in mind, we focus now on solvability of the boundary integral equation. The following proposition glues together the papers [26] and [14] and applies them to the complex potential by making use of the uniqueness of the interior problem in this scenario obtained in Proposition 3.

Proposition 4 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let q be a complex-valued potential in $L^\infty(\Omega)$ and suppose that* 0 *is not Dirichlet eigenvalue of $-\Delta+q$ in Ω. We define $K_\zeta = S_\zeta\Lambda_q - B_\zeta - \frac{1}{2}I$ and for any $\zeta \in \mathcal{V}$ it holds*

(a) The operators K_0, K_ζ are compact on $H^{3/2}(\partial\Omega)$.
(b) If Re $q \geq 0$*, then $I + K_0$ is invertible in $H^{3/2}(\partial\Omega)$.*
(c) If Re $q \geq 0$ *there exists an $\epsilon > 0$ with $|\zeta| < \epsilon$ for which the operator $I + K_\zeta$ is invertible in $H^{3/2}(\Omega)$.*
(d) There exists an $R > 0$ such that for all $|\zeta| > R$ the operator $I + K_\zeta$ is invertible in $H^{3/2}(\partial\Omega)$.

Proof Part (a) follows by a compactness embedding. Let $f \in H^{3/2}(\partial\Omega)$ and set $w = P_q f$ as the solution of interior Dirichlet problem (19). Let $x \in \Omega$ we use the Green's formula to obtain

$$\int_\Omega G_\zeta(x-y)\Delta w(y)\,dy + w(x) = \left[S_\zeta \Lambda_q - D_\zeta\right] f(x)$$

which is equivalent to

$$\int_\Omega G_\zeta(x-y)q(y)P_q f(y)\,dy + w(x) = \left[S_\zeta \Lambda_q - D_\zeta\right] f(x)$$

By letting x approach the boundary non-tangentially from the inside we thus obtain

$$\text{tr}\,\left(G_\zeta * (qP_q f)\right) + f(x) = S_\zeta \Lambda_q f(x) - \left[-\frac{1}{2}f(x) + B_\zeta f(x)\right]$$

and therefore

$$\left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I\right] f = \text{tr}\,\left(G_\zeta * (qP_q f)\right).$$

Hence, our desired operator satisfies this factorization, where the following mapping properties hold

$P_q : H^{3/2}(\partial\Omega) \to H^2(\Omega)$;
$\imath : H^2(\Omega) \to L^2(\Omega)$ is a compact embedding;
$M_q : L^2(\Omega) \to L^2(\Omega)$;
$\mathbf{G}_\zeta : L^2(\Omega) \to H^2(\Omega)$ convolution with G_ζ, which we prove up next;
$\text{tr} : H^2(\Omega) \to H^{3/2}(\partial\Omega)$.

And the compactness of the embedding implies compactness of the desired operator.

(b) Let $\zeta = 0$. In this case G_0 is the classical fundamental solution and the corresponding operators are the classical ones. By part (a), we already know that $S_0\Lambda_q - B_0 - \frac{1}{2}I$ is compact on $H^{3/2}(\partial\Omega)$. Then $I + K_0 = \left[\frac{1}{2}I + S_0\Lambda_q - B_0\right]$ is Fredholm of index zero on $H^{3/2}(\partial\Omega)$. Therefore, it is enough to show injectivity.

Let $h \in H^{3/2}(\partial\Omega)$ such that $\left[\frac{1}{2}I + S_0\Lambda_q - B_0\right] h = 0$. Define $w = -S_0\Lambda_q h + D_0 h$. Then w is harmonic in $\mathbb{R}^n$, $w \in H^2(\Omega)$ and $w \in H^2(\Omega'_\rho)$ by Lemmas 2 and 3. Moreover, approaching the boundary non-tangentially by the inside we obtain

$$w_- = -S_0\Lambda_q h + \left(-\frac{1}{2}h + B_0 h\right) = -\left[\frac{1}{2}I + S_0\Lambda_q - B_0\right] h = 0.$$

Since, the problem $-\Delta w = 0,\;\; w|_{\partial\Omega} = 0$ is uniquely solvable in $H^2(\Omega)$ it follows that $w \equiv 0$ in Ω and thus $\frac{\partial w}{\partial \nu_-} = 0$ on $\partial\Omega$.

By noticing the jump relations for the single and double layer operator (see [25]), we can deduce that

$$[w] = w_+ - w_- = w_+ = [D_0 h] = h$$

and

$$\left[\frac{\partial w}{\partial \nu}\right] = \frac{\partial w}{\partial \nu_+} = -\left[\frac{\partial}{\partial \nu} S_0 \Lambda_q h\right] = \Lambda_q h.$$

Now, by Proposition 3 there is a unique solution $u \in H^2(\Omega)$ of

$$\begin{cases} (-\Delta + q)\, u = 0, \\ u|_{\partial\Omega} = h, \end{cases}$$

such that $\Lambda_q h = \frac{\partial u}{\partial \nu_-}\Big|_{\partial\Omega}$. We set

$$v = \begin{cases} u, \text{ in } \ \Omega \\ w, \text{ in } \ \mathbb{R}^n \setminus \Omega \end{cases}$$

and see that $u_- = w_+ = h$ and $\frac{\partial u}{\partial \nu_-} = \frac{\partial w}{\partial \nu_+} = \Lambda_q$, thus it holds that v and $\frac{\partial v}{\partial \nu}$ are continuous over the boundary $\partial\Omega$. Therefore $v \in H^2(B_\rho(0)),\ \rho > 0$ and it solves $-\Delta v + qv = 0$ in $\mathbb{R}^n$, since $q \equiv 0$, in $\mathbb{R}^n \setminus \Omega$.

Let $\chi_\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ in $B_{\rho-\epsilon}(0)$ and $\chi \equiv 0$ in $\rho - \epsilon < |x| < \rho$, for $\epsilon > 0$ small enough.

Then for $\phi \in H^1(\mathbb{R}^n)$ it follows by Green's identity

$$\int_{|x|<\rho} (-\Delta v + qv)\,(\chi\phi)\ dx = 0,$$

which is equivalent to

$$\int_{|x|<\rho} \nabla v \cdot \nabla(\chi\phi) + qv(\chi\phi)\, dx = 0,$$

as well as

$$\int_\Omega \nabla v \cdot \nabla\phi + qv\phi\, dx + \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla(\chi\phi)\, dx = 0.$$

In particular we can take $\phi = \bar{v}$ and since w is given through the classical single and double layer it follows that $\nabla w \in L^2(B_\rho(0) \setminus \bar{\Omega})$. Thus taking the limit as $\rho \to \infty$

$$\int_\Omega |\nabla v|^2 \phi + q|v|^2\,dx + \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla(\chi \bar{w})\,dx = 0,$$
$$\int_\Omega |\nabla v|^2 \phi + q|v|^2\,dx + \int_{B_\rho(0)\setminus\Omega} |\nabla w|^2\,dx = \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla((1-\chi)\bar{w})\,dx,$$

which yields

$$\int_{\mathbb{R}^n} |\nabla v|^2 + q|v|^2\,dx = 0$$

and therefore

$$\int_{\mathbb{R}^n} |\nabla v|^2 + (\operatorname{Re} q)|v|^2\,dx = 0.$$

Now, we can apply Hardy's inequality for $H^1(\mathbb{R}^n)$

$$\frac{(d-2)^2}{4}\int_{\mathbb{R}^n} |x|^{-2}|v|^2\,dx \le \int_{\mathbb{R}^n} |\nabla v|^2\,dx$$

to finally obtain the condition

$$\int_{\mathbb{R}^n} \left[\frac{(d-2)^2}{4|x|^2} + (\operatorname{Re} q(x))\right]|v|^2\,dx \le 0.$$

Hence, for $\operatorname{Re} q \ge 0$ this implies that $v \equiv 0$ in $\mathbb{R}^n$. Thus $h \equiv 0$ in $\partial\Omega$. Hence we obtain invertibility in the case $\zeta = 0$. Notice that we have been loose on the requirement for q, since this will be enough for the complex-conductivity purposes, but this proof works for potentials that satisfy the estimate $\operatorname{Re} q \ge -\frac{(d-2)^2}{4|x|^2}$.

Part (c) follows quite easily by the fact that the set of invertible operators is open. However, we present the result with the help of some estimates and Neumann series.

For $h \in H^{3/2}(\partial\Omega)$ it holds $K_\zeta h = S_\zeta(\Lambda_q - \Lambda_0)h$, since due to Green's formula we have $B_\zeta = -\frac{1}{2}I + S_\zeta \Lambda_0$.

Moreover, by Lemma 4 and for $h \in H^{3/2}(\partial\Omega)$ we have the decomposition $S_\zeta(\Lambda_q - \Lambda_0)h = S_0(\Lambda_q - \Lambda_0)h + \mathcal{H}_\zeta(\Lambda_q - \Lambda_0)h$. Moreover, we also have by the lemma the estimate

$$\begin{aligned}\|\mathcal{H}_\zeta(\Lambda_q - \Lambda_0)h\|_{H^{3/2}(\partial\Omega)} &\le C|\zeta|^{n-2}\|(\Lambda_q - \Lambda_0)h\|_{H^{1/2}(\partial\Omega)}\\ &\le C|\zeta|^{n-2}\|h\|_{H^{3/2}(\partial\Omega)}.\end{aligned}$$

From the invertibility of $I + K_0$ we obtain the decomposition

$$\begin{aligned}[I + K_\zeta] &= I + K_0 + \mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right) \\ &= (I + K_0) \left(I + (I + K_0)^{-1} \mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right)\right)\end{aligned}$$

and if

$$\| (I + K_0)^{-1} \mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right) \|_{\mathcal{L}(H^{3/2}(\partial\Omega))} < 1$$

we obtain invertibility for $I + K_\zeta$ in $H^{3/2}(\partial\Omega)$. This norm can be translated to an estimate for ζ by the above on $\mathcal{H}_\zeta$. We have

$$\begin{aligned}&\| (I + K_0)^{-1} \mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right) \|_{\mathcal{L}(H^{3/2}(\partial\Omega))} \\ &\qquad \le C|\zeta|^{n-2} \left\|(I + K_0)^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))} \left\|\mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right)\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))} < 1.\end{aligned}$$

Hence, for

$$|\zeta| < \left[\frac{1}{\left\|(I + K_0)^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))} \left\|\mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right)\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}}\right]^{1/(n-2)} =: \epsilon,$$

invertibility is obtained by Neumann series.

Part (d) uses the existence of exponential growing solutions for large values of $|\zeta|$.

Let $R > 0$ be large enough such that for $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0,\ |\zeta| > R$ we have unique exponential growing solutions of (9), Corollary 1. Under this conditions, we have showed that $K_\zeta := S_\zeta \Lambda_q - B_\zeta - \frac{1}{2} I$ is compact in $H^{3/2}(\partial\Omega)$. Therefore, $I + K_\zeta$ is a Fredholm operator of index zero in $H^{3/2}(\partial\Omega)$. We need to show that the kernel is empty to prove that it is invertible.

Let $g \in H^{3/2}(\partial\Omega)$ be in $\ker K$. Then $h = [-S_\zeta \Lambda_q + D_\zeta] g$ solves the exterior problem (i), (ii), (iv) and fulfills the radiation condition (15) (the proof is analogous to Lemma 5).

Moreover, we can extend h to a solution $\tilde{h}$ of $\tilde{h} = -\int_{\mathbb{R}^n} G_\zeta(x - y) q(y) \tilde{h}(y)\, dy$ in all of $\mathbb{R}^n$ (analogous to the previous lemma). By the estimates on G_ζ we note that $e^{-ix\cdot\zeta} \tilde{h} \in L^2_{\delta-1}(\mathbb{R}^n),\ 0 < \delta < 1$ and

$$e^{-ix\cdot\zeta} \tilde{h} = -A_\zeta(e^{-ix\cdot\zeta} \tilde{h})$$

with A_ζ defined as in (13). Since, we took $R > 0$ large enough then A_ζ is a contraction in $L^2_{\delta-1}(\mathbb{R}^n)$ and this forces $\tilde{h} \equiv 0$. Therefore,

$$g \equiv 0 \text{ and } I + K_\zeta \text{ is invertible in } H^{3/2}(\partial\Omega)$$

and the statement is proven. □

Therefore, we can solve the boundary integral equation for small and large values of $|\zeta|$ and obtain ψ on $\partial\Omega$ by

$$\psi(x,\zeta) = \left[\frac{1}{2}I + S_\zeta \Lambda_q - B_\zeta\right]^{-1} \left(e^{ix\cdot\zeta}\right)$$

This allows us to obtain the scattering transform from the boundary data

Theorem 3 *Suppose that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in L^\infty(\Omega)$ be complex-valued and suppose that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω.*

We define the scattering transform for non-exceptional points $\zeta \in \mathcal{V}$ by

$$\mathbf{t}(\xi,\zeta) = \int_{\mathbb{R}^3} e^{-ix\cdot(\zeta+\xi)} q(x)\psi(x,\zeta)\,dx, \ \xi \in \mathbb{R}^n. \tag{30}$$

Then, for each $\xi \in \mathbb{R}^n$ we can compute the scattering transform for the non-exceptional points $\zeta \in \mathcal{V}_\xi := \left\{\zeta \in \mathbb{C}^n \setminus \{0\} : \zeta\cdot\zeta = 0,\ |\xi|^2 + 2\zeta\cdot\xi = 0\right\}$ from the solutions of the boundary integral equation by

$$\mathbf{t}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\zeta+\xi)} \left[\Lambda_q + i(\xi+\zeta)\cdot\nu\right] \psi(x,\zeta)\,ds(x), \ \xi \in \mathbb{R}^n. \tag{31}$$

Proof From Lemmas 5 and 6 we obtain unique exponentially growing solutions of (9) by the one-to-one relation with the boundary integral (22). Therefore, by Green identity it holds

$$\begin{aligned}
\mathbf{t}(\xi,\zeta) &= \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x,\zeta)\,dx \\
&= \int_\Omega e^{-ix\cdot(\xi+\zeta)} \Delta\psi(x,\zeta) - \left(\Delta e^{-ix\cdot(\zeta+\xi)}\right)\psi(x,\zeta)\,dx \\
&= \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)} \left[\Lambda_q + i(\xi+\zeta)\cdot\nu\right] \psi(x,\zeta)\,ds(x)
\end{aligned}$$

for $\xi \in \mathbb{R}^n$ and $\zeta \in \mathcal{V}_\xi$ such that the boundary integral equation has a unique solution. □

5 From t to γ

From the scattering transform we can obtain the Fourier transform of the potential through large asymptotics. Unfortunately for this we need to solve the boundary integral equation for large values ζ, which is makes this method very unstable. In [20] they avoid the boundary integral equation by using the approximation $\psi(x,\zeta) \approx e^{ix\cdot\zeta}$ to compute the scattering transform. This simplified version was even applied for complex conductivities in order to obtain a stable reconstruction procedure.

This method is based on the following asymptotic

Theorem 4 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in L^\infty(\Omega)$ be a complex-valued potential extended to zero outside Ω, such that 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$. Then for $|\zeta| > R$ and $0 < \delta < 1$*

$$|\mathbf{t}(\xi,\zeta) - \hat{q}(\xi)| \leq \frac{\tilde{c}(\delta, R)}{|\zeta|}\|q\|_\delta^2 \tag{32}$$

for all $\xi \in \mathbb{R}^n$.

Proof The proof follows trivially by Corollary 1. If $q \in L^\infty(\Omega)$ is a complex-valued and compactly supported potential it follows that $\hat{q}$ is well-defined and

$$\begin{aligned}\left|\mathbf{t}(\xi,\zeta) - \hat{q}(\xi)\right| &= \left|\int e^{-ix\cdot\xi} q(x)\left[e^{-ix\cdot\zeta}\psi(x,\zeta) - 1\right]dx\right| \\ &\leq \|q\|_{1-\delta}\|e^{-ix\cdot\zeta}\psi(x,\zeta) - 1\|_{\delta-1} \leq \frac{\tilde{c}(\delta,R)}{|\zeta|}\|q\|_\infty^2\end{aligned}$$

holds true. □

One of the ways to obtain a more stable reconstruction is the so-called d-bar method. Following the $\bar{\partial}$ compatibility equations satisfied by $\mathbf{t}$ known from [4, 21, 28], Nachman was able to derive a d-bar equation in three dimensions, which allows to obtain solutions μ that eventually permit the computation of $\hat{q}$ from $\mathbf{t}(\xi,\zeta)$ for $\xi \in \mathbb{R}^n$, $|\zeta| \geq M$, $(\xi+\zeta)^2 = 0$ and its derivative in ζ. Although more elaborate than in two dimensions, this method does not require taking the limit of $|\zeta| \to \infty$. In essence, the proof follows through for the complex-potential as well!

For such, let $\psi(x,\zeta)$ be the solution of (9) with $e^{-ix\cdot\zeta}\psi(x,\zeta) - 1 \in L^2_{\delta-1}(\mathbb{R}^n)$, that is, ζ is not an exceptional point. Define,

$$\mu(x,\zeta) := |q(x)|e^{-ix\cdot\zeta}\psi(x,\zeta) \tag{33}$$

then μ solves the following integral equation

$$\mu(x,\zeta) = |q(x)| - |q(x)|\int_{\mathbb{R}^n} g_\zeta(x-y)\tilde{q}(y)\mu(y,\zeta)\,dy \tag{34}$$

Hereby, we set

$$\tilde{A}_\zeta f(x) := |q(x)| \int_{\mathbb{R}^n} g_\zeta(x-y)\tilde{q}(y) f(y)\,dy$$

with $\tilde{q}(x) = q(x)/|q(x)|$ in the support of q and 0 otherwise. Moreover, the scattering transform is given through

$$\mathbf{t}(\xi,\zeta) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\tilde{q}(y)\mu(x,\zeta)\,dx. \tag{35}$$

Lemma 7 *Suppose $q \in L^\infty(\mathbb{R}^n)$ with compact support. Let $R > c(\delta,a)\|q(x)\langle x\rangle\|_{L^\infty}$ with $\delta \in (0,1)$ and $c(\delta,a)$ as in Proposition 1.*

(a) If $\zeta \geq R$, $\zeta\cdot\zeta = 0$, then (34) has a unique solution $\mu(\cdot,\zeta)$ in $L^2(\mathbb{R}^n)$ with compact support.
(b) For $|\zeta| > M$, $\zeta\cdot\zeta = 0$ and all $w \in \mathbb{C}^n$ with $w\cdot\bar{\zeta} = 0$,

$$w\cdot\frac{\partial\mu}{\partial\bar{\zeta}}(x,\zeta) = \frac{-1}{(2\pi)^{n-1}}\int e^{ix\cdot\xi} w\cdot\xi\,\delta(|\xi|^2 + 2\zeta\cdot\xi)t(\xi,\zeta)\mu(x,\zeta+\xi)\,d\xi. \tag{36}$$

(c) For $\zeta \in \mathcal{V}_\xi$ with $|\zeta| > M$ and all $w \in \mathbb{C}^n$ with $w\cdot\bar{\zeta} = 0$ and $w\cdot\xi = 0$,

$$w\cdot\frac{\partial\mathbf{t}}{\partial\bar{\zeta}}(\xi,\zeta) = \frac{-1}{(2\pi)^n}\int w\cdot\eta\,\delta(\eta^2 + 2\zeta\cdot\eta)\mathbf{t}(\xi-\eta,\zeta+\eta)\mathbf{t}(\eta,\zeta)\,d\eta. \tag{37}$$

Proof For details see Nachman [26]. □

We keep it short here and refer to Nachman [26] for the formula to obtain q without taking limits of the scattering transform.

Our interest resides now in the behavior of exponential growing solutions for ζ close to zero. Due to invertibility of the boundary integral equation we can in fact show that there are no exceptional points near 0. Therefore, analogously to [14] we are able to obtain the following estimate

Lemma 8 *Let $\gamma \in C^{1,1}(\Omega)$ be the complex-conductivity with $\sigma \geq c > 0$, $\epsilon \geq 0$, $\omega \in \mathbb{R}^+$ and suppose $\gamma \equiv 1$ near $\partial\Omega$. Set $q = (\Delta\gamma^{1/2})/\gamma^{1/2} \in L^\infty(\Omega)$.*

For $\zeta \in \mathcal{V}$ sufficiently small and $\phi \in H^{3/2}(\partial\Omega)$ the corresponding boundary integral solution of (22), it holds

$$\|\phi(\cdot,\zeta) - 1\|_{H^{3/2}(\partial\Omega)} \leq C|\zeta|. \tag{38}$$

Proof Let $K_\zeta = S_\zeta\left(\Lambda_q - \Lambda_0\right)$. Solutions of the boundary integral equation fulfill

$$\phi(x,\zeta) - 1 = \left(e^{ix\cdot\zeta} - 1\right) - K_\zeta\left(\phi(x,\zeta) - 1\right),$$

which follows by $\Lambda_q 1 = 0$ and $\Lambda_0 1 = 0$, since the unique H^2-solution of $(-\Delta + q)u = 0,\ u|_{\partial\Omega} = 1$ is $\gamma^{1/2}$ and $w = 1$ is the unique harmonic function in $H^2(\Omega)$ with boundary value 1.

Under the conditions on γ it holds that Re $q > 0$ and hence by Proposition 4 it holds that $[I + K_\zeta]$ is invertible in $H^{3/2}(\partial\Omega)$ for small ζ and hence,

$$\phi - 1 = \left[I + K_\zeta\right]^{-1}\left(e^{ix\cdot\zeta} - 1\right).$$

It clearly holds that by Taylor series that $\|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial\Omega)} \leq C_1|\zeta|$ and $\left\|[I + K_\zeta]^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}$ is uniformly bounded for small $|\zeta|$ due to Neumann series inversion. Thus,

$$\|\phi - 1\|_{H^{3/2}(\partial\Omega)} \leq C_2\|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial\Omega)} \leq C_3|\zeta|$$

and the statement follows. □

Theorem 5 *Let $\gamma \in C^{1,1}(\Omega)$ be the complex-conductivity with $\sigma \geq c > 0$, $\epsilon \geq 0$, $\omega \in \mathbb{R}^+$ and suppose $\gamma \equiv 1$ near $\partial\Omega$. Set $q = (\Delta\gamma^{1/2})/\gamma^{1/2} \in L^\infty(\Omega)$.*

For $\zeta \in \mathcal{V}$ small enough such that (9) has unique exponentially growing solutions $\psi(x,\zeta)$, it holds

$$\|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{L^2(\Omega)} \leq C|\zeta|. \tag{39}$$

Proof Since $\gamma = 1$ near the boundary $\partial\Omega$ we have that $\gamma^{1/2}$ is the unique $H^2(\Omega)$ solution of

$$\begin{cases} -\Delta u + qu = 0, \text{ in } \Omega \\ u|_{\partial\Omega} = 1. \end{cases}$$

By the elliptic estimates, we obtain that

$$\begin{aligned} \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{L^2(\Omega)} &\leq \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{H^2(\Omega)} \\ &\leq \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{H^{3/2}(\partial\Omega)} \leq C|\zeta| \end{aligned}$$

and the statement follows. □

This theorem states that we can reconstruct the complex-conductivity from the exponential growing solutions by

$$\gamma(x) = \lim_{|\zeta|\to 0} \psi(x,\zeta), \quad \text{for a.e. } x \in \Omega. \tag{40}$$

However, recall that for small ζ we only know how to obtain the boundary values of the exponential growing solutions from the boundary measurements. To provide a reconstruction of γ in Ω it is necessary to compute these solutions for all ζ small inside Ω from the scattering data. This might be possible by the $\bar{\partial}$-equation.

In order to obtain a $\bar{\partial}$ reconstruction method complex conductivities the following problems need to be solved

1. **Uniqueness of (34) for ζ non-exceptional and considerably small:** A first step is to show that this equation is uniquely solvable for small values of ζ. In Nachman's proof invertibility of the operator $I + \tilde{A}_\zeta$ in $L^2(\mathbb{R}^n)$ follows by that of $I + A_\zeta$ obtained in Corollary 1 for ζ large. In the case of small values, we showed that the integral equation (9) is uniquely solvable due to the unique solvability of the boundary integral equation. However, this will not imply that the operator $I + A_\zeta$ is invertible and therefore we can follow the same approach to prove the existence of a unique solution μ to (34).
2. **Solvability of $\bar{\partial}$-equation:** In [26] there is no proof that $\bar{\partial}-$equation is uniquely solvable, but this is essential since this would be the only equation fully independent of q and where its information is given through the scattering transform. In this sense, we need to study the equation in the space $\mathcal{V} \setminus \{\zeta \in \mathbb{C}^n : \epsilon \leq |\zeta| < R\}$. In the work of [23] they establish this approach in a two-dimensional positive energy setting and intuition could lead to a similar work in our case.

6 Reconstruction of Λ_q from the Boundary Measurements Λ_γ

The Dirichlet-to-Neumann map Λ_γ is bounded from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. Moreover, it is properly defined through

$$\langle \Lambda_\gamma f, g\rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx, \tag{41}$$

where u is the unique $H^1(\Omega)$ solution of the interior problem $\nabla \cdot (\gamma \nabla u) = 0$ in Ω and $u|_{\partial\Omega} = f$ and $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$.

We can also define the Dirichlet-to-Neumann map for the Schrödinger operator by

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$\langle \Lambda_q \tilde{f}, \tilde{g}\rangle = \int_\Omega \nabla\tilde{u}\cdot\nabla\tilde{v} + q\tilde{u}\tilde{v}\,dx, \quad \forall\tilde{v}\in H^1(\Omega), \text{ s.t. } \tilde{v}|_{\partial\Omega} = \tilde{g},$$

and $\tilde{u} \in H^1(\Omega)$ is the unique solution to $(-\Delta + q)\tilde{u} = 0,$ in $\Omega,$ $\tilde{u}|_{\partial\Omega} = \tilde{f}.$

As in the real case, since both problems are interconnected we can obtain Λ_q from Λ_γ by

$$\Lambda_q = \gamma^{-1/2}\left[\Lambda_\gamma + \frac{1}{2}\frac{\partial\gamma}{\partial\nu}\right]\gamma^{-1/2}. \tag{42}$$

This brings to light that we can determine Λ_q from Λ_γ and the boundary values $\gamma|_{\partial\Omega}$ and $\left.\frac{\partial\gamma}{\partial\nu}\right|_{\partial\Omega}$. Thus, if $\gamma \equiv 1$ near $\partial\Omega$ then for $\gamma \in W^{2,\infty}(\Omega)$ it holds that $\Lambda_q = \Lambda_\gamma$. Otherwise, we need to obtain a method to reconstruct these boundary values.

There are many results to compute these boundary values. However, most of them need further smoothness. Still Nachman holds the best result for our case. In [27] he showed that the boundary values can be obtained without further smoothness assumptions. Following his proof we see that there is no explicit requirement of γ being real, besides the fact that $\gamma \geq c > 0$ and uniqueness of the Dirichlet problem in $H^1(\Omega)$. Hence, we can quickly extend the result for complex-conductivities in $W^{2,\infty}(\Omega)$ with Re $\gamma \geq c > 0$.

The result is obtained through the following lemmas.

Lemma 9 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$. Assume $\gamma \in W^{1,r}(\Omega)$ for $r > n$ and Re $\gamma \geq c > 0$.*

Then for any $f \in H^{1/2}(\partial\Omega)$ and

$$h \in \mathring{H}^{-1/2}(\partial\Omega) := \left\{h \in H^{-1/2}(\partial\Omega) : \langle h, 1\rangle_{\partial\Omega_j} = 0,\ j = 1,\ldots,N\right\}$$

the identity holds

$$\langle h, (\gamma - \mathcal{R}\Lambda_\gamma)f\rangle = \int_\Omega u\nabla w\cdot\nabla\gamma, \tag{43}$$

where $u \in H^1(\Omega)$ solution of $\nabla\cdot(\gamma\nabla u) = 0,$ $u|_{\partial\Omega} = f$, and $w \in H^1(\Omega)$ is a weak solution of $\Delta w = 0$ in Ω with $\frac{\partial w}{\partial\nu} = h$ and $\mathcal{R}$ denotes the Neumann-to-Dirichlet map.

Lemma 10 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n, n \geq 2$. Assume γ is in $W^{2,p}(\Omega)$, $p > n/2$ and Re $\gamma \geq c > 0$.*

For any $f,\ g \in H^{1/2}(\partial\Omega)$ the identity holds

$$\langle g, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial\nu}\right) f\rangle = \int_\Omega 2v\nabla(u-u_0)\cdot\nabla\gamma + v(2u-u_0)\Delta\gamma\, dx$$

where u, u_0, v are respectively the $H^1(\Omega)$ solutions of $L_\gamma(u) = 0$, $\Delta u_0 = 0$ and $\Delta v = 0$, in Ω, with $u|_{\partial\Omega} = f$, $u_0|_{\partial\Omega} = f$ and $v|_{\partial\Omega} = g$.

From this we obtain the boundary reconstruction formulas.

Theorem 6 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n, n \geq 2$. Suppose $\gamma \in W^{1,r}(\Omega),\ r > n$ and Re $\gamma \geq c > 0$.*

(i) *$\gamma|_{\partial\Omega\cap U}$ can be recovered from Λ_γ by*

$$\langle h, \gamma f\rangle = \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, \mathcal{R}\Lambda_\gamma e^{-i\langle\cdot,\eta\rangle} f\rangle, \tag{44}$$

with $f \in H^{1/2}(\partial\Omega)\cap C(\partial\Omega)$ and $h \in L^2(\Omega)$ supported in $U\cap\partial\Omega$ and h_η is defined as zero outside $\partial\Omega\cap U$ and

$$h_\eta(x) = h(x)e^{-ix\cdot\eta} - \frac{1}{|\partial\Omega\cap U|}\int_{\partial\Omega\cap U} h(y)e^{-iy\cdot\eta}\, dy, \quad \text{for } x \in \partial\Omega\cap U.$$

(ii) *If $\gamma \in W^{2,r}$, $r > n/2$, then for any continuous function f, g in $H^{1/2}(\partial\Omega)$ with support in $\partial\Omega\cap\partial\Omega$ holds*

$$\langle g, \frac{\partial\gamma}{\partial\nu} f\rangle = \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle g, e^{-i\langle\cdot,\eta\rangle}\left(\gamma\Lambda_1 + \Lambda_1\gamma - 2\Lambda_\gamma\right)e^{i\langle\cdot,\eta\rangle} f\rangle. \tag{45}$$

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