Chapter 2
Polynomials of Degree *n*

Abstract This chapter focuses on the study of the geometry of the unit ball of the space of polynomials in one variable of degree at most $n \in \mathbb{N}$ endowed with the supremum norm defined on the interval $[-1, 1]$ (when the polynomial is defined over $\mathbb{R}$) or on the unit disk (when the polynomial is defined over $\mathbb{C}$). More precisely, we are interested on the parametrization of the unit ball as well as the extreme points when we are dealing with the space of polynomials of degree at most 2. For the space of polynomials of arbitrary degree with the supremum norm defined on $[-1, 1]$, we are only interested on the extreme polynomials of the unit ball.

2.1 On the Real Line

Let us endow the vector space of real polynomials of the degree at most $n \in \mathbb{N}$, that is, of the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$ where $a_i \in \mathbb{R}$ for every $i \in \{1, \ldots, n\}$ and $x \in \mathbb{R}$, with the supremum norm

$$\|P\|_{\mathbb{R}} = \max\{|P(x)| \colon x \in [-1, 1]\}.$$

We denote this normed space by $\mathcal{P}_n(\mathbb{R})$. Now consider the following construction: let us define the mapping T from $\mathcal{P}_n(\mathbb{R})$ to $\mathbb{R}^{n+1}$ that assigns to each polynomial $a_n x^n + \cdots + a_1 x + a_0$ the vector $(a_n, \ldots, a_1, a_0)$, i.e., each polynomial is mapped into the vector formed by its coefficients. This mapping T is a topological isomorphism between $\mathcal{P}_n(\mathbb{R})$ and $\mathbb{R}^{n+1}$ when we endow $\mathbb{R}^{n+1}$ with the norm

$$\|(a_n, \ldots, a_1, a_0)\|_{\mathbb{R}} := \|a_n x^n + \cdots + a_1 x + a_0\|_{\mathbb{R}}.$$

Let us denote the unit ball and the unit sphere of $(\mathbb{R}^{n+1}, \|\cdot\|_{\mathbb{R}})$ by $\mathsf{B}_n(\mathbb{R})$ and $\mathsf{S}_n(\mathbb{R})$, respectively. Thus, in particular, on the space $\mathcal{P}_2(\mathbb{R})$, we can give a visual representation of the unit ball.

The geometry of $\mathcal{P}_n(\mathbb{R})$ was already studied by A. G. Konheim and T. J. Rivlin in 1966 [40]. They were able to characterize when a polynomial of degree at most $n \in$

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$\mathbb{N}$ that belongs to the unit ball is an extreme polynomial based on the multiplicity of intersection of the polynomial with 1 and -1.

Definition 2.1 Let P be a real polynomial of degree at most n. We denote by $N(P, y)$ the total multiplicity with which the value y is assumed by P and, in particular, let us define the multiplicity of P by the number $N(P) := N(P, 1) + N(P, -1)$.

Theorem 2.1 (Konheim and Rivlin [40]) *Let $P \in \mathcal{P}_n(\mathbb{R})$ with $\|P\| \leq 1$. We have that P is an extreme polynomial if, and only if, $N(P) > n$.*

Although Konheim and Rivlin gave a characterization of the extreme polynomials of the unit ball of $\mathcal{P}_n(\mathbb{R})$, they do not give an explicit formula for the values of the extreme polynomials. However, R. M. Aron and M. Klimek [5] were able to obtain an explicit formula for the extreme polynomials in the unit ball of $\mathcal{P}_2(\mathbb{R})$ by using an approach that will appear in many results of this survey. Firstly, they gave an explicit formula for the norm of a polynomial of degree at most 2. Secondly, they found the projection of the unit ball onto a plane. And finally, using this information, they were able to parametrize the unit ball and, in the process, find the extreme polynomials of the unit ball. The results that Aron and Klimek provided are shown below.

Theorem 2.2 (Aron and Klimek [5]) *Let $P(x) = ax^2 + bx + c$. We have*

$$\|(a, b, c)\|_{\mathbb{R}} = \begin{cases} \left|\frac{b^2}{4a} - c\right| & \text{if } |b| < 2|a|t \text{ and } \frac{c}{a} + 1 < \frac{1}{2}\left(\left|\frac{b}{2a}\right| - 1\right)^2, \\ |a + c| + |b| & \text{otherwise.} \end{cases}$$

Let us define the sets

$$\begin{aligned} U &= \left\{(a, b) \in \mathbb{R}^2 \colon a \leq 0 \text{ and } |b| \leq \min\left\{2|a|, 2\left(\sqrt{2|a|} - |a|\right)\right\}\right\}, \\ V &= \left\{(a, b) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1] \colon |b| \geq 2|a|\right\}, \\ W &= \left\{(a, b) \in \mathbb{R}^2 \colon a \geq 0 \text{ and } |b| \leq \min\left\{2|a|, 2\left(\sqrt{2|a|} - |a|\right)\right\}\right\}. \end{aligned}$$

Theorem 2.3 (Aron and Klimek [5]) *The projection of $\mathsf{B}_2(\mathbb{R})$ onto the ab-plane is the set $U \cup V \cup W$ (see Fig. 3.5 for a representation of $U \cup V \cup W$ with $n = 1$).*

Theorem 2.4 (Aron and Klimek [5]) *Let us define the functions*

$$\begin{aligned} f_+(a, b) &= 1 - |b| - |a|, \\ g_+(a, b) &= \frac{b^2}{4a} - 1, \end{aligned}$$

and also the functions $f_-(a, b) = -f_+(-a, b)$ and $g_-(a, b) = -g_+(-a, b)$. We have *(i)* $\mathsf{S}_2(\mathbb{R}) = \text{graph}\left(f_+|_{(V\cup W)}\right) \cup \text{graph}\left(f_-|_{(U\cup V)}\right) \cup \text{graph}\,(g_+|_W) \cup \text{graph}\,(g_-|_U)$ *(see Fig. 3.6 for a representation of* $\mathsf{B}_2(\mathbb{R})$ *with* $n = 1$*).*

(ii) The set of extreme points (denoted by ext*) is*

$$\text{ext}\,(\mathsf{B}_2(\mathbb{R})) = \left\{\pm\left(t, \pm 2(\sqrt{2t} - t), 1 + t - 2\sqrt{2t}\right) : t \in \left[\frac{1}{2}, 2\right]\right\} \bigcup \{\pm(0, 0, 1)\}.$$

The following results of this section are devoted to the study of extreme polynomials of degree at most 3.

Theorem 2.5 (Araújo et al. [4]) *The extreme polynomials of the unit ball of* $\mathcal{P}_3(\mathbb{R})$ *are given by*

(i) $P_1(x) = \pm 1$*;*

(ii) $P_2(x) = \pm\left[1 - \frac{1}{4}(\pm x + 1)^3\right]$*;*

(iii) $P_3(x) = \pm(2x^2 - 1)$*;*

(iv) $P_4(x) = \pm\left[1 - \frac{1}{(1-q^2)^2}(x - q)^2(4qx + 2 + 2q^2)\right]$ *and*
$P_5(x) = \pm\left[1 + \frac{1}{(1-q^2)^2}(x + q)^2(4qx - 2 - 2q^2)\right]$*, for every* $q \in \left(-\frac{1}{3}, 0\right)$*;*

(v) $P_6(x) = \pm\left[1 + \frac{1}{(1+t)^2}(x - t)^2(x - 1)\right]$ *and*
$P_7(x) = \pm\left[1 - \frac{1}{(1+t)^2}(x + t)^2(x + 1)\right]$*, for every* $t \in \left(-\frac{1}{2}, 1\right)$*;*

(vi) $P_8(x) = \pm\left[1 + \frac{4}{(s-r)^3}(x - r)^2\left(x - \frac{3s-r}{2}\right)\right]$ *and*
$P_9(x) = \pm\left[1 - \frac{4}{(s-r)^3}(x + r)^2\left(x + \frac{3s-r}{2}\right)\right]$*, for every* $-1 \le r < s \le 1$ *such that* $s \ge \min\left\{3r + 2, \frac{r+2}{3}\right\}$.

2.1.1 Polynomials Bounded by a Majorant

Assume that P is a polynomial of degree at most n such that P is constrained on the interval $[-1, 1]$ by a mapping $\varphi\colon [-1, 1] \to [0, +\infty)$ called the majorant, i.e., $|P(x)| \le \varphi(x)$ for every $x \in [-1, 1]$. We will denote by $\mathcal{P}_n^{\varphi}(\mathbb{R})$ the space of polynomials on the real line of degree at most n that are bounded by a majorant φ endowed with the supremum norm over the interval $[-1, 1]$. In this section we are interested in studying the extreme points of the unit ball of the space $\mathcal{P}_3^{\varphi}(\mathbb{R})$ when φ is a circular majorant, that is, $\varphi(x) = \sqrt{1 - x^2}$ for any $x \in [-1, 1]$.

Notice that if a polynomial P belongs to $\mathcal{P}_3^{\varphi}(\mathbb{R})$, where φ is a circular majorant, then P has roots at ± 1. Hence all polynomials of degree not greater than 3 bounded by a circular majorant are of the form $P_{a,b}(x) = (1-x^2)(ax+b)$ for some $a, b \in \mathbb{R}$. Thus, in fact, we have the following inequality $|(1-x^2)(ax+b)| \le \sqrt{1 - x^2}$ for any

$x \in [-1, 1]$, which is equivalent to $\left|\sqrt{1-x^2}(ax+b)\right| \leq 1$ for any $x \in [-1, 1]$. The latter shows that we can study the unit ball of the space $\mathcal{P}_3^{\varphi}(\mathbb{R})$, when φ is a circular majorant, by studying the unit ball of the norm space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$, where

$$\|(a,b)\|_{\infty,\varphi} = \sup\left\{\left|\sqrt{1-x^2}(ax+b)\right| : x \in [-1,1]\right\}.$$

We begin by showing an explicit formula for the norm $\|\cdot\|_{\infty,\varphi}$.

Theorem 2.6 (Muñoz et al. [47]) *If $\varphi\colon [-1,1] \to [0,+\infty)$ is defined by $\varphi(x) = \sqrt{1-x^2}$, then for every $(a,b) \in \mathbb{R}^2$ we have*

$$\|(a,b)\|_{\infty,\varphi} = \begin{cases} \frac{(3|b|+\sqrt{8a^2+b^2})\sqrt{4a^2-b^2+|b|\sqrt{8a^2+b^2}}}{8\sqrt{2}|a|} & \text{if } a \neq 0, \\ |b| & \text{if } a = 0. \end{cases}$$

As an easy consequence of Theorem 2.6 we have the following characterization of the unit ball of $\mathcal{P}_3^{\varphi}(\mathbb{R})$.

Theorem 2.7 (Muñoz et al. [47]) *Let $\varphi\colon [-1,1] \to [0,+\infty)$ be defined by $\varphi(x) = \sqrt{1-x^2}$. If $(a,b) \in \mathbb{R}^2$, then $\|(a,b)\|_{\infty,\varphi} \leq 1$ if, and only if,*

$$\left(\sqrt{8a^2+b^2}+3|b|\right)^3 \leq 32\left(\sqrt{8a^2+b^2}+|b|\right),$$

where equality is satisfied if, and only if, $\|(a,b)\|_{\infty,\varphi} = 1$. Moreover, the set of extreme points of the unit ball of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$ are the points of the unit sphere.

Figure 2.1 shows an approximate representation of the unit sphere of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$.

2.2 On the Complex Plane

Let us consider now the vector space of complex polynomials with real coefficients of degree at most $n \in \mathbb{N}$, that is, we have polynomials of the form $P(z) = a_n z^n + \ldots + a_1 z + a_0$ where $a_i \in \mathbb{R}$ and $z \in \mathbb{C}$, endowed with the following norm

$$\|P\|_{\mathbb{C}} = \sup_{|z|\leq 1} |P(z)|.$$

We denote this normed space by $\mathcal{P}_{\mathbb{R},n}(\mathbb{C})$. Using the mapping T defined on Sect. 2.1, there is a topological isomorphism between the space $\mathcal{P}_{\mathbb{R},n}(\mathbb{C})$ and $\mathbb{R}^{n+1}$ endowed with the norm

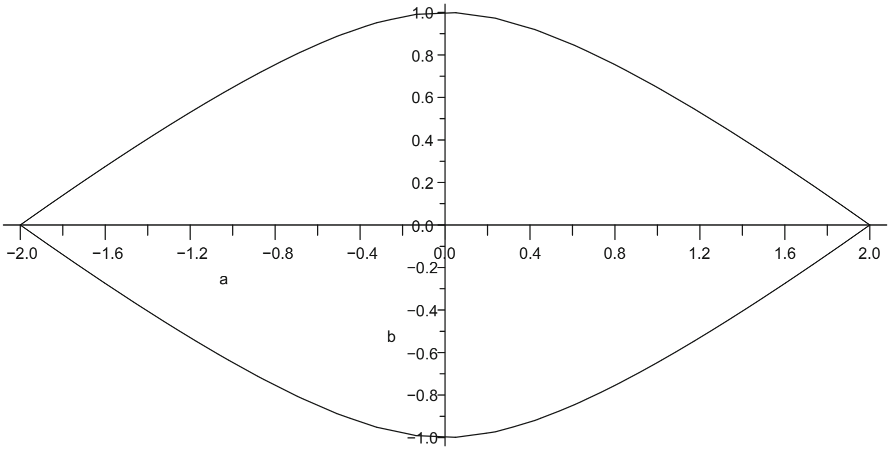

Fig. 2.1 Unit sphere of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$

$$\|(a_n, \ldots, a_1, a_0)\|_{\mathbb{C}} = \|a_n z^n + \ldots + a_1 z + a_0\|_{\mathbb{C}}.$$

Let us denote the unit ball and the unit sphere of $(\mathbb{R}^{n+1}, \|\cdot\|_{\mathbb{C}})$ by $\mathsf{B}_{\mathbb{R},n}(\mathbb{C})$ and $\mathsf{S}_{\mathbb{R},n}(\mathbb{C})$, respectively. Furthermore, we can a give a visual representation of the unit ball of the space $\mathcal{P}_{\mathbb{R},2}(\mathbb{C})$ on $\mathbb{R}^3$. We use the same approach as in the previous section. We begin by showing an explicit formula for the norm of the space $\mathcal{P}_{\mathbb{R},2}(\mathbb{C})$.

Theorem 2.8 (Aron and Klimek [5]) *If* $P(z) = az^2 + bz + c \in \mathcal{P}_{\mathbb{R},2}(\mathbb{C})$, *then*

$$\|(a, b, c)\|_{\mathbb{C}} = \begin{cases} |a+c| + |b| & \text{if } ac \geq 0 \text{ or } |b(a+c)| > 4|ac|, \\ (|a| + |c|)\sqrt{1 + \frac{b^2}{4|ac|}} & \text{otherwise.} \end{cases}$$

We continue by showing the projection of the unit ball onto a coordinate plane. To do so, we define the following sets

$$A = \left\{(a, c) \in \mathbb{R}^2 \colon |a| + |c| \leq 1 \text{ and } |a + c| \leq (|a| + |c|)^2\right\},$$

$$B = \left\{(a, c) \in \mathbb{R}^2 \colon |a| + |c| \leq 1 \text{ and } |a + c| > (|a| + |c|)^2\right\}.$$

Figure 2.2 shows a representation of A and B.

Theorem 2.9 (Aron and Klimek [5]) *The projection of* $\mathsf{B}_{\mathbb{R},2}(\mathbb{C})$ *onto the ac-plane is the set* $A \cup B$.

Finally, we show a parametrization of $\mathsf{S}_{\mathbb{R},2}(\mathbb{C})$ as well as the extreme points of $\mathsf{B}_{\mathbb{R},2}(\mathbb{C})$.

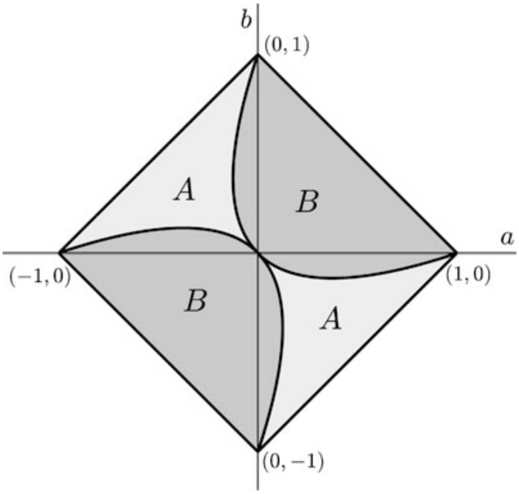

Fig. 2.2 Projection of $\mathsf{B}_{\mathbb{R},2}(\mathbb{C})$ onto the *ac*-plane

Theorem 2.10 (Aron and Klimek [5]) *Let us define the function*

$$f(a,c) = \begin{cases} \sqrt{4|ac|\left(\frac{1}{(|a|+|c|)^2} - 1\right)} & \textit{if } (a,c) \in A, \\ 1 - |a+c| & \textit{if } (a,c) \in B. \end{cases}$$

We have

(i) $\mathsf{S}_{\mathbb{R},2}(\mathbb{C}) = \text{graph}(f) \cup \text{graph}(-f)$.

(ii)

$$\text{ext}(\mathsf{B}_{\mathbb{R},2}(\mathbb{C})) = \left\{ \left(a, \pm\sqrt{4|ac|\left(\frac{1}{(|a|+|c|)^2} - 1\right)}, c\right) : a, c \neq 0,\ |a| + |c| < 1 \textit{ and } |a+c| \leq (|a|+|c|)^2 \right\}.$$