Chapter 7
Free Localized Vibrations of a Thin Elastic Composite Panel

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Abstract Free boundary and interfacial vibrations of a composite cylindrical panel with free edges comprised of two finite orthotropic thin cylindrical panels with different elastic properties and full contact along the generators are studied. Starting from the formulation of the classical theory of orthotropic cylindrical shells, dispersion relations and asymptotic approximations for eigenfrequencies of interfacial and boundary vibrations of such composite cylindrical panels are derived. An algorithm for separating the interfacial and boundary vibrations is presented. Asymptotic connections between the dispersion relations of the problem at hand and the analogous problems for a composite rectangular plate are established. Examples of cylindrical panel with different widths of constituents are considered, and approximate values of dimensionless eigenfrequencies are obtained.

Keywords Cylindrical panel · Boundary and interfacial vibrations · Full contact

7.1 Introduction

Investigation of free vibrations of composite cylindrical panels plays an important role in the studies of dynamics of deformable solids. Such studies contribute to the development of the theory itself and are also required for the practical needs of different branches of engineering and industry, e.g. construction, instrument-engineering, seismic surveys, etc. [1]. In many cases the objects of investigations are finite thin-walled composite cylindrical panels. For such structures, attention is often attracted to free vibrations localized near the edges of the panels, i.e. edge

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vibrations, as well as vibrations localized near the interface of material properties, i.e. interfacial vibrations.

It is known that, at the free edge of an orthotropic plate planar and flexural vibrations can occur independently of each other [1–4]. When the plate is bent, these types of vibration are coupled, giving two new types of vibrations localized at the free edge: predominantly tangential and predominantly bending vibrations. Moreover, for thin cylindrical elastic panel the transformation of one type of vibration into the other occurs at the free edge of the panel. For this transformation of vibrations complex distribution picture of frequencies of natural vibrations occur, depending on the geometrical and mechanical parameters of the finite and infinite cylindrical panels [4–15]. By increasing the number of free edges of a cylindrical panel, this picture becomes more complex, see [15–18] and also [28, 29]. Therefore, investigation of edge resonance phenomena in composite plates and cylindrical panels with free edges is among the most challenging problems in the theory of vibrations of plates and shells [8]. These difficulties can be resolved by using a combination of analytical and asymptotic theories, as well as by numerical methods.

For studies of free interfacial vibrations, the reader is referred to contributions [16–18]. Transverse vibrations occurring along the contact line of two semi-infinite plates and concentrated close to it are studied in [16]. The plane interfacial vibrations near the interface of two joined semi-strips with different elastic properties are investigated in [17]. We also mention important contributions to edge and interfacial vibrations in shells, using special asymptotic methods, [5–8], see also a review paper [9].

In the present paper, free interfacial and edge vibrations of a composite cylindrical panel with free edges, consisting of finite orthotropic cylindrical panels with different elastic properties and longitudinal section of material properties are studied. On the line of material interface, the full contact conditions are imposed. Such type of structural elements are important components of modern constructions; therefore, an issue of free vibrations of such elements is important and deserves attention.

The dispersion relations which determine the appropriate frequencies of interfacial and edge vibrations of the considered composite cylindrical panel with free edges are derived. An asymptotic link between the dispersion relations of the considered problem and the analogous problem for a plate with free edges, composed of orthotropic plates with different elastic properties is established. The derived dispersion relations and related asymptotic formulas can be used for controlling the spectrum of frequencies of the formulated problem by varying the geometry of the panel and mechanical properties of materials. In particular, one can control the spectrum by shifting either the origin of the spectrum or the points of condensation from the undesirable resonance region.

1. **Statement of the Problem and Basic Equations.** Let the generatrices of the cylindrical panel be orthogonal to the edge of the panel. On the middle surface of the panel, the curvilinear coordinates (α, β) are introduced, where $\alpha(0 \leq \alpha \leq l)$ and $\beta(-s^{(2)} \leq \beta \leq s^{(1)})$ are the oriented length of the generatrix and the length of the arc of the directing circle, respectively. l is the length of the cylindrical panel (Fig. 7.1). $\beta = 0$ corresponds to the interface between materials with

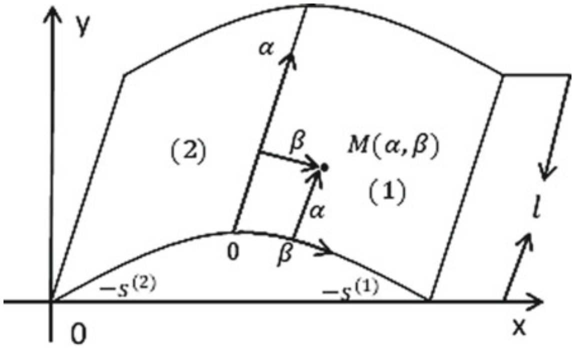

Fig. 7.1 The interface between materials with different properties

different properties. All the values corresponding to the right panel ($0 \le \beta \le s^{(1)}$) on (Fig. 7.1) are marked with superscript (1). Similarly, for the left panel ($-s^{(2)} \le \beta \le 0$) superscript (2) is used.

As the initial equations describing vibrations of the left and right sides cylindrical panels, we will use the equations corresponding to the classical theory of orthotropic cylindrical shells written in the curvilinear coordinates for given α and β (Fig. 7.1) [19]:

$$
\begin{aligned}
&-B_{11}^{(r)}\frac{\partial^2 u_1^{(r)}}{\partial \alpha^2} - B_{66}^{(r)}\frac{\partial^2 u_1^{(r)}}{\partial \beta^2} - \left(B_{12}^{(r)} + B_{66}^{(r)}\right)\frac{\partial^2 u_2^{(r)}}{\partial\alpha\partial\beta} + \frac{B_{12}^{(r)}}{R}\frac{\partial u_3^{(r)}}{\partial\alpha} = \rho^{(r)}\omega^2 u_1^{(r)},\\
&-\left(B_{12}^{(r)} + B_{66}^{(r)}\right)\frac{\partial^2 u_1^{(r)}}{\partial\alpha\partial\beta} - B_{66}^{(r)}\frac{\partial^2 u_2^{(r)}}{\partial \alpha^2} - B_{22}^{(r)}\frac{\partial^2 u_2^{(r)}}{\partial\beta^2} + \frac{B_{22}^{(r)}}{R}\frac{\partial u_3^{(r)}}{\partial\beta} - \frac{\mu^4}{R^2}X\\
&\left(4B_{66}^{(r)}\frac{\partial^2 u_2^{(r)}}{\partial \alpha^2} + B_{22}^{(r)}\frac{\partial^2 u_2^{(r)}}{\partial\beta^2}\right) - \frac{\mu^4}{R}\left(B_{22}^{(r)}\frac{\partial^3 u_3^{(r)}}{\partial\beta^3} + \left(B_{12}^{(r)} + 4B_{66}^{(r)}\right)\frac{\partial^3 u_3^{(r)}}{\partial\alpha^2\partial\beta}\right)\\
&= \rho^{(r)}\omega^2 u_2^{(r)},\\
&\mu^4\left(B_{11}^{(r)}\frac{\partial^4 u_3^{(r)}}{\partial\alpha^4} + 2\left(B_{12}^{(r)} + 2B_{66}^{(r)}\right)\frac{\partial^4 u_3^{(r)}}{\partial\alpha^2\partial\beta^2} + B_{22}^{(r)}\frac{\partial^4 u_3^{(r)}}{\partial\beta^4}\right) + \frac{\mu^4}{R}\left(B_{22}^{(r)}\frac{\partial^3 u_2^{(r)}}{\partial\beta^3}\right.\\
&\left.+\left(B_{12}^{(r)} + 4B_{66}^{(r)}\right)\frac{\partial^3 u_2^{(r)}}{\partial\alpha^2\partial\beta}\right) - \frac{B_{12}^{(r)}}{R}\frac{\partial u_1^{(r)}}{\partial\alpha} - \frac{B_{22}^{(r)}}{R}\frac{\partial u_2^{(r)}}{\partial\beta} + \frac{B_{22}^{(r)}}{R^2}u_3^{(r)} = \rho^{(r)}\omega^2 u_3^{(r)},\\
&r = 1, 2.
\end{aligned}
\tag{7.1}
$$

Here $u_1^{(r)}$, $u_2^{(r)}$ and $u_3^{(r)}(r = 1, 2)$ are projections of the displacement vector on the directions α, β and on the normal to the median surface of the shell, respectively; R is the radius of the directing circumference of the median surface; $\mu^4 = h^2/12$ (h is shell thickness); ω is the angular frequency, $\rho^{(r)}(r = 1, 2)$ are densities of materials; $B_{ij}^{(r)}(r = 1, 2)$ are elasticity coefficients. The boundary conditions for orthotropic cantilever cylindrical panel have the form [19]

$$T_2^{(1)}\Big|_{\beta=0} = T_2^{(2)}\Big|_{\beta=0},\ S_{21}^{(1)}\Big|_{\beta=0} = S_{21}^{(2)}\Big|_{\beta=0},\ N_2^{(1)} + \frac{\partial H^{(1)}}{\partial\alpha}\Big|_{\beta=0} = N_2^{(2)} + \frac{\partial H^{(2)}}{\partial\alpha}\Big|_{\beta=0},$$
$$M_2^{(1)}\Big|_{\beta=0} = M_2^{(2)}\Big|_{\beta=0},\ u_1^{(1)}\Big|_{\beta=0} = u_1^{(2)}\Big|_{\beta=0},\ u_2^{(1)}\Big|_{\beta=0} = u_2^{(2)}\Big|_{\beta=0}$$
$$u_3^{(1)}\Big|_{\beta=0} = u_3^{(2)}\Big|_{\beta=0},\ \frac{\partial u_3^{(1)}}{\partial\beta}\Big|_{\beta=0} = \frac{\partial u_3^{(2)}}{\partial\beta}\Big|_{\beta=0}. \tag{7.2}$$

$$T_2^{(r)}\Big|_{\beta=(-1)^{r-1}s^{(r)}} = S_{21}^{(r)}\Big|_{\beta=(-1)^{r-1}s^{(r)}} = N_2^{(r)} + \frac{\partial H^{(r)}}{\partial\alpha}\Big|_{\beta=(-1)^{r-1}s^{(r)}} =$$
$$M_2^{(r)}\Big|_{\beta=(-1)^{r-1}s^{(r)}} = 0, r = 1, 2. \tag{7.3}$$

$$T_1^{(r)}\Big|_{\alpha=0,l} = S_{12}^{(r)} + \frac{H^{(r)}}{R}\Big|_{\alpha=0,l} = N_1^{(r)} + \frac{\partial H^{(r)}}{\partial\beta}\Big|_{\alpha=0,l} = M_1^{(r)}\Big|_{\alpha=0,l} = 0, r = 1, 2.$$
$$T_1^{(r)} = hB_{11}^{(r)}\left[\frac{\partial u_1^{(r)}}{\partial\alpha} + \frac{B_{12}^{(r)}}{B_{11}^{(r)}}\left(\frac{\partial u_2^{(r)}}{\partial\beta} - \frac{u_3^{(r)}}{R}\right)\right],$$
$$T_2^{(r)} = hB_{22}^{(r)}\left[\frac{B_{12}^{(r)}}{B_{22}^{(r)}}\frac{\partial u_1^{(r)}}{\partial\alpha} + \frac{\partial u_2^{(r)}}{\partial\beta} - \frac{u_3^{(r)}}{R}\right], \tag{7.4}$$

$$M_1^{(r)} = \frac{h^3}{12}B_{11}^{(r)}\left[\frac{\partial^2 u_3^{(r)}}{\partial\,\propto^2} + \frac{B_{12}^{(r)}}{B_{11}^{(r)}}\left(\frac{\partial^2 u_3^{(r)}}{\partial\beta^2} + \frac{1}{R}\frac{\partial u_2^{(r)}}{\partial\beta}\right)\right],$$
$$M_2^{(r)} = \frac{h^3}{12}B_{22}^{(r)}\left[\frac{B_{12}^{(r)}}{B_{22}^{(r)}}\frac{\partial^2 u_3^{(r)}}{\partial\,\propto^2} + \frac{\partial^2 u_3^{(r)}}{\partial\beta^2} + \frac{1}{R}\frac{\partial u_2^{(r)}}{\partial\beta}\right],$$
$$S_{12}^{(r)} + \frac{H^{(r)}}{R} = hB_{66}^{(r)}\left[\frac{\partial u_1^{(r)}}{\partial\beta} + \frac{\partial u_2^{(r)}}{\partial\alpha} + \frac{h^2}{3R}\left(\frac{\partial^2 u_3^{(r)}}{\partial\alpha\partial\beta} + \frac{1}{R}\frac{\partial u_2^{(r)}}{\partial\alpha}\right)\right],$$
$$S_{21}^{(r)} = hB_{66}^{(r)}\left(\frac{\partial u_1^{(r)}}{\partial\beta} + \frac{\partial u_2^{(r)}}{\partial\alpha}\right), \tag{7.5}$$
$$N_1^{(r)} + \frac{\partial H^{(r)}}{\partial\beta} = \frac{h^3}{12}B_{11}^{(r)}\left[\frac{\partial^3 u_3^{(r)}}{\partial\,\propto^3} + \frac{B_{12}^{(r)} + 4B_{66}^{(r)}}{B_{11}^{(r)}}\left(\frac{\partial^3 u_3^{(r)}}{\partial\beta^2\partial\alpha} + \frac{1}{R}\frac{\partial^2 u_2^{(r)}}{\partial\alpha\partial\beta}\right)\right],$$
$$N_2^{(r)} + \frac{\partial H^{(r)}}{\partial\alpha} = \frac{h^3}{12}B_{22}^{(r)}\left[\frac{\partial^3 u_3^{(r)}}{\partial\beta^3} + \frac{B_{12}^{(r)} + 4B_{66}^{(r)}}{B_{22}^{(r)}}\frac{\partial^3 u_3^{(r)}}{\partial\alpha^2\partial\beta} + \frac{1}{R}\frac{\partial^2 u_2^{(r)}}{\partial\beta^2} + \frac{4B_{66}^{(r)}}{B_{22}^{(r)}}\frac{1}{R}\frac{\partial^2 u_2^{(r)}}{\partial\,\propto^2}\right].$$

Relations (7.2) are complete contact conditions at $\beta = 0$. Relations (7.3) and (7.4) are the conditions of free edges at $\beta = -s^{(2)}, s^{(1)}$ and $\alpha = 0, l$, respectively (Fig. 7.1). It can be proved that the problem (7.1)–(7.4) is self–conjugate and has a non–negative discrete spectrum with a limit point at $+\infty$.

The problem (7.1)–(7.4) does not allow separation of variables. Therefore, based on the nonnegative definiteness of the corresponding operator of problem (7.1)–(7.4), for finding natural frequencies and corresponding natural forms a generalized

Kantorovich—Vlasov method of reduction to ordinary differential equations can be applied [21–25].

The eigenfunctions of the problem

$$\mathrm{w}^{VIII} = \theta^8 \mathrm{w}, \quad \mathrm{w}|_{\alpha=0,l} = \mathrm{w}'\big|_{\alpha=0,l} = \mathrm{w}''\big|_{\alpha=0,l} = \mathrm{w}'''\big|_{\alpha=0,l} = 0, 0 \le \alpha \le l \tag{7.6}$$

are used as the basic functions. The problem (7.6) is self-conjugate and positive definite. The eigenfunctions corresponding to the eigenvalues $\theta_m^8, m = \overline{1, \infty}$ of the problem (7.6) have the form.

$$\mathrm{w}_m(\theta_m\alpha) = \frac{\Delta_1}{\Delta} x_1(\theta_m\alpha) + \frac{\Delta_2}{\Delta} x_2(\theta_m\alpha) + \frac{\Delta_3}{\Delta} x_3(\theta_m\alpha) + x_4(\theta_m\alpha), 0 \le \alpha \le l.$$

$$x_1(\theta_m\alpha) = ch(\theta_m\alpha) - ch\frac{\theta_m\alpha}{\sqrt{2}} cos\frac{\theta_m\alpha}{\sqrt{2}} - sh\frac{\theta_m\alpha}{\sqrt{2}} sin\frac{\theta_m\alpha}{\sqrt{2}}, \tag{7.7}$$

$$\begin{aligned} x_2(\theta_m\alpha) &= sh(\theta_m\alpha) - \sqrt{2}ch\frac{\theta_m\alpha}{\sqrt{2}} sin\frac{\theta_m\alpha}{\sqrt{2}}, x_3(\theta_m\alpha) = sin(\theta_m\alpha) \\ &\quad - \sqrt{2}sh\frac{\theta_m\alpha}{\sqrt{2}} cos\frac{\theta_m\alpha}{\sqrt{2}}, x_4(\theta_m\alpha) = cos(\theta_m\alpha) - ch\frac{\theta_m\alpha}{\sqrt{2}} cos\frac{\theta_m\alpha}{\sqrt{2}} \\ &\quad + sh\frac{\theta_m\alpha}{\sqrt{2}} sin\frac{\theta_m\alpha}{\sqrt{2}}, \quad m = \overline{1, \infty.} \end{aligned} \tag{7.8}$$

In above

$$\Delta = \begin{vmatrix} x_1(\theta_m\alpha) & x_2(\theta_m\alpha) & x_3(\theta_m\alpha) \\ x_1'(\theta_m\alpha) & x_2'(\theta_m\alpha) & x_3'(\theta_m\alpha) \\ x_1''(\theta_m\alpha) & x_2''(\theta_m\alpha) & x_3''(\theta_m\alpha) \end{vmatrix},$$

$$\Delta_1 = -\begin{vmatrix} x_4(\theta_m\alpha) & x_2(\theta_m\alpha) & x_3(\theta_m\alpha) \\ x_4'(\theta_m\alpha) & x_2'(\theta_m\alpha) & x_3'(\theta_m\alpha) \\ x_4''(\theta_m\alpha) & x_2''(\theta_m\alpha) & x_3''(\theta_m\alpha) \end{vmatrix},$$

$$\Delta_2 = -\begin{vmatrix} x_1(\theta_m\alpha) & x_4(\theta_m\alpha) & x_3(\theta_m\alpha) \\ x_1'(\theta_m\alpha) & x_4'(\theta_m\alpha) & x_3'(\theta_m\alpha) \\ x_1''(\theta_m\alpha) & x_4''(\theta_m\alpha) & x_3''(\theta_m\alpha) \end{vmatrix},$$

$$\Delta_3 = -\begin{vmatrix} x_1(\theta_m\alpha) & x_2(\theta_m\alpha) & x_4(\theta_m\alpha) \\ x_1'(\theta_m\alpha) & x_2'(\theta_m\alpha) & x_4'(\theta_m\alpha) \\ x_1''(\theta_m\alpha) & x_2''(\theta_m\alpha) & x_4''(\theta_m\alpha) \end{vmatrix}.$$

Also, these functions form an orthogonal basis in the Hilbert space $L_2[0, l]$ with their first and second derivatives [25]. Here $\theta_m, m = \overline{1, \infty}$, are the positive zeros of the Wronskian of functions (7.8) at the point $\alpha = l$.

Let us introduce the designations

$$\beta_m' = \int_0^l \left(\mathrm{w}_m'(\theta_m\alpha)\right)^2 \mathrm{d}\alpha \quad \Big/ \int_0^l \left(\mathrm{w}_m(\theta_m\alpha)\right)^2 \mathrm{d}\alpha,$$

$$\beta_m'' = \int_0^l \left(\mathrm{w}_m''(\theta_m\alpha)\right)^2 \mathrm{d}\alpha \quad \Big/ \int_0^l \left(\mathrm{w}_m'(\theta_m\alpha)\right)^2 \mathrm{d}\alpha. \tag{7.9}$$

In formulas (7.8) and (7.9), derivatives are taken with respect to $\theta_m\alpha$ and $\beta_m^{'} \to 1$, $\beta_m^{''} \to 1 atm \to +\infty$.

2. **Derivation and Analysis of Characteristic Equations.** In the first, second, and third equations of system (7.1), the angular frequency ω is formally replaced by ω_1, ω_2, and ω_3, respectively. The solution of system (7.1) is sought in the form.

$$\begin{aligned}&\left(u_1^{(r)}, u_2^{(r)}, u_3^{(r)}\right) = \left(u_m^{(r)}\mathrm{w}_m'(\theta_m\alpha), v_m^{(r)}\mathrm{w}_m(\theta_m\alpha), \mathrm{w}_m(\theta_m\alpha)\right)\\ &\times \exp\left((-1)^r\chi^{(r)}\theta_m\beta + \chi^{(r)}\theta_m s^{(r)}\right), r = 1, 2; m = \overline{1, \infty}.\end{aligned} \tag{7.10}$$

Here, $\mathrm{w}_m(\theta_m\alpha)$, $m = \overline{1, \infty}$, are determined from formula (7.7) and $u_m^{(r)}$, $v_m^{(r)}$ and $\chi^{(r)}$ are undetermined constants. In this case, conditions (7.4) are satisfied automatically. Let us insert Eq. (7.10) into Eq. (7.1). The equations found are scalarly multiplied by the vector-function

$$\left(\mathrm{w}_m'(\theta_m\alpha), \mathrm{w}_m(\theta_m\alpha), \mathrm{w}_m(\theta_m\alpha), \mathrm{w}_m'(\theta_m\alpha), \mathrm{w}_m(\theta_m\alpha), \mathrm{w}_m(\theta_m\alpha)\right) \tag{7.11}$$

and then integrated between the limits from 0 to l. From the first two pairs of equations we deduce

$$\begin{aligned}&\left(c_m^{(r)} + \varepsilon_m^2 a^2 g_m^{(r)} d_m^{(r)}\right)u_m^{(r)}\\ &= \varepsilon_m\left\{a_m^{(r)} + a^2\frac{B_{22}^{(r)}\left(B_{12}^{(r)} + B_{66}^{(r)}\right)}{B_{11}^{(r)}B_{66}^{(r)}}\left(\chi^{(r)}\right)^2 l_m^{(r)} + \varepsilon_m^2 a^2\frac{B_{22}^{(r)}B_{12}^{(r)}}{B_{11}^{(r)}B_{66}^{(r)}}d_m^{(r)}\right\}\end{aligned} \tag{7.12}$$

$$\left(c_m^{(r)} + \varepsilon_m^2 a^2 g_m^{(r)} d_m^{(r)}\right)v_m^{(r)} = (-1)^r\varepsilon_m\chi^{(r)}\left\{b_m^{(r)} - a^2 g_m^{(r)} l_m^{(r)}\right\}, r = 1, 2. \tag{7.13}$$

From the third equation, by taking into account the relations (7.12) and (7.13), we obtain the characteristic equations

$$\begin{aligned}&R_{mm}^{(r)}c_m^{(r)} + \varepsilon_m^2\{c_m^{(r)} - b_m^{(r)}\left(\chi^{(r)}\right)^2 + \frac{B_{12}^{(r)}}{B_{22}^{(r)}}\beta_m' a_m^{(r)} + a^2(R_{mm}^{(r)}g_m^{(r)}d_m^{(r)}\\ &+ 2b_m^{(r)}l_m^{(r)}\left(\chi^{(r)}\right)^2) + \varepsilon_m^2 a^2 d_m^{(r)}\left(b_m^{(r)} - \frac{B_{12}^{(r)}}{B_{22}^{(r)}}\beta_m'\right) - a^4\left(\chi^{(r)}\right)^2 g_m^{(r)}\left(l_m^{(r)}\right)^2\} = 0,\\ &r = 1, 2; m = \overline{1, \infty}\end{aligned} \tag{7.14}$$

$$a_m^{(r)} = -\left(\frac{B_{12}^{(r)}}{B_{11}^{(r)}}\left(\chi^{(r)}\right)^2 + \frac{B_{12}^{(r)}}{B_{11}^{(r)}}\beta_m' - \frac{B_{12}^{(r)}}{B_{11}^{(r)}}\left(\eta_{2m}^{(r)}\right)^2\right),$$

$$b_m^{(r)} = \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\chi^{(r)}\right)^2 + \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\eta_{1m}^{(r)}\right)^2 - B_1^{(r)}, a^2 = \mu^4\theta_m^2,$$

$$c_m^{(r)} = \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\chi^{(r)}\right)^4 - B_2^{(r)}\left(\chi^{(r)}\right)^2 + \left(\frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\eta_{1m}^{(r)}\right)^2 + \frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_{2m}^{(r)}\right)^2\right)\left(\chi^{(r)}\right)^2$$

$$+\left(\beta_m' - \left(\eta_{2m}^{(r)}\right)^2\right)\left(\beta_m'' - \frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_{1m}^{(r)}\right)^2\right), d_m^{(r)} = \left(\chi^{(r)}\right)^2 - \frac{4\,B_{66}^{(r)}}{B_{11}^{(r)}}\beta_m',$$

$$g_m^{(r)} = \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\chi^{(r)}\right)^2 + \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\eta_{1m}^{(r)}\right)^2 - \frac{B_{22}^{(r)}}{B_{66}^{(r)}}\beta_m'', l_m^{(r)} = \left(\chi^{(r)}\right)^2 - \frac{B_{12}^{(r)} + 4B_{66}^{(r)}}{B_{22}^{(r)}}\beta_m', \tag{7.15}$$

$$R_{mm}^{(r)} = a^2\left(\left(\chi^{(r)}\right)^4 - \frac{2\left(B_{12}^{(r)} + 2B_{66}^{(r)}\right)}{B_{22}^{(r)}}\beta_m'\left(\chi^{(r)}\right)^2 + \frac{B_{11}^{(r)}}{B_{22}^{(r)}}\beta_m'\beta_m''\right) - \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_{3m}^{(r)}\right)^2,$$

$$\left(\eta_{im}^{(r)}\right)^2 = \frac{\rho^{(r)}\omega_i}{B_{66}^{(r)}\theta_m^2}, i = 1, 2, 3; , \ \varepsilon_m = \frac{1}{R\theta_m}.$$

$$B_1^{(r)} = \frac{B_{11}^{(r)}B_{11}^{(r)}\beta_m'' - \left(B_{12}^{(r)}\right)^2\beta_m' - B_{12}^{(r)}B_{66}^{(r)}\beta_m'}{B_{11}^{(r)}B_{66}^{(r)}},$$

$$B_2^{(r)} = \frac{B_{11}^{(r)}B_{11}^{(r)}\beta_m'' - \left(B_{12}^{(r)}\right)^2\beta_m' - 2B_{12}^{(r)}B_{66}^{(r)}\beta_m'}{B_{11}^{(r)}B_{66}^{(r)}}$$

Let $\chi^{(r)}(j = 1, 2, 3, 4)$, be pairwise different roots of the Eq. (7.14) with nonnegative real parts and $\chi_{j+4}^{(r)} = -\chi_j^{(r)}$, $j = 1, 2, 3, 4$. Let $\left(u_{1j}^{(r)}, u_{2j}^{(r)}, u_{3j}^{(r)}\right)$, , be nontrivial solutions of type (7.10) of system (7.1) at $\chi^{(r)} = \chi_j^{(r)}$, $j = 1, 2, \ldots, 8$, respectively. The solution of the problem (7.1)–(7.4) is sought for in the form

$$u_i^{(r)} = \sum_{j=1}^{8} u_{ij}^{(r)} w_j^{(r)}, i = 1, 2, 3; r = 1, 2. \tag{7.16}$$

Let us insert Eq. (7.16) into the boundary conditions (7.2)−(7.4). As a result, we obtain the system of equations

$$\sum_{j=1}^{8}\frac{M_{ij}^{(1)}\exp\left(\chi_j^{(1)}\theta_m s^{(1)}\right)w_j^{(1)}}{c_{mj}^{(1)} + \varepsilon_m^2 a^2 g_{mj}^{(1)} d_{mj}^{(1)}} - \sum_{j=1}^{8}\frac{c\,M_{ij}^{(2)}\exp\left(\chi_j^{(2)}\theta_m s^{(2)}\right)w_j^{(2)}}{c_{mj}^{(2)} + \varepsilon_m^2 a^2 g_{mj}^{(2)} d_{mj}^{(2)}}, i = \overline{1, 4};$$

$$\sum_{j=1}^{8} \frac{M_{ij}^{(1)} \exp\left(\chi_j^{(1)} \theta_m s^{(1)}\right) w_j^{(1)}}{c_{mj}^{(1)} + \varepsilon_m^2 a^2 g_{mj}^{(1)} d_{mj}^{(1)}} - \sum_{j=1}^{8} \frac{c M_{ij}^{(2)} \exp\left(\chi_j^{(2)} \theta_m s^{(2)}\right) w_j^{(2)}}{c_{mj}^{(2)} + \varepsilon_m^2 a^2 g_{mj}^{(2)} d_{mj}^{(2)}}, i = \overline{5, 8}; \tag{7.17}$$

$$\sum_{j=1}^{8} \frac{M_{ij}^{(1)} w_j^{(1)}}{c_{mj}^{(1)} + \varepsilon_m^2 a^2 g_{mj}^{(1)} d_{mj}^{(1)}} = 0, i = \overline{9, 12}; \sum_{j=1}^{8} \frac{M_{ij}^{(2)} w_j^{(2)}}{c_{mj}^{(2)} + \varepsilon_m^2 a^2 g_{mj}^{(2)} d_{mj}^{(2)}}, i = \overline{13, 16};$$

$$\begin{aligned}
M_{1j}^{(r)} &= \left(\chi_j^{(r)}\right)^2 b_{mj}^{(r)} - \frac{B_{12}^{(r)}}{B_{22}^{(r)}} \beta_m' a_{mj}^{(r)} - c_{mj}^{(r)} - a^2 l_{mj}^{(r)} b_{mj}^{(r)} \left(\chi_j^{(r)}\right)^2 \\
&- \varepsilon_m^2 a^2 g_{mj}^{(r)} \left(b_{mj}^{(r)} - \frac{B_{12}^{(r)}}{B_{22}^{(r)}} \beta_m'\right), M_{2j}^{(r)} = \frac{B_{66}^{(r)}}{B_{11}^{(r)}} \chi_j^{(r)} \{a_{mj}^{(r)} + b_{mj}^{(r)} \\
&+ a^2 l_{mj}^{(r)} \left[\frac{B_{22}^{(r)} B_{12}^{(r)}}{B_{11}^{(r)} B_{66}^{(r)}} \left(\chi_j^{(r)}\right)^2 + \frac{B_{22}^{(r)}}{B_{66}^{(r)}} \beta_m'' - \frac{B_{22}^{(r)}}{B_{11}^{(r)}} \left(\eta_{1m}^{(r)}\right)^2\right] \\
&+ \varepsilon_m^2 a^2 \frac{B_{22}^{(r)} B_{12}^{(r)}}{B_{11}^{(r)} B_{66}^{(r)}} d_{mj}^{(r)}\},
\end{aligned}$$

$$M_{3j}^{(r)} = \left(\left(\chi_j^{(r)}\right)^2 - \frac{B_{12}^{(r)}}{B_{22}^{(r)}} \beta_m'\right) c_{mj}^{(r)} + \varepsilon_m^2 \left[b_{mj}^{(r)} \left(\chi_j^{(r)}\right)^2 + 4a^2 \frac{B_{66}^{(r)} B_{12}^{(r)}}{\left(B_{22}^{(r)}\right)^2} g_{mj}^{(r)} (\beta_m')^2\right]$$

$$M_{4j}^{(r)} = \chi_j^{(r)} \left(l_{mj}^{(r)} c_{mj}^{(r)} + \varepsilon_m^2 b_{mj}^{(r)} d_{mj}^{(r)}\right), c = \frac{B_{22}^{(2)}}{B_{22}^{(1)}}, \tag{7.18}$$

$$\begin{aligned}
M_{5j}^{(r)} &= a_{mj}^{(r)} + a^2 \beta_m' \frac{B_{22}^{(r)} (B_{12}^{(r)} + B_{66}^{(r)}}{B_{11}^{(r)} B_{66}^{(r)}} l_{mj}^{(r)} \left(\chi_j^{(r)}\right)^2 + \varepsilon_m^2 a^2 \frac{B_{22}^{(r)} B_{12}^{(r)}}{B_{11}^{(r)} B_{66}^{(r)}} d_{mj}^{(r)}, \\
M_{6j}^{(r)} &= \chi_j^{(r)} \left(b_{mj}^{(r)} - a^2 g_{mj}^{(r)} l_{mj}^{(r)}\right), M_{7j}^{(r)} = c_{mj}^{(r)} + \varepsilon_m^2 a^2 g_{mj}^{(r)} d_{mj}^{(r)}, \\
M_{8j}^{(r)} &= \chi_j^{(r)} \left(c_{mj}^{(r)} + \varepsilon_m^2 a^2 g_{mj}^{(r)} d_{mj}^{(r)}\right); \\
M_{9j}^{(1)} &= M_{1j}^{(1)}, M_{10j}^{(1)} = M_{2j}^{(1)}, M_{11j}^{(1)} = M_{3j}^{(1)}, M_{12j}^{(1)} = M_{4j}^{(1)}, j = \overline{1, 8}; \\
M_{13j}^{(2)} &= M_{1j}^{(2)}, M_{14j}^{(2)} = M_{2j}^{(2)}, M_{15j}^{(2)} = M_{3j}^{(2)}, M_{16j}^{(2)} = M_{4j}^{(2)}, j = \overline{1, 8}.
\end{aligned}$$

The superscript j means that the corresponding function is taken at $\chi^{(r)} = \chi_j^{(r)}$. In order for the system (7.17) to possess a nontrivial solution, it is necessary and sufficient that

$$\Delta = exp\left(-\sum_{r=1}^{2} \sum_{j=1}^{4} z_j^{(r)}\right) Det \left\| T_{ij} \right\|_{i,j=1}^{4} = 0, m = \overline{1, \infty} \tag{7.19}$$

$$
\begin{aligned}
&T_{11} = \left\| M_{ij}^{(1)} \right\|_{i,j=1}^{4}, T_{12} = \left\| (-1)^{i-1} M_{ij}^{(1)} \exp\left(z_j^{(1)}\right) \right\|_{i,j=1}^{4}, \\
&T_{13} = -c\left\| M_{ij}^{(2)} \right\|_{i,j=1}^{4}, T_{14} = \left\| (-1)^{i} M_{ij}^{(2)} \exp\left(z_j^{(2)}\right) \right\|_{i,j=1}^{4}, \\
&T_{21} = \left\| M_{ij}^{(1)} \right\|_{i=5,j=1}^{8,4}, T_{22} = \left\| (-1)^{i-1} M_{ij}^{(1)} \exp\left(z_j^{(1)}\right) \right\|_{i=5,j=1}^{8,4}, \\
&T_{23} = -\left\| M_{ij}^{(2)} \right\|_{i=5,j=1}^{8,4}, T_{24} = \left\| (-1)^{i} M_{ij}^{(2)} \exp\left(z_j^{(2)}\right) \right\|_{i=5,j=1}^{8,4}, \\
&T_{31} = T_{12}, T_{32} = T_{11}, T_{33} = 0, T_{34} = 0; \\
&T_{41} = 0,\ T_{42} = 0,\ T_{43} = T_{14},\ T_{44} = T_{13};\ z_j^{(r)} = -\theta_m \chi_j^{(r)} s^{(r)}.
\end{aligned}
\tag{7.20}
$$

Performing elementary operations with columns of determinant (7.19), we obtain

$$
\Delta = exp\left(-\sum_{r=1}^{2}\sum_{j=1}^{4} z_j^{(r)}\right)\left(K^{(1)}\right)^2\left(K^{(2)}\right)^2 Det\left\| t_{ij} \right\|_{i,j=1}^{4} = 0, m = \overline{1,\infty}; \tag{7.21}
$$

$$
\begin{aligned}
K^{(r)} = &\left(\chi_1^{(r)} - \chi_2^{(r)}\right)\left(\chi_1^{(r)} - \chi_3^{(r)}\right)\left(\chi_1^{(r)} - \chi_4^{(r)}\right)\left(\chi_2^{(r)} - \chi_3^{(r)}\right)\left(\chi_2^{(r)} - \chi_4^{(r)}\right) \\
&\left(\chi_3^{(r)} - \chi_4^{(r)}\right) r = 1, 2.
\end{aligned}
\tag{7.22}
$$

$$
t_{11} = \left\| m_{ij} \right\|_{i,j=1}^{4}, t_{12} = \left\| m_{ij} \right\|_{i=1,j=5}^{4,8}, t_{13} = c\left\| m_{ij} \right\|_{i=1,j=9}^{4,12}, t_{14} = c\left\| m_{ij} \right\|_{i=1,j=13}^{4,16};
$$

$$
t_{21} = \left\| m_{ij} \right\|_{i=5,j=1}^{8,4}, t_{22} = \left\| m_{ij} \right\|_{i,j=5}^{8}, t_{23} = c\left\| m_{ij} \right\|_{i=5,j=9}^{8,12}, t_{24} = c\left\| m_{ij} \right\|_{i=5,j=13}^{8,16};
$$

$$
t_{31} = t_{12}, t_{32} = t_{11}, t_{33} = 0, t_{34} = 0; T_{41} = 0, t_{42} = 0, t_{43} = t_{14}, t_{44} = t_{13}.
$$

Expressions for m_{ij} can be obtained in a similar way as in [24, 25].

It follows from (7.21) that the Eq. (7.19) are equivalent to the following

$$
\nabla = Det\left\| t_{ij} \right\|_{i,j=1}^{4} = 0, m = \overline{1,\infty} \tag{7.23}
$$

Taking into account the relations between $\eta_{1m}^{(r)}, \eta_{2m}^{(r)}$ and $\eta_{3m}^{(r)}$, we conclude that Eqs. (7.23) determine frequencies of the interfacial and boundary types of corresponding vibrations. At $\eta_{1m}^{(r)} = \eta_{2m}^{(r)} = \eta_{3m}^{(r)} = \eta_m^{(r)}$ the Eqs. (7.14) are the characteristic equations of system (7.1), and Eqs. (7.23) at $\theta = \theta_{m,} m \in \mathrm{N}$ the dispersion equations of problem (7.1)–(7.4).

In Sect. 7.50, we will study the asymptotic behavior of the dispersion relations (7.23) at $\varepsilon_m = 1/(\theta_m R) \to 0$ (transition to a rectangular composed plate with free edges or to vibrations localized at the free edges and at the interface of materials with different properties of the cylindrical panel) and at $\theta_m s^{(r)} \to \infty$ transition to a wide enough cylindrical panel or to vibrations localized at the free edges and at the interface of materials with different properties of the cylindrical panel). To verify the reliability of the asymptotic relations found in Sect. 7.5, we will investigate the free planar and flexural vibrations of a rectangular composed plate in the next sections.

3. **Planar vibrations of a composite rectangular plate with free edges.** Free interfacial and edge vibrations of a rectangular plate composed of finite thin elastic orthotropic rectangular plates with different elastic coefficients are considered. Let us introduce rectilinear oriented orthogonal coordinates (α, β) on the mid-plane, where $0 \le \alpha \le l, -s_0^{(2)} \le \beta \le s_0^{(1)}$. The line $\beta = 0$ corresponds to the interface of material properties. All values related to the right plate $0 \le \beta \le s_0^{(1)}$ are indicated by superscript (1), and to the left plate $-s_0^{(2)} \le \beta \le 0$ by (2), respectively. As the initial equations, the equations of small planar vibrations of the left and right plates are used, which correspond to the classical theory of orthotropic plates [19].

$$-B_{11}^{(r)}\frac{\partial^2 u_1^{(r)}}{\partial \propto^2} - B_{66}^{(r)}\frac{\partial^2 u_1^{(r)}}{\partial \beta^2} - \left(B_{12}^{(r)} + B_{66}^{(r)}\right)\frac{\partial^2 u_2^{(r)}}{\partial\alpha\partial\beta} = \rho^{(r)}\omega^2 u_1^{(r)}, \tag{7.24}$$

$$-\left(B_{12}^{(r)} + B_{66}^{(r)}\right)\frac{\partial^2 u_1^{(r)}}{\partial\alpha\partial\beta} - B_{66}^{(r)}\frac{\partial^2 u_2^{(r)}}{\partial \propto^2} - B_{22}^{(r)}\frac{\partial^2 u_2^{(r)}}{\partial \beta^2} = \rho^{(r)}\omega^2 u_2^{(r)}, r = 1, 2.$$

Here $\alpha(0 \le \alpha \le l)$ and $\beta(-s_0^{(2)} \le \beta \le s_0^{(1)})$ are the orthogonal rectilinear coordinates of a point of the mid-plane;$u_1^{(r)}, u_2^{(r)}(r = 1, 2)$ are the projections of the displacements vector in the directions α and β, respectively; $B_{ik}^{(r)}, i, k = 1, 2, 6(r = 1, 2)$, are coefficients of elasticity; ω- is the natural frequency; $\rho^{(r)}(r = 1, 2)$–are the density of the materials. The boundary conditions are written as

$$T_2^{(1)}\Big|_{\beta=0} = T_2^{(2)}\Big|_{\beta=0}, \; S_{21}^{(1)}\Big|_{\beta=0} = S_{21}^{(2)}\Big|_{\beta=0},$$

$$u_1^{(1)}\Big|_{\beta=0} = u_1^{(2)}\Big|_{\beta=0}, \; u_2^{(1)}\Big|_{\beta=0} = u_2^{(2)}\Big|_{\beta=0}, \tag{7.25}$$

$$T_2^{(r)}\Big|_{\beta=(-1)^{r-1}s_0^{(r)}} = 0, \; S_{21}^{(r)}\Big|_{\beta=(-1)^{r-1}s_0^{(r)}} = 0, r = 1, 2. \tag{7.26}$$

$$T_1^{(r)}\Big|_{\alpha=0,l} = 0, \; S_{12}^{(r)}\Big|_{\alpha=0,l} = 0, r = 1, 2. \tag{7.27}$$

$$T_1^{(r)} = hB_{11}^{(r)}\left[\frac{\partial u_1^{(r)}}{\partial\alpha} + \frac{B_{12}^{(r)}}{B_{11}^{(r)}}\frac{\partial u_2^{(r)}}{\partial\beta}\right], T_2^{(r)} = hB_{22}^{(r)}\left[\frac{B_{12}^{(r)}}{B_{22}^{(r)}}\frac{\partial u_1^{(r)}}{\partial\alpha} + \frac{\partial u_2^{(r)}}{\partial\beta}\right], \tag{7.28}$$

$$S_{21}^{(r)} = S_{12}^{(r)} = hB_{66}^{(r)}\left[\frac{\partial u_1^{(r)}}{\partial\beta} + \frac{\partial u_2^{(r)}}{\partial\alpha}\right].$$

The relations (7.25) describe the full contact conditions at$\beta = 0$. Relations (7.26) and (7.27) are the conditions of free edges at $= -s_0^{(2)}, s_0^{(1)}$ and $\alpha = 0, l$, respectively, see Fig. 7.2. The problem (7.24)–(7.27) does not allow the separation of variables. The differential operator corresponding to this problem is self-conjugate and nonnegative definite. Therefore, the generalized Kantorovich-Vlasov method of reduction to ordinary differential equations can be used to find vibration eigenfrequencies and eigenmodes [21–25]. The solution of system (7.24) is sought for in the form

$$\left(u_1^{(r)}, u_2^{(r)}\right) = \left(u_m^{(r)} w_m'(\theta_m\alpha),\, v_m^{(r)} w_m(\theta_m\alpha)\right) \exp\left((-1)^r y^{(r)}\theta_m\beta + y^{(r)}\theta_m s_0^{(r)}\right),$$
$$r = 1, 2;\; m = \overline{1, \infty}. \tag{7.29}$$

Here $w_m(\theta_m\alpha)$ are determined from formula (7.7) and $u_1^{(r)}, u_2^{(r)}, y^{(r)}$ are undetermined constants. In this case, the conditions (7.27) are satisfied automatically. Let us insert (7.29) into Eq. (7.24). As a result, we obtain the system of equations

$$\left(\frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(y^{(r)}\right)^2 - \beta_m'' + \frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_m^{(r)}\right)^2\right)u_m^{(r)} + (-1)^r y^{(r)}\frac{B_{12}^{(r)} + B_{66}^{(r)}}{B_{11}^{(r)}} v_m^{(r)} = 0, \tag{7.30}$$

$$(-1)^r y^{(r)}\frac{B_{12}^{(r)} + B_{66}^{(r)}}{B_{22}^{(r)}}\beta_m' u_m^{(r)} - \left(\left(y^{(r)}\right)^2 - \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\beta_m' + \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_m^{(r)}\right)^2\right)v_m^{(r)} = 0,$$

$$r = 1, 2,$$

where $\eta_m^{(r)}, \beta_m', \beta_m''$ are defined in (7.15) and (7.9), respectively. Equating the determinant of system (7.30) to zero, the following characteristic equation of the system (7.24) is found:

$$c_m^{(r)} = \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(y^{(r)}\right)^4 - B_2^{(r)}\left(y^{(r)}\right)^2 + \frac{B_{22}^{(r)} + B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_m^{(r)}\right)^2\left(y^{(r)}\right)^2$$

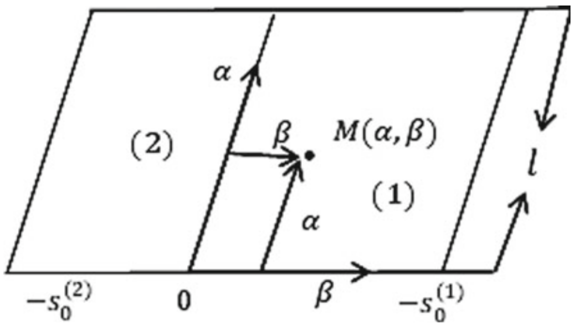

Fig. 7.2 Kantorovich-Vlasov method of reduction to ordinary differential equations can be used to find vibration eigenfrequencies and eigenmodes

$$+\left(\beta_m' - \left(\eta_m^{(r)}\right)^2\right)\left(\frac{B_{22}^{(r)}}{B_{11}^{(r)}}\beta_m'' - \frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_m^{(r)}\right)^2\right),$$

$$r = 1, 2; m = \overline{1, \infty}. \tag{7.31}$$

Let $y_1^{(r)}, y_2^{(r)} (r = 1, 2)$ be various roots of Eq. (7.31) with nonnegative real parts and $y_{j+2}^{(r)} = -y_j^{(r)} (j = 1, 2)$. As a solution of Eq. (7.30) for $y_j^{(r)}\left(j = \overline{1, 4}\right)$ we take

$$u_{mj}^{(r)} = \left(y_j^{(r)}\right)^2 - \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\beta_m' + \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_m^{(r)}\right)^2, \quad v_{mj}^{(r)} = (-1)^r y^{(r)} \frac{B_{12}^{(r)} + B_{66}^{(r)}}{B_{22}^{(r)}}\beta_m' \tag{7.32}$$

The solution of the problem (7.24)–(7.27) can be presented in the form

$$u_1^{(r)} = \sum_{j=1}^{4} u_{mj}^{(r)} w_m'(\theta_m \alpha) \exp\left((-1)^r y_j^{(r)} \theta_m \beta + y_j^{(r)} \theta_m s_0^{(r)}\right) w_j^{(r)}, \tag{7.33}$$

$$u_2^{(r)} = \sum_{j=1}^{4} v_{mj}^{(r)} w_m(\theta_m \alpha) \exp\left((-1)^r y_j^{(r)} \theta_m \beta + y_j^{(r)} \theta_m s_0^{(r)}\right) w_j^{(r)}, r = 1, 2.$$

Let us insert Eq. (7.33) into boundary conditions (7.25) and (7.26). As a result, we arrive at the following system of equations

$$\sum_1^4 R_{1j}^{(1)} \exp\left(y_j^{(1)} \theta_m s_0^{(1)}\right) w_j^{(1)} - \mathrm{c} \sum_1^4 R_{1j}^{(2)} \exp\left(y_j^{(2)} \theta_m s_0^{(2)}\right) w_j^{(2)} = 0,$$

$$\sum_1^4 R_{2j}^{(1)} \exp\left(y_j^{(1)} \theta_m s_0^{(1)}\right) w_j^{(1)} + \mathrm{c} \sum_1^4 R_{2j}^{(2)} \exp\left(y_j^{(2)} \theta_m s_0^{(2)}\right) w_j^{(2)} = 0,$$

$$\sum_1^4 R_{3j}^{(1)} \exp\left(y_j^{(1)} \theta_m s_0^{(1)}\right) w_j^{(1)} - \sum_1^4 R_{3j}^{(2)} \exp\left(y_j^{(2)} \theta_m s_0^{(2)}\right) w_j^{(2)} = 0,$$

$$\sum_1^4 R_{4j}^{(1)} \exp\left(y_j^{(1)} \theta_m s_0^{(1)}\right) w_j^{(1)} + \sum_1^4 R_{4j}^{(2)} \exp\left(y_j^{(2)} \theta_m s_0^{(2)}\right) w_j^{(2)} = 0,$$

$$\sum_1^4 R_{1j}^{(r)} w_j^{(r)} = 0, \sum_1^4 R_{2j}^{(r)} w_j^{(r)} = 0, r = 1, 2. \tag{7.34}$$

$$R_{1j}^{(r)} = \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\left(y_j^{(r)}\right)^2 + \frac{B_{12}^{(r)}}{B_{22}^{(r)}}\left(\beta_m' - \left(\eta_m^{(r)}\right)^2\right)\right),$$

$$R_{2j}^{(r)} = \frac{B_{66}^{(r)}}{B_{22}^{(r)}} y_j^{(r)} \left(\left(y_j^{(r)} \right)^2 + \frac{B_{12}^{(r)}}{B_{22}^{(r)}} \beta'_m + \frac{B_{66}^{(r)}}{B_{22}^{(r)}} \left(\eta_m^{(r)} \right)^2 \right),$$
$$R_{3j}^{(r)} = \left(y_j^{(r)} \right)^2 - \frac{B_{66}^{(r)}}{B_{22}^{(r)}} \left(\beta'_m - \left(\eta_m^{(r)} \right)^2 \right),$$
$$R_{4j}^{(r)} = y_j^{(r)} \frac{B_{12}^{(r)} + B_{66}^{(r)}}{B_{22}^{(r)}}, c = \frac{B_{22}^{(2)}}{B_{22}^{(1)}}, r = 1, 2. \tag{7.35}$$

Equating the determinant Δ_e of system (7.34) to zero and performing elementary operations with columns of the determinant, we obtain the dispersion equations

$$\Delta_e = \left(y_2^{(1)} - y_1^{(1)} \right)^2 \left(y_2^{(2)} - y_1^{(2)} \right)^2 exp \left(\theta_m \sum_{r=1}^{2} s_0^{(r)} \left(y_1^{(r)} + y_2^{(r)} \right) \right)$$
$$Det \left\| e_{ij} \right\|_{i,j=1}^{8} = 0, m = \overline{1, \infty}. \tag{7.36}$$

$$e_{11} = \frac{B_{66}^{(1)}}{B_{22}^{(1)}} \left(\left(y_j^{(1)} \right)^2 + \frac{B_{12}^{(1)}}{B_{22}^{(1)}} \left(\beta'_m - \left(\eta_m^{(1)} \right)^2 \right) \right), e_{12} = \frac{B_{66}^{(1)}}{B_{22}^{(1)}} \left(y_1^{(1)} + y_2^{(1)} \right),$$
$$e_{13} = e_{11} exp \left(z_1^{(1)} \right), e_{14} = e_{12} exp \left(z_2^{(1)} \right) + e_{11} \left[z_1^{(1)} z_2^{(1)} \right],$$
$$e_{15} = \frac{B_{66}^{(2)}}{B_{22}^{(2)}} \left(\left(y_j^{(2)} \right)^2 + \frac{B_{12}^{(2)}}{B_{22}^{(2)}} \left(\beta'_m - \left(\eta_m^{(2)} \right)^2 \right) \right),$$
$$e_{16} = -c \frac{B_{66}^{(2)}}{B_{22}^{(2)}} \left(y_1^{(2)} + y_2^{(2)} \right), e_{17} = c e_{15} exp \left(z_1^{(2)} \right),$$
$$e_{18} = e_{16} exp \left(z_2^{(2)} \right) + e_{15} \left[z_1^{(2)} z_2^{(2)} \right];$$

$$e_{21}^{(1)} = \frac{B_{66}^{(1)}}{B_{22}^{(1)}} y_1^{(1)} \left(\left(y_1^{(1)} \right)^2 + \frac{B_{12}^{(1)}}{B_{22}^{(1)}} \beta'_m + \frac{B_{66}^{(1)}}{B_{22}^{(1)}} \left(\eta_m^{(1)} \right)^2 \right),$$
$$e_{22}^{(1)} = \frac{B_{66}^{(1)}}{B_{22}^{(1)}} \left(y_1^{(1)} y_2^{(1)} + B^{(1)} - \left(\eta_m^{(1)} \right)^2 \right),$$
$$e_{23} = -e_{21} exp \left(z_1^{(1)} \right), e_{24} = -e_{22} exp \left(z_2^{(1)} \right) - e_{21} \left[z_1^{(1)} z_2^{(1)} \right],$$
$$e_{25} = \frac{B_{66}^{(2)}}{B_{22}^{(2)}} y_1^{(2)} \left(\left(y_1^{(2)} \right)^2 + \frac{B_{12}^{(2)}}{B_{22}^{(2)}} \beta'_m + \frac{B_{66}^{(2)}}{B_{22}^{(2)}} \left(\eta_m^{(2)} \right)^2 \right),$$
$$e_{26}^{(1)} = \frac{B_{66}^{(2)}}{B_{22}^{(2)}} \left(y_1^{(2)} y_2^{(2)} + B^{(2)} - \left(\eta_m^{(2)} \right)^2 \right), e_{27} = -e_{25} exp \left(z_1^{(2)} \right),$$
$$e_{28} = -e_{26} exp \left(z_2^{(2)} \right) - e_{25} \left[z_1^{(2)} z_2^{(2)} \right]; \tag{7.37}$$

$$e_{31}=\left(y_1^1\right)^2-\frac{B_{66}^{(1)}}{B_{22}^{(1)}}\left(\beta_m'-\left(\eta_m^{(1)}\right)^2\right), e_{32}=y_1^{(1)}+y_2^{(1)}, e_{33}=e_{31}exp\left(z_1^{(1)}\right),$$
$$e_{34}=e_{32}exp\left(z_2^{(1)}\right)+e_{31}\left[z_1^{(1)}z_2^{(1)}\right], e_{35}=-\left(\left(y_1^2\right)^2-\frac{B_{66}^{(2)}}{B_{22}^{(2)}}\left(\beta_m'-\left(\eta_m^{(2)}\right)^2\right)\right),$$
$$e_{36}=-\left(y_1^{(2)}+y_2^{(2)}\right), e_{37}=e_{35}exp\left(z_1^{(2)}\right), e_{38}=e_{36}exp\left(z_2^{(2)}\right)+e_{35}\left[z_1^{(2)}z_2^{(2)}\right];$$

$$e_{41}=y_1^{(1)}\frac{B_{12}^{(1)}+B_{66}^{(1)}}{B_{22}^{(1)}}, e_{42}=\frac{B_{12}^{(1)}+B_{66}^{(1)}}{B_{22}^{(1)}}, e_{43}=-e_{41}exp\left(z_1^{(1)}\right),$$
$$e_{44}=-e_{42}exp\left(z_2^{(1)}\right)-e_{41}\left[z_1^{(1)}z_2^{(1)}\right],$$
$$e_{45}=y_1^{(2)}\frac{B_{12}^{(2)}+B_{66}^{(2)}}{B_{22}^{(2)}}, e_{46}=\frac{B_{12}^{(2)}+B_{66}^{(2)}}{B_{22}^{(2)}},$$
$$e_{47}=-e_{45}exp\left(z_1^{(2)}\right), e_{48}=-e_{46}exp\left(z_2^{(2)}\right)-e_{45}\left[z_1^{(2)}z_2^{(2)}\right];$$

$$e_{51}=e_{13}, e_{52}=e_{14}, e_{53}=e_{11,}e_{54}=e_{12}, e_{55}=e_{56}=e_{57}=e_{58}=0,$$
$$e_{61}=e_{23}, e_{62}=e_{24}, e_{63}=e_{21,}e_{64}=e_{22}, e_{65}=e_{66}=e_{67}=e_{68}=0,$$
$$e_{71}=e_{72}=e_{73}=e_{74}=0, e_{75}=e_{17}, e_{76}=e_{18}, e_{77}=e_{15,}e_{78}=e_{16},$$
$$e_{81}=e_{82}=e_{83}=e_{84}=0, e_{85}=e_{27}, e_{86}=e_{28}, e_{87}=e_{25,}e_{88}=e_{26};$$
$$z_j^{(r)}=-y_j^{(r)}\theta_m s_0^{(r)}, \left[z_1^{(r)}z_2^{(r)}\right]=-\frac{\theta_m s_0^{(r)}\left(exp\left(z_2^{(r)}\right)-exp\left(z_1^{(r)}\right)\right)}{\left(z_2^{(r)}-z_1^{(r)}\right)}, j=1,2; r=1,2.$$

Equation (7.36) are equivalent to the following

$$Det\left\|e_{ij}\right\|_{i,j=1}^{8}=0, m=\overline{1,\infty}. \tag{7.38}$$

For $\theta_m s_0^{(1)}\to\infty$ and $\theta_m s_0^{(2)}\to\infty$ the set of Eq. (7.38) have the form

$$Det\left\|e_{ij}\right\|_{i,j=1}^{8}=$$
$$\left(\frac{B_{66}^{(1)}}{B_{22}^{(1)}}\right)^2\left(\frac{B_{66}^{(2)}}{B_{22}^{(1)}}\right)^2\frac{B_{12}^{(1)}+B_{66}^{(1)}}{B_{22}^{(1)}}\frac{B_{12}^{(2)}+B_{66}^{(2)}}{B_{22}^{(2)}}K_2^{(1)}\left(\eta_m^{(1)}\right)K_2^{(2)}\left(\eta_m^{(2)}\right)L\left(\eta_m^{(1)},\eta_m^{(2)}\right)+$$
$$\sum_{r=1}^{2}\sum_{j=1}^{2}O\left(exp\left(z_j^{(r)}\right)\right)=0, m=\overline{1,\infty} \tag{7.39}$$
$$L\left(\eta_m^{(1)},\eta_m^{(2)}\right)=\frac{B_{12}^{(1)}+B_{66}^{(1)}}{B_{11}^{(1)}}\frac{B_{12}^{(2)}+B_{66}^{(2)}}{B_{11}^{(2)}}\left\{\left(\frac{B_{66}^{(1)}}{B_{22}^{(1)}}\right)^2K_2^{(1)}\left(\eta_m^{(1)}\right)Q^{(2)}\left(\eta_m^{(2)}\right)\right.$$

$$
\begin{aligned}
&+\left(\frac{B_{22}^{(2)}}{B_{22}^{(1)}}\right)^2\left(\frac{B_{66}^{(2)}}{B_{22}^{(1)}}\right)^2 K_2^{(2)}\left(\eta_m^{(2)}\right)Q^{(1)}\left(\eta_m^{(1)}\right)\Bigg\}\\
&+\frac{B_{22}^{(2)}}{B_{22}^{(1)}}\Big[\left(l_{11}^{(1)}l_{22}^{(2)}+l_{21}^{(1)}l_{12}^{(2)}\right)\left(l_{32}^{(1)}l_{41}^{(2)}+l_{31}^{(1)}l_{42}^{(2)}\right)\\
&+\left(l_{11}^{(2)}l_{22}^{(1)}+l_{21}^{(2)}l_{12}^{(1)}\right)\left(l_{32}^{(2)}l_{41}^{(1)}+l_{31}^{(2)}l_{42}^{(1)}\right)-\left(l_{11}^{(1)}l_{21}^{(2)}+l_{11}^{(2)}l_{21}^{(1)}\right)\\
&\left(l_{32}^{(1)}l_{42}^{(2)}+l_{31}^{(2)}l_{42}^{(1)}\right)-\left(l_{12}^{(1)}l_{22}^{(2)}+l_{12}^{(2)}l_{22}^{(1)}\right)\left(l_{31}^{(1)}l_{41}^{(2)}+l_{31}^{(2)}l_{41}^{(1)}\right)\Big],
\end{aligned}
\tag{7.40}
$$

$$
K_2^{(r)}\left(\eta_m^{(r)}\right)=\left(\beta_m'-\left(\eta_m^{(r)}\right)^2\right)\left(\frac{B_{11}^{(r)}B_{22}^{(r)}\beta_m''-\left(B_{12}^{(r)}\right)^2\beta_m'}{B_{11}^{(r)}B_{66}^{(r)}}-\left(\eta_m^{r}\right)^2\right)-\left(\eta_m^{r}\right)^2 y_1^{(r)}y_2^{(r)},
$$

$$
Q^{(r)}\left(\eta_m^{(r)}\right)=y_1^{(r)}y_2^{(r)}+\frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\beta_m'-\left(\eta_m^{(r)}\right)^2\right)
$$

$$
\frac{l_{11}^{(r)}}{=}\frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\left(y_j^{(r)}\right)^2+\frac{B_{12}^{(r)}}{B_{22}^{(r)}}\left(\beta_m'-\left(\eta_m^{(r)}\right)^2\right)\right),\ \frac{l_{12}^{(r)}}{=}\frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(y_1^{(r)}+y_2^{(r)}\right),
$$

$$
\frac{l_{21}^{(r)}}{=}\frac{B_{66}^{(r)}}{B_{22}^{(r)}}y_1^{(r)}\left(\left(y_1^{(r)}\right)^2+\frac{B_{12}^{(r)}}{B_{22}^{(r)}}\beta_m'+\frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_m^{(r)}\right)^2\right),
$$

$$
l_{22}^{(r)}=\frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(y_1^{(r)}y_2^{(r)}+B^{(r)}-\left(\eta_m^{(r)}\right)^2\right)
$$

$$
l_{31}^{(r)}=\left(y_1^{r}\right)^2-\frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\beta_m'-\left(\eta_m^{(r)}\right)^2\right), l_{32}^{(r)}=y_1^{(r)}+y_2^{(r)},
$$

$$
l_{41}^{(r)}=y_1^{(r)}\frac{B_{12}^{(r)}+B_{66}^{(r)}}{B_{22}^{(r)}}, l_{42}^{(r)}=\frac{B_{12}^{(r)}+B_{66}^{(r)}}{B_{22}^{(r)}}; r=1,2.
$$

It follows from (7.39) that for $\theta_m s_0^{(1)}\to\infty$ and $\theta_m s_0^{(2)}\to\infty$ the set of equations from (7.38) splits into sets of equations

$$
\begin{aligned}
&K_2^{(1)}\left(\eta_m^{(1)}\right)=0, m=\overline{1,\infty}; K_2^{(2)}\left(\eta_m^{2}\right)=0, m=\overline{1,\infty};\\
&L\left(\eta_m^{(1)},\eta_m^{(2)}\right)=0, m=\overline{1,\infty}.
\end{aligned}
\tag{7.41}
$$

The first and second sets of equations from (7.41) are analogs of the Rayleigh equation, which determine the frequencies of the edge oscillations of the generators $\beta = s_0^{(1)}, -s_0^{(2)}$ and the third set of equations from (7.41) is an analog of Stoneley's dispersion relations, which determines the frequencies of the interface planar vibrations at the interface line of material properties $\beta=0$.

4. **Bending vibrations of composite rectangular plate with free edges**. Existence of free bending interface and edge vibrations of a rectangular plate composed of

thin elastic orthotropic rectangular plates with different elastic properties is investigated. We introduce rectilinear orthogonal coordinates (α, β) on the middle plane of a rectangular plate, where, $0 \leq \alpha \leq l,\ -s_0^{(2)} \leq \beta \leq s_0^{(1)}$. The straight line $\beta = 0$ corresponds to the interface of material properties. All values related to the right plate $0 \leq \beta \leq s_0^{(1)}$ are indicated by the superscript (1), and to the left plate $-s_0^{(2)} \leq \beta \leq 0$by (2), respectively. As the initial equations, the equations of small bending vibrations of the left and right plates are used, which correspond to the classical theory of orthotropic plates [19].

$$\mu^4\left(B_{11}^{(r)}\frac{\partial^4 u_3^{(r)}}{\partial\alpha^4} + 2\left(B_{12}^{(r)} + 2B_{66}^{(r)}\right)\frac{\partial^4 u_3^{(r)}}{\partial\alpha^2\partial\beta^2} + B_{22}^{(r)}\frac{\partial^4 u_3^{(r)}}{\partial\beta^4}\right) = \rho^{(r)}\omega^2 u_3^{(r)},\ r = 1, 2. \tag{7.42}$$

Here $u_3^{(r)}, r = 1, 2$ are the normal components of the displacement vector of the right and left plates; $B_{ij}^{(r)}, r = 1, 2$—elasticity coefficients; ω–angular frequency; $\rho^{(r)}, r = 1, 2$—the density of materials; $\mu^4 = h^2/12$ (h–plate thickness). The following boundary conditions are considered.

$$M_2^{(1)}\Big|_{\beta=0} = M_2^{(2)}\Big|_{\beta=0},\ N_2^{(1)} + \frac{\partial H^{(1)}}{\partial\alpha}\Big|_{\beta=0} = N_2^{(2)} + \frac{\partial H^{(2)}}{\partial\alpha}\Big|_{\beta=0},$$
$$u_3^{(1)}\Big|_{\beta=0} = u_3^{(2)}\Big|_{\beta=0},\ \frac{\partial u_3^{(1)}}{\partial\beta}\Big|_{\beta=0} = \frac{\partial u_3^{(2)}}{\partial\beta}\Big|_{\beta=0}; \tag{7.43}$$

$$N_2^{(r)} + \frac{\partial H^{(r)}}{\partial\alpha}\Big|_{\beta=(-1)^{r-1}s^{(r)}} = 0,\ M_2^{(r)}\Big|_{\beta=(-1)^{r-1}s^{(r)}} = 0, r = 1, 2. \tag{7.44}$$

$$N_1^{(r)} + \frac{\partial H^{(r)}}{\partial\beta}\Big|_{\alpha=0,l} = 0,\ M_1^{(r)}\Big|_{\alpha=0,l} = 0, r = 1, 2. \tag{7.45}$$

$$M_1^{(r)} = \frac{h^3}{12}B_{11}^{(r)}\left[\frac{\partial^2 u_3^{(r)}}{\partial\propto^2} + \frac{B_{12}^{(r)}}{B_{11}^{(r)}}\frac{\partial^2 u_3^{(r)}}{\partial\beta^2}\right], M_2^{(r)} = \frac{h^3}{12}B_{22}^{(r)}\left[\frac{B_{12}^{(r)}}{B_{22}^{(r)}}\frac{\partial^2 u_3^{(r)}}{\partial\propto^2} + \frac{\partial^2 u_3^{(r)}}{\partial\beta^2}\right],$$
$$N_1^{(r)} + \frac{\partial H^{(r)}}{\partial\beta} = \frac{h^3}{12}B_{11}^{(r)}\left[\frac{\partial^3 u_3^{(r)}}{\partial\propto^3} + \frac{B_{12}^{(r)} + 4B_{66}^{(r)}}{B_{11}^{(r)}}\frac{\partial^3 u_3^{(r)}}{\partial\beta^2\partial\alpha}\right],$$
$$N_2^{(r)} + \frac{\partial H^{(r)}}{\partial\alpha} = \frac{h^3}{12}B_{22}^{(r)}\left[\frac{\partial^3 u_3^{(r)}}{\partial\beta^3} + \frac{B_{12}^{(r)} + 4B_{66}^{(r)}}{B_{22}^{(r)}}\frac{\partial^3 u_3^{(r)}}{\partial\alpha^2\partial\beta}\right], \mathrm{r} = 1, 2. \tag{7.46}$$

Here relations (7.43) are complete contact conditions at $\beta = 0$. Relations (7.44) and (7.45) are the conditions of free edges at $\beta = s_0^{(1)}, -s_0^{(2)}$ and $\alpha = 0, l$, where l is the length of the plate, respectively (Fig. 7.2.) Problem (7.42)–(7.45) does not allow separation of variables. The differential operator corresponding to problem (7.42)–(7.45) is self-conjugate and nonnegative definite. Therefore, the generalized Kantorovich-Vlasov method of reduction to ordinary differential equations can be used to find vibration eigenfrequencies and eigenmodes [25]. The solution of system (7.42) is sought in the form

$$\begin{aligned} u_3^{(r)} =& w_m(\theta_m\alpha)\exp\Big((-1)^r y^{(r)}\theta_m\beta + y^{(r)}\theta_m s_0^{(r)}\Big), \\ & r = 1,2;\, m = \overline{1,\infty}. \end{aligned} \tag{7.47}$$

Here $w_m(\theta_m\alpha)$ defined in (7.7), $y^{(r)}$, $r = 1,2$ are undetermined coefficients. In this case, conditions (7.45) are satisfied automatically. We substitute (7.47) into (7.42). As a result, we obtain the characteristic equations

$$\begin{aligned} R_{mm}^{(r)} = a^2 & \left(\left(y^{(r)}\right)^4 - \frac{2\left(B_{12}^{(r)} + 2B_{66}^{(r)}\right)}{B_{22}^{(r)}}\beta'_m\left(y^{(r)}\right)^2 + \frac{B_{11}^{(r)}}{B_{22}^{(r)}}\beta'_m\beta''_m \right) \\ & - \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_m^{(r)}\right)^2 = 0, \quad r = 1,2;\, m = \overline{1,\infty}, \end{aligned} \tag{7.48}$$

where a^2, $\eta_m^{(r)}$, β'_m, β''_m are defined in (7.15), (7.9).

Let $y_3^{(r)}$, $y_4^{(r)}$ be the different roots of Eq. (7.48) with positive real parts and $y_{j+2}^{(r)} = -y_j^{(r)}$ $j = 3,4$. We seek solutions to problem (7.42)–(7.45) in the form

$$\begin{aligned} u_3^{(r)} &= \sum_{j=1}^{4} w_m(\theta_m\alpha)\exp\Big((-1)^r y_j^{(r)}\theta_m\beta + y_j^{(r)}\theta_m s_0^{(r)}\Big) w_j^{(r)}, \\ r &= 1,2;\, m = \overline{1,\infty}. \end{aligned} \tag{7.49}$$

We substitute (7.49) into the boundary conditions (7.43)–(7.44). As a result, we arrive at the system of equations

$$\begin{aligned} &\sum_3^6 P_{1j}^{(1)}\exp\Big(y_j^{(1)}\theta_m s_0^{(1)}\Big)w_j^{(1)} - \mathrm{c}\sum_3^6 P_{1j}^{(2)}\exp\Big(y_j^{(2)}\theta_m s_0^{(2)}\Big)w_j^{(2)} = 0, \\ &\sum_3^6 P_{2j}^{(1)}\exp\Big(y_j^{(1)}\theta_m s_0^{(1)}\Big)w_j^{(1)} + \mathrm{c}\sum_3^6 P_{2j}^{(2)}\exp\Big(y_j^{(2)}\theta_m s_0^{(2)}\Big)w_j^{(2)} = 0, \\ &\sum_3^6 \exp\Big(y_j^{(1)}\theta_m s_0^{(1)}\Big)w_j^{(1)} - \sum_3^6 R_{3j}^{(2)}\exp\Big(y_j^{(2)}\theta_m s_0^{(2)}\Big)w_j^{(2)} = 0, \end{aligned} \tag{7.50}$$

$$\sum_{3}^{6} y_j^{(1)} \exp\left(y_j^{(1)}\theta_m s_0^{(1)}\right) w_j^{(1)} + \sum_{3}^{6} y_j^{(2)} \exp\left(y_j^{(2)}\theta_m s_0^{(2)}\right) w_j^{(2)} = 0,$$

$$\sum_{3}^{6} P_{1j}^{(r)} w_j^{(r)} = 0, \sum_{3}^{6} P_{2j}^{(r)} w_j^{(r)} = 0, r = 1, 2.$$

$$P_{1j}^{(r)} = \left(y_j^{(r)}\right)^2 - \frac{B_{12}^{(r)}}{B_{22}^{(r)}}\beta'_m, P_{2j}^{(r)} = \left(y_j^{(r)}\right)^3 - \frac{B_{12}^{(r)} + 4B_{66}^{(r)}}{B_{22}^{(r)}}\beta'_m y_j^{(r)},$$

$$c = \frac{B_{22}^{(2)}}{B_{22}^{(1)}}, r = 1, 2;\ j = \overline{3, 6}. \tag{7.51}$$

Equating the determinant of system (7.50) to zero and performing elementary operations on the columns of the determinant, we obtain the dispersion equations

$$\Delta_p = \left(y_4^{(1)} - y_3^{(1)}\right)^2 \left(y_4^{(2)} - y_3^{(2)}\right)^2 exp\left(\theta_m \sum_{r=1}^{2} s_0^{(r)}\left(y_3^{(r)} + y_4^{(r)}\right)\right)$$

$$Det\left\|b_{ij}\right\|_{i,j=1}^{8} = 0, m = \overline{1, \infty}. \tag{7.52}$$

$$b_{11} = \left(y_3^{(1)}\right)^2 - \frac{B_{12}^{(1)}}{B_{22}^{(1)}}\beta'_m, b_{12} = y_3^{(1)} + y_4^{(1)}$$

$$b_{13} = b_{11} exp\left(z_3^{(1)}\right), b_{14} = b_{12} exp\left(z_4^{(1)}\right) + b_{11}\left[z_3^{(1)} z_4^{(1)}\right],$$

$$b_{15} = -c\left(\left(y_j^{(2)}\right)^2 - \frac{B_{12}^{(2)}}{B_{22}^{(2)}}\beta'_m\right), b_{16} = -c\left(y_3^{(2)} + y_4^{(2)}\right),$$

$$b_{17} = b_{15} exp\left(z_3^{(2)}\right), b_{18} = b_{16} exp\left(z_4^{(2)}\right) + b_{15}\left[z_3^{(2)} z_4^{(2)}\right];$$

$$b_{21} = \left(y_3^{(1)}\right)^3 - \frac{B_{12}^{(1)} + 4B_{66}^{(1)}}{B_{22}^{(1)}}\beta'_m y_3^{(1)}, b_{22} = y_3^{(1)} y_4^{(1)} + \frac{B_{12}^{(1)}}{B_{22}^{(1)}}\beta'_m,$$

$$b_{23} = -b_{21} exp\left(z_3^{(1)}\right), b_{24} = -b_{22} exp\left(z_4^{(1)}\right) - b_{21}\left[z_3^{(1)} z_4^{(1)}\right],$$

$$b_{25} = c\left(\left(y_3^{(2)}\right)^3 - \frac{B_{12}^{(2)} + 4B_{66}^{(2)}}{B_{22}^{(2)}}\beta'_m y_3^2\right),$$

$$b_{26} = c\left(y_3^{(2)} y_4^{(2)} + \frac{B_{12}^{(2)}}{B_{22}^{(2)}}\beta'_m\right), b_{27} = -b_{25} exp\left(z_3^{(2)}\right),$$

$$b_{28} = -b_{26} exp\left(z_4^{(2)}\right) - b_{25}\left[z_3^{(2)} z_4^{(2)}\right]; \tag{7.53}$$

$$b_{31} = 1, b_{32} = 0, , b_{33} = exp\left(z_3^{(1)}\right), b_{34} = \left[z_3^{(1)} z_4^{(1)}\right],$$

$$b_{35} = -1, b_{36} = 0, b_{37} = -exp\left(z_3^{(2)}\right), b_{38} = -\left[z_3^{(2)} z_4^{(2)}\right];$$

$$b_{41} = y_3^{(1)}, b_{42} = 1, b_{43} = -y_3^{(1)} exp\left(z_3^{(1)}\right), b_{44} = -exp\left(z_4^{(1)}\right) - y_3^{(1)}\left[z_3^{(1)} z_4^{(1)}\right],$$

$$b_{45} = y_3^{(2)}, b_{46} = 1, b_{47} = -y_3^{(2)} exp\left(z_3^{(2)}\right), b_{48} = -b_{46} exp\left(z_4^{(2)}\right) - b_{45}\left[z_3^{(2)} z_4^{(2)}\right];$$

$$b_{51} = b_{13}, b_{52} = b_{14}, b_{53} = b_{11,} b_{54} = b_{12}, b_{55} = b_{56} = b_{57} = b_{58} = 0,$$
$$b_{61} = b_{23}, b_{62} = b_{24}, b_{63} = b_{21,} b_{64} = b_{22}, b_{65} = b_{66} = b_{67} = b_{68} = 0,$$
$$b_{71} = b_{72} = b_{73} = b_{74} = 0, b_{75} = b_{17}, b_{76} = b_{18}, b_{77} = b_{15,} b_{78} = b_{16},$$
$$b_{81} = b_{82} = b_{83} = b_{84} = 0, b_{85} = b_{27}, b_{86} = b_{28}, b_{87} = b_{25,} b_{88} = b_{26};$$

$$z_j^{(r)} = -y_j^{(r)} \theta_m s_0^{(r)}, \left[z_3^{(r)} z_4^{(r)}\right] = -\frac{\theta_m s_0^{(r)}\left(exp\left(z_4^{(r)}\right) - exp\left(z_3^{(r)}\right)\right)}{\left(z_4^{(r)} - z_3^{(r)}\right)},$$
$$j = 3, 4; r = 1, 2:$$

Equation (7.52) are equivalent to the equations

$$Det\left\|b_{ij}\right\|_{i,j=1}^{8} = 0, m = \overline{1, \infty}. \tag{7.54}$$

For $\theta_m s_0^{(1)} \to \infty$ and $\theta_m s_0^{(2)} \to \infty$ the set of Eq. (7.54) have the form

$$Det\left\|b_{ij}\right\|_{i,j=1}^{8} = -\left(\frac{B_{22}^{(2)}}{B_{22}^{(1)}}\right)^2 G\left(\eta_m^{(1)}, \eta_m^{(2)}\right) K_1^{(1)}\left(\eta_m^{(1)}\right) K_1^{(2)}\left(\eta_m^{(2)}\right)$$
$$+ \sum_{r=1}^{2} \sum_{j=3}^{4} O\left(exp\left(z_j^{(r)}\right)\right) = 0, m = \overline{1, \infty}. \tag{7.55}$$

$$G\left(\eta_m^{(1)}, \eta_m^{(2)}\right) = -K_1^{(1)}\left(\eta_m^{(1)}\right) - c^2 K_1^{(2)}\left(\eta_m^{(2)}\right) + c[\left(b_{11}^{(1)} b_{22}^{(2)} + b_{22}^{(1)} b_{11}^{(2)}\right)$$
$$+ \left(l_{21}^{(1)} l_{12}^{(2)} + l_{12}^{(1)} l_{21}^{(2)}\right) - \left(y_3^{(1)} + y_3^{(2)}\right)\left(b_{12}^{(1)} b_{22}^{(2)} + b_{22}^{(1)} b_{12}^{(2)}\right)], m = \overline{1, \infty} \tag{7.56}$$

$$K_1^{(r)}\left(\eta_m^{(r)}\right) = \left(y_3^{(r)} y_4^{(r)}\right)^2 + \frac{4B_{66}^{(r)}}{B_{22}^{(r)}} \beta'_m y_3^{(r)} y_4^{(r)} - \left(\frac{B_{12}^{(r)}}{B_{22}^{(r)}}\right)^2 \left(\beta'_m\right)^2, r = 1, 2;$$

$$b_{11}^{(r)} = \left(y_3^{(r)}\right)^2 - \frac{B_{12}^{(r)}}{B_{22}^{(r)}} \beta'_m, b_{12}^{(r)} = y_3^{(r)} + y_4^{(r)}, b_{21}^{(r)} = \left(y_3^{(r)}\right)^3 -$$
$$\frac{B_{12}^{(r)} + 4B_{66}^{(r)}}{B_{22}^{(r)}} \beta'_m y_3^{(r)}, b_{22}^{(r)} = y_3^{(r)} y_4^{(r)} + \frac{B_{12}^{(r)}}{B_{22}^{(r)}} \beta'_m;$$

It follows from (7.55) that, for $\theta_m s_0^{(1)} \to \infty$ and $\theta_m s_0^{(2)} \to \infty$, Eq. (7.54) splits into sets of equations

$$G\left(\eta_m^{(1)}, \eta_m^{(2)}\right) = 0, m = \overline{1, \infty};\ K_1^{(1)}\left(\eta_m^{(1)}\right) = 0, m = \overline{1, \infty};$$
$$K_1^{(2)}\left(\eta_m^2\right) = 0, m = \overline{1, \infty}. \tag{7.57}$$

The first set of equations from (7.57) is a set of dispersion equations of the bending type of the interface vibration of a plate composed of two orthotropic sufficiently wide plates with different elastic properties when the edges are free. The second and third sets of equations from (7.57) are analogs of the Konenkov type equations for the bending vibration of a plate made of the material of the right and left plates, respectively, with free edges [8].

5. **Asymptotics of dispersion Eq. (7.23) for $\boldsymbol{\varepsilon_m \to 0}$**

Using the formulae from Sect. 7.2, we assume that $\eta_1^{(r)} = \eta_2^{(r)} = \eta_3^{(r)} = \eta^{(r)}$, $(r = 1, 2)$. Then, when $\varepsilon_m \to 0$ Eq. (7.14) is transformed into a set of equations

$$c_m^{(r)} = \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\chi^{(r)}\right)^4 - B_2^{(r)}\left(\chi^{(r)}\right)^2 + \frac{B_{22}^{(r)} + B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_m^{(r)}\right)^2\left(\chi^{(r)}\right)^2$$
$$+ \left(\beta_m' - \left(\eta_{2m}^{(r)}\right)^2\right)\left(\beta_m'' - \frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_{1m}^{(r)}\right)^2\right) r = 1, 2, m = \overline{1, \infty} \tag{7.58}$$

$$R_{mm}^{(r)} = a^2\left(\left(\chi^{(r)}\right)^4 - \frac{2\left(B_{12}^{(r)} + 2B_{66}^{(r)}\right)}{B_{22}^{(r)}}\beta_m'\left(\chi^{(r)}\right)^2 + \frac{B_{11}^{(r)}}{B_{22}^{(r)}}\beta_m'\beta_m''\right) -$$
$$\frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_{3m}^{(r)}\right)^2 = 0, r = 1, 2,\ m = \overline{1, \infty} \tag{7.59}$$

which are sets of characteristic equations for the equations of planar and bending vibrations of the left and right rectangular components with two opposite free edges, respectively [10]. The roots of Eqs. (7.58) and (7.59) with positive real parts are denoted by $y_1^{(r)}, y_2^{(r)}$ and $y_3^{(r)}, y_4^{(r)}$ respectively.

In the same way as in [27], it may be proved that under the condition

$$\varepsilon_m \ll 1, y_1^{(r)}, y_j^{(r)} \neq y_i^{(r)}, \quad i \neq j, \tag{7.60}$$

the roots $\left(\chi^{(r)}\right)^2$ of Eq. (7.14) can be represented as

$$\left(\chi_j^{(r)}\right)^2 = \left(y_j^{(r)}\right)^2 + \alpha_{jm}^{(r)}\varepsilon_m^2 + \beta_{jm}^{(r)}\varepsilon_m^4 + \cdots, j = \overline{1, 4}. \tag{7.61}$$

Under conditions (7.60), taking into account relations (7.18), (7.61) and the fact that

$$M_{3j}^{(r)} = M_{4j}^{(r)} = M_{7j}^{(r)} = M_{8j}^{(r)} = O\left(\varepsilon_m^2\right),\ j = 1, 2;\ r = 1, 2, \tag{7.62}$$

the set of Eq. (7.23) is reduced to the form

$$\begin{aligned} &Det\left\|t_{ij}\right\|_{i,j=1}^{4} = \left(N^{(1)}\left(\eta_m^{(1)}\right)N^{(2)}\left(\eta_m^{(2)}\right)\right)^2\left(K_3^{(1)}\left(\eta_m^{(1)}\right)K_3^{(2)}\left(\eta_m^{(2)}\right)\right)^2 X \\ &Det\left\|e_{ij}\right\|_{i,j=1}^{8}.Det\left\|b_{ij}\right\|_{i,j=1}^{8} + O\left(\varepsilon_m^2\right) = 0, \quad m = \overline{1,\infty}, \end{aligned} \tag{7.63}$$

$$\begin{aligned} &N^{(r)}\left(\eta_m^{(r)}\right) = \left(y_1^{(r)} + y_3^{(r)}\right)\left(y_1^{(r)} + y_4^{(r)}\right)\left(y_2^{(r)} + y_3^{(r)}\right)\left(y_2^{(r)} + y_3^{(r)}\right), \\ &K_3^{(r)}\left(\eta_m^{(r)}\right) = (\left(\beta_m' - \left(\eta_m^{(r)}\right)^2\right)\left(\frac{B_{11}^{(r)}}{B_{22}^{(r)}}\beta_m'' - \frac{B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_m^{(r)}\right)^2\right)\left(p^{(r)}\right)^2 \\ &+ \left(\frac{B_{11}^{(r)}}{B_{22}^{(r)}}B_2^{(r)} - \frac{B_{22}^{(r)} + B_{66}^{(r)}}{B_{22}^{(r)}}\left(\eta_m^{(r)}\right)^2\right)p^{(r)}q^{(r)} + \left(p^{(r)}\right)^2) \\ &\left(\frac{B_{22}^{(r)}}{B_{22}^{(r)} + B_{66}^{(r)}}\right)^2\left(\frac{1}{\beta_m'}\right)^2, \end{aligned}$$

$$p^{(r)} = \frac{B_{22}^{(r)}}{B_{11}^{(r)}} + a^2\left(\frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_m^{(r)}\right)^2 + \frac{\left(B_{12}^{(r)}\right)^2 + 3B_{12}^{(r)}B_{66}^{(r)} + 4\left(B_{66}^{(r)}\right)^2}{B_{11}^{(r)}B_{66}^{(r)}}\beta_m'\right), \tag{7.64}$$

$$q^{(r)} = \frac{B_{22}^{(r)}}{B_{11}^{(r)}}\left(\eta_m^{(r)}\right)^2 - B_1^{(r)} + a^2\left(\frac{B_{66}^{(r)}}{B_{11}^{(r)}}\left(\eta_m^{(r)}\right)^2 - \beta_m''\right)\left(\left(\eta_m^{(r)}\right)^2 + \frac{B_{12}^{(r)} + 3B_{66}^{(r)}}{B_{66}^{(r)}}\beta_m'\right),$$

and determinants $Det\left\|e_{ij}\right\|_{i,j=1}^{8}.Det\left\|b_{ij}\right\|_{i,j=1}^{8}$ are defined in (7.38) and (7.54), respectively.

It follows from Eq. (7.63) that, when $\varepsilon_m \to 0$ the set of Eq. (7.23) splits into sets of equations

$$\begin{aligned} &Det\left\|e_{ij}\right\|_{i,j=1}^{8} = 0, m = \overline{1,\infty};\ Det\left\|b_{ij}\right\|_{i,j=1}^{8} = 0, \\ &m = \overline{1,\infty};\ K_3^{(r)}\left(\eta_m^{(r)}\right) = 0, r = 1, 2;\ m = \overline{1,\infty}. \end{aligned} \tag{7.65}$$

The first and second sets of equations from (7.65) are the dispersion equations of planar and bending interface and edge vibrations of a composite plate, respectively. The roots of the third and fourth set of equations correspond to planar vibrations of the components of the cylindrical panel. They appear as a result of using the equation of the corresponding classical theory of orthotropic cylindrical shells.

For $\varepsilon_m \to 0$, $\theta_m s_0^{(1)} \to \infty$ and $\theta_m s_0^{(2)} \to \infty$, relations (7.23) take the form

$$Det\left\|t_{ij}\right\|_{i,j=1}^{4}=\left(\frac{B_{66}^{(1)}}{B_{22}^{(1)}}\right)^2\left(\frac{B_{66}^{(2)}}{B_{22}^{(2)}}\right)^2\left(\frac{B_{22}^{(2)}}{B_{22}^{(1)}}\right)^2\frac{B_{12}^{(1)}+B_{66}^{(1)}}{B_{22}^{(2)}}\frac{B_{12}^{(2)}+B_{66}^{(2)}}{B_{22}^{(2)}}X$$

$$\left(N^{(1)}\left(\eta_m^{(1)}\right)N^{(2)}\left(\eta_m^{(2)}\right)\right)^2\left(K_3^{(1)}\left(\eta_m^{(1)}\right)K_3^{(2)}\left(\eta_m^{(2)}\right)\right)^2K_1^{(1)}\left(\eta_m^{(1)}\right)K_2^{(1)}\left(\eta_m^{(1)}\right)X$$

$$K_1^{(2)}\left(\eta_m^{(2)}\right)K_2^{(2)}\left(\eta_m^{(2)}\right)L\left(\eta_m^{(1)},\eta_m^{(2)}\right)G\left(\eta_m^{(1)},\eta_m^{(2)}\right)+O\left(\varepsilon_m^2\right)$$

$$+\sum_{r=1}^{2}\sum_{j=1}^{4}O\left(exp\left(z_j^{(r)}\right)\right)=0,\ m=\overline{1,\infty} \tag{7.66}$$

From (7.66) it follows that for, $\varepsilon_m\to 0$, $\theta_m s_0^{(1)}\to\infty$ and $\theta_m s_0^{(2)}\to\infty$ the set of Eqs. (7.23) splits into the totality of the equations

$$L\left(\eta_m^{(1)},\eta_m^{(2)}\right)=0,\ m=\overline{1,\infty};\ G\left(\eta_m^{(1)},\eta_m^{(2)}\right)=0,\ m=\overline{1,\infty};$$

$$K_2^{(r)}\left(\eta_m^{(r)}\right)=0,\ m=\overline{1,\infty};\ K_1^{(r)}\left(\eta_m^{(r)}\right)=0,\ m=\overline{1,\infty};$$

$$K_3^{(r)}\left(\eta_m^{(r)}\right)=0,\ m=\overline{1,\infty};\ r=1,2. \tag{7.67}$$

The first and second sets of equations from (7.67) are, respectively, the dispersion equations of the planar and bending interface vibrations for a sufficiently wide composite plate with free edges.

The third and fourth sets of equations from (7.67) are, respectively, analogs to the Rayleigh and Konenkov equations for vibrations of a plate made of material (1) and (2) localized at the free edges.

6. **Numerical investigation.** Tables 7.1 and 7.2 shows some of the roots of Eqs. (7.41), (7.57) and the dispersion Eqs. (7.38) and (7.54) of planar and bending vibrations for a composite rectangular plate, with free edges, made of boroplastic and special paper with mechanical parameters [19, 30]

$$\underline{Boroplastic}\ \rho^{(1)}=2.10^3\frac{kg}{m^3},\ E_1^{(1)}=2.646.10^{11}\frac{N}{m^2},\ E_2^{(1)}=1.323.10^{10}\frac{N}{m^2},$$

$$G^{(1)}=9.604.10^9\frac{N}{m^2},\ \gamma_1^{(1)}=0.2,\ \gamma_2^{(1)}=0.01; \tag{7.68}$$

$$\underline{Paper}\ \rho^{(2)}=0.16\frac{kg}{m^3},\ E_1^{(2)}=2.95281.10^9\frac{N}{m^2},\ E_2^{(2)}=2.210.10^9\frac{N}{m^2},$$

$$G^{(2)}=9.77076.10^8\frac{N}{m^2},\ \gamma_1^{(2)}=\frac{\gamma_2^{(2)}E_1^{(2)}}{E_2^{(2)}},\ \gamma_2^{(2)}=0.23 \tag{7.69}$$

Table 7.1 Characteristics of the natural frequencies for planar vibrations of a rectangular plate

N	θ_m	$K_2^{(1)}\left(\eta_m^{(1)}\right)=0, L\left(\eta_m^{(1)},\eta_m^{(2)}\right)=0$ $K_2^{(2)}\left(\eta_m^{(2)}\right)=0$	$Det\Vert e_{ij}\Vert_{i,j=1}^{8}=0$
1	1.95473		0.91143 $e^{(1)}$ 0.92511 ^{ine} 27.5679 $e^{(2)}$
2	2.74891	0.98307 0.98398 32.9634	0.98171 $e^{(1)}$ 0.98311 ^{ine} 30.7910 $e^{(2)}$
3	3.52957	1.00099 1.00225 33.3045	0.99965 $e^{(1)}$ 1.00100 ^{ine} 30.6812 $e^{(2)}$
4	4.27693	0.97897 0.98012 32.6340	0.977771 $e^{(1)}$ 0.997897^{ine} 30.1650 $e^{(2)}$
11	11.6577	0.98580 0.98696 32.8645	0.98436 $e^{(1)}$ 0.98580 ^{ine} 30.3826 $e^{(2)}$
15	16.0962	0.98580 0.98696 32.8645	0.98436 $e^{(1)}$ 0.98580 ^{ine} 30.3826 $e^{(2)}$
16	17.1935	0.98580 0.98696 32.8645	0.98436 $e^{(1)}$ 0.98580 ^{ine} 30.3826 $e^{(2)}$

and geometric parameters: $l = 4, h = \frac{1}{50}, s_0^{(1)} = 15, s_0^{(2)} = 5$. The roots of the dispersion Eqs. (7.41) and (7.57) of planar and bending edge and interface vibrations of a rectangular plate are given.

Note that the connection between $\eta_m^{(1)}$ and $\eta_m^{(2)}$ has the form

$$\eta_m^{(2)} = \frac{\rho^{(2)} B_{66}^{(1)}}{\rho^{(1)} B_{66}^{(2)}} \tag{7.70}$$

Tables 7.3 and 7.4 shows dimensionless characteristics $\eta_{1m}^{(1)}$ of the natural frequencies of interface and edge vibrations for a composite cylindrical panel, made of boroplastic and special paper, with mechanical parameters (7.68), (7.69) and geometric parameters:

$$R = 45, l = 4, h = 1/50, s^{(1)} = 15.2927, s^{(2)} = 5.01035.$$

Table 7.2 Characteristics of the natural frequencies for bending vibrations of a rectangular plate

N	θ_m	$K_1^{(1)}\left(\eta_m^{(1)}\right)=0,\ G\left(\eta_m^{(1)},\eta_m^{(2)}\right)=0$ $K_1^{(2)}\left(\eta_m^{(2)}\right)=0$	$Det\lVert b_{ij}\rVert_{i,j=1}^{8}=0$
1	1.95473	0.04872 0.04874 0.59552	0.04872 $b^{(1)}$ 0.04926 ^{inb} 0.31068 $b^{(2)}$
2	2.74891	0.08576 0.08580 1.04836	0.08580 $b^{(1)}$ 0.08638 ^{inb} 0.82623 $b^{(2)}$
3	3.52957	0.10921 0.10927 1.33494	0.10921 $b^{(1)}$ 0.10949 ^{inb} 1.11285 $b^{(2)}$
4	4.27693	0.12789 0.12795 1.56329	0.12789 $b^{(1)}$ 0.12830 ^{inb} 1.31165 $b^{(2)}$
11	11.6577	0.35363 0.35382 4.32282	0.35363 $b^{(1)}$ 0.35410 ^{inb} 4.11623 $b^{(2)}$
15	16.0962	0.48828 0.488531 5.96867	0.48828 $b^{(1)}$ 0.49196 ^{inb} 5.80026 $b^{(2)}$
16	17.1935	0.52156 0.52183 6.37556	0.52156 $b^{(1)}$ 0.52389 ^{inb} 6.00390 $b^{(2)}$

In Tables 7.1, 7.2, 7.3 and 7.4 after the characteristics of the natural frequencies, the type of interface vibrations is indicated: *ine* is predominantly planar, *inb* is predominantly bending; edge vibrations are: $e^{(r)}$, $r = 1{,}2$—predominantly planar types, $b^{(r)}$, $r = 1{,}2$—predominantly bending types, $n^{(r)}$, $r = 1{,}2$—new types of vibrations corresponding to materials (1) and (2), respectively. Note that the new types of vibrations are predominantly planar types. The latest manifests itself as a result of using the basic equations corresponding to the classical theory of orthotropic cylindrical shells.

In Table 7.3 the case $\eta_{1m}^{(r)} = \eta_{2m}^{(r)} = \eta_m^{(r)}, \eta_{3m}^{(r)} = 0, r = 1,2$ corresponds to problem (7.1)–(7.4), in which there is no normal component of the inertia force, i.e. we have a predominantly planar type of interface and edge vibrations. Similarly, the case $\eta_{1m}^{(r)} = \eta_{2m}^{(r)} = 0, \eta_{3m}^{(r)} = \eta_m^{(r)}, r = 1,2$ corresponds predominantly to bending type.

In Table 7.4, the case $\eta_{1m}^{(r)} = \eta_{2m}^{(r)} = \eta_{3m}^{(r)} = \eta_m^{(r)}$ corresponds to the problem (7.1)–(7.4).

Table 7.3 Characteristics of the natural frequencies for predominantly planar and predominantly bending vibrations of cylindrical panel

N	θ_m	$K_3^{(1)}\left(\eta_m^{(1)}\right)=0$ $K_3^{(2)}\left(\eta_m^{(2)}\right)=0$	$\eta_{1m}^{(r)}=\eta_{2m}^{(r)}=\eta_m^{(r)}$ $\eta_{3m}^{(r)}=0, r=1,2$	$\eta_{1m}^{(r)}=\eta_{2m}^{(r)}=0$ $\eta_{3m}^{(r)}=\eta_m^{(r)}, r=1,2$
1	1.95391	4.25538 41.3822	0.91110 $e^{(1)}$ 4.25538 $n^{(1)}$ 0.92492 ^{ine} 30.8139 $e^{(2)}$ 41.4274 $n^{(2)}$	0.04987 $b^{(1)}$ 0.05037 ^{inb} 0.31296 $b^{(2)}$
2	2.74776	4.94711 44.7479	0.98479 $e^{(1)}$ 4.94711 $n^{(1)}$ 0.99608 ^{ine} 30.8149 $e^{(2)}$ 44.7652 $n^{(2)}$	0.08584 $b^{(1)}$ 0.08625 ^{inb} 0.81451 $b^{(2)}$
3	3.52810	4.81097 44.0542	1.00349 $e^{(1)}$ 4.81005 $n^{(1)}$ 1.01586 ^{ine} 30.6635 $e^{(2)}$ 43.9950 $n^{(2)}$	0.10888 $b^{(1)}$ 0.10965 ^{inb} 1.10953 $b^{(2)}$
4	4.27542	4.75564 43.8201	0.98210 $e^{(1)}$ 4.75564 $n^{(1)}$ 0.99330 ine 30.5387 $e^{(2)}$ 43.8119 $n^{(2)}$	0.12759 $b^{(1)}$ 0.12803 inb 1.31880 $b^{(2)}$
11	11.6577	4.78850 44.0115	0.98696 $e^{(1)}$ 4.78850 $n^{(1)}$ 0.99970 ^{ine} 30.3831 $e^{(2)}$ 44.0155 $n^{(2)}$	0.35377 $b^{(1)}$ 0.35430 ^{inb} 4.12242 $b^{(2)}$
15	16.1102	4.78555 44.0396	0.98696 $e^{(1)}$ 4.78555 $n^{(1)}$ 0.99970 ^{ine} 30.3822 $e^{(2)}$ 44.0403 $n^{(2)}$	0.48841 $b^{(1)}$ 0.50299 ^{inb} 5.80563 $b^{(2)}$
16	17.2065	4.78465 44.0479	0.98696 $e^{(1)}$ 4.78465 $n^{(1)}$ 0.99970 ^{ine} 30.3829 $e^{(2)}$ 44.0506 $n^{(2)}$	0.52158 $b^{(1)}$ 0.52392 ^{inb} 5.99851 $b^{(2)}$

For $\varepsilon_m \to 0$ the interface and edge vibrations of problem (7.1)–(7.4), are splitting on quasi-transverse and quasi tangential vibrations. Meanwhile, the frequencies of this problem tend to the frequencies of a similar problem for a composite plate.

In vibrations of a predominantly tangential type$\eta_{1m}^{(r)}=\eta_{2m}^{(r)}=\eta_m^{(r)}$, $\eta_{3m}^{(r)}=0$, $r=1,2$, in addition to planar interface and edge vibrations of the Stoneley and Rayleigh type, new vibrations can also appear due to the longitudinal and torsional components of the inertia force [14].

7.2 Conclusions

Using the system of equations of dynamic equilibrium of orthotropic cylindrical shells of the corresponding classical theory, dispersion equations are obtained to determine the eigenfrequencies of interfacial and edge vibrations of the composite cylindrical panel with free edges.

Table 7.4 Characteristics of the natural frequencies of a cylindrical panel

N	θ_m	$\eta_{1m}^{(r)} = \eta_{2m}^{(r)} = \eta_{3m}^{(r)} = \eta_m^{(r)}$
1	1.95391	0.91201 $e^{(1)}$ 0.04986 $b^{(1)}$ 4.25908 $n^{(1)}$ 0.92503 ^{ine} 0.05037 ^{inb} 30.8092 $e^{(2)}$ 0.31295 $b^{(2)}$ 41.4275 $n^{(2)}$
2	2.74776	0.98431 $e^{(1)}$ 0.08621 $b^{(1)}$ 4.94711 $n^{(1)}$ 0.99667 ^{ine} 0.086631 ^{inb} 30.7821 $e^{(2)}$ 0.81451 $b^{(2)}$ 44.7696 $n^{(2)}$
3	3.52810	1.00225 $e^{(1)}$ 0.10932 $b^{(1)}$ 4.81097 $n^{(1)}$ 1.01914 ^{ine} 0.10991 ^{inb} 30.6693 $e^{(2)}$ 1.10952 $b^{(2)}$ 43.9950 $n^{(2)}$
4	4.27542	0.98291 $e^{(1)}$ 0.12803 $b^{(1)}$ 4.75564 $n^{(1)}$ 0.99309 ^{ine} 0.12868 ^{inb} 30.5395 $e^{(2)}$ 1.33719 $b^{(2)}$ 43.8120 $n^{(2)}$
11	11.6577	0.98696 $e^{(1)}$ 0.35351 $b^{(1)}$ 4.78850 $n^{(1)}$ 0.99670 ^{ine} 0.35380 ^{inb} 30.3831 $e^{(2)}$ 4.11548 $b^{(2)}$ 44.0155 $n^{(2)}$
15	16.1102	0.98696 $e^{(1)}$ 0.48828 $b^{(1)}$ 4.78555 $n^{(1)}$ 0.99970 ^{ine} 0.49098 ^{inb} 30.3822 $e^{(2)}$ 5.79800 $b^{(2)}$ 44.0403 $n^{(2)}$
16	17.2065	0.98696 $e^{(1)}$ 0.52160 $b^{(1)}$ 4.78465 $n^{(1)}$ 0.99970 ^{ine} 0.52339 ^{inb} 30.3829 $e^{(2)}$ 5.98166 $b^{(2)}$ 44.0465 $n^{(2)}$

The frequencies of intrinsic interfacial and edge vibrations of a composite cylindrical panel composed of two orthotropic thin elastic cylindrical panels with different elastic coefficients and having full contact along the generators are determined by the set of Eq. (7.23).

The frequencies of natural interface and edge vibrations for the composite rectangular plate with free edges are determined by the sets of Eqs. (7.38) and (7.54).

The existence of interfacial and boundary vibrations depends on the radius of the circle, the length and width of the components of the cylindrical panels and the elastic coefficients.

The obtained asymptotic formulas and numerical analysis show that for large θ_m or small curvature of the composite panel, all the characteristics of the intrinsic interface and edge vibrations of the cylindrical panel tend to the characteristics of the interface and edge vibrations of the composed rectangular plate, respectively.

The first frequencies of natural vibrations depend on the selected basis functions satisfying the same boundary conditions, and also, when $\varepsilon_m \to 0$ the vibration frequencies at the free generators become independent of the basis functions and the boundary conditions on the ends [29].

Numerical results show that the asymptotic formulas (7.63), (7.66) of the dispersion Eq. (7.23) provide an efficient approximation for finding the eigenfrequencies of problem (7.1)–(7.4).

Further possible generalizations may include accounting for the effects of prestress [31] and elastic foundations [32].

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