Chapter 9
Applications

Abstract As we know, one of the main goals of this book has been to find a parametrization of the unit sphere of spaces of polynomials endowed with different norms whose unit balls can be described in $\mathbb{R}^3$, but mainly we have tried to obtain the extreme polynomials of the unit balls. We have also studied some of the extreme polynomials in arbitrary dimensions and we have even described some of the extreme polynomials of arbitrary degree. The reason behind this is that a full description of the extreme polynomials of the unit ball has, as a matter of fact, can be applied to obtain many sharp polynomial inequalities (as we will see in this final chapter).

If the extreme polynomials of the unit ball are known, then we can simplify the problems that involve finding sharp inequalities between norms that depend on polynomials by using a simple consequence of the Krein-Milman Theorem.

Theorem 9.1 (Krein-Milman Theorem [41]) *Let X be a normed space. If C is a compact convex subset of X, then C coincides with the closed convex hull of its extreme points.*

Corollary 9.1 *If C is a convex body in a normed space X and $f: C \to \mathbb{R}$ is a convex function that attains its maximum, then there exists an extreme point $p \in C$ such that $f(p) = \max\{f(x): x \in C\}$.*

The main idea to apply Corollary 9.1 is the following: Let $\mathbf{B}$ be a convex body in a normed space of polynomials and f be a convex function defined on $\mathbf{B}$ which attains its maximum and takes real values, then f attains its maximum at an extreme point of $\mathbf{B}$ by Corollary 9.1. Furthermore, if we have a full description of the extreme points of $\mathbf{B}$, then we can find the maximum of f by evaluating f in the extreme points of $\mathbf{B}$ (this is the *Krein-Milman Approach*). This can be used in the case of norms of polynomials since it is known that the norm function is convex.

The rest of this chapter involves finding well known sharp inequalities for norms of polynomials that have appeared in this survey.

Let $(X, \|\cdot\|)$ be a normed space and consider the normed space $\mathcal{P}(^n X)$ (see the beginning of Sect. 5.5). Now, let us also consider the space of continuous symmetric

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n-linear forms of X denoted by $\mathcal{L}_s({}^nX)$ and endowed with the following norm:

$$\|L\| = \sup\{|L(x_1, \ldots, x_n)| \colon \|x_i\| \leq 1, \text{for every } i \in \{1, \ldots, n\}\},$$

for every $L \in \mathcal{L}_s({}^nX)$. By the beginning of Sect. 5.5, for every $P \in \mathcal{P}({}^nX)$, there exists a unique $L \in \mathcal{L}_s({}^nX)$ such that $P(x) = L(x, \ldots, x)$, for every $x \in X$, the polar of P.

9.1 Bernstein-Markov Type Inequalities

Bernstein type inequalities for polynomials are inequalities of the following form: if $P \in \mathcal{P}({}^nX)$, there exists a function $\Psi(\mathbf{x})$ defined over $\mathbf{C}$ such that

$$\|D^k P(\mathbf{x})\| \leq \Psi(\mathbf{x})\|P\|,$$

where $D^k P$ denotes the k-th derivative of P (the optimal function $\Psi(\mathbf{x})$ is known as *the Bernstein function*). On the other hand, Markov type inequalities are of the same fashion as Bernstein type inequalities but we are also taking the supremum of $\|D^k P(\mathbf{x})\|$ over all $\mathbf{x} \in \mathbf{C}$ (the optimal constant in Markov type inequalities is known as *the Markov constant*). The results of this section focus on finding the Bernstein function and the Markov constant that are known for the spaces that have been presented in this survey.

Theorem 9.2 (Araújo et al. [4]) *Take $\mathcal{P}_3(\mathbb{R})$ (see Sect. 2.1). The Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{\mathbb{R}}$$

is given by

$$\begin{cases} 3(1-4x^2) & \text{if } 0 \leq |x| \leq \frac{\sqrt{7}-2}{6}, \\ \frac{7\sqrt{7}+10}{9(|x|+1)} & \text{if } \frac{\sqrt{7}-2}{6} \leq |x| \leq \frac{2\sqrt{7}-1}{9}, \\ \frac{-16x^3}{(1-9x^2)(1-x^2)} & \text{if } \frac{2\sqrt{7}-1}{9} \leq |x| \leq \frac{1+2\sqrt{7}}{9}, \\ \frac{7\sqrt{7}-10}{9(1-|x|)} & \text{if } \frac{1+2\sqrt{7}}{9} \leq |x| \leq \frac{\sqrt{7}+2}{6}, \\ 3(4x^2-1) & \text{if } |x| \geq \frac{\sqrt{7}+2}{6}. \end{cases}$$

The Bernstein function for the inequality

$$|P''(x)| \leq \Psi(x)\|P\|_{\mathbb{R}}$$

is given by

$$\begin{cases} \frac{4}{1-9x^2} & \text{if } 0 \leq |x| \leq \frac{1}{9}, \\ \frac{32}{9(|x|-1)^2} & \text{if } \frac{1}{9} \leq |x| \leq \frac{1}{3}, \\ 24|x| & \text{if } |x| \geq \frac{1}{3}. \end{cases}$$

Theorem 9.3 (Muñoz et al. [47]) *Let* $\varphi\colon [-1,1] \to [0,+\infty)$ *be defined by* $\varphi(x) = \sqrt{1-x^2}$. *On the space* $\mathcal{P}_3^{\varphi}(\mathbb{R})$ *(see Sect. 2.1.1), the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{\mathbb{R}}$$

is given by

$$\begin{cases} 2|1-3x^2| & \text{if } |x| \in \left[0, \frac{\sqrt{4-\sqrt{7}}}{3}\right] \cup \left[\frac{\sqrt{4+\sqrt{7}}}{3}, 1\right], \\ \frac{4x^2}{\sqrt{-9x^4+10x^2-1}} & \text{if } |x| \in \left[\frac{\sqrt{4-\sqrt{7}}}{3}, \frac{\sqrt{4+\sqrt{7}}}{3}\right]. \end{cases}$$

Theorem 9.4 (Muñoz et al. [48]) *Let* $m, n \in \mathbb{N}$ *be odd and such that* $m > n$. *On the space* $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ *(see Sect. 3.1), the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} \frac{mn}{n+m\lambda_0} \cdot x^{n-1} \cdot |x^{m-n} + \lambda_0| & \text{if } |x| \in [0,1] \setminus I_{m,n}, \\ n\left(\frac{n}{m}\right)^{\frac{n}{m-n}} \cdot \frac{1}{|x|} & \text{if } |x| \in I_{m,n}, \end{cases}$$

where λ_0 *comes from Theorem 3.1 and*

$$I_{m,n} = \left[\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right].$$

The Markov constant is given by

$$\frac{mn(1+\lambda_0)}{n+m\lambda_0}$$

and equality is attained for the polynomials

$$P(x) = \pm\frac{1}{n+m\lambda_0}(nx^m + \lambda_0 m x^n).$$

In order to prove Theorem 9.4, we will prove first the following technical lemmas.

Lemma 9.1 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be odd and such that $m > n$ and let λ_0 be the number from Theorem 3.1. We have*

$$|\lambda_0|\frac{n}{m} < |\lambda_0|\frac{1-|\lambda_0|^{\frac{n}{m-n}}}{1-|\lambda_0|^{\frac{m}{m-n}}} < \frac{n}{m}.$$

Proof Recall from Lemma 3.1 that $|\lambda_0| < \frac{n}{m} < 1$ and consider the inequality

$$\frac{n}{m} < \frac{1-x^n}{1-x^m}. \tag{9.1}$$

We will show when (9.1) holds. If $0 < x < 1$, then inequality (9.1) is equivalent to $m - n > mx^n - nx^m$. Now, since the function $x \mapsto mx^n - nx^m$ is strictly increasing on $(0, 1)$, the curves $y = mx^n - nx^m$ and $y = m - n$ intersect in, at most, one point which is $x = 1$. Hence, it is easy to check that the inequality $m - n > mx^n - nx^m$ is satisfied on $(0, 1)$, which implies that $m - n > mx^n - nx^m$ holds when $x \in \left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$ and we have proven the first inequality of the lemma. The second inequality follows after doing some simple calculations and using the fact that λ_0 satisfies $n + m\lambda_0 = (m-n)|\lambda_0|^{\frac{m}{m-n}}$. □

Lemma 9.2 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be odd and such that $m > n$ and let λ_0 be the number from Theorem 3.1. If we define the functions*

$$f(x) = \frac{mn}{m-n}x^{n-1}|x^{m-m} - 1|,$$

$$g(x) = \frac{mn}{n+m\lambda_0}x^{n-1}|x^{m-m} + \lambda_0|,$$

then $g(x) \geq f(x)$ provided x satisfies

$$0 \leq |x| \leq \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \quad or \quad \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq 1.$$

Proof By symmetry, assume that $x > 0$. After some calculations, it is easy to check that the functions f and g intersect at the points $x_1 = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$ and $x_2 = \left(|\lambda_0|\frac{1-|\lambda_0|^{\frac{n}{m-n}}}{1-|\lambda_0|^{\frac{m}{m-n}}}\right)^{\frac{1}{m-n}}$. By Lemma 9.1, the points x_1 and x_2 are not in the intervals $\left(0, \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right)$ or $\left(\left(\frac{n}{m}\right)^{\frac{1}{m-n}}, 1\right)$. Hence, either $f \geq g$ or $f \leq g$ in each one of the previous intervals. Now, notice that $f(1) < g(1)$ and $f\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right) < g\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right)$. Indeed, the former is trivial and the latter is true because of the following reasoning. Notice that the inequality $f\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right) < g\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right)$ is equivalent to $\left|\frac{\lambda_0 n}{m} + 1\right| < \frac{1}{|\lambda_0|^{\frac{m}{m-n}}}\left|\frac{\lambda_0 n}{m} - \lambda_0\right|$. Moreover, it is also equivalent to $|\lambda_0|^{\frac{m}{m-n}} < |\lambda_0|$ which is satisfied since $-1 < -\frac{n}{m} < \lambda_0 < 0$ (see Lemma 3.1) and the proof is complete. □

Lemma 9.3 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be odd and such that $m > n$ and let λ_0 be the number from Theorem 3.1. If we define the functions*

$$f(x) = \frac{mn}{m-n}x^{n-1}|x^{m-m} - 1|,$$

$$g(x) = \frac{mn}{n+m\lambda_0}x^{n-1}|x^{m-m} + \lambda_0|,$$

$$h(x) = n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|},$$

then $h(x) \geq \max\{f(x), g(x)\}$ provided x satisfies

$$\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}.$$

Proof Assume that $\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$ holds, then it is enough to show that $h(x) \geq f(x)$ and $h(x) \geq g(x)$.

Firstly, notice that the function $x^n - x^m$ is strictly increasing on the interval $\left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$ since the derivative is positive. Hence, the maximum of $x \mapsto x^n - x^m$ on $\left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$ is attained at $x = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$ with value $\frac{(m-n)n^{\frac{n}{m-n}}}{m^{\frac{m}{m-n}}}$. Thus, $x^n - x^m \leq \frac{(m-n)n^{\frac{n}{m-n}}}{m^{\frac{m}{m-n}}}$ for $\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$, which implies after rearranging the inequality that $f(x) \leq h(x)$.

Secondly, notice that the inequality $\frac{mn}{n+m\lambda_0}x^{n-1}|x^{m-m} + \lambda_0| \leq n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|}$ is equivalent to $\frac{m}{n+m\lambda_0}|x^m + \lambda_0 x^n| \leq \left(\frac{n}{m}\right)^{\frac{n}{m-n}}$. Since the derivative of $x^m + \lambda_0 x^n$ is only 0 when $x = 0$ or $x = \pm\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}$, we have that $x^n + \lambda_0 x^n$ is monotone on the interval $\left[\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right]$. Hence, it is enough to evaluate $x^n + \lambda_0 x^n$ at the endpoints of the interval and after some simple evaluations notice that the proof is complete. □

Proof (of Theorem 9.4) Notice that the Bernstein function on the space $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ is given by

$$\mathcal{B}_{m,n,\infty}(x) = \sup\{|P'(x)|\colon\ P \text{ belongs to the unit sphere of } \mathcal{P}_{m,n,\infty}(\mathbb{R})\}.$$

However it is enough to find the above supremum over the set of extreme points of the unit ball by Corollary 9.1.

We know from Theorem 3.3 that the set of extreme points of $\mathsf{B}_{m,n,\infty}$ is

$$\left\{\pm\left(t, -\frac{m}{(m-n)^{\frac{m-n}{m}} n^{\frac{n}{m}}}\cdot t^{\frac{n}{m}}, 0\right) : \frac{n}{m-n} \leq t \leq \frac{n}{n+m\lambda_0}\right\} \bigcup \{\pm(0,0,1)\}.$$

Observe that the extreme polynomials $P(x) = \pm 1$ are irrelevant to find the Bernstein function. Hence we focus our attention on the extreme polynomials

$$P_t(x) = \pm\left(tx^m - \frac{m}{(m-n)^{\frac{m-n}{n}} n^{\frac{n}{m}}} t^{\frac{n}{m}} x^n\right),$$

where $t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]$. Thus,

$$\begin{aligned}
\mathcal{B}_{m,n,\infty}(x) &= \sup\left\{|P_t'(x)|\colon t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\} \\
&= \sup\left\{\left|mtx^{m-1} - \frac{mnt^{\frac{n}{m}}}{(m-n)^{\frac{m-n}{n}} n^{\frac{n}{m}}} x^{n-1}\right| : t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\} \\
&= \sup\left\{\left|mx^{n-1}\left[tx^{m-n} - \left(\frac{n}{m-n}\right)^{\frac{m-n}{m}} t^{\frac{n}{m}}\right]\right| : \right. \\
&\qquad \left. t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\}.
\end{aligned}$$

Let us define $R(t) = mx^{n-1}\left[tx^{m-n} - \left(\frac{n}{m-n}\right)^{\frac{m-n}{m}} t^{\frac{n}{m}}\right]$. Notice that the above supremum is attained at either $t = \frac{n}{m-n}$, or $t = \frac{n}{n+m\lambda_0}$, or at a critical point of $R(t)$ inside the open interval $\left(\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right)$. It is easy to show that there exists only one critical point of $R(t)$ which is $t_0 = \frac{n}{m-n}\left(\frac{n}{m}\right)^{\frac{m}{m-n}} \frac{1}{|x|^m}$ and

$$R(t_0) = n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|}.$$

Now, notice that the series of inequalities $\frac{n}{m-n} \leq t_0 \leq \frac{n}{n+m\lambda_0}$ is equivalent to

$$\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}.$$

Hence, after some easy calculations, we have

$$\mathcal{B}_{m,n,\infty}(x) = \sup\left\{|R(t)| : t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\}$$
$$= \begin{cases} \max\left\{\left|R\left(\frac{n}{m-n}\right)\right|, \left|R\left(\frac{n}{n+m\lambda_0}\right)\right|, n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|}\right\} \text{ if } \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}, \\ \max\left\{\left|R\left(\frac{n}{m-n}\right)\right|, \left|R\left(\frac{n}{n+m\lambda_0}\right)\right|\right\} \text{ if } |x| \leq \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \text{ or } \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq 1, \end{cases}$$

where, after evaluating the function R in the above points, we have

$$R\left(\frac{n}{m-n}\right) = \frac{mnx^{n-1}}{m-n}|x^{m-n} - 1|$$

and

$$R\left(\frac{n}{n+m\lambda_0}\right) = \frac{mnx^{n-1}}{n+m\lambda_0}|x^{m-n} + \lambda_0|.$$

By applying Lemmas 9.2 and 9.3 the result follows. □

Theorem 9.5 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be such that $m > n$, m is odd and n is even. On the space $\mathcal{P}_{m,n,\infty}(\mathbb{R})$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} 2n|x|^{n-1} & \text{if } |x| \in \left[0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right], \\ mx^{m-1} + n|x|^{n-1} & \text{if } |x| \in \left[\left(\frac{n}{m}\right)^{\frac{1}{m-n}}, 1\right]. \end{cases}$$

The Markov constant is given by

$$m + n$$

and equality is attained for the polynomials

$$P(x) = \pm(x^m \pm x^n - 1).$$

Theorem 9.6 (Muñoz et al. [48]) *Let $n \in \mathbb{N}$ be odd. On the space $\mathcal{P}_{2n,n,\infty}(\mathbb{R})$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{2n,n,\infty}$$

is given by

$$\begin{cases} \frac{n|x|^{n-1}}{1-x^n} & \text{if } |x| \in \left[0, \frac{1}{\sqrt[n]{2}}\right], \\ 4n|x|^{2n-1} & \text{if } |x| \in \left[\frac{1}{\sqrt[n]{2}}, 1\right]. \end{cases}$$

The Markov constant is given by $4n$ and equality is attained for the polynomials

$$P(x) = \pm(2x^{2n} - 1).$$

Theorem 9.7 (Muñoz et al. [48]) *Let $n \in \mathbb{N}$ be even. On the space $\mathcal{P}_{2n,n,\infty}(\mathbb{R})$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{2n,n,\infty}$$

is given by

$$\begin{cases} 8n(-2|x|^{2n-1} + |x|^{n-1}) & \text{if } |x| \in \left[0, \left(\frac{1}{4}\right)^{\frac{1}{n}}\right], \\ \frac{n}{|x|} & \text{if } |x| \in \left[\left(\frac{1}{4}\right)^{\frac{1}{n}}, \left(\frac{1}{2}\right)^{\frac{1}{n}}\right], \\ \frac{n|x|^{n-1}}{1-x^n} & \text{if } |x| \in \left[\left(\frac{1}{2}\right)^{\frac{1}{n}}, \left(\frac{3}{4}\right)^{\frac{1}{n}}\right], \\ 8n(2|x|^{2n-1} - |x|^{n-1}) & \text{if } |x| \in \left[\left(\frac{3}{4}\right)^{\frac{1}{n}}, 1\right]. \end{cases}$$

The Markov constant is given by $8n$ and equality is attained for the polynomials

$$P(x) = \pm(8x^{2n} - 8x^n + 1).$$

Theorem 9.8 (Muñoz et al. [47]) *Let $m, n \in \mathbb{N}$ be such that m is odd, n is even and $m > n$. On the normed subspace of $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ given by trinomials that are bounded by the linear mapping $\varphi(x) = |x|$ over the interval $[-1, 1]$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} (m+1)|x|^m - (n+1)x^n + 1 & \text{if } |x| \leq t_1, \\ 2(n+1)x^n - 1 & \text{if } t_1 \leq |x| \leq \sqrt[m-n]{\frac{n+1}{m+1}}, \\ (m+1)|x|^m + (n+1)x^n - 1 & \text{if } \sqrt[m-n]{\frac{n+1}{m+1}} \leq |x| \leq 1, \end{cases}$$

where $t_1 \in \mathbb{R}$ is the unique solution of

$$(m+1)x^m - 3(n+1)x^n + 2 = 0$$

on the interval $\left(\frac{1}{\sqrt[n]{2(n+1)}}, \frac{1}{\sqrt[n]{n+1}}\right)$. The Markov constant is given by $m+n+1$ and equality is attained for the polynomials

$$P(x) = \pm[x^m \pm (x^n - 1)].$$

Theorem 9.9 (Muñoz et al. [47]) *On the normed subspace of $\mathcal{P}_{2,1,\infty}(\mathbb{R})$ given by trinomials that are bounded by the linear mapping $\varphi(x) = |x|$ over the interval $[-1, 1]$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} \left|\frac{3x^2-1}{2}\right| + 2|x| & \text{if } |x| \in \left[\frac{\sqrt{13}-2}{9}, \frac{\sqrt{13}+2}{9}\right], \\ |6x^2 - 1| & \text{if } |x| \in \left[0, \frac{\sqrt{13}-2}{9}\right] \cup \left[\frac{\sqrt{13}+2}{9}, 1\right]. \end{cases}$$

Theorem 9.10 (Muñoz et al. [49]) *Let $m, n \in \mathbb{N}$ be with different parity and such that $m > n$. On the space $\mathcal{P}_{m,n,2}(\mathbb{R})$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,2}$$

is given by

$$\begin{cases} \sqrt{\frac{n^2(2n+1)x^{2(n-1)}+(m+1)^2(2m+1)x^{2(m-1)}}{2}} & \text{if } m \text{ is even and } n \text{ is odd}, \\ \sqrt{\frac{m^2(2m+1)x^{2(m-1)}+(n+1)^2(2n+1)x^{2(n-1)}}{2}} & \text{if } m \text{ is odd and } n \text{ is even}. \end{cases}$$

The Markov constant is given by

$$\begin{cases} (m+1)\sqrt{\frac{2m+1}{2m-1}} & \textit{if } m \textit{ is even and } n \textit{ is odd,} \\ m\sqrt{\frac{2m+1}{2m-1}} & \textit{if } m \textit{ is odd, } n \textit{ is even and } m > n+1, \\ m\sqrt{\frac{2m-1}{2m-3}} & \textit{if } m \textit{ is odd and } n = m-1. \end{cases}$$

Remark 9.1 On Theorem 9.10, notice that if we consider $n = 1$, then we have Bernstein's function and Markov's constant for the space $\mathcal{P}_2(\mathbb{R})$ (see Sect. 2.1) which are given, respectively, by

$$\begin{cases} \frac{1}{1-|x|} & \text{if } 0 \le |x| \le \frac{1}{2}, \\ 4|x| & \text{if } |x| \ge \frac{1}{2}, \end{cases}$$

and

$$4,$$

with equality attained for the polynomials

$$P(x) = \pm(1 - 2x^2).$$

Theorem 9.11 (Muñoz et al. [46]) *Take $\mathcal{P}(^2\Delta)$ (see Sect. 4.1). The Markov constant for the inequality*

$$\|DP(x, y)\|_{\ell_2} \le \Psi(x, y)\|P\|_\Delta$$

is given by

$$2\sqrt{10}$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 - 6xy + y^2).$$

The Bernstein function for the inequality

$$\|DP(x, y)\|_\Delta \le \Psi(x, y)\|P\|_\Delta$$

is given by

$$\begin{cases} |2x-6y| & \textit{if } x=0 \textit{ or } x\neq 0 \textit{ and } \left(\frac{y}{x}\leq -1 \textit{ or } \frac{y}{x}\geq 2\right), \\ \left|2x+2y+\frac{y^2}{x}\right| & \textit{if } x\neq 0 \textit{ and } \frac{y}{x}\in[1,2], \\ \left|2x+2y+\frac{x^2}{y}\right| & \textit{if } y\neq 0 \textit{ and } \frac{x}{y}\in[1,2], \\ |6x-2y| & \textit{if } y=0 \textit{ or } y\neq 0 \textit{ and } \left(\frac{x}{y}\leq -1 \textit{ or } \frac{x}{y}\geq 2\right). \end{cases}$$

The Markov constant is given by 6 *and equality is attained for the polynomials*

$$P(x,y)=\pm(x^2-6xy+y^2).$$

Theorem 9.12 (Gámez et al. [23]) *Take $\mathcal{P}(^2\square)$ (see Sect. 4.2). The Bernstein function for the inequality*

$$\|DP(x,y)\|_{\ell_2}\leq \mathcal{M}(x,y)\|P\|_\square$$

is given by

$$\begin{cases} \sqrt{\frac{24y^4+12x^2y^2+x^4+x(8y^2+x^2)^{\frac{3}{2}}}{8y^2}} & \textit{if } 0<\alpha_0 x\leq y\leq x, \\ \sqrt{\frac{24x^4+12x^2y^2+y^4+y(8x^2+y^2)^{\frac{3}{2}}}{8x^2}} & \textit{if } 0<x\leq y\leq \frac{x}{\alpha_0}, \\ \sqrt{13x^2-24xy+13y^2} & \textit{otherwise}, \end{cases}$$

where α_0 is the unique root of the equation

$$80\alpha^4-192\alpha^3+92\alpha^2-1=(8\alpha^2+1)^{\frac{3}{2}}$$

in the interval $\left[\frac{3-\sqrt{5}}{2},\frac{12-3\sqrt{3}}{13}\right]$. The Markov constant is given by

$$\sqrt{13}$$

and equality is attained for the polynomials

$$P(x,y)=\pm(x^2-3xy+y^2).$$

The Bernstein function for the inequality

$$\|DP(x,y)\|_\square\leq \Psi(x,y)\|P\|_\square$$

is given by

$$\begin{cases} 3x-2y & \text{if } 0 \le y \le (\sqrt{2}-1)x, \\ \frac{5}{2}x - y + \frac{y^2}{2x} & \text{if } x \ne 0 \text{ and } (\sqrt{2}-1)x \le y \le \frac{1}{2}x, \\ 2x + \frac{y^2}{2x} & \text{if } x \ne 0 \text{ and } \frac{1}{2}x \le y \le x, \\ 2y + \frac{x^2}{2y} & \text{if } y \ne 0 \text{ and } x \le y \le 2x, \\ \frac{5}{2}y - x + \frac{x^2}{2y} & \text{if } y \ne 0 \text{ and } 2x \le y \le (\sqrt{2}+1)x, \\ 3y-2x & \text{if } (\sqrt{2}+1)x \le y \le 1. \end{cases}$$

The Markov constant is given by 3 *and equality is attained for the polynomials*

$$P(x, y) = \pm(x^2 - 3xy + y^2).$$

Theorem 9.13 (Araújo et al. [2]) *Take* $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ *(see Sect. 4.3). The Bernstein function for the inequality*

$$\|DP(x, y)\|_{\ell_2} \le \Psi(x, y)\|P\|_{D(\frac{\pi}{4})}$$

is given by

$$\begin{cases} 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right] & \text{if } (a), \\ \frac{x^2}{y^2} + 4\left(x^2+y^2\right) & \text{if } (b), \\ \frac{\left(3x^2-2xy+3y^2\right)^2}{2(x-y)^2} & \text{if } (c), \end{cases}$$

where

(a) $0 \le y \le \frac{\sqrt{2}-1}{2}x$ *or* $\left(4\sqrt{2}-5\right)x \le y \le x$,
(b) $\frac{\sqrt{2}-1}{2}x \le y \le \left(\sqrt{2}-1\right)x$,
(c) $\left(\sqrt{2}-1\right)x \le y \le \left(4\sqrt{2}-5\right)x$.

The Markov constant is

$$4\left(13+8\sqrt{2}\right)$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + (5+4\sqrt{2})y^2 - 4(1+\sqrt{2})xy).$$

The Bernstein function for the inequality

$$\|DP(x, y)\|_{D(\frac{\pi}{4})} \le \Psi(x, y)\|P\|_{D(\frac{\pi}{4})}$$

is given by

$$\begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \textit{if } 0 \le y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \textit{if } \frac{2\sqrt{2}-1}{7}x \le y < \left(\sqrt{2}-1\right)x, \\ 2\left(x+\frac{y^2}{x-y}\right) & \textit{if } \left(\sqrt{2}-1\right)x \le y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y-2x & \textit{if } \left(2-\sqrt{2}\right)x \le y \le x. \end{cases}$$

The Markov constant is given by

$$4+\sqrt{2}$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + (5+4\sqrt{2})y^2 - 4(1+\sqrt{2})xy).$$

Theorem 9.14 (Jiménez et al. [34]) *Take $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$. The Bernstein function for the inequality*

$$\|DP(x, y)\|_{\ell_2} \le \Phi(x, y)\|P\|_{D(\frac{\pi}{2})}$$

is given by

$$\begin{cases} \sqrt{16\,(x-y)^2+4\left(x^2+y^2\right)} & \textit{if } 0 \le y \le \frac{x}{2}, \\ \sqrt{\frac{x^4}{y^2}+4\left(x^2+y^2\right)} & \textit{if } 0 < \frac{x}{2} < y \le x, \\ \sqrt{\frac{y^4}{x^2}+4\left(x^2+y^2\right)} & \textit{if } 0 < x < y \le 2x, \\ \sqrt{16\,(y-x)^2+4\left(x^2+y^2\right)} & \textit{if } 2x < y \le 1. \end{cases}$$

The Markov constant is given by $2\sqrt{5}$ and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + y^2 - 4xy).$$

The Bernstein function for the inequality

$$\|DP(x, y)\|_{D(\frac{\pi}{2})} \le \Psi(x, y)\|P\|_{D(\frac{\pi}{2})}$$

is given by

$$\begin{cases} 2(2x-y) & \textit{if } 0 \le y < \frac{x}{2}, \\ 2\left(y+\frac{x^2}{2y}\right) & \textit{if } \frac{x}{2} \le y < x, \\ 2\left(x+\frac{y^2}{2x}\right) & \textit{if } x \le y < 2x, \\ 2(2y-x) & \textit{if } y \ge 2x. \end{cases}$$

The Markov constant is given by 4 *and equality is attained for the polynomials*

$$P(x, y) = \pm(x^2 + y^2 - 4xy).$$

Theorem 9.15 (Jiménez et al. [34]) *On* $\mathcal{P}(^2\ell_p^2)$ *for* $p \in \{1, 2, \infty\}$ *(see Sects. 4.3, 5.1, and 5.2), the Markov constant in the inequality*

$$\|DP(x, y)\|_{\ell_p^2} \leq \Psi(x, y)\|P\|_{\ell_p^2}$$

is

(i) 4 *if* $p = 1$,
(ii) 2 *if* $p = 2$,
(iii) $2\sqrt{2}$ *if* $p = \infty$.

9.2 Polarization Constants

It is easy to see just by the definition of the norms defined on $\mathcal{P}(^nX)$ and $\mathcal{L}_s(^nX)$ that: for every $P \in \mathcal{P}(^nX)$,

$$\|P\| \leq \|L\|,$$

where L is the polar of P. But furthermore, the converse is also true, i.e., there exists $C \geq 1$ such that $\|L\| \leq C\|P\|$. In particular, we have the following result that can be applied for any normed space X.

Theorem 9.16 (Martin [42]) *Let* X *be a normed space. If* $P \in \mathcal{P}(^nX)$, *then*

$$\|P\| \leq \|L\| \leq \frac{n^n}{n!}\|P\|,$$

where L *is the polar of* P.

Notice that throughout this survey we have considered the norm over the space of n-homogeneous polynomials to be, not only defined over the unit ball of a certain normed space, but also over a convex body of a normed space. To be more precise, let X be a normed space and take $\mathbf{C}$ a convex body in X. We define the following norm over the space of continuous n-homogeneous polynomials of X: for every continuous n-homogeneous polynomial P,

$$\|P\|_{\mathbf{C}} = \sup\{|P(x)| : x \in \mathbf{C}\};$$

and we also define the following norm over the space of symmetric n-linear forms of X: for every symmetric n-linear form L,

$$\|L\|_{\mathbf{C}} = \sup\{|L(x_1, \ldots, x_n)|\colon x_i \in \mathbf{C},\ \text{for every } i \in \{1, \ldots, n\}\}.$$

Notice that the condition "every continuous n-homogeneous polynomial P has a unique continuous symmetric n-linear form L (the polar of P) such that $P(x) = L(x, \ldots, x)$" is purely algebraic. Therefore, it does not depend on the topology that we consider over the space of n-homogenous polynomials or over the space of symmetric n-linear forms.

It is easy to see by the definition of the above norms that $\|P\|_{\mathbf{C}} \leq \|L\|_{\mathbf{C}}$. However, the reverse inequality as in Martin's Theorem is not true as it can be seen later on. Furthermore, there is not yet an analogous version of Martin's Theorem when the norm is defined over an arbitrary convex body. Thus it is still an open problem to find a result similar to the one of Martin's Theorem when we consider the norm defined over other convex bodies apart from the unit ball of X.

We are able to define now what is known as the n-polarization constant of a space of continuous n-homogeneous polynomials on a convex body. Let X be a normed space and $\mathbf{C} \subset X$ a convex body. Let $\mathcal{P}(^n\mathbf{C})$ be the space of n-homogeneous polynomials on X bounded on $\mathbf{C}$ endowed with the norm defined by

$$\|P\|_{\mathbf{C}} = \sup\{|P(x)| : x \in \mathbf{C}\}.$$

Similarly, if L is the polar of $P \in \mathcal{P}(^n\mathbf{C})$ we define

$$\|L\|_{\mathbf{C}} = \sup\{|L(x_1, \ldots, x_n)| : x_1, \ldots, x_n \in \mathbf{C}\}.$$

We define the n-polarization constant $\mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\mathbf{C}))$ of $\mathcal{P}(^n\mathbf{C})$ as the following value:

$$\inf\left\{K\colon \|L\|_{\mathbf{C}} \leq K\|P\|_{\mathbf{C}},\ \text{where } P \in \mathcal{P}(^n\mathbf{C}) \text{ and } L \text{ is the polar of } P\right\}.$$

Furthermore, assume that there exists $P \in \mathcal{P}(^n\mathbf{C})$ such that

$$\|L\|_{\mathbf{C}} = \mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\mathbf{C}))\|P\|_{\mathbf{C}},$$

where L is the polar of P, then we say that P is an extremal polynomial for $\mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\mathbf{C}))$.

The following results show the known exact values of the polarization constants of the spaces of homogeneous polynomials that have been dealt with in this survey (most of them use the Krein-Milman approach, specially those whose norm involve convex bodies different from the unit ball).

Theorem 9.17 (Muñoz et al. [46]) *If Δ is the simplex defined in Sect. 4.1, then $c_{pol}(\mathcal{P}(^2\Delta)) = 3$. Furthermore, $P(x, y) = \pm(x^2 + y^2 - 6xy)$ are extremal polynomials for $c_{pol}(\mathcal{P}(^2\Delta))$.*

Proof The result follows from the Markov constant in Theorem 9.11 for the inequality $\|DP(x, y)\|_{\Delta} \leq \Psi(x, y)\|P\|_{\Delta}$ since

$$DP(x, y)(u, v) = 2L((x, y), (u, v))$$

for all $(x, y), (u, v) \in \mathbb{R}^2$ and where L is the polar of P. $\square$

Theorem 9.18 (Gámez et al. [23]) *If $\square$ is the unit square defined in Sect. 4.2, then $c_{pol}(\mathcal{P}(^2\square)) = \frac{3}{2}$. Furthermore, $P(x, y) = \pm(x^2 + y^2 - 3xy)$ are extremal polynomials for $c_{pol}(\mathcal{P}(^2\square))$.*

Theorem 9.19 (Araújo et al. [2]) *If $D\left(\frac{\pi}{4}\right)$ is the circular sector defined in Sect. 4.3, then $c_{pol}\left(\mathcal{P}\left(^2 D\left(\frac{\pi}{4}\right)\right)\right) = 2 + \frac{\sqrt{2}}{2}$. Furthermore, $P(x, y) = \pm(x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy)$ are extremal polynomials for $c_{pol}\left(\mathcal{P}\left(^2 D\left(\frac{\pi}{4}\right)\right)\right)$.*

Theorem 9.20 (Jiménez et al. [34]) *If $D\left(\frac{\pi}{2}\right)$ is the circular sector defined in Sect. 4.3, then $c_{pol}\left(\mathcal{P}\left(^2 D\left(\frac{\pi}{2}\right)\right)\right) = 2$. Furthermore, $P(x, y) = \pm(x^2 + y^2 - 4xy)$ are extremal polynomials for $c_{pol}\left(\mathcal{P}\left(^2 D\left(\frac{\pi}{2}\right)\right)\right)$.*

Theorem 9.21 (Sarantopoulos [53]) *Let $1 \leq p \leq \infty$. We have $c_{pol}\left(\mathcal{P}\left(^2\ell_p^2\right)\right) = 2^{\frac{|p-2|}{2}}$ (see Sect. 5). Furthermore, $P(x, y) = \pm(x^2 - y^2)$ are extremal polynomials for $c_{pol}\left(\mathcal{P}\left(^2\ell_p^2\right)\right)$.*

Remark 9.2 It is important to mention that, although we know the extreme polynomials on the spaces ℓ_p^2, the proof of Theorem 9.21 in [53] does not use the Krein-Milman approach but a direct approach. It involves obtaining a sharper bound C than that of Martin's bound for every polynomial and then finding a polynomial P such that $\|L\|_{\mathbf{C}} = C\|P\|_{\mathbf{C}}$, where L is the polar of P.

An interesting question started by Harris in 1975 related to polarization constants for polynomials on ℓ_p spaces states that, in a complex setting we have

$$\mathrm{c_{pol}}(\mathcal{P}(^n\ell_\infty^n(\mathbb{C}))) \leq \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{2^n n!}.$$

For the previous estimate consult [32] or [20] for a more modern and accessible exposition. The question as to whether $\mathrm{c_{pol}}(\mathcal{P}(^n\ell_\infty^n(\mathbb{C}))) = \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{2^n n!}$ remains unsolved nowadays.

Theorem 9.22 (Kim [37]) *Let $w \in (0, 1)$.*

(a) If $w \leq \sqrt{2} - 1$, then $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right) = \frac{2(1+w^2)}{(1+w)^2}$ (see Sect. 6.1). Furthermore, $P(x, y) = \pm\left(\frac{4}{(1+w)^2}xy\right)$ are extremal polynomials for $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right)$.

(b) If $\sqrt{2} - 1 < w$, then $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right) = 1 + w^2$. Furthermore, $P(x, y) = \pm\left(x^2 - y^2\right)$ are extremal polynomials for $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right)$.

Theorem 9.23 (Kim [39]) *Let* $w = \frac{1}{2}$. *We have* $c_{pol}\left(\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)\right) = \frac{5}{4}$ *(see Sect. 6.2). Furthermore,*

$$P(x, y) = \pm\left(x^2 - y^2\right)$$

and

$$Q(x, y) = \pm\left(\frac{3}{4}x^2 - \frac{5}{16}y^2 \pm \frac{7}{4}\right)$$

are extremal polynomials for $c_{pol}\left(\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)\right)$.

9.3 Unconditional Constants

Let us denote by $\mathbf{x}^\alpha$ the monomial

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m},$$

where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{K}^m$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_k \in \mathbb{N} \cup \{0\}$ for every $k \in \{1, \ldots, m\}$. For $P(\mathbf{x}) = \sum_{|\alpha| \leq n} a_\alpha \mathbf{x}^\alpha$ (where $|\alpha| = \alpha_1 + \cdots + \alpha_m$) a polynomial of degree n on $\mathbb{K}^m$, we define the modulus $|\cdot|$ of P by $|P|(\mathbf{x}) = \sum_{|\alpha| \leq n} |a_\alpha| \mathbf{x}^\alpha$. If $\mathbf{C}$ is a convex body in $\mathbb{R}^m$, we denote by $\mathcal{P}({}^n\mathbf{C})$ the space of n-homogeneous polynomials on $\mathbb{R}^m$ endowed with the norm $\|P\|_{\mathbf{C}}$ (see Sect. 9.2). Let $\mathcal{B}_n = \{\mathbf{x}^\alpha : |\alpha| \leq n\}$ be the canonical basis of $\mathcal{P}({}^n\mathbf{C})$. The unconditional constant of $\mathcal{B}_n$ is equal to the best possible constant C (denoted by $C_{\mathrm{unc}}(\mathcal{P}({}^n\mathbf{C}))$) in the inequality

$$\||P|\|_{\mathbf{C}} \leq C\|P\|_{\mathbf{C}}.$$

The following results show all the exact values of the unconditional constants that are known of the spaces that have been presented on this survey.

Theorem 9.24 (Grecu et al. [30]) *If* $m, n \in \mathbb{N}$ *with* $m > n$, *then*

$$C_{unc}(\mathcal{P}_{m,n,\infty}(\mathbb{R})) = \begin{cases} 3 & \text{if } m \text{ and } n \text{ have different parity,} \\ 1 + \frac{4}{m-n}\left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}} & \text{if } m \text{ and } n \text{ are even,} \\ \frac{n-\lambda_0 m}{n+\lambda_0 m} & \text{if } m \text{ and } n \text{ are odd,} \end{cases}$$

(see Sect. 3.1) where λ_0 *comes from Theorem 3.1, and equality is attained for the polynomials*

$$P(x) = \begin{cases} \pm(2x^m - 1), \\ \pm(-\gamma_0 x^m + \gamma_0 x^n + 1) \text{ where } \gamma_0 = -\frac{2}{m-n} \cdot \left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}}, \\ \pm\left(\frac{nx^m}{n+m\lambda_0} - \frac{m|\lambda_0|x^n}{n+m\lambda_0}\right), \end{cases}$$

respectively.

Remark 9.3 (Grecu et al. [30]) In Theorem 9.24 it can be seen that for every $k \in \mathbb{N}$ with $k > 1$ and every $n \in \mathbb{N}$ even we have

$$C_{\text{unc}}(\mathcal{P}_{kn,n,\infty}(\mathbb{R})) = 1 + \frac{4}{k-1} \cdot k^{\frac{k}{k-1}},$$

which is independent of n.

Theorem 9.25 (Grecu et al. [30]) *On the space $\mathcal{P}(^2\Delta)$ (see Sect. 4.1) we have*

$$C_{unc}(\mathcal{P}(^2\Delta)) = 2$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 - 6xy + y^2).$$

Theorem 9.26 (Gámez et al. [23]) *On the space $\mathcal{P}(^2\square)$ (see Sect. 4.2) we have*

$$C_{unc}(\mathcal{P}(^2\square)) = 5$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 - 3xy + y^2).$$

Theorem 9.27 (Gámez et al. [23]) *On the space $\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$ (see Sect. 4.3) we have*

$$C_{unc}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)\right) = 5 + 4\sqrt{2}$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy)).$$

Theorem 9.28 (Jiménez et al. [34]) *On the space $\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$ we have*

$$C_{unc}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)\right) = 3$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + y^2 - 4xy).$$

Theorem 9.29 (Grecu et al. [30]) *On the spaces $\mathcal{P}(^2\ell_1^2)$, $\mathcal{P}(^2\ell_2^2)$ and $\mathcal{P}(^2\ell_\infty^2)$ (see Sects. 4.3, 5.1, and 5.2) we have, respectively, the unconditional constants given by*

$$\begin{cases} \frac{1+\sqrt{2}}{2}, \\ \sqrt{2}, \\ 1+\sqrt{2}, \end{cases}$$

with equality attained for the polynomials

$$\begin{cases} \pm\frac{\sqrt{2}}{2}(x^2 - y^2) \pm (2+\sqrt{2})xy, \\ \pm(x^2 + y^2 + 2xy), \\ \frac{2+\sqrt{2}}{4}(x^2 - y^2) \pm \frac{\sqrt{2}}{2}xy, \end{cases}$$

respectively.

Proof We will prove the result for the space $\mathcal{P}(^2\ell_1^2)$ since the other cases can be done analogously. By Theorem 5.2, we know that the extreme polynomials of the unit ball of $\mathcal{P}(^2\ell_1^2)$ are

(a) $P(x, y) = \pm x^2 \pm y^2 \pm 2xy$,
(b) $P(x, y) = \pm\frac{\sqrt{4|t|-t^2}}{2}(x^2 - y^2) + txy$, where $|t| \in (2, 4]$.

Notice that if P is as in (a), then $|||P|||_{\ell_1^2} = \|P\|_{\ell_1^2} = 1$. Hence, it is enough to consider polynomials of type (b). If P is as in (b), then P attains its norm in ℓ_1^2 at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus,

$$\begin{aligned} C_{\mathrm{unc}}(\mathcal{P}\left(^2\ell_1^2\right)) &= \sup\left\{\left\|\frac{\sqrt{4|t|-t^2}}{2}(x^2 + y^2) + |t|xy\right\|_{\ell_1^2} : |t| \in (2, 4]\right\} \\ &= \sup\left\{\left\|\frac{\sqrt{4s-s^2}}{2}(x^2 + y^2) + sxy\right\|_{\ell_1^2} : s \in (2, 4]\right\} \\ &= \sup\left\{\frac{\sqrt{4s-s^2}+s}{4} : s \in (2, 4]\right\} \\ &= 2+\sqrt{2}. \end{aligned}$$

$\square$

Theorem 9.30 (Araújo et al. [2]) *Let $1 < p < \infty$ with $p \neq 2$ and take $\mathcal{P}(^2\ell_p^2)$ (see Sects. 5.3 and 5.4). Let us define the function*

$$f(\alpha) = \frac{2^{\frac{p-2}{p}}\left[\alpha(1-\alpha^p)\left(\alpha - (1-\alpha^p)^{\frac{1}{p}}\right) + \alpha^p\,(1-\alpha^p)^{\frac{1}{p}}\left(\alpha + (1-\alpha^p)^{\frac{1}{p}}\right)\right]}{\alpha\,(1-\alpha^p)^{\frac{1}{p}}\left(\alpha^2 + (1-\alpha^p)^{\frac{2}{p}}\right)}$$

and set $M_f = \sup\left\{f(\alpha)\colon \alpha \in \left[2^{-\frac{1}{p}}, 1\right]\right\}$, we have that $C_{unc}(\mathcal{P}(^2\ell_p^2)) = M_f$.

Theorem 9.31 (Kim [37]) *Let $0 < w < 1$.*

(a) If $w \leq \sqrt{2} - 1$, then $c_{unc}\left(\mathcal{P}\left(^2O_w^2\right)\right) = \frac{1+w^2+\sqrt{2(1+w^4)}}{(1+w)^2}$ (see Sect. 6.1) and equality is attained for the polynomials $P(x, y) = \pm\left(\frac{4}{(1+w)^2}xy\right)$.

(b) If $\sqrt{2} - 1 < w$, then $c_{unc}\left(\mathcal{P}\left(^2O_w^2\right)\right) = \frac{1+w^2+\sqrt{(1+w^2)^2+4w^2}}{2}$ and equality is attained for the polynomials

$$P(x, y) = \pm(\alpha x^2 - \alpha y^2 \pm \sqrt{\alpha(1-\alpha)}xy),$$

where $\alpha = \frac{1}{2} + \frac{1+w^2}{2\sqrt{(1+w^2)^2+4w^2}}$.

Theorem 9.32 (Kim [39]) *Let $w = \frac{1}{2}$. Then, $c_{unc}\left(\mathcal{P}\left(^2\mathcal{H}_{1/2}^2\right)\right) = \frac{3}{2}$ (see Sect. 6.2) and equality is attained for the polynomials $P(x, y) = \pm\left(x^2 + \frac{1}{4}y^2 + xy\right)$ and $Q(x, y) = \pm\left(x^2 + \frac{3}{4}y^2 + xy\right)$.*

9.4 Bohnenblust–Hille and Hardy–Littlewood Constants

We begin by considering the following constants which are closely related to the Bohnenblust–Hille and Hardy–Littlewood constants as we will see. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $n \in \mathbb{N}$ and let us consider the standard notation $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$. Let $\mathcal{P}(^m\mathbb{K}^n)$ denote the vector space of m-homogeneous polynomials on $\mathbb{K}^n$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Notice that if $P \in \mathcal{P}(^m\mathbb{K}^n)$, then P can be written as

$$P(\mathbf{x}) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

where $a_\alpha \in \mathbb{K}$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$. If $|\cdot|$ is a norm on $\mathbb{K}^n$, then $|\cdot|$ induces a norm on $\mathcal{P}(^m\mathbb{K}^n)$ called the polynomial norm and it is given by

$$\|P\| = \sup\{|P(\mathbf{x})| : \mathbf{x} \in \mathsf{B}_X\},$$

where B_X is the unit ball of the normed space $X = (\mathbb{K}^n, |\cdot|)$. The space $\mathcal{P}(^m\mathbb{K}^n)$ endowed with the polynomial norm is denoted by $\mathcal{P}(^m X)$. Besides the polynomial norm, there are other interesting norms on $\mathcal{P}(^m\mathbb{K}^n)$ such as the ℓ_q-norms on the coefficients, i.e., if $P \in \mathcal{P}(^m\mathbb{K}^n)$ and $1 \leq q \leq \infty$, then

$$|P|_q = \begin{cases} \left(\sum_{|\alpha|=m} |a_\alpha|^q\right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty, \\ \max\{|a_\alpha| : |\alpha| = m\} & \text{if } q = \infty. \end{cases}$$

Let us represent by $\|\cdot\|_p$ the polynomial norm of the space $\mathcal{P}(^m\ell_p^n(\mathbb{K}))$, where $1 \leq p \leq \infty$. Since the space $\mathcal{P}(^m\mathbb{K}^n)$ is finite dimensional, we have that the norms $|\cdot|_q$ and $\|\cdot\|_p$ $(1 \leq q, p \leq \infty)$ are equivalent, i.e., there exist $k, K > 0$ such that

$$k\|P\|_p \leq |P|_q \leq K\|P\|_p,$$

for any $P \in \mathcal{P}(^m\mathbb{K}^n)$. Notice that the unit balls of the spaces $(\mathcal{P}(^m\mathbb{K}^n), |\cdot|_q)$ and $\mathcal{P}(^m\ell_p^n(\mathbb{K}))$, denoted by $\mathsf{B}_{|\cdot|_q}$ and $\mathsf{B}_{\|\cdot\|_p}$, respectively, satisfy that the mapping $\mathsf{B}_{|\cdot|_q} \ni P \to \|P\|_p$ is bounded by $\frac{1}{k}$ and the mapping $\mathsf{B}_{\|\cdot\|_p} \ni P \to |P|_q$ is bounded by K. Moreover, the continuity of such mappings and the compactness of $\mathsf{B}_{|\cdot|_q}$ and $\mathsf{B}_{\|\cdot\|_p}$ satisfy the following maxima.

Definition 9.1 Let $1 \leq q, p \leq \infty$. We define the following constants

$$k_{m,n,q,p} = \frac{1}{\max\left\{\|P\|_p : P \in \mathsf{B}_{|\cdot|_q}\right\}},$$
$$K_{m,n,q,p} = \max\left\{|P|_q : P \in \mathsf{B}_{\|\cdot\|_p}\right\}.$$

From now on, we are interested in calculating the exact values of $k_{m,n,q,p}$ and $K_{m,n,q,p}$ when we are considering polynomials whose coefficients are real numbers (we will consider real polynomials and complex polynomials with real coefficients separately). To do so, we will be applying the Krein-Milman approach to the mappings $\mathsf{B}_{|\cdot|_q} \ni P \to \|P\|_p$ and $\mathsf{B}_{\|\cdot\|_p} \ni P \to |P|_q$. Hence, we will need, for instance, the extreme points of the unit ball $\mathsf{B}_{|\cdot|_q}$. It is well known that the extreme points of $\mathsf{B}_{|\cdot|_q}$ are

$$\begin{cases} \{\pm e_k : 1 \leq k \leq m+1\} & \text{if } q = 1, \\ \left\{\sum_{k=1}^{m+1} \varepsilon_k e_k : \varepsilon_k = \pm 1\right\} & \text{if } q = \infty, \\ \mathsf{S}_{|\cdot|_q} & \text{if } 1 < q < \infty, \end{cases}$$

where $\{e_1, \ldots, e_{m+1}\}$ stands for the canonical basis of $\mathbb{R}^{m+1}$ and $\mathsf{S}_{|\cdot|_q}$ is the unit sphere of $(\mathbb{R}^{m+1}, |\cdot|_q)$.

The above problem is an extension of the polynomial Bohnenblust–Hille and Hardy–Littlewood constants problem. The m-Bohnenblust–Hille constant for polynomials is, in fact, an upper bound on $K_{m,n,\frac{2m}{m+1},\infty}$. It was proved in [8] that if $q \geq \frac{2m}{m+1}$, then there exists a constant $D_{m,q} > 0$ depending only on m and q such that

$$|P|_q \leq D_{m,q} \|P\|_\infty,$$

for any $P \in \mathcal{P}({}^m\ell_\infty^n(\mathbb{K}))$ and every $n \in \mathbb{N}$. Furthermore, any constant in the latter inequality for $q < \frac{2m}{m+1}$ depends necessarily on n. By construction, notice that any viable choice of $D_{m,q}$ satisfies $D_{m,q} \geq \sup\{K_{m,n,q,\infty} \colon n \in \mathbb{N}\}$. This construction allows us to define the Bohnenblust-Hille constants depending on the field ($\mathbb{R}$ or $\mathbb{C}$) since there are substantial differences.

Definition 9.2 The m-Bohnenblust-Hille constant for polynomials on $\mathbb{K}$ is defined as

$$D_{\mathbb{K},m} = \inf\left\{D_m \colon |P|_{\frac{2m}{m+1}} \leq D_m \|P\|_\infty, \text{ for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}({}^m\ell_\infty^n(\mathbb{K}))\right\}.$$

If $n \in \mathbb{N}$ is fixed, then we define (m, n)-Bohnenblust-Hille constant for polynomials on $\mathbb{K}$ as

$$D_{\mathbb{K},m}(n) = \inf\left\{D_m(n) \colon |P|_{\frac{2m}{m+1}} \leq D_m(n)\|P\|_\infty, \text{ for all } P \in \mathcal{P}({}^m\ell_\infty^n(\mathbb{K}))\right\}.$$

Also, if we consider a subset E of $\mathcal{P}({}^m\ell_\infty^n(\mathbb{K}))$ for some $n \in \mathbb{N}$, then we define the (m, E)-Bohnenblust-Hille constant for polynomials on $\mathbb{K}$ as

$$D_{\mathbb{K},m}(E) = \inf\left\{D_m(E) \colon |P|_{\frac{2m}{m+1}} \leq D_m(E)\|P\|_\infty, \text{ for all } P \in E\right\}.$$

It is easy to see that

$$1 \leq D_{\mathbb{K},m}(n) \leq D_{\mathbb{K},m},$$

for all $n \in \mathbb{N}$. A similar result to that of Bohnenblust-Hille for values of p different from ∞ can also be obtained. The proofs of the following results can be found in [1, 18]. There exist constants $C_{m,p}$ and $D_{m,p}$ independent of n such that

$$|P|_{\frac{p}{p-m}} \leq C_{m,p}\|P\|_p \text{ for } m < p \leq 2m,$$

$$|P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}\|P\|_p \text{ for } 2m \leq p \leq \infty,$$

for all $P \in ({}^m\ell_p^n(\mathbb{K}))$ and every $n \in \mathbb{N}$. If $p = \infty$, then we simply put $\frac{2mp}{mp+p-2m} = \frac{2m}{m+1}$. Moreover, the exponents $\frac{p}{p-m}$ for $m < p \leq 2m$ and $\frac{2mp}{mp+p-2m}$ for $2m \leq p \leq$

∞ are optimal in the sense that any constant H that satisfies

$$|P|_q \leq H\|P\|_p,$$

for all $P \in ({}^m\ell_p^n(\mathbb{K}))$ depends necessarily on n. The above construction allows us to define the following constants.

Definition 9.3 Let $m < p \leq \infty$. The (m, p)-Hardy-Littlewood constant for polynomials on $\mathbb{K}$ is defined as

$$C_{\mathbb{K},m,p} = \inf\left\{C_{m,p} \colon |P|_{\frac{p}{p-m}} \leq C_{m,p}\|P\|_p,\ \text{for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}({}^m\ell_p^n(\mathbb{K}))\right\},$$

for $m < p \leq 2m$, and

$$D_{\mathbb{K},m,p} = \inf\Big\{D_{m,p} \colon |P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}\|P\|_p,$$
$$\text{for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}({}^m\ell_p^n(\mathbb{K}))\Big\},$$

for $2m \leq p \leq \infty$. If $n \in \mathbb{N}$ is fixed, then we define the (m, n, p)-Hardy-Littlewood constant for polynomials on $\mathbb{K}$ as

$$C_{\mathbb{K},m,p}(n) = \inf\left\{C_{m,p}(n) \colon |P|_{\frac{p}{p-m}} \leq C_{m,p}(n)\|P\|_p,\ \text{for all } P \in \mathcal{P}({}^m\ell_p^n(\mathbb{K}))\right\},$$

for $m < p \leq 2m$, and

$$D_{\mathbb{K},m,p}(n) = \inf\Big\{D_{m,p}(n) \colon |P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}(n)\|P\|_p,$$
$$\text{for all } P \in \mathcal{P}({}^m\ell_p^n(\mathbb{K}))\Big\},$$

for $2m \leq p \leq \infty$. Also, if we consider a subset E of $\mathcal{P}({}^m\ell_\infty^n(\mathbb{K}))$ for some $n \in \mathbb{N}$, then we define

$$C_{\mathbb{K},m,p}(E) = \inf\left\{C_{m,p}(E) \colon |P|_{\frac{p}{p-m}} \leq C_{m,p}(E)\|P\|_p,\ \text{for all } P \in \mathcal{P}({}^mE)\right\},$$

for $m < p \leq 2m$, and

$$D_{\mathbb{K},m,p}(E) = \inf\left\{D_{m,p}(E) \colon |P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}(E)\|P\|_p,\ \text{for all } P \in \mathcal{P}({}^mE)\right\}$$

for $2m \leq p \leq \infty$.

Notice that $D_{\mathbb{K},m} = D_{\mathbb{K},m,\infty}$. So essentially the Hardy-Littlewood constants are in fact a generalization of the Bohnenblust-Hille constants. But furthermore, the constants $K_{m,n,q,p}$ are also a generalization of the Hardy-Littlewood constants since $C_{\mathbb{K},m,p}(n) = K_{m,n,\frac{p}{p-m},p}$ for $m < p \leq 2m$ and $D_{\mathbb{K},m,p}(n) = K_{m,n,\frac{2mp}{mp+p-2m},p}$ for $2m \leq p \leq \infty$. Hence we have

$$\begin{cases} C_{\mathbb{K},m,p} \geq \sup\left\{K_{m,n,\frac{p}{p-m},p} : n \in \mathbb{N}\right\} & \text{for } m < p \leq 2m, \\ D_{\mathbb{K},m,p} \geq \sup\left\{K_{m,n,\frac{2mp}{mp+p-2m},p} : n \in \mathbb{N}\right\} & \text{for } 2m \leq p \leq \infty. \end{cases}$$

This section is about providing some of the constants $k_{m,n,q,p}$, $K_{m,n,q,p}$, and in particular, the Hardy-Littlewood and Bohnenblust-Hille constants, that have been obtained through the Krein-Milman approach.

9.4.1 On the Complex Case

Assume that $\mathbb{K} = \mathbb{C}$.

Theorem 9.33 (Jiménez et al. [33]) *Let $E_{\mathbb{R}}$ be the real subspace of $\mathcal{P}(^2\ell_\infty^2(\mathbb{C}))$ given by $\{az^2 + bw^2 + czw : (a, b, c) \in \mathbb{R}^3\}$. We have*

$$D_{\mathbb{C},2}(E_{\mathbb{R}}) = D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}}$$

with extremal polynomials

$$P(x, y) = \pm\left(\frac{\sqrt{3}}{6}z^2 - \frac{\sqrt{3}}{6}w^2 \pm \sqrt{\frac{2}{3}}zw\right).$$

9.4.2 On the Real Case

Assume that $\mathbb{K} = \mathbb{R}$. All the results that are presented have been obtained for the cases when $m = n = 2$.

Theorem 9.34 (Jiménez et al. [33]) *Let $f : \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ be given by*

$$f(t) = \left[2t^{\frac{4}{3}} + \left(2\sqrt{t(1-t)}\right)^{\frac{4}{3}}\right]^{\frac{3}{4}}.$$

We have

$$D_{\mathbb{R},2}(2) = f(t_0),$$

where

$$t_0 = \frac{1}{36}\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right).$$

In particular, the exact value of $f(t_0)$ *is given by*

$$(A + B)^{\frac{3}{4}},$$

where

$$A = \frac{\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right)^{\frac{4}{3}}}{186^{\frac{2}{3}}}$$

and

$$B = \frac{1}{9\left(-\frac{3}{-2\sqrt[3]{107+9\sqrt{129}}+\left(107+9\sqrt{129}\right)^{\frac{2}{3}}-2\sqrt[3]{107-9\sqrt{129}}+\left(107-9\sqrt{129}\right)^{\frac{2}{3}}-60}\right)^{\frac{2}{3}}}.$$

Moreover, the following polynomials are extremal

$$P(x, y) = \pm\left(t_0x^2 - t_0y^2 \pm 2\sqrt{t_0(1 - t_0)}xy\right).$$

Theorem 9.35 (Araújo et al. [3]) *If* $q, p \in \{1, \infty\}$, *then*

$$k_{2,2,q,p} = \begin{cases} 1 & \text{if } q = p = 1, \\ 1 & \text{if } q = 1 \text{ and } p = \infty, \\ 1 & \text{if } q = \infty \text{ and } p = 1, \\ \frac{1}{3} & \text{if } q = p = \infty, \end{cases}$$

with extremal polynomials given, respectively, by

$$P_{1,1}(x, y) = \pm x^2,\ \pm y^2,$$

$$P_{1,\infty}(x, y) = \pm x^2,\ \pm y^2,\ \pm xy,$$

$$P_{\infty,1}(x, y) = \pm x^2 \pm y^2 \pm xy,$$

$$P_{\infty,\infty}(x, y) = \pm(x^2 + y^2 \pm xy).$$

Theorem 9.36 (Araújo et al. [3]) *If* $q, p \in \{1, \infty\}$, *then*

$$K_{2,2,q,p} = \begin{cases} 2+2\sqrt{2} & \text{if } q = p = 1, \\ 1+\sqrt{2} & \text{if } q = 1 \text{ and } p = \infty, \\ 4 & \text{if } q = \infty \text{ and } p = 1, \\ 1 & \text{if } q = p = \infty, \end{cases}$$

with extremal polynomials given, respectively, by

$$\begin{aligned} P_{1,1}(x, y) &= \pm\frac{\sqrt{2}}{2}(x^2 - y^2) + (2+\sqrt{2})xy, \\ P_{1,\infty}(x, y) &= \pm\left(\frac{2+\sqrt{2}}{4}x^2 - \frac{2+\sqrt{2}}{4}y^2 \pm \frac{\sqrt{2}}{2}xy\right), \\ P_{\infty,1}(x, y) &= \pm 4xy, \\ P_{\infty,\infty}(x, y) &= \pm x^2,\ \pm y^2,\ \pm\left(\frac{1}{2}x^2 - \frac{1}{2}y^2 \pm xy\right). \end{aligned}$$

Theorem 9.37 (Araújo et al. [3]) *For every* $q \in [1, \infty)$, *let* $f_{q,1}\colon [2, 4] \to \mathbb{R}$ *and* $f_{q,\infty}\colon \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ *be given by*

$$\begin{aligned} f_{q,1}(t) &= \left(2^{1-q}(4t - t^2)^{\frac{q}{2}} + t^q\right)^{\frac{1}{q}}, \\ f_{q,\infty}(t) &= \left(2t^q + 2^q(t - t^2)^{\frac{q}{2}}\right)^{\frac{1}{q}}. \end{aligned}$$

We have

$$\begin{aligned} K_{2,2,q,1} &= \max\left\{f_{q,1}(t)\colon t \in [2, 4]\right\}, \\ K_{2,2,q,\infty} &= \max\left\{f_{q,\infty}(t)\colon t \in \left[\frac{1}{2}, 1\right]\right\}. \end{aligned}$$

In particular, $K_{2,2,q,1} = 4$ *and* $K_{2,2,q,\infty} = 2^{\frac{1}{q}}$ *for every* $q \geq 2$, *with extremal polynomials given, respectively, by*

$$\begin{aligned} P_{q,1}(x, y) &= \pm 4xy, \\ P_{q,\infty}(x, y) &= \pm(x^2 - y^2). \end{aligned}$$

Remark 9.4 (Araújo et al. [3]) The exact value of the maximum of the functions $f_{q,1}$ and $f_{q,\infty}$ or the points of attainment of the maximum seems to be a much harder task. However, by using the symbolic calculus tool of MATLAB, we are able to obtain the exact values where the functions reach its maximum for certain values of q. For instance, for $q = \frac{4}{3}$, the maximum of $f_{q,1}(t)$ and $f_{q,\infty}(t)$ is attained at

$$t = \frac{1}{9}\left(2\sqrt[3]{181 + 9\sqrt{273}} + \sqrt[3]{1448 - 72\sqrt{273}} + 14\right).$$

and

$$t = \frac{1}{36}\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right),$$

respectively. Also, for $q = \frac{3}{2}$, the maximum of $f_{q,1}(t)$ is attained at

$$t = \frac{1}{15}\left(\sqrt{6\,(A + 24)} + \sqrt{6\left(-A + 204\sqrt{\frac{6}{A + 24}} + 48\right) + 18}\right),$$

where

$$A = -10 \cdot 3^{2/3}\sqrt[3]{\frac{2}{9 + \sqrt{93}}} + 5 \cdot 2^{2/3}\sqrt[3]{3\left(9 + \sqrt{93}\right)}.$$

And also for $q = \frac{3}{2}$, the maximum of $f_{q,\infty}(t)$ is attained at

$$t = \frac{1}{20}\sqrt{B} + \frac{1}{2}\sqrt{C + D} + \frac{9}{20},$$

where

$$B = \frac{10\sqrt[3]{9 + \sqrt{273}}}{3^{2/3}} - \frac{40}{\sqrt[3]{3\left(9 + \sqrt{273}\right)}} + 1,$$

$$C = -\frac{\sqrt[3]{9 + \sqrt{273}}}{10 \cdot 3^{2/3}} + \frac{1}{50} + \frac{2}{5\sqrt[3]{3\left(9 + \sqrt{273}\right)}}$$

and

$$D = \frac{40}{50\sqrt{\frac{10\sqrt[3]{9+\sqrt{273}}}{3^{2/3}} - \frac{40}{\sqrt[3]{3\left(9+\sqrt{273}\right)}} + 1}}.$$

Theorem 9.38 (Araújo et al. [3]) *If $p \in (1, \infty)$, then*

$$k_{2,2,q,p} = \begin{cases} 1 & \text{if } q = 1, \\ \frac{2^{\frac{2}{p}}}{3} & \text{if } q = \infty \text{ and } p \geq \frac{4}{3}, \\ \frac{1}{\max\left\{x^2+(1-x^p)^{\frac{2}{p}}+x(1-x^p)^{\frac{1}{p}} : x\in[0,1]\right\}} & \text{if } q = \infty \text{ and } 1 < p < \frac{4}{3}, \end{cases}$$

with extremal polynomials given, respectively, by

$$P_{1,p}(x, y) = \pm x^2, \ \pm y^2,$$
$$P_{\infty,p}(x, y) = \pm \left(x^2 + y^2 + xy\right),$$
$$Q_{\infty,p}(x, y) = \pm \left(x^2 + y^2 + xy\right).$$

Theorem 9.39 (Araújo et al. [3]) *For every $q \geq 1$ and $p \geq 2$, let $f_{q,p} \colon [0, 1] \to \mathbb{R}$ be given by*

$$f_{q,p}(s) = \begin{cases} \left(2(1-s)^{\frac{q}{2}} + 2^q s^{\frac{q}{2}}\right)^{\frac{1}{q}} & \text{if } p = 2, \\ \frac{\left\{2|1-2s|^q+2^q\left[(1-s)^{1-\frac{1}{p}}s^{\frac{1}{p}}+(1-s)^{\frac{1}{p}}s^{1-\frac{1}{p}}\right]^q\right\}^{\frac{1}{q}}}{(1-s)^{\frac{2}{p}}+s^{\frac{2}{p}}} & \text{if } p \neq 2. \end{cases}$$

We have

$$K_{2,2,q,p} = \max\left\{f_{q,p}(t) \colon t \in [0, 1]\right\}.$$

See also [13] in connection to the previous result.

Corollary 9.2 (Araújo et al. [3]) *For $4 \leq p \leq \infty$, we have*

$$\begin{aligned} D_{\mathbb{R},2,p}(2) &= K_{2,2,\frac{4p}{3p-4},p} \\ &= \max_{s\in\left[0,\frac{1}{2}\right]} \frac{\left\{2|1-2s|^{\frac{4p}{3p-4}} + 2^{\frac{4p}{3p-4}}\left[(1-s)^{1-\frac{1}{p}}s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}}s^{1-\frac{1}{p}}\right]^{\frac{4p}{3p-4}}\right\}^{\frac{3p-4}{4p}}}{(1-s)^{\frac{2}{p}} + s^{\frac{2}{p}}}. \end{aligned}$$

Theorem 9.40 (Araújo et al. [3]) *If $q > 1$, then*

$$K_{2,2,q,2} = \begin{cases} 2 & \textit{if } q \geq 2, \\ \dfrac{2\left(1+2^{\frac{1}{q-2}}\right)^{\frac{1}{q}}}{\left(1+2^{\frac{2(q-1)}{q-2}}\right)^{\frac{1}{2}}} & \textit{if } 1 < q < 2, \end{cases}$$

with extremal polynomials given by

$$P_{q,2}(x, y) = \begin{cases} \pm(x^2 - y^2) & \textit{if } q \geq 2, \\ \pm\left(a_0 x^2 - a_0 y^2 + 2\sqrt{1 - a_0^2}\, xy\right) & \textit{if } 1 < q < 2, \end{cases}$$

where $a_0 = \left(1 + 2^{\frac{2(1-q)}{q-2}}\right)^{-\frac{1}{2}}$.

Theorem 9.41 (Araújo et al. [3]) *If* $q, p > 2$, *then*

$$K_{2,2,q,p} = 2^{\max\left\{\frac{1}{q}, \frac{2}{p}\right\}}.$$

If $f_{q,p}$ *is as in Theorem 9.39 and* $q, p > 2$, *then the following polynomials are extremal*

$$P_{q,p}(x, y) = \begin{cases} \pm 2^{\frac{2}{p}} xy & \textit{if } q \geq \frac{p}{2}, \\ \pm(x^2 - y^2) & \textit{if } q < \frac{p}{2}. \end{cases}$$

Corollary 9.3 (Araújo et al. [3]) *If* $p \geq 2$, *then*

$$K_{2,2,\infty,p} = 2^{\frac{2}{p}}$$

with extremal polynomials given by

$$P_{\infty,p}(x, y) = \begin{cases} \pm(x^2 - y^2) & \textit{if } p = 2, \\ \pm 2^{\frac{2}{p}} xy & \textit{if } p > 2. \end{cases}$$

Corollary 9.4 (Araújo et al. [3]) *For* $2 < p \leq 4$, *we have*

$$C_{\mathbb{R},2,p}(2) = K_{2,2,\frac{p}{p-2},p} = 2^{\frac{2}{p}}.$$

It is important to mention that Corollary 9.4 was first proven in [13].

Corollary 9.5 (Araújo et al. [3]) *We have*

$$D_{\mathbb{R},2,4}(2) = C_{\mathbb{R},2,4}(2) = K_{2,2,4,p} = \sqrt{2}$$

with all extremal polynomials given by

$$\begin{aligned} P(x, y) &= \pm(x^2 - y^2), \\ Q(x, y) &= \pm\left((\alpha^2 - \beta^2)(x^2 - y^2) + 2\alpha\beta xy\right), \end{aligned}$$

with $\alpha, \beta \geq 0$ *and* $\alpha^4 + \beta^4 = 1$.

Theorem 9.42 (Araújo et al. [3]) *For* $p > 2$, *let* $f_{1,p} \colon \left[0, \frac{1}{2}\right] \to \mathbb{R}$ *be defined by*

$$f_{1,p}(s) = \frac{2(1-2s) + 2\left[(1-s)^{1-\frac{1}{p}} s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}} s^{1-\frac{1}{p}}\right]}{(1-s)^{\frac{2}{p}} + s^{\frac{2}{p}}}.$$

We have

$$K_{2,2,1,p} = \sup\left\{ f_{1,p}(t) \colon t \in \left[0, \frac{1}{2}\right] \right\}.$$

Remark 9.5 (Araújo et al. [3]) The exact calculation of the above supremum seems to be a harder task. However, by using the symbolic calculus tool of MATLAB, we can obtain the exact value of the supremum of $f_{1,p}(t)$ as well as the point where it attains its maximum for certain values of p. For $p = 4$, the function $f_{1,4}(t)$ attains its maximum on $\left[0, \frac{1}{2}\right]$ at $t = \frac{3-2\sqrt{2}}{6}$ and, therefore, $K_{2,1,4} = \sqrt{6}$.