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Michael Ruzhansky
Jens Wirth
Editors

Harmonic Analysis and Partial Differential Equations

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Harmonic Analysis and Partial Differential Equations

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Preface

Both, harmonic analysis and the analysis of partial differential equations are closely intertwined fields of mathematical research. The aim of this volume is to bring together a broad range of current research topics and emerging ideas, providing insights into novel approaches and highlighting some of the connections between seemingly different areas of pure and applied mathematics.

Most of the contributions in this volume are related to talks given and results presented at the 13th ISAAC Congress in Ghent in August 2021. The special session *Harmonic Analysis and PDEs* attracted 42 participants from 16 countries.

Ghent, Belgium | Michael Ruzhansky
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The Wave Resolvent for Compactly Supported Perturbations of Minkowski Space

Michał Wrochna and Ruben Zeitoun

Abstract In this note, we consider the wave operator $\Box_g$ in the case of globally hyperbolic, compactly supported perturbations of static spacetimes. We give an elementary proof of the essential self-adjointness of $\Box_g$ and of uniform microlocal estimates for the resolvent in this setting. This provides a model for studying Lorentzian spectral zeta functions which is particularly simple, yet sufficiently general for locally deriving Einstein equations from a spectral Lagrangian.

1 Introduction

1.1 Motivation

Let $P = \Box_g$ be the wave operator on a Lorentzian manifold (M, g). It was shown by Vasy [26] that if (M, g) is a *non-trapping Lorentzian scattering space* then $\Box_g$ is essentially self-adjoint in the sense of the canonical $L^2(M, g)$ space. This result was then generalized by Nakamura–Taira [16–18] to *long-range perturbations of Minkowski space*, higher order operators and *asymptotically static spacetimes* with compact Cauchy surface. In consequence, in each of these settings one can define complex powers $(\Box_g - i\varepsilon)^{-\alpha}$ by functional calculus for all $\varepsilon > 0$.

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In the first situation, it was shown in [2] that under the extra hypothesis that $n \geq 4$ is even and (M, g) is *globally hyperbolic*, the Schwartz kernel of $(\Box_g - i\varepsilon)^{-\alpha}$ has for $\operatorname{Re}\alpha > \frac{n}{2}$ a well-defined on-diagonal restriction $(\Box_g - i\varepsilon)^{-\alpha}(x, x)$, which extends to a meromorphic function of $\alpha \in \mathbb{C}$ (called the *Lorentzian spectral zeta function density*). Furthermore, the residues can be expressed in terms of the metric g, in particular:

$$\lim_{\varepsilon \to 0^+} \operatorname*{res}_{\alpha = \frac{n}{2} - 1} \left(\Box_g - i\varepsilon\right)^{-\alpha}(x, x) = \frac{R_g(x)}{i6(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2} - 1\right)}, \tag{1}$$

where $R_g(x)$ is the scalar curvature at $x \in M$. Since the variational principle $\delta_g R_g = 0$ is equivalent to vacuum Einstein equations and the l.h.s. refers to spectral theory, this gives a *spectral action* (or strictly speaking, Lagrangian) for gravity.

The proofs of essential self-adjointness and formula (1) rely on *microlocal radial estimates* [8, 15, 24–26], which are nowadays broadly used in hyperbolic problems. The non-expert reader might however not be familiar with the required formalism, nor with the various technical issues that arise from the combination of microlocal and global aspects (even the definition of non-trapping Lorentzian scattering spaces requires some familiarity).

In this note, our objective is to present a much simpler model in which it is possible to give more elementary proofs. This is motivated first of all by pedagogical reasons, but also by the need of having a toy model for testing various ideas that go beyond formula (1).

The easiest case is without doubt the class of *ultra-static* spacetimes (M, g) (Minkowski space being the primary example). In this situation, the wave operator $\Box_g$ is of the form $\partial_t^2 - \Delta_h$ for some t-independent Riemannian metric h. Essential self-adjointness is then almost immediate (provided that Δ_h is essentially self-adjoint), and it can also be easily proved for more general *static* metrics (see Dereziński–Siemssen [5]) in which case there are extra multiplication operators in the expression for $\Box_g$. The proof of (1) simplifies as well, at least for ultra-static metrics [2]. However, this type of assumptions is in practice too restrictive because it narrows down the allowed metric variations to time-independent ones.

This leads us to consider *compactly supported perturbations* of static spacetimes. Such perturbations are indeed sufficient for formulating a variational principle and for the purpose of illustrating propagation phenomena arising in greater generality. On the other hand, the assumption that the perturbation has compact support allows us to largely bypass the asymptotic analysis, and we can give proofs based almost exclusively on variants of Hörmander's classical propagation of singularities theorem.

1.2 Main Result and Sketch of Proof

More precisely, let (Y, h) be a Riemannian metric of dimension $n-1$ (where $n \geqslant 2$), and let (M, g_0) be $M = \mathbb{R} \times Y$ equipped with a Lorentzian metric of the form

$$g_0 = \beta\, dt^2 - h = \beta^2(y)dt^2 - h_{ij}(y)dy^i dy^j,$$

for some positive $\beta \in C^\infty(Y)$. A metric of this form is called *static*, or more precisely, *standard static* (see e.g. [21] for more remarks on the terminology). In the special case $\beta = 1$ the metric is said to be *ultra-static*; the latter is the natural Lorentzian analogue of a Riemannian product-type metric.

Let g be another smooth Lorentzian metric on M. We make the following assumptions.

Hypothesis 1 We assume that:

1. the Riemannian manifold (Y, h) is complete;
2. g is a compactly supported perturbation of g_0, i.e.

$$\text{supp}(g - g_0) \text{ is compact;}$$

3. there exists a constant $C > 0$ such that $C < \beta(y) < C^{-1}$ for all $y \in Y$;
4. (M, g_0) and (M, g) are globally hyperbolic spacetimes.

We recall that a Lorentzian manifold (M, g) is a *globally hyperbolic spacetime* if it is time oriented and there exists a Cauchy surface, i.e. a closed subset of M intersected exactly once by each maximally extended time-like curve. We remark that when (M, g_0) is (for instance) Minkowski space, then global hyperbolicity of the perturbed spacetime (M, g) is equivalent to a non-trapping condition, see [10, Prop. 4.3].

Let $\Box_g$ be the wave operator, or d'Alembertian on (M, g), i.e. the Laplace–Beltrami operator for the Lorentzian metric g. More explicitly, denoting $|g| = |\det g|$ for brevity, we have

$$\Box_g = |g(x)|^{-\frac{1}{2}}\, \partial_{x^j}\, |g(x)|^{\frac{1}{2}}\, g^{jk}(x)\partial_{x^k}$$

where we sum over repeated indices. In this setting, we prove the following result.

Theorem 1 *Assume Hypothesis 1. Then the wave operator $\Box_g$ is essentially self-adjoint on $C_c^\infty(M)$ in $L^2(M, g)$.*

Furthermore, we show *uniform microlocal resolvent estimates* for the wave operator $\Box_g$ (strictly speaking, its closure). In [2] they are a key ingredient in the analysis of complex powers of $\Box_g$. We give here an analogue in our setting.

Theorem 2 *Assume Hypothesis 1. Then the wave resolvent $(\Box_g - z)^{-1}$ has Feynman wavefront set. More precisely, let $s \in \mathbb{R}$, $\varepsilon > 0$ and $\theta \in\,]0, \pi/2[$. Then* *for* $|\arg z - \pi/2| < \theta$, $|z| \geq \varepsilon$, *the uniform operator wavefront set of* $(\Box_g - z)^{-1}$ *of order* s *and weight* $\langle z \rangle^{-\frac{1}{2}}$ *(see Definition 1) satisfies*

$$\mathrm{WF}'^{(s)}_{\langle z \rangle^{-\frac{1}{2}}}\big((\Box_g - z)^{-1}\big) \subset \Lambda,$$

where Λ *is the (primed) Feynman wavefront set (see Definition 2).*

This type of estimates is used in [2] to show that the resolvent and complex powers of $\Box_g$ are sufficiently well approximated by a *Hadamard parametrix*, which in turn can be used to extract the scalar curvature R_g (see [4] for a brief review). That subsequent analysis is completely general, and so by combining Theorem 2 with [2, §§4–8] we obtain the following result (see also [3] for further consequences).

Corollary 1 *Assume Hypothesis 1. Then the identity* (1) *holds true in even dimension* $n = \dim M \geqslant 4$.

We remark that while our assumptions are certainly restrictive, our results are not exclusively special cases of [2, 16–18, 26] because we allow for more general behaviour in the spatial directions. Together with the recent work [18], this provides further evidence for Dereziński's conjecture [6] that essential self-adjointness may hold true on a large class of *asymptotically static* spacetimes (with possibly general behaviour in the spatial directions). We conjecture that the statement of Theorem 2 would remain valid as well.

1.3 Structure of Paper

Essential self-adjointness, i.e. Theorem 1, is proved in Sect. 2, preceded by various preliminaries on propagation of singularities. Theorem 2 is proved in Sect. 3; that section also contains the necessary background on operator wavefront sets.

2 Essential Self-Adjointness

2.1 Preliminaries on Self-Adjointness

Let us first consider the ultra-static case $\beta = 1$. Let

$$P_0 = \partial_t^2 - \Delta_h$$

be the unperturbed wave operator, i.e. the wave operator on the static spacetime (M, g_0). In that case there is an argument that gives its essential self-adjointness immediately.

Lemma 1 *P_0 is essentially self-adjoint on $C_c^\infty(M)$ in $L^2(M, g_0)$.*

Proof We quote the argument from [5] for the reader's convenience. We know that D_t^2 is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$, and Δ_h is essentially self-adjoint on $C_c^\infty(Y)$ in $L^2(Y, h)$ [1]. Therefore by Reed and Simon [20, § VIII.10], $P_0 = -D_t^2 \otimes \mathbf{1} - \mathbf{1} \otimes \Delta_h$ is essentially self-adjoint on the algebraic tensor product of $C_c^\infty(\mathbb{R})$ with $C_c^\infty(Y)$, which is dense in $C_c^\infty(M)$ in $L^2(M, dt^2 + h) = L^2(M, dt^2 - h) = L^2(M, g_0)$. □

Denoting also by P_0 the closure, the resolvent $(P_0 - z)^{-1}$ exists for $z \in \mathbb{C} \setminus \mathbb{R}$.

Let us denote by L_0 the closure of minus the Laplace–Beltrami operator $\partial_t^2 + \Delta_h$ on the complete Riemannian metric $dt^2 + h$. We use it to introduce a global Sobolev space of order $s \in \mathbb{R}$:

$$H^s(M) := (\mathbf{1} + L_0)^{-\frac{s}{2}} L^2(M, g_0),$$

i.e. the norm is given by $\|u\|_{H^s} = \left\| (\mathbf{1} + L_0)^{\frac{s}{2}} u \right\|_{L^2}$ in terms of the norm of $L^2(M, g_0)$. We will also frequently write $L^2(M)$ instead of $L^2(M, g_0)$ for the sake of brevity. Since P_0 commutes with L_0, for all $m \in \mathbb{R}$ we can extend the resolvent to an operator $(P_0 - z)^{-1} \in B(H^m(M), H^m(M))$ which satisfies $(P_0 - z)(P_0 - z)^{-1} = \mathbf{1}$ on $H^m(M)$. By a direct computation one can check the formula

$$\big((P_0 - z)^{-1} f\big)(t) = -\frac{1}{2} \int_{\mathbb{R}} \frac{e^{-i|t-s|\sqrt{-\Delta_h - z}}}{\sqrt{-\Delta_h - z}} f(s) ds, \tag{2}$$

for $\operatorname{Im} z > 0$ and $f \in L^2(M)$, where the r.h.s. is defined using Fourier transform and functional calculus.

Let us now focus on the wave operator $\Box_g$ for the perturbed metric g. Let $U : L^2(M, g_0) \to L^2(M, g)$ be the multiplication operator by $|g|^{-\frac{1}{4}} |g_0|^{\frac{1}{4}}$, and let

$$P := U^* \Box_g U.$$

Then, $\operatorname{supp}(P - P_0)$ is compact, and since U is bounded and boundedly invertible, essential self-adjointness of $\Box_g$ in $L^2(M, g)$ is equivalent to essential self-adjointness of P in $L^2(M, g_0)$.

Recall that the standard criterion for essential self-adjointness says that it suffices to show the implication

$$\forall u \in L^2(M) \text{ s.t. } (P \pm i)u = 0, \; u = 0, \tag{3}$$

where $(P \pm i)u = 0$ is meant in the sense of distributions. While the two conditions with different signs are needed, they are largely analogous so we will only consider the '−' case.

The basic argument consists in writing for all $u \in L^2(M)$ such that $(P-i)u = 0$,

$$2i \, \|u\|^2_{L^2} = (Pu|u)_{L^2} - (u|Pu)_{L^2}.$$

If $u \in H^2(M)$, by integration by parts the latter expression vanishes, and we conclude in that case $u = 0$. For this reason it suffices to prove

$$\forall u \in L^2(M) \text{ s.t. } (P \pm i)u = 0, \; u \in H^2(M). \tag{4}$$

As shown by Nakamura–Taira [16], in the case of compactly supported perturbations, global aspects can be dealt with relatively easily.

We denote by $\Psi^m(M)$ the set of pseudo-differential operators of order $m \in \mathbb{R}$ on M (in the sense of the general pseudo-differential calculus on manifolds, see e.g. [22, §4.3]).

Proposition 1 *Assume* $\beta = 1$. *Let* $k \in \mathbb{N}_{\geq 0}$ *and suppose* $u \in L^2(M) \cap H^{k+1}_{\mathrm{loc}}(M)$ *satisfies* $(P - i)u = 0$. *Then* $u \in H^k(M)$.

Proof The proof of [16, Prop. C.1] applies verbatim to our case; we repeat it for the reader's convenience. Set $N_\varepsilon = (\mathbf{1} + L_0)^{\frac{1}{2}}(\mathbf{1} + \varepsilon L_0)^{-\frac{1}{2}}$, $\varepsilon \geq 0$. For $\varepsilon > 0$, $N_\varepsilon \in \Psi^0(M) \cap B(L^2(M))$, hence $N_\varepsilon^{2k} u \in L^2(M) \cap H^{k+1}_{\mathrm{loc}}(M)$. Let $\psi \in C^\infty(M)$ be such that $\psi = 0$ in a neighborhood of $\mathrm{supp}(P - P_0)$ and $\psi = 1$ on the complement of some compact set.

Then,

$$P_0(\psi u) = P(\psi u) = \psi P u + [P, \psi]u = -i\psi u + Bu, \tag{5}$$

where $B := [P, \psi]$ is of order 1 and has compactly supported coefficients. The latter implies $Bu \in H^k(M)$, so by (5) we get $P_0(\psi u) \in L^2(M)$. We can now compute

$$\begin{aligned} 2i \,\mathrm{Im}(N_\varepsilon^{2k}(\psi u)|P_0(\psi u))_{L^2} &= 2i \,\mathrm{Im}(N_\varepsilon^{2k}(\psi u)| - i\psi u + Bu)_{L^2} \\ &= 2\|N_\varepsilon^k(\psi u)\|^2_{L^2} + 2i \,\mathrm{Im}(N_\varepsilon^{2k}(\psi u)|Bu)_{L^2}. \end{aligned} \tag{6}$$

On the other hand, $[N_\varepsilon, P_0] = 0$, N_ε is bounded and $P_0(\psi u) \in L^2(M) \cap H^k_{\mathrm{loc}}(M)$, so $P_0(N_\varepsilon^{2k}(\psi u)) = N_\varepsilon^{2k}(P_0(\psi u)) \in L^2(M) \cap H^k_{\mathrm{loc}}(M)$. In consequence,

$$2i \,\mathrm{Im}(N_\varepsilon^{2k}(\psi u)|P_0(\psi u))_{L^2} = (N_\varepsilon^{2k}(\psi u)|P_0(\psi u))_{L^2} - (P_0(\psi u)|N_\varepsilon^{2k}(\psi u))_{L^2} = 0. \tag{7}$$

Thus, we have

$$\|N_\varepsilon^k(\psi u)\|^2_{L^2} = \left|\mathrm{Im}(N_\varepsilon^{2k}(\psi u), Bu)_{L^2}\right| \leqslant \|N_\varepsilon^k(\psi u)\|_{L^2} \|N_\varepsilon^k Bu\|_{L^2}, \tag{8}$$

hence $\|N_\varepsilon^k(\psi u)\|_{L^2} \leqslant \|N_\varepsilon^k Bu\|_{L^2}$. Since $L_0 \geq 0$, $N_\varepsilon \leqslant N_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Moreover, $N_0^k Bu \in L^2(M)$ since $Bu \in H^k(M)$. Therefore, by monotone convergence, as $\varepsilon \to 0^+$ we get $\|N_0^k(\psi u)\|_{L^2} \leqslant \|N_0^k Bu\|_{L^2} < +\infty$. Since $N_0 = \langle L_0 \rangle$, this implies $\psi u \in H^k(M)$ as claimed. □

2.2 Preliminaries on Microlocal Analysis

In view of Proposition 1 we are left with the task of proving sufficient local regularity of L^2 solutions of $(P - i)u = 0$. To that end we will need several basic notions from microlocal analysis.

We will write $(x; \xi) = (t, y; \tau, \eta)$ for points in T^*M and o for the zero section. Let $p(x; \xi)$ be the principal symbol of P, and let $\Sigma = p^{-1}(\{0\})$ be its characteristic set. It splits into two connected components, $\Sigma = \Sigma^+ \cup \Sigma^-$, where the sign convention is fixed by saying that in the special case when $p(x; \xi) = p_0(x; \xi) = -\tau^2 + \eta^2$, $\Sigma^\pm$ equals

$$\Sigma_0^\pm = \{(t, y; \tau, \eta) \in T^*M \setminus o \mid \tau = \pm|\eta|\}.$$

Let us recall that *bicharacteristics* are integral curves of the *Hamilton vector field* H_p of p, defined in terms of the Poisson bracket by $H_p = \{p, \cdot\}$. For a pair of points $(x_i; \xi_i) \in T^*M \setminus o$, $i = 1, 2$, we write $(x_1; \xi_1) \sim (x_2; \xi_2)$ if $(x_1; \xi_1) \in \Sigma$ and $(x_2; \xi_2)$ can be joined from $(x_1; \xi_1)$ by a bicharacteristic in Σ.

Recall that given $u \in \mathcal{D}'(M)$, its *Sobolev wavefront set* $\mathrm{WF}^{(s)}(u)$ of order $s \in \mathbb{R}$ is defined as follows: $(x; \xi) \in T^*M \setminus o$ is *not* in $\mathrm{WF}^{(s)}(u)$ if and only if there exists a properly supported $B \in \Psi^0(M)$ (or equivalently, $B \in \Psi^m(M)$ for some $m \in \mathbb{R}$) such that $Bu \in H^s_{\mathrm{loc}}(M)$ (resp. $Bu \in H^{s-m}_{\mathrm{loc}}(M)$).

Let us recall a special case of Hörmander's classical propagation of singularities theorem for real principal type operators ($P - z$ is of real principal type by global hyperbolicity of (M, g), see e.g. [19, Prop. 4.3]), formulated here in terms of the Sobolev wavefront set.

Proposition 2 (Duistermaat and Hörmander [7, §6.3]) *Let $z \in \mathbb{C}$ and suppose $u \in \mathcal{D}'(M)$ satisfies $f := (P - z)u \in H^{s-1}_{\mathrm{loc}}(M)$. If $(x; \xi) \in \mathrm{WF}^{(s)}(u)$, then $(x; \xi) \in \Sigma$, and furthermore $(x'; \xi') \in \mathrm{WF}^{(s)}(u)$ for all $(x'; \xi') \in T^*M \setminus o$ such that $(x; \xi) \sim (x'; \xi')$.*

Strictly speaking, the basic statement that $(x; \xi) \in \Sigma$ is referred to as *microlocal elliptic regularity* or the *elliptic estimate*, as it can indeed be written in the form of a uniform estimate.

2.3 Proof of Local Regularity

Let $V = P - P_0$. By hypothesis, V is a second order differential operator with compactly supported coefficients. Let $T > 0$ be large enough so that $\operatorname{supp} V \subset [-T, T] \times Y$.

We start by showing a key lemma about microlocal regularity for large times. Although in the proof of essential self-adjointness we will only need a particular case with fixed z and $f = 0$, the general statement will be useful in the next section. For further reference the lemma is stated for general P obtained with compactly supported perturbations of P_0.

Lemma 2 *Let P be a second order differential operator such that $V = P - P_0$ has compactly supported coefficients. Assume $\beta = 1$. Let $(x_1; \xi_1) = (t_1, y_1; \tau_1, \eta_1) \in \Sigma^{\pm}$ be such that $\pm t_1 > T$. Then for $\operatorname{Im} z \geqslant \varepsilon > 0$, there exists a bounded family of properly supported pseudo-differential operators $B_{\pm}(z) \in \Psi^0(M)$, each elliptic at $(x_1; \xi_1)$ and such that for all $u \in L^2(M)$ satisfying $f := (P - z)u \in L^2_{\mathrm{c}}(M)$,*

$$B_{\pm}(z)(u - (P_0 - z)^{-1} f) = 0. \tag{9}$$

If in addition $\operatorname{supp} f \subset [-T_-, T_+] \times Y$ *for some $T_+, T_- > 0$ and $\pm t_1 > \pm T_{\pm}$ then*

$$B_{\pm}(z)u = 0. \tag{10}$$

Proof For all $u \in L^2(M)$, if $f = (P - z)u \in L^2_{\mathrm{c}}(M)$ then $(P_0 - z)u = f - Vu$ as elements of $H^{-2}_{\mathrm{loc}}(M)$, and

$$u - (P_0 - z)^{-1} f = -(P_0 - z)^{-1} V u. \tag{11}$$

Let $A(z) = (-\Delta_h - z)^{1/2}$. Then $A(z) \in \Psi^1(M)$, and its principal symbol is $|\eta|_h^{\frac{1}{2}}$ (cf. the last paragraph in the proof of Lemma 3). Setting $v = (\mathbf{1} \otimes A(z)^{-1}) V u$ and using the formula (2) for $(P_0 - z)^{-1}$, extended to elements $H^{-2}_{\mathrm{loc}}(M)$ supported in a finite time interval, we obtain

$$\big(u - (P_0 - z)^{-1} f\big)(t) = \frac{1}{2} \int_{\mathbb{R}} e^{-i|t-s|A(z)} v(s) ds.$$

Since $\operatorname{supp} v \subset [-T, T] \times Y$, this implies that

$$\big(u - (P_0 - z)^{-1} f\big)(t) = \frac{1}{2} e^{\mp i t A(z)} \int_{\mathbb{R}} e^{\pm i s A(z)} v(s) ds \ \text{ for } \pm t > T. \tag{12}$$

In consequence,

$$(D_t \pm A(z))\big(u - (P_0 - z)^{-1} f\big)(t) = 0 \ \text{ for } \pm t > T.$$

If in addition $\operatorname{supp} f \subset [T_-, T_+] \times Y$, then we can represent $(P_0 - z)^{-1} f$ similarly as the r.h.s. of (12). Hence $(D_t \pm A(z))(P_0 - z)^{-1} f = 0$ for $\pm t > \pm T_\pm$ and we conclude

$$(D_t \pm A(z))u(t) = 0 \ \text{ for } \ \pm t > \max\{T, \pm T_\pm\}. \tag{13}$$

Now, let $(x_1; \xi_1) = (t_1, y_1; \tau_1, \eta_1) \in \Sigma^\pm$ be such that $\pm t_1 > T$. Although $D_t \pm A(z) = (D_t \otimes \mathbf{1}) \pm (\mathbf{1} \otimes A(z))$ is not a pseudo-differential operator in $\Psi^1(M)$ (instead, it is in some larger class with rather bad properties), there exists $B_0 \in \Psi^0(M)$ properly supported such that

$$B_{\pm,1}(z) := B_0(D_t \pm A(z)) \in \Psi^1(M),$$

and such that $B_{\pm,1}(z)$ is elliptic at $(x_1; \xi_1)$. In fact, since $(t_1, y_1; \tau_1, \eta_1) \in \Sigma$, we have $\eta_1 \neq 0$ (as $\eta_1 = 0$ would imply $\tau_1 = 0$), so we can choose $B_0 = 0$ microlocally in a conic neighborhood of $\{\eta_1 = 0\}$ and $B_0 = 1$ in a punctured neighborhood of it, see [13, Thm. 18.1.35], cf. the proof of [9, (3), Prop. 6.8]. Finally, by composing $B_{\pm,1}(z)$ with a suitable family $C(z) \in \Psi^{-1}(M)$, vanishing for $\pm t \leqslant T$ (resp. for $\pm t \leqslant \max\{T, \pm T_\pm\}$), we obtain $B_\pm(z) := C(z)B_{\pm,1}(z)$ with the desired uniformity in $\Psi^0(M)$ and satisfying (9) (resp. (10)). □

Remark 1 Lemma 2 is a microlocal regularity statement at large, but *finite* times, and then our next step will be to deduce a corresponding statement for arbitrary times by Hörmander's propagation of singularity theorem. In more general situations, one needs to start with a regularity statement at *infinite* times, which motivates the use of radial propagation estimates or related methods [16–18, 26]. In these settings, the asymptotic analogues of the two conditions (13) can be thought as boundary conditions at infinity [10]: these were shown by Taira to be satisfied in the case of the wave resolvent on asymptotically Minkowski spacetimes [23].

Proposition 3 *Assume $\beta = 1$, and suppose $u \in L^2(M)$ satisfies $(P - i)u = 0$. Then $u \in C^\infty(M)$.*

Proof For any $(x; \xi) \in \Sigma^\pm \cap \{\pm t > T\}$ we use Lemma 2 with $z = i$ and $f = 0$, which gives existence of $B_\pm \in \Psi^0(M)$ elliptic at $(x; \xi)$ such that $B_\pm u = 0$. Thus, $(x; \xi) \notin \mathrm{WF}^{(s)}(u)$ for all $s \in \mathbb{R}$. We conclude

$$\mathrm{WF}^{(s)}(u) \cap \Sigma^\pm \cap \{\pm t > T\} = \emptyset.$$

By propagation of singularities, this implies $\mathrm{WF}^{(s)}(u) \cap \Sigma^\pm = \emptyset$. Since $\mathrm{WF}^{(s)}(u) \subset \Sigma = \Sigma^+ \cup \Sigma^-$ we deduce immediately $\mathrm{WF}^{(s)}(u) = \emptyset$ for all $s \in \mathbb{R}$, hence $u \in C^\infty(M)$. □

Proposition 3 combined with Proposition 1 implies (4). This concludes the proof of essential self-adjointness of P, hence the self-adjointness of $\Box_g$ stated in Theorem 1 in the case $\beta = 1$.

2.4 Generalization to Static Spacetimes

Let us now discuss the adaptations needed to prove the essential self-adjointness in the case when the spacetime is not necessarily ultra-static, i.e. when β is not necessarily 1.

The unperturbed wave operator is then

$$P_0 = \beta^{-1}\partial_t^2 - \Delta_h.$$

Thanks to the assumption $C < \beta < C^{-1}$, the multiplication operator β is bounded with bounded inverse. Let

$$\tilde{P}_0 = \beta^{\frac{1}{2}} P_0 \beta^{\frac{1}{2}}, \quad \tilde{P} = \beta^{\frac{1}{2}} P \beta^{\frac{1}{2}}, \quad \tilde{\Delta}_h = \beta^{\frac{1}{2}} \Delta_h \beta^{\frac{1}{2}}.$$

Then, as observed in [5], essential self-adjointness of P is equivalent to essential self-adjointness of $\tilde{P}$. Furthermore,

$$\tilde{P}_0 = \partial_t^2 - \tilde{\Delta}_h$$

with $\tilde{\Delta}_h$ essentially self-adjoint, and the coefficients of $\tilde{P} - \tilde{P}_0$ are compactly supported. Therefore, we can repeat the arguments from Sects. 2.1–2.3 to show the essential self-adjointness of $\tilde{P}_0$ and $\tilde{P}$, and hence of P.

This concludes the proof of Theorem 1.

In the next section we will be interested in the resolvent $(P - z)^{-1}$, which is not related in a straightforward way with the resolvent of $(\tilde{P} - z)^{-1}$. For this reason we will need a more direct approach. The key fact is that Lemma 2 remains valid for P with $\beta \neq 1$, as shown below.

Lemma 3 *The assertion of Lemma 2 holds true for P_0 and P without the assumption $\beta = 1$.*

Proof Let $\operatorname{Im} z \geq \varepsilon > 0$. In comparison with the case $\beta = 1$, the main difference is that the formula for the unperturbed resolvent $(P_0 - z)^{-1}$ needs to be modified. We have indeed

$$(P_0 - z)^{-1} = \beta^{\frac{1}{2}} (\tilde{P}_0 - z\beta)^{-1} \beta^{\frac{1}{2}}, \tag{14}$$

provided that we check that $\tilde{P}_0 - z\beta = \partial_t^2 - \tilde{\Delta}_h - z\beta$ is boundedly invertible.

Let us first define

$$L(z) := i(-\tilde{\Delta}_h - z\beta), \text{ with domain } \operatorname{Dom} L(z) := \operatorname{Dom}(-\tilde{\Delta}_h).$$

Since β is bounded, the operator $L(z)$ is closed. Furthermore,

$$\operatorname{Re}(u|L(z)u) = (\operatorname{Im} z)(u|\beta u) \geq \frac{1}{2} C^{-1} \varepsilon \|u\|_{L^2}, \quad u \in \operatorname{Dom} L(z), \tag{15}$$

so $L(z)$ is m-accretive and $0 \notin \mathrm{sp}(L(z))$. By [14, §3, Thm. 3.35], $L(z)$ has a unique m-accretive square root $L(z)^{\frac{1}{2}}$, which in addition is sectorial of angle $\frac{\pi}{4}$ and satisfies $0 \notin \mathrm{sp}(L(z)^{\frac{1}{2}})$. It follows that if we set

$$A(z) := e^{-i\pi/4} L(z)^{\frac{1}{2}}, \tag{16}$$

then $0 \notin \mathrm{sp}(A(z))$ and moreover, $iA(z)$ is m-accretive. In consequence, $-iA(z)$ is the generator of a strongly continuous contraction semigroup denoted by $\mathbb{R}_+ \ni t \mapsto e^{-itA(z)} \in B(L^2(M))$. Let now

$$(R(z)f)(t) = \int_{\mathbb{R}} e^{-i|t-s|A(z)} A(z)^{-1} f(s) ds$$

for $f \in L^2_{\mathrm{c}}(M)$. Since $(\mathbf{1} \otimes A(z)^{-1}) f \in L^2_{\mathrm{c}}(\mathbb{R}; \mathrm{Dom}\, A(z))$, standard semigroup theory applies, and we get easily $R(z)f \in C^0(\mathbb{R}; \mathrm{Dom}\, A(z))$, in particular $R(z)f$ is a distribution. In the sense of distributions,

$$\begin{aligned} (\tilde{P}_0 - z\beta) R(z) f &= (\partial_t^2 + A(z)^2) R(z) f \\ &= (\partial_t - iA(z))(\partial_t + iA(z)) R(z) f = f \end{aligned} \tag{17}$$

for all $f \in L^2_{\mathrm{c}}(M)$. On the other hand, by a computation analogous to (15) we obtain that the operator $i(\tilde{P}_0 - z\beta)$ with domain $\mathrm{Dom}\, \tilde{P}_0$ is m-accretive and boundedly invertible. By applying its inverse to both sides of (17) we obtain $(\tilde{P}_0 - z\beta)^{-1} = R(z)$ on $L^2_{\mathrm{c}}(M)$. We conclude that $(P_0 - z)^{-1} = \beta^{\frac{1}{2}} (\tilde{P}_0 - z\beta)^{-1} \beta^{\frac{1}{2}} = \beta^{\frac{1}{2}} R(z) \beta^{\frac{1}{2}}$ on $L^2_{\mathrm{c}}(M)$. In summary,

$$((P_0 - z)^{-1} f)(t) = \beta^{\frac{1}{2}} \int_{\mathbb{R}} e^{-i|t-s|A(z)} A(z)^{-1} \beta^{\frac{1}{2}} f(s) ds, \quad f \in L^2_{\mathrm{c}}(M).$$

From that point on we can repeat the proof of Lemma 2 with $(D_t \pm A(z))$ replaced by $(D_t \pm A(z))\beta^{-\frac{1}{2}}$, where $A(z)$ is defined in (16).

This requires us to check that $A(z) \in \Psi^1(M)$. In fact, we can show in analogy to the proof of [11, Prop. 4.7] that the resolvent $(L(z) - \lambda)^{-1}$ of $L(z)$ satisfies a variant of the Beals criterion in global Sobolev spaces defined using $-\tilde{\Delta}_h$. Then, for all $\chi_1, \chi_2 \in C^\infty_{\mathrm{c}}(M)$, $\chi_1 A(z) \chi_2$ can be expressed as an integral of $\chi_1 (L(z) - \lambda)^{-1}) \chi_2$ (see the proof of [14, §3, Thm. 3.35]). By repeating the arguments in the proof of [11, Thm. 4.8] (with all relevant formulas multiplied by χ_1 and χ_2) we conclude that $A(z) \in \Psi^1(M)$, and its principal symbol equals $\sigma_{\mathrm{pr}}\big(A(z)\big)(y; \eta) = |\eta|_h^{\frac{1}{2}}(y)$. □

3 Uniform Microlocal Estimates

3.1 Uniform Wavefront Set

Throughout this section we will write $P = \Box_g$ (rather than $P = U^*\Box_g U$).

We start by introducing the uniform wavefront set which appears in the formulation of Theorem 2.

Definition 1 Let $Z \subset \mathbb{C}$ and suppose $\{G(z)\}_{z\in Z}$ is for all $m \in \mathbb{R}$ a bounded family of operators in $B(H^m_{\rm c}(M), H^m_{\rm loc}(M))$. The *uniform operator wavefront set of order* $s \in \mathbb{R}$ *and weight* $\langle z\rangle^{-\frac{1}{2}}$ of $\{G(z)\}_{z\in Z}$ is the set

$$\mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big(G(z)\big) \subset (T^*M \setminus o) \times (T^*M \setminus o) \tag{18}$$

defined as follows: $((x_1;\xi_1),(x_2;\xi_2))$ is *not* in (18) if and only if for all $\varepsilon > 0$ there exists a uniformly bounded family $B_i(z) \in \Psi^0(M)$ of properly supported operators, each elliptic at $(x_i;\xi_i)$ and such that for all $r \in \mathbb{R}$, the family

$$\langle z\rangle^{\frac{1}{2}} B_1(z)G(z)B_2(z)^* \text{ for } z \in Z \text{ is bounded in } B(H^r_{\rm c}(M), H^{r+s}_{\rm loc}(M)).$$

We define the *uniform operator wavefront set of order* $s \in \mathbb{R}$ *and weight* 1 in the same way, with $\langle z\rangle^{\frac{1}{2}}$ replaced by 1, and we denote that set $\mathrm{WF}'^{(s)}\big(G(z)\big)$ for simplicity. Definition 1 is similar to the definition from [2, §3], with the only difference that we allow B_i to depend on z (which is easier to verify in practice).

Let us denote by Δ^* be the diagonal in $(T^*M \setminus o)^{\times 2}$, i.e.

$$\Delta^* = \{((x_1;\xi_1),(x_2;\xi_2)) \,|\, x_1 = x_2,\ \xi_1 = \xi_2\} \subset (T^*M \setminus o)^{\times 2}.$$

Definition 2 The *Feynman wavefront set* $\Lambda \subset (T^*M \setminus o)^{\times 2}$ is defined by

$$\begin{aligned}\Lambda := &\big((\Sigma^+)^{\times 2} \cap \{((x_1;\xi_1),(x_2;\xi_2)) \,|\, (x_1;\xi_1) \sim (x_2;\xi_2) \text{ and } x_1 \in J_-(x_2)\}\big)\\ &\cup \big((\Sigma^-)^{\times 2} \cap \{((x_1;\xi_1),(x_2;\xi_2)) \,|\, (x_1;\xi_1) \sim (x_2;\xi_2)\\ &\text{and } x_1 \in J_+(x_2)\}\big) \cup \Delta^*.\end{aligned}$$

In the definition we employed the convention which corresponds to considering *primed* wavefront sets (as opposed to wavefront sets of Schwartz kernels). We caution the reader that beside the choice of working with 'primed' or 'non-primed' wavefront sets, in the context of QFT there are two sign conventions possible.

As in [2] we will use the following version of Hörmander's propagation of singularities theorem, formulated in terms of the uniform wavefront set.

Proposition 4 *Let $Z \subset \{z \in \mathbb{C} \mid \operatorname{Im} z \geqslant 0\}$. Suppose that for all $m \in \mathbb{R}$, $G(z)$ and $(P - z)G(z)$ are bounded families of operators in $B(H^m_{\rm c}(M), H^m_{\rm loc}(M))$ for $z \in Z$. Suppose*

$$((x_1;\xi_1),(x_2;\xi_2)) \in \mathrm{WF}'^{(s)}\big(G(z)\big) \setminus \mathrm{WF}'^{(s-1)}\big((P-z)G(z)\big). \tag{19}$$

Then $(x_1;\xi_1) \in \Sigma$. Furthermore, $((x_1';\xi_1'),(x_2;\xi_2)) \in \mathrm{WF}'^{(s)}\big(G(z)\big)$ for all $(x_1';\xi_1')$ such that $(x_1';\xi_1') \sim (x_1;\xi_1)$ and $(x_1';\xi_1')$ precedes $(x_1;\xi_1)$ along the bicharacteristic flow, provided that $((x;\xi),(x_2;\xi_2)) \notin \mathrm{WF}'^{(s-1)}\big((P-z)G(z)\big)$ for all $(x;\xi)$ on the bicharacteristic connecting $(x_1;\xi_1)$ and $(x_1';\xi_1')$.

Proof We explain the relationship to better known formulations for the sake of completeness, see [2] for more details. In what follows, all pseudo-differential operators are assumed compactly supported.

The proof of propagation of singularities by positive commutator arguments [12] gives a uniform estimate of the following form. Let $s \in \mathbb{R}$, $N \ll 0$. For any $B_1' \in \Psi^0(M)$ elliptic at $(x_1;\xi_1)$, and any $B \in \Psi^0(M)$ elliptic in a neighborhood of the bicharacteristic from $(x_1';\xi_1')$ to $(x;\xi)$, we have

$$\|B_1 u\|_s \leqslant C(\|B_1' u\|_s + \|B(P-z)u\|_{s-1} + \|\chi u\|_N) \tag{20}$$

uniformly for $u \in H^{-N}_{\rm loc}(M)$ and $z \in Z$, where $B_1 \in \Psi^0(M)$ is some ΨDO elliptic at $(x_1;\xi_1)$ and $\chi \in C^\infty_{\rm c}(M)$. Now, suppose $((x_1';\xi_1'),(x_2;\xi_2)) \notin \mathrm{WF}'^{(s)}(G(z))$. Then there exist $B_1'(z), B_2(z) \in \Psi^0(M)$ elliptic at respectively $(x_1';\xi_1')$, $(x_2;\xi_2)$ such that for any bounded subset $\mathcal{U} \subset H^l_{\rm c}(M)$, the set $B_1'(z)G(z)B_2^*(z)\mathcal{U}$ is uniformly bounded in $H^{l+s}_{\rm loc}(M)$. By (20) applied to elements of $G(z)B_2^*\mathcal{U}$, $B_1 G(z) B_2^*(z)\mathcal{U}$ is bounded in $H^{l+s}_{\rm loc}(M)$, hence $((x_1;\xi_1),(x_2;\xi_2)) \notin \mathrm{WF}'^{(s)}(G_z)$. □

Note that $\mathrm{WF}'^{(s)}(\mathbf{1}) = \Delta^*$ for large $s \in \mathbb{R}$. Thus, if $(P-z)G(z) = \mathbf{1}$, then Proposition 4 says that we can propagate singularities (or equivalently, regularity) of $G(z)$ along bicharacteristics in the first factor as long as they do not hit Δ^*.

There is an analogous statement for propagation in the second factor of $(T^*M \setminus o)^{\times 2}$ if $G(z)(P-z)$ is bounded in $B(H^m_{\rm c}(M), H^m_{\rm loc}(M))$. Namely, if

$$((x_1;\xi_1),(x_2;\xi_2)) \in \mathrm{WF}'^{(s)}\big(G(z)\big) \setminus \mathrm{WF}'^{(s-1)}\big(G(z)(P-z)\big), \tag{21}$$

then $(x_2;\xi_2) \in \Sigma$. Furthermore, $((x_1;\xi_1),(x_2';\xi_2')) \in \mathrm{WF}'^{(s)}(G(z))$ for all $(x_2';\xi_2')$ such that $(x_2';\xi_2') \sim (x_2;\xi_2)$, provided that $((x_1;\xi_1),(x;\xi)) \notin \mathrm{WF}'^{(s-1)}\big(G(z)(P-z)\big)$ for all $(x;\xi)$ on the bicharacteristic connecting $(x_2;\xi_2)$ and $(x_2';\xi_2')$.

For $\varepsilon > 0$, let $Z_\varepsilon \subset \mathbb{C}$ be a "punctured sector" in the upper half-plane of the form

$$Z_\varepsilon := \{z \in \mathbb{C} \mid \ |\arg z - \pi/2| < \theta,\ |z| \geq \varepsilon\} \tag{22}$$

for some arbitrarily chosen $\theta \in \,]0, \pi/2[$.

Proposition 5 *If* $Z = Z_\varepsilon$ *with* $\varepsilon > 0$ *then in Proposition 4 we can replace* $((x_1'; \xi_1'), (x_2; \xi_2)) \in \mathrm{WF}'^{(s)}\big(G(z)\big)$ *by* $((x_1'; \xi_1'), (x_2; \xi_2)) \in \mathrm{WF}'^{(s-1/2)}_{\langle z\rangle^{-\frac{1}{2}}}\big(G(z)\big)$

Proof The positive commutator argument used to prove (20) gives actually the stronger estimate

$$\|B_1 u\|_s + (\mathrm{Im}\, z)^{\frac{1}{2}} \|B_1 u\|_{s-\frac{1}{2}} \leqslant C(\|B_1' u\|_s + \|B(P-z)u\|_{s-1} + \|\chi u\|_N), \tag{23}$$

see [2] for more details. Furthermore,

$$\|B_1 u\|_{s-\frac{1}{2}} \leqslant C_1 \langle z\rangle^{-\frac{1}{2}} (\|B_1 u\|_s + (\mathrm{Im}\, z)^{\frac{1}{2}} \|B_1 u\|_{s-\frac{1}{2}})$$

for some $C_1 > 0$ uniformly in $z \in Z_\varepsilon$. Hence,

$$\|B_1 u\|_{s-\frac{1}{2}} \leqslant C_2 \langle z\rangle^{-\frac{1}{2}} (\|B_1' u\|_s + \|B(P-z)u\|_{s-1} + \|\chi u\|_N),$$

and from that point on we can apply the argument recalled after (20). □

3.2 *Uniform Resolvent Estimate*

We first prove a basic estimate on regularity properties of $(P-z)^{-1}$, which later on enables us to use the operator formulation of propagation of singularities.

Lemma 4 *For all* $m \geqslant 0$*, the family of operators* $(P-z)^{-1}$, $\mathrm{Im}\, z > 0$*, is bounded in* $B(H^m_{\mathrm{c}}(M), H^{m+1}_{\mathrm{loc}}(M))$.

Proof This can be shown in a similar vein as Proposition 3. Namely, let $f \in H^m_{\mathrm{c}}(M)$. By Lemma 2, for every $(x_1; \xi_1) = (t_1, y_1; \tau_1, \eta_1) \in \Sigma^\pm$ with $\pm t_1$ sufficiently large there exists a bounded family $B_\pm(z) \in \Psi^0(M)$ such that $B_\pm(z)$ is elliptic at $(x_1; \xi_1)$ and

$$B_\pm(z)(P-z)^{-1} f = 0, \tag{24}$$

hence $(x_1; \xi_1) \notin \mathrm{WF}^{(m+1)}((P-z)^{-1} f)$ by (24). Since $\mathrm{WF}^{(m)}(f) = \emptyset$, by propagation of singularities applied to $(P-z)^{-1} f$ we get $(x; \xi) \notin \mathrm{WF}^{(m+1)}((P-z)^{-1} f)$ for all $(x; \xi)$ such that $(x; \xi) \sim (x_1; \xi_1)$ and $\pm t \leqslant \pm t_1$. In conclusion, $\Sigma^\pm \cap \mathrm{WF}^{(m+1)}((P-z)^{-1} f) = \emptyset$. On the other hand $\mathrm{WF}^{(m+1)}((P-z)^{-1} f) \subset \Sigma = \Sigma^+ \cup \Sigma^-$ by elliptic regularity. Hence $\mathrm{WF}^{(m+1)}((P-z)^{-1} f) = \emptyset$, which yields $(P-z)^{-1} f \in H^{m+1}_{\mathrm{loc}}(M)$. By the uniformity of propagation estimates and of the elliptic estimate, $H^{m+1}_{\mathrm{loc}}(M)$-seminorms of $(P-z)^{-1} f$ are bounded by $H^m_{\mathrm{c}}(M)$-seminorms of f, uniformly in z. □

We are now ready to prove that the uniform operator wavefront set of $(P - z)^{-1}$ in Z_ε is contained in the Feynman wavefront Λ.

Proof of Theorem 2

Step 1. Let $(x_1; \xi_1) = (t_1, y_1; \tau_1, \eta_1) \in \Sigma^\pm$ with $\pm t_1 > T$ (where T is as in Sect. 2.3) and let $(x_2; \xi_2) = (t_2, y_2; \tau_2, \eta_2) \in T^*M \setminus o$ be such that $\pm t_1 > \pm t_2$. Then by Lemma 2, there exists a bounded family $B_\pm(z) \in \Psi^0(M)$ such that $B_\pm(z)$ is elliptic at $(x_1; \xi_1)$ and

$$B_\pm(z) \circ (P - z)^{-1} \circ \chi = 0$$

for some $\chi \in C_c^\infty(M)$ with $\chi(x_2) \neq 0$, provided that $\operatorname{supp}\chi$ is a sufficiently small neighborhood of x_2. This implies $((x_1; \xi_1), (x_2; \xi_2)) \notin \mathrm{WF}'^{(s)}\big((P - z)^{-1}\big)$ for all $s \in \mathbb{R}$. In conclusion,

$$\big(\Sigma^\pm \times (T^*M \setminus o)\big) \cap \{\pm t_1 > T,\ \pm t_1 > \pm t_2\} \cap \mathrm{WF}'^{(s)}\big((P - z)^{-1}\big) = \emptyset. \quad (25)$$

Step 2. Next, we use propagation of singularities to deduce

$$\big(\Sigma^\pm \times (T^*M \setminus o)\big) \cap \mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big((P - z)^{-1}\big) \subset \Lambda. \quad (26)$$

More precisely, let $(x; \xi) \in \Sigma^\pm$ and suppose $(x_2; \xi_2) \in T^*M \setminus o$ is such that

$$((x; \xi), (x_2; \xi_2)) \in \mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big((P - z)^{-1}\big) \setminus \Lambda. \quad (27)$$

Since $(x; \xi) \in \Sigma^\pm$ and $((x; \xi), (x_2; \xi_2)) \notin \Lambda$, we can find $(x_1; \xi_1) = (t_1, y_1; \tau_1, \eta_1) \in \Sigma^\pm$ with $\pm t_1 > \max\{T, \pm t_2\}$ such that $(x_1; \xi_1) \sim (x; \xi)$ and $(x_2; \xi_2)$ does not intersect the bicharacteristic connecting $(x_1; \xi_1)$ and $(x; \xi)$. By (25), $((x_1; \xi_1), (x_2; \xi_2)) \notin \mathrm{WF}'^{(s)}\big((P - z)^{-1}\big)$. By propagation of singularities in the form given in Proposition 4 this implies $((x; \xi), (x_2; \xi_2)) \notin \mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big((P - z)^{-1}\big)$, which contradicts (27). The argument is valid for any $(x; \xi) \in \Sigma^\pm$, so we conclude (26).

Step 3. By proceeding analogously in the second factor, we obtain

$$\big((T^*M \setminus o) \times \Sigma^\pm\big) \cap \mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big((P - z)^{-1}\big) \subset \Lambda. \quad (28)$$

In combination with the two versions of identity (26) this yields

$$(\Sigma \times \Sigma) \cap \mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big((P - z)^{-1}\big) \subset \Lambda. \quad (29)$$

On the other hand, by the elliptic regularity statement in Proposition 4 and its analogue in the second factor, we have

$$\mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big((P-z)^{-1}\big) \subset (\Sigma\times\Sigma)\cup\Delta^*.$$

Thus (29) implies $\mathrm{WF}'^{(s)}_{\langle z\rangle^{-\frac{1}{2}}}\big((P-z)^{-1}\big) \subset \Lambda\cup\Delta^* = \Lambda$, which concludes the proof. □

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Smoothing Effect and Strichartz Estimates for Some Time-Degenerate Schrödinger Equations

Serena Federico

Abstract In this paper we present recent results about the smoothing properties of some Schrödinger operators with time degeneracies. More specifically, we will show that time-weighted smoothing and Strichartz estimates hold true for the operators under consideration. Finally, by means of the aforementioned estimates, we will derive local well-posedness results for the suitable corresponding nonlinear initial value problem.

1 Introduction

In this paper we shall investigate the smoothing properties of some time-degenerate Schrödinger operators of the form

$$\mathcal{L}_{\alpha,c} := i\partial_t + t^\alpha \Delta + c(t,x) \cdot \nabla_x \tag{1}$$

and

$$\mathcal{L}_b := i\partial_t + b'(t)\Delta, \tag{2}$$

where $\alpha > 0$, $c(t,x) = (c_1(t,x), \dots, c_n(t,x))$ is such that, for all $j = 1, \dots, n$, $c_j(t,x)$ is a complex valued function satisfying suitable dacy assumptions, while $b \in C^1(\mathbb{R})$ and satisfies $b(0) = b'(0) = 0$. We will go through the analysis of two kind of smoothing properties characterizing the solutions of Schrödinger equations in the Euclidean setting, that is, those described by smoothing and Strichartz estimates. More specifically, we will prove that local weighted smoothing estimates are satisfied by $\mathcal{L}_{\alpha,c}$, while local weighted Strichartz estimates are satisfied by $\mathcal{L}_b$. Once these results will be at our disposal, we will consider suitable nonlinear initial

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value problems for $\mathcal{L}_{\alpha,c}$ and $\mathcal{L}_b$, and give the corresponding local well-posedness results in each case.

Considering what previously mentioned, it should be clear that the estimates object of this work constitute a crucial tool to attack nonlinear IVPs (initial value problems) for dispersive equations.

Smoothing estimates are used to show that the solution of a certain equation gains regularity (in terms of derivatives) with respect to the regularity of the initial datum (homogeneous smoothing estimate) and/or with respect to the regularity of the inhomogeneous term of the equation (inhomogeneous smoothing estimate). Therefore these estimates are the suitable ones to be used when dealing with nonlinear problems with derivative nonlinearities.

Strichartz estimates, instead, allow to obtain a gain of integrability of the solution of a certain equation with respect to the integrability property of the initial datum (homogeneous Strichartz estimate) and/or with respect to the integrability property of the inhomogeneous term of the equation (inhomogeneous Strichartz estimate). These are the estimates to be used to solve semilinear IVPs.

Results concerning smoothing and Strichartz estimates for constant coefficient Schrödinger equations, but also for general constant coefficient dispersive equations, are by now classical (see [3, 5, 6, 17–21, 31, 32]). As for the variable coefficients case where the Laplacian is replaced by a variable coefficient (elliptic) operator (the constant case with potentials is also well understood and widely studied) the situation is much different, and the results available are quite limited.

The smoothing effect of Schrödinger equations with nondegenerate space-variable coefficients was proved in [22] by Kenig et al., where the authors considered and solved the ultrahyperbolic case too. Important achievements in the study of smoothing estimates are due to Doi (see [8] and [7]), who considered the problem in the general manifold setting. As regards Strichartz estimates, Staffilani and Tataru proved in [30] the validity of such estimates for Schrödinger equations with nonsmooth coefficients (with compactly supported perturbations of the Laplacian), while in [26] Robbiano and Zuily obtained these estimates for Schrödinger equations with small perturbations of the Laplacian. Let us mention that several results have been proved for equations with potentials and in the manifold setting, and we refer the interested reader to [1, 2, 9, 10, 24, 27, 28] and references therein. As for uniqueness properties of Schrödinger operators with nondegenerate space variable coefficients, we refer to the recent work [15] and references therein.

Our analysis here, despite the aforementioned results, focuses on time-degenerate Schrödinger operators of the form (1) and (2). It is worth to mention that the class of operators (1) was first considered by Cicognani and Reissig in [4], who studied the linear problem and proved the local well-posedness of the linear IVP both in Sobolev and Gevrey spaces. The results about the local smoothing effect of the class (1), proved by the author and Staffilani in [12], will be presented below in a selfcontained way. Some results about the homogeneous smoothing effect of ***nondegenerate*** operators of the form (2) were proved by Sugimoto and Ruzhansky in [29]. As for Strichartz estimates and local well-posedness for the class $\mathcal{L}_b$ on the one and on the two-dimensional torus, and possibily generalizable to general compact Riemaniann manifolds, they were proved by the author and Staffilani in [13] (see also [14]), where some nondegenerate space-variable coefficient Schrödinger operators on the one and on the two-dimensional torus were also studied.

Concerning the Strichartz estimates for (2) treated in this paper, they were proved by the author and Ruzhansky in [11] (see also [14]), where some homogeneous smoothing results were also established by means of comparison principles.

Let us now conclude this introduction by giving the plan of the paper. In Sect. 2 we shall analyze the local smoothing effect of $\mathcal{L}_{\alpha,c}$ in two cases: when $c \equiv 0$ (in Sect. 2.1) and when c is not necessarily identically 0 (in Sect. 2.2). In each case we also give the local well-posedness result for the corresponding nonlinear IVP.

In Sect. 3 we focus on the class $\mathcal{L}_b$ and on the validity of local Strichartz estimates in this case. A local well-posedness result for a semilinear IVP for $\mathcal{L}_b$ will also be given.

Notations We use the notation $A \lesssim B$ to indicate that there exists an absolute constant $c > 0$ such that $A \leq cB$. We shall denote by Λ^s the pseudo-differential operator of order s whose symbol is given by $\Lambda^s(\xi) = \langle\xi\rangle^s = (1 + |\xi|^2)^{s/2}$.

The mixed norm space $L^p_x L^q_t(\mathbb{R}^n \times [0, T])$, $1 \leq p, q \leq \infty$, is the space of functions $f(t, x)$ that are L^q in time on the interval $[0, T]$ and are L^p in space. The norm is taken in the right to left order. In a similar manner we define the spaces $L^p([0, T]; H^s(\mathbb{R}^n))$, $1 \leq p \leq \infty$, of functions that are L^p in time and in the Sobolev space $H^s(\mathbb{R}^n)$ in space. Finally we shall denote by $S^m := S^m_{1,0}$ the class of classical symbols of order $m \in \mathbb{R}$ defined by

$$S^m := \{p(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n); |p|^{(j)}_{S^m} < \infty\},$$

where

$$|p|^{(j)}_{S^m} := \sup_{|\alpha+\beta|=j} \|\langle\xi\rangle^{-m+|\alpha|\partial^\alpha_\xi \partial^\beta_x p(x,\xi)}\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}.$$

Finally, by writing $g \not\equiv 0$ we will mean that a function $g = g(t, x)$ is not necessarily identically 0.

2 Smoothing Effect and Local Well-Posedness for the Class $\mathcal{L}_{\alpha,c}$

This section is devoted to the study of the class $\mathcal{L}_{\alpha,c}$ as in (1). Below we will discuss the cases $c \equiv 0$ and $c \not\equiv 0$ separately, in Sects. 2.1 and 2.2 respectively . This distinction is done in order to show that one can use standard techniques in the first case $c \equiv 0$, and that in the more general case $c \not\equiv 0$ the usual technique does not work anymore (the case $c \equiv 0$ is always contained in the case $c \not\equiv 0$ according to our notation). For the reader convenience we shall state our main results for the class under consideration at the beginning of each subsection. As explained in the introduction, these results will be about the local smoothing and about the local well-posedness of the nonlinear IVP.

2.1 The Class $\mathcal{L}_\alpha$

In the sequel we will use the notation $\mathcal{L}_\alpha := \mathcal{L}_{\alpha,0} := i\partial_t + t^\alpha \Delta_x$. The operator $W_\alpha(t,s)$ in the statements below is the operator defined as in (9) giving the solution at time t of the homogeneous IVP for $\mathcal{L}_\alpha$ with initial condition $u(s,x) = u_s(x)$ at time s. Our main results for $\mathcal{L}_\alpha$ are the following.

Theorem 1 *Let $W_\alpha(t) := W_\alpha(t,0)$, with $\alpha > 0$, then*
If $n = 1$ for all $f \in L^2(\mathbb{R})$,

$$\sup_x \|t^{\alpha/2} D_x^{1/2} W_\alpha(t) f\|^2_{L^2_t([0,T])} \lesssim \|f\|^2_{L^2(\mathbb{R})}. \tag{3}$$

If $n \geq 2$, on denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ the family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then for all $f \in L^2_x(\mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \int_0^T |t^{\alpha/2} D_x^{1/2} W_\alpha(t) f(x)|^2 dt\, dx \right)^{1/2} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \tag{4}$$

where $D_x^\gamma f(x) = (|\xi|^\gamma \widehat{f}(\xi))^\vee(x)$.

Theorem 2 *Let $n = 1$ and $g \in L^1_x L^2_t(\mathbb{R} \times [0,T])$, then*

$$\|D_x^{1/2} \int_{\mathbb{R}_+} t^{\alpha/2} W_\alpha(0,t) g(t) dt\|_{L^2_x(\mathbb{R})} \lesssim \|g\|_{L^1_x L^2_t(\mathbb{R}\times[0,T])}, \tag{5}$$

and, for all $g \in L^1_t L^2_x([0,T] \times \mathbb{R})$,

$$\|t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau\|_{L^\infty_x(\mathbb{R}) L^2_t([0,T])} \lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R})}. \tag{6}$$

If $n \geq 2$, on denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ a family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then, for all $g \in L^1_t L^2_x([0, T] \times \mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \left\| t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t, \tau) g(\tau) d\tau \right\|^2_{L^2_t([0,T])} dx \right)^{1/2} \lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^n)}. \tag{7}$$

Theorem 3 *Let $k \geq 1$, then the IVP*

$$\begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0, x) = u_0(x), \end{cases} \tag{8}$$

is locally well-posed in H^s for $s > n/2$ and its solution satisfies smoothing estimates.

Remark 1 Notice that Theorem 2 amounts to the validity of the homogeneous and inhomogeneous *weighted* smoothing estimates with a gain of $1/2$ derivative for $\mathcal{L}_\alpha$.

When $\alpha = 0$ one has actually an inhomogeneous smoothing effect better than the one described in (7), that is the inhomogeneous part of the solution gains 1 instead of $1/2$ derivative with respect to the inhomogeneious part of the equation (in other words, when $\alpha = 0$, one can replace $D_x^{1/2}$ by D_x^1 in (7), see [20]).

When $\alpha \neq 0$ the suitable corresponding weighted estimate still holds. This property is described in Theorem 4 part (iii) below for the general case $\mathcal{L}_{\alpha,c}$, with c being not necessarily identically 0, directly.

We stress that the proofs of the results of this subsection rely on the explicit knowledge of the solution of the inhomogeneous IVP for $\mathcal{L}_\alpha$. Indeed, by using classical Fourier analysis methods and Duhamel's principle (that still holds in this case, see [12]), we get that the solution of the IVP

$$\begin{cases} \mathcal{L}_\alpha u = f(t, x) \\ u(s, x) = u_s(x), \end{cases}$$

for $s < t$, is given by

$$u(t, x) = W_\alpha(t, s)u_s(x) + \int_s^t W_\alpha(t, t') f(t', x) dt',$$

where

$$W_\alpha(t, s)u_s(x) := e^{i\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}\Delta_x} u_s(x) := \int_{\mathbb{R}^n} e^{-i(\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}|\xi|^2 - x\cdot\xi)} \widehat{u_s}(\xi) d\xi \tag{9}$$

is the so called *solution operator*, that is the operator giving the solution of the homogeneous problem at time t with initial condition at time s. This is a two-parameter family of unitary operators satisfying:

1. $W_\alpha(t,t) = I$;
2. $W_\alpha(t,s) = W_\alpha(t,r)W_\alpha(r,s)$ for every $s, t, r \in [0, T]$;
3. $W_\alpha(t,s)\Delta_x u = \Delta_x W_\alpha(t,s)u$;
4. $\|W_\alpha(t,s)u_s\|_{H^s_x} = \|u_s\|_{H^s_x}$.

Let us remark that in the case $\alpha = 0$ the operator above coincides with the well known Schrödinger group.

Now we can finally give the proofs of Theorems 2 and 3.

Proof of Theorem 1 First note that (3) and (4) are true when $\alpha = 0$, that is, when $W_\alpha(t) = W_0(t) = e^{it\Delta_x}$ (see, for instance, [20]). Then it suffices to prove that

$$\|t^{\alpha/2}D_x^{1/2}W_\alpha(t)f\|^2_{L^2_t([0,T])} = \|D_x^{1/2}W_0(t)f\|^2_{L^2_t([0,T'])}.$$

To prove that the identity above is satisfied, we use the change of variables $t^{\alpha+1}/(\alpha+1) = s$, and get

$$\begin{aligned}
&\|t^{\alpha/2}D_x^{1/2}W_\alpha(t)f\|^2_{L^2_t([0,T])}\\
&= \int_0^T \left| t^{\alpha/2}\int_{\mathbb{R}^n} e^{-i\left(t^{\alpha+1}|\xi|^2/(\alpha+1)-x\cdot\xi\right)}|\xi|^{1/2}\widehat{f}(\xi)d\xi\right|^2 dt\\
&= \int_0^{\frac{T^{\alpha+1}}{\alpha+1}} \left|\int_{\mathbb{R}^n} e^{-i\left(s|\xi|^2-x\cdot\xi\right)}|\xi|^{1/2}\widehat{f}(\xi)d\xi\right|^2 ds\\
&= \|D_x^{1/2}W_0(t)f\|^2_{L^2_t([0,T^{\alpha+1}/(\alpha+1)])}.
\end{aligned}$$

Finally, by application of the smoothing estimates for $W_0(t) = e^{it\Delta_x}$, we conclude (3) and (4) (see [20], Theorem 2.1). □

Proof of Theorem 2 Inequality (5) follows directly from (3) by duality.

As for (6), on denoting by $L^p_x := L^p_x(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\|t^{\alpha/2}D_x^{1/2}\int_0^t W_\alpha(t,\tau)g(\tau)\tau\|_{L^\infty_x L^2_t([0,T])}\\
&\quad \underset{\text{Minkowski}}{\leq} \left\| \int_0^T \left(\int_0^T \left|t^{\alpha/2}D_x^{1/2}W_\alpha(t,\tau)g(\tau)\right|^2 dt\right)^{1/2} d\tau\right\|_{L^\infty_x}\\
&\quad \underset{\text{by (3)}}{\leq} \int_0^T \|W_\alpha(0,\tau)g(\tau)\|_{L^2_x}d\tau = \|g\|_{L^1_t([0,T])L^2_x}
\end{aligned}$$

which gives (6).

To prove (7) we first observe that, by Minkowsky inequality,

$$\left\| t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau \right\|_{L_t^2([0,T])} \leq \int_0^T \| t^{\alpha/2} D_x^{1/2} W_\alpha(t,0) W_\alpha(0,\tau) g(\tau) \|_{L_t^2([0,T])} d\tau,$$

therefore

$$\left(\int_{Q_\beta} \left\| t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau \right\|_{L_t^2([0,T])}^2 dx \right)^{1/2}$$
$$\leq \left[\int_{Q_\beta} \left(\int_0^T \| t^{\alpha/2} D_x^{1/2} W_\alpha(t) W_\alpha(0,\tau) g(\tau) \|_{L_t^2([0,T])} d\tau \right)^2 dx \right]^{1/2}$$
$$\underset{\text{Minkowski}}{\leq} \int_0^T \left(\int_{Q_\beta} \| t^{\alpha/2} D_x^{1/2} W_\alpha(t,0) W_\alpha(0,\tau) g(\tau) \|_{L_t^2([0,T)]}^2 dx \right)^{1/2} d\tau.$$

Then we apply the $\sup_{\beta\in\mathbb{Z}^n}$ on both the RHS and the LHS of the previous inequality and get

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \left\| t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau \right\|_{L_t^2([0,T])}^2 dx \right)^{1/2}$$
$$\leq \int_0^T \sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \| t^{\alpha/2} D_x^{1/2} W_\alpha(t,0) W_\alpha(0,\tau) g(\tau) \|_{L_t^2([0,T])}^2 dx \right)^{1/2} d\tau$$
$$\underset{\text{by (4)}}{\leq} \int_0^T \| W_\alpha(0,\tau) g(\tau) \|_{L_x^2(\mathbb{R}^n)} d\tau = \int_0^T \| g(\tau) \|_{L_x^2(\mathbb{R}^n)} d\tau,$$

which gives (7) and concludes the proof. □

We are almost ready to prove our well-posedness result, but first let us recall what we mean by saying that the IVP (8) is locally well-posed.

Definition 1 We say that the IVP (8) is locally well-posed (l.w.p) in $H^s(\mathbb{R}^n)$ if for any ball B in the space $H^s(\mathbb{R}^n)$ there exist a time T and a Banach space of functions $X \subset L^\infty([0,T], H^s(\mathbb{R}^n))$ such that, for each initial datum $u_0 \in B$, there exists a unique solution $u \in X \subset C([0,T], H^s(\mathbb{R}^n))$ of the integral equation

$$u(x,t) = W_\alpha(t) u_0 + \int_0^t W_\alpha(t,\tau) |u|^{2k} u(\tau) d\tau.$$

Furthermore the map $u_0 \mapsto u$ is continuous as a map from $H^s(\mathbb{R}^n)$ into $C([0,T], H^s(\mathbb{R}^n))$.

Proof of Theorem 3 The proof is based on the standard contraction argument. We summarize below the main steps of the proof. For further details we refer the interested reader to [12].

Let us first assume that $n = 1$, and let us define the metric space X as

$$X := \{u : [0,T]\times\mathbb{R} \to \mathbb{C};\ \|t^{\alpha/2}D_x^{1/2+s}u\|_{L_x^\infty L_t^2([0,T])} < \infty,\ \|u\|_{L_t^\infty([0,T])H_x^s} < \infty\},$$

which we equip with the distance

$$d(u,v) = \|t^{\alpha/2}D_x^{1/2+s}(u-v)\|_{L_x^\infty L_t^2([0,T])} + \|u-v\|_{L_t^\infty([0,T])\dot{H}_x^s} + \|u-v\|_{L_t^\infty([0,T])L_x^2},$$

where $\dot{H}_x^s$ stands for the homogeneous Sobolev space. We then consider the map

$$\Phi : X \to X, \quad \Phi(u) = W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau)u|u|^{2k}(\tau)d\tau,$$

and prove that it is a contraction on a ball of X, that is on $B_R := \{u \in X;\ \|u\|_X \le R\} \subset X$ for a suitable R.

By using the estimates in Theorem 2 and in Theorem 3 we get that

$$\|\Phi(u)\|_X \le 3\|u_0\|_{H_x^s} + C_1 T\|u\|_X^{2k+1},$$

which, for $R = 6\|u_0\|_{H_x^s}$ and $T = \frac{1}{C_1 R^{2k}}$, gives that Φ sends B_R into itself. Now, fixing $R = 6\|u_0\|_{H_x^s}$, and by using arguments similar to those used above, we can conclude that Φ is a contraction. Indeed, for all $u, v \in B_R$, we have

$$\|\Phi(u) - \Phi(v)\|_X \le C_2 T R^{2k}\|u - v\|_X,$$

therefore, by choosing T such that $T = \min\{\frac{1}{C_1 R^{2k}}, \frac{1}{C_2 R^{2k}}\}$, we obtain that Φ is a contraction, and the result follows by the fixed point theorem.

Let us now assume that $n > 1$. In this case we define X to be the space

$$X := \{u : [0,T] \times \mathbb{R}^n \to \mathbb{C};\ |||t^{\alpha/2}D_x^{s+1/2}u|||_T < \infty,\ \|u\|_{L^\infty_{[0,T]}H_x^s} < \infty\},$$

where

$$|||\cdot|||_T = \sup_{\beta\in\mathbb{Z}^n} \|\cdot\|_{L_x^2(Q_\beta)L_t^2([0,T])},$$

and

$$d_X(u,v) = |||t^{\alpha/2}D_x^{s+1/2}(u-v)|||_T + \|u-v\|_{L_t^\infty([0,T])\dot{H}_x^s} + \|u-v\|_{L_t^\infty([0,T])L_x^2}.$$

Then, considering the map Φ as before but now defined on the new space X, we can exploit the estimates in Theorem 2 and in Theorem 3 holding in the high dimensional case to get the same estimates and properties as in the case $n = 1$. The result then follows again by the fixed point theorem. For more details and explicit computations see [12]. □

Remark 2 Let us remark that the methods applied above in the case $\mathcal{L}_\alpha := \mathcal{L}_{\alpha,0}$ can also be applied to the case $\mathcal{L}_{\alpha,c} = \mathcal{L}_{\alpha,t^\alpha v}$, with v being a complex vector $v \in \mathbb{C}^n$.

2.2 The Class $\mathcal{L}_{\alpha,c}$

This section focuses on the study of the more general case $\mathcal{L}_{\alpha,c}$ with c being not necessarily identically zero ($c \not\equiv 0$ in our notation). We stress that the results of this subsection hold true in the case $c \equiv 0$ as well, and that in the latter case a direct proof can be performed. However, due to the presence of the variable coefficients $c(t,x)$, whose properties will be stated soon (see Theorem 4), the strategy to be used to analyze the problem for $\mathcal{L}_{\alpha,c}$ is different than the one used before for $\mathcal{L}_\alpha$. The key tools of our analysis will be the use of the pseudodifferential calculus and the application of a lemma due to Doi in [8], that we shall call Doi's Lemma, that we recall in Lemma 2 in the Appendix.

We shall state in Theorem 4 below our result about the smoothing properties of the solution of the IVP

$$\begin{cases} \partial_t u = it^\alpha \Delta_x u + ic(t,x) \cdot \nabla_x u + f(t,x), \\ u(0,x) = u_0(x). \end{cases} \tag{10}$$

Moreover, we will give in Theorems 5 and 6 local well-posedness results for the IVP (10) when $f = \pm|u|^{2k}u$, $k \geq 1$, and when $f = \pm t^\beta \sum_{j=1}^n (\partial_{x_j} u)u$, with $\beta \geq \alpha > 0$, respectively.

Theorem 4 *Let $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Assume that, for all $j = 1, \dots, n$, c_j is such that $c_j \in C([0,T], C_b^\infty(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that*

$$|\mathrm{Im}\, \partial_x^\gamma c_j(t,x)|, |\mathrm{Re}\, \partial_x^\gamma c_j(t,x)| \lesssim t^\alpha \langle x \rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n. \tag{11}$$

Then, denoting by $\lambda(|x|) := \langle x \rangle^{-\sigma}$, we have the following properties:

(i) If $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ then the IVP (10) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_s \leq C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt \right);$$

(ii) *If* $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ *then the IVP* (10) *has a unique solution* $u \in C([0,T]; H^s(\mathbb{R}^n))$ *and there exist two positive constants* C_1, C_2 *such that*

$$\sup_{0\le t\le T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)dx\,dt$$

$$\le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt\right);$$

(iii) *If* $\Lambda^{s-1/2} f \in L^2([0,T] \times \mathbb{R}^n; t^{-\alpha}\lambda(|x|)^{-1} dt dx)$ *then the IVP* (10) *has a unique solution* $u \in C([0,T]; H^s(\mathbb{R}^n))$ *and there exist positive constants* C_1, C_2 *such that*

$$\sup_{0\le t\le T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2}u\right|^2 \lambda(|x|)dx\,dt$$

$$\le C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{-\alpha}\lambda(|x|)^{-1} \left|\Lambda^{s-1/2} f\right|^2 dx\,dt\right).$$

Above we abbreviated the norm $\|f\|_{H^s(\mathbb{R}^n)} =: \|f\|_s$.

Theorem 5 *Let* $\mathcal{L}_\alpha$ *be such that condition* (11) *is satisfied. Then the IVP* (10) *with* $f(t,x) = \pm|u|^{2k}u$ *is locally well posed in* H^s *for* $s > n/2$ *and the solution satisfies smoothing estimates.*

Theorem 6 *Let* $\mathcal{L}_\alpha$ *be such that condition* (11) *is satisfied with* $\sigma = 2N$ *(thus* $\lambda(|x|) = \langle x\rangle^{-2N}$*) for some* $N \ge 1$*, and let* $s > n+4N+3$ *be such that* $s-1/2 \in 2\mathbb{N}$*. Then, the IVP* (10) *with* $f = \pm t^\beta \sum_{j=1}^n (\partial_{x_j} u)u$*, where* $\beta \ge \alpha > 0$*, is locally well posed in* $H^s_\lambda := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$ *and the solution satisfies smoothing estimates.*

Remark 3 Let us stress that it is natural to require the coefficients c_j of the first order term to satisfy some decay conditions, usually called Levi conditions. Indeed such kind of conditions were proved to be necessary to have the local well-posedness of the linear IVP in the case $\alpha = 0$. To be precise, it is enough to impose some decay on $\operatorname{Re} \partial_x^\gamma c_j(t,x)$ only (for all $j = 1, \dots, n$,) to conclude the local well-posedness of the linear IVP. However, the additional condition on $\operatorname{Im} \partial_x^\gamma c_j(t,x)$, for all $j = 1, \dots, n$, appears in order to get estimates with "gain of derivatives", namely smoothing estimates, needed to deal with the nonlinear problem with derivative nonlinearities.

Remark 4 Notice that part (ii) and (iii) in Theorem 4 correspond to the weighted homogeneous and inhomogeneous smoothing estimate for $\mathcal{L}_{\alpha,c}$ with a gain of $1/2$ and 1 derivative, respectively. When $\alpha = 0$, these results coincide with the classical ones for $\mathcal{L}_{0,c}$ (see, for instance, [20] and [22]).

The proof of Theorem 4 is based on the results in Lemma 1 below. The proof of Lemma 1, instead, relies deeply on the use of Lemma 2, also called Doi's lemma. The crucial result due to Doi in [8] is needed to define a new norm N, equivalent to the H^s-Sobolev norm, which is used to perform the energy estimate from which the smoothing estimates are derived. We explain below the way we use Doi's lemma, that is Lemma 2, to define N.

We apply Lemma 2 on the symbol $a^w := a = a_2 + ia_1 + a_0$ with $a_2(x, \xi) = |\xi|^2$ and $a_1 = a_0 = 0$. In this case conditions (B1) and (B2) of Lemma 2 are trivially satisfied, while (A6) holds with $q(x, \xi) = x \cdot \xi \langle \xi \rangle^{-1}$. Therefore, by Lemma 2 with $\lambda'(|x|) = C' \langle x \rangle^{-\sigma}$ (see Remark 6), with C' to be chosen later, we get that there exists $p \in S^0$ and $C > 0$ such that (37) holds.

We then consider the pseudo-differential operator K with symbol $K(x, \xi) = e^{p(x,\xi)} \Lambda^s$, where $\Lambda^s := \langle \xi \rangle^s$ and $p(x, \xi)$ is the symbol given by Doi's lemma, and define the norm N on $H^s(\mathbb{R}^n)$, equivalent to the standard one (see [22] for the proof of the equivalence), as

$$N(u)^2 = \|Ku\|_0^2 + \|u\|_{s-1}^2, \tag{12}$$

where $\| \cdot \|_s$ stands for the standard norm in the Sobolev space $H^s(\mathbb{R}^n)$.

With the norm $N(\cdot)$ in (12) at our disposal we can prove Lemma 1 from which Theorem 4 will follow. To prove Lemma 1 we employ the technique used in [22].

Lemma 1 *Let $s \in \mathbb{R}$, $\lambda(|x|) := \langle x \rangle^{-\sigma}$, $P_\alpha := \partial_t - it^\alpha \Delta_x - ic(t, x) \cdot \nabla_x$, and $\sigma > 1$ such that (11) holds. Then there exists $C_1, C_2 > 0$ such that, for all $u \in C([0, T]; H^{s+2}(\mathbb{R}^n)) \bigcap C^1([0, T]; H^s(\mathbb{R}^n))$, we have*

$$\sup_{0 \le t \le T} \|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1} + T)} \left(\|u_0\|_s + \int_0^T \|P_\alpha u(t, \cdot)\|_s dt \right); \tag{13}$$

$$\sup_{0 \le t \le T} \|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1} + T)} \left(\|u(\cdot, T)\|_s + \int_0^T \|P_\alpha^* u(t, \cdot)\|_s dt \right); \tag{14}$$

$$\begin{aligned} \sup_{0 \le t \le T} \|u(t)\|_s^2 &+ \int_0^T \int_{\mathbb{R}^n} t^\alpha \left| \Lambda^{s+1/2} u \right|^2 \lambda(|x|) dx\, dt \\ &\le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1} + T)} \left(\|u_0\|_s^2 + \int_0^T \|P_\alpha u(t, \cdot)\|_s^2 dt \right); \end{aligned} \tag{15}$$

$$\begin{aligned} \sup_{0 \le t \le T} \|u(t)\|_s^2 &+ \int_0^T \int_{\mathbb{R}^n} t^\alpha \left| \Lambda^{s+1/2} u \right|^2 \lambda(|x|) dx\, dt \\ &\le C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{-\alpha} \lambda(|x|)^{-1} \left| \Lambda^{s-1/2} P_\alpha u(t, \cdot) \right|^2 dx dt \right). \end{aligned} \tag{16}$$

Proof The proof is based on an energy estimate in terms of the norm $N(\cdot)$ in (12). We recall that $P_\alpha := \partial_t - it^\alpha \Delta_x - ic(t,x)\cdot\nabla_x$, $D_x = (D_{x_1},\dots,D_{x_n}) := (-i\partial_{x_1},\dots,-i\partial_{x_n})$, and that $\langle\cdot,\cdot\rangle$ stands for the $L^2(\mathbb{R}^n)$-scalar product. We then consider

$$\partial_t N(u)^2 = \partial_t \|Ku\|_0^2 + \partial_t \|u\|_{s-1}^2 = I + II,$$

and estimate I and II separately.

We start by estimating term II, for which we get

$$\begin{aligned} II = \partial_t \|u\|_{s-1}^2 &= 2\mathrm{Re}\langle \Lambda^{s-1}\partial_t u, \Lambda^{s-1}u\rangle = 2\mathrm{Re}\langle \Lambda^{s-1}P_\alpha u, \Lambda^{s-1}u\rangle \\ &= -2\mathrm{Re}\langle \Lambda^{s-1}c(t,x)\cdot D_x u, \Lambda^{s-1}u\rangle + 2\mathrm{Re}\langle \Lambda^{s-1}f, \Lambda^{s-1}u\rangle \\ &\le Ct^\alpha \|u\|_s^2 + 2\mathrm{Re}\langle \Lambda^{s-1}f, \Lambda^{s-1}u\rangle. \end{aligned}$$

Now, since

$$2\mathrm{Re}\langle \Lambda^{s-1}f, \Lambda^{s-1}u\rangle \le 2\|f\|_{s-1}\|u\|_{s-1} \le CN(f)N(u) \tag{17}$$

and

$$\begin{aligned} 2\mathrm{Re}\langle \Lambda^{s-1}f, \Lambda^{s-1}u\rangle &= 2\mathrm{Re}\langle t^{-\alpha/2}\lambda(|x|)^{-1/2}\Lambda^{s-1/2}f, t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s-3/2}u\rangle \\ &\le \|t^{-\alpha/2}\lambda(|x|)^{-1/2}\Lambda^{s-1/2}f\|_0^2 + \|t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s-3/2}u\|_0^2 \\ &\le \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f, \Lambda^{s-1/2}f\rangle + t^\alpha N(u)^2, \end{aligned} \tag{18}$$

it follows that

$$II \le Ct^\alpha N(u)^2 + C' \min\{N(f)N(u); \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f, \Lambda^{s-1/2}f\rangle\}, \tag{19}$$

with C and C' new suitable constants.

As for term I we have that

$$\begin{aligned} \partial_t \|Ku\|_0^2 &= 2\mathrm{Re}\langle \partial_t Ku, Ku\rangle = 2\mathrm{Re}\langle K\partial_t u, Ku\rangle \\ &= 2\mathrm{Re}\langle KP_\alpha u, Ku\rangle + 2\mathrm{Re}\langle Kf, Ku\rangle \\ &= 2\mathrm{Re}\langle it^\alpha[K,\Delta_x]u, Ku\rangle + \underbrace{2\mathrm{Re}\langle it^\alpha \Delta_x Ku, Ku\rangle}_{=0} \\ &\quad - 2\mathrm{Re}\langle K\, b(t,x)\cdot D_x u, Ku\rangle + 2\mathrm{Re}\langle Kf, Ku\rangle \\ &= 2\mathrm{Re}\langle it^\alpha[K,\Delta_x]u, Ku\rangle - 2\mathrm{Re}\langle [K, c(t,x)\cdot D_x]u, Ku\rangle \\ &\quad - 2\mathrm{Re}\langle c(t,x)\cdot D_x\, Ku, Ku\rangle + 2\mathrm{Re}\langle Kf, Ku\rangle, \end{aligned} \tag{20}$$

therefore, in order to estimate I, it is crucial to prove suitable upper bounds for the quantities $2\mathrm{Re}\langle it^\alpha[K, \Delta_x]u, Ku\rangle$ and $2\mathrm{Re}\langle[K, c(t,x)\cdot D_x]u, Ku\rangle$ in the fifth line of (20).

By using the pseudodifferential calculus we can compute the symbol of the commutator $[K, c(t,x)\cdot D_x]$, which is an operator of order s, and get, thanks to the properties of c (recall that $c \in C_b^\infty$ and is bounded, together with its derivatives in space, by $t^\alpha\lambda(|x|)$), that

$$-2\mathrm{Re}\langle[K, b(t,x)D_x]u, Ku\rangle \le Ct^\alpha\|u\|_s^2.$$

For more details about how to get to this estimate see Lemma 5.0.1 in [12].

For the term $2\mathrm{Re}\langle it^\alpha[K, \Delta_x]u, Ku\rangle$, once more by using the pseudodifferential calculus, we have that $[K, \Delta_x](x, D) = [p, \Delta_x]K(x, D) + r_s(x, D)$, where r_s is an operator of order s, while $p = p(x, D)$ is the operator of order 0 appearing in the definition of the norm $N(\cdot)$.

These considerations lead to

$$\begin{aligned}(20) &\le Ct^\alpha\|u\|_s^2 + 2\mathrm{Re}\langle(it^\alpha[p, \Delta_x](x, D) - c(t,x)\cdot D_x)Ku, Ku\rangle \\ &\quad + |2\mathrm{Re}\langle it^\alpha r_s(x, D)u, Ku\rangle| \\ &\le Ct^\alpha\|u\|_s^2 + 2\mathrm{Re}\langle(it^\alpha[p, \Delta_x](x, D) - c(t,x)D_x)Ku, Ku\rangle, \end{aligned} \tag{21}$$

where C is a new suitable positive constant.

Now we denote by $Q(x, D) := it^\alpha[p, \Delta_x](x, D) - b(t,x)\cdot D_x$ the operator whose symbol satisfies

$$\begin{aligned}\mathrm{Re}\, Q(x,\xi) &= \mathrm{Re}\left(it^\alpha(-i)\{p, -|\xi|^2\}(x,\xi) - b(t,x)\cdot\xi\right) + r_0 \\ &\le -t^\alpha\{p, |\xi|^2\}(x,\xi) + |\mathrm{Re}\, b(t,x)\cdot\xi| + r_0 \\ &\underset{\text{by (37)}}{\le} -C't^\alpha\lambda(|x|)|\xi| + C_2t^\alpha + C_0t^\alpha\lambda(|x|)|\xi| + C \\ &\le -Ct^\alpha\lambda(|x|)|\xi| + C_2t^\alpha + C_4 \\ &\le -Ct^\alpha\lambda(|x|)(1+|\xi|^2)^{1/2} + C_3t^\alpha + C_4 \\ &= t^\alpha(-C\lambda(|x|)(1+|\xi|^2)^{1/2} + C_3) + C_4,\end{aligned}$$

where we chose C' (which is possible by Doi's lemma, see Remark 6) in order to have $C_0 - C' < 0$.

The property of the symbol of Q allow us to apply the sharp Gårding inequality and to conclude that

$$\begin{aligned} 2\mathrm{Re}\langle Q(x,D)Ku, Ku\rangle &\le -Ct^{\alpha}\langle \lambda(|x|)\Lambda^{1}Ku, Ku\rangle + C_3 t^{\alpha}\|Ku\|_0^2 + C_4\|Ku\|_0^2 \\ &\le -Ct^{\alpha}\langle \lambda(|x|)\Lambda^{1}Ku, Ku\rangle + C_3 t^{\alpha}\|u\|_s^2 + C_4\|u\|_s^2 \\ &\le Ct^{\alpha}\|\lambda(|x|)^{1/2}\Lambda^{1/2}Ku\|_0^2 + C_3 t^{\alpha}\|u\|_s^2 + C_4\|u\|_s^2, \quad (22) \end{aligned}$$

where $C > 0$ is a new suitable constant.

By plugging (22) in (21) we get

$$\partial_t\|Ku\|_0 \le Ct^{\alpha}N(u)^2 + C'N(u)^2 - C''t^{\alpha}\|\lambda(|x|)^{1/2}\Lambda^{1/2}Ku\|_0^2 + C'''N(f)N(u). \quad (23)$$

Finally, (19) and the equivalence of the norms $\|\cdot\|_s$ and $N(\cdot)$ (see [22] pag.390) yield

$$\begin{aligned} \partial_t N(u)^2 &= \partial_t\|Ku\|^2 + \partial_t\|u\|_{s-1}^2 \\ &\le Ct^{\alpha}N(u)^2 + C'N(u)^2 - C''t^{\alpha}\|\lambda(|x|)^{1/2}\Lambda^{1/2}Ku\|_0^2 + C'''N(f)N(u) \\ &+ C_3\min\{N(f)N(u); \langle t^{-\alpha}\lambda(|x|)^{-1}\Lambda^{s-1/2}f, \Lambda^{s-1/2}f\rangle\}, \quad (24) \end{aligned}$$

where the constants are (eventually) new suitable constants. □

Estimate (24) is now the starting point to get (13), (14) and (15).

Proof of (13) From (24) we have

$$\partial_t N(u)^2 \le C_1(t^{\alpha}+1)N(u)^2 + C_2 N(u)N(f)$$

(again with C_1 and C_2 new constants), which gives

$$2\partial_t N(u) \le C_1(t^{\alpha}+1)N(u) + C_2 N(f)$$

and

$$\partial_t\left(2e^{-\frac{1}{2}C_1(t^{\alpha+1}/(\alpha+1)+t)}N(u)\right) \le C_2 e^{-\frac{1}{2}C_1(t^{\alpha+1}/(\alpha+1)+t)}N(f).$$

Hence, by integrating in time from 0 to t and using the equivalence of the norms $N(\cdot)$ and $\|\cdot\|_s$, (13) follows. □

Proof of (14) The proof of (14) follows from (13) applied to the adjoint operator and with $u(t,\cdot)$ replaced by $u(T-t,\cdot)$. □

Proof of (15) Here we use the fact that there exists a pseudodifferential operator $\tilde{K}$ such that

$$I = \tilde{K}K + \Psi_{r_{-1}},$$

where $\Psi_{r_{-1}}$ is a pseudodifferential operator with symbol r_{-1} of order -1 (see [22] pag.390 for the proof of this property). This gives that

$$\|\lambda(|x|)^{1/2}\Lambda^{s+1/2}u\|_0 \le \|(\lambda(|x|)^{1/2}\Lambda^{1/2})(\Lambda^s\tilde{K})Ku\|_0 + O(N(u))$$

$$\le \|(\Lambda^s\tilde{K})(\lambda(|x|)^{1/2}\Lambda^{1/2})Ku\|_0 + cN(u) \le c\left(\|(\lambda(|x|)^{1/2}\Lambda^{1/2})Ku\|_0 + N(u)\right), \tag{25}$$

since $[\Lambda^s\tilde{K}, \lambda(|x|)^{1/2}\Lambda^{1/2}]K\Lambda^{1/2}$ is a pseudo-differential operator of order s. Therefore, (24) and (25) yield

$$\begin{aligned}&\partial_t N(u)^2 + C_2\langle t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u, t^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u\rangle\\ &\quad\le C_1(t^\alpha+1)N(u)^2 + C_4N(f)^2.\end{aligned}$$

Now, integrating in time from 0 to t the previous inequality, using (13) and the estimate

$$\begin{aligned}e^{\frac{1}{2}C_1(t^{\alpha+1}/(\alpha+1)+t)}\int_0^t &e^{-\frac{1}{2}C_1(s^{\alpha+1}/(\alpha+1)+s)}\langle s^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u, s^{\alpha/2}\lambda(|x|)^{1/2}\\ &\Lambda^{s+1/2}u\rangle ds\\ &\ge \int_0^t \langle s^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u, s^{\alpha/2}\lambda(|x|)^{1/2}\Lambda^{s+1/2}u\rangle ds,\end{aligned}$$

(15) follows (for further details see [12]). □

Proof of (16) To prove (16) we exploit the following estimate

$$\begin{aligned}2\mathrm{Re}\langle Kf, Ku\rangle &= 2\mathrm{Re}\langle t^{\alpha/2}\lambda^{1/2}\Lambda^{1/2}Kf, t^{-\alpha/2}\lambda^{-1/2}\Lambda^{-1/2}Ku\rangle \\ &\le c_1\varepsilon\|t^{\alpha/2}\lambda^{1/2}\Lambda^{s+1/2}u\|_0^2 + c_2\frac{1}{\varepsilon}\|t^{-\alpha/2}\lambda^{-1/2}\Lambda^{s-1/2}f\|_0^2\\ &\quad + c_3t^\alpha\|u\|_s^2.\end{aligned} \tag{26}$$

By using (25) and (26) in (24) and the equivalence of $N(\cdot)$ and $\|\cdot\|_s$, we obtain

$$\partial_t N(u)^2 + (c_0 - c_1\varepsilon)\|t^{\alpha/2}\lambda^{1/2}\Lambda^{s+1/2}u\|_0^2 \le c_3 t^\alpha N(u)^2 + c_2\frac{1}{\varepsilon}\|t^{-\alpha/2}\lambda^{-1/2}\Lambda^{s-1/2}f\|_0^2,$$

where c_j, $j = 0, 1, 2, 3$, are new suitable constants, and where $\varepsilon > 0$ can be chosen in such a way that $c_0 - c_1\varepsilon \ge c > 0$. Finally, integrating in time from 0 to t, and arguing as in the proof of (15), the result follows. This concludes the proof. □

Proof of Theorem 4 Estimate (13) of Lemma 1 gives readily the uniqueness of the solution. In fact, let u be a solution of the homogeneous IVP for $\mathcal{L}_{\alpha,c}$ with initial datum $u_0 = 0$. Then, by (13) of Lemma 1, $u = 0$, which proves the uniqueness (even in the general inhomogeneous IVP where $f \neq 0$ and $u_0 \neq 0$).

As for the existence, it will follow by using density arguments.

Case 1: $f \in \mathcal{S}(\mathbb{R}^{n+1})$ and $u_0 \in \mathcal{S}(\mathbb{R}^n)$.

We consider the subspace $E \subset L^1([0,T]; H^{-s}(\mathbb{R}^n)$

$$E = \{P^*\varphi;\ \varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))\} = (\partial_t - it^\alpha\Delta_x + b(t,x)\cdot D_x)^*(C_0^\infty(\mathbb{R}^{n+1}))$$

and the linear functional

$$\ell^* : E \to \mathbb{C}, \quad \ell^*(P^*\varphi) = \int_0^T \langle f, \varphi\rangle_{L^2\times L^2} dt + \langle u_0, \varphi(\cdot,0)\rangle_{L^2\times L^2}.$$

Now inequality (14) of Lemma 1 (applied to φ) with s replaced by $-s$ gives, for $\eta = P^*\varphi$ and $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T))$,

$$|\ell^*(\eta)| \le \|f\|_{(L^1[0,T];H_x^s)} \sup_{t\in[0,T]} \|\varphi\|_{H_x^{-s}} + \|u_0\|_{H_x^s}\|\varphi(0)\|_{H_x^{-s}}$$

$$\le e^{C(T^{\alpha+1}/(\alpha+1)+T)}\left(\|f\|_{L_t^1([0,T];H_x^s)} + \|u_0\|_{H_x^s}\right)\|\eta\|_{L_t^1([0,T];H_x^{-s})},$$

which implies the continuity of ℓ^* on E. Then, by the Hahn-Banach theorem we can extend ℓ^* on $L^1([0,T] : H^{-s}(\mathbb{R}^n))$ and finally get the existence of $u \in L^1([0,T]; H^{-s}(\mathbb{R}^n))^* = L^\infty([0,T]; H^s(\mathbb{R}^n))$ such that

$$\ell^*(P^*\varphi) = \langle u, P^*\varphi\rangle_{L^2\times L^2} = \int_0^T \langle f, \varphi\rangle_{L^2\times L^2} dt + \langle u_0, \varphi(\cdot,0)\rangle_{L^2\times L^2},$$

and thus $Pu = f$ in the sense of distributions for $0 < t < T$.

Notice that $Pu \overset{\mathcal{D}'}{=} f$ means that $(\partial_t - it^\alpha\Delta_x + b(t,x)\cdot D_x)u \overset{\mathcal{D}'}{=} f$ (as distributions on $C_0^\infty([0,T) \times \mathbb{R}^n)$). Therefore, since $f \in \mathcal{S}(\mathbb{R}^{n+1})$, we have that $\partial_t u \in (L^\infty[0,T) : H^{s-2}(\mathbb{R}^n))$, which gives $u \in (C([0,T) : H^{s-2}(\mathbb{R}^n))$. We then use the equation once more, that is $\partial_t u = it^\alpha \Delta_x + b(t,x)\cdot D_x u + f$, and get, by the same consideration, that $u \in (C^1[0,T) : H^{s-4}(\mathbb{R}^n))$ and $u(x,0) = u_0(x)$. Finally, since $u_0 \in H^s(\mathbb{R}^n)$, repeating the previous argument with $s+4$ in place of s we conclude that there exists a solution u of the IVP associated to (10) to which parts (i)–(iv) of Lemma 1 apply.

Case 2: $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ and $u_0 \in H^s(\mathbb{R}^n)$.

In this case we take two sequences $f_j \in \mathcal{S}(\mathbb{R}^{n+1})$, $v_j \in \mathcal{S}(\mathbb{R}^n)$ such that $f_j \to f$ in $(L^1([0,T]) : H^s(\mathbb{R}^n)$ and $v_j \to u_0$ in $H^s(\mathbb{R}^n)$.

By the arguments of case 1 we find a solution u_j of (10) with f_j and v_j in place of f and u_0 respectively. Since u_j satisfies (13) of Lemma 1, we have that u_j is a Cauchy sequence, therefore, passing to the limit, we get that $u = \lim_{j\to} u_j$ is a solution of the IVP with inhomogeneous term f and with initial datum u_0 satisfying (14) of Lemma 1, which proves part (ii) of the theorem.

Case 3: $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ and $u_0 \in H^s(\mathbb{R}^n)$.

Here we proceed as in case 2 but with $f_j \in \mathcal{S}(\mathbb{R}^{n+1})$ being such that $f_j \to f$ in $(L^2([0,T]); H^s(\mathbb{R}^n)$. Under this hypothesis we obtain point (ii) of the theorem, that is, it exists a solution $u \in (C[0,T) : H^s(\mathbb{R}^n))$ satisfying (15) of Lemma 1.

Case 4: $\Lambda^{s-1/2} f \in (L^2(\mathbb{R}^n \times [0,T]) : t^{-\alpha}\lambda(|x|)^{-1}dxdt)$ and $u_0 \in H^s(\mathbb{R}^n)$.

In this case it is possible to prove that there exists $g_j \in \mathcal{S}(\mathbb{R}^{n+1})$ such that $g_j \to \Lambda^{s-1/2} f$ in $(L^2(\mathbb{R}^n) \times [0,T] : t^{-\alpha}\lambda(|x|)^{-1}dxdt)$. Applying once again the strategy used in case 1 with f_j replaced by $\Lambda^{-s+1/2} g_j$ in (16) of Lemma 1, and passing to the limit, we finally obtain point (iii) of Theorem 4. □

As a consequence of Theorem 4 one gets the local well-posedness results stated in Theorem 5 and in Theorem 6. We will not give a complete proof of these results here, and we refer the interested reader to [12] for detailed proofs. However, we give below a sketch of the proof listing the main ingredients of the argument.

Sketch of the Proof of Theorem 5 As in the case $c \equiv 0$, the proof is based on the standard contraction argument.

According to Theorem 4 we have the local well-posedness in H^s, $s > n/2$, for the linear IVP (10) for a general function f satisfying the assumptions. We now write the solution of (10) as

$$u(t,x) = W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau) f(\tau,x)d\tau, \tag{27}$$

where $W_\alpha(t,\tau)$ is a new suitable two-parameter family of unitary operators representing the solution operator.

Because of the previous assumption, solving the IVP (10) with $f = u|u|^{2k}$ is equivalent to find the solution of the integral equation

$$u(t,x) = W_\alpha(t)u_0(x) + \int_0^t W_\alpha(t,\tau)u|u|^{2k}(\tau,x)d\tau.$$

Hence, as in the proof of Theorem 4, we look for the solution given by the fixed point of the map

$$\Phi_{u_0}(u) := W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau)u|u|^{2k}d\tau,$$

defined on

$$X_T^s := \{u : [0,T]\times\mathbb{R}^n \to \mathbb{C};\ \|u\|_{L_t^\infty H_x^s} < \infty,$$
$$\left(\int_0^T\int_{\mathbb{R}^n} t^\alpha \lambda(|x|)|\Lambda^{s+1/2}u|^2 dx\,dt\right)^{1/2} < \infty\},$$

where, recall, $\lambda(|x|) := \langle x\rangle^\sigma$, with $\sigma > 1$ being such that (11) holds. Notice that the choice of the space X_T^s is dictated by the smoothing estimates we proved in Theorem 4. To conclude that Φ_{u_0} is a contraction on the space X_T^s, we apply the estimates in Theorem 4 together with Sobolev embeddings and a few technical lemmas taken from [22]. Finally, the application of the fixed point theorem then gives the result. Notice that the solution will belong to the space X_T^s, and, consequently, will satisfy smoothing estimates. □

Sketch of the Proof of Theorem 6 Their proof of this result follows by using the same arguments as before. Here the contraction argument is performed on a different space, that is, specifically, on the space

$$X_T^s := \{u : [0,T]\times\mathbb{R}^n \to \mathbb{C};\ \|u\|_{L_t^\infty H_x^s} < \infty,$$
$$\left(\int_0^T\int_{\mathbb{R}^n} t^\alpha \lambda(|x|)|\Lambda^{s+1/2}u|^2 dx\,dt\right)^{1/2} < \infty,$$
$$\|\lambda(|x|)^{-1}u\|_{L_t^\infty H_x^{s-2N-3/2}} < \infty\},$$

where

$$\|u\|_{X_T^s}^2 = \|u\|_{L_t^\infty H_x^s}^2 + \int_0^T\int_{\mathbb{R}^n} t^\alpha\lambda(|x|)|\Lambda^{s+1/2}u|^2 dx\,dt + \|\lambda(|x|)^{-1}u\|_{L_t^\infty H_x^{s-2N-3/2}}^2.$$

We repeat the assumption that the solution of (10) is given in terms of a solution operator $W_\alpha(t,s)$, so we look for the solution of the nonlinear problem as the fixed point of a map Φ_{u_0} as before, but now with $f = t^\beta \sum_{j=1}^n \partial_{\xi_j} u|u|^2$, with $\beta \geq \alpha$. We then use the smoothing estimates in Lemma 1, more precisely (16), together with Lemma 6.0.1 in [12] and some technical lemmas taken form [22], and conclude the result via the standard contraction argument. Once again the solution satisfies smoothing estimates. For the complete proof see [12]. □

Let us remark once again that the previous results still hold true in the case $c \equiv 0$. Moreover, more general nonlinearities can be considered in the IVP for $\mathcal{L}_{\alpha,c}$, that is, for instance, nonlinearities containing polynomials in u, in the derivatives of order one of u, and in their complex conjugates. The specific choices we made for the nonlinear terms were to keep the exposition simpler and shorter.

We finally conclude by saying that the smoothing and well-posedness results presented here are very likely still true for some generalizations of $\mathcal{L}_{\alpha,c}$, that is for equations containing first order terms in $\bar{u}$ and with time degeneracies different than t^α (for more details about these generalizations see Section 7 in [12]).

3 Strichartz Estimates and Local Well-Posedness for $\mathcal{L}_b$

This section is devoted to the study of the class $\mathcal{L}_b$ as in (2), for which, as we shall show below, local weighted Strichartz estimates hold true. Additionally, we will employ such estimates to prove the local well-posedness of a semilinear IVP associated with $\mathcal{L}_b$, where the form of the nonlinear term is dictated by the inhomogenous Strichartz estimate at our disposal. The results of this section were proved in [11] where results other than local weighted Strichartz estimates are proved. In particular, in [11] also global weighted Strichartz estimates are derived, as well as homogeneous smoothing estimates for time-degenerate operators of any order by means of comparison principles. Our choice to treat the local estimates only is due to the fact that these inequalities, because of their different form with respect to the global counterpart, are the ones to be used to get the well-posedness of the semilinear IVP. For more details and results about the class $\mathcal{L}_b$ we refer the interested reader to [11].

The semilinear IVP we will study in this section is

$$\begin{cases} \partial_t u + ib'(t)\Delta u = \mu|b'(t)||u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \tag{28}$$

with $p > 1$ suitable, $\mu \in \mathbb{R}$, and b satisfying the following condition (H):

(H) $b \in C^1(\mathbb{R})$, $b(0) = b'(0) = 0$, and, for any $\tilde{T} < \infty$, $\sharp\{t \in [0,\tilde{T}], b'(t) = 0\} = k < \infty$.

Since we are interested in the time-degenerate case, we assume $k \geq 1$ in condition (H), that is, $b(0) = b'(0) = 0$. However, our results are applicable in the nondegenerate case $b'(t) \neq 0$, $t \in [0,T]$, as well.

Notice that, as for $\mathcal{L}_\alpha$, the solution operator for $\mathcal{L}_b$ (giving the solution of the homogeneous IVP at time t starting at time s) can be computed explicitly, and is given, for $s < t$, by

$$e^{i(b(t)-b(s))\Delta}u_s(x) := W(t,s)u_s(x) := \int_s^t e^{ix\cdot\xi - i(b(t)-b(s))|\xi|^2}\widehat{u_s}(\xi)d\xi,$$

which coincides with the Schrödinger group $e^{i(t-s)\Delta}$ when $b(t) = t$. Moreover, Duhamel's principle still holds true in this case.

As we will make use of the so called *admissible pairs*, we recall this notion here for completeness.

Given $n \geq 1$ we shall call a pair of exponents (q, p) *admissible* if $2 \leq q, p \leq \infty$, and

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \quad \text{with} \quad (q, p, n) \neq (2, \infty, 2).$$

With this definition in mind we can now state the main results of this section.

Theorem 7 (Local Weighted Strichartz Estimates) *Let $b \in C^1([0, T])$ be such that it satisfies condition (H). Then, on denoting by $L^q_t L^p_x := L^q([0, T]; L^p(\mathbb{R}^n))$, we have that for any (q, p) admissible pair, with $2 < q, p < \infty$, the following estimates hold*

$$\| |b'(t)|^{1/q} e^{ib(t)\Delta} \varphi \|_{L^q_t L^p_x} \leq C \|\varphi\|_{L^2_x(\mathbb{R}^n)}, \tag{29}$$

$$\| e^{ib(t)\Delta} \varphi \|_{L^\infty_t L^2_x} \leq \|\varphi\|_{L^2_x(\mathbb{R}^n)}, \tag{30}$$

$$\| |b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \|_{L^q_t L^p_x} \leq C \| |b'|^{1/q'} g \|_{L^{q'}_t L^{p'}_x}, \tag{31}$$

and

$$\| \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \|_{L^\infty_t L^2_x} \leq C \| |b'|^{1/q'} g \|_{L^{q'}_t L^{p'}_x}, \tag{32}$$

with $C = C(k, n, q, p)$.

Remark 5 Observe that, as opposed to the classical statement of Strichartz estimates, that is in the case when $b(t) = t$, we have estimates involving only one admissible pair (q, p), and not two arbitrary admissible pairs (q, p) and $(\tilde{q}, \tilde{p})$. However, this is enough to derive the following well-posedness result.

Theorem 8 *Let $1 < p < \frac{4}{n} + 1$ and $b \in C^1([0, +\infty))$ satisfying condition (H). Then, for all $u_0 \in L^2(\mathbb{R}^n)$, there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (28) in the time interval $[0, T]$ with*

$$u \in C([0, T]; L^2(\mathbb{R}^n)) \bigcap L^q_t([0, T]; L^{p+1}_x(\mathbb{R}^n))$$

and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot, t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0, T); L^2(\mathbb{R}^n))$, is continuous.

Proof of Theorem 7 Estimate (30) is immediate and follows by the unitary of $e^{ib(t)\Delta}$. As for (31), we consider $0 = T_0 \leq T_1 < T_2 < \ldots < T_k \leq T_{k+1} = T$ such that $b'(T_j) = 0$ for $j = 1, \ldots k$, so that b is strictly monotone on $[T_j, T_{j+1}]$, and we have

$$\begin{aligned}
\||b'(t)|^{1/q} e^{ib(t)\Delta}\varphi\|_{L^q_t L^p_x} &= \left(\sum_{j=0}^{k} \||b'(t)|^{1/q} e^{ib(t)\Delta}\varphi\|^q_{L^q([T_j,T_{j+1}];L^p_x)} \right)^{1/q} \\
&\leq \sum_{j=0}^{k} \||b'(t)|^{1/q} e^{ib(t)\Delta}\varphi\|_{L^q([T_j,T_{j+1}];L^p_x)} \\
&\underset{b(t)=t'}{\leq} \sum_{j=0}^{k} \|e^{it\Delta}\varphi\|_{L^q([\tilde{T}_j,\tilde{T}_{j+1}];L^p_x)} \\
&\leq (k+1) C(n,q,p) \|\varphi\|_{L^2_x},
\end{aligned}$$

which proves the estimate.

To prove (29) we split the time interval again, and get

$$\begin{aligned}
&\||b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s)ds\|_{L^q_t L^p_x} \\
&\leq \sum_{j=0}^{k} \||b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s)ds\|_{L^q_t([T_j,T_{j+1}];L^p_x)}.
\end{aligned} \tag{33}$$

Now, by using the changes of variables $t' = b(t)$ and $s' = b(s)$, each therm in the sum above satisfies

$$\begin{aligned}
&\||b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s)ds\|_{L^q_t([T_j,T_{j+1}];L^p_x)} \\
&\leq \| \int_0^{b(t)} e^{i(t'-s')\Delta} \tilde{g}(s')ds'\|_{L^q_{t'}([\tilde{T}_j,\tilde{T}_{j+1}];L^p_x)} \\
&= \| \int_0^{b(T)} e^{i(t'-s')\Delta} \chi(s')\tilde{g}(s')ds'\|_{L^q_{t'}([\tilde{T}_j,\tilde{T}_{j+1}];L^p_x)},
\end{aligned} \tag{34}$$

where $\tilde{g} = g \circ b^{-1}$, $\tilde{T}_j = b(T_j)$ and $\chi = 1_{[0,b(t)]}$. We then analyze the last quantity, and, by using the properties of the Schrödinger group $e^{it\Delta}$, we have

$$\| \int_0^{b(T)} e^{i(t'-s')\Delta}\chi(s')\tilde{g}(s')ds' \|_{L^q_{t'}([\tilde{T}_j,\tilde{T}_{j+1}];L^p_x)}$$

$$\leq \| \int_0^{b(T)} \|e^{i(t'-s')\Delta}\chi(s)\tilde{g}(s')\|_{L^p_x} ds' \|_{L^q_{t'}([\tilde{T}_j,\tilde{T}_{j+1}])}$$

$$\leq \| \int_0^{b(T)} \frac{1}{|t'-s'|^{n(1/2-1/p)}} \|\chi(s')\tilde{g}(s')\|_{L^{p'}_x} ds' \|_{L^q_{t'}([\tilde{T}_j,\tilde{T}_{j+1}])}$$

$$\underset{\text{H-L-S}}{\leq} C(n,q,p)\|\tilde{g}\|_{L^{q'}_{t'}([\tilde{T}_j,\tilde{T}_{j+1}];L^{p'}_x)} \underset{t=b^{-1}(t')}{\leq} C(n,q,p)\||b'|^{1/q'}g\|_{L^{q'}_t([T_j,T_{j+1}];L^{p'}_x)},$$

where H-L-S stands for the application of the Hardy-Littlewood-Sobolev inequality. Summarizing, we have proved that

$$\||b'(t)|^{1/q}\int_0^t |b'(s)|e^{i(b(t)-b(s))\Delta}g(s)ds\|_{L^q_t([T_j,T_{j+1}];L^p_x)}$$

$$\lesssim \||b'|^{1/q'}g\|_{L^{q'}_t([T_j,T_{j+1}];L^{p'}_x)} \lesssim \||b'|^{1/q'}g\|_{L^{q'}_t([0,T];L^{p'}_x)},$$

which, together with (33), gives

$$\||b'(t)|^{1/q}e^{ib(t)\Delta}\varphi\|_{L^q_tL^p_x} \leq (k+1)C(n,q,p)\||b'|^{1/q'}g\|_{L^{q'}_tL^{p'}_x},$$

and thus (31).

We are now left with the proof of (32). By using the fact that $e^{ib(t)\Delta}$ is unitary, we have

$$\| \int_0^t |b'(s)|e^{i(b(t)-b(s))\Delta}g(s)ds\|^2_{L^2_x} = \| \int_0^t |b'(s)|e^{-ib(s))\Delta}g(s)ds\|^2_{L^2_x}$$

$$= \int_{\mathbb{R}^n}\left(\int_0^t |b'(s)|e^{-ib(s)\Delta}g(s)ds\right)\overline{\left(\int_0^t |b'(s')|e^{-ib(s')\Delta}g(s')ds'\right)}dx$$

$$\leq \int_0^t \||b'(s)|^{1/q'}g(s)\|_{L^{p'}_x}\||b'(s)|^{1/q}\int_0^t |b'(s')|e^{i(b(s)-b(s'))\Delta}g(s')ds'\|_{L^p_x}ds$$

$$\underset{\text{by (31)}}{\leq} (k+1)C(n,q,p)\||b'|^{1/q'}g\|^2_{L^{q'}_tL^{p'}_x},$$

which, in particular, gives (32). This concludes the proof . □

Proof of Theorem 8 The proof is standard and based on the fixed point argument. Here the space where the contraction argument is performed is

$$X_T := \{u \in C([0,T]; L^2(\mathbb{R}^n) \bigcap L_t^q([0,T]; L_x^{p+1}(\mathbb{R}^n)); \|u\|_{X_T} < \infty\},$$

where

$$\|u\|_{X_T} := \|u\|_{L_t^\infty L_x^2} + \||b'(t)|^{1/q} u\|_{L_t^q L_x^{p+1}},$$

with $L_t^q L_x^p := L^q([0,T]; L_x^p(\mathbb{R}^n))$, and the map Φ_{u_0} is

$$\Phi_{u_0}(u) := e^{ib(t)\Delta} u_0 + \mu \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} |u|^{p-1} u \, ds.$$

Then we take $q = \frac{4(p+1)}{n(p-1)}$ so that $(q, p+1)$ is an admissible pair, and we prove that the map above is a contraction on a suitable ball of X_T (with sufficiently small radius depending on $\|u_0\|_{L_x^2}$) by using the estimates in Theorem 7. Finally, the application of the fixed point theorem gives the result. For a detailed proof see [11]. □

We conclude this section by giving a few examples of operators to which Theorem 8 for the IVP (28) applies.

Example 1 $\mathcal{L}_b = \mathcal{L}_{\frac{t^{\alpha+1}}{\alpha+1}} = \partial_t + it^\alpha \Delta, \quad \alpha \geq 0;$
Example 2 $\mathcal{L}_b = \mathcal{L}_{e^t - t - 1} = \partial_t + i(e^t - 1)\Delta;$
Example 3 $\mathcal{L}_b = \mathcal{L}_{\cos(t)} := \partial_t u - i \sin(t)\Delta.$

Notice that in the first two examples we have only one degenerate point, that is at time $t = 0$. Example 3, instead, is more interesting, since we have $k \geq 1$ degenerate points on any finite time interval $[0, T]$. Since Theorem 8 applies to all the cases listed above, this gives that, if the time of existence T in Theorem 8 is large enough, then in Example 3 we will cross more than one degenerate point.

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Appendix

We use this section to give the statement of a key result used in this paper, that is, specifically, that of the so called Doi's lemma (Lemma 2.3 in [8]). But first, let us make clear the conditions needed to apply the aforementioned lemma.

In the sequel we will use the notations used by Doi in [8], so we shall denote by (B1), (B2) and (A6) the following conditions:

Let $a^w(t, x, \xi)$ *be the Weyl symbol of a pseudo-differential operator* $A = A(t, x, D_x)$ *(see [16]). We shall say that* $a^w := a$ *satisfies* $(B1)$, $(B2)$ *and* $(A6)$ *if*

(B1) $a(t, x, \xi) = ia_2(x, \xi) + a_1(t, x, \xi) + a_0(t, x, \xi)$, *where* $a_2 \in S^2_{1,0}$ *is real-valued and* $a_j \in S^j_{1,0}$, for $j = 0, 1$;

(B2) $|a_2(x, \xi)| \geq \delta|\xi|^2$ *with* $x \in \mathbb{R}^n$, $|\xi|^2 \geq C$, *and* $\delta, C > 0$;

(A6) *There exists a real-valued function* $q \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *such that, with* $C_{\alpha\beta}, C_1, C_2 > 0$,

$$|\partial_\xi^\alpha \partial_x^\beta q(x, \xi)| \leq C_{\alpha\beta}\langle x\rangle\langle\xi\rangle^{-|\alpha|}, \quad x, \xi \in \mathbb{R}^n,$$

$$H_{a_2}q(x, \xi) = \{a_2, q\}(x, \xi) \geq C_1|\xi| - C_2, \quad x, \xi \in \mathbb{R}^n,$$

where we denoted by $S^j_{1,0} = S^j_{\rho=1,\delta=0} =: S^j$ *the standard class of pseudo-differential symbols of order j, and by* $\{\cdot, \cdot\}$ *the Poisson bracket.*

Lemma 2 (Doi [8], Lemma 2.3) *Assume* $(B1)$, $(B2)$ *and* $(A6)$. *Let* $\lambda(s)$ *be a positive non increasing function in* $C([0, \infty))$. *Then*

1. *If* $\lambda \in L^1([0, \infty))$ *there exists a real-valued symbol* $p \in S^0$ *and* $C > 0$ *such that*

$$H_{a_2}p \geq \lambda(|x|)|\xi| - C, \quad x, \xi \in \mathbb{R}^n; \tag{35}$$

2. *If* $\int_0^t \lambda(\tau)d\tau \leq C\log(t+1)+C'$, $t \geq 0$, $C, C' > 0$, *then there exists a real-valued symbol* $p \in S^0_1(\log\langle\xi\rangle)$ *such that*

$$H_{a_2}p \geq \lambda(|x|)|\xi| - C_1\log\langle\xi\rangle - C_2, \quad x, \xi \in \mathbb{R}^n. \tag{36}$$

Remark 6 We remark that, by taking $\lambda'(|x|) = C'\lambda(|x|)$ in Doi's lemma, where C' is any positive constant and λ is as in Lemma 2, then we get that there exists a real-valued symbol $p \in S^0$ and a constant $C > 0$ such that

$$H_{a_2}p \geq C'\lambda(|x|)|\xi| - C, \quad x, \xi \in \mathbb{R}^n. \tag{37}$$

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On the Cauchy Problem for the Nonlinear Wave Equation with Damping and Potential

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Abstract In this note, we study the Cauchy problem for the nonlinear wave equation with damping and potential terms. The aim of this study is to generalize the result in Georgiev et al. (J. Differ. Equ. 267(5):3271–3288, 2019) into two directions. One is to relax the condition which characterizes the behavior of the coefficient of the damping term at spatial infinity as in (6). The other is to treat the slowly decreasing initial data. The decaying rate of the data affects the global behavior of the solutions even if the nonlinear exponent lies in the super-critical regime (see Theorem 5 below).

1 Introduction

This paper is concerned with the Cauchy problem for the nonlinear wave equation with damping and potential:

$$\begin{cases} (\partial_t^2 + 2w(r)\partial_t - \Delta + V(r))U = |U|^p & \text{in } (0, T) \times \mathbb{R}^3, \\ U(0, x) = \varepsilon f_0(r), \quad (\partial_t U)(0, x) = \varepsilon f_1(r) & \text{for } x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $r = |x|$ and $p > 1$. In the earlier work [8], the coefficients of damping and potential terms are supposed to satisfy the relation:

$$V(r) = -w'(r) + w(r)^2 \quad \text{for } r > 0, \tag{2}$$

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where

$$w(r) = 1/r \quad \text{for } r \geq 1.$$

Keeping such a relation between the coefficients of damping and potential terms, we relax the assumption on the initial data at spatial infinity. Actually, we obtain upper bound of the lifespan for slowly decreasing initial data in Theorems 1 and 5 below. Moreover, we are able to broaden the choice of the damping coefficient, essentially, as $w(r) = \mu/(2r)$ for $\mu \geq 0$ and $r \geq 1$. The number μ affects on the shift of the critical exponent of the Strauss type, as we shall see below.

Before going into further details, we recall some known results. The case without any damping term, i.e. the case when $w = V = 0$, has been intensively studied for few decades (see [4, 6, 9, 11, 14, 17, 20], or references in [5]) and in this case there is a critical nonlinear exponent known as Strauss critical exponent that separates the global existence and blow-up of the small data solutions. This critical exponent $p_0(n)$ is given by the positive root of

$$\gamma(p, n) := 2 + (n + 1)p - (n - 1)p^2 = 0.$$

For the semilinear wave equation with potential

$$(\partial_t^2 - \Delta + V(x))U = |U|^p \quad \text{in } (0, T) \times \mathbb{R}^3,$$

one can find blow up result in [18] or global existence part in [7].

In the case where the coefficient of the damping term is a function of time variable, D'Abbicco et al. [3] derived the critical exponent for the Cauchy problem to

$$\left(\partial_t^2 + \frac{2}{1+t}\partial_t - \Delta\right) U = |U|^p \quad \text{in } (0, T) \times \mathbb{R}^3, \tag{3}$$

by assuming the radial symmetry. Indeed, they proved that the problem admits a global solution for sufficiently small initial data if $p > p_0(5)$, and that the solution blows up in finite time if $1 < p < p_0(5)$. This result can be interpreted as an effect of the damping term in (3) that shifts the critical exponent for small data solutions from $p_0(3)$ to $p_0(5)$. The assumption about the radial symmetry posed in [3] was removed by Ikeda and Sobajima [10] for the blow-up part (actually, they treated more general damping term $\mu(1 + t)^{-1}\partial_t u$ with $\mu > 0$), and by Kato and Sakuraba [12] and Lai [16] for the existence part, independently.

In the next section, we formulate our problem and describe the statements to the problem.

2 Formulation of the Problem and Results

Since we are interested in spherically symmetric solutions to the problem (1), we set

$$u(t,r) = rU(t, r\omega) \quad \text{with } r = |x|,\ \omega = x/|x|.$$

Then, by the relation (2) we obtain

$$\begin{cases} (\partial_t - \partial_r + w(r))(\partial_t + \partial_r + w(r))u = |u|^p/r^{p-1} & \text{in } (0,T)\times(0,\infty), \\ u(0,r) = \varepsilon\varphi(r), \quad (\partial_t u)(0,r) = \varepsilon\psi(r) & \text{for } r > 0, \\ u(t,0) = 0 \quad \text{for } t \in (0,T), \end{cases} \tag{4}$$

where $\varphi(r) = rf_0(r)$ and $\psi(r) = rf_1(r)$.

In order to express the solution of (4), we set $W(r) = \int_0^r w(\tau)d\tau$ for $r \geq 0$ and define

$$E_-(t,r,y) = e^{-W(r)}e^{2W(2^{-1}(y-t+r))}e^{-W(y)} \quad \text{for } t, r \geq 0,\ y \geq t-r. \tag{5}$$

We suppose that $w(r)$ is a function in $C([0,\infty)) \cap C^1(0,\infty)$ satisfying

$$w(r) = \frac{\mu}{2r} + \tilde{w}(r), \quad |\tilde{w}(r)| \lesssim r^{-1-\delta} \quad \text{for } r \geq r_0 \tag{6}$$

with some positive number r_0, $\mu \geq 0$, and $\delta > 0$. This assumption implies

$$e^{W(r)} \sim \langle r\rangle^{\mu/2}, \quad r > 0.$$

Then the definition (5) of E_- implies

$$E_-(t,r,y) \sim \frac{\langle r-t+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}}. \tag{7}$$

Following the argument in [8], we see that the problem (4) can be written in the integral form

$$u(t,r) = \varepsilon u_L(t,r) + \frac{1}{2}\iint_{\Delta_-(t,r)} E_-(t-\sigma,r,y)\frac{|u(\sigma,y)|^p}{y^{p-1}}dyd\sigma \tag{8}$$

for $t > 0$, $r > 0$, where we have set

$$\Delta_-(t,r) = \{(\sigma,y) \in (0,\infty)\times(0,\infty);\ |t-r| < \sigma + y < t + r,\ \sigma - y < t - r\}.$$

Besides, we put

$$u_L(t,r) = \frac{1}{2}\int_{|t-r|}^{t+r} E_-(t,r,y)\left(\psi(y)+\varphi'(y)+w(y)\varphi(y)\right)dy \\ + \chi(r-t)E_-(t,r,r-t)\varphi(r-t), \tag{9}$$

where $\chi(s)=1$ for $s\geq 0$, and $\chi(s)=0$ for $s<0$.

Then, the blow-up result in [8] where the case of $\mu = 2$ is handled can be extended as follows.

Theorem 1 *Suppose that* (6) *holds. Let* φ, $\psi \in C([0,\infty))$ *satisfy*

$$\varphi(r)\equiv 0, \quad \psi(r)\geq 0, \quad \psi(r)\not\equiv 0 \quad \text{for } r\geq 0. \tag{10}$$

If $1 < p \leq p_0(3+\mu)$, *then*

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & \text{if} \quad p = p_0(3+\mu), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & \text{if} \quad 1<p<p_0(3+\mu). \end{cases}$$

Here $T(\varepsilon)$ *denotes the lifespan of the problem* (4).

On the other hand, when $p > p_0(3+\mu)$, we expect that the solution exists globally. Actually, when the initial data decays rapid enough, one can show the following result analogously to [8]. But the pointwise estimate (12) is improved in the region away from the light cone, due to the factor $\langle t+r\rangle^{-1}$.

Theorem 2 *Suppose that* (6) *holds. Assume* $p > p_0(3+\mu)$ *and* $\kappa \geq (\mu/2+1)p-1$. *Let* $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$ *satisfy*

$$|\varphi(r)| \leq r\langle r\rangle^{-\kappa}, \quad |\varphi'(r)|+|\psi(r)| \leq r\langle r\rangle^{-\kappa-1} \quad \text{for } r\geq 0. \tag{11}$$

Then there exists $\varepsilon_0 > 0$ *so that the corresponding integral Eq.* (8) *to the problem* (4) *has a unique global solution satisfying*

$$|u(t,r)| \lesssim \varepsilon r\,\langle r\rangle^{-\mu/2}\langle t+r\rangle^{-1}\langle t-r\rangle^{-\eta}, \quad \eta := (\mu/2+1)(p-1)-1 \tag{12}$$

for $t>0$, $r>0$ *and any* $\varepsilon \in (0,\varepsilon_0]$.

This theorem leads us to one natural question, that is, what will happen when the initial data decays more slowly. In view of the work of Asakura [1], the self-similarity comes into play (see also [2, 13, 15, 19]). Namely, the global behavior would be different between the cases $\kappa \geq 2/(p-1)$ and $\kappa < 2/(p-1)$. Indeed, we are able to show the global existence result in the former case.

Theorem 3 *Let* $\kappa > \mu/2$. *Suppose that* (6) *holds. Assume* $p > p_0(3+\mu)$ *and* $\kappa \geq 2/(p-1)$. *Let* $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$ *satisfy* (11). *Then there exists* $\varepsilon_0 > 0$ *so that the integral Eq.* (8) *has a unique global solution for* $\varepsilon \in (0, \varepsilon_0]$.

The proof of Theorem 3 is based on the contraction mapping principle in a suitable weighted L^∞-space, similarly to the proof of Theorem 2. But we need to replace the weight function according to the size of κ as

$$w(r,t) = \frac{r}{\langle r \rangle^{\mu/2}} \times \begin{cases} \langle t+r \rangle^{-(\kappa-\mu/2)} & (\mu/2 < \kappa < \mu/2+1), \\ \langle t+r \rangle^{-1} \left(1 + \log \dfrac{1+t+r}{1+|t-r|}\right) & (\kappa = \mu/2+1), \\ \langle t+r \rangle^{-1} \langle t-r \rangle^{-(\kappa-\mu/2-1)} & (\mu/2+1 < \kappa \leq (\mu/2+1)p-1). \end{cases}$$

for $t > 0$, $r > 0$. Note that $w(r,t)$ coincides with the upper bound appeared in (12) when $\kappa = (\mu/2+1)p - 1$.

When either $p > p_0(3+\mu)$ and $\kappa < 2/(p-1)$ or $1 < p \leq p_0(3+\mu)$, we obtain the following lower bounds of the lifespan.

Theorem 4 *Let* $\kappa > \mu/2$ *and set* $\kappa_1 := \mu/2 + 1 + 1/p$. *Suppose that* (6) *holds. Let* $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$ *satisfy (11). Then there exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$

$$T(\varepsilon) \geq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & (p = p_0(3+\mu) \text{ and } \kappa > \kappa_1), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & (1 < p < p_0(3+\mu) \text{ and } \kappa > \kappa_1), \\ \exp(C\varepsilon^{-(p-1)}) & (p = 1 + 2/\kappa = p_0(3+\mu)), \\ Cb(\varepsilon) & (1 < p < 1 + 2/\kappa \text{ and } \kappa = \kappa_1), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1 + 2/\kappa \text{ and } \kappa < \kappa_1). \end{cases}$$

Here $b(\varepsilon)$ *is defined by*

$$\varepsilon^{p(p-1)} b^{\gamma(p,3+\mu)/2} (\log(1+b))^{p-1} = 1.$$

In order to prove Theorem 4, we reformulate the integral Eq. (8) to the following one:

$$v(t,r) = \frac{1}{2} \iint_{\Delta_-(t,r)} E_-(t-\sigma, r, y) \frac{|\varepsilon u_L(\sigma, y) + v(\sigma, y)|^p}{y^{p-1}} dy d\sigma \qquad (13)$$

for $t > 0$, $r > 0$, by introducing the new unknown function $v = u - \varepsilon u_L$, as in the proof of Theorem 2.3 in [14]. It is rather easy to treat the integral Eq. (13) than the original one. Indeed, the solution v can be presumably assumed to satisfy the essentially same upper bound as in (12), although the solution u_L to the homogeneous equation does not satisfy such an estimate if the size of κ is small. Moreover, since u_L exists globally in time, the maximal existence time of the solution u of (8) is the same as that of the solution v of (13), so that the desired conclusion follows from the study of (13).

To conclude the optimality of those lower bounds in Theorem 4 with respect to ε, the upper bounds given in Theorem 1 are not enough for the last three cases. However, the following result enable us to conclude the optimality in these cases.

Theorem 5 *Suppose that* (6) *holds. Let* φ, $\psi \in C([0,\infty))$ *satisfy*

$$\varphi(r) \equiv 0, \quad \psi(r) \geq (1+r)^{-\kappa} \quad \textit{for } r \geq 0 \tag{14}$$

for some $0 < \kappa \leq \kappa_1$. *Then there exist* $C > 0$, $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-(p-1)}) & (p = 1 + 2/\kappa = p_0(3+\mu)), \\ Cb(\varepsilon) & (1 < p < 1 + 2/\kappa \textit{ and } \kappa = \kappa_1), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1 + 2/\kappa \textit{ and } \kappa < \kappa_1). \end{cases}$$

Thanks to the assumption (14), if the solution of (8) exists globally in time, then we can prove that for any (t, r) satisfying $0 < t \leq 2r$ and $t - r \geq b$ with a positive number b, and for any natural number n, the following type of lower bound of the solution:

$$u(t,r) \geq \frac{(t-r)^{\mu/2+1}}{r^{\mu/2}(t-r-b)^{2/(p-1)}} \exp(p^n \log J(t,r)), \tag{15}$$

$$J(t,r) = \varepsilon E\,(t-r-b)^{2/(p-1)+\mu/2+1}(t-r)^{-\kappa-(\mu/2)-1} \tag{16}$$

holds, when $1 < p < 1 + 2/\kappa$, for instance. Here E is a positive constant independent of t, r, n, and ε. By choosing (t, r) far away from the origin on the line $t = 2r$ so that $\log J(t, r)$ is strictly positive, we find that the value of $u(t, r)$ becomes unbounded as $n \to \infty$. This gives a contradiction together with the upper bound of the lifespan.

This paper is organized as follows. We shall prove only Theorems 1 and 2 in this note, because the proofs of other theorems are rather technical and will appear elsewhere. In the Sect. 3, we give preliminary facts. The Sect. 4 is devoted to the proof of a blow-up result given in Theorem 1. In the Sect. 5, we derive a priori upper bounds and complete the proof of Theorem 2.

3 Preliminaries

In this section we prepare a couple of lemmas which will be used in the proofs of Theorems 1 and 2. For the proofs of Lemmas 1 and 2, see [14], Lemma 2.2 and Lemma 2.3.

Lemma 1 *Let* $0 < a < b$ *and* $\mu, \nu \ge 0$. *Then there exists* $C = C(\mu, \nu) > 0$ *such that*

$$\int_a^b \frac{(\rho - a)^\nu}{\rho^\mu} d\rho \ge \frac{C}{a^{\mu-\nu-1}} \left(1 - \frac{a}{b}\right)^{\nu+1}.$$

Lemma 2 *Let* $C_1, C_2 > 0$, $\alpha, \beta \ge 0$, $\theta \le 1$, $\varepsilon \in (0, 1]$, *and* $p > 1$. *Suppose that* $f(y)$ *satisfies*

$$f(y) \ge C_1 \varepsilon^\alpha, \quad f(y) \ge C_2 \varepsilon^\beta \int_1^y \left(1 - \frac{\eta}{y}\right) \frac{f(\eta)^p}{\eta^\theta} d\eta, \quad y \ge 1.$$

Then, $f(y)$ *blows up in a finite time* $T_*(\varepsilon)$. *Moreover, there exists a constant* $C^* = C^*(C_1, C_2, p, \theta) > 0$ *such that*

$$T_*(\varepsilon) \le \begin{cases} \exp(C^* \varepsilon^{-\{(p-1)\alpha+\beta\}}) & \text{if } \theta = 1, \\ C^* \varepsilon^{-\{(p-1)\alpha+\beta\}/(1-\theta)} & \text{if } \theta < 1. \end{cases}$$

Lemma 3 *Let* $0 \le a \le b$ *and* $k \in \mathbb{R}$. *Then we have*

$$\int_a^b \langle x \rangle^{-k} dx \lesssim (b - a) \times \begin{cases} \langle b \rangle^{-k} & (k < 1), \\ \langle b \rangle^{-1} \langle a \rangle^{-k+1} & (k > 1), \\ \langle b \rangle^{-1} \Psi(a, b) & (k = 1). \end{cases} \tag{17}$$

Here, for $0 \le a \le b$, *we put*

$$\Psi(a, b) := 2 + \log \frac{1 + b}{1 + a}. \tag{18}$$

Proof

(i) When $k > 1$, we have

$$\begin{aligned} \int_a^b \langle x \rangle^{-k} dx &\lesssim \frac{1}{k-1} \left\{ \frac{1}{(1+a)^{k-1}} - \frac{1}{(1+b)^{k-1}} \right\} \\ &\lesssim \frac{1}{(1+a)^{k-1}} \left\{ 1 - \left(\frac{1+a}{1+b} \right)^{k-1} \right\}. \end{aligned}$$

Note that

$$1 - s^l \le \max\{1, l\}(1 - s) \quad \text{for} \quad l \ge 0, \ 0 \le s \le 1. \tag{19}$$

Hence we obtain (17) for $k > 1$.

(ii) When $k < 1$, we have in the similar manner

$$\begin{aligned}\int_a^b \langle x\rangle^{-k}dx &\lesssim \frac{1}{1-k}\left\{\frac{1}{(1+b)^{k-1}}-\frac{1}{(1+a)^{k-1}}\right\}\\ &\lesssim \frac{1}{(1+b)^{k-1}}\left\{1-\left(\frac{1+a}{1+b}\right)^{1-k}\right\}\\ &\lesssim (b-a)\langle b\rangle^{-k}.\end{aligned}$$

(iii) When $k = 1$, It follows that

$$\int_a^b \langle x\rangle^{-1}dx \lesssim \log\left(\frac{1+b}{1+a}\right). \tag{20}$$

If $a \geq b/2$, since $\log(1+s) \leq s$ $(s \geq 0)$, we find that

$$\int_a^b \langle x\rangle^{-1}dx \lesssim \log\left(1+\frac{b-a}{1+a}\right) \lesssim \frac{b-a}{1+a} \lesssim (b-a)\langle b\rangle^{-1}.$$

If $a \leq b/2$ and $b \geq 1$, we find that $b-a \geq b/2$. Hence we have from (20)

$$\int_a^b \langle x\rangle^{-1}dx \lesssim \frac{b-a}{b}\log\left(\frac{1+b}{1+a}\right) \lesssim (b-a)\langle b\rangle^{-1}\log\left(\frac{1+b}{1+a}\right).$$

If $0 < b \leq 1$, we obtain

$$\int_a^b \langle x\rangle^{-1}dx \lesssim b-a \sim (b-a)\langle b\rangle^{-1}.$$

Therefore we get (17). This completes the proof. □

Lemma 4 *Let* $k_1, k_2, k_3 \geq 0$ *and* $\alpha \geq 0$. *Then we have*

$$\int_{-\alpha}^{\alpha} \langle\alpha+\beta\rangle^{-k_1-k_2}\langle\beta\rangle^{-k_1-k_3}d\beta \lesssim \langle\alpha\rangle^{-k_1} \times \begin{cases}\langle\alpha\rangle^{1-(k_1+k_2+k_3)} & (k_1+k_2+k_3 < 1),\\ 1 & (k_1+k_2+k_3 > 1),\\ \log(2+\alpha) & (k_1+k_2+k_3 = 1),\end{cases}$$

Proof First of all, we prove for $a, b \geq 0$ and $\alpha \geq 0$

$$\int_{-\alpha}^{\alpha} \langle\alpha+\beta\rangle^{-a}\langle\beta\rangle^{-b}d\beta \lesssim \begin{cases}\langle\alpha\rangle^{1-(a+b)} & (a+b < 1),\\ 1 & (a+b > 1),\\ \log(2+\alpha) & (a+b = 1).\end{cases} \tag{21}$$

We note that for $-\alpha < \beta < -\alpha/2$, we have $|\beta| > \alpha+\beta$ and that for $-\alpha/2 < \beta < \alpha$, we have $|\beta| < \alpha + \beta$. Then we see that the β-integral is bounded by the sum of

$$\int_{-\alpha}^{-\alpha/2} \langle \alpha + \beta \rangle^{-(a+b)} d\beta \le \int_{-\alpha}^{0} \langle \alpha + \beta \rangle^{-(a+b)} d\beta,$$
$$\int_{-\alpha/2}^{\alpha} \langle \beta \rangle^{-(a+b)} d\beta \le 2 \int_{0}^{\alpha} \langle \beta \rangle^{-(a+b)} d\beta.$$

Then we get (21) by a direct computation.

We now divide the β-integral into I_1 and I_2:

$$I_1 := \int_{-\alpha}^{-\alpha/2} \langle \alpha + \beta \rangle^{-k_1-k_2} \langle \beta \rangle^{-k_1-k_3} d\beta,$$
$$I_2 := \int_{-\alpha/2}^{\alpha} \langle \alpha + \beta \rangle^{-k_1-k_2} \langle \beta \rangle^{-k_1-k_3} d\beta.$$

Then we get from (21)

$$I_1 \lesssim \langle \alpha/2 \rangle^{-k_1} \int_{-\alpha}^{\alpha} \langle \alpha + \beta \rangle^{-k_1-k_2} \langle \beta \rangle^{-k_3} d\beta$$
$$\lesssim \langle \alpha \rangle^{-k_1} \times \begin{cases} \langle \alpha \rangle^{1-(k_1+k_2+k_3)} & (k_1+k_2+k_3 < 1), \\ 1 & (k_1+k_2+k_3 > 1), \\ \log(2+\alpha) & (k_1+k_2+k_3 = 1). \end{cases}$$

As to I_1, we have

$$I_2 \lesssim \langle \alpha/2 \rangle^{-k_1} \int_{-\alpha}^{\alpha} \langle \alpha + \beta \rangle^{-k_2} \langle \beta \rangle^{-k_1-k_3} d\beta,$$

which implies the desired estimate by (21). This completes the proof. □

4 Proof of Theorem 1

Let u denote the solution of the problem (4) in what follows. When $\varphi \equiv 0$, it follows from (8), (9) and (7) that

$$u(t,r) \gtrsim \varepsilon u_L(t,r) + \widetilde{I_-}(|u|^p/y^{p-1})(t,r), \tag{22}$$
$$u_L(t,r) \gtrsim \widetilde{J_-}(\psi)(t,r) \tag{23}$$

holds for $t, r > 0$, where we put

$$\widetilde{I_-}(F)(t,r) = \iint_{\Delta_-(t,r)} \frac{\langle -t+\sigma+r+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} F(\sigma, y)dyd\sigma, \tag{24}$$

$$\widetilde{J_-}(\psi)(t,r) = \int_{|t-r|}^{t+r} \frac{\langle r-t+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \psi(y)dy. \tag{25}$$

Our first step is to obtain basic lower bounds of the solution to the problem (4). By (10), we may assume that ψ is strictly positive in an interval $[a, b]$.

Lemma 5 *We assume* (10) *holds. Then we have*

$$u_L(t,r) \gtrsim \frac{c_0}{\langle r\rangle^{\mu/2}}, \quad c_0 := \min_{a\le r\le b} \psi(r) \tag{26}$$

for

$$t < r < t+a, \ t+r > b. \tag{27}$$

Moreover, if u is the solution to (4)*, then we have*

$$u(t,r) \gtrsim \frac{\varepsilon^p}{\langle r\rangle^{\mu/2}(t-r)^{\eta}}, \quad \eta = (\mu/2+1)(p-1)-1 \tag{28}$$

for $0 < t < 2r$ *and* $t - r > b$*.*

Proof First, we show (26). Let (t, r) satisfy (27). Then, from (23) we have

$$\begin{aligned} u_L(t,r) &\gtrsim \int_{r-t}^{t+r} \frac{\langle r-t+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \psi(y)dy \\ &\gtrsim c_0 \int_a^b \frac{1}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} dy, \end{aligned}$$

which implies (26).

Next we show (28). Let $0 < t < 2r$ and $t - r > b$. If we set

$$\widetilde{\Sigma}(t,r) = \{(\sigma, y) \in (0,\infty)\times(0,\infty);\ 0 \le y-\sigma \le a,\ t-r < \sigma+y < t+r\},$$

then $\widetilde{\Sigma}(t,r) \subset \Delta_-(t,r)$. In addition, we see from (22) and (26) that $u(\sigma, y) \gtrsim \varepsilon\langle y\rangle^{-\mu/2}$ for $(\sigma, y) \in \widetilde{\Sigma}(t,r)$. Therefore, from (22) we get

$$u(t,r) \gtrsim \varepsilon^p \iint_{\widetilde{\Sigma}(t,r)} \frac{\langle -t+\sigma+r+y\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \frac{1}{\langle y\rangle^{(\mu/2+1)p-1}} dyd\sigma.$$

Now, introducing the coordinates $\alpha = \sigma + y$, $\beta = \sigma - y$, we obtain

$$u(t,r) \gtrsim \varepsilon^p \int_{t-r}^{t+r} d\alpha \int_{-a}^{0} \frac{\langle \alpha - t + r\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle \alpha - \beta\rangle^{(\mu/2+1)p+\mu/2-1}} d\beta$$
$$\gtrsim \varepsilon^p \int_{t-r}^{t+r} \frac{\langle \alpha - t + r\rangle^{\mu}}{\langle r\rangle^{\mu/2}\langle \alpha\rangle^{(\mu/2+1)p+\mu/2-1}} d\alpha.$$

Since $t < 2r$, we have $t + r > 3(t - r)$, so that

$$\langle r\rangle^{\mu/2} u(t,r) \gtrsim \varepsilon^p \int_{t-r}^{3(t-r)} \frac{(\alpha - t + r)^{\mu}}{\langle \alpha\rangle^{(\mu/2+1)p+\mu/2-1}} d\alpha$$
$$\gtrsim \varepsilon^p \langle t - r\rangle^{-((\mu/2+1)p+\mu/2-1)} \int_{t-r}^{3(t-r)} (\alpha - t + r)^{\mu} d\alpha$$
$$\gtrsim \varepsilon^p (t - r)^{-\eta}$$

for $t - r > b$. This completes the proof. □

For $\rho > 0$, we introduce the following quantity:

$$\langle u\rangle(\rho) = \inf\{\langle y\rangle^{\mu/2}(\sigma - y)^{\eta}|u(\sigma, y)|;\ (\sigma, y) \in \Sigma(\rho)\},$$

where we set

$$\Sigma(\rho) = \{(\sigma, y);\ 0 \le \sigma \le 2y,\ \sigma - y \ge \rho\}.$$

For simplicity, we assume $0 < b \le 1$. Then, (28) yields

$$\langle u\rangle(y) \ge C_1 \varepsilon^p \quad \text{for } y \ge 1. \tag{29}$$

Let $\xi \ge 1$ and $(t, r) \in \Sigma(\xi)$, so that $t - r \ge 1$. For $\rho > 0$ we set

$$\tilde{\Sigma}(\rho, t - r) = \{(\sigma, y);\ y \ge t - r,\ \sigma + y \le 3(t - r),\ \sigma - y \ge \rho\}.$$

It is easy to see that $\tilde{\Sigma}(\rho, t - r) \subset \Delta_-(t, r)$ for any $\eta > 0$ and $(t, r) \in \Sigma(\xi)$ and that $(\sigma, y) \in \tilde{\Sigma}(1, t - r)$ implies $(\sigma, y) \in \Sigma(\sigma - y)$. Therefore, from (22) we have

$$u(t,r)$$
$$\gtrsim \iint_{\tilde{\Sigma}(1,t-r)} \frac{(-t + \sigma + r + y)^{\mu}}{\langle r\rangle^{\mu/2}\langle y\rangle^{\mu/2}} \frac{[\langle u\rangle(\sigma - y)]^p}{\langle y\rangle^{(\mu/2+1)p-1}(\sigma - y)^{p\eta}} dy\, d\sigma$$
$$\gtrsim \frac{(t - r)^{\mu}}{\langle r\rangle^{\mu/2}} \iint_{\tilde{\Sigma}(1,t-r)} \frac{[\langle u\rangle(\sigma - y)]^p}{\langle y\rangle^{(\mu/2+1)p+\mu/2-1}(\sigma - y)^{p\eta}} dy\, d\sigma,$$

because $-t+\sigma+r+y=-t+r+(\sigma-y)+2y\geq 1+(t-r)$ for $(\sigma,y)\in\tilde{\Sigma}(1,t-r)$. Changing the variables by $\beta=\sigma-y$, $z=y$, we have

$$\begin{aligned}u(t,r)&\gtrsim\frac{(t-r)^{\mu}}{\langle r\rangle^{\mu/2}}\int_1^{t-r}d\beta\int_{t-r}^{(3(t-r)-\beta)/2}\frac{[\langle u\rangle(\beta)]^p}{\langle z\rangle^{(\mu/2+1)p+(\mu/2-1)}\beta^{p\eta}}dz\\&\gtrsim\frac{1}{\langle r\rangle^{\mu/2}(t-r)^{(\mu/2+1)(p-1)}}\int_1^{t-r}\frac{t-r-\beta}{2}\frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}}\,d\beta\\&\gtrsim\frac{1}{\langle r\rangle^{\mu/2}(t-r)^{(\mu/2+1)p-(\mu/2+2)}}\int_1^{t-r}\left(1-\frac{\beta}{t-r}\right)\frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}}\,d\beta.\end{aligned}$$

Since the function

$$y\mapsto\int_1^{y}\left(1-\frac{\beta}{y}\right)\frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}}\,d\beta$$

is non-decreasing, for any $(t,r)\in\Sigma(\xi)$, we have

$$\langle r\rangle^{\mu/2}(t-r)^{(\mu/2+1)p-(\mu/2+2)}u(t,r)\geq C_2\int_1^{\xi}\left(1-\frac{\beta}{\xi}\right)\frac{[\langle u\rangle(\beta)]^p}{\beta^{p\eta}}\,d\beta.$$

Thus, recalling $\eta=(\mu/2+1)p-(\mu/2+2)$ from (28), we obtain

$$\langle u\rangle(\xi)\geq C_2\int_1^{\xi}\left(1-\frac{\beta}{\xi}\right)\frac{[\langle u\rangle(\beta)]^p}{\eta^{p\eta}}\,d\beta,\quad \xi\geq 1 \tag{30}$$

Proof of Theorem 1 By (29) and (30), we can apply Lemma 2 as $\alpha=p$, $\beta=0$, and $\theta=p\eta$. Since $1<p\leq p_0(3+\mu)$ implies $\theta\leq 1$, the maximal existence time $T_*(\varepsilon)$ of $\langle u\rangle(\xi)$ satisfies the following estimates:

$$T_*(\varepsilon)\leq\begin{cases}\exp(C\varepsilon^{-p(p-1)}) & \text{if } \theta=1,\\ C\varepsilon^{-p(p-1)/(1-\theta)} & \text{if } \theta<1.\end{cases}$$

Since $\theta=1$ and $\theta<1$ correspond to $p=p_0(3+\mu)$ and $1<p<p_0(3+\mu)$, respectively, we obtain the desired conclusion. □

5 Proof of Theorem 2

Our first step is to obtain the following estimates for the homogeneous part of the solution to the problem (8).

Lemma 6 *Assume that* (6) *holds and* $\varphi \in C^1([0,\infty))$, $\psi \in C^0([0,\infty))$ *satisfy* (11)*, so that*

$$|\varphi(r)| \lesssim r\,\langle r\rangle^{-\kappa}, \quad |\psi(r)+\varphi'(r)+w(r)\varphi(r)| \lesssim \langle r\rangle^{-\kappa} \text{ for } r \geq 0 \tag{31}$$

holds with some positive constant κ. *We put*

$$\nu := \kappa - \mu/2 - 1. \tag{32}$$

Then we have

$$\begin{aligned}&\left|\int_{|t-r|}^{t+r} E_-(t,r,y)\big(\psi(y)+\varphi'(y)+w(y)\varphi(y)\big)\,dy\right|\\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases}\langle t+r\rangle^{-\kappa+\mu/2} & (\nu<0),\\ \langle t+r\rangle^{-1}\Psi(|t-r|,t+r) & (\nu=0),\\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu>0)\end{cases}\end{aligned} \tag{33}$$

for $t>0$, $r>0$, *where* $\Psi(a,b)$ *is defined in* (18)*. Moreover, for* $0<t\leq r$ *we have*

$$|E_-(t,r,r-t)\varphi(r-t)| \lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases}\langle t+r\rangle^{-\kappa+\mu/2} & (\nu<0),\\ \langle t+r\rangle^{-1}\Psi(r-t,t+r) & (\nu=0),\\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu>0).\end{cases} \tag{34}$$

Proof We begin with the proof of (33). In the following, let $t>0$, $r>0$. Since $0 \leq r-t+y \leq 2y$ for $y \geq |t-r|$, from (7) we have

$$|E_-(t,r,y)| \lesssim \langle y\rangle^{\mu/2}/\langle r\rangle^{\mu/2} \quad \text{for } y \geq |t-r|. \tag{35}$$

Therefore, by using the assumptions on the data, the left hand side of (33) is estimated by

$$\langle r\rangle^{-\mu/2}\int_{|r-t|}^{t+r}\langle y\rangle^{\mu/2}\left|\psi(y)+\varphi'(y)+w(y)\varphi(y)\right|dy \lesssim \langle r\rangle^{-\mu/2}\int_{|r-t|}^{t+r}\langle y\rangle^{-\kappa+\mu/2}dy.$$

From (32) and Lemma 3, the last integral is estimated as follows:

$$\int_{|r-t|}^{t+r}\langle y\rangle^{-\kappa+\mu/2}dy \lesssim r \times \begin{cases}\langle t+r\rangle^{-\kappa+\mu/2} & (\nu<0),\\ \langle t+r\rangle^{-1}\Psi(|t-r|,t+r) & (\nu=0),\\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu>0).\end{cases}$$

Therefore we obtain (33).

Next we prove (34), by assuming $0 < t \le r$. From (31) and (35) we have

$$|E_-(t,r,y)\varphi(y)| \lesssim \frac{y}{\langle r\rangle^{\mu/2}\langle y\rangle^{\kappa-\mu/2}} \quad \text{for } y \ge |t-r|. \tag{36}$$

Let $r \ge 1$. It follows from (32) and (36) that

$$\begin{aligned} |E_-(t,r,r-t)\varphi(r-t)| &\lesssim \frac{r-t}{\langle r\rangle^{\mu/2}\langle r-t\rangle^{\kappa-\mu/2}} \\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2+1}\langle r-t\rangle^{\kappa-\mu/2-1}} \\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases} \langle r+t\rangle^{-\kappa+\mu/2} & (\nu \le 0), \\ \langle r+t\rangle^{-1}\langle r-t\rangle^{-\nu} & (\nu > 0). \end{cases} \end{aligned}$$

Let $0 < r \le 1$. We obtain from (36)

$$\begin{aligned} |E_-(t,r,r-t)\varphi(r-t)| &\lesssim \frac{r}{\langle r\rangle^{\mu/2}\langle r-t\rangle^{\kappa-\mu/2}} \\ &\lesssim \frac{r}{\langle r\rangle^{\mu/2}} \times \begin{cases} \langle r+t\rangle^{-\kappa+\mu/2} & (\nu \le 0), \\ \langle r+t\rangle^{-1}\langle r-t\rangle^{-\kappa+\mu/2} & (\nu > 0). \end{cases} \end{aligned}$$

Hence, we obtain the desired estimate (34). This completes the proof. □

For $t > 0$, $r > 0$, it follows from (9) and Lemma 6 that

$$|u_L(t,r)| \le \tilde{C}_0 r\,\langle r\rangle^{-\mu/2} \times \begin{cases} \langle t+r\rangle^{-\kappa+\mu/2} & (\nu < 0), \\ \langle t+r\rangle^{-1}\Psi(|t-r|,t+r) & (\nu = 0), \\ \langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu} & (\nu > 0) \end{cases} \tag{37}$$

with some positive constant $\tilde{C}_0$, provided (6) and (11) hold.

Our next step is to consider the integral operator appeared in (8):

$$I_-(F)(t,r) := \frac{1}{2}\iint_{\Delta_-(t,r)} E_-(t-\sigma,r,y)F(\sigma,y)dyd\sigma.$$

For $(\sigma,y) \in \Delta_-(t,r)$ we have $y \ge |t-r-\sigma|$, so that (35) yields

$$E_-(t-\sigma,r,y) \lesssim \langle r\rangle^{-\mu/2}\langle y\rangle^{\mu/2} \quad \text{for} \quad (\sigma,y) \in \Delta_-(t,r).$$

Hence we get

$$|I_-(F)(t,r)| \lesssim \langle r\rangle^{-\mu/2}\iint_{\Delta_-(t,r)} \langle y\rangle^{\mu/2}|F(\sigma,y)|dyd\sigma. \tag{38}$$

In order to derive an apriori estimate, we introduce the following weighted L^∞-norm:

$$\|u\|_1 = \sup_{(t,r)\in[0,\infty)\times[0,\infty)} \{w_1(t,r)^{-1}|u(t,r)|\}, \tag{39}$$

where we put

$$w_1(t,r) := r\,\langle r\rangle^{-\mu/2}\langle t+r\rangle^{-1}\langle t-r\rangle^{-\eta}. \tag{40}$$

Here we choose

$$\eta = (\mu/2+1)(p-1)-1$$

as in (28), so that $\eta > 1/p$, by the assumption $p > p_0(3+\mu)$.

Lemma 7 *Let $\eta > 0$ be as above. Then, there exists a positive constant $\widetilde{C}_1$ such that*

$$\|I_-(F)\|_1 \le \widetilde{C}_1\|u\|_1^p \tag{41}$$

with $F(t,r) = |u(t,r)|^p/r^{p-1}$.

Proof From (39) and (40), we obtain

$$\langle r\rangle^{\mu p/2}\langle t+r\rangle^p\langle t-r\rangle^{\eta p}|F(t,r)| \le r\|u\|_1^p.$$

It follows from (38) that

$$|I_-(F)(t,r)| \lesssim \langle r\rangle^{-\mu/2}\|u\|_1^p I(t,r),$$

where we put

$$I(t,r) := \iint_{\Delta_-(t,r)} \frac{y}{\langle y\rangle^{\mu(p-1)/2}\langle\sigma+y\rangle^p\langle\sigma-y\rangle^{\eta p}}dyd\sigma. \tag{42}$$

In order to show (41), it is enough to prove

$$I(t,r) \lesssim \frac{r}{\langle t+r\rangle\langle t-r\rangle^\eta}.$$

To evaluate the integral (42), we pass to the coordinates

$$\alpha = \sigma + y, \quad \beta = y - \sigma$$

and deduce

$$I(t,r) \lesssim \int_{|r-t|}^{t-r} \int_{r-t}^{\alpha} \frac{1}{\langle\alpha\rangle^{p}\langle\alpha+\beta\rangle^{\mu(p-1)/2-1}\langle\beta\rangle^{\eta p}} d\beta d\alpha. \tag{43}$$

First, suppose $r \geq t$. Then we get

$$I(t,r) \lesssim \int_{r-t}^{t-r} \frac{d\alpha}{\langle\alpha\rangle^{\eta+1}} \int_{r-t}^{\alpha} \frac{1}{\langle\beta\rangle^{\eta p}} d\beta.$$

Since $p\eta > 1$, we have from Lemma 3

$$I(t,r) \lesssim \int_{r-t}^{t+r} \frac{1}{\langle\alpha\rangle^{\eta+1}} d\alpha \lesssim \frac{r}{\langle t+r\rangle\langle t-r\rangle^{\eta}}.$$

Next, suppose $r < t$. Since $p\eta > 1$, we have from (43), Lemma 3, and Lemma 4 with $k_1 = \eta$, $k_2 = 0$, and $k_3 = \eta(p-1)$

$$\begin{aligned} I(t,r) &\lesssim \int_{t-r}^{t+r} \frac{d\alpha}{\langle\alpha\rangle} \int_{-\alpha}^{\alpha} \frac{1}{\langle\alpha+\beta\rangle^{\eta}\langle\beta\rangle^{\eta p}} d\beta \\ &\lesssim \int_{t-r}^{t+r} \frac{1}{\langle\alpha\rangle^{\eta+1}} d\alpha \lesssim \frac{r}{\langle t+r\rangle\langle t-r\rangle^{\eta}}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2 Let X be the linear space defined by

$$X = \{u(t,r) \in C([0,\infty) \times [0,\infty))\ ;\ \|u\|_1 < \infty\}.$$

We can verify easily that X is complete with respect the norm $\|\cdot\|_1$. We define the sequence of functions $\{u_n\}$ by

$$u_0 = \varepsilon u_L, \quad u_{n+1} = \varepsilon u_L + I_-(|u_n|^p/r^{p-1}) \quad (n = 0, 1, 2, \cdots).$$

Since $\kappa \geq (\mu/2+1)p - 1$ and $\nu = \kappa - \mu/2 - 1$, we have $\nu \geq \eta$. Therefore, it follows from (37), (39) and (40) that $\|u_0\|_1 \leq \varepsilon\widetilde{C}_0$. Hence $u_0 \in X$.

Now, by choosing ε is sufficiently small, we find from Lemma 7 that $\{u_n\} \in X$ for all n. Moreover, we can prove that $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that u_n converges uniformly to u as $n \to \infty$. Clearly, u satisfies (8). This completes the proof. □

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Local Well-Posedness for the Scale-Critical Semilinear Heat Equation with a Weighted Gradient Term

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Abstract The purpose of this paper is to prove local well-posedness for the Cauchy problem of the scale-critical semilinear heat equation with a weighted gradient term in the framework of weighted Lebesgue spaces and weighted Sobolev spaces.

1 Introduction and Main Results

We consider the Cauchy problem of the following semilinear heat equation:

$$\begin{cases} \partial_t u - \Delta u = |x|^{\gamma} |\nabla u|^{\alpha}, & t > 0, \ x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $d \in \mathbb{N}$, $\gamma > -1$ and $\alpha > 1$. The Eq. (1) in the case $\gamma = 0$ is known as the viscous Hamilton-Jacobi equation, and has been well studied in the framework of Lebesgue space $L^q(\mathbb{R}^d)$ and Sobolev space $W^{1,q}(\mathbb{R}^d)$ (see [1, 3, 4, 11, 12, 14, 16, 18]

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and references therein). A similar problem to (1) with $\gamma \neq 0$ is the Hardy-Hénon parabolic equation

$$\begin{cases} \partial_t u - \Delta u = |x|^{\gamma} |u|^{\alpha-1} u, & t > 0,\ x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

which has been studied by many authors (see [5–10, 13, 15, 17, 19]). In particular, local well-posedness has been established for all $\gamma > -\min\{2, d\}$ in the framework of the weighted Lebesgue spaces (see [9]).

In this paper, we prove the local well-posedness for (1) in the scale-critical case. Following the idea of [9], we employ the weighted Lebesgue spaces $L^q_s(\mathbb{R}^d)$ and weighted Sobolev space $W^{1,q}_s(\mathbb{R}^d)$ as the solution spaces. Here, for $s \in \mathbb{R}$ and $q \in [1, \infty]$, $L^q_s(\mathbb{R}^d)$ and $W^{1,q}_s(\mathbb{R}^d)$ are defined by the spaces of Lebesgue measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{L^q_s} := \left(\int_{\mathbb{R}^d} (|x|^s |f(x)|)^q \, d\tau \right)^{\frac{1}{q}} < \infty,$$

$$\|f\|_{W^{1,q}_s} := \|f\|_{L^q_s} + \|\nabla f\|_{L^q_s} < \infty$$

with the usual modification from integral to supremum for $q = \infty$, respectively. We introduce some exponents from the point of view of scale invariance. We define the scale transformation

$$u_\lambda(t, x) := \lambda^{\frac{2+\gamma-\alpha}{\alpha-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

under which (1) is invariant. More precisely, if u is a classical solution to (1), then so is u_λ with the rescaled initial data $\lambda^{\frac{2+\gamma-\alpha}{\alpha-1}} u_0(\lambda x)$. Moreover, we calculate

$$\|u_\lambda(0)\|_{L^q_s} = \lambda^{-s+\frac{2+\gamma-\alpha}{\alpha-1}-\frac{d}{q}} \|u_0\|_{L^q_s},$$

$$\|\nabla u_\lambda(0)\|_{L^q_s} = \lambda^{-s+\frac{1+\gamma}{\alpha-1}-\frac{d}{q}} \|\nabla u_0\|_{L^q_s}$$

for $\lambda > 0$. Hence, we define

$$s_{c,1} = s_{c,1}(d, \gamma, \alpha, q) := \frac{2+\gamma-\alpha}{\alpha-1} - \frac{d}{q},$$

$$s_{c,2} = s_{c,2}(d, \gamma, \alpha, q) := \frac{1+\gamma}{\alpha-1} - \frac{d}{q},$$

and then,

$$\|u_\lambda(0)\|_{L^q_{s_{c,1}}} = \|u_0\|_{L^q_{s_{c,1}}},$$

$$\|\nabla u_\lambda(0)\|_{L^q_{s_{c,2}}} = \|\nabla u_0\|_{L^q_{s_{c,2}}}$$

for $\lambda > 0$, i.e., the norms $\|u_\lambda(0)\|_{L^q_{s_{c,1}}}$ and $\|\nabla u_\lambda(0)\|_{L^q_{s_{c,2}}}$ are invariant with respect to λ. Thus, the exponents $s_{c,1}$ and $s_{c,2}$ are called scale-critical exponents with respect to $\|\cdot\|_{L^q_s}$ and $\|\nabla\cdot\|_{L^q_s}$, respectively. Note that $s_{c,2} = s_{c,1} + 1$ and

$$W^{1,q}_{s_{c,2}}(\mathbb{R}^d) \hookrightarrow L^q_{s_{c,1}}(\mathbb{R}^d)$$

holds if $2 + \gamma - \alpha > 0$ and $1 \le q < \infty$ by the weighted Hardy inequality (see, e.g., (1.2) in [2]). Moreover, we define the exponent α_F as

$$\alpha_F = \alpha_F(d, \gamma) := 1 + \frac{1+\gamma}{d+1},$$

which is the scale-critical exponent in $L^1(\mathbb{R}^d)$ and the Fujita exponent for (1). From the point of view of integrability of the nonlinear term, we introduce the exponent S_c defined by

$$S_c = S_c(d, \gamma, \alpha, q) := \frac{d+\gamma}{\alpha} - \frac{d}{q}.$$

In fact, this S_c is the critical exponent in the sense that

$$|x|^\gamma |v|^\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{for any } v \in L^q_s(\mathbb{R}^d)$$

if and only if

$$s < S_c \quad \text{or} \quad s = S_c \text{ and } q \le \alpha.$$

Let us give a notion of solution in this paper.

Definition 1.1 Let $X = L^q_s(\mathbb{R}^d)$ or $W^{1,q}_s(\mathbb{R}^d)$. Let $u_0 \in X$ and $T \in (0, \infty]$. We say that a function u is a mild solution to (1) with initial data u_0 if $u \in C([0, T); X)$ satisfies the integral equation

$$u(t, x) = (e^{t\Delta}u_0)(x) + \int_0^t e^{(t-\tau)\Delta}(|\cdot|^\gamma |\nabla u(\tau)|^\alpha)(x)\, d\tau$$

for almost everywhere $(t, x) \in (0, T) \times \mathbb{R}^d$, where $\{e^{t\Delta}\}_{t>0}$ is the heat semigroup whose elements are defined by

$$e^{t\Delta} f := G_t * f, \quad f \in \mathcal{S}'(\mathbb{R}^d)$$

with the Gaussian kernel

$$G_t(x) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \ x \in \mathbb{R}^d.$$

Here, $*$ denotes the convolution operator and $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions. We denote by $T_{\max} = T_{\max}(u_0)$ the maximal existence time of u. □

Our results are the following:

Theorem 1.2 *Let $d \in \mathbb{N}$, $-1 < \gamma < d$, $\alpha_F < \alpha < \min\{2, 2+\gamma\}$, $\alpha \le q < \infty$ and $s = s_{c,1}$. Assume that*

$$\frac{1}{q} < \min\left\{\frac{2-\alpha}{d(\alpha-1)}, \frac{1}{d(\alpha-1)} + \frac{(d-1)\alpha - d - \gamma}{d\alpha(\alpha-1)}, \frac{1}{d(\alpha-1)} + \frac{(d-1)\alpha - \gamma - 1}{d(\alpha-1)^2}\right\},$$

$$\max\left\{-\frac{d}{q}, s_{c,2} - \frac{2}{\alpha}, \frac{\gamma}{\alpha-1}, \frac{s_{c,1}+\gamma}{\alpha}\right\} < \tilde{s} < \min\left\{S_c, s_{c,1}\right\}.$$

Then, for any $u_0 \in L^q_{s_{c,1}}(\mathbb{R}^d)$, there exist a time $T > 0$ and a unique mild solution u to (1) in the class

$$C([0, T]; L^q_{s_{c,1}}(\mathbb{R}^d)) \cap C((0, T]; W^{1,q}_{\tilde{s}}(\mathbb{R}^d)).$$

Moreover, there exists an $\varepsilon_0 > 0$ such that if the initial data u_0 satisfies

$$\max\left\{\sup_{0<t<T_{\max}} t^{\beta_1} \|e^{t\Delta}u_0\|_{L^q_{\tilde{s}}}, \sup_{0<t<T_{\max}} t^{\beta_1+\frac{1}{2}} \|\nabla e^{t\Delta}u_0\|_{L^q_{\tilde{s}}}\right\} \le \varepsilon_0,$$

then $T_{\max} = \infty$ and $\lim_{t\to\infty} \|u(t)\|_{L^q_{s_{c,1}}} = 0$. Here, $\beta_1 = (s_{c,1} - \tilde{s})/2$. □

Theorem 1.3 *Let $d \in \mathbb{N}$, $\gamma > -1$, $\alpha > \alpha_F$, $\alpha \le q < \infty$ and $s = s_{c,2}$. Assume that*

$$\frac{1}{q} < \min\left\{\frac{1}{d(\alpha-1)}, \frac{1}{d(\alpha-1)} + \frac{(d-1)\alpha - d - \gamma}{d\alpha(\alpha-1)}, \frac{1}{d(\alpha-1)} + \frac{(d-1)\alpha - \gamma - \alpha}{d(\alpha-1)^2}\right\},$$

$$\max\left\{-\frac{d}{q}, s_{c,2} - \frac{2}{\alpha}, \frac{\gamma}{\alpha-1}, \frac{s_{c,2}+\gamma}{\alpha}\right\} < \tilde{s} < \min\left\{S_c, s_{c,2}\right\}$$

Then, for any $u_0 \in W^{1,q}_{s_{c,2}}(\mathbb{R}^d)$, there exist a time $T > 0$ and a unique mild solution u to (1) in the class

$$C([0,T]; W^{1,q}_{s_{c,2}}(\mathbb{R}^d)) \cap C((0,T]; W^{1,q}_{\tilde{s}}(\mathbb{R}^d)).$$

Moreover, there exists an $\varepsilon_0 > 0$ such that if the initial data u_0 satisfies

$$\max\left\{\sup_{0<t<T_{\max}} t^{\beta_2}\|e^{t\Delta}u_0\|_{L^q_{\tilde{s}}}, \sup_{0<t<T_{\max}} t^{\beta_2}\|\nabla e^{t\Delta}u_0\|_{L^q_{\tilde{s}}}\right\} \le \varepsilon_0,$$

then $T_{\max} = \infty$ and $\lim_{t\to\infty}\|u(t)\|_{W^{1,q}_{s_{c,2}}} = 0$. Here, $\beta_2 = (s_{c,2} - \tilde{s})/2$. □

Remark 1.4 Let us give some remarks on the theorems.

(a) Theorems 1.2 and 1.3 with $\gamma = 0$ correspond to the results on local well-posedness for the viscous Hamilton-Jacobi equation in the scale-critical case (see Theorem 2.1 and Proposition 2.4 in [4]).
(b) It is not clear whether the restrictions $\alpha < \min\{2, 2+\gamma\}$ and $\gamma < d$ in Theorem 1.2 are sharp or not. In the case $\gamma = 0$, it is known that the restriction $\alpha < 2$ is sharp (see Proposition 3.1 in [4]).
(c) The conditions on $\tilde{s}$ come from the nonlinear estimates (Lemmas 2.2 and 2.3 below) and $\beta_1, \beta_2 > 0$. The upper bounds of $1/q$ are conditions to guarantee the existence of $\tilde{s}$.

2 Proofs of Theorems 1.2 and 1.3

We use the following weighted estimates for the heat semigroup $\{e^{t\Delta}\}_{t>0}$.

Lemma 2.1 (Chikami et al. [10]) *Let $d \in \mathbb{N}$, $1 \le q_1, q_2 \le \infty$, $s_1, s_2 \in \mathbb{R}$ and $k = 0, 1$. Assume that q_1, q_2, s_1, s_2 satisfy*

$$\begin{cases} -\frac{d}{q_2} < s_2 \le s_1 < d\left(1 - \frac{1}{q_1}\right), \\ d\left(\frac{1}{q_1} - \frac{1}{q_2}\right) + s_1 - s_2 \ge 0, \\ q_1 \le q_2 \quad \text{if } d\left(\frac{1}{q_1} - \frac{1}{q_2}\right) + s_1 - s_2 = 0. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$\|\nabla^k e^{t\Delta} f\|_{L^{q_2}_{s_2}} \le C t^{-\frac{d}{2}(\frac{1}{q_1} - \frac{1}{q_2}) - \frac{s_1 - s_2}{2} - \frac{k}{2}} \|f\|_{L^{q_1}_{s_1}}$$

for any $t > 0$ and $f \in L^{q_1}_{s_1}(\mathbb{R}^d)$. □

To prove local well-posedness for (1) in $L^q_{s_{c,1}}(\mathbb{R}^d)$, we prepare the following nonlinear estimates.

Lemma 2.2 *Let $d \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $\alpha > 1$, $q \in [1, \infty)$, $\tilde{s} \in \mathbb{R}$ and $k = 0, 1$, and let β_1 be defined by*

$$\beta_1 := \frac{s_{c,1} - \tilde{s}}{2}. \tag{2}$$

Then the following assertions hold:

(i) Assume that

$$\alpha_F < \alpha \le q, \quad -\frac{d}{q} < \tilde{s}, \quad \frac{\gamma}{\alpha - 1} \le \tilde{s} < S_c, \tag{3}$$

$$s_{c,2} - \frac{2}{\alpha} < \tilde{s} < s_{c,2} + \frac{1-k}{\alpha - 1}, \tag{4}$$

Then there exists a constant $C > 0$ such that

$$t^{\beta_1 + \frac{k}{2}} \|\nabla^k (N(u_1)(t) - N(u_2)(t))\|_{L^q_{\tilde{s}}}$$
$$\le C \max_{i=1,2} \left(\sup_{0<\tau<t} \tau^{\beta_1 + \frac{1}{2}} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} \right)^{\alpha - 1} \sup_{0<\tau<t} \tau^{\beta_1 + \frac{1}{2}} \|\nabla (u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}}$$

for any functions u_1, u_2 satisfying

$$\sup_{0<\tau<t} \tau^{\beta_1 + \frac{1}{2}} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} < \infty, \quad i = 1, 2. \tag{5}$$

(ii) Assume that

$$\alpha_F < \alpha \le q, \quad \frac{s_{c,1}+\gamma}{\alpha} \le \tilde{s} < S_c, \tag{6}$$

$$s_{c,2} - \frac{2}{\alpha} < \tilde{s} < s_{c,2}. \tag{7}$$

Then there exists a constant $C > 0$ such that

$$\|N(u_1)(t) - N(u_2)(t)\|_{L^q_{s_{c,1}}}$$
$$\le C \max_{i=1,2} \left(\sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} \right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}}$$

for any functions u_1, u_2 satisfying (5).

Proof Let $\alpha > 1$, $\gamma \in \mathbb{R}$, $q \in [1,\infty)$, $\tilde{s} \in \mathbb{R}$, $k = 0, 1$ and $\sigma := \alpha\tilde{s} - \gamma$. By the definition of $N(u)$, we have

$$|\nabla^k (N(u_1)(t) - N(u_2)(t))| \le \int_0^t \nabla^k e^{(t-\tau)\Delta} \left(|\cdot|^\gamma F(u_1,u_2)(\tau)\right) d\tau, \tag{8}$$

where

$$F(u_1,u_2)(\tau) := \left(|\nabla u_1(\tau)|^{\alpha-1} + |\nabla u_2(\tau)|^{\alpha-1}\right) |\nabla(u_1(\tau) - u_2(\tau))|.$$

First, we prove the assertion (i). Assume (3). Then, by Lemma 2.1 with $(q_1, s_1) = (q/\alpha, \sigma)$ and $(q_2, s_2) = (q, \tilde{s})$, we have

$$\|\nabla^k (N(u_1)(t) - N(u_2)(t))\|_{L^q_{\tilde{s}}}$$
$$\le C \int_0^t \|\nabla^k e^{(t-\tau)\Delta} \left(|\cdot|^\gamma F(u_1,u_2)(\tau)\right)\|_{L^q_{\tilde{s}}} d\tau$$
$$\le C \int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-\tilde{s}}{2}-\frac{k}{2}} \||\cdot|^\gamma F(u_1,u_2)(\tau)\|_{L^{\frac{q}{\alpha}}_\sigma} d\tau.$$

By Hölder's inequality, we have

$$\||\cdot|^\gamma F(u_1,u_2)(\tau)\|_{L^{\frac{q}{\alpha}}_\sigma} = \||\cdot|^{\alpha\tilde{s}} F(u_1,u_2)(\tau)\|_{L^{\frac{q}{\alpha}}}$$
$$\le C \max_{i=1,2} \|\nabla u_i(\tau)\|^{\alpha-1}_{L^q_{\tilde{s}}} \|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}}$$

$$\leq C\tau^{-(\beta_1+\frac{1}{2})\alpha} \max_{i=1,2} \left(\sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} \right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}},$$

where β_1 is given in (2). Hence,

$$\|\nabla^k (N(u_1)(t) - N(u_2)(t))\|_{L^q_{\tilde{s}}} \leq C \int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-\tilde{s}}{2}-\frac{k}{2}} \tau^{-(\beta_1+\frac{1}{2})\alpha}\, d\tau \times \max_{i=1,2} \left(\sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} \right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}}.$$

Assume further (4). Then, the integral in the right-hand side can be calculated as follows:

$$\int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-\tilde{s}}{2}-\frac{k}{2}} \tau^{-(\beta_1+\frac{1}{2})\alpha}\, d\tau = t^{-\beta_1-\frac{k}{2}} B\left(-\frac{d}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right) - \frac{\sigma-\tilde{s}}{2} - \frac{k}{2} + 1, -\left(\beta_1+\frac{1}{2}\right)\alpha + 1 \right),$$

where $B(a, b)$ is the beta function for $a, b > 0$. Here, we note that

$$-\frac{d}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right) - \frac{\sigma-\tilde{s}}{2} - \frac{k}{2} + 1 > 0 \quad \text{and} \quad -\left(\beta_1+\frac{1}{2}\right)\alpha + 1 > 0.$$

Thus, the assertion (i) is proved.

Similarly, we can prove the assertion (ii). Assume (6) and (7). Then, by Lemma 2.1 with $(q_1, s_1) = (q/\alpha, \sigma)$ and $(q_2, s_2) = (q, s_{c,1})$, we have

$$\begin{aligned}
&\|N(u_1)(t) - N(u_2)(t)\|_{L^q_{s_{c,1}}} \\
&\leq C \int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-s_{c,1}}{2}} \||\cdot|^{\gamma} F(u_1, u_2)(\tau)\|_{L^{\frac{q}{\alpha}}_{\sigma}}\, d\tau \\
&\leq C \int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-s_{c,1}}{2}} \tau^{-(\beta_1+\frac{1}{2})\alpha}\, d\tau \\
&\quad \times \max_{i=1,2} \left(\sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} \right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}} \\
&\leq CB\left(-\frac{d}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right) - \frac{\sigma-s_{c,1}}{2} + 1, -\left(\beta_1+\frac{1}{2}\right)\alpha + 1 \right) \\
&\quad \times \max_{i=1,2} \left(\sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} \right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_1+\frac{1}{2}} \|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}},
\end{aligned}$$

where we note that

$$-\frac{d}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right)-\frac{\sigma-s_{c,1}}{2}+1>0 \quad \text{and} \quad -\left(\beta_1+\frac{1}{2}\right)\alpha+1>0.$$

Thus, the assertion (ii) is also proved. □

Proof of Theorem 1.2 Let $d \in \mathbb{N}$, $-1<\gamma<d$ and $\alpha_F<\alpha<\min\{2, 2+\gamma\}$, and let q and $\tilde{s}$ be as in Theorem 1.2. Let us take $\rho>0$ and $M>0$ as

$$\rho + C_0 M^\alpha \le M \quad \text{and} \quad 2C_1 M^{\alpha-1}<1,$$

where C_0 and C_1 are given in (9) and (10) below. For $T>0$, let $X(T)$ be the complete metric space defined by

$$X(T):=\left\{u:(0,T]\to W^{1,q}_{\tilde{s}}(\mathbb{R}^d)\,;\ \|u\|_{X(T)}\le M\right\},$$

$$\|u\|_{X(T)}:=\max\left\{\sup_{0<t<T} t^{\beta_1}\|u(t)\|_{L^q_{\tilde{s}}},\ \sup_{0<t<T} t^{\beta_1+\frac{1}{2}}\|\nabla u(t)\|_{L^q_{\tilde{s}}}\right\}$$

endowed with the metric

$$\mathrm{d}(u_1,u_2):=\|u_1-u_2\|_{X(T)}.$$

Let $u_0\in\mathcal{S}'(\mathbb{R}^d)$ be such that

$$\|e^{t\Delta}u_0\|_{X(T)}\le\rho.$$

Define the map Φ by

$$\Phi(u):=e^{t\Delta}u_0+\int_0^t e^{(t-\tau)\Delta}(|x|^\gamma|\nabla u|^\alpha)\,d\tau.$$

Thanks to Lemmas 2.1 and 2.2 (i), we have

$$\|\Phi(u)\|_{X(T)}\le\|e^{t\Delta}u_0\|_{X(T)}+C_0\|u\|_{X(T)}^\alpha\le\rho+C_1M^\alpha\le M \tag{9}$$

and

$$\begin{aligned}\|\Phi(u_1)-\Phi(u_2)\|_{X(T)}&\le C_1\max_{i=1,2}\|u_i\|_{X(T)}^{\alpha-1}\|u_1-u_2\|_{X(T)}\\&\le C_1M^{\alpha-1}\|u_1-u_2\|_{X(T)}\end{aligned} \tag{10}$$

for any $u,u_1,u_2\in X(T)$, where $C_1M^{\alpha-1}<1$. These show that Φ is a contraction mapping from $X(T)$ into itself. Thus, Banach's fixed point theorem ensures the existence of a unique fixed point $u \in X(T)$ of Φ, and u satisfies $u \in C([0,T]; L^q_{s_{c,1}}(\mathbb{R}^d))$ by Lemma 2.2 (ii). The proof of small data global existence is similar to the above argument, and the proof of small data dissipation can be found in Proposition A.10 in [8] for instance. Hence, we may omit the details. □

Next, to prove local well-posedness in $W^{1,q}_{s_{c,2}}(\mathbb{R}^d)$, we prepare the following nonlinear estimates.

Lemma 2.3 *Let* $d \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $\alpha > 1$, $q \in [1,\infty)$, $\tilde{s} \in \mathbb{R}$ *and* $k = 0, 1$, *and let* β_2 *be defined by*

$$\beta_2 := \frac{s_{c,2} - \tilde{s}}{2}. \tag{11}$$

Then the following assertions hold:

(i) Assume that

$$\alpha_F < \alpha \le q, \quad -\frac{d}{q} < \tilde{s}, \quad \frac{\gamma}{\alpha - 1} \le \tilde{s} < S_c, \tag{12}$$

$$s_{c,2} - \frac{2}{\alpha} < \tilde{s} < s_{c,2} + \frac{1-k}{\alpha - 1}, \tag{13}$$

Then there exists a constant $C > 0$ *such that*

$$\begin{aligned} &t^{\beta_2} \|\nabla^k (N(u_1)(t) - N(u_2)(t))\|_{L^q_{\tilde{s}}} \\ &\le C t^{\frac{1-k}{2}} \max_{i=1,2} \left(\sup_{0<\tau<t} \tau^{\beta_2} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} \right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_2} \|\nabla (u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}} \end{aligned}$$

for any functions u_1, u_2 *satisfying*

$$\sup_{0<\tau<t} \tau^{\beta_2} \|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}} < \infty, \quad i = 1, 2. \tag{14}$$

(ii) Assume that

$$\alpha \le q, \quad \frac{s_{c,2} + \gamma}{\alpha} \le \tilde{s} < S_c, \tag{15}$$

$$s_{c,2} - \frac{2}{\alpha} < \tilde{s} < s_{c,2} + \frac{1-k}{\alpha - 1}. \tag{16}$$

Then there exists a constant $C > 0$ such that

$$\|\nabla^k(N(u_1)(t) - N(u_2)(t))\|_{L^q_{s_{c,2}}}$$
$$\leq Ct^{\frac{1-k}{2}} \max_{i=1,2}\left(\sup_{0<\tau<t} \tau^{\beta_2}\|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}}\right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_2}\|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}}$$

for any functions u_1, u_2 satisfying (14)*.*

Proof The proof is similar to that of Lemma 2.2. Let $\alpha > 1$, $\gamma \in \mathbb{R}$, $q \in [1, \infty)$, $\tilde{s} \in \mathbb{R}$, $k = 0, 1$ and $\sigma := \alpha\tilde{s} - \gamma$. First, we prove the assertion (i). Assume (12). Then, by (8) and Lemma 2.1 with $(q_1, s_1) = (q/\alpha, \sigma)$ and $(q_2, s_2) = (q, \tilde{s})$, we have

$$\|\nabla^k(N(u_1)(t) - N(u_2)(t))\|_{L^q_{\tilde{s}}}$$
$$\leq C\int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-\tilde{s}}{2}-\frac{k}{2}}\||\cdot|^\gamma F(u_1, u_2)(\tau)\|_{L^{\frac{q}{\alpha}}_\sigma}\, d\tau.$$

By Hölder's inequality, we have

$$\||\cdot|^\gamma F(u_1, u_2)(\tau)\|_{L^{\frac{q}{\alpha}}_\sigma}$$
$$\leq C\tau^{-\beta_2\alpha} \max_{i=1,2}\left(\sup_{0<\tau<t} \tau^{\beta_2}\|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}}\right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_2}\|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}},$$

where β_2 is given in (11). Hence,

$$\|\nabla^k(N(u_1)(t) - N(u_2)(t))\|_{L^q_{\tilde{s}}} \leq C\int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-\tilde{s}}{2}-\frac{k}{2}}\tau^{-\beta_2\alpha}\, d\tau$$
$$\times \max_{i=1,2}\left(\sup_{0<\tau<t} \tau^{\beta_2}\|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}}\right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_2}\|\nabla(u_1(\tau) - u_2(\tau))\|_{L^q_{\tilde{s}}}.$$

Assume further (13). Then, the integral in the right-hand side can be calculated as follows:

$$\int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-\tilde{s}}{2}-\frac{k}{2}}\tau^{-\beta_2\alpha}\, d\tau$$
$$= t^{-\beta_2+\frac{1-k}{2}} B\left(-\frac{d}{2}\left(\frac{\alpha}{q} - \frac{1}{q}\right) - \frac{\sigma - \tilde{s}}{2} - \frac{k}{2} + 1, -\beta_2\alpha + 1\right),$$

Here, we note that

$$-\frac{d}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right)-\frac{\sigma-\tilde{s}}{2}-\frac{k}{2}+1>0 \quad \text{and} \quad -\beta_2\alpha+1>0.$$

Thus, the assertion (i) is proved.

Similarly, we can prove the assertion (ii). Assume (15) and (16). Then, by Lemma 2.1 with $(q_1, s_1) = (q/\alpha, \sigma)$ and $(q_2, s_2) = (q, s_{c,2})$, we have

$$\begin{aligned}
&\|\nabla^k N(u_1)(t) - N(u_2)(t)\|_{L^q_{s_{c,2}}} \\
&\le C\int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-s_{c,2}}{2}-\frac{k}{2}} \||\cdot|^{\gamma} F(u_1,u_2)(\tau)\|_{L^{\frac{q}{\alpha}}_{\sigma}}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{d}{2}(\frac{\alpha}{q}-\frac{1}{q})-\frac{\sigma-s_{c,2}}{2}-\frac{k}{2}} \tau^{-\beta_2\alpha}\, d\tau \\
&\quad \times \max_{i=1,2}\left(\sup_{0<\tau<t} \tau^{\beta_2}\|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}}\right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_2}\|\nabla(u_1(\tau)-u_2(\tau))\|_{L^q_{\tilde{s}}} \\
&\le Ct^{\frac{1-k}{2}} B\left(-\frac{d}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right)-\frac{\sigma-s_{c,2}}{2}-\frac{k}{2}+1, -\beta_2\alpha+1\right) \\
&\quad \times \max_{i=1,2}\left(\sup_{0<\tau<t} \tau^{\beta_2}\|\nabla u_i(\tau)\|_{L^q_{\tilde{s}}}\right)^{\alpha-1} \sup_{0<\tau<t} \tau^{\beta_2}\|\nabla(u_1(\tau)-u_2(\tau))\|_{L^q_{\tilde{s}}},
\end{aligned}$$

where we note that

$$-\frac{d}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right)-\frac{\sigma-s_{c,2}}{2}-\frac{k}{2}+1>0 \quad \text{and} \quad -\beta_2\alpha+1>0.$$

Thus, the assertion (ii) is also proved. □

The proof of Theorem 1.3 is done by proceeding the argument in the proof of Theorem 1.2 together with Lemma 2.3. So, we may omit the proof.

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On the Rellich Type Inequality for Schrödinger Operators with Singular Potential

Vladimir Georgiev and Hideo Kubo

Dedicated to Professor Tohru Ozawa on his 60th birthday

Abstract The aim of this note is to derive the Rellich type inequality for Schrödinger operators with singular potential of the inverse-square type. In general, the standard Rellich inequality holds in the case of the space dimensions n is larger than 5. It seems interesting that the inequality still holds by perturbing the minus Laplacian by means of inverse-square positive potential even if $n = 3, 4$.

1 Introduction

The Hardy inequality:

$$\frac{n-2}{2}\left\|\frac{u}{|\cdot|}\right\|_{L^2(\mathbb{R}^n)} \leq \|\nabla u\|_{L^2(\mathbb{R}^n)} \text{ for } u \in H^1(\mathbb{R}^n),\ n \geq 3$$

and the Rellich inequality:

$$\frac{n^2-4n}{4}\left\|\frac{u}{|\cdot|^2}\right\|_{L^2(\mathbb{R}^n)} \leq \|\Delta u\|_{L^2(\mathbb{R}^n)} \text{ for } u \in H^2(\mathbb{R}^n),\ n \geq 5$$

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are known to be useful in the study of partial differential equations. In this note, we prove analogous inequalities for Schrödinger operators with singular potential of the form:

$$A_0u = -\Delta u + \frac{\chi(r)}{r^2}u, \quad u \in C_0^\infty(\mathbb{R}^n),$$

where $\chi \in L^\infty([0,\infty))$ is a given real-valued function. The reason why we are interesting in this inverse-square type potential is the fact that it has the same scaling as the classical Laplacian. In this critical case, the size of χ affects the self-adjointness of the operator A_0. Indeed, in order to get the Friedrichs extension of A_0, it suffices to assume

$$\chi(r) > -\left(\frac{n-2}{2}\right)^2, \quad r \geq 0.$$

However, we do need the essential self-adjointness of A_0 for getting the Rellich type inequality. As such, we require the following stronger condition on χ:

$$\chi(r) > 1 - \left(\frac{n-2}{2}\right)^2 = \frac{n^2-4n}{4}, \quad r \geq 0. \tag{1}$$

Under this assumption, the operator A_0 can be extended as an essentially self-adjoint operator A (by using the result of Simon in [10]).

Theorem 1 *Let $n \geq 3$. If a real-valued function $\chi \in L^\infty([0,\infty))$ satisfies (1), then there exists a constant C such that*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2}dx \leq C \int_{\mathbb{R}^n} |A^{1/2}u(x)|^2 dx \tag{2}$$

for $u \in \mathcal{D}(A^{1/2})$, and

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^4}dx \leq C \int_{\mathbb{R}^n} |Au(x)|^2 dx \tag{3}$$

for $u \in \mathcal{D}(A)$.

We underline that when $n \geq 5$, (3) recovers the standard Rellich inequality, because (1) is fulfilled by $\chi \equiv 0$, and that even when $n = 3, 4$, the Rellich type inequality is still valid by shifting $-\Delta$ by $\chi(r)/r^2$ in the positive direction. We also remark that a similar approach to the Rellich inequality is found in Evans and Lewis [3], where Δ is replaced by the magnetic Laplacian $(\nabla - i\mathbf{A})^2$ with curl $(\mathbf{A})$ being of Aharonov-Bohm type.

In our previous work [4], we proved the inequalities in Theorem 1 for $n = 3$, and used them for treating the following Cauchy problem to the power-type nonlinear wave equations with the potential:

$$\begin{aligned}&(\partial_t^2 - \Delta + \frac{\chi(r)}{r^2})u = |u|^{p-1}u, \quad \text{for } (t,x) \in [0,T) \times \mathbb{R}^3,\\ &u(0,x) = f(x),\ (\partial_t u)(0,x) = g(x) \quad \text{for } x \in \mathbb{R}^3.\end{aligned} \tag{4}$$

More precisely, we established the existence of global strong solution $u(t)$, i.e., there exists

$$u \in C^2([0,\infty) : L^2(\mathbb{R}^3)) \cap C([0,\infty) : \mathcal{D}(A))$$

which satisfies the associated integral equation:

$$u(t) = (\cos tA^{1/2})f + \frac{\sin tA^{1/2}}{A^{1/2}}g + \int_0^t \frac{\sin(t-\tau)A^{1/2}}{A^{1/2}}|u(\tau)|^{p-1}u(\tau)d\tau,$$

provided that $p > 1 + \sqrt{2}$ and the initial data are sufficiently small and compactly supported (see also [1, 2, 7, 8] for the case of the inverse-square potential a/r^2 with a constant $a > -(n-2)^2/4$).

This note is organized as follows. In the Sect. 2, we describe an overview about the basic facts for Schrödinger operators with singular potential. We give a new characterization of the domain of the closure of A_0 in the Sect. 3, and finish the proof of Theorem 1.

2 Preliminary Observations

This section is an overview on the basic facts for Schrödinger operators with singular potential, without specifying the concrete form of the potential. In order to handle such an operator, it is useful to consider the quadratic form, as we shall see in Proposition 1 for instance.

Let X be a Hilbert space equipped with an inner product $(\cdot,\cdot)$ and the norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. Let $\mathcal{D}$ be a dense subspace of X. We shall consider the Hermitian form $q : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$, that is, q is linear in the first variable and $q(u,v) = \overline{q(v,u)}$.

We say that a Hermitian form q is bounded from below, if there exists $\alpha \in \mathbb{R}$ such that

$$q[u] := q(u,u) \geq \alpha\|u\|^2, \quad u \in \mathcal{D}. \tag{5}$$

When (5) holds, if we define

$$(u,v)_\alpha = q(u,v) + (1-\alpha)(u,v), \quad u,v \in \mathcal{D},$$

then it is an inner product on $\mathcal{D}$. If $\mathcal{D}$ is complete with respect to the norm $\|u\|_\alpha = (u, u)_\alpha$, then q is called an α-closed form.

As is well-known, such a Hermitian form generates a self-adjoint operator, due to Friedrichs (see for example Theorem X.23 in [9]).

Theorem 2 *(Friedrichs extension) Let $\mathcal{D}$ be a dense subspace of X. If a Hermitian form q on $\mathcal{D}$ satisfies (5) and is α-closed, then there exists a unique self-adjoint operator Q_F on X such that*

- $\mathcal{D}(Q_F) \subset \mathcal{D}$,
- $(Q_F u, v) = q(u, v)$ *for* $u \in \mathcal{D}(Q_F)$, $v \in \mathcal{D}$,
- $(Q_F u, u) \geq \alpha \|u\|^2$ *for* $u \in \mathcal{D}(Q_F)$,
- $\mathcal{D} = \mathcal{D}((Q_F - \alpha)^{1/2})$.

Proof Since it is easy to see that the uniqueness holds, we only consider the existence part. First of all, we note that

$$\|u\|_\alpha^2 = q[u] + (1 - \alpha)\|u\|^2 \geq \|u\|^2, \quad u \in \mathcal{D}.$$

For a fixed $v \in X$, we define a functional $F_v(u) := (u, v)$ for $u \in \mathcal{D}$. Then, we have

$$|F_v(u)| \leq \|u\|\|v\| \leq \|u\|_\alpha \|v\|, \quad u \in \mathcal{D},$$

so that $\|F_v\|_{\mathcal{L}(\mathcal{D},\mathbb{C})} \leq \|v\|$. Since $\mathcal{D}$ is a Hilbert space equipped with the norm $\|\cdot\|_\alpha$, by the Riesz theorem, there exists a unique $w_v \in \mathcal{D}$ such that $F_v(u) = (u, w_v)_\alpha$ and $\|w_v\|_\alpha = \|F_v\|_{\mathcal{L}(\mathcal{D},\mathbb{C})}$. Therefore, we can define a map $T : X \to \mathcal{D}$ by $Tv := w_v$. It is easy to see that T is a bounded operator on X. Since

$$(u, v) = (u, Tv)_\alpha, \quad u \in \mathcal{D}, \ v \in X,$$

we find that T is self-adjoint and injective. Moreover, the range of T denoted by $\mathcal{R}(T)$ is dense in $\mathcal{D}$ with respect to $\|\cdot\|_\alpha$. Indeed, if $u \in \mathcal{D}$ satisfies $(u, v)_\alpha = 0$ for any $v \in \mathcal{R}(T)$, then we have $(u, w) = 0$ for any $w \in X$, so that $u = 0$. This implies the density of $\mathcal{R}(T)$ in $\mathcal{D}$. This fact further means that $\mathcal{R}(T)$ is dense in X with respect to $\|\cdot\|$.

Then, one can verify that $S := T^{-1}$ is a self-adjoint operator with a domain $\mathcal{D}(S) = \mathcal{R}(T)$. Moreover, for $u \in \mathcal{D}(S)$, $v \in \mathcal{D}$

$$(Su, v) = (TSu, v)_\alpha = (u, v)_\alpha,$$

because

$$(u, v) = (Tu, v)_\alpha, \quad u \in X, \ v \in \mathcal{D}.$$

Now, we define $Q_F := S - (1-\alpha)$. Then we have

$$\mathcal{D}(Q_F) = \mathcal{D}(S) = \mathcal{R}(T) \subset \mathcal{D},$$

and for $u \in \mathcal{D}(Q_F)$, $v \in \mathcal{D}$

$$(Q_F u, v) = (Su, v) - (1-\alpha)(u, v) = (u, v)_\alpha - (1-\alpha)(u, v) = q(u, v),$$

so that $(Q_F u, u) = q[u] \geq \alpha \|u\|^2$ for $u \in \mathcal{D}(Q_F)$.

Finally, we prove $\mathcal{D} = \mathcal{D}(Q_\alpha^{1/2})$ with $Q_\alpha := Q_F - \alpha$. Let $u \in \mathcal{D}$. Then there exists $\{u_j\} \subset \mathcal{D}(Q_F)$ such that $\|u_j - u\|_\alpha \to 0$, because of the density of $\mathcal{D}(Q_F)$ in $\mathcal{D}$. Since

$$\|u_j - u_k\|_\alpha^2 - (1-\alpha)\|u_j - u_k\|^2 = (Q_F(u_j - u_k), u_j - u_k),$$

and $\mathcal{D}(Q_F) \subset \mathcal{D}(Q_F^{1/2})$ by the functional calculus, we get

$$\|Q_\alpha^{1/2}(u_j - u_k)\|^2 = \|u_j - u_k\|_\alpha^2 - \|u_j - u_k\|^2 \to 0.$$

Since any self-adjoint operator is closed, so is $Q_\alpha^{1/2}$. Therefore, the fact that $\{u_j\}$ converges to u and $\{Q_\alpha^{1/2} u_j\}$ is Cauchy in X implies $u \in \mathcal{D}(Q_\alpha^{1/2})$. Hence $\mathcal{D} \subset \mathcal{D}(Q_\alpha^{1/2})$.

In order to derive the opposite inclusion, we use the fact that $\mathcal{D}(Q_\alpha)$ is a core of $Q_\alpha^{1/2}$. Indeed, since Q_α is a self-adjoint operator satisfying $(Q_\alpha u, u) \geq 0$ for any $u \in \mathcal{D}(Q_\alpha)$, (i.e. non-negative), we can apply a general result stated in Theorem 3 below. Now, let $u \in \mathcal{D}(Q_\alpha^{1/2})$. Then, we find a sequence $\{u_j\} \subset \mathcal{D}(Q_\alpha)$ satisfying $u_j \to u$ and $Q_\alpha^{1/2} u_j \to Q_\alpha^{1/2} u$ in X, as $\overline{Q_\alpha^{1/2}|_{\mathcal{D}(Q_\alpha)}} = Q_\alpha^{1/2}$. Since $\{u_j\} \subset \mathcal{D}(Q_F)$, we have

$$\|u_j - u_k\|_\alpha^2 = \|Q_\alpha^{1/2}(u_j - u_k)\|^2 + \|u_j - u_k\|^2 \to 0.$$

By the completeness of $\mathcal{D}$ with respect to $\|\cdot\|_\alpha$, there exists $v \in \mathcal{D}$ such that $\|u_j - v\|_\alpha \to 0$, which implies $u = v \in \mathcal{D}$. Thus we have done the proof. □

The following result was used in the proof of Theorem 2. For the proof, we refer to Theorem 3.35 in Kato [6] (notice that if T is self-adjoint and non-negative, then T is m-accretive).

Theorem 3 *Let T be a non-negative self-adjoint operator. Then there exists a unique non-negative self-adjoint operator $T^{1/2}$ such that $(T^{1/2})^2 = T$, $\mathcal{D}(T)$ is a core of $T^{1/2}$, and $T^{1/2}$ commutes with any bounded operator on X that commutes with T.*

As an application of Theorem 2, we consider the following Hermitian form:

$$q(u,v) := (\nabla u, \nabla v)_{L^2} + \int_{\mathbb{R}^n} V u \bar{v} dx, \quad \mathcal{D} := H^1(\mathbb{R}^n) \cap \mathcal{D}(\sqrt{V}),$$

where $V \in L^1_{loc}(\mathbb{R}^n)$ is a non-negative function. It is clear that $q[u] \geq 0$ for all $u \in \mathcal{D}$. So, once we could show that q is 0-closed, we obtain an associated self-adjoint operator Q_F with the above Hermitian form q.

Let $\{u_j\}$ be Cauchy in $\mathcal{D}$ with respect to $\|\cdot\|_0$, that is

$$\|\nabla(u_j - u_k)\|_{L^2} + \int_{\mathbb{R}^n} V|u_j - u_k|^2 dx + \|u_j - u_k\|_{L^2} \to 0$$

as $j, k \to \infty$. In particular, $\{u_j\}$ is Cauchy in $H^1(\mathbb{R}^n)$, so that there exists $u \in H^1(\mathbb{R}^n)$ such that $\|u_j - u\|_{H^1} \to 0$ as $j \to \infty$. Moreover, since $\{\sqrt{V}u_j\}$ is Cauchy in $L^2(\mathbb{R}^n)$, there exists $v \in L^2(\mathbb{R}^n)$ such that $\|\sqrt{V}u_j - v\|_{L^2} \to 0$ as $j \to \infty$. Then, we can choose a subsequence of $\{u_j\}$, still denoted by the same letter, such that

$$u_j \to u, \quad \sqrt{V}u_j \to v, \quad \text{a.e. } x \in \mathbb{R}^n,$$

which means that $v = \sqrt{V}u$. Therefore, we find that $\|\sqrt{V}u_j - v\|_0 \to 0$ and $u \in \mathcal{D}$. This shows the 0-closedness of q.

As a consequence, we obtain the following.

Proposition 1 *Let $V \in L^1_{loc}(\mathbb{R}^n)$ be a non-negative function. Then there exists a unique non-negative self-adjoint operator Q_F in $L^2(\mathbb{R}^n)$ such that for $u \in \mathcal{D}(Q_F)$, $v \in H^1(\mathbb{R}^n) \cap \mathcal{D}(\sqrt{V})$*

$$(\nabla u, \nabla v)_{L^2} + \int_{\mathbb{R}^n} V u \bar{v} dx = (Q_F u, v)_{L^2} \tag{6}$$

holds and $\mathcal{D}(Q_F) \subset \mathcal{D}(Q_F^{1/2}) = H^1(\mathbb{R}^n) \cap \mathcal{D}(\sqrt{V})$.

However, the concrete description of $\mathcal{D}(Q_F)$ is not clear. To clarify this issue, we introduce an operator H in $L^2(\mathbb{R}^n)$ by

$$\begin{aligned} &Hu = -\Delta u + Vu, \ u \in \mathcal{D}(H), \\ &\mathcal{D}(H) = \{u \in L^2(\mathbb{R}^n) |\, Vu \in L^1_{loc}(\mathbb{R}^n), \ -\Delta u + Vu \in L^2(\mathbb{R}^n)\}. \end{aligned} \tag{7}$$

Here Δu should be understood in the distribution sense, but we may regard it as a function in $L^1_{loc}(\mathbb{R}^n)$ when $u \in \mathcal{D}(H)$.

Proposition 2 *Let $V \in L^1_{loc}(\mathbb{R}^n)$ be a non-negative function and let Q_F be the one obtained in Proposition 1. Then, $Q_F = H$.*

Proof First, we show $Q_F \subset H$. Let $u \in \mathcal{D}(Q_F)$. Note that $C_0^\infty(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \cap \mathcal{D}(\sqrt{V})$ and that $Vu = \sqrt{V} \cdot \sqrt{V} u \in L^1_{loc}(\mathbb{R}^n)$ for $u \in \mathcal{D}(Q_F)$. Therefore, from (6), we have

$$- (u, \Delta\phi)_{L^2} + (Vu, \phi)_{L^2} = (Q_F u, \phi)_{L^2}$$

for any $\phi \in C_0^\infty(\mathbb{R}^n)$. This means that $\Delta u = -Q_F u + Vu$ holds in $L^1_{loc}(\mathbb{R}^n)$, and hence $-\Delta u + Vu = Q_F u \in L^2(\mathbb{R}^n)$. Therefore, $u \in \mathcal{D}(H)$ and $Q_F u = Hu$ holds.

Next we prove $\mathcal{D}(H) \subset \mathcal{D}(Q_F)$. Let $v \in \mathcal{D}(H)$. Since Q_F is non-negative, we see that -1 is in the resolvent set of Q_F. Therefore, there exists $u \in \mathcal{D}(Q_F)$ such that $(Q_F + 1)u = v + Hv$. Since $Q_F \subset H$, we have $H(u - v) = v - u$, that is

$$\Delta(u - v) = (V + 1)(u - v).$$

Since the right hand side is in $L^1_{loc}(\mathbb{R}^n)$, we are in a position to apply the Kato inequality, which is stated in Theorem 4 below, to get

$$\begin{aligned} \Delta|u - v| &\geq \mathrm{Re}\,(\mathrm{sgn}\,(\overline{u - v})\,\Delta(u - v)) \\ &= (V + 1)|u - v| \geq |u - v|, \end{aligned} \tag{8}$$

as $V \geq 0$. From this inequality, we shall deduce that

$$\int_{\mathbb{R}^n} |v - u|\psi dx = 0 \tag{9}$$

for any rapidly decreasing function ψ satisfying $\psi \geq 0$. Once we could prove it, we can conclude that $|u(x) - v(x)| = 0$ for a.e. $x \in \mathbb{R}^n$, i.e., $v = u \in \mathcal{D}(Q_F)$.

For a given $\psi \in \mathcal{S}$, we can solve the equation $(1 - \Delta)\phi = \psi$. Indeed, it is clear that $\phi \in \mathcal{S}$ and it has the integral representation with the positive kernel. Therefore, if $\psi \geq 0$, then $\phi \geq 0$. On the other hand, there exists a sequence $\{\phi_j\}$ such that $\phi_j \geq 0$ and $(1 - \Delta)\phi_j \to (1 - \Delta)\phi = \psi$ as $j \to \infty$. Then we see from (8) that

$$(|u - v|, (1 - \Delta)\phi_j)_{L^2} \leq 0.$$

Taking the limit as $j \to \infty$, we get (9), This completes the proof. □

The following inequality is found by the work of Kato [5].

Theorem 4 *Let* $u \in L^1_{loc}(\mathbb{R}^n)$ *satisfy* $\Delta u \in L^1_{loc}(\mathbb{R}^n)$. *Then*

$$(\Delta|u|, \phi)_{L^2} \geq (Re\,(sgn\,\overline{u}\,\Delta u), \phi)_{L^2} \tag{10}$$

holds for any $\phi \in C_0^\infty(\mathbb{R}^n)$ *satisfying* $\phi \geq 0$. *Here we put*

$$sgn\,\overline{u}(x) = \begin{cases} \dfrac{\overline{u(x)}}{|u(x)|}, & u(x) \neq 0, \\ 0, & u(x) = 0. \end{cases}$$

Now, the domain of Q_F and its action become clear. However, it is still not clear whether $-\Delta u$ and Vu are in L^2 for $u \in \mathcal{D}(H)$ or not, although their sum is in L^2. In order to consider this problem, we shall make use of the following result, due to Simon in [10].

Theorem 5 *Let* $V \in L^2_{loc}(\mathbb{R}^n \setminus \{0\})$ *satisfy*

$$V(x) \geq \frac{3-(n-1)(n-3)}{4|x|^2}, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{11}$$

We define H_0 *by*

$$H_0 = -\Delta + V \text{ with domain } \mathcal{D}(H_0) = C_0^\infty(\mathbb{R}^3 \setminus \{0\}).$$

Then H_0 *is essentially self-adjoint, i.e., its closure* $\overline{H_0}$ *is self-adjoint.*

As a corollary, we show that $\overline{H_0}$ coincides with the Friedrichs extension Q_F obtained in Proposition 1, if $1 \leq n \leq 4$.

Corollary 1 *Let* $1 \leq n \leq 4$ *and let* $V \in L^2_{loc}(\mathbb{R}^n \setminus \{0\})$ *satisfy* (11). *Then,* $\overline{H_0} = H = Q_F$. *Here* H *is the one defined by* (7). *Moreover,*

$$\mathcal{D}(\overline{H_0}) = \{u \in L^2 |\, Vu \in L^1_{loc},\ -\Delta u + Vu \in L^2\},$$
$$\mathcal{D}((\overline{H_0})^{1/2}) = H^1(\mathbb{R}^n) \cap \mathcal{D}(\sqrt{V}).$$

Proof Note that (11) implies $V \geq 0$ for $1 \leq n \leq 4$. Therefore, the existence of Q_F is assured by Proposition 1 and we have $H = Q_F$ by Proposition 2, so that H is self-adjoint. Hence, we have only to show $\overline{H_0} = H$.

It is easy to see that $C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset \mathcal{D}(H)$ for $V \in L^2_{loc}(\mathbb{R}^n \setminus \{0\})$ and that $H_0 = H|_{C_0^\infty(\mathbb{R}^n \setminus \{0\})}$. This means that $H_0 \subset H$, and hence $\overline{H_0} \subset H$ by the self-adjointness of H. But, since $\overline{H_0}$ is also self-adjoint by Theorem 5, we get $\overline{H_0} = H$. This completes the proof. □

3 Proof of Theorem 1

We now turn back to the case where the potential takes the following form:

$$V(x) = \frac{\chi(r)}{r^2}, \; x \in \mathbb{R}^n \setminus \{0\}, \; r = |x|,$$

and consider the associated Schrödinger operator:

$$A_0 = -\Delta + \frac{\chi(r)}{r^2} \text{ with domain } \mathcal{D}(A_0) = C_0^\infty(\mathbb{R}^3 \setminus \{0\}).$$

Here χ is a real-valued bounded function on $[0, \infty)$ satisfying

$$c := \inf_{r \geq 0} \chi(r) > \frac{3 - (n-1)(n-3)}{4} = 1 - \left(\frac{n-2}{2}\right)^2. \tag{12}$$

We shall give now simple characterization of the domain of the closure $\overline{A_0}$ of A_0, based on Theorem 5. Once we could prove Lemma 1 below, then it is easy to see that Theorem 1 is valid.

Lemma 1 *We assume $n \geq 3$. Let χ be a real-valued bounded function on $[0, \infty)$ satisfying (12). Then we have*

a) $\overline{A_0}u = -\Delta u + \frac{\chi(r)}{r^2} u$ *for $u \in \mathcal{D}(\overline{A_0})$ and*

$$\mathcal{D}(\overline{A_0}) = \{u \in H^2(\mathbb{R}^n) | \, u/|x|^2 \in L^2(\mathbb{R}^n)\},$$

b) for $u \in \mathcal{D}(\overline{A_0})$ we have

$$\left\| \frac{u}{|\cdot|^2} \right\|_{L^2} \lesssim \|\overline{A_0}u\|_{L^2}, \quad \|\Delta u\|_{L^2} \lesssim \|\overline{A_0}u\|_{L^2}. \tag{13}$$

c) for $u \in \mathcal{D}((\overline{A_0})^{1/2})$ we have

$$\left\| \frac{u}{|\cdot|} \right\|_{L^2} \lesssim \|(\overline{A_0})^{1/2}u\|_{L^2}, \quad \|\nabla u\|_{L^2} \lesssim \|(\overline{A_0})^{1/2}u\|_{L^2}. \tag{14}$$

Proof For $u \in \mathcal{D}(\overline{A_0})$, there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ in $\mathcal{D}(A_0) = C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$\lim_{k \to \infty} \|u_k - u\|_{L^2} = 0 \text{ and } \lim_{k \to \infty} \|A_0 u_k - \overline{A_0}u\|_{L^2} = 0. \tag{15}$$

We set

$$g_k := A_0 u_k = -\Delta u_k + \frac{\chi(r)}{r^2} u_k. \tag{16}$$

Using radial coordinates $r = |x|$, $\omega = x/|x|$, we rewrite the above relation as:

$$-\partial_r^2 u_k - \frac{n-1}{r}\partial_r u_k - \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} u_k + \frac{\chi(r)}{r^2} u_k = g_k.$$

We shall multiply this equation by $r^{n-3}u_k(r\omega)$ and integrate over $r \in (0,\infty)$, $\omega \in \mathbb{S}^{n-1}$ by using the measure $drd\omega$. By the integration by parts, it follows that

$$\int_0^\infty \partial_r^2 u_k u_k r^{n-3} dr = -\int_0^\infty (\partial_r (u_k r^{(n-3)/2}))^2 dr + \left(\frac{n-3}{2}\right)^2 \int_0^\infty u_k^2 r^{n-5} dr,$$

$$\int_0^\infty \frac{n-1}{r}\partial_r u_k u_k r^{n-3} dr = -\frac{(n-1)(n-4)}{2}\int_0^\infty u_k^2 r^{n-5} dr,$$

because $u_k(r\omega) = 0$ for r close to 0. Therefore we have

$$\begin{cases} \int_0^\infty \int_{\mathbb{S}^{n-1}} |\partial_r (u_k(r\omega) r^{(n-3)/2})|^2 d\omega dr + \\ + \frac{n^2-4n-1}{4}\int_0^\infty \int_{\mathbb{S}^{n-1}} |u_k(r\omega)|^2 r^{n-5} d\omega dr + \\ + \int_0^\infty \int_{\mathbb{S}^{n-1}} |\nabla_\omega u_k(r\omega)|^2 r^{n-5} d\omega dr + \\ + \int_0^\infty \int_{\mathbb{S}^{n-1}} \chi(r)|u_k(r\omega)|^2 r^{n-5} d\omega dr = \\ = \int_0^\infty \int_{\mathbb{S}^{n-1}} g_k(r\omega) u_k(r\omega) r^{n-3} d\omega dr. \end{cases} \tag{17}$$

Notice that for $f \in C^1_{(0)}((0,\infty))$, i.e. for $f \in C^1((0,\infty))$ with compact support in $(0,\infty)$ the Hardy inequality:

$$\int_0^\infty |\partial_r f(r)|^2 dr \geq \frac{1}{4}\int_0^\infty \frac{|f(r)|^2}{r^2} dr$$

is valid. Applying this to the first term on the left-hand side of (17), we see that the left-hand side can be estimated from below by

$$\left(c + \frac{n^2-4n}{4}\right)\int_0^\infty \int_{\mathbb{S}^{n-1}} |u_k(r\omega)|^2 r^{n-5} d\omega dr.$$

We can bound the right-hand side (17) from above by using Cauchy inequality by

$$\|g_k\|_{L^2(\mathbb{R}^n)}\left(\int_0^\infty\int_{\mathbb{S}^{n-1}}|u_k(r\omega)|^2r^{n-5}d\omega dr\right)^{1/2}.$$

Hence we have the estimate

$$\left(c+\frac{n^2-4n}{4}\right)\left(\int_0^\infty\int_{\mathbb{S}^{n-1}}|u_k(r\omega)|^2r^{n-5}d\omega dr\right)^{1/2}\le\|g_k\|_{L^2(\mathbb{R}^n)},$$

or

$$\left(c+\left(\frac{n-2}{2}\right)^2-1\right)\left\|\frac{u_k}{|\cdot|^2}\right\|_{L^2(\mathbb{R}^n)}\le\|g_k\|_{L^2(\mathbb{R}^n)}. \tag{18}$$

Since this estimate with u_k being replaced by u_k-u_m is valid, using the fact that g_k is a Cauchy sequence in $L^2(\mathbb{R}^n)$, we see that $u_k/|x|^2$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Combining this with the fact that u_k converges to u in $L^2(\mathbb{R}^n)$, we find that $u_k/|x|^2$ tends to $u/|x|^2$ in $L^2(\mathbb{R}^n)$. Since χ is bounded, from (16) we obtain

$$-\Delta u+\frac{\chi(r)u}{r^2}=\overline{A_0}u \tag{19}$$

in the sense of distribution. Thus we deduce $u\in H^2(\mathbb{R}^n)$.

In conclusion, we have established that

$$\mathcal{D}(\overline{A_0})\subseteq\{u\in H^2(R^3)|\ u/|x|^2\in L^2(\mathbb{R}^n)\}$$

and (taking the limit as $k\to\infty$ in (18))

$$\left\|\frac{u}{|\cdot|^2}\right\|_{L^2(\mathbb{R}^n)}\lesssim\|\overline{A_0}u\|_{L^2(\mathbb{R}^n)} \tag{20}$$

for any $u\in\mathcal{D}(\overline{A_0})$, by (12).

The opposite inclusion easily follows. This completes the proof of a). The proof of b) follows directly from (20) and the Eq. (19).

Finally, we prove c). For $u\in\mathcal{D}((\overline{A_0})^{1/2})$, there exists $\{u_j\}\subset\mathcal{D}(\overline{A_0})$ such that $u_j\to u$ and $(\overline{A_0})^{1/2}u_j\to(\overline{A_0})^{1/2}u$ in $L^2(\mathbb{R}^n)$, because $\mathcal{D}(\overline{A_0})$ is a core of the closed operator $(\overline{A_0})^{1/2}$. Then we see from a) that

$$(\overline{A_0}u_j,\phi)_{L^2}=(-\Delta u_j,\phi)_{L^2}+\int_{\mathbb{R}^n}\frac{\chi(r)}{r^2}u_j\overline{\phi}dx$$

for any $\phi \in C_0^\infty(\mathbb{R}^n)$. By definition of $\overline{A_0}$, for each u_j with fixed j, there exists $\{\phi_j^k\}_k \subset C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\phi_j^k \to u_j$ as $k \to \infty$ in $L^2(\mathbb{R}^n)$. Therefore, we have

$$(\overline{A_0}u_j, u_j)_{L^2} = (-\Delta u_j, u_j)_{L^2} + \int_{\mathbb{R}^n} \frac{\chi(r)}{r^2}|u_j|^2 dx, \tag{21}$$

as $u_j \in \mathcal{D}(\overline{A_0})$. Note that we may replace u_j by $u_j - u_m$ in this identity.

Let us denote the positive and negative parts of χ by χ_+ and χ_-, respectively. Namely,

$$\chi = \chi_+ - \chi_-, \;\; \chi_+ = (|\chi| + \chi)/2, \;\; \chi_- = (|\chi| - \chi)/2.$$

Then we have

$$\|(\overline{A_0})^{1/2}(u_j - u_m)\|_{L^2} = \|\nabla(u_j - u_m)\|_{L^2} + \int_{\mathbb{R}^n} \frac{\chi_+(r)}{r^2}|u_j - u_m|^2 dx$$
$$- \int_{\mathbb{R}^n} \frac{\chi_-(r)}{r^2}|u_j - u_m|^2 dx.$$

If $c = \inf \chi(r) \geq 0$, then we have $\chi_- \equiv 0$, so that

$$\|(\overline{A_0})^{1/2}(u_j - u_m)\|_{L^2} = \|\nabla(u_j - u_m)\|_{L^2} + \int_{\mathbb{R}^n} \frac{\chi_+(r)}{r^2}|u_j - u_m|^2 dx,$$

which implies that $\{u_j\}$ converges in $H^1(\mathbb{R}^n)$ and $\{\sqrt{\chi_+(r)}u_j/r\}$ converges in $L^2(\mathbb{R}^n)$, since $\{u_j\}$ converges to u in $L^2(\mathbb{R}^n)$. Thus, from (21), we find

$$\|(\overline{A_0})^{1/2}u\|_{L^2} = \|\nabla u\|_{L^2} + \int_{\mathbb{R}^n} \frac{\chi_+(r)}{r^2}|u|^2 dx.$$

On the other hand, if $c < 0$, then we have $0 \leq \chi_-(r) \leq -c$ for $r \geq 0$, so that

$$0 \leq \int_{\mathbb{R}^n} \frac{\chi_-(r)}{r^2}|u_j - u_m|^2 dx \leq -c\left(\frac{2}{(n-2)}\right)^2 \|\nabla(u_j - u_m)\|_{L^2}^2,$$

by the Hardy inequality. Since $c > -(\frac{2}{(n-2)})^2$ by (12), similarly to the above, we see that $\{u_j\}$ converges in $H^1(\mathbb{R}^n)$ and $\{\sqrt{\chi_\pm(r)}u_j/r\}$ converges in $L^2(\mathbb{R}^n)$. Since $\{u_j\}$ converges to u in $L^2(\mathbb{R}^n)$, we obtain

$$\|(\overline{A_0})^{1/2}u\|_{L^2} = \|\nabla u\|_{L^2} + \int_{\mathbb{R}^n} \frac{\chi(r)}{r^2}|u|^2 dx$$
$$\geq (1 + c\left(\frac{2}{(n-2)}\right)^2)\|\nabla u\|_{L^2} + \int_{\mathbb{R}^n} \frac{\chi_+(r)}{r^2}|u|^2 dx.$$

In both cases, we have established $\|\nabla u\|_{L^2} \lesssim \|(\overline{A_0})^{1/2}u\|_{L^2}$, and also $\|u/r\|_{L^2} \lesssim \|(\overline{A_0})^{1/2}u\|_{L^2}$, thanks to the Hardy inequality. This finishes the proof. □

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Global Solutions to the Nonlinear Maxwell-Schrödinger System

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Abstract We study the existence of global-in-time solutions to a nonlinear Maxwell-Schrödinger system in three spatial dimensions. We combine suitable dispersive estimate in Besov spaces for the magnetic Schrödinger flow with a refined Brezis-Gallouet inequality, in order to prove global bounds in the case of a sub-cubic nonlinearity.

In this note we consider the nonlinear Maxwell-Schrödinger system

$$\begin{cases} i\partial_t u = -\Delta_A u + \phi u + |u|^{\gamma-1}u \\ \Box A = \mathbb{P}J, \end{cases} \qquad t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{1}$$

in the unknown $(u, A) : \mathbb{R}_t \times \mathbb{R}^3 \to \mathbb{C} \times \mathbb{R}^3$, with initial conditions

$$(u(0), A(0), \partial_t A(0)) = (u_0, A_0, A_1), \quad \operatorname{div}_x A_0 = \operatorname{div}_x A_1 = 0. \tag{2}$$

Here $\Delta_A := \nabla_A^2 = (\nabla - iA)^2$ is the magnetic Laplacian, $\phi = \phi(u) := (-\Delta)^{-1}|u|^2$, $J = J(u, A) := 2Im(\overline{u}(\nabla - iA)u)$, and $\mathbb{P} := \mathbb{I} - \nabla\mathrm{div}\Delta^{-1}$ is the Helmholtz-Leray projection onto divergence free vector fields. The choice of the Coulomb gauge ($\operatorname{div}_x A = 0$) is convenient for our analysis, but other gauges are possible as well, see e.g. the discussion in [3, Section 6].

The total charge $Q(t) := \|u\|_{L_x^2}^2$ and the energy

$$\mathcal{E}(t) := \int_{\mathbb{R}^3} |\nabla_A u|^2 + \frac{1}{2}\big(|\partial_t A|^2 + |\nabla A|^2 + |\nabla\phi|^2\big) + \frac{2}{\gamma+1}|u|^{\gamma+1}dx \tag{3}$$

are formally conserved by solutions to (1).

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The solution theory for the classical Maxwell-Schrödinger system (i.e. system (1) without the pure-power nonlinearity) has been widely investigated by several authors—we refer to [3] for a comprehensive overview of the literature. Global well-posedness at the $H^{\frac{11}{8}+}$-level of regularity for the wave function u has been obtained by Nakamura and Wada in [9]. The well-posedness at lower regularity, and in particular at the finite energy level $(u, A) \in H^1 \times H^1$, has been recently proved by Bejenaru and Tataru [4] by exploiting the tool of Bourgain-type spaces adapted to the magnetic-Schrödinger evolution.

For system (1) there are fewer results available. *Local* well-posedness at the $H^{\frac{11}{8}+}$-level of regularity for u has been proved in [5], by adapting the approach by Nakamura and Wada. On the other hand, it is unclear whether the result of Bejenaru and Tataru in the energy space could be extended to the case of a pure-power nonlinearity (some attempts in this direction can be found in [5]). The existence of *weak* solutions in the energy space has been proved in [1] by means of a regularization-compactness argument (see also [2, 6] for related results). *Global* well-posedness in the sub-cubic case (i.e. $\gamma \in (1, 3)$), at the H^2-level of regularity for u, has been proved in [3] by means of a modified energy argument.

The purpose of this note is to extend the above global well-posedness result for (1) to the whole regime for which a local theory is available.

In order to state our main theorem, we preliminary introduce suitable functional spaces. Given $s, \sigma \in \mathbb{R}$, we define

$$\Sigma^\sigma := \{(A_0, A_1) \in H^\sigma(\mathbb{R}^3, \mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3, \mathbb{R}^3) \mid \mathrm{div}_x A_0 = \mathrm{div}_x A_1 = 0\},$$
$$M^{s,\sigma} := H^s(\mathbb{R}^3, \mathbb{C}) \times \Sigma^\sigma.$$

Without loss of generality, we restrict our attention to forward-in-time solutions. We have the following result.

Theorem 1 *Let us fix* $s \in [\frac{11}{8}, 2]$, $\sigma \in (1, \min\{\frac{3}{2}s, 2s - \frac{3}{4}\})$*, and* $\gamma \in (s, 3)$. *Let us fix moreover an initial data* $(u_0, A_0, A_1) \in M^{s,\sigma}$. *Then the Cauchy problem* (1)–(2) *admits a unique, global solution* $u \in C(\mathbb{R}^+, M^{s,\sigma})$*, satisfying for* $T > 0$ *the double exponential upper bound*

$$\|(u, A, \partial_t A)\|_{L^\infty_T M^{s,\sigma}} \lesssim \exp\exp T. \tag{4}$$

Some remarks on Theorem 1 are in order.

- The restriction $s \leq 2$ is not strictly necessary for the well-posedness, and with some extra analysis (in the same spirit of [9, Proposition 4.2]) one could extend the result to the whole regime $s \geq \frac{11}{8}$ (with suitable upper bounds on σ).
- The restriction $\sigma > 1$ is more subtle, and it is related to the failure of the endpoint Strichartz estimates for the wave equation in three dimensions. It could be possible to assume the extra regularity on the *angular* part of (A_0, A_1) only,

as in this regime the endpoint Strichartz are recovered [8]. For the sake of concreteness, we do not purse this direction here.

- The lower bound $\gamma > s$ is required so that the pure-power nonlinearity is smooth enough to preserve the H^s-regularity. When $s = 2$ this can be sharpened to $\gamma > 1$ (see [3, Section 4]), and it is conceivable that the same improvement can be obtained for a generic $s \in [\frac{11}{8}, 2]$ by adapting the argument of Pecher [10].
- In the special case $s = 2$, as already mentioned, Theorem 1 can be proved via a modified energy method [3]. This allows, as byproduct, to deduce a polynomial upper bound for the growth of the $M^{2,\sigma}$-norm of the solutions (for $s = 2$ also the cubic case $\gamma = 3$ is covered, with an exponential upper bound). In the general case of fractional regularity, it is difficult to reproduce the modified energy argument, which is based on explicit integrations by parts. Our approach use instead refined a priori estimates in Besov spaces for u (see estimate (7) below) and sharp logarithmic corrections to critical Sobolev embedding (Lemma 2), which allows to cover the sub-cubic case with a double exponential bound.

Before starting our analysis, we briefly comment on our notation. Given an interval $I \subseteq \mathbb{R}$, we often write L^p_I to denote the space $L^p(I)$; if $I = [0, T]$ we just write L^p_T. We use the Japanese bracket $\langle\lambda\rangle := \sqrt{1+\lambda^2}$. Given two positive quantities A, B, we write $A \lesssim B$ if there exists $C > 0$ such that $A \leq CB$; if the constant C depends on some parameter k, we write $A \lesssim_k B$. When not specified otherwise, m denotes a positive integer constant, which may change at each occurrence. We consider the shifted logarithm $\ln_+(x) := \ln(e + x)$. We will use the classical Besov space $B^s_{p,q}$, that is the space of all tempered distributions f such that

$$\|f\|_{B^s_{p,q}} := \|2^{ks} f_k\|_{\ell^q_k L^p_x} < \infty,$$

where $\{f_k\}_{k\geq 0}$ is the Littlewood-Paley decomposition of f.

One of the main tool we shall exploit is the following version of Koch-Tzvetkov estimates for the inhomogeneous Schrödinger in the context of Besov spaces.

Lemma 1 *Let $s \in \mathbb{R}$, $\alpha, T > 0$, and let $\psi \in L^\infty_T H^s$ be a weak solution to the equation $i\partial_t\psi = -\Delta\psi + F$. Then we have the estimate*

$$\|\psi\|_{L^2_T B^{s-\alpha}_{6,2}} \lesssim_T \|\psi\|_{L^\infty_T H^s} + \|F\|_{L^2_T H^{s-2\alpha}}. \tag{5}$$

Estimate (5) has been proved in [9, Lemma 2.4], even though the result in [9] is stated for the Sobolev (Bessel potential) space $H^{s-\alpha,6}(\mathbb{R}^3)$ instead of the refined Besov space $B^{s-\alpha}_{6,2}(\mathbb{R}^3)$.

Using Lemma 1, and arguing as in the proof of [3, Proposition 3.1], one deduces the following a priori bounds for the Maxwell-Scrödinger system (1).

Proposition 1 *Let $\gamma \in (1, 3)$, $s \in [1, 2]$, $\sigma > 1$, $T > 0$, and set $\tilde\sigma := \min\{\sigma, \frac{7}{6}\}$. Fix moreover an initial data $(u_0, A_0, A_1) \in M^{s,\sigma}$, and let (u, A) be a weak solution* *to the Cauchy problem* (1)–(2)*. Then the following estimates hold true.*

$$\|A\|_{L^2_T L^\infty} + \|(A, \partial_t A)\|_{L^\infty_T \Sigma^{\tilde\sigma}} \lesssim_T \langle\|(u, A)\|^m_{L^\infty_T(H^1\times H^1)}\rangle\langle\|(A_0, A_1)\|^m_{\Sigma^\sigma}\rangle, \tag{6}$$

$$\|u\|_{L^2_T B^{s-1/2}_{6,2}} \lesssim_T \langle\|(u, A)\|^m_{L^\infty_T(H^1\times H^1)}\rangle\langle\|(A_0, A_1)\|^m_{\Sigma^\sigma}\rangle\|u\|_{L^\infty_T H^s}. \tag{7}$$

For our purposes it is relevant to consider the refined bound (7) for u in a Besov space, as the second scale exponent is strictly related to the logarithmic corrections of critical Sobolev embeddings (which in turn will be crucial for the globalization argument). In particular, we consider the following generalization [7, Theorem 2.1] of the classical Brezis-Gallouet-Weinger inequality.

Lemma 2 *Let $p, q, r \in [1, \infty)$, and $s > \frac{3}{q}$. Then we have the estimate*

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq D\big(1 + \|f\|_{\dot B^{3/p}_{p,r}(\mathbb{R}^3)} \ln_+^{1-1/r}\|f\|_{\dot B^s_{q,\infty}(\mathbb{R}^3)}\big), \tag{8}$$

for a suitable constant $D > 0$.

Observe that, when $r = 1$, estimate (8) is essentially equivalent to the well-known embedding $\dot B^{3/p}_{p,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. When $r = 2$, which is the second scale exponent arising from the refined estimate (7), one gets a $\ln_+^{1/2}$ correction to the critical embedding into $L^\infty(\mathbb{R}^3)$, which will be sufficient to globalize the solutions (in the regime $\gamma < 3$) through a nonlinear Grönwall-type argument.

We are now able to prove the main result of this note.

Proof of Theorem 1 The existence of a unique, strong $M^{s,\sigma}$-solution to (1), defined on a maximal existence interval $[0, T_{\max})$, together with conservation of mass and energy and the blow-up alternative, has been proved in [5, Theorem 2.1]. We need to show that the solution can be extended globally in time, namely $T_{\max} = +\infty$. To this aim, let us fix an arbitrary $T \in (0, T_{\max})$.

In view of the a priori estimates (6)–(7), the conservation of mass and energy, and the equivalence between classical and magnetic H^1-norms [9, Lemma 2.2], we deduce the bound

$$\|u\|_{L^\infty_T H^1 \cap L^2_T B^{1/2}_{6,2}} + \|(A, \partial_t A)\|_{L^\infty_T \Sigma^{\min\{\sigma, \frac{7}{6}\}}} \lesssim_T 1. \tag{9}$$

As a consequence of (9) and the a priori bound (7) we also get

$$\|u\|_{L^2_I B^{s-1/2}_{6,2}} \leq C_1\|u\|_{L^\infty_I H^s}, \tag{10}$$

for some positive constant $C_1 := C_1(T)$, valid for every interval $I \subseteq [0, T]$.

In addition, using estimate [9, Lemma 3.3] for the linear magnetic Schrödinger propagator and the bound (9), we obtain

$$\|u\|_{L_I^\infty H^s} \le C_2\big(\|u(\widetilde{t})\|_{H^s} + \||u|^{\gamma-1}u\|_{L_I^1 H^s}\big) \tag{11}$$

for some constant $C_2 := C_2(T) > 1$, valid for every interval $I \subseteq [0, T]$ and $\widetilde{t} \in \overline{I}$.

Next we consider a sequence of times $t_0 = 0 < t_1 \ldots < t_{N-1} < t_N = T$ such that, setting $I_j = [t_{j-1}, t_j]$ and

$$M_j := \ln_+^{(\gamma-1)/2}(T^{\frac{1}{2}} C_1 2^j C_2^j),$$

we have $|I_j| < (4CD)^{-1}$ and

$$\|u\|^{\gamma-1}_{L^2_{I_j} B^{1/2}_{6,2}} \le \frac{1}{4C_2 D M_j}, \qquad j = 1, \ldots N, \tag{12}$$

where $D > 0$ is the constant appearing in estimate (8) (with $p = 6$, $r = 2$). The existence of such sequence is guaranteed by the bound (9) for the $L^2_T B^{1/2}_{6,2}$-norm of u, the asymptotic $M(j) \sim j^{(\gamma-1)/2}$ for large j, and the divergence of the harmonic series (observe that $N \approx \exp T$). Now we have

$$\begin{aligned}
\||u|^{\gamma-1}\|_{L^1_{I_j} L^\infty} &\le D \int_{t_{j-1}}^{t_j} 1 + \|u(t)\|^{\gamma-1}_{B^{1/2}_{6,2}} \ln_+^{(\gamma-1)/2} \|u(t)\|_{B^{s-1/2}_{6,2}} dt \\
&\le (4C_2)^{-1} + D\|u\|^{\gamma-1}_{L^2_{I_j} B^{1/2}_{6,2}} \Big(\int_{t_{j-1}}^{t_j} \ln_+^{(\gamma-1)/(3-\gamma)} \|u\|_{B^{s-1/2}_{6,2}} dt\Big)^{\frac{3-\gamma}{2}} \\
&\le (4C_2)^{-1} + D\|u\|^{\gamma-1}_{L^2_{I_j} B^{1/2}_{6,2}} \ln_+^{(\gamma-1)/2} \|u\|_{L^1_{I_j} B^{s-1/2}_{6,2}} \\
&\le (4C_2)^{-1} + (4C_2 M_j)^{-1} \ln_+^{(\gamma-1)/2} (T^{\frac{1}{2}} C_1 \|u\|_{L^\infty_{I_j} H^s}),
\end{aligned}$$

where we used the generalized Brezis-Gallouet inequality (8) in the first step, Hölder inequality in time and the bound $|I_j| \le (4CD)^{-1}$ in the second step, Jensen inequality in the third step ($\ln_+^\alpha$ is concave for any $\alpha > 0$), and the bounds (10) and (12) in the last step. Combining the estimate above with (11) we obtain

$$\begin{aligned}
\|u\|_{L_{I_j} H^s} &\le C_2 \|u(t_{j-1})\|_{H^s} + \tfrac{1}{4}\|u\|_{L^\infty_{I_j} H^s} \\
&\quad + (4M_j)^{-1} \ln_+^{(\gamma-1)/2} \big(T^{\frac{1}{2}} C_1 \|u\|_{L^\infty_{I_j} H^s}\big) \|u\|_{L^\infty_{I_j} H^s}.
\end{aligned} \tag{13}$$

Using the definition of M_j, and applying the bound (13) iteratively, we get $\|u(t_j)\|_{H^s} \leq (2C_2)^j \|u_0\|_{H^s}$ for $j = 1, \dots, N$. Since $N \approx \exp T$, we deduce

$$\|u\|_{L^\infty_T H^s} \lesssim \exp\exp T. \tag{14}$$

Once the bound (14) for the H^s-norm of u is available, one can argue as in the proof of [3, Proposition 3.5] (which treats the case $\sigma = 2$) to infer

$$\|(A, \partial_t A)\|_{L^\infty_T \Sigma^\sigma} \lesssim \exp\exp T. \tag{15}$$

Combining (14)–(15) and using the blow-up alternative we then get $T_{\max} = +\infty$, together with the double exponential bound (4). The proof is complete. □

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On the Plate Equation with Exponentially Degenerating Stochastic Coefficients on the Torus

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Abstract This paper aims to investigate the plate equation with time-dependent stochastic coefficients on the torus, which is used for modeling the vibration of beams with random perturbations from various sources. We mainly study the joint influence from the exponentially degenerating and strong oscillating coefficients on the biharmonic and Laplace-Beltrami operators to explore the upper bound of loss of regularity by applying important techniques from microlocal analysis and stochastic analysis. More importantly, the critical case for loss of regularity has been deduced by the exquisite normal form diagonalization process. Furthermore, appropriate counter-examples with periodic coefficients are constructed in order to demonstrate the optimality of the estimates by the application of instability arguments.

1 Introduction

Plate equations are important mathematical models in the theoretical and practical research of vibrations of beams and deflection of plates. Study from various fields, such as mechanics of biochemical soft matters, inspection of polymeric material defects by piezoelectric smart beams, safety assessment of mega projects (cross-sea bridges, channel tunnels), etc. reveals that plenty of random influencing factors from both internal and external environments will inevitably make a significant impact on the vibration mechanics. To incorporate certain random perturbations, we discuss the plate equations with exponentially degenerating and strong oscillating coefficients of second-order moment stochastic processes. It is interesting to explore the well-posedness of the solutions of such kind of stochastic equations.

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This paper mainly addresses a degenerate 2-evolution plate equation described as follows,

$$\begin{cases} u_{tt} + D^2(t,\omega)\Delta_T^2 u - B(t,\omega)\Delta_T u = 0, & \text{in } (0,T] \times \mathbb{T}^N \times \Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & \text{on } \mathbb{T}^N. \end{cases} \tag{1}$$

- Here, Δ_T represents the Laplace-Beltrami operator on the torus $\mathbb{T}^N$. Let $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$ be an integer vector. As is known, the spectrum set $\text{spec}(-\Delta_T)$ on $L^2(\mathbb{T}^N)$ is composed of nonnegative discrete points

$$\text{spec}(-\Delta_T) = \{|\,\mathbf{m}|^2 : \mathbf{m} \in \mathbb{Z}^N\}.$$

In effect, the dimension of the eigenspace associated to $|\mathbf{m}|^2$ is calculated by the permutations and combinations of $(m_1, m_2, \cdots, m_N) \in \mathbb{Z}^N$ such that $\sum_{j=1}^N m_j^2 = |\mathbf{m}|^2$. Moreover, on $L^2(\mathbb{T}^N)$, there exists a corresponding complete orthonormal basis in the following form,

$$\Big\{\phi_{|\mathbf{m}|}^{(m_1,\cdots,m_N)}(x) = (2\pi)^{-N} \exp\Big(-i\sum_{j=1}^N m_j x_j\Big) :\ \sum_{j=1}^N m_j^2 = |\mathbf{m}|^2,\ \mathbf{m} \in \mathbb{Z}^N\Big\}.$$

Next, we begin to define the generalized Fourier transform and Fourier inverse transform by using the eigenvalues (spectrum) and eigenfunctions of the Laplace-Beltrami operator Δ_T on the torus $\mathbb{T}^N$.

Definition 1 Given the Laplace-Beltrami operator Δ_T on the torus $\mathbb{T}^N$, the associated generalized Fourier transform and Fourier inverse transform of a function $f \in L^2(\mathbb{T}^N)$ are defined as follows:

$$\hat{f}^{(m_1,\cdots,m_N)}(|\mathbf{m}|^2) := (f, \phi_{|\mathbf{m}|}^{(m_1,\cdots,m_N)})_{L^2}, \quad \mathbf{m} \in \mathbb{Z}^N,$$

$$f(x) := \sum_{\mathbf{m}\in\mathbb{Z}^N} \sum_{\sum_{i=1}^N m_i^2 = |\mathbf{m}|^2} \hat{f}^{(m_1,\cdots,m_N)}(|\,\mathbf{m}|^2)\phi_{|\mathbf{m}|}^{(m_1,\cdots,m_N)}(x).$$

The L^2-norm for f is defined accordingly in the sense of Fourier analysis,

$$\|f\|_{L^2} := \Big(\sum_{\mathbf{m}\in\mathbb{Z}^N} \sum_{\sum_{i=1}^N m_i^2 = |\mathbf{m}|^2} \Big|\hat{f}^{(m_1,\cdots,m_N)}(|\,\mathbf{m}|^2)\Big|^2\Big)^{\frac{1}{2}}.$$

Definition 2 Let F be a real-valued function on $[0, +\infty)$. We define a pseudodifferential operator as follows:

$$F(\sqrt{-\Delta_T}) : \mathrm{Dom}(F(\sqrt{-\Delta_T})) \subset L^2(\mathbb{T}^N) \to L^2(\mathbb{T}^N),$$

$$F(\sqrt{-\Delta_T})u(x) := \sum_{\mathbf{m}\in\mathbb{Z}^N} F(|\,\mathbf{m}|) \Big(\sum_{\sum_{i=1}^N m_i^2 = |\,\mathbf{m}|^2} \hat{u}^{(m_1,\cdots,m_N)}(|\,\mathbf{m}|^2)\phi_{|\mathbf{m}|}^{(m_1,\cdots,m_N)}(x)\Big),$$

for $u \in \mathrm{Dom}(F(\sqrt{-\Delta_T}))$. Furthermore, for any $s > 0$, the nonhomogeneous Sobolev space $\mathscr{H}^s(\mathbb{T}^N)$ is defined as follows,

$$\mathscr{H}^s(\mathbb{T}^N) := \{u \in L^2(\mathbb{T}^N) : (I_d - \Delta_T)^{\frac{s}{2}} u \in L^2(\mathbb{T}^N)\}.$$

And the corresponding $\mathscr{H}^s$-norm is defined by means of the general Fourier transform and inverse transform, ($\langle \mathbf{m}\rangle := \sqrt{1 + |\mathbf{m}|^2}$)

$$\|u\|_{\mathscr{H}^s} := \Big(\sum_{\mathbf{m}\in\mathbb{Z}^N} \langle \mathbf{m}\rangle^{2s} \Big(\sum_{\sum_{i=1}^N m_i^2 = |\,\mathbf{m}|^2} \Big|\hat{u}^{(m_1,\cdots,m_N)}(|\,\mathbf{m}|^2)\Big|^2\Big)\Big)^{\frac{1}{2}}.$$

Spacial fractional order pseudodifferential operators are effective in modeling nuclear magnetic resonance diffusometry in percolative and porous media, diffusion of a scalar tracer in an array of convection rolls, charge carrier transport in amorphous semiconductors, etc. For more interesting industrial applications and modeling processes, please refer to [12, 14, 18] and references therein.

- Due to the complex physical structures and random perturbations [2–4, 8, 9, 13, 14, 19], the mathematical model has both degenerating and randomly oscillating coefficients for the principle biharmonic operator. This coupling effect poses lots of challenges and it is our prior mission to explore the joint influence of both types of coefficients on the well-posedness of the solution. The degenerating and oscillating coefficient on the biharmonic operator part, that is, $D(t, \omega) = \exp(-t^{-1})A(t, \omega)$ is a second-order moment stochastic process and $A : (0, T] \times \Omega \to \mathbb{R}$, $T < 1$ is a second-order moment stochastic process, which measures the oscillation on the principal biharmonic operator about the starting time 0. In addition, for any $t \in (0, T]$, the oscillating process A is bounded almost surely, namely, there exist positive constants b_0 and b_1 such that for any $t \in (0, T]$,

$$\mathbb{P}\Big(\Omega_1 := \{\omega \in \Omega \mid b_0 \le A(t, \omega) \le b_1\}\Big) = 1. \tag{2}$$

Assume that for $\forall\ \omega \in \Omega_1$, $A \in C^2((0,T])$ and there exist two nonnegative random variables $C_k : \Omega \to \mathbb{R}, k = 1, 2$, such that the first and second derivatives of $A(t, \omega)$ satisfy

$$|A^{(k)}(t,\omega)| \leq C_k(\omega)(\nu(t)t^{-2})^k, k = 1, 2, \tag{3}$$

where $\nu \in C(0, T]$ is a decreasing and positive measure function of oscillation near the starting time 0. For instance, in constructing counterexamples, we will focus on the general case $\lim_{t\to 0^+} \nu(t) = +\infty$. More importantly, for $k = 1, 2$, C_k are uniformly bounded almost surely, that is to say, there exists an $M > 0$, such that

$$\mathbb{P}\Big(\Omega_2 := \{\omega \in \Omega \mid 0 \leq C_k(\omega) \leq M,\ \ k = 1, 2\}\Big) = 1. \tag{4}$$

Actually, from (2) and (4), we have $\mathbb{P}(\Omega_1 \cap \Omega_2) = 1$. Recall [17, 20] and let $\{X(t), t \in \mathscr{T}\}$ be a second-order moment stochastic process. $X \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ is called the limit in mean square (l.i.m) of $X(t)$ at $t = t_0$, denoted as $\text{l.i.m}_{t\to t_0} X(t) = X$, if $\lim_{t\to t_0} \|X(t) - X\|_2 = 0$. Here, t_0 can also be ∞. Due to the almost sure boundedness of $A(t, \omega)$, it is clear that $\text{l.i.m}_{t\to 0^+} D(t, \omega) = 0$, and $D(t, \omega)$ is a nonstationary stochastic process since the corresponding autocorrelation function $R(s, t)$ does not depend on the difference of s and t. As a matter of fact, our model covers lots of second-order stochastic processes in mechanical applications. For instance, $A(t, \omega) = F(t^{-1}(\ln(t^{-1}))^\kappa + \Theta),\ \ \kappa \in (1, +\infty)$, where $F \in C^2(\mathbb{R})$ is a bounded positive periodic function. It is evident that $A(t, \omega)$ is a nonstationary process. In this case, for any random variable Θ, such as Bernoulli trials, simple symmetric random walk, $(0 - 1)$ distribution, Poisson distribution, uniform distribution, normal distribution or Γ distribution, the corresponding stochastic processes $A(t, \omega)$ satisfy the above assumptions with $\nu(t) = (\ln(t^{-1}))^\kappa$.

- The degenerating and oscillating coefficient on the Laplace-Beltrami operator part $B : [0, T] \times \Omega \to \mathbb{R}$ is expressed as

$$B(t, \omega) = \gamma t^\beta \exp(-t^{-1})K(t, \omega),$$

where $\beta \geq -2$, $\gamma \in \mathbb{R}$ and $K(t, \omega)$ is an oscillating second-order moment stochastic process. Here we are interested in the C^∞ type Levi conditions. For any given $\omega \in \Omega$, K is continuous with respect to t and furthermore, K is uniformly bounded with respect to t and ω, and there exist two positive constants δ_0 and δ, such that

$$0 < \delta_0 \leq \inf_{(t,\omega)\in(0,T]\times\Omega} |K(t, \omega)| \leq \sup_{(t,\omega)\in(0,T]\times\Omega} |K(t, \omega)| \leq \delta.$$

Now, it is ready to show the main results concerned with the regularity behavior. Under the above assumptions, we have the following regularity statement for the Cauchy problem (1):

Theorem 1 *Let us consider the Cauchy problem* (1) *on* $[0, T] \times \mathbb{T}^N \times \Omega$. *Assume that the initial Cauchy data satisfy*

$$u_0 \in \mathscr{H}^s(\mathbb{T}^N), \quad u_1 \in \frac{1}{I^{-1}(\frac{2^{P_1}}{I_d - \Delta_T})} \mathscr{H}^s(\mathbb{T}^N),$$

where the Sobolev exponent $s \geq 2$, P_1 *is an appropriate positive integer to be detailed in the proof and* I^{-1} *stands for the inverse of the strictly increasing function* $I : (0, T] \to \mathbb{R}^N$, *defined by*

$$I(t) := \exp(-t^{-1})t^2.$$

I_d *stands for the identity operator. Then, there exists a unique stochastic process* u *whose regularity behavior is described by means of pseudodifferential operators as follows,*

- *For the following three cases:*

  1. $\gamma = 0$
  2. $\gamma \neq 0$ *and* $\beta > -2$
  3. $\gamma \neq 0$, $\beta = -2$ *and* $0 < |\gamma|\delta \leq b_0$

  we have

$$u \in \mathbb{C}_{\text{ms}}\Big([0, T]; J(I_d - \Delta_T)\mathscr{H}^s(\mathbb{T}^N)\Big),$$

$$u_t \in \mathbb{C}_{\text{ms}}\Big([0, T]; J(I_d - \Delta_T)\mathscr{H}^{s-2}(\mathbb{T}^N)\Big).$$

- *For* $\gamma \neq 0$, $\beta = -2$ *and* $|\gamma|\delta > b_0$, *we have*

$$u \in \mathbb{C}_{\text{ms}}\Big([0, T]; L\Big(G^{-1}(\frac{2^{P_2}}{I_d - \Delta_T})\Big) J(I_d - \Delta_T)\mathscr{H}^s(\mathbb{T}^N)\Big),$$

$$u_t \in \mathbb{C}_{\text{ms}}\Big([0, T]; L\Big(G^{-1}(\frac{2^{P_2}}{I_d - \Delta_T})\Big) J(I_d - \Delta_T)\mathscr{H}^{s-2}(\mathbb{T}^N)\Big).$$

Here, P_2 *is an appropriate positive integer. The function* G^{-1} *is the inverse of the strictly increasing function* $G : (0, T] \to \mathbb{R}$, *defined by*

$$G(t) := \frac{t^2 \exp(-t^{-1})}{\nu(t)}.$$

The strictly decreasing function $L : (0, T] \to \mathbb{R}^N$ *is defined as*

$$L(t) := \exp\Big(\frac{|\gamma|\delta - b_0}{2b_0 t}\Big) \text{ for } |\gamma|\delta > b_0.$$

And the function $J : (0, +\infty) \to \mathbb{R}$ *is defined as*

$$J(z) := \exp\Big(C\nu(G^{-1}(\frac{2^{P_2}}{z}))\Big),$$

where C *is a sufficiently large positive constant and* $\mathbb{C}_{\text{ms}}(\mathscr{T})$ *is defined as the set of all continuous in mean square second-order moment stochastic processes on* $\mathscr{T}$.

Remark 1 Here, $|\gamma|\delta = b_0$ is a critical case of loss of regularity, since there usually appears an abrupt jump of loss from $|\gamma|\delta < b_0$ to $|\gamma|\delta > b_0$. It is interesting to observe that, if $|\gamma|\delta < b_0$, then, loss of regularity comes directly from the biharmonic part. While in the other case $|\gamma|\delta > b_0$, the interaction of all the time-dependent coefficients determines the loss of regularity. As for the case $\gamma \neq 0$, $\beta < -2$, which does not satisfy the Levi-condition[8–10], the techniques we used in the proof does not work any more. However, when we consider a mild case in an approximate manner, that is, we let $\delta \to +\infty$, actually we obtain the infinite loss of regularity. This fact gives us a glimpse of the infinite loss of regularity for the case $\beta < -2$. In comparison with the results in [11], it is clear the difference of regularity for initial Cauchy data is significantly influenced by the degenerating part of the coefficients. More importantly, the oscillation of the stochastic coefficient $K(t, \omega)$ on the Laplace-Beltrami part plays an insignificant role in the regularity behavior of the solution.

Remark 2 Continuity in mean square $\mathbb{C}_{\text{ms}}$ with respect to t is a much stronger notion in comparison with the deterministic classic continuity with respect to t. That is to say, the deterministic continuity is the simplest case in the sense of continuity in mean square. More often, it is not correct to deduce the continuity in mean square purely from the classic continuity for a specific given ω. For instance, let us consider a discrete stochastic process $F_1(t, \Theta)$, $t \in \mathbb{R}$ as follows:

Obviously, $\Theta \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ since the series (Table 1)

$$\sum_{n=1}^{\infty} n^2 \cdot \frac{1}{2^n} < \infty.$$

For each θ_n, $F_1(\cdot, \theta_n) \in C^\infty(\mathbb{R})$. It is easy to verify that $F_1(t, \Theta)$ is definitely not a second-order moment stochastic process. Indeed, for each $t \neq 0$,

$$\mathbb{E}[|F_1(t, \Theta)|^2] = \sum_{n=1}^{\infty} |F_1(t, \theta_n)|^2 \mathbb{P}(\theta_n) = \sum_{n=1}^{\infty} t^2 \frac{25^n}{4^n} \frac{1}{2^n} = \infty.$$

Table 1 Distribution of a stochastic process $F_1(t, \Theta)$

Θ	$\mathbb{P}(\Theta)$	$F_1(t, \Theta)$
$\theta_1 = 1$	$\frac{1}{2}$	$\frac{5}{2}t$
$\theta_2 = 2$	$\frac{1}{2^2}$	$\frac{5^2}{2^2}t$
$\vdots$	$\vdots$	$\vdots$
$\theta_n = n$	$\frac{1}{2^n}$	$\frac{5^n}{2^n}t$
$\vdots$	$\vdots$	$\vdots$

Table 2 Distribution of stochastic processes $F_2(t, \Theta)$ and $F_3(t, \Theta)$

Θ	$\mathbb{P}(\Theta)$	$F_2(t, \Theta)$	$F_3(t, \Theta)$
$\theta_1 = 2$	$\frac{4}{5}$	$\sin(2^5 t)$	$2^5 \cos(2^5 t)$
$\theta_2 = 2^2$	$\frac{4}{5^2}$	$\sin(2^{10} t)$	$2^{10} \cos(2^{10} t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\theta_n = 2^n$	$\frac{4}{5^n}$	$\sin(2^{5n} t)$	$2^{5n} \cos(2^{5n} t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Subsequently, $F_1 \notin \mathbb{C}_{\text{ms}}(\mathbb{R})$. In the proof, we will apply the apriori micro-energy estimates and theory of second-order moment stochastic processes to deal with this property.

Remark 3 It is important to emphasize that In Theorem 1, the derivative u_t is in the sense of classic derivative for given ω. We attach more importance upon the discussion of continuity in mean square for $\|\frac{1}{J(I_d-\Delta_T)}u_t(t,\cdot,\omega)\|_{\mathscr{H}^{s-2}}$, since the above stochastic process is not necessarily the derivative in mean square of $\|\frac{1}{J(I_d-\Delta_T)}u(t,\cdot,\omega)\|_{\mathscr{H}^s}$ due to loss of regularity and stochastic phenomenon. In order to elucidate this point, let us consider two stochastic processes $F_2(t, \Theta)$ and $F_3(t, \Theta)$ displayed below (Table 2).

Clearly, $\Theta \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ since the series

$$\sum_{n=1}^{\infty} 2^{2n} \cdot \frac{4}{5^n} < \infty.$$

Here, $F_2(t, \Theta)$ is a second-order moment stochastic process for $t \in \mathbb{R}$ due to the fact

$$\mathbb{E}[|F_2(t,\Theta)|^2] = \sum_{n=1}^{\infty} |F_2(t,\theta_n)|^2 \mathbb{P}(\theta_n) = \sum_{n=1}^{\infty} \sin^2(2^{5n}t) \cdot \frac{4}{5^n} < \infty.$$

As is known, for each θ_n,

$$\frac{\mathrm{d}}{\mathrm{d}t} F_2(t,\theta_n) = F_3(t,\theta_n).$$

However, $F_3(t, \Theta)$ is not a derivative in mean square of $F_2(t, \Theta)$ since $F_3(t, \Theta)$ is not a second-order moment stochastic process for $t \in \mathbb{R}$. Indeed, for each $t = k\pi$, $k \in \mathbb{N}_+$, $h \neq 0$ sufficiently small, there exists a $P \in \mathbb{N}$ such that $2^{5P}h > 10$. By systematic calculation, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\frac{\sin(\Theta(k\pi + h)) - \sin(\Theta k\pi)}{h} - \Theta \cos(\Theta k\pi)\Big)^2\Big] \\
&= \sum_{n=1}^{\infty} \Big(\frac{\sin(2^{5n}(k\pi + h)) - \sin(2^{5n}k\pi)}{h} - 2^{5n}\cos(2^{5n}k\pi)\Big)^2 \cdot \frac{4}{5^n} \\
&= \sum_{n=1}^{\infty} \Big(\frac{\sin(2^{5n}k\pi)(\cos(2^{5n}h) - 1)}{h} + \frac{\cos(2^{5n}k\pi)(\sin(2^{5n}h) - 2^{5n}h)}{h}\Big)^2 \cdot \frac{4}{5^n} \\
&= \sum_{n=1}^{\infty} \Big(\frac{\sin(2^{5n}h) - 2^{5n}h}{h}\Big)^2 \cdot \frac{4}{5^n} \\
&\geq \sum_{n=P}^{\infty} \Big(\frac{2^{5n-1}h}{h}\Big)^2 \cdot \frac{4}{5^n} = \sum_{n=P}^{\infty} \frac{2^{10n}}{5^n}.
\end{aligned}$$

As a result,

$$\lim_{h\to 0} \mathbb{E}\Big[\Big(\frac{\sin(\Theta(k\pi + h)) - \sin(\Theta k\pi)}{h} - \Theta \cos(\Theta k\pi)\Big)^2\Big] \geq \lim_{P\to\infty} \sum_{n=P}^{\infty} \frac{2^{10n}}{5^n} = \infty.$$

Further analysis of the autocorrelation function for $F_2(t, \Theta)$ tells that $F_2(t, \Theta)$ is not differentiable in mean square for $t \in \mathbb{R}$.

It remains to discuss the optimality of the apriori estimates in Theorem 1. As a matter of fact, the method of instability argument was introduced in [2] to deal with well-posedness for equations with non-Lipschitz coefficients. Now we further develop this idea to demonstrate that the precise ν-loss of regularity really appears for plate equations with both degenerating and oscillating stochastic coefficients. Indeed, from Theorem 1, it is natural to deduce the following corollary which shows that at most a ν-loss for a family of Cauchy problems. In this part, we focus on the deterministic case with the general assumption $\lim_{t\to 0^+} \nu(t) = +\infty$.

Corollary 1 *Let us consider a family of Cauchy problems on* $[0, T] \times \mathbb{T}^N$.

$$\begin{cases} u_{k,tt} + \exp(-2t^{-1})A_k^2(t,\omega)\Delta_T^2 u_k - B(t,\omega)\Delta_T u_k = 0, & \text{in } (0,T] \times \mathbb{T}^N, \\ u_k(0,x) = u_{k,0}(x),\ u_{k,t}(0,x) = u_{k,1}(x), & \text{on } \mathbb{T}^N. \end{cases} \tag{5}$$

where the sequence $\{A_k\}_k$ satisfies all assumptions for the coefficient of the principal biharmonic operator in Theorem 1. If the initial Cauchy data satisfy

$$u_{k,0} \in \mathscr{H}^s(\mathbb{T}^N), \quad u_{k,1} \in \frac{1}{I^{-1}(\frac{2^{P_1}}{I_d - \Delta_T})}\mathscr{H}^s(\mathbb{T}^N),$$

then, there exists a unique sequence of solutions $\{u_k\}_k$ in the following function spaces.

- *For the following three cases:*
  1. $\gamma = 0$
  2. $\gamma \neq 0$ *and* $\beta > -2$
  3. $\gamma \neq 0$, $\beta = -2$ *and* $0 < |\gamma|\delta \leq b_0$

  we have

$$u_k \in \mathbb{C}_{\text{ms}}\Big([0, T]; J(I_d - \Delta_T)\mathscr{H}^s(\mathbb{T}^N)\Big),$$

$$u_{k,t} \in \mathbb{C}_{\text{ms}}\Big([0, T]; J(I_d - \Delta_T)\mathscr{H}^{s-2}(\mathbb{T}^N)\Big).$$

- *For $\gamma \neq 0$, $\beta = -2$ and $|\gamma|\delta > b_0$, we have*

$$u_k \in \mathbb{C}_{\text{ms}}\Big([0, T]; L\Big(G^{-1}(\frac{2^{P_2}}{I_d - \Delta_T})\Big)J(I_d - \Delta_T)\mathscr{H}^s(\mathbb{T}^N)\Big),$$

$$u_{k,t} \in \mathbb{C}_{\text{ms}}\Big([0, T]; L\Big(G^{-1}(\frac{2^{P_2}}{I_d - \Delta_T})\Big)J(I_d - \Delta_T)\mathscr{H}^{s-2}(\mathbb{T}^N)\Big).$$

In the following, we mainly focus on the loss operator $J(I_d - \Delta_T)$ in the exponential form. In order to construct a counter-example showing at least a ν-loss appears, we apply the following nonhomogeneous energy.

Definition 3 For w belonging to the function spaces

$$w \in C\Big([0, T]; \mathscr{H}^s(\mathbb{T}^N)\Big), \quad w_t \in C\Big([0, T]; \mathscr{H}^{s-2}(\mathbb{T}^N)\Big),$$

we introduce the nonhomogeneous energy for $s \geq 2$,

$$\mathbf{E}_s(w)(t) := \exp(-2t^{-1})\|w(t, \cdot)\|^2_{\mathscr{H}^s} + \|w_t(t, \cdot)\|^2_{\mathscr{H}^{s-2}}. \tag{6}$$

As a matter of fact, the difference of regularity for the initial Cauchy data originates from the degenerating part $\exp(-2t^{-1})$. Here, we apply the above definition since the difference of regularity has been fully embodied in this energy form. Without loss of generality, we mainly consider the case $\gamma = 0$. In a word, our optimality argument can be expressed in the following statement.

Theorem 2 *For $\gamma = 0$, there exists*

- *a sequence of oscillating coefficients $\{b_k(t)\}_k$ satisfying all assumptions of Theorem 1 with constants independent of k;*
- *a sequence of initial Cauchy data $\{(u_{k,0}(x), u_{k,1}(x))\}_k$ belonging to the function space*

$$\mathscr{H}^s(\mathbb{T}^N) \times \frac{1}{I^{-1}(\frac{2^{P_1}}{I_d - \Delta_T})} \mathscr{H}^s(\mathbb{T}^N),$$

such that the sequence of the corresponding solutions $\{u_k\}_k$ of (5) satisfies

$$\sup_k \mathbf{E}_2(u_k)(0) \le C(\varepsilon), \tag{7}$$

$$\sup_k \mathbf{E}_2\Big(\exp\Big(- c_1(\varepsilon)\nu(G^{-1}(\frac{2^{P_2}}{I_d - \Delta_T}))\Big) u_k\Big)(t) = +\infty, \tag{8}$$

for $t \in (0, T]$, where $C(\varepsilon)$ and $c_1(\varepsilon)$ depend on the sufficiently small positive constant ε.

Remark 4 The idea of the above argument lies in the essential regularity behavior which emphasizes that, de facto, the constant C in the pseudodifferential operator $J(I_d - \Delta_T)$ must be larger than 0. In this case, we are endowed with the possibility to search for a smaller positive constant $c_1(\varepsilon)$ which will lead to infinity of the corresponding $\mathscr{H}^s$-norms.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. Some important tools from microlocal analysis will be taken to obtain the precise ν-loss of regularity. In Sect. 3, we construct suitable counter-examples for Theorem 2 and discuss the optimality of the apriori estimates in Theorem 2 by the application of harmonic analysis and instability arguments.

2 Proof of Theorem 1

For all $\omega \in \Omega_1 \bigcap \Omega_2$ and $\mathbf{m} \in \mathbb{Z}^N$, by applying the generalized Fourier transform from Definition 1 on the Cauchy problem (1), we have, for $t \in [0, T]$,

$$\begin{cases} \hat{u}_{tt}^{(m_1,\cdots,m_N)} + D^2(t,\omega)|\mathbf{m}|^4 \hat{u}^{(m_1,\cdots,m_N)} + B(t,\omega)|\mathbf{m}|^2 \hat{u}^{(m_1,\cdots,m_N)} = 0, \\ \hat{u}^{(m_1,\cdots,m_N)}(0, |\mathbf{m}|^2) = \hat{u}_0^{(m_1,\cdots,m_N)}(|\mathbf{m}|^2), \\ \hat{u}_t^{(m_1,\cdots,m_N)}(0, |\mathbf{m}|^2) = \hat{u}_1^{(m_1,\cdots,m_N)}(|\mathbf{m}|^2). \end{cases} \tag{9}$$

For convenience's sake, as no further confusion arises, we denote

$$\lambda := |\mathbf{m}|^2,$$

$$\hat{u}(t, \lambda, \omega) := \hat{u}^{(m_1, \cdots, m_N)}(t, |\mathbf{m}|^2, \omega) = (u, \phi_{|\mathbf{m}|}^{(m_1, \cdots, m_N)})_{L^2}.$$

Therefore, (9) can be simplified as

$$\begin{cases} \hat{u}_{tt} + D^2(t, \omega)\lambda^2 \hat{u} + B(t, \omega)\lambda \hat{u} = 0, \\ \hat{u}(0, \lambda) = \hat{u}_0(\lambda), \ \hat{u}_t(0, \lambda) = \hat{u}_1(\lambda). \end{cases} \tag{10}$$

2.1 Introduction to Some Tools from Microlocal Analysis

First and foremost, we introduce some important tools from microlocal analysis which will facilitate our discussion. Similar as the I-Method for Schrödinger type equations introduced in [1, 14], we give three zones in the quantized time-frequency (spectrum) space.

Definition 4 We divide the quantized time-frequency space into three zones $\mathbb{Z}_i$, $i = 1, 2, 3$, namely,

$$\begin{aligned} \mathbb{Z}_1 &:= \left\{(t, \lambda) \in [0, T] \times \mathrm{spec}(-\Delta_T) : t^2 \exp(-t^{-1})(1 + \lambda) \in (0, 2^{P_1})\right\}; \\ \mathbb{Z}_2 &:= \left\{(t, \lambda) \in (0, T] \times \mathrm{spec}(-\Delta_T) : t^2 \exp(-t^{-1})(1 + \lambda) \in [2^{P_1}, 2^{P_2}\nu(t))\right\}; \\ \mathbb{Z}_3 &:= \left\{(t, \lambda) \in (0, T] \times \mathrm{spec}(-\Delta_T) : t^2 \exp(-t^{-1})(1 + \lambda) \in [2^{P_2}\nu(t), +\infty)\right\}, \end{aligned}$$

where $P_1, P_2 \in \mathbb{N}_+$ are constants which will be detailed later in the proof.

In fact, when we study the regularity behavior of solutions, large λ is of prime consideration. Therefore, in the definition of zones, we exclude the case $\lambda = 0$. Here, for large λ, $|\mathbf{m}|^2 \approx \langle \mathbf{m} \rangle^2$.

Definition 5 Due to the monotonicity of the function ν, $t_\lambda^{(i)}$, $i = 1, 2$ are defined as the solutions of

$$t^4 \exp(-2t^{-1})(1 + \lambda)^2 = 2^{2P_i} \nu^{2i-2}(t), \ \ i = 1, 2,$$

respectively. And

$$(t_\lambda^{(i)})^4 \exp(-2(t_\lambda^{(i)})^{-1})(1 + \lambda)^2 = 2^{2P_i} \nu^{2i-2}(t_\lambda^{(i)}), \ \ i = 1, 2$$

are called the first and second separating curves in the quantized time-frequency space, respectively.

Definition 6 In each zone, for given $\omega \in \Omega_1 \bigcap \Omega_2$, we define the corresponding micro-energy matrix uniformly as

$$V(t,\lambda,\omega) := (V_1(t,\lambda,\omega), V_2(t,\lambda,\omega))^T.$$

More specifically, let ϕ be the positive root of

$$\phi^2 := 1 + t^{-2}\exp(-t^{-1})\lambda.$$

The precise micro-energies are listed as, ($D_t := \dfrac{1}{i}\dfrac{\partial}{\partial t}$)

- In $\mathbb{Z}_1$, $V(t,\lambda,\omega) := (\phi\hat{u}, D_t\hat{u})^T$;
- In $\mathbb{Z}_2$, $V(t,\lambda,\omega) := \exp(-t^{-1})\lambda\hat{u}, D_t\hat{u})^T$;
- In $\mathbb{Z}_3$, $V(t,\lambda,\omega) := ((D(t,\omega)\lambda\hat{u}, D_t\hat{u})^T$.

Indeed, the choice of the auxiliary function ϕ is very important. A good choice can lead to exquisite estimates. Up to now, the above ϕ is very effective from the empirical point of view. For given $\omega \in \Omega_1 \bigcap \Omega_2$, in $\mathbb{Z}_3$, we define a generalized Schwarz symbol class $S^r_\omega\{m_1, m_2, m_3\}$ and introduce several rules of symbol calculus.

Definition 7 For given $\omega \in \Omega_1 \bigcap \Omega_2$ and $r \in \mathbb{N}$, we say that

$$a(t,\lambda,\omega) \in S^r_\omega\{m_1, m_2, m_3\}$$

if for all $k = 1, \cdots, r$,

$$|a^{(k)}(t,\lambda,\omega)| \le C_k(\omega)\lambda^{\frac{m_1}{2}}\exp(-m_2 t^{-1})\Big(t^{-2}\nu(t)\Big)^{m_3+k}$$

hold, where $m_i \in \mathbb{R}$, $i = 1, 2, 3$, $C_k(\omega)$, $k = 1, \cdots, r$ are positive constants depending upon ω. And $a^{(k)}$ stands for the k-th partial derivative with respect to the time.

Indeed, by using Definitions 4 and 7, we can easily verify the following facts about symbol calculus, which are very crucial for the normal form diagonalization processes.

Lemma 1 *For given $\omega \in \Omega_1 \bigcap \Omega_2$, the subsequent rules of symbol calculus hold:*

- $S^{r+1}_\omega\{m_1, m_2, m_3\} \subset S^r_\omega\{m_1, m_2, m_3\}$;
- *If $a \in S^r_\omega\{m_1, m_2, m_3\}$, $b \in S^r_\omega\{k_1, k_2, k_3\}$, then, $ab \in S^r_\omega\{m_1+k_1, m_2+k_2, m_3+k_3\}$;*
- *If $a \in S^r_\omega\{m_1, m_2, m_3\}$, then, $a^{(k)} \in S^{r-k}_\omega\{m_1, m_2, m_3+k\}$, where $k = 1, \cdots, r$;*
- *In $\mathbb{Z}_3$, $S^r_\omega\{m_1, m_2, m_3\} \subset S^r_\omega\{m_1+2p, m_2+p, m_3-p\}$, where $p \ge 0$;*
- *In $\mathbb{Z}_3$, if $a \in S^0_\omega\{-2, -1, 2\}$, then, $\left|\int_{t^{(2)}_\lambda}^t a(\tau,\lambda,\omega)\mathrm{d}\tau\right| \le C(\omega)\nu(t^{(2)}_\lambda)$.*

2.2 Micro-Energy Estimates in $\mathbb{Z}_1$

Lemma 2 *Given $\omega \in \Omega_1 \bigcap \Omega_2$, for $(t, \lambda) \in \mathbb{Z}_1$, we have the following uniform apriori estimates with a positive constant C depending upon P_1 and b_1:*

$$|\hat{u}(t, \lambda, \omega)| \leq C\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big);$$

$$|D_t\hat{u}(t, \lambda, \omega)| \leq C\lambda \exp(-(t_\lambda^{(1)})^{-1})\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).$$

Proof Let us study the first order system

$$D_t V = \mathscr{C}(t, \lambda, \omega)V := \begin{pmatrix} \dfrac{D_t\phi}{\phi} & \phi \\ \dfrac{D^2(t, \omega)\lambda^2 + \gamma t^\beta \exp(-t^{-1})K(t, \omega)\lambda}{\phi} & 0 \end{pmatrix} V.$$

And we explore the fundamental solution $\mathscr{E} = \mathscr{E}(t, s, \lambda, \omega)$ of the above system by means of the method in [6],

$$\begin{cases} D_t\mathscr{E}(t, s, \lambda, \omega) = \mathscr{C}(t, \lambda, \omega)\mathscr{E}(t, s, \lambda, \omega), \\ \mathscr{E}(s, s, \lambda, \omega) = I. \end{cases} \tag{11}$$

Actually, the following facts hold

$$\begin{aligned} D_t\mathscr{E}_{11}(t, s, \lambda, \omega) &= \frac{D_t\phi}{\phi}\mathscr{E}_{11}(t, s, \lambda, \omega) + \phi\mathscr{E}_{21}(t, s, \lambda, \omega), \\ D_t\mathscr{E}_{12}(t, s, \lambda, \omega) &= \frac{D_t\phi}{\phi}\mathscr{E}_{12}(t, s, \lambda, \omega) + \phi\mathscr{E}_{22}(t, s, \lambda, \omega), \\ D_t\mathscr{E}_{21}(t, s, \lambda, \omega) &= \frac{D^2(t, \omega)\lambda^2 + \gamma t^\beta \exp(-t^{-1})K(t, \omega)\lambda}{\phi}\mathscr{E}_{11}(t, s, \lambda, \omega), \\ D_t\mathscr{E}_{22}(t, s, \lambda, \omega) &= \frac{D^2(t, \omega)\lambda^2 + \gamma t^\beta \exp(-t^{-1})K(t, \omega)\lambda}{\phi}\mathscr{E}_{12}(t, s, \lambda, \omega), \end{aligned} \tag{12}$$

with the initial value matrix

$$\begin{pmatrix} \mathscr{E}_{11}(s, s, \lambda, \omega) & \mathscr{E}_{12}(s, s, \lambda, \omega) \\ \mathscr{E}_{21}(s, s, \lambda, \omega) & \mathscr{E}_{22}(s, s, \lambda, \omega) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

Moreover, we have the following estimates for the fundamental solution of (11) in $\mathbb{Z}_1$:

Lemma 3 *The entries of the fundamental solution matrix $\mathscr{E} = \mathscr{E}(t, s, \lambda, \omega)$ can be estimated as (Here, the constant $C(P_1, b_1)$ maybe different from line to line)*

$$|\mathscr{E}_{11}(t, s, \lambda, \omega)| \leq C(P_1, b_1)\frac{\phi(t, \lambda)}{\phi(s, \lambda)},$$

$$|\mathscr{E}_{12}(t, s, \lambda, \omega)| \leq C(P_1, b_1)\phi(t, \lambda)(t - s),$$

$$|\mathscr{E}_{21}(t, s, \lambda, \omega)| \leq C(P_1, b_1)\frac{(\exp(-t^{-1}) - \exp(-s^{-1}))\lambda}{\phi(s, \lambda)},$$

$$|\mathscr{E}_{22}(t, s, \lambda, \omega)| \leq C(P_1, b_1)\Big(1 + (t - s)(\exp(-t^{-1}) - \exp(-s^{-1}))\lambda\Big).$$

Proof Let us denote

$$\mathscr{C}(t, \lambda, \omega) = \begin{pmatrix} \mathscr{C}_{11} & \mathscr{C}_{12} \\ \mathscr{C}_{21} & \mathscr{C}_{22} \end{pmatrix},$$

$$\mathscr{C}_{21}^0(t, \lambda, \omega) := D^2(t, \omega)\lambda^2 + \gamma t^\beta \exp(-t^{-1})K(t, \omega)\lambda.$$

By taking into account the definition of $\mathbb{Z}_1$, we deduce that, there exists a positive constant $C(P_1, b_1)$ depending upon P_1 and b_1 such that

$$\|\mathscr{C}(t, \lambda, \omega)\| \leq C(P_1, b_1)\phi + \frac{\phi_t}{\phi}.$$

By the matrizant representation method in [12, 15], one has, in $\mathbb{Z}_1$

$$\|\mathscr{E}(t, s, \lambda, \omega)\| \leq C(P_1, b_1)\frac{\phi(t, \lambda)}{\phi(s, \lambda)}.$$

From (12), we deduce that\$°

$$\begin{aligned} |\mathscr{E}_{21}(t, s, \lambda, \omega)| &\leq C(P_1, b_1)\Big|\int_s^t \frac{\mathscr{C}_{21}^0(\tau, \lambda, \omega)}{\phi(s, \lambda)} \mathrm{d}\tau\Big| \\ &\leq C(P_1, b_1)\frac{(\exp(-t^{-1}) - \exp(-s^{-1}))\lambda}{\phi(s, \lambda)}. \end{aligned}$$

As for $\mathscr{E}_{12}(t, s, \lambda, \omega)$, indeed, once more from (12), we obtain the following representation

$$\mathscr{E}_{12}(t, s, \lambda, \omega) = i\phi(t, \lambda) \int_s^t \mathscr{E}_{22}(\tau, s, \lambda, \omega)\mathrm{d}\tau.$$

Let us denote $f(t, s, \lambda, \omega) := \int_s^t \mathscr{E}_{22}(\tau, s, \lambda, \omega)\mathrm{d}\tau$. Then, we get

$$\begin{cases} f_t(t, s, \lambda, \omega) = \mathscr{E}_{22}(t, s, \lambda, \omega), \\ f_{tt}(t, s, \lambda, \omega) = \mathscr{E}_{22,t}(t, s, \lambda, \omega), \\ f(s, s, \lambda, \omega) = 0, \\ f_t(s, s, \lambda, \omega) = 1. \end{cases}$$

Simple calculation shows

$$f_{tt} = -\mathscr{C}_{21}^0(t, \lambda, \omega) f.$$

In order to estimate f in a profound way, we apply a comparison lemma from [5].

□

Lemma 4 *If $h, g \in C^2[s, T]$ are the solutions of*

$$\begin{cases} h_{tt}(t, s) = A_1(t)h(t, s), \\ h(s, s) = H_0 \geq 0, \ h_t(s, s) = H_1 \geq 0, \end{cases}$$

and

$$\begin{cases} g_{tt}(t, s) = B_1(t)g(t, s), \\ g(s, s) = G_0 \geq 0, \ g_t(s, s) = G_1 \geq 0, \end{cases}$$

respectively, where $A_1, \ B_1 \in C[s, T]$ and $|B_1(t)| \leq A_1(t)$, $G_0 \leq H_0$, $G_1 \leq H_1$. Then, it holds

$$|g(t, s)| \leq h(t, s), t \in [s, T].$$

For convenience' sake, without any confusion, let $h = h(t, s)$ be the solution of the second-order differential equation with Cauchy data

$$\begin{cases} h_{tt}(t, s) = C(P_1, b_1)t^{-2} \exp(-t^{-1})\lambda h(t, s), \\ h(s, s) = 0, \ h_t(s, s) = 1. \end{cases}$$

Clearly, for T sufficiently small, since

$$|\mathscr{C}_{21}^{0}(t,\lambda,\omega)| \le C(P_1,b_1)t^{-2}\exp(-t^{-1})\lambda,$$

then, $h(t,s) \ge 0$ and $h_t(t,s) > 0$, which lead to

$$h_{tt}(t,s) \le C(P_1,b_1)\Big(\exp(-t^{-1})\lambda h(t,s)\Big)_t.$$

By integrating from s to t, we have

$$h_t(t,s) - 1 \le C(P_1,b_1)\exp(-t^{-1})\lambda h(t,s).$$

Furthermore, keeping in mind the definition of $\mathbb{Z}_1$, we apply Gronwall's inequality and obtain

$$h(t,s) \le C(P_1,b_1)(t-s).$$

Subsequently, by means of the comparison result in Lemma 4, we have

$$\Big|\int_s^t \mathscr{E}_{22}(\tau,s,\lambda,\omega)\mathrm{d}\tau\Big| \le C(P_1,b_1)(t-s).$$

Consequently,

$$|\mathscr{E}_{12}(t,s,\lambda,\omega)| \le C(P_1,b_1)\phi(t,\lambda)(t-s).$$

It remains to estimate $\mathscr{E}_{22}(t,s,\lambda,\omega)$. Actually, it holds that

$$\begin{aligned}
&|\mathscr{E}_{22}(t,s,\lambda,\omega) - 1| \\
&= \Big|i\int_s^t \frac{\mathscr{C}_{21}^{0}(t,\lambda)}{\phi(\tau,\lambda)}\mathscr{E}_{12}(\tau,s,\lambda,\omega)\mathrm{d}\tau\Big| \\
&\le C(P_1,b_1)(t-s)(\exp(-t^{-1}) - \exp(-s^{-1}))\lambda.
\end{aligned}$$

Lemma 3 is concluded. □

Using the entry estimates for the fundament solution $\mathscr{E}$, we have

$$|\phi(t,\lambda)\hat{u}(t,\lambda,\omega)| \le C(P_1,b_1)\Big(\phi(t,\lambda)|\hat{u}_0(\lambda)| + \phi(t,\lambda)t|\hat{u}_1(\lambda)|\Big);$$

$$|D_t\hat{u}(t,\lambda,\omega)| \le C(P_1,b_1)\Big(\lambda\exp(-t^{-1})|\hat{u}_0(\lambda)| + (1+t\exp(-t^{-1})\lambda)|\hat{u}_1(\lambda)|\Big).$$

Lemma 2 is deduced immediately when we recall the definitions of $\mathbb{Z}_1$ and the first separating curve $(t_\lambda^{(1)})^2 \exp(-(t_\lambda^{(1)})^{-1})\lambda = 2^{P_1}$. The proof is concluded. □

2.3 Micro-Energy Estimates in $\mathbb{Z}_2$

Lemma 5 *Given $\omega \in \Omega_1 \bigcap \Omega_2$, for $(t,\lambda) \in \mathbb{Z}_2$, we have the following uniform apriori estimates with positive constants C_1 and C_2 depending upon P_1, P_2 and b_1:*

$$|\hat{u}(t,\lambda,\omega)| \leq C_1 \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big);$$

$$|D_t\hat{u}(t,\lambda,\omega)| \leq C_1 \lambda \exp(-(t_\lambda^{(1)})^{-1}) \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).$$

Proof In this zone, we consider the following first order system

$$\begin{aligned} D_t V &= \mathscr{H}(t,\lambda,\omega)V \\ &:= \begin{pmatrix} \dfrac{1}{it^2} & \exp(-t^{-1})\lambda \\ \exp(-t^{-1})A^2(t,\omega)\lambda + \gamma t^\beta K(t,\omega) & 0 \end{pmatrix} V. \end{aligned}$$

Without any confusion, we still denote the fundamental solution of the above system by $\mathscr{E} = \mathscr{E}(t,s,\lambda,\omega)$,

$$\begin{cases} D_t\mathscr{E}(t,s,\lambda,\omega) = \mathscr{H}(t,s,\lambda,\omega)\mathscr{E}(t,s,\lambda,\omega), \\ \mathscr{E}(s,s,\lambda,\omega) = I. \end{cases}$$

As a matter of fact, the fundamental solution $\mathscr{E} = \mathscr{E}(t,s,\lambda,\omega)$ can be given in the matrizant representation form

$$\mathscr{E} = I + \sum_{k=1}^{\infty} i^k \int_s^t \mathscr{H}(t_1,\lambda,\omega) \int_s^{t_1} \mathscr{H}(t_2,\lambda,\omega) \cdots \int_s^{t_{k-1}} \mathscr{H}(t_k,\lambda,\omega) dt_k \cdots dt_1.$$

Indeed, by the induction method, it holds that

$$
\begin{aligned}
&\|\mathscr{E}(t_\lambda^{(2)}, t_\lambda^{(1)}, \lambda, \omega)\| \\
&\le \exp\Big(\int_{t_\lambda^{(1)}}^{t_\lambda^{(2)}} \|\mathscr{H}(\tau, \lambda, \omega)\| \mathrm{d}\tau\Big) \\
&\le \exp\Big(\int_{t_\lambda^{(1)}}^{t_\lambda^{(2)}} C(b_1)(\exp(-\tau^{-1})\lambda + \frac{1}{\tau^2})\mathrm{d}\tau\Big) \\
&\le \exp\Big(C(P_1, P_2, b_1)\nu(t_\lambda^{(2)})\Big).
\end{aligned}
$$

As a result, we have the following estimates,

$$
\begin{aligned}
|\exp(-t^{-1})\lambda\hat{u}(t, \lambda, \omega)| \le \exp\Big(C\nu(t_\lambda^{(2)})\Big)& \\
\Big(\exp(-(t_\lambda^{(1)})^{-1})\lambda|\hat{u}(t_\lambda^{(1)}, \lambda, \omega)| &+ |\hat{u}_t(t_\lambda^{(1)}, \lambda, \omega)|\Big);
\end{aligned}
$$

$$
|D_t\hat{u}(t, \lambda, \omega)| \le \exp\Big(C\nu(t_\lambda^{(2)})\Big)\Big(\exp(-(t_\lambda^{(1)})^{-1})\lambda|\hat{u}(t_\lambda^{(1)}, \lambda, \omega)| + |\hat{u}_t(t_\lambda^{(1)}, \lambda, \omega)|\Big).
$$

Taking into account Lemma 2, we conclude the proof immediately. □

2.4 Micro-Energy Estimates in $\mathbb{Z}_3$

Lemma 6 *Given $\omega \in \Omega_1 \bigcap \Omega_2$, for $(t, \lambda) \in \mathbb{Z}_3$, we have the following uniform apriori estimates with positive constants C_1 and C_2 depending upon P_1, P_2, b_0 and b_1:*

- *For the following three cases:*
  1. $\gamma = 0$
  2. $\gamma \ne 0$ *and* $\beta > -2$
  3. $\gamma \ne 0$, $\beta = -2$ *and* $0 < |\gamma|\delta \le b_0$

$$
|\hat{u}(t, \lambda, \omega)| \le C_1 \exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big);
$$

$$
|D_t\hat{u}(t, \lambda, \omega)| \le C_1\lambda \exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).
$$

- *For* $\gamma \neq 0$, $\beta = -2$ *and* $|\gamma|\delta > b_0$

$$|\hat{u}(t,\lambda,\omega)| \le C_1 L(t_\lambda^{(2)}) \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big);$$

$$|D_t\hat{u}(t,\lambda,\omega)| \le C_1 \lambda L(t_\lambda^{(2)}) \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).$$

Proof The whole proof is based on two steps of diagonalization and the estimates of fundamental solutions. By taking into account the definition of micro-energy in this zone, we study the following first order system:

$$D_t V - \begin{pmatrix} 0 & D(t,\omega)\lambda \\ D(t,\omega)\lambda & 0 \end{pmatrix} V - \begin{pmatrix} \frac{1}{it^2} + \frac{A'(t,\omega)}{iA(t,\omega)} & 0 \\ \frac{\gamma t^\beta K(t,\omega)}{A(t,\omega)} & 0 \end{pmatrix} V = 0.$$

Step 1: First Step of Diagonalization

For our purposes, let the diagonalizer matrix be

$$\mathscr{M} := \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Applying the change of variables $V_0 = \mathscr{M}V$, we obtain

$$\begin{aligned} D_t V_0 - \mathscr{D}V_0 + \mathscr{K}V_0 &= D_t V_0 - \mathscr{D}V_0 + \mathscr{K}_1 V_0 - \mathscr{K}_2 V_0 \\ := D_t V_0 &- \begin{pmatrix} -D(t,\omega)\lambda & 0 \\ 0 & D(t,\omega)\lambda \end{pmatrix} V_0 \\ &+ \frac{\gamma t^\beta K(t,\omega)}{2A(t,\omega)}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} V_0 - \Big(\frac{1}{2it^2} + \frac{A'(t,\omega)}{2iA(t,\omega)}\Big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} V_0 \\ = \ 0, \end{aligned}$$

where $\mathscr{D} \in S_\omega^2\{2,1,0\}$ and $\mathscr{K} = \mathscr{K}_1 - \mathscr{K}_2 \in S_\omega^1\{0,0,1\}$.

Step 2: Second Step of Normal Form Diagonalization

To carry out this step of diagonalization, or the so-called normal form diagonalization, we follow the procedure of asymptotic theory for differential equations. That is to say, we aim to construct an invertible matrix

$$\mathscr{N}_1(t,\lambda,\omega) := I + \mathscr{N}^{(1)}(t,\lambda,\omega)$$

for our purposes. To further the discussion, let

$$\mathscr{N}^{(0)} := I,\ \mathscr{K}^{(0)} := \mathscr{K},\ \mathscr{F}^{(0)} := \mathrm{diag}(\mathscr{K}^{(0)});$$

$$\mathscr{N}^{(1)}_{qr} := \frac{\mathscr{K}^{(0)}_{qr}}{\tau_q - \tau_r},\ q \neq r;$$

$$\mathscr{N}^{(1)}_{qq} := 0;$$

$$\tau_k := (-1)^k D(t,\omega)\lambda,\ k = 1, 2;$$

$$\mathscr{K}^{(1)} := (D_t - \mathscr{D} + \mathscr{K})(I + \mathscr{N}^{(1)}) - (I + \mathscr{N}^{(1)})(D_t - \mathscr{D} + \mathscr{F}^{(0)}).$$

According to the calculus properties of the Schwarz symbol class, we deduce that $\mathscr{N}^{(1)} \in S^1_\omega\{-2,-1,1\}$ and $\mathscr{F}^{(0)} \in S^1_\omega\{0,0,1\}$. Moreover, it holds that

$$\mathscr{K}^{(1)} = \mathscr{K} + [\mathscr{N}^{(1)}, \mathscr{D}] - \mathscr{F}^{(0)} + D_t\mathscr{N}^{(1)} + \mathscr{K}\mathscr{N}^{(1)} - \mathscr{N}^{(1)}\mathscr{F}^{(0)},$$

where $[\cdot,\cdot]$ is the Lie bracket. Indeed, the construction principle indicates that the first three terms of $\mathscr{K}^{(1)}$ vanish. Hence, $\mathscr{K}^{(1)} \in S^0_\omega\{-2,-1,2\}$. Subsequently, let us define

$$\mathscr{R}_1 := \mathscr{N}_1^{-1}((D_t - \mathscr{D} + \mathscr{K})(I + \mathscr{N}^{(1)}) - (I + \mathscr{N}^{(1)})(D_t - \mathscr{D} + \mathscr{F}^{(0)})).$$

By virtue of the principles of symbol calculus, this definition means $\mathscr{R}_1 \in S^0_\omega\{-2,-1,2\}$. As a matter of fact, keeping in mind the definition of $\mathbb{Z}_3$, from $\mathscr{N}^{(1)} \in S^1_\omega\{-2,-1,1\}$, we deduce that

$$|\mathscr{N}^{(1)}_{qr}| \leq \frac{C_1}{2^{P_2}},$$

where C_1 is a positive constant independent of ω. Evidently, an appropriate large integer P_2 assures that

$$\|\mathscr{N}_1 - I\| < \frac{1}{2},$$

which implies the invertibility of $\mathscr{N}_1$.

As a result, we have the following system after the second step of diagonalization:

$$(D_t - \mathscr{D} + \mathscr{K})\mathscr{N}_1 = \mathscr{N}_1(D_t - \mathscr{D} + \mathrm{diag}(\mathscr{K}) + \mathscr{R}_1), \text{ where } \mathscr{R}_1 \in S^0_\omega\{-2, -1, 2\}.$$

Step 3: Estimates of the Fundamental Solutions

We apply the change of variables $V_1 = \mathscr{N}_1^{-1} V_0$ and consider the following first order system

$$\Big(D_t - \mathscr{D} + \mathrm{diag}(\mathscr{K}) + \mathscr{R}_1\Big)V_1 = 0.$$

After a careful rearrangement of the terms, we obtain another representation form

$$(D_t - \widetilde{\mathscr{D}} + \widetilde{\mathscr{K}} + \mathscr{R}_1)V_1 = 0, \tag{13}$$

with

$$\widetilde{\mathscr{D}} := \begin{pmatrix} -D(t,\omega)\lambda & 0 \\ 0 & D(t,\omega)\lambda \end{pmatrix} + \begin{pmatrix} \frac{1}{2it^2} + \frac{A'(t,\omega)}{2iA(t,\omega)} & 0 \\ 0 & \frac{1}{2it^2} + \frac{A'(t,\omega)}{2iA(t,\omega)} \end{pmatrix}$$

and

$$\widetilde{\mathscr{K}} := \begin{pmatrix} \frac{\gamma t^\beta K(t,\omega)}{2A(t,\omega)} & 0 \\ 0 & -\frac{\gamma t^\beta K(t,\omega)}{2A(t,\omega)} \end{pmatrix}.$$

Without any confusion, we still denote the fundamental solution of (13) as $\mathscr{E}$. In fact, the fundamental solution for the system (13) is represented as $\mathscr{E} = \mathscr{E}_1\mathscr{S}$, where

$$\mathscr{E}_1^{(11)}(t,s,\lambda,\omega) = \exp\Big(-i\int_s^t D(\tau,\omega)\lambda \mathrm{d}\tau + \frac{1}{2}\ln(\frac{\exp(-t^{-1})}{\exp(-s^{-1})}) + \frac{1}{2}\ln(\frac{A(t,\omega)}{A(s,\omega)})\Big),$$

$$\mathscr{E}_1^{(22)}(t,s,\lambda,\omega) = \exp\Big(i\int_s^t D(\tau,\omega)\lambda \mathrm{d}\tau + \frac{1}{2}\ln(\frac{\exp(-t^{-1})}{\exp(-s^{-1})}) + \frac{1}{2}\ln(\frac{A(t,\omega)}{A(s,\omega)})\Big),$$

$$\mathscr{E}_1^{(12)}(t,s,\lambda,\omega) = \mathscr{E}_1^{(21)}(t,s,\lambda,\omega) = 0,$$

and $\mathscr{S}$ is the fundamental solution of the following system

$$D_t\mathscr{S} + \underbrace{\mathscr{E}_1(t,s,\lambda,\omega)^{-1}(\widetilde{\mathscr{K}}(t,\omega) + \mathscr{R}_1(t,\lambda,\omega))\mathscr{E}_1(t,s,\lambda,\omega)}_{\widetilde{\mathscr{R}}_1(t,s,\lambda,\omega)}\mathscr{S} = 0.$$

Indeed, we have

$$\|\mathscr{E}_1(t,s,\lambda)\| \le C\sqrt{\frac{\exp(-t^{-1})}{\exp(-s^{-1})}},$$

where the positive constant C is independent of ω. Since $\mathscr{R}_1 \in S^0_\omega\{-2,-1,2\}$, then, it holds that

$$\begin{aligned}\|\widetilde{\mathscr{R}}_1(t,s,\lambda,\omega)\| &\le \Big|\frac{\gamma t^\beta K(t,\omega)}{2A(t,\omega)}\Big| + C\frac{t^{-4}\exp(t^{-1})\nu^2(t)}{\lambda}\\ &\le \frac{|\gamma|\delta}{2b_0 t^{-\beta}} + C\frac{t^{-4}\exp(t^{-1})\nu^2(t)}{\lambda},\end{aligned}$$

where C is independent of ω. Consequently, we have

$$\|\mathscr{S}(t,s,\lambda,\omega)\| \le \exp\Big(\int_s^t \|\widetilde{\mathscr{R}}_1(\tau,s,\lambda,\omega)\|\mathrm{d}\tau\Big)$$

$$\le \begin{cases}\exp\Big(C\nu(s)\Big)\Big(\frac{t}{s}\Big)^{\frac{|\gamma|\delta}{2b_0}}, & \beta = -1,\\ \exp\Big(C\nu(s)\Big)\exp\Big(\frac{|\gamma|\delta}{2b_0}(\frac{t^{\beta+1}}{\beta+1} - \frac{s^{\beta+1}}{\beta+1})\Big), & \beta \ne -1,\end{cases}$$

with C independent of ω. Therefore, the fundamental solution of (13) can be estimated as:

for $\beta = -1$,

$$\|\mathscr{E}(t,t_\lambda^{(2)},\lambda,\omega)\| \le C_1\sqrt{\frac{\exp(-t^{-1})}{\exp(-(t_\lambda^{(2)})^{-1})}}\Big(\frac{t}{t_\lambda^{(2)}}\Big)^{\frac{|\gamma|\delta}{2b_0}}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big),$$

for $\beta \neq -1$,

$$\|\mathscr{E}(t, t_\lambda^{(2)}, \lambda, \omega)\| \leq C_1 \sqrt{\frac{\exp(-t^{-1})}{\exp(-(t_\lambda^{(2)})^{-1})}} \exp\Big(C_2 \nu(t_\lambda^{(2)}) + \frac{|\gamma|\delta}{2b_0} \frac{t^{\beta+1} - (t_\lambda^{(2)})^{\beta+1}}{\beta+1}\Big).$$

Step 4: Estimates of the Micro-Energy

As a matter of fact,

$$V_1(t, \lambda, \omega) = \mathscr{E}(t, t_\lambda^{(2)}, \lambda, \omega) V_1(t_\lambda^{(2)}, \lambda, \omega).$$

For $\gamma = 0$, by recalling the invertible transforms

$$V_1 = \mathscr{N}_1^{-1} V_0, \quad V_0 = \mathscr{M} V_1,$$

we have the following estimates for both cases $\beta = -1$ and $\beta \neq -1$,

$$|\hat{u}(t, \lambda, \omega)| \leq C_1 \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)|\Big);$$

$$|D_t \hat{u}(t, \lambda, \omega)| \leq C_1 \lambda \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)|\Big),$$

where C_1 and C_2 are positive constants depending upon P_1, P_2, b_0 and b_1.

Now we discuss the further estimates for $\gamma \neq 0$ in three cases.

Case I $\gamma \neq 0, \beta = -1$

In this case, we have

$$\|V(t, \lambda, \omega)\| \leq C_1 \sqrt{\frac{\exp(-t^{-1})}{\exp(-(t_\lambda^{(2)})^{-1})} \Big(\frac{t}{t_\lambda^{(2)}}\Big)^{\frac{|\gamma|\delta}{2b_0}}} \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big) \|V(t_\lambda^{(2)}, \lambda, \omega)\|, \tag{14}$$

where C_1 and C_2 are positive constants depending upon P_1, P_2, b_0 and b_1. Actually, from Lemma 5, we deduce that,

$$|\hat{u}(t_\lambda^{(2)}, \lambda, \omega)| \leq C_1 \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)|\Big);$$

$$|D_t \hat{u}(t_\lambda^{(2)}, \lambda, \omega)| \leq C_1 \lambda \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big) \exp\Big(-(t_\lambda^{(1)})^{-1}\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)|\Big). \tag{15}$$

By combining the estimates (14) and (15), we derive the following estimates in $\mathbb{Z}_3$,

$$\begin{aligned}
&|\hat{u}(t,\lambda,\omega)| \\
&\le C_1\Big(\frac{\exp(-(t_\lambda^{(2)})^{-1})}{\exp(-t^{-1})}\Big)^{\frac{1}{2}}\Big(\frac{t}{t_\lambda^{(2)}}\Big)^{\frac{|\gamma|\delta}{2b_0}}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)|+t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big) \\
&\le C_1\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)|+t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big); \\
&|D_t\hat{u}(t,\lambda,\omega)| \\
&\le C_1\lambda\exp(-t^{-1})\Big(\frac{\exp(-(t_\lambda^{(2)})^{-1})}{\exp(-t^{-1})}\Big)^{\frac{1}{2}}\Big(\frac{t}{t_\lambda^{(2)}}\Big)^{\frac{|\gamma|\delta}{2b_0}}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big) \\
&\cdot\Big(|\hat{u}_0(\lambda)|+t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big) \\
&\le C_1\lambda\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)|+t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).
\end{aligned} \tag{16}$$

Case II $\gamma \neq 0, \beta > -1$

For this case, we have

$$\begin{aligned}
&|\hat{u}(t,\lambda,\omega)| \\
&\le C_1\Big(\frac{\exp(-(t_\lambda^{(2)})^{-1})}{\exp(-t^{-1})}\Big)^{\frac{1}{2}}\exp\Big(C_2\nu(t_\lambda^{(2)})+\frac{|\gamma|\delta}{2b_0}\frac{t^{\beta+1}-(t_\lambda^{(2)})^{\beta+1}}{\beta+1}\Big) \\
&\cdot\Big(|\hat{u}_0(\lambda)|+t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big) \\
&\le C_1\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)|+t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big); \\
&|D_t\hat{u}(t,\lambda,\omega)|
\end{aligned}$$

$$\leq C_1 \lambda \exp(-t^{-1}) \Big(\frac{\exp(-(t_\lambda^{(2)})^{-1})}{\exp(-t^{-1})} \Big)^{\frac{1}{2}} \exp \Big(C_2 \nu(t_\lambda^{(2)}) + \frac{|\gamma|\delta}{2b_0} \frac{t^{\beta+1} - (t_\lambda^{(2)})^{\beta+1}}{\beta+1} \Big)$$

$$\cdot \Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)| \Big)$$

$$\leq C_1 \lambda \exp \Big(C_2 \nu(t_\lambda^{(2)}) \Big) \Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)| \Big). \tag{17}$$

Case III $\gamma \neq 0, -2 \leq \beta < -1$

Let $S : (0, T] \to \mathbb{R}$ be defined as

$$S(t) := \exp(-t^{-1}) \exp \Big(\frac{|\gamma|\delta}{b_0(-\beta - 1)t^{-\beta-1}} \Big).$$

In the case $-2 < \beta < -1$, since $S'(t) > 0$, then, we have

$$\begin{aligned} &|\hat{u}(t, \lambda, \omega)| \\ &\leq C_1 \sqrt{\frac{S(t_\lambda^{(2)})}{S(t)}} \exp \Big(C_2 \nu(t_\lambda^{(2)}) \Big) \Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)| \Big) \\ &\leq C_1 \exp \Big(C_2 \nu(t_\lambda^{(2)}) \Big) \Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)| \Big); \\ &|D_t \hat{u}(t, \lambda, \omega)| \\ &\leq C_1 \lambda \exp(-t^{-1}) C_1 \sqrt{\frac{S(t_\lambda^{(2)})}{S(t)}} \exp \Big(C_2 \nu(t_\lambda^{(2)}) \Big) \Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)| \Big) \\ &\leq C_1 \lambda \exp \Big(C_2 \nu(t_\lambda^{(2)}) \Big) \Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)} |\hat{u}_1(\lambda)| \Big). \end{aligned} \tag{18}$$

In the case $\beta = -2$, since $S'(t) > 0$ when $\dfrac{|\gamma|\delta}{b_0} \leq 1$, then, we have

$$\begin{aligned}
&|\hat{u}(t,\lambda,\omega)| \\
&\leq C_1\sqrt{\frac{S(t_\lambda^{(2)})}{S(t)}}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big) \\
&\leq C_1\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big); \\
&|D_t\hat{u}(t,\lambda,\omega)| \\
&\leq C_1\lambda\exp(-t^{-1})C_1\sqrt{\frac{S(t_\lambda^{(2)})}{S(t)}}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big) \\
&\leq C_1\lambda\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).
\end{aligned} \tag{19}$$

In the case $\beta = -2$, since $S'(t) < 0$ when $\dfrac{|\gamma|\delta}{b_0} > 1$, then, we have

$$\begin{aligned}
&|\hat{u}(t,\lambda,\omega)| \\
&\leq C_1\sqrt{\frac{S(t_\lambda^{(2)})}{S(t)}}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big) \\
&\leq C_1\sqrt{S(t_\lambda^{(2)})}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big); \\
&|D_t\hat{u}(t,\lambda,\omega)| \\
&\leq C_1\lambda\exp(-t^{-1})C_1\sqrt{\frac{S(t_\lambda^{(2)})}{S(t)}}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big) \\
&\leq C_1\lambda\sqrt{S(t_\lambda^{(2)})}\exp\Big(C_2\nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).
\end{aligned} \tag{20}$$

Lemma 6 follows immediately when we recall the results from Lemma 5.

□

2.5 Conclusion

By considering Lemmas 2, 5 and 6 in a comprehensive manner, we have the following uniform estimates with respect to ω. Pay attention that, the positive constants C_1 and C_2 depend upon P_1, P_2, b_0 and b_1.

- For the following three cases:
    1. $\gamma = 0$
    2. $\gamma \neq 0$ and $\beta > -2$
    3. $\gamma \neq 0$, $\beta = -2$ and $0 < |\gamma|\delta \leq b_0$

$$
\begin{aligned}
|\hat{u}(t,\lambda,\omega)| &\leq C_1 \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big);\\
|D_t\hat{u}(t,\lambda,\omega)| &\leq C_1 \lambda \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).
\end{aligned}
\tag{21}
$$

- For $\gamma \neq 0$, $\beta = -2$ and $|\gamma|\delta > b_0$

$$
\begin{aligned}
|\hat{u}(t,\lambda,\omega)| &\leq C_1 L(t_\lambda^{(2)}) \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big);\\
|D_t\hat{u}(t,\lambda,\omega)| &\leq C_1 \lambda L(t_\lambda^{(2)}) \exp\Big(C_2 \nu(t_\lambda^{(2)})\Big)\Big(|\hat{u}_0(\lambda)| + t_\lambda^{(1)}|\hat{u}_1(\lambda)|\Big).
\end{aligned}
\tag{22}
$$

Taking into account the definitions of two separating curves and definitions of pseudodifferential operators associated to the Laplace-Beltrami operator on the torus, we may conclude the regularity results of Theorem 1. It is worth noticing that the first separating curve determines the difference of regularity of initial Cauchy data, while the second separating curve determines the loss of regularity.

It remains to discuss the continuity in mean square of the solutions. Without loss of generality, we consider the discrete stochastic processes with $\gamma = 0$. For the discussion of the case $\gamma \neq 0$, we can replace the operator $J(I_d - \Delta_T)$ by corresponding operators in other cases since the time interval is bounded. The continuous cases can be discussed in a similar manner. Let

$$
u_i(t,x) := u(t,\omega_i,x).
$$

As a matter of fact, from (21), we know the continuity properties for each $\omega_i \in \Omega_1 \bigcap \Omega_2$,

$$
\begin{aligned}
u_i &\in C\Big([0,T]; J(I_d - \Delta_T)\mathscr{H}^s(\mathbb{T}^N)\Big),\\
u_{i,t} &\in C\Big([0,T]; J(I_d - \Delta_T)\mathscr{H}^{s-2}(\mathbb{T}^N)\Big).
\end{aligned}
\tag{23}
$$

At the same time, for the initial Cauchy data satisfying

$$u_{i,0} \in \mathscr{H}^s(\mathbb{T}^N), \quad u_{i,1} \in \frac{1}{I^{-1}(\frac{2^{P_1}}{I_d - \Delta_T})}\mathscr{H}^s(\mathbb{T}^N),$$

we have the following uniform estimates with a positive constant D independent of t and ω_i,

$$\Big\|\frac{1}{J(I_d - \Delta_T)}u_i(t,\cdot)\Big\|_{\mathscr{H}^s} \leq C\Big(\|u_0\|_{\mathscr{H}^s} + \Big\|I^{-1}(\frac{2^{P_1}}{I_d - \Delta_T})u_1\Big\|_{\mathscr{H}^s}\Big) \leq D,$$

$$\Big\|\frac{1}{J(I_d - \Delta_T)}u_{i,t}(t,\cdot)\Big\|_{\mathscr{H}^{s-2}} \leq C\Big(\|u_0\|_{\mathscr{H}^s} + \Big\|I^{-1}(\frac{2^{P_1}}{I_d - \Delta_T})u_1\Big\|_{\mathscr{H}^s}\Big) \leq D. \tag{24}$$

In order to prove the continuity in mean square for $\|\frac{1}{J(I_d - \Delta_T)}u(t,\cdot,\omega)\|_{\mathscr{H}^s}$ and $\|\frac{1}{J(I_d - \Delta_T)}u_t(t,\cdot,\omega)\|_{\mathscr{H}^{s-2}}$, respectively, it suffices to consider the autocorrelation functions for both stochastic processes, in respectively. In the following, we focus on the continuity in mean square for $\|\frac{1}{J(I_d - \Delta_T)}u(t,\cdot,\omega)\|_{\mathscr{H}^s}$. For the sake of simplicity, we denote

$$X(t) := \Big\|\frac{1}{J(I_d - \Delta_T)}u(t,\cdot,\omega)\Big\|_{\mathscr{H}^s},$$

$$X(t,\omega_i) := \Big\|\frac{1}{J(I_d - \Delta_T)}u_i(t,\cdot)\Big\|_{\mathscr{H}^s}.$$

As a matter of fact, $X(t)$ is a second-order moment stochastic process. Indeed, from the estimates in (24), we have

$$\|X(t)\|_2^2 = \mathbb{E}\Big[\Big\|\frac{1}{J(I_d - \Delta_T)}u(t,\cdot,\omega)\Big\|^2_{\mathscr{H}^s}\Big]$$

$$= \sum_i \Big\|\frac{1}{J(I_d - \Delta_T)}u_i(t,\cdot)\Big\|^2_{\mathscr{H}^s}\mathbb{P}(\omega_i) \leq D^2.$$

Similarly, $\|\frac{1}{J(I_d - \Delta_T)}u_t(t,\cdot,\omega)\|_{\mathscr{H}^{s-2}} \in L^2(\Omega,\mathscr{F},\mathbb{P})$.

In effect, we deduce that, for $(s, t) \in [0, T] \times [0, T]$, the corresponding autocorrelation function $R(s, t)$ of $X(t)$ satisfies

$$R(s,t) = \mathbb{E}[X(s)X(t)] = \sum_i X(s,\omega_i)X(t,\omega_i)\mathbb{P}(\omega_i) \leq D^2.$$

Actually, for $|\Omega| < +\infty$, the continuity in mean square for $X(t)$ is directly deduced from the continuity of $X(t, \omega_i)$. Indeed, the corresponding autocorrelation function $R(s, t)$ is continuous with respect to $(s, t) \in [0, T] \times [0, T]$. Now we turn our attention to the general case $|\Omega| = \infty$. For each $t_0 \in [0, T]$, by using Schwarz inequality, we have

$$\begin{aligned}
&|R(s,t) - R(t_0,t_0)|^2 \\
&= \Big|\mathbb{E}[X(s)X(t)] - \mathbb{E}[X(t_0)X(t_0)]\Big|^2 \\
&\leq 2\Big|\mathbb{E}[(X(s) - X(t_0))X(t)]\Big|^2 + 2\Big|\mathbb{E}[X(t_0)(X(t) - X(t_0))]\Big|^2 \\
&\leq 2\mathbb{E}[|X(s) - X(t_0)|^2]\mathbb{E}[|X(t)|^2] + 2\mathbb{E}[|X(t) - X(t_0)|^2]\mathbb{E}[|X(t_0)|^2] \\
&\leq 2D^2\mathbb{E}[|X(s) - X(t_0)|^2] + 2D^2\mathbb{E}[|X(t) - X(t_0)|^2].
\end{aligned}$$

Furthermore, Weierstrass theorem assures that the series

$$\begin{aligned}
&\mathbb{E}[|X(s) - X(t_0)|^2] \\
&= \sum_i |X(s,\omega_i) - X(t_0,\omega_i)|^2\mathbb{P}(\omega_i) \\
&= \sum_i \Big(|X(s,\omega_i)|^2 + |X(t_0,\omega_i)|^2 - 2X(s,\omega_i)X(t_0,\omega_i)\Big)\mathbb{P}(\omega_i)
\end{aligned}$$

converges uniformly, which gives the continuity of $\mathbb{E}[|X(s) - X(t_0)|^2]$ with respect to s together with the results in (23). Consequently, we have

$$\lim_{s\to t_0} \mathbb{E}[|X(s) - X(t_0)|^2] = 0.$$

In the final analysis, we have

$$\lim_{(s,t)\to(t_0,t_0)} R(s,t) = R(t_0,t_0),$$

which demonstrates the property of continuity in mean square in the interval $[0, T]$ for $\|\frac{1}{J(I_d - \Delta_T)}u(t,\cdot,\omega)\|_{\mathscr{H}^s}$. In a similar manner, continuity in mean square for the stochastic process $\|\frac{1}{J(I_d - \Delta_T)}u_t(t,\cdot,\omega)\|_{\mathscr{H}^{s-2}}$ in the interval $[0, T]$ can be proved.

In particular, one can prove that

$$\frac{\partial}{\partial s}R(s,t),\ \frac{\partial}{\partial t}R(s,t),\ \frac{\partial^2}{\partial s\partial t}R(s,t),\ \frac{\partial^2}{\partial t\partial s}R(s,t)$$

all exist and are continuous on $[0, T] \times [0, T]$ due to the uniform continuity properties and estimates in (24). Therefore, the general second-order differentiability of $R(s,t)$ is assured, which indicates the differentiability in mean square of $\|\frac{1}{J(I_d - \Delta_T)}u(t,\cdot,\omega)\|_{\mathscr{H}^s}$ with respect to $t \in [0, T]$. It is worth noticing that the derivative in mean square for $\|\frac{1}{J(I_d - \Delta_T)}u(t,\cdot,\omega)\|_{\mathscr{H}^s}$ cannot be directly deduced as in the deterministic case, since $\|\frac{1}{J(I_d - \Delta_T)}u_t(t,\cdot,\omega)\|_{\mathscr{H}^s}$ may possibly tend to infinity. This phenomenon is not so surprising since the derivative with respect to time t is inevitably influenced by the spacial structures of the system stochastic and phenomenon as shown in Remark 3.

The other cases can be proved in a similar manner. And we conclude the whole proof of Theorem 1.

3 Proof of Theorem 2

Similar as in [7, 11, 14–16, 19], we divide the proof into four steps.

3.1 Step 1: Introduction of Some Auxiliary Functions and Series

In order to construct appropriate coefficients leading to the designated loss of regularity, it is necessary to introduce some proper oscillating functions.

Definition 8 For a sufficiently small $\varepsilon > 0$, we define two oscillating functions on $\mathbb{R}$:

$$w_\varepsilon(t) := \sin t \exp\Big(2\varepsilon \int_0^t \psi(\tau)\sin^2\tau \mathrm{d}\tau\Big),$$

$$a_\varepsilon(t) := 1 - 4\varepsilon\psi(t)\sin(2t) - 2\varepsilon\psi'(t)\sin^2 t - 4\varepsilon^2\psi^2(t)\sin^4 t,$$

where the non-negative real-valued smooth function ψ is 2π-periodic on $\mathbb{R}$ and identically 0 in a neighborhood of 0. Furthermore, it satisfies

$$\int_0^{2\pi} \psi(\tau)\sin^2(\tau)\mathrm{d}\tau = \pi.$$

Based on the above definition, it is easy to verify the following fact through simple calculations.

Lemma 7 *As are defined in Definition 8, both a_ε and w_ε are smooth functions on $\mathbb{R}$. Particularly, w_ε is the unique solution of the following Cauchy problem*

$$\begin{cases} w_\varepsilon''(t) + a_\varepsilon(t)w_\varepsilon(t) = 0, \\ w_\varepsilon(0) = 0, \quad w_\varepsilon'(0) = 1. \end{cases}$$

Next, we begin to construct the oscillating and degenerating intervals.

Definition 9 Three sequences are defined as follows:

$$\{\delta_k\}_k := \Big\{\exp(-t_k^{-1})\Big\}_k,$$

$$\{h_k\}_k := \Big\{\frac{2^{P_2}\nu(t_k)}{t_k^2}\Big\}_k,$$

$$\{\rho_k\}_k := \Big\{\frac{2^{-P_2+2}\pi t_k^2[\nu(t_k)]}{\nu(t_k)}\Big\}_k,$$

where $\{t_k\}_k$ is a sequence tending to 0 and lies on the following curve, namely, $2^{P_2}\nu(t_k) = t_k^2\exp(-t_k^{-1})\langle \mathbf{m}_k\rangle^2$, $k \in \mathbb{N}$. $[a]$ represents the integer part of a. Furthermore, the sequence $\{h_k\}_k$ tends to $+\infty$ and $\dfrac{h_k\rho_k}{4\pi} \in \mathbb{N}_+$.

Definition 10 Subsequently, another two time sequences are defined: $\{t'_k\} := \{t_k + \rho_k\}_k$ and $\{t''_k\} := \{t_k - \rho_k\}_k$. Accordingly, we define three time intervals I_k, I'_k and I''_k,

$$I_k := \Big[t_k - \frac{\rho_k}{2}, t_k + \frac{\rho_k}{2}\Big],$$

$$I'_k := \Big[t'_k - \frac{\rho_k}{2}, t'_k + \frac{\rho_k}{2}\Big],$$

$$I''_k := \Big[t''_k - \frac{\rho_k}{2}, t''_k + \frac{\rho_k}{2}\Big].$$

Remark 5 $\{I_k\}_k$ is called the sequence of oscillating intervals, $\{I'_k\}_k$ is the sequence of right buffer intervals, and $\{I''_k\}_k$ is the sequence of left buffer intervals. It is evident that both the sequences $\{t_k\}_k$ and $\{\rho_k\}_k$ tend to 0 with $t_k >> \rho_k$ since P_2 is sufficiently large. Such choice of ρ_k guarantees that $I_k \subset (0, T]$ for all $k \in \mathbb{N}$.

3.2 Step 2: Construction of a Sequence of Oscillating Coefficients

First, we define a smooth increasing function on $\mathbb{R}$ as

$$\mu(x) := \begin{cases} 0, & x \in (-\infty, -1/3], \\ \text{strictly increasing}, & x \in (-1/3, 1/3), \\ 1, & x \in [1/3, +\infty). \end{cases}$$

Next, we introduce a sequence of oscillating coefficients $\{b_k(t)\}_k$:

$$b_k^2(t) := \begin{cases} 1, & t \in [0, T] \setminus (I'_k \cup I_k \cup I''_k); \\[2ex] \dfrac{\delta_k^2 a_\varepsilon\Big(h_k(t - t_k)\Big)}{\exp(-2t^{-1})}, & t \in I_k; \\[2ex] \dfrac{\delta_k^2\Big\{1 - \mu\Big(\dfrac{t - t'_k}{\rho_k}\Big)\Big\}}{\exp(-2t^{-1})} + \mu\Big(\dfrac{t - t'_k}{\rho_k}\Big), & t \in I'_k; \\[2ex] \dfrac{\delta_k^2 \mu\Big(\dfrac{t - t''_k}{\rho_k}\Big)}{\exp(-2t^{-1})} + 1 - \mu\Big(\dfrac{t - t''_k}{\rho_k}\Big), & t \in I''_k. \end{cases}$$

In addition, according to the definition of $\{\rho_k\}_k$, we have the following uniform estimates,

$$d_0^+ \leq \inf_k \frac{\exp(-t_k^{-1})}{\exp(-(t_k-\rho_k)^{-1})} \leq \sup_k \frac{\exp(-t_k^{-1})}{\exp(-(t_k-\rho_k)^{-1})} \leq d_1^+,$$

$$d_0^- \leq \inf_k \frac{\exp(-t_k^{-1})}{\exp(-(t_k+\rho_k)^{-1})} \leq \sup_k \frac{\exp(-t_k^{-1})}{\exp(-(t_k+\rho_k)^{-1})} \leq d_1^-,$$

where d_0^+, d_0^-, d_1^+ and d_1^- are all positive constants independent of k. Indeed, since P_2 is sufficiently large, then, t_k is always the dominating part. Simple calculations lead to

$$0 < b_0 \leq \inf_{t\in(0,T]} b_k(t) \leq \sup_{t\in(0,T]} b_k(t) \leq b_1 < \infty,$$

where b_0 and b_1 are independent of k. Moreover, in the union $I_k \bigcup I_k' \bigcup I_k''$, we have

$$|b_k'(t)| \leq C\frac{\nu(t)}{t^2}, \quad |b_k''(t)| \leq C\Big(\frac{\nu(t)}{t^2}\Big)^2,$$

where C is independent of k. More importantly, one can check that, for all $k \in \mathbb{N}_+$, the coefficient $\exp(-2t^{-1})b_k^2(t)$ is increasing except for the interval I_k.

3.3 Step 3: Construction of Auxiliary Functions

Succeedingly, we study the sequence of Cauchy problems in $I_k \times \mathbb{T}^N$,

$$\begin{cases} u_{k,tt} + \exp(-2t_k^{-1})a_\varepsilon(h_k(t-t_k))\Delta_T^2 u_k = 0, \\ u_k(t_k,x) = 0, \ u_{k,t}(t_k,x) = u_{k,1}(x). \end{cases} \tag{25}$$

Let the initial data be

$$u_{k,1}(x) = \phi_{|\mathbf{m}_k|}^{(m_1,\cdots,m_N)}(x)$$

and apply the change of variables

$$s = h_k(t-t_k).$$

At the same time, we define

$$v_k(s,x) := u_k(t(s),x).$$

Then, for $s \in [-\frac{h_k \rho_k}{2}, \frac{h_k \rho_k}{2}]$, we obtain the following Cauchy problem

$$\begin{cases} v_{k,ss} + \exp(-2t_k^{-1}) h_k^{-2} a_\varepsilon(s) \Delta_T^2 v_k = 0, \\ v_k(0, x) = 0, \ v_{k,s}(0, x) = h_k^{-1} u_{k,1}(x). \end{cases} \tag{26}$$

As a matter of fact, there exists a unique solution for (26) in the form of

$$v_k(s, x) = h_k^{-1} u_{k,1}(x) w_\varepsilon(s).$$

By transforming back to $u_k(t, x)$, we obtain

$$u_k(t, x) = h_k^{-1} \phi_{|\mathbf{m}_k|}^{(m_1, \cdots, m_N)}(x) w_\varepsilon(h_k(t - t_k))$$

in I_k. Further calculations lead to

$$\begin{aligned} &u_k(t_k - \frac{\rho_k}{2}, x) = 0, \ u_{k,t}\big(t_k - \frac{\rho_k}{2}, x\big) = \phi_{|\mathbf{m}_k|}^{(m_1, \cdots, m_N)}(x) \exp(-\frac{\varepsilon \rho_k h_k}{2}), \\ &u_k\big(t_k + \frac{\rho_k}{2}, x\big) = 0, \ u_{k,t}\big(t_k + \frac{\rho_k}{2}, x\big) = \phi_{|\mathbf{m}_k|}^{(m_1, \cdots, m_N)}(x) \exp(\frac{\varepsilon \rho_k h_k}{2}). \end{aligned} \tag{27}$$

3.4 Step 4: Existence of ν-Loss of Regularity

Now, we introduce an energy increasing property for the following time-dependent 2-evolution problem.

Lemma 8 *For the Cauchy problem in* $[t_0, T_0) \times \mathbb{T}^N$,

$$\begin{cases} u_{tt} + z^2(t) \Delta_T^2 u = 0, \\ u(t_0, x) = 0, \ \ u_t(t_0, x) = c \phi_{|\mathbf{m}_k|}^{(m_1, \cdots, m_N)}(x), \end{cases} \tag{28}$$

with $c \in \mathbb{R}$. *If* $z(t)$ *is nonnegative and* $z_t(t) \geq 0$, *then, for sufficiently large* $|\mathbf{m}_k|$ *and sufficiently small time interval* $T_0 - t_0 > 0$, *the nonhomogeneous energy does not decrease, that is,*

$$\mathbf{E}_s(u)(t) \geq \mathbf{E}_s(u)(t_0).$$

Proof In effect, by means of separation of variables, we have the following explicit representation of the unique solution

$$u(t,x) = cy(t)\phi_{|\mathbf{m}_k|}^{(m_1,\cdots,m_N)}(x),$$

where $y(t)$ satisfies the following Cauchy problem:

$$\begin{cases} y_{tt} + |\mathbf{m}_k|^4 z^2(t) y = 0, \\ y(t_0) = 0, \quad y_t(t_0) = 1. \end{cases} \tag{29}$$

By applying the definition of the nonhomogeneous Sobolev spaces $\mathscr{H}^s(\mathbb{T}^N)$, $s \geq 2$, we calculate the nonhomogeneous energy for the solution u. It holds that

$$\begin{aligned}
\mathbf{E}'_s(u)(t) &= \left(z^2(t)\|u(t,\cdot)\|^2_{\mathscr{H}^s} + \|u_t(t,\cdot)\|^2_{\mathscr{H}^{s-2}}\right)' \\
&= \Big\{z^2(t)\Big(\sum_{\mathbf{m}\in\mathbb{Z}^N} \langle \mathbf{m}\rangle^{2s}\Big(\sum_{\sum_{i=1}^N m_i^2 = |\mathbf{m}|^2} \left|\hat{u}_t^{(m_1,\cdots,m_N)}(|\mathbf{m}|^2)\right|^2\Big)\Big) \\
&\quad + \sum_{\mathbf{m}\in\mathbb{Z}^N} \langle \mathbf{m}\rangle^{2(s-2)}\Big(\sum_{\sum_{i=1}^N m_i^2 = |\mathbf{m}|^2} \left|\hat{u}^{(m_1,\cdots,m_N)}(|\mathbf{m}|^2)\right|^2\Big)\Big\}' \\
&= \left(c^2 z^2(t) y^2(t)\langle \mathbf{m}_k\rangle^{2s} + c^2 y_t^2(t)\langle \mathbf{m}_k\rangle^{2(s-2)}\right)' \\
&= 2c^2\langle \mathbf{m}_k\rangle^{2s} z z_t y^2 + 2c^2 z^2 y y_t \langle \mathbf{m}_k\rangle^{2(s-2)}(\langle \mathbf{m}_k\rangle^4 - |\mathbf{m}_k|^4) \\
&\geq 0.
\end{aligned}$$

The proof is concluded. □

By keeping in mind the definition of $b_k(t)$ and the monotonicity of $\exp(-2t^{-1})b_k^2(t)$ on various intervals except for I_k, we have, according to Lemma 8,

$$\mathbf{E}_2(u_k)(t) \leq \exp(-\varepsilon\rho_k h_k), \quad \text{for } t \in [0, t_k - \frac{\rho_k}{2}]; \tag{30}$$

$$\mathbf{E}_2(u_k)(t) \geq \exp(\varepsilon\rho_k h_k), \quad \text{for } t \in [t_k + \frac{\rho_k}{2}, T]. \tag{31}$$

It is evident that (7) follows directly from (30). While for $t = t_k + \frac{\rho_k}{2}$, by applying the theory of pseudodifferential operators, we have

$$\begin{aligned} &\mathbf{E}_2\Big(\exp\Big(-c_1(\varepsilon)\nu(G^{-1}(\frac{2^{P_2}}{1-\Delta_T}))\Big)u_k\Big)(t) \\ &= \exp\Big(-2c_1(\varepsilon)\nu(G^{-1}(\frac{2^{P_2}\delta_k}{h_k})) + \varepsilon\rho_k h_k\Big) \\ &= \exp\Big(-2c_1(\varepsilon)\nu(t_k) + \varepsilon\rho_k h_k\Big). \end{aligned} \tag{32}$$

Taking into account the choice of ρ_k, h_k, we can choose a sufficiently small $c_1(\varepsilon)$ satisfying

$$0 < c_1(\varepsilon) < \pi,$$

which is independent of k such that (8) holds. This concludes our proof.

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Existence Results for Critical Problems Involving p-Sub-Laplacians on Carnot Groups

Annunziata Loiudice

Abstract We provide existence results for the quasilinear subelliptic Dirichlet problem

$$-\Delta_{p,\mathbb{G}}u = |u|^{p^*-2}u + g(\xi, u) \quad \text{in } \Omega, \quad u \in S_0^{1,p}(\Omega),$$

where $\Delta_{p,\mathbb{G}}$ is the p-sub-Laplacian on a Carnot group $\mathbb{G}$, $p^* = pQ/(Q-p)$ is the critical Sobolev exponent in this context, Ω is a bounded domain of $\mathbb{G}$ and $g(\xi, u)$ is a subcritical perturbation. By means of standard variational methods adapted to the stratified context, we prove the existence of solutions both in the mountain pass and in the linking case. A crucial ingredient in this abstract framework is the knowledge of the exact rate of decay of the p-Sobolev extremals on Carnot groups.

1 Introduction

In this paper we provide existence results for the following quasilinear subelliptic problem with critical Sobolev exponent

$$\begin{cases} -\Delta_{p,\mathbb{G}}u = |u|^{p^*-2}u + g(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_g}$$

under suitable subcritical assumptions on the lower order perturbation $g(\xi, u)$. Here, $\Delta_{p,\mathbb{G}}$ is the p-sub-Laplacian operator on a Carnot group $\mathbb{G}$ of homogeneous dimension Q, where $1 < p < Q$, the exponent $p^* = pQ/(Q-p)$ is the critical Sobolev exponent in this context, Ω is a bounded domain of $\mathbb{G}$ and the lower order

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term satisfies subcritical growth assumptions. In particular, we obtain existence results for the case $g(\xi, u) = \lambda|u|^{p-2}u$, with $\lambda \in \mathbb{R}$.

The present results extend to the p-sub-Laplacian case the existence results obtained by the author in [30] for the semilinear Carnot case $p = 2$, subsequently generalized in [31] to the semilinear case with a Hardy-Sobolev nonlinearity and in [35] to the case with a Hardy-type perturbation.

We recall that a great deal of interest has been paid in the literature to the topic of subelliptic equations with critical Sobolev exponent or general power-type nonlinearities on stratified Lie groups. See e.g. [4, 5, 12, 22, 23, 25, 29–37, 39, 41, 44, 45, 48] and the references therein. In particular, in the recent papers [44, 45], interesting generalizations of variational-type results are obtained for Rockland operators on general graded Lie groups (see [15] for an overview on this functional setting). Concerning the quasilinear case, we recall that Vassilev in [48] studies the main aspects of the critical equation

$$-\Delta_{p,\mathbb{G}}u = |u|^{p^*-2}u, \qquad u \in S_0^{1,p}(\Omega),$$

where Ω is an arbitrary open subset of $\mathbb{G}$. Precisely, he proves global boundedness and interior regularity of solutions, discusses the problem of the regularity near the characteristic set of the boundary and, in the case $\Omega = \mathbb{G}$, obtains the existence of ground state solutions.

In [33], the author establishes the decay of positive entire solutions $u \in S^{1,p}(\mathbb{G})$ of the critical equation

$$-\Delta_{p,\mathbb{G}}u = u^{p^*-1} \quad \text{in } \mathbb{G},$$

obtaining that they have the following asymptotic behavior at infinity

$$u(\xi) \sim \frac{1}{d(\xi)^{(Q-p)/(p-1)}} \quad \text{as } d(\xi) \to \infty, \tag{1}$$

where d is any fixed homogeneous norm on $\mathbb{G}$. This result applies, in particular, to the extremals of the p-Sobolev inequality on Carnot groups (4) and it turns out to be a useful tool in adapting the well-known Brezis-Nirenberg type techniques [8] to problems of the type (P_g), in absence of the explicit knowledge of Sobolev minimizers. In fact, such minimizers are only known when $\mathbb{G}$ is a Iwasawa-type group and $p = 2$.

Further recent results for quasilinear equations and systems on Carnot groups can be found e.g. in [7, 13, 14, 19, 39, 41, 43, 46, 47]. In particular, Pucci-Temperini in [41] obtain existence of entire solutions to the problem

$$-\Delta_{p,H}u = k(\xi)|u|^{p^*-2}u + \lambda w(\xi)|u|^{q-2}u, \quad u \in S^1(\mathbb{H}^n),$$

in the model case of the Heisenberg group $\mathbb{G} = \mathbb{H}^n$, where $p \le q < p^*$, under appropriate hypotheses on k and w.

Let us, now, introduce our existence results on bounded domains for problem (P_g). Let $\mathbb{G}$ be a Carnot group of homogeneous dimension Q and, for $1 < p < Q$, let

$$\Delta_{p,\mathbb{G}} u = \sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u)$$

be the p-sub-Laplacian operator on $\mathbb{G}$ (see Sect. 2 for the definition). We denote by $S_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{S_0^{1,p}(\Omega)} := (\int_\Omega |Xu|^p \, d\xi)^{1/p}.$$

We shall deal with weak solutions of problem (P_g), i.e. functions $u \in S_0^{1,p}(\Omega)$ such that

$$\int_\Omega |Xu|^{p-2} < Xu, X\phi > d\xi = \int_\Omega |u|^{p^*-2} u\phi \, d\xi + \int_\Omega g(\xi, u)\phi \, d\xi, \ \ \forall \phi \in C_0^\infty(\Omega).$$

Let the functional $J : S_0^{1,p}(\Omega) \to \mathbb{R}$ be defined as

$$J(u) = \frac{1}{p} \int_\Omega |Xu|^p \, d\xi - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, d\xi - \int_\Omega G(\xi, u) \, d\xi,$$

where $G(\xi, s) = \int_0^s g(\xi, t) dt$. If g is continuous, then $J \in C^1(S_0^{1,p}(\Omega), \mathbb{R})$ and the critical points of J corresponds to weak solutions of Eq. (P_g).

Concerning the lower order term, following [1, 24], we assume that g is subcritical in the following sense

$$\begin{cases} g : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function satisfying:} \\ \qquad \forall \varepsilon > 0, \exists \, a_\varepsilon \in L^{\frac{pQ}{Q(p-1)+p}} \text{ such that} \\ |g(\xi, s)| \le a_\varepsilon(\xi) + \varepsilon |s|^{\frac{Q(p-1)+p}{Q-p}} \text{ for a.e. } \xi \in \Omega, \forall s \in \mathbb{R}. \end{cases} \tag{2}$$

Moreover, other assumptions will be required on the primitive $G(\xi, s) = \int_0^s g(\xi, t) \, dt$. In particular, we assume that

$$G(\xi, s) \ge 0 \quad \text{for a.e. } \xi \in \Omega, \quad \forall s \in \mathbb{R}, \tag{3}$$

while $g(\xi, s)$ is allowed to change sign. Further assumptions on G will be required, according to the different cases to be considered, namely the case when J has a mountain pass geometry or the case where J has a linking structure, with or without resonance. Roughly speaking, these three cases correspond to

$$0 \leq \lim_{s \to 0^+} \frac{G(\xi, s)}{s^p} < \frac{\lambda_1}{p}, \quad \lim_{s \to 0^+} \frac{G(\xi, s)}{s^p} = \frac{\lambda_1}{p}, \quad \frac{\lambda_1}{p} < \lim_{s \to 0^+} \frac{G(\xi, s)}{s^p},$$

where λ_1 denotes the first eigenvalue of $-\Delta_{p,\mathbb{G}}$ with Dirichlet boundary condition, that is,

$$\lambda_1 = \min_{u \in S_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|Xu\|_p^p}{\|u\|_p^p}.$$

As in the Euclidean case, in view of Lemma 1 below, the existence results in the different three cases will be obtained by constructing suitable Palais-Smale sequences (in short PS-sequences) for J at a level $c \in (0, \frac{S^{Q/p}}{Q})$, where S denotes the best constant in the Sobolev inequality (4). To this aim, as in the semilinear cases in [30, 31, 35] the behavior at infinity of the extremals for the Sobolev inequality on groups recalled in (1) will be used in a crucial way. We finally quote that analogous considerations have been used in the Euclidean setting to treat the quasilinear nonlocal case (see [38]).

The paper is organized as follows. In Sect. 2, we introduce the functional framework of Carnot groups; in Sect. 3 we treat the case when the functional J has a mountain pass geometry: we state the existence results in Theorems 1, 2 and 3, introducing the appropriate additional hypotheses on G, and we give a sketch of the proofs; in Sect. 4, we consider the case when J has a linking geometry, treating both the resonance and the non resonance case; the related existence results are contained in Theorems 4 and 5.

2 The Functional Setting

Let us briefly introduce the functional setting of Carnot groups. For a complete overview, we refer the reader to the monographs [6, 15] and the classical papers [16, 17].

A Carnot group $(\mathbb{G}, \circ)$ (or Stratified Lie group) is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \bigoplus_{j=1}^r V_j$, such that $[V_1, V_j] = V_{j+1}$ for $1 \leq j < r$, and $[V_1, V_r] = \{0\}$. The number r is called the *step* of the group $\mathbb{G}$ and the integer $Q = \sum_{i=1}^r i \dim V_i$ is the *homogeneous dimension* of $\mathbb{G}$. Note that, if $Q \leq 3$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $\mathbb{G} = (\mathbb{R}^N, +)$.

The simplest non-abelian Carnot group is the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$, which is a two-step Carnot group with homogeneous dimension $Q = 2n + 2$ and composition law given by $\xi \circ \xi' = (x + x', y + y', t + t' +$

$2(\langle x', y\rangle - \langle x, y'\rangle))$, for every $\xi = (x, y, t)$, $\xi' = (x', y', t') \in R^{2n+1}$, where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map, which we shall assume throughout, it is not restrictive to suppose that $\mathbb{G}$ is a homogeneous Carnot group, according to the definition in [6], i.e. $\mathbb{G} = \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r}$, where $N_i = \dim V_i$, endowed with dilations δ_λ of the form

$$\delta_\lambda(\xi) = (\lambda\, \xi^{(1)}, \lambda^2\, \xi^{(2)}, \cdots, \lambda^r\, \xi^{(r)}),$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \ldots, r$. Let $m := N_1$ and let $X_1, \ldots, X_m$ be the set of left invariant vector fields of V_1 that coincide at the origin with the first m partial derivatives. It holds that $rank(Lie\{X_1, \ldots, X_m\}) = N$ at any point of $\mathbb{G}$. We shall denote by $X = (X_1, \ldots, X_m)$ such system of vector fields. Then, the differential operator

$$\Delta_{p,\mathbb{G}} u := \sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u)$$

is called the canonical *p-sub-Laplacian* on $\mathbb{G}$. Note that for any $c > 0$, one has $\Delta_{p,\mathbb{G}}(cu) = c^{p-1} \Delta_{p,\mathbb{G}} u$ and furthermore, since the X_j's are homogeneous of degree one with respect to the dilations δ_λ, the operator $\Delta_{p,\mathbb{G}}$ is homogeneous of degree p with respect to δ_λ, namely

$$\Delta_{p,\mathbb{G}}(u \circ \delta_\lambda) = \lambda^p \Delta_{p,\mathbb{G}} u \circ \delta_\lambda.$$

By definition, a homogeneous norm on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \to [0, +\infty)$, smooth away from the origin, such that $d(\delta_\lambda(\xi)) = \lambda\, d(\xi)$, for every $\lambda > 0$ and $\xi \in \mathbb{G}$,$d(\xi) = 0$ if and only if $\xi = 0$. We recall that any two homogeneous norms on a Carnot group $\mathbb{G}$ are equivalent, as observed in [6, Prop. 5.1.4]. If we let $d(\xi, \eta) := d(\eta^{-1} \circ \xi)$, d is a pseudo-distance on $\mathbb{G}$. Throughout the paper, d will indicate a fixed homogeneous norm on $\mathbb{G}$; we shall denote by $B(\xi, r)$ the d-ball with center at ξ and radius r, i.e.

$$B(\xi, r) = \{\eta \in \mathbb{G} \,|\, d(\xi^{-1} \circ \eta) < r\},$$

and we will simply denote by B_r the d-ball centered at 0 with radius r.

The starting point of the variational formulation of our problem is the validity of the following Sobolev-type inequality on $\mathbb{G}$ (see Folland [16]): for any $p \in (1, Q)$, there exists $S > 0$, depending on p and $\mathbb{G}$, such that

$$\int_{\mathbb{G}} |Xu|^p \, d\xi \geq S \left(\int_{\mathbb{G}} |u|^{p^*} \, d\xi \right)^{p/p^*}, \quad \forall\, u \in C_0^\infty(\mathbb{G}). \tag{4}$$

It is know that the best constant in (4) is achieved (see [23, 48]); however, the explicit form of the extremal functions is not known, except for the case when $p = 2$ and $\mathbb{G}$ is a group of Iwasawa type (see Jerison-Lee [28], Frank-Lieb [18] for the Heisenberg case, Ivanov-Minchev-Vassilev [27] and Christ-Liu-Zhang [11] for the remaining cases). Nevertheless, relevant qualitative properties of such extremals in the general case have been obtained by the author in [33].

Concerning the main regularity tools for quasilinear subelliptic equations, such as Moser-type estimates and Harnack-type inequality, we refer to Capogna-Danielli-Garofalo [9]. Moreover, we indicate the paper [2] for an overview on the main aspects of nonlinear potential theory on Carnot groups. We also quote [3] for a strong maximum principle for quasilinear equations involving Hörmander vector fields.

3 The Mountain Pass Case

In this section, we treat the case when J has a mountain pass geometry. We introduce the additional needed assumptions on G and we state the related existence results in Theorems 1, 2 and 3 below. Finally, we sketch the proof of the theorems.

Before introducing the additional assumptions which ensure the mountain pass geometry, we state a compactness result which is valid under the only assumption (2). Recall here that a sequence $\{u_n\} \subset S_0^1(\Omega)$ is called a Palais-Smale sequence (PS sequence in short) for J at level c if $J(u_n) \to c$ and $J'(u_n) \to 0$ in $S^{-1,p'}(\Omega)$.

Lemma 1 *Assume that (2) holds. If $\{u_n\} \subset S_0^1(\Omega)$ is a PS sequence for J at level $c \in (0, \frac{S^{Q/p}}{Q})$, there exists $u \in S_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightharpoonup u$ up to a subsequence and $J'(u) = 0$.*

The proof is standard and it will be omitted, referring to the Euclidean counterpart (see, for instance, Lemma 1 in [1]). In view of the above result, the solutions to problem (P_g), both in the mountain pass and in the linking case, will be obtained by constructing a PS sequence at a level $c \in (0, \frac{S^{Q/p}}{Q})$.

Assume, now, that there exist an open subset $\Omega_0 \subset \Omega$ and some constants $\sigma, \delta, \mu > 0$ and $a, b > 0$, $a < b$, such that

$$G(\xi, s) \leq \frac{1}{p}(\lambda_1 - \sigma)|s|^p \qquad \text{for a.e.}\, \xi \in \Omega, \quad \forall |s| \leq \delta \tag{5}$$

and

$$G(\xi, s) \geq \mu \qquad \text{for a.e.}\, \xi \in \Omega_0, \forall s \in [a, b]. \tag{6}$$

Under these assumptions, the following existence results hold.

Theorem 1 *Assume that* (2), (3), (5), (6) *hold.*
If $1 < p^2 < Q$*, then problem* (P_g) *admits a positive solution.*
If $Q = p^2$ *and* μ *in* (6) *is large enough, then problem* (P_g) *admits a positive solution.*

From the above theorem, we obtain, for the particular case $g(\xi, u) = \lambda|u|^{p-2}u$, the following result, which was proved in the Euclidean context by Garcìa Azorero and Peral [20] (see also [26] for related regularity and nonexistence results).

Corollary 1 *Let* $1 < p^2 \leq Q$*. Then, problem*

$$\begin{cases} -\Delta_{p,\mathbb{G}}u = |u|^{p^*-2}u + \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P_λ}$$

admits a positive solution for any $\lambda \in (0, \lambda_1)$.

If, instead, $p < Q < p^2$, in the ordinary Euclidean setting we are in the case of *critical dimensions* in the sense of Pucci-Serrin [40]. Therefore, in order to get existence of solutions for problem (P_g), the assumption (6) is no longer sufficient; we require that there exists a nonempty open set $\Omega_0 \subset \Omega$ such that

$$\lim_{s\to+\infty} \frac{G(\xi, s)}{s^\beta} \quad \text{uniformly in } \Omega_0, \tag{7}$$

where $\beta = \frac{p(Qp+p-2Q)}{(p-1)(Q-p)}$. Under this additional assumption, the following result holds.

Theorem 2 *Let* $1 < p < Q < p^2$*. Assume that conditions* (2), (3), (5) *and* (7) *hold. Then problem* (P_g) *admits a positive solution.*

We notice that, in the Euclidean setting, Theorems 1 and 2 were generalization of results proved in [21].

Finally, in the same range of dimensions considered in Theorem 2, we can also prove the following result about problem (P_λ), which provides existence of solutions in a left neighborhood of λ_1. For the semilinear subelliptic counterpart, see [30, Theorem 1.2].

Theorem 3 *Let* $\Lambda = S|\Omega|^{-p/Q}$ *and assume that* $1 < p < Q < p^2$ *and* $\lambda \in (\lambda_1 - \Lambda, \lambda_1)$*. Then, problem* ($P_\lambda$) *admits a positive solution.*

In what follows, we prove the existence results stated above. The idea of the proofs is to find a nonnegative function $v \in S_0^{1,p}(\Omega)$ such that $\max_{t\geq 0} J(tv) < \frac{S^{Q/p}}{Q}$. Indeed, noting that there exists $t_v > 0$ such that $J(t_v v) < 0$, consider the set

$$\Gamma_v = \{\gamma \in C([0, 1])|\gamma(0) = 0, \gamma(1) = t_v v\}$$

and the inf-max value

$$c := \inf_{\gamma \in \Gamma_v} \max_{t \in [0,1]} J(\gamma(t)).$$

By standard variational arguments (see, for instance, [42]), if such v exists, we obtain a PS sequence at level $c \in (0, \frac{S^{Q/p}}{Q})$. In the proofs of Theorems 1, 2 and 3, a different choice of v will be done.

Proof of Theorem 1 Let $U > 0$ be a fixed extremal function for (4). We can assume, up to a normalization, that $\|XU\|_p^p = \|U\|_{p^*}^{p^*} = S^{Q/p}$. For $\epsilon > 0$, define

$$U_\epsilon(\xi) = \epsilon^{-(Q-p)/p} U(\delta_{1/\epsilon}(\xi)), \quad \xi \in \mathbb{G}. \tag{8}$$

Of course, U_ϵ are also minimizers and verify $\|XU_\epsilon\|_p^p = \|U_\epsilon\|_{p^*}^{p^*} = S^{Q/p}$. Now, observe that it is not restrictive to suppose $0 \in \Omega$. Let $R > 0$ be such that $B_R \subset \Omega$ and let $\varphi \in C_0^\infty(B_R)$ be a cut-off function, $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_{R/2}$. Define

$$u_\epsilon(\xi) = \varphi(\xi) U_\epsilon(\xi). \tag{9}$$

Reasoning as in [30], by exploiting the asymptotic estimate (1) proved in [33], we are able to prove that

$$\|Xu_\epsilon\|_p^p \le S^{Q/p} + C\epsilon^{(Q-p)/(p-1)}, \quad \|u_\epsilon\|_{p^*}^{p^*} \ge S^{Q/p} - C\epsilon^{Q/(p-1)}. \tag{10}$$

We claim that, for ϵ sufficiently small, it holds

$$\max_{t \ge 0} J(tu_\epsilon) < \frac{1}{Q} S^{Q/p}. \tag{11}$$

Indeed, assume by contradiction that for all $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$J(t_\epsilon u_\epsilon) \ge \frac{1}{Q} S^{Q/p}. \tag{12}$$

It is easily seen that, as $\epsilon \to 0$, the sequence $\{t_\epsilon\}$ is upper and lower bounded by two positive constants; moreover, by the expansions (10), as $\epsilon \to 0$ we have

$$\begin{aligned} \frac{\|X(t_\epsilon u_\epsilon)\|_p^p}{p} - \frac{\|t_\epsilon u_\epsilon\|_{p^*}^{p^*}}{p^*} &\le \frac{S^{Q/p}}{Q} + \left(t_\epsilon^p - 1 - \frac{Q-p}{Q}(t_\epsilon^{p^*} - 1)\right) \frac{S^{Q/p}}{p} \\ &\quad + O(\epsilon^{(Q-p)/(p-1)}) \\ &\le \frac{S^{Q/p}}{Q} + O(\epsilon^{(Q-p)/(p-1)}). \end{aligned} \tag{13}$$

It can be also verified that there exists $c_1, c_2 > 0$ such that, for ϵ sufficiently small

$$c_1\epsilon^{1/p} < d(\xi) < c_2\epsilon^{1/p} \text{ implies } a < t_\epsilon u_\epsilon(\xi) < b,$$

where a, b are as in (6). Hence, since $B_\epsilon \subset \Omega_0$ for small ϵ, by (3) and (6) we have

$$\int_\Omega G(\xi, t_\epsilon u_\epsilon) \geq c\mu \int_{c_1\epsilon^{1/p}}^{c_2\epsilon^{1/p}} \rho^{Q-1}\mathrm{d}\rho \geq c\mu\epsilon^{Q/p}, \tag{14}$$

where we used the appropriate polar coordinates formula. So, if $Q > p^2$, we get that there exists a function $\zeta = \zeta(\epsilon)$ such that $\lim_{\epsilon\to 0}\zeta(\epsilon) = +\infty$ such that, for ϵ small,

$$\int_\Omega G(\xi, t_\epsilon u_\epsilon) \geq \zeta(\epsilon)\cdot\epsilon^{(Q-p)/(p-1)}.$$

Hence, from (13) and the latter estimate, we get that, for ϵ small enough,

$$J(t_\epsilon u_\epsilon) < \frac{S^{Q/p}}{Q}.$$

Analogously, if $Q = p^2$, from (13) and (14), we get

$$J(t_\epsilon u_\epsilon) \leq \frac{S^{Q/p}}{Q} + O(\epsilon^p) - c\mu\epsilon^p < \frac{S^{Q/p}}{Q},$$

for suitable small ϵ and μ large enough. Hence,(12) cannot hold. So, from (11), we obtain a Palais-Smale sequence for J, belonging to the cone of nonnegative functions in $S_0^{1,p}(\Omega)$, at level $c \in (0, \frac{S^{Q/p}}{Q})$: its weak limit is nonnegative, nontrivial and it solves (P_g).

Finally, such solution turns out to be strictly positive by the nonlinear strong maximum principle in [3]. □

Proof of Theorem 2 The proof follows the scheme of the proof of Theorem 1, except for estimate (14), which is replaced by the following considerations. From (7), there exists an increasing function ζ such that $\lim_{t\to+\infty}\zeta(t) = +\infty$ such that $G(\xi, s) \geq \zeta(s)\cdot s^\beta$ for a.e. $\xi \in \Omega_0$ and all $s > 0$. Therefore,

$$\begin{aligned}\int_\Omega G(\xi, t_\epsilon u_\epsilon) &\geq \zeta\left(c\epsilon^{(p-Q)/p}\right)\epsilon^{\beta(p-Q)/p}\int_0^\epsilon \rho^{Q-1}\mathrm{d}\rho \\ &\geq \zeta\left(c\epsilon^{(p-Q)/p}\right)\epsilon^{(Q-p)/(p-1)},\end{aligned} \tag{15}$$

where it is used that $\min_{d(\xi)\leq\epsilon} t_\epsilon u_\epsilon \geq c\epsilon^{(p-Q)/p}$. □

Proof of Theorem 3 Also in this case, we prove that the PS sequence obtained by the mountain pass argument is at level below the compactness threshold. Following the idea in [10], let e_1 be the first positive eigenfunction of $-\Delta_{p,\mathbb{G}}$ in Ω and let us estimate $J(te_1)$, where $t > 0$. We have

$$\begin{aligned} J(te_1) &= \frac{\lambda_1 - \lambda}{p} t^p \|e_1\|_p^p - \frac{Q-p}{Qp} t^{p^*} \|e_1\|_{p^*}^{p^*} \\ &\leq \frac{\lambda_1 - \lambda}{p} |\Omega|^{p/Q} t^p \|e_1\|_{p^*}^p - \frac{Q-p}{Qp} t^{p^*} \|e_1\|_{p^*}^{p^*} \\ &\leq \frac{(\lambda_1 - \lambda)^{Q/p}}{Q} |\Omega|, \end{aligned} \tag{16}$$

where in the last inequality we have maximized with respect to $t \geq 0$. So, if $\lambda \in (\lambda_1 - \Lambda, \lambda_1)$, then

$$\max_{t \geq 0} J(te_1) < \frac{1}{Q} S^{Q/p},$$

and the existence of a solution follows as in the preceding proofs. □

4 The Case with Linking Geometry

This section is devoted to the case with linking geometry. We introduce the necessary notation and, under the appropriate hypotheses on G, we state the related existence results (see Theorems 4 and 5 below). Finally, after some preliminary lemmas, we sketch the proofs.

4.1 Statement of the Results

Let us introduce some further notation. Let $w \in S^{-1,p'}(\Omega)$, the dual space of $S_0^{1,p}(\Omega)$. We denote by $E_w^\perp$ the subspace of $S_0^{1,p}(\Omega)$ orthogonal to w, i.e.

$$E_w^\perp = \{u \in S_0^{1,p}(\Omega) \mid \langle w, u \rangle = 0\},$$

where $\langle , \rangle$ is the duality product between $S^{-1,p'}(\Omega)$ and $S_0^{1,p}(\Omega)$; denote by $B^1 = \{u \in S_0^{1,p}(\Omega) \mid \|u\|_p = 1\}$. Let

$$\overline{\lambda} = \sup_{w \in S^{-1,p'}(\Omega)} \inf_{u \in E_w^\perp \cap B^1} \|Xu\|_p^p .$$

It is possible to verify that $\overline{\lambda} \le \lambda_2$, where λ_2 is the second eigenvalue of $-\Delta_{p,\mathbb{G}}$. If $p = 2$, then $\overline{\lambda} = \lambda_2$; if $p \ne 2$, it is not clear whether the equality holds or not. However, it holds that $\overline{\lambda} > \lambda_1$ (see Lemma 2 below).

The non-resonance case corresponds to requiring the following assumptions on $G(\xi, u)$:

$$\begin{aligned} \frac{1}{p}(\lambda_1 + \sigma)|s|^p \le G(\xi, s) \le \frac{1}{p}(\overline{\lambda} - \sigma)|s|^p \quad &\text{for a.e. } \xi \in \Omega, \forall |s| \le \delta \\ G(\xi, s) \ge \frac{1}{p}(\lambda_1 + \sigma)|s|^p - \frac{1}{p^*}|s|^{p^*} \quad &\text{for a.e. } \xi \in \Omega, \forall s \ne 0. \end{aligned} \tag{17}$$

We observe that (5) and (17) imply $g(\xi, 0) = 0$ a. e. in Ω and $u = 0$ is a solution of (P_g). In this case, we prove the following result.

Theorem 4 *If $1 < p^2 \le Q$, assume that* (2), (3) *and* (17) *hold; if $1 < p < Q < p^2$, assume that* (2), (3), (7), (17) *hold. Then, problem* (P_g) *admits a nontrivial solution.*

We conclude with the case of resonance near the origin. We assume that there exists $\delta > 0$ and $\sigma \in (0, 1/p^*)$ such that

$$\begin{aligned} \frac{1}{p}\lambda_1|s|^p \le G(\xi, s) \le \frac{1}{p}(\overline{\lambda} - \sigma)|s|^p \quad &\text{for a.e. } \xi \in \Omega, \forall |s| \le \delta \\ G(\xi, s) \ge \frac{1}{p}\lambda_1|s|^p - (\frac{1}{p^*} - \sigma)|s|^{p^*} \quad &\text{for a.e. } \xi \in \Omega, \forall s \in \mathbb{R}. \end{aligned} \tag{18}$$

We also need the following condition on $G(\xi, s)$ at infinity: there exists an open nonempty set $\Omega_0 \subset \Omega$ such that

$$\lim_{s \to +\infty} \frac{G(\xi, s)}{s^\gamma} = +\infty \quad \text{uniformly in} \Omega_0, \tag{19}$$

where $\gamma = \frac{Qp(Qp+2p-2Q)}{(Q-p)(Qp+p-Q)}$. The following result holds.

Theorem 5 *Let $1 < p < Q$ and assume that* (2), (3), (18), (19) *hold. Then, problem* (P_g) *admits a nontrivial solution.*

We observe that $\gamma < p^*$ for any $1 < p < Q$ and $\gamma > 0$ for $p > \frac{2Q}{Q+2}$. From Theorem 5, we deduce the following result for problem (P_λ).

Corollary 2 *Let $p > 1$ and Q such that $\frac{Q^2}{Q+1} > p^2$. Then, problem* (P_λ) *admits a nontrivial solution for $\lambda = \lambda_1$.*

4.2 Proof of the Results

In this subsection, we sketch the proof of Theorems 4 and 5. We begin with some preliminary lemmas. We first need a lemma which provides a sufficient condition for the linking geometry to hold.

Lemma 2 *For any $w \in S^{-1,p'}(\Omega)$ such that $\langle w, e_1 \rangle \neq 0$ there exists a constant $c_w > 0$ depending on w such that if $u \in E_w^{\perp}$, then $\|Xu\|_p^p - \lambda_1 \|u\|_p^p \geq c_w \|Xu\|_p^p$; therefore, $\overline{\lambda} > \lambda_1$.*

Proof The proof follows the Euclidean outline in [1], so we omit it. □

Now, let e_1 denote the positive eigenfunction relative to λ_1 and such that $\|e_1\|_p = 1$ and let Ω_0 be as in (7) or (19). Without restriction, we can assume that $0 \in \Omega_0 \subset \Omega$. Denote by B_r the d-ball centered at 0 with radius r. For $m \in \mathbb{N}$ sufficiently large so that $B_{2/m} \subset \Omega_0$, define the functions $\phi_m : \Omega \to \mathbb{R}$ as follows

$$\phi_m(\xi) := \begin{cases} 0 & \text{if } \xi \in B_{1/m} \\ m\, d(\xi) - 1 & \text{if } \xi \in B_{2/m} \setminus B_{1/m} \\ 1 & \text{if } \xi \in \Omega \setminus B_{2/m}. \end{cases} \tag{20}$$

Let $e_1^m := \phi_m e_1$ and let $E^m := \text{span}\{e_1^m\}$. We prove the following approximation result.

Lemma 3 *As $m \to \infty$, there holds*

$$e_1^m \to e_1 \quad in\ S_0^{1,p}(\Omega) \quad and \quad \|X(e_1^m)\|_p^p \leq \lambda_1 + \nu m^{p-Q}, \tag{21}$$

for a suitable $\nu > 0$.

Proof Let us compute

$$\begin{aligned} \|X(e_1^m - e_1)\|_p &= \|e_1 X\phi_m + (\phi_m - 1)Xe_1\|_p \\ &\leq \|e_1 X\phi_m\|_p + \|(\phi_m - 1)Xe_1\|_p \\ &\leq c(m^{p-Q} + m^{-Q}) \to 0, \end{aligned} \tag{22}$$

where we have used that

$$\int_\Omega |X\phi_m|^p = m^p \int_{B_{2/m}\setminus B_{1/m}} |Xd|^p \leq Cm^p |B_{2/m} \setminus B_{1/m}| = Cm^{p-Q},$$

due to the boundedness of $\psi = |Xd|$. Hence, $e_1^m \to e_1$ and by the definition of e_1^m, the second estimate in (21) follows. □

Let now observe that, for all $\delta > 0$, there exists $w \in S^{-1,p'}(\Omega)$ such that $\min_{u \in E_w^\perp \cap B^1} \|Xu\|_p \geq \overline{\lambda} - \delta$. Let, for such w, $E_\delta^\perp := E_w^\perp$.

Lemma 4 *Assume that* (2), (3) *and either* (17) *or* (18) *hold. Then, there exists* $\alpha, \delta, \rho > 0$ *such that*

$$J(u) \geq \alpha \quad \forall u \in \partial B_\rho \cap E_\delta^\perp.$$

Now, consider the family of Sobolev minimizers U_ϵ defined in (8) and, for $m \in \mathbb{N}$, take a cut-off function $\eta_m \in C_0^\infty(B_{1/m})$, $0 \leq \eta \leq 1$, such that $\eta_m \equiv 1$ in $B_{1/2m}$ and $\|X\eta_m\|_\infty \leq 3m$. Then, for $\epsilon > 0$, define

$$u_\epsilon^m(\xi) = \eta_m(\xi) U_\epsilon(\xi), \quad \xi \in \mathbb{G}. \tag{23}$$

Then, as $\epsilon m \to 0$, analogous estimates as in (10) hold

$$\|Xu_\epsilon^m\|_p^p \leq S^{Q/p} + C(\epsilon m)^{(Q-p)/(p-1)}, \quad \|u_\epsilon^m\|_{p^*}^{p^*} \geq S^{Q/p} - C(\epsilon m)^{Q/(p-1)}. \tag{24}$$

Note that, by construction, for all $\epsilon > 0$ and $m \in \mathbb{N}$ we get

$$\mathrm{supp}(u_\epsilon^m) \cap \mathrm{supp}(e_1^m) = \emptyset. \tag{25}$$

Now, define

$$Q_m^\epsilon = \{u \in S_0^{1,p}(\Omega) \mid u = ae_1^m + bu_\epsilon^m, |a| \leq R, 0 \leq b \leq R\}.$$

It can be verified that ∂Q_m^ϵ and $\partial B_\rho \cap E^\perp$ link (see [42]) if $R > \rho$, where ρ is as in Lemma 4. Moreover, by (25), if R and m are large enough, then $J(u) \leq 0$ for all $u \in \partial Q_m^\epsilon$. By these choices on R and m, if we let

$$\Gamma = \{h \in C(Q_m^\epsilon, S_0^{1,p}(\Omega) \mid h(u) = u, \forall u \in \partial Q_m^\epsilon\},$$

by standard arguments, we obtain a PS sequence for J at level

$$c = \inf_{h \in \Gamma} \max_{u \in Q_m^\epsilon} J(h(u)).$$

Then, the conclusions of Theorems 4 and 5 will follow by showing that, for ϵ sufficiently small, $c < \frac{S^{Q/p}}{Q}$.

Proof of Theorem 4 Let $1 < p^2 \leq Q$. Choose m large enough so that $\nu m^{p-Q} < \sigma$, where ν is as in Lemma 3 and σ is as in (17). It follows that

$$\forall w \in E^m \quad J(w) \leq 0. \tag{26}$$

We prove that there exists $\epsilon > 0$ such that

$$\max_{u \in Q_m^\epsilon} J(u) < \frac{1}{Q} S^{Q/p}. \tag{27}$$

Arguing by contradiction, assume that

$$\forall \epsilon > 0, \quad \max_{u \in Q_m^\epsilon} J(u) \geq \frac{1}{Q} S^{Q/p}.$$

By the compactness of the set $\{u \in Q_m^\epsilon | J(u) \geq 0\}$, for all $\epsilon > 0$ there exist $w_\epsilon \in E^m$ and $t_\epsilon \geq 0$ such that, letting $v_\epsilon = w_\epsilon + t_\epsilon u_\epsilon^m$, it holds

$$J(v_\epsilon) = \max_{u \in Q_m^\epsilon} J(u) \geq \frac{1}{Q} S^{Q/p},$$

that is

$$\frac{1}{p} \|X v_\epsilon\|_p^p - \int_\Omega G(\xi, v_\epsilon) - \frac{1}{p^*} \|v_\epsilon\|_{p^*}^{p^*} \geq \frac{1}{Q} S^{Q/p}, \quad \forall \epsilon > 0. \tag{28}$$

As in Theorem 1, it follows that t_ϵ is bounded between two positive constants. We now estimate the term $\int_\Omega G(\xi, t_\epsilon u_\epsilon)$. We claim that there exists a function $\zeta = \zeta(\epsilon)$ satisfying $\lim_{\epsilon \to 0} \zeta(\epsilon) = +\infty$, such that for ϵ sufficiently small, it holds

$$\int_\Omega G(\xi, t_\epsilon u_\epsilon) \geq \zeta(\epsilon) \cdot \epsilon^{(Q-p)/(p-1)}. \tag{29}$$

The above estimate can be seen as follows. For ϵ sufficiently small, there exists a constant $c_1 > 0$ such that $t_\epsilon U_\epsilon(\xi) \in (0, \delta)$ for all ξ such that $d(\xi) \geq c_1 \epsilon^{(p-1)/p^2}$; we also observe that, if $\xi \in B_{1/2m}$, then $u_\epsilon^m = U_\epsilon$. Therefore, by (17) and (1), we get

$$\begin{aligned} \int_\Omega G(\xi, t_\epsilon u_\epsilon) &\geq c \int_{c_1 \epsilon^{1/p}}^{1/2m} U_\epsilon^p(\xi) \\ &\geq c\epsilon^{(Q-p)/(p-1)} \int_{c_1 \epsilon^{1/p}}^{1/2m} \rho^{(p^2-Q-p+1)/(p-1)} d\rho \\ &= c\epsilon^{(Q-p)/(p-1)} \begin{cases} \epsilon^{(p^2-Q)/(p-1)} & \text{if } Q > p^2 \\ |\log \epsilon| & \text{if } Q = p^2. \end{cases} \end{aligned} \tag{30}$$

Hence, the function $\zeta(\epsilon)$ in (29) is obtained.

So, from (13), (25), (26) and (30), we have

$$J(v_\epsilon) \leq J(t_\epsilon u_\epsilon) \leq \frac{S^{Q/p}}{Q} + (c - \zeta(\epsilon))\epsilon^{(Q-p)/(p-1)},$$

and choosing ϵ small, we get a contradiction with (28).

The case $p < Q < p^2$ can be treated analogously, taking into account estimate (15). □

Proof of Theorem 5 The proof follows the scheme of that of Theorem 4: as before, we can show that, (27) holds for sufficiently large $m \in \mathbb{N}$, under the asymptotic assumption (19) on G. We omit the details, referring to the Euclidean outline in [1]. □

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The Wodzicki Residue for Pseudo-Differential Operators on Compact Lie Groups

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Abstract Let G be an arbitrary compact Lie group. In this work we apply the method of the analytic continuation of traces in order to compute the Wodzicki residue for a classical pseudo-differential operator on G in terms of its matrix-valued symbol (which is globally defined on the non-commutative phase space $G\times\widehat{G}$, with $\widehat{G}$ being the unitary dual of G). Our main theorem is complementary to the results in Cardona et al. (Dixmier traces, Wodzicki residues, and determinants on compact Lie groups: the paradigm of the global quantisation. arXiv:2105.14949), where we remove the ellipticity hypothesis when the operators belong to the Hörmander classes on G defined by local coordinate systems.

1 Introduction

Let $\Psi^{\bullet}_{\mathrm{cl}}(M) := \cup_m \Psi^{m}_{\mathrm{cl}}(M)$ be the algebra of classical pseudo-differential operators on a closed manifold M. Of particular interest for this work, is the case where $M = G$ is a compact Lie group endowed with its unique bi-invariant Riemannian metric. Indeed, although much work has been done when computing regularised traces (as the Wodzicki reside, se e.g. [5–7, 9–11, 16]) on the algebra $\Psi^{\bullet}_{\mathrm{cl}}(M)$, in this paper we address the problem of computing the (non-commutative) Wodzicki residue on $\Psi^{\bullet}_{\mathrm{cl}}(G)$, in terms of the representation theory of G, where very little appears to be known, see [1–3].

In order to accomplish our goal, we are going to apply the algebraic formalism developed by Ruzhansky and Turunen [12], where the notion of a global symbol has been introduced for describing the Hörmander classes of pseudo-differential operators, see Hörmander [8]. In such a formalism, the symbol of an operator is globally defined on the non-commutative phase space $G \times \widehat{G}$, where $\widehat{G}$ is the unitary dual of G, instead of the classical local notion of a symbol defined by

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charts, which is defined on the cotangent space T^*M. In particular the paper [14] by Ruzhansky et al. is a source of many open problems, between them, the problem of computing geometric invariants of G in terms of the matrix-valued symbols, where the Wodzicki residue is a fundamental one on the list.

In order to present the contribution of this note, we recall the definition of the Wodzicki residue, which is a trace, that measures the locality of an operator. Indeed, if $A \in \Psi^m_{\mathrm{cl}}(M)$ is a classical pseudo-differential operator of order m, its Hörmander symbol (defined by local coordinate systems) admits a decomposition of the form

$$\sigma_H(x,\xi) \sim \sum_{k=0}^{\infty} a_{m-k}(x,\xi),\ (x,\xi) \in T^*M,$$

in components a_{m-j}, of homogeneous degree $m-j$ at $\xi \neq 0$. Then, Wodzicki in his seminal paper [17] proved that, up to a constant, the functional

$$\mathrm{res}(A) := \frac{1}{n(2\pi)^n} \int_{M\times\{\xi\in T^*_xM:|\xi|=1\}} a_{-n}(x,\xi)|d\xi\, dx|,\ n = \dim(M), \tag{1}$$

is the unique trace on the algebra of classical pseudo-differential operators $\Psi^{\bullet}_{\mathrm{cl}}(M) := \cup_m \Psi^m_{\mathrm{cl}}(M)$. It is a remarkable fact that, although the complete symbol does not have an invariant meaning, the right hand side of (1) is well-defined. Indeed, the invariance of the Wodzicki residue shows that $x \mapsto \mathrm{res}_x(A) : M \to \mathbb{C}$ is a density on M, where $\mathrm{res}_x(A) := \frac{1}{n(2\pi)^n}\int_{\{\xi\in T^*_xM:|\xi|=1\}} a_{-n}(x,\xi)d\xi$. In the picture below (see Fig. 1) we illustrate how any open covering of M allows us to define $\mathrm{res}_x(A)$ in a local way and how this functional can be glued over the whole manifold.

There are several ways of computing the Wodzicki residue. There is a global spectral approach which shows the delicate relation between ζ-functions and heat kernel expansions that is as follows. For $A \in \Psi^m_{\mathrm{cl}}(M)$, and for any elliptic operator $E \in \Psi^e(M)$, of positive order $e > 0$, we have the complex formula (see e.g. [9, 10, 15])

$$\mathrm{res}(A) = \frac{1}{n(2\pi)^n} e \cdot \mathrm{res}_{z=0}\mathrm{Tr}(AP^{-z}), \tag{2}$$

Fig. 1 Here we illustrate the local construction of the Wodzicki residue in the case of the torus. We start by choosing an atlas of the manifold, and on each open set of this covering we define a local density res_x. After doing this process, we construct a global density using the compatibility of the open subsets of the covering

and the identity via the heat kernel

$$\mathrm{res}(A) = c_{n,m} \text{ coefficient of } \log(t) \text{ in the expansion of } \mathrm{Tr}(Ae^{-tP}), \text{ as } t \to 0^+, \tag{3}$$

where $c_{n,m} := -m/n(2\pi)^n$. Our main Theorem 1, proved by using the method of the analytic continuation of traces, removes the ellipticity hypothesis in [3] when computing the Wodzicki residue for the Hörmander classes defined by charts, where the results were derived by using the real variable methods of the spectral calculus of subelliptic operators.

Now we present our main result. For a bounded operator T on a separable Hilbert space H, we use the following notation for the decomposition of T into its real and imaginary part,

$$\mathrm{Re}(T) := \frac{T + T^*}{2}, \; \mathrm{Im}(T) := \frac{T - T^*}{2i},$$

and the decomposition of $\mathrm{Re}(T)$ and $\mathrm{Im}(T)$ into their positive and negative parts,

$$\mathrm{Re}(T)^+ := \frac{\mathrm{Re}(T) + |\mathrm{Re}(T)|}{2}, \; \mathrm{Re}(T)^- := \frac{|\mathrm{Re}(T)| - \mathrm{Re}(T)}{2},$$

and

$$\mathrm{Im}(T)^+ := \frac{\mathrm{Im}(T) + |\mathrm{Im}(T)|}{2}, \; \mathrm{Im}(T)^- := \frac{|\mathrm{Im}(T)| - \mathrm{Im}(T)}{2},$$

the mapping $\sigma_A : (x, [\xi]) \in G \times \widehat{G} \mapsto \sigma_A(x, [\xi])$ denotes the (global) matrix-valued symbol of a pseudo-differential operator A in the algebraic formalism of Ruzhansky and Turunen [12], and $\|L\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})}$ denotes the weak-ℓ^1 quasi-norm of a matrix-valued function L on the unitary dual $\widehat{G}$, see Sect. 2.3 for details.

Theorem 1 *Let $A \in \Psi^{-n}_{\mathrm{cl}}(G)$ be a classical pseudo-differential operator on G. Assume that the symbol of A admits an asymptotic expansion*

$$\sigma_A(x, [\xi]) \sim \sum_{k=-\infty}^{-n} \sigma_k(x, [\xi]), \tag{4}$$

in components with decreasing order, which means that, for any $N \in \mathbb{N}$,

$$\sigma_A(x, [\xi]) - \sum_{k=-N-n}^{-n} \sigma_k(x, [\xi]) \in S^{-(N+1)-n}_{1,0}(G \times \widehat{G}). \tag{5}$$

Then, the Wodzicki residue of A can be computed according to the formula

$$\mathrm{res}(A) = \int_G \left(\|\mathrm{Re}(\sigma_{-n}(x,[\xi]))^{+}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} - \|\mathrm{Re}(\sigma_{-n}(x,[\xi]))^{-}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} \right) dx$$
$$+ i \int_G \left(\|\mathrm{Im}(\sigma_{-n}(x,[\xi]))^{+}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} - \|\mathrm{Im}(\sigma_{-n}(x,[\xi]))^{-}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} \right) dx.$$

This paper is organised as follows. In Sect. 2 we record the preliminaries of this work. More precisely, Sect. 2.1 is dedicated to presenting the basics on the Hörmander classes defined by local coordinate systems. In Sect. 2.2 we record the construction of the global matrix-valued symbols in the Ruzhansky–Turunen formalism. In Sect. 2.3 we present the weak ℓ^1-space on the unitary dual $\widehat{G}$ that we will use in Sect. 3 in order to prove our main Theorem 1.

2 Preliminaries

2.1 *Pseudo-Differential Operators via Localisations*

Pseudo-differential operators on compact manifolds, and consequently on compact Lie groups, can be defined by using local coordinate charts, see Hörmander [8]. Let us briefly introduce these classes starting with the definition in the Euclidean setting. Let U be an open subset of $\mathbb{R}^n$. We say that the "symbol" $a \in C^{\infty}(U \times \mathbb{R}^n, \mathbb{C})$ belongs to the Hörmander class of order m and of (ρ,δ)-type, $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, $0 \leqslant \rho, \delta \leqslant 1$, if for every compact subset $K \subset U$ and for all $\alpha, \beta \in \mathbb{N}_0^n$, the symbol inequalities

$$|\partial_x^{\beta}\partial_{\xi}^{\alpha} a(x,\xi)| \leqslant C_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

hold true uniformly in $x \in K$ for all $\xi \in \mathbb{R}^n$. Then, a continuous linear operator $A : C_0^{\infty}(U) \to C^{\infty}(U)$ is a pseudo-differential operator of order m of (ρ,δ)-type, if there exists a symbol $a \in S^m_{\rho,\delta}(U \times \mathbb{R}^n)$ such that

$$Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} a(x,\xi)(\mathscr{F}_{\mathbb{R}^n} f)(\xi) d\xi,$$

for all $f \in C_0^{\infty}(U)$, where

$$(\mathscr{F}_{\mathbb{R}^n} f)(\xi) := \int_U e^{-i2\pi x\cdot\xi} f(x) dx$$

is the Euclidean Fourier transform of f at $\xi \in \mathbb{R}^n$.

Once the definition of Hörmander classes on open subsets of $\mathbb{R}^n$ is established, it can be extended to smooth manifolds as follows. Given a C^∞-manifold M, a linear continuous operator $A : C_0^\infty(M) \to C^\infty(M)$ is a pseudo-differential operator of order m of (ρ, δ)-type, with $\rho \geqslant 1 - \delta$, and $0 \leq \delta < \rho \leq 1$, if for every local coordinate patch $\omega : M_\omega \subset M \to U_\omega \subset \mathbb{R}^n$, and for every $\phi, \psi \in C_0^\infty(U_\omega)$, the operator

$$Tu := \psi(\omega^{-1})^* A \omega^*(\phi u), \ \ u \in C^\infty(U_\omega),$$

is a standard pseudo-differential operator with symbol $a_T \in S^m_{\rho,\delta}(U_\omega \times \mathbb{R}^n)$. In this case we write $A \in \Psi^m_{\rho,\delta}(M, \text{loc})$. In particular for $(\rho, \delta) = (1, 0)$ we denote $\Psi^m(M) := \Psi^m_{1,0}(M, \text{loc})$.[1]

2.2 The Global Symbol in the Ruzhansky–Turunen Formalism

Let A be a continuous linear operator from $C^\infty(G)$ into $C^\infty(G)$, and let $\widehat{G}$ be the algebraic unitary dual of G. Then, it was established by Ruzhansky and Turunen in [12] that there is a function

$$\sigma_A : G \times \widehat{G} \to \cup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell \times \ell}, \tag{6}$$

that we call the matrix-valued symbol of A, such that $\sigma_A(x, \xi) := \sigma_A(x, [\xi]) \in \mathbb{C}^{d_\xi \times d_\xi}$ for every equivalence class $[\xi] \in \widehat{G}$, where $\xi : G \to \text{Hom}(H_\xi)$, $H_\xi \cong \mathbb{C}^{d_\xi}$, and such that

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x) \sigma_A(x, \xi) \widehat{f}(\xi)], \ \ \forall f \in C^\infty(G). \tag{7}$$

We have denoted by

$$\widehat{f}(\xi) \equiv (\mathscr{F} f)(\xi) := \int_G f(x) \xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi}, \ \ [\xi] \in \widehat{G},$$

the Fourier transform of f at $\xi \cong (\xi_{ij})_{i,j=1}^{d_\xi}$, where the matrix representation of ξ is induced by an orthonormal basis of the representation space H_ξ. The function σ_A in (6) satisfying (7) is unique, and satisfies the identity

$$a(x, \xi) = \xi(x)^* (A\xi)(x), \ \ A\xi := (A\xi_{ij})_{i,j=1}^{d_\xi}, \ \ [\xi] \in \widehat{G}.$$

[1] As usually, ω^* and $(\omega^{-1})^*$ are the pullbacks, induced by the maps ω and ω^{-1} respectively.

Note that the previous identity is well defined. Indeed, it is well known that the functions ξ_{ij}, which are of C^∞-class, are the eigenfunctions of the positive Laplace operator $\mathcal{L}_G$, that is $\mathcal{L}_G\xi_{ij} = \lambda_{[\xi]}\xi_{ij}$ for some non-negative real number $\lambda_{[\xi]} \geq 0$ depending only of the equivalence class $[\xi]$ and not on the representation ξ.

In general, we refer to the function a as the (global or full) *symbol* of the operator A, and we will use the notation $A = \mathrm{Op}(a)$ to indicate that $a := \sigma_A$ is the symbol associated with the operator A. We will use the notation

$$S^m_{\mathrm{cl}}(G \times \widehat{G}) := \{\sigma_A : G \times \widehat{G} \to \bigcup_{\ell} \mathbb{C}^{\ell\times\ell} |\, A \in \Psi^m_{\mathrm{cl}}(G)\},$$

for the class of matrix-valued symbols of the classical operators of order $m \in \mathbb{R}$ on M.

One of the main contributions of the works [12, 14] was to use the notion of *difference operators*, which endows $\widehat{G}$ with a difference structure making possible to classify the Hörmander classes [8]. Using the class of functions

$$\Sigma(\widehat{G}) := \{\sigma : \widehat{G} \to \cup_{\ell\in\mathbb{N}}\mathbb{C}^{\ell\times\ell}\},$$

the space of distributions on $\widehat{G}$ is exactly the set

$$\mathscr{D}'(\widehat{G}) := \{\sigma \in \Sigma(\widehat{G}) : \exists K \in \mathscr{D}'(G) \text{ such that } \sigma = \widehat{K}\}, \tag{8}$$

where $\mathscr{D}'(G)$ is the topological dual of $C^\infty(G)$ endowed with its natural Fréchet structure. Moreover, The Schwartz class of distributions on $\widehat{G}$ is defined via

$$\mathscr{S}(\widehat{G}) := \{\sigma \in \Sigma(\widehat{G}) : \exists K \in C^\infty(G) \text{ such that } \sigma = \widehat{K}\}. \tag{9}$$

Following [14], we will say that a difference operator $Q_\xi : \mathscr{D}'(\widehat{G}) \to \mathscr{D}'(\widehat{G})$ is of order k if

$$Q_\xi \widehat{f}(\xi) = \widehat{qf}(\xi),\ [\xi] \in \widehat{G}, \tag{10}$$

for some function q vanishing of order k at the neutral element $e = e_G$. We will denote by $\mathrm{diff}^k(\widehat{G})$ the class of all difference operators of order k. For a fixed smooth function q, the associated difference operator will be denoted by $\Delta_q \equiv Q_\xi$. A system of difference operators (see e.g. [13])

$$\Delta^\alpha_\xi := \Delta^{\alpha_1}_{q_{(1)}} \cdots \Delta^{\alpha_i}_{q_{(i)}},\ \alpha = (\alpha_j)_{1\leqslant j\leqslant i},$$

with $i \geq n$, is called admissible if, for any orthonormal basis

$$X = \{X_1, X_2, \cdots, X_n\}$$

of the Lie algebra $\mathfrak{g}$, and for its respective gradient $\nabla_X = (X_1, \cdots, X_n)$, we have that

$$\operatorname{rank}\{\nabla_X q_{(j)}(e) : 1 \leqslant j \leqslant i\} = \dim(G), \text{ and } \Delta_{q_{(j)}} \in \operatorname{diff}^1(\widehat{G}).$$

Then, it is useful to introduce an admissible collection of difference operators, which is admissible and additionally,

$$\bigcap_{j=1}^{i} \{x \in G : q_{(j)}(x) = 0\} = \{e_G\}.$$

Remark 1 Matrix components of unitary representations induce difference operators, [13]. Indeed, if $\xi_1, \xi_2, \cdots, \xi_k$, are fixed irreducible and unitary representation of G, which not necessarily belong to the same equivalence class, then each coefficient of the matrix

$$\xi_\ell(g) - I_{d_{\xi_\ell}} = [\xi_\ell(g)_{ij} - \delta_{ij}]_{i,j=1}^{d_{\xi_\ell}}, \quad g \in G, \ 1 \leq \ell \leq k, \tag{11}$$

that is each function $q_{ij}^{\ell}(g) := \xi_\ell(g)_{ij} - \delta_{ij}$, $g \in G$, defines a difference operator

$$\mathbb{D}_{\xi_\ell, i, j} := \mathscr{F}(\xi_\ell(g)_{ij} - \delta_{ij})\mathscr{F}^{-1}. \tag{12}$$

We can fix $k \geq \dim(G)$ of these representations in such a way that the corresponding family of difference operators is admissible, that is,

$$\operatorname{rank}\{\nabla q_{i,j}^{\ell}(e) : 1 \leqslant \ell \leqslant k\} = \dim(G).$$

To define higher order difference operators of this kind, let us fix a unitary irreducible representation ξ_ℓ. Since the representation is fixed we omit the index ℓ of the representations ξ_ℓ in the notation that will follow. Then, for any given multi-index $\alpha \in \mathbb{N}_0^{d_{\xi_\ell}^2}$, with $|\alpha| = \sum_{i,j=1}^{d_{\xi_\ell}} \alpha_{i,j}$, we write

$$\mathbb{D}^{\alpha} := \mathbb{D}_{1,1}^{\alpha_{11}} \cdots \mathbb{D}_{d_{\xi_\ell} d_{\xi_\ell}}^{\alpha_{d_{\xi_\ell}, d_{\xi_\ell}}}$$

for a difference operator of order $|\alpha|$.

We are now going to introduce the global Hörmander classes of symbols defined in [12]. First let us recall that every left-invariant vector field $Y \in \mathfrak{g}$ can be identified with the first order differential operator $\partial_Y : C^\infty(G) \to \mathscr{D}'(G)$ given by

$$\partial_Y f(x) = (Y_x f)(x) = \frac{d}{dt} f(x \exp(tY))|_{t=0}.$$

If $\{X_1, \cdots, X_n\}$ is a basis of the Lie algebra $\mathfrak{g}$, then we will use the standard multi-index notation

$$\partial_X^\alpha = X_x^\alpha = \partial_{X_1}^{\alpha_1} \cdots \partial_{X_n}^{\alpha_n},$$

for a canonical left-invariant differential operator of order $|\alpha|$.

By using this property, together with the following notation for the so-called elliptic weight

$$\langle \xi \rangle := (1 + \lambda_{[\xi]})^{1/2}, \; [\xi] \in \widehat{G},$$

we can finally give the definition of global symbol classes, as finally it was adopted in [4].

Definition 1 Let G be a compact Lie group and let $0 \leqslant \delta, \rho \leqslant 1$. Let

$$\sigma : G \times \widehat{G} \to \bigcup_{[\xi] \in \widehat{G}} \mathbb{C}^{d_\xi \times d_\xi},$$

be a matrix-valued function such that for any $[\xi] \in \widehat{G}$, $\sigma(\cdot, [\xi])$ is of C^∞-class, and such that, for any given $x \in G$ there is a distribution $k_x \in \mathscr{D}'(G)$, smooth in x, satisfying $\sigma(x, \xi) = \widehat{k}_x(\xi)$, $[\xi] \in \widehat{G}$. We say that $\sigma \in \mathscr{S}^m_{\rho,\delta}(G)$ if the following symbol inequalities

$$\|\partial_X^\beta \Delta_\xi^\gamma \sigma_A(x, \xi)\|_{\text{op}} \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m - \rho|\gamma| + \delta|\beta|}, \tag{13}$$

are satisfied for all β and γ multi-indices and for all $(x, [\xi]) \in G \times \widehat{G}$. For $\sigma_A \in \mathscr{S}^m_{\rho,\delta}(G)$ we will write $A \in \Psi^m_{\rho,\delta}(G) \equiv \text{Op}(\mathscr{S}^m_{\rho,\delta}(G))$.

The global Hörmander classes on compact Lie groups can be used to describe the Hörmander classes defined by local coordinate systems. This is one of the depth applications of the Ruzhansky–Turunen formalism. We present the corresponding statement as follows.

Theorem 2 (Equivalence of Classes, [12, 14]) *Let* $A : C^\infty(G) \to \mathscr{D}'(G)$ *be a continuous linear operator and let* $0 \leq \delta < \rho \leq 1$, *with* $\rho \geq 1 - \delta$. *Then,* $A \in \Psi^m_{\rho,\delta}(G, \text{loc})$, *if and only if* $\sigma_A \in \mathscr{S}^m_{\rho,\delta}(G)$. *Consequently,*

$$\text{Op}(\mathscr{S}^m_{\rho,\delta}(G)) = \Psi^m_{\rho,\delta}(G, \text{loc}), \;\; 0 \leqslant \delta < \rho \leqslant 1, \; \rho \geqslant 1 - \delta. \tag{14}$$

2.3 The Weak ℓ^1 Space $\mathcal{L}^{(1,\infty)}(\widehat{G})$

It was introduced by the author and C. Del Corral in [1] the weak ℓ^1 space on the unitary dual $\widehat{G}$ for the study of the Dixmier trace of left-invariant operators. To introduce this space let us consider the family of matrix-valued symbols

$$\Sigma(\widehat{G})_{\text{mod}} := \{\sigma : \widehat{G} \to \cup_{\ell\in\mathbb{N}}\mathbb{C}^{\ell\times\ell} \text{ and size}(\sigma([\xi])) = d_\xi \times d_\xi\}.$$

Then, the weak ℓ^1-space on $\widehat{G}$ is defined by the subset of functions σ in $\Sigma(\widehat{G})_{\text{mod}}$ such that

$$\|\sigma(\xi)\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} := \lim_{N\to\infty} \frac{1}{\log N} \sum_{[\xi]:\langle\xi\rangle\leq N} d_\xi \mathbf{Tr}(|\sigma_A(\xi)|) < \infty, \tag{15}$$

where the elliptic weight $\langle\xi\rangle := (1+\lambda_{[\xi]})^{\frac{1}{2}}$ is the elliptic weight associated to the positive Laplace operator $\mathcal{L}_G$.

3 Proof of Theorem 1

Proof First, let us assume that the matrix symbol σ of A is positive on any representation space, i.e.

$$\forall(x,[\xi]) \in G\times\widehat{G},\ \forall L\in\mathbb{C}^{d_\xi},\ (\sigma(x,[\xi])L, L)_{\mathbb{C}^{d_\xi}} \geq 0, \tag{16}$$

where $(L, M)_{\mathbb{C}^{d_\xi}} := \sum_{j=1}^{d_\xi} L_j\overline{M}_j$, stands for the inner product on $\mathbb{C}^{d_\xi}$. For any $x\in G$, we denote by

$$A_x := \text{Op}[[\xi]\mapsto\sigma(x,[\xi])]$$

the operator associated to the matrix symbol $\sigma(x,\cdot)$. In view of the compactness of G, for any $x\in G$, $A_x \in \Psi^{-n}_{cl}(G,\text{loc})$ is also a classical pseudo-differential operator, see [14, Page 482]. Let us assume also that for any x, A_x is self-adjoint on $L^2(G)$. Note that for any $z\in\mathbb{C}$,

$$A(1+\mathcal{L}_G)^{\frac{z}{2}} \in \Psi^{-n+\text{Re}(z)}_{cl}(G,\text{loc}),\ \forall x\in G,\ A_x(1+\mathcal{L}_G)^{\frac{z}{2}} \in \Psi^{-n+\text{Re}(z)}_{cl}(G,\text{loc}), \tag{17}$$

in view of the composition properties of the Hörmander calculus, see [8, Chapter XVIII]. The functions

$$z\mapsto f(z) := \text{Tr}[A(1+\mathcal{L}_G)^{\frac{z}{2}}] \tag{18}$$

and

$$z \mapsto f(x,z) := \mathrm{Tr}[A_x(1+\mathcal{L}_G)^{\frac{z}{2}}], \quad x \in G, \tag{19}$$

are analytic mappings on the left semi-plane

$$\mathbb{C}_{<0} := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}. \tag{20}$$

It is straightforward the existence of holomorphic extensions $\tilde{f}$ and $\tilde{f}(x,\cdot)$ of f and $f(x,\cdot)$, respectively, on the right semi-plane (see e.g. Lesch [9])

$$\mathbb{C}_{\geq 0} := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}. \tag{21}$$

The analytic continuations $\tilde{f}$ and $\tilde{f}(x,\cdot)$ have simple poles at $\mathbb{N}_0 := \{j \in \mathbb{Z} : j \geq 0\}$. Let us make the principal observation of this proof. For $z \in \mathbb{C}_{<0}$, we have the identity

$$f(z) := \mathrm{Tr}[A(1+\mathcal{L}_G)^{\frac{z}{2}}] = \int_G \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[a(x,[\xi])\langle\xi\rangle^z]dx = \int_G f(x,z)dx.$$

Note that the function

$$\tilde{F}(z) := \int_G \tilde{f}(x,z)dx$$

is an analytic extension of

$$f(z) = \int_G f(x,z)dx,$$

and that in consequence $\tilde{F}|_{\mathbb{C}_{<0}} = \tilde{f}|_{\mathbb{C}_{<0}} = f$. Because the domain $\mathbb{C}_{<0}$ clearly contains accumulation points in its interior, in view of the identity theorem for analytic functions, we conclude that $\tilde{F}$ and $\tilde{f}$ agree on the domain $\mathbb{C}_{\geq 0} \setminus \{j \in \mathbb{Z} : j \geq 0\}$. All these facts allow us to compute the Wodzicki residue of A. Indeed, by fixing $0 < \varepsilon < 1/2$, using that for any classical operator B of order $-n$, on G, the Wodzicki residue can be computed as the residue at $z = 0$ of the analytic extension of the complex function $\mathrm{Tr}[B(1+\mathcal{L}_G)^{\frac{z}{2}}]$, see e.g. Lesch [9, Eq. (1.2)], that is

$$\mathrm{res}(B) = \mathrm{res}_{z=0}\mathrm{Tr}[B(1+\mathcal{L}_G)^{\frac{z}{2}}],\ B \in \Psi^m_{\mathrm{cl}}(G,\mathrm{loc}),\ m \in \mathbb{R}, \tag{22}$$

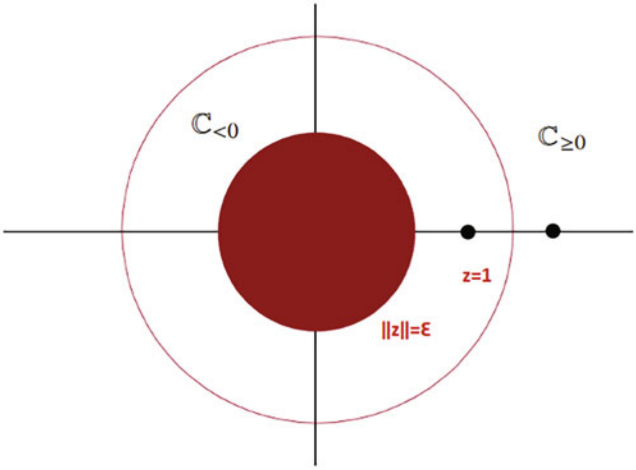

Fig. 2 The figure illustrates the curve of integration on the complex plane used in the proof of the theorem

and by using Fubini theorem, we have

$$\begin{aligned}\mathrm{res}(A) = \mathrm{res}_{z=0}\tilde{f}(z) &= \frac{1}{2\pi i}\int_{|z|=\varepsilon}\tilde{f}(z)dz = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\int_{G}\tilde{f}(x,z)dxdz\\ &= \frac{1}{2\pi i}\int_{G}\int_{|z|=\varepsilon}\tilde{f}(x,z)dxdz\\ &= \int_{G}\mathrm{res}_{z=0}\tilde{f}(x,z)dx\\ &= \int_{G}\mathrm{res}(A_x)dx,\end{aligned}$$

where we have used the compactness of G. We illustrate the sector of integration on the complex plane in the figure below (see Fig. 2).

The positivity of the symbol $a(x,\cdot)$ implies the positivity of the operator A_x. Indeed, for any $x_0 \in G$,

$$\mathrm{Spect}[A_{x_0}] = \bigcup_{[\xi]\in\widehat{G}}\mathrm{Spect}[a(x_0,[\xi])] \subset \mathbb{R}_0^+. \tag{23}$$

Indeed, the spectrum of any matrix $a(x_0,[\xi])$ is positive. This fact implies that A_{x_0} being self-adjoint is positive on $L^2(G)$. Then, using Theorem 1.1 in [1], we have that

$$\mathrm{res}(A_x) = \|a(x,\cdot)\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})}.$$

In consequence,

$$\operatorname{res}(A) = \int_G \|a(x,\cdot)\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})}dx. \tag{24}$$

Now, we are going to use the decomposition strategy in [3] for extending the formula in (24) to the general case where the symbol is not necessarily a positive matrix on every representation space.

We use the decomposition of A into its (self-adjoint) real and imaginary part,

$$\operatorname{Re}(A) := \frac{A + A^*}{2},\ \operatorname{Im}(A) := \frac{A - A^*}{2i},$$

and the decomposition of $\operatorname{Re}(A)$ and $\operatorname{Im}(A)$ into their (self-adjoint) positive and negative parts,

$$\operatorname{Re}(A)^+ := \frac{\operatorname{Re}(A) + |\operatorname{Re}(A)|}{2},\ \operatorname{Re}(A)^- := \frac{|\operatorname{Re}(A)| - \operatorname{Re}(A)}{2},$$

and

$$\operatorname{Im}(A)^+ := \frac{\operatorname{Im}(A) + |\operatorname{Im}(A)|}{2},\ \operatorname{Im}(A)^- := \frac{|\operatorname{Im}(A)| - \operatorname{Im}(A)}{2}.$$

Now, the operator A can be written as

$$\begin{aligned} A &= \operatorname{Re}(A) + i\operatorname{Im}(A) \\ &= \left(\operatorname{Re}(A)^+ - \operatorname{Re}(A)^-\right) + i\left(\operatorname{Im}(A)^+ - \operatorname{Im}(A)^-\right). \end{aligned}$$

So, by the linearity of the Wodzicki residue Tr_ω we have

$$\begin{aligned} \operatorname{res}(A) &= \operatorname{res}(\operatorname{Re}(A)) + i\operatorname{res}(\operatorname{Im}(A)) \\ &= \left(\operatorname{res}(\operatorname{Re}(A)^+) - \operatorname{res}(\operatorname{Re}(A)^-)\right) + i\left(\operatorname{res}(\operatorname{Im}(A)^+) - \operatorname{res}(\operatorname{Im}(A)^-)\right). \end{aligned}$$

Now, we will exploit the subelliptic functional calculus in [2] in order to compute the symbols of the positive operators $\operatorname{Re}(A)^+$, $\operatorname{Re}(A)^-$, $\operatorname{Im}(A)^+$, $\operatorname{Im}(A)^-$. Indeed, we have

$$\sigma_{\operatorname{Re}(A)}(x,[\xi]) = \sigma_{\frac{A+A^*}{2}}(x,[\xi]) = \operatorname{Re}(\sigma_{-n}(x,[\xi])) + \text{lower order terms},$$

$$\sigma_{\operatorname{Im}(A)}(x,[\xi]) = \sigma_{\frac{A-A^*}{2i}}(x,[\xi]) = \operatorname{Im}(\sigma_{-n}(x,[\xi])) + \text{lower order terms},$$

$$\sigma_{\mathrm{Re}(A)^+}(x,[\xi]) = \sigma_{\frac{\mathrm{Re}(A)+|\mathrm{Re}(A)|}{2}}(x,[\xi]) = \mathrm{Re}(\sigma_{-n}(x,[\xi]))^+ + \text{lower order terms},$$

$$\sigma_{\mathrm{Re}(A)^-}(x,[\xi]) = \sigma_{\frac{|\mathrm{Re}(A)|-\mathrm{Re}(A)}{2}}(x,[\xi]) = \mathrm{Re}(\sigma_{-n}(x,[\xi]))^- + \text{lower order terms},$$

and

$$\sigma_{\mathrm{Im}(A)^+}(x,[\xi]) = \sigma_{\frac{\mathrm{Im}(A)+|\mathrm{Im}(A)|}{2}}(x,[\xi]) = \mathrm{Im}(\sigma_{-n}(x,[\xi]))^+ + \text{lower order terms},$$

$$\sigma_{\mathrm{Im}(A)^-}(x,[\xi]) = \sigma_{\frac{|\mathrm{Im}(A)|-\mathrm{Im}(A)}{2}}(x,[\xi]) = \mathrm{Im}(\sigma_{-n}(x,[\xi]))^- + \text{lower order terms}.$$

Now, by applying (24), we can eliminate the lower terms when computing the Wodzicki residue, and we have the following formulae

$$\mathrm{res}(\mathrm{Re}(A)^+) = \int_G \|\mathrm{Re}(\sigma_{-n}(x,[\xi]))^+\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})}dx$$

$$\mathrm{res}(\mathrm{Re}(A)^-) = \int_G \|\mathrm{Re}(\sigma_{-n}(x,[\xi]))^-\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})}dx$$

$$\mathrm{res}(\mathrm{Im}(A)^+) = \int_G \|\mathrm{Im}(\sigma_{-n}(x,[\xi]))^+\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})}dx$$

$$\mathrm{res}(\mathrm{Im}(A)^-) = \int_G \|\mathrm{Im}(\sigma_{-n}(x,[\xi]))^-\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})}dx.$$

In view of the linearity of the Wodzicki residue we end the proof. □

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New Characterizations of Harmonic Hardy Spaces

Joel E. Restrepo and Durvudkhan Suragan

Abstract We present new equivalent descriptions of the harmonic Hardy spaces in the unit disc and in the upper half plane. Such descriptions are found as applications of a generalized Hadamard operator of a standard function kernel.

1 Introduction

The generalized Hadamard operator $L^{(\omega)}$ of M. M. Djrbashian (see [17], pp. xxxi, xxxvi, 344–346, 432, 435) or, as it is also called Djrbashian's generalized fractional integral, was introduced and used to construct the factorization theory of the Nevanlinna type classes $N\{\omega\}$ [4, 5] of meromorphic functions in the unit disc of the complex plane. Some of these classes contain Nevanlinna's class N of functions of bounded type and all meromorphic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, while the others are contained in $N\{\omega\}$ and possess better boundary properties [8]. In Sect. 2, we discuss appearance of the operator $L^{(\omega)}$ in an equivalent description of the harmonic Hardy space h^1 in $\mathbb{D}$, i.e. harmonic functions in $\mathbb{D}$ with bounded integral means. We also establish several results for its inverse operator. Notice that the integral means give a measure of growth and lead to a fruitful theory with many applications. For more details we refer to the books [2, 6].

Substantial progress was made on the half-plane analogue of a part of the theory of M. M. Djrbashian–V. S. Zakaryan [4, 5] over the space of harmonic and delta-subharmonic functions in $G^+ = \{z : \text{Im}\, z > 0\}$. Namely, similar results

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were obtained in [12] for ω-weighted classes of harmonic functions possessing nonnegative harmonic majorants in the upper half-plane, which extend the results established in [10]. Further studies were done for delta-subharmonic in G^+ [13]. The following growth condition is used in the upper half plane G^+:

$$\sup_{y>0} \int_{-\infty}^{+\infty} |u(x+iy)|dx < +\infty, \tag{1}$$

which represents an analogue of the bounded integral means condition used in the unit disc. Notice that the harmonic Hardy space H^1 over G^+ is the set of harmonic functions in G^+ which satisfy condition (1). In the latter mentioned papers, a generalized integro-differential operator L_{ω^*} (ω^* is taken in a certain function space) was used, which becomes, in a particular case, the Liouville integro-differential operator [14, 17]. The operator L_{ω^*}, which is defined in Sect. 3, is a half plane analog of the operator $L^{(\omega)}$ introduced in the unit disc. The operator L_{ω^*} was first introduced in [11]. In Sect. 3, we show the important role which plays the operator L_{ω^*} over the harmonic Hardy space H^1. Moreover, we give the explicit form of the inverse operator of L_{ω^*} and some results related to it.

2 A Description of the Harmonic Hardy Space h^1 in $\mathbb{D}$

In this section we recall some basic statements on the harmonic Hardy space h^1 in the unit disc. In fact, we show that the functions of h^1 can be represented by the generalized fractional operator $L^{(\omega)}$ of M.M. Djrbashian to an integral which depends on the Schwarz' type kernel. We also recall an important result where it is proved the existence and explicit form of a left inverse operator [15, Lemma 2.1], which is, in fact, as it is proved here a right inverse operator as well.

We begin by recalling preliminaries on the generalized fractional operator $L^{(\omega)}$ of M.M. Djrbashian.

A function $\omega(t)$ is said to be of the class Ω, if

(i) $\omega(t) > 0$ and is continuous and nondecreasing in $[0, 1)$,
(ii) $\omega(0) = 1$ and $\int_0^1 \omega(t)dt < +\infty$.

Sometimes, see e.g. [4, 5, 15], it is also assumed that $\omega(t)$ is of a Lipschitz class $\lambda_t \in (0, 1]$ for all $t \in [0, 1)$. Nevertheless, the Lipschitz condition is used to prove continuity and convergence for some logarithmic Blaschke type factors under the application of $L^{(\omega)}$. Here we do not require it since we do not use such factors.
For a functional parameter $\omega(t) \in \Omega$, we use the following operators which are formally defines on functions $u(z)$ given in the unit disc $\mathbb{D}$:

$$L^{(\omega_1)}u(z) := -\int_0^1 u(zt)d\omega_1(t), \quad \text{where} \quad \omega_1(t) = \int_t^1 \frac{\omega(x)}{x}dx,$$

and

$$L^{(\omega)}u(z) = u(0) + L^{(\omega_1)}U(z), \quad \text{where} \quad U(z) = |z|\frac{\partial}{\partial |z|}u(z), \quad z \in \mathbb{D}.$$

The operator $L^{(\omega)}u(r)$ is an essential generalization of the Riemann-Liouville integro-differential operators I^{α}_{a+} and D^{α}_{a+} $(-1 < \alpha < +\infty,\ a \in \mathbb{R})$ [14, 17], since

$$L^{(\omega)}u(r) = \Gamma(1+\alpha)r^{-\alpha}I^{1+\alpha}_{0+}u'(r), \quad -1 < \alpha < +\infty,$$

$$L^{(\omega)}u(r) = \Gamma(1+\alpha)r^{-\alpha}D^{-(1+\alpha)}_{0+}u(r), \quad -1 < \alpha < 0,$$

for $\omega(x) = (1-x)^{\alpha}$ when $0 \leqslant x \leqslant 1$ and it is also assumed $u(0) = 0$. Notice that the operator $L^{(\omega)}$ is a simplified form of the operator used in [4, 5] (see Lemma 1.1 in [8]). Besides, we use M. M. Djrbashian's Cauchy type kernel

$$C(z;\omega) := \sum_{k=0}^{+\infty}\frac{z^k}{\Delta_k}, \quad \Delta_0 = 1, \quad \Delta_k = k\int_0^1 t^{k-1}\omega(t)dt \quad (k = 1, 2, \ldots), \tag{2}$$

which is holomorphic in $\mathbb{D}$ for any $\omega(t) \in \Omega$ and in the particular case of power functions $\omega(x) = (1-x)^{\alpha}$, $-1 < \alpha < 0$, it is the $1+\alpha$ order of the ordinary Cauchy kernel:

$$C(z;\omega) = \frac{1}{(1-z)^{1+\alpha}}, \quad \text{and} \quad C_0(z)\Big|_{\omega\equiv 1} = \frac{1}{1-z}, \quad z \in \mathbb{D}.$$

Also, we use the Schwarz type kernel

$$S(z;\omega) := 2C(z;\omega) - 1 = 1 + 2\sum_{k=1}^{+\infty}\frac{z^k}{\Delta_k}, \quad z \in \mathbb{D},$$

for which

$$S(z;\omega)\Big|_{\omega(x)=(1-x)^{\alpha}} = \frac{2}{(1-z)^{1+\alpha}} - 1, \quad S_0(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

One can see that

$$L^{(\omega)}[r^k] = r^k\Delta_k, \quad r \in [0, 1], \quad k = 0, 1, 2, \ldots, \tag{3}$$

$$L^{(\omega)}u(z)\Big|_{u(z)\equiv 1} = 1, \quad z \in \mathbb{D},$$

and by (2) and (3)

$$L^{(\omega)}C(z;\omega) = C_0(z), \quad z \in \mathbb{D}.$$

The next lemma is about the inversion operator of $L^{(\omega)}$ [15, Lemma 2.1].

Lemma 2.1 *If $\omega \in \Omega$, then the Volterra equation*

$$-\int_x^1 \omega\left(\frac{x}{t}\right) d\widetilde{\omega}(t) \equiv 1 \quad \textit{for a.e.} \quad x \in (0,1)$$

has a non-increasing solution $\widetilde{\omega}(x)$ such that $\widetilde{\omega}(1) = 0$, $\widetilde{\omega}(+0) = 1$ and $\widetilde{\omega}(x) \leq [\omega(x)]^{-1}$ for all $0 < x < 1$. Moreover, the operator $L^{(\omega)}$ is one-to-one mapping of harmonic functions in any disc $|z| < R < +\infty$ to harmonic functions in the same disc, and $\left(L^{(\omega)}\right)^{-1}$ is the operator

$$L^{(\widetilde{\omega})}u(z) := -\int_{+0}^1 u(z\sigma)d\widetilde{\omega}(\sigma), \quad \textit{i.e.} \quad L^{(\widetilde{\omega})}L^{(\omega)}u(z) = \left(L^{(\omega)}\right)^{-1}L^{(\omega)}u(z) = u(z).$$

Remark Notice that the application of the operator $L^{(\widetilde{\omega})}$ to a function harmonic $v(z)$ in a disc $|z| < R < +\infty$ means a multiplication of the coefficients of the harmonic series of $v(z)$ by $\Delta_k := \Delta_k(\widetilde{\omega})$. Therefore, $L^{(\widetilde{\omega})}$ is a one-to-one operator. □

Let us now give the definition of bounded variation function which will be used through the whole chapter.

Definition 2.2 Let $f : [a,b] \to \mathbb{C}$ be a given function. Given any finite partition of $[a,b]$, i.e. $P = \{a = y_0 < \cdots < y_n = b\}$, and setting

$$S_P = \sum_{k=1}^{n} |f(x_i) - f(x_{i-1})|,$$

we have that the variation of f on $[a,b]$ is

$$\bigvee_a^b f = V(f,a,b) = \sup\{S_P : \ P \text{ is a partition of } [a,b]\}.$$

Therefore, a function is said to be of bounded variation on $[a,b]$ if $\bigvee_a^b f < \infty$.

Similarly, for a function $f : \mathbb{R} \to \mathbb{C}$, we have $\bigvee_{-\infty}^{+\infty} f = \sup_{a<b} V(f,a,b)$. □

Now we recall an important relation between the Poisson kernel, the generalized fractional operator $L^{(\omega)}$ and the Schwarz type kernel [5, formula (4.10)]:

$$P(\theta-\varphi, r) = \frac{1-|z|^2}{|e^{i\theta}-z|^2} = L^{(\omega)}\big\{\mathrm{Re}\, S(re^{i(\theta-\varphi)};\omega)\big\}, \quad z = re^{i\varphi} \in \mathbb{D}. \tag{4}$$

We also recall the harmonic Hardy space h^1, which is the class of harmonic functions in $\mathbb{D}$ such that

$$\sup_{0\le r<1}\left\{\int_0^{2\pi}\big|u(re^{i\varphi})\big|d\varphi\right\} < +\infty.$$

Let us denote

$$G_{\omega,\sigma}(z) := -\mathrm{Re}\,\frac{1}{2\pi}\int_0^{2\pi} S\big(re^{i(\theta-\varphi)};\omega\big)d\sigma(\theta), \quad z = re^{i\varphi} \in \mathbb{D},$$

where $\sigma(t)$ is a function of bounded variation in $[0, 2\pi]$. One can see that $G_{\omega,\sigma}(z)$ is a harmonic function in $\mathbb{D}$ since the kernel $S(z;\omega)$ is holomorphic in $\mathbb{D}$. It is also easy to show that $G_{\omega,\sigma} \in h^1$.

Observe that the class of functions $u \in h^1$ coincides with the set of functions representable in the form

$$u(z) = L^{(\omega)}G_{\omega,\sigma}(z), \quad z \in \mathbb{D}, \tag{5}$$

where $\sigma(t)$ is a function of bounded variation in $[0, 2\pi]$ and it can be found by the Stieltjes inversion formula

$$\sigma(\theta) = \lim_{\rho\to 1-0}\int_0^{\theta} u(\rho e^{i\varphi})d\varphi, \quad \text{a.e.} \quad 0 \leqslant \theta \leqslant 2\pi. \tag{6}$$

In fact, recall that a function in h^1 can be written as [6, Theorem 1.1]

$$u(re^{\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} P(\theta-\varphi, r)d\sigma(\theta),$$

where $\sigma(\theta)$ is a function of bounded variation in $[0, 2\pi]$. By (4) we have

$$\begin{aligned} u(re^{\varphi}) &= \frac{1}{2\pi}\int_0^{2\pi} L^{(\omega)}\big\{\mathrm{Re}\, S(re^{i(\theta-\varphi)};\omega)\big\}d\sigma(\theta) \\ &= L^{(\omega)}\left(\mathrm{Re}\,\frac{1}{2\pi}\int_0^{2\pi} S\big(re^{i(\theta-\varphi)};\omega\big)d\sigma(\theta)\right). \end{aligned}$$

Corollary 2.3 *The set of functions $u \in h^1$ coincides with the set of functions representable in the form:*

$$u(z) = L^{(\omega)} G_{\omega,\sigma}(z) = L^{(\omega)} L^{(\widetilde{\omega})} u(z) \quad \textit{and} \quad L^{(\widetilde{\omega})} u(z) = G_{\omega,\sigma}(z), \quad z \in \mathbb{D},$$

therefore $L^{(\widetilde{\omega})}$ is a left and right inverse operator of $L^{(\omega)}$ over the harmonic Hardy space h^1 (Lemma 2.1). □

Proof By (5) we have $u(z) = L^{(\omega)} G_{\omega,\sigma}(z)$. Now applying the operator $L^{(\widetilde{\omega})}$ to both sides of the latter equality and using Lemma 2.1 we get $L^{(\widetilde{\omega})} u(z) = L^{(\widetilde{\omega})} L^{(\omega)} G_{\omega,\sigma}(z) = G_{\omega,\sigma}(z)$. □

3 A Description of the Harmonic Hardy Space H^1 in G^+

This section is devoted to prove that any function u of the harmonic Hardy space H^1 over G^+ can be represented as $u(z) = L_\omega K_{\omega,\mu}(z)$, where L_ω is a Hadamard-Liouville type operator and $K_{\omega,\mu}$ is an integral of a holomorphic Cauchy type kernel in G^+. We also show the existence and explicit form of the inverse operator which improves some results from [12]. An expository survey of the harmonic Hardy space in a half space can be found in [1] and for specific studies, see e.g. [3, 7, 16, 19, 20].

In this section, we assume that $\omega(x)$ is a function of the class Δ, i.e. $\omega(x) > 0$, is non-increasing in $(0, +\infty)$ and

$$\omega_1(x) = \int_0^x \omega(t)dt < +\infty, \quad 0 < x < +\infty.$$

For $\omega(x) \in \Delta$ and $u(z)$ given in G^+ we recall the Hadamard-Liouville type operator [11]

$$L_\omega u(z) = -L_{\omega_1} \frac{\partial}{\partial y} u(z), \quad \text{where} \quad L_{\omega_1} u(z) = \int_0^{+\infty} u(z + i\lambda) d\omega_1(\lambda).$$

Besides, we use the Cauchy type kernel

$$C_\omega(z) = \int_0^{+\infty} e^{izt} \frac{dt}{I_\omega(t)}, \quad \text{where} \quad I_\omega(t) = t \int_0^{+\infty} e^{-t\lambda} \omega(\lambda) d\lambda,$$

which is a holomorphic function in G^+, under the restriction [12, Lemma 2.1]:

$$\omega(x) \asymp x^\alpha \quad \text{for some} \quad -1 < \alpha < 0 \quad \text{and any} \quad x \geqslant \Delta_0 > 0$$

($\omega(x) \asymp x^\alpha$ means that $C_1 x^\alpha \leqslant \omega(x) \leqslant C_2 x^\alpha$ for some constants $C_{1,2} > 0$). Here we give an alternative proof of the holomorphicity of $C_\omega(z)$ without requiring the latter strong condition.

Lemma 3.1 *If $\omega(x) \in \Delta$ and $\int_{\Delta_0}^{+\infty} \frac{d\sigma}{\sigma^2\omega(\sigma)} < +\infty$ for some $\Delta_0 \in (0,+\infty)$, then the function $C_\omega(z)$ is holomorphic in G^+, and for any $\rho > 0$ there exists a constant $M_{\rho,\omega} > 0$ depending only on ρ and ω, such that*

$$|C_\omega(x+iy)| \leqslant M_{\rho,\omega}, \quad -\infty < x < +\infty, \quad y > \rho. \tag{7}$$

Proof Let $0 < \rho < y < +\infty$. Then for $0 < t < 1/\Delta_0$, we have

$$\begin{aligned}\left|\frac{e^{i(x+iy)t}}{I_\omega(t)}\right| &= \frac{e^{-yt}}{t\int_0^{+\infty} e^{-ut}\omega(u)du} \leqslant \frac{e^{-\rho t}}{t\int_0^{1/t} e^{-ut}\omega(u)du} \\ &\leqslant \frac{e^{-\rho t}}{\omega(1/t)t\int_0^{1/t} e^{-ut}du} = \frac{e^{-\rho t}}{\omega(1/t)(1-e^{-1})},\end{aligned}$$

while for $1/\Delta_0 < t < +\infty$, we obtain

$$\begin{aligned}\left|\frac{e^{i(x+iy)t}}{I_\omega(t)}\right| &= \frac{e^{-yt}}{t\int_0^{+\infty} e^{-xt}\omega(x)dx} \leq \frac{e^{-\rho t}}{t\int_0^{\Delta_0} e^{-xt}\omega(x)dx} \\ &\leqslant \frac{1}{\omega(\Delta_0)}\frac{e^{-\rho t}}{t\int_0^{\Delta_0} e^{-xt}dx} \leq \frac{1}{\omega(\Delta_0)}\frac{e^{-\rho t}}{1-e^{-t\Delta_0}}.\end{aligned}$$

Therefore, it yields

$$\begin{aligned}|C_\omega(z)| &\leqslant \frac{1}{(1-e^{-1})}\int_0^{1/\Delta_0}\frac{e^{-\rho t}dt}{\omega(1/t)} + \frac{1}{\omega(\Delta_0)}\int_{1/\Delta_0}^{+\infty}\frac{e^{-\rho t}dt}{1-e^{-t\Delta_0}} \\ &\leqslant \frac{1}{(1-e^{-1})}\int_{\Delta_0}^{+\infty}\frac{d\sigma}{\sigma^2\omega(\sigma)} + \frac{1}{\omega(\Delta_0)(1-e^{-1})}\frac{e^{-\rho/\Delta_0}}{\rho} < +\infty,\end{aligned}$$

for any $\rho > 0$ and then the integrand of the kernel C_ω has a summable majorant, which implies that is uniformly convergent in the half-plane $y > \rho$, and thus $C_\omega(z)$ is holomorphic in G^+. □

It is easy to verify that for any $z \in G^+$

$$\frac{\partial}{\partial \operatorname{Im} z}C_\omega(z) = -C_{\omega_1}(z) \quad \text{and} \quad C_\omega(z) = \int_{\operatorname{Im} z}^{+\infty} C_{\omega_1}(\operatorname{Re} z + it)dt,$$

where, $C_{\omega_1}(z)$ is assumed to be the Cauchy type kernel from [11]. Besides, in the particular case of power functions $\omega(x) = x^\alpha$, $-1 < \alpha < 0$, we have

$$C_\omega(z) = \frac{1}{(-iz)^{1+\alpha}} \quad \text{and} \quad C_\omega(z)\Big|_{\omega(x)=1} = \frac{1}{-iz} \quad z \in G^+,$$

i.e. $C_\omega(z)$ becomes the ordinary Cauchy kernel, and also

$$L_\omega C_\omega(z) = \frac{1}{-iz}, \quad z \in G^+.$$

We now recall an important theorem established by Solomentsev on subharmonic functions in G^+ which satisfy the condition (1).

Theorem 3.2 (E. D. Solomentsev [18]) *The class of subharmonic functions $u(z)$ in G^+ satisfying the condition (1) coincides with the set of functions representable in the form*

$$u(z) = \iint_{G^+} \log\left|\frac{z-\zeta}{z-\overline{\zeta}}\right| d\nu(\zeta) + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(x-t)^2+y^2}, \quad z = x+iy \in G^+,$$

where $\mu(t)$ is a function of bounded variation on $(-\infty, +\infty)$ and $\nu(\xi) \geq 0$ is a Borel measure on G^+, such that

$$\iint_{G^+} \operatorname{Im}\zeta \, d\nu(\zeta) < +\infty.$$

Let us now denote

$$K_{\omega,\mu}(z) := -\operatorname{Re}\frac{1}{\pi}\int_{-\infty}^{+\infty} C_\omega(z-t)d\mu(t), \quad z = x+iy \in G^+,$$

where $\mu(t)$ is a function of bounded variation on $(-\infty, +\infty)$. Notice that $K_{\omega,\mu}(z)$ is a harmonic function in G^+ by Lemma 3.1.

Let us now establish the main results of this section. Notice that for harmonic functions Solomentsev's theorem gives the representation with only the second integral since the Borel measure (so called charge) does not exist. Indeed, if one has such measure, then the first integral will be a subharmonic function, which yields a contradiction.

Theorem 3.3 *The harmonic Hardy space H^1 coincides with the class of functions representable in the form*

$$u(z) = L_\omega K_{\omega,\mu}(z), \quad z = x + iy \in G^+ \quad \text{and} \quad \omega \in \Delta,$$

where $\mu(t)$ *is a function of bounded variation on* $(-\infty, +\infty)$. □

Proof By Solomentsev's theorem we have

$$\begin{aligned}
u(z) &= \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(x-t)^2+y^2} = \text{Re}\,\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{t-z}\\
&= \text{Re}\,\frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\int_0^{+\infty} e^{i(z-t)\xi}d\xi\right]d\mu(t)\\
&= \text{Re}\,\frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\int_0^{+\infty} e^{i(z-t)\xi}\left(\frac{\int_0^{+\infty} e^{-s\xi}d\omega_1(s)}{\int_0^{+\infty} e^{-\sigma\xi}d\omega_1(\sigma)}\right)d\xi\right]d\mu(t)\\
&= \text{Re}\,\frac{1}{\pi}\int_0^{+\infty}\left[\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}\frac{e^{i(z+is-t)\xi}d\xi}{\int_0^{+\infty} e^{-\sigma\xi}d\omega_1(\sigma)}\right)d\mu(t)\right]d\omega_1(s)\\
&= \text{Re}\,\frac{1}{\pi}\int_0^{+\infty}\left[\int_{-\infty}^{+\infty} C_{\omega_1}(z+is-t)d\mu(t)\right]d\omega_1(s)\\
&= -\text{Re}\,\frac{1}{\pi}\int_0^{+\infty}\left[\frac{\partial}{\partial y}\int_{-\infty}^{+\infty} C_{\omega}(z+is-t)d\mu(t)\right]d\omega_1(s)\\
&= -\int_0^{+\infty}\left[\frac{\partial}{\partial y}K_{\omega,\mu}(z+is)\right]d\omega_1(s),
\end{aligned}$$

since the above integrals are absolutely and uniformly convergent inside G^+ as we now prove. First, notice that

$$\int_0^{+\infty} e^{-s\xi}d\omega_1(s) \geqslant \int_0^{y/2} e^{-s\xi}d\omega_1(s) \geqslant e^{-(y/2)\xi}\omega_1(y/2),$$

and

$$\begin{aligned}
\int_0^{+\infty} e^{-s\xi}d\omega_1(s) &= \int_0^{y/2} e^{-s\xi}d\omega_1(s) + \int_{y/2}^{+\infty} e^{-s\xi}d\omega_1(s)\\
&\leqslant \omega_1(y/2) + \omega(y/2)\frac{e^{-(y/2)\xi}}{\xi}.
\end{aligned} \tag{8}$$

Therefore, using the fact that $(y/2)\omega(y/2) \leqslant \int_0^{y/2} \omega(r)dr$, we get

$$\left| \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} e^{i(z-t)\xi} \left(\frac{\int_0^{+\infty} e^{-s\xi} d\omega_1(s)}{\int_0^{+\infty} e^{-\sigma\xi} d\omega_1(\sigma)} \right) d\xi \right] d\mu(t) \right|$$
$$\leqslant \bigvee_{-\infty}^{+\infty} \mu \cdot \int_0^{+\infty} \frac{e^{-y\xi}}{e^{-(y/2)\xi}\omega_1(y/2)} \left(\omega_1(y/2) + \omega(y/2) \frac{e^{-(y/2)\xi}}{\xi} \right) d\xi$$
$$\leqslant \bigvee_{-\infty}^{+\infty} \mu \cdot \int_0^{+\infty} e^{-(y/2)\xi} \left(1 + \frac{2}{y} \frac{e^{-(y/2)\xi}}{\xi} \right) d\xi \leqslant \frac{C}{y} \bigvee_{-\infty}^{+\infty} \mu \leqslant C_{\Delta_0} \bigvee_{-\infty}^{+\infty} \mu < +\infty,$$

for some fixed Δ_0 such that $y > \Delta_0 > 0$ and some positive constants C, C_{Δ_0}. □

The Hadamard–Liouville type operator has a left inverse operator, which is stated in [12, Lemma 4]. In the proof of the latter lemma it was required that

$$\omega(x) \asymp x^{\alpha} \quad \text{for some} \quad -1 < \alpha < 0 \quad \text{and any} \quad x \geqslant \Delta_0 > 0$$

which is a strong condition to control the behavior of the functional parameters ω near infinity. We now prove that the above condition is not necessary if we require that

$$\int_{\Delta_0}^{+\infty} \frac{d\sigma}{\sigma^2\omega(\sigma)} < +\infty, \text{ for some } \Delta_0 > 1, \text{ and } \lim_{\sigma\to+\infty} \frac{1}{\sigma\omega(\sigma)} = 0. \tag{9}$$

By $x^2 \leqslant e^{2x}$ (for any $x \geqslant 0$) and the convergence of the above integral, it implies that

$$\int_{\Delta_0}^{+\infty} \frac{e^{-t\sigma}d\sigma}{\omega(\sigma)} < +\infty, \tag{10}$$

is absolutely convergent for any $t > 0$. We also point out that the condition

$$\int_{\Delta_0}^{+\infty} \frac{d\sigma}{\sigma\omega(\sigma)} < +\infty, \text{ for some } \Delta_0 > 1, \tag{11}$$

is stronger than (9). The absolute convergence of the above integral implies that $\lim\limits_{\sigma\to+\infty} \dfrac{1}{\sigma\omega(\sigma)} = 0$ and also the absolute convergence of integral in (9). The simple example of $\omega \equiv 1$ satisfies the condition (9) but it fails for the condition (11).

The statement of the following lemma is proved in [12, Lemma 4] under some stronger conditions which we have avoided in this paper, and the proof remains almost the same, but we give it for the sake of completeness.

Lemma 3.4 *If $\omega \in \Delta$ satisfies the conditions (9), then the Volterra equation*

$$\int_0^x \omega(x-t)d\widetilde{\omega}(t) \equiv 1, \quad 0 < x < +\infty,$$

has a nondecreasing solution $\widetilde{\omega}(x)$ such that $\widetilde{\omega}(0) = 0$ and $\widetilde{\omega}(x) \leqslant [\omega(x)]^{-1}$ for all $0 < x < +\infty$. Besides,

$$C_\omega(z) = L_{\widetilde{\omega}}\left(\frac{1}{-iz}\right) = L_{\widetilde{\omega}} L_\omega C_\omega(z), \quad z \in G^+, \tag{12}$$

for the operator

$$L_{\widetilde{\omega}} f(z) \equiv \int_0^{+\infty} f(z + i\sigma)d\widetilde{\omega}(\sigma).$$

Proof By [9, Theorem 1.2] with $\Omega_1(x) \equiv 1$, $\Omega_2(x) \equiv \omega(x)$, we get the existence of the function $\widetilde{\omega}(x)$ with the desired properties. Therefore, we get

$$\int_0^{+\infty} e^{-t\mu}\left(\int_0^{\mu} \omega(\mu-\sigma)d\widetilde{\omega}(\sigma)\right)d\mu = 1/t, \quad 0 < t < +\infty. \tag{13}$$

Notice now that for any $t > 0$ we have

$$\begin{aligned}\int_0^{+\infty} e^{-t\sigma}d\widetilde{\omega}(\sigma)\int_0^{+\infty} e^{-t\lambda}\omega(\lambda)d\lambda &= \int_0^{+\infty} d\widetilde{\omega}(\sigma)\int_\sigma^{+\infty} e^{-t\mu}\omega(\mu-\sigma)d\mu \\ &= \int_0^{+\infty} e^{-t\mu}d\mu\int_0^{\mu}\omega(\mu-\sigma)d\widetilde{\omega}(\sigma)\end{aligned} \tag{14}$$

due to the absolute convergence of these integrals, which follows by

$$\begin{aligned}\int_0^{+\infty} e^{-t\sigma}d\widetilde{\omega}(\sigma) &= e^{-t\sigma}\widetilde{\omega}(\sigma)\Big|_{\sigma=0}^{+\infty} + t\int_0^{+\infty} e^{-t\sigma}\widetilde{\omega}(\sigma)d\sigma \\ &\leqslant t\int_0^{+\infty}\frac{e^{-t\sigma}d\sigma}{\omega(\sigma)} = t\left(\int_0^{\Delta_0} + \int_{\Delta_0}^{+\infty}\right)\frac{e^{-t\sigma}d\sigma}{\omega(\sigma)} \\ &\leqslant \frac{t}{\omega(\Delta_0)}\int_0^{\Delta_0} e^{-t\sigma}d\sigma + t\int_{\Delta_0}^{+\infty}\frac{e^{-t\sigma}d\sigma}{\omega(\sigma)}d\sigma < +\infty,\end{aligned}$$

since $\frac{e^{-t\sigma}}{\omega(\sigma)} \leqslant \frac{1}{t\sigma\omega(\sigma)}$, $\lim\limits_{\sigma\to+\infty}\dfrac{1}{\sigma\omega(\sigma)} = 0$, $\widetilde{\omega}(x) \leq [\omega(x)]^{-1}$ for all $0 < x < +\infty$ and the absolute convergence of integral (10) for any $t \in (0, +\infty)$. On the other hand, by (8) we also have

$$\int_0^{+\infty} e^{-t\lambda}\omega(\lambda)d\lambda \leqslant \omega_1(\delta_0) + \omega(\delta_0)\frac{e^{-\delta_0 t}}{t}, \quad \text{for some} \quad \delta_0 > 0.$$

By (13) and (14)

$$\frac{1}{I_\omega(t)} = \left(t\int_0^{+\infty} e^{-t\lambda}\omega(\lambda)d\lambda\right)^{-1} = \int_0^{+\infty} e^{-t\sigma}d\widetilde{\omega}(\sigma) \tag{15}$$

for any $t \in (0, +\infty)$, and consequently

$$C_\omega(z) = \int_0^{+\infty} e^{izt}\frac{dt}{I_\omega(t)} = \int_0^{+\infty} e^{izt}\left(\int_0^{+\infty} e^{-t\sigma}d\widetilde{\omega}(\sigma)\right)dt.$$

To complete the proof, it is necessary to see that for any $z \in G^+$

$$L_{\widetilde{\omega}}\left(\frac{1}{-iz}\right) = \int_0^{+\infty} d\widetilde{\omega}(\sigma)\int_0^{+\infty} e^{i(z+i\sigma)t}dt = \int_0^{+\infty} e^{izt}\left(\int_0^{+\infty} e^{-t\sigma}d\widetilde{\omega}(\sigma)\right)dt,$$

where the integrals are absolutely convergent since

$$\int_0^{+\infty} d\widetilde{\omega}(\sigma)\int_0^{+\infty}\left|e^{i(z+i\sigma)t}\right|dt \leqslant \int_0^{+\infty}\frac{d\widetilde{\omega}(\sigma)}{y+\sigma} < +\infty, \quad z = x+iy \in G^+,$$

due to the conditions in (9), we have

$$\begin{aligned}\int_0^{+\infty}\frac{d\widetilde{\omega}(\sigma)}{y+\sigma} &= \frac{\widetilde{\omega}(\sigma)}{y+\sigma}\bigg|_{\sigma=0}^{+\infty} + \int_0^{+\infty}\frac{\widetilde{\omega}(\sigma)d\sigma}{(y+\sigma)^2} \leqslant \left(\int_0^{\Delta_0} + \int_{\Delta_0}^{+\infty}\right)\frac{d\sigma}{(y+\sigma)^2\omega(\sigma)} \\ &\leqslant \frac{1}{\omega(\Delta_0)}\int_0^{\Delta_0}\frac{d\sigma}{(y+\sigma)^2} + \int_{\Delta_0}^{+\infty}\frac{d\sigma}{\sigma^2\omega(\sigma)} < +\infty.\end{aligned}$$

□

Remark It is clear that functions like $\omega(x) = 1/x^\alpha$ for $0 < \alpha < 1$ and $x > 0$ satisfy the conditions of Lemma 3.4. But, we also have functions like $\omega(x) = \frac{1}{x^\alpha \log(1+x)}$ for $0 < \alpha < 1$ and $x > 0$. In fact, Lemma 3.4 allows us to consider functions which possesses a different behavior as $\omega(x) = 1/x^\alpha$ for $0 < \alpha < 1$ and $x > 0$ in infinity. □

Remark Since the condition (11) is stronger than (9), it is obvious that:

If $\omega \in \Delta$ satisfy the condition (11), then the Volterra equation

$$\int_0^x \omega(x-t)d\widetilde{\omega}(t) \equiv 1, \quad 0 < x < +\infty,$$

has a nondecreasing solution $\widetilde{\omega}(x)$ such that $\widetilde{\omega}(0) = 0$ and $\widetilde{\omega}(x) \leqslant [\omega(x)]^{-1}$ for all $0 < x < +\infty$, which satisfies the equality (12). □

Theorem 3.5 *Let $\omega \in \Delta$ be such that the conditions (9) hold. For any function $u \in H^1$ we have*

$$u(z) = L_\omega L_{\widetilde{\omega}} u(z) = L_{\widetilde{\omega}} L_\omega u(z), \quad L_{\widetilde{\omega}} u(z) = K_{\omega,\mu}(z), \quad z = x + iy \in G^+,$$

i.e. $L_{\widetilde{\omega}}$ is a left and right inverse operator of L_ω over the harmonic Hardy space H^1, where $\mu(t)$ is a function of bounded variation on $(-\infty, +\infty)$. □

Proof By Theorem 3.3 we have

$$u(z) = L_\omega K_{\omega,\mu}(z) = -\mathrm{Re}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} L_\omega C_\omega(z-t) d\mu(t).$$

Applying the operator $L_{\widetilde{\omega}}$ to both sides of the above equality and using (12) we get

$$\begin{aligned} L_{\widetilde{\omega}} u(z) &= -\mathrm{Re}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} L_{\widetilde{\omega}} L_\omega C_\omega(z-t) d\mu(t) \\ &= -\mathrm{Re}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} C_\omega(z-t) d\mu(t) = K_{\omega,\mu}(z), \end{aligned}$$

which implies $u(z) = L_\omega L_{\widetilde{\omega}} u(z)$. On the other hand, as it was obtained in the proof of Theorem 3.3, we have

$$\begin{aligned} u(z) &= \mathrm{Re}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} e^{i(z-t)\xi} d\xi \right) d\mu(t) \\ &= \mathrm{Re}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} e^{i(z-t)\xi} \left(\frac{\int_0^{+\infty} e^{-s\xi} d\omega_1(s)}{\int_0^{+\infty} e^{-\sigma\xi} d\omega_1(\sigma)} \right) d\xi \right] d\mu(t). \end{aligned}$$

By equality (15) it follows that

$$\begin{aligned} u(z) &= \mathrm{Re}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} e^{i(z-t)\xi} \left(\frac{\int_0^{+\infty} e^{-s\xi} d\omega_1(s)}{\int_0^{+\infty} e^{-\sigma\xi} d\omega_1(\sigma)} \right) d\xi \right] d\mu(t) \\ &= \mathrm{Re}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} \\ &\left[\int_0^{+\infty} e^{i(z-t)\xi} \int_0^{+\infty} e^{-s\xi} d\omega_1(s) \left(\xi \int_0^{+\infty} e^{-\sigma\xi} d\widetilde{\omega}(\sigma) \right) d\xi \right] d\mu(t) \\ &= \mathrm{Re}\, \frac{1}{\pi} \int_0^{+\infty} d\widetilde{\omega}(\sigma) \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} \xi d\xi \int_0^{+\infty} e^{i(z-t+is+i\sigma)\xi} d\omega_1(s) \right] d\mu(t) \end{aligned}$$

$$= -\mathrm{Re}\,\frac{1}{\pi}\int_0^{+\infty} d\widetilde{\omega}(\sigma)\int_{-\infty}^{+\infty}$$

$$\left[\int_0^{+\infty} d\xi \int_0^{+\infty} \frac{\partial}{\partial y} e^{i(z-t+is+i\sigma)\xi} d\omega_1(s)\right] d\mu(t)$$

$$= -\int_0^{+\infty} d\widetilde{\omega}(\sigma)\int_0^{+\infty}\frac{\partial}{\partial y}$$

$$\left[\mathrm{Re}\,\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} e^{i(z-t+is+i\sigma)\xi} d\xi\right) d\mu(t)\right] d\omega_1(s)$$

$$= \int_0^{+\infty} d\widetilde{\omega}(\sigma)\cdot L_\omega\left(\mathrm{Re}\,\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} e^{i(z-t+i\sigma)\xi} d\xi\right) d\mu(t)\right)$$

$$= L_{\widetilde{\omega}} L_\omega\left(\mathrm{Re}\,\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} e^{i(z-t)\xi} d\xi\right) d\mu(t)\right) = L_{\widetilde{\omega}} L_\omega u(z).$$

□

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On the Solvability of the Synthesis Problem for Optimal Control Systems with Distributed Parameters

Akylbek Kerimbekov, Elmira Abdyldaeva, and Aitolkun Anarbekova

Abstract The solvability of synthesis problem of external and boundary controls is investigated for optimization of oscillation process, described by partial differential equations with Fredholm integral operator. Functions of the external and boundary actions are nonlinearly with respect to control. An integro-differential equation is obtained in the specific type for Bellman functional.

An algorithm is developed for constructing solutions to synthesis problem of external and boundary controls.

Keywords Generalized solution · Bellman functional · Frechet differential · Integro-differential equation · Fredholm operator · Optimal control synthesis

1 Introduction

In the study of applied problems by methods of the theory of optimal control of systems with distributed parameters, a special place is occupied by the problem of constructing positional control (the problem of control synthesis), where the desired control is defined as a function (or functional) of the state of the controlled process. The first studies in this direction were carried out for controlled processes described by partial differential equations in [1, 2]. After the appearance of the work

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of Professor A.I. Egorov [2], where the procedure for deriving a functional equation of the Bellman type was described using the example of thermal process control, it became possible to study the synthesis problem for controlled systems with distributed parameters. In doing so, he used the definition of a generalized solution of a boundary value problem of a controlled process and the Fréchet differential of the Bellman functional. According to the Bellman-Egorov scheme, studies were carried out in works [3–7].

This article investigates the solvability of the problem of synthesis of boundary and distributed (special structure) controls in the case of nonlinear optimization of an oscillatory process described by a partial integro-differential equation with a Fredholm integral operator. An algorithm for constructing boundary and distributed positional controls has been developed.

2 Formulation of Synthesis Problem

Consider a controlled oscillation process described by the following boundary value problem

$$v_{tt} - Av = \lambda \int_0^T K(t,\tau)v(\tau,x)d\tau + g(t,x)f[t,u(t)], \tag{1}$$

for $(t,x) \in Q_T = (0,T) \times Q$ with

$$v(0,x) = \psi_1(x), \quad v_t(0,x) = \psi_2(x) \tag{2}$$

for all $x \in Q$ and

$$\Gamma v(t,x) \equiv \sum_{i,k=1}^{n} a_{ik}(x) v_{x_k}(t,x)\cos(\mu, x_i) + a(x)v(t,x) = b(t,x)p[t,\vartheta(t)], \tag{3}$$

for $x \in \gamma$ and $0 < t < T$. The operator A is elliptic and defined by the formula

$$Av(t,x) = \sum_{i,k}^{n} (a_{ik}(x)v_{x_k}(t,x))_{x_i} - c(x)v(t,x), \tag{4}$$

and Q is a domain of $\mathbb{R}^n$ bounded by a piecewise smooth curve $\gamma = \partial Q$ and $a(x) \geq 0$ and $c(x) \geq 0$ are measurable functions. We denote $Q_T = (0, T) \times Q$ and $\gamma_T = (0, T) \times \gamma$. The functions $K(t, \tau) \in H(D)$ on $D = \{0 \leq t, \tau \leq T\}$ and $\psi_1(x) \in H_1(Q)$, $\psi_2(x) \in H(Q)$, $a_{ik}(x)$ are known. We denote by μ the outer normal vector from the point $x \in \gamma$. Furthermore, $f[t, u(t)] \in H(0, T)$ for all external controls $u(t) \in H(0, T)$ and $p[t, \vartheta(t)] \in H(0, T)$ for all boundary controls $\vartheta(t) \in H(0, T)$, where $H(Y)$ denotes the Hilbert space of square-summable functions defined on the set Y and $H_1(Y)$ is the Sobolev space of the first order.

The functions of external and boundary influences are assumed to be monotonic with regard to functional variable, i.e.

$$f_u[t, u(t)] \neq 0, \quad \forall t \in (0, T); \quad p_\vartheta[t, \vartheta(t)] \neq 0, \quad \forall t \in (0, T), \tag{5}$$

and $g(t, x) \in H(Q_T)$, $b(t, x) \in H(\gamma_T)$ are given functions.

Note that, according to conditions (5) a one-to-one correspondence is established between the elements of the control space $\{[u(t), \vartheta(t)]\}$ and the space of controlled process states $\{v(t, x)\}$.

It is required to find such controls $u^0(t) \in H(0, T)$ and $\vartheta^0(t) \in H(0, T)$ that minimize the functional

$$\begin{aligned} J[u(t), \vartheta(t)] = &\int_Q \{[v(T, x) - \xi_1(x)]^2 + [v_t(T, x) - \xi_2(x)]^2\} dx \\ &+ \int_0^T \{\alpha M^2[t, u(t)] + \beta N^2[t, \vartheta(t)]\} dt, \end{aligned} \tag{6}$$

where $\alpha, \beta > 0$ and $\xi_1(x) \in H(Q)$ $\xi_2(x) \in H(Q)$ are given functions, defined on set of solutions of boundary value problem (1)–(5).

The sought controls $u^0(t)$ and $\vartheta^0(t)$should be found as a functional of the controlled process's state, i.e. as

$$u^0(t) = u[t, v(t, x), v_t(t, x)], \quad t \in (0, T),$$

$$\vartheta^0(t) = \vartheta[t, v(t, x), v_t(t, x)], \quad t \in (0, T).$$

3 Generalized Solution of Boundary Value Problem

As it is known, in the study of applied control problems it is advisable to use the concept of a generalized solution of a boundary value problem.

Definition 1 A generalized solution of the boundary value problem (1)–(5) is a function $v(t,x) \in H(Q_T)$, that together with the generalized derivatives $v_t(t,x)$ and $v_{x_i}(t,x)$ satisfies the integral identity

$$\int_Q (v_t(t,x)\phi(t,x))\Big|_{t_1}^{t_2} dx = \int_{t_1}^{t_2} \{ \int_Q [v_t(t,x)\phi_t(t,x) - \sum_{i,k=1}^{n} a_{ik}(x) v_{x_k}(t,x)\phi_{x_i}(t,x) \\ - c(x)v(t,x)\phi(t,x) \\ + \big(\lambda \int_0^T K(t,\tau)v(\tau,x)d\tau \\ + g(t,x)f[t,u(t)]\big)\phi(t,x)]dx \\ + \int_\gamma (b(t,x)p[t,\vartheta(t)] - a(x)v(t,x))\phi(t,x)dx\}dt \tag{7}$$

for any t_1 and t_2 $(0 < t_1 \le t \le t_2 \le T)$ and any function $\phi(t,x) \in H_1(\overline{Q}_T)$, and satisfies the initial conditions in the weak sense, i.e. we have the next equalities

$$\lim_{t \to t_0} \int_Q [v(t,x) - \psi_1(x)]\phi_0(x)dx = 0,$$

$$\lim_{t \to t_0} \int_Q [v_t(t,x) - \psi_1(x)]\phi_1(x)dx = 0$$

for $\phi_0(x) \in H(Q)$ and $\phi_1(x) \in H(Q)$.

The solution of problem (1)–(5) we will seek in the form of

$$v(t,x) = \sum_{n=1}^{\infty} v_n(t) z_n(x), \quad v_n(t) = \int_Q v(t,x) z_n(x) dx, \tag{8}$$

where $z_j(x)$ are generalized eigenfunctions of the boundary value problem

$$B[\phi(t,x), z_j(x)] \equiv \int_Q \Big[\sum_{i,k=1}^{n} a_{ik}(x)\phi_{x_k}(t,x) z_{j_{x_i}}(x) + c(x) z_j(x)\phi(t,x) \Big] dx \\ + \int_\gamma a(x) z_j(x)\phi(t,x)dx \\ = \lambda_j^2 \int_Q \phi(t,x) z_j(x) dx, \\ \Gamma z_j(x) = 0, \ (t,x) \in \gamma_T, \quad j = 1,2,3,\ldots, \tag{9}$$

and they are form a complete orthonormal system in the Hilbert space $H(Q)$ and the corresponding eigenvalues λ_j satisfy the conditions $\lambda_j \leq \lambda_{j+1}$, $j = 1, 2, 3, \ldots$, $\lim \lambda_j = \infty$ at $j \to \infty$.

Using Liouville's method,we find the Fourier coefficients $v_n(t)$ for each fixed $n = 1, 2, 3, \ldots$, as a solution of a linear Fredholm integral equation of the second kind of the following form

$$v_n(t) = \lambda \int_0^T K_n(t, s)v_n(s)ds + a_n(t), \tag{10}$$

where

$$K_n(t, s) = \frac{1}{\lambda_n} \int_0^t \sin \lambda_n(t - \tau)K(\tau, s)d\tau, \quad K_n(0, s) = 0, \quad n = 1, 2, 3, \ldots,$$

and

$$\begin{aligned} a_n(t) =&\psi_{1n} \cos \lambda_n t + \frac{1}{\lambda_n}\psi_{2n} \sin \lambda_n t \\ &+ \frac{1}{\lambda_n} \int_0^t \sin \lambda_n(t - \tau)(g_n(\tau) f[\tau, u(\tau)] + b_n(\tau)p[\tau, \vartheta(\tau)])d\tau, \end{aligned} \tag{11}$$

together with

$$g_n(\tau) = \int_Q g[\tau, x]z_n(x)dx, \quad b_n(\tau) = \int_\gamma b[\tau, x]z_n(x)dx. \tag{12}$$

The solution to integral equation (10) is found by formula

$$v_n(t) = \lambda \int_0^T R_n(t, s, \lambda)a_n(s)ds + a_n(t), \tag{13}$$

where $R_n(t, s, \lambda)$ is the resolvent of kernel $K_n(t, s)$. The resolvent is a continuous function for the values of the parameter λ satisfying the following estimates for any $n = 1, 2, 3, \ldots$,

$$|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}, \tag{14}$$

where

$$K_0 = \int_0^T \int_0^T K^2(t, \tau)d\tau dt,$$

and

$$\int_0^T R_n^2(t,s,\lambda)ds \le \frac{K_0 T}{(\lambda_n - |\lambda| T\sqrt{K_0})^2}. \tag{15}$$

Thus, the solution to the boundary value problem (1)–(5) according to (8)–(9) has the form

$$v(t,x) = \sum_{n=1}^{\infty} \Big[\lambda \int_0^T R_n(t,s,\lambda)a_n(s)ds + a_n(t)\Big] z_n(x) \tag{16}$$

differentiating with respect to t, we obtain

$$v_t(t,x) = \sum_{n=1}^{\infty} \Big[\lambda \int_0^T R'_{nt}(t,s,\lambda)a_n(s)ds + a'_n(t)\Big] z_n(x).$$

Taking into account (11)–(15) and the inequality

$$\int_0^T R_{nt}^2(t,s,\lambda)ds \le \frac{T K_0 \lambda_n^2}{(\lambda_n - |\lambda| T\sqrt{K_0})^2}. \tag{17}$$

Based on these calculations, one can prove $v(t,x)$, $v_t(t,x) \in H(Q_T)$.

The solution (16) of the boundary value problem (1)–(5) we rewrite in the form

$$\begin{aligned} v(t,x) = \sum_{n=1}^{\infty} \Big\{ & \psi_n(t,\lambda) \\ & + \frac{1}{\lambda_n} \int_0^T \epsilon_n(t,\eta,\lambda)[g_n(\eta) f(\eta,u(\eta)) + b_n(\eta) p(\eta,\vartheta(\eta))]d\eta \Big\} z_n(x), \end{aligned} \tag{18}$$

where

$$\epsilon_n(t,\eta,\lambda) = \begin{cases} \sin\lambda(t-\eta) + \lambda \int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0 \le \eta \le t, \\ \lambda \int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t \le \eta \le T, \end{cases}$$

is continuous at $\eta = t$ and

$$\begin{aligned} \psi_n(t,\lambda) = & \psi_{1n}[\cos\lambda_n t + \lambda \int_0^T R_n(t,s,\lambda)\cos\lambda_n s ds] \\ & + \frac{\psi_{2n}}{\lambda_n}[\sin\lambda_n t + \lambda \int_0^T R_n(t,s,\lambda)\sin\lambda_n s ds]. \end{aligned} \tag{19}$$

Lemma 1 *The function $v(t,x)$ is an element of the Hilbert space $H(Q_T)$.*

Proof The assertion of the lemma follows from the inequality

$$\int_0^T \int_Q v^2(t,x)dxdt = \int_0^T \sum_{n=1}^{\infty} v_n^2(t)dt$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty} \Big\{\psi_n^2(t,\lambda) + \frac{2}{\lambda_n^2}[\int_0^T \epsilon_0^2(t,\eta,\lambda)g_n^2(\eta)d\eta \int_0^T f^2(\eta,u(\eta))d\eta$$

$$+\int_0^T \varepsilon_n^2(t,\eta,\lambda)b_n^2(\eta)d\eta \int_0^T p^2(\eta,\vartheta(\eta))]\Big\}d\eta$$

$$\leq 4T\sum_{n=1}^{\infty}\Big(1+\frac{\lambda^2 K_0 T^2}{(\lambda_n^2-|\lambda|\sqrt{K_0T^2})^2}\Big)\Big\{\psi_{1n}^2+\frac{\psi_{2n}^2}{\lambda_n^2}$$

$$+\frac{2}{\lambda_n^2}[\int_0^T g_n^2(\tau)d\tau\,\|f(t,u(t))\|^2_{H(0,T)} + \int_o^T b_n^2(\tau)d\tau\cdot\|p(t,\upsilon(t))\|^2_{H(0,T)}]\Big\}$$

$$\leq 4T\Big(1+\frac{\lambda^2K_0T^2}{(\lambda_1^2-|\lambda|\sqrt{K_0T^2})^2}\Big)\Big\{\|\psi_1(x)\|^2_{H(Q)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)}$$

$$+\frac{2}{\lambda_n^2}[\|g(t,x)\|^2_{H(0,T)}\,\|f(t,u(t))\|^2_{H(0,T)}$$

$$+\|b(t,x)\|^2_{H(0,T)}\,\|p(t,\vartheta(t))\|^2_{H(0,T)}]\Big\}<\infty,$$

i.e. $v(t,x) \in H(Q_T)$. □

Lemma 2 *The function $v_t(t,x)$ is an element of the Hilbert space $H(Q_T)$.*

Proof Differentiating by t the function (16) $v(t,x)$ we get

$$v_t(t,x) = \sum_{n=1}^{\infty} v_n'(t)z_n(x) = \sum_{n=1}^{\infty}\Big\{\psi_n'(t,\lambda)$$

$$+\frac{1}{\lambda_n}\int_0^T \epsilon_{nt}'(t,\eta,\lambda)[g_n(\eta)f(\eta,u(\eta))+b_n(\eta)p(\eta,\vartheta(\eta))]d\eta\}z_n(x)\Big\}$$

$$=\sum_{n=1}^{\infty}\Big\{\psi_{1n}\Big[-\lambda_n\sin\lambda_n t+\lambda\int_0^T R_{n_t}'(t,s,\lambda)\cos\lambda_n s ds\Big]$$

$$+\frac{\psi_{2n}}{\lambda_n}\Big[\lambda_n\cos\lambda_n t+\lambda\int_0^T R_{n_t}'(t,s,\lambda)\sin\lambda_n s ds\Big]$$

$$+\frac{1}{\lambda_n}\int_0^T \epsilon_{nt}'(t,\eta,\lambda)[g_n(\eta)f(\eta,u(\eta))+b_n(\eta)p(\eta,\vartheta(\eta))]d\eta\Big\}z_n(x),$$

where now

$$\epsilon'_{nt}(t,\eta,\lambda)=\begin{cases}\lambda_n\cos\lambda_n(t-\eta)+\lambda\int_\eta^T R'_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0\le\eta\le t,\\ \lambda\int_\eta^T R'_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t\le\eta\le T,\end{cases}$$

has at $\eta=t$ a jump equal to λ_n. The assertion of the lemma follows from the inequality

$$\int_0^T\int_Q v_t^2(t,x)dxdt=\int_0^T\int_Q\Big(\sum_{n=1}^\infty v'_n(t)z_n(x)\Big)^2dxdt=\int_0^T\sum_{n=1}^\infty v_n'^2(t)dt$$

$$\le 2\int_0^T\sum_{n=1}^\infty\Big\{\psi_{n_t}'^2(t,\lambda)+\frac{2}{\lambda^2}\int_0^T\epsilon_{n_t}'^2(t,\eta,\lambda)g_n^2(\eta)d\tau\int_0^T f^2(\eta,u(\eta))d\eta$$

$$+\int_0^T\varepsilon'_n(t,\eta,\lambda)b_n^2(\eta)\int_0^T p^2(\eta,\vartheta(\eta))\}d\eta\Big\}dt$$

$$\le 2\int_0^T\sum_{n=1}^\infty\Big\{2\psi_{1n}^2\Big[\lambda_n^2+\lambda^2\int_0^T R_{n_t}'^2(t,s,\lambda)ds\int_0^T\cos^2\lambda_n sds\Big]$$

$$+2\frac{\psi_{2n}^2}{\lambda_n^2}\Big[\lambda_n^2+\lambda^2\int_0^T R_{n_t}'^2(t,s,\lambda)ds\int_0^T\sin^2\lambda_n sds\Big]$$

$$+\frac{2}{\lambda_n^2}\int_0^T\epsilon_{n_t}'^2(t,\eta,\lambda)g_n^2(\eta)d\eta\int_0^T f^2(\eta,u(\eta))d\eta$$

$$+\int_0^T\varepsilon'_n(t,\eta,\lambda)b_n^2(\eta)\int_0^T p^2(\eta,\vartheta(\eta))\}d\eta\Big\}dt$$

$$\le 4\int_0^T\sum_{n=1}^\infty\Big\{\psi_{1n}^2\Big[\lambda_n^2+\lambda^2\frac{K_0T^2\lambda_n^2}{(\sqrt{\lambda_n^2}-|\lambda|\sqrt{K_0T^2}}\Big)^2\Big]$$

$$+\frac{\psi_{2n}^2}{\lambda_n^2}\Big[\lambda_n^2+\lambda^2\frac{K_0T^2\lambda_n^2}{(\sqrt{\lambda_n^2}-|\lambda|\sqrt{K_0T^2})^2}\Big]$$

$$+\frac{2}{\lambda_n^2}\Big[\lambda_n^2+\lambda^2\frac{K_0T^2\lambda_n^2}{(\sqrt{\lambda_n^2}-|\lambda|\sqrt{K_0T^2})^2}\Big]\int_0^T g_n^2(\eta)d\eta\,\|f(t,u(t))\|^2_{H(0,T)}$$

$$+\int_0^T b_n^2(\eta)d\eta\,\|p(t,\vartheta(t))\|^2_{H(0,T)}\Big\}dt$$

$$\le 4T\sum_{n=1}^\infty\Big(1+\frac{\lambda^2K_0T^2}{(\sqrt{\lambda_n^2}-|\lambda|\sqrt{K_0T^2})^2}\Big)\Big\{\lambda_n^2\psi_{1n}^2+\psi_{2n}^2+$$

$$+ \int_0^T g_n^2(\eta)d\eta \, \|f(t,u(t))\|^2_{H(0,T)}$$

$$+ \int_0^T b_n^2(\eta)d\eta \, \|p(t,\vartheta(t))\|^2_{H(0,T)} \Big\}$$

$$\leq 4T\left(1 + \frac{\lambda^2 K_0 T^2}{(\sqrt{\lambda_1^2} - |\lambda|\sqrt{K_0 T^2})^2}\right)\Big\{\|\psi_1(x)\|^2_{H_1(Q)}$$

$$+ \|\psi_2(x)\|^2_{H(Q)} + \|g(t,x)\|^2_{H(Q)} \, \|f(t,u(t))\|^2_{H(0,T)}$$

$$+ \|b(t,x)\|^2_{H(0,T)} \, \|p(t,\vartheta(t))\|^2_{H(0,T)} \Big\} < \infty$$

i.e. $v_t(t,x) \in H(Q_T)$. □

4 On Solvability of the Synthesis Problem

For functional (6), the Bellman functional is defined in the form

$$S[t,\omega(t,x)] = \min_{u\in U,\vartheta\in V} \Big\{ \int_t^T \big\{\alpha M^2[\tau,u(\tau)] + \beta N^2[\tau,\vartheta(\tau)]\big\}d\tau + \int_Q \| \omega(T,x) - \xi(x) \|^2 \, dx \Big\}, \tag{20}$$

where $\omega(t,x) = \{v(t,x), v_t(t,x)\}$ is the vector function of states; $\xi(x) = \{\xi_1(x), \xi_2(x)\}$ is the vector function of the desired state of the controlled process at the moment of time T; $\| \, . \, \|$ is the norm of vector; U is the set of allowed values of control $u(t)$, $t \in (0,T)$; V is the set of allowed values of control $\vartheta(t)$, $t \in (0,T)$. According to the Bellman-Egorov scheme, assuming that $S[t,\omega(t,x)]$ is differentiable function with respect to t, it is Frechet differentiable functional and can be rewritten as

$$-\frac{\partial S[t,\omega(t,x)]}{\partial t}\Delta t = \min_{u\in U,\vartheta\in V} \Big\{ \int_t^{t+\Delta t} \Big(\alpha M^2[\tau,u(\tau)] + \beta N^2[\tau,\vartheta(\tau)]\Big)d\tau + ds[t,\omega(t,x);\Delta\omega(t,x)] + o(\Delta t) + \delta[t,\omega(t,x);\Delta\omega(t,x)]\Big\}, \tag{21}$$

where $\Delta\omega(t,x) = \omega[t+\Delta t,x] - \omega[t,x]$, $ds[t,\omega(t,x);\Delta\omega(t,x)]$ is a Frechet differential, $\delta[t,\omega(t,x);\Delta\omega(t,x)]$ are infinitesimal values with respect to Δt.

As the Frechet differential is linear functional with respect to $\Delta\omega(t,x) \in H^2(Q_T) = H(Q_T) \times H(Q_T)$ the following equality holds

$$\begin{aligned} ds[t,\omega(t,x);\,\Delta\omega(t,x)] &= \int_Q m^*(t,x)\Delta\omega(t,x)dx \\ &\equiv \int_Q \Big\{ m_1(t,x)\Delta v(t,x) + m_2(t,x)\Delta v_t(t,x) \Big\} dx, \end{aligned} \tag{22}$$

where * is the transpose symbol; the vector-function $m(t,x) = \{m_1(t,x), m_2(t,x)\}$ is the gradient of the functional $S[t,\omega(t,x)]$ and belongs to the space $H^2(Q_T)$ in almost all $(t,x) \in Q_T$. Note, that $m(t,x)$ is defined depending on the functional $S[t,\omega(t,x)]$, i.e.

$$m(t,x) = m(t,x,S[t,\omega(t,x)]). \tag{23}$$

The following identity holds

$$\begin{aligned} \int_Q m^*(t,x)\Delta\omega(t,x)dx &= \int_Q (m_2(t,x)\Delta v_t(t,x))_t^{t+\Delta t} dx \\ &\quad + \int_Q m_1(t,x)\Delta v(t,x)dx \\ &\quad - \int_Q \Delta m_2(t,x) v_t(t+\Delta t,x)dx. \end{aligned} \tag{24}$$

Taking (22)–(24) into account equality (21) can be rewritten in form of

$$\begin{aligned} -\frac{\partial S[t,\omega(t,x)]}{\partial t}\Delta t = \min_{u\in U,\vartheta\in V} \Big\{ \int_t^{t+\Delta t} \Big[\alpha M^2[\tau,u(\tau)] \\ + \beta N^2[\tau,\vartheta(\tau)] \Big] d\tau + \int_Q (m_2(\tau,x)v_t(\tau,x))_t^{t+\Delta t} dx \\ + \int_Q [m_1(t,x)\Delta v(t,x) - \Delta m_2(t,x) v_t(t+\Delta t,x)]dx \\ + o(\Delta t) + \delta[t,\omega(t,x);\,\Delta\omega(t,x)] \Big\}. \end{aligned} \tag{25}$$

Let $m_2(t,x) \in H_1(Q_T)$. Then in the integral identity (7) assuming that $\phi(t,x) \equiv m_2(t,x)$ and $t_1 = t, t_2 = t + \Delta t$ we have

$$\int_Q (m_2(\tau,x)v_t(\tau,x))\Big|_t^{t+\Delta t} dt \equiv \int_t^{t+\Delta t} \Big\{ \int_Q \Big[m_{2t}(\tau,x)v_t(\tau,x)$$

$$- \sum_{i,k=1}^n a_{ik}(x)v_{x_k}(\tau,x)m_{2x_i}(\tau,x) - c(x)v(\tau,x)m_2(\tau,x)$$

$$+ \Big(\lambda \int_0^T K(\tau,\sigma)v(\sigma,x)d\sigma + g(\tau,x)f[\tau,u(\tau)]\Big)m_2(\tau,x)\Big]dx$$

$$+ \int_\gamma \Big(b(\tau,x)p[\tau,\vartheta(\tau)] - a(x)v(\tau,x)\Big)m_2(\tau,x)dx\Big\}d\tau.$$

Taking this identity into account, we rewrite the equality (25) in the form

$$-\frac{\partial S[t,\omega(t,x)]}{\partial t} = \min_{u\in U,\vartheta\in V} \Big\{ \frac{1}{\Delta t}\int_t^{t+\Delta t} \Big[\alpha M^2[\tau,u(\tau)] + \beta N^2[\tau,\vartheta(\tau)]\Big]d\tau$$

$$+ \frac{1}{\Delta t}\int_t^{t+\Delta t}\Big(\int_Q \Big[m_{2t}(\tau,x)v_t(\tau,x) - \sum_{i,k=1}^n a_{ik}(x)v_{x_k}(\tau,x)m_{2x_i}(\tau,x) -$$

$$- c(x)v(\tau,x)m_2(\tau,x) + \Big(\lambda\int_0^T K(\tau,\sigma)v(\sigma,x)d\sigma +$$

$$+ g(\tau,x)f[\tau,u(\tau)]\Big)m_2(\tau,x)\Big]dx + \int_\gamma \Big(b(\tau,x)p[\tau,\vartheta(\tau)] -$$

$$- a(x)v(\tau,x)\Big)m_2(\tau,x)dx\Big)d\tau + \int_Q \Big[m_1(t,x)\frac{\Delta v(t,x)}{\Delta t} -$$

$$- \frac{\Delta m_2(t,x)}{\Delta t}v_t(t+\Delta t,x)\Big]dx + \frac{o(\Delta t)}{\Delta t} + \frac{\delta[t,\omega(t,x);\Delta\omega(t,x)]}{\Delta t}\Big\}.$$

According the following relation

$$\lim_{t\to+0}\frac{o(\Delta t)}{\Delta t} = 0, \quad \lim_{t\to+0}\frac{\delta[t,\omega;\Delta\omega]}{\Delta t} = 0,$$

we obtain nonlinear integro-differential equation of Bellman-type

$$-\frac{\partial S[t,\omega(t,x)]}{\partial t} = \min_{u\in U,\vartheta\in Y}\Big\{\alpha M^2[t,u(t)] + \int_Q m_2(t,x)g(t,x)f[t,u(t)]dx +$$

$$+ \beta N^2[t,\vartheta(t)] + \int_\gamma m_2(t,x)b(t,x)p[t,\vartheta(t)]dx +$$

$$+\int_Q\left(\lambda\int_0^T K(t,\tau)v(\tau,x)d\tau\right)m_2(t,x)dx+\int_Q m_1(t,x)v_t(t,x)dx-$$
$$-\int_Q\Big[\sum_{i,k=1}^n a_{ik}(x)v_{x_k}(t,x)m_{2_{x_i}}(t,x)+c(x)v(t,x)m_2(t,x)\Big]dx-$$
$$-\int_\gamma a(x)v(t,x)m_2(t,x)dx\Big\}. \qquad (26)$$

According to (20) this equation should be considered with the condition

$$S[T,\omega(T,x)]=\int_Q \parallel \omega(T,x)-\xi(x)\parallel^2 dx. \qquad (27)$$

Thus, function $S[t,\omega(t,x)]$ should be found as a solution to problem (26)–(27), which is called the Cauchy-Bellman problem. To solve this problem, we first solve the minimization problem in the right-hand side of Eq. (25).

Consider the case when U and V are open sets. Using the classical method for solving the extremum problem, we find that the "controls suspicious for optimality" $u^0(t)$ are obtained as follows.

The desired control $u^0(t)$ is determined according to the optimality conditions in the form of the equality

$$2\alpha M[t,u(t)]M_u[t,u(t)]+\int_Q m_2(t,x)g(t,x)dxf_u[t,u(t)]=0 \qquad (28)$$

and a differential inequality

$$2\alpha\Big(M[t,u(t)]M_u[t,u(t)]\Big)_u+\int_Q m_2(t,x)g(t,x)dxf_{uu}[t,u(t)]>0$$

which are fulfilled simultaneously for almost all $(t,x)\in Q_T$. A differential inequality is a difficult condition to verify. However, it can be transformed to the form of

$$f_u[t,u(t)]\Big(\frac{M[t,u(t)]M_u[t,u(t)]}{f_u[t,u(t)]}\Big)_u>0. \qquad (29)$$

Let the optimality conditions (28) and (29) be satisfied. Then, according to the implicit function theorem from equality (28), the control $u(t)$ is uniquely determined, that is, there exists a function φ_1 such that

$$u^0(t)=\varphi_1[t,\int_Q m_2(t,x)g(t,x)dx,\alpha]$$

for $t\in(0,T)$.

Similarly, the "boundary controls suspicious for optimality" $\vartheta^0(t)$ are obtained as follows.

According to the optimality conditions in the form of the equality

$$2\beta N[t, \vartheta(t)]N_\vartheta[t, \vartheta(t)] + \int_\gamma m_2(t, x)\epsilon(t, x)p[t, \vartheta(t)]dx = 0$$

and the differential inequality

$$p_\vartheta[t, \vartheta(t)]\Big(\frac{N[t, \vartheta(t)]N_\vartheta[t, \vartheta(t)]}{p_\vartheta[t, \vartheta(t)]}\Big)_\vartheta > 0$$

the desired control $\vartheta^0(t)$ is determined by the formula

$$\vartheta^0(t) = h_1[t, \int_\gamma m_2(t, x)\epsilon(t, x)dx, \beta]$$

for $t \in (0, T)$ and a suitable function h_1.

5 Conclusion

This article shows some features of the considered synthesis problem, in particular, the presence of the Fredholm integral operator in the boundary value problem significantly affects the solvability of the Cauchy-Bellman problem, and as well as on the construction of an algorithm for synthesizing controls depending on the state of the controlled process. The results obtained can be used in the development of new research methods and methods for solving nonlinear synthesis problems.

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On the Determination of a Coefficient of an Elliptic Equation via Partial Boundary Measurement

Hyeonbae Kang, June-Yub Lee, and Igor Trooshin

Abstract We consider an inverse problem to identify coefficient of elliptic equation via partial boundary measurement when the given domain is a rectangle and the coefficient depends only on one variable. We prove unique identifiability and provide reconstruction procedure in this case using classical results of the inverse Sturm–Liouville theory.

1 Introduction

In this paper we are interested in the inverse problem to identify a unknown potential of stationary Schrödinger equation by means of the partial Dirichlet-to-Neumann (DtN) map.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, with the connected boundary $\partial\Omega$. Let Γ_1 and Γ_2 be open connected subsets of $\partial\Omega$. Γ_1 is the part of $\partial\Omega$ where the input is assigned and Γ_2 is where measurement is made. The partial DtN map Λ_q is defined by

$$\Lambda_q(f) := \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_2}, \quad \text{supp}(f) \subset \Gamma_1, \tag{1}$$

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where u is the solution of the following problem:

$$\begin{cases} \Delta u + qu = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f \in H^1(\partial\Omega). \end{cases} \tag{2}$$

The inverse problem is to determine q by means of Λ_q. This inverse problem is closely related to the inverse problem to identify an unknown conductivity by means of the partial Dirichlet-to-Neumann (DtN) map.

The conductivity distribution γ of Ω is a continuous function on Ω satisfying $\gamma(x) > c_0, x \in \Omega$, for some $c_0 > 0$. The partial DtN map $\Lambda_\gamma^{\Gamma_1,\Gamma_2}$ is defined to be

$$\Lambda_\gamma^{\Gamma_1,\Gamma_2}(f) := \gamma \frac{\partial u}{\partial \nu}\bigg|_{\Gamma_2}, \quad f \in H^{1/2}(\partial\Omega), \quad \text{supp}(f) \subset \Gamma_1, \tag{3}$$

where u is the solution of the problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f. \end{cases} \tag{4}$$

By the well-known Liouville transform $q = -\Delta\sqrt{\gamma}/\sqrt{\gamma}$, the inverse conductivity problem can be transformed to our problem under investigation and the corresponding DtN maps are related via

$$\Lambda_q(f) = \frac{1}{2}\gamma^{-1}\frac{\partial \gamma}{\partial \nu} f + \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2} f). \tag{5}$$

If $\Gamma_1 = \Gamma_2 = \partial\Omega$, then there is a well-established theory on uniqueness and reconstruction of the conductivity [1, 2, 7, 11, 12, 16–18]. However, in practice some part of $\partial\Omega$ is inaccessible, and hence the problem of identification via a partial DtN map has practical significance [3, 5, 6, 8–10]. These results mentioned are mainly concerned with infinitely many measurements, but in our paper we are dealing with a single partial measurement.

Let us now turn to a very special situation. Let $\Omega = (0, \pi) \times (0, L) \subset \mathbb{R}^2$ for some $L > 0$. We assume that the conductivity γ depends only on x, i.e., $\gamma(x, y) = \gamma(x)$. Let

$$\Gamma_l = \{0\} \times [0, L], \quad \Gamma_r = \{\pi\} \times [0, L], \quad \text{and} \quad \Gamma_b = [0, \pi] \times \{0\}.$$

When $\Gamma_1 = \Gamma_l \cup \Gamma_r$ and $\Gamma_2 = \Gamma_b$, it is well known that uniqueness holds. In fact, $\Lambda_\gamma^{\Gamma_1,\Gamma_2}(f)$ for a single nonzero f determines γ in an explicit simple formula [11]. One probable reason for this is that one can see inside Ω through the boundary portion Γ_b since γ depends only on x. Thus in this case the genuine boundaries seem to be Γ_l and Γ_r.

In this paper we deal with the Schrödinger equation when $q(x, y) = q(x)$ and $\Gamma_1 = \Gamma_2 = \Gamma_l = \{0\} \times [0, L]$ in terms of the measurement corresponding to a single Dirichlet datum.

We specify the Dirichlet data to be used: $\text{supp} f \subset \Gamma_l$ and $f \in H^1(\partial\Omega)$. We expand the given Dirichlet data f in terms of the Fourier sine series

$$f(y) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi y}{L}$$

and denote by K the family of those functions $f \in H^1(\partial\Omega)$ with support in Γ_l such that there exists a sequence of natural numbers $k(n)$, $n \geq N$ for some fixed N, with the property

$$f_{k(n)} \neq 0, \qquad \sum_{n=N}^{\infty} \frac{1}{k(n)} = \infty. \tag{6}$$

For example, we can take the following sequence

$$k(n) = \frac{n}{\sigma}(1 + \epsilon_n), \quad 0 < \sigma \leq 1, \quad \epsilon_n \to 0.$$

The set K is not empty for arbitrary small support of Dirichlet data f. For example, the function $f(x) = (x - x_0)^2 - \epsilon^2$ for $x \in [x_0 - \epsilon, x_0 + \epsilon]$ and $f(x) = 0$ for $x \in [0, x_0 - \epsilon] \cup [x_0 + \epsilon, L]$ belongs to K and its support can be arbitrary small. Let us also mention that linear combinations of basic trigonometric functions do not belong to K.

The main result of this paper is the following:

Theorem 1 *Suppose that $q(x) \in L^2(0, \pi)$ and $\tilde{q}(x) \in L^2(0, \pi)$ are functions depending only on x-variable, and that $\Gamma_1 = \Gamma_2 = \Gamma_l$. If $\Lambda_q(f) = \Lambda_{\tilde{q}}(f)$ for a single $f \in K$, then $q(x) = \tilde{q}(x)$ almost everywhere in Ω.*

To prove this theorem we use the fact (see appendix) that the solution u of (28) is given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \frac{n\pi y}{L},$$

where $u_n(x)$, $n \in \mathbb{N}$, are the solutions of the problems

$$\begin{cases} u_n'' + q(x)u_n = \dfrac{n^2\pi^2}{L^2} u_n, \\ u_n(0) = f_n, \quad u_n(\pi) = 0. \end{cases} \tag{7}$$

Then DtN map is then represented as

$$\Lambda_q(f)(0, y) = -\sum_{n=1}^{\infty} u_n'(0) \sin \frac{n\pi y}{L}. \tag{8}$$

Coefficients $u_n'(0)$ can be found uniquely from $\Lambda_q(f)(0, y)$. Then the inverse problem is to determine $q(x)$ in terms of the data $\{u_n(0) = f_n,\ u_n'(0) : n \in \mathbb{N}\}$ and Theorem 1 follows directly from the following theorem

Theorem 2 *Let $u_n(x)$, $n \in \mathbb{N}$, be the solutions of (7), and $\tilde{u}_n(x)$, $n \in \mathbb{N}$, be the solution of (7) with $q(x)$ replaced with $\tilde{q}(x)$. If $k(n)$ is defined by (6), $f_{k(n)} \neq 0$ and $u_{k(n)}'(0) = \tilde{u}_{k(n)}'(0)$ for all $n \in \mathbb{N}$, then $q(x) = \tilde{q}(x)$ almost everywhere in $(0, \pi)$.*

In the case when the coefficient $q(x)$ is an analytic function we can use the method of standard models [4] to reconstruct the coefficient $q(x)$. Let us suppose that it is known a priori, that

$$q(x) = \sum_{k=0}^{\infty} q_k \frac{x^k}{k!} \tag{9}$$

with the radius of convergence $\geq \pi$. In such situation it is sufficient to reconstruct the sequence of coefficients $q_k,\ \ k = 0, \ldots, \infty$. To do so, we can employ the following algorithm: First

$$q_0 = 2 \lim_{l \to \infty} \left(\frac{k(l)\pi}{L} \right)^3 \left(\frac{u_{k(l)}(0)}{u_{k(l)}'(0)} + \frac{L}{k(l)\pi} \tanh \frac{k(l)\pi}{L} \right). \tag{10}$$

Next we find recurrently

$$q_n = 2^{n+1} \lim_{l \to \infty} \left(\frac{k(l)\pi}{L} \right)^{n+3} \left[\frac{u_{k(l)}(0)}{u_{k(l)}'(0)} - m_n \left(- \left(\frac{k(l)\pi}{L} \right)^2 \right) \right] \tag{11}$$

Here $m_n(\lambda) = -s_n(\pi, \lambda)/c_n(\pi, \lambda)$, where $s_n(x, \lambda)$ and $c_n(x, \lambda)$ are solutions to the equation $-y'' - q_n(x)y = \lambda y$ subject to initial conditions $c_n(0, \lambda) = s_n'(0, \lambda) = 1$, $c_n'(0, \lambda) = s_n(0, \lambda) = 0$ with

$$q_n(x) = \sum_{k=0}^{n-1} q_k \frac{x^k}{k!}. \tag{12}$$

Theorem 2 is proved in the next section. In Sect. 3 we justify the reconstruction algorithm (10)–(11).

2 Inverse Spectral Problem

Here we prove the Theorem 2. For $\lambda \in \mathbb{C}$, let $F(x, \lambda)$ be the solution to the boundary value problem

$$\begin{cases} -F''(x, \lambda) - q(x)F(x, \lambda) = \lambda F(x, \lambda), \\ F'(0, \lambda) = 1, \; F(\pi, \lambda) = 0. \end{cases} \tag{13}$$

The function $F(x, \lambda)$ is called the Weyl solution and the function $m(\lambda) = F(0, \lambda)$ is called the Weyl function of the boundary problem

$$\begin{cases} -y'' - q(x)y = \lambda y, \\ y'(0) = y(\pi) = 0. \end{cases} \tag{14}$$

It is well known that the Weyl function $m(\lambda)$, $\lambda \in \mathbb{C}$, determines the potential $q(x)$ of the problem (14) uniquely. (See, for example, [4, pp. 29–31]).

The Weyl function m can be represented as

$$m(\lambda) = -\frac{s(\pi, \lambda)}{c(\pi, \lambda)}, \tag{15}$$

where $s(x, \lambda)$ and $c(x, \lambda)$ are solutions to the equation

$$-y'' - q(x)y = \lambda y \tag{16}$$

subject to initial conditions

$$c(0, \lambda) = s'(0, \lambda) = 1, \quad c'(0, \lambda) = s(0, \lambda) = 0. \tag{17}$$

To show (15) we introduce the solution $\chi(x, \lambda)$ to (16) subject to initial conditions

$$\chi(\pi, \lambda) = 0, \quad \chi'(\pi, \lambda) = -1.$$

Since functions $F(x, \lambda)$ and $\chi(x, \lambda)$ satisfy the same boundary condition on the right side of the interval, we have

$$F(x, \lambda) = a(\lambda)\chi(x, \lambda)$$

for every $\lambda \in \mathbb{C}$. Let us find that $a(\lambda)$. By differentiating $\delta(x, \lambda) = \chi(x, \lambda)c'(x, \lambda) - \chi'(x, \lambda)c(x, \lambda)$ with respect to x, one can see immediately that $\delta(x, \lambda)$ is independent of x, and hence

$$c(\pi, \lambda) = -\chi'(0, \lambda).$$

Since

$$1 = F'(0, \lambda) = a(\lambda)\chi'(0, \lambda) = -a(\lambda)c(\pi, \lambda),$$

we get

$$F(x, \lambda) = -\chi(x, \lambda)/c(\pi, \lambda), \tag{18}$$

and hence

$$m(\lambda) = F(0, \lambda) = -\chi(0, \lambda)/c(\pi, \lambda). \tag{19}$$

On the other hand, one can also see that $\chi(x, \lambda)s'(x, \lambda) - \chi'(x, \lambda)s(x, \lambda)$ is independent of x, and hence

$$s(\pi, \lambda) = \chi(0, \lambda).$$

Thus we get (15) from (19).

The solutions $u_n(x)$ and $F(x, -\left(\frac{n\pi}{L}\right)^2)$ satisfy the same boundary condition $u_n(\pi) = F(\pi, -\left(\frac{n\pi}{L}\right)^2) = 0$. As result, the solution $u_n(x)$ to (7) may be represented as

$$u_n(x) = u_n'(0)F\left(x, -\left(\frac{n\pi}{L}\right)^2\right). \tag{20}$$

The Sturm-Liouville problem (14) possesses only finitely many negative eigenvalues (see [15, p. 38], [14, p. 7]).

So we may assume, by excluding corresponding $k(n)$ from further consideration, that all of $-\left(\frac{k(n)\pi}{L}\right)^2$ are not eigenvalues of (14) (and the corresponding problem, where the potential $q(x)$ is replaced with $\tilde{q}(x)$).

After such adjustment we have $u'_{k(n)}(0)u_{k(n)}(0) \neq 0$ for all n. Then it follows from the coincidence $u'_{k(n)}(0) = \tilde{u}'_{k(n)}(0)$ and $u_{k(n)}(0) = \tilde{u}_{k(n)}(0)$ and (20) that

$$\frac{u_{k(n)}(0)}{u'_{k(n)}(0)} = F\left(0, -\left(\frac{n\pi}{L}\right)^2\right) = m\left(-\frac{k(n)^2\pi^2}{L^2}\right) = \tilde{m}\left(-\frac{k(n)^2\pi^2}{L^2}\right), \quad n \in \mathbb{N}, \tag{21}$$

where $\tilde{m}$ is the Weyl function corresponding to the potential $\tilde{q}$. Thus, to complete the proof it is sufficient to show that (21) implies that

$$m(\lambda) = \tilde{m}(\lambda) \quad \text{for all } \lambda \in \mathbb{C}. \tag{22}$$

Let $\tilde{c}(x, \lambda)$ and $\tilde{s}(x, \lambda)$ be the solutions of (16) with $q(x)$ replaced with $\tilde{q}(x)$ satisfying the same initial conditions as $c(x, \lambda)$ and $s(x, \lambda)$, respectively. Then $\tilde{m}(\lambda) = -\tilde{s}(\pi, \lambda)/\tilde{c}(\pi, \lambda)$. Let

$$G(\rho) := [\tilde{m}(\rho^2) - m(\rho^2)]c(\pi, \rho^2)\tilde{c}(\pi, \rho^2). \tag{23}$$

By (15) we get

$$G(\rho) = s(\pi, \rho^2)\tilde{c}(\pi, \rho^2) - c(\pi, \rho^2)\tilde{s}(\pi, \rho^2). \tag{24}$$

and we see, that $G(\rho)$ has $i\frac{k(n)\pi}{L}$ as its zeros.

It is well known (see, for example, [4, pp. 8, 13]) that $s(\pi, \rho^2)$, $c(\pi, \rho^2)$ and consequently $G(\rho)$ is entire function of exponential type and is bounded on the real axis.

To show that $G(\rho) \equiv 0$ for $\rho \in \mathbb{C}$, we now invoke the following corollary of Carleman's lemma (see [13, pp. 222–223]).

Theorem 3 (Corollary of Carleman's Lemma) *If $f(z)$ is an entire nonzero function of exponential type, bounded on the real axis, and if $z_n = r_n e^{i\theta_n}$ $(n = 1, 2, 3, \ldots)$ are the zeros of $f(z)$ other than $z = 0$, then the series*

$$\sum_{n=1}^{\infty} \frac{\sin \theta_n}{r_n}$$

is absolutely convergent.

Since $G(\rho)$ has $i\frac{k(n)\pi}{L}$ as its zeros and the series $\sum_{n=1}^{\infty} \frac{L}{k(n)\pi}$ does not converge because of (6), we have $G(\rho) \equiv 0$ and consequently $\tilde{m}(\rho^2) \equiv m(\rho^2)$ and it completes the proof of the Theorem 2.

3 Reconstruction Procedure

In the case of an analytic coefficient $q(x)$ we apply the method of standard models [4, pp. 74–77] to reconstruct the coefficient $q(x)$. Let us suppose that it is known a priori, that

$$q(x) = \sum_{k=0}^{\infty} q_k \frac{x^k}{k!}$$

with radius of convergence $\geq \pi$. In such situation it is sufficient to reconstruct the sequence of coefficients q_k, $k = 0, \ldots, \infty$.

We now take arbitrary analytic coefficient $\tilde{q}_0(x)$ and sequence of arbitrary analytic coefficients $\tilde{q}_n(x),\ \ n \geq 1$, such that the first n Maclaurin coefficients of $q(x)$ and $\tilde{q}_n(x)$ coincide. As the coefficients $\tilde{q}_{nk}$, $k \geq n$ are arbitrary, we can put $\tilde{q}_{nk} = 0$, $k \geq n$, and obtain

$$\tilde{q}_n(x) = \sum_{k=0}^{n-1} q_k \frac{x^k}{k!} + \sum_{k=n}^{\infty} \tilde{q}_{nk} \frac{x^k}{k!}.$$

Denote $\tilde{F}_n(x, \lambda)$ and $\tilde{m}_n(\lambda)$ the Weyl solution and Weyl function respectively of problem (14) with q replaced with $\tilde{q}_n$

$$\begin{cases} -\tilde{F}_n''(x, \lambda) - \tilde{q}_n(x)\tilde{F}_n(x, \lambda) = \lambda \tilde{F}_n(x, \lambda), \\ \tilde{F}_n'(0, \lambda) = 1, \, \tilde{F}_n(\pi, \lambda) = 0. \end{cases} \tag{25}$$

Multiplying (13) by $\tilde{F}_n(x, \lambda)$ and (25) by $F(x, \lambda)$, subtracting and integrating we see, that

$$\int_0^{\pi} (q(x) - \tilde{q}_n(x)) F(x, \lambda) \tilde{F}_n(x, \lambda) dx = \tilde{m}_n(\lambda) - m(\lambda). \tag{26}$$

Denote $D = \{\rho : \arg \rho \in [\delta, \pi - \delta]\}$, $\delta > 0$. It follows directly from representation (18), that Weyl solution has the following asymptotic behavior,

$$F(x, \lambda) = \frac{1}{i\rho} e^{i\rho x} \left(1 + \frac{\psi(x, \rho)}{\rho}\right),$$

where the function $\psi(x, \rho)$ is continuous and bounded for $x \in [0, \pi]$, $\rho \in D_\delta = \{\rho : \arg \rho \in [\delta, \pi - \delta]\}$, $|\rho| \geq \rho^\star$ with fixed sufficiently large $\rho^\star > 0$ and arbitrary fixed $\delta > 0$.

By the choice of the $\tilde{q}_n(x)$ we have

$$q(x) - \tilde{q}_n(x) = \frac{x^n}{n!}(q_n - \tilde{q}_{nn} + p(x)), \quad p(x) \in C[0, 1], \quad p(0) = 0.$$

Then we can calculate (see [4, p. 77])

$$\int_0^{\pi} (q(x) - \tilde{q}_n(x)) F(x, \lambda) \tilde{F}_n(x, \lambda) dx = \frac{(-1)^n 4}{(2i\rho)^{n+3}} (q_n - \tilde{q}_{nn} + o(1))$$

for $\rho \in D_\delta,\ |\rho| \to \infty$ and, consequently, from (26)

$$q_n = \tilde{q}_{nn} + \frac{(-1)^{n+1}}{4} \lim_{|\rho| \to \infty, \rho \in D} (2i\rho)^{n+3} (m(\lambda) - \tilde{m}_n(\lambda)).$$

Let us remind now (see (21)), that

$$m\left(-\frac{k(n)^2\pi^2}{L^2}\right) = \frac{u_{k(n)}(0)}{u'_{k(n)}(0)}.$$

It allows us to find immediately

$$q_n = \tilde{q}_{nn} + 2^{n+1} \lim_{l\to\infty} \left(\frac{k(l)\pi}{L}\right)^{n+3} \left[\frac{u_{k(l)}(0)}{u'_{k(l)}(0)} - \tilde{m}_n\left(-\left(\frac{k(l)\pi}{L}\right)^2\right)\right].$$

If we take $\tilde{q}_0(x) = 0$, we can calculate directly

$$q_0 = 2 \lim_{l\to\infty} \left(\frac{k(l)\pi}{L}\right)^3 \left(\frac{u_{k(l)}(0)}{u'_{k(l)}(0)} + \frac{L}{k(l)\pi} \tanh \frac{k(l)\pi}{L}\right)$$

and then, taking $\tilde{q}_{nk} = 0$, $k \geq n$ we come to the recurrent formulae (10)–(11).

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Appendix: Representation of Solution to Schrödinger Equation

In the case $f \in H^1(\partial\Omega)$, supp$f \subset \Gamma_l = \{0\} \times [0, L]$ we expand

$$f(0, y) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi y}{L}, \tag{27}$$

with series uniformly convergent over $[0, L]$.

Theorem 4 *In the case* $f \in H^1(\partial\Omega)$, *supp*$f \subset \Gamma_l = \{0\} \times [0, L]$ *the solution* $u(x, y)$ *of*

$$\begin{cases} \Delta u + qu = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases} \tag{28}$$

could be represented as uniformly convergent in Ω *series*

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \frac{n\pi y}{L}, \tag{29}$$

where $u_n(x)$, $n \in \mathbb{N}$, are the unique solutions of the boundary problems

$$\begin{cases} u_n'' + q(x)u_n = \dfrac{n^2\pi^2}{L^2}u_n, \\ u_n(0) = f_n, \quad u_n(\pi) = 0. \end{cases} \tag{30}$$

Proof We put

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y), \ u_n(x, y) = u_n(x) \sin \frac{n\pi y}{L}, \tag{31}$$

As $u_n(x, y)$ satisfy Eq. (28), to show that $u(x, y_1, \dots, y_{N-1})$ satisfy Eq. (28) it is sufficient to prove that series consisting of $u_n(x, y)$ and its corresponding partial derivatives converge uniformly on $[\epsilon, \pi] \times [0, L]$ for every $\epsilon > 0$.

To show that we note that

$$u_n(x) = f_n \Phi\left(x, -\left(\frac{n\pi}{L}\right)^2\right)$$

where $\Phi(x, \lambda)$ is the solution to the boundary value problem

$$\begin{cases} -\Phi''(x, \lambda) - q(x)\Phi(x, \lambda) = \lambda\Phi(x, \lambda), \\ \Phi(0, \lambda) = 1, \ \Phi(\pi, \lambda) = 0. \end{cases}$$

As in the proof of the Theorem 2, we show that

$$\Phi(x, \lambda) = -\chi(x, \lambda)/s(\pi, \lambda),$$

with $s(x, \lambda)$ and $\chi(x, \lambda)$ being solutions to the equation

$$-y'' - q(x)y = \lambda y, \tag{32}$$

subject to initial conditions

$$\begin{aligned} s(0, \lambda) &= 0, \quad s'(0, \lambda) = 1, \\ \chi(\pi, \lambda) &= 0, \quad \chi'(\pi, \lambda) = -1. \end{aligned} \tag{33}$$

and hence

$$u_n(x) = f_n \chi\left(x, -\left(\frac{n\pi}{L}\right)^2\right) / s\left(\pi, -\left(\frac{n\pi}{L}\right)^2\right).$$

There are known asymptotic formulas (see e.g. [4, pp. 8, 13])

$$\begin{aligned} \chi(x,\lambda) &= \cos(\rho(\pi - x)) + O\left(\frac{1}{\rho} e^{|\tau|(\pi - x)}\right), \\ \chi'(x,\lambda) &= \rho \sin(\rho(\pi - x)) + O\left(e^{|\tau|(\pi - x)}\right) \end{aligned} \tag{34}$$

and

$$s(x,\lambda) = \frac{\sin(\rho x)}{\rho} + O\left(\frac{1}{\rho^2} e^{|\tau| x}\right). \tag{35}$$

as $|\rho| \to \infty$ uniformly with respect to $x \in [0, \pi]$. Here $\lambda = \rho^2, \ \tau = \mathrm{Im}\rho$.

So for sufficiently large n there are uniform with respect to $x \in [0, \pi]$ estimates

$$|u_n(x)| \le Cne^{-xn\pi/L} \tag{36}$$

$$|u_n'(x)| \le Cn^2 e^{-xn\pi/L} \tag{37}$$

$$|u_n''(x) + q(x)u_n(x)| = |\left(\frac{n\pi}{L}\right)^2 u_n(x)| \le Cn^3 e^{-xn\pi/L} \tag{38}$$

Hence, the series (29) converges uniformly on $[\epsilon, \pi] \times [0, L]$ for every $\epsilon > 0$ and represents a solution of Eq. (28). Boundary conditions are satisfied according to the maximum principle.

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Reconstruction from Boundary Measurements: Complex Conductivities

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Abstract In this paper we show that following Nachman's method we can still reconstruct complex conductivities in $C^{1,1}$ from its Dirichlet-to-Neumann map in three and higher dimensions. For such, we analyze all of his results and pinpoint what really needs to be shown for complex conductivities. Moreover, we show the existence of non-exceptional points for low frequency and $C^{1,1}$-domains. As far as we are aware, this is the first reconstruction procedure for complex conductivities, even though the proof follows easily by extending some of the theorems obtained by Nachman to the complex case.

1 Introduction

In Electrical Impedance Tomography (EIT) we determine the interior impedance inside a bounded domain Ω by applying alternating electrical currents and measuring the corresponding voltages at the boundary $\partial\Omega$, or vice-versa. Impedance is the inverse of admittance which is defined through $\gamma = \sigma + i\omega\epsilon$, where ω is the angular frequency, σ, ϵ are the electrical conductivity and permittivity of materials inside Ω, respectively.

Our working assumptions are

$$\gamma \in C^{1,1}(\bar{\Omega}) \text{ and isotropic}, \quad \sigma \geq c > 0, \quad \epsilon \geq 0, \quad \omega \in \mathbb{R}^+, \tag{1}$$

$$\Omega \text{ is a bounded domain with } C^{1,1} \text{ boundary in } \mathbb{R}^n,\ n \geq 3 \tag{2}$$

In applications, most data acquisition systems and respective algorithms focus on computing the conductivity σ. However, in certain applications it is highly valuable to also obtain permittivity from boundary measurements. It brings extra

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knowledge to the table and allows us to distinguish more clinical conditions than is possible with the conductivity alone. An example is the ability to distinguish between pneumothorax and hyperinflation. Both scenarios correspond to regions of low resistivity, which implies high conductivity, but the pneumothorax has zero permittivity while the hyperinflation corresponds to low yet positive permittivity. Other possible application is in multi-frequency EIT since the properties σ and ϵ vary with the applied angular frequency ω, while in the real case the frequency is somewhat discarded.

Mathematically, the direct problem concerns the unique determination of the electrical potential $u \in H^1(\Omega)$ given a voltage $f \in H^{1/2}(\partial\Omega)$ set at the boundary, modelled by

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, & \text{in } \Omega \\ u|_{\partial\Omega} = f \end{cases} \tag{3}$$

Uniqueness in $H^1(\Omega)$ holds from the assumption $\mathrm{Re}\,\gamma > 0$, which implies by the weak formulation that 0 is not a Dirichlet eigenvalue of the operator $\nabla \cdot (\gamma \nabla u)$ in Ω.

Formally, from each voltage $f \in H^{1/2}(\partial\Omega)$ and each corresponding electrical potential $u \in H^1(\Omega)$ we can determine the electrical current measured at the boundary given by $\gamma \frac{\partial u}{\partial \nu}$. In essence, we can define for $\gamma \in L^\infty(\Omega)$ the Dirichlet-to-Neumann map $\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}$, which holds weakly by

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega), \\ f \quad &\mapsto \quad \langle \Lambda_\gamma f, g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx, \end{aligned} \tag{4}$$

where $v \in H^1(\Omega)$ has trace $g \in H^{1/2}(\partial\Omega)$.

In 1980 A.P. Calderón [11] was the first to pose the mathematical problem whether the conductivity $\sigma \in L^\infty(\Omega)$ can be uniquely determined by boundary measurements, Λ_γ, and if so how to reconstruct it. He showed that the linearized problem at constant conductivities has a unique solution. In mathematical literature, this is designated as Calderón's problem or inverse conductivity problem. In medical imaging the problem is know by Electrical Impedance Tomography (EIT).

After the initial work of Calderón there were many extensions to global uniqueness results. In [34], Sylvester and Uhlmann used ideas of scattering theory, namely the exponential growing solutions of Faddeev [15] to obtain global uniqueness in dimensions $n \geq 3$ for smooth conductivities. Using this foundations the uniqueness for lesser regular conductivities was further generalized for dimensions $n \geq 3$ in the works of [1, 7, 8, 12, 13, 19, 26, 29, 32]. Currently, the best know result is due to Haberman [18] for conductivities $\gamma \in W^{1,3}(\Omega)$. The reconstruction procedure for $n \geq 3$ was obtained in both [26] and [30] independently. As far as we are aware, there seems to be no literature concerning reconstruction for conductivities with less than two derivatives.

In two dimensions the problem seems to be of a different nature and tools of complex analysis were used to establish uniqueness. Nachman [27] obtained uniqueness and a reconstruction method for conductivities with two derivatives. The uniqueness result was soon extend for once-differentiable conductivities in [9] and a corresponding reconstruction method was obtained in [22]. In 2006, Astala and Päivärinta [2] gave a positive answer Calderón's problem for $\sigma \in L^\infty(\Omega)$, $\sigma \geq c > 0$, by providing the uniqueness proof through the reconstruction process.

The first extension to admittances, and here forward also designated by complex-conductivities, was in two-dimensions in [16]. Francini extended the work of Brown and Uhlmann [9] in two-dimensions by proving uniqueness for small angular frequencies and $\gamma \in W^{2,\infty}$. Afterwards, Bukgheim influential paper [10] proved the general result in two-dimensions for complex-conductivities in $W^{2,\infty}$. He reduced the (3) to a Schrödinger equation and shows uniqueness through the stationary phase method (based on is work many extensions followed [3, 5, 31]). Recently, by mixing techniques of [9] and [10], Lakshtanov et al. obtained in [24] uniqueness for Lipschitz complex-conductivities in $\mathbb{R}^2$. In [33], the author followed up their work to show that it is possible to reconstruct complex-conductivity with a jump at least in a certain set of points.

As far as we are aware, in dimensions higher than 2 there is no explicit literature for complex-conductivities. As stated in [6], it is possible to obtain uniqueness for twice differentiable complex-conductivities by the approach obtained in [34] and [29]. Furthermore, there is reference of a theoretical work for direct reconstruction method in the case of complex conductivities, since most works restrict themselves to the real scenario.

However, in [20] Nachman's reconstruction method is used to find complex conductivities from boundary measurements. This is a promising result that requires a theoretical background to support it, even if for some researchers it seems folklore.

Hence, in this paper, we show that Nachman's reconstruction method holds for complex conductivities. To be precise, the main result of this paper is the following:

Theorem 1 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $\gamma_1, \gamma_2 \in C^{1,1}(\Omega)$ be complex-valued conductivities, such that Re $\gamma_j \geq c > 0$ for $j = 1, 2$. Further, let Λ_1, Λ_2 be their corresponding Dirichlet-to-Neumann maps.*

If $\Lambda_1 = \Lambda_2$, then $\gamma_1 = \gamma_2$ in Ω.

Under careful examination of [26], we highlight here that the only requirement for the reconstruction method to hold concerns the uniqueness of boundary value problems with complex coefficient for $f \in H^{3/2}(\partial\Omega)$. For convenience of the reader, we present here the most essential results of Nachman's magnificent work, taking a sequential tour through the pieces needed to make this work. Hence, in essence this paper works as a review of Nachman's procedure highlighting the requirements for it to work for complex-conductivities.

Furthermore, following the work of [14] we show that the complex-conductivity can also be obtained from low-frequency asymptotics through the exponential growing solutions.

2 Uniqueness of Schrödinger Inverse Problem

The recurring idea on Calderón's problem is to convert our equation into one that has the coefficient in the lowest order terms. Here, we transform into the Schrödinger equation with complex-potential. We start by following the uniqueness result presented in [29], under the assumption of complex-valued potentials in $L^\infty(\bar{\Omega})$. In their work, there is no mention and need of the potential to be real, therefore we present their proof in its entirety.

Let $u \in H^1(\Omega)$ be the unique solution of (3) with trace $f \in H^{1/2}(\partial\Omega)$ at the boundary. Then the substitution $u = \gamma^{-1/2} w$ yields with $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$,

$$\begin{cases} -\Delta w + qw = 0, \text{ in } \Omega, \\ w|_{\partial\Omega} = \gamma^{1/2} f. \end{cases} \tag{5}$$

Notice that if $\gamma \in C^{1,1}(\Omega)$ and $\sigma \geq c > 0$ then $\gamma^{1/2}$ is well-defined and twice weakly differentiable. Therefore, q is well-defined and in $L^\infty(\Omega)$. As previously stated, the assumptions on γ lead to 0 not being a Dirichlet eigenvalue of $\nabla \cdot (\gamma \nabla u)$. The relation above implies a bijection between solutions of the (3) and of (5). Therefore, 0 is also not a Dirichlet eigenvalue of the Schrödinger problem.

In general, if 0 is not a Dirichlet eigenvalue of the Schrödinger operator in Ω, then the Dirichlet-to-Neumann map, Λ_q, is well-defined from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ and formally is given by

$$\Lambda_q f = \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega}$$

for w being the unique solution of $(-\Delta + q)w = 0$, in Ω and $w|_{\partial\Omega} = f$. Hence, here the corresponding inverse problem is to determine q from the boundary measurements Λ_q uniquely.

In this manner, we can cast our focus into the Schrödinger equation. First, we can extend q to zero outside the domain and study solutions of

$$-\Delta w + qw = 0, \quad \text{in } \mathbb{R}^n \tag{6}$$

which behave like

$$w = e^{ix\cdot\zeta}\left(1 + \psi(x, \zeta)\right), \text{ for } \zeta \in \mathbb{C}^n, \ \zeta \cdot \zeta = 0.$$

In Calderón's paper [11] he already uses the family of exponential harmonic functions, $e^{ix\cdot\zeta}$ in its proof, but it was Sylvester and Uhlmann [34] that first used this type of solutions to dispense the requirement of σ be close to a constant. Substituting into (6), it follows that ψ must satisfy

$$-\Delta\psi - 2i\zeta\cdot\nabla\psi + q\psi = -q \tag{7}$$

From scattering theory, we inherited the Faddeev-Green's function (see [15]) which takes a principal role in the study of the above Eq. (7). For $\zeta \in \mathbb{C}^n$ with $\zeta\cdot\zeta = 0$ it is given by $G_\zeta(x) = e^{ix\cdot\zeta} g_\zeta(x)$, where g_ζ is the fundamental solution of operator $(-\Delta - 2i\zeta\cdot\nabla)$ and is defined as:

$$g_\zeta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\zeta\cdot\xi}\, d\xi, \tag{8}$$

Recall, that as a fundamental solution G_ζ differs from the classical one G_0 by an harmonic function H_ζ.

From this, solutions of (6) with the desired asymptotics can be obtained by solving the integral equation

$$w(x,\zeta) = e^{ix\cdot\zeta} - \int G_\zeta(x-y)q(y)w(y,\zeta)\, dy \tag{9}$$

with ψ solving

$$\psi + g_\zeta * (q\psi) = -g_\zeta * q. \tag{10}$$

The study of these integral equations follows by a weighted L^2 estimate for g_ζ obtained in [34], which guarantees unique solvability of (9) for $|\zeta|$ large, even for complex conductivities. This estimate is one of the most important elements in scattering works, since it allows the existence and uniqueness of solutions and already puts into light their behavior in terms of ζ.

Let $\langle x\rangle = (1+|x|^2)^{1/2}$. We define the weighted L^2-space for $\delta\in\mathbb{R}$ as

$$L^2_\delta(\mathbb{R}^n) := \left\{ f : \|f\|_\delta := \|\langle x\rangle^\delta f\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$

Then the convolution operators with g_ζ and G_ζ satisfy the following estimates

Proposition 1 *For all $\zeta\in\mathbb{C}^n$ with $\zeta\cdot\zeta = 0$ and $|\zeta|\geq a$ the operator of convolution with g_ζ satisfies*

$$\left\|g_\zeta * f\right\|_{\delta-1} \leq \frac{c(\delta,a)}{|\zeta|}\|f\|_\delta, \quad \textit{for}\ \ 0<\delta<1 \tag{11}$$

Moreover, let $H^2_\delta(\Omega) := \left\{f : D^\alpha f \in L^2_{-\delta}(\mathbb{R}^n), 0 \le |\alpha| \le 2\right\}$ *be the weighted Sobolev space with norm*

$$\|f\|_{2,\delta} = \left(\sum_{|\alpha|\le 2} \|D^\alpha f\|^2_\delta\right)^{1/2}.$$

Then, for any $\zeta \in \mathbb{C}^n$ *with* $\zeta \cdot \zeta = 0$ *it holds for* $\delta \in (1/2, 1)$ *that*

$$\|g_\zeta * w\|_{2,-\delta} \le c(\delta, \zeta)\|w\|_{2,\delta}.$$

Furthermore, under the definition

$$\mathbf{G}_\zeta w(x) = \int_\Omega G_\zeta(x-y)w(y)\,dy$$

it holds that

$$\|\mathbf{G}_\zeta w\|_{H^2(\Omega)} \le c(\zeta, \Omega)\|w\|_{L^2(\Omega)}.$$

Proof The first estimate can be found in [34, Corollary 2.2], while the rest is in [26, Lemma 2.11]. □

For the uniqueness proof our interest resides in studying the exponential growing solutions given through Eq. (9)

Corollary 1 *Let* $0 < \delta < 1$ *and* $q \in L^\infty(\Omega)$ *be complex-valued and extended to zero outside* Ω*. Then there exists an* $R > 0$ *such that for all* $\zeta \in \mathbb{C}^n$ *with* $\zeta \cdot \zeta = 0$ *and* $|\zeta| > R$ *the integral equation (9) is uniquely solvable with* $e^{-ix\cdot\zeta}w(x,\zeta) - 1 \in L^2_{\delta-1}(\mathbb{R}^n)$*. Furthermore, it holds*

$$\|e^{-ix\cdot\zeta}w(x,\zeta) - 1\|_{\delta-1} \le \frac{\tilde{c}(R,\delta)}{|\zeta|}\|q\|_\delta. \tag{12}$$

Proof Let $M_q\phi = q\phi$, i.e., the operator of multiplication with q. We show that for $q \in L^\infty(\mathbb{R}^n)$ with compact support, $M_q : L^2_{\delta-1}(\mathbb{R}^n) \to L^2_\delta(\mathbb{R}^n)$ is a bounded operator.

Let $f \in L^2_{\delta-1}(\mathbb{R}^n)$. Then

$$\begin{aligned}\|M_q f\|_\delta &= \left[\int_{\mathbb{R}^n} (1+|x|^2)^\delta |q(x)f(x)|^2\,dx\right]^{1/2}\\ &= \left[\int_{\mathbb{R}^n} \left(1+|x|^2\right)|q(x)|^2\left(1+|x|^2\right)^{\delta-1}|f(x)|^2\,dx\right]^{1/2}\\ &\le \|\langle x\rangle q\|_\infty \|f\|_{\delta-1}.\end{aligned}$$

We define the operator $A_\zeta = C_\zeta M_q$, where C_ζ is the convolution with g_ζ, that is

$$A_\zeta f(x) = \int_{\mathbb{R}^n} g_\zeta(x-y)q(y)f(y)\,dy = C_\zeta M_q f \tag{13}$$

By Proposition 1 for $|\zeta| \geq R$ we obtain

$$\|A_\zeta f\|_{\delta-1} = \|C_\zeta M_q f\|_{\delta-1} \leq \frac{c(\delta, R)}{|\zeta|}\|M_q f\|_\delta \leq \frac{c(\delta, R)}{|\zeta|}\,\|\langle x\rangle q\|_\infty\,\|f\|_{\delta-1}$$

Therefore, A_ζ is bounded in $L^2_{\delta-1}(\mathbb{R}^n)$. Furthermore, if we consider

$$|\zeta| > R := c(\delta, R)\,\|\langle x\rangle q\|_\infty$$

then A_ζ is a contraction and $I + A_\zeta$ is invertible.

Since $q \in L^\infty$ and as compact support then it is in L^2_δ and therefore the right-hand side of (10) is in $L^2_{\delta-1}$. Hence, the unique solution to (7) is given by

$$\psi(x, \zeta) = -\left[I + A_\zeta\right]^{-1}\left(g_\zeta * q\right).$$

From here, we already know that

$$w = e^{ix\cdot\zeta}\left(1 - \left[I + A_\zeta\right]^{-1}\left(g_\zeta * q\right)\right)$$

solves the integral equation (9). Furthermore, the estimate (12) easily follows from $[I + A_\zeta]^{-1}$ being bounded in $L^2_{\delta-1}$, Proposition 1 and $g_\zeta * q \in L^2_{\delta-1}$.

Now, let us suppose that there exist two solutions w_1, w_2 of (9) such that

$$\phi_j = e^{-ix\cdot\zeta} w_j - 1 \in L^2_{\delta-1}.$$

Then, their difference is also in $L^2_{\delta-1}$ and both fulfill the equation

$$[I + A_\zeta]\phi_j = -g_\zeta * q.$$

This implies $[I + A_\zeta]\left(e^{-ix\cdot\zeta}(w_1 - w_2)\right) = 0$ and thus $w_1 \equiv w_2$ by the invertibility of $I + A_\zeta$ in $L^2_{\delta-1}$. Hence, uniqueness of the integral equation (9) for exponential growing solutions follows. □

We designate the values ζ for which this solution does not exist or is not unique as exceptional points.

Definition 1 Let $q \in L^\infty(\Omega)$ complex-valued and extended to zero outside Ω.

Let $\zeta \in \mathcal{V} := \{\zeta \in \mathbb{C}^n \setminus \{0\}\zeta \cdot \zeta = 0\}$. Then we call $\zeta \in \mathcal{V}$ an exceptional point for q if there is no unique exponential growing solution of $(-\Delta + q)w = 0$ in $\mathbb{R}^n$, that is, there is no unique solution of the type

$$w(x,\zeta) := e^{ix\cdot\zeta}\,(1+\mu(x,\zeta)),\ \text{with}\ \mu \in L^2_{\delta-1}(\mathbb{R}^n),\ 0<\delta<1.$$

The uniqueness proof of this section given by Nachman et al. [29] and Nachman's reconstruction method [26], only require large non-exceptional points ζ. However, in this sense the reconstruction process is very unstable. Hence, one of our desires is to mimic the theory in two-dimensions, where we are able to reconstruct γ from small values of non-exceptional points ζ by the D-bar method. We will see in a further section how under circumstances one can still apply a version of it.

Now a first step in the uniqueness proof is to show that the exponential growing solutions outside Ω are uniquely identified by the Dirichlet-to-Neumann map.

Lemma 1 *Let $q_1, q_2 \in L^\infty(\Omega)$ and extended to zero outside Ω, such that 0 is not a Dirichlet eigenvalue of $-\Delta + q_j$ in Ω for $j=1,2$. Further, let $\zeta \in \mathcal{V}$ a non-exceptional point for q_1, q_2. Suppose that $\Lambda_{q_1} = \Lambda_{q_2}$ and w_1, w_2 are the unique solutions of $(-\Delta+q_j)w_j = 0$ in $\mathbb{R}^n$ of the form $e^{ix\cdot\zeta}\left(1+\mu_j\right)$. Then*

$$w_1 = w_2, \qquad \text{in } \mathbb{R}^n \setminus \Omega.$$

Proof Let $v \in H^1(\Omega)$ be the unique solution of

$$-\Delta v + q_2\, v = 0, \qquad \text{in } \Omega$$
$$v|_{\partial\Omega} = w_1|_{\partial\Omega}\,.$$

Then we define

$$h = \begin{cases} v, & \text{in } \Omega \\ w_1, & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Since, $\Lambda_{q_1} = \Lambda_{q_2}$ it holds that $\Lambda_{q_1}\, w_1|_{\partial\Omega} = \Lambda_{q_2}\, w_1|_{\partial\Omega}$ and thus $\frac{\partial w_1}{\partial \nu} = \frac{\partial v}{\partial \nu}$. This implies that h is continuous over $\partial\Omega$, as well as, $\frac{\partial h}{\partial \nu}$. Therefore, h solves $-\Delta h + q_2\, h = 0$ in $\mathbb{R}^n$ and has the appropriate asymptotics since w_1 has them. By the uniqueness theorem it follows that $h = w_2$ and thus $w_1 = w_2$ in $\mathbb{R}^n \setminus \Omega$. □

Now the uniqueness theorem obtained in [29] follows also for complex-conductivities directly.

Theorem 2 *Let $q_1, q_2 \in L^\infty(\Omega)$ extended to zero outside Ω. Suppose that 0 is not a Dirichlet eigenvalue of $-\Delta + q_j$, $j = 1, 2$ on Ω. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$.*

Proof Let $k \in \mathbb{R}^n$ be fixed and for $m, s \in \mathbb{R}^n$ we set

$$\zeta = \frac{1}{2}\,((k+s)+im) \ \text{ and } \ \tilde{\zeta} = \frac{1}{2}\,((k-s)-im)$$

with $k \cdot s = k \cdot m = s \cdot m = 0$ and $|k|^2 + |s|^2 = |m|^2$. The ζ, $\tilde{\zeta}$ are in $\mathbb{C}^n$ and fulfill the condition $\zeta \cdot \zeta = 0$. Hence, taking s, m large enough we obtain solutions w_j of the integral equation (9) for their respective potentials for $\tilde{\zeta}$. By Green's identity it holds that

$$\int_\Omega e^{ix\cdot\zeta} q_j(x) w_j(x)\, dx = \int_\Omega e^{ix\cdot\zeta} \Delta w_j(x) - w_j \Delta e^{ix\cdot\zeta}\, dx$$
$$= \int_{\partial\Omega} e^{ix\cdot\zeta} \frac{\partial w_j}{\partial \nu} - w_j\, (\nu \cdot i\zeta)\, e^{ix\cdot\zeta}\, d\sigma(x).$$

By hypothesis and the previous lemma $\Lambda_{q_1} = \Lambda_{q_2}$ and thus $w_1|_{\partial\Omega} = w_2|_{\partial\Omega}$. Therefore, it also holds that

$$\left.\frac{\partial w_1}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial w_2}{\partial \nu}\right|_{\partial\Omega}$$

as w_j solve the interior problem $(-\Delta + q_j)w_j = 0$. Hence, the right-hand side of the integral above is equal for both q_j and assuming the asymptotics of w_j w.r.t. $\tilde{\zeta}$ it follows

$$\int_\Omega e^{ix\cdot\zeta}\, (q_1 w_1 - q_2 w_2)\, dx = 0$$

which is equivalent to

$$\int_\Omega e^{ix\cdot(\zeta+\tilde{\zeta})}\, (q_1 - q_2)\, dx = \int_\Omega e^{ix\cdot(\zeta+\tilde{\zeta})}\, (q_1\psi_1 - q_2\psi_2)\, dx$$

Using $\zeta + \tilde{\zeta} = k$ and taking modulus we obtain by Cauchy-Schwarz inequality and Corollary 1

$$\left|\int_\Omega e^{ix\cdot k}\, (q_1 - q_2)\, dx\right| \le \sum_{j=1}^{2} \int_\Omega \left|q_j \psi_j\right| \le \sum_{j=1}^{2} \|q_j\|_{1-\delta} \|\psi_j\|_{\delta-1}$$
$$\le \sum_{j=1}^{2} \frac{C}{|\tilde{\zeta}|} \|q_j\|_{1-\delta} \|q_j\|_{\delta}.$$

Since $\tilde{\zeta}$ was arbitrarily depending on s, we can take the limit as $|s| \to \infty$. This implies that the left-hand side equals to zero for each fixed $k \in \mathbb{R}^n$. Given that the proof holds for all k we have

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2)\, dx = 0, \quad \forall k \in \mathbb{R}^n$$

Therefore, by Fourier inversion theorem we obtain $q_1 = q_2$ in Ω. □

Note We do not require more assumptions for q being complex; second the following uniqueness proof only works for $n \geq 3$ due to the required choice of $\zeta, \tilde{\zeta}$.

Hence, uniqueness is extended for complex-potentials in $L^\infty(\Omega)$ with 0 not a Dirichlet eigenvalue of $(-\Delta + q)$, directly from the work of [29]. To extend uniqueness for admittivities $\gamma \in C^{1,1}(\bar{\Omega})$ it is still necessary to establish a relation between Λ_γ and Λ_q. We will present this in a later section. Now, we proceed to explain how Nachman's reconstruction method equally holds for complex-conductivities.

3 Preliminaries for Reconstruction

In the following section, we present the necessary results to follow Nachman's approach [26]. The two main pinpoints in our extension to complex-conductivities are the Lemma 4 and Proposition 3.

The first result concerns an estimate for the single layer operator which will help us prove that there are no-exceptional points close to zero for complex-conductivities. The estimate was obtained in [14].

The second result concerns the uniqueness of the interior Schrödinger problem. In essence, this is were Nachman's work needs to be extended, since is version of this proposition was proven by estimates for real-coefficients. Even though, the machinery to prove a complex-coefficient version does not require anything novel it seems essential to provide a clear statement into why Nachman's method still works.

Analogously to the classical single and double layer potentials we define the respective operators for G_ζ. The single layer operator is defined as

$$S_\zeta f(x) = \int_{\partial\Omega} G_\zeta(x-y) f(y)\, ds(y)$$

and the double layer as

$$D_\zeta f(x) = \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial \nu}(x-y) f(y)\, ds(y).$$

Moreover, taking the trace of double layer potential it holds

$$B_\zeta f(x) := \text{ p.v.} \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial \nu}(x-y) f(y)\, ds(y), \text{ for } x \in \partial\Omega.$$

Since the singularity of G_ζ for x near y is the same as G_0, it is locally integrable on $\partial\Omega$ and the trace of S_ζ is still "itself".

We state here the properties that Nachman established and are essential for the later proofs.

Proposition 2 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$.*

(i) For $0 \leq s \leq 1$

$$\left\| S_\zeta f \right\|_{H^{s+1}(\partial\Omega)} \leq c(\zeta, s) \| f \|_{H^s(\partial\Omega)}. \tag{14}$$

(ii) For $0 \leq s \leq \frac{3}{2}$ we have that B_ζ is bounded in $H^s(\partial\Omega)$.

Let ρ_0 be a number large enough so that $\bar{\Omega} \subset \{x : |x| < \rho_0\}$. For any $\rho > \rho_0$ we define $\Omega'_\rho = \{x : x \notin \bar{\Omega}, \ |x| < \rho\}$.

Lemma 2 *If $f \in H^{1/2}(\partial\Omega)$, the function $\phi = S_\zeta f$ has the following properties*

(i) $\Delta\phi = 0$ in $\mathbb{R}^n \setminus \partial\Omega$.
(ii) $\phi \in H^2(\Omega)$ and $\phi \in H^2(\Omega'_\rho)$ for any $\rho > \rho_0$.
(iii) ϕ satisfies an analogue to the Sommerfeld radiation condition. For almost every x it holds

$$\lim_{\rho\to\infty} \int_{|y|=\rho} \left[G_\zeta(x-y) \frac{\partial\phi}{\partial\nu(y)} - \phi(y) \frac{\partial G_\zeta}{\partial\nu(y)}(x-y) \right] ds(y) = 0. \tag{15}$$

In fact, for $\rho > \rho_0$ the above identity holds for $|x| < \rho$ even without taking the limit.

(iv) Let $B_\zeta^\dagger$ denote the operator on the boundary

$$B_\zeta^\dagger f(x) = \text{ p.v.} \int_{\partial\Omega} \frac{\partial G_\zeta}{\partial\nu(x)}(x-y) f(y)\, ds(y). \tag{16}$$

It follows that the (nontangential) limits $\partial\phi/\partial\nu_+$, $\partial\phi/\partial\nu_-$ of the normal derivative of ϕ as the boundary is approached from the outside and inside Ω, respectively, are given by

$$\frac{\partial\phi}{\partial\nu_\pm} = \mp\frac{1}{2} f(x) + B_\zeta^\dagger f(x), \quad \textit{for almost every } x \in \partial\Omega. \tag{17}$$

(v) The boundary values ϕ_+, ϕ_- of ϕ from outside and inside of Ω, respectively, are identical as elements of $H^{3/2}(\partial\Omega)$ and agree with the trace of the single layer potential $S_\zeta f$.

Lemma 3 *If $f \in H^{3/2}(\partial\Omega)$ the function $\psi = D_\zeta f$ defined in $\mathbb{R}^n \setminus \partial\Omega$ has the properties (i), (ii) and (iii) of the Lemma 2.*

Moreover, the non-tangential limits ψ_+, ψ_- of ψ as we approach the boundary from outside and inside of Ω, respectively, exist and satisfy

$$\psi_\pm(x) = \pm\frac{1}{2} f(x) + B_\zeta f(x), \textit{ for almost every } x \in \partial\Omega. \tag{18}$$

Lemma 4 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. The Faddeev fundamental solution G_ζ can be given through the decomposition*

$$G_\zeta(x) = G_0(x) + H_\zeta(x),$$

where G_0 is the classical fundamental solution and H_ζ is an harmonic function.

Moreover, the single and double layer operators have a similar decomposition and, for our own convenience, we present here the case for the single layer. For $f \in H^{1/2}(\partial\Omega)$ we have

$$S_\zeta f(x) = S_0 f(x) + \int_{\partial\Omega} H_\zeta(x-y) f(y)\, ds(y) =: S_0 f(x) + \mathcal{H}_\zeta f(x).$$

Further, it holds

$$\|\mathcal{H}_\zeta\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{3/2}(\partial\Omega)} \leq C|\zeta|^{n-2},$$

where the constant C only depends on the domain.

Proof See [14] for further details. □

Now we provide the proof for uniqueness of the interior problem with a complex-potential. This is the only "new" and required statement to bring forth the proof of Nachman's reconstruction to complex-admittivities.

Proposition 3 *Let Ω be a bounded $C^{1,1}$−domain in $\mathbb{R}^n$, $n \geq 3$. Suppose that $q \in L^\infty(\bar{\Omega})$ is complex-valued and that 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$ in Ω. Then for every $f \in H^{3/2}(\partial\Omega)$ there is a unique $w \in H^2(\Omega)$ such that*

$$\begin{cases} (-\Delta + q)\, w = 0 \text{ in } \Omega \\ w|_{\partial\Omega} = f. \end{cases} \tag{19}$$

The solution operator is defined by $P_q f := w$ and has the mapping property

$$P_q : H^{3/2}(\partial\Omega) \to H^2(\Omega).$$

Moreover, the Dirichlet-to-Neumann map operator has the mapping property

$$\Lambda_q : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega).$$

Proof The proof follows by studying first the Laplacian and showing that multiplication by q is a compact operator from $H^2(\Omega)$ to $L^2(\Omega)$.

Thus, let

$$P_0 : H^2(\Omega) \to L^2(\Omega) \times H^{3/2}(\partial\Omega), \quad u \mapsto (-\Delta u, \operatorname{tr} u).$$

By the definition of $H^2(\Omega)$ and the trace properties on this space and $C^{1,1}-$domains the operator P_0 is linear and bounded. By Theorem 9.15. of [17] and under our conditions on the domain, there always exists a unique solution in $H^2(\Omega)$ of

$$\begin{cases} -\Delta u = f, \\ u|_{\partial\Omega} = g. \end{cases}$$

Therefore, the operator P_0 is bijective and invertible. In particular is Fredholm of index zero.

Analogously, we define the operator

$$P_q : H^2(\Omega) \to L^2(\partial\Omega) \times H^{3/2}(\partial\Omega), \quad u \mapsto ([-\Delta + q]u, \ \mathrm{tr}\, u) \,.$$

Then, the difference of the operators $P_q - P_0$ maps u to $(qu, \ 0)$ between the same spaces. Since, the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it immediately follows that multiplication by $q \in L^\infty(\Omega)$ is a compact operator. Hence, by definition $P_q - P_0$ is a compact and since $P_q = P_0 + (P_q - P_0)$ is the sum of a Fredholm operator of index zero and a compact operator, it still is Fredholm of index zero. Thus, to show invertibility we prove that $\ker P_q = \{0\}$. Let $w \in \ker P_q$. By definition this implies w is a solution in $H^2(\Omega)$ of

$$\begin{cases} -\Delta w + qw = 0, \\ w|_{\partial\Omega} = 0, \end{cases}$$

but due to the assumption of 0 not being a Dirichlet eigenvalue of $(-\Delta + q)$ in Ω it follows that $w \equiv 0$. □

Our main assumption is that $\gamma \in C^{1,1}(\bar{\Omega})$, thus it is in $H^2(\Omega)$. Therefore, for potentials q given by the complex-conductivity the following statement holds.

Corollary 2 *Let Ω be a bounded $C^{1,1}-$domain in $\mathbb{R}^n$, $n \geq 3$. For $\gamma \in C^{1,1}(\bar{\Omega})$ such that Re $\gamma \geq c > 0$.*

Then $q \in L^\infty(\Omega)$ given by $q = \Delta(\gamma^{1/2})/\gamma^{1/2}$ is well-defined and 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$. Then the unique solution $w \in H^2(\Omega)$ of

$$\begin{cases} -\Delta w + qw = 0 \\ w|_{\partial\Omega} = \gamma^{1/2} \end{cases} \tag{20}$$

is $w \equiv \gamma^{1/2}$.

This corollary brings to light that if we know the boundary values of the conductivity and the potential q in Ω, we can find γ by solving the above boundary value problem.

4 Boundary Integral Equation

The properties of the previous section allows us to establish a one-to-one correspondence between the solution of a boundary integral equation and of the following exterior problem

$$\begin{aligned} &(i)\ \Delta\psi = 0,\ \text{in } \Omega' := \mathbb{R}^n \setminus \bar{\Omega}, \\ &(ii)\ \psi \in H^2(\Omega'_\rho),\ \text{for any } \rho > \rho_0, \\ &(iii)\ \psi(x,\zeta) - e^{ix\cdot\zeta} \text{ satisfies } (15), \\ &(iv)\ \frac{\partial \psi}{\partial \nu_+} = \Lambda_q \psi \text{ on } \partial\Omega. \end{aligned} \tag{21}$$

In this section, we assume that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$ and $q \in L^\infty(\Omega)$ is a complex-potential for which 0 is not a Dirichlet eigenvalue. Further, most proofs follow directly from Nachman's work [26], but we provide them here for convenience of the reader. We will highlight the new pieces needed to put the puzzle together.

Lemma 5 *Let $\zeta \in \mathcal{V}$.*

(a) Suppose ψ solves the exterior problem (21). Then its trace $f_\zeta = \psi_+ = \psi|_{\partial\Omega}$ solves the boundary integral equation

$$f_\zeta = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2} I \right] f_\zeta. \tag{22}$$

(b) Conversely, suppose $f_\zeta \in H^{3/2}(\partial\Omega)$ solves (22). Then the function $\psi(x,\zeta)$ defined for $x \in \Omega'$ by

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \left(S_\zeta \Lambda_q - D_\zeta \right) f_\zeta(x) \tag{23}$$

solves the above exterior problem under all conditions. Furthermore, $\psi|_{\partial\Omega} = f_\zeta$.

Proof

(a) Assume ψ solves (21). We apply Green's identity to G_ζ and ψ in Ω'_ρ, $\rho > \rho_0$. It holds

$$\left(\int_{|y|=\rho} - \int_{\partial\Omega} \right) \left[G_\zeta(x-y) \frac{\partial \psi}{\partial \nu_+} - \psi_+(y,\zeta) \frac{\partial G_\zeta}{\partial \nu_+(y)}(x-y) \right] ds(y) \tag{24}$$

$$= \int_{\Omega'_\rho} \left[G_\zeta(x-y) \Delta\psi(y,\zeta) - \psi(y,\zeta) \Delta_y G_\zeta(x-y) \right] dy.$$

Since, ψ is harmonic on Ω'_ρ and G_ζ is the fundamental solution of $-\Delta$ we obtain for arbitrary $x \in \Omega'_\rho$

$$\begin{aligned}\psi(x,\zeta) = \int_{|y|=\rho} \Bigg[G_\zeta(x-y) \frac{\partial\left(\psi - e^{iy\cdot\zeta}\right)}{\partial \nu} \\ - \left(\psi - e^{iy\cdot\zeta}\right) \frac{\partial G_\zeta}{\partial \nu_+(y)}(x-y) \Bigg] ds(y) \qquad (25) \\ + \int_{|y|=\rho} \left[G_\zeta(x-y) \frac{\partial e^{iy\cdot\zeta}}{\partial \nu} - e^{iy\cdot\zeta} \frac{\partial G_\zeta}{\partial \nu(y)}(x-y) \right] ds(y) \\ - \int_{\partial\Omega} \left[G_\zeta(x-y) \frac{\partial \psi}{\partial \nu_+} ds(y) - \int_{\partial\Omega} \psi_+(y,\zeta) \frac{\partial G_\zeta}{\partial \nu(y)}(x-y) \right] ds(y)\end{aligned}$$

By hypothesis (21-iii), the first integral vanishes. The function $e^{iy\cdot\zeta}$ is harmonic and a re-application of Green's identity to the second integral on $|y| < \rho$ equals $e^{ix\cdot\zeta}$. Finally, due to (21-iv) the last integral is $\left[S_\zeta \Lambda_q - D_\zeta\right]\psi$. Thus, the function ψ fulfills for $x \in \Omega'$ the identity

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - D_\zeta\right] f_\zeta.$$

Taking the non-tangential limit to the boundary from the outside we obtain by Lemmas 2 and 3

$$f_\zeta(x) = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2} I \right] f_\zeta(x).$$

(b) Conversely, suppose $f_\zeta \in H^{3/2}(\partial\Omega)$ solves the boundary integral equation (22). Define a function ψ in Ω' by

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - D_\zeta\right] f_\zeta(x). \qquad (26)$$

We show that this ψ solves the exterior problem (21) from properties of the single and double layer (Lemmas 2 and 3).

It is immediate to see that ψ fulfills the property (i) of (21), since for $\zeta \cdot \zeta = 0$ the exponential $e^{ix\cdot\zeta}$ is harmonic, and $S_\zeta \Lambda_q f_\zeta$, $D_\zeta f_\zeta$ are harmonic in Ω' by the above mentioned lemmas. Moreover, it holds that $S_\zeta \Lambda_q f_\zeta,\ D_\zeta f_\zeta \in H^2(\Omega'_\rho),\ \rho > \rho_0$ and further the identity (15) also holds. Hence, the property (ii) and (iii) of the exterior problem follow.

To show the last property, we approach the boundary $\partial\Omega$ non-tangentially from the outside and we obtain, as in part (a),

$$\psi|_{\partial\Omega} = e^{ix\cdot\zeta} - \left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I\right] f_\zeta.$$

By virtue of f_ζ fulfilling the boundary integral equation the right-hand side equals f_ζ and therefore $\psi|_{\partial\Omega} = f_\zeta$. From this and the first three properties of (21), that we already showed ψ fulfills, we can obtain analogously to part (a)

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - S_\zeta\left(\frac{\partial\psi}{\partial\nu_+}\right) + D_\zeta f_\zeta, \quad \text{for } x \in \Omega'. \tag{27}$$

Subtracting both formulations of ψ, (27) and (26), the following equality holds throughout Ω'

$$S_\zeta\left[\Lambda_q f_\zeta - \frac{\partial\psi}{\partial\nu_+}\right] = 0. \tag{28}$$

By taking traces from the outside, it actually holds on the boundary $\partial\Omega$. We are reminded that $S_\zeta\left[\Lambda_q f_\zeta - \frac{\partial\psi}{\partial\nu_+}\right]$ is harmonic in $\mathbb{R}^n \setminus \partial\Omega$ and since the trace is 0 on $\partial\Omega$ uniqueness of the interior problem for $q \equiv 0$ implies that the equality (28) holds everywhere. Then, its normal derivatives will be zero and subtracting them on $\partial\Omega$ with the help of (17) we obtain

$$\left[\Lambda_q - \partial\psi/\partial\nu_+\right] = \frac{\partial S_\zeta\left[\Lambda_q - \partial\psi/\partial\nu_+\right]}{\partial\nu_-} - \frac{\partial S_\zeta\left[\Lambda_q - \partial\psi/\partial\nu_+\right]}{\partial\nu_+} = 0. \tag{29}$$

Thus the last property of the exterior problem follows. □

Furthermore, we are able to obtain a relation between the exterior problem and the solutions of integral equation (9).

Lemma 6 *Let $\zeta \in \mathcal{V}$.*

(a) Suppose $\psi \in L^2_{\text{loc}}(\mathbb{R}^n)$ is a solution of

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\mathbb{R}^n} G_\zeta(x-y)q(y)\psi(y,\zeta).$$

Then the restriction of ψ to Ω' solves the exterior problem (21) and fulfills the respective properties (i)–(iv).

(b) Conversely, if ψ solves the exterior problem (21), there is a unique solution $\tilde{\psi} \in L^2_{\text{loc}}(\mathbb{R}^n)$ of the integral equation (9), such that $\tilde{\psi} = \psi$ in Ω'.

Proof

(a) From the Proposition 1 it follows $\psi \in H^2_{\rm loc}(\mathbb{R}^n)$, which immediately implies property (ii) of the exterior problem. Moreover, in $\mathbb{R}^n$ it holds $(-\Delta + q)\psi = 0$, thus due to $q \equiv 0$ on Ω' the property (i) holds, i.e., $-\Delta\psi = 0$ in Ω'.

Applying Green identity on $|y| < \rho$ yields

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial\psi}{\partial\nu(y)} - \psi(y,\zeta)\frac{\partial G_\zeta}{\partial\nu(y)}(x-y)\right] ds(y)$$
$$= \int_{|y|<\rho} G_\zeta(x-y)q(y)\psi(y,\zeta)\,dy + \psi(x,\zeta), \text{ for a.e. } x \text{ with } |x| < \rho.$$

Now, we can choose ρ large in order to contain the supp of q. Since ψ solves integral equation this means that the right-hand side equals $e^{ix\cdot\zeta}$. Moreover, we already showed that

$$e^{ix\cdot\zeta} = \int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial e^{iy\cdot\zeta}}{\partial\nu(y)} - e^{iy\cdot\zeta}\frac{\partial G_\zeta}{\partial\nu(y)}(x-y)\right] ds(y).$$

Then passing the exponential to the right-hand side, we obtain

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial\left(\psi - e^{iy\cdot\zeta}\right)}{\partial\nu(y)} - \left(\psi(y,\zeta) - e^{iy\cdot\zeta}\right)\frac{\partial G_\zeta}{\partial\nu(y)}(x-y)\right]$$
$$\times\, ds(y) = 0$$

for all $\rho > \rho_0$ Thus property iii) follows by taking the limit as $\rho \to \infty$.

Immediately, we can see that $\Lambda_q\psi_- = \frac{\partial\psi}{\partial\nu_-}$ and since $\psi \in H^2$ in a two-sided neighborhood of $\partial\Omega$ it holds that $\psi_- = \psi_+$ and $\frac{\partial\psi}{\partial\nu_-} = \frac{\partial\psi}{\partial\nu_+}$. This leads to ψ fulfilling the iv) property. Therefore, the restriction of ψ to Ω' solves the exterior problem (21).

(b) Suppose ψ defined in Ω' solves the exterior problem (21). Set $\tilde\psi$ by $\tilde\psi = P_q\psi_+$ in Ω and $\tilde\psi = \psi$ in Ω'. Then on $\partial\Omega$,

$$\tilde\psi_- = (P_q\psi_+) = \psi_+ = \tilde\psi_+$$

and

$$\frac{\partial\tilde\psi}{\partial\nu_-} = \Lambda_q\psi_+ = \frac{\partial\psi}{\partial\nu_+} = \frac{\partial\tilde\psi}{\partial\nu_+}$$

due to (iv). Thus $\tilde{\psi}$ solves $(-\Delta + q)\tilde{\psi} = 0$ on $\mathbb{R}^n$. Applying Green's formula in $|y| < \rho$ yields

$$\int_{|y|=\rho} \left[G_\zeta(x-y)\frac{\partial \psi}{\partial \nu(y)} - \psi(y,\zeta)\frac{\partial G_\zeta}{\partial \nu(y)}(x-y) \right] ds(y)$$
$$= \int_{|y|<\rho} G_\zeta(x-y)q(y)\tilde{\psi}(y,\zeta)\,dy + \tilde{\psi}(x,\zeta)$$

for almost every x with $|x| < \rho$. Thus by letting $\rho \to \infty$ the radiation condition (iii) implies that the left-hand side is $e^{ix\cdot\zeta}$. Hence $\tilde{\psi}$ verifies the desired integral equation in $\mathbb{R}^n$.

To finalize we prove that this extension is unique. Suppose that we have two extensions $\tilde{\psi}^1, \tilde{\psi}^2 \in L^2_{\text{loc}}(\mathbb{R}^n)$ of ψ which agree in Ω' and solve the integral equation everywhere. As in part (a), we see that $\tilde{\psi}^1, \tilde{\psi}^2 \in H^2_{\text{loc}}(\mathbb{R}^n)$ and $(-\Delta + q)\tilde{\psi}^j = 0$ in $\mathbb{R}^n$ for $j = 1, 2$. Hence, they are in H^2 on a two-sided neighborhood of $\partial\Omega$. This implies that $\tilde{\psi}^j_+ = \tilde{\psi}^j_-$, for $j = 1, 2$, which promptly leads to $\tilde{\psi}^1_- = \tilde{\psi}^2_-$ since they agree on Ω'. Now, from the uniqueness of the interior problem it follows that $\tilde{\psi}^1 = \tilde{\psi}^2$. □

Remark The two previous lemmas establish that a solution of the boundary integral equation is equivalent to a exponential growing solution of the Schrödinger equation in $\mathbb{R}^n$. The interesting remark is that there was no explicit requirement of ζ being large. Hence, by showing that the boundary integral equation is uniquely solvable for small values of ζ we guarantee the existence of exponential growing solutions for these ζ.

Moreover, on all the proves above there is no explicit difference of q being real or complex.

Keeping this in mind, we focus now on solvability of the boundary integral equation. The following proposition glues together the papers [26] and [14] and applies them to the complex potential by making use of the uniqueness of the interior problem in this scenario obtained in Proposition 3.

Proposition 4 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let q be a complex-valued potential in $L^\infty(\Omega)$ and suppose that* 0 *is not Dirichlet eigenvalue of* $-\Delta + q$ *in Ω. We define $K_\zeta = S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I$ and for any $\zeta \in \mathcal{V}$ it holds*

(a) The operators K_0, K_ζ are compact on $H^{3/2}(\partial\Omega)$.
(b) If $\operatorname{Re} q \geq 0$, *then $I + K_0$ is invertible in $H^{3/2}(\partial\Omega)$.*
(c) If $\operatorname{Re} q \geq 0$ *there exists an $\epsilon > 0$ with $|\zeta| < \epsilon$ for which the operator $I + K_\zeta$ is invertible in $H^{3/2}(\Omega)$.*
(d) There exists an $R > 0$ such that for all $|\zeta| > R$ the operator $I + K_\zeta$ is invertible in $H^{3/2}(\partial\Omega)$.

Proof Part (a) follows by a compactness embedding. Let $f \in H^{3/2}(\partial\Omega)$ and set $w = P_q f$ as the solution of interior Dirichlet problem (19). Let $x \in \Omega$ we use the Green's formula to obtain

$$\int_\Omega G_\zeta(x-y)\Delta w(y)\,dy + w(x) = \left[S_\zeta \Lambda_q - D_\zeta\right] f(x)$$

which is equivalent to

$$\int_\Omega G_\zeta(x-y)q(y)P_q f(y)\,dy + w(x) = \left[S_\zeta \Lambda_q - D_\zeta\right] f(x)$$

By letting x approach the boundary non-tangentially from the inside we thus obtain

$$\text{tr}\,\left(G_\zeta * (qP_q f)\right) + f(x) = S_\zeta \Lambda_q f(x) - \left[-\frac{1}{2} f(x) + B_\zeta f(x)\right]$$

and therefore

$$\left[S_\zeta \Lambda_q - B_\zeta - \frac{1}{2} I\right] f = \text{tr}\,\left(G_\zeta * (qP_q f)\right).$$

Hence, our desired operator satisfies this factorization, where the following mapping properties hold

$P_q : H^{3/2}(\partial\Omega) \to H^2(\Omega)$;
$\imath : H^2(\Omega) \to L^2(\Omega)$ is a compact embedding;
$M_q : L^2(\Omega) \to L^2(\Omega)$;
$\mathbf{G}_\zeta : L^2(\Omega) \to H^2(\Omega)$ convolution with G_ζ, which we prove up next;
$\text{tr} : H^2(\Omega) \to H^{3/2}(\partial\Omega)$.

And the compactness of the embedding implies compactness of the desired operator.

(b) Let $\zeta = 0$. In this case G_0 is the classical fundamental solution and the corresponding operators are the classical ones. By part (a), we already know that $S_0\Lambda_q - B_0 - \frac{1}{2}I$ is compact on $H^{3/2}(\partial\Omega)$. Then $I + K_0 = \left[\frac{1}{2}I + S_0\Lambda_q - B_0\right]$ is Fredholm of index zero on $H^{3/2}(\partial\Omega)$. Therefore, it is enough to show injectivity.

Let $h \in H^{3/2}(\partial\Omega)$ such that $\left[\frac{1}{2}I + S_0\Lambda_q - B_0\right] h = 0$. Define $w = -S_0\Lambda_q h + D_0 h$. Then w is harmonic in $\mathbb{R}^n$, $w \in H^2(\Omega)$ and $w \in H^2(\Omega'_\rho)$ by Lemmas 2 and 3. Moreover, approaching the boundary non-tangentially by the inside we obtain

$$w_- = -S_0\Lambda_q h + \left(-\frac{1}{2}h + B_0 h\right) = -\left[\frac{1}{2}I + S_0\Lambda_q - B_0\right] h = 0.$$

Since, the problem $-\Delta w = 0,\ \ w|_{\partial\Omega} = 0$ is uniquely solvable in $H^2(\Omega)$ it follows that $w \equiv 0$ in Ω and thus $\frac{\partial w}{\partial \nu_-} = 0$ on $\partial\Omega$.

By noticing the jump relations for the single and double layer operator (see [25]), we can deduce that

$$[w] = w_+ - w_- = w_+ = [D_0 h] = h$$

and

$$\left[\frac{\partial w}{\partial \nu}\right] = \frac{\partial w}{\partial \nu_+} = -\left[\frac{\partial}{\partial \nu} S_0 \Lambda_q h\right] = \Lambda_q h.$$

Now, by Proposition 3 there is a unique solution $u \in H^2(\Omega)$ of

$$\begin{cases} (-\Delta + q)\, u = 0, \\ u|_{\partial\Omega} = h, \end{cases}$$

such that $\Lambda_q h = \frac{\partial u}{\partial \nu_-}\Big|_{\partial\Omega}$. We set

$$v = \begin{cases} u, \text{ in } \Omega \\ w, \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}$$

and see that $u_- = w_+ = h$ and $\frac{\partial u}{\partial \nu_-} = \frac{\partial w}{\partial \nu_+} = \Lambda_q$, thus it holds that v and $\frac{\partial v}{\partial \nu}$ are continuous over the boundary $\partial\Omega$. Therefore $v \in H^2(B_\rho(0)),\ \rho > 0$ and it solves $-\Delta v + qv = 0$ in $\mathbb{R}^n$, since $q \equiv 0$, in $\mathbb{R}^n \setminus \Omega$.

Let $\chi_\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ in $B_{\rho-\epsilon}(0)$ and $\chi \equiv 0$ in $\rho - \epsilon < |x| < \rho$, for $\epsilon > 0$ small enough.

Then for $\phi \in H^1(\mathbb{R}^n)$ it follows by Green's identity

$$\int_{|x|<\rho} (-\Delta v + qv)\,(\chi\phi)\; dx = 0,$$

which is equivalent to

$$\int_{|x|<\rho} \nabla v \cdot \nabla(\chi\phi) + qv(\chi\phi)\, dx = 0,$$

as well as

$$\int_\Omega \nabla v \cdot \nabla\phi + qv\phi\, dx + \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla(\chi\phi)\, dx = 0.$$

In particular we can take $\phi = \bar{v}$ and since w is given through the classical single and double layer it follows that $\nabla w \in L^2(B_\rho(0) \setminus \bar{\Omega})$. Thus taking the limit as $\rho \to \infty$

$$\int_\Omega |\nabla v|^2 \phi + q|v|^2\,dx + \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla(\chi \bar{w})\,dx = 0,$$
$$\int_\Omega |\nabla v|^2 \phi + q|v|^2\,dx + \int_{B_\rho(0)\setminus\Omega} |\nabla w|^2\,dx = \int_{B_\rho(0)\setminus\Omega} \nabla w \cdot \nabla((1-\chi)\bar{w})\,dx,$$

which yields

$$\int_{\mathbb{R}^n} |\nabla v|^2 + q|v|^2\,dx = 0$$

and therefore

$$\int_{\mathbb{R}^n} |\nabla v|^2 + (\operatorname{Re} q)|v|^2\,dx = 0.$$

Now, we can apply Hardy's inequality for $H^1(\mathbb{R}^n)$

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^n} |x|^{-2}|v|^2\,dx \le \int_{\mathbb{R}^n} |\nabla v|^2\,dx$$

to finally obtain the condition

$$\int_{\mathbb{R}^n} \left[\frac{(d-2)^2}{4|x|^2} + (\operatorname{Re} q(x))\right] |v|^2\,dx \le 0.$$

Hence, for $\operatorname{Re} q \ge 0$ this implies that $v \equiv 0$ in $\mathbb{R}^n$. Thus $h \equiv 0$ in $\partial\Omega$. Hence we obtain invertibility in the case $\zeta = 0$. Notice that we have been loose on the requirement for q, since this will be enough for the complex-conductivity purposes, but this proof works for potentials that satisfy the estimate $\operatorname{Re} q \ge -\frac{(d-2)^2}{4|x|^2}$.

Part (c) follows quite easily by the fact that the set of invertible operators is open. However, we present the result with the help of some estimates and Neumann series.

For $h \in H^{3/2}(\partial\Omega)$ it holds $K_\zeta h = S_\zeta(\Lambda_q - \Lambda_0)h$, since due to Green's formula we have $B_\zeta = -\frac{1}{2}I + S_\zeta \Lambda_0$.

Moreover, by Lemma 4 and for $h \in H^{3/2}(\partial\Omega)$ we have the decomposition $S_\zeta(\Lambda_q - \Lambda_0)h = S_0(\Lambda_q - \Lambda_0)h + \mathcal{H}_\zeta(\Lambda_q - \Lambda_0)h$. Moreover, we also have by the lemma the estimate

$$\begin{aligned}\|\mathcal{H}_\zeta(\Lambda_q - \Lambda_0)h\|_{H^{3/2}(\partial\Omega)} &\le C|\zeta|^{n-2}\|(\Lambda_q - \Lambda_0)h\|_{H^{1/2}(\partial\Omega)}\\ &\le C|\zeta|^{n-2}\|h\|_{H^{3/2}(\partial\Omega)}.\end{aligned}$$

From the invertibility of $I + K_0$ we obtain the decomposition

$$\begin{aligned}[I + K_\zeta] &= I + K_0 + \mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right) \\ &= (I + K_0)\left(I + (I + K_0)^{-1}\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\right)\end{aligned}$$

and if

$$\| (I + K_0)^{-1}\mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right) \|_{\mathcal{L}(H^{3/2}(\partial\Omega))} < 1$$

we obtain invertibility for $I + K_\zeta$ in $H^{3/2}(\partial\Omega)$. This norm can be translated to an estimate for ζ by the above on $\mathcal{H}_\zeta$. We have

$$\begin{aligned}&\| (I + K_0)^{-1}\mathcal{H}_\zeta \left(\Lambda_q - \Lambda_0\right) \|_{\mathcal{L}(H^{3/2}(\partial\Omega))} \\ &\qquad \leq C|\zeta|^{n-2}\left\|(I + K_0)^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}\left\|\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))} < 1.\end{aligned}$$

Hence, for

$$|\zeta| < \left[\frac{1}{\left\|(I + K_0)^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}\left\|\mathcal{H}_\zeta\left(\Lambda_q - \Lambda_0\right)\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}}\right]^{1/(n-2)} =: \epsilon,$$

invertibility is obtained by Neumann series.

Part (d) uses the existence of exponential growing solutions for large values of $|\zeta|$.

Let $R > 0$ be large enough such that for $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0,\ |\zeta| > R$ we have unique exponential growing solutions of (9), Corollary 1. Under this conditions, we have showed that $K_\zeta := S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I$ is compact in $H^{3/2}(\partial\Omega)$. Therefore, $I + K_\zeta$ is a Fredholm operator of index zero in $H^{3/2}(\partial\Omega)$. We need to show that the kernel is empty to prove that it is invertible.

Let $g \in H^{3/2}(\partial\Omega)$ be in $\ker K$. Then $h = [-S_\zeta\Lambda_q + D_\zeta]g$ solves the exterior problem (i), (ii), (iv) and fulfills the radiation condition (15) (the proof is analogous to Lemma 5).

Moreover, we can extend h to a solution $\tilde{h}$ of $\tilde{h} = -\int_{\mathbb{R}^n} G_\zeta(x-y)q(y)\tilde{h}(y)\,dy$ in all of $\mathbb{R}^n$ (analogous to the previous lemma). By the estimates on G_ζ we note that $e^{-ix\cdot\zeta}\tilde{h} \in L^2_{\delta-1}(\mathbb{R}^n),\ 0 < \delta < 1$ and

$$e^{-ix\cdot\zeta}\tilde{h} = -A_\zeta(e^{-ix\cdot\zeta}\tilde{h})$$

with A_ζ defined as in (13). Since, we took $R > 0$ large enough then A_ζ is a contraction in $L^2_{\delta-1}(\mathbb{R}^n)$ and this forces $\tilde{h} \equiv 0$. Therefore,

$$g \equiv 0 \text{ and } I + K_\zeta \text{ is invertible in } H^{3/2}(\partial\Omega)$$

and the statement is proven. □

Therefore, we can solve the boundary integral equation for small and large values of $|\zeta|$ and obtain ψ on $\partial\Omega$ by

$$\psi(x,\zeta) = \left[\frac{1}{2}I + S_\zeta \Lambda_q - B_\zeta\right]^{-1} \left(e^{ix\cdot\zeta}\right)$$

This allows us to obtain the scattering transform from the boundary data

Theorem 3 *Suppose that Ω is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in L^\infty(\Omega)$ be complex-valued and suppose that* 0 *is not a Dirichlet eigenvalue of* $-\Delta + q$ *in* Ω.

We define the scattering transform for non-exceptional points $\zeta \in \mathcal{V}$ by

$$\mathbf{t}(\xi,\zeta) = \int_{\mathbb{R}^3} e^{-ix\cdot(\zeta+\xi)} q(x)\psi(x,\zeta)\,dx,\ \ \xi \in \mathbb{R}^n. \tag{30}$$

Then, for each $\xi \in \mathbb{R}^n$ we can compute the scattering transform for the non-exceptional points $\zeta \in \mathcal{V}_\xi := \left\{\zeta \in \mathbb{C}^n \setminus \{0\} : \zeta\cdot\zeta = 0,\ |\xi|^2 + 2\zeta\cdot\xi = 0\right\}$ from the solutions of the boundary integral equation by

$$\mathbf{t}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\zeta+\xi)} \left[\Lambda_q + i(\xi+\zeta)\cdot\nu\right]\psi(x,\zeta)\,ds(x),\ \ \xi \in \mathbb{R}^n. \tag{31}$$

Proof From Lemmas 5 and 6 we obtain unique exponentially growing solutions of (9) by the one-to-one relation with the boundary integral (22). Therefore, by Green identity it holds

$$\begin{aligned}
\mathbf{t}(\xi,\zeta) &= \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x,\zeta)\,dx \\
&= \int_\Omega e^{-ix\cdot(\xi+\zeta)} \Delta\psi(x,\zeta) - \left(\Delta e^{-ix\cdot(\zeta+\xi)}\right)\psi(x,\zeta)\,dx \\
&= \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)} \left[\Lambda_q + i(\xi+\zeta)\cdot\nu\right]\psi(x,\zeta)\,ds(x)
\end{aligned}$$

for $\xi \in \mathbb{R}^n$ and $\zeta \in \mathcal{V}_\xi$ such that the boundary integral equation has a unique solution. □

5 From t to γ

From the scattering transform we can obtain the Fourier transform of the potential through large asymptotics. Unfortunately for this we need to solve the boundary integral equation for large values ζ, which is makes this method very unstable. In [20] they avoid the boundary integral equation by using the approximation $\psi(x,\zeta) \approx e^{ix\cdot\zeta}$ to compute the scattering transform. This simplified version was even applied for complex conductivities in order to obtain a stable reconstruction procedure.

This method is based on the following asymptotic

Theorem 4 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in L^\infty(\Omega)$ be a complex-valued potential extended to zero outside Ω, such that 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$. Then for $|\zeta| > R$ and $0 < \delta < 1$*

$$|\mathbf{t}(\xi,\zeta) - \hat{q}(\xi)| \leq \frac{\tilde{c}(\delta, R)}{|\zeta|}\|q\|_\delta^2 \tag{32}$$

for all $\xi \in \mathbb{R}^n$.

Proof The proof follows trivially by Corollary 1. If $q \in L^\infty(\Omega)$ is a complex-valued and compactly supported potential it follows that $\hat{q}$ is well-defined and

$$\begin{aligned}\left|\mathbf{t}(\xi,\zeta) - \hat{q}(\xi)\right| &= \left|\int e^{-ix\cdot\xi} q(x)\left[e^{-ix\cdot\zeta}\psi(x,\zeta) - 1\right] dx\right| \\ &\leq \|q\|_{1-\delta}\|e^{-ix\cdot\zeta}\psi(x,\zeta) - 1\|_{\delta-1} \leq \frac{\tilde{c}(\delta,R)}{|\zeta|}\|q\|_\infty^2\end{aligned}$$

holds true. □

One of the ways to obtain a more stable reconstruction is the so-called d-bar method. Following the $\bar{\partial}$ compatibility equations satisfied by $\mathbf{t}$ known from [4, 21, 28], Nachman was able to derive a d-bar equation in three dimensions, which allows to obtain solutions μ that eventually permit the computation of $\hat{q}$ from $\mathbf{t}(\xi,\zeta)$ for $\xi \in \mathbb{R}^n$, $|\zeta| \geq M$, $(\xi + \zeta)^2 = 0$ and its derivative in ζ. Although more elaborate than in two dimensions, this method does not require taking the limit of $|\zeta| \to \infty$. In essence, the proof follows through for the complex-potential as well!

For such, let $\psi(x,\zeta)$ be the solution of (9) with $e^{-ix\cdot\zeta}\psi(x,\zeta) - 1 \in L^2_{\delta-1}(\mathbb{R}^n)$, that is, ζ is not an exceptional point. Define,

$$\mu(x,\zeta) := |q(x)|e^{-ix\cdot\zeta}\psi(x,\zeta) \tag{33}$$

then μ solves the following integral equation

$$\mu(x,\zeta) = |q(x)| - |q(x)|\int_{\mathbb{R}^n} g_\zeta(x-y)\tilde{q}(y)\mu(y,\zeta)\,dy \tag{34}$$

Hereby, we set

$$\tilde{A}_\zeta f(x) := |q(x)| \int_{\mathbb{R}^n} g_\zeta(x-y)\tilde{q}(y) f(y)\, dy$$

with $\tilde{q}(x) = q(x)/|q(x)|$ in the support of q and 0 otherwise. Moreover, the scattering transform is given through

$$\mathbf{t}(\xi,\zeta) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\tilde{q}(y)\mu(x,\zeta)\, dx. \tag{35}$$

Lemma 7 *Suppose $q \in L^\infty(\mathbb{R}^n)$ with compact support. Let $R > c(\delta,a)\|q(x)\langle x\rangle\|_{L^\infty}$ with $\delta \in (0,1)$ and $c(\delta,a)$ as in Proposition 1.*

(a) If $\zeta \geq R$, $\zeta\cdot\zeta = 0$, then (34) has a unique solution $\mu(\cdot,\zeta)$ in $L^2(\mathbb{R}^n)$ with compact support.
(b) For $|\zeta| > M$, $\zeta\cdot\zeta = 0$ and all $w \in \mathbb{C}^n$ with $w\cdot\bar{\zeta} = 0$,

$$w\cdot\frac{\partial\mu}{\partial\bar{\zeta}}(x,\zeta) = \frac{-1}{(2\pi)^{n-1}}\int e^{ix\cdot\xi} w\cdot\xi\delta(|\xi|^2 + 2\zeta\cdot\xi)t(\xi,\zeta)\mu(x,\zeta+\xi)\, d\xi. \tag{36}$$

(c) For $\zeta \in \mathcal{V}_\xi$ with $|\zeta| > M$ and all $w \in \mathbb{C}^n$ with $w\cdot\bar{\zeta} = 0$ and $w\cdot\xi = 0$,

$$w\cdot\frac{\partial\mathbf{t}}{\partial\bar{\zeta}}(\xi,\zeta) = \frac{-1}{(2\pi)^n}\int w\cdot\eta\delta(\eta^2 + 2\zeta\cdot\eta)\mathbf{t}(\xi-\eta,\zeta+\eta)\mathbf{t}(\eta,\zeta)\, d\eta. \tag{37}$$

Proof For details see Nachman [26]. □

We keep it short here and refer to Nachman [26] for the formula to obtain q without taking limits of the scattering transform.

Our interest resides now in the behavior of exponential growing solutions for ζ close to zero. Due to invertibility of the boundary integral equation we can in fact show that there are no exceptional points near 0. Therefore, analogously to [14] we are able to obtain the following estimate

Lemma 8 *Let $\gamma \in C^{1,1}(\Omega)$ be the complex-conductivity with $\sigma \geq c > 0$, $\epsilon \geq 0$, $\omega \in \mathbb{R}^+$ and suppose $\gamma \equiv 1$ near $\partial\Omega$. Set $q = (\Delta\gamma^{1/2})/\gamma^{1/2} \in L^\infty(\Omega)$.*

For $\zeta \in \mathcal{V}$ sufficiently small and $\phi \in H^{3/2}(\partial\Omega)$ the corresponding boundary integral solution of (22), it holds

$$\|\phi(\cdot,\zeta) - 1\|_{H^{3/2}(\partial\Omega)} \leq C|\zeta|. \tag{38}$$

Proof Let $K_\zeta = S_\zeta \left(\Lambda_q - \Lambda_0\right)$. Solutions of the boundary integral equation fulfill

$$\phi(x,\zeta) - 1 = \left(e^{ix\cdot\zeta} - 1\right) - K_\zeta\left(\phi(x,\zeta) - 1\right),$$

which follows by $\Lambda_q 1 = 0$ and $\Lambda_0 1 = 0$, since the unique H^2-solution of $(-\Delta + q)u = 0,\ u|_{\partial\Omega} = 1$ is $\gamma^{1/2}$ and $w = 1$ is the unique harmonic function in $H^2(\Omega)$ with boundary value 1.

Under the conditions on γ it holds that Re $q > 0$ and hence by Proposition 4 it holds that $[I + K_\zeta]$ is invertible in $H^{3/2}(\partial\Omega)$ for small ζ and hence,

$$\phi - 1 = \left[I + K_\zeta\right]^{-1}\left(e^{ix\cdot\zeta} - 1\right).$$

It clearly holds that by Taylor series that $\|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial\Omega)} \leq C_1|\zeta|$ and $\left\|[I + K_\zeta]^{-1}\right\|_{\mathcal{L}(H^{3/2}(\partial\Omega))}$ is uniformly bounded for small $|\zeta|$ due to Neumann series inversion. Thus,

$$\|\phi - 1\|_{H^{3/2}(\partial\Omega)} \leq C_2\|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial\Omega)} \leq C_3|\zeta|$$

and the statement follows. □

Theorem 5 *Let $\gamma \in C^{1,1}(\Omega)$ be the complex-conductivity with $\sigma \geq c > 0$, $\epsilon \geq 0$, $\omega \in \mathbb{R}^+$ and suppose $\gamma \equiv 1$ near $\partial\Omega$. Set $q = (\Delta\gamma^{1/2})/\gamma^{1/2} \in L^\infty(\Omega)$.*

For $\zeta \in \mathcal{V}$ small enough such that (9) has unique exponentially growing solutions $\psi(x,\zeta)$, it holds

$$\|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{L^2(\Omega)} \leq C|\zeta|. \tag{39}$$

Proof Since $\gamma = 1$ near the boundary $\partial\Omega$ we have that $\gamma^{1/2}$ is the unique $H^2(\Omega)$ solution of

$$\begin{cases} -\Delta u + qu = 0, \text{ in } \Omega \\ u|_{\partial\Omega} = 1. \end{cases}$$

By the elliptic estimates, we obtain that

$$\begin{aligned} \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{L^2(\Omega)} &\leq \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{H^2(\Omega)} \\ &\leq \|\psi(\cdot,\zeta) - \gamma^{1/2}(\cdot)\|_{H^{3/2}(\partial\Omega)} \leq C|\zeta| \end{aligned}$$

and the statement follows. □

This theorem states that we can reconstruct the complex-conductivity from the exponential growing solutions by

$$\gamma(x) = \lim_{|\zeta|\to 0} \psi(x,\zeta), \quad \text{for a.e. } x \in \Omega. \tag{40}$$

However, recall that for small ζ we only know how to obtain the boundary values of the exponential growing solutions from the boundary measurements. To provide a reconstruction of γ in Ω it is necessary to compute these solutions for all ζ small inside Ω from the scattering data. This might be possible by the $\bar{\partial}$-equation.

In order to obtain a $\bar{\partial}$ reconstruction method complex conductivities the following problems need to be solved

1. **Uniqueness of (34) for ζ non-exceptional and considerably small:** A first step is to show that this equation is uniquely solvable for small values of ζ. In Nachman's proof invertibility of the operator $I + \tilde{A}_\zeta$ in $L^2(\mathbb{R}^n)$ follows by that of $I + A_\zeta$ obtained in Corollary 1 for ζ large. In the case of small values, we showed that the integral equation (9) is uniquely solvable due to the unique solvability of the boundary integral equation. However, this will not imply that the operator $I + A_\zeta$ is invertible and therefore we can follow the same approach to prove the existence of a unique solution μ to (34).
2. **Solvability of $\bar{\partial}$-equation:** In [26] there is no proof that $\bar{\partial}-$equation is uniquely solvable, but this is essential since this would be the only equation fully independent of q and where its information is given through the scattering transform. In this sense, we need to study the equation in the space $\mathcal{V} \setminus \{\zeta \in \mathbb{C}^n : \epsilon \leq |\zeta| < R\}$. In the work of [23] they establish this approach in a two-dimensional positive energy setting and intuition could lead to a similar work in our case.

6 Reconstruction of Λ_q from the Boundary Measurements Λ_γ

The Dirichlet-to-Neumann map Λ_γ is bounded from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. Moreover, it is properly defined through

$$\langle \Lambda_\gamma f, g\rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx, \tag{41}$$

where u is the unique $H^1(\Omega)$ solution of the interior problem $\nabla \cdot (\gamma \nabla u) = 0$ in Ω and $u|_{\partial\Omega} = f$ and $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$.

We can also define the Dirichlet-to-Neumann map for the Schrödinger operator by

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$\langle \Lambda_q \tilde{f}, \tilde{g} \rangle = \int_\Omega \nabla\tilde{u} \cdot \nabla\tilde{v} + q\tilde{u}\tilde{v}\, dx, \quad \forall \tilde{v} \in H^1(\Omega), \text{ s.t. } \tilde{v}|_{\partial\Omega} = \tilde{g},$$

and $\tilde{u} \in H^1(\Omega)$ is the unique solution to $(-\Delta + q)\tilde{u} = 0, \text{ in } \Omega, \ \tilde{u}|_{\partial\Omega} = \tilde{f}$.

As in the real case, since both problems are interconnected we can obtain Λ_q from Λ_γ by

$$\Lambda_q = \gamma^{-1/2}\left[\Lambda_\gamma + \frac{1}{2}\frac{\partial\gamma}{\partial\nu}\right]\gamma^{-1/2}. \tag{42}$$

This brings to light that we can determine Λ_q from Λ_γ and the boundary values $\gamma|_{\partial\Omega}$ and $\left.\frac{\partial\gamma}{\partial\nu}\right|_{\partial\Omega}$. Thus, if $\gamma \equiv 1$ near $\partial\Omega$ then for $\gamma \in W^{2,\infty}(\Omega)$ it holds that $\Lambda_q = \Lambda_\gamma$. Otherwise, we need to obtain a method to reconstruct these boundary values.

There are many results to compute these boundary values. However, most of them need further smoothness. Still Nachman holds the best result for our case. In [27] he showed that the boundary values can be obtained without further smoothness assumptions. Following his proof we see that there is no explicit requirement of γ being real, besides the fact that $\gamma \geq c > 0$ and uniqueness of the Dirichlet problem in $H^1(\Omega)$. Hence, we can quickly extend the result for complex-conductivities in $W^{2,\infty}(\Omega)$ with $\operatorname{Re}\gamma \geq c > 0$.

The result is obtained through the following lemmas.

Lemma 9 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$. Assume $\gamma \in W^{1,r}(\Omega)$ for $r > n$ and Re $\gamma \geq c > 0$.*

Then for any $f \in H^{1/2}(\partial\Omega)$ and

$$h \in \mathring{H}^{-1/2}(\partial\Omega) := \left\{h \in H^{-1/2}(\partial\Omega) : \langle h, 1\rangle_{\partial\Omega_j} = 0, \ j = 1, \ldots, N\right\}$$

the identity holds

$$\langle h, (\gamma - \mathcal{R}\Lambda_\gamma)f\rangle = \int_\Omega u\nabla w \cdot \nabla\gamma, \tag{43}$$

where $u \in H^1(\Omega)$ solution of $\nabla\cdot(\gamma\nabla u) = 0, \ u|_{\partial\Omega} = f$, and $w \in H^1(\Omega)$ is a weak solution of $\Delta w = 0$ in Ω with $\frac{\partial w}{\partial\nu} = h$ and $\mathcal{R}$ denotes the Neumann-to-Dirichlet map.

Lemma 10 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$. Assume γ is in $W^{2,p}(\Omega)$, $p > n/2$ and Re $\gamma \geq c > 0$.*

For any $f,\ g \in H^{1/2}(\partial\Omega)$ the identity holds

$$\langle g, \left(2\Lambda_\gamma - \Lambda_1\gamma - \gamma\Lambda_1 + \frac{\partial\gamma}{\partial\nu}\right) f\rangle = \int_\Omega 2v\nabla(u-u_0)\cdot\nabla\gamma + v(2u-u_0)\Delta\gamma\, dx$$

where u, u_0, v are respectively the $H^1(\Omega)$ solutions of $L_\gamma(u) = 0$, $\Delta u_0 = 0$ and $\Delta v = 0$, in Ω, with $u|_{\partial\Omega} = f$, $u_0|_{\partial\Omega} = f$ and $v|_{\partial\Omega} = g$.

From this we obtain the boundary reconstruction formulas.

Theorem 6 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n, n \geq 2$. Suppose $\gamma \in W^{1,r}(\Omega)$, $r > n$ and Re $\gamma \geq c > 0$.*

(i) $\gamma|_{\partial\Omega\cap U}$ can be recovered from Λ_γ by

$$\langle h, \gamma f\rangle = \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle h_\eta, \mathcal{R}\Lambda_\gamma e^{-i\langle\cdot,\eta\rangle} f\rangle, \tag{44}$$

with $f \in H^{1/2}(\partial\Omega) \cap C(\partial\Omega)$ and $h \in L^2(\Omega)$ supported in $U \cap \partial\Omega$ and h_η is defined as zero outside $\partial\Omega \cap U$ and

$$h_\eta(x) = h(x)e^{-ix\cdot\eta} - \frac{1}{|\partial\Omega\cap U|}\int_{\partial\Omega\cap U} h(y)e^{-iy\cdot\eta}\, dy, \quad \text{for } x \in \partial\Omega\cap U.$$

(ii) If $\gamma \in W^{2,r}$, $r > n/2$, then for any continuous function f, g in $H^{1/2}(\partial\Omega)$ with support in $\partial\Omega \cap \partial\Omega$ holds

$$\langle g, \frac{\partial\gamma}{\partial\nu} f\rangle = \lim_{\substack{|\eta|\to\infty \\ \eta\in\mathbb{R}^{n-1}\times\{0\}}} \langle g, e^{-i\langle\cdot,\eta\rangle}\left(\gamma\Lambda_1 + \Lambda_1\gamma - 2\Lambda_\gamma\right) e^{i\langle\cdot,\eta\rangle} f\rangle. \tag{45}$$

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