SpringerBriefs in Mathematics

Jesús Ferrer · Domingo García · Manuel Maestre ·
Gustavo A. Muñoz · Daniel L. Rodríguez ·
Juan B. Seoane

Geometry of the Unit Sphere in Polynomial Spaces

Springer

SpringerBriefs in Mathematics

SpringerBriefs present concise summaries of cutting-edge research and practical applications across a wide spectrum of fields. Featuring compact volumes of 50 to 125 pages, the series covers a range of content from professional to academic. Briefs are characterized by fast, global electronic dissemination, standard publishing contracts, standardized manuscript preparation and formatting guidelines, and expedited production schedules.

Typical topics might include:

- A timely report of state-of-the art techniques
- A bridge between new research results, as published in journal articles, and a contextual literature review
- A snapshot of a hot or emerging topic
- An in-depth case study
- A presentation of core concepts that students must understand in order to make independent contributions

SpringerBriefs in Mathematics showcases expositions in all areas of mathematics and applied mathematics. Manuscripts presenting new results or a single new result in a classical field, new field, or an emerging topic, applications, or bridges between new results and already published works, are encouraged. The series is intended for mathematicians and applied mathematicians. All works are peer-reviewed to meet the highest standards of scientific literature.

Titles from this series are indexed by Scopus, Web of Science, Mathematical Reviews, and zbMATH.

Jesús Ferrer • Domingo García • Manuel Maestre •
Gustavo A. Muñoz • Daniel L. Rodríguez •
Juan B. Seoane

Geometry of the Unit Sphere in Polynomial Spaces

Springer

Jesús Ferrer (*deceased*)
Valencia, Valencia, Spain

Domingo García
Departamento de Análisis Matemático
Universidad de Valencia
Valencia, Valencia, Spain

Manuel Maestre
Departamento de Análisis Matemático
Universidad de Valencia
Valencia, Valencia, Spain

Gustavo A. Muñoz
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
Madrid, Madrid, Spain

Daniel L. Rodríguez
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
Madrid, Madrid, Spain

Juan B. Seoane
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
Madrid, Madrid, Spain

ISSN 2191-8198 ISSN 2191-8201 (electronic)
SpringerBriefs in Mathematics
ISBN 978-3-031-23675-4 ISBN 978-3-031-23676-1 (eBook)
https://doi.org/10.1007/978-3-031-23676-1

Mathematics Subject Classification: 46B20

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Contents

Chapter 1
Introduction

This book was completed after the passing of the first named author. The rest of authors would like to dedicate the book to the loving memory of their friend and colleague Jesús Ferrer (1952–2022).

The study and classification of the extreme points of the unit ball of a Banach space is a classical problem in functional analysis. This question is particularly interesting in the case of Banach spaces of polynomials. The case of integral, nuclear or orthogonally additive polynomials in Banach spaces have been studied, for instance, in [10, 11, 17, 20]. We devote Chap. 8 to show a selection of results where extreme integral, nuclear or orthogonally additive polynomials have been characterized in several different settings. As a matter of fact the geometry of the unit ball of polynomial spaces has been studied intensively for decades. Special attention has to be given to polynomial spaces of finite dimension. The case of polynomials on the real line of degree at most n endowed with the norm

$$\|P\| = \sup\{|P(x)| : x \in [-1, 1]\},$$

which we will represent by $\mathcal{P}_n(\mathbb{R})$, was solved by Konheim and Rivlin in [40] as early as in 1966 providing a characterization of the extreme polynomials of the unit ball B_n of $\mathcal{P}_n(\mathbb{R})$. The search for characterizations of the extreme polynomials of other finite dimensional polynomial spaces has been intensified since the late 90's of the twentieth century, motivating dozens of publications. In this paper we present a thorough revision of the most relevant results in this topic with special emphasis in the polynomial spaces of dimension 3. The fact that in dimension three we are able to provide a visual representation of the unit ball of a polynomial space is in itself a powerful tool in the study of the geometry of polynomial spaces. Although Konheim and Rivlin characterization of the extreme polynomials in B_n is not explicit, we will see in the next chapters that in many finite-dimensional Banach spaces of polynomials extreme polynomials can be fully described. Some

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_1

representative examples of the spaces which have been studied so far are listed below:

- The subspaces $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$ of $\mathcal{P}_n(\mathbb{R})$ (see [4, 5]).
- The space of the quadratic polynomials on the complex plane with real coefficients, $\mathcal{P}_2(\mathbb{C})$, endowed with the sup norm over the unit disk $\mathbb{D}$ (see [5]).
- The subspace $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ $(m > n)$ of $\mathcal{P}_m(\mathbb{R})$ consisting of all the trinomials of the form $ax^m + bx^n + c$ (see [50]).
- The trinomials $ax^m + bx^n y^{m-n} + cy^m$ $(m > n)$ on $\mathbb{R}^2$, represented by $\mathcal{P}^h_{m,n,\infty}(\mathbb{R}^2)$, endowed with the sup norm on the unit ball of $\ell^2_\infty(\mathbb{R})$ (see [35]).
- The spaces of quadratic forms on $\ell^2_p(\mathbb{R})$ $(1 \leq p \leq \infty)$, namely $\mathcal{P}(^2\ell^2_p)$, endowed with the sup norm over the unit ball of $\ell^2_p(\mathbb{R})$ (see for instance [14–16, 25–28]).
- The space $\mathcal{P}(^3\ell^2_2)$, of 3-homogeneous polynomials on $\mathbb{R}^2$ endowed with the sup norm over the unit ball of $\ell^2_2(\mathbb{R})$ (see [29]).
- The spaces $\mathcal{P}(^2\Delta)$ and $\mathcal{P}(^2\square)$ of the quadratic forms on $\mathbb{R}^2$ endowed with the sup norm over the simplex Δ and the square $\square = [0, 1]^2$ respectively (see [23, 46]).
- The space $\mathcal{P}_2(\Delta)$ of polynomials of degree at most 2 on $\mathbb{R}^2$ endowed with the supremum norm over the simplex Δ (see[43]).
- The space $\mathcal{P}(^2D(\alpha, \beta))$ with $\alpha \leq \beta$ (see [6, 45]) of the quadratic forms on $\mathbb{R}^2$ endowed with the sup norm on the sectors

$$D(\alpha, \beta) = \{re^{i\theta} : r \in [0, 1] \text{ and } \theta \in [\alpha, \beta]\}.$$

- The space $\mathcal{P}(^2\mathcal{O}^2_w)$ of the quadratic forms on $\mathbb{R}^2$ endowed with the norm

$$\|P\|_{\mathcal{O}^2_w} = \sup\{|P(x, y)| : \|(x, y)\|_{\mathrm{oct}(w)} \leq 1\},$$

  where

$$\|(x, y)\|_{\mathrm{oct}(w)} = \max\left\{|x|, |y|, \frac{|x| + |y|}{1 + w}\right\}$$

  for a fixed $w \in [0, 1]$ (see [38]).

Having an explicit description of the extreme points of the unit ball of a polynomial space has many interesting applications. The Krein-Milman approach allows us to prove many sharp polynomial inequalities. Recall that, as a direct consequence of the Krein-Milman theorem, any convex function on a convex body of a finite dimensional Banach space attains its maximum at an extreme point. Using this idea combined with a description of the extreme points of a polynomial space one can derive a number of polynomial inequalities. Sharp Bernstein and Markov inequalities are among the applications of the Krein-Milman approach. Other problems of interest where the geometry of the unit ball of polynomial spaces yield excellent results are the calculation of exact unconditional constants in polynomial spaces, the calculation of polarization constants or the calculation of sharp Bohnenblust-Hile and Hardy-Littlewood constants. Chapter 9 is devoted to present a selection of the many achievements that can be obtained by using the Krein-Milman approach.

In this book we pursuit three main achievements. The first is to provide the reader with a visual perspective of each of the Banach spaces of polynomials we study by representing their unit spheres. To this end the following steps are implemented in most of the cases:

1. First we give an explicit formula to calculate the polynomial norm.
2. Then we parametrize the unit sphere of the space, for which it might be of help to calculate the projection of the unit ball onto a plane.
3. The parametrization of the unit sphere is a valuable source of information that allows us to identify and classify the extreme points of the unit ball of each polynomial space.

The third point above accomplishes the second of the main objective of this monograph, providing the reader with explicit characterizations of the extreme polynomials in several Banach spaces of polynomials. The third objective is to highlight the many applications of having an explicit classification of the extreme points of the unit ball of a space of polynomials. In particular, we will show a number of interesting sharp Bernstein and Markov type inequalities and Bohnenblust-Hille inequalities obtained using the already mentioned Krein-Milman approach. We can also obtain exact unconditional constants, polarization constants and other related results.

This book is arranged as follows: In Chap. 2 we study the spaces $\mathcal{P}_n(\mathbb{R})$ with $n = 2, 3$. Polynomials with majorants are also considered. In Chap. 3 we study several spaces of trinomials including $\mathcal{P}_{m,n}(\mathbb{R})$ and $\mathcal{P}^h_{m,n}(\mathbb{R}^2)$ $(m > n)$ defined above, but also other related problems. Trinomials with the L_p-norm or trinomials on the complex plane are studied as well. In Chap. 4 we consider several polynomial spaces where the norm is calculated as the supremum over a non-symmetric convex body. In particular, Chap. 4 comprises the spaces $\mathcal{P}(^2\Delta)$, $\mathcal{P}(^2\square)$ and $\mathcal{P}(^2D(\alpha,\beta))$. In Chap. 5 we treat the case of polynomials defined on several ℓ_p-spaces. More specifically we investigate the spaces $\mathcal{P}(^2\ell_p^2)$ for all $p \in [1,\infty]$ and the spaces of quadratic forms on c_0, ℓ_1 and ℓ_2 for $p > 2$. In Chap. 6 we consider the space of quadratic forms in $\mathbb{R}^2$ with the sup norm over an octagon, represented as $\mathcal{P}(^2O_w^2)$ above, and with the sup norm over the hexagon defined by

$$\|(x, y)\|_{\text{hex}(w)} := \max\{|y|, |x| + (1-w)|y|\} = 1$$

for $w \in [0, 1]$. In Chap. 7 we study polynomials on real or complex Hilbert spaces. In Chap. 8 extreme integral, nuclear or orthogonally additive polynomials are regarded. Finally, in Chap. 9 we gather a number of applications of the geometrical results included in Chaps. 2–8.

Chapter 2
Polynomials of Degree *n*

Abstract This chapter focuses on the study of the geometry of the unit ball of the space of polynomials in one variable of degree at most $n \in \mathbb{N}$ endowed with the supremum norm defined on the interval $[-1, 1]$ (when the polynomial is defined over $\mathbb{R}$) or on the unit disk (when the polynomial is defined over $\mathbb{C}$). More precisely, we are interested on the parametrization of the unit ball as well as the extreme points when we are dealing with the space of polynomials of degree at most 2. For the space of polynomials of arbitrary degree with the supremum norm defined on $[-1, 1]$, we are only interested on the extreme polynomials of the unit ball.

2.1 On the Real Line

Let us endow the vector space of real polynomials of the degree at most $n \in \mathbb{N}$, that is, of the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$ where $a_i \in \mathbb{R}$ for every $i \in \{1, \ldots, n\}$ and $x \in \mathbb{R}$, with the supremum norm

$$\|P\|_{\mathbb{R}} = \max\{|P(x)| : x \in [-1, 1]\}.$$

We denote this normed space by $\mathcal{P}_n(\mathbb{R})$. Now consider the following construction: let us define the mapping T from $\mathcal{P}_n(\mathbb{R})$ to $\mathbb{R}^{n+1}$ that assigns to each polynomial $a_n x^n + \cdots + a_1 x + a_0$ the vector $(a_n, \ldots, a_1, a_0)$, i.e., each polynomial is mapped into the vector formed by its coefficients. This mapping T is a topological isomorphism between $\mathcal{P}_n(\mathbb{R})$ and $\mathbb{R}^{n+1}$ when we endow $\mathbb{R}^{n+1}$ with the norm

$$\|(a_n, \ldots, a_1, a_0)\|_{\mathbb{R}} := \|a_n x^n + \cdots + a_1 x + a_0\|_{\mathbb{R}}.$$

Let us denote the unit ball and the unit sphere of $(\mathbb{R}^{n+1}, \|\cdot\|_{\mathbb{R}})$ by $\mathsf{B}_n(\mathbb{R})$ and $\mathsf{S}_n(\mathbb{R})$, respectively. Thus, in particular, on the space $\mathcal{P}_2(\mathbb{R})$, we can give a visual representation of the unit ball.

The geometry of $\mathcal{P}_n(\mathbb{R})$ was already studied by A. G. Konheim and T. J. Rivlin in 1966 [40]. They were able to characterize when a polynomial of degree at most $n \in$

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_2

$\mathbb{N}$ that belongs to the unit ball is an extreme polynomial based on the multiplicity of intersection of the polynomial with 1 and -1.

Definition 2.1 Let P be a real polynomial of degree at most n. We denote by $N(P, y)$ the total multiplicity with which the value y is assumed by P and, in particular, let us define the multiplicity of P by the number $N(P) := N(P, 1) + N(P, -1)$.

Theorem 2.1 (Konheim and Rivlin [40]) *Let $P \in \mathcal{P}_n(\mathbb{R})$ with $\|P\| \leq 1$. We have that P is an extreme polynomial if, and only if, $N(P) > n$.*

Although Konheim and Rivlin gave a characterization of the extreme polynomials of the unit ball of $\mathcal{P}_n(\mathbb{R})$, they do not give an explicit formula for the values of the extreme polynomials. However, R. M. Aron and M. Klimek [5] were able to obtain an explicit formula for the extreme polynomials in the unit ball of $\mathcal{P}_2(\mathbb{R})$ by using an approach that will appear in many results of this survey. Firstly, they gave an explicit formula for the norm of a polynomial of degree at most 2. Secondly, they found the projection of the unit ball onto a plane. And finally, using this information, they were able to parametrize the unit ball and, in the process, find the extreme polynomials of the unit ball. The results that Aron and Klimek provided are shown below.

Theorem 2.2 (Aron and Klimek [5]) *Let $P(x) = ax^2 + bx + c$. We have*

$$\|(a, b, c)\|_{\mathbb{R}} = \begin{cases} \left|\frac{b^2}{4a} - c\right| & \text{if } |b| < 2|a|t \text{ and } \frac{c}{a} + 1 < \frac{1}{2}\left(\left|\frac{b}{2a}\right| - 1\right)^2, \\ |a + c| + |b| & \text{otherwise.} \end{cases}$$

Let us define the sets

$$U = \left\{(a, b) \in \mathbb{R}^2 \colon a \leq 0 \text{ and } |b| \leq \min\left\{2|a|, 2\left(\sqrt{2|a|} - |a|\right)\right\}\right\},$$
$$V = \left\{(a, b) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1] \colon |b| \geq 2|a|\right\},$$
$$W = \left\{(a, b) \in \mathbb{R}^2 \colon a \geq 0 \text{ and } |b| \leq \min\left\{2|a|, 2\left(\sqrt{2|a|} - |a|\right)\right\}\right\}.$$

Theorem 2.3 (Aron and Klimek [5]) *The projection of $\mathsf{B}_2(\mathbb{R})$ onto the ab-plane is the set $U \cup V \cup W$ (see Fig. 3.5 for a representation of $U \cup V \cup W$ with $n = 1$).*

Theorem 2.4 (Aron and Klimek [5]) *Let us define the functions*

$$f_+(a, b) = 1 - |b| - |a|,$$
$$g_+(a, b) = \frac{b^2}{4a} - 1,$$

and also the functions $f_-(a, b) = -f_+(-a, b)$ and $g_-(a, b) = -g_+(-a, b)$. We have *(i)* $\mathsf{S}_2(\mathbb{R}) = \text{graph}\left(f_+|_{(V\cup W)}\right) \cup \text{graph}\left(f_-|_{(U\cup V)}\right) \cup \text{graph}\,(g_+|_W) \cup \text{graph}\,(g_-|_U)$ *(see Fig. 3.6 for a representation of* $\mathsf{B}_2(\mathbb{R})$ *with* $n = 1$*).*

(ii) The set of extreme points (denoted by ext*) is*

$$\text{ext}\,(\mathsf{B}_2(\mathbb{R})) = \left\{\pm\left(t, \pm 2(\sqrt{2t} - t), 1 + t - 2\sqrt{2t}\right) : t \in \left[\frac{1}{2}, 2\right]\right\} \bigcup \{\pm(0, 0, 1)\}.$$

The following results of this section are devoted to the study of extreme polynomials of degree at most 3.

Theorem 2.5 (Araújo et al. [4]) *The extreme polynomials of the unit ball of* $\mathcal{P}_3(\mathbb{R})$ *are given by*

(i) $P_1(x) = \pm 1$*;*
(ii) $P_2(x) = \pm\left[1 - \frac{1}{4}(\pm x + 1)^3\right]$*;*
(iii) $P_3(x) = \pm(2x^2 - 1)$*;*
(iv) $P_4(x) = \pm\left[1 - \frac{1}{(1-q^2)^2}(x - q)^2(4qx + 2 + 2q^2)\right]$ *and* $P_5(x) = \pm\left[1 + \frac{1}{(1-q^2)^2}(x + q)^2(4qx - 2 - 2q^2)\right]$*, for every* $q \in \left(-\frac{1}{3}, 0\right)$*;*
(v) $P_6(x) = \pm\left[1 + \frac{1}{(1+t)^2}(x - t)^2(x - 1)\right]$ *and* $P_7(x) = \pm\left[1 - \frac{1}{(1+t)^2}(x + t)^2(x + 1)\right]$*, for every* $t \in \left(-\frac{1}{2}, 1\right)$*;*
(vi) $P_8(x) = \pm\left[1 + \frac{4}{(s-r)^3}(x - r)^2\left(x - \frac{3s-r}{2}\right)\right]$ *and* $P_9(x) = \pm\left[1 - \frac{4}{(s-r)^3}(x + r)^2\left(x + \frac{3s-r}{2}\right)\right]$*, for every* $-1 \le r < s \le 1$ *such that* $s \ge \min\left\{3r + 2, \frac{r+2}{3}\right\}$.

2.1.1 Polynomials Bounded by a Majorant

Assume that P is a polynomial of degree at most n such that P is constrained on the interval $[-1, 1]$ by a mapping $\varphi\colon [-1, 1] \to [0, +\infty)$ called the majorant, i.e., $|P(x)| \le \varphi(x)$ for every $x \in [-1, 1]$. We will denote by $\mathcal{P}_n^{\varphi}(\mathbb{R})$ the space of polynomials on the real line of degree at most n that are bounded by a majorant φ endowed with the supremum norm over the interval $[-1, 1]$. In this section we are interested in studying the extreme points of the unit ball of the space $\mathcal{P}_3^{\varphi}(\mathbb{R})$ when φ is a circular majorant, that is, $\varphi(x) = \sqrt{1 - x^2}$ for any $x \in [-1, 1]$.

Notice that if a polynomial P belongs to $\mathcal{P}_3^{\varphi}(\mathbb{R})$, where φ is a circular majorant, then P has roots at ± 1. Hence all polynomials of degree not greater than 3 bounded by a circular majorant are of the form $P_{a,b}(x) = (1-x^2)(ax+b)$ for some $a, b \in \mathbb{R}$. Thus, in fact, we have the following inequality $|(1-x^2)(ax+b)| \le \sqrt{1 - x^2}$ for any

$x \in [-1,1]$, which is equivalent to $\left|\sqrt{1-x^2}(ax+b)\right| \le 1$ for any $x \in [-1,1]$. The latter shows that we can study the unit ball of the space $\mathcal{P}_3^{\varphi}(\mathbb{R})$, when φ is a circular majorant, by studying the unit ball of the norm space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$, where

$$\|(a,b)\|_{\infty,\varphi} = \sup\left\{\left|\sqrt{1-x^2}(ax+b)\right| : x \in [-1,1]\right\}.$$

We begin by showing an explicit formula for the norm $\|\cdot\|_{\infty,\varphi}$.

Theorem 2.6 (Muñoz et al. [47]) *If $\varphi\colon [-1,1] \to [0,+\infty)$ is defined by $\varphi(x) = \sqrt{1-x^2}$, then for every $(a,b) \in \mathbb{R}^2$ we have*

$$\|(a,b)\|_{\infty,\varphi} = \begin{cases} \frac{(3|b|+\sqrt{8a^2+b^2})\sqrt{4a^2-b^2+|b|\sqrt{8a^2+b^2}}}{8\sqrt{2}|a|} & \text{if } a \neq 0, \\ |b| & \text{if } a = 0. \end{cases}$$

As an easy consequence of Theorem 2.6 we have the following characterization of the unit ball of $\mathcal{P}_3^{\varphi}(\mathbb{R})$.

Theorem 2.7 (Muñoz et al. [47]) *Let $\varphi\colon [-1,1] \to [0,+\infty)$ be defined by $\varphi(x) = \sqrt{1-x^2}$. If $(a,b) \in \mathbb{R}^2$, then $\|(a,b)\|_{\infty,\varphi} \le 1$ if, and only if,*

$$\left(\sqrt{8a^2+b^2}+3|b|\right)^3 \le 32\left(\sqrt{8a^2+b^2}+|b|\right),$$

where equality is satisfied if, and only if, $\|(a,b)\|_{\infty,\varphi} = 1$. Moreover, the set of extreme points of the unit ball of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$ are the points of the unit sphere.

Figure 2.1 shows an approximate representation of the unit sphere of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$.

2.2 On the Complex Plane

Let us consider now the vector space of complex polynomials with real coefficients of degree at most $n \in \mathbb{N}$, that is, we have polynomials of the form $P(z) = a_n z^n + \ldots + a_1 z + a_0$ where $a_i \in \mathbb{R}$ and $z \in \mathbb{C}$, endowed with the following norm

$$\|P\|_{\mathbb{C}} = \sup_{|z|\le 1} |P(z)|.$$

We denote this normed space by $\mathcal{P}_{\mathbb{R},n}(\mathbb{C})$. Using the mapping T defined on Sect. 2.1, there is a topological isomorphism between the space $\mathcal{P}_{\mathbb{R},n}(\mathbb{C})$ and $\mathbb{R}^{n+1}$ endowed with the norm

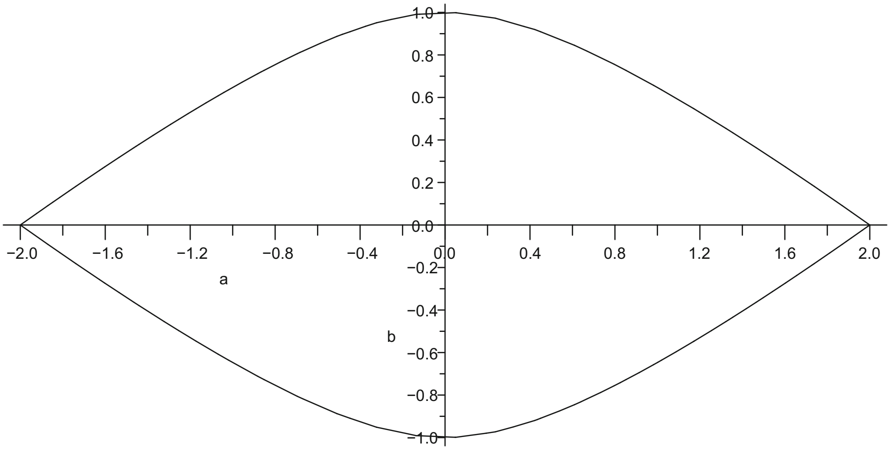


Fig. 2.1 Unit sphere of the space $(\mathbb{R}^2, \|\cdot\|_{\infty,\varphi})$

$$\|(a_n, \ldots, a_1, a_0)\|_{\mathbb{C}} = \|a_n z^n + \ldots + a_1 z + a_0\|_{\mathbb{C}}.$$

Let us denote the unit ball and the unit sphere of $(\mathbb{R}^{n+1}, \|\cdot\|_{\mathbb{C}})$ by $\mathsf{B}_{\mathbb{R},n}(\mathbb{C})$ and $\mathsf{S}_{\mathbb{R},n}(\mathbb{C})$, respectively. Furthermore, we can a give a visual representation of the unit ball of the space $\mathcal{P}_{\mathbb{R},2}(\mathbb{C})$ on $\mathbb{R}^3$. We use the same approach as in the previous section. We begin by showing an explicit formula for the norm of the space $\mathcal{P}_{\mathbb{R},2}(\mathbb{C})$.

Theorem 2.8 (Aron and Klimek [5]) *If $P(z) = az^2 + bz + c \in \mathcal{P}_{\mathbb{R},2}(\mathbb{C})$, then*

$$\|(a, b, c)\|_{\mathbb{C}} = \begin{cases} |a+c| + |b| & \text{if } ac \geq 0 \text{ or } |b(a+c)| > 4|ac|, \\ (|a| + |c|)\sqrt{1 + \frac{b^2}{4|ac|}} & \text{otherwise.} \end{cases}$$

We continue by showing the projection of the unit ball onto a coordinate plane. To do so, we define the following sets

$$A = \left\{(a, c) \in \mathbb{R}^2 \colon |a| + |c| \leq 1 \text{ and } |a+c| \leq (|a| + |c|)^2\right\},$$

$$B = \left\{(a, c) \in \mathbb{R}^2 \colon |a| + |c| \leq 1 \text{ and } |a+c| > (|a| + |c|)^2\right\}.$$

Figure 2.2 shows a representation of A and B.

Theorem 2.9 (Aron and Klimek [5]) *The projection of $\mathsf{B}_{\mathbb{R},2}(\mathbb{C})$ onto the ac-plane is the set $A \cup B$.*

Finally, we show a parametrization of $\mathsf{S}_{\mathbb{R},2}(\mathbb{C})$ as well as the extreme points of $\mathsf{B}_{\mathbb{R},2}(\mathbb{C})$.

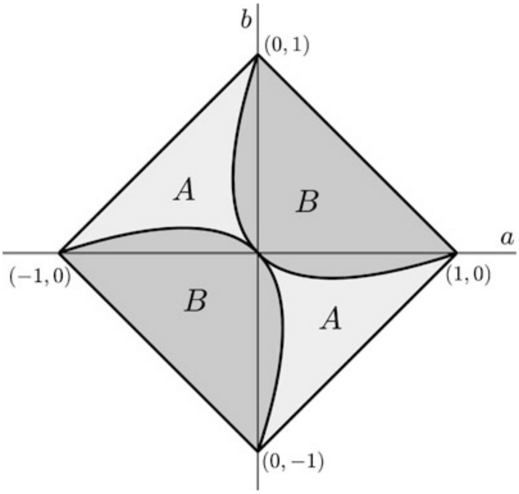


Fig. 2.2 Projection of $\mathsf{B}_{\mathbb{R},2}(\mathbb{C})$ onto the ac-plane

Theorem 2.10 (Aron and Klimek [5]) *Let us define the function*

$$f(a,c) = \begin{cases} \sqrt{4|ac|\left(\frac{1}{(|a|+|c|)^2} - 1\right)} & \text{if } (a,c) \in A, \\ 1 - |a+c| & \text{if } (a,c) \in B. \end{cases}$$

We have

(i) $\mathsf{S}_{\mathbb{R},2}(\mathbb{C}) = \operatorname{graph}(f) \cup \operatorname{graph}(-f)$.
(ii)

$$\operatorname{ext}(\mathsf{B}_{\mathbb{R},2}(\mathbb{C})) = \left\{ \left(a, \pm\sqrt{4|ac|\left(\frac{1}{(|a|+|c|)^2} - 1\right)}, c\right) : \right.$$
$$\left. a, c \neq 0,\ |a| + |c| < 1 \text{ and } |a+c| \leq (|a|+|c|)^2 \right\}.$$

Chapter 3
Spaces of Trinomials

Abstract A trinomial is a polynomial that consists of three monomials. This chapter is about studying the geometry of the normed space of trinomials on different scenarios. To be more precise, we will study the geometry of the space of real trinomials in one variable with the supremum norm and the L^p norm, the space of real trinomials in two variables with the supremum norm and finally the space of complex trinomials with the supremum norm.

3.1 On the Real Line with the Supremum Norm

We are using trinomials of the form $ax^m + bx^n + c$ with $m, n \in \mathbb{N}$, $m > n$ and $a, b, c \in \mathbb{R}$. Let $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ denote the vector space of trinomials of the previous form endowed with the supremum norm on the unit interval $[-1, 1]$ where $m, n \in \mathbb{N}$ with $m > n$. Notice that the space $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ is a 3-dimensional space because the set $\{x^m, x^n, 1\}$ is a basis of $\mathcal{P}_{m,n,\infty}(\mathbb{R})$.

Now consider the mapping T (defined in Chap. 2) from $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ to $\mathbb{R}^3$. This mapping T is a topological isomorphism between $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ and $\mathbb{R}^3$ endowed with the norm

$$\|(a, b, c)\|_{m,n,\infty} := \max\{\|ax^m + bx^n + c\| : x \in [-1, 1]\}.$$

Therefore we can give a geometrical representation of the unit ball of the space $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ in $\mathbb{R}^3$.

To do so, we begin by showing an explicit formula for the norm $\|\cdot\|_{m,n,\infty}$. The explicit formula of $\|\cdot\|_{m,n,\infty}$ depends on the four different kinds of parity of $m, n \in \mathbb{N}$ which will be treated separately. This formula is used to obtain a parametrization of the unit sphere and, therefore, a sketch of the unit sphere of the normed space $(\mathbb{R}^3, \|\cdot\|_{m,n,\infty})$. We denote by $\mathsf{S}_{m,n,\infty}$ and $\mathsf{B}_{m,n,\infty}$ the unit sphere and unit ball of the space $(\mathbb{R}^3, \|\cdot\|_{m,n,\infty})$, respectively.

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_3

3.1.1 The Geometry of $\mathsf{B}_{m,n,\infty}$ for Odd Numbers m, n

In this case, to illustrate in a better way how this approach of finding a parametrization and the extreme polynomials of the unit sphere is done, we provide the complete proofs of all the results that are needed. On the other hand, many results of this paper follow the same pattern and, therefore, most of the time the proofs will be omitted. In order to show the formula for the norm, we begin by proving the following lemma.

Lemma 3.1 (Muñoz and Seoane [50]) *If $m, n \in \mathbb{N}$ with $m > n$ odd, then the equation*

$$|n + my| = (m - n)|y|^{\frac{m}{m-n}}$$

has only three solutions, one is $y = -1$, another is at a point $\lambda_0 \in \left(-\frac{n}{m}, 0\right)$ and the last one at a point $\lambda_1 > 0$. Furthermore, the inequality

$$|n + my| < (m - n)|y|^{\frac{m}{m-n}}$$

is satisfied if, and only if, $y \in (-\infty, \lambda_0) \cup (\lambda_1, \infty)$.

Proof Let us define the functions $f(y) = |n + my|$ and $g(y) = (m - n)|y|^{\frac{m}{m-n}}$ for every $y \in \mathbb{R}$. Notice that the function f consists of two straight lines and g is a convex function. Therefore f and g intersect in at most four points. First of all, since $m > n$, if $y = -1$, then $f(-1) = g(-1)$. Also, it is easy to see that $f\left(-\frac{n}{m}\right) = 0$, $f(0) > 0$, $g\left(-\frac{n}{m}\right) > 0$ and $g(0) = 0$, thus $f\left(-\frac{n}{m}\right) < g\left(-\frac{n}{m}\right)$ and $f(0) > g(0)$. This implies by continuity that there exists $\lambda_0 \in \left(-\frac{n}{m}, 0\right)$ such that $f(\lambda_0) = g(\lambda_0)$. On the other hand, as $\lim_{y\to\infty}(g(y) - f(y)) = \infty$ and $f(0) > g(0)$ we have that there exists $\lambda_1 > 0$ such that $f(\lambda_1) = g(\lambda_1)$. The inequality $f(y) < g(y)$ on the set $(-\infty, \lambda_0) \cup (\lambda_1, \infty)$ follows from the strict convexity of g. □

Theorem 3.1 (Muñoz and Seoane [50]) *If $m, n \in \mathbb{N}$ are odd with $m > n$, then*

$$\|(a, b, c)\|_{m,n,\infty} = \begin{cases} \frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}} + |c| & \text{if } a \neq 0 \text{ and } -1 < \frac{nb}{ma} < \lambda_0, \\ |a + b| + |c| & \text{otherwise,} \end{cases}$$

where λ_0 is one of the three roots of the equation

$$|n + my| = (m - n)|y|^{\frac{m}{m-n}}$$

such that $-\frac{n}{m} < \lambda_0 < 0$, with the other two roots at $\lambda_1 = -1$ and $\lambda_2 > 0$.

Remark 3.1 (Muñoz and Seoane [50]) Notice that by the definition of the norm $\|\cdot\|_{m,n,\infty}$ and the assumption that m and n are odd natural numbers, then we have

$$\|(a, b, c)\|_{m,n,\infty} = \|(a, b, 0)\|_{m,n,\infty} + |c|$$

for all $a, b, c \in \mathbb{R}$. This means that the norm $\|\cdot\|_{m,n,\infty}$ is symmetric with respect to the ab-plane. However in the following cases related to the parity of m and n, the property of being symmetric to a certain plane is not always satisfied (for example, when m and n are even, there is no symmetry). But if m and n have different parity, then we have symmetry with respect to some coordinate plane.

Proof Let $(a, b, c) \in \mathbb{R}^3$ and take $P(x) = ax^m + bx^n$. By Remark 3.1 it suffices to prove that

$$\|(a, b, 0)\|_{m,n,\infty} = \begin{cases} \frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}} & \text{if } a \neq 0 \text{ and } -1 < \frac{nb}{ma} < \lambda_0, \\ |a+b| & \text{otherwise.} \end{cases}$$

Notice that the polynomial P is symmetric with respect to the origin, which implies that $\|P\|_{m,n,\infty} = \max_{x\in[-1,0]} |P(x)|$. Furthermore, since $P(0) = 0$, the latter maximum is attained either at -1 or at a critical point of P in the interval $(-1, 0)$. We are looking now for the critical points of the polynomial P. To do so, we solve the equation $P'(x) = 0$, that is, since $m > n$, we solve the equation

$$P'(x) = amx^{m-1} + bnx^{n-1} = x^{n-1}(amx^{m-n} + bn) = 0. \tag{3.1}$$

It can be easily seen that Eq. (3.1) has, at most, one solution in the interval $(-1, 0)$ which is $\overline{x} = -\left|\frac{nb}{ma}\right|^{\frac{1}{m-n}}$ provided that $a \neq 0$ and $-1 < \frac{nb}{ma} < 0$. Hence, P has a critical point in the interval $(-1, 0)$ at $\overline{x} = -\left|\frac{nb}{ma}\right|^{\frac{1}{m-n}}$ provided that $a \neq 0$ and $-1 < \frac{nb}{ma} < 0$, which implies that

$$\begin{aligned} &\|(a, b, 0)\|_{m,n,\infty} \\ &\quad= \begin{cases} \max\{|P(1)|, |P(\overline{x})|\} & \text{if } a \neq 0 \text{ and } -1 < \frac{nb}{ma} < 0, \\ |P(1)| & \text{otherwise,} \end{cases} \\ &\quad= \begin{cases} \max\left\{|a+b|, \frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}}\right\} & \text{if } a \neq 0 \text{ and } -1 < \frac{nb}{ma} < 0, \\ |a+b| & \text{otherwise.} \end{cases} \end{aligned}$$

It suffices to show when $|a + b| < \frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}}$ provided that $a \neq 0$ and $-1 < \frac{nb}{ma} < 0$. If $a \neq 0$, then by multiplying inequality $|a+b| < \frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}}$ by $\frac{n}{a}$ we have, equivalently, that $\left|n + \frac{nb}{a}\right| < (m-n)\left|\frac{nb}{ma}\right|^{\frac{m}{m-n}}$, which can be seen as $|n+my| < (m-n)|y|^{\frac{m}{m-n}}$, where $y = \frac{nb}{ma}$. Since $-1 < y = \frac{nb}{ma} < 0$, by Lemma 3.1 we have that $-1 < \frac{nb}{ma} < \lambda_0$ and therefore

$$\|(a,b,0)\|_{m,n,\infty} = \begin{cases} \frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}} & \text{if } a \neq 0 \text{ and } -1 < \frac{nb}{ma} < \lambda_0, \\ |a+b| & \text{otherwise,} \end{cases}$$

which finishes the proof. □

We know by Remark 3.1 that $\mathsf{B}_{m,n,\infty}$ is symmetric with respect to the ab-plane. Hence the projection of $\mathsf{B}_{m,n,\infty}$ will be onto the ab-plane. Let us define the function $\Gamma(a) = \frac{m}{(m-n)^{\frac{m-n}{m}} n^{\frac{n}{m}}} \cdot |a|^{\frac{n}{m}}$ and the sets

$$V = \left\{(a,b) \in \mathbb{R}^2 : a \neq 0,\ -1 \leq \frac{nb}{ma} \leq \lambda_0 \text{ and } |b| \leq \Gamma(a)\right\},$$

$$W_1 = \left\{(a,b) \in \mathbb{R}^2 : b \geq -\frac{m}{n}a,\ b \geq \lambda_0 \frac{ma}{n} \text{ and } b \leq 1-a\right\},$$

$$W_2 = \left\{(a,b) \in \mathbb{R}^2 : b \leq -\frac{m}{n}a,\ b \leq \lambda_0 \frac{ma}{n} \text{ and } b \geq -1-a\right\},$$

$$W = W_1 \cup W_2.$$

Figure 3.1 shows a representation of V and W when $m = 3$ and $n = 1$.

Theorem 3.2 (Muñoz and Seoane [50]) *If $m, n \in \mathbb{N}$ are odd with $m > n$, then the projection onto the ab-plane of $\mathsf{B}_{m,n,\infty}$ is the set $V \cup W$.*

Proof By Theorem 3.1, notice that $\|(a,b,c)\|_{m,n,\infty} = \|(a,b,-c)\|_{m,n,\infty}$ for every $a, b, c \in \mathbb{R}$. This symmetric property shows that the projection of the unit ball onto the ab-plane is just the intersection of the unit ball with the ab-plane, i.e., the projection is the set $\{(a,b) \in \mathbb{R}^2 \colon \|(a,b,0)\|_{m,n,\infty} \leq 1\}$ and, furthermore, the projection is bounded by the curved defined implicitly by $\|(a,b,0)\|_{m,n,\infty} = 1$. Thus, if $\|(a,b,0)\|_{m,n,\infty} = 1$, $a \neq 0$ and $-1 < \frac{nb}{ma} < \lambda_0$, then $\frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}} = 1$. Solving b in terms of a in the latter we have $b = \pm\Gamma(a)$ and, in particular, using the restrictions on a and b we see that

$$b = \Gamma(a) \text{ provided } a \in \left(-\frac{n}{m+m\lambda_0}, -\frac{n}{m-n}\right),$$

and

$$b = -\Gamma(a) \text{ provided } a \in \left(\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right),$$

where the limits in the first and second interval are obtained by intersecting the straight lines $b = -\frac{m}{n}a$ and $b = \frac{m\lambda_0}{n}a$ with $b = -1-a$ and $b = 1-a$.

On the other hand, if $\|(a,b,0)\|_{m,n,\infty} = 1$ and either $a = 0$ or $\frac{nb}{ma} \in (-\infty, -1] \cup [\lambda_0, \infty)$, then $|a+b| = 1$. Now we have that $b = \pm 1 - a$ and, furthermore, using

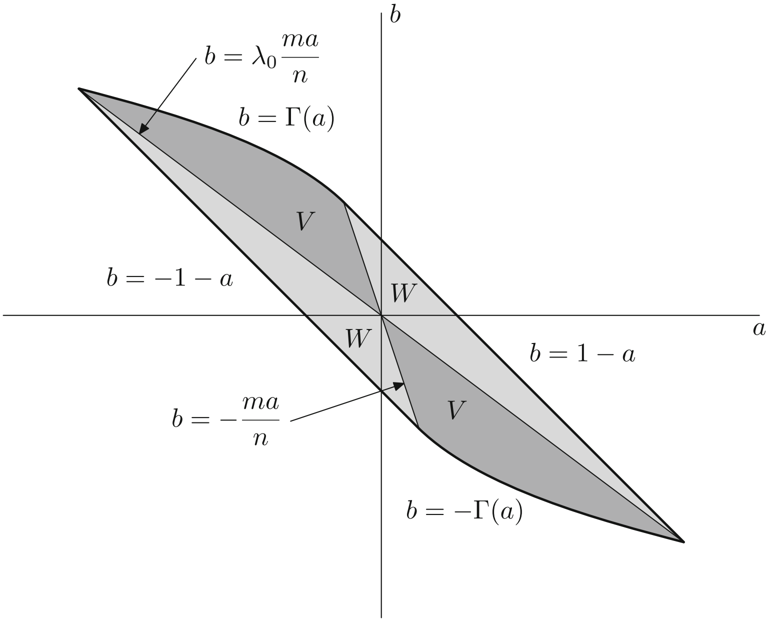


Fig. 3.1 Projection of $\mathsf{B}_{3,1,\infty}$ onto the ab-plane. The general case when $m, n \in \mathbb{N}$ are odd with $m > n$ is of similar shape

the restrictions on a and b we see that

$$b = -1 - a \text{ provided } a \in \left[-\frac{n}{n + m\lambda_0}, \frac{n}{m - n}\right],$$

or

$$b = 1 - a \text{ provided } a \in \left[-\frac{n}{m - n}, \frac{n}{n + m\lambda_0}\right],$$

where the limits in the intervals are obtained in the same way as before. This completes the proof. □

So far we have given an explicit formula for the norm $\| \cdot \|_{m,n,\infty}$ and a projection of $\mathsf{B}_{m,n,\infty}$ onto the ab-plane. Now we are capable of finding a parametrization of $\mathsf{S}_{m,n,\infty}$ and also we can obtain the values of the extreme polynomials of $\mathsf{B}_{m,n,\infty}$.

For the following theorem, it is very useful to know a more general result on real normed spaces that states the following.

Lemma 3.2 (Muñoz and Seoane [50]) *Let E be a real normed space with norm $\| \cdot \|_E$ and define $\widetilde{E} = E \oplus \mathbb{R}$ as the space of pairs $(x, \lambda) \in E \times \mathbb{R}$ endowed with the norm given by $\|(x, \lambda)\|_{\widetilde{E}} = \|x\|_E + |\lambda|$. Then, if $f_+(x) = 1 - \|x\|_E$ for $x \in E$ and $f_- = -f_+$, we have*

(i) $\mathsf{S}_{\widetilde{E}} = \mathrm{graph}\left(f_+|_{\mathsf{B}_E}\right) \bigcup \mathrm{graph}\left(f_-|_{\mathsf{B}_E}\right)$.
(ii) $\mathrm{ext}\left(\mathsf{B}_{\widetilde{E}}\right) = \{(x, 0) : x \in \mathrm{ext}(\mathsf{B}_E)\} \bigcup \{\pm(\mathbf{0}, 1)\}$, *where* $\mathbf{0}$ *denotes the null vector in* E.

Proof It is easy to see part (i) of the theorem. On the other hand, for part (ii), notice that the graphs of f_+ and f_- are affine on half-lines coming from the origin in E. Thus, the extreme points of the unit ball of $\widetilde{E}$ are inside the set of points where the graphs of the functions f_+ and f_- intersect, i.e., the extreme points are either in the set $\{(x, 0) \colon x \in \mathsf{S}_E\}$ or they are one of the points $\pm(\mathbf{0}, 1)$. It is easy to prove that if $x \notin \mathrm{ext}(\mathsf{B}_E)$, then $(x, 0) \notin \mathsf{B}_{\widetilde{E}}$. Hence, it is enough to prove now that $\mathrm{ext}(\mathsf{B}_{\widetilde{E}}) = \{(x, 0) \colon x \in \mathrm{ext}(\mathsf{B}_E)\} \cup \{\pm(\mathbf{0}, 1)\}$. Firstly, since the hyperplane $M = \{(x, \pm 1) \colon x \in E\}$ intersects the unit ball of $\widetilde{E}$ at $\pm(\mathbf{0}, 1)$, we have that the points $\pm(\mathbf{0}, 1)$ are extreme points of $\mathsf{B}_{\widetilde{E}}$. Indeed, if $(y, \pm 1) \in M \cap \mathsf{B}_{\widetilde{E}}$, then $1 \leq \|(y, \pm 1)\|_{\widetilde{E}} = \|y\|_E + 1$, which implies that $y = 0$. Lastly, if $x \in \mathrm{ext}(\mathsf{B}_E)$, then $(x, 0)$ is an extreme point of $\mathsf{B}_{\widetilde{E}}$ by definition of extreme point. □

Theorem 3.3 (Muñoz and Seoane [50]) *Let* $m, n \in \mathbb{N}$ *be odd with* $m > n$. *If for every* $(a, b) \in \mathbb{R}^2$ *we define*

$$f_+(a, b) = \begin{cases} 1 - \frac{(m-n)|a|}{n} \cdot \left|\frac{nb}{ma}\right|^{\frac{m}{m-n}} & \text{if } a \neq 0 \text{ and } -1 < \frac{nb}{ma} < \lambda_0, \\ 1 - |a + b| & \text{otherwise}, \end{cases}$$

and $f_- = -f_+$ *(notice that* $f_+(a, b) = 1 - \|(a, b, 0)\|_{m,n,\infty}$*), then*

(i) $\mathsf{S}_{m,n,\infty} = \mathrm{graph}\left(f_+|_V\right) \bigcup \mathrm{graph}\left(f_-|_W\right)$.
(ii)

$$\mathrm{ext}\left(\mathsf{B}_{m,n,\infty}\right) = \left\{\pm\left(t, -\frac{m}{(m-n)^{\frac{m-n}{m}} n^{\frac{n}{m}}} \cdot t^{\frac{n}{m}}, 0\right) : \frac{n}{m-n} \leq t \leq \frac{n}{n + m\lambda_0}\right\}$$
$$\bigcup \{\pm(0, 0, 1)\}.$$

Proof Let us endow the space $E = \mathbb{R}^2$ with the following norm: $\|(a, b)\| = \|(a, b, 0)\|_{m,n,\infty}$ for every $(a, b) \in E$. It is straightforward to prove that the unit ball of E is the projection of $\mathsf{B}_{m,n,\infty}$ onto the ab-plane and the set of extreme of points of $\mathsf{B}_{m,n,\infty}$ is the set $\left\{\pm\left(t, -\Gamma(t)\right) : \frac{n}{m-n} \leq t \leq \frac{n}{n+m\lambda_0}\right\}$. Thus, by Lemma 3.2 applied to the normed space E and the fact that $\|(a, b, c)\|_{m,n,\infty} = \|(a, b, 0)\|_{m,n,\infty} + |c|$ for every $(a, b, c) \in \mathbb{R}^3$, we have the desired result. □

Figure 3.2 is a sketch of $\mathsf{B}_{m,n,\infty}$ when $m = 3$ and $n = 1$.

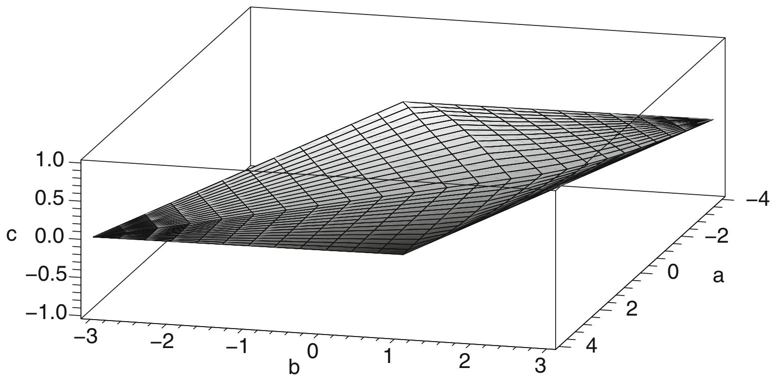


Fig. 3.2 Unit ball of $\mathcal{P}_{3,1,\infty}(\mathbb{R})$. In the general case the unit ball of $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ with $m, n \in \mathbb{N}$ odd and $m > n$ has a similar shape

3.1.2 The Geometry of $\mathsf{B}_{m,n,\infty}$ for m Odd and n Even

In this case we consider the space $\mathbb{R}^3$ endowed with the norm $\|\cdot\|_{m,n,\infty}$ where $m \in \mathbb{N}$ is odd and $n \in \mathbb{N}$ is even. As in the previous case we begin by showing an explicit formula for the norm $\|\cdot\|_{m,n,\infty}$.

Theorem 3.4 (Muñoz and Seoane [50]) *Let $m, n \in \mathbb{N}$ with m odd, n even and $m > n$. For every $(a, b, c) \in \mathbb{R}^3$ we have*

$$\|(a, b, c)\|_{m,n,\infty} = \max\{|c|, |a| + |b + c|\},$$

that is

$$\|(a, b, c)\|_{m,n,\infty} = \begin{cases} |c| & \text{if } c \neq 0 \text{ and } \left|\frac{b}{c} + 1\right| \leq 1 - \left|\frac{a}{c}\right|, \\ |a| + |b + c| & \text{otherwise.} \end{cases} \tag{3.2}$$

In this case, $\mathsf{B}_{m,n,\infty}$ is symmetric with respect to the bc-plane. But instead of projecting $\mathsf{B}_{m,n,\infty}$ onto the bc-plane (in contrast with the argument used in the previous case), we are going to project $\mathsf{B}_{m,n,\infty}$ onto the ab-plane.

Let us define the sets

$$U = \{(a, b) \in \mathbb{R}^2 \colon |a| + |b + 1| \leq 1\},$$
$$V = \{(a, b) \in \mathbb{R}^2 \colon |b| \leq |a| \leq 1\}, \text{ and}$$
$$W = \{(a, b) \in \mathbb{R}^2 \colon |a| + |b - 1| \leq 1\}.$$

Figure 3.3 shows what U, V and W look like when $m = 3$ and $n = 2$.

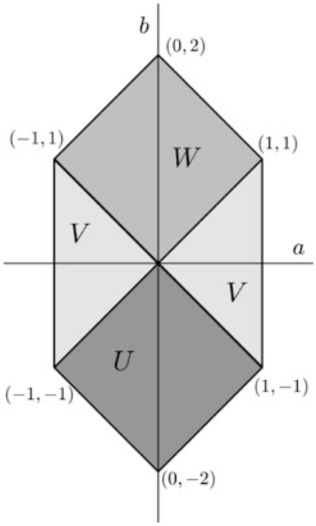


Fig. 3.3 Projection of $\mathsf{B}_{3,2,\infty}$ onto the ab-plane. The general case when m is odd and n is even has a similar form

Theorem 3.5 (Muñoz and Seoane [50]) *Let $m, n \in \mathbb{N}$ with m odd, n even and $m > n$. The projection of $\mathsf{B}_{m,n,\infty}$ onto the ab-plane is the set $U \cup V \cup W$.*

We have an explicit formula of the norm $\| \cdot \|_{m,n,\infty}$ and we know the projection of $\mathsf{B}_{m,n,\infty}$ onto the ab-plane. Now, we can find a parametrization of $\mathsf{S}_{m,n,\infty}$ and obtain the extreme points of $\mathsf{B}_{m,n,\infty}$.

Theorem 3.6 (Muñoz and Seoane [50]) *Let $m, n \in \mathbb{N}$ with m odd, n even and $m > n$. For every $(a, b) \in \mathbb{R}^2$, we define*

$$f_+(a, b) = 1 - |a| - b,$$
$$g_+(a, b) = 1,$$

and also the functions $f_-(a, b) = -f_+(a, -b)$ and $g_-(a, b) = -g_+(a, b)$. We have that

(i) $\mathsf{S}_{m,n,\infty} = \text{graph}\,(g_+|_U) \cup \text{graph}\,(g_-|_W) \cup \text{graph}\,(f_+|_{V\cup W}) \cup \text{graph}\,(f_-|_{U\cup V})$.
(ii) $\text{ext}\left(\mathsf{B}_{m,n,\infty}\right) = \{\pm(0, 2, -1),\ \pm(1, 1, -1),\ \pm(-1, 1, -1),\ \pm(0, 0, -1)\}$.

Figure 3.4 shows a sketch of the unit ball of $\mathcal{P}_{3,2,\infty}(\mathbb{R})$.

3.1.3 The Geometry of $\mathsf{B}_{m,n,\infty}$ for m Even and n Odd

In this case we study the space $\mathbb{R}^3$ endowed with the norm $\| \cdot \|_{m,n,\infty}$ where m is even and n is odd. We start off with the explicit formula of the norm $\| \cdot \|_{m,n,\infty}$.

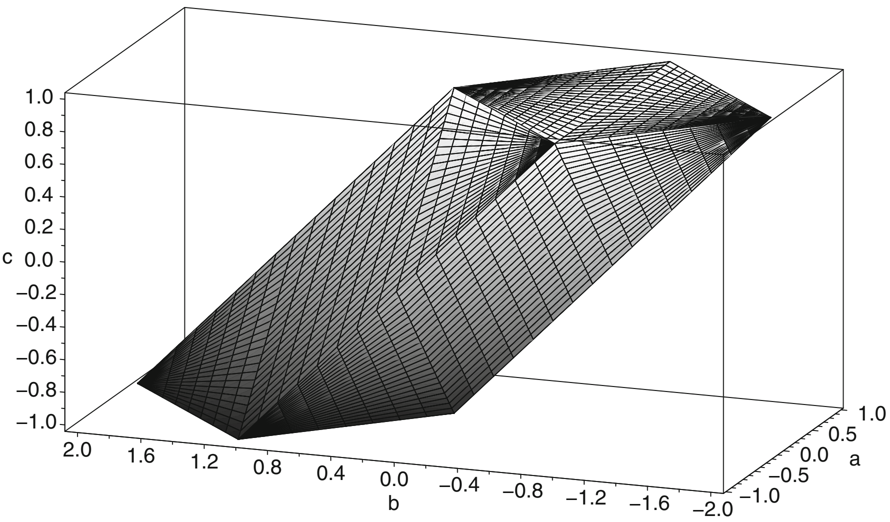


Fig. 3.4 Unit ball of $\mathcal{P}_{3,2,\infty}(\mathbb{R})$. The general case for m odd and n even looks similar

Theorem 3.7 (Muñoz and Seoane [50]) *If $m, n \in \mathbb{N}$ are such that m is even, n is odd and $m > n$, defining $\mathcal{I}_{m,n}$ as the set of triples $(a, b, c) \in \mathbb{R}^3$ such that*

$$a \neq 0, \ \left|\frac{nb}{ma}\right| < 1 \text{ and } 1 + \frac{c}{a} < \frac{1}{2}\left[\frac{m-n}{n}\left(\frac{nb}{ma}\right)^{\frac{m}{m-n}} - \left|\frac{b}{a}\right| + 1\right],$$

we have that

$$\|(a, b, c)\|_{m,n,\infty} = \begin{cases} \left|\frac{(m-n)a}{n}\left(\frac{nb}{ma}\right)^{\frac{m}{m-n}} - c\right| & \text{if } (a, b, c) \in \mathcal{I}_{m,n}, \\ |a + c| + |b| & \text{otherwise.} \end{cases} \tag{3.3}$$

Remark 3.2 (Muñoz and Seoane [50]) In Theorem 3.7, if $n \in \mathbb{N}$ is odd and $m = 2n$, then the formula (3.3) is given by

$$\|(a, b, c)\|_{2n,n,\infty} = \begin{cases} \left|\frac{b^2}{4a} - c\right| & \text{if } a \neq 0, \ \left|\frac{b}{2a}\right| < 1 \text{ and } \frac{c}{a} + 1 < \frac{1}{2}\left(\left|\frac{b}{2a}\right| - 1\right)^2, \\ |a + c| + |b| & \text{otherwise.} \end{cases} \tag{3.4}$$

for all $(a, b, c) \in \mathbb{R}^3$. Moreover, if $n \in \mathbb{N}$ is odd and $(a, b, c) \in \mathbb{R}^3$, then, by using the fact that x^n is a bijection from the interval $[-1, 1]$ into itself, we obtain the following equality

$$\|(a, b, c)\|_{2n,n,\infty} = \max\{|ax^{2n} + bx^n + c| : x \in [-1, 1]\}$$

$$= \max\{|ax^2 + bx + c| : x \in [-1, 1]\} = \|(a, b, c)\|_{\mathbb{R}}.$$

Consequently the calculations done with the norm $\|\cdot\|_{2n,n,\infty}$ when n is odd can be obtained simply by using the norm $\|\cdot\|_{\mathbb{R}}$.

We are interested in obtaining a parametrization of $\mathsf{S}_{m,n,\infty}$ to show a sketch of $\mathsf{S}_{m,n,\infty}$ and to find the extreme points of $\mathsf{B}_{m,n,\infty}$. To do so, we project $\mathsf{B}_{m,n,\infty}$ onto the ab-plane.

Consider the following equation

$$\frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} = 2 - a - b, \tag{3.5}$$

where $m, n \in \mathbb{N}$ with $m > n$. The Implicit Function Theorem states that formula (3.5) defines implicitly a unique differentiable curve $b = \Gamma_{m,n}(a)$ on $(0, \infty)$ such that $\Gamma_{m,n}(2) = 0$ and $\Gamma_{m,n}(n/m) = 1$. Also, if n is odd and $m = 2n$, then

$$\Gamma_{2n,n}(a) = 2(\sqrt{2a} - a).$$

Now consider the following sets in the ab-plane

$$U_{m,n} = \left\{(a, b) \in \mathbb{R}^2 : a < 0 \text{ and } |b| \le \min\left\{\frac{m|a|}{n}, \Gamma_{m,n}(|a|)\right\}\right\},$$

$$V_{m,n} = \left\{(a, b) \in \left[-\frac{n}{m}, \frac{n}{m}\right] \times [-1, 1] : |b| \ge \frac{m|a|}{n}\right\},$$

$$W_{m,n} = \left\{(a, b) \in \mathbb{R}^2 : a > 0 \text{ and } |b| \le \min\left\{\frac{m|a|}{n}, \Gamma_{m,n}(|a|)\right\}\right\}.$$

Figure 3.5 shows what $U_{2n,n}$, $V_{2n,n}$ and $W_{2n,n}$ look like when $n \in \mathbb{N}$ is odd.

Theorem 3.8 (Muñoz and Seoane [50]) *Let $m, n \in \mathbb{N}$ with m even, n odd and $m > n$. The projection of $\mathsf{B}_{m,n,\infty}$ onto the ab-plane is the set $U_{m,n} \cup V_{m,n} \cup W_{m,n}$.*

Finally, we can give a parametrization of $\mathsf{S}_{m,n,\infty}$ and show the extreme points of $\mathsf{B}_{m,n,\infty}$.

Theorem 3.9 (Muñoz and Seoane [50]) *Let $m, n \in \mathbb{N}$ with m even, n odd and $m > n$. If for every $(a, b) \in \mathbb{R}^2$ we define*

$$f_+(a, b) = 1 - a - |b|, \quad f_-(a, b) = -f_+(-a, b),$$

and for every $(a, b) \in \mathbb{R}^2$ with $a \neq 0$ we define

$$g_+(a, b) = \frac{(m-n)a}{n} \cdot \left(\frac{nb}{ma}\right)^{\frac{m}{m-n}} - 1, \quad g_-(a, b) = -g_+(-a, b),$$

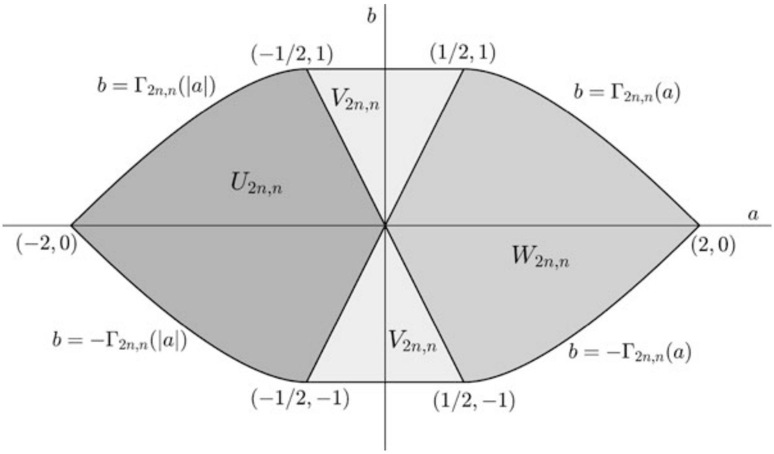


Fig. 3.5 Projection of $\mathsf{B}_{2n,n,\infty}$ onto the ab-plane with $n \in \mathbb{N}$ odd

then

(i) $\mathsf{S}_{m,n,\infty} = \text{graph}\left(f_+|_{W_{m,n}\cup V_{m,n}}\right) \cup \text{graph}\left(f_-|_{U_{m,n}\cup V_{m,n}}\right) \cup \text{graph}\left(g_+|_{W_{m,n}}\right) \cup \text{graph}\left(g_-|_{U_{m,n}}\right)$.

(ii)

$$\text{ext}\left(\mathsf{B}_{m,n,\infty}\right) = \left\{\pm(t, \pm\Gamma_{m,n}(t), 1 - t - \Gamma_{m,n}(t)) \colon t \in \left[\frac{n}{m}, 2\right]\right\} \bigcup\{\pm(0, 0, 1)\}.$$

Corollary 3.1 (Muñoz and Seoane [50]) *Let $n \in \mathbb{N}$ be odd. If for every $(a, b) \in \mathbb{R}^2$ we define*

$$f_+(a, b) = 1 - |b| - a, \quad f_-(a, b) = -f_+(-a, b),$$

and for every $(a, b) \in \mathbb{R}^2$ with $a \neq 0$ we define

$$g_+(a, b) = \frac{b^2}{4a} - 1, \quad g_-(a, b) = -g_+(-a, b),$$

then

(i)

$$\mathsf{S}_{2n,n,\infty} = \text{graph}\left(f_+|_{V_{2n,n}\cup W_{2n,n}}\right) \cup \text{graph}\left(f_-|_{U_{2n,n}\cup V_{2n,n}}\right) \cup \text{graph}\left(g_+|_{W_{2n,n}}\right) \cup \text{graph}\left(g_-|_{U_{2n,n}}\right).$$

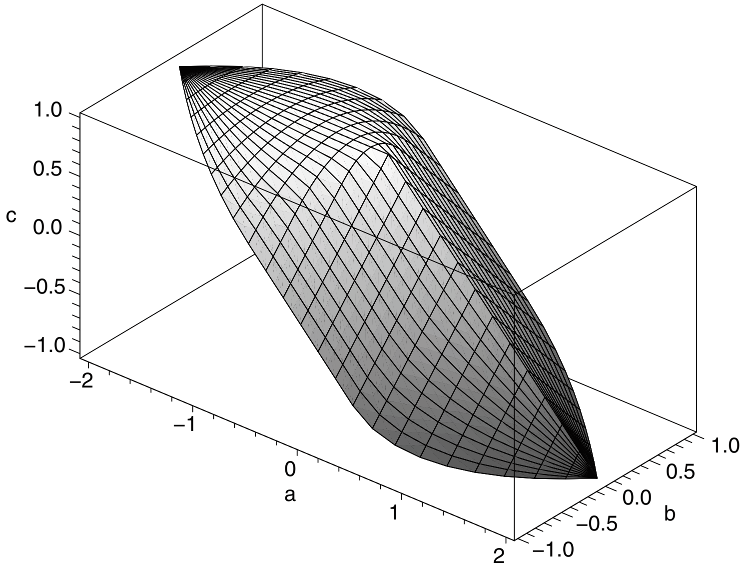


Fig. 3.6 Unit ball of $\mathcal{P}_{2n,n,\infty}(\mathbb{R})$ when $n \in \mathbb{N}$ is odd

(ii)

$$\operatorname{ext}\left(\mathsf{B}_{2n,n,\infty}\right) = \left\{ \pm \left(t, \pm 2 \left(\sqrt{2t} - t \right), 1 + t - 2\sqrt{2t} \right) : t \in \left[\frac{1}{2}, 2 \right] \right\} \bigcup \{ \pm (0, 0, 1) \}.$$

Figure 3.6 shows an approximation of how $\mathsf{B}_{2n,n,\infty}$ looks like when $n \in \mathbb{N}$ is odd.

3.1.4 The Geometry of $\mathsf{B}_{m,n,\infty}$ for Even Numbers m, n

Consider the space $\mathbb{R}^3$ endowed with the norm $\| \cdot \|_{m,n,\infty}$ where $m, n \in \mathbb{N}$ are even and $m > n$. The biggest difference in this case with respect to the previous ones is that there is no symmetry with some coordinate plane, however the way to tackle this case is very similar to the immediate previous one. Let us begin by giving an explicit formula of $\| \cdot \|_{m,n,\infty}$.

Theorem 3.10 (Muñoz and Seoane [50]) *For every $m, n \in \mathbb{N}$ with m and n even numbers and $m > n$, let us define $\mathcal{J}_{m,n}$ as the set of triples $(a, b, c) \in \mathbb{R}^3$ such that*

$$a \neq 0,\ 0 < -\frac{nb}{ma} < 1 \textit{ and } \frac{1}{2}\left[1 + \frac{b}{a} + \left|1 + \frac{b}{a}\right|\right] + \frac{2c}{a} < \frac{m-n}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}}.$$

We have that

$$\|(a,b,c)\|_{m,n,\infty} = \begin{cases} \left|\frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} - c\right| & \textit{if } (a,b,c) \in \mathcal{J}_{m,n}, \\ \left|\frac{a+b}{2} + c\right| + \left|\frac{a+b}{2}\right| & \textit{otherwise.} \end{cases} \tag{3.6}$$

Proof Let $(a, b, c) \in \mathbb{R}^3$ and let $P(x) = ax^m + bx^n + c$. Notice that if $a = 0$ or $b = 0$, then we have the desired result. Thus, assume that $a \neq 0$ and $b \neq 0$. Since $\|\cdot\|_{m,n,\infty}$ is symmetric with respect to the origin, we can also assume that $a > 0$. On the other hand, since $P(x) = P(-x)$ (m and n are even), we can restrict our attention to the interval $[0, 1]$ instead of the interval $[-1, 1]$ in order to find the maximum of $|P(x)|$ in the interval $[-1, 1]$. Thus, the maximum of $|P(x)|$ is attained at either one of the endpoints of the interval $[0, 1]$ or at one of the critical points of P in $(0, 1)$. In fact, there is only one critical point $\overline{x} = \left(-\frac{nb}{ma}\right)^{\frac{1}{m-n}}$ provided that $-\frac{nb}{ma} \in (0, 1)$ (this critical point has been found by solving the equation $P'(x) = 0$). Furthermore,

$$P(\overline{x}) = c - \frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}},$$

which shows that if $P(\overline{x}) \geq 0$, then $\max\{|P(0)|, |P(1)|\} \geq |P(\overline{x})|$. Hence,

$$\|P\|_{m,n,\infty} = \begin{cases} \max\{|P(0)|, |P(1)|, |P(\overline{x})|\} & \text{if } 0 < \overline{x} < 1 \text{ and } P(\overline{x}) < 0, \\ \max\{|P(0)|, |P(1)|\} & \text{otherwise.} \end{cases}$$

Notice that $\max\{|P(0)|, |P(1)|\} = \max\{|c|, |a + b + c|\} = \left|\frac{a+b}{2} + c\right| + \left|\frac{a+b}{2}\right|$. It is enough to show when $\max\{|P(0)|, |P(1)|\} < |P(\overline{x})| = -P(\overline{x})$ or, equivalently, when

$$\left|\frac{a+b}{2} + c\right| + \left|\frac{a+b}{2}\right| < \frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} - c.$$

Since $a \neq 0$, we can multiply the previous inequality by $\frac{2}{a}$ and, then, add $1 + \frac{b}{a} + \frac{2c}{2a}$ to both sides, which gives us

$$1 + \frac{b}{a} + \frac{2c}{a} + \left|1 + \frac{b}{a} + \frac{2c}{a}\right| < 2\frac{m-n}{n}\cdot\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} + 1 + \frac{b}{a} - \left|1 + \frac{b}{a}\right|. \tag{3.7}$$

Notice that the right-hand side of inequality (3.7) is non-negative since it is equal to

$$\begin{cases} 2\left[\frac{m-n}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} - \frac{m}{n}\left|\frac{nb}{ma}\right| + 1\right] & \text{if } 1+\frac{b}{a} < 0, \\ 2\frac{m-n}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} & \text{if } 1+\frac{b}{a} \geq 0, \end{cases}$$

and the mapping $h(x) = \frac{m-n}{n}x^{\frac{m}{m-n}} - \frac{m}{n}x + 1$ defined on the interval $(0, 1)$ is positive. Hence, inequality (3.7) is in fact of the following form

$$\frac{1}{2}\left[1 + \frac{b}{a} + \left|1 + \frac{b}{a}\right|\right] + \frac{2c}{a} < \frac{m-n}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}},$$

which finishes the proof. □

Now let us define the following two curves

$$\Upsilon_{m,n}(a) = \left(\frac{2m}{m-n}\right)^{\frac{m-n}{m}} \cdot \left(\frac{|a|m}{n}\right)^{\frac{n}{m}}$$

and

$$\Lambda_{m,n}(a) = -\Gamma_{m,n}(|a|),$$

where $b = \Gamma_{m,n}(a)$ for every $a \in (0, \infty)$ is the curve given by (3.5) using the Implicit Function Theorem. Notice that according to the Implicit Function Theorem, $b = \Lambda_{m,n}(a)$ for every $a \in (-\infty, 0)$ is the unique differentiable curve passing through $(-2, 0)$, satisfying the equation

$$\frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} = -2 - a - b, \tag{3.8}$$

If $m = 2n$, then both curves $\Upsilon_{m,n}(a)$ and $\Lambda_{m,n}(a)$ have an explicit formula given by

$$\Upsilon_{2n,n}(a) = 2\sqrt{2|a|} \text{ and } \Lambda_{2n,n}(a) = 2\left(|a| - \sqrt{2|a|}\right).$$

The curves $b = \Upsilon_{m,n}(a)$ and $b = \Lambda_{m,n}(a)$ with $a \in (-\infty, 0)$ intersect in one single point $(\gamma_0, -\gamma_0)$ such that

$$\gamma_0 = -\frac{2}{m-n} \cdot \left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}} < -2.$$

In fact, if $m = 2n$, then $\gamma_0 = -8$. Now if we also consider the curve $b = 2 - a$, then this curve intersects $b = \Upsilon_{m,n}(a)$ with $a \in (-\infty, 0)$ at one single point $(\gamma_1, \upsilon_1) = \left(\frac{-2n}{m-n}, \frac{2m}{m-n}\right)$. Notice that, if $m = 2n$, then $(\gamma_1, \upsilon_1) = (-2, 4)$.

Now, let us define the following sets contained in the ab-plane

$$U_{m,n} = \left\{(a,b)\in\mathbb{R}^2 : a\in[\gamma_0,0),\, \max\{0,\Lambda_{m,n}(a)\}\le b\le \min\left\{\frac{-m}{n}a,\, \Upsilon_{m,n}(a)\right\}\right\},$$

$$W_{m,n} = \left\{(a,b)\in\mathbb{R}^2 : a\in(0,-\gamma_0],\, -\max\{0,\Lambda_{m,n}(a)\}\right.$$
$$\left.\le b\le -\min\left\{\frac{-m}{n}a,\, \Upsilon_{m,n}(a)\right\}\right\},$$

$$V^1_{m,n} = \left\{(a,b)\in\mathbb{R}^2 : a\in[\gamma_1,2],\, \max\left\{0,\frac{-m}{n}a\right\}\le b\le 2-a\right\},$$

$$V^2_{m,n} = \left\{(a,b)\in\mathbb{R}^2 : a\in[\gamma_1,2],\, -2-a\le b\le \min\left\{0,\frac{-m}{n}a\right\}\right\},$$

$$V_{m,n} = V^1_{m,n}\cup V^2_{m,n}.$$

Figure 3.7 shows what $U_{2n,n}$, $V_{2n,n}$ and $W_{2n,n}$, when $n\in\mathbb{N}$ is even, look like. The projection of $\mathsf{B}_{m,n,\infty}$ with $m,n\in\mathbb{N}$ even can be stated in the following way.

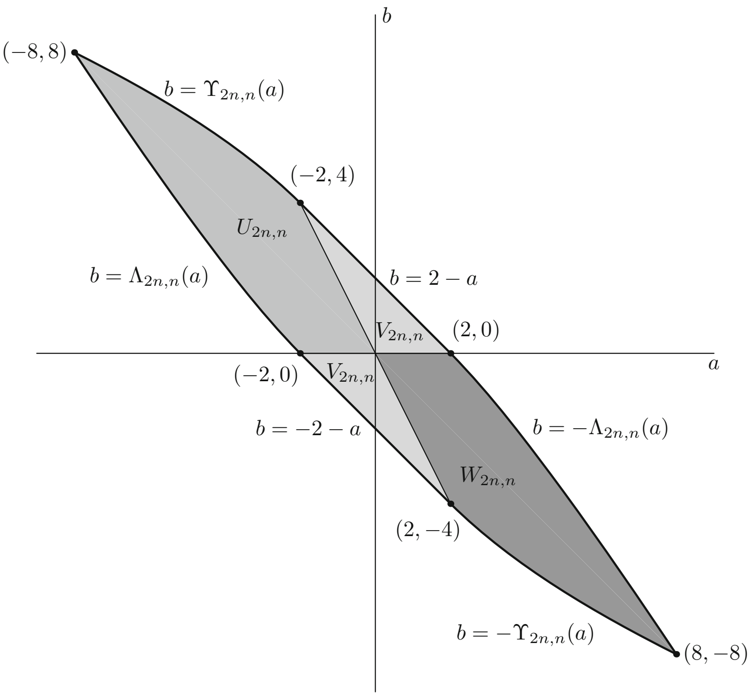


Fig. 3.7 Projection of $\mathsf{B}_{2n,n,\infty}$ onto the ab-plane with $n\in\mathbb{N}$ even

Theorem 3.11 (Muñoz and Seoane [50]) *If $m, n \in \mathbb{N}$ are even with $m > n$, then the projection of $\mathsf{B}_{m,n,\infty}$ onto the ab-plane is the set $U_{m,n} \cup V_{m,n} \cup W_{m,n}$.*

Proof By symmetry, we can restrict our attention to three different cases depending on the value of $(a, b) \in \mathbb{R}^2$:

(a) $A_1 = \{(a, b) \in \mathbb{R}^2 : b = 2 - a \text{ and } a \in [\gamma_1, 2]\}$,
(b) $A_2 = \{(a, b) \in \mathbb{R}^2 : b = \Upsilon_{m,n}(a) \text{ and } a \in [\gamma_0, \gamma_1)\}$,
(c) $A_3 = \{(a, b) \in \mathbb{R}^2 : b = \Lambda_{m,n}(a) \text{ and } a \in (\gamma_0, -2)\}$.

We want to prove that if (a, b) belongs to one of these sets and $c \in \mathbb{R}$ is such that $\|(a, b, c)\|_{m,n,\infty} = 1$, then c is unique.

Firstly, assume that $(a, b) \in A_1$. Notice that if $a < 0$, then $-\frac{nb}{ma} \geq 1$ which implies that $(a, b, c) \notin \mathcal{J}_{m,n}$. Furthermore, if $a = 0$, then we also have that $(a, b, c) \notin \mathcal{J}_{m,n}$. Hence, on the one hand, if $a \leq 0$, notice that $1 = \|(a, b, c)\|_{m,n,\infty} = \left|\frac{a+b}{2} + c\right| + \left|\frac{a+b}{2}\right|$. Thus, $c = -1$ since $a + b = 2$. On the other hand, if $a > 0$, then $-\frac{nb}{ma} < 1$. Suppose that $(a, b, c) \in \mathcal{J}_{m,n}$, then $\left|\frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} - c\right| = 1$ if, and only if,

$$c = \pm 1 + \frac{(m-n)a}{n}\left(\frac{nb}{ma}\right)^{\frac{m}{m-n}}.$$

Since $a, b \geq 0$, $b = 2 - a$ and $(a, b, c) \in \mathcal{J}_{m,n}$, notice that we have the following inequality

$$\frac{2}{a}(1 + c) < \frac{m-n}{n}\left(\frac{nb}{ma}\right)^{\frac{m}{m-n}}.$$

It can be easily seen that the last inequality is not satisfied for any of the two possible values of c. We have reached a contradiction, thus $(a, b, c) \notin \mathcal{J}_{m,n}$. This implies that since $\|(a, b, c)\|_{m,n,\infty} = 1$ and $(a, b, c) \notin \mathcal{J}_{m,n}$, then $\left|\frac{a+b}{2} + c\right| + \left|\frac{a+b}{2}\right| = 1$, and, by the calculations previously done, we have that $c = -1$.

Secondly, assume that $(a, b) \in A_2$. Notice that in this case $-\frac{nb}{ma} \in (0, 1)$. Assume first that $(a, b, c) \in \mathcal{J}_{m,n}$, then, since $\|(a, b, c)\|_{m,n,\infty} = 1$, $c = \frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} \pm 1 = -2 \pm 1$. However, in both cases of c, the inequality $\frac{1}{2}\left[1 + \frac{b}{a} + \left|1 + \frac{b}{a}\right|\right] + \frac{2c}{a} < \frac{m-n}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}}$ is not satisfied and we have a contradiction. Thus, $(a, b, c) \notin \mathcal{J}_{m,n}$. Notice that in this case $a + b \geq 0$, and therefore, by solving the equation $\left|\frac{a+b}{2} + c\right| + \left|\frac{a+b}{2}\right| = 1$, we have that $c = -1$ or $c = -1 - a - b$. But the latter form of c guarantees that $(a, b, c) \in \mathcal{J}_{m,n}$, which is a contradiction. So, $c = -1$.

Finally, assume that $(a, b) \in A_3$. Once again we have that $-\frac{nb}{ma} \in (0, 1)$. Assume that $(a, b, c) \notin \mathcal{J}_{m,n}$, then in this case, using the same procedure as in the previous case, but now using the fact that $a + b < 0$, we have that $c = 1$ or $c = -1 - a - b$. However, both forms of c guarantee that $(a, b, c) \in \mathcal{J}_{m,n}$ which is a contradiction.

Therefore, assume that $(a, b, c) \in \mathcal{J}_{m,n}$. Since $\|(a, b, c)\|_{m,n,\infty} = 1$, we have that $c = -1 - a - b$ or $c = -3 - a - b$. However, the latter form of c implies that $(a, b, c) \notin \mathcal{J}_{m,n}$ and we conclude that $c = -1 - a - b$. □

To this end let us give a parametrization of $\mathsf{S}_{m,n,\infty}$ when $m, n \in \mathbb{N}$ are even, which is used to sketch $\mathsf{B}_{m,n,\infty}$ and also show the extreme points of $\mathsf{B}_{m,n,\infty}$.

Theorem 3.12 (Muñoz and Seoane [50]) *Let $m, n \in \mathbb{N}$ be even with $m > n$. If for every $(a, b) \in \mathbb{R}^2$ we define*

$$f_+(a, b) = 1 - \left|\frac{a+b}{2}\right| - \frac{a+b}{2}, \quad f_-(a, b) = -f_+(-a, -b),$$

and for every $(a, b) \in \mathbb{R}^2$ with $a \neq 0$ we define

$$g_+(a, b) = \frac{(m-n)a}{n}\left(\left|\frac{nb}{ma}\right|\right)^{\frac{m}{m-n}} - 1, \quad g_-(a, b) = -g_+(-a, -b),$$

then

(i) $\mathsf{S}_{m,n,\infty} = \operatorname{graph}\left(f_+|_{W_{m,n} \cup V_{m,n}}\right) \cup \operatorname{graph}\left(f_-|_{U_{m,n} \cup V_{m,n}}\right) \cup \operatorname{graph}\left(g_+|_{W_{m,n}}\right) \cup \operatorname{graph}\left(g_-|_{U_{m,n}}\right)$.

(ii)

$$\begin{aligned}\operatorname{ext}\left(\mathsf{B}_{m,n,\infty}\right) = &\left\{\pm(s, \Lambda_{m,n}(s), -1 - s - \Lambda_{m,n}(s)) : s \in [\gamma_0, -2]\right\}\\ &\bigcup\left\{\pm(t, -\Upsilon_{m,n}(t), 1) : t \in [-\gamma_1, -\gamma_0]\right\}\bigcup\{\pm(0, 0, 1)\}.\end{aligned}$$

Proof Notice that part (i) is a direct consequence of Theorems 3.10 and 3.11. For part (ii), it is easy to see that f_+, f_-, g_+ and g_- are affine half-lines radiating from the origin, which directly implies that the candidates to being extreme points are the points $(0, 0, f_+(0, 0)) = (0, 0, 1)$, $(0, 0, f_-(0, 0)) = (0, 0, -1)$ and the points where the graphs of the functions $f_+|_{W_{m,n} \cup V_{m,n}}$, $f_-|_{U_{m,n} \cup V_{m,n}}$, $g_+|_{W_{m,n}}$ and $g_-|_{U_{m,n}}$ intersect along a non-affine line. Thus, we only need to find the intersection of the graphs of the functions. To do so, notice that by symmetry, we can suppose that $a + b \leq 0$. On the one hand, it is easy to prove that $\operatorname{graph}\left(f_+|_{W_{m,n} \cup V_{m,n}}\right) \cap \operatorname{graph}\left(f_-|_{U_{m,n} \cup V_{m,n}}\right)$, $\operatorname{graph}\left(f_+|_{W_{m,n} \cup V_{m,n}}\right) \cap \operatorname{graph}\left(g_-|_{U_{m,n}}\right)$ and $\operatorname{graph}\left(f_-|_{U_{m,n} \cup V_{m,n}}\right) \cap \operatorname{graph}\left(g_+|_{W_{m,n}}\right)$ are the segments (thus contained in affine lines) $\{(a, -2 - a, 1) \in \mathbb{R}^3 : a \in [\gamma_1, 2)\}$, $\{(a, 0, 1) \in \mathbb{R}^3 : a \in [\gamma_1, 0)\}$ and $\left\{\left(a, -\frac{m}{n}a, -1 + \frac{m-n}{n}a\right) \in \mathbb{R}^3 : a \in (0, 2]\right\}$, respectively. On the other hand, it is also easy to prove that the intersection of the graphs of $g_+|_{W_{m,n}}$ and $g_-|_{U_{m,n}}$ is the empty set. Thus, the only other possible extreme points in $\mathsf{B}_{m,n,\infty}$ are the following cases:

(a) $\operatorname{graph}\left(f_+|_{W_{m,n} \cup V_{m,n}}\right) \cap \operatorname{graph}\left(g_+|_{W_{m,n}}\right) = \operatorname{graph}(f_+) \cap \operatorname{graph}(g_+) \cap W_{m,n}$,
(b) $\operatorname{graph}\left(f_-|_{U_{m,n} \cup V_{m,n}}\right) \cap \operatorname{graph}\left(g_-|_{U_{m,n}}\right) = \operatorname{graph}(f_-) \cap \operatorname{graph}(g_-) \cap U_{m,n}$.

In case (a), from equation $f_+ = g_+$ along the set $W_{m,n}$ (which guarantees that the b coordinate is negative), we have that $b = -\Upsilon_{m.n}(a)$ with $a \in [-\gamma_1, -\gamma_0]$. Therefore, the graphs of the functions $f_+|_{W_{m,n}\cup V_{m,n}}$ and $g_+|_{W_{m,n}}$ intersect along a non-affine curve of the form $\left(t, -\Upsilon_{m,n}(t), f_+\left(t, -\Upsilon_{m,n}(t)\right)\right) = (t, -\Upsilon_{m,n}(t), 1)$, where $t \in [-\gamma_1, -\gamma_0]$. In case (b), from $f_- = g_-$ restricted to the set $U_{m,n}$, which is equivalent to Eq. (3.8), we have that the graphs intersect along a non-affine curve of the form $\left(s, \Lambda_{m,n}(s), f_-\left(s, \Lambda_{m,n}(s)\right)\right) = \left(s, \Lambda_{m,n}(s), -1 - s - \Lambda_{m,n}(s)\right)$, where $s \in [\gamma_0, 2]$.

Finally, for each one of the previous points p that are candidates to extreme points, it is easy to construct a plane $\Pi \subset \mathbb{R}^3$ such that $\Pi \cap \mathsf{B}_{m,n,\infty} = \{p\}$, which guarantees that p is an extreme point (this is left as an exercise to the reader) and the proof is complete. □

Corollary 3.2 (Muñoz and Seoane [50]) *Let $n \in \mathbb{N}$ be even. If for every $(a, b) \in \mathbb{R}^2$ we define*

$$f_+(a, b) = 1 - \left|\frac{a+b}{2}\right| - \frac{a+b}{2}, \quad f_-(a, b) = -f_+(-a, -b),$$

and for every $(a, b) \in \mathbb{R}^2$ with $a \neq 0$ we define

$$g_+(a, b) = \frac{b^2}{4a} - 1, \quad g_-(a, b) = -g_+(-a, -b),$$

then

(i) $\mathsf{S}_{2n,n,\infty} = \mathrm{graph}(f_+|_{W_{2n,n}\cup V_{2n,n}}) \cup \mathrm{graph}(f_-|_{U_{2n,n}\cup V_{2n,n}}) \cup \mathrm{graph}(g_+|_{W_{2n,n}}) \cup \mathrm{graph}(g_-|_{U_{2n,n}})$.

(ii)

$$\mathrm{ext}\left(\mathsf{B}_{2n,n,\infty}\right) = \left\{\pm\left(t, 2\left(\sqrt{2t} - t\right), 1 + t - 2\sqrt{2t}\right) : t \in [2, 8]\right\}$$
$$\bigcup\left\{\pm\left(s, -2\sqrt{2s}, 1\right) : s \in [2, 8]\right\}\bigcup\{\pm(0, 0, 1)\}.$$

Using the previous parametrization we show in Fig. 3.8 a sketch of $\mathsf{B}_{2n,n,\infty}$ with $n \in \mathbb{N}$ even.

3.2 On the Real Line with the L_p-Norm

Let us consider the space of trinomials with real coefficients of the form $P(x) = ax^m + bx^n + c$, endowed with the L_p-norm (where $1 \leq p < \infty$). We denote this normed space of polynomials by $\mathcal{P}_{m,n,p}(\mathbb{R})$. Once again, the mapping T defined at the beginning of Sect. 2.1, can be used to represent each trinomial $ax^m + bx^n + c$

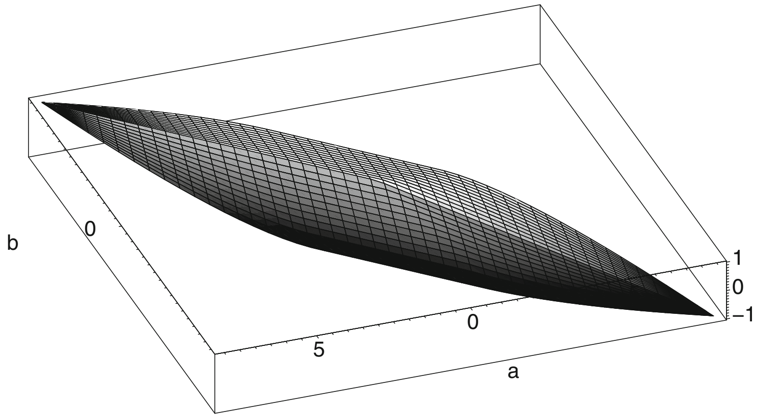


Fig. 3.8 Unit ball of $\mathcal{P}_{2n,n,\infty}(\mathbb{R})$ with $n \in \mathbb{N}$ even

of the space $\mathcal{P}_{m,n,p}(\mathbb{R})$ in $\mathbb{R}^3$. To do so, we begin by defining the norm $\|\cdot\|_{m,n,p}$ in $\mathbb{R}^3$ by

$$\|(a,b,c)\|_{m,n,p} = \left(\int_{-1}^{1} |ax^m + bx^n + c|^p dx\right)^{\frac{1}{p}}$$

for every $a, b, c \in \mathbb{R}$. In this case once again, the mapping T is a topological isomorphism between $\mathcal{P}_{m,n,p}(\mathbb{R})$ and $(\mathbb{R}^3, \|\cdot\|_{m,n,p})$.

In this section, we give a parametrization of the unit sphere $\mathsf{S}_{m,n,p}$ of the normed space $(\mathbb{R}^3, \|\cdot\|_{m,n,p})$.

The main results of this section are from [48] where the authors obtain an explicit formula for $\|\cdot\|_{m,n,2}$. In this survey we go deeper in this study by finding an explicit formula for $\|\cdot\|_{m,n,p}$ when p is even. It is interesting to observe that the norms $\|\cdot\|_{m,n,p}$ are uniformly convex and therefore all the elements of the unit sphere of the space $(\mathbb{R}^3, \|\cdot\|_{m,n,p})$ are extreme points of the closed unit ball.

Theorem 3.13 *Let $m, n \in \mathbb{N}$ and $p \in \mathbb{N}$ even. For every $(a,b,c) \in \mathbb{R}^3$ we have that*

$$\|(a,b,c)\|_{m,n,p} = \left(2\sum_{i,j,k}\binom{p}{i,j,k}\frac{a^i b^j c^k}{mi+nj+1}\right)^{\frac{1}{p}},$$

where $\binom{p}{i,j,k}$ denotes $\frac{p!}{i!j!k!}$ and the sum is taken over all $i, j, k \in \mathbb{N}\cup\{0\}$ such that $i+j+k = p$ and $mi+nj$ is even.

Proof Let $(a,b,c) \in \mathbb{R}^3$, then

$$\begin{aligned}\|(a,b,c)\|_{m,n,p} &= \left(\int_{-1}^{1}|ax^m+bx^n+c|^p dx\right)^{\frac{1}{p}}\\ &= \left(\int_{-1}^{1}(ax^m+bx^n+c)^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

By the trinomial expansion, we have

$$\begin{aligned}\|(a,b,c)\|_{m,n,p} &= \left(\int_{-1}^{1}\left[\sum_{\substack{i,j,k\in\mathbb{N}\cup\{0\}\\ i+j+k=p}}\binom{p}{i,\,j,\,k}(ax^m)^i(bx^n)^jc^k\right]dx\right)^{\frac{1}{p}}\\ &= \left(\sum_{\substack{i,j,k\in\mathbb{N}\cup\{0\}\\ i+j+k=p}}\binom{p}{i,\,j,\,k}a^ib^jc^k\int_{-1}^{1}x^{mi+nj}dx\right)^{\frac{1}{p}}\\ &= \left(\sum_{\substack{i,j,k\in\mathbb{N}\cup\{0\}\\ i+j+k=p}}\binom{p}{i,\,j,\,k}\frac{a^ib^jc^k}{mi+nj+1}\left[x^{mi+nj+1}\right]_{-1}^{1}dx\right)^{\frac{1}{p}}\\ &= \left(\sum_{\substack{i,j,k\in\mathbb{N}\cup\{0\}\\ i+j+k=p}}\binom{p}{i,\,j,\,k}\frac{a^ib^jc^k}{mi+nj+1}\left[1-(-1)^{mi+nj+1}\right]dx\right)^{\frac{1}{p}}\\ &= \left(2\sum_{i,j,k}\binom{p}{i,\,j,\,k}\frac{a^ib^jc^k}{mi+nj+1}\right)^{\frac{1}{p}},\end{aligned}$$

where the last sum is taken over all $i, j, k \in \mathbb{N}\cup\{0\}$ such that $i+j+k=p$ and $mi+nj$ is even. □

Corollary 3.3 (Muñoz et al. [48]) *For every $m,n\in\mathbb{N}$ with $m>n$ and every $a,b,c\in\mathbb{R}$, we have that $\|(a,b,c)\|_{m,n,2}^2$ is equal to*

$$\begin{cases}\frac{2a^2}{2m+1}+\frac{2b^2}{2n+1}+2c^2+\frac{4ab}{m+n+1} & \text{if } m \text{ and } n \text{ are odd,}\\ \frac{2a^2}{2m+1}+\frac{2b^2}{2n+1}+2c^2+\frac{4ac}{m+1} & \text{if } m \text{ is even and } n \text{ is odd,}\\ \frac{2a^2}{2m+1}+\frac{2b^2}{2n+1}+2c^2+\frac{4bc}{n+1} & \text{if } m \text{ is odd and } n \text{ is even,}\\ \frac{2a^2}{2m+1}+\frac{2b^2}{2n+1}+2c^2+\frac{4ab}{m+n+1}+\frac{4ac}{m+1}+\frac{4bc}{n+1} & \text{if } m \text{ and } n \text{ are even.}\end{cases}$$

We also have an explicit formula for $\|\cdot\|_{m,n,1}$.

Theorem 3.14 (Muñoz et al. [48]) *If $a, b, c \in \mathbb{R}$, $\Delta = b^2 - 4ac$ and, when $\Delta > 0$, $r_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $r_2 = \frac{-b+\sqrt{\Delta}}{2a}$, then*

$$\|(a,b,c)\|_{2,1,1} = \begin{cases} \left|\frac{2a+6c}{3}\right| & \text{if } a=0 \text{ or } \Delta \le 0 \text{ or } \min\{|r_1|,|r_2|\} \ge 1, \\ \frac{\operatorname{sign}(a)(2a^3+6a^2c)+\Delta^{\frac{3}{2}}}{3a^2} & \text{if } a \ne 0,\ \Delta > 0 \text{ and } \max\{|r_1|,|r_2|\} > 1, \\ \frac{\operatorname{sign}(b)(-b^3+6a^2b+6abc)+\Delta^{\frac{3}{2}}}{6a^2} & \text{otherwise.} \end{cases}$$

We now proceed to show the projection onto a coordinate plane of the unit ball of the spaces $\mathcal{P}_{m,n,2}(\mathbb{R})$ by defining the following sets:

$$E_k = \left\{(\alpha,\beta) \in \mathbb{R}^2 \colon \frac{2\alpha^2}{2k+1} + 2\beta^2 + \frac{4\alpha\beta}{k+1} \le 1\right\},$$

where $k \in \mathbb{N}$,

$$F_{m,n} = \left\{(a,b) \in \mathbb{R}^2 \colon \frac{a^2}{2m+1} + \frac{b^2}{2n+1} + \frac{2ab}{m+n+1} \le \frac{1}{2}\right\},$$

and

$$G_{m,n} = \left\{(a,b) \in \mathbb{R}^2 \colon \frac{m^2}{(m+1)^2(2m+1)}a^2 + \frac{n^2}{(n+1)^2(2n+1)}b^2 + \frac{2mn}{(m+n+1)(m+1)(n+1)}ab \le \frac{1}{2}\right\}.$$

Notice that E_k, $F_{m,n}$ and $G_{m,n}$ are in fact ellipses.

It is important to mention at this point that the following two results can be found in [48] but only for the cases when m and n have different parity. The case when m and n have the same parity, can be easily deduced from Corollary 3.3 and is presented in Theorems 3.17 and 3.18.

Theorem 3.15 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ with $m > n$.*

(i) If m is even and n is odd, then the projection of $\mathsf{B}_{m,n,2}$ onto the ac-plane is the set E_m.
(ii) If m is odd and n is even, then the projection of $\mathsf{B}_{m,n,2}$ onto the bc-plane is the set E_n.

By knowing the projection of the unit ball onto some coordinate plane, we can find a parametrization of the unit sphere and therefore the extreme points of the unit ball. As pointed out right before Theorem 3.13 recall that we always have $\operatorname{ext}\left(\mathsf{B}_{m,n,p}\right) = \mathsf{S}_{m,n,p}$ due to the uniform convexity of the spaces $(\mathbb{R}^3, \|\cdot\|_{m,n,p})$.

Theorem 3.16 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ with $m > n$, and let us define the mappings*

$$f_{m,n}(a, c) = \sqrt{\frac{2n+1}{2}\left(1 - \frac{2a^2}{2m+1} - 2c^2 - \frac{4ac}{m+1}\right)},$$

$$g_{m,n}(b, c) = \sqrt{\frac{2n+1}{2}\left(1 - \frac{2b^2}{2m+1} - 2c^2 - \frac{4bc}{m+1}\right)}.$$

We have

(i) If m is even and n is odd, then

$$\mathsf{S}_{m,n,2} = \operatorname{graph}\left(f_{m,n}|_{E_m}\right) \cup \operatorname{graph}\left(-f_{m,n}|_{E_m}\right)$$

and $\operatorname{ext}\left(\mathsf{B}_{m,n,2}\right) = \mathsf{S}_{m,n,2}$.

(ii) If m is odd and n is even, then

$$\mathsf{S}_{m,n,2} = \operatorname{graph}\left(g_{m,n}|_{E_n}\right) \cup \operatorname{graph}\left(-g_{m,n}|_{E_n}\right)$$

and $\operatorname{ext}\left(\mathsf{B}_{m,n,2}\right) = \mathsf{S}_{m,n,2}$.

Theorem 3.17 *Let $m, n \in \mathbb{N}$ with $m > n$.*

(i) If m and n are odd, then the projection of $\mathsf{B}_{m,n,2}$ onto the ab-plane is the set $F_{m,n}$.

(ii) If m and n are even, then the projection of $\mathsf{B}_{m,n,2}$ onto the ab-plane is the set $G_{m,n}$.

Proof Assume that m and n are odd natural numbers (the case when m and n are even is done in a similar way). We want prove that if (a, b) belongs to $F_{m,n}$ and $c \in \mathbb{R}$ is such that $\|(a, b, c)\|_{m,n,2} = 1$, then $|c|$ is unique (notice that we have considered $|c|$ instead of c since $\|(a, b, c)\|_{m,n,2} = \|(a, b, -c)\|_{m,n,2}$ by Theorem 3.3). Thus, assume that $(a, b) \in F_{m,n}$ and $c \in \mathbb{R}$ is such that $\|(a, b, c)\|_{m,n,2} = 1$. Then, $1 = \|(a, b, c)\|^2_{m,n,2} = \frac{2a^2}{2m+1} + \frac{2b^2}{2n+1} + 2c^2 + \frac{4ab}{m+n+1}$. Solving c in terms of a and b in the latter equation we have that

$$c = \pm\sqrt{\frac{1}{2} - \frac{a^2}{2m+1} - \frac{b^2}{2n+1} - \frac{2ab}{m+n+1}}.$$

Since $(a, b) \in F_{m,n}$, we have that $c \in \mathbb{R}$ and $|c|$ is unique. □

Theorem 3.18 *Let $m, n \in \mathbb{N}$ with $m > n$, and let us define the mappings*

$$
\begin{aligned}
h_+(a,b) &= \sqrt{\frac{1}{2} - \frac{a^2}{2m+1} - \frac{b^2}{2n+1} - \frac{2ab}{m+n+1}},\\
k_+(a,b) &= -\frac{a}{m+1} - \frac{b}{n+1} + \sqrt{\delta_{m,n}},\\
k_-(a,b) &= -\frac{a}{m+1} - \frac{b}{n+1} - \sqrt{\delta_{m,n}},
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_{m,n} =& \frac{1}{2} - \frac{m^2}{(m+1)^2(2m+1)}a^2 - \frac{m^2}{(n+1)^2(2n+1)}b^2\\
&- \frac{2mn}{(m+n+1)(m+1)(n+1)}ab,
\end{aligned}
$$

and $h_- = -h_+$. *We have*

(i) If m and n are odd, then $\mathsf{S}_{m,n,2} = \operatorname{graph}\left(h_+|_{F_{m,n}}\right) \cup \operatorname{graph}\left(h_-|_{F_{m,n}}\right)$ *and* $\operatorname{ext}\left(\mathsf{B}_{m,n,2}\right) = \mathsf{S}_{m,n,2}$.

(ii) If m and n are even, then $\mathsf{S}_{m,n,2} = \operatorname{graph}\left(k_+|_{G_{m,n}}\right) \cup \operatorname{graph}\left(k_-|_{G_{m,n}}\right)$ *and* $\operatorname{ext}\left(\mathsf{B}_{m,n,2}\right) = \mathsf{S}_{m,n,2}$.

Proof It is easy to see that the result is a direct consequence of Corollary 3.3 and Theorem 3.17. □

3.3 On the Real Plane

In this section we will study the geometry and extreme polynomials of the space of real homogeneous trinomials over the real plane, i.e., polynomials of the form

$$
P(x,y) = ax^m + bx^{m-n}y^n + cy^m,
$$

where $(x,y) \in \mathbb{R}^2$, $a,b,c \in \mathbb{R}$ and $m,n \in \mathbb{N}$ are such that $m > n$; endowed with the supremum norm over the unit square $[-1,1]^2$. We will denote this space by $\mathcal{P}^h_{m,n,\infty}(\mathbb{R}^2)$. As in the previous sections, the mapping T defined at the beginning of Sect. 2.1 and considered over $\mathcal{P}^h_{m,n,\infty}(\mathbb{R}^2)$ is a topological isomorphism between $\mathcal{P}^h_{m,n,\infty}(\mathbb{R}^2)$ and $(\mathbb{R}^3, \|\cdot\|_{m,n,\infty,2,h})$, where

$$
\|(a,b,c)\|_{m,n,\infty,2,h} = \sup\{|ax^m + bx^{m-n}y^n + cy^m| \colon (x,y) \in [-1,1]^2\}.
$$

We will denote the unit ball and the unit sphere of $(\mathbb{R}^3, \|\cdot\|_{m,n,\infty,2,h})$ by $\mathsf{B}^h_{m,n,\infty,2}$ and $\mathsf{S}^h_{m,n,\infty,2}$, respectively.

Analogously to Sect. 3.1, we will distinguish several cases depending on the parity of m and n. To be more precise, we will study the cases when $m = 2n$ and when m is odd.

3.3.1 The Geometry of $\mathsf{B}_{2n,n,\infty,2}$ for n Odd

Let $n \in \mathbb{N}$ be odd. It is straightforward to see that

$$R\colon [-1,1]^2 \to [-1,1]^2 \text{ with } R(x,y) = (x^n, y^n)$$

is a bijection since n is odd. Hence,

$$\|(a,b,c)\|_{2n,n,\infty,2,h} = \sup\{|ax^2 + bxy + cy^2| : (x,y) \in [-1,1]^2\}.$$

Therefore observe that the space $\mathcal{P}^h_{2n,n,\infty}(\mathbb{R}^2)$ is isometric to $\mathcal{P}\left({}^2\ell^2_\infty\right)$ which will studied in Sect. 5.2.

3.3.2 The Geometry of $\mathsf{B}_{2n,n,\infty,2}$ for n Even

Let $n \in \mathbb{N}$ be even. Although the function R defined in Sect. 3.3.1 is not a bijection in the case when n is even, note that R maps $[-1,1]^2$ onto $[0,1]^2$. Thus, we have that

$$\|(a,b,c)\|_{2n,n,\infty,2,h} = \sup\{|ax^2 + bxy + cy^2| : (x,y) \in [0,1]^2\}.$$

In this case the space $\mathcal{P}^h_{2n,n,\infty}(\mathbb{R}^2)$ is isometric to the space of polynomials $\mathcal{P}\left({}^2\square\right)$ which will be analyzed in Sect. 4.2.

3.3.3 The Geometry of $\mathsf{B}_{m,n,\infty,2}$ for m Odd

Let $m, n \in \mathbb{N}$ be such that $m > n$ and m is odd. The case when n is odd can be reduced to the case when n is even since it is easy to see that

$$\|(a,b,c)\|_{m,n,\infty,2,h} = \|(c,b,a)\|_{m,m-n,\infty,2,h},$$

for every $(a,b,c) \in \mathbb{R}^3$.

Assume that m is odd and n is even with $m > n$. Recall from Lemma 3.1 that the equation

$$|n + my| = (m - n)|y|^{\frac{m}{m-n}}$$

has a root $\lambda_0 \in \left(-\frac{n}{m}, 0\right)$. Clearly, λ_0 depends on the values of m and n which justifies the notation $\lambda_0(m, n)$. Let us consider $\mu_0 = \mu_0(m, n) = \lambda_0(m, m - n)$ which, by definition, belongs to the interval $\left(\frac{n-m}{m}, 0\right)$ and is a root of

$$|m - n + my| = n|y|^{\frac{m}{n}}.$$

The following theorem provides an explicit formula for the norm $\|\cdot\|_{m,n,\infty,2}$.

Theorem 3.19 (Jiménez et al. [35]) *Let $m, n \in \mathbb{N}$ be such that $m > n$, m is odd and n is even. Take the number $K_{m,n} = \frac{n}{m-n}\left(\frac{m-n}{m}\right)^{\frac{m}{n}}$, the interval $I_{m,n} = [\eta_1, \eta_2]$, where $\eta_1 = -\frac{m}{m-n}$, $\eta_2 = \frac{m}{m-n}\mu_0$, and the sets $A_{m,n}$, $F_{m,n}$, $B_{m,n}$ and $\mathcal{B}$ (see Figs. 3.9 and 3.10) defined as*

$$A_{m,n} = \left\{(x, y) \in \mathbb{R}^2 : x \in I_{m,n} \text{ and } |y| \geq 1 - K_{m,n}|x|^{\frac{m}{n}}\right\},$$

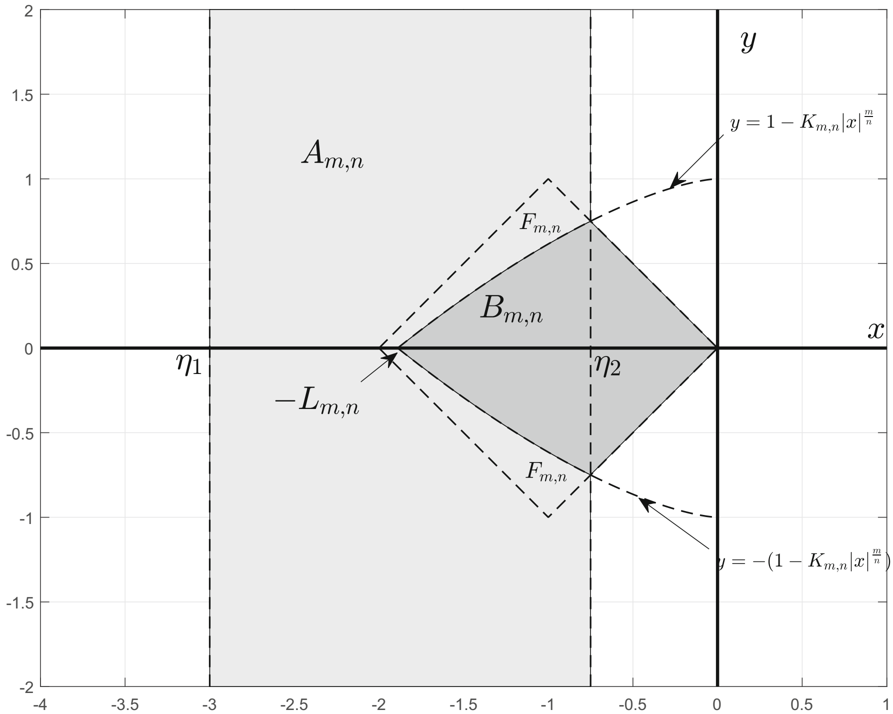


Fig. 3.9 Regions appearing in the definition of $\|\cdot\|_{m,n,\infty,2}$ when m is odd, n is even, $m > n$ and $\frac{m}{n} < 2$. The figure corresponds to the values $m = 3$ and $n = 2$

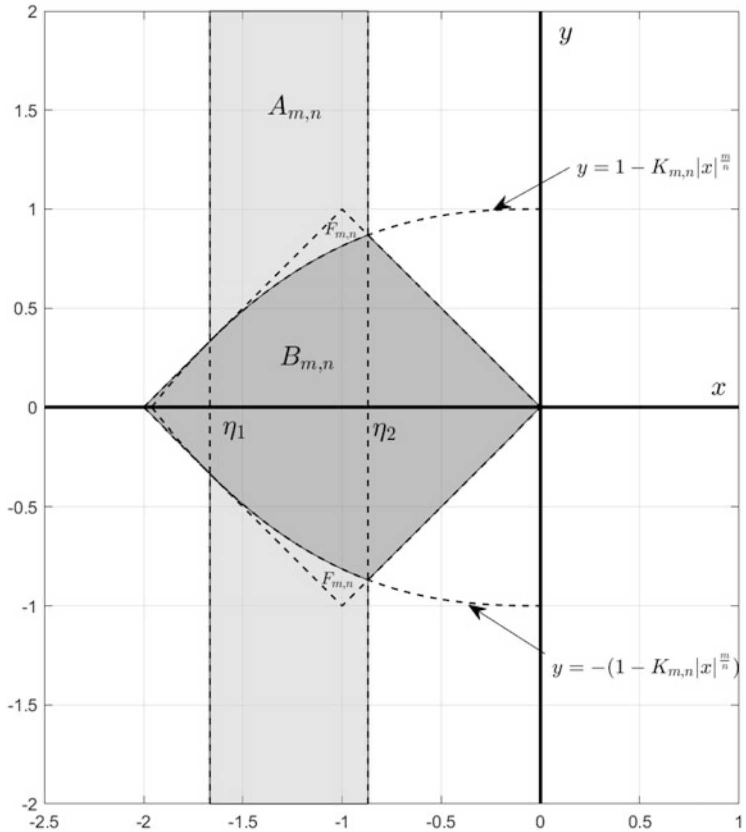


Fig. 3.10 Regions appearing in the definition of $\|\cdot\|_{m,n,\infty,2}$ when m is odd, n is even, $m > n$ and $\frac{m}{n} > 2$. The figure corresponds to the values $m = 5$ and $n = 2$

$$F_{m,n} = \{(x, y) \in \mathbb{R}^2 : x \in I_{m,n} \text{ and } 1 - K_{m,n}|x|^{\frac{m}{n}} < |y| < 1 - |1 + x|\},$$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 : |x + 1| + |y| < 1\},$$

$$B_{m,n} = \mathcal{B} \setminus F_{m,n}.$$

Then,

$$\|(a, b, c)\|_{m,n,\infty,2,h} = \begin{cases} \frac{n|a|}{m-n} \cdot \left|\frac{(m-n)b}{ma}\right|^{\frac{m}{n}} + |c| & \text{if } a \neq 0 \text{ and } \left(\frac{b}{a}, \frac{c}{a}\right) \in A_{m,n}, \\ |a| & \text{if } a \neq 0 \text{ and } \left(\frac{b}{a}, \frac{c}{a}\right) \in B_{m,n}, \\ |a + b| + |c| & \text{otherwise}. \end{cases}$$

Theorem 3.20 (Jiménez et al. [35]) *Let $m, n \in \mathbb{N}$ be such that $m > n$, m is odd and n is even. Consider the numbers $a_0 = \frac{m-n}{n}$ and $L_{m,n} = \frac{m}{m-n}\left(\frac{m-n}{n}\right)^{\frac{n}{m}}$. If $\frac{m}{n} < 2$, let*

$$R_{m,n} = \left\{(a,b) \in \mathbb{R}^2 :\ -1 \le a \le 0 \text{ and } \eta_2 a < b < \min\left\{\eta_1 a, L_{m,n}|a|^{\frac{m-n}{m}}\right\}\right\},$$
$$U_{m,n} = \left\{(a,b) \in \mathbb{R}^2 :\ -a_0 \le a \le 1 \text{ and } \max\{\eta_1 a, \eta_2 a\} \le b \le 1-a\right\},$$
$$S_{m,n} = -R_{m,n},$$
$$V_{m,n} = -U_{m,n}.$$

If $\frac{m}{n} > 2$, let

$$R_{m,n} = \left\{(a,b) \in \mathbb{R}^2 :\ -1 \le a \le 0 \text{ and } \eta_2 a < b < \eta_1 a\right\},$$
$$U_{m,n} = \left\{(a,b) \in \mathbb{R}^2 :\ -1 \le a \le 1 \text{ and } \max\{\eta_1 a, \eta_2 a\} \le b \le 1-a\right\},$$
$$S_{m,n} = -R_{m,n},$$
$$V_{m,n} = -U_{m,n}.$$

Then, the projection of $\mathsf{B}^h_{m,n,\infty,2}$ onto the ab-plane is the set $R_{m,n} \cup S_{m,n} \cup U_{m,n} \cup V_{m,n}$ (see Figs. 3.11 and 3.12).

Finally, the following theorem shows a parametrization of $\mathsf{S}^h_{m,n,\infty,2}$ as well as the extreme points of $\mathsf{B}^h_{m,n,\infty,2}$.

Theorem 3.21 *Let $m, n \in \mathbb{N}$ be such that $m > n$, m is odd and n is even. Define the function*

$$G_{m,n}(a,b) = \begin{cases} 1 - K_{m,n}|a|\left|\frac{b}{a}\right|^{\frac{m}{n}} & \text{if } (a,b) \in R_{m,n} \cup S_{m,n}, \\ 1 - |a+b| & \text{if } (a,b) \in U_{m,n} \cup V_{m,n}, \end{cases}$$

and the set

$$\Gamma_{m,n} = \begin{cases} \{(-1,b,c) \in \mathbb{R}^3 : 0 \le b \le L_{m,n} \text{ and } |c| \le G_{m,n}(-1,b)\} & \text{if } \frac{m}{n} < 2, \\ \{(-1,b,c) \in \mathbb{R}^3 : 0 \le b \le 2 \text{ and } |c| \le G_{m,n}(-1,b)\} & \text{if } \frac{m}{n} > 2, \end{cases}$$

where $K_{m,n}$, $L_{m,n}$, a_0, η_1 and η_2 are as in Theorems 3.19 and 3.20. Then,

(i) $\mathsf{S}^h_{m,n,\infty,2} = graph(G_{m,n}) \cup graph(-G_{m,n}) \cup \Gamma_{m,n} \cup (-\Gamma_{m,n})$.
(ii) *If $\frac{m}{n} < 2$, then*

$$\operatorname{ext}(\mathsf{B}^h_{m,n,\infty,2}) = \left\{\pm\left(-1, t, \pm(1 - K_{m,n}|t|^{\frac{m}{n}})\right) : t \in [-\eta_2, L_{m,n}]\right\}$$

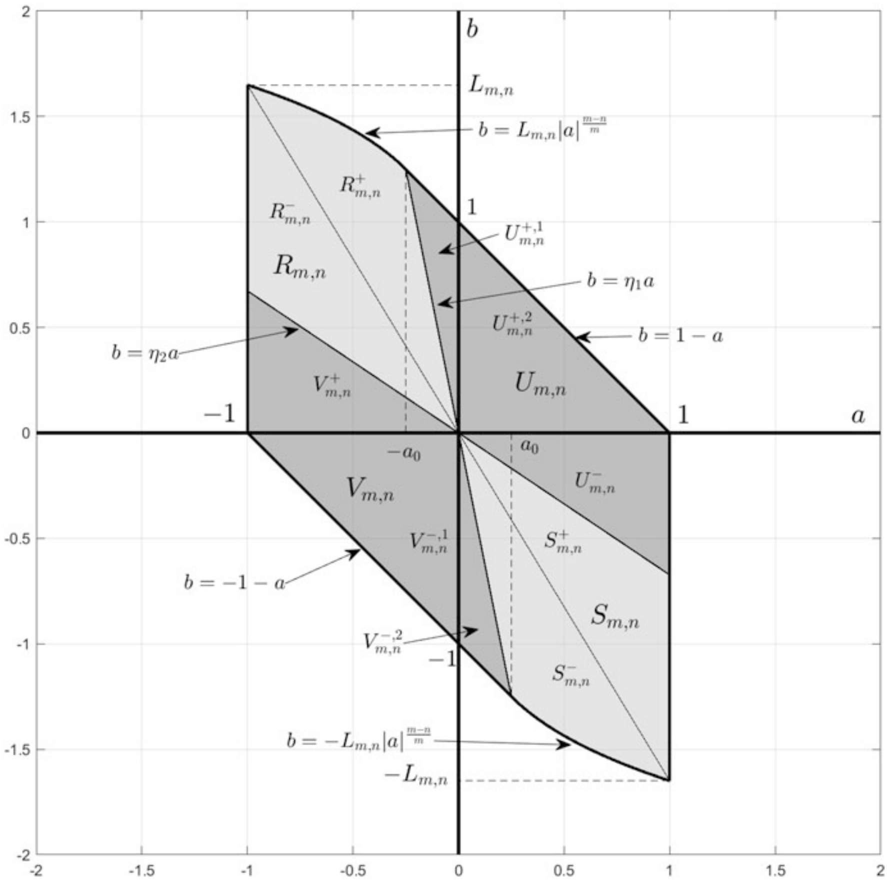


Fig. 3.11 Projection of $\mathsf{B}^h_{m,n,\infty,2}$ onto the ab-plane with m odd, n even, $m > n$ and $\frac{m}{n} < 2$. The picture corresponds to the case when $m = 5$, $n = 4$

$$\bigcup \left\{ \pm(0, s, L_{m,n}|s|^{\frac{m}{n}}) : s \in [-1, -a_0] \right\}$$
$$\bigcup \{(\pm 1, 0, 0), (0, 0 \pm 1)\}.$$

If $\frac{m}{n} > 2$, *then*

$$\mathrm{ext}(\mathsf{B}^h_{m,n,\infty,2}) = \left\{ \pm \left(-1, t, \pm(1 - K_{m,n}|t|^{\frac{m}{n}}) \right) : t \in [-\eta_2, -\eta_1] \right\}$$
$$\bigcup \{(\pm 1, 0, 0), (0, 0 \pm 1), \pm(1, -2, 0)\}.$$

See Figs. 3.13 and 3.14.

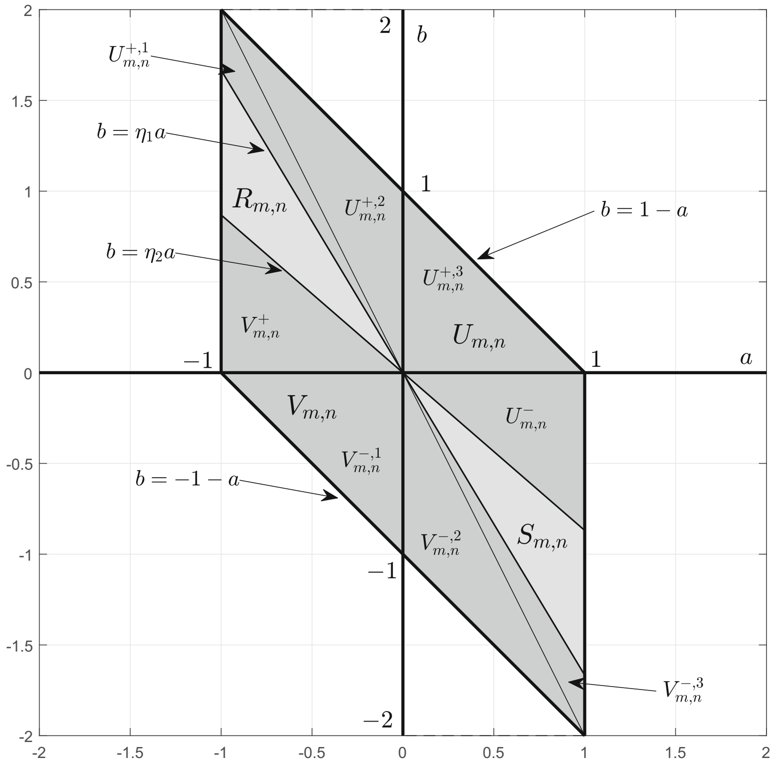


Fig. 3.12 Projection of $\mathsf{B}^h_{m,n,\infty,2}$ over the ab-plane with $\frac{m}{n} > 2$. The picture corresponds to the case when $m = 5, n = 2$

3.4 On the Complex Plane

A (trigonometric) trinomial in the field of complex numbers $\mathbb{C}$ is a trinomial of the form:

$$P(z) = a\mathrm{e}^{\mathrm{i}\lambda_1 z} + b\mathrm{e}^{\mathrm{i}\lambda_2 z} + c\mathrm{e}^{\mathrm{i}\lambda_3 z}, \tag{3.9}$$

where a, b, c are real numbers such that $a, b, c \geq 0$ and $\lambda_i \in \mathbb{Z}$ for every $i \in \{1, 2, 3\}$. Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and consider $\mathcal{P}_\Lambda(\mathbb{C})$ the vector space of trinomials spanned by $\{\mathrm{e}^{\mathrm{i}\lambda} : \lambda \in \Lambda\}$ where e_λ denotes the function $z \mapsto \mathrm{e}^{\lambda z}$. Let us also endow $\mathcal{P}_\Lambda(\mathbb{C})$ with the maximum modulus norm, that is, if P is a polynomial of the form (3.9), then

$$||P||_\Lambda := \max\{|P(z)| : |z| \leq 1\}.$$

We will denote the unit ball of $(\mathcal{P}_\Lambda(\mathbb{C}), || \cdot ||_\Lambda)$ by B_Λ.

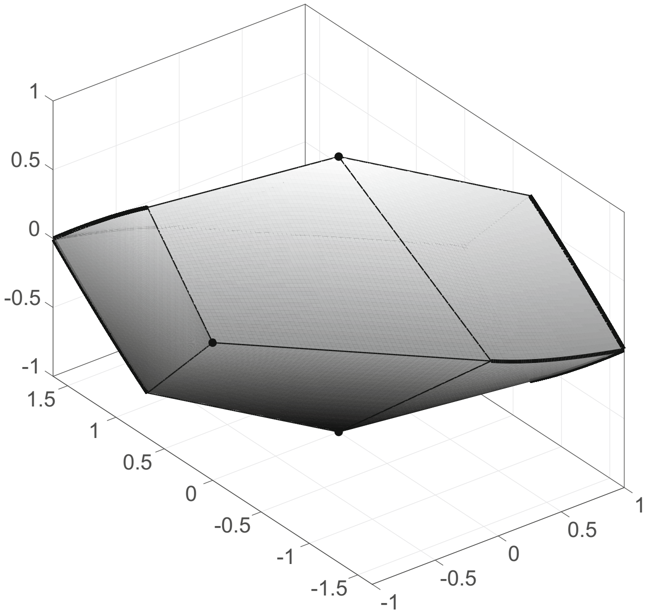


Fig. 3.13 Sketch of $\mathsf{S}^h_{m,n,\infty,2}$ with m odd, n even, $m > n$ and $\frac{m}{n} < 2$. The picture corresponds to the case when $m = 5$ and $n = 4$. The extreme points appear with a thicker line or big dots. The surfaces that form $\mathsf{S}^h_{m,n,\infty,2}$ are delimited by thin lines

Theorem 3.22 (Neuwirth [51]) *A polynomial $P \in \mathcal{P}_\Lambda(\mathbb{C})$ of the form* (3.9) *is an extreme point of* B_Λ *if, and only if, P is either a trigonometric monomial of the form $e^{i(\alpha+\lambda z)}$ with $\alpha \in \mathbb{R}$ and $\lambda \in \Lambda$ or P satisfies that $1 - |P|^2$ has four zeros that are multiples of $\frac{2\pi}{d}$, counted with multiplicities, and where $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$.*

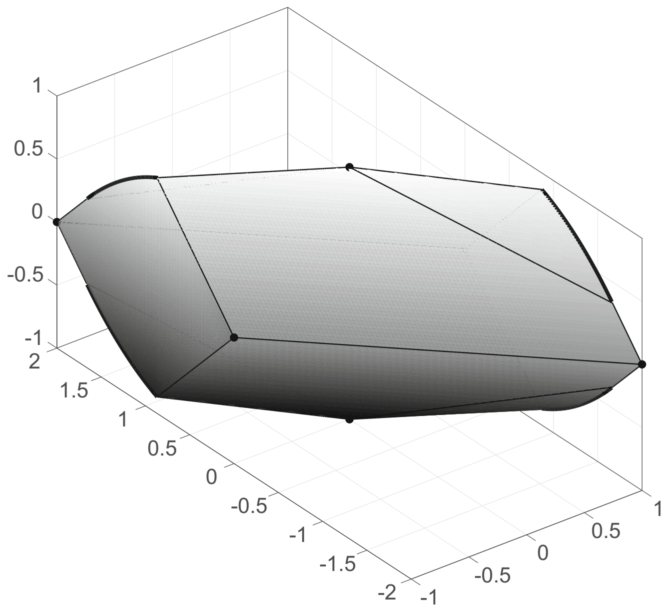


Fig. 3.14 Sketch of $\mathsf{S}^h_{m,n,\infty,2}$ with m odd, n even, $m > n$ and $\frac{m}{n} > 2$. The picture corresponds to the case when $m = 5$ and $n = 2$. The extreme points appear with a thicker line or big dots. The surfaces that form $\mathsf{S}^h_{m,n,\infty,2}$ are delimited by thin lines

Chapter 4
Polynomials on Non-Balanced Convex Bodies

Abstract We investigate some geometrical properties of polynomials of degree 2 on non-balanced convex bodies with respect to the origin in $\mathbb{R}^2$, providing an explicit formula to calculate their norm and a full description of the extreme points of the corresponding unit balls. We review all the cases considered up to now in the literature in this context.

A convex body in a topological space is a closed convex bounded set with nonempty interior. It is well known that in finite dimensional normed vector spaces, closed bounded sets are, in fact, compact (in infinite dimensional normed vector spaces this is not true in general). Therefore, a convex body over a finite dimensional normed vector space is a convex compact set with nonempty interior. A symmetric (with respect to the origin) convex body K in a normed vector space is a convex body such that K satisfies the following condition: $x \in K$ if, and only if, $-x \in K$.

Most of the norms that we are considering for polynomials in this expository work are taken over the unit ball of a normed vector space. However, in this chapter, we are interested in studying the geometry of normed vector spaces of polynomials where the norm is taken over a non-balanced convex body in $\mathbb{R}^2$.

In a real finite dimensional space, we say that a polynomial P is a 2-homogeneous polynomial on $\mathbb{R}^2$ if $P(x, y) = ax^2 + by^2 + cxy$ where $a, b, c \in \mathbb{R}$. Let K be a non-balanced convex body and let P be a 2-homogeneous polynomial on $\mathbb{R}^2$ endowed with the following norm

$$\|P\|_K = \max\{|P(\mathbf{x})| : \mathbf{x} \in K\}.$$

The space of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_K$ will be denoted by $\mathcal{P}(^2K)$.

Adapting the definition of the mapping T defined in Sect. 2.1 to 2-homogenous polynomials, we have a topological isomorphism between $\mathcal{P}(^2K)$ and the normed space $(\mathbb{R}^3, \|\cdot\|_K)$, where

$$\|(a, b, c)\|_K = \|ax^2 + by + cxy\|_K,$$

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_4

for any $(a, b, c) \in \mathbb{R}^3$. This means that the unit sphere can be represented in $\mathbb{R}^3$. From now on, the sets S_K and B_K will denote, respectively, the unit sphere and the unit ball of the space $(\mathbb{R}^3, \|\cdot\|_K)$.

Many of the results of this chapter deal with 2-homogeneous polynomials of degree 2. However, in Sect. 4.1, we will also study the extreme points of the unit ball of the vector space of polynomials of degree at most 2 in $\mathbb{R}^2$ endowed with the supremum norm over a non-balanced convex body K. We will denote this space by $\mathcal{P}_2(K)$.

4.1 The Simplex Δ

Let Δ be the region in $\mathbb{R}^2$ enclosed by the triangle of vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ called the simplex. Notice that the simplex is a non-balanced convex body of $\mathbb{R}^2$. We use the same approach that appeared in the previous chapters by showing first an explicit formula for $\|\cdot\|_\Delta$.

Theorem 4.1 (Muñoz et al. [46]) *Let $a, b, c \in \mathbb{R}$. We have*

$$\|(a,b,c)\|_\Delta = \begin{cases} \max\left\{|a|, |c|, \left|\frac{b^2-4ac}{4(a-b+c)}\right|\right\} & \textit{if } a-b+c \neq 0 \textit{ and } 0 < \frac{2c-b}{2(a-b+c)} < 1, \\ \max\{|a|, |c|\} & \textit{otherwise.} \end{cases}$$

Proof Let $(a, b, c) \in \mathbb{R}^3$ and take $P(x, y) = ax^2 + by^2 + cxy$. Notice that the maximum of $|P|$ defined over Δ is attained at the boundary of Δ or at an interior point of Δ.

We will analyze first $|P|$ over the boundary of Δ. On the one hand, notice that the maximum of $|P|$ over the segment $\{(t, 1-t) : t \in [0, 1]\}$ (which is one of the sides of Δ) is equal to

$$\begin{aligned} M &= \max\{(a-b+c)t^2 + (b-2c)t + c : t \in [0,1]\} \\ &= \begin{cases} \max\{|a|, |c|, \left|\frac{b^2-4ac}{4(a-b+c)}\right|\} & \text{if } a-b+c \neq 0 \text{ and } 0 < \frac{2c-b}{2(a-b+c)} < 1, \\ \max\{|a|, |c|\} & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, it is easy to see that the maximum of $|P|$ over the segments $\{(t, 0) : t \in [0, 1]\}$ and $\{(0, t) : t \in [0, 1]\}$ are $|a|$ and $|c|$, respectively. It is also easy to see that $\max\{|a|, |c|\} \le M$, hence the maximum of $|P|$ over the boundary of Δ is equal to M.

On the interior of Δ, notice that if P has a critical point $(\bar{x}, \bar{y})$ different from $(0, 0)$, then P along the line $\{t(\bar{x}, \bar{y}) : t \in \mathbb{R}\}$ has the form $P(t\bar{x}, t\bar{y}) = at^2\bar{x}^2 + bt^2\bar{y}^2 + ct^2\bar{x}\bar{y} = \alpha t^2$, where $\alpha = a\bar{x}^2 + b\bar{y}^2 + c\bar{x}\bar{y}$. Hence, necessarily we have that

$\alpha = 0$ which implies that $P(\bar{x}, \bar{y}) = 0$. Therefore $(\bar{x}, \bar{y})$ is neither a local maximum nor minimum unless $P \equiv 0$. This concludes the proof. $\square$

The second thing that is often used to parametrize S_Δ is to project B_Δ onto a plane. In this case, we project B_Δ onto the ac-plane. Interestingly, the projection of B_Δ onto the ac-plane is none other than the unit ball on $\mathbb{R}^2$ endowed with the supremum norm. The space $\mathbb{R}^2$ endowed with the supremum norm will be denoted by ℓ_∞^2 and the unit ball and the unit sphere in ℓ_∞^2 will be denoted by $\mathsf{B}_{\ell_\infty^2}$ and $\mathsf{S}_{\ell_\infty^2}$, respectively.

Theorem 4.2 (Muñoz et al. [46]) *The projection of* B_Δ *onto the* ac*-plane is* $\mathsf{B}_{\ell_\infty^2}$.

Proof Let $(a, b, c) \in \mathsf{S}_\Delta$, then $\|(a, c)\|_\infty \leq \|(a, b, c)\|_\Delta = 1$. Hence, the projection of B_Δ onto the ac-plane is contained in $\mathsf{B}_{\ell_\infty^2}$. Furthermore, since it is easy to see that the projection of B_Δ onto the ac-plane is a convex subset of the plane $\{(a, b, c)\colon b = 0\}$, it is enough to prove that $\mathsf{S}_{\ell_\infty^2}$ is contained in the projection of B_Δ onto the ac-plane.

If $(a, c) \in \mathsf{S}_{\ell_\infty^2}$, then

$$\|(a, 0, c)\|_\Delta = \begin{cases} \max\left\{1, \left|\frac{ac}{a+c}\right|\right\} & \text{if } a + c \neq 0 \text{ and } 0 < \frac{c}{a+c} < 1, \\ 1 & \text{otherwise.} \end{cases}$$

Also, notice that $\left|\frac{ac}{a+c}\right| < |a| \leq 1$ provided that $a + c \neq 0$ and $0 < \frac{c}{a+c} < 1$, which concludes the proof since this implies that $\|(a, 0, c)\|_\Delta = 1$. $\square$

Now we have the tools to give a parametrization of S_Δ and a characterization of the extreme points of B_Δ. Figure 4.1 shows an approximate representation of the unit ball of $\mathcal{P}(^2\Delta)$.

Theorem 4.3 (Muñoz et al. [46]) *If we define the mappings*

$$f_+(a, c) = 2 + 2\sqrt{(1 - a)(1 - c)}$$

and

$$f_-(a, c) = -f_+(-a, -c) = -2 - 2\sqrt{(1 + a)(1 + c)},$$

for every $(a, c) \in \mathsf{B}_{\ell_\infty^2}$ *and the set*

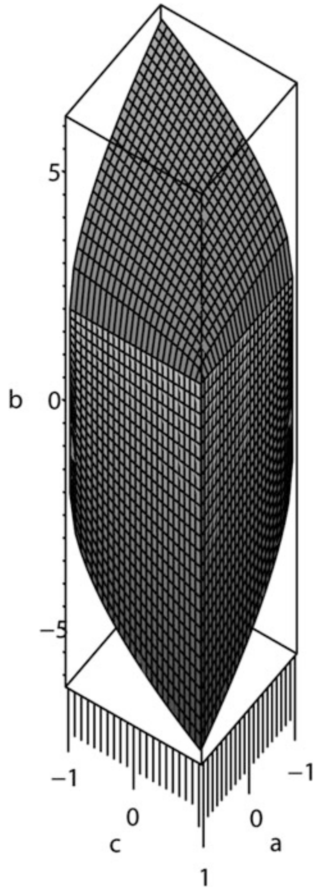
$$F = \{(a, b, c) \in \mathbb{R}^3 : (a, c) \in \mathsf{S}_{\ell_\infty^2} \text{ and } f_-(a, c) \leq b \leq f_+(a, c)\},$$

then

(i) $\mathsf{S}_\Delta = \operatorname{graph}\left(f_+|_{\mathsf{B}_{\ell_\infty^2}}\right) \cup \operatorname{graph}\left(f_-|_{\mathsf{B}_{\ell_\infty^2}}\right) \cup F$.

(ii)

Fig. 4.1 Unit ball of $\mathcal{P}(^2\Delta)$



$$\operatorname{ext}(\mathsf{B}_\Delta) = \left\{\pm\left(1, -2-2\sqrt{2(1+t)}, t\right) : t \in [-1,1]\right\}$$
$$\bigcup\left\{\pm\left(s, -2-2\sqrt{2(1+s)}, 1\right) : s \in [-1,1]\right\}$$
$$\bigcup\{\pm(1,1,1)\}.$$

Proof Part (i) is a direct consequence of Theorems 4.1 and 4.2. For part (ii), on one hand, notice that every $(a_0, b_0, c_0) \in F$ with $f_-(a_0, c_0) < b_0 < f_+(a_0, c_0)$ is in the interior of the segment $\{(a, b, c) \in \mathbb{R}^3 : (a, c) \in \mathsf{S}_{\ell_\infty^2}$ and $f_-(a_0, c_0) \leq b \leq f_+(a_0, c_0)\}$. On the other hand, for every $(a_0, c_0) \in \mathsf{S}_{\ell_\infty^2}$, the graph of the function f_+ (respectively f_-) is affine along the straight line $(1-c_0)a = (1-a_0)c + a_0 - c_0$ (respectively $(1+c_0)a = (1+a_0)c - a_0 + c_0$). This shows that the extreme points are at the points where graph$\left(f_+|_{\mathsf{B}_{\ell_\infty^2}}\right)$, graph$\left(f_-|_{\mathsf{B}_{\ell_\infty^2}}\right)$ and F intersect along a non-affine curve. It can be easily proved that this happens only at the points $\left(1, -2-2\sqrt{2(1+t)}, t\right)$ and $\pm\left(t, -2-2\sqrt{2(1+t)}, 1\right)$, where $t \in [-1, 1]$. □

4.1.1 Polynomials of Degree at Most 2

We will study now the extreme points of the unit ball of the space of polynomials of degree at most 2 in 2 real variables endowed with the supremum norm over Δ. We say that a polynomial P in $\mathbb{R}^2$ has degree at most 2 provided that $P(x, y) = a + bx + cy + dx^2 + exy + fy^2$, where $a, b, c, d, e, f \in \mathbb{R}$.

In order to show the extreme points of the unit ball of $\mathcal{P}_2(\Delta)$, we will distinguish between strictly definite, semidefinite and indefinite polynomials of degree exactly 2. Notice that in this case one of the eigenvalues of $M = \begin{pmatrix} d & e/2 \\ e/2 & f \end{pmatrix}$ is non-zero.

Definition 4.1 Let $P(x, y) = a + bx + cy + dx^2 + exy + fy^2$, where $a, b, c, d, e, f \in \mathbb{R}$, be a polynomial of degree 2 and take $M = \begin{pmatrix} d & e/2 \\ e/2 & f \end{pmatrix}$.

(i) We say that P is strictly positive (resp. negative) definite provided that the eigenvalues of M are positive (resp. negative).
(ii) We say that P is positive (resp. negative) semidefinite provided that one eigenvalue of M is positive (resp. negative) and the other is 0.
(iii) We say that P is indefinite provided that the eigenvalues of M are non-zero and have distinct sign.

Let us consider now the following construction and notation. Let $\{T_i\}_{i=1}^6$ be the affine transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ that map Δ onto itself given by $\{T_i(x, y)\}_{i=1}^6 = \{(x, y), (y, x), (1-x-y, y), (x, 1-x-y), (1-x-y, x), (y, 1-x-y)\}$. Notice that if P is an extreme polynomial of the unit ball of $\mathcal{P}_2(\Delta)$, then the polynomials $\{\pm P(T_i(x, y))\}_{i=1}^6$, known as *symmetrical* to P, are also extreme points. Given a polynomial P, we denote by $\mathsf{M}(P)$ the set of points (x, y) in Δ such that $|P(x, y)| = \|P\|_\Delta$.

Theorem 4.4 (Milev and Naidenov [43, 44]) *Let P be a polynomial of degree at most 2 in $\mathbb{R}^2$. If P has degree at most 1, then P is an extreme point of the unit ball of $\mathcal{P}_2(\Delta)$ if, and only if, $P \equiv \pm 1$. Assume now that P has degree 2.*

(i) If P is strictly (positive or negative) definite, then P is an extreme point of the unit ball of $\mathcal{P}_2(\Delta)$ if, and only if, P is of the form

$$P(x, y) = \pm[1 + \alpha(x - x_0)^2 + \beta(x - x_0)(y - y_0) + \gamma(y - y_0)^2],$$

where $\alpha = \frac{2(2y_0-1)}{x_0(1-x_0-y_0)}$, $\beta = -\frac{2(2x_0-1)(2y_0-1)}{x_0 y_0(1-x_0-y_0)}$, $\gamma = \frac{2(2x_0-1)}{y_0(1-x_0-y_0)}$ and (x_0, y_0) belongs to the interior of the triangle with vertices $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$.

(ii) *Assume that P is negative semidefinite (all positive semidefinite extreme points of the unit ball of $\mathcal{P}_2(\Delta)$ have the form $-P$, where P is negative semidefinite).*

(a) *If there exists (x_0, y_0) in the interior of Δ such that $P(x_0, y_0) = 1$, then P is an extreme point of the unit ball of $\mathcal{P}_2(\Delta)$ if, and only if, P satisfies the following conditions*

(1) $P(x, y) = 1 - [\alpha(x - x_0) + \beta(y - y_0)]^2, \ (\alpha, \beta) \neq (0, 0),$
(2) $\min\{P(0, 0), P(1, 0), P(0, 1)\} = -1.$

(b) *If $P(x, y) \neq 1$ for every (x, y) in the interior of Δ, then P is an extreme point of the unit ball of $\mathcal{P}_2(\Delta)$ if, and only if, P is one of the following polynomials*

(1) $P_1(x, y) = 1 - 2(x + y)^2,$
(2) $P_2(x, y) = 1 - 2(x - 1)^2,$
(3) $P_3(x, y) = 1 - 2(y - 1)^2,$
(4) $P_4(x, y) = 1 - 2(x + y - 1)^2,$
(5) $P_5(x, y) = 1 - 2x^2,$
(6) $P_6(x, y) = 1 - 2y^2.$

(iii) *Assume that P is indefinite.*

(a) *If $\mathsf{M}(P)$ is infinite, then P is an extreme point of the unit ball of $\mathcal{P}_2(\Delta)$ if, and only if, P is symmetrical to*

$$Q(x, y) = 1 - \frac{4}{\alpha}xy + \frac{2(1 - 2\alpha)}{\alpha^2}y^2,$$

where $\alpha = \frac{\sqrt{2}}{\sqrt{2}+\sqrt{1+\beta}}$ *and* $\beta \in [-1, 1]$.

(b) *If $\mathsf{M}(P)$ is finite, then P is an extreme point of the unit ball of $\mathcal{P}_2(\Delta)$ if, and only if, P is symmetrical to*

$$Q(x, y) = a + bx + cy + dx^2 + exy + fy^2,$$

where

$$\begin{aligned}
a &= \gamma, \\
b &= 2\sqrt{1-\gamma}(\sqrt{1-\alpha} + \sqrt{1-\gamma}), \\
c &= 2\sqrt{1-\gamma}(\sqrt{1-\beta} + \sqrt{1-\gamma}), \\
d &= -(\sqrt{1-\alpha} + \sqrt{1-\gamma})^2, \\
e &= -(\sqrt{1+\alpha} + \sqrt{1+\beta})^2 - (\sqrt{1-\alpha} + \sqrt{1-\gamma})^2 \\
&\quad - (\sqrt{1-\beta} + \sqrt{1-\gamma})^2,
\end{aligned}$$

$$f = -(\sqrt{1-\beta} + \sqrt{1-\gamma})^2,$$

with $(\alpha, \beta, \gamma) \in \bigcup_{i=1}^{4} \mathsf{P}_i$ *and*

$$\begin{aligned}
\mathsf{P}_1 &= \{(\alpha, \beta, \gamma) \colon \alpha, \beta, \gamma \in (-1, 1),\ \alpha \neq \beta\},\\
\mathsf{P}_2 &= \{(\pm 1, \beta, \gamma) \colon \beta, \gamma \in (-1, 1)\} \cup \{(\alpha, \pm 1, \gamma)\\
&\quad \colon \alpha, \gamma \in (-1, 1)\} \cup \{(\alpha, \beta, -1) \colon \alpha, \beta \in (-1, 1)\},\\
\mathsf{P}_3 &= \{(\alpha, \pm 1, -1) \colon \alpha \in (-1, 1)\} \cup \{(\pm 1, \beta, -1)\\
&\quad \colon \beta \in (-1, 1)\} \cup \{(\pm 1, \mp 1, \gamma) \colon \gamma \in (-1, 1)\},\\
\mathsf{P}_4 &= \{(\pm 1, \mp 1, -1), (1, 1, -1)\}.
\end{aligned}$$

4.2 The Unit Square

Consider the quadrilateral region in $\mathbb{R}^2$ enclosed by the vertices (0, 0), (1, 0), (0, 1) and (1, 1). We will denote this region by $\square$. It is easy to see that the set $\square$ is a non-balanced convex body. We want to sketch the set $\mathsf{S}_\square$ on $\mathbb{R}^3$, so we begin by showing an explicit formula for $\|\cdot\|_\square$.

Theorem 4.5 (Gámez et al. [23]) *If* $(a, b, c) \in \mathbb{R}^3$*, then* $\|(a, b, c)\|_\square$ *is equal to*

$$\begin{cases}
\max\left\{|a|, |c|, |a+b+c|, \frac{b^2-4ac}{4|c|}\right\} & \textit{if } b^2 - 4ac > 0,\ c \neq 0 \textit{ and } -\frac{b}{2c} \in (0, 1);\\
\max\left\{|a|, |c|, |a+b+c|, \frac{b^2-4ac}{4|a|}\right\} & \textit{if } b^2 - 4ac > 0,\ a \neq 0 \textit{ and } -\frac{b}{2a} \in (0, 1);\\
\max\{|a|, |c|, |a+b+c|\} & \textit{otherwise.}
\end{cases}$$

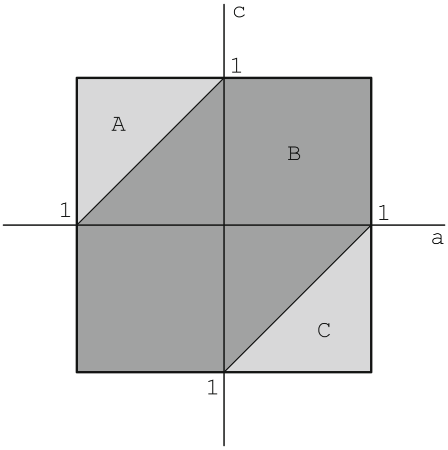
Just as we did in the previous case, we project $\mathsf{B}_\square$ onto the ac-plane. This projection of $\mathsf{B}_\square$ onto the ac-plane is once again the set $\mathsf{B}_{\ell_\infty^2}$. However, due to technical difficulties in the proof of this theorem, the set $\mathsf{B}_{\ell_\infty^2}$ is divided into three regions defined by

$$\begin{aligned}
A &= \{(a, c) \in \mathsf{B}_{\ell_\infty^2} \colon -1 \leq a \leq 0 \text{ and } a + 1 \leq c \leq 1\},\\
B &= \left\{(a, c) \in \mathsf{B}_{\ell_\infty^2} \colon -1 \leq a \leq 1 \text{ and } \max\{-1, a-1\} \leq c \leq \min\{1, a+1\}\right\},\\
C &= \{(a, c) \in \mathsf{B}_{\ell_\infty^2} \colon 0 \leq a \leq 1 \text{ and } -1 \leq c \leq a - 1\}.
\end{aligned}$$

Figure 4.2 shows how the set $\mathsf{B}_{\ell_\infty^2}$ is decomposed into the sets A, B and C.

Theorem 4.6 (Gámez et al. [23]) *The projection of* $\mathsf{B}_\square$ *onto the* ac*-plane is* $\mathsf{B}_{\ell_\infty^2}$.

Fig. 4.2 The projection of $\mathsf{B}_\square$ onto the ac-plane



Given the explicit formula for $\|\cdot\|_\square$ and the projection of $\mathsf{B}_\square$ onto the ac-plane, we can show a parametrization of $\mathsf{S}_\square$ as well as the extreme points of $\mathsf{S}_\square$. Figure 4.3 shows a sketch of $\mathsf{S}_\square$.

Theorem 4.7 (Gámez et al. [23]) *If for every $(a, c) \in \mathsf{B}_{\ell_\infty^2}$ we define the mappings*

$$F(a,c) = \begin{cases} 2\sqrt{ac + |a|} & \text{if } (a,c) \in A, \\ 2\sqrt{ac + |c|} & \text{if } (a,c) \in C, \\ 1 - a - c & \text{if } (a,c) \in B, \end{cases}$$

$$G(a,c) = -F(-a,-c),$$

where A, B and C are as in Fig. 4.2 and the set

$$H = \left\{(a,b,c) \in \mathbb{R}^3 : (a,c) \in \partial\mathsf{B}_{\ell_\infty^2} \text{ and } G(a,c) \le b \le F(a,c)\right\},$$

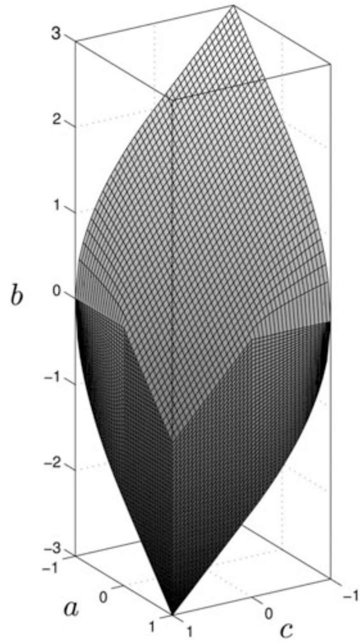
where $\partial\mathsf{B}_{\ell_\infty^2}$ is the boundary of $\mathsf{B}_{\ell_\infty^2}$, then

(a) $\mathsf{S}_\square = \mathrm{graph}(F) \cup \mathrm{graph}(G) \cup H.$

(b)

$$\begin{aligned} \mathrm{ext}(\mathsf{B}_\square) = &\left\{\pm\left(t, 2\sqrt{1-t}, -1\right) : t \in [0,1]\right\} \\ &\bigcup \left\{\pm\left(-1, 2\sqrt{1-s}, s\right) : s \in [0,1]\right\} \\ &\bigcup\{\pm(1,-1,1)\}\bigcup\{\pm(1-3,1)\}\bigcup\{\pm(1,0,0)\}\bigcup\{\pm(0,0,1)\}. \end{aligned}$$

Fig. 4.3 Unit ball of $\mathcal{P}(^2\square)$



4.3 Circular Sectors

For every $\alpha, \beta \in [0, 2\pi]$ with $\alpha \leq \beta$ we define the sector $D(\alpha, \beta)$ as

$$D(\alpha, \beta) := \left\{ re^{i\theta} : 0 \leq r \leq 1,\ \alpha \leq \theta \leq \beta \right\}.$$

If $\alpha = 0$, we use $D(\beta)$ instead of $D(0, \beta)$. Notice that $D(\alpha, \beta)$ is a non-balanced convex body in $\mathbb{R}^2$.

The mapping T defined in Sect. 2.1 over the space $\mathcal{P}(^2D(\alpha, \beta))$ is a topological isomorphism between the space $\mathcal{P}(^2D(\alpha, \beta))$ and $(\mathbb{R}^3, \|\cdot\|_{D(\alpha,\beta)})$, where

$$\|(a, b, c)\|_{D(\alpha,\beta)} = \|P\|_{D(\alpha,\beta)}.$$

The sets $\mathsf{B}_{D(\alpha,\beta)}$ and $\mathsf{S}_{D(\alpha,\beta)}$ denote the unit ball and the unit sphere of $(\mathbb{R}^3, \|\cdot\|_{D(\alpha,\beta)})$, respectively.

Remark 4.1 For every $\alpha, \beta \in [0, 2\pi]$, the spaces $\mathcal{P}(^2D(\alpha, \alpha + \beta))$ and $\mathcal{P}(^2D(\beta))$ are isometric. Indeed, the mapping

$$\begin{aligned} Q : \mathcal{P}(^2D(\alpha, \alpha + \beta)) &\to \mathcal{P}(^2D(\beta)) \\ P &\mapsto P \circ \Phi \end{aligned}$$

where Φ is a rotation of angle α in $\mathbb{R}^2$ and with center the origin given by

$$\Phi(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha),$$

for all $(x, y) \in \mathbb{R}^2$, is an isometry between $\mathcal{P}(^2D(\alpha, \alpha+\beta))$ and $\mathcal{P}(^2D(\beta))$. In fact, the rotation Φ is a bijection from $D(\alpha, \alpha + \beta)$ onto $D(\beta)$. It can be easily seen that the mapping Q is defined by the matrix

$$\begin{pmatrix} \cos^2\alpha & \sin^2\alpha & \frac{\sin 2\alpha}{2} \\ \sin^2\alpha & \cos^2\alpha & -\frac{\sin 2\alpha}{2} \\ -\sin 2\alpha & \sin 2\alpha & \cos 2\alpha \end{pmatrix}.$$

Hence, it suffices to study the geometry of $\mathsf{B}_{D(\beta)}$.

In this section, we are interested in studying the geometry of the unit sphere of $\mathcal{P}(^2D(\beta))$ where $\beta \geq 0$. In particular, we study different cases. The case when $\beta \geq \pi$, the extreme cases when $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and finally the remaining cases.

4.3.1 *The Geometry of* $\mathsf{B}_{D(\beta)}$ *When* $\beta \geq \pi$

As in the previous sections, we begin by giving an explicit formula for $\|\cdot\|_{D(\beta)}$ when $\beta \geq \pi$.

Theorem 4.8 (Muñoz et al. [45]) *Let $(a, b, c) \in \mathbb{R}^3$. If $\beta \geq \pi$, then*

$$\|(a, b, c)\|_{D(\beta)} = \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2 + c^2}\right).$$

Moreover, the norm does not depend on the angle β.

Proof Let $(a, b, c) \in \mathbb{R}^3$ and take $P(x, y) = ax^2 + by^2 + cxy$. Since $|P(-x, -y)| = |P(x, y)|$, notice that the supremum of $|P|$ over the whole unit disk is equal to the supremum of $|P|$ taken over $D(\beta)$. Hence, we can calculate the supremum of $|P|$ over the whole unit disk in order to obtain the desired result. The polynomial P restricted to the unit circle parametrized by $\{(\cos\theta, \sin\theta) : \theta \in [0, 2\pi]\}$ is of the form

$$\begin{aligned} f(\theta) &= P(\cos\theta, \sin\theta) = a\cos^2\theta + b\sin^2\theta + c\sin\cos\theta \\ &= a\frac{1+\cos(2\theta)}{2} + b\frac{1-\cos(2\theta)}{2} + c\frac{\sin(2\theta)}{2} \\ &= \frac{1}{2}\left[a + b + (a-b)\cos(2\theta) + c\sin(2\theta)\right] \end{aligned}$$

for every $\theta \in [0, 2\pi]$. Thus, it is easy to see that

$$\|(a,b,c)\|_{D(\beta)} = \sup\{|f(\theta)| : \theta \in [0, 2\pi]\} = \frac{1}{2}\left(|a+b| + \|(a-b,c)\|_{\ell_2^2}\right)$$
$$= \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2 + c^2}\right).$$

□

Remark 4.2 (Muñoz et al. [45]) By the proof of Theorem 4.8, notice that the case for $\beta \geq \pi$ is the same as the case for the space ℓ_2^2, that is:

If $P(x,y) = ax^2 + by^2 + cxy$ where $(x,y) \in \ell_2^2$, then the norm of P over ℓ_2^2 is defined as

$$\|P\|_{\ell_2^2} = \sup\{|P(x,y)| : \|(x,y)\|_{\ell_2^2} \leq 1\}.$$

Notice that $\|P\|_{\ell_2^2} = \|P\|_{D(\beta)}$ for $\beta \geq \pi$. Thus, $\mathsf{B}_{D(\beta)} = \mathsf{B}_{\ell_2^2}$. This case for the space ℓ_2^2 has been done in a different way in [15].

The easy explicit formula for the norm allows us to simplify the calculations and instead of using a projection onto a plane, we can give directly an explicit parametrization of $\mathsf{S}_{D(\beta)}$. A sketch of the unit ball of $\mathcal{P}(^2D(\beta))$ can be seen in Fig. 4.4.

Theorem 4.9 (Muñoz et al. [45]) *Let $\beta \geq \pi$. If we define*

$$f(a,b) = 2\sqrt{1 + ab - |a+b|},$$

for all $(a,b) \in [-1,1]^2$, then

(i) $\mathsf{S}_{D(\beta)} = \text{graph}(f) \cup \text{graph}(-f)$.
(ii)

$$\text{ext}\left(\mathsf{B}_{D(\beta)}\right) = \left\{\pm\left(a, -a, \sqrt{1-a^2}\right) : a \in [-1,1]\} \cup \{\pm(1,1,0)\right\}.$$

Proof Part (i) of the proof is an easy consequence of solving c in terms of a and b in the equation $1 = \|(a,b,c)\|_{D(\beta)} = \frac{1}{2}\left(|a+b| + \sqrt{(a-b)^2 + c^2}\right)$. For part (ii), let $a \in [-1,1]$ and consider the segment that joins $(-1,1)$ and $(a,-a)$ which has the form $S = \{(-1 + \lambda(a+1), -1 + \lambda(-a+1)) : \lambda \in [0,1]\}$. If we restrict f (and also $-f$) to S, then we have

$$f(-1 + \lambda(a+1), -1 + \lambda(-a+1)) = 2\lambda\sqrt{1-a^2},$$

where $\lambda \in [0,1]$. Notice that $f(-1 + \lambda(a+1), -1 + \lambda(-a+1))$, where $\lambda \in [0,1]$, is a segment. Analogously, if we restrict f (and also $-f$) to the segment joining $(1,1)$ with the point $(a,-a)$, then we have also a segment. And this is true for every $a \in [-1,1]$. Thus, the extreme points of $\mathsf{B}_{D(\beta)}$ are contained in the set

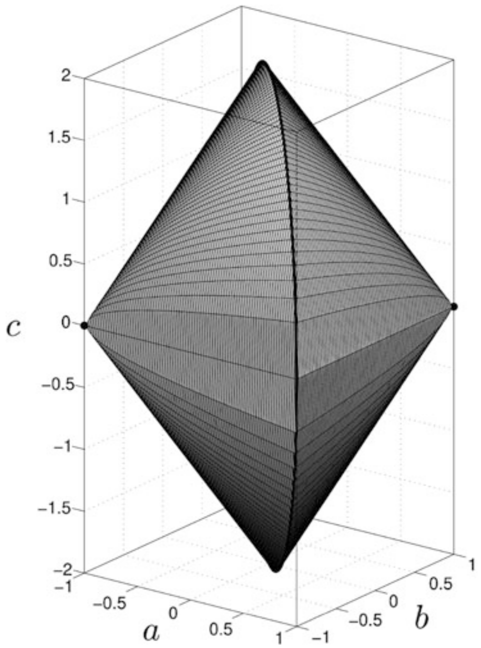


Fig. 4.4 Unit ball of $\mathcal{P}(^2D(\beta))$ when $\beta \geq \pi$. The extreme points of the unit ball are drawn with a thicker line and dots

$$M = \{(a, -a, \pm f(a, -a)) \colon a \in [-1, 1]\} \cup \{\pm(1, 1, 0)\}.$$

It is easy to prove that for every $p \in M$, there exists a plane $\Pi \subset \mathbb{R}^3$ such that $\Pi \cap M = \{p\}$ and this concludes the proof. $\square$

4.3.2 The Geometry of $\mathsf{B}_{D(\beta)}$ *When* $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

We begin by giving an explicit formula of $\|\cdot\|_{D(\beta)}$.

Theorem 4.10 (Muñoz et al. [45]) *For all* $a, b, c \in \mathbb{R}$, *we have that* $\|(a, b, c)\|_{D(\frac{\pi}{4})}$, $\|(a, b, c)\|_{D(\frac{\pi}{2})}$ *and* $\|(a, b, c)\|_{D\left(\frac{3\pi}{4}\right)}$ *are given, respectively, by*

$$\begin{cases} \max\left\{|a|, \frac{1}{2}|a+b+c|, \frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\} & \textit{if } c(a-b) \geq 0, \\ \max\{|a|, \frac{1}{2}|a+b+c|\} & \textit{if } c(a-b) \leq 0, \end{cases}$$

$$\max\left\{|a|, |b|, \frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\}, \qquad \textit{and}$$

$$\begin{cases} \frac{1}{2}\left(|a+b|+\sqrt{(a-b)^2+c^2}\right) & \text{if } c(a-b)\geq 0, \\ \max\left\{|a|, \frac{1}{2}|a+b-c|, \frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\} & \text{if } c(a-b)\leq 0. \end{cases}$$

In contrast with the previous case, we proceed as in other sections by projecting $\mathsf{B}_{D(\frac{\pi}{4})}$, $\mathsf{B}_{D(\frac{\pi}{2})}$ and $\mathsf{B}_{D\left(\frac{3\pi}{4}\right)}$ onto the ab-plane. Once we have the projection, we can give a parametrization of $\mathsf{S}_{D(\frac{\pi}{4})}$, $\mathsf{S}_{D(\frac{\pi}{2})}$ and $\mathsf{S}_{D\left(\frac{3\pi}{4}\right)}$. Let us begin with the case $\mathsf{B}_{D(\frac{\pi}{4})}$. To do so, let us define first the following sets

$$A = \{(a,b):\ a\in[-1,1],\ a<b\leq \gamma_1(a)\}, \tag{4.1}$$

$$B = \{(a,b):\ a\in[-1,1],\ \gamma_2(a)\leq b\leq a\}, \tag{4.2}$$

where γ_1, γ_2 are functions defined by

$$\gamma_1(a) = 4+a+4\sqrt{1+a},$$
$$\gamma_2(a) = -\gamma_1(-a) = -4+a-4\sqrt{1-a},$$

where $a\in[-1,1]$.

Theorem 4.11 (Muñoz et al. [45]) *The projection of* $\mathsf{B}_{D(\frac{\pi}{4})}$ *onto the ab-plane is the set* $\{(a,b):\ a\in[-1,1],\ \gamma_2(a)\leq b\leq\gamma_1(a)\}$.

An approximate representation of the projection of $\mathsf{B}_{D(\frac{\pi}{4})}$ onto the ab-plane can be seen in Fig. 4.5.

Theorem 4.12 (Muñoz et al. [45]) *Let A and B be defined as before Theorem 4.11 and let us define the mappings*

$$F_1(a,b) = \begin{cases} 2-a-b & \text{if } (a,b)\in A, \\ 2\sqrt{(1-a)(1-b)} & \text{if } (a,b)\in B, \end{cases}$$

and $F_2(a,b) = -F_1(-a,-b)$ *for all* $(a,b)\in\pi_{ab}\left(\mathsf{S}_{D(\frac{\pi}{4})}\right)$. *If*

$$\Gamma = \left\{(\pm1,b,c)\in\mathbb{R}^2:\ (\pm1,b)\in\partial\pi_{ab}\left(\mathsf{B}_{D(\frac{\pi}{4})}\right),\ F_2(\pm1,b)\leq c\leq F_1(\pm1,b)\right\},$$

then

(i) $\mathsf{S}_{D(\frac{\pi}{4})} = \operatorname{graph}(F_1)\cup\operatorname{graph}(F_2)\cup\Gamma$.
(i)

$$\operatorname{ext}\left(\mathsf{B}_{D(\frac{\pi}{4})}\right) = \left\{\pm\left(t, 4+t+4\sqrt{1+t}, -2-2t-4\sqrt{1+t}\right):\ t\in[-1,1]\right\}$$

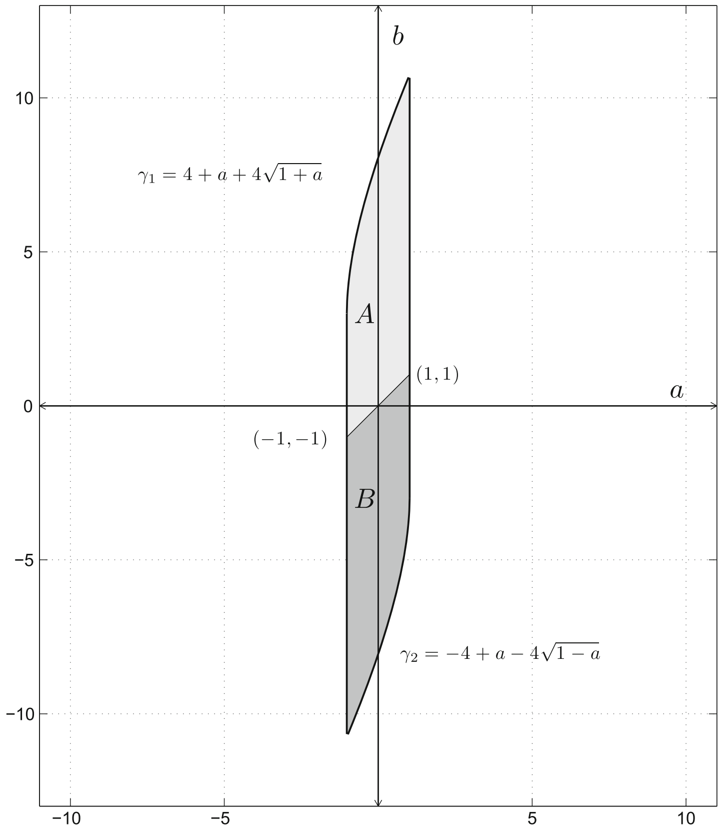


Fig. 4.5 Projection of $\mathsf{B}_{D(\frac{\pi}{4})}$ onto the *ab*-plane

$$\bigcup \left\{ \pm \left(1, s, -2\sqrt{2(1+s)}\right) : s \in \left[1, 5 + 4\sqrt{2}\right] \right\}$$
$$\bigcup \{\pm(1, 1, 0)\}.$$

A representation of the unit sphere of $\mathcal{P}\left({}^2 D\left(\frac{\pi}{4}\right)\right)$ appears in Fig. 4.6.

Now we turn our attention to the space $\mathcal{P}\left({}^2 D\left(\frac{\pi}{2}\right)\right)$.

Theorem 4.13 (Muñoz et al. [45]) *The projection of* $\mathsf{B}_{D(\frac{\pi}{2})}$ *onto the ab-plane is* $\mathsf{B}_{\ell_\infty^2}$.

Theorem 4.14 (Muñoz et al. [45]) *If we define the mappings*

$$G_1(a, b) = 2\sqrt{(1-a)(1-b)}$$

and

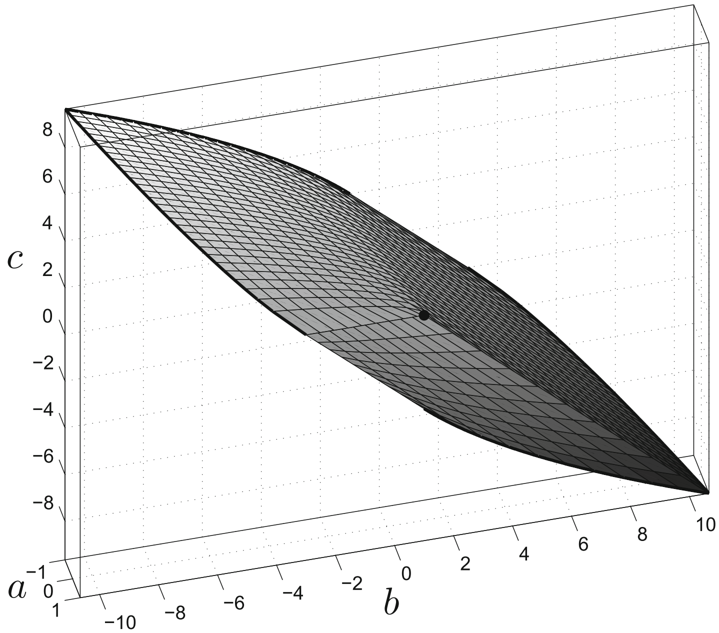


Fig. 4.6 Unit ball of $\mathcal{P}(^2D(\beta))$. The extreme points of the unit ball are drawn with a thicker line and dots

$$G_2(a,b) = -f_+(-a,-b) = -2\sqrt{(1+a)(1+b)},$$

for every $(a,b) \in \mathsf{B}_{\ell_\infty^2}$ *and the set*

$$\Upsilon = \left\{(a,b,c) \in \mathbb{R}^3 : (a,b) \in \partial\mathsf{B}_{\ell_\infty^2} \text{ and } G_2(a,b) \le c \le G_1(a,b)\right\},$$

then

(i) $\mathsf{S}_{D(\frac{\pi}{2})} = \operatorname{graph}(G_1) \cup \operatorname{graph}(G_2) \cup \Upsilon.$

(ii)

$$\begin{aligned}\operatorname{ext}\left(\mathsf{B}_{D(\frac{\pi}{2})}\right) = &\left\{\pm\left(1, t, -2\sqrt{2(1+t)}\right) : t \in [-1,1]\right\}\\ &\bigcup\left\{\pm\left(s, 1, -2\sqrt{2(1+s)}\right) : s \in [-1,1]\right\}\bigcup\{\pm(1,1,0)\}.\end{aligned}$$

Figure 4.7 shows what the unit sphere of $\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$ looks like.

Due to difficult calculations, to prove that the projection of $\mathsf{B}_{D\left(\frac{3\pi}{4}\right)}$ onto the *ab*-plane is $\mathsf{B}_{\ell_\infty^2}$, we define the following sets: Let C, D and F be as in Fig. 4.8, namely

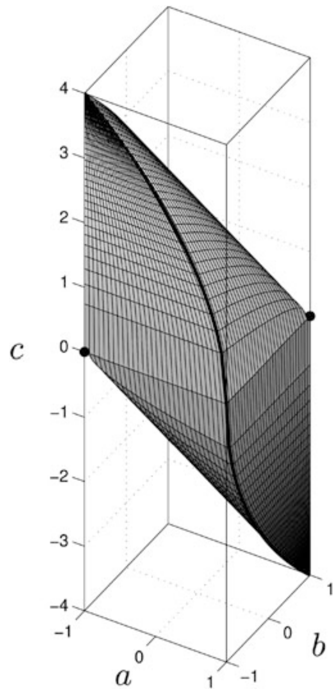


Fig. 4.7 Unit ball of $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$. The extreme points of the unit ball are drawn with a thicker line and dots

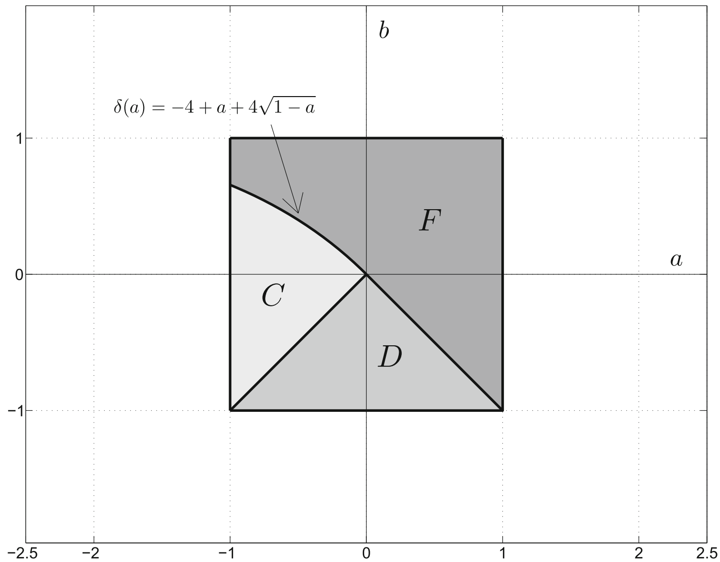


Fig. 4.8 Projection of $\mathsf{B}_{D(\frac{3\pi}{4})}$ onto the ab-plane

$$C = \{(a, b) : a \in [-1, 0], \ 0 < b \le \delta(a)\},$$
$$D = \{(a, b) : a \in [-1, 1], \ -1 \le b \le -|a|\},$$
$$F = [-1, 1]^2 \setminus (C \cup D),$$

where $\delta(a) = -4 + a + 4\sqrt{1-a}$ for $a \in [-1, 0]$. Notice that $\mathsf{B}_{\ell_\infty^2} = C \cup D \cup F$.

Theorem 4.15 (Muñoz et al. [45]) *The projection of* $\mathsf{B}_{D\left(\frac{3\pi}{4}\right)}$ *onto the ab-plane is* $\mathsf{B}_{\ell_\infty^2}$.

Theorem 4.16 (Muñoz et al. [45]) *Let us define the mappings*

$$H_1(a, b) = \begin{cases} 2 + a + b & \text{if } (a, b) \in C, \\ 2\sqrt{(1+a)(1+b)} & \text{if } (a, b) \in D, \\ 2\sqrt{(1-a)(1-b)} & \text{if } (a, b) \in F, \end{cases}$$

where C*,* D *and* F *were defined before Theorem 4.15, and* $H_2(a, b) = -H_1(-a, -b)$ *for all* $(a, b) \in \mathsf{B}_{\ell_\infty^2}$*. If*

$$\Omega = \{\pm(-1, b, c) \in \mathbb{R}^3 : \ -1 \le b \le 1, \ 0 \le c \le H_1(-1, b)\},$$

then

(i) $\mathsf{S}_{D\left(\frac{3\pi}{4}\right)} = \text{graph}(H_1) \cup \text{graph}(H_2) \cup \Omega.$

(ii)

$$\begin{aligned} \text{ext}\left(\mathsf{B}_{D(3\frac{\pi}{4})}\right) = &\left\{\pm\left(t, -4 + t + 4\sqrt{1-t}, -2 + 2t + 4\sqrt{1-t}\right) : t \in [-1, 0]\right\} \\ &\bigcup \left\{\pm\left(s, -s, 2\sqrt{1-s^2}\right) : s \in [0, 1]\right\} \\ &\bigcup \left\{\pm\left(-1, r, 2\sqrt{2(1-r)}\right) : r \in \left[-5 + 4\sqrt{2}, 1\right]\right\} \\ &\bigcup \{\pm(1, 1, 0)\}. \end{aligned}$$

Figure 4.9 shows two different points of view of $\mathsf{B}_{D\left(\frac{3\pi}{4}\right)}$.

4.3.3 The General Case of $\mathsf{B}_{D(\beta)}$

We have considered in Sects. 4.3.1 and 4.3.2, the cases of $\mathcal{P}(^2D(\beta))$ when $\beta \ge \pi$ and $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. In this last segment we are interested in the geometry of

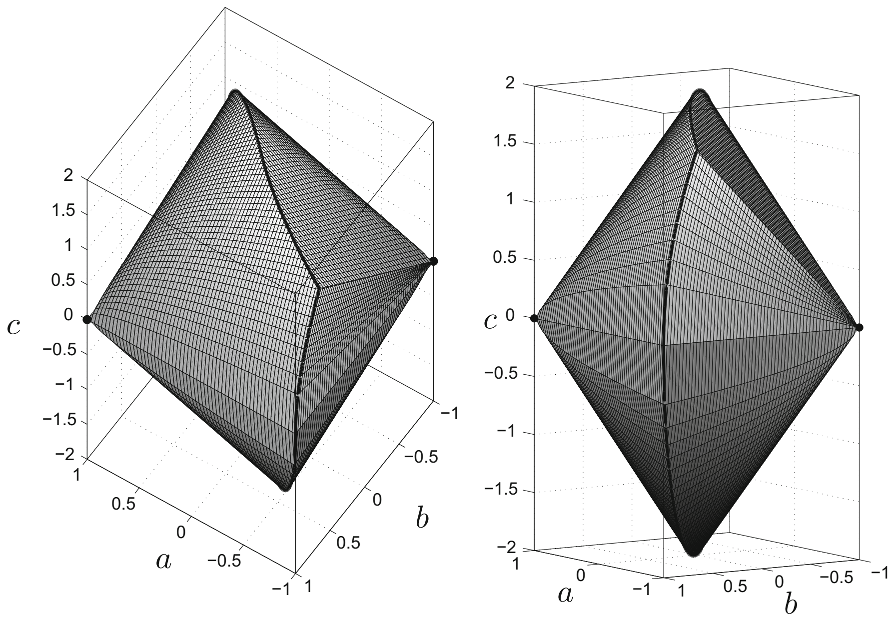


Fig. 4.9 Two different perspectives of the unit ball of $\mathcal{P}\left({}^2D\left(\frac{3\pi}{4}\right)\right)$. The extreme points of the unit ball have been drawn with a thicker line and dots

$\mathcal{P}({}^2D(\beta))$ when $\beta \in \left(0, \frac{\pi}{4}\right)$, $\beta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$, or $\beta \in \left(\frac{3\pi}{4}, \pi\right)$. In fact, it is easy to notice in the following results that the particular cases $\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ are just the limit cases of the previous ranges of β. To simplify the notation, we are using N and M to denote $\cos(2\beta)$ and $\sin(2\beta)$, respectively. We begin by showing an explicit formula for the norm in each one of these four ranges of β.

Theorem 4.17 (Bernal et al. [6]) *If $a, b, c \in \mathbb{R}$, then $\|(a, b, c)\|_{D(\beta)}$ where $0 < \beta < \frac{\pi}{4}$, $\frac{\pi}{4} < \beta < \frac{\pi}{2}$, $\frac{\pi}{2} < \beta < \frac{3\pi}{4}$ or $\frac{3\pi}{4} < \beta < \pi$ is given, respectively, by*

$$\begin{cases} \max\Big\{|a|,\ \frac{1}{2}\left|(1+M)a + (1-M)b + cN\right|, & \\ \qquad \frac{1}{2}\left|a + b + \operatorname{sign}(c)\sqrt{(a-b)^2 + c^2}\right|\Big\}, & \textit{(1)} \\ \max\{|a|,\ \frac{1}{2}\left|(1+M)a + (1-M)b + cN\right|\}, & \textit{(2)} \end{cases}$$

(1) if $c(a-b) > 0$ *and* $\tan(2\beta) > \dfrac{c}{a-b}$,

(2) if $\left(c(a-b) > 0 \text{ and } \tan(2\beta) \le \dfrac{c}{a-b}\right)$ *or* $(c(a-b) \le 0)$,

$$\begin{cases} \max\left\{|a|,\ \frac{1}{2}\left|(1+M)a+(1-M)b+cN\right|,\right. & \\ \qquad \left.\frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\}, & \textit{(3)} \\ \max\{|a|,\ \frac{1}{2}\left|(1+M)a+(1-M)b+cN\right|\}, & \textit{(4)} \end{cases}$$

(3) if $\left(c(a-b)<0 \textit{ and } \tan(2\beta)>\dfrac{c}{a-b}\right)$ *or* $(c(a-b)>0)$ *or* $(a=b)$,

(4) if $\left(c(a-b)<0 \textit{ and } \tan(2\beta)\le\dfrac{c}{a-b}\right)$ *or* $(c=0)$,

$$\begin{cases} \frac{1}{2}\left(|a+b|+\sqrt{(a-b)^2+c^2}\right), & \textit{(5)} \\ \max\left\{|a|,\ \frac{1}{2}\left|(1+M)a+(1-M)b+cN\right|,\right. & \\ \qquad \left.\frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\}, & \textit{(6)} \end{cases}$$

(5) if $\left(c(a-b)>0 \textit{ and } \tan(2\beta)>\dfrac{c}{a-b}\right)$ *or* $(c=0)$,

(6) if $\left(c(a-b)>0 \textit{ and } \tan(2\beta)\le\dfrac{c}{a-b}\right)$ *or* $(c(a-b)<0)$ *or* $(a=b)$,

$$\begin{cases} \frac{1}{2}\left(|a+b|+\sqrt{(a-b)^2+c^2}\right), & \textit{(7)} \\ \max\left\{|a|,\ \frac{1}{2}\left|(1+M)a+(1-M)b+cN\right|,\right. & \\ \qquad \left.\frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right|\right\}, & \textit{(8)} \end{cases}$$

(7) if $\left(c(a-b)<0 \textit{ and } \tan(2\beta)>\dfrac{c}{a-b}\right)$ *or* $(c(a-b)\ge 0)$,

(8) if $c(a-b)<0 \textit{ and } \tan(2\beta)\le\dfrac{c}{a-b}$.

Since we know an explicit formula of the norm, we proceed to show the projection of the unit ball onto the ab-plane which is used to obtain a parametrization of the unit sphere.

We begin with the cases when $\beta\in\left(0,\frac{\pi}{4}\right)$ and $\beta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$, because its approach is not the same as the other two cases. Let us define first the following functions from $[-1,1]$ to $\mathbb{R}$ and sets in $\mathbb{R}^2$:

$$\gamma_1(a)=\frac{(1+M)a+2(2+M)+4\sqrt{(1+M)(1+a)}}{1-M},$$

$$\gamma_2(a)=-\gamma_1(-a)=\frac{(1+M)a-2(2+M)-4\sqrt{(1+M)(1-a)}}{1-M},$$

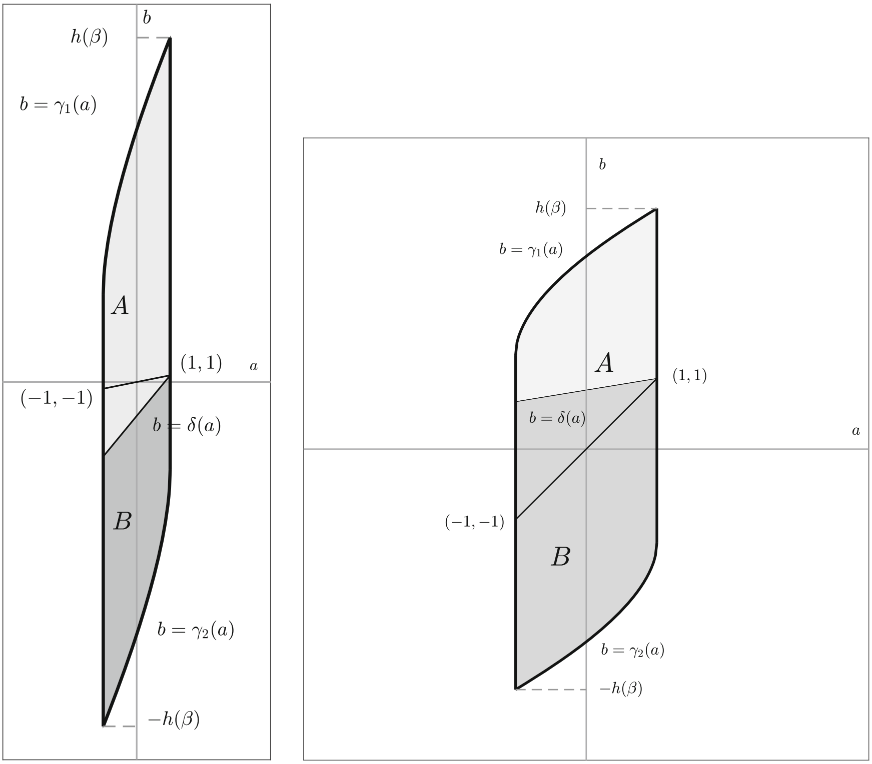


Fig. 4.10 Sketch of the projection of $\mathsf{B}_{D(\beta)}$ onto the ab-plane with $\beta \in (0, \frac{\pi}{2})$. On the left we have the case $\beta = \frac{\pi}{4} - 0.4$. The b axis has been rescaled by a factor 0.2. On the right we have represented the case $\beta = \frac{\pi}{4} + 0.4$. The axis here are scaled. In both cases $h(\beta) = \frac{5+3M+4\sqrt{2(1+M)}}{1-M}$

$$\delta(a) = \frac{(1+M)a - 2M}{1-M},$$

$$A = \{(a, b) \in \mathbb{R}^2 : a \in [-1, 1], \ \delta(a) \leq b \leq \gamma_1(a)\},$$

$$B = \{(a, b) \in \mathbb{R}^2 : a \in [-1, 1], \ \gamma_2(a) \leq b \leq \delta(a)\}.$$

The definition of these sets is to simplify the parametrization of the unit sphere.

Theorem 4.18 (Bernal et al. [6]) *Let $\beta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. The projection of $\mathsf{B}_{D(\beta)}$ onto the ab-plane is the set $\mathsf{B}_{D(\beta)} = \{(a, b) \in \mathbb{R}^2 : a \in [-1, 1], \ \gamma_2(a) \leq b \leq \gamma_1(a)\}$. See Fig. 4.10 for a sketch of the projections.*

The next result shows a parametrization of $\mathsf{S}_{D(\beta)}$ when $\beta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Theorem 4.19 (Bernal et al. [6]) *Let $\beta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and let A and B be defined as before Theorem 4.18. Define the mappings*

$$F_1(a,b) = \begin{cases} \frac{1}{N}[2-(1+M)a-(1-M)b] & \textit{if } (a,b) \in A, \\ 2\sqrt{(1-a)(1-b)} & \textit{if } (a,b) \in B, \end{cases}$$

and $F_2(a,b) = -F_1(-a,-b)$ *for all* $(a,b) \in \pi_{ab}(\mathsf{B}_{D(\beta)})$. *If*

$$\Gamma_F = \left\{(\pm 1, b, c) \in \mathbb{R}^2 : \ (\pm 1, b) \in \partial\pi_{ab}(\mathsf{B}_{D(\beta)}), \ F_2(\pm 1, b) \le c \le F_1(\pm 1, b)\right\},$$

then

(i) $\mathsf{S}_{D(\beta)} = \mathrm{graph}(F_1) \cup \mathrm{graph}(F_2) \cup \Gamma_F$. *See Figs. 4.11 and 4.12 for a sketch of* $\mathsf{S}_{D(\beta)}$.
(ii) The set $\mathrm{ext}(\mathsf{B}_{D(\beta)})$ *consists of the elements*

$$\pm\Bigg(t, \frac{(1+M)t + 2(2+M) + 4\sqrt{(1+M)(1+t)}}{1-M},$$
$$\frac{1}{N}\left[-2(1+M)(1+t) - 4\sqrt{(1+M)(1+t)}\right]\Bigg) \quad \textit{for} \quad t \in [-1,1],$$

$$\pm\left(1, s, -2\sqrt{2(1+s)}\right) \quad \textit{for} \quad s \in \left[\frac{1+3M}{1-M}, \frac{5+3M+4\sqrt{2(1+M)}}{1-M}\right],$$

and

$$\pm\,(1,1,0).$$

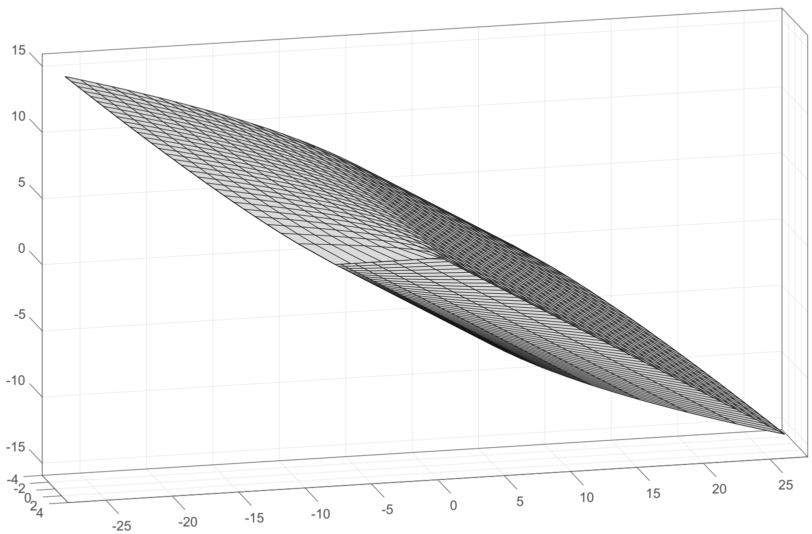


Fig. 4.11 Unit ball of $\mathcal{P}_{D(\beta)}$ with $\beta = \pi/6$

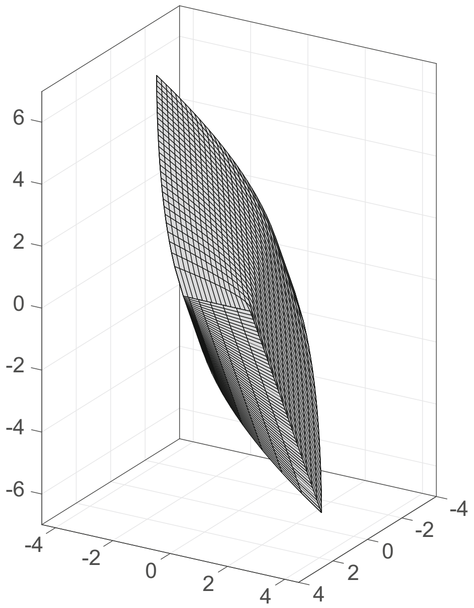


Fig. 4.12 Unit ball of $\mathcal{P}_{D(\beta)}$ with $\beta = 3\pi/8$

To finish the section, we show the projection of the unit ball, a parametrization of the unit sphere and its extreme points when $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ and $\beta \in \left(\frac{3\pi}{4}, \pi\right)$. Using the same arguments as in the previous cases, we begin by defining the following sets in $\mathbb{R}^2$:

$$C = \left\{(a,b) \in \mathbb{R}^2 \colon a \in [-1,1],\ b \le \gamma_3(a) \text{ and } b \ge \widehat{\delta}(a)\right\},$$

$$D = \left\{(a,b) \in \mathbb{R}^2 \colon a \in [-1,1],\ b \le -a \text{ and } b \le \widehat{\delta}(a)\right\},$$

$$E = [-1,1]^2 \backslash (C \cup D),$$

where

$$\gamma_3(a) = \frac{(1+M)a - 2(2+M) + 4\sqrt{(1+M)(1-a)}}{1-M},$$

$$\widehat{\delta}(a) = \frac{(1+M)a + 2M}{1-M}.$$

Notice that $\mathsf{B}_{\ell_\infty^2} = C \cup D \cup E$. The reason why we define these sets is just to simplify the parametrization of the unit sphere.

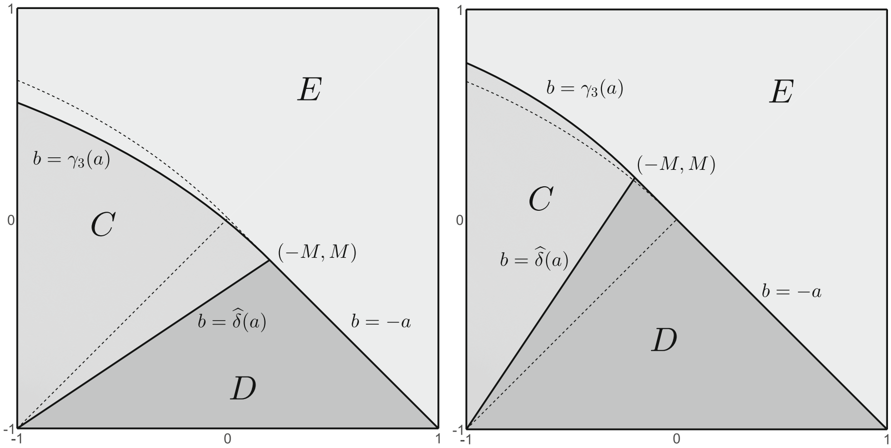


Fig. 4.13 Scaled pictures of the projections onto the ab-plane of $\mathsf{B}_{D(\beta)}$ with $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$. On the left we depict the sets C, D and E for $\beta = \pi/4 - 0.1$. On the right, we have C, D and E for $\beta = \pi/4 + 0.1$. In both cases, we draw the case $\beta = \pi/4$ with a dashed line

Theorem 4.20 (Bernal et al. [6]) *Let* $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$. *The projection of* $\mathsf{B}_{D(\beta)}$ *onto the ab-plane is* $\mathsf{B}_{\ell_\infty^2}$. *See Fig. 4.13 for a better understanding of the projection.*

Theorem 4.21 (Bernal et al. [6]) *Let* $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$ *and let* C, D *and* E *be the sets defined before Theorem 4.20. Define*

$$G_1(a,b) = \begin{cases} -\frac{1}{N}[2 + (1+M)a + (1-M)b] & \text{if } (a,b) \in C, \\ 2\sqrt{(1+a)(1+b)} & \text{if } (a,b) \in D, \\ 2\sqrt{(1-a)(1-b)} & \text{if } (a,b) \in E, \end{cases}$$

and $G_2(a,b) = -G_1(-a,-b)$ *for all* $(a,b) \in \pi_{ab}(\mathsf{B}_{D(\beta)})$. *Define also the set*

$$\Omega_G = \left\{(1,b,c) \in \mathbb{R}^2 : \ (1,b) \in \partial\pi_{ab}(\mathsf{B}_{D(\beta)}), \ 0 \geq c \geq G_1(1,b)\right\}$$
$$\bigcup \left\{(-1,b,c) \in \mathbb{R}^2 : \ (-1,b) \in \partial\pi_{ab}(\mathsf{B}_{D(\beta)}), \ 0 \leq c \leq G_1(-1,b)\right\}.$$

We have

(a) $\mathsf{S}_{D(\beta)} = \mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \cup \Omega_G$. *See Fig. 4.14 for a representation of the unit sphere.*
(b) The set $\mathrm{ext}(\mathsf{B}_{D(\beta)})$ *consists of the elements*

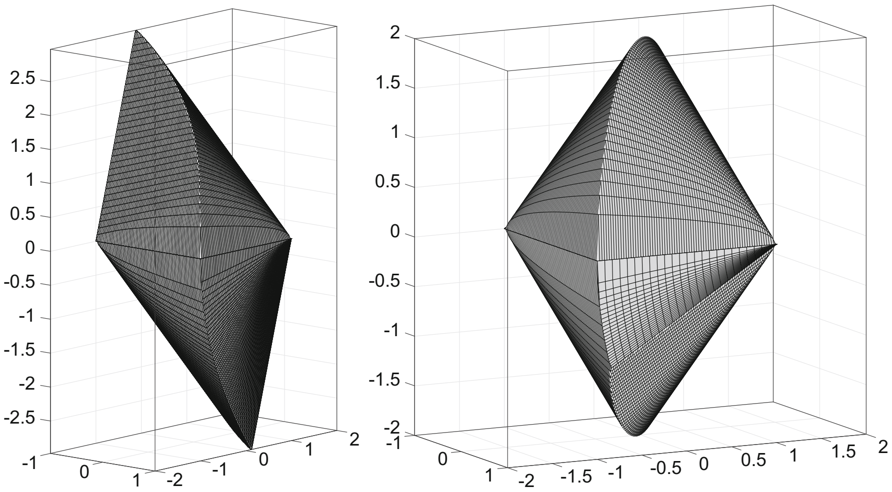


Fig. 4.14 Unit ball of $\mathcal{P}_{D(\beta)}$ with $\beta = 3\pi/4 - 0.5$ and unit ball of $\mathcal{P}_{D(\beta)}$ with $\beta = 3\pi/4 + 0.5$

$$\pm\Bigg(t, \frac{(1+M)t - 2(2+M) + 4\sqrt{(1+M)(1-t)}}{1-M},$$

$$-\frac{1}{N}\left[2(1+M)(t-1) + 4\sqrt{(1+M)(1-t)}\right]\Bigg) \quad \textit{for} \quad t \in [-1, -M],$$

$$\pm\left(s, -s, 2\sqrt{1-s^2}\right) \quad \textit{for} \quad s \in [-M, 1],$$

$$\pm\left(-1, r, 2\sqrt{2(1-r)}\right) \quad \textit{for} \quad r \in \left[\frac{-5-3M+4\sqrt{2(1+M)}}{1-M}, 1\right],$$

and

$$\pm(1, 1, 0).$$

Remark 4.3 To finish this chapter, notice that the projections obtained, as well as the unit spheres, when the angle β varies in the interval $(0, \infty)$ are continuous transformations since the norm is a continuous operator. This explains in an informal way why the projections and unit spheres are so similar when the angles are close.

Chapter 5
Sequence Banach Spaces

Abstract This chapter is dedicated to the study of the geometry of polynomial spaces on ℓ_p^q for certain values of p, q, presenting all known results for these classes of spaces.

5.1 The Space ℓ_1^2

Let $P(x, y) = ax^2 + by^2 + cxy$ be a 2-homogeneous polynomial where $(x, y) \in \ell_1^2$ and ℓ_1^2 is considered over the real or complex numbers. The supremum norm of P over ℓ_1 is denoted by

$$\|P\|_{\ell_1^2} = \sup\{|P(x, y)| : \|(x, y)\|_{\ell_1^2} \leq 1\}.$$

The space of 2-homogeneous polynomials over ℓ_1^2 endowed with the norm $\|\cdot\|_{\ell_1^2}$ is denoted by $\mathcal{P}\left({}^2\ell_1^2\right)$. If ℓ_1^2 is defined over the real numbers, then (using the same arguments as in the previous chapters) the adaptation of the mapping T defined in Sect. 2.1 helps us to give a visual representation of the unit ball of the space $\mathcal{P}\left({}^2\ell_1^2\right)$ on $\mathbb{R}^3$ endowed with the norm $\|\cdot\|_{\mathcal{P}({}^2\ell_1^2)}$ defined by

$$\|(a, b, c)\|_{\mathcal{P}({}^2\ell_1^2)} = \|ax^2 + by^2 + cxy\|_{\ell_1^2}$$

for every $(a, b, c) \in \mathbb{R}^3$. We denote the unit ball and the unit sphere of $(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}({}^2\ell_1^2)})$ by $\mathsf{B}_{\mathcal{P}({}^2\ell_1^2)}$ and $\mathsf{S}_{\mathcal{P}({}^2\ell_1^2)}$, respectively.

In this section we are using a different approach. Until now, we have given an explicit formula for the norm of the polynomial. However, in this case, the problem can be solved using a more direct approach. We can find directly the projection and parametrization of the unit sphere in the real case as well as the extreme points of the unit ball. In the complex case we are going to show a parametrization of the unit sphere as well as the extreme points when c is a pure imaginary number.

In the real case let us consider the following sets

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_5

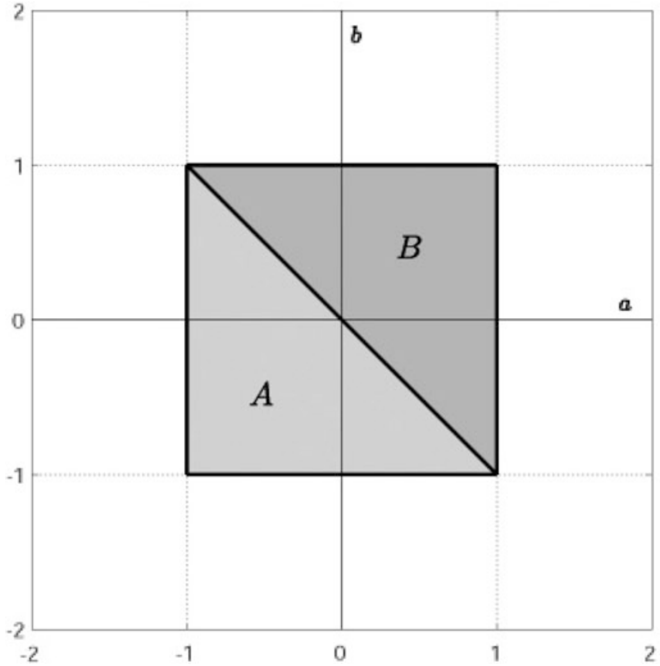


Fig. 5.1 Projection of $\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}$ onto the ab-plane and where ℓ_1^2 is defined over the real numbers

$$A = \left\{(a,b) \in \mathbb{R}^2 \colon b < -a\right\},$$

$$B = \left\{(a,b) \in \mathbb{R}^2 \colon b \geq -a\right\}.$$

Notice that $\mathsf{B}_{\ell_\infty^2} = A \cup B$.

Theorem 5.1 (Choi et al. [16]) *Let ℓ_1^2 be defined over the real numbers. The projection of $\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}$ onto the ab-plane is $\mathsf{B}_{\ell_\infty^2}$.*

Figure 5.1 shows a representation of the projection of the unit ball onto the ab-plane.

Notice that we distinguish between the sets A and B. The explanation is given in the following result that gives a parametrization of the unit ball. It is important to mention that the result comes from Y. S. Choi et al. [16], but in the sense that they gave an implicit characterization of when a polynomial belongs to the unit ball and, furthermore, when such polynomial is an extreme polynomial of the unit ball. In this survey we go further by giving a parametrization of the unit ball and by showing the explicit forms of the extreme polynomials of the unit ball. We omit the proofs of such constructions since they are easy to obtain from [16].

Theorem 5.2 *Let ℓ_1^2 be defined over the real or complex numbers and $P(x, y) = ax^2 + by^2 + cxy$ be a 2-homogeneous polynomial in $\mathcal{P}\left({}^2\ell_1^2\right)$. If $\|P\|_{\ell_1^2} \leq 1$, then $|a| \leq 1$, $|b| \leq 1$ and $|c| \leq 4$.*

Assume that ℓ_1^2 is defined over the real numbers, and let us define

$$F_1(a, b) = \begin{cases} 2\left(1 + \sqrt{(1+a)(1+b)}\right) & \text{if } (a, b) \in A, \\ 2\left(1 + \sqrt{(1-a)(1-b)}\right) & \text{if } (a, b) \in B, \end{cases}$$

$F_2(a, b) = -F_1(a, b)$, and

$$\Gamma_{\ell_1^2} = \{(a, b, c) : ((|a| = 1 \text{ and } |b| \leq 1) \text{ or } (|b| = 1 \text{ and } |a| \leq 1)) \,\&\, (|c| \leq 2)\}.$$

Then,

(i) $\mathsf{S}_{\mathcal{P}({}^2\ell_1^2)} = \operatorname{graph}(F_1) \cup \operatorname{graph}(F_2) \cup \Gamma_{\ell_1^2}$ *(see Fig. 5.2).*

(ii)

$$\begin{aligned} \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}({}^2\ell_1^2)}\right) = & \left\{\left(\frac{\sqrt{4|t| - t^2}}{2}, -\frac{\sqrt{4|t| - t^2}}{2}, t\right) : |t| \in (2, 4]\right\} \\ & \bigcup \left\{\left(-\frac{\sqrt{4|t| - t^2}}{2}, +\frac{\sqrt{4|t| - t^2}}{2}, t\right) : |t| \in (2, 4]\right\} \\ & \bigcup \{(\pm 1, \pm 1, \pm 2)\}. \end{aligned}$$

Assume now that ℓ_1^2 is defined over the complex numbers, $a, b \in \mathbb{R}$ and c is a pure imaginary number. We have the following results:

(a) If $|c| \leq 2$, then $\|P\|_{\ell_1^2} = 1$ if, and only if, $|a| = 1$ or $|b| = 1$.

(b) If $2 < |c| \leq 4$, then

$$\|P\|_{\ell_1^2} = 1 \text{ if, and only if, } 4|c| - |c|^2 = 4(|a + b| + ab).$$

Remark 5.1 Notice that we could have also given a parametrization of the unit sphere in Theorem 5.2 in the complex case when c is a pure imaginary number (which would be the same as in the real case). The reason why we have not done this is that the assumption that a and b are real numbers can be avoided since the same result is also true when a and b are complex numbers (to see this simply rotate the complex variables x and y).

Furthermore, due to difficult calculations in [16], we do not know what happens when the real part of c is different than zero.

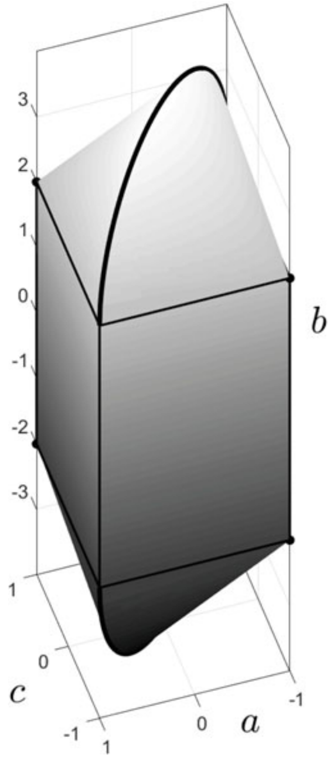


Fig. 5.2 $\mathsf{S}_{\mathcal{P}(^2\ell_1^2)}$. The extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}$ are drawn with a thicker line and dots

5.2 The Space ℓ_∞^2

Assume that ℓ_∞^2 is defined over the real numbers or complex numbers. Let $P(x, y) = ax^2 + by^2 + cxy$ be a 2-homogeneous polynomial, where $(a, b, c) \in \mathbb{R}^3$ and $(x, y) \in \ell_\infty^2$. The supremum norm of P over ℓ_∞^2 is defined by

$$\|P\|_{\ell_\infty^2} = \sup\{|P(x, y)| \colon \|(x, y)\|_{\ell_\infty^2} \leq 1\}.$$

The space of 2-homogeneous polynomials over ℓ_∞^2 with the supremum norm is denoted by $\mathcal{P}\left(^2\ell_\infty^2\right)$. The mapping T defined in Sect. 2.1 helps us to give a visual representation of the unit ball of the space $\mathcal{P}\left(^2\ell_\infty^2\right)$ on $\mathbb{R}^3$ endowed with the norm $\|\cdot\|_{\mathcal{P}(^2\ell_\infty^2)}$ defined by

$$\|(a, b, c)\|_{\mathcal{P}(^2\ell_\infty^2)} := \|ax^2 + by^2 + cxy\|_{\ell_\infty^2}$$

for every $(a, b, c) \in \mathbb{R}^3$. We denote the unit ball and the unit sphere of $(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}(^2\ell_\infty^2)})$ by $\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2)}$ and $\mathsf{S}_{\mathcal{P}(^2\ell_\infty^2)}$, respectively.

Notice that the case when ℓ_∞^2 is defined over the complex numbers has already been tackled in Sect. 2.2. Thus, assume that ℓ_∞^2 is defined over $\mathbb{R}$.

Remark 5.2 The case of ℓ_∞^2 is a simple consequence of the case ℓ_1^2. The reason why comes from the following approach:

Let us define the function Q on $\mathbb{R}^2$ by

$$Q(x, y) = \frac{1}{2}(x - y, x + y).$$

Notice that this function Q is just the rotation of angle $\frac{\pi}{4}$ on $\mathbb{R}^2$ scaled by $\frac{\sqrt{2}}{2}$ and it transforms $\mathcal{P}\left({}^2\ell_\infty^2\right)$ isometrically onto $\mathcal{P}\left({}^2\ell_1^2\right)$. Therefore, if $P \in \mathcal{P}\left({}^2\ell_\infty^2\right)$, then $P \circ Q \in \mathcal{P}\left({}^2\ell_1^2\right)$. Using this transformation in Sect. 5.1, we can give the projection of $\mathsf{B}_{\mathcal{P}({}^2\ell_\infty^2)}$ onto the ab-plane, the parametrization of $\mathsf{S}_{\mathcal{P}({}^2\ell_\infty^2)}$ and its extreme points.

The reader can check easily that: if $P(x, y) = ax^2 + by^2 + cxy$, then

$$\begin{aligned} P \circ Q(x, y) &= \frac{1}{4}P(x - y, x + y) \\ &= \frac{1}{4}(a + b + c)x^2 + \frac{1}{4}(a + b - c)y^2 + \frac{1}{2}(b - a)xy. \end{aligned}$$

Using the same procedure from the space ℓ_1^2, we begin by showing the projection of $\mathsf{B}_{\mathcal{P}({}^2\ell_\infty^2)}$ onto the ab-plane. Let us define the following sets in $\mathbb{R}^2$:

$$A \equiv \text{Triangle of vertices } (-1, 1),\ (0, 1),\ \left(-\frac{1}{2}, \frac{1}{2}\right).$$

$$B \equiv \text{Triangle of vertices } (-1, 1),\ \left(-\frac{1}{2}, \frac{1}{2}\right),\ (-1, 0).$$

$$C \equiv \text{Square of vertices } (0, 1),\ (1, 0),\ (0, -1),\ (-1, 0).$$

$$D \equiv \text{Triangle of vertices } (1, 0),\ (1, -1),\ \left(\frac{1}{2}, -\frac{1}{2}\right).$$

$$E \equiv \text{Triangle of vertices } (1, -1),\ (0, -1),\ \left(\frac{1}{2}, -\frac{1}{2}\right).$$

A sketch of A, B, C, D and E can be found in Fig. 5.3.

Theorem 5.3 (Jiménez et al. [35]) *The projection of* $\mathsf{B}_{\mathcal{P}({}^2\ell_\infty^2)}$ *onto the ac-plane is* $A \cup B \cup C \cup D \cup E$.

Finally, we show a parametrization of $\mathsf{S}_{\mathcal{P}({}^2\ell_\infty^2)}$ in $\mathbb{R}^3$ as well as the extreme points. See Fig. 5.4 for a sketch of $\mathsf{B}_{\mathcal{P}({}^2\ell_\infty^2)}$. Notice that $\mathsf{B}_{\mathcal{P}({}^2\ell_\infty^2)}$ is just a scaled rotation of $\mathsf{B}_{\mathcal{P}({}^2\ell_1^2)}$.

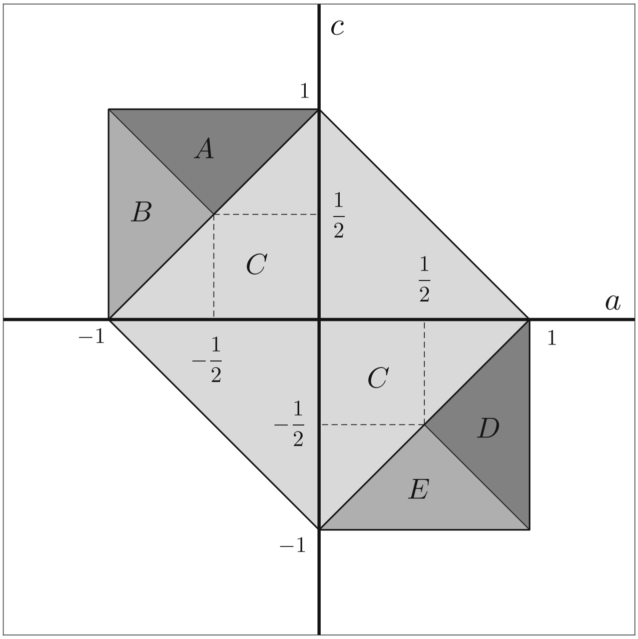


Fig. 5.3 Projection of $\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2)}$ onto the ac-plane

Theorem 5.4 (Choi and Kim [15]; Jiménez et al. [35]) *Let G be the mapping defined on $A \cup B \cup C \cup D \cup E$ by*

$$G(a,c) = \begin{cases} 2\sqrt{a(c-1)} & \textit{if } (a,c) \in A, \\ 2\sqrt{c(a+1)} & \textit{if } (a,c) \in B, \\ 1 - |a+c| & \textit{if } (a,c) \in C, \\ 2\sqrt{c(a-1)} & \textit{if } (a,c) \in D, \\ 2\sqrt{a(c+1)} & \textit{if } (a,c) \in E. \end{cases}$$

We have

(i) $\mathsf{S}_{\mathcal{P}(^2\ell_\infty^2)} = \mathrm{graph}(G) \cup \mathrm{graph}(-G)$.

(ii)

$$\mathrm{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2)}\right) = \left\{\pm\left(-t, t, \pm 2\sqrt{t(1-t)}\right) : t \in \left[\frac{1}{2}, 1\right]\right\} \bigcup \{\pm(1,0,0)\} \bigcup \{\pm(0,1,0)\}.$$

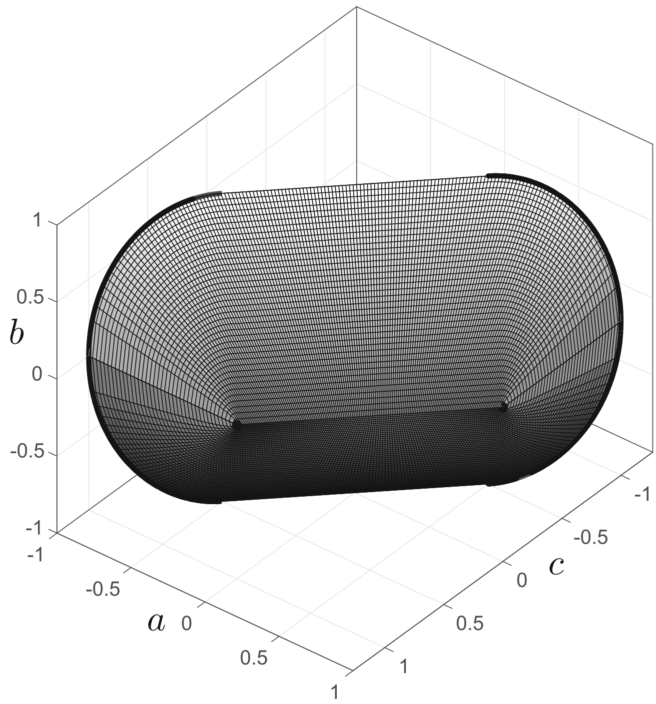


Fig. 5.4 Unit ball $\mathcal{P}(^2\ell_\infty^2)$. The extreme points are drawn with a thicker line and dots

5.3 The Space ℓ_p^2 when $1 < p < 2$

Let $1 < p < 2$ and let $P(x, y) = ax^2 + by^2 + cxy$ be a 2-homogeneous polynomial such that $(x, y) \in \ell_p^2$ where ℓ_p^2 is defined over $\mathbb{R}$. We define the norm of P over ℓ_p^2 as

$$\|P\|_{\ell_p^2} = \sup\{|P(x, y)|\colon \|(x, y)\|_{\ell_p^2} \leq 1\}.$$

Let $\mathcal{P}\left(^2\ell_p^2\right)$ denote the space of all 2-homogeneous polynomials in ℓ_p^2 endowed with the norm $\|\cdot\|_{\ell_p^2}$. Just like in the previous sections we can identify the space $\mathcal{P}\left(^2\ell_p^2\right)$ with $\mathbb{R}^3$ endowed with the norm

$$\|(a, b, c)\|_{\mathcal{P}\left(^2\ell_p^2\right)} = \|ax^2 + by^2 + cxy\|_{\ell_p^2},$$

for every $(a, b, c) \in \mathbb{R}^3$, via the mapping T from Sect. 2.1. Therefore, we can give a representation in $\mathbb{R}^3$ of the unit ball of $\mathcal{P}\left({}^2\ell_p^2\right)$. Let us denote by $\mathsf{B}_{\mathcal{P}\left({}^2\ell_p^2\right)}$ and $\mathsf{S}_{\mathcal{P}\left({}^2\ell_p^2\right)}$ the unit ball and the unit sphere of $(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}\left({}^2\ell_p^2\right)})$, respectively.

Now, in this case the approach is very different to the previous ones. In this section we are going to begin by providing the extreme points of the unit ball of $\mathcal{P}\left({}^2\ell_p^2\right)$. Interestingly enough this can be used to find a parametrization of the unit sphere by noticing that by definition every point of the unit sphere that is not an extreme point lies in the interior of the segment that joins two extreme points. Therefore we only need to find the segments of the unit sphere that join two extreme points.

Proposition 5.1 (Grecu [26]) *If* $1 < p < 2$*, then the 2-homogeneous polynomials* $x^2 \pm y^2$ *are extreme points of the unit ball of* $\mathcal{P}\left({}^2\ell_p^2\right)$.

Proposition 5.2 (Grecu [26]) *Let* $1 < p < 2$ *and* $\alpha, \beta \geq 0$ *with* $\alpha^p + \beta^p = 1$*. The 2-homogeneous polynomials* $\pm P$ *with* $P(x, y) = a(x^2 - y^2) + cxy$ *where* $a = \frac{\alpha^p - \beta^p}{\alpha^2 + \beta^2}$ *and* $c = 2\frac{\alpha^{p-1}\beta + \alpha\beta^{p-1}}{\alpha^2 + \beta^2}$ *have norm one and are extreme points of the unit ball of* $\mathcal{P}\left({}^2\ell_p^2\right)$*. Furthermore,* $\pm(\alpha, \beta)$ *are the only points where* P *takes the value* 1 *and* $\pm(-\beta, \alpha)$ *are the only points where* P *takes the value* -1.

Proposition 5.3 (Grecu [26]) *Let* $1 < p < 2$ *and* $\alpha, \beta \geq 0$ *whith* $\alpha^p + \beta^p = 1$ *and* $\alpha \neq \beta$*. The 2-homogeneous polynomial* $P(x, y) = a(x^2 + y^2) + cxy$ *where* $a = \frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}$ *and* $c = 2\frac{\alpha\beta^{p-1} - \alpha^{p-1}\beta}{\alpha^2 - \beta^2}$ *has norm one and is an extreme point of the unit ball of* $\mathcal{P}\left({}^2\ell_p^2\right)$*. Furthermore, the only points where* P *takes the value* 1 *are* (α, β) *and* (β, α).

Proposition 5.4 (Grecu [26]) *Let* $1 < p < 2$*. The 2-homogeneous polynomial* $P(x, y) = 2^{\frac{2}{p-2}} p(x^2 + y^2) + 2^{\frac{2}{p-1}}(2 - p)xy$ *has norm one and is an extreme point of the unit ball of* $\mathcal{P}\left({}^2\ell_p^2\right)$*. Furthermore, the only point where* P *takes the value* 1 *is* $(2^{-\frac{1}{p}}, 2^{-\frac{1}{p}})$.

Proposition 5.5 (Grecu [26]) *Let* $1 < p < 2$ *and* $\alpha, \beta \geq 0$ *with* $\alpha^p + \beta^p = 1$ *and* $\alpha \neq \beta$*. The 2-homogeneous polynomials* $\pm P$ *with* $P(x, y) = a(x^2 + y^2) \pm cxy$ *where* $a = \frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}$ *and* $c = 2\frac{\alpha\beta^{p-1} - \alpha^{p-1}\beta}{\alpha^2 - \beta^2}$ *have norm one and are extreme points of the unit ball of* $\mathcal{P}\left({}^2\ell_p^2\right)$*. Furthermore, the only points where* P *takes the value* 1 *are* $\pm(\alpha, \pm\beta)$.

The following result shows the form of the polynomials that belong to the line segments in the unit sphere that join two extreme points, and although the result itself does not appear in [26], it can be deduced from the proofs.

Theorem 5.5 (Grecu [26]) *Let $1 < p < 2$ and P be a polynomial that belongs to the unit sphere of $\mathcal{P}\left({}^2\ell_p^2\right)$. We have that P lies in the segment that joins two extreme points P_1 and P_2 of the unit ball of $\mathcal{P}\left({}^2\ell_p^2\right)$, where the pair (P_1, P_2) is of the following forms*

(i)

$$P_1(x, y) = \frac{\alpha^p - \beta^p}{\alpha^2 + \beta^2}(x^2 - y^2) + 2\frac{\alpha\beta^{p-1} + \alpha^{p-1}\beta}{\alpha^2 + \beta^2}xy,$$
$$P_2(x, y) = \frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) + 2\frac{\alpha\beta^{p-1} - \alpha^{p-1}\beta}{\alpha^2 - \beta^2}xy;$$

(ii)

$$P_1(x, y) = \frac{\alpha^p - \beta^p}{\alpha^2 + \beta^2}(x^2 - y^2) - 2\frac{\alpha\beta^{p-1} + \alpha^{p-1}\beta}{\alpha^2 + \beta^2}xy,$$
$$P_2(x, y) = \frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) - 2\frac{\alpha\beta^{p-1} - \alpha^{p-1}\beta}{\alpha^2 - \beta^2}xy;$$

(iii)

$$P_1(x, y) = -\frac{\alpha^p - \beta^p}{\alpha^2 + \beta^2}(x^2 - y^2) + 2\frac{\alpha\beta^{p-1} + \alpha^{p-1}\beta}{\alpha^2 + \beta^2}xy,$$
$$P_2(x, y) = -\frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) + 2\frac{\alpha\beta^{p-1} - \alpha^{p-1}\beta}{\alpha^2 - \beta^2}xy;$$

(iv)

$$P_1(x, y) = -\frac{\alpha^p - \beta^p}{\alpha^2 + \beta^2}(x^2 - y^2) - 2\frac{\alpha\beta^{p-1} + \alpha^{p-1}\beta}{\alpha^2 + \beta^2}xy,$$
$$P_2(x, y) = -\frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) - 2\frac{\alpha\beta^{p-1} - \alpha^{p-1}\beta}{\alpha^2 - \beta^2}xy;$$

where $\alpha, \beta \geq 0$ satisfy that $\alpha^p + \beta^p = 1$.

We can use Theorem 5.5 to give a sketch of one of these unit spheres which can be found in Fig. 5.5.

Finally, we provide a characterization of the extreme points of the unit ball of $\mathcal{P}\left({}^2\ell_p^2\right)$.

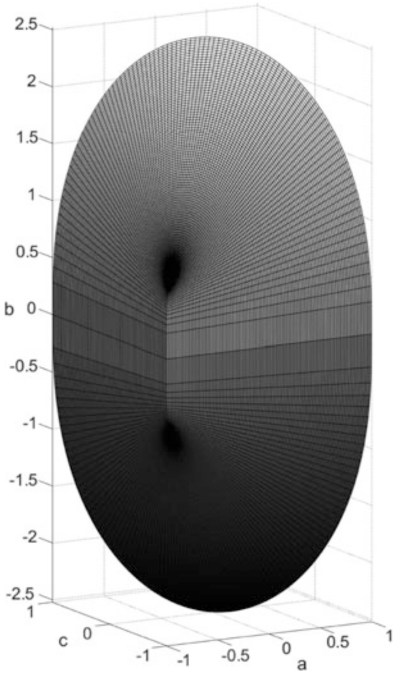


Fig. 5.5 Unit ball of $\mathcal{P}(^2\ell_{3/2}^2)$

Theorem 5.6 (Grecu [26]) *Let $1 < p < 2$. A 2-homogeneous polynomial of unit norm $P(x, y) = ax^2 + by^2 + cxy$ is an extreme point of the unit ball of $\mathcal{P}\left(^2\ell_p^2\right)$ if, and only if,*

(i) $a + b = 0$, *or*
(ii) $a = b \geq 2^{\frac{2}{p-2}} p$.

5.4 The Space ℓ_p^2 when $2 < p < \infty$

In this section we consider the space of 2-homogeneous polynomials on ℓ_p^2 defined over $\mathbb{R}$ with $2 < p < \infty$. Following the same techniques as in the case of the space ℓ_p^2 with $1 < p < 2$, we begin by showing the extreme points of the unit ball of $\mathcal{P}\left(^2\ell_p^2\right)$. Then we find all the polynomials that belong to the unit sphere by providing the line segments contained in the unit sphere that join two extreme points, and finally we show a characterization of when a polynomial in the unit ball of $\mathcal{P}\left(^2\ell_p^2\right)$ is an extreme polynomial.

Proposition 5.6 (Grecu [26]) *Let $2 < p < \infty$. A 2-homogeneous polynomial $P(x, y) = ax^2 + by^2$ is an extreme point of the unit ball $\mathcal{P}\left({}^2\ell_p^2\right)$ if, and only if, $ab \geq 0$ and $|a|^{\frac{p}{p-2}} + |b|^{\frac{p}{p-2}} = 1$ or $ab < 0$ and $P(x, y) = \pm(x^2 - y^2)$.*

Notice that Proposition 5.6 can be used to find a projection onto the ac-plane as it is shown in the following result which does not appear in [26]. Let us consider the sets

$$A = \left\{(a, c) \in \mathbb{R}^2 : a \in [0, 1],\ c \in \left[0, \left(1 - a^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\right]\right\}$$
$$\bigcup \left\{(a, c) \in \mathbb{R}^2 : a \in [-1, 0],\ c \in \left[-\left(1 - a^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}, 0\right]\right\},$$
$$B = \left\{(a, c) \in \mathbb{R}^2 : a \in [0, 1],\ b \in [-1, 0]\right\}$$
$$\bigcup \left\{(a, c) \in \mathbb{R}^2 : a \in [-1, 0],\ b \in [0, 1]\right\}.$$

Corollary 5.1 *The projection of $\mathsf{B}_{\mathcal{P}\left({}^2\ell_p^2\right)}$ onto the ac-plane is $A \cup B$. A sketch can be found in Fig. 5.6.*

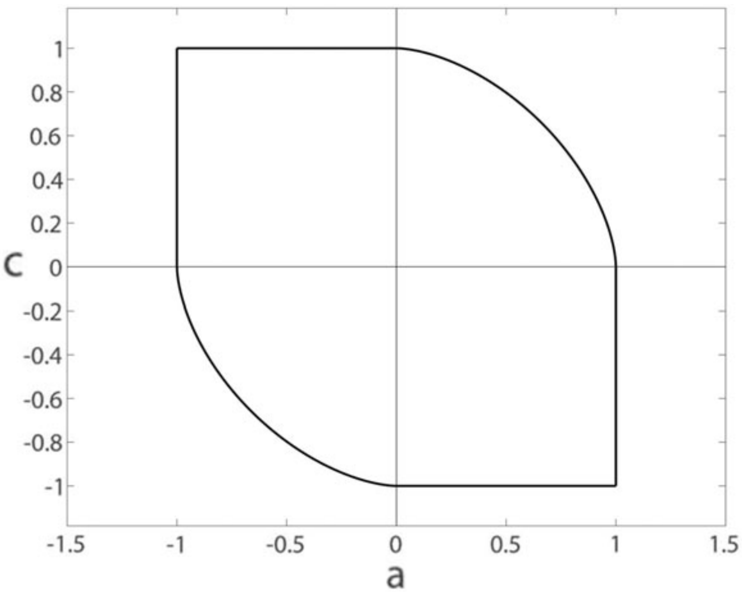


Fig. 5.6 Sketch of the projection of the unit ball of $\mathcal{P}({}^2\ell_5^2)$ onto the ac-plane

Proposition 5.7 (Grecu [26]) *Let $2 < p < \infty$ and $\alpha \geq \beta \geq 0$ with $\alpha^p + \beta^p = 1$. The 2-homogeneous polynomial $P(x, y) = a(x^2 - y^2) + cxy$ where $a = \frac{\alpha^p - \beta^p}{\alpha^2 + \beta^2}$ and $c = 2\alpha\beta\frac{\alpha^{p-2} + \beta^{p-2}}{\alpha^2 + \beta^2}$ has norm one and is an extreme point of the unit ball of $\mathcal{P}\left({}^2\ell_p^2\right)$. Furthermore, $\pm(\alpha, \beta)$ are the only points where P takes the value 1.*

Proposition 5.8 (Grecu [26]) *Let $2 < p < \infty$ and $\alpha, \beta \geq 0$ such that $\alpha^p + \beta^p = 1$ and $a = \frac{\alpha^p - \beta^p}{\alpha^+ \beta^2}$, $c = 2\alpha\beta\frac{\alpha^{p-2} + \beta^{p-2}}{\alpha^2 + \beta^2}$. The polynomials $\pm P$ where $P(x, y) = a(x^2 - y^2) + cxy$ are extreme points of the unit ball of $\mathcal{P}\left({}^2\ell_p^2\right)$ with $P(\alpha, \beta) = -P(-\beta, \alpha) = 1$.*

Theorem 5.7 (Grecu [26]) *Let $2 < p < \infty$ and P be a polynomial of the unit sphere of $\mathcal{P}\left({}^2\ell_p^2\right)$. We have that P lies in the segment that joins two extreme points P_1 and P_2 of the unit ball of $\mathcal{P}\left({}^2\ell_p^2\right)$, where the pair (P_1, P_2) is of the following forms*

(i)

$$\begin{aligned} P_1(x, y) =& \frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) + 2\alpha\beta\frac{\alpha^{p-2} + \beta^{p-2}}{\alpha^2 + \beta^2}xy, \\ P_2(x, y) =& \alpha^{p-2}x^2 + \beta^{p-2}y^2; \end{aligned}$$

(ii)

$$\begin{aligned} P_1(x, y) =& -\frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) - 2\alpha\beta\frac{\alpha^{p-2} + \beta^{p-2}}{\alpha^2 + \beta^2}xy, \\ P_2(x, y) =& \beta^{p-2}x^2 + \alpha^{p-2}y^2; \end{aligned}$$

(iii)

$$\begin{aligned} P_1(x, y) =& \frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) + 2\alpha\beta\frac{\alpha^{p-2} + \beta^{p-2}}{\alpha^2 + \beta^2}xy, \\ P_2(x, y) =& -\beta^{p-2}x^2 - \alpha^{p-2}y^2; \end{aligned}$$

(iv)

$$\begin{aligned} P_1(x, y) =& -\frac{\alpha^p - \beta^p}{\alpha^2 - \beta^2}(x^2 + y^2) - 2\alpha\beta\frac{\alpha^{p-2} + \beta^{p-2}}{\alpha^2 + \beta^2}xy, \\ P_2(x, y) =& -\alpha^{p-2}x^2 - \beta^{p-2}y^2. \end{aligned}$$

where $\alpha, \beta \geq 0$ satisfy that $\alpha^p + \beta^p = 1$ (Fig. 5.7).

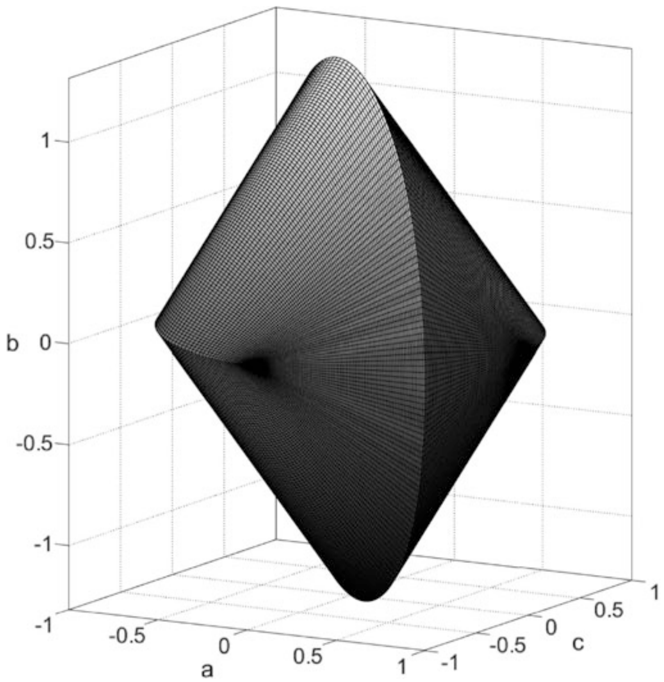


Fig. 5.7 Sketch of the unit sphere of $\mathcal{P}(^2\ell_5^2)$

By using Theorem 5.7, we can give a visual representation of the unit sphere of $\mathcal{P}\left(^2\ell_p^2\right)$. A sketch of one of these unit spheres can be found in Fig. 5.7.

Theorem 5.8 (Grecu [26]) *Let $2 < p < \infty$. A 2-homogeneous polynomial of the unit ball of $\mathcal{P}\left(^2\ell_p^2\right)$ of the form $P(x, y) = ax^2 + by^2 + cxy$ is an extreme point of the unit ball of $\mathcal{P}\left(^2\ell_p^2\right)$ if, and only if,*

(i) $a + b = 0$, *or*
(ii) $c = 0$ *and* $ab \geq 0$ *with* $|a|^{\frac{p}{p-2}} + |b|^{\frac{p}{p-2}} = 1.$

5.5 The Space c_0

Notice that until now all the vector spaces that we have considered have been finite dimensional. However, in this section, we focus our attention for the first time on the infinite dimensional case. To do so, we begin by defining an n-homogeneous polynomial over an arbitrary normed space.

Let E be a normed space defined over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). For every $n \in \mathbb{N}$, a continuous n-homogeneous polynomial on E is a mapping defined on E that takes values on $\mathbb{K}$ such that there exists a continuous symmetric n-linear form L from E^n to $\mathbb{K}$ such that $P(x) = L(x, \ldots, x)$, for every $x \in E$. Furthermore, L is unique by the Polarization Formula (see, for instance, [19]) and we call L the polar of P. At first glance this definition of a homogeneous polynomial of degree n on a normed space is a bit puzzling but, on the one hand, it is technically very efficient and, on the other, it extends the classical notion of an n-homogeneous polynomial in several variables in a very natural way since it is a very well-known result in the theory of polynomials that a continuous mapping $P : E \to \mathbb{K}$ is an n-homogeneous polynomial on a real or complex normed space E, if and only if the restriction of P to any finite dimensional subspace F of E is a homogeneous polynomial of degree n in real or complex variables with respect to the coefficients of any basis of F.

Regarding the continuity of polynomials on normed spaces we have to say that all homogeneous polynomials on a finite dimensional Banach are continuous. This is far from being true for polynomials on infinite dimensional normed spaces. As a matter of fact the set of noncontinuous homogeneous polynomials on any infinite dimensional normed space is not only infinite, but also has an enormous size in terms of algebraic genericity (see, for instance, [21] and [22]). In any case, another conventional result in the theory of polynomials on normed spaces states that the n-homogeneous polynomial $P : E \to \mathbb{K}$ on the normed space E is continuous if and only if P is bounded on the unit ball of E. This allows us to define the following norm in the space of n-homogeneous polynomials on E:

$$\|P\|_E = \sup\{|P(x)| : \|x\| \le 1\}.$$

We denote this normed space by $\mathcal{P}(^n E)$ and also we denote the unit ball of $\mathcal{P}(^n E)$ by $\mathsf{B}_{\mathcal{P}(^n E)}$.

In this section, we provide some insight regarding the extreme polynomials of $\mathsf{B}_{\mathcal{P}(^n c_0)}$. In particular, we are interested in studying when a homogenous polynomial defined on the infinite dimensional Banach space c_0 that takes values over $\mathbb{C}$ or $\mathbb{R}$ is an extreme polynomial of the unit ball.

Theorem 5.9 (Choi and Kim [14]) *Assume that c_0 is defined over $\mathbb{C}$. Let $P(x) = \sum_{i_1 \le \cdots \le i_n} a_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n} \in \mathcal{P}(^n c_0)$ be such that $\|P\|_{c_0} = 1$ and $\sum_{i_1 \le \cdots \le i_n} |a_{i_1,\ldots,i_n}| = 1$. We have that P is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^n c_0)}$ if, and only if, P is a monomial.*

Proof Firstly, we will prove that if P is an extreme polynomial of the unit ball of $\mathcal{P}(^n c_0)$, then P is a monomial. By way of contradiction, assume that P is not a monomial. We will show that there exist Q and R in $\mathsf{S}_{\mathcal{P}(^n c_0)}$ such that $P = \frac{1}{2}(Q + R)$. Since P is not a monomial, there exist $j_1 \le \cdots \le j_n$ and $k_1 \le \cdots \le k_n$ with $(j_1, \ldots, j_n) \ne (k_1, \ldots, k_n)$ such that $0 < |a_{j_1,\ldots,j_n}| < 1$ and $0 < |a_{k_1,\ldots,k_n}| < 1$. Choose $\varepsilon > 0$ such that $0 < |a_{j_1,\ldots,j_n}| \pm \varepsilon < 1$ and $0 < |a_{k_1,\ldots,k_n}| \pm \varepsilon < 1$, and let us define

$$Q(x) = \sum_{i_1 \le \cdots \le i_n} b_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n}$$

and

$$R(x) = \sum_{i_1 \le \cdots \le i_n} c_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n},$$

where $b_{j_1,\ldots,j_n} = a_{j_1,\ldots,j_n} - \operatorname{sign}(a_{j_1,\ldots,j_n})\varepsilon$, $b_{k_1,\ldots,k_n} = a_{k_1,\ldots,k_n} + \operatorname{sign}(a_{k_1,\ldots,k_n})\varepsilon$, $c_{j_1,\ldots,j_n} = a_{j_1,\ldots,j_n} + \operatorname{sign}(a_{j_1,\ldots,j_n})\varepsilon$, $c_{k_1,\ldots,k_n} = a_{k_1,\ldots,k_n} - \operatorname{sign}(a_{k_1,\ldots,k_n})\varepsilon$ and $b_{i_1,\ldots,i_n} = c_{i_1,\ldots,i_n} = a_{i_1,\ldots,i_n}$ for any other $i_1 \le \cdots \le i_n$ with $(j_1,\ldots,j_n) \neq (i_1,\ldots,i_n) \neq (k_1,\ldots,k_n)$. Observe that here $\operatorname{sign}(z) = z/|z|$ for every $z \in \mathbb{C} \setminus \{0\}$, that is, $\operatorname{sign}(z) = e^{i\theta}$ where θ is an argument for z. By construction, it is straightforward that $Q \neq R$, $\|Q\|_{c_0} = \|R\|_{c_0} = 1$ and $P = \frac{1}{2}(Q + R)$, which is a contradiction.

Finally, assume that P is a monomial. Without loss of generality, we can assume that $P(x) = x_{i_1} \cdots x_{i_n}$. Fix $\varepsilon > 0$ and let $\delta = \frac{\varepsilon^2}{3}$. We will prove that P is in fact a strong extreme polynomial (that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|P \pm Q\|_{c_0} \le 1+\delta$ for some polynomial Q, then $\|Q\|_{c_0} \le \varepsilon$). If $\|P \pm Q\|_{c_0} \le 1+\delta$ for some polynomial Q then, by the maximum modulus theorem, we have that

$$\begin{aligned}\|P \pm Q\|_{c_0} = \sup\{|x_{i_1} \cdots x_{i_n} \pm Q(x)| \colon x = (x_m)_{m\in\mathbb{N}} \in c_0,\ \text{with } \|x\|_{c_0} \le 1 \\ \text{and } |x_{i_1}| = \cdots = |x_{i_n}| = 1\}\end{aligned}$$

and

$$\begin{aligned}\|Q\|_{c_0} = \sup\{|Q(x)| \colon x = (x_m)_{m\in\mathbb{N}} \in c_0,\ \text{with } \|x\|_{c_0} \le 1 \\ \text{and } |x_{i_1}| = \cdots = |x_{i_n}| = 1\}.\end{aligned}$$

Thus, by Choi et al. [16, lemma 2.1], notice that $\|Q\|_{c_0} \le \varepsilon$. It is easy to see that every strong extreme point is an extreme point and this concludes the proof. □

If we just focus our attention on 2-homogeneous polynomials, then we can find some relations with other Banach spaces such as the finite dimensional space ℓ_∞^k.

Theorem 5.10 (Choi and Kim [14]) *Identifying ℓ_∞^k with the subspace of c_0 generated by $e_{n_1},\ldots,e_{n_k}$, we have that* $\operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2c_0)}\right) \cap \mathcal{P}\left(^2\ell_\infty^k\right) \subset \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^k)}\right)$. *Furthermore, if c_0 is defined over $\mathbb{C}$, then* $\operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^k)}\right) \subset \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2c_0)}\right)$.

Remark 5.3 Unfortunately, we do not have yet a characterization of the extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_\infty^k)}$ in the complex case.

In the real case, we do not know also if $\operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^k)}\right)$ is a subset of $\operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2c_0)}\right)$, except for $k = 2$ (Sect. 5.2) since it is shown in [14] (see also [15, section 4]) that $\operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2)}\right) \subset \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2c_0)}\right)$.

5.6 The Space ℓ_1

In Sect. 5.5, we studied several cases of when a polynomial in the infinite dimensional space c_0 that has norm one is an extreme polynomial of the unit ball. In this section, we study similar results but considering the infinite dimensional space ℓ_1 instead of c_0.

The following results show how to obtain extreme polynomials in $\mathsf{B}_{\mathcal{P}(^2\ell_1)}$ by using extreme polynomials in $\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}$ (see Sect. 5.1) independently of the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

For every $i < j$ and every $P \in \mathcal{P}(^2\ell_1)$, let us define $P_{ij}(x, y) = P(xe_i + ye_j)$. It is easy to see that $P_{ij} \in \mathcal{P}(^2\ell_1^2)$. The following result guarantees when the polynomial P that defines the polynomial P_{ij} is an extreme polynomial.

Theorem 5.11 (Choi et al. [16]) *Let $P \in \mathsf{B}_{\mathcal{P}(^2\ell_1)}$. If $P_{ij} \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}\right)$ for every $i < j$, then $P \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1)}\right)$.*

Proof Assume that there exists $Q \in \mathcal{P}(^2\ell_1)$ of the form $Q(x) = \sum_{i\leq j} b_{ij}x_ix_j$ such that $\|P \pm Q\|_{\ell_1} = 1$. Clearly, we have $\|P_{ij} \pm Q_{ij}\|_{\ell_1^2} \leq 1$ for all $i < j$. Hence, since P_{ij} is an extreme polynomial of the unit ball of $\mathcal{P}(^2\ell_1^2)$, we have that $Q_{ij} = 0$ for any $i < j$, which implies that Q is the zero polynomial. □

Remark 5.4 Using Theorem 5.11 and [24, theorem 2.1], it follows that every polynomial of the form

$$P(x) = a\sum_{i=1}^{\infty} x_i^2 + b\sum_{i<j} x_ix_j$$

where $|a| = 1$ and $|b| = 2$ is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^2\ell_1)}$.

Let A and B be disjoint subsets of $\mathbb{N}$ and take $P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}\left(^2\ell_1^2\right)$, where $a, b, c \in \mathbb{K}$. Take $(a_i) \in \mathbb{K}^{\mathbb{N}}$ such that $|a_1| = 1$, and let us define

$$\overline{P}((x_i)_{i\in\mathbb{N}}) = P\left(\sum_{i\in A} a_ix_i, \sum_{i\in B} a_ix_I\right).$$

It is easy to prove that $\|P\|_{\ell_1^2} = \|\overline{P}\|_{\ell_1}$.

Theorem 5.12 (Choi et al. [16]) *If A and B form a partition of $\mathbb{N}$, then $P \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}\right)$ if, and only if, $\overline{P} \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1)}\right)$.*

Proof Assume first that $\overline{P}$ is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^2\ell_1)}$ but P is not an extreme point of $\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}$. Then, there exist Q and R in the unit ball of $\mathcal{P}\left(^2\ell_1^2\right)$ with $Q \neq R$ such that $P = \frac{1}{2}(Q + R)$. Hence, $\overline{P} = \frac{1}{2}(\overline{Q} + \overline{R})$, where clearly $\overline{Q}$ $\overline{R}$ are distinct polynomials of the unit ball of $\mathcal{P}\left(^2\ell_1\right)$. Thus, $\overline{P}$ is not an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^2\ell_1)}$, which is a contradiction.

Suppose now that $P \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}\right)$. Say $P(x, y) = ax^2 + by^2 + cxy$. First, assume that there exists $x \in \mathbb{K}$ with $|x| = 1$ such that $|P(x, 0)| = 1$, then $|a| = 1$ (the same can be said if $|P(0, x)| = 1$ but in this case $|b| = 1$). Hence, since $P \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}\right)$, by Theorem 5.2 (ii), we have that $|b| = 1$ and $|c| = 2$. Therefore, by Theorem 5.11, we have that $\overline{P} \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1)}\right)$. Thus, assume without loss of generality that there exist $x_0, y_0 \in \mathbb{K} \setminus \{0\}$ with $|x_0| + |y_0| = 1$ such that $|P(x_0, y_0)| = 1$.

By way of contradiction, suppose that there exist distinct $\overline{Q}$ and $\overline{R}$ in $\mathsf{B}_{\mathcal{P}(^2\ell_1)}$ such that $\overline{P} = \frac{1}{2}(\overline{Q} + \overline{R})$. Let us decompose $\overline{Q}$ and $\overline{R}$ into $\overline{Q} = \overline{P} + S$ and $\overline{R} = \overline{P} - S$, where $S((x_i)_{i\in\mathbb{N}}) = \sum_{i\leq j} b_{ij}x_i x_j \in \mathcal{P}(^2\ell_1)$ with $b_{ij} \in \mathbb{K}$ for every $i \leq j$. The latter can be done by construction of $\overline{Q}$ and $\overline{R}$.

We will prove that $b_{kk} = b_{ll} = b_{kl} = 0$ for any $k \in A$ and $l \in B$, or $l \in A$ and $k \in B$. Assume that $k \in A$ and $l \in B$ (the following reasoning can be applied also in the case when $l \in A$ and $k \in B$). Take $\alpha = \operatorname{sign}(a_k)$ and $\beta = \operatorname{sign}(a_l)$. Then, since $P \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}\right)$ and

$$\begin{aligned}
&\|P(x_k, x_l) \pm (\alpha^2 b_{kk}x_k^2 + \beta^2 b_{ll}x_l^2 + \alpha\beta b_{kl}x_k x_l)\|_{\ell_1^2}\\
&= \sup\{|(\overline{P} \pm S)(\alpha x_k e_k + \beta x_l e_l)| : x_0, y_0 \in \mathbb{K} \setminus \{0\} \text{ with } |x_0| + |y_0| = 1\} \leq 1,
\end{aligned}$$

we have that $b_{kk} = b_{ll} = b_{kl} = 0$.

Now take $k, l \in A$ with $k < l$ and fix $m \in B$. Let us take $x_k, x_l \in \mathbb{K} \setminus \{0\}$ such that $x_0 = x_k + x_l$ and $|x_0| = |x_k| + |x_l|$, and let us define $\alpha = \operatorname{sign}(a_k)$, $\beta = \operatorname{sign}(a_l)$ and $\gamma = \operatorname{sign}(a_m)$. Then,

$$\begin{aligned}
1 &\geq |(\overline{P} \pm S)(\alpha x_k e_k + \beta x_l e_l + \gamma y_0 e_m)|\\
&= |\overline{P}(\alpha x_k e_k + \beta x_l e_l + \gamma y_0 e_m) \pm (\alpha\beta b_{kl}x_k x_l)|\\
&= |P(x_0, y_0) \pm (\alpha\beta b_{kl}x_k x_l)|.
\end{aligned}$$

Hence, $b_{kl} = 0$ for any $k, l \in A$ with $k < l$. Since the above can also be applied in the case when $k, l \in B$, it follows that $S = 0$ and the proof is complete. $\square$

5.7 The Space ℓ_p when $p > 2$

To finish this chapter of polynomials defined over sequence Banach spaces, we analyze when a polynomial defined over ℓ_p when $p > 2$ is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^2\ell_p)}$. In particular, we will study when a diagonal polynomial is an extreme point of the unit ball of $\mathcal{P}\left(^2\ell_p\right)$.

Theorem 5.13 (Grecu [26]) *If $p > 2$ and $(\alpha_n)_{n\in\mathbb{N}} \in \ell_{p/(p-2)}$ has unit norm with all $\alpha_n \geq 0$ or all $\alpha_n \leq 0$, then $P(x) = \sum_{n\in\mathbb{N}} \alpha_n x_n^2$ is an extreme point of $\mathsf{B}(\mathcal{P}(^2\ell_p))$.*

Proof Let $(\alpha_n)_{n\in\mathbb{N}} \in \ell_{p/(p-2)}$ be with unit norm and all $\alpha_n \geq 0$. Notice that if we prove the result for all $\alpha_n \geq 0$, then we clearly have the case when all $\alpha_n \leq 0$.

Firstly, the polynomial $P(x) = \sum_{n\in\mathbb{N}} \alpha_n x_n^2$ is in the unit sphere of $\mathcal{P}(^2\ell_p)$. Indeed, clearly we have that $\overline{\alpha} = \left(\alpha_n^{1/(p-2)}\right)_{n\in\mathbb{N}}$ has unit norm in ℓ_p and

$$P(\overline{\alpha}) = \sum_{n\in\mathbb{N}} \alpha_n \alpha_n^{2/(p-2)} = \sum_{n\in\mathbb{N}} \alpha_n^{p/(p-2)} = 1.$$

Furthermore, by Holder's inequality,

$$|P(x)| \leq \left(\sum_{n\in\mathbb{N}} |\alpha_n|^{p/(p-2)}\right)^{(p-2)/p} \left(\sum_{n\in\mathbb{N}} (x_n^2)^{p/2}\right)^{2/p} = \|x\|_{\ell_p}.$$

Hence, $\|P\|_{\ell_p} = 1$.

Now suppose that there exist polynomials P_1 and P_2 with unit norm in $\mathcal{P}(^2\ell_p)$ such that $P = \frac{1}{2}(P_1 + P_2)$. It is enough to show that $P = P_1 = P_2$. Let $P_k = \sum_{n\in\mathbb{N}} \alpha_n^{(k)} x_n^2 + \sum_{1\leq n<m} \alpha_{nm}^{(k)} x_n x_m$ with $k = 1, 2$. Fix $r \in \mathbb{N}$ and take $x_n = 0$ for every $n > r$, then

$$\left|\sum_{n=1}^{r} \alpha_n^{(k)} x_n^2 + \sum_{1\leq n<m\leq r} \alpha_{nm}^{k} x_n x_m\right| \leq \|x\|_{\ell_p}^2.$$

If we replace now x_r by $-x_r$, then we have by the triangle inequality that

$$\left|\sum_{n=1}^{r} \alpha_n^{(k)} x_n^2 + \sum_{1\leq n<m\leq r-1} \alpha_{nm}^{k} x_n x_m\right| \leq \|x\|_{\ell_p}^2.$$

Hence, repeating the same argument with x_{r-1} down to x_2, we have

$$\left|\sum_{n=1}^{r} \alpha_n^{(k)} x_n^2\right| \leq \|x\|_{\ell_p}^2,$$

for any $r \in \mathbb{N}$. Thus the polynomials $Q_k(x) = \sum_{n\in\mathbb{N}} \alpha_n^{(k)} x_n^2$ with $k = 1, 2$ are in the unit ball of $\mathcal{P}(^2\ell_p)$. Moreover, the latter polynomials are in the unit sphere of $\mathcal{P}(^2\ell_p)$ because, since $\alpha_n = \frac{1}{2}\left(\alpha_n^{(1)} + \alpha_n^{(2)}\right)$ for every $n \in \mathbb{N}$, we have $P = \frac{1}{2}(Q_1 + Q_2)$, which implies that $P(\overline{\alpha}) = 1$ only if both Q_1 and Q_2 are in the unit sphere. Notice that we also have $Q_1(\overline{a}) = Q_2(\overline{a}) = 1$, which implies that $\alpha_n^{(k)} \geq 0$ for any $n \in \mathbb{N}$ and $k = 1, 2$. Indeed, take $k = 1$ and let $N_P = \left\{n : \alpha_n^{(1)} \geq 0\right\}$ and $N_N = \left\{n : \alpha_n^{(1)} \leq 0\right\}$. If $N_N \neq \emptyset$, then

$$1 < 1 - \sum_{n\in N_N} \alpha_n^{(1)} \alpha_n^{2/(p-2)} = \sum_{n\in N_p} \alpha_n^{(1)} \alpha_n^{2/(p-2)} \leq 1,$$

which is a contradiction.

As we have already proven that $\left|\sum_{n\in\mathbb{N}} \alpha_n^{(k)} x_n^2\right| \leq \|x\|_{\ell_p}^2$, it is obvious that $\left|\sum_{n\in\mathbb{N}} \alpha_n^{(k)} b_n\right| \leq \|b\|_{\ell_{p/2}}$ for every $b = (b_n)_{n\in\mathbb{N}} \in \ell_{p/2}$. The latter implies that $\alpha^{(k)} = \left(\alpha_n^{(k)}\right)_{n\in\mathbb{N}} \in \ell_{p/2}^* = \ell_{p/(p-2)}$ with $\|\alpha^{(k)}\|_{\ell_{p/(p-2)}} \leq 1$ for every $k = 1, 2$. Hence, $\alpha = \frac{1}{2}\left(\alpha^{(1)} + \alpha^{(2)}\right)$ in ℓ_p which yields $\alpha = \alpha^{(1)} = \alpha^{(2)}$. It remains to prove that $\alpha_{nm}^{(k)} = 0$ for every $1 \leq n < m$ with $k = 1, 2$.

Applying the same techniques of fixing $r \in \mathbb{N}$, taking $x_n = 0$ for every $n > r$ and replacing the coordinates from x_r down to x_3 by their opposites, we have that

$$\left|\sum_{n\in\mathbb{N}} \alpha_n^{p/(p-2)} \pm \alpha_{12}^{(k)} \alpha_1^{(1/(p-2))} \alpha_2^{1/(p-2)}\right| = \left|1 \pm \alpha_{12}^{(k)} \alpha_1^{1/(p-2)} \alpha_2^{1/(p-2)}\right| \leq 1,$$

which implies $\alpha_{12}^{(k)} \alpha_1^{1/(p-2)} \alpha_2^{1/(p-2)} = 0$. If α_1 and α_2 are not 0, then $\alpha_{12}^{(k)} = 0$ for every $k = 1, 2$. Assume that both α_1 and α_2 are 0 (a similar argument can be applied in the case when only one is 0), then P_k can be written as

$$\begin{aligned} P_k(x) =& \alpha_{12}^{(k)} x_1 x_2 + \sum_{n\geq 3} \alpha_{1n}^{(k)} x_1 x_i + \sum_{n\geq 3} \alpha_{2n}^{(k)} x_2 x_i \\ &+ \sum_{n\geq 3} \alpha_n x_n^2 + \sum_{3\leq n<m} \alpha_{nm}^{(k)} x_n x_m \end{aligned}$$

with

$$\sum_{n\geq 3} \alpha_n^{p/(p-2)} + \sum_{3\leq n<m} \alpha_{nm}^{(k)} \alpha_n^{1/(p-2)} \alpha_m^{1/(p-2)} = 1.$$

By taking $x_1 = x_2 = 1$ and $x_n = \pm r\alpha_n^{1/(p-2)}$ for $n \geq 3$ and tending then r to ∞ it is easy to see that $\alpha_{12}^{(k)} \leq 0$. By construction $\alpha_{12}^{(1)} + \alpha_{12}^{(2)} = 0$, which yields $\alpha_{nm}^{(k)} = 0$ for any $1 \leq n < m$ and $k = 1, 2$. □

Corollary 5.2 (Grecu [26]) *Let $p > 2$. If A and B are disjoint sets of $\mathbb{N}$, $(\alpha_a)_{a\in A} \in \ell_{p/(p-2)}(A)$, $(\beta_b)_{b\in B} \in \ell_{p/(p-2)}(B)$ with $\alpha_a > 0$ for every $a \in A$, $\beta_b > 0$ for every $b \in B$, $\|(\alpha_a)_{a\in A}\|_{\ell_{p/(p-2)}(A)} = 1$ and $\|(\alpha_b)_{b\in B}\|_{\ell_{p/(p-2)}(B)} = 1$, then the polynomial $P(x) = \sum_{a\in A} \alpha_a x_a^2 - \sum_{b\in B} \beta_b x_b^2$ is an extreme point of $\mathsf{B}_{\mathcal{P}(^2\ell_p)}$.*

The following result shows that the polynomials described in Corollary 5.2 are the only diagonal extreme polynomials of $\mathsf{B}_{\mathcal{P}(^2\ell_p)}$.

Theorem 5.14 (Grecu [26]) *If $p > 2$ and $P(x) = \sum_{n\in\mathbb{N}} \gamma_n x_n^2$ where $x = (x_n)_{n\in\mathbb{N}} \in \ell_p$, then $P \in \mathcal{P}(^2\ell_p)$. Furthermore, P is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^2\ell_p)}$ if, and only if, there exist A and B disjoint sets of natural numbers and $(\alpha_a)_{a\in A} \in \mathsf{B}_{\ell_{p/(p-2)}(A)}$ and $(\beta_b)_{b\in B} \in \mathsf{B}_{\ell_{p/(p-2)}(B)}$ positive sequences of unit norm such that $P(x) = \sum_{a\in A} \alpha_i x_i^2 - \sum_{b\in B} \beta_b x_b^2$.*

Proof By Corollary 5.2, given $P(x) = \sum_{n\in\mathbb{N}}^{\infty} \gamma_n x_n^2$ where $x = (x_n)_{n\in\mathbb{N}} \in \ell_p$ an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^2\ell_p)}$, it is enough to find A and B disjoint sets of natural numbers and $(\alpha_a)_{a\in A} \in \mathsf{B}_{\ell_{p/(p-2)}(A)}$ and $(\beta_b)_{b\in B} \in \mathsf{B}_{\ell_{p/(p-2)}(B)}$ positive sequences of unit norm such that $P(x) = \sum_{a\in A} \alpha_i x_i^2 - \sum_{b\in B} \beta_b x_b^2$.

Let $A = \{n \in \mathbb{N} \colon \gamma_n \geq 0\}$ and $B = \{n \in \mathbb{N} \colon \gamma_n < 0\}$. Take $\alpha_a = \gamma_a$ provided $a \in A$, and $\beta_b = -\gamma_b$ provided $b \in B$. Notice that by construction we have $\alpha_a \geq 0$ for every $a \in A$, $\beta_b \geq 0$ for every $b \in B$ and $P(x) = \sum_{a\in A} \alpha_i x_i^2 - \sum_{b\in B} \beta_b x_b^2$. It suffices to show that $\alpha = (\alpha_a)_{a\in A}$ is an extreme point of the unit ball of $\ell_{p/(p-2)}(A)$ and has unit norm.

First, by using a duality argument (see the proof of Theorem 5.13), α belongs to the unit ball of $\ell_{p/(p-2)}(A)$. The same reasoning shows that $\beta = (\beta_b)_{b\in B}$ belongs to the unit ball of $\ell_{p/(p-2)}(B)$. Suppose that there exist $\alpha^{(1)}$ and $\alpha^{(2)}$ in the unit ball of $\ell_{p/(p-2)}(A)$ such that $\alpha = \frac{1}{2}\left(\alpha^{(1)} + \alpha^{(2)}\right)$, and let us define $P_k(x) = \sum_{a\in A} \alpha_a^{(k)} x_a^2 - \sum_{b\in B} \beta_b x_b^2$ for $k = 1, 2$. By Holder's inequality and the triangle inequality, notice that $\|P\|_{\ell_p} \leq 1$. Also, by construction, we have $P = \frac{1}{2}(P_1 + P_2)$. Hence, $P = P_1 = P_2$ since P is an extreme polynomial of the unit ball of $\mathcal{P}(^2\ell_p)$. Thus, we have $\alpha = \alpha^{(1)} = \alpha^{(2)}$, which yields that α is an extreme point of $\mathsf{B}_{\ell_{p/(p-2)}(A)}$ and, therefore, α is in the unit sphere of $\ell_{p/(p-2)}(A)$. Analogously, β is the unit sphere of $\ell_{p/(p-2)}(B)$ and is an extreme point of $\mathsf{B}_{\ell_{p/(p-2)}(B)}$. □

Chapter 6
Polynomials with the Hexagonal and Octagonal Norms

Abstract In this chapter we focus on the extreme points of the unit ball of quadratic forms on $\mathbb{R}^2$ endowed with the octagonal and hexagonal norms.

6.1 Octagonal Norm

Let us endow the vector space $\mathbb{R}^2$ with the following octagonal norm with weight $w \in [0, 1]$: for every $(x, y) \in \mathbb{R}^2$,

$$\|(x, y)\|_{\mathrm{oct}(w)} = \max \left\{ |x|, |y|, \frac{|x| + |y|}{1 + w} \right\}.$$

For the rest of this section, we denote by $\mathcal{O}_w^2$ the space $\mathbb{R}^2$ endowed with the octagonal norm $\| \cdot \|_{\mathrm{oct}(w)}$.

Let us endow the space of 2-homogeneous real polynomials with the following norm: for every $P(x, y) = ax^2 + by^2 + cxy$, where $a, b, c \in \mathbb{R}^3$,

$$\|P\|_{\mathcal{O}_w^2} = \sup\{|P(x, y)| \colon \|(x, y)\|_{\mathrm{oct}(w)} \leq 1\}.$$

We will denote by $\mathcal{P}(^2\mathcal{O}_w^2)$ the space of 2-homogeneous real polynomials endowed with the norm $\| \cdot \|_{\mathcal{O}_w^2}$. As in previous chapters, we have that the mapping T defined on Sect. 2.1 and considered on the space $\mathcal{P}(^2\mathcal{O}_w^2)$ is a topological isomorphism from $\mathcal{P}(^2\mathcal{O}_w^2)$ to the normed space $(\mathbb{R}^3, \| \cdot \|_{\mathrm{oct}(w)})$, where $\| \cdot \|_{\mathcal{P}(^2\mathcal{O}_w^2)}$ is defined as follows: for every $(a, b, c) \in \mathbb{R}^3$,

$$\|(a, b, c)\|_{\mathcal{P}(^2\mathcal{O}_w^2)} = \|ax^2 + by^2 + cxy\|_{\mathcal{O}_w^2}.$$

The unit ball of $(\mathbb{R}^3, \| \cdot \|_{\mathcal{P}(^2\mathcal{O}_w^2)})$ will be denoted by $\mathsf{B}_{\mathcal{P}\left(^2\mathcal{O}_w^2\right)}$.

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_6

We are interested in studying the extreme points of the unit ball of $\mathcal{P}({}^2O_w^2)$. Notice that $O_0^2 = \ell_1^2$ and $O_1^2 = \ell_\infty^2$ which have already been analyzed in Sects. 5.1 and 5.2.

We begin by providing an explicit formula for the norm of the space $\mathcal{P}({}^2O_w^2)$. First of all, notice that

$$\|ax^2 + by^2 + cxy\|_{O_w^2} = \|bx^2 + ay^2 \pm cxy\|_{O_w^2} = \| - bx^2 - ay^2 \pm cxy\|_{O_w^2}.$$

Therefore we may assume that $a \geq |b|$ and $c \geq 0$ since the remaining cases can be easily deduced.

Theorem 6.1 (Kim [38]) *Let $P(x, y) = ax^2 + by^2 + cxy$ where $(a, b, c) \in \mathbb{R}^3$, $a \geq |b|$ and $c \geq 0$. We have*

$$\|P\|_{O_w^2} = \begin{cases} a + \frac{c^2}{4|b|} & \text{if } b < 0,\ 0 \leq c < -2b \text{ and } -\frac{c}{2b} \leq w, \\ bw^2 + cw + a & \text{if } (2|b| \leq c \leq 2a),\ \text{or } (b > 0 \text{ and } 0 \leq c < 2b), \\ & \text{or } \left(2a < c \text{ and } \frac{c-2a}{c-2b} < w\right), \\ & \text{or } \left(b < 0,\ 0 \leq c < -2b \text{ and } -\frac{c}{2b} > w\right), \\ \frac{(c^2-4ab)(1+w)^2}{4(c-a-b)} & \text{if } 2a < c \text{ and } \frac{c-2a}{c-2b} \geq w. \end{cases}$$

Although the following result does not appear explicitly in [38], it is remarked at the beginning of [38, section 2].

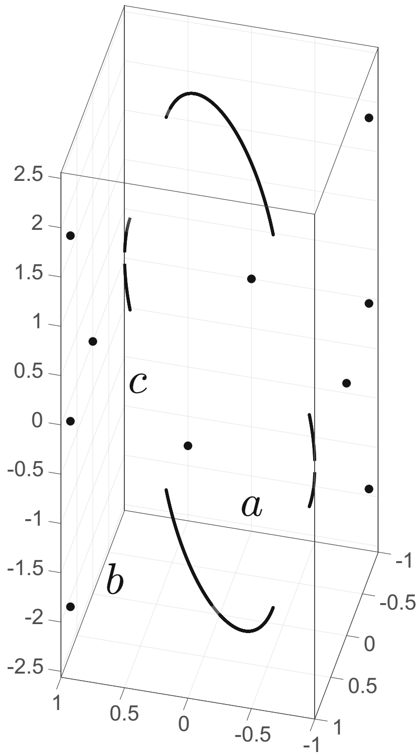
Theorem 6.2 (Kim [38]) *The projection of $\mathsf{B}_{\mathcal{P}\left({}^2O_w^2\right)}$ onto the ab-plane is $\mathsf{B}_{\ell_\infty^2}$.*

Now we are ready to show the extreme points of $\mathsf{B}_{\mathcal{P}({}^2O_w^2)}$.

Theorem 6.3 (Kim [38]) *The set of extreme points of $\mathsf{B}_{\mathcal{P}({}^2O_w^2)}$ consist of the elements*

$$\begin{aligned} \operatorname{ext}\left(\mathsf{B}_{\mathcal{P}\left({}^2O_w^2\right)}\right) = &\left\{ \pm\left(t, -t, \pm\left[\frac{2}{(1+w)^2} + 2\sqrt{\frac{1}{(1+w)^4} - t^2}\right]\right) : \right. \\ &\left. t \in \left[0, \frac{1-w}{(1+w)(1+w^2)}\right]\right\} \\ &\bigcup \left\{\pm\left(s, -s, \pm 2\sqrt{s(1-s)}\right) : s \in \left[\frac{1}{1+w^2}, 1\right]\right\} \\ &\bigcup \left\{\pm\frac{1}{(1+w)^2}(1, 1, \pm 2)\right\} \\ &\bigcup \left\{\pm\frac{1}{1+w^2}(1, 1, 0)\right\} \bigcup \{\pm(0, 1, 0)\} \bigcup \{\pm(1, 0, 0)\}. \end{aligned}$$

Fig. 6.1 Extreme points of $\mathsf{B}_{\mathcal{P}(^2\mathcal{O}_w^2)}$ with $w = 1/4$



Unfortunately the authors have not been able to obtain a parametrization of the unit sphere of $(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}(^2\mathcal{O}_w^2)})$. However, we provide a visual representation of the extreme points of the unit ball $\mathsf{B}_{\mathcal{P}(^2\mathcal{O}_w^2)}$ in Figs. 6.1 and 6.2.

6.2 Hexagonal Norm

Let us endow $\mathbb{R}^2$ with the following norm known as the hexagonal norm with weight $w \in [0, 1]$: for every $(x, y) \in \mathbb{R}^2$,

$$\|(x, y)\|_{\text{hex}(w)} := \max\{|y|, |x| + (1 - w)|y|\}.$$

The space $\mathbb{R}^2$ endowed with $\|\cdot\|_{\text{hex}(w)}$ is denoted by $\mathcal{H}_w^2$. Notice that $\mathcal{H}_0^2 = \ell_1^2$ and $\mathcal{H}_1^2 = \ell_\infty^2$, which have already been treated in Sects. 5.1 and 5.2. Nonetheless, in this section we are interested in studying the case when $w = \frac{1}{2}$.

Let P be a 2-homogeneous real polynomial of the form $P(x, y) = ax^2 + by^2 + cxy$ and consider the following norm

$$\|P\|_{\mathcal{H}_{1/2}^2} = \sup\{|P(x, y)| : \|(x, y)\|_{\text{hex}(1/2)} \le 1\}.$$

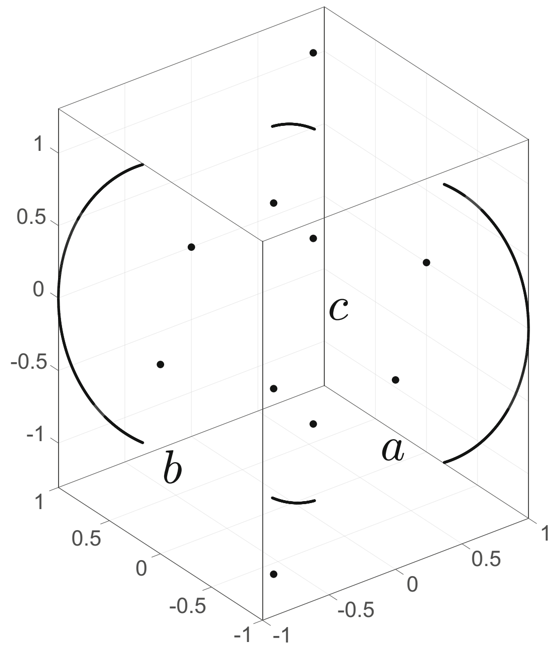


Fig. 6.2 Extreme points of $\mathsf{B}_{\mathcal{P}(^2\mathcal{O}_w^2)}$ with $w = 3/4$

Let us denote by $\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)$ the space of 2-homogeneous real polynomials endowed with the norm $\|\cdot\|_{\mathcal{H}_{1/2}^2}$. As in the previous cases treated in this survey, the mapping T defined on Sect. 2.1 and restricted to the space $\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)$ is a topological isomorphism from $\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)$ to the normed space $\left(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)}\right)$, where

$$\|(a, b, c)\|_{\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)} = \|ax^2 + by^2 + cxy\|_{\mathcal{H}_{1/2}^2},$$

for every $(a, b, c) \in \mathbb{R}^3$. Let us denote the unit ball of $\left(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)}\right)$ by $\mathsf{B}_{\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)}$. Once again, just as in the case of the octagonal norm (Sect. 6.1), we begin by showing an explicit formula for the hexagonal norm $\|\cdot\|_{\mathcal{H}_{1/2}^2}$.

Theorem 6.4 (Kim [39]) *Let $P(x, y) = ax^2 + by^2 + cxy$ with $a \geq 0$, $c \geq 0$ and $a^2 + b^2 + c^2 \neq 0$. We have*

$$\|P\|_{\mathcal{H}_{1/2}^2} = \begin{cases} \max\left\{a, \frac{1}{4}a + b + \frac{1}{2}c\right\} & \text{if } c < a \text{ and } a \leq 4b, \\ \max\left\{a, |b|, \left|\frac{1}{4}a + b\right| + \frac{1}{2}c, \frac{|c^2-4ab|}{4a}\right\} & \text{if } c < a \text{ and } a > 4b, \\ \max\left\{a, \frac{1}{4}a + b + \frac{1}{2}c, \frac{|c^2-4ab|}{2c+a+4b}\right\} & \text{if } c \geq a \text{ and } a \leq 4b, \\ \max\left\{a, |b|, \left|\frac{1}{4}a + b\right| + \frac{1}{2}c, \frac{c^2-4ab}{2c-a-4b}\right\} & \text{if } c \geq a \text{ and } a > 4b. \end{cases}$$

We now proceed to show the projection of $\mathsf{B}_{\mathcal{P}\left({}^2\mathcal{H}^2_{1/2}\right)}$ onto the ab-plane.

Theorem 6.5 (Kim [38]) *The projection of* $\mathsf{B}_{\mathcal{P}\left({}^2\mathcal{H}^2_{1/2}\right)}$ *onto the ab-plane is* $\mathsf{B}_{\ell^2_\infty}$.

Theorem 6.6 (Kim [39]) *The set of extreme points of* $\mathsf{B}_{\mathcal{P}({}^2\mathcal{H}^2_{1/2})}$ *consists of*

$$\begin{aligned}\operatorname{ext}(\mathsf{B}_{\mathcal{P}({}^2\mathcal{H}^2_{1/2})}) = &\left\{\pm\left(t, \frac{t+4\sqrt{1-t}}{4} - 1, \pm\left[t+2\sqrt{1-t}\right]\right) : t \in [0,1]\right\}\\ &\bigcup\left\{\pm\left(1, \frac{s^2}{4} - 1, \pm s\right) : s \in [0,1]\right\}\\ &\bigcup\left\{\pm\left(1, \frac{3}{4}, 0\right)\right\}\bigcup\left\{\pm\left(1, \frac{1}{4}, \pm 1\right)\right\}\bigcup\{\pm(0,1,0)\}.\end{aligned}$$

Just as in the case of the octagonal norm (Sect. 6.1), the authors were not able to provide a parametrization of the unit sphere of $\left(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}\left({}^2\mathcal{H}^2_{1/2}\right)}\right)$, but Fig. 6.3 shows the extreme points of $\mathsf{B}_{\mathcal{P}\left({}^2\mathcal{H}^2_{1/2}\right)}$.

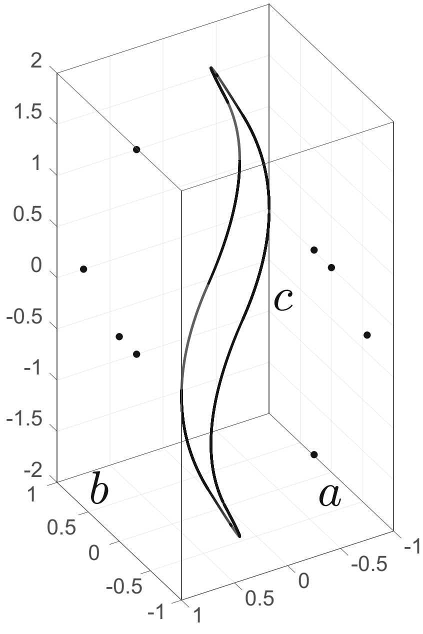


Fig. 6.3 The extreme points of the unit ball of $\mathcal{P}\left({}^2\mathcal{H}^2_{1/2}\right)$

Chapter 7
Hilbert Spaces

Abstract Let H denote a Hilbert space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), i.e., H is a Banach space which norm $\|\cdot\|_H$ comes from a bilinear product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ which verifies that $\|x\|_H = \sqrt{|\langle x, x\rangle|}$ for every $x \in H$. In this chapter, we are interested in studying the extreme points of 2-homogeneous polynomials defined over Hilbert spaces.

Recall that the spaces of polynomials mentioned above have already been studied when H is a 2-dimensional Hilbert space. Indeed, given two Hilbert spaces of dimension 2, H_1 and H_2, there exists an isometry between H_1 and H_2. Therefore, the results follow from Sect. 4.3.1. Let us denote by $\mathcal{P}(^n H)$ the space of n-homogeneous polynomials over H, where the norm of an n-homogeneous polynomial P over H is defined as

$$||P||_H = \sup\{|P(\mathbf{x})|: ||\mathbf{x}||_H \leq 1\}$$

Also, let $\mathsf{B}_{\mathcal{P}(^n H)}$ and $\mathsf{S}_{\mathcal{P}(^n H)}$ denote the unit ball and the unit sphere of $\mathcal{P}(^n H)$, respectively.

7.1 The Real and Complex Case for 2-Homogeneous Polynomials

Assume first that $\mathbb{K} = \mathbb{R}$. We begin by providing a characterization of the set of extreme points of $\mathsf{B}_{\mathcal{P}(^n H)}$ where H is defined over the real numbers and is finite dimensional.

Theorem 7.1 (Sundaresan [54]) *Let H be a Hilbert space of dimension $n \geq 2$. A 2-homogeneous polynomial P is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^2 H)}$ if, and only if, the matrix $A = (a_{ij})$ defined by $a_{ij} = \frac{1}{2}(P(e_i + e_j) - P(e_i) - P(e_j))$ with $1 \leq i, j \leq n$ has only unimodular eigenvalues.*

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_7

Proof By Sundaresan [54, section 1], there exists a linear isometry

$$F\colon (S_n, \|\cdot\|_\infty) \to (S_n, \|\cdot\|_\infty),$$

where S_n is the real vector space of symmetric $n \times n$ matrices and

$$\|A\|_\infty = \sup\{\|Ax\|_H \colon \|x\|_H \le 1\}$$

for every $A \in S_n$, such that $F(A)$ is a diagonal matrix $B = (b_{ij})_{1\le i,j\le n}$ with b_{ii} being the eigenvalues of A in no particular order. Clearly, since F is a linear isometry, we have that A is an extreme point of the unit ball of $(S_n, \|\cdot\|_\infty)$ if, and only if, B is an extreme point of the unit ball of $(S_n, \|\cdot\|_\infty)$. Hence, it is enough to show that B is an extreme point of the unit ball of $(S_n, \|\cdot\|_\infty)$ if, and only if, the diagonal entries of B are unimodular.

Assume first that $|b_{ii}| = 1$ for every $1 \le i \le n$. By way of contradiction, suppose that there exist distinct matrices $C = (c_{ij})$ and $D = (d_{ij})$ in $(S_n, \|\cdot\|_\infty)$ of norm one such that $B = \frac{1}{2}(C + D)$. Hence,

$$n = \sum_{i=1}^{n} b_{ii}^2 = \sum_{i=1}^{n} \frac{1}{2}(c_{ii} + d_{ii}) \le \frac{1}{2}\left(\sum_{i=1}^{n} c_{ii}^2 + \sum_{i=1}^{n} d_{ii}^2\right) \le n.$$

Clearly we have $\sum_{i=1}^{n} \frac{1}{2}(c_{ii} + d_{ii}) = \frac{1}{2}\left(\sum_{i=1}^{n} c_{ii}^2 + \sum_{i=1}^{n} d_{ii}^2\right)$ and therefore $p_{ii} = q_{ii}$ for every $1 \le i \le n$. The latter implies that $b_{ii} = c_{ii} = d_{ii}$ for any $1 \le i \le n$. To reach a contradiction, it is enough to show that the non-diagonal entries of C and D are 0. Let $(\lambda_i)_{i=1}^n$ be the eigenvalues of C. Then, by the Frobenius equation and using the fact that $\|C\|_\infty = \sup\{|\lambda_i| \colon 1 \le i \le n\} = 1$, we have

$$\sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n} \lambda_i^2 \le n.$$

Hence,

$$\sum_{1\le i,j\le n,\ i\ne j} c_{ij}^2 + n \le n,$$

which implies that $c_{ij} = 0$ for every $1 \le i, j \le n$ with $i \ne j$.

Assume now that B is an extreme point of the unit ball of $(S_n, \|\cdot\|_\infty)$. Suppose that there exists $i_0 \in \{1, \ldots, n\}$ such that $|b_{i_0 i_0}| < 1$. Choose $\delta > 0$ such that $|d_{i_0 i_0} \pm \delta| \le 1$, and choose $P = (p_{ij})$ and $Q = (q_{ij})$ diagonal matrices such that $p_{jj} = q_{jj} = b_{jj}$ if $j \ne i_0$, $p_{i_0 i_0} = b_{i_0 i_0} + \delta$ and $q_{i_0 i_0} = b_{i_0 i_0} - \delta$. By construction it is easy to see that $P \ne Q$, $B = \frac{1}{2}(P + Q)$ and also P and Q belong to the unit ball of $(S_n, \|\cdot\|_\infty)$, which is a contradiction. $\square$

For the rest of this section, assume that H is a Hilbert space of finite or infinite dimension. The following theorem is an extension of Theorem 7.1 to arbitrary real Hilbert spaces, but in order to prove it we will use another result related to the self-adjoint operator of a polynomial.

Remark 7.1 Let P be a 2-homogeneous polynomial on H and L the polar of P (see the beginning of Sect. 5.5). Notice that for a fixed $y \in H$, the mapping $H \ni x \mapsto L(x, y)$ is linear and continuous. Hence, by Riesz representation theorem, there exists a unique bounded linear operator $T : H \to H$ such that $L(x, y) = \langle x, Ty\rangle$, T is self-adjoint and $\|T\| = \|L\| = \|P\|$. We say that T is the self-adjoint linear operator of P.

Proposition 7.1 (Grecu [25]) *Let H be a real Hilbert space and P be an extreme 2-homogeneous real polynomial of the unit ball of $\mathcal{P}(^2H)$ with T the self-adjoint linear operator of P. We have that $T^2 = Id$.*

Proof As $\|P\| = 1$, we have that $\|T\| = 1$ by Remark 7.1. Furthermore, as $T : H \to H$ is self-adjoint, there exist a positive measure μ, $g \in L_\infty(\mu)$ with $\|g\|_\infty = \|T\|$ and a unitary operator $U : H \to L_2(\mu)$ such that $UT = M_g U$, where $M_g : L_2(\mu) \to L_2(\mu)$ is defined by $M_g(f) = gf$ for every $f \in L_2(\mu)$ (see [31]).

Hence

$$\begin{aligned} P(x) &= \langle Tx, x\rangle = \langle Tx, x\rangle \\ &= \langle gUx, Ux\rangle = \int g(Ux)^2 d\mu. \end{aligned}$$

Now suppose that there exist $g_1, g_2 \in L_\infty(\mu)$ with $\|g_1\|_\infty = \|g_2\|_\infty$ such that $g = \frac{1}{2}(g_1 + g_2)$, and let us define the 2-homogeneous real polynomials $P_i(x) = \int g_i(Ux)^2 d\mu$ for $i = 1, 2$. By construction, it is easy to see that $P = \frac{1}{2}(P_1 + P_2)$ with $0 < \|P_i\|_H \leq \|g_i\|_\infty \leq 1$. As P is an extreme polynomial, the latter implies that $P_1 = P_2$. Now take $f \in L_1(\mu)$ arbitrary and decompose f as $f = f^+ - f^-$, where $f^+, f^- \geq 0$. Clearly, $\sqrt{f^+}, \sqrt{f^-} \in L_2(\mu)$, and so there exists $x \in H$ such that $Ux = \sqrt{f^+}$. Thus,

$$\int (g_1 - g_2) f^+ d\mu = \int (g_1 - g_2)(Ux)^2 d\mu = P_1(x) - P_2(x) = 0.$$

Applying the same arguments for f^-, we have that $\int (g_1 - g_2) f d\mu = 0$ for every $f \in L_1(\mu)$, which proves that $g_1 = g_2$.

We have proven that g is an extreme point of $L_\infty(\mu)$ and therefore we have that $|g| = 1$ a.e. Thus, we have that

$$U(T^2x) = gU(Tx) = g^2Ux = Ux.$$

Since U is injective (as it preserves the inner product), we have that $T^2 = Id$. $\square$

Theorem 7.2 (Grecu [25]) *Let H be a real Hilbert space. A 2-homogeneous real polynomial P is an extreme point of $\mathsf{B}_{\mathcal{P}(^2H)}$ if, and only if, there exists an orthogonal decomposition of $H = H_1 \oplus H_2$ such that $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$ for every $x \in H$, where π_1 and π_2 denote the orthogonal projections of H onto H_1 and H_2, respectively.*

Proof Firstly, assume that there exists an orthogonal decomposition of $H = H_1 \oplus H_2$ such that $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$ for every $x \in H$. Clearly, $-\|\pi_2(x)\|^2 \leq P(x) \leq \|\pi_1(x)\|^2$ for any $x \in H$ and $P(x) = \|x\|^2$ when $x \in H_1$. Hence, $\|P\|_H = 1$. Assume that there exist P_1 and P_2 in $\mathsf{B}_{\mathcal{P}(^2H)}$ such that $P = \frac{1}{2}(P_1 + P_2)$. Then, for any $x \in H_1$, we have that $\|x\|^2 = \frac{1}{2}(P_1(x) + P_2(x)) \leq \|x\|^2$, which implies that $P_1(x) = P_2(x) = \|x\|^2$ on H_1. Analogously, we have $P_1(x) = P_2(x) = -\|x\|^2$ on H_2. Let L_k and T_k be the polar and self-adjoint linear operator of P_k for $k = 1, 2$, respectively. Then, for every $x = x_1 + x_2 \in H_1 \oplus H_2$, we have

$$\begin{aligned} P_1(x) &= P_1(x_1 + x_2) = L_1(x_1 + x_2, x_1 + x_2) \\ &= P_1(x_1) + 2L_1(x_1, x_2) + P_1(x_2) \\ &= \|x_1\|^2 + 2L_1(x_1, x_2) - \|x_2\|^2. \end{aligned}$$

As $P_1(x_2) = -\|x_2\|^2$ we have $\langle x_2, T_1x_2\rangle = -\|x_2\|^2$. Hence, we have $T_1x_2 = -x_2$ because $\|T_1\| = \|P_1\|_H \leq 1$. Therefore, $L_1(x_1, x_2) = \langle x_1, -x_2\rangle = 0$ since H_1 and H_2 are an orthogonal decomposition of H. The latter implies that $P_1 = P$ and the proof is complete.

Secondly, assume that P is an extreme point of $\mathsf{B}_{\mathcal{P}(^2H)}$ and let T be the associated self-adjoint linear operator of P. By Proposition 7.1, $T^2 = Id$. Let us define the continuous linear operators $\pi_1 = (I + T)/2$ and $\pi_2 = (I - T)/2$, and take $H_1 = \pi_1(H)$ and $H_2 = \pi_2(H)$. On the one hand, since T is self-adjoint and $T^2 = Id$,

$$\begin{aligned} \langle \pi_1(x), \pi_2(x)\rangle &= \frac{1}{4}\langle x + Tx, x - Tx\rangle \\ &= \frac{1}{4}(\langle x, x\rangle + \langle Tx, x\rangle - \langle x, Tx\rangle - \langle Tx, Tx\rangle) \\ &= \frac{1}{4}(\langle x, x\rangle - \langle T^2x, x\rangle) = 0 \end{aligned}$$

for any $x \in H$. On the other hand $x = (x + Tx)/2 + (x - Tx)/2 = \pi_1(x) + \pi_2(x)$. Hence, $H = H_1 \oplus H_2$. Moreover, $Tx = \pi_1(x) - \pi_2(x)$, which implies that $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$. □

Assume now that $\mathbb{K} = \mathbb{C}$. We are going to show a characterization of the set of extreme points of $\mathsf{B}_{\mathcal{P}(^2H)}$.

Theorem 7.3 (Grecu [25]) *Let H be a Hilbert space defined over $\mathbb{C}$. A 2-homogeneous complex polynomial P is an extreme point of $\mathsf{B}_{\mathcal{P}(^2H)}$ if, and only* *if, there exists an orthonormal basis $\{e_j\}_{j\in J}$ of H such that $P(x) = \sum_{j\in J} x_j^2$ for every $x = \sum_{j\in J} a_j x_j \in H$.*

Last but not least, we show a characterization of the extreme points of $\mathsf{B}_{\mathcal{P}(^2H)}$, when H is defined over $\mathbb{R}$ or $\mathbb{C}$, but in a more practical way since it uses the coefficients of the polynomial. If $\{e_j\}_{j\in J}$ is an orthonormal basis of H, then a 2-homogeneous polynomial P with polar L is of the form $P(x) = \sum_{i,j\in J} a_{ij} x_i x_j$ with $a_{ij} = a_{ji} = L(e_i, e_j)$.

Theorem 7.4 (Grecu [25]) *Let P be a 2-homogeneous polynomial of unit norm on a separable Hilbert space H defined over $\mathbb{R}$ or $\mathbb{C}$ and let L be the polar of P. If $\{e_j\}_{j\in J}$ is an orthonormal basis for H and A is the matrix whose entries are the coefficients $a_{ij} = L(e_i, e_j)$ of P, then the polynomial P is an extreme point of $\mathsf{B}_{\mathcal{P}(^2H)}$ if, and only if, $\overline{A}A = I$ (where I is the identity matrix).*

7.2 Polynomials of Degree *n*

Now we aim to give a more general approach. In most of the results that we have given, we are mostly interested in 2-homogeneous polynomials. But in this section we go further by showing a characterization of the extreme points on $\mathsf{B}_{\mathcal{P}(^nH)}$ where $n \in \{3, 4\}$ and H is a real Hilbert space. Unfortunately this characterization is not true when $n \geq 5$ but we are able to give some insight nonetheless.

Theorem 7.5 (Grecu [28]) *Let P be a 3-homogeneous real polynomial of unit norm on a two dimensional real Hilbert space H. We have that P is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^3H)}$ if, and only if, for any orthonormal basis $\{e_1, e_2\}$ of H such that $P((x_1, x_2)) = x_1^3 + 3bx_1x_2^2 + cx_2^3$, the coefficients b and c satisfy the condition $c^2 = (b+1)^2(2b-1)$.*

Theorem 7.6 (Grecu [27]) *Let P be a 4-homogeneous real polynomial of unit norm on a two dimensional real Hilbert space H. We have that P is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^4H)}$ if, and only if, either $P(x) = \pm\|x\|_H^4$ or for any orthonormal basis $\{e_j\}_{j\in J}$ of H such that $P((x_1, x_2)) = x_1^4 + 6bx_1^2x_2^2 + 4cx_1x_2^3 + dx_2^4$, the coefficients b, c and d satisfy one of the following conditions:*

(i) $b = \frac{4\sqrt{\frac{1-d}{2}}-3}{3}$ *and* $c = \pm 2\sqrt{\frac{1-d}{2}}\sqrt{1-\sqrt{\frac{1-d}{2}}}$,

(ii) $b = \frac{-1-\sqrt{2d+2}}{3}$ *and* $c = 0$,

where in both cases $d \in [-1, 1]$.

Let us show now the general results for arbitrary n.

Theorem 7.7 (Grecu [27]) *Let H be a two dimensional real Hilbert space.*

(i) If n is odd and P is an n-homogeneous real polynomial of unit norm such that $|P(x)| = 1$ at $n + 1$ distinct points of the unit ball of H, then P is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^n H)}$.
(ii) If n is even and P is an n-homogeneous real polynomial of unit norm such that $|P(x)| = 1$ at $n + 2$ distinct points of the unit ball of H, then P is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^n H)}$.

The following result in this general setting gives exact values for some extreme points.

Theorem 7.8 (Grecu [27]) *Let H be a two dimensional real Hilbert space and $n > 5$.*

(i) If n is odd, then the polynomial

$$P(x) = \sum_{l=0}^{\frac{n-1}{2}} \frac{(n-2l+2)(n-2l+4)\cdots n}{2\cdot 4\cdots 2l} x_1^{n-2l} x_2^{2l}$$

is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^n H)}$.
(ii) If n is even, then the polynomials

$$P(x) = \|x\|_H^n$$

and

$$P(x) = \sum_{l=0}^{n/2-1} \binom{n/2}{l} x_1^{n-2l} x_2^{2l} - x_2^n$$

are extreme polynomials of $\mathsf{B}_{\mathcal{P}(^n H)}$.

To finish this section, we show a characterization for the extreme points of 3-homogeneous polynomials on a two dimensional complex Hilbert space.

Theorem 7.9 (Grecu et al. [29]) *Let H be a two dimensional complex Hilbert space. A 3-homogeneous complex polynomial $P \in \mathsf{B}_{\mathcal{P}(^3 H)}$ of unit norm is an extreme polynomial of $\mathsf{B}_{\mathcal{P}(^3 H)}$ if, and only if, P attains its norm at two or more linearly independent points.*

Chapter 8
Banach Spaces

Abstract In this chapter we will show some results on the extreme points of the unit ball of certain polynomial spaces in arbitrary Banach spaces. More particularly, we are interested in studying integral, nuclear and orthogonally additive polynomials.

8.1 Integral and Nuclear Polynomials

First, we begin by defining what are known as n-homogeneous integral polynomials on Banach spaces.

Definition 8.1 Let X be a Banach space. We say that a continuous n-homogeneous polynomial P is integral if there is a regular Borel measure μ of total variation on $(\mathsf{B}_{X^*}, \omega^*)$ such that

$$P(x) = \int_{\mathsf{B}_{X^*}} \varphi(x)^n d\mu(\varphi), \tag{8.1}$$

for every $x \in X$, where ω^* stands for the weak*-topology. We denote by $\mathcal{P}_I(^n X)$ the space of continuous n-homogeneous integral polynomials.

Let X be a Banach space, we endow the space $\mathcal{P}_I(^n X)$ with the norm given by

$$\|P\|_I = \inf\{|\mu|(\mathsf{B}_{X^*}) \colon \mu \text{ satisfies } (8.1)\}.$$

The normed space $(\mathcal{P}_I(^n X), \|\cdot\|_I)$ is in fact a Banach space. For simplicity, let us denote $(\mathcal{P}_I(^n X), \|\cdot\|_I)$ by $\mathcal{P}_I(^n X)$. We will denote the unit balls of X and $(\mathcal{P}_I(^n X), \|\cdot\|_I)$, respectively, by B_X and $\mathsf{B}_{\mathcal{P}_I(^n X)}$.

The following result characterizes the set of extreme polynomials of $\mathsf{B}_{\mathcal{P}_I(^n X)}$ when X is real Banach space. Although such characterization is proven in [17], it is worth mentioning that the study of the extreme polynomials of $\mathsf{B}_{\mathcal{P}_I(^n X)}$ has also been done by several authors prior to the final characterization (see [9, 10, 52]).

Theorem 8.1 (Dimant et al. [17]) *If X is a real Banach space and $n \geq 2$, then*

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_8

$$\mathrm{ext}(\mathsf{B}_{\mathcal{P}_I(^nX)}) = \{\pm\varphi^n : \varphi \in \mathsf{S}_{X^*}\}.$$

The following results shed some light on the extreme complex polynomials on arbitrary Banach spaces. We begin by defining what is known as an extreme complex point on a normed space. The standard definition that we have considered for a point on a normed space X to be an extreme point of B_X is the following: given a normed space X defined over $\mathbb{R}$ or $\mathbb{C}$, we say that $x \in \mathsf{B}_X$ is a (real) extreme point of B_X provided that if $x + \lambda y \in \mathsf{B}_X$ for some $\lambda \in \mathbb{C}$ with $\lambda \in [-1, 1]$, then $y = 0$. Now, analogously, given X a normed space defined over $\mathbb{C}$, we say that $x \in \mathsf{B}_X$ is a *complex extreme point* of B_X provided that if $x + \lambda y \in \mathsf{B}_X$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, then $y = 0$.

Given a normed space X defined over $\mathbb{C}$, let us denote for the rest of this section the set of real and complex extreme points of B_X by $\mathrm{ext}_{\mathbb{R}}(\mathsf{B}_X)$ and $\mathrm{ext}_{\mathbb{C}}(\mathsf{B}_X)$, respectively. Clearly, $\mathrm{ext}_{\mathbb{R}}(\mathsf{B}_X) \subseteq \mathrm{ext}_{\mathbb{C}}(\mathsf{B}_X)$.

Theorem 8.2 (Boyd and Ryan [10]) *If X is a complex Banach space and $n \geq 2$, then*

$$\mathrm{ext}_{\mathbb{R}}(\mathsf{B}_{\mathcal{P}_I(^nX)}) \subseteq \{\pm\varphi^n : \varphi \in \mathsf{S}_{X^*}\}.$$

For the following result we need the next definition.

Definition 8.2 Let X be a Banach space, we say that $A \subset X^*$ is X-transitive if for all $\varphi, \psi \in A$, there exists an isometry T of X onto itself such that $\psi \circ T = \varphi$.

Theorem 8.3 (Dineen [20]) *Let X be a complex Banach space. If $\mathrm{ext}_{\mathbb{C}}(X^*)$ is X-transitive and $n \geq 1$, then*

$$\mathrm{ext}_{\mathbb{R}}(\mathsf{B}_{\mathcal{P}_I(^nX)}) = \{\pm\varphi^n : \varphi \in \mathrm{ext}_{\mathbb{C}}(\mathsf{B}_{X^*})\} = \{\pm\varphi^n : \varphi \in \mathrm{ext}_{\mathbb{R}}(\mathsf{B}_{X^*})\}.$$

Theorem 8.4 (Dineen [20]) *If X is a finite dimensional complex Banach space and $n \geq 1$, then*

$$\mathrm{ext}_{\mathbb{R}}(\mathsf{B}_{\mathcal{P}_I(^nX)}) \supseteq \{\pm\varphi^n : \varphi \in \mathrm{ext}_{\mathbb{R}}(\mathsf{B}_{X^*})\}.$$

Theorem 8.5 (Dineen [20]) *If X^* is a strictly convex finite dimensional complex Banach space and $n \geq 1$, then*

$$\mathrm{ext}_{\mathbb{R}}(\mathsf{B}_{\mathcal{P}_I(^nX)}) = \{\pm\varphi^n : \varphi \in \mathsf{B}_{X^*}\}.$$

Finally, we provide some insight on arbitrary infinite dimensional complex Banach spaces. To do so, we begin by providing the definition of weak*-exposed points of complex Banach spaces.

Definition 8.3 Let X be a Banach space. We say that $x \in \mathsf{B}_X$ is an exposed point of B_X if there exists $\varphi \in \mathsf{B}_{X^*}$ such that $\varphi(x) = 1$ and $\varphi(y) < 1$ for every $y \in \mathsf{B}_{X^*}$. We say that $x \in X$ is a weak*-exposed point of the unit ball of X if x is an exposed point of the weak*-closure of B_X in X^{**}. We denote by $\exp_{\omega^*}(\mathsf{B}_X)$ the set of weak*-exposed points of B_X.

Theorem 8.6 (Dineen [20]) *If X is a complex Banach space and $n \geq 1$, then*

$$\operatorname{ext}_{\mathbb{R}}(\mathsf{B}_{\mathcal{P}_I(^n X)}) \supseteq \{\pm\varphi^n : \varphi \in \exp_{\omega^*}(\mathsf{B}_{X^*})\}.$$

The inclusion in Theorem 8.6 can be improved in the sense that the equality is satisfied for specific Banach spaces such as ℓ_1 [20].

Finally, we will study the extreme nuclear polynomials. Let us begin by defining what is known as a nuclear polynomial on a Banach space.

Definition 8.4 Let X be a Banach space. We say that a continuous n-homogeneous polynomial P on X is nuclear if there exists a bounded sequence $(\varphi_i)_{i=1}^{\infty} \subset X^*$ and $(\lambda_i)_{i=1}^{\infty} \in \ell_1$ such that

$$P(x) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x)^n, \tag{8.2}$$

for every $n \in \mathbb{N}$. We denote by $\mathcal{P}_N(^n X)$ the space of nuclear n-homogeneous polynomials on X.

The space $\mathcal{P}_N(^n X)$ is a Banach space when it is endowed with the norm $\|P\|_N$ defined as the infimum of $\sum_{i=1}^{\infty} |\lambda_i| \|\varphi_i\|^n$ taken over all representations of P of the form (8.2). It is not difficult to prove that given a Banach space X we have $\mathcal{P}_N(^n X) \subseteq \mathcal{P}_I(^n X) \subseteq \mathcal{P}(^n X)$ and for every $P \in \mathcal{P}_N(^n X)$ we have $\|P\| \leq \|P\|_I \leq \|P\|_N$. Moreover, if $\varphi \in X^*$, then φ^n is an n-homogeneous nuclear polynomial and $\|\varphi^n\|_N = \|\varphi^n\|_I = \|\varphi^n\|$.

Theorem 8.7 (Boyd and Ryan [10]) *If X is a Banach space, then*

$$\operatorname{ext}_{\mathbb{R}}(\mathsf{B}_{\mathcal{P}_I(^n X)}) \subseteq \operatorname{ext}_{\mathbb{R}}(\mathsf{B}_{\mathcal{P}_N(^n X)}),$$

where $\mathsf{B}_{\mathcal{P}_N(^n X)}$ is the unit ball of $\mathcal{P}_N(^n X)$.

8.2 Orthogonally Additive Polynomials

In this section we will be interested in studying the extreme polynomials of the unit ball of the space of orthogonally additive polynomials on Banach lattices endowed with two different norms. First, we begin by defining orthogonally additive n-homogeneous polynomials on Banach lattices.

Definition 8.5 Let X be a Banach lattice. We say that a continuous n-homogeneous polynomial P on X is orthogonally additive if $P(x + y) = P(x) + P(y)$ whenever

x and y are disjoint. We denote the space of orthogonally additive n-homogeneous polynomials by $\mathcal{P}_{OA}(^nX)$.

One of the main advantages when working with Banach lattices in this scenario is that we only have to study the space of orthogonally additive n-homogeneous polynomials on $C(K)$ for every compact Hausdorff topological space K. Indeed, let X be a Banach lattice. For every positive $a \in X$, we can consider the principal ideal

$$X_a = \{x \in X\colon |x| \leq na \text{ for some } n \in \mathbb{N}\},$$

with lattice structure inherited from X. The space X_a is a Banach lattice endowed with the norm $\|x\|_a = \inf\{C > 0\colon |x| \leq Ca\}$. By the Kakutani representation theorem [36], the Banach lattice X_a is canonically a Banach lattice isometrically isomorphic to $C(K)$ for some compact Hausdorff topological space K, with a being identified with the unit function on K. It is known that the Banach lattice structure of X is uniquely determined by its principal ideals. Hence, the study of the space $\mathcal{P}_{OA}(^nC(K))$ is crucial to understanding the behaviour of $\mathcal{P}_{OA}(^nX)$ for arbitrary Banach lattices X.

We will consider in $\mathcal{P}_{OA}(^nX)$ two natural ways to norm the space. The standard one is the uniform convergence norm on the unit ball of X, i.e.,

$$\|P\|_\infty = \sup\{|P(x)|\colon x \in \mathsf{B}_X\},$$

which makes the normed space $(\mathcal{P}_{OA}(^nX), \|\cdot\|_\infty)$ into a Banach space (in fact, notice that $(\mathcal{P}_{OA}(^nX), \|\cdot\|_\infty)$ is a closed subspace of $(\mathcal{P}(^nX), \|\cdot\|_\infty)$).

Another way to norm $\mathcal{P}_{OA}(^nX)$ is developed below. Since X is a Banach lattice we can define a partial order $\leq$ on $\mathcal{P}(^nX)$: we say that $P \leq Q$ with $P, Q \in \mathcal{P}(^nX)$ if, and only if, $L(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n)$, for every $x_1, \ldots, x_n \in X$ and where L and M are the polars of P and Q, respectively. Hence, we say that an n-homogeneous polynomial is positive if $P \geq 0$ with this partial order $\leq$. Furthermore, we can define the absolute value of an n-homogeneous polynomial.

Definition 8.6 Let $P \in \mathcal{P}(^nX)$. We say that P is regular if P is the difference of two positive n-homogeneous polynomials. We denote by $\mathcal{P}_r(^nX)$ the space of regular n-homogeneous polynomials

All regular n-homogeneous polynomials are those that have absolute value given by the formula

$$|P|(x) = \sup\left\{\sum_{i_1,\ldots,i_n} |L(x^1_{i_1}, \ldots, x^n_{i_n})|\colon x^1, \ldots, x^n \in \Pi(x)\right\},$$

for every $x \geq 0$ and where $\Pi(x)$ denotes all finite sets of positive vectors of X whose sum is x. Notice that the vector space $\mathcal{P}_r(^nX)$ is in fact a Banach lattice when it is endowed with the regular norm

$$\|P\|_r = \||P|\|_\infty.$$

For more information on regular polynomials see [7].

Now, every orthogonally additive n-homogeneous polynomial is regular. This was first proved by M. A. Toumi in [55, theorem 1] but another proof can be found in [11]. Hence, we can consider the space $\mathcal{P}_{OA}(^nX)$ in two scenarios: endowed with the supremum norm or the regular norm. It is important to mention that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_r$ are equivalent in $\mathcal{P}_{OA}(^nX)$ for any Banach lattice X as the following result shows, but the geometric properties of the two Banach spaces are not the same as we will see.

Theorem 8.8 (Boyd et al. [11]) *Let X be a Banach lattice. If $P \in \mathcal{P}_{OA}(^nX)$, then $\|P\|_r = \|P\|_\infty$ if n is odd and $\|P\|_\infty \le \|P\|_r \le 2\|P\|_\infty$ if n is even. Moreover, the inequalities are sharp.*

See [11] for more information on the supremum and regular norm defined on $\mathcal{P}_{OA}(^nX)$.

We now proceed to state the main results of this section which are an extension of [12].

Theorem 8.9 (Boyd et al. [11]) *Let K be a compact Hausdorff topological space. A polynomial $P \in \mathcal{P}_{OA}(^nC(K))$ is an extreme polynomial of the unit ball of the space $(\mathcal{P}_{OA}(^nC(K)), \|\cdot\|_r)$ if, and only if, $P(x) = \pm\delta_t^n(x)$, where $t \in K$ and $\delta_t^n(x) = x(t)^n$.*

Theorem 8.10 (Boyd et al. [11]) *Let K be a compact Hausdorff topological space.*

(i) If n is odd, then a polynomial $P \in \mathcal{P}_{OA}(^nC(K))$ is an extreme polynomial of the unit ball of the space $(\mathcal{P}_{OA}(^nC(K)), \|\cdot\|_\infty)$ if, and only if, $P(x) = \pm\delta_t^n(x)$, where $t \in K$ and $\delta_t^n(x) = x(t)^n$.

(ii) If n is even, then a polynomial $P \in \mathcal{P}_{OA}(^nC(K))$ is an extreme polynomial of the unit ball of the space $(\mathcal{P}_{OA}(^nC(K)), \|\cdot\|_\infty)$ if, and only if, P is one of the following polynomials:

(a) $P(x) = \pm\delta_t^n(x)$, where $t \in K$ and $\delta_t^n(x) = x(t)^n$;

(b) $P(x) = (\delta_s^n - \delta_t^n)(x)$, where $s, t \in K$ and $(\delta_s^n - \delta_t^n)(x) = x(s)^n - x(t)^n$.

Chapter 9
Applications

Abstract As we know, one of the main goals of this book has been to find a parametrization of the unit sphere of spaces of polynomials endowed with different norms whose unit balls can be described in $\mathbb{R}^3$, but mainly we have tried to obtain the extreme polynomials of the unit balls. We have also studied some of the extreme polynomials in arbitrary dimensions and we have even described some of the extreme polynomials of arbitrary degree. The reason behind this is that a full description of the extreme polynomials of the unit ball has, as a matter of fact, can be applied to obtain many sharp polynomial inequalities (as we will see in this final chapter).

If the extreme polynomials of the unit ball are known, then we can simplify the problems that involve finding sharp inequalities between norms that depend on polynomials by using a simple consequence of the Krein-Milman Theorem.

Theorem 9.1 (Krein-Milman Theorem [41]) *Let X be a normed space. If C is a compact convex subset of X, then C coincides with the closed convex hull of its extreme points.*

Corollary 9.1 *If C is a convex body in a normed space X and $f : C \to \mathbb{R}$ is a convex function that attains its maximum, then there exists an extreme point $p \in C$ such that $f(p) = \max\{f(x) : x \in C\}$.*

The main idea to apply Corollary 9.1 is the following: Let $\mathbf{B}$ be a convex body in a normed space of polynomials and f be a convex function defined on $\mathbf{B}$ which attains its maximum and takes real values, then f attains its maximum at an extreme point of $\mathbf{B}$ by Corollary 9.1. Furthermore, if we have a full description of the extreme points of $\mathbf{B}$, then we can find the maximum of f by evaluating f in the extreme points of $\mathbf{B}$ (this is the *Krein-Milman Approach*). This can be used in the case of norms of polynomials since it is known that the norm function is convex.

The rest of this chapter involves finding well known sharp inequalities for norms of polynomials that have appeared in this survey.

Let $(X, \|\cdot\|)$ be a normed space and consider the normed space $\mathcal{P}(^nX)$ (see the beginning of Sect. 5.5). Now, let us also consider the space of continuous symmetric

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1_9

n-linear forms of X denoted by $\mathcal{L}_s({}^n X)$ and endowed with the following norm:

$$\|L\| = \sup\{|L(x_1, \ldots, x_n)| \colon \|x_i\| \leq 1, \text{for every } i \in \{1, \ldots, n\}\},$$

for every $L \in \mathcal{L}_s({}^n X)$. By the beginning of Sect. 5.5, for every $P \in \mathcal{P}({}^n X)$, there exists a unique $L \in \mathcal{L}_s({}^n X)$ such that $P(x) = L(x, \ldots, x)$, for every $x \in X$, the polar of P.

9.1 Bernstein-Markov Type Inequalities

Bernstein type inequalities for polynomials are inequalities of the following form: if $P \in \mathcal{P}({}^n X)$, there exists a function $\Psi(\mathbf{x})$ defined over $\mathbf{C}$ such that

$$\|D^k P(\mathbf{x})\| \leq \Psi(\mathbf{x})\|P\|,$$

where $D^k P$ denotes the k-th derivative of P (the optimal function $\Psi(\mathbf{x})$ is known as *the Bernstein function*). On the other hand, Markov type inequalities are of the same fashion as Bernstein type inequalities but we are also taking the supremum of $\|D^k P(\mathbf{x})\|$ over all $\mathbf{x} \in \mathbf{C}$ (the optimal constant in Markov type inequalities is known as *the Markov constant*). The results of this section focus on finding the Bernstein function and the Markov constant that are known for the spaces that have been presented in this survey.

Theorem 9.2 (Araújo et al. [4]) *Take $\mathcal{P}_3(\mathbb{R})$ (see Sect. 2.1). The Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{\mathbb{R}}$$

is given by

$$\begin{cases} 3(1 - 4x^2) & \textit{if } 0 \leq |x| \leq \frac{\sqrt{7}-2}{6}, \\ \frac{7\sqrt{7}+10}{9(|x|+1)} & \textit{if } \frac{\sqrt{7}-2}{6} \leq |x| \leq \frac{2\sqrt{7}-1}{9}, \\ \frac{-16x^3}{(1-9x^2)(1-x^2)} & \textit{if } \frac{2\sqrt{7}-1}{9} \leq |x| \leq \frac{1+2\sqrt{7}}{9}, \\ \frac{7\sqrt{7}-10}{9(1-|x|)} & \textit{if } \frac{1+2\sqrt{7}}{9} \leq |x| \leq \frac{\sqrt{7}+2}{6}, \\ 3(4x^2 - 1) & \textit{if } |x| \geq \frac{\sqrt{7}+2}{6}. \end{cases}$$

The Bernstein function for the inequality

$$|P''(x)| \leq \Psi(x)\|P\|_{\mathbb{R}}$$

is given by

$$\begin{cases} \frac{4}{1-9x^2} & \textit{if } 0 \leq |x| \leq \frac{1}{9}, \\ \frac{32}{9(|x|-1)^2} & \textit{if } \frac{1}{9} \leq |x| \leq \frac{1}{3}, \\ 24|x| & \textit{if } |x| \geq \frac{1}{3}. \end{cases}$$

Theorem 9.3 (Muñoz et al. [47]) *Let* $\varphi\colon [-1, 1] \to [0, +\infty)$ *be defined by* $\varphi(x) = \sqrt{1-x^2}$*. On the space* $\mathcal{P}_3^{\varphi}(\mathbb{R})$ *(see Sect. 2.1.1), the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{\mathbb{R}}$$

is given by

$$\begin{cases} 2|1-3x^2| & \textit{if } |x| \in \left[0, \frac{\sqrt{4-\sqrt{7}}}{3}\right] \cup \left[\frac{\sqrt{4+\sqrt{7}}}{3}, 1\right], \\ \frac{4x^2}{\sqrt{-9x^4+10x^2-1}} & \textit{if } |x| \in \left[\frac{\sqrt{4-\sqrt{7}}}{3}, \frac{\sqrt{4+\sqrt{7}}}{3}\right]. \end{cases}$$

Theorem 9.4 (Muñoz et al. [48]) *Let* $m, n \in \mathbb{N}$ *be odd and such that* $m > n$*. On the space* $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ *(see Sect. 3.1), the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} \frac{mn}{n+m\lambda_0} \cdot x^{n-1} \cdot |x^{m-n} + \lambda_0| & \textit{if } |x| \in [0, 1] \setminus I_{m,n}, \\ n\left(\frac{n}{m}\right)^{\frac{n}{m-n}} \cdot \frac{1}{|x|} & \textit{if } |x| \in I_{m,n}, \end{cases}$$

where λ_0 *comes from Theorem 3.1 and*

$$I_{m,n} = \left[\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right].$$

The Markov constant is given by

$$\frac{mn(1+\lambda_0)}{n+m\lambda_0}$$

and equality is attained for the polynomials

$$P(x) = \pm\frac{1}{n+m\lambda_0}(nx^m + \lambda_0 mx^n).$$

In order to prove Theorem 9.4, we will prove first the following technical lemmas.

Lemma 9.1 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be odd and such that $m > n$ and let λ_0 be the number from Theorem 3.1. We have*

$$|\lambda_0|\frac{n}{m} < |\lambda_0|\frac{1-|\lambda_0|^{\frac{n}{m-n}}}{1-|\lambda_0|^{\frac{m}{m-n}}} < \frac{n}{m}.$$

Proof Recall from Lemma 3.1 that $|\lambda_0| < \frac{n}{m} < 1$ and consider the inequality

$$\frac{n}{m} < \frac{1-x^n}{1-x^m}. \tag{9.1}$$

We will show when (9.1) holds. If $0 < x < 1$, then inequality (9.1) is equivalent to $m - n > mx^n - nx^m$. Now, since the function $x \mapsto mx^n - nx^m$ is strictly increasing on $(0, 1)$, the curves $y = mx^n - nx^m$ and $y = m - n$ intersect in, at most, one point which is $x = 1$. Hence, it is easy to check that the inequality $m - n > mx^n - nx^m$ is satisfied on $(0, 1)$, which implies that $m - n > mx^n - nx^m$ holds when $x \in \left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$ and we have proven the first inequality of the lemma. The second inequality follows after doing some simple calculations and using the fact that λ_0 satisfies $n + m\lambda_0 = (m-n)|\lambda_0|^{\frac{m}{m-n}}$. $\square$

Lemma 9.2 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be odd and such that $m > n$ and let λ_0 be the number from Theorem 3.1. If we define the functions*

$$f(x) = \frac{mn}{m-n}x^{n-1}|x^{m-m} - 1|,$$

$$g(x) = \frac{mn}{n+m\lambda_0}x^{n-1}|x^{m-m} + \lambda_0|,$$

then $g(x) \geq f(x)$ provided x satisfies

$$0 \leq |x| \leq \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \quad or \quad \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq 1.$$

Proof By symmetry, assume that $x > 0$. After some calculations, it is easy to check that the functions f and g intersect at the points $x_1 = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$ and $x_2 = \left(|\lambda_0|\frac{1-|\lambda_0|^{\frac{n}{m-n}}}{1-|\lambda_0|^{\frac{m}{m-n}}}\right)^{\frac{1}{m-n}}$. By Lemma 9.1, the points x_1 and x_2 are not in the intervals $\left(0, \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right)$ or $\left(\left(\frac{n}{m}\right)^{\frac{1}{m-n}}, 1\right)$. Hence, either $f \geq g$ or $f \leq g$ in each one of the previous intervals. Now, notice that $f(1) < g(1)$ and $f\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right) < g\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right)$. Indeed, the former is trivial and the latter is true because of the following reasoning. Notice that the inequality $f\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right) < g\left(\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}\right)$ is equivalent to $\left|\frac{\lambda_0 n}{m} + 1\right| < \frac{1}{|\lambda_0|^{\frac{m}{m-n}}}\left|\frac{\lambda_0 n}{m} - \lambda_0\right|$. Moreover, it is also equivalent to $|\lambda_0|^{\frac{m}{m-n}} < |\lambda_0|$ which is satisfied since $-1 < -\frac{n}{m} < \lambda_0 < 0$ (see Lemma 3.1) and the proof is complete. □

Lemma 9.3 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be odd and such that $m > n$ and let λ_0 be the number from Theorem 3.1. If we define the functions*

$$f(x) = \frac{mn}{m-n}x^{n-1}|x^{m-m} - 1|,$$
$$g(x) = \frac{mn}{n+m\lambda_0}x^{n-1}|x^{m-m} + \lambda_0|,$$
$$h(x) = n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|},$$

then $h(x) \geq \max\{f(x), g(x)\}$ provided x satisfies

$$\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}.$$

Proof Assume that $\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$ holds, then it is enough to show that $h(x) \geq f(x)$ and $h(x) \geq g(x)$.

Firstly, notice that the function $x^n - x^m$ is strictly increasing on the interval $\left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$ since the derivative is positive. Hence, the maximum of $x \mapsto x^n - x^m$ on $\left(0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right)$ is attained at $x = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$ with value $\frac{(m-n)n^{\frac{n}{m-n}}}{m^{\frac{m}{m-n}}}$. Thus, $x^n - x^m \leq \frac{(m-n)n^{\frac{n}{m-n}}}{m^{\frac{m}{m-n}}}$ for $\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$, which implies after rearranging the inequality that $f(x) \leq h(x)$.

Secondly, notice that the inequality $\frac{mn}{n+m\lambda_0}x^{n-1}|x^{m-m} + \lambda_0| \leq n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|}$ is equivalent to $\frac{m}{n+m\lambda_0}|x^m + \lambda_0 x^n| \leq \left(\frac{n}{m}\right)^{\frac{n}{m-n}}$. Since the derivative of $x^m + \lambda_0 x^n$ is only 0 when $x = 0$ or $x = \pm\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}$, we have that $x^n + \lambda_0 x^n$ is monotone on the interval $\left[\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}}, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right]$. Hence, it is enough to evaluate $x^n + \lambda_0 x^n$ at the endpoints of the interval and after some simple evaluations notice that the proof is complete. □

Proof (of Theorem 9.4) Notice that the Bernstein function on the space $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ is given by

$$\mathcal{B}_{m,n,\infty}(x) = \sup\{|P'(x)|\colon P \text{ belongs to the unit sphere of } \mathcal{P}_{m,n,\infty}(\mathbb{R})\}.$$

However it is enough to find the above supremum over the set of extreme points of the unit ball by Corollary 9.1.

We know from Theorem 3.3 that the set of extreme points of $\mathsf{B}_{m,n,\infty}$ is

$$\left\{\pm\left(t, -\frac{m}{(m-n)^{\frac{m-n}{m}} n^{\frac{n}{m}}} \cdot t^{\frac{n}{m}}, 0\right) : \frac{n}{m-n} \leq t \leq \frac{n}{n+m\lambda_0}\right\} \bigcup \{\pm(0,0,1)\}.$$

Observe that the extreme polynomials $P(x) = \pm 1$ are irrelevant to find the Bernstein function. Hence we focus our attention on the extreme polynomials

$$P_t(x) = \pm\left(tx^m - \frac{m}{(m-n)^{\frac{m-n}{n}} n^{\frac{n}{m}}} t^{\frac{n}{m}} x^n\right),$$

where $t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]$. Thus,

$$\begin{aligned}
\mathcal{B}_{m,n,\infty}(x) &= \sup\left\{|P_t'(x)|\colon t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\} \\
&= \sup\left\{\left|mtx^{m-1} - \frac{mnt^{\frac{n}{m}}}{(m-n)^{\frac{m-n}{n}} n^{\frac{n}{m}}} x^{n-1}\right| : t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\} \\
&= \sup\left\{\left|mx^{n-1}\left[tx^{m-n} - \left(\frac{n}{m-n}\right)^{\frac{m-n}{m}} t^{\frac{n}{m}}\right]\right| : \right. \\
&\qquad \left. t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\}.
\end{aligned}$$

Let us define $R(t) = mx^{n-1}\left[tx^{m-n} - \left(\frac{n}{m-n}\right)^{\frac{m-n}{m}} t^{\frac{n}{m}}\right]$. Notice that the above supremum is attained at either $t = \frac{n}{m-n}$, or $t = \frac{n}{n+m\lambda_0}$, or at a critical point of $R(t)$ inside the open interval $\left(\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right)$. It is easy to show that there exists only one critical point of $R(t)$ which is $t_0 = \frac{n}{m-n}\left(\frac{n}{m}\right)^{\frac{m}{m-n}} \frac{1}{|x|^m}$ and

$$R(t_0) = n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|}.$$

Now, notice that the series of inequalities $\frac{n}{m-n} \leq t_0 \leq \frac{n}{n+m\lambda_0}$ is equivalent to

$$\left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}.$$

Hence, after some easy calculations, we have

$$\mathcal{B}_{m,n,\infty}(x) = \sup\left\{|R(t)|: t \in \left[\frac{n}{m-n}, \frac{n}{n+m\lambda_0}\right]\right\}$$
$$= \begin{cases} \max\left\{\left|R\left(\frac{n}{m-n}\right)\right|, \left|R\left(\frac{n}{n+m\lambda_0}\right)\right|, n\left(\frac{n}{m}\right)^{\frac{n}{m-n}}\frac{1}{|x|}\right\} \text{ if } \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq \left(\frac{n}{m}\right)^{\frac{1}{m-n}}, \\ \max\left\{\left|R\left(\frac{n}{m-n}\right)\right|, \left|R\left(\frac{n}{n+m\lambda_0}\right)\right|\right\} \text{ if } |x| \leq \left(\frac{|\lambda_0|n}{m}\right)^{\frac{1}{m-n}} \text{ or } \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \leq |x| \leq 1, \end{cases}$$

where, after evaluating the function R in the above points, we have

$$R\left(\frac{n}{m-n}\right) = \frac{mnx^{n-1}}{m-n}|x^{m-n} - 1|$$

and

$$R\left(\frac{n}{n+m\lambda_0}\right) = \frac{mnx^{n-1}}{n+m\lambda_0}|x^{m-n} + \lambda_0|.$$

By applying Lemmas 9.2 and 9.3 the result follows. □

Theorem 9.5 (Muñoz et al. [48]) *Let $m, n \in \mathbb{N}$ be such that $m > n$, m is odd and n is even. On the space $\mathcal{P}_{m,n,\infty}(\mathbb{R})$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} 2n|x|^{n-1} & \text{if } |x| \in \left[0, \left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right], \\ mx^{m-1} + n|x|^{n-1} & \text{if } |x| \in \left[\left(\frac{n}{m}\right)^{\frac{1}{m-n}}, 1\right]. \end{cases}$$

The Markov constant is given by

$$m + n$$

and equality is attained for the polynomials

$$P(x) = \pm(x^m \pm x^n - 1).$$

Theorem 9.6 (Muñoz et al. [48]) *Let $n \in \mathbb{N}$ be odd. On the space $\mathcal{P}_{2n,n,\infty}(\mathbb{R})$, the Bernstein function for the inequality*

$$|P'(x)| \le \Psi(x)\|P\|_{2n,n,\infty}$$

is given by

$$\begin{cases} \frac{n|x|^{n-1}}{1-x^n} & \text{if } |x| \in \left[0, \frac{1}{\sqrt[n]{2}}\right], \\ 4n|x|^{2n-1} & \text{if } |x| \in \left[\frac{1}{\sqrt[n]{2}}, 1\right]. \end{cases}$$

The Markov constant is given by $4n$ and equality is attained for the polynomials

$$P(x) = \pm(2x^{2n} - 1).$$

Theorem 9.7 (Muñoz et al. [48]) *Let $n \in \mathbb{N}$ be even. On the space $\mathcal{P}_{2n,n,\infty}(\mathbb{R})$, the Bernstein function for the inequality*

$$|P'(x)| \le \Psi(x)\|P\|_{2n,n,\infty}$$

is given by

$$\begin{cases} 8n(-2|x|^{2n-1} + |x|^{n-1}) & \text{if } |x| \in \left[0, \left(\frac{1}{4}\right)^{\frac{1}{n}}\right], \\ \frac{n}{|x|} & \text{if } |x| \in \left[\left(\frac{1}{4}\right)^{\frac{1}{n}}, \left(\frac{1}{2}\right)^{\frac{1}{n}}\right], \\ \frac{n|x|^{n-1}}{1-x^n} & \text{if } |x| \in \left[\left(\frac{1}{2}\right)^{\frac{1}{n}}, \left(\frac{3}{4}\right)^{\frac{1}{n}}\right], \\ 8n(2|x|^{2n-1} - |x|^{n-1}) & \text{if } |x| \in \left[\left(\frac{3}{4}\right)^{\frac{1}{n}}, 1\right]. \end{cases}$$

The Markov constant is given by $8n$ and equality is attained for the polynomials

$$P(x) = \pm(8x^{2n} - 8x^n + 1).$$

Theorem 9.8 (Muñoz et al. [47]) *Let $m, n \in \mathbb{N}$ be such that m is odd, n is even and $m > n$. On the normed subspace of $\mathcal{P}_{m,n,\infty}(\mathbb{R})$ given by trinomials that are bounded by the linear mapping $\varphi(x) = |x|$ over the interval $[-1, 1]$, the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} (m+1)|x|^m - (n+1)x^n + 1 & \textit{if } |x| \leq t_1, \\ 2(n+1)x^n - 1 & \textit{if } t_1 \leq |x| \leq \sqrt[m-n]{\frac{n+1}{m+1}}, \\ (m+1)|x|^m + (n+1)x^n - 1 & \textit{if } \sqrt[m-n]{\frac{n+1}{m+1}} \leq |x| \leq 1, \end{cases}$$

where $t_1 \in \mathbb{R}$ *is the unique solution of*

$$(m+1)x^m - 3(n+1)x^n + 2 = 0$$

on the interval $\left(\frac{1}{\sqrt[n]{2(n+1)}}, \frac{1}{\sqrt[n]{n+1}}\right)$. *The Markov constant is given by* $m+n+1$ *and equality is attained for the polynomials*

$$P(x) = \pm[x^m \pm (x^n - 1)].$$

Theorem 9.9 (Muñoz et al. [47]) *On the normed subspace of* $\mathcal{P}_{2,1,\infty}(\mathbb{R})$ *given by trinomials that are bounded by the linear mapping* $\varphi(x) = |x|$ *over the interval* $[-1, 1]$, *the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,\infty}$$

is given by

$$\begin{cases} \left|\frac{3x^2-1}{2}\right| + 2|x| & \textit{if } |x| \in \left[\frac{\sqrt{13}-2}{9}, \frac{\sqrt{13}+2}{9}\right], \\ |6x^2 - 1| & \textit{if } |x| \in \left[0, \frac{\sqrt{13}-2}{9}\right] \cup \left[\frac{\sqrt{13}+2}{9}, 1\right]. \end{cases}$$

Theorem 9.10 (Muñoz et al. [49]) *Let* $m, n \in \mathbb{N}$ *be with different parity and such that* $m > n$. *On the space* $\mathcal{P}_{m,n,2}(\mathbb{R})$, *the Bernstein function for the inequality*

$$|P'(x)| \leq \Psi(x)\|P\|_{m,n,2}$$

is given by

$$\begin{cases} \sqrt{\frac{n^2(2n+1)x^{2(n-1)} + (m+1)^2(2m+1)x^{2(m-1)}}{2}} & \textit{if } m \textit{ is even and } n \textit{ is odd}, \\ \sqrt{\frac{m^2(2m+1)x^{2(m-1)} + (n+1)^2(2n+1)x^{2(n-1)}}{2}} & \textit{if } m \textit{ is odd and } n \textit{ is even}. \end{cases}$$

The Markov constant is given by

$$\begin{cases} (m+1)\sqrt{\frac{2m+1}{2m-1}} & \textit{if } m \textit{ is even and } n \textit{ is odd,} \\ m\sqrt{\frac{2m+1}{2m-1}} & \textit{if } m \textit{ is odd, } n \textit{ is even and } m > n+1, \\ m\sqrt{\frac{2m-1}{2m-3}} & \textit{if } m \textit{ is odd and } n = m-1. \end{cases}$$

Remark 9.1 On Theorem 9.10, notice that if we consider $n = 1$, then we have Bernstein's function and Markov's constant for the space $\mathcal{P}_2(\mathbb{R})$ (see Sect. 2.1) which are given, respectively, by

$$\begin{cases} \frac{1}{1-|x|} & \text{if } 0 \leq |x| \leq \frac{1}{2}, \\ 4|x| & \text{if } |x| \geq \frac{1}{2}, \end{cases}$$

and

$$4,$$

with equality attained for the polynomials

$$P(x) = \pm(1 - 2x^2).$$

Theorem 9.11 (Muñoz et al. [46]) *Take $\mathcal{P}(^2\Delta)$ (see Sect. 4.1). The Markov constant for the inequality*

$$\|DP(x, y)\|_{\ell_2} \leq \Psi(x, y)\|P\|_\Delta$$

is given by

$$2\sqrt{10}$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 - 6xy + y^2).$$

The Bernstein function for the inequality

$$\|DP(x, y)\|_\Delta \leq \Psi(x, y)\|P\|_\Delta$$

is given by

$$\begin{cases} |2x-6y| & \textit{if } x=0 \textit{ or } x\neq 0 \textit{ and } \left(\frac{y}{x}\leq -1 \textit{ or } \frac{y}{x}\geq 2\right), \\ \left|2x+2y+\frac{y^2}{x}\right| & \textit{if } x\neq 0 \textit{ and } \frac{y}{x}\in[1,2], \\ \left|2x+2y+\frac{x^2}{y}\right| & \textit{if } y\neq 0 \textit{ and } \frac{x}{y}\in[1,2], \\ |6x-2y| & \textit{if } y=0 \textit{ or } y\neq 0 \textit{ and } \left(\frac{x}{y}\leq -1 \textit{ or } \frac{x}{y}\geq 2\right). \end{cases}$$

The Markov constant is given by 6 *and equality is attained for the polynomials*

$$P(x,y)=\pm(x^2-6xy+y^2).$$

Theorem 9.12 (Gámez et al. [23]) *Take $\mathcal{P}(^2\square)$ (see Sect. 4.2). The Bernstein function for the inequality*

$$\|DP(x,y)\|_{\ell_2}\leq \mathcal{M}(x,y)\|P\|_{\square}$$

is given by

$$\begin{cases} \sqrt{\frac{24y^4+12x^2y^2+x^4+x(8y^2+x^2)^{\frac{3}{2}}}{8y^2}} & \textit{if } 0<\alpha_0 x\leq y\leq x, \\ \sqrt{\frac{24x^4+12x^2y^2+y^4+y(8x^2+y^2)^{\frac{3}{2}}}{8x^2}} & \textit{if } 0<x\leq y\leq \frac{x}{\alpha_0}, \\ \sqrt{13x^2-24xy+13y^2} & \textit{otherwise}, \end{cases}$$

where α_0 is the unique root of the equation

$$80\alpha^4-192\alpha^3+92\alpha^2-1=(8\alpha^2+1)^{\frac{3}{2}}$$

in the interval $\left[\frac{3-\sqrt{5}}{2},\frac{12-3\sqrt{3}}{13}\right]$. The Markov constant is given by

$$\sqrt{13}$$

and equality is attained for the polynomials

$$P(x,y)=\pm(x^2-3xy+y^2).$$

The Bernstein function for the inequality

$$\|DP(x,y)\|_{\square}\leq \Psi(x,y)\|P\|_{\square}$$

is given by

$$\begin{cases} 3x-2y & \text{if } 0 \le y \le (\sqrt{2}-1)x, \\ \frac{5}{2}x - y + \frac{y^2}{2x} & \text{if } x \ne 0 \text{ and } (\sqrt{2}-1)x \le y \le \frac{1}{2}x, \\ 2x + \frac{y^2}{2x} & \text{if } x \ne 0 \text{ and } \frac{1}{2}x \le y \le x, \\ 2y + \frac{x^2}{2y} & \text{if } y \ne 0 \text{ and } x \le y \le 2x, \\ \frac{5}{2}y - x + \frac{x^2}{2y} & \text{if } y \ne 0 \text{ and } 2x \le y \le (\sqrt{2}+1)x, \\ 3y-2x & \text{if } (\sqrt{2}+1)x \le y \le 1. \end{cases}$$

The Markov constant is given by 3 *and equality is attained for the polynomials*

$$P(x,y) = \pm(x^2 - 3xy + y^2).$$

Theorem 9.13 (Araújo et al. [2]) *Take $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ (see Sect. 4.3). The Bernstein function for the inequality*

$$\|DP(x,y)\|_{\ell_2} \le \Psi(x,y)\|P\|_{D(\frac{\pi}{4})}$$

is given by

$$\begin{cases} 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right] & \text{if } (a), \\ \frac{x^2}{y^2} + 4\left(x^2+y^2\right) & \text{if } (b), \\ \frac{\left(3x^2-2xy+3y^2\right)^2}{2(x-y)^2} & \text{if } (c), \end{cases}$$

where

(a) $0 \le y \le \frac{\sqrt{2}-1}{2}x$ *or* $\left(4\sqrt{2}-5\right)x \le y \le x$*,*

(b) $\frac{\sqrt{2}-1}{2}x \le y \le \left(\sqrt{2}-1\right)x$*,*

(c) $\left(\sqrt{2}-1\right)x \le y \le \left(4\sqrt{2}-5\right)x$*.*

The Markov constant is

$$4\left(13+8\sqrt{2}\right)$$

and equality is attained for the polynomials

$$P(x,y) = \pm(x^2 + (5+4\sqrt{2})y^2 - 4(1+\sqrt{2})xy).$$

The Bernstein function for the inequality

$$\|DP(x,y)\|_{D(\frac{\pi}{4})} \le \Psi(x,y)\|P\|_{D(\frac{\pi}{4})}$$

is given by

$$\begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \le y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \le y < \left(\sqrt{2}-1\right)x, \\ 2\left(x+\frac{y^2}{x-y}\right) & \text{if } \left(\sqrt{2}-1\right)x \le y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \left(2-\sqrt{2}\right)x \le y \le x. \end{cases}$$

The Markov constant is given by

$$4+\sqrt{2}$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + (5+4\sqrt{2})y^2 - 4(1+\sqrt{2})xy).$$

Theorem 9.14 (Jiménez et al. [34]) *Take $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$. The Bernstein function for the inequality*

$$\|DP(x, y)\|_{\ell_2} \le \Phi(x, y)\|P\|_{D(\frac{\pi}{2})}$$

is given by

$$\begin{cases} \sqrt{16(x-y)^2+4\left(x^2+y^2\right)} & \text{if } 0 \le y \le \frac{x}{2}, \\ \sqrt{\frac{x^4}{y^2}+4\left(x^2+y^2\right)} & \text{if } 0 < \frac{x}{2} < y \le x, \\ \sqrt{\frac{y^4}{x^2}+4\left(x^2+y^2\right)} & \text{if } 0 < x < y \le 2x, \\ \sqrt{16(y-x)^2+4\left(x^2+y^2\right)} & \text{if } 2x < y \le 1. \end{cases}$$

The Markov constant is given by $2\sqrt{5}$ and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + y^2 - 4xy).$$

The Bernstein function for the inequality

$$\|DP(x, y)\|_{D(\frac{\pi}{2})} \le \Psi(x, y)\|P\|_{D(\frac{\pi}{2})}$$

is given by

$$\begin{cases} 2(2x-y) & \text{if } 0 \le y < \frac{x}{2}, \\ 2\left(y+\frac{x^2}{2y}\right) & \text{if } \frac{x}{2} \le y < x, \\ 2\left(x+\frac{y^2}{2x}\right) & \text{if } x \le y < 2x, \\ 2(2y-x) & \text{if } y \ge 2x. \end{cases}$$

The Markov constant is given by 4 *and equality is attained for the polynomials*

$$P(x, y) = \pm(x^2 + y^2 - 4xy).$$

Theorem 9.15 (Jiménez et al. [34]) *On* $\mathcal{P}({}^2\ell_p^2)$ *for* $p \in \{1, 2, \infty\}$ *(see Sects. 4.3, 5.1, and 5.2), the Markov constant in the inequality*

$$\|DP(x, y)\|_{\ell_p^2} \leq \Psi(x, y)\|P\|_{\ell_p^2}$$

is

(i) 4 *if* $p = 1$,
(ii) 2 *if* $p = 2$,
(iii) $2\sqrt{2}$ *if* $p = \infty$.

9.2 Polarization Constants

It is easy to see just by the definition of the norms defined on $\mathcal{P}({}^nX)$ and $\mathcal{L}_s({}^nX)$ that: for every $P \in \mathcal{P}({}^nX)$,

$$\|P\| \leq \|L\|,$$

where L is the polar of P. But furthermore, the converse is also true, i.e., there exists $C \geq 1$ such that $\|L\| \leq C\|P\|$. In particular, we have the following result that can be applied for any normed space X.

Theorem 9.16 (Martin [42]) *Let* X *be a normed space. If* $P \in \mathcal{P}({}^nX)$*, then*

$$\|P\| \leq \|L\| \leq \frac{n^n}{n!}\|P\|,$$

where L *is the polar of* P.

Notice that throughout this survey we have considered the norm over the space of n-homogeneous polynomials to be, not only defined over the unit ball of a certain normed space, but also over a convex body of a normed space. To be more precise, let X be a normed space and take $\mathbf{C}$ a convex body in X. We define the following norm over the space of continuous n-homogeneous polynomials of X: for every continuous n-homogeneous polynomial P,

$$\|P\|_{\mathbf{C}} = \sup\{|P(x)| : x \in \mathbf{C}\};$$

and we also define the following norm over the space of symmetric n-linear forms of X: for every symmetric n-linear form L,

$$\|L\|_{\mathbf{C}} = \sup\{|L(x_1, \ldots, x_n)| \colon x_i \in \mathbf{C}, \text{ for every } i \in \{1, \ldots, n\}\}.$$

Notice that the condition "every continuous n-homogeneous polynomial P has a unique continuous symmetric n-linear form L (the polar of P) such that $P(x) = L(x, \ldots, x)$" is purely algebraic. Therefore, it does not depend on the topology that we consider over the space of n-homogenous polynomials or over the space of symmetric n-linear forms.

It is easy to see by the definition of the above norms that $\|P\|_{\mathbf{C}} \leq \|L\|_{\mathbf{C}}$. However, the reverse inequality as in Martin's Theorem is not true as it can be seen later on. Furthermore, there is not yet an analogous version of Martin's Theorem when the norm is defined over an arbitrary convex body. Thus it is still an open problem to find a result similar to the one of Martin's Theorem when we consider the norm defined over other convex bodies apart from the unit ball of X.

We are able to define now what is known as the n-polarization constant of a space of continuous n-homogeneous polynomials on a convex body. Let X be a normed space and $\mathbf{C} \subset X$ a convex body. Let $\mathcal{P}(^n\mathbf{C})$ be the space of n-homogeneous polynomials on X bounded on $\mathbf{C}$ endowed with the norm defined by

$$\|P\|_{\mathbf{C}} = \sup\{|P(x)| : x \in \mathbf{C}\}.$$

Similarly, if L is the polar of $P \in \mathcal{P}(^n\mathbf{C})$ we define

$$\|L\|_{\mathbf{C}} = \sup\{|L(x_1, \ldots, x_n)| : x_1, \ldots, x_n \in \mathbf{C}\}.$$

We define the n-polarization constant $\mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\mathbf{C}))$ of $\mathcal{P}(^n\mathbf{C})$ as the following value:

$$\inf\left\{K \colon \|L\|_{\mathbf{C}} \leq K\|P\|_{\mathbf{C}}, \text{ where } P \in \mathcal{P}(^n\mathbf{C}) \text{ and } L \text{ is the polar of } P\right\}.$$

Furthermore, assume that there exists $P \in \mathcal{P}(^n\mathbf{C})$ such that

$$\|L\|_{\mathbf{C}} = \mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\mathbf{C}))\|P\|_{\mathbf{C}},$$

where L is the polar of P, then we say that P is an extremal polynomial for $\mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\mathbf{C}))$.

The following results show the known exact values of the polarization constants of the spaces of homogeneous polynomials that have been dealt with in this survey (most of them use the Krein-Milman approach, specially those whose norm involve convex bodies different from the unit ball).

Theorem 9.17 (Muñoz et al. [46]) *If Δ is the simplex defined in Sect. 4.1, then $c_{pol}(\mathcal{P}(^2\Delta)) = 3$. Furthermore, $P(x, y) = \pm(x^2 + y^2 - 6xy)$ are extremal polynomials for $c_{pol}(\mathcal{P}(^2\Delta))$.*

Proof The result follows from the Markov constant in Theorem 9.11 for the inequality $\|DP(x, y)\|_\Delta \leq \Psi(x, y)\|P\|_\Delta$ since

$$DP(x,y)(u,v) = 2L((x,y),(u,v))$$

for all $(x,y),(u,v) \in \mathbb{R}^2$ and where L is the polar of P. □

Theorem 9.18 (Gámez et al. [23]) *If* $\square$ *is the unit square defined in Sect. 4.2, then* $c_{pol}(\mathcal{P}(^2\square)) = \frac{3}{2}$. *Furthermore,* $P(x,y) = \pm(x^2 + y^2 - 3xy)$ *are extremal polynomials for* $c_{pol}(\mathcal{P}(^2\square))$.

Theorem 9.19 (Araújo et al. [2]) *If* $D\left(\frac{\pi}{4}\right)$ *is the circular sector defined in Sect. 4.3, then* $c_{pol}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)\right) = 2 + \frac{\sqrt{2}}{2}$. *Furthermore,* $P(x,y) = \pm(x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy)$ *are extremal polynomials for* $c_{pol}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)\right)$.

Theorem 9.20 (Jiménez et al. [34]) *If* $D\left(\frac{\pi}{2}\right)$ *is the circular sector defined in Sect. 4.3, then* $c_{pol}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)\right) = 2$. *Furthermore,* $P(x,y) = \pm(x^2 + y^2 - 4xy)$ *are extremal polynomials for* $c_{pol}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)\right)$.

Theorem 9.21 (Sarantopoulos [53]) *Let* $1 \leq p \leq \infty$. *We have* $c_{pol}\left(\mathcal{P}\left(^2\ell_p^2\right)\right) = 2^{\frac{|p-2|}{2}}$ *(see Sect. 5). Furthermore,* $P(x,y) = \pm(x^2 - y^2)$ *are extremal polynomials for* $c_{pol}\left(\mathcal{P}\left(^2\ell_p^2\right)\right)$.

Remark 9.2 It is important to mention that, although we know the extreme polynomials on the spaces ℓ_p^2, the proof of Theorem 9.21 in [53] does not use the Krein-Milman approach but a direct approach. It involves obtaining a sharper bound C than that of Martin's bound for every polynomial and then finding a polynomial P such that $\|L\|_{\mathbf{C}} = C\|P\|_{\mathbf{C}}$, where L is the polar of P.

An interesting question started by Harris in 1975 related to polarization constants for polynomials on ℓ_p spaces states that, in a complex setting we have

$$\mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\ell_\infty^n(\mathbb{C}))) \leq \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{2^n n!}.$$

For the previous estimate consult [32] or [20] for a more modern and accessible exposition. The question as to whether $\mathrm{c}_{\mathrm{pol}}(\mathcal{P}(^n\ell_\infty^n(\mathbb{C}))) = \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{2^n n!}$ remains unsolved nowadays.

Theorem 9.22 (Kim [37]) *Let* $w \in (0,1)$.

(a) If $w \leq \sqrt{2} - 1$, *then* $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right) = \frac{2(1+w^2)}{(1+w)^2}$ *(see Sect. 6.1). Furthermore,* $P(x,y) = \pm\left(\frac{4}{(1+w)^2}xy\right)$ *are extremal polynomials for* $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right)$.

(b) If $\sqrt{2} - 1 < w$, *then* $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right) = 1 + w^2$. *Furthermore,* $P(x,y) = \pm\left(x^2 - y^2\right)$ *are extremal polynomials for* $c_{pol}\left(\mathcal{P}\left(^2O_w^2\right)\right)$.

Theorem 9.23 (Kim [39]) *Let $w = \frac{1}{2}$. We have $c_{pol}\left(\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)\right) = \frac{5}{4}$ (see Sect. 6.2). Furthermore,*

$$P(x, y) = \pm\left(x^2 - y^2\right)$$

and

$$Q(x, y) = \pm\left(\frac{3}{4}x^2 - \frac{5}{16}y^2 \pm \frac{7}{4}\right)$$

are extremal polynomials for $c_{pol}\left(\mathcal{P}\left({}^2\mathcal{H}_{1/2}^2\right)\right)$.

9.3 Unconditional Constants

Let us denote by $\mathbf{x}^\alpha$ the monomial

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m},$$

where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{K}^m$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_k \in \mathbb{N} \cup \{0\}$ for every $k \in \{1, \ldots, m\}$. For $P(\mathbf{x}) = \sum_{|\alpha|\leq n} a_\alpha \mathbf{x}^\alpha$ (where $|\alpha| = \alpha_1 + \cdots + \alpha_m$) a polynomial of degree n on $\mathbb{K}^m$, we define the modulus $|\cdot|$ of P by $|P|(\mathbf{x}) = \sum_{|\alpha|\leq n} |a_\alpha| \mathbf{x}^\alpha$. If $\mathbf{C}$ is a convex body in $\mathbb{R}^m$, we denote by $\mathcal{P}({}^n\mathbf{C})$ the space of n-homogeneous polynomials on $\mathbb{R}^m$ endowed with the norm $\|P\|_{\mathbf{C}}$ (see Sect. 9.2). Let $\mathcal{B}_n = \{\mathbf{x}^\alpha : |\alpha| \leq n\}$ be the canonical basis of $\mathcal{P}({}^n\mathbf{C})$. The unconditional constant of $\mathcal{B}_n$ is equal to the best possible constant C (denoted by $C_{\mathrm{unc}}(\mathcal{P}({}^n\mathbf{C}))$) in the inequality

$$\||P|\|_{\mathbf{C}} \leq C\|P\|_{\mathbf{C}}.$$

The following results show all the exact values of the unconditional constants that are known of the spaces that have been presented on this survey.

Theorem 9.24 (Grecu et al. [30]) *If $m, n \in \mathbb{N}$ with $m > n$, then*

$$C_{unc}(\mathcal{P}_{m,n,\infty}(\mathbb{R})) = \begin{cases} 3 & \textit{if m and n have different parity,} \\ 1 + \frac{4}{m-n}\left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}} & \textit{if m and n are even,} \\ \frac{n-\lambda_0 m}{n+\lambda_0 m} & \textit{if m and n are odd,} \end{cases}$$

(see Sect. 3.1) where λ_0 comes from Theorem 3.1, and equality is attained for the polynomials

$$P(x) = \begin{cases} \pm(2x^m - 1), \\ \pm(-\gamma_0 x^m + \gamma_0 x^n + 1) \text{ where } \gamma_0 = -\frac{2}{m-n} \cdot \left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}}, \\ \pm\left(\frac{nx^m}{n+m\lambda_0} - \frac{m|\lambda_0|x^n}{n+m\lambda_0}\right), \end{cases}$$

respectively.

Remark 9.3 (Grecu et al. [30]) In Theorem 9.24 it can be seen that for every $k \in \mathbb{N}$ with $k > 1$ and every $n \in \mathbb{N}$ even we have

$$C_{\text{unc}}(\mathcal{P}_{kn,n,\infty}(\mathbb{R})) = 1 + \frac{4}{k-1} \cdot k^{\frac{k}{k-1}},$$

which is independent of n.

Theorem 9.25 (Grecu et al. [30]) *On the space $\mathcal{P}(^2\Delta)$ (see Sect. 4.1) we have*

$$C_{unc}(\mathcal{P}(^2\Delta)) = 2$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 - 6xy + y^2).$$

Theorem 9.26 (Gámez et al. [23]) *On the space $\mathcal{P}(^2\square)$ (see Sect. 4.2) we have*

$$C_{unc}(\mathcal{P}(^2\square)) = 5$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 - 3xy + y^2).$$

Theorem 9.27 (Gámez et al. [23]) *On the space $\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$ (see Sect. 4.3) we have*

$$C_{unc}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)\right) = 5 + 4\sqrt{2}$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy)).$$

Theorem 9.28 (Jiménez et al. [34]) *On the space $\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$ we have*

$$C_{unc}\left(\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)\right) = 3$$

and equality is attained for the polynomials

$$P(x, y) = \pm(x^2 + y^2 - 4xy).$$

Theorem 9.29 (Grecu et al. [30]) *On the spaces $\mathcal{P}(^2\ell_1^2)$, $\mathcal{P}(^2\ell_2^2)$ and $\mathcal{P}(^2\ell_\infty^2)$ (see Sects. 4.3, 5.1, and 5.2) we have, respectively, the unconditional constants given by*

$$\begin{cases} \frac{1+\sqrt{2}}{2}, \\ \sqrt{2}, \\ 1+\sqrt{2}, \end{cases}$$

with equality attained for the polynomials

$$\begin{cases} \pm\frac{\sqrt{2}}{2}(x^2 - y^2) \pm (2+\sqrt{2})xy, \\ \pm(x^2 + y^2 + 2xy), \\ \frac{2+\sqrt{2}}{4}(x^2 - y^2) \pm \frac{\sqrt{2}}{2}xy, \end{cases}$$

respectively.

Proof We will prove the result for the space $\mathcal{P}(^2\ell_1^2)$ since the other cases can be done analogously. By Theorem 5.2, we know that the extreme polynomials of the unit ball of $\mathcal{P}(^2\ell_1^2)$ are

(a) $P(x, y) = \pm x^2 \pm y^2 \pm 2xy$,
(b) $P(x, y) = \pm\frac{\sqrt{4|t|-t^2}}{2}(x^2 - y^2) + txy$, where $|t| \in (2, 4]$.

Notice that if P is as in (a), then $|||P|||_{\ell_1^2} = \|P\|_{\ell_1^2} = 1$. Hence, it is enough to consider polynomials of type (b). If P is as in (b), then P attains its norm in ℓ_1^2 at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus,

$$\begin{aligned} C_{\mathrm{unc}}(\mathcal{P}\left(^2\ell_1^2\right)) &= \sup\left\{\left\|\frac{\sqrt{4|t|-t^2}}{2}(x^2 + y^2) + |t|xy\right\|_{\ell_1^2} : |t| \in (2, 4]\right\} \\ &= \sup\left\{\left\|\frac{\sqrt{4s-s^2}}{2}(x^2 + y^2) + sxy\right\|_{\ell_1^2} : s \in (2, 4]\right\} \\ &= \sup\left\{\frac{\sqrt{4s-s^2}+s}{4} : s \in (2, 4]\right\} \\ &= 2+\sqrt{2}. \end{aligned}$$

$\square$

Theorem 9.30 (Araújo et al. [2]) *Let $1 < p < \infty$ with $p \neq 2$ and take $\mathcal{P}(^2\ell_p^2)$ (see Sects. 5.3 and 5.4). Let us define the function*

$$f(\alpha) = \frac{2^{\frac{p-2}{p}}\left[\alpha(1-\alpha^p)\left(\alpha-(1-\alpha^p)^{\frac{1}{p}}\right)+\alpha^p\,(1-\alpha^p)^{\frac{1}{p}}\left(\alpha+(1-\alpha^p)^{\frac{1}{p}}\right)\right]}{\alpha\,(1-\alpha^p)^{\frac{1}{p}}\left(\alpha^2+(1-\alpha^p)^{\frac{2}{p}}\right)}$$

and set $M_f = \sup\left\{f(\alpha) \colon \alpha \in \left[2^{-\frac{1}{p}}, 1\right]\right\}$, we have that $C_{unc}(\mathcal{P}(^2\ell_p^2)) = M_f$.

Theorem 9.31 (Kim [37]) *Let $0 < w < 1$.*

(a) If $w \leq \sqrt{2}-1$, then $c_{unc}\left(\mathcal{P}\left(^2O_w^2\right)\right) = \frac{1+w^2+\sqrt{2(1+w^4)}}{(1+w)^2}$ (see Sect. 6.1) and equality is attained for the polynomials $P(x,y) = \pm\left(\frac{4}{(1+w)^2}xy\right)$.

(b) If $\sqrt{2}-1 < w$, then $c_{unc}\left(\mathcal{P}\left(^2O_w^2\right)\right) = \frac{1+w^2+\sqrt{(1+w^2)^2+4w^2}}{2}$ and equality is attained for the polynomials

$$P(x,y) = \pm(\alpha x^2 - \alpha y^2 \pm \sqrt{\alpha(1-\alpha)}xy),$$

where $\alpha = \frac{1}{2} + \frac{1+w^2}{2\sqrt{(1+w^2)^2+4w^2}}$.

Theorem 9.32 (Kim [39]) *Let $w = \frac{1}{2}$. Then, $c_{unc}\left(\mathcal{P}\left(^2\mathcal{H}_{1/2}^2\right)\right) = \frac{3}{2}$ (see Sect. 6.2) and equality is attained for the polynomials $P(x,y) = \pm\left(x^2+\frac{1}{4}y^2+xy\right)$ and $Q(x,y) = \pm\left(x^2+\frac{3}{4}y^2+xy\right)$.*

9.4 Bohnenblust–Hille and Hardy–Littlewood Constants

We begin by considering the following constants which are closely related to the Bohnenblust–Hille and Hardy–Littlewood constants as we will see. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $n \in \mathbb{N}$ and let us consider the standard notation $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$. Let $\mathcal{P}(^m\mathbb{K}^n)$ denote the vector space of m-homogeneous polynomials on $\mathbb{K}^n$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Notice that if $P \in \mathcal{P}(^m\mathbb{K}^n)$, then P can be written as

$$P(\mathbf{x}) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

where $a_\alpha \in \mathbb{K}$ and $\mathbf{x}^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$. If $|\cdot|$ is a norm on $\mathbb{K}^n$, then $|\cdot|$ induces a norm on $\mathcal{P}(^m\mathbb{K}^n)$ called the polynomial norm and it is given by

$$\|P\| = \sup\{|P(\mathbf{x})| \colon \mathbf{x} \in \mathsf{B}_X\},$$

where B_X is the unit ball of the normed space $X = (\mathbb{K}^n, |\cdot|)$. The space $\mathcal{P}(^m\mathbb{K}^n)$ endowed with the polynomial norm is denoted by $\mathcal{P}(^m X)$. Besides the polynomial norm, there are other interesting norms on $\mathcal{P}(^m\mathbb{K}^n)$ such as the ℓ_q-norms on the coefficients, i.e., if $P \in \mathcal{P}(^m\mathbb{K}^n)$ and $1 \le q \le \infty$, then

$$|P|_q = \begin{cases} \left(\sum_{|\alpha|=m} |a_\alpha|^q\right)^{\frac{1}{q}} & \text{if } 1 \le q < \infty, \\ \max\{|a_\alpha| \colon |\alpha| = m\} & \text{if } q = \infty. \end{cases}$$

Let us represent by $\|\cdot\|_p$ the polynomial norm of the space $\mathcal{P}(^m\ell_p^n(\mathbb{K}))$, where $1 \le p \le \infty$. Since the space $\mathcal{P}(^m\mathbb{K}^n)$ is finite dimensional, we have that the norms $|\cdot|_q$ and $\|\cdot\|_p$ $(1 \le q, p \le \infty)$ are equivalent, i.e., there exist $k, K > 0$ such that

$$k\|P\|_p \le |P|_q \le K\|P\|_p,$$

for any $P \in \mathcal{P}(^m\mathbb{K}^n)$. Notice that the unit balls of the spaces $(\mathcal{P}(^m\mathbb{K}^n), |\cdot|_q)$ and $\mathcal{P}(^m\ell_p^n(\mathbb{K}))$, denoted by $\mathsf{B}_{|\cdot|_q}$ and $\mathsf{B}_{\|\cdot\|_p}$, respectively, satisfy that the mapping $\mathsf{B}_{|\cdot|_q} \ni P \to \|P\|_p$ is bounded by $\frac{1}{k}$ and the mapping $\mathsf{B}_{\|\cdot\|_p} \ni P \to |P|_q$ is bounded by K. Moreover, the continuity of such mappings and the compactness of $\mathsf{B}_{|\cdot|_q}$ and $\mathsf{B}_{\|\cdot\|_p}$ satisfy the following maxima.

Definition 9.1 Let $1 \le q, p \le \infty$. We define the following constants

$$k_{m,n,q,p} = \frac{1}{\max\left\{\|P\|_p \colon P \in \mathsf{B}_{|\cdot|_q}\right\}},$$

$$K_{m,n,q,p} = \max\left\{|P|_q \colon P \in \mathsf{B}_{\|\cdot\|_p}\right\}.$$

From now on, we are interested in calculating the exact values of $k_{m,n,q,p}$ and $K_{m,n,q,p}$ when we are considering polynomials whose coefficients are real numbers (we will consider real polynomials and complex polynomials with real coefficients separately). To do so, we will be applying the Krein-Milman approach to the mappings $\mathsf{B}_{|\cdot|_q} \ni P \to \|P\|_p$ and $\mathsf{B}_{\|\cdot\|_p} \ni P \to |P|_q$. Hence, we will need, for instance, the extreme points of the unit ball $\mathsf{B}_{|\cdot|_q}$. It is well known that the extreme points of $\mathsf{B}_{|\cdot|_q}$ are

$$\begin{cases} \{\pm e_k \colon 1 \le k \le m+1\} & \text{if } q = 1, \\ \left\{\sum_{k=1}^{m+1} \varepsilon_k e_k \colon \varepsilon_k = \pm 1\right\} & \text{if } q = \infty, \\ \mathsf{S}_{|\cdot|_q} & \text{if } 1 < q < \infty, \end{cases}$$

where $\{e_1, \ldots, e_{m+1}\}$ stands for the canonical basis of $\mathbb{R}^{m+1}$ and $\mathsf{S}_{|\cdot|_q}$ is the unit sphere of $(\mathbb{R}^{m+1}, |\cdot|_q)$.

The above problem is an extension of the polynomial Bohnenblust–Hille and Hardy–Littlewood constants problem. The m-Bohnenblust–Hille constant for polynomials is, in fact, an upper bound on $K_{m,n,\frac{2m}{m+1},\infty}$. It was proved in [8] that if $q \geq \frac{2m}{m+1}$, then there exists a constant $D_{m,q} > 0$ depending only on m and q such that

$$|P|_q \leq D_{m,q}\|P\|_\infty,$$

for any $P \in \mathcal{P}(^m\ell_\infty^n(\mathbb{K}))$ and every $n \in \mathbb{N}$. Furthermore, any constant in the latter inequality for $q < \frac{2m}{m+1}$ depends necessarily on n. By construction, notice that any viable choice of $D_{m,q}$ satisfies $D_{m,q} \geq \sup\{K_{m,n,q,\infty} \colon n \in \mathbb{N}\}$. This construction allows us to define the Bohnenblust-Hille constants depending on the field ($\mathbb{R}$ or $\mathbb{C}$) since there are substantial differences.

Definition 9.2 The m-Bohnenblust-Hille constant for polynomials on $\mathbb{K}$ is defined as

$$D_{\mathbb{K},m} = \inf\left\{D_m \colon |P|_{\frac{2m}{m+1}} \leq D_m\|P\|_\infty, \text{ for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}(^m\ell_\infty^n(\mathbb{K}))\right\}.$$

If $n \in \mathbb{N}$ is fixed, then we define (m, n)-Bohnenblust-Hille constant for polynomials on $\mathbb{K}$ as

$$D_{\mathbb{K},m}(n) = \inf\left\{D_m(n) \colon |P|_{\frac{2m}{m+1}} \leq D_m(n)\|P\|_\infty, \text{ for all } P \in \mathcal{P}(^m\ell_\infty^n(\mathbb{K}))\right\}.$$

Also, if we consider a subset E of $\mathcal{P}(^m\ell_\infty^n(\mathbb{K}))$ for some $n \in \mathbb{N}$, then we define the (m, E)-Bohnenblust-Hille constant for polynomials on $\mathbb{K}$ as

$$D_{\mathbb{K},m}(E) = \inf\left\{D_m(E) \colon |P|_{\frac{2m}{m+1}} \leq D_m(E)\|P\|_\infty, \text{ for all } P \in E\right\}.$$

It is easy to see that

$$1 \leq D_{\mathbb{K},m}(n) \leq D_{\mathbb{K},m},$$

for all $n \in \mathbb{N}$. A similar result to that of Bohnenblust-Hille for values of p different from ∞ can also be obtained. The proofs of the following results can be found in [1, 18]. There exist constants $C_{m,p}$ and $D_{m,p}$ independent of n such that

$$|P|_{\frac{p}{p-m}} \leq C_{m,p}\|P\|_p \text{ for } m < p \leq 2m,$$

$$|P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}\|P\|_p \text{ for } 2m \leq p \leq \infty,$$

for all $P \in (^m\ell_p^n(\mathbb{K}))$ and every $n \in \mathbb{N}$. If $p = \infty$, then we simply put $\frac{2mp}{mp+p-2m} = \frac{2m}{m+1}$. Moreover, the exponents $\frac{p}{p-m}$ for $m < p \leq 2m$ and $\frac{2mp}{mp+p-2m}$ for $2m \leq p \leq$

∞ are optimal in the sense that any constant H that satisfies

$$|P|_q \leq H\|P\|_p,$$

for all $P \in (^m\ell_p^n(\mathbb{K}))$ depends necessarily on n. The above construction allows us to define the following constants.

Definition 9.3 Let $m < p \leq \infty$. The (m, p)-Hardy-Littlewood constant for polynomials on $\mathbb{K}$ is defined as

$$C_{\mathbb{K},m,p} = \inf\left\{C_{m,p} \colon |P|_{\frac{p}{p-m}} \leq C_{m,p}\|P\|_p,\ \text{for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}(^m\ell_p^n(\mathbb{K}))\right\},$$

for $m < p \leq 2m$, and

$$D_{\mathbb{K},m,p} = \inf\Big\{D_{m,p} \colon |P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}\|P\|_p,$$
$$\text{for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}(^m\ell_p^n(\mathbb{K}))\Big\},$$

for $2m \leq p \leq \infty$. If $n \in \mathbb{N}$ is fixed, then we define the (m, n, p)-Hardy-Littlewood constant for polynomials on $\mathbb{K}$ as

$$C_{\mathbb{K},m,p}(n) = \inf\left\{C_{m,p}(n) \colon |P|_{\frac{p}{p-m}} \leq C_{m,p}(n)\|P\|_p,\ \text{for all } P \in \mathcal{P}(^m\ell_p^n(\mathbb{K}))\right\},$$

for $m < p \leq 2m$, and

$$D_{\mathbb{K},m,p}(n) = \inf\Big\{D_{m,p}(n) \colon |P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}(n)\|P\|_p,$$
$$\text{for all } P \in \mathcal{P}(^m\ell_p^n(\mathbb{K}))\Big\},$$

for $2m \leq p \leq \infty$. Also, if we consider a subset E of $\mathcal{P}(^m\ell_\infty^n(\mathbb{K}))$ for some $n \in \mathbb{N}$, then we define

$$C_{\mathbb{K},m,p}(E) = \inf\left\{C_{m,p}(E) \colon |P|_{\frac{p}{p-m}} \leq C_{m,p}(E)\|P\|_p,\ \text{for all } P \in \mathcal{P}(^m E)\right\},$$

for $m < p \leq 2m$, and

$$D_{\mathbb{K},m,p}(E) = \inf\left\{D_{m,p}(E) \colon |P|_{\frac{2mp}{mp+p-2m}} \leq D_{m,p}(E)\|P\|_p,\ \text{for all } P \in \mathcal{P}(^m E)\right\}$$

for $2m \leq p \leq \infty$.

Notice that $D_{\mathbb{K},m} = D_{\mathbb{K},m,\infty}$. So essentially the Hardy-Littlewood constants are in fact a generalization of the Bohnenblust-Hille constants. But furthermore, the constants $K_{m,n,q,p}$ are also a generalization of the Hardy-Littlewood constants since $C_{\mathbb{K},m,p}(n) = K_{m,n,\frac{p}{p-m},p}$ for $m < p \leq 2m$ and $D_{\mathbb{K},m,p}(n) = K_{m,n,\frac{2mp}{mp+p-2m},p}$ for $2m \leq p \leq \infty$. Hence we have

$$\begin{cases} C_{\mathbb{K},m,p} \geq \sup\left\{K_{m,n,\frac{p}{p-m},p} \colon n \in \mathbb{N}\right\} & \text{for } m < p \leq 2m, \\ D_{\mathbb{K},m,p} \geq \sup\left\{K_{m,n,\frac{2mp}{mp+p-2m},p} \colon n \in \mathbb{N}\right\} & \text{for } 2m \leq p \leq \infty. \end{cases}$$

This section is about providing some of the constants $k_{m,n,q,p}$, $K_{m,n,q,p}$, and in particular, the Hardy-Littlewood and Bohnenblust-Hille constants, that have been obtained through the Krein-Milman approach.

9.4.1 On the Complex Case

Assume that $\mathbb{K} = \mathbb{C}$.

Theorem 9.33 (Jiménez et al. [33]) *Let $E_{\mathbb{R}}$ be the real subspace of $\mathcal{P}(^2\ell_\infty^2(\mathbb{C}))$ given by $\{az^2 + bw^2 + czw \colon (a, b, c) \in \mathbb{R}^3\}$. We have*

$$D_{\mathbb{C},2}(E_{\mathbb{R}}) = D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}}$$

with extremal polynomials

$$P(x, y) = \pm\left(\frac{\sqrt{3}}{6}z^2 - \frac{\sqrt{3}}{6}w^2 \pm \sqrt{\frac{2}{3}}zw\right).$$

9.4.2 On the Real Case

Assume that $\mathbb{K} = \mathbb{R}$. All the results that are presented have been obtained for the cases when $m = n = 2$.

Theorem 9.34 (Jiménez et al. [33]) *Let $f \colon \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ be given by*

$$f(t) = \left[2t^{\frac{4}{3}} + \left(2\sqrt{t(1-t)}\right)^{\frac{4}{3}}\right]^{\frac{3}{4}}.$$

We have

$$D_{\mathbb{R},2}(2) = f(t_0),$$

where

$$t_0 = \frac{1}{36}\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right).$$

In particular, the exact value of $f(t_0)$ *is given by*

$$(A + B)^{\frac{3}{4}},$$

where

$$A = \frac{\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right)^{\frac{4}{3}}}{186^{\frac{2}{3}}}$$

and

$$B = \frac{1}{9\left(-\frac{3}{-2\sqrt[3]{107+9\sqrt{129}}+\left(107+9\sqrt{129}\right)^{\frac{2}{3}}-2\sqrt[3]{107-9\sqrt{129}}+\left(107-9\sqrt{129}\right)^{\frac{2}{3}}-60}\right)^{\frac{2}{3}}}.$$

Moreover, the following polynomials are extremal

$$P(x, y) = \pm\left(t_0x^2 - t_0y^2 \pm 2\sqrt{t_0(1 - t_0)}xy\right).$$

Theorem 9.35 (Araújo et al. [3]) *If* $q, p \in \{1, \infty\}$, *then*

$$k_{2,2,q,p} = \begin{cases} 1 & \text{if } q = p = 1, \\ 1 & \text{if } q = 1 \text{ and } p = \infty, \\ 1 & \text{if } q = \infty \text{ and } p = 1, \\ \frac{1}{3} & \text{if } q = p = \infty, \end{cases}$$

with extremal polynomials given, respectively, by

$$P_{1,1}(x, y) = \pm x^2,\ \pm y^2,$$

$$P_{1,\infty}(x, y) = \pm x^2,\ \pm y^2,\ \pm xy,$$

$$P_{\infty,1}(x, y) = \pm x^2 \pm y^2 \pm xy,$$

$$P_{\infty,\infty}(x, y) = \pm(x^2 + y^2 \pm xy).$$

Theorem 9.36 (Araújo et al. [3]) *If $q, p \in \{1, \infty\}$, then*

$$K_{2,2,q,p} = \begin{cases} 2+2\sqrt{2} & \text{if } q = p = 1, \\ 1+\sqrt{2} & \text{if } q = 1 \text{ and } p = \infty, \\ 4 & \text{if } q = \infty \text{ and } p = 1, \\ 1 & \text{if } q = p = \infty, \end{cases}$$

with extremal polynomials given, respectively, by

$$\begin{aligned} P_{1,1}(x, y) &= \pm\frac{\sqrt{2}}{2}(x^2 - y^2) + (2+\sqrt{2})xy, \\ P_{1,\infty}(x, y) &= \pm\left(\frac{2+\sqrt{2}}{4}x^2 - \frac{2+\sqrt{2}}{4}y^2 \pm \frac{\sqrt{2}}{2}xy\right), \\ P_{\infty,1}(x, y) &= \pm 4xy, \\ P_{\infty,\infty}(x, y) &= \pm x^2,\ \pm y^2,\ \pm\left(\frac{1}{2}x^2 - \frac{1}{2}y^2 \pm xy\right). \end{aligned}$$

Theorem 9.37 (Araújo et al. [3]) *For every $q \in [1, \infty)$, let $f_{q,1}\colon [2, 4] \to \mathbb{R}$ and $f_{q,\infty}\colon \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ be given by*

$$\begin{aligned} f_{q,1}(t) &= \left(2^{1-q}(4t - t^2)^{\frac{q}{2}} + t^q\right)^{\frac{1}{q}}, \\ f_{q,\infty}(t) &= \left(2t^q + 2^q(t - t^2)^{\frac{q}{2}}\right)^{\frac{1}{q}}. \end{aligned}$$

We have

$$\begin{aligned} K_{2,2,q,1} &= \max\left\{f_{q,1}(t)\colon t \in [2, 4]\right\}, \\ K_{2,2,q,\infty} &= \max\left\{f_{q,\infty}(t)\colon t \in \left[\frac{1}{2}, 1\right]\right\}. \end{aligned}$$

In particular, $K_{2,2,q,1} = 4$ and $K_{2,2,q,\infty} = 2^{\frac{1}{q}}$ for every $q \geq 2$, with extremal polynomials given, respectively, by

$$\begin{aligned} P_{q,1}(x, y) &= \pm 4xy, \\ P_{q,\infty}(x, y) &= \pm(x^2 - y^2). \end{aligned}$$

Remark 9.4 (Araújo et al. [3]) The exact value of the maximum of the functions $f_{q,1}$ and $f_{q,\infty}$ or the points of attainment of the maximum seems to be a much harder task. However, by using the symbolic calculus tool of MATLAB, we are able to obtain the exact values where the functions reach its maximum for certain values of q. For instance, for $q = \frac{4}{3}$, the maximum of $f_{q,1}(t)$ and $f_{q,\infty}(t)$ is attained at

$$t = \frac{1}{9}\left(2\sqrt[3]{181 + 9\sqrt{273}} + \sqrt[3]{1448 - 72\sqrt{273}} + 14\right).$$

and

$$t = \frac{1}{36}\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right),$$

respectively. Also, for $q = \frac{3}{2}$, the maximum of $f_{q,1}(t)$ is attained at

$$t = \frac{1}{15}\left(\sqrt{6\,(A+24)} + \sqrt{6\left(-A + 204\sqrt{\frac{6}{A+24}} + 48\right) + 18}\right),$$

where

$$A = -10 \cdot 3^{2/3}\sqrt[3]{\frac{2}{9+\sqrt{93}}} + 5 \cdot 2^{2/3}\sqrt[3]{3\left(9+\sqrt{93}\right)}.$$

And also for $q = \frac{3}{2}$, the maximum of $f_{q,\infty}(t)$ is attained at

$$t = \frac{1}{20}\sqrt{B} + \frac{1}{2}\sqrt{C+D} + \frac{9}{20},$$

where

$$B = \frac{10\sqrt[3]{9+\sqrt{273}}}{3^{2/3}} - \frac{40}{\sqrt[3]{3\left(9+\sqrt{273}\right)}} + 1,$$

$$C = -\frac{\sqrt[3]{9+\sqrt{273}}}{10 \cdot 3^{2/3}} + \frac{1}{50} + \frac{2}{5\sqrt[3]{3\left(9+\sqrt{273}\right)}}$$

and

$$D = \frac{40}{50\sqrt{\frac{10\sqrt[3]{9+\sqrt{273}}}{3^{2/3}} - \frac{40}{\sqrt[3]{3\left(9+\sqrt{273}\right)}} + 1}}.$$

Theorem 9.38 (Araújo et al. [3]) *If $p \in (1, \infty)$, then*

$$k_{2,2,q,p} = \begin{cases} 1 & \text{if } q = 1, \\ \frac{2^{\frac{2}{p}}}{3} & \text{if } q = \infty \text{ and } p \geq \frac{4}{3}, \\ \frac{1}{\max\left\{x^2 + (1-x^p)^{\frac{2}{p}} + x(1-x^p)^{\frac{1}{p}} : x \in [0,1]\right\}} & \text{if } q = \infty \text{ and } 1 < p < \frac{4}{3}, \end{cases}$$

with extremal polynomials given, respectively, by

$$P_{1,p}(x, y) = \pm x^2, \ \pm y^2,$$
$$P_{\infty,p}(x, y) = \pm \left(x^2 + y^2 + xy\right),$$
$$Q_{\infty,p}(x, y) = \pm \left(x^2 + y^2 + xy\right).$$

Theorem 9.39 (Araújo et al. [3]) *For every $q \geq 1$ and $p \geq 2$, let $f_{q,p} \colon [0, 1] \to \mathbb{R}$ be given by*

$$f_{q,p}(s) = \begin{cases} \left(2(1-s)^{\frac{q}{2}} + 2^q s^{\frac{q}{2}}\right)^{\frac{1}{q}} & \text{if } p = 2, \\ \frac{\left\{2|1-2s|^q + 2^q\left[(1-s)^{1-\frac{1}{p}} s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}} s^{1-\frac{1}{p}}\right]^q\right\}^{\frac{1}{q}}}{(1-s)^{\frac{2}{p}} + s^{\frac{2}{p}}} & \text{if } p \neq 2. \end{cases}$$

We have

$$K_{2,2,q,p} = \max\left\{f_{q,p}(t) \colon t \in [0, 1]\right\}.$$

See also [13] in connection to the previous result.

Corollary 9.2 (Araújo et al. [3]) *For $4 \leq p \leq \infty$, we have*

$$\begin{aligned} D_{\mathbb{R},2,p}(2) &= K_{2,2,\frac{4p}{3p-4},p} \\ &= \max_{s \in \left[0, \frac{1}{2}\right]} \frac{\left\{2|1-2s|^{\frac{4p}{3p-4}} + 2^{\frac{4p}{3p-4}}\left[(1-s)^{1-\frac{1}{p}} s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}} s^{1-\frac{1}{p}}\right]^{\frac{4p}{3p-4}}\right\}^{\frac{3p-4}{4p}}}{(1-s)^{\frac{2}{p}} + s^{\frac{2}{p}}}. \end{aligned}$$

Theorem 9.40 (Araújo et al. [3]) *If $q > 1$, then*

$$K_{2,2,q,2} = \begin{cases} 2 & \text{if } q \geq 2, \\ \dfrac{2\left(1+2^{\frac{1}{q-2}}\right)^{\frac{1}{q}}}{\left(1+2^{\frac{2(q-1)}{q-2}}\right)^{\frac{1}{2}}} & \text{if } 1 < q < 2, \end{cases}$$

with extremal polynomials given by

$$P_{q,2}(x, y) = \begin{cases} \pm(x^2 - y^2) & \text{if } q \geq 2, \\ \pm\left(a_0 x^2 - a_0 y^2 + 2\sqrt{1 - a_0^2}\, xy\right) & \text{if } 1 < q < 2, \end{cases}$$

where $a_0 = \left(1 + 2^{\frac{2(1-q)}{q-2}}\right)^{-\frac{1}{2}}$.

Theorem 9.41 (Araújo et al. [3]) *If* $q, p > 2$, *then*

$$K_{2,2,q,p} = 2^{\max\left\{\frac{1}{q}, \frac{2}{p}\right\}}.$$

If $f_{q,p}$ *is as in Theorem 9.39 and* $q, p > 2$, *then the following polynomials are extremal*

$$P_{q,p}(x, y) = \begin{cases} \pm 2^{\frac{2}{p}} xy & \text{if } q \geq \frac{p}{2}, \\ \pm(x^2 - y^2) & \text{if } q < \frac{p}{2}. \end{cases}$$

Corollary 9.3 (Araújo et al. [3]) *If* $p \geq 2$, *then*

$$K_{2,2,\infty,p} = 2^{\frac{2}{p}}$$

with extremal polynomials given by

$$P_{\infty,p}(x, y) = \begin{cases} \pm(x^2 - y^2) & \text{if } p = 2, \\ \pm 2^{\frac{2}{p}} xy & \text{if } p > 2. \end{cases}$$

Corollary 9.4 (Araújo et al. [3]) *For* $2 < p \leq 4$, *we have*

$$C_{\mathbb{R},2,p}(2) = K_{2,2,\frac{p}{p-2},p} = 2^{\frac{2}{p}}.$$

It is important to mention that Corollary 9.4 was first proven in [13].

Corollary 9.5 (Araújo et al. [3]) *We have*

$$D_{\mathbb{R},2,4}(2) = C_{\mathbb{R},2,4}(2) = K_{2,2,4,p} = \sqrt{2}$$

with all extremal polynomials given by

$$\begin{aligned} P(x, y) &= \pm(x^2 - y^2), \\ Q(x, y) &= \pm\left((\alpha^2 - \beta^2)(x^2 - y^2) + 2\alpha\beta xy\right), \end{aligned}$$

with $\alpha, \beta \geq 0$ *and* $\alpha^4 + \beta^4 = 1$.

Theorem 9.42 (Araújo et al. [3]) *For* $p > 2$*, let* $f_{1,p} \colon \left[0, \frac{1}{2}\right] \to \mathbb{R}$ *be defined by*

$$f_{1,p}(s) = \frac{2(1-2s) + 2\left[(1-s)^{1-\frac{1}{p}} s^{\frac{1}{p}} + (1-s)^{\frac{1}{p}} s^{1-\frac{1}{p}}\right]}{(1-s)^{\frac{2}{p}} + s^{\frac{2}{p}}}.$$

We have

$$K_{2,2,1,p} = \sup\left\{ f_{1,p}(t) \colon t \in \left[0, \frac{1}{2}\right]\right\}.$$

Remark 9.5 (Araújo et al. [3]) The exact calculation of the above supremum seems to be a harder task. However, by using the symbolic calculus tool of MATLAB, we can obtain the exact value of the supremum of $f_{1,p}(t)$ as well as the point where it attains its maximum for certain values of p. For $p = 4$, the function $f_{1,4}(t)$ attains its maximum on $\left[0, \frac{1}{2}\right]$ at $t = \frac{3-2\sqrt{2}}{6}$ and, therefore, $K_{2,1,4} = \sqrt{6}$.

References

1. N. Albuquerque, F. Bayart, D. Pellegrino, J.B. Seoane-Sepúlveda, Optimal Hardy-Littlewood type inequalities for polynomials and multilinear operators. Israel J. Math. **211**(1), 197–220 (2016)
2. G. Araújo, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, Polynomial inequalities on the $\pi/4$-circle sector. J. Convex Anal. **24**(3), 927–953 (2017)
3. G. Araújo, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, Equivalent norms in polynomial spaces and applications. J. Math. Anal. **445**, 1200–1220 (2017)
4. G. Araújo, G.A. Muñoz-Fernández, D.L. RodríguezVidanes, J.B. Seoane-Sepúlveda, Sharp Bernstein inequalities using convex analysis techniques. Math. Inequal. Appl. **23**(2), 725–750 (2020). https://doi.org/10.7153/mia-2020-23-61
5. R.M. Aron, M. Klimek, Supremum norms for quadratic polynomials. Arch. Math. (Basel) **76**(1), 73–80 (2001). https://doi.org/10.1007/s000130050544
6. L. Bernal-González, G.A. Muñoz-Fernández, D.L. RodríguezVidanes, J.B. Seoane-Sepúlveda, A complete study of the geometry of 2-homogeneous polynomials on circle sectors, Preprint (2019)
7. Q. Bu, G. Buskes, Polynomials on Banach lattices and positive tensor products. J. Math. Anal. Appl. **388**(2), 845–862 (2012)
8. H.F. Bohnenblust, E. Hille, On the absolute convergence of Dirichlet series. Ann. Math. (2) **32**(3), 600–622 (1931)
9. C. Boyd, S. Lassalle, Extreme and exposed points of spaces of integral polynomials. Proc. Am. Math. Soc. **138**(4), 1415–1420 (2010)
10. C. Boyd, R.A. Ryan, Geometric theory of spaces of integral polynomials and symmetric tensor products. J. Funct. Anal. **179**(1), 18–42 (2001)
11. C. Boyd, R.A. Ryan, N. Snigireva, Geometry of spaces of orthogonally additive polynomials on $C(K)$. J. Geom. Anal. (2020). https://doi.org/10.1007/s12220-019-00240-0
12. D. Carando, S. Lassalle, I. Zalduendo, Orthogonally additive polynomials over $C(K)$ are measures—a short proof. Integral Equations Oper. Theory **56**(4), 597–602 (2006)
13. W. Cavalcante, D. NúñezAlarcón, D. Pellegrino, The optimal Hardy-Littlewood constants for 2-homogeneous polynomials on $\ell_p(\mathbb{R}^2)$ for $2 < p < 4$ are $2^{2/p}$. J. Math. Anal. **445**, 1200–1220 (2017)
14. Y.S. Choi, S.G. Kim, Extreme polynomials on c_0. Indian J. Pure Appl. Math. **29**(10), 983–989 (1998)
15. Y.S. Choi, S.G. Kim, The unit ball of $\mathcal{P}(^2l_2^2)$. Arch. Math. (Basel) **71**(6), 472–480 (1998). https://doi.org/10.1007/s000130050292

J. Ferrer et al., *Geometry of the Unit Sphere in Polynomial Spaces*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-3-031-23676-1

16. Y.S. Choi, S.G. Kim, H. Ki, Extreme polynomials and multilinear forms on l_1. J. Math. Anal. Appl. **228**(2), 467–482 (1998). https://doi.org/10.1006/jmaa.1998.6161
17. V. Dimant, D. Galicer, R. García, Geometry of integral polynomials, M-ideals and unique norm preserving extensions. J. Funct. Anal. **262**(5), 1987–2012 (2012)
18. V. Dimant, P. Sevilla-Peris, Summation of coefficients of polynomials on ℓ_p spaces. Publ. Mat. **60**(2), 289–310 (2016)
19. S. Dineen, *Complex Analysis on Infinite-dimensional Spaces*, Springer Monographs in Mathematics (Springer-Verlag London, London, 1999)
20. S. Dineen, Extreme integral polynomials on a complex Banach space. Math. Scand. **92**(1), 129–140 (2003)
21. V.V. Fávaro, D.M. Pellegrino, P. Rueda, On the size of the set of unbounded multilinear operators between Banach spaces. Linear Algebra Appl. **606**, 144–158 (2020)
22. J.L. Gámez-Merino, G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, Bounded and unbounded polynomials and multilinear forms: characterizing continuity. Linear Algebra Appl. **436**(1), 237–242 (2012)
23. J.L. Gámez-Merino, G.A. Muñoz-Fernández, V.M. Sánchez, J.B. Seoane-Sepúlveda, Inequalities for polynomials on the unit square via the Krein-Milman theorem. J. Convex Anal. **20**(1), 125–142 (2013)
24. R. Gonzalo, Multilinear forms, subsymmetric polynomials and spreading models on Banach spaces. J. Math. Anal. Appl. **202**, 379–397 (1996)
25. B.C. Grecu, Extreme 2-homogeneous polynomials on Hilbert spaces. Quaest. Math. **25**(4), 421–435 (2002). https://doi.org/10.2989/16073600209486027
26. B.C. Grecu, Geometry of 2-homogeneous polynomials on l_p spaces, $1 < p < \infty$. J. Math. Anal. Appl. **273**(2), 262–282 (2002). https://doi.org/10.1016/S0022-247X(02)00217-2
27. B.C. Grecu, Geometry of homogeneous polynomials on two-dimensional real Hilbert spaces. J. Math. Anal. Appl. **293**(2), 578–588 (2004). https://doi.org/10.1016/j.jmaa.2004.01.020
28. B.C. Grecu, Geometry of three-homogeneous polynomials on real Hilbert spaces. J. Math. Anal. Appl. **246**(1), 217–229 (2000). https://doi.org/10.1006/jmaa.2000.6783
29. B.C. Grecu, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, The unit ball of the complex $\mathcal{P}(^3H)$. Math. Z. **263**(4), 775–785 (2009). https://doi.org/10.1007/s00209-008-0438-y
30. B.C. Grecu, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, Unconditional constants and polynomial inequalities. J. Approx. Theory **161**(2), 706–722 (2009). https://doi.org/10.1016/j.jat.2008.12.001
31. P.R. Halmos, What does the spectral theorem say? Am. Math. Mon. **70**, 241–247 (1963)
32. L.A. Harris, Bounds on the derivatives of holomorphic functions of vectors, in *Analyse fonctionnelle et applications* (Comptes Rendus Colloq. Analyse, Inst. Mat., Univ Federal Rio de Janeiro, Rio de Janeiro, 1972) (Hermann, Paris, 1975), pp. 145–163. Actualités Aci. Indust., No. 1367. MR0477773
33. P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, M. MurilloArcila, J.B. Seoane- Sepúlveda, Sharp values for the constants in the polynomial Bohnenblust-Hille inequality. Linear Multilinear Algebra **64**(9), 1731–1749 (2015)
34. P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, Bernstein-Markov type inequalities and other interesting estimates for polynomials on circle sectors. Math. Inequal. Appl. **20**(1), 285–300 (2017). https://doi.org/10.7153/mia-20-21
35. P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, D.L. RodríguezVidanes, Geometry of spaces of homogeneous trinomials on $\mathbb{R}^2$. Banach J. Math. Anal. **15**(4), Paper No. 61, 22 (2021). https://doi.org/10.1007/s43037-021-00144-8
36. S. Kakutani, Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.). Ann. Math. (2) **42**, 994–1024 (1941)
37. S.G. Kim, Polarization and unconditional constants of $\mathcal{P}(^2d_*(1,w)^2)$. Commun. Korean Math. Soc. **29**(3), 421–428 (2014). https://doi.org/10.4134/CKMS.2014.29.3.421
38. S.G. Kim, The unit ball of $\mathcal{P}(^2D_*(1,W)^2)$. Math. Proc. R. Irish Acad. **111A**(2), 79–94 (2011)

39. S.G. Kim, Extreme 2-homogeneous polynomials on the plane with a hexagonal norm and applications to the polarization and unconditional constants. Stud. Sci. Math. Hungar. **54**(3), 362–393 (2017)
40. A.G. Konheim, T.J. Rivlin, Extreme points of the unit ball in a space of real polynomials. Am. Math. Mon. **73**, 505–507 (1966). https://doi.org/10.2307/2315472
41. M. Krein, D. Milman, On extreme points of regular convex sets. Stud. Math. **9**(1), 133–138 (1940)
42. R.S. Martin, Contributions to the Theory of Functionals. Dissertation (Ph.D.), Cal. Inst. of Tech. (1932). https:/doi.org/10.7907/9rtv-x659
43. L. Milev, N. Naidenov, Semidefinite extreme points of the unit ball in a polynomial space. J. Math. Anal. Appl. **405**(2), 631–641 (2013)
44. L. Milev, N. Naidenov, Strictly definite extreme points of the unit ball in a polynomial space. C. R. Acad. Bulg. Sci. **61**(11), 1393–1400 (2008)
45. G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, A. Weber, Supremum norms for 2-homogeneous polynomials on circle sectors. J. Convex Anal. **21**(3), 745–764 (2014)
46. G.A. Muñoz-Fernández, S. Gy Révész, J.B. Seoane-Sepúlveda, Geometry of homogeneous polynomials on non symmetric convex bodies. Math. Scand. **105**(1), 147–160 (2009). https://doi.org/10.7146/math.scand.a-15111
47. G.A. Muñoz-Fernández, V.M. Sánchez, J.B. Seoane-Sepúlveda, Estimates on the derivative of a polynomial with a curved majorant using convex techniques. J. Convex Anal. **17**(1), 241–252 (2010)
48. G.A. Muñoz-Fernández, V.M. Sánchez, J.B. Seoane-Sepúlveda, L^p-analogues of Bernstein and Markov inequalities. Math. Inequal. Appl. **14**(1), 135–145 (2011). https://doi.org/10.7153/mia-14-11
49. G.A. Muñoz-Fernández, Y. Sarantopoulos, J.B. Seoane-Sepúlveda, An application of the Krein-Milman theorem to Bernstein and Markov inequalities. J. Convex Anal. **15**(2), 299–312 (2008)
50. G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, Geometry of Banach spaces of trinomials. J. Math. Anal. Appl. **340**(2), 1069–1087 (2008). https://doi.org/10.1016/j.jmaa.2007.09.010
51. S. Neuwirth, The maximum modulus of a trigonometric trinomial. J. Anal. Math. **104**, 371–396 (2008). https://doi.org/10.1007/s11854-008-0028-2
52. R.A. Ryan, B. Turett, Geometry of spaces of polynomials. J. Math. Anal. Appl. **221**(2), 698–711 (1998)
53. Y. Sarantopoulos, Estimates for polynomial norms on $L_p(\mu)$ spaces. Math. Proc. Camb. Philos. Soc. **99**(2), 263–271 (1986)
54. K. Sundaresan, Geometry of spaces of quadratic forms, in *Inner Product Spaces and Applications*, Pitman Res. Notes Math. Ser., vol. 376 (Longman, Harlow, 1997), pp. 261–265
55. M.A. Toumi, A decomposition of orthogonally additive polynomials on Archimedean vector lattices. Bull. Belg. Math. Soc. Simon Stevin **20**(4), 621–638 (2013)