

Lexicographic Ranking Based on Minimal Winning Coalitions



M. Aleandri, V. Fragnelli, and S. Moretti

1 Introduction

The *desirability relation* for players in simple games Carreras and Freixas (1995); Freixas et al. (2012); Isbell (1958); Taylor and Zwicker (1999) has been widely studied also in connection with the property-driven analysis of power indices Diffo Lambo and Moulen (2002); Freixas and Gambarelli (1997); Holler and Nurmi (2013). A player i is in desirability relation with a player j (meaning that i is at least as desirable as j) if we can replace player j with player i in any winning coalition without changing the outcome, i.e. for any winning coalition S with $j \in S$ and $i \notin S$, we have that $S \setminus \{j\} \cup \{i\}$ is still a winning coalition. So, the desirability relation between i and j suggests that player i is at least as influential as player j , for it is never harmful for coalitions to replace j by i . Clearly, the desirability relation is not necessarily a total relation on the set of players, as two players may not be in any desirability relation (see, for example, Freixas and Pons (2005, 2008); Holler and Nurmi (2013) for an analysis of properties of the desirability relation on simple games). When the desirability relation is a total preorder the simple game is called complete and, for example, weighted games are complete, see Alonso-Meijide and Freixas (2010).

A classical property for power indices based on the desirability relation, is the monotonicity property: a power index is monotone if, whenever a player i is at least as desirable as player j , then the power of i is at least as much as the power of player j (see Remark 1). For instance, the Shapley-Shubik index Shapley and Shubik (1954), the Banzhaf index Banzhaf (1965), the Johnston index Johnston (1978), the

M. Aleandri
Luiss University, Viale Romania, 32, 00197 Rome, Italy
e-mail: maleandri@luiss.it

V. Fragnelli (✉)
Department of Sciences and Innovative Technologies (DISIT), University of Eastern Piedmont,
Viale T. Michel 11, 15121 Alessandria, Italy
e-mail: franco.fragnelli@gmail.com

S. Moretti
LAMSADE, CNRS, Université Paris-Dauphine, Université PSL, 75016 Paris, France
e-mail: stefano.moretti@dauphine.fr

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2023
M. A. Leroch and F. Rupp (eds.), *Power and Responsibility*,
https://doi.org/10.1007/978-3-031-23015-8_13

239

nucleolus Schmeidler (1969) and many other power indices satisfy the monotonicity property (see, for instance, Freixas and Gambarelli (1997)). Instead, it is easy to provide examples of simple games showing this property is not satisfied by other famous indices that take into account exclusively minimal winning coalitions, like the Deegan-Packel index (DPI) Deegan and Packel (1980) or the Public Good Index (PGI) Holler (1982); Holler and Packel (1983) (see Sect. 3 for some examples from the literature Deegan and Packel (1980); Freixas and Gambarelli (1997)).

The main objective of this paper is to show that it is possible to rank players consistently with the desirability relation and using exclusively minimal winning coalitions in an ordinal way. In fact, while classical power indices convert the information about coalitions into a numerical personal score representing players' relevance in a simple game, in many practical situations, having a reliable ranking to select the top players is enough and the information provided by players' score is only marginal. For instance, in the application of power indices to computational biology, the goal is short-listing the most relevant genes on complex networks with a huge number of nodes Moretti et al. (2007). In a similar way, ranking players are essential for the analysis of centrality of network elements with the goal to select the most critical or sensible parts of a system Lindelauf et al. (2013), or in studies aimed at establishing which agents are the strongest or the weakest in a voting system Fertö et al. (2020).

To that purpose, we introduce a *ranking solution* (formally, a map that associates to any simple game with player-set N , a total preorder on N) aimed at ranking players in a simple game according to their influence and in a way that is compatible with the desirability relation. Our ranking solution contains elements of both the DPI and the PGI, taking into account the minimal winning coalitions an individual belongs to. More exactly, given $n = |N|$ players, we first compute for each player a vector of n real numbers, where the k -th component of each vector is the number of minimal winning coalitions including the player of size k , with $k = 1, \dots, n$; second, our ranking solution lexicographically compares those real-valued vectors. Due to the similarity with the PGI to compute vectors components, we called such a ranking solution the *Lexicographic Ranking based on Minimal winning coalitions* (shortly, the *LRM*).

We show that the *LRM* is monotonic, and we prove that it is the unique solution satisfying (strong) monotonicity with respect to the desirability relation together with two other axioms: (1) the *coalitional anonymity* property, saying that the relative ranking between two players i and j in two different simple games should be independent of the identity of other players in minimal winning coalitions, provided that the number of minimal winning coalitions to which they belong in the two games is the same; (2) the property of *independence of larger minimal winning coalitions*, saying that once a player i is considered more influential than a player j in a simple game, player i will continue to be considered more influential than j in any simple game obtained by adding new "larger" minimal winning coalitions to the original game.

As a side-product of our analysis, we also point out some connections between our ranking solution and the criticality-based ranking provided in Aleandri et al. (2021) to compare, in an ordinal way, the blocking power of players and we explore some

similarities with the axioms used to characterize the dual version of the criticality-based ranking.

The paper is structured as follows. In Sect. 2 we provide some basic notions and notation. The definition of the *LRM* is then introduced in Sect. 3 together with some examples comparing it with the ranking defined by other solutions from the literature. An axiomatic characterization of the *LRM* is then presented and discussed in Sect. 4. A connection between the criticality-based ranking and the *LRM* is investigated in Sect. 5 using the desirability relation on dual games. Section 6 concludes.

2 Preliminaries and Notation

Given a finite set N , we denote by $|N|$ its *cardinality* and by $2^N = \{S \subseteq N\}$ its *power set*. A *simple game* is a pair (N, v) , where $N = \{1, 2, \dots, n\}$ denotes a finite set of players and $v : 2^N \rightarrow \{0, 1\}$ is a *characteristic function*, with $v(\emptyset) = 0$, $v(N) = 1$ and $v(S) \leq v(T)$ for all sets S, T such that $S \subseteq T \subseteq N$. A coalition $S \subseteq N$ such that $v(S) = 0$ is said a *losing coalition*, whereas a coalition $S \subseteq N$ such that $v(S) = 1$ is said a *winning coalition*. The class of simple games with N as the set of players is denoted by \mathcal{SG}^N .

Let \mathcal{W}^v be the set of winning coalitions in (N, v)

$$\mathcal{W}^v = \{S \subseteq N : v(S) = 1\}$$

and let \mathcal{W}_{\min}^v be the set of minimal winning coalitions in (N, v)

$$\mathcal{W}_{\min}^v = \text{Min } \mathcal{W}^v$$

where, for any family of sets \mathcal{F} , the Min operator on \mathcal{F} removes all non-inclusion-minimal sets of \mathcal{F} :

$$\text{Min } \mathcal{F} = \{F \in \mathcal{F} \mid \nexists G \in \mathcal{F} : G \subset F\}.$$

A simple game (N, v) is a *weighted majority game* if there exists a vector of non-negative real numbers $w \in \mathbb{R}_{\geq 0}^N$ and a quota $q \in \mathbb{R}_{\geq 0}$ such that a coalition $S \subseteq N$ is winning if and only if $\sum_{i \in S} w_i \geq q$.

In Holler (1982) the author introduced the *Public Good index (PGI)* of a player in a simple game, as the quotient between the number of minimal winning coalitions containing that player and the sum of cardinalities of all the minimal winning coalitions. Let (N, v) be a simple game, the PGI of player $i \in N$:

$$h_v(i) = \frac{|\mathcal{W}_{\min}^v(i)|}{\sum_{j \in N} |\mathcal{W}_{\min}^v(j)|}$$

where $\mathcal{W}_{\min}^v(i) = \{W \in \mathcal{W}_{\min}^v : i \in W\}$.

Example 1 Let $N = \{1, 2, 3, 4, 5\}$, and let

$$\mathcal{W}_{\min}^v = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{2, 4, 5\}, \{1, 4, 5\}\}.$$

We have

$$h_v(1) = \frac{3}{12}, h_v(2) = \frac{2}{12}, h_v(3) = \frac{2}{12}, h_v(4) = \frac{3}{12}, h_v(5) = \frac{2}{12}.$$

In Example 1 we can observe that player 3 and 5 have the same PGI, but they belong to minimal winning coalitions of different cardinality and whenever player 5 is winning together with some coalition $S \subseteq N \setminus \{3, 5\}$ then player 3 is winning together with the same coalition.

In Deegan and Packel (1980) the authors measure the power of a player according to the size of the minimal winning coalitions she belongs to. So, the *Deegan-Packel index (DPI)* for player i is defined as:

$$\delta_v(i) = \sum_{W \in \mathcal{W}_{\min}^v(i)} \frac{1}{|\mathcal{W}_{\min}^v|} \frac{1}{|W|}.$$

Example 2 Taking the same simple game (N, v) of Example 1 we have:

$$\delta_v(1) = \frac{8}{30}, \delta_v(2) = \frac{5}{30}, \delta_v(3) = \frac{6}{30}, \delta_v(4) = \frac{7}{30}, \delta_v(5) = \frac{4}{30}.$$

According to the DPI, player 3, for instance, has more power than player 5 because it belongs to two minimal winning coalitions of size smaller than the two minimal winning coalitions containing player 5.

3 A Ranking Solution and the Desirability Relation

Let us start recalling that a binary relation on N is a subset of $N \times N$. A reflexive, transitive and total binary relation on N is a *total preorder* (also called, a *ranking*) on N . We denote by \mathcal{T}^N the set of all total preorders on N . For instance, consider the lexicographic total preorder among vectors of real numbers:

$$\mathbf{x} \geq_L \mathbf{y} \quad \text{if either } \mathbf{x} = \mathbf{y} \quad \text{or } \exists k : x_t = y_t, t = 1, \dots, k - 1 \quad \text{and } x_k > y_k.$$

We define a *ranking solution* or, simply, a *solution*, as a map $R : \mathcal{SG}^N \rightarrow \mathcal{T}^N$ that associates to each simple game $v \in \mathcal{SG}^N$ a total preorder on N . The value assumed by a map R on a simple game v is the ranking on N denoted by R^v . We use the

notation $i R^v j$ to say that $(i, j) \in R^v$, and it means that i is at least as important as j according to ranking R^v , for all $i, j \in N$. We denote by I^v the symmetric part of R^v , i.e. $i I^v j$ means that $(i, j) \in R^v$ and $(j, i) \in R^v$ (i and j are equivalent), and by P^v its asymmetric part, i.e. $i P^v j$ means that $(i, j) \in R^v$ and $(j, i) \notin R^v$ (i is strictly more important than j).

Clearly, any real-valued N -vector numerically represents a total preorder over the player set N . Consequently, any power index $\phi : \mathcal{SG}^N \rightarrow \mathbb{R}^N$ underpins a ranking solution denoted by R_ϕ and such that $i R_\phi j \Leftrightarrow \phi_i(v) \geq \phi_j(v)$.

In this section, we introduce a new ranking solution for simple games based on minimal winning coalitions. The main idea of the new solution is that the smaller is the size of a minimal winning coalition, the larger is the power of its members. Therefore, the ranking of a player is positively correlated first to the size of minimal winning coalitions the player belongs to and, second, to their number.

To define the ranking solution, we need to introduce the notation i_k representing the number of minimal winning coalitions of size k containing i in a simple game (N, v) : $i_k = |\{S \in \mathcal{W}_{\min}^v : i \in S, |S| = k\}|$ for all $k \in \{1, \dots, n\}$. For each $i \in N$, let $\theta_v(i)$ be the n -dimensional vector $\theta_v(i) = (i_1, \dots, i_n)$ associated to v .

Definition 1 [Lexicographic Ranking based on Minimal winning coalitions (LRM)]
 The *Lexicographic Ranking based on Minimal winning coalitions (LRM)* solution is the function $R_l : \mathcal{SG}^N \rightarrow \mathcal{T}^N$ defined for any simple game $v \in \mathcal{SG}^N$ as

$$i R_l^v j \quad \text{if} \quad \theta_v(i) \geq_L \theta_v(j).$$

Let I_l^v and P_l^v be the symmetric part and the asymmetric part of R_l^v , respectively.

Example 3 Consider the simple game of Example 1. We have that

$$\begin{aligned} \theta_v(1) &= (0, 2, 1, 0, 0), & \theta_v(2) &= (0, 1, 1, 0, 0), \\ \theta_v(3) &= (0, 2, 0, 0, 0), & \theta_v(4) &= (0, 1, 2, 0, 0), \\ \theta_v(5) &= (0, 0, 2, 0, 0). \end{aligned}$$

So, the LRM solution R_l^v ranks the players as follows

$$1 P_l^v 3 P_l^v 4 P_l^v 2 P_l^v 5.$$

Notice that the ranking provided by the PGI and the DPI do not coincide with the ranking R_l^v on this example. In fact, for instance, $h_v(4) > h_v(3)$ and $\delta_v(4) > \delta_v(3)$, while $3 P_l^v 4$.

The LRM solution always provides a total preorder over the player set N for any simple game (N, v) . Instead, given a simple game (N, v) , the desirability relation Isbell (1958) is a preorder over the elements of N and is defined as follows.

Definition 2 Let (N, v) be a simple game. For any pair of players $i, j \in N$, the *desirability relation* $\geq^v \subseteq N \times N$ is defined as follows:

$$i \succeq^v j \Leftrightarrow [S \cup \{j\} \in \mathcal{W}^v \Rightarrow S \cup \{i\} \in \mathcal{W}^v \text{ for all } S \subseteq N \setminus \{i, j\}].$$

In the following, if the game v on which \succeq^v is defined is clear from the context, we denote relation \succeq^v simply by \succeq . For any $i, j \in N$, $i \succeq j$ is interpreted as player i is at least as desirable as player j (as a coalitional member); $i \succ j$ means that $i \succeq j$ and there exists a coalition $T \subseteq N \setminus \{i, j\}$ such that $T \cup \{i\} \in \mathcal{W}^v$ but $T \cup \{j\} \notin \mathcal{W}^v$, and it is interpreted as player i is (strictly) more desirable than player j ; $i \sim j$ means that $i \succeq j$ and $j \succeq i$, i.e. it is true that $S \cup \{j\} \in \mathcal{W}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}^v$ for all $S \subseteq N \setminus \{i, j\}$, and it is interpreted as players i and j are equally desirable.

As discussed in Sect. 1, the desirability relation, when it holds, represents a criterion to select between two players the most influential one, that is the player winning the maximum number of times. So, it is interesting to require the following basic property for ranking solutions.

Property 1 (*Desirable Monotonicity (DM)*) Let $i, j \in N$. For any $v \in \mathcal{SG}^N$, a solution R satisfies the desirable monotonicity property if

$$i \sim^v j \Rightarrow iI^v j,$$

and

$$i \succ^v j \Rightarrow iP^v j.$$

A solution satisfying the desirable monotonicity should strictly obey to the desirability relation: if the desirability relation between two players is strict (i.e., $i \succ^v j$), then a ranking solution should put such players in a strict relation too (i.e., $iP^v j$); of course, if two players are equally desirable (i.e., $i \sim^v j$) then the ranking solution must define the same kind of relation (i.e., $iI^v j$). Notice that this kind of “strong” monotonicity relation is not satisfied by the ranking over players represented by the nucleolus Schmeidler (1969), as it is easy to find examples of simple games having players in the symmetric part of the desirable relation and such that the allocation provided by the nucleolus is different (see, for instance, Freixas and Gambarelli (1997) page 600).

The rankings over players represented by the DPI and the PGI do not satisfy the desirable monotonicity property, as shown by the following example.

Example 4 Consider a weighted majority game (N, v) , $N = \{1, 2, 3, 4, 5\}$, with weight function $(4, 2, 1, 1, 1)$ and quota $q = 6$. So, the minimal winning coalitions are

$$\mathcal{W}_{\min}^v = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}.$$

We have that $1 \succ^v 2 \succ^v 3 \sim^v 4 \sim^v 5$. However, according to the PGI $h_2(v) = \frac{1}{11} < \frac{2}{11} = h_3(v)$, while according to the DPI we have $\delta_2(v) = \frac{1}{8} < \frac{1}{6} = \delta_3(v)$. So, according to the rankings underpinned by both indices, player 3 is ranked strictly higher than player 2.

On the other hand,

$$\begin{aligned}\theta_v(1) &= (0, 1, 3, 0, 0), & \theta_v(2) &= (0, 1, 0, 0, 0), \\ \theta_v(3) &= (0, 0, 2, 0, 0), & \theta_v(4) &= (0, 0, 2, 0, 0), \\ \theta_v(5) &= (0, 0, 2, 0, 0).\end{aligned}$$

So, $1 P_l^v 2 P_l^v 3 I_l^v 4 I_l^v 5: R_l^v$ and \succeq^v coincide.

In general, a total preorder provided by the LRM solution coincides with the desirability relation on any simple game where the desirability relation is total. This fact is an immediate consequence of the following proposition.

Proposition 1 *The LRM solution R_l fulfils the desirable monotonicity property.*

Proof Let (N, v) be a simple game. It is easy to verify that the condition

$$S \cup \{j\} \in \mathcal{W}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}^v \text{ for all } S \subseteq N \setminus \{i, j\} \quad (1)$$

is equivalent to the condition

$$S \cup \{j\} \in \mathcal{W}_{\min}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}_{\min}^v \text{ for all } S \subseteq N \setminus \{i, j\}. \quad (2)$$

We prove that $i \sim^v j \Rightarrow i I_l^v j$.

Since $i \sim^v j$, according to the equivalence between relations (1) and (2), we immediately have that $S \cup \{j\} \in \mathcal{W}_{\min}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}_{\min}^v$ for all $S \subseteq N \setminus \{i, j\}$. So, $\theta_v(i) = \theta_v(j)$, and therefore $i I_l^v j$.

Now, we prove that $i \succ^v j \Rightarrow i P_l^v j$.

Let be $i \succ^v j$ and define $\mathcal{T} = \{T \subseteq N \setminus \{i, j\} : T \cup \{i\} \in \mathcal{W}_{\min}^v, T \cup \{j\} \notin \mathcal{W}_{\min}^v\}$.

We first need to prove that $\mathcal{T} \neq \emptyset$. Since $i \succ^v j$, it must exist $T \subseteq N \setminus \{i, j\}$ such that $T \cup \{i\} \in \mathcal{W}^v$ and $T \cup \{j\} \notin \mathcal{W}^v$ and, by the equivalence between relation (1) and (2), it is not possible that $S \cup \{j\} \in \mathcal{W}_{\min}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}_{\min}^v$ for all $S \subseteq N \setminus \{i, j\}$. Moreover, again for $i \succ^v j$, it is not possible that there exists $S \subseteq N \setminus \{i, j\}$ such that $S \cup \{i\} \notin \mathcal{W}_{\min}^v$ and $S \cup \{j\} \in \mathcal{W}_{\min}^v$. So, it must exist $T \subseteq N \setminus \{i, j\}$ such that $T \cup \{i\} \in \mathcal{W}_{\min}^v$ and $T \cup \{j\} \notin \mathcal{W}_{\min}^v$.

Now, let $k = \min\{|T| : T \in \mathcal{T}\}$. If $k = 0$, we immediately have that $\{i\} \in \mathcal{W}_{\min}^v$ and $\{j\} \notin \mathcal{W}_{\min}^v$, so $i P_l^v j$.

Consider the case $k > 0$. By the minimality of k we have that $i_t = j_t$ for all $t = 0, \dots, k - 1$ and $i_k > j_k$ and so $i P_l^v j$. \square

Remark 1 It is well known from the literature that the desirability relation on weighted majority games is a total preorder and that the following monotonicity condition w.r.t. weights holds for a weighted majority game (N, v) with weights (w_1, \dots, w_n) :

$$w_i \geq w_j \Rightarrow i \succeq^v j,$$

for all $i, j \in N$ (see for instance Freixas and Gambarelli (1997)). As a direct consequence of Proposition 1 we have that also the LRM solution on weighted majority games is monotonic w.r.t. weights, that is $w_i \geq w_j \Rightarrow i R_l^j$ for all $i, j \in N$.

4 An Axiomatic Characterization of the LRM Solution

Now, we introduce two new properties for ranking solutions that are inspired by similar properties introduced in Aleandri et al. (2021) on the sets of blocking coalitions.

The next property says that winning coalitions of the same size should have the same impact on the ranking, independently of their members.

Property 2 (*Anonymity of Minimal Winning Coalitions (AMWC)*) Let $i, j \in N$, $v, v_\pi \in \mathcal{SG}^N$ and let π be a bijection on $2^{N \setminus \{i,j\}}$ with $|\pi(S)| = |S|$ and such that

$$S \cup \{i\} \in \mathcal{W}_{\min}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}_{\min}^{v_\pi}$$

and

$$S \cup \{j\} \in \mathcal{W}_{\min}^v \Leftrightarrow \pi(S) \cup \{j\} \in \mathcal{W}_{\min}^{v_\pi},$$

for all $S \in 2^{N \setminus \{i,j\}}$. A solution R satisfies the *anonymity of minimal winning coalitions* property if

$$i R^v j \Leftrightarrow i R^{v_\pi} j.$$

Example 5 Consider the weighted majority game (N, v) of Example 4 and the players 3 and 4 in the role of players i and j of the definition of Property 2. Define a bijection π on $2^{\{1,2,5\}}$ such that $\pi(\{1, 5\}) = \{2, 5\}$. So the simple game (N, v_π) is such that

$$\mathcal{W}_{\min}^{v_\pi} = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 4, 5\}\}.$$

Game v differs from v_π in terms of minimal winning coalitions just for coalition $\{1, 4, 5\}$ which is replaced in v_π by the minimal winning coalition $\{2, 4, 5\}$. Nevertheless, the number of minimal winning coalitions of each size containing player 4 in game in v_π is precisely as in game in v , so her capacity to form minimal winning coalitions should not be affected (assuming that the other players are equally inclined to form minimal winning coalitions with 4). So, the property of Anonymity of Minimal Winning Coalitions says that the relative ranking between 3 and 4 in v should be the same as in v_π .

Property 2 reflects a broadly adopted principle, satisfied by classical power indices like the Shapley-Shubik index Shapley and Shubik (1954), the Banzhaf index Banzhaf (1965) and all semivalues Dubey et al. (1981), saying that coalitions of the same size are equally likely. So, it seems compelling to assume that the relative

position of two players is not affected by permutations preserving the size of minimal winning coalitions containing them, as it is required by Property 2.

Another property we consider in our analysis is the one of independence of larger minimal winning coalitions, saying that, once a solution exists, in which a player i is ranked strictly better than a player j , adding “larger” minimal winning coalitions should not affect the relative ranking between i and j .

Property 3 (*Independence of Larger Minimal Winning Coalitions (ILMWC)*) Let $i, j \in N$. For any $v \in \mathcal{SG}^N$, let $h = \max\{|S| : S \in \mathcal{W}_{\min}^v \text{ and } S \cap \{i, j\} \neq \emptyset\}$ be the highest cardinality of coalitions in the set \mathcal{W}_{\min}^v containing either i or j . Let \mathcal{S}_h be a collection of (minimal) winning coalitions with cardinality strictly larger than h , i.e., $\mathcal{S}_h = \{S_1, \dots, S_r\}$ such that $S_k \subseteq N$, $|S_k| > h$ for $k = 1, \dots, r$ and there is no $Q \in \mathcal{W}_{\min}^v \cup \mathcal{S}_h$ with $Q \subset S_k$, for all $k \in \{1, \dots, r\}$. A solution R satisfies the *independence of larger minimal winning coalitions* property if

$$i P^v j \Rightarrow i P^{v'} j,$$

where v' is a simple game such that the set of minimal winning coalitions is obtained as $\mathcal{W}_{\min}^{v'} = \mathcal{W}_{\min}^v \cup \mathcal{S}_h$.

Example 6 Consider again the weighted majority game (N, v) of Example 4 and the player 1 and 2 in the role of players i and j of the definition of Property 3. Let $\mathcal{S}_h = \{\{2, 3, 4, 5\}\}$ and consider a new simple game (N, v') such that

$$\mathcal{W}_{\min}^{v'} = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4, 5\}\}.$$

Notice that the new simple game v' contains one more minimal winning coalition containing 2 but not 1, but the size of such a minimal winning coalition in v' is strictly larger than the size of any minimal winning coalition in v , and therefore is considered less likely to form. If a solution satisfying the property of independence of larger minimal winning coalitions ranks 1 strictly better than 2 in the simple game v , in v' the solution also must rank 1 strictly better than 2: the new (and larger) minimal winning coalition does not affect the strict ranking decided on the basis of smaller minimal winning coalitions.

In collective decision-making bodies, forming large winning coalitions in practice may result more difficult than forming small ones due to many factors, like the presence of complex institutional rules, the need of mediators in the decision-making process, higher negotiation costs or other “psychological” aspects, like contrasting political positions of their members. As a consequence, it is crucial to emphasize the impact of minimal winning coalitions of small size, as demanded by Property 3, which preserves strict rankings after the addition of large minimal winning coalitions.

Proposition 2 Let R be a solution satisfying Properties 1 (DM) and 2 (AMWC). Then for any simple game v and $i, j \in N$ such that $\theta_v(i) = \theta_v(j)$ we have that $i I^v j$.

Proof Since $\theta_v(i) = \theta_v(j)$, we have that $i_k = j_k$ for all $k \in \{1, \dots, n\}$. Define a bijection π on $2^{N \setminus \{i, j\}}$ such that for each $k \in \{1, \dots, n - 1\}$ and for each coalition $S \in 2^{N \setminus \{i, j\}}$ of size $k - 1$ with $S \cup \{j\} \in \mathcal{W}_{\min}^v$, $\pi(S) = T$, where $T \in 2^{N \setminus \{i, j\}}$ is a coalition of size $k - 1$, with $T \cup \{i\} \in \mathcal{W}_{\min}^v$. Consider a game v_π such that $S \cup \{i\} \in \mathcal{W}_{\min}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}_{\min}^{v_\pi}$ and $S \cup \{j\} \in \mathcal{W}_{\min}^v \Leftrightarrow \pi(S) \cup \{j\} \in \mathcal{W}_{\min}^{v_\pi}$. So, we have that for all $k \in \{1, \dots, n\}$ and all coalitions $T \in 2^{N \setminus \{i, j\}}$ of size $k - 1$ with $T \cup \{i\} \in \mathcal{W}_{\min}^v$

$$T \cup \{i\} \in \mathcal{W}_{\min}^{v_\pi} \Leftrightarrow T \cup \{j\} \in \mathcal{W}_{\min}^{v_\pi}.$$

Then, $i \sim^{v_\pi} j$ (players i and j are equally desirable in v_π) and, by Property 1

$$i I^{v_\pi} j. \tag{3}$$

Notice that i, j, π, v and v_π satisfy the conditions for bijections demanded in the statement of Property 2, with v_π such that the set of minimal winning coalitions of v_π is

$$\mathcal{W}_{\min}^{v_\pi} = \bigcup_{T \in 2^{N \setminus \{i, j\}} \text{ s.t. } T \cup \{i\} \in \mathcal{W}_{\min}^v} \{T \cup \{i\}, T \cup \{j\}\}$$

(notice that the minimality of the elements in $\mathcal{W}_{\min}^{v_\pi}$ is guaranteed by the minimality of the elements in \mathcal{W}_{\min}^v). So, since R satisfies Property 2, we have that

$$i I^v j \Leftrightarrow i I^{v_\pi} j. \tag{4}$$

So, by relation (3), we have $i I^v j$, which concludes the proof. \square

Theorem 1 *The LRM solution R_l is the unique solution that fulfils Properties 1 (DM), 2 (AMWC) and 3 (ILMWC).*

Proof By Proposition 1 we have that R_l fulfils Property 1 (DM). It is easy to check that it also fulfils Properties 2 (AMWC) and 3 (ILMWC) (it directly follows from Definition 1 and the lexicographic relation).

To show that R_l is the unique index fulfilling Properties 1 (DM), 2 (AMWC) and 3 (ILMWC), we need to prove that, if a solution $R : \mathcal{SG}^N \rightarrow \mathcal{T}^N$ satisfies Properties 1 (DM), 2 (AMWC) and 3 (ILMWC), then $i R_l^v j \Leftrightarrow i R^v j$ or, equivalently, $i P_l^v j \Leftrightarrow i P^v j$ and $i I_l^v j \Leftrightarrow i I^v j$.

We first prove that $i P_l^v j \Leftrightarrow i P^v j$:

(\Rightarrow)

Let $i P_l^v j$. By Definition 3, let k' be the smallest integer in $\{1, \dots, n\}$ with $i_{k'} > j_{k'}$. Let $s = i_{k'} - j_{k'}$ and $\mathcal{S}_{k'}^i = \{S \in \mathcal{W}_{\min}^v : |S| = k' \text{ and } S \cap \{i, j\} = i\}$ be a subset of coalitions in \mathcal{W}_{\min}^v of size k' containing i but not j such that $|\mathcal{S}_{k'}^i| = s$. Moreover, let $\Sigma = \{S \in \mathcal{W}_{\min}^v : |S| > k'\}$ be the set of coalitions in \mathcal{W}_{\min}^v with cardinality strictly larger than k' .

Consider a new simple game v' such that $\mathcal{W}_{\min}^{v'} = \mathcal{W}_{\min}^v \setminus \Sigma$, and the set of minimal winning coalitions containing j (of size at most k') in $\mathcal{W}_{\min}^{v'}$:

$$\mathcal{S}^j = \{S \cup \{j\} : S \in 2^{N \setminus \{i,j\}} \text{ with } S \cup \{j\} \in \mathcal{W}_{\min}^{v'}\}.$$

Define a bijection π on $2^{N \setminus \{i,j\}}$ such that for each $t \in \{1, \dots, k'\}$ and for each coalition $S \in 2^{N \setminus \{i,j\}}$ of size $t - 1$ with $S \cup \{j\} \in \mathcal{W}_{\min}^{v'}$, $\pi(S) = T$, where $T \in 2^{N \setminus \{i,j\}}$ is a coalition of size $t - 1$, with $T \cup \{i\} \in \mathcal{W}_{\min}^{v'}$. So, the set of minimal winning coalitions contained in \mathcal{S}^j after the transformation via π is:

$$\mathcal{T}^j = \{\pi(S) \cup \{j\} : S \in 2^{N \setminus \{i,j\}} \text{ with } S \cup \{j\} \in \mathcal{S}^j\}.$$

Consider a new game \hat{v}_π such that

$$\mathcal{W}_{\min}^{\hat{v}_\pi} = \left(\mathcal{W}_{\min}^{v'} \setminus \mathcal{S}^j \right) \cup \mathcal{T}^j.$$

So, we have that for all coalitions $T \in 2^{N \setminus \{i,j\}}$ of size $t - 1$, $t \in \{1, \dots, k'\}$,

$$T \cup \{j\} \in \mathcal{W}_{\min}^{\hat{v}_\pi} \Rightarrow T \cup \{i\} \in \mathcal{W}_{\min}^{\hat{v}_\pi},$$

and, consequently, for all $S \in 2^{N \setminus \{i,j\}}$,

$$S \cup \{j\} \in \mathcal{W}_{\min}^{\hat{v}_\pi} \Rightarrow S \cup \{i\} \in \mathcal{W}_{\min}^{\hat{v}_\pi},$$

which means that $i \succeq_{\hat{v}_\pi} j$. So, by Property 1 (DM), we have that $i P^{\hat{v}_\pi} j$.

On the other hand,

$$S \cup \{i\} \in \mathcal{W}_{\min}^{v'} \Leftrightarrow S \cup \{i\} \in \mathcal{W}_{\min}^{\hat{v}_\pi}$$

and

$$S \cup \{j\} \in \mathcal{W}_{\min}^{v'} \Leftrightarrow \pi(S) \cup \{j\} \in \mathcal{W}_{\min}^{\hat{v}_\pi},$$

for all $S \in 2^{N \setminus \{i,j\}}$, and therefore, by Property 2 on R applied to v' and \hat{v}_π , we also have $i P^{v'} j$.

Finally, by Property 3 on R (with v' in the role of v in the statement of Property 3), we have that $i P^v j$, as $\mathcal{W}_{\min}^v = \mathcal{W}_{\min}^{v'} \cup \Sigma$.

(\Leftarrow)

Let $i P^v j$. Suppose that $i I_1^v j$. Then, by Definition 3, $\theta_v(i) = \theta_v(j)$. So, by Proposition 2, $i I^v j$, which yields a contradiction with $i P^v j$. Since it can't even be $j P_1^v i$ (by the other implication proved above), and by the fact that P_1^v is a total relation, it must be $i P_1^v j$.

We now prove that $i I_1^v j \Leftrightarrow i I^v j$:

(\Rightarrow)

Let $i I_1^v j$. Then, by Definition 3, $\theta_v(i) = \theta_v(j)$. So, by Proposition 2 and the fact that R^v satisfies Properties 1 and 2, $i I^v j$.

(\Leftarrow)

Let $i I^v j$. As we have shown previously, $i P_l^v j \Leftrightarrow i P^v j$. So it is not possible that $i P_l^v j$ or $j P_l^v i$. Since P_l is a total relation, it must be $i I_l^v j$, which concludes the proof. \square

We end this section showing the logical independence of Properties 1, 2 and 3.

Example 7 [No Property 1] Given $i, j \in N$, consider the ranking solution R_{DM} defined by

$$i R_{DM}^v j \quad \text{iff} \quad v(\{i\}) \geq v(\{j\}).$$

This solution satisfies all the Properties but Property 1.

Example 8 [No Property 2] For any $i \in N$, let $B(i)$ the the largest player index within minimal winning coalitions containing player i , i.e.

$$B^v(i) = \max_{S \in \mathcal{W}_{\min}^v, i \in S} \left(\min_{j \in S \setminus \{i\}} j \right).$$

Consider the ranking solution R_{AMWC}^v such that

$$\begin{cases} i I_{AMWC}^v j & \text{if } i \sim^v j, \\ i P_{AMWC}^v j & \text{if } \theta_v(i) >_L \theta_v(j), \\ i P_{AMWC}^v j & \text{if } (i, j) \notin \succeq^v, (j, i) \notin \succeq^v, \theta_v(i) = \theta_v(j) \text{ and } B^v(i) > B^v(j), \end{cases}$$

This solution satisfies all the Properties but Property 2.

[It is clear that R_{AMWC}^v satisfies properties 1 and 3. To see that R_{AMWC}^v does not satisfy Property 2, consider, for instance, games v and v_π of Example 5. As we noticed, a solution satisfying Property 2 should rank players 3 and 4 in the same way in both games v and v_π . However, since $B^v(3) = B^v(4) = 1$ in v and $B^{v_\pi}(3) = 1$ and $B^{v_\pi}(4) = 2$ (and the two players are not in a desirable relation in both games) we have that $3 I_{AMWC}^v 4$, while $4 P_{AMWC}^{v_\pi} 3$.]

Example 9 [No Property 3] For each $i \in N$, let $\bar{\theta}_v$ be the n -dimensional vector $\bar{\theta}_v(i) = (i_n, \dots, i_1)$ associated to v . Given $i, j \in N$, consider the vector ranking solution R_{ILMWC}^v such that

$$\begin{cases} i P_{ILMWC}^v j & \text{if } i \succ^v j, \\ i I_{ILMWC}^v j & \text{if } \theta_v(i) = \theta_v(j), \\ i P_{ILMWC}^v j & \text{if } (i, j) \notin \succeq^v, (j, i) \notin \succeq^v \text{ and } \bar{\theta}_v(i) >_L \bar{\theta}_v(j), \end{cases}$$

This solution satisfies all the Properties but Property 3.

[It is easy to verify that R_{ILMWC}^v satisfies Properties 1 and 2. To see that R_{ILMWC}^v does not satisfy property 3, consider, for instance, games v and v' of Example 6. Notice that $1 \succ^v 2$ and, so, $1 P_{ILMWC}^v 2$. However, in game v' , $(1, 2) \notin \succeq^{v'}$ and $(2, 1) \notin \succeq^{v'}$ (1 and 2 are not in desirable relation), while

$$\bar{\theta}_{v'}(2) = (0, 1, 0, 1, 0) >_L (0, 0, 3, 1, 0) = \bar{\theta}_{v'}(1)$$

and therefore $2 P_{ILMWC}^{v'} 1$.]

5 Duality

In this section we investigate the connections between the LRM solution and the criticality-based ranking introduced in Aleandri et al. (2021) to rank players in a simple game. In Aleandri et al. (2021) a ranking over players is defined according to the power of blocking the grand coalition to be winning. Given a simple game (N, v) a coalition $B \subseteq N$ is called *blocking* coalition for N if $v(N \setminus B) = 0$. Let \mathcal{B}^v be the set of all blocking coalitions in the game (N, v) and let \mathcal{B}_{\min}^v be the set of all minimal blocking coalitions $\mathcal{B}_{\min}^v = \text{Min } \mathcal{B}^v$. Denote by i_k^* the number of minimal blocking coalitions (for N) of size k containing player i , so $i_k^* = |\{B \in \mathcal{B}_{\min}^v : i \in B, |B| = k\}|$ for all $k \in \{1, \dots, n\}$. For each $i \in N$, let $\theta_v^*(i)$ be the n -dimensional vector $\theta_v^*(i) = (i_1^*, \dots, i_n^*)$ associated to v .

The criticality-based ranking is based on the idea that the smaller is the size of a blocking coalition, the larger is the influence on the blocking power of its members; the ranking of a player in terms of blocking power is positively correlated first to the size of minimal blocking coalitions the player belongs to and second to their number.

Definition 3 The *criticality-based solution* is the function $R_c : \mathcal{SG}^N \longrightarrow \mathcal{T}^N$ defined for any simple game $v \in \mathcal{SG}^N$ as

$$i R_c^v j \quad \text{if} \quad \theta_v^*(i) \geq_L \theta_v^*(j).$$

Let I_c^v and P_c^v be the symmetric part and the asymmetric part of R_c^v , respectively.

Example 10 Consider the simple game of Example 1 then we have that

$$\mathcal{B}_{\min}^v = \{\{1, 4\}, \{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}.$$

Therefore,

$$\begin{aligned} \theta_v^*(1) &= (0, 1, 2, 0, 0), & \theta_v^*(2) &= (0, 0, 3, 0, 0), \\ \theta_v^*(3) &= (0, 0, 4, 0, 0), & \theta_v^*(4) &= (0, 1, 1, 0, 0), \\ \theta_v^*(5) &= (0, 0, 2, 0, 0). \end{aligned}$$

So, the criticality-based ranking is such that

$$1 P_c^v 4 P_c^v 3 P_c^v 2 P_c^v 5.$$

We first show that the LRM coincides with the criticality-based ranking of the dual game.

Proposition 3 *Let (N, v) be a simple game. Then $R_l^v = R_c^{v^*}$.*

Proof Given the simple game (N, v) its dual v^* is defined by

$$v^*(S) = v(N) - v(N \setminus S), \quad (5)$$

for each coalition $S \in 2^N$. The proposition follows recalling that $\mathcal{W}_{\min}^{v^*} = \mathcal{B}_{\min}^v$, as proved in Proposition 3 in Aleandri et al. (2021), and then $\theta_v = \theta_{v^*}$. \square

On the other hand, it is also interesting to study under which conditions the LRM and the criticality-based ranking coincide. To this purpose, we analyse the behaviour of the desirability relation on a simple game v and its dual v^* .

Proposition 4 *Given a simple game (N, v) and the dual game (N, v^*) then, $\forall i, j \in N, i \neq j$*

$$i \succeq^v j \iff i \succeq^{v^*} j.$$

Proof \Rightarrow By hypothesis, for all $S \subseteq N \setminus \{i, j\}$, $S \cup \{j\} \in \mathcal{W}^v$ implies that $S \cup \{i\} \in \mathcal{W}^v$. We want to prove that for all $S \subseteq N \setminus \{i, j\}$ such that $N \setminus \{S \cup \{j\}\} \notin \mathcal{W}^v$ implies that $N \setminus \{S \cup \{i\}\} \notin \mathcal{W}^v$. Suppose that $N \setminus \{S \cup \{i\}\}$ is winning then define $T := N \setminus \{S \cup \{i, j\}\}$. We observe that $T \cup \{j\}$ is winning then $T \cup \{i\} = N \setminus \{S \cup \{j\}\}$ is winning, i.e. a contradiction.

\Leftarrow By hypothesis, $\forall S \subseteq N \setminus \{i, j\}$, $N \setminus \{S \cup \{j\}\} \notin \mathcal{W}^v$ implies that $N \setminus \{S \cup \{i\}\} \notin \mathcal{W}^v$. We want to prove that for all $S \subseteq N \setminus \{i, j\}$, $S \cup \{j\} \in \mathcal{W}^v$ implies that $S \cup \{i\} \in \mathcal{W}^v$. Suppose $S \cup \{i\} \notin \mathcal{W}^v$ and let $T = N \setminus \{S \cup \{i, j\}\}$. We observe that on the one hand $N \setminus \{T \cup \{i\}\}$ is not winning, but on the other hand $N \setminus \{T \cup \{i\}\} = S \cup \{j\}$ is winning, i.e. a contradiction, and the proof is complete. \square

Corollary 1 *Let (N, v) be a simple game such that the desirability relation is total. Then, $R_l^v = R_c^v$. Moreover the LRM and the criticality-based ranking are self-dual.*

Example 11 Consider the weighted majority game (N, v) of Example 4. The minimal winning and blocking coalitions are

$$\mathcal{W}_{\min}^v = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\},$$

$$\mathcal{B}_{\min}^v = \{\{1\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\}.$$

We have that

$$\begin{aligned} \theta_v^*(1) &= (1, 0, 0, 0, 0), & \theta_v^*(2) &= (0, 0, 3, 0, 0), \\ \theta_v^*(3) &= (0, 0, 2, 0, 0), & \theta_v^*(4) &= (0, 0, 2, 0, 0), \\ \theta_v^*(5) &= (0, 0, 2, 0, 0). \end{aligned}$$

The criticality-based ranking is such that

$$1 P_c^v \ 2 P_c^v \ 3 I_c^v \ 4 I_c^v \ 5.$$

So, as expected for a weighted majority game in which the desirable relation is total, $R_c^v = R_c^v$.

With the purpose of ranking players in a simple game according to their influence in the process of forming blocking coalitions, it seems natural to look at a dual version of Property 1.

Property 4 (*Dual Desirable Monotonicity (DDM)*) Let $i, j \in N$. For any $v \in \mathcal{SG}^N$, a solution R satisfies the *dual desirable monotonicity* property if

$$i \sim^{v^*} j \Rightarrow i I^v j,$$

and

$$i \succ^{v^*} j \Rightarrow i P^v j.$$

A solution satisfying Property 4 obeys to the desirability relation defined on dual games, specifying that i is at least as desirable as j if we can replace player j with player i in any blocking coalition (instead of in any winning one). In a similar fashion, Properties 2 and 3 can be reformulated as their following dual counterparts.

Property 5 (*Dual Anonymity of Minimal Winning Coalitions (DAMWC)*) Let $i, j \in N$, $v, v_\pi \in \mathcal{SG}^N$ and let π be a bijection on $2^{N \setminus \{i, j\}}$ with $|\pi(S)| = |S|$ and such that

$$S \cup \{i\} \in \mathcal{W}_{\min}^{v^*} \Leftrightarrow S \cup \{i\} \in \mathcal{W}_{\min}^{v_\pi^*}$$

and

$$S \cup \{j\} \in \mathcal{W}_{\min}^{v^*} \Leftrightarrow \pi(S) \cup \{j\} \in \mathcal{W}_{\min}^{v_\pi^*},$$

for all $S \in 2^{N \setminus \{i, j\}}$. A solution R satisfies the *anonymity of minimal winning coalitions* property if

$$i R^v j \Leftrightarrow i R^{v_\pi} j.$$

Property 6 (*Independence of Larger Minimal Winning Coalitions in the Dual (ILMWCD)*) Let $i, j \in N$. For any $v \in \mathcal{SG}^N$, let $h = \max\{|S| : S \in \mathcal{W}_{\min}^v \text{ and } S \cap \{i, j\} \neq \emptyset\}$ be the highest cardinality of coalitions in the set $\mathcal{W}_{\min}^{v^*}$ containing either i or j . Let \mathcal{S}_h be a collection of (minimal) winning coalitions in the dual game v^* with cardinality strictly larger than h , i.e., $\mathcal{S}_h = \{S_1, \dots, S_r\}$ such that $S_k \subseteq N$, $|S_k| > h$ for $k = 1, \dots, r$ and there is no $Q \in \mathcal{W}_{\min}^{v^*} \cup \mathcal{S}_h$ with $Q \subset S_k$, for all $k \in \{1, \dots, r\}$. A solution R satisfies the property of *independence of larger minimal winning coalitions in the dual* if

$$i P^v j \Rightarrow i P^{v'} j,$$

where v' is a simple game such that the set of minimal winning coalitions is obtained as $\mathcal{W}_{\min}^{v'^*} = \mathcal{W}_{\min}^{v^*} \cup \mathcal{S}_h$.

We can state the following result.

Theorem 2 *The solution R_V such that $R_V^v = R_V^{v^*}$ for all $v \in \mathcal{W}_{\min}^v$ is the unique solution that fulfils Properties 4, 5 and 6.*

Proof The proof follows the same steps of the proof of Theorem 1, with v^* in the role of v . □

By Proposition 3, and the fact that $(v^*)^* = v$ (the dual of the dual of a game v equals game v), we have that $R_V = R_C$. Moreover, by Proposition 4, we have that Properties 1 and 4 are equivalent. So, the following corollary holds.

Corollary 2 *The criticality-based solution R_C is the unique solution that fulfils Properties 1, 5 and 6.*

In Aleandri et al. (2021), the criticality-based solution has been axiomatically characterized using four properties, namely, *Players' Anonymity, Dual Coalitional Anonymity, Dual Monotonicity and Independence of Higher Cardinalities*(see Sects. 4 in Aleandri et al. (2021) for a formal definitions of these axioms). Notice that Property 5 coincides with the property of *Dual Coalitional Anonymity* in Aleandri et al. (2021), while Property 6 coincides with the property of *Independence of Higher Cardinalities* in Aleandri et al. (2021). So, according to Corollary 2, Property 1 replaces properties of *Players' Anonymity and Dual Monotonicity* in the axiomatic characterization of the criticality-based solution presented in Aleandri et al. (2021).

Example 12 Consider the simple game (N, v) in Example 2.7 in Alonso-Meijide and Freixas (2010):

$$\mathcal{W}_{\min}^v = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{3, 4, 5\}\};$$

$$\mathcal{B}_{\min}^v = \{\{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{3, 4, 5\}\};$$

The LRM solution in v is

$$1 P_\ell^v \ 2 I_\ell^v \ 3 I_\ell^v \ 4 P_\ell^v \ 5.$$

and the LRM solution in v^* is

$$1 P_\ell^{v^*} \ 3 I_\ell^{v^*} \ 4 P_\ell^{v^*} \ 2 P_\ell^{v^*} \ 5.$$

Then if the desirability relation is not a total preorder the LRM solution in not self-dual. The same result holds for the criticality-based solution.

We conclude this section pointing out that the axioms of *Dual Coalitional Monotonicity* and of *Players' Anonymity* are replaced by *Desirable Monotonicity* (Property 1) in the characterization of the criticality-based solution in Aleandri et al. (2021), as

shown by Corollary 2, but the two axioms do not imply Property 1. In fact, as shown by Example 4, the ranking over players represented by the PGI does not satisfy Property 1 (we leave to the reader to check that such a ranking satisfy both Dual Coalitional Monotonicity and Players' Anonymity axioms).

6 Conclusions

In this paper, using the minimal winning coalitions of a simple game, we introduced a new ranking among players that satisfies the desirability relation: the Lexicographic Ranking based on Minimal winning coalitions. The players are ranked according to the size of the minimal winning coalitions they belong to and then to the number of such coalitions. The ranking solution satisfies the coalitional anonymity property and the independence of larger minimal winning coalitions property that together with a monotonicity property rooted on the desirability relation uniquely characterized it. Looking at the dual game, we prove that there is a relation between the Lexicographic Ranking based on Minimal winning coalitions and the criticality-based ranking and, consequently, between ranking players according to their power to win and to their power to block the grand coalition. In particular, if the desirable relation is total the two rankings coincide.

Following this line of research, it would be interesting to delve more into the connection between the power to initiate and the power to block a winning coalition Deegan and Packel (1980), in particular, when the desirability relation between two players does not hold.

Acknowledgements The authors want to thank professor Marco Dall'Aglio for the valuable discussions and the anonymous referee for the useful comments. S. Moretti gratefully acknowledges the support of the ANR project THEMIS (ANR-20-CE23-0018).

References

- Aleandri, M., Dall'Aglio, M., Fragnelli, V., & Moretti, S. (2021). Minimal winning coalitions and orders of criticality. *Annals of Operations Research*. <https://doi.org/10.1007/s10479-021-04199-6>
- Alonso-Mejide, J. M., & Freixas, J. (2010). A new power index based on minimal winning coalitions without any surplus. *Decision Support Systems*, 49(1), 70–76.
- Banzhaf, J. (1965). Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19, 317–343.
- Carreras, F., & Freixas, J. (1995). Complete simple games. *Mathematical Social Sciences*, 32, 139–155.
- Deegan, J., & Packel, E. W. (1980). An axiomated family of power indices for simple n-person games. *Public Choice*, 35, 229–239.
- Deegan, J., & Packel, E. W. (1980). A new index of power for simple n-person games. *International Journal of Game Theory*, 7, 113–123.

- Diffo Lambo, L., & Moulen, J. (2002). Ordinal equivalence of power notions in voting games. *Theory and Decision*, 53, 313–325.
- Dubey, P., Neyman, A., & Weber, R. (1981). Value theory without efficiency. *Mathematics of Operations Research*, 6(1), 122–128.
- Fertő, I., Kóczy, L. Á., Kovács, A., & Sziklai, B. R. (2020). The power ranking of the members of the Agricultural Committee of the European Parliament. *European Review of Agricultural Economics*, 47(5), 1897–1919.
- Freixas, J., & Gambarelli, G. (1997). Common internal properties among power indices. *Control and Cybernetics*, 26, 591–604.
- Freixas, J., Marciniak, D., & Pons, M. (2012). On the ordinal equivalence of the Johnston, Banzhaf and Shapley power indices. *European Journal of Operational Research*, 216(2), 367–375.
- Freixas, J., & Pons, M. (2005). Two measures of circumstantial power: Influences and bribes. *Homo Oeconomicus*, 22, 569–588.
- Freixas, J., & Pons, M. (2008). Circumstantial power: Optimal persuadable voters. *European Journal of Operational Research*, 186(3), 1114–1126.
- Holler, M. J. (1982). Forming coalitions and measuring voting power. *Political Studies*, 30, 262–271.
- Holler, M. J., & Packel, E. W. (1983). Power, luck and the right index. *Journal of Economics*, 43, 21–29.
- Holler, M. J., & Nurmi, H. (Eds.). (2013). *Power, voting, and voting power: 30 years after*. Springer.
- Isbell, J. R. (1958). A class of simple games. *Duke Mathematical Journal*, 25, 423–439.
- Johnston, R. J. (1978). On the measurement of power: some reactions to Laver. *Environment and Planning A*, 10(8), 907–914.
- Lindelauf, R. H., Hamers, H. J., & Husslage, B. G. M. (2013). Cooperative game theoretic centrality analysis of terrorist networks: The cases of jemaah islamiyah and al Qaeda. *European Journal of Operational Research*, 229(1), 230–238.
- Moretti, S., Patrone, F., & Bonassi, S. (2007). The class of microarray games and the relevance index for genes. *Top*, 15(2), 256–280.
- Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal of Applied Mathematics*, 17, 1163–1170.
- Shapley, L. S., & Shubik, M. (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48, 787–792.
- Taylor, A. D., & Zwicker, W. S. (1999). *Simple games: Desirability relations, trading, pseudoweightings*. Princeton University Press.