Nonintegrability of the Problem of Motion of an Ellipsoidal Body with a Fixed Point in a Flow of Particles

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Abstract. The problem of motion in the free molecular flow of particles of a rigid body with a fixed point, bounded by the surface of an ellipsoid of revolution is considered. This problem is similar in many aspects to the classical problem of motion of a heavy rigid body about a fixed point. In particular, this problem possesses the integrable cases, correspond to the classical Euler – Poinsot, Lagrange and Hess cases of integrability of equations of motion of a heavy rigid body with a fixed point. Equations of motion of the body in the flow of particles are presented in hamiltonian form. Using the theorem on the Liouville – type nonintegrability of Hamiltonian systems near elliptic equilibrium positions we present the necessary conditions for the existence in the considered problem of an additional analytic first integral independent of the energy integral. We proved that the obtained necessary conditions are not fulfilled for the rigid body with a mass distribution corresponding to the classical Kovalevskaya integrable case in the problem of motion of a heavy rigid body with a fixed point.

Keywords: Rigid body with a fixed point · Free molecular flow of particles · Hamiltonian system · Nonintegrability

1 Introduction. V. V. Kozlov's Theorem on the Nonexistence of Analytic First Integral Near the Equilibrium Position of Hamiltonian System

In 1976 V.V. Kozlov in his paper [1] (see also [2,3]), proved the theorem, which gives the sufficient conditions of the nonexistence for the Hamiltonian system the analytic with respect to canonical variables first integral, independent with Hamilton function H. Below we give the statement of the problem using the notations from [1] and the formulation of the corresponding theorem.

Let us consider the system of canonical equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots n, \quad n \geq 2 \tag{1}$$

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with the Hamilton function $H(y_1, \dots, y_n, x_1, \dots, x_n, \boldsymbol{\varepsilon})$, depending analytically on the variables $\boldsymbol{y} = (y_1, \dots, y_n)$, $\boldsymbol{x} = (x_1, \dots, x_n)$ and on the parameter $\boldsymbol{\varepsilon}$, which takes values in some connected domain $D \in \mathbb{R}^r$. Suppose that for all $\boldsymbol{\varepsilon}$ the point $y_i = 0$, $x_i = 0$, $(i = 1, \dots, n)$ be an equilibrium position of the system (1). In the vicinity of an equilibrium position $y_i = 0$, $x_i = 0$, $(i = 1, \dots, n)$ the Hamilton function H can be represented as follows:

$$H = H^{(2)} + H^{(3)} + \cdots ,$$

where $H^{(s)}$ is a homogeneous form of degree s with respect to $\boldsymbol{y} = (y_1, \dots, y_n)$ and $\boldsymbol{x} = (x_1, \dots, x_n)$. The coefficients of this expansion are analytic functions of the parameter $\boldsymbol{\varepsilon}$. Let us assume that for all $\boldsymbol{\varepsilon} \in D$ the frequencies of linear oscillations $\boldsymbol{\omega}(\boldsymbol{\varepsilon}) = (\omega_1(\boldsymbol{\varepsilon}), \dots, \omega_n(\boldsymbol{\varepsilon}))$ do not satisfy any resonant relation

$$(\boldsymbol{m} \cdot \boldsymbol{\omega}) = m_1\omega_1 + \cdots + m_n\omega_n = 0$$

of order $|m_1| + \cdots + |m_n| \le m - 1$. Using Birkhoff's normalization method (see, for example [4,5]), we can find a canonical transformation $(\boldsymbol{y}, \boldsymbol{x}) \to (\boldsymbol{p}, \boldsymbol{q})$, such that in the new variables

$$H^{(2)} = \frac{1}{2}\sum_{i=1}^{n} \omega_i \rho_i, \quad H^{(k)} = H^{(k)}(\rho_1, \dots, \rho_n, \boldsymbol{\varepsilon}), \quad k \le m - 1,$$

where $\rho_i = p_i^2 + q_i^2$. The corresponding transformation is analytic in $\boldsymbol{\varepsilon}$. Now we introduce the canonical action – angle variables $(\boldsymbol{I}, \boldsymbol{\varphi})$ by the formulas:

$$I_i = \frac{\rho_i}{2}, \quad \varphi_i = \arctan\frac{p_i}{q_i}, \quad (1 \le i \le n).$$

In the canonical variables $(\boldsymbol{I}, \boldsymbol{\varphi})$ we have

$$H = H^{(2)}(\boldsymbol{I}, \boldsymbol{\varepsilon}) + \cdots + H^{(m-1)}(\boldsymbol{I}, \boldsymbol{\varepsilon}) + H^{(m)}(\boldsymbol{I}, \boldsymbol{\varphi}, \boldsymbol{\varepsilon}) + \cdots$$

We represent the trigonometric polynomial $H^{(m)}$ as a finite Fourier series

$$H^{(m)} = \sum h_{\boldsymbol{k}}^{(m)}(\boldsymbol{I}, \boldsymbol{\varepsilon}) \exp(i(\boldsymbol{k} \cdot \boldsymbol{\varphi})).$$

Theorem 1 (V. V. Kozlov [1–3]). *Let $(\boldsymbol{k} \cdot \boldsymbol{\omega}(\boldsymbol{\varepsilon})) \not\equiv 0$ for all $\boldsymbol{k} \in \mathbb{Z}^n \backslash \boldsymbol{0}$. Suppose that for some $\boldsymbol{\varepsilon}_0 \in D$ the resonant relation $(\boldsymbol{k}_0 \cdot \boldsymbol{\omega}(\boldsymbol{\varepsilon}_0)) = 0$, $|\boldsymbol{k}_0| = m$ is satisfied and $h_{\boldsymbol{k}_0}^{(m)} \not\equiv 0$. Then the canonical Eqs. (1) with Hamilton function $H = \sum H^{(s)}$ do not have a complete set of (formal) integrals $F_j = \sum F_j^{(s)}$, whose quadratic terms $F_j^{(2)}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\varepsilon})$ are independent for all $\boldsymbol{\varepsilon} \in D$.* □

Remark 1. Note that under the assumptions of the V. V. Kozlov's Theorem 1 there may exist independent integrals with dependent (for certain values of $\boldsymbol{\varepsilon}$)

quadratic parts of their Maclaurin expansions. Here is a simple example: the canonical equations with Hamilton function

$$H = \frac{1}{2}\left(x_1^2 + y_1^2\right) + \frac{\alpha}{2}\left(x_2^2 + y_2^2\right) + 2x_1y_1y_2 - x_2y_1^2 + x_1^2x_2$$

have a first integral

$$F = x_1^2 + y_1^2 + 2\left(x_2^2 + y_2^2\right).$$

For $\alpha = 2$, it is dependent on the quadratic form $H^{(2)}$. However, all conditions of the Theorem 1 are satisfied. □

The advantage of the V. V. Kozlov's Theorem 1 consists in the absence of preliminary restrictive assumptions regarding the parameters of the system. This advantage substantially compensates for the fact that the additional integral must belong to the class of analytic functions, the quadratic part of which are functionally independent with the quadratic part of the Hamilton function.

V. V. Kozlov's Theorem 1 was successfully applied for proving the nonexistence of an additional first integral in the plane circular restricted three body problem [1–3]; for studying the integrability of the problem of motion about a fixed point of a dynamically symmetric rigid body with the center of mass lies in the equatorial plane of the ellipsoid of inertia [1,3,6]; for proving the nonexistence of an additional integral in the problem of motion of a plane heavy double pendulum [6–8]; for obtaining the necessary conditions for the existence of an additional first integral in the problem of motion of a dynamically symmetric ellipsoid on a smooth horizontal plane [9]; for the study of nonintegrability of the Kirchhoff equations of motion of a rigid body in a fluid [10,11].

In this paper V. V. Kozlov's Theorem 1 is used to derive necessary conditions for the existence of an additional analytic integral in the problem of motion in the flow of particles of a rigid body with a fixed point bounded by the surface of an ellipsoid of revolution.

2 Formulation of the Problem. Hamilton Function of the Problem

Equations of motion of a rigid body with a fixed point, bounded by the surface of an ellipsoid and exposed by the flow of particles, have the form [12,13]:

$$\begin{aligned}
&A_1\dot{\omega}_1 + (A_3 - A_2)\,\omega_2\omega_3 = \rho v_0^2 \pi a_1 a_2 a_3 \sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\,(h_2\gamma_3 - h_3\gamma_2)\,,\\
&A_2\dot{\omega}_2 + (A_1 - A_3)\,\omega_1\omega_3 = \rho v_0^2 \pi a_1 a_2 a_3 \sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\,(h_3\gamma_1 - h_1\gamma_3)\,,\\
&A_3\dot{\omega}_3 + (A_2 - A_1)\,\omega_1\omega_2 = \rho v_0^2 \pi a_1 a_2 a_3 \sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\,(h_1\gamma_2 - h_2\gamma_1)\,;\\
&\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \quad \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \quad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2.
\end{aligned} \tag{2}$$

Here A_1, A_2, A_3 are the moments of inertia of the body about the principal axes of inertia $Ox_1x_2x_3$ with the origin at the fixed point O; $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector of the body; $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector directed along the flow of particles; ρ is the constant density of the flow of particles; v_0 is the constant velocity of particles in the flow, a_1, a_2, a_3 are the lengths of the semiaxes of the ellipsoid, bounding a rigid body; $\boldsymbol{h} = (h_1, h_2, h_3)$ is the vector directed from a fixed point to the center of the ellipsoid bounding the rigid body.

For any values of parameters Eqs. (2) possess the first integrals:

$$J_1 = A_1\omega_1\gamma_1 + A_2\omega_2\gamma_2 + A_3\omega_3\gamma_3 = \mathrm{c}_1 = \mathrm{const}, \quad J_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \quad (3)$$

Let us assume that the center of the ellipsoid lies on the first principal axis of inertia Ox_1 with the origin at the fixed point O, at a distance l from the fixed point. In other words, in the Eqs. (2) we put

$$h_1 = l, \quad h_2 = 0, \quad h_3 = 0.$$

We also assume that the ellipsoid bounding the rigid body is an ellipsoid of revolution with the axis of symmetry passing through the fixed point O. Therefore in the Eq. (2) we put

$$a_1 = b, \quad a_2 = a_3 = a.$$

In addition we assume, that the body is dynamically symmetric, and the axis of dynamical symmetry of the body does not coincide with the axis of symmetry of the ellipsoid, that bounds the body. In other words we assume, that

$$A_1 = A_2 = A, \quad A_3 = C.$$

Then the equations of motion in the flow of particles of a rigid body with a fixed point bounded by the surface of an ellipsoid of revolution will be rewritten as follows:

$$\begin{aligned}
&A\dot{\omega}_1 + (C - A)\,\omega_2\omega_3 = 0,\\
&A\dot{\omega}_2 + (A - C)\,\omega_1\omega_3 = -\rho v_0^2 \pi a^2 bl\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}\gamma_3,\\
&C\dot{\omega}_3 = \rho v_0^2 \pi a^2 bl\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}\gamma_2;\\
&\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \quad \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \quad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2.
\end{aligned} \quad (4)$$

We multiply the first equation of system (4) by ω_1, the second—by ω_2, the third—by ω_3 and add them. As a result we get the following equation:

$$A\,(\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2) + C\omega_3\dot{\omega}_3 = \rho v_0^2\pi a^2 bl\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}\,(\omega_3\gamma_2 - \omega_2\gamma_3) = \rho v_0^2\pi a^2 bl\dot{\gamma}_1\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}.$$

Thus we can conclude that Eqs. (4) admit in addition to first integrals (3) the energy type first integral

$$H = \frac{A}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{C}{2}\omega_3^2 - G\,(\gamma_1) = h = \mathrm{const}.$$

The function $G(\gamma_1)$ is written differently depending on whether the ellipsoid, bounding the rigid body is prolate $(b > a)$ or oblate $(a > b)$. For a prolate ellipsoid of revolution $(b > a)$, the function $G(\gamma_1)$ has the form:

$$G(\gamma_1) = \frac{\rho v_0^2 \pi a^2 bl}{2} \gamma_1 \sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}} + \frac{\rho v_0^2 \pi bl}{2\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}} \arctan\left(\frac{\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}\,\gamma_1}{\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}}} \right).$$

For an oblate ellipsoid of revolution $(a > b)$, the function $G(\gamma_1)$ has the form:

$$G(\gamma_1) = \frac{\rho v_0^2 \pi a^2 bl}{2} \gamma_1 \sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}} + \frac{\rho v_0^2 \pi bl}{2\sqrt{\frac{1}{b^2} - \frac{1}{a^2}}} \ln\left(a\sqrt{\frac{1}{b^2} - \frac{1}{a^2}}\,\gamma_1 + a\sqrt{\frac{1-\gamma_1^2}{a^2} + \frac{\gamma_1^2}{b^2}} \right).$$

Further we will consider the case of a prolate ellipsoid of revolution (the case of an oblate ellipsoid of revolution is considered in a similar way and gives the same result). As generalized coordinates in this problem we introduce the standard Euler angles θ, ψ and φ. Then we have

$$\gamma_1 = \sin\theta \sin\varphi, \quad \gamma_2 = \sin\theta \cos\varphi, \quad \gamma_3 = \cos\theta$$

and the Hamilton function of the problem in standard notations has the form:

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{A} + \frac{p_\varphi^2}{C} + \frac{(p_\psi - p_\varphi \cos\theta)^2}{A \sin^2\theta} \right) - \frac{\rho v_0^2 \pi a^2 bl}{2} \sin\theta \sin\varphi \sqrt{\frac{1 - \sin^2\theta \sin^2\varphi}{a^2} + \frac{\sin^2\theta \sin^2\varphi}{b^2}} -$$
$$- \frac{\rho v_0^2 \pi bl}{2\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}} \arctan\left(\frac{\sqrt{\frac{1}{a^2} - \frac{1}{b^2}} \sin\theta \sin\varphi}{\sqrt{\frac{1 - \sin^2\theta \sin^2\varphi}{a^2} + \frac{\sin^2\theta \sin^2\varphi}{b^2}}} \right). \tag{5}$$

Obviously, the Hamilton function H does not depend on the generalized coordinate ψ, that is the generalized momentum p_ψ is a constant. The generalized momentum p_ψ is the area integral J_1 (see (3)). The equations of motion of the body have a hamiltonian form with the Hamilton function (5), in which p_ψ is a parameter. We will assume that the parameter p_ψ is the parameter that was mentioned in the statement of the V. V. Kozlov's Theorem 1. Let us obtain the necessary conditions for the existence of an additional first integral, analytic in p_ψ and independent of the Hamilton function H.

3 Application of V. V. Kozlov's Theorem 1

For any value of p_ψ the point

$$(p_\theta,\, p_\varphi,\, \theta,\, \varphi) = \left(0,\, 0,\, \frac{\pi}{2},\, \frac{\pi}{2}\right) -$$

is the equilibrium of the considered Hamiltonian system. We denote

$$p_\theta = y_1, \quad p_\varphi = y_2, \quad \theta = \frac{\pi}{2} + x_1, \quad \varphi = \frac{\pi}{2} + x_2.$$

The units of measurement can always be chosen so, that

$$\pi \rho v_0^2 l a^2 = 1, \quad A = 1.$$

We introduce also the following parameters:

$$p_\psi = \sqrt{x}, \quad \frac{1}{C} = y, \quad \frac{b^2}{a^2} = z.$$

Then $(x,\, y,\, z)$ are change in the domain $\mathbb{R}^3_+ = \{x,\, y,\, z : x > 0,\, y > 0,\, z > 0\}$. In a neighborhood of the equilibrium point $y_1 = 0,\, y_2 = 0,\, x_1 = 0,\, x_2 = 0$ the expansion of the Hamilton function (5) has the form:

$$H = H^{(2)} + H^{(3)} + H^{(4)} + \cdots,$$

$$H^{(2)}(y_1,\, y_2,\, x_1,\, x_2) = \frac{1}{2} y_1^2 + \frac{y}{2} y_2^2 + \sqrt{x} x_1 y_2 + \frac{(1+x)}{2} x_1^2 + \frac{1}{2} x_2^2,$$

$$H^{(3)}(y_1,\, y_2,\, x_1,\, x_2) = 0,$$

$$H^{(4)}(y_1,\, y_2,\, x_1,\, x_2) = \frac{1}{2} x_1^2 y_2^2 + \frac{5}{6}\sqrt{x} x_1^3 y_2 + \left(\frac{z}{4} - \frac{1}{2}\right) x_1^2 x_2^2 + \left(\frac{x}{3} + \frac{z}{8} - \frac{1}{6}\right) x_1^4 + \left(\frac{z}{8} - \frac{1}{6}\right) x_2^4.$$

Note that in the case of $z = 1$, i.e. when the rigid body is bounded by the sphere, the expressions $H^{(2)}(y_1,\, y_2,\, x_1,\, x_2)$ and $H^{(4)}(y_1,\, y_2,\, x_1,\, x_2)$ exactly coincide with the corresponding expressions obtained by V. V. Kozlov [1–3] when studying the problem of motion of a heavy dynamically symmetric rigid body with a fixed point, with the center of mass situated in the equatorial plane of the ellipsoid of inertia.

Equations of motion of the system with the Hamilton function $H^{(2)}$ has the form of the linearized equations of the system, namely

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{x} & -(x+1) & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & y & \sqrt{x} & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix}. \tag{6}$$

The characteristic equation for determining the natural frequencies of the linear system (6) with the Hamilton function $H = H^{(2)}$ is written as follows:

$$\lambda^4 + (1 + x + y)\,\lambda^2 + y\,(1 + x) - x = 0. \tag{7}$$

Obviously, the roots of the characteristic equation are purely imaginary if

$$y > \frac{x}{1+x}.$$

Let us denote by E the subset of $\mathbb{R}_+^2$, where this inequality is satisfied. The characteristic Eq. (7) is biquadratic, therefore, if the frequency ratio is three, then the ratio of the squares of the frequencies should be nine. Calculating the squares of the frequencies and equating their ratio to nine, we obtain the following condition for the parameters x and y:

$$4\,(1 + x + y) = 5\sqrt{1 + 6x - 2y + x^2 - 2xy + y^2}. \tag{8}$$

Therefore, squaring both sides of this equation and subtracting the left side from the right side, we find that the ratio of the frequencies $\lambda_1/\lambda_2 = 3$ if the parameters x and y are connected by the following equation

$$9x^2 - 82xy + 9y^2 + 118x - 82y + 9 = 0. \tag{9}$$

This is the equation of a hyperbola; for $x > 0$ and $y > 0$ its branches are entirely in E.

From the triangle inequality for the moments of inertia ($A_1 + A_2 \geq A_3$) it follows, that $y \geq 1/2$. For any fixed $y_0 \geq 1/2$, there exists $x_0 > 0$, such that the point $(x_0,\, y_0)$ satisfies Eq. (9). Consider a small interval $(a,\, b)$ of variation of the parameter x, including the point x_0. For $x \in (a,\, b)$, $y = y_0$ the roots of the characteristic equation are purely imaginary and distinct. When $x = x_0$, then the frequencies λ_1 and λ_2 are connected by the equation $\lambda_1 - 3\lambda_2 = 0$. It remains to find out, when the secular coefficient $h^{(4)}_{1,-3}$ is zero.

To calculate the coefficient $h^{(4)}_{1,-3}$ let us make the canonical change of variables $(y_1,\, y_2,\, x_1,\, x_2) \to (p_1,\, p_2,\, q_1,\, q_2)$ such, that in the new variables the quadratic part $H^{(2)}$ of the Hamilton function H is represented in the form:

$$H^{(2)} = \frac{B_1}{2}p_1^2 + \frac{K_1}{2}q_1^2 + \frac{B_2}{2}p_2^2 + \frac{K_2}{2}q_2^2,$$

where B_i and K_i, $(i = 1, 2)$ are coefficients to be determined.

The required change of variables in linear with respect to the variables p_1, p_2, q_1, q_2. Let us represent it in the most general form, namely:

$$\begin{aligned} y_1 &= \alpha_1 p_1 + \beta_1 p_2 + \xi_1 q_1 + \eta_1 q_2, \quad & y_2 &= \alpha_2 p_1 + \beta_2 p_2 + \xi_2 q_1 + \eta_2 q_2, \\ x_1 &= \alpha_3 p_1 + \beta_3 p_2 + \xi_3 q_1 + \eta_3 q_2, \quad & x_2 &= \alpha_4 p_1 + \beta_4 p_2 + \xi_4 q_1 + \eta_4 q_2. \end{aligned} \tag{10}$$

This change of variables must satisfy two properties:

1. it should be a canonical transformation;
2. in the new variables the expression $H^{(2)}$ do not contain the mixed products $p_1 p_2,\, p_1 q_1,\, p_1 q_2,\, p_2 q_1,\, p_2 q_2,\, q_1 q_2$.

Using the standard condition of the canonicity of the change of variables in the Hamiltonian system (see, for example, [14,15])

$$p_1 dq_1 + p_2 dq_2 - y_1 dx_1 - y_2 dx_2 = -dF$$

it can be shown that a linear change of variables (10) will be canonical transformation if the following conditions are satisfied:

$$\begin{aligned}
&\beta_1\alpha_3 + \beta_2\alpha_4 - \beta_3\alpha_1 - \beta_4\alpha_2 = 0, \quad \xi_1\alpha_3 + \xi_2\alpha_4 - \xi_3\alpha_1 - \xi_4\alpha_2 + 1 = 0,\\
&\eta_1\alpha_3 + \eta_2\alpha_4 - \eta_3\alpha_1 - \eta_4\alpha_2 = 0, \quad \xi_1\beta_3 + \xi_2\beta_4 - \xi_3\beta_1 - \xi_4\beta_2 = 0,\\
&\eta_1\beta_3 + \eta_2\beta_4 - \eta_3\beta_1 - \eta_4\beta_2 + 1 = 0, \quad \eta_1\xi_3 + \eta_2\xi_4 - \eta_3\xi_1 - \eta_4\xi_2 = 0.
\end{aligned} \tag{11}$$

In addition to these six equations, we should write down the condition for the vanishing of the coefficients of the mixed terms in the Hamilton function $H^{(2)}$, written in the variables p_1, p_2, q_1, q_2 (there also be six such mixed members: p_1p_2, p_1q_1, p_1q_2, p_2q_1, p_2q_2, q_1q_2). These conditions are as follows:

$$\begin{aligned}
&\xi_1\eta_1 + \xi_3\eta_3 + \xi_4\eta_4 + \sqrt{x}\left(\xi_2\eta_3 + \xi_3\eta_2\right) + x\xi_3\eta_3 + y\xi_2\eta_2 = 0,\\
&\alpha_1\xi_1 + \alpha_3\xi_3 + \alpha_4\xi_4 + \sqrt{x}\left(\alpha_2\xi_3 + \alpha_3\xi_2\right) + x\alpha_3\xi_3 + y\alpha_2\xi_2 = 0,\\
&\beta_1\xi_1 + \beta_3\xi_3 + \beta_4\xi_4 + \sqrt{x}\left(\beta_2\xi_3 + \beta_3\xi_2\right) + x\beta_3\xi_3 + y\beta_2\xi_2 = 0,\\
&\alpha_1\eta_1 + \alpha_3\eta_3 + \alpha_4\eta_4 + \sqrt{x}\left(\alpha_2\eta_3 + \alpha_3\eta_2\right) + x\alpha_3\eta_3 + y\alpha_2\eta_2 = 0,\\
&\alpha_1\beta_1 + \alpha_3\beta_3 + \alpha_4\beta_4 + \sqrt{x}\left(\alpha_2\beta_3 + \alpha_3\beta_2\right) + x\alpha_3\beta_3 + y\alpha_2\beta_2 = 0,\\
&\beta_1\eta_1 + \beta_3\eta_3 + \beta_4\eta_4 + \sqrt{x}\left(\beta_2\eta_3 + \beta_3\eta_2\right) + x\beta_3\eta_3 + y\beta_2\eta_2 = 0.
\end{aligned} \tag{12}$$

Thus we have 12 Eqs. (11)–(12) on the 16 unknown coefficients α_i, β_i, ξ_i and η_i, $i = 1, \ldots, 4$. In order for the number of equations to be equal to the number of unknown coefficients, we assume from the very beginning that

$$\beta_1 = 0, \quad \alpha_2 = 0, \quad \eta_3 = 0, \quad \xi_4 = 0.$$

The solution of the obtained system of 12 Eqs. (11)–(12) with respect to 12 unknown coefficients α_1, α_3, α_4, β_2, β_3, β_4, ξ_1, ξ_2, ξ_3 and η_1, η_2, η_4 was found using the software for symbolic computations MAPLE 7. It turned out, that the solution has the form:

$$\xi_1 = 0, \quad \eta_2 = 0, \quad \alpha_3 = 0, \quad \beta_4 = 0, \quad \xi_2 = \Delta\xi_3,$$

$$\alpha_1 = \frac{\sqrt{x}}{\xi_3\left(2\sqrt{x} + (y-1-x)\,\Delta\right)}, \quad \eta_1 = \frac{\Delta\sqrt{x}}{\beta_2\left(2\sqrt{x} + (y-1-x)\,\Delta\right)},$$

$$\beta_3 = -\frac{\beta_2\left(\sqrt{x} + (y-1-x)\,\Delta\right)}{\Delta\sqrt{x}}, \quad \alpha_4 = -\frac{\sqrt{x} + (y-1-x)\,\Delta}{\xi_3\Delta\left(2\sqrt{x} + (y-1-x)\,\Delta\right)},$$

$$\eta_4 = \frac{\sqrt{x}}{\beta_2\left(2\sqrt{x} + (y-1-x)\,\Delta\right)},$$

where β_2 and ξ_3 are free parameters, and $\varDelta$ is the positive root of the quadratic equation:

$$\sqrt{x}\varDelta^2 + (x+1-y)\,\varDelta - \sqrt{x} = 0$$

We will assume that the free parameters take the following values:

$$\beta_2 = \frac{\varDelta\sqrt{x}}{\left(\sqrt{x} + (y-1-x)\,\varDelta\right)}, \quad \xi_3 = 1.$$

For these values of the free parameters, the linear canonical transformation $(y_1,\, y_2,\, x_1,\, x_2) \to (p_1,\, p_2,\, q_1,\, q_2)$ takes the most simple form

$$y_1 = \frac{1}{1+\varDelta^2}p_1 + \frac{\varDelta^2}{1+\varDelta^2}q_2, \quad y_2 = \frac{1}{\varDelta}p_2 + \varDelta q_1, \quad x_1 = q_1 - p_2, \quad x_2 = \frac{\varDelta}{1+\varDelta^2}\left(q_2 - p_1\right)$$

The quadratic part $H^{(2)}$ of the Hamilton function H is represented as follows:

$$H^{(2)} = \frac{B_1}{2}p_1^2 + \frac{K_1}{2}q_1^2 + \frac{B_2}{2}p_2^2 + \frac{K_2}{2}q_2^2,$$

$$B_1 = \frac{1}{1+\varDelta^2}, \quad B_2 = \frac{y - 2\varDelta\sqrt{x} + (1+x)\,\varDelta^2}{\varDelta^2} = \frac{\left(1+\varDelta^2\right)\left(y - \sqrt{x}\varDelta\right)}{\varDelta^2},$$

$$K_1 = \varDelta^2 y + 2\varDelta\sqrt{x} + 1 + x = \left(1+\varDelta^2\right)\left(y + \frac{\sqrt{x}}{\varDelta}\right), \quad K_2 = \frac{\varDelta^2}{1+\varDelta^2}.$$

Now we introduce action – angle variables $(\boldsymbol{I},\, \boldsymbol{\varphi})$ by the formulas:

$$q_1 = i\sqrt{\frac{I_1}{2}\sqrt{\frac{B_1}{K_1}}}\left(\exp\left(-i\varphi_1\right) - \exp\left(i\varphi_1\right)\right), \quad p_1 = \sqrt{\frac{I_1}{2}\sqrt{\frac{K_1}{B_1}}}\left(\exp\left(i\varphi_1\right) + \exp\left(-i\varphi_1\right)\right),$$

$$q_2 = i\sqrt{\frac{I_2}{2}\sqrt{\frac{B_2}{K_2}}}\left(\exp\left(-i\varphi_2\right) - \exp\left(i\varphi_2\right)\right), \quad p_2 = \sqrt{\frac{I_2}{2}\sqrt{\frac{K_2}{B_2}}}\left(\exp\left(i\varphi_2\right) + \exp\left(-i\varphi_2\right)\right).$$

Here i is the unit imaginary number. In the new variables the form $H^{(4)}$ will be written as follows:

$$H^{(4)} = \sum_{0 \le |m_1| + |m_2| \le 4} h^{(4)}_{m_1,m_2} \exp\left(i(m_1\varphi_1 + m_2\varphi_2)\right).$$

Let us calculate now the coefficient $h^{(4)}_{1,-3}$ explicitly. Note, that the exponent $\exp\left(i\left(\varphi_1 - 3\varphi_2\right)\right)$ can only appear in the following expressions: $p_1p_2^3$, $p_1p_2^2q_2$, $p_1p_2q_2^2$, $p_1q_2^3$, $q_1p_2^3$, $q_1p_2^2q_2$, $q_1p_2q_2^2$, $q_1q_2^3$.

This remark greatly simplifies the process of calculating the coefficient $h^{(4)}_{1,-3}$. The condition for this coefficient to be zero can be written as follows:

$$\begin{aligned} &5\sqrt{x}\varDelta^3 + (3z + 8x - 10)\,\varDelta^2 + 3\,(z-7)\,\sqrt{x}\varDelta + 6y - 3zy + 6 = \\ &= \left((4-3z)\left(y - \varDelta\sqrt{x}\right) + 3\,(z-2)\,\varDelta^2\right)\sqrt{xy + y - x}. \end{aligned} \tag{13}$$

Further simplifications of the Eq. (13) are based on the Eqs. (8)–(9) and also on the equations

$$\sqrt{xy + y - x} = \frac{3}{10}\left(1 + x + y\right), \quad \varDelta = \frac{9y - x - 1}{10\sqrt{x}},$$

which can be derived by direct calculations from Eqs. (8)–(9) and from the definition of the parameter $\varDelta$.

Finally, the condition for vanishing of the coefficient $h^{(4)}_{1,-3}$ in the expansion of the function $H^{(4)}$ can be reduced to the following form:

$$27x^3z + 111x^2yz - 159xy^2z - 243y^3z - 9x^3 - 617x^2y - 39x^2z + 2093xy^2 - 118xyz + 1701y^3 + \\ + 621y^2z + 653x^2 - 4374xy - 59xz - 2727y^2 - 129yz + 2633x + 543y + 7z - 29 = 0. \tag{14}$$

Thus, the following theorem is valid.

Theorem 2. *Necessary conditions for the existence of an additional integral, analytic in canonical variables and the parameter x and independent with the Hamilton function H, in the problem of motion in the flow of particles of a dynamically symmetric rigid body with a fixed point, bounded by the surface of an ellipsoid of revolution, whose center lies in the equatorial plane of the ellipsoid of inertia, have the form of Eqs.* (9), (14).

Remark 2. For $z = 1$ i.e. in the case when the rigid body is bounded by a sphere, the conditions (9), (14) take the form

$$9x^2 - 82xy + 9y^2 + 118x - 82y + 9 = 0, \tag{15}$$

$$18x^3 - 506x^2y + 1934xy^2 + 1458y^3 + 614x^2 - 4492xy - 2106y^2 + 2574x + 414y - 22 = 0, \tag{16}$$

and coincide with the necessary conditions for the existence of an additional integral in the problem of motion of a heavy dynamically symmetric rigid body with a fixed point and with the center of mass situated in the equatorial plane of the ellipsoid of inertia, obtained by V. V. Kozlov [1–3,6]. Algebraic curves (15) and (16) intersect at two points (x, y):

$$\left(\frac{4}{3}, 1\right) \quad \text{and} \quad (7, 2),$$

which correspond to the Lagrange integrable case ($A = C$) and Kovalevskaya integrable case ($A = 2C$). □

Let us put in the conditions (9), (14) $y = 2$, i.e. consider a rigid body with the mass distribution corresponding to the Kovalevskaya integrable case in the problem of motion of a heavy rigid body with a fixed point. Then the condition (9) takes the form:

$$(9x + 17)(x - 7) = 0,$$

and can only be valid if $x = 7$. Substituting the values $x = 7$ and $y = 2$ into the condition (14) gives

$$12000\,(z-1) = 0.$$

Thus, for a rigid body with a mass distribution corresponding to the Kovalevskaya case, an additional first integral, independent of the energy integral, can exist only when the rigid body is bounded by a sphere. In the case, when a rigid body, exposed by the flow of particles, is bounded by the ellipsoid, there is no additional first integral.

Analysis of Eqs. (9), (14), performed using MAPLE 7 symbolic computations software, shows that this system has solutions

$$x = 0,\ y = \frac{1}{9}; \quad x = -\frac{16}{3},\ y = 1; \quad x = \frac{4}{3},\ y = 1. \tag{17}$$

existing for any value of the parameter z. The first two of the solutions (17) do not satisfy the conditions

$$x > 0, \quad y \geq \frac{1}{2}$$

and therefore they have no physical meaning. As for the third solution, it corresponds to the Lagrange integrable case ($A = C$). Thus, in this problem, for any shape of the ellipsoid (both when it is prolate and when it is oblate), there is an integrable case, corresponding to the Lagrange case.

In addition to the three solutions (17), Eqs. (9), (14) admit a z − dependent solution, in which y is a root of the quadratic equation with coefficients, depending on z, and x is expressed in terms of y and z:

$$(3z-4)\,(7z-52)\,y^2 - \left(76z^2 - 632z + 736\right) y + 20z^2 - 432z + 592 = 0,$$

$$x = \frac{\left(4048z^2 - 471z^3 - 3200 - 2672z\right) y + 3252z^2 - 54z^3 - 17424z + 18816}{2\,(3z-4)\,(7z-52)\,((23z-32)\,y - 38z + 56)}.$$

Among the parameters (x, y, z) that belong to this solution, one can find such parameters, that have a physical meaning. These are, for example, the parameters

$$x = \frac{57}{23},\ y = \frac{30}{23},\ z = \frac{1}{5}.$$

Thus for some values of parameters, the necessary conditions for the existence of an additional first integral in the problem of motion of a rigid body with a fixed point in the flow of particles are satisfied. The study of existence of an additional first integral for such values of parameters is a problem, which we will try to investigate in the future.

4 Conclusions

In this paper we presented necessary conditions for the existence of an additional analytic first integral independent of the energy integral in the problem of motion of a rigid body with a fixed point in the flow of particles. The obtained necessary conditions is always fulfilled in the case of motion of a dynamically symmetric rigid body with the center of mass situated on the axis of dynamical symmetry of the body (the case similar to the Lagrange integrable case of the classical problem of motion of a heavy rigid body with a fixed point) and these conditions is not fulfilled for the dynamically symmetric rigid body with the center of mass situated in the equatorial plane of the ellipsoid of inertia (the mass distribution similar to the Kovalevskaya integrable case in the classical problem of motion of a heavy rigid body with a fixed point). Thereby we proved the nonexistence of the integrable case similar to the Kovalevskaya integrable case in the problem of motion in the flow of particles of a rigid body with a fixed point.

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