Discrete Neural Networks with Maximum and Non-instantaneous Impulses with Computer Simulation

Snezhana Hristova and Kremena Stefanova

Abstract A nonlinear non-instantaneous impulsive difference equations with maximum of the state variable over a past time interval is investigated. The exponential stability concept is studied and some criteria are derived. These results are also applied for a neural networks with switching topology at certain moments and long time lasting impulses. It is considered the general case of time varying connection weights. The equilibrium is defined and exponential stability is studied. The obtained results are illustrated on examples.

Keywords Difference equations · Non-instantaneous impulses · Maximum · Exponential stability · Discrete neural networks · Switching topology

1 Introduction

One of the most important problems in the theory and application of difference equations is stability (see, for example, [3, 6, 7, 9, 11, 12]). At the same time impulses are a very useful mathematical apparatus to model some instantaneous perturbations in the process. In the case when the acting time of the impulses is not possible to be neglected, these impulses are called non-instantaneous impulses (for continuous case, see, [4]).

In this paper we study nonlinear difference equations with a special type of delay in the case there are some impulses starting at initially given points and acting on finite time intervals. The delay is presented as the maximum value of the unknown function over a past time discrete interval. By utilizing the Lyapunov stability theory

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and discrete-time Gronwall inequality, we establish some sufficient conditions for exponential stability of the zero solution.

Neural networks have received extensive interests in recent years in connection with their potential applications in signal processing, content addressable memory, pattern recognition, combinatorial optimization. It is well known that the existence of delays in neural networks causes undesirable complex dynamical behaviors such as instability, oscillation and chaotic phenomena. In practice, for computation convenience, continuous-time neural networks are often discretized to generate discrete-time neural networks. Thus, the study of discrete-time neural networks attracts more and more interests.

In this paper, we deal with a class of discrete-time neural networks with a special type of delay subject to long time lasting impulsive perturbations. The delay is presented by the maximum value of the state variables ove a past time interval with fixed interval. The basic characteristic of these perturbations is that the time of their action is not negligible small, so we consider the so called non-instantaneous impulses. We consider the general case when the connection weights between neurons are changeable in time. We apply the obtained theoretical results to obtain exponential stability criteria and new exponential convergence rate for non-instantaneous impulsive discrete-time neural networks with delays and variable connection weights.

Some discrete neural networks are considered and the theoretical results are applied. The example is computer realized by the help of Wolfram Mathematica. Following the theoretical schemes for solving the problems, the corresponding algorithms are coded to calculate the values of the solution for each step. The graphs are generated by CAS Wolfram Mathematica.

2   Statement of the Problem and Definition of Solution

We will introduce basic notations used in this paper. Most of them are well known and used in the literature. Let $\mathbb{Z}_+$ be the set of all nonnegative integers; the increasing sequence $\{n_i\}_{i=0}^{\infty}:\ n_0 = 0,\ n_i \in \mathbb{Z}_+,\ n_i \geq n_{i-1} + 3, i = 1, 2, \ldots$ and the sequence $\{d_i\}_{i=1}^{\infty}:\ d_i \in \mathbb{Z}_+,\ 1 \leq d_i \leq n_{i+1} - n_i - 2, i = 1, 2, \ldots$ be given; $\mathbb{Z}[a, b] = \{z \in \mathbb{Z}_+ : a \leq z \leq b\},\ a,\ b \in \mathbb{Z}_+,\ a < b,\ \mathbb{Z}_a = \{z \in \mathbb{Z}_+ : z \geq a\}$ and $I_k = \mathbb{Z}[n_k + d_k, n_{k+1} - 1],\ k \in \mathbb{Z}_+,$ and $J_k = \mathbb{Z}[n_k + 1, n_k + d_k],\ k \in \mathbb{Z}_1,$ where $d_0 = 0$.

Let $\phi \in \mathbb{Z}(-h, 0) \to \mathbb{R}^N$ with $\|\phi\|_0 = \max\limits_{\sigma \in [-h,0]} \|\phi(\sigma)\|$, where $\|.\|$ is a norm in $\mathbb{R}^N$.

Consider the *initial value problem (IVP)* for the system of nonlinear *difference equation with non-instantaneous impulses*

$$x(n + 1) = Ax(n) + F\left(n, \max_{\xi \in \mathbb{Z}[n-h,n]} x(\xi)\right) \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,$$

$$x(n) = P_k(n, x(n_k)), \quad \text{for } n \in J_k, \ k \in \mathbb{Z}_1, \tag{1}$$

$$x(n) = \phi(n), \quad n \in \mathbb{Z}[-h, 0],$$

where $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, A is $N \times N$ square matrix, $F = (F_1, F_2, \ldots, F_N)$, $F_i : \bigcup_{k=0}^{\infty} I_k \times \mathbb{R}^N \to \mathbb{R}$, $P_k = (P_{k,1}, P_{k,2}, \ldots, P_{k,N})$, $P_{k,i} : J_k \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots$, and h is a natural number.

Denote by $\mathbb{M}_N$ the set of all quadratic $N \times N$ dimensional matrices with the spectral norm $|A| = \sqrt{\lambda_{max}(A^T A)}$, and for any vector $x \in \mathbb{R}^N$ we will use the norm $|x| = \sqrt{\sum_{i=1}^{N} x_i^2}$. Moreover, denote by $\lambda_{min}(A)$ and $\lambda_{max}(A)$ the minimum and the maximum eigenvalue of a positive definite symmetric matrix A and $\Phi(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$.

Usually, the difference equation describes the development of a certain phenomenon by recursively defining a sequence, each of whose terms is defined as a function of the preceding terms, once one or more initial terms are known (see, for example, [10]). Differently than that, we consider a difference equation in which the present state is also involved nonlinearly in the right side part. It makes the answer of the question about the existence of the solution more complicated.

Definition 1 The trivial solution of the system (1) is called *globally exponentially stable*, if there exist constants $L > 0$ and $\alpha \in (0, 1)$ such that for any initial function ϕ the inequality $|x(n)| \leq L\alpha^n ||\phi||_0$, $n = 1, 2, \ldots$ holds.

The constant α is called the *exponential convergence rate*.

Consider the Lyapunov equation

$$A^T H A - H = -C, \tag{2}$$

where A, H, $C \in \mathcal{M}_N$.

3 Exponential Stability of Linear Delay Discrete Equations

We will study the exponential stability of the linear system (1).

Theorem 1 (Exponential stability results). *Let*

1. *The matrix $A \in \mathbb{M}_N$ and $C \in \mathbb{M}_N$ be a positive definite matrix.*
2. *The function $F \in C(\mathbb{Z}_+ \times \mathbb{R}^N, \mathbb{R}^N)$, $F(n, 0) = 0$ for any $n \in \mathbb{Z}_+$ and there exists a constant $K > 0$ such that $|F(n, u)| \leq \sqrt{K}|u|$ for $u \in \mathbb{R}^N$, $n \in \mathbb{Z}_+$.*
4. *The functions $P_k \in C(\mathbb{Z}_+ \times \mathbb{R}^N, \mathbb{R}^N)$, $P_k(n, 0) = 0$, $k = 1, 2, \ldots$ and there exist constants $M_k > 0$ such that $|P_k(n, u)| \leq \sqrt{M_k}|u|$ for any $u \in \mathbb{R}^N$, $n \in J_k$ and $k = 1, 2, \ldots$.*

5. *There exists a solution $H \in \mathbb{M}_n$ of ([2](#)) such that $|H| M_k < 1$, $k = 1, 2, \ldots$,*
   $L_1(H) - L_2(H) < \lambda_{max}(H) - \lambda_{min}(H)$, and

$$\Phi(H)\left(|A^T H| + |HA| + K\lambda_{max}(H)\right)$$
$$min(C) < \lambda_{max}(H) + 0.5\Phi(H)(|A^T H| + |HA|)$$

where $L_1(H) = \lambda_{max}(H) - \lambda_{min}(C) + 0.5\Phi(H)(|A^T H| + |HA|)$, and $L_2(H)$
$= \lambda_{min}(H) - \Phi(H)K\left(\lambda_{max}(H) + 0.5|HA| + 0.5|A^T H|\right)$.

Then the zero solution of ([1](#)) is exponentially stable.

Proof Denote $\quad \Theta = \max\left\{\Lambda, \frac{L_1(H)}{\lambda_{max}(H)} - L_2 + \lambda_{min}(H)\right\} < 1, \quad$ where $\quad \Lambda = \sup_{k \geq 1} |M_k^T H M_k|$.

Consider the function $V(x) = x^T H x$ for $x \in \mathbb{R}^N$. Then $\lambda_{min}(H)|x|^2 \leq V(x) \leq \lambda_{max}(H)|x|^2$.

Let $x(n)$, $n \in \mathbb{Z}[-h + 1, \infty)$, be a solution of the IVP ([1](#)) with the initial function ϕ.

Let $n \in \bigcup_{k=0}^{p} I_k$. Then we have

$$\begin{aligned}
V&(x(n+1)) - V(x(n)) \\
&\leq \left(-\lambda_{min}(C) + 0.5|A^T H| + 0.5|HA|\right)|x(n)|^2 \\
&\quad + \left(\lambda_{max}(H) + 0.5|HA| + 0.5|A^T H|\right) K|\max_{\xi \in \mathbb{Z}[n-h,n]} x(\xi)|^2 \qquad (3) \\
&\leq -\lambda_{min}(C)|x(n)|^2 + |A^T HB||x(n)|^2 \\
&\quad + \left(|A^T HB| + |B^T HB|\right)|\max_{\xi \in \mathbb{Z}[n-h,n]} x(\xi)|^2.
\end{aligned}$$

Apply the inequalities $-|x(n)|^2 \leq -\frac{V(x(n))}{\lambda_{max}(H)}$, and

$$\left|\max_{\xi \in \mathbb{Z}[n-h,n]} x(\xi)\right|^2 = |x(\eta)|^2 \leq \frac{V(x(\eta))}{\lambda_{min}(H)} \leq \frac{\lambda_{max}(H)}{\lambda_{min}(H)}\|\phi\|_0^2$$

to ([3](#)) and obtain

$$\begin{aligned}
V(x(n+1)) &\leq \left(1 - \frac{\lambda_{min}(C)}{\lambda_{max}(H)} + \frac{|A^T HB|}{\lambda_{min}(H)}\right)V(x(n)) \\
&\quad + \left(|A^T HB| + |B^T HB|\right)\frac{\lambda_{max}(H)}{\lambda_{min}(H)}\|\phi\|_0^2.
\end{aligned} \qquad (4)$$

From equalities

$$1 - \frac{\lambda_{min}(C)}{\lambda_{max}(H)} + 0.5\frac{|A^T H| + |HA|}{\lambda_{min}(H)}$$
$$= \frac{1}{\lambda_{max}(H)} \left(\lambda_{max}(H) - \lambda_{min}(C) + 0.5\Phi(H)(|A^T H| + |HA|) \right)$$

and

$$\left(\lambda_{max}(H) + 0.5|HA| + 0.5|A^T H| \right) \frac{K}{\lambda_{min}(H)}$$
$$= \Phi(H) \left(\lambda_{max}(H) + 0.5|HA| + 0.5|A^T H| \right) \frac{K}{\lambda_{max}(H)}$$

and inequality (4) we get

$$V(x(n+1)) \leq \frac{L_1(H)}{\lambda_{max}(H)} V(x(n-1)) + \left(\lambda_{min}(H) - L_2(H) \right) \|\phi\|_0^2. \qquad (5)$$

Let $n = 0$. Then from inequality (5) we obtain

$$V(x(1)) \leq \left(L_1(H) - L_2(H) + \lambda_{min}(H) \right) \|\phi\|_0^2 < \sqrt[1+h]{\Theta(H)} \|\phi\|_0^2. \qquad (6)$$

Let $n = 1$. Then from inequalities (5) and (6) we get

$$V(x(2)) \leq \frac{L_1(H)}{\lambda_{max}(H)} V(x(1)) + \left(\lambda_{min}(H) - L_2(H) \right) \|\phi\|_0^2$$
$$\leq \Theta(H)\|\phi\|_0^2 < \sqrt[1+h]{\Theta^2(H)} \|\phi\|_0^2. \qquad (7)$$

Consider the following two possible cases:

Case 1. Let $m \geq n_1 - 1$. Then using induction, the inequalities $\Theta < \sqrt[p]{\Theta}$ for $p > 1$, $n \leq n_1 - 1 < m + 1$, i.e. $\frac{m+1}{n} > 1$ for $n \in I_0$ and inequality (5), we prove that

$$V(x(n+1)) \leq \sqrt[1+h]{\Theta^n(H)} \|\phi\|_0^2, \quad \text{for } n \in \mathbb{Z}[0, n_1 - 1].$$

Case 2. Let $h < n_1 - 1$. Then using induction and inequality (5) we prove that

$$V(x(n+1)) \leq \sqrt[1+h]{\Theta^n(H)} \|\phi\|_0^2, \text{ for } n = 1, 2, \ldots, h.$$

Then for $k = 1, 2, \ldots, n_1 - h$ we get

$$V(x(h+k)) \leq \frac{L_1(H)}{\lambda_{max}(H)} V(x(h)) + \left(\lambda_{min}(H) - L_2(H) \right) \|\phi\|_0^2$$
$$\leq \Theta(H) \sqrt[1+h]{\Theta^k(H)} \|\phi\|_0^2 < \sqrt[1+h]{\Theta^{h+k}(H)} \|\phi\|_0^2. \qquad (8)$$

By induction we prove that

$$V(x(n_1 + k)) < \sqrt[1+h]{\Theta^{n_1+k-1}} v_0, \quad k = 0, 1, \ldots, d_1.$$

Let $n = n_1 + d_1$. Then using the inequalities $n_1 + d_1 + 1 - h > 0$ and (5) we get

$$V(x(n_1 + d_1 + 1)) \leq \frac{1}{\lambda_{max}(H)} L_1(H) \sqrt[1+h]{\Theta^{n_1+d_1}} \|\phi\|_0^2$$
$$+ \left(-L_2(H) + \lambda_{min}(H) \right) \sqrt[1+h]{\Theta^{n_1+d_1+1-h}} \|\phi\|_0^2 \qquad (9)$$
$$< \Theta^{\frac{n_1+d_1+1}{h+1}} \|\phi\|_0^2.$$

Similarly, $V(x(n_1 + d_1 + 2)) < \Theta^{\frac{n_1+d_1+2}{h+1}} \|\phi\|_0^2$.
By induction process we prove the validity of the inequality

$$V(x(n)) < \Theta^{\frac{n}{h+1}} \|\phi\|_0^2 \quad \text{for all } n \in \mathbb{Z}_1. \qquad (10)$$

Therefore, we get $|x(n)| < L\|\phi\|_0 \alpha^n$, for all $n \in \mathbb{Z}_1$ with $\alpha = \sqrt[2(h+1)]{\Theta} < 1$, and $L = \sqrt{\Phi(H)} = \sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}}$.

$\square$

4 Exponential Stability of Discrete Neural Networks with Maximum, Non-instantaneous Impulses and Time Variable Connection Weights

Consider the following neural network modeled by discrete system with maximum, non-instantaneous impulses and time variable connection weights

$$u_i(n + 1) = a_i u_i(n) + \sum_{j=1}^{n} \Psi_{ij}(n) f_j \left(\max_{\xi \in \mathbb{Z}[n-h,n]} u_j(\xi) \right) + G_i$$

$$\text{for } n \in \bigcup_{k=0}^{\infty} I_k, \quad i \in \mathbb{Z}[1, N],$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (11)$$

$$u_i(n) = M_{ik} u_i(n_k) + \sum_{j=1}^{n} \Phi_{ij}^k(n) S_j(u_j(n_k)) + Q_{ik},$$

$$\text{for } n \in J_k, \ k \in \mathbb{Z}_1, \ i \in \mathbb{Z}[1, N],$$

$$u_i(n) = \phi_i(n), \ n \in \mathbb{Z}[-h, 0], \ i \in \mathbb{Z}[1, N],$$

where $u_i(n)$, $i \in \mathbb{Z}[1, N]$, denotes the state of the i-th neuron at discrete time n, a_i, $i \in \mathbb{Z}[1, N]$, represents the passive decay rate, f_j is the neuron activation function with $f_j(0) = 0$, G_i is the exogenous input, P_j is the neuron output signal function

which is a continuous function, $\Psi_{ij}(n)$ and $\Phi_{ij}^k(n)$ denote the connection weight from the neuron j to the neuron i at time n, $m \in \mathbb{Z}(1)$ is the transmission delay, $\phi_i(n), n \in \mathbb{Z}[-h, 0]$ is the initial function for the i-th neuron.

For any $u = (u_1, u_2, \ldots, u_N)$ we denote $f(u) = \big(f_1(u_1), f_2(u_2), \ldots, f_N(u_N)\big)$ and $S(u) = \big(S_1(u_1), S_2(u_2), \ldots, S_N(u_N)\big)$.

We will introduce the following assumptions:

A1. The functions $f_i \in C(\mathbb{R}, \mathbb{R})$, $i \in \mathbb{Z}[1, N]$, and there exist positive constants $L_i, i \in \mathbb{Z}[1, N]$, such that $|f_i(u) - f_i(v)| \le L_i|u - v|, u, v \in \mathbb{R}$.

A2. The functions $S_i \in C(\mathbb{R}, \mathbb{R})$, $i \in \mathbb{Z}[1, N]$, and there exist positive constants $K_i, i \in \mathbb{Z}[1, N]$, such that $|S_i(u) - S_i(v)| \le K_i|u - v|, u, v \in \mathbb{R}$.

A3. The functions $\Psi_{ij} : \bigcup\limits_{k=0}^{\infty} I_k \to \mathbb{R}, i, j \in \mathbb{Z}[1, N]$, and $\Phi_{ij}^k : J_k \to \mathbb{R}, i, j \in \mathbb{Z}[1, N], k \in \mathbb{Z}_+$, are bounded, i.e. there exists constants $\beta_{ij}^k > 0, \gamma_{ij} > 0$ such that $|\Psi_{ij}(n)| \le \beta_{ij}$ for $n \in \bigcup\limits_{k=0}^{\infty} I_k$ and $|\Phi_{ij}^k(n)| \le \gamma_{ij}^k$ for $n \in J_k, k \in \mathbb{Z}_+, i, j \in \mathbb{Z}[1, N]$.

In the non-homogeneous case we will define an equilibrium of the model (11):

Definition 2 ([2]) A vector $u^* \in \mathbb{R}^N : u^* = (u_1^*, u_2^*, \ldots, u_N^*)$ is said to be an equilibrium point of the impulsive discrete-time neural network (11) if it satisfies the equalities

$$u_i^* = a_i u_i^* + \sum_{j=1}^{N} \Psi_{ij}(n) f_j(u_j^*) + G_i \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k, \ i \in \mathbb{Z}[1, N],$$
$$u_i^* = M_{ik} u_i^* + \sum_{j=1}^{N} \Phi_{ij}^k(n) S_j(u_j^*) + Q_{ik} \quad \text{for } n \in J_k, \ k \in \mathbb{Z}_1. \tag{12}$$

Let (11) has an equilibrium $u^* \in \mathbb{R}^N$. Substitute $x = u - u^* \in \mathbb{R}^N$ in (11) and obtain

$$x_i(n+1) = a_i x_i(n) + \sum_{j=1}^{N} \Psi_{ij}(n) \mathcal{F}_j \left(\max_{\xi \in \mathbb{Z}[n-h,n]} x_j(\xi) \right), \ n \in \bigcup_{k=0}^{\infty} I_k,$$
$$x_i(n) = M_{ik} x_i(n_k)) + \sum_{j=1}^{N} \Phi_{ij}^k(n) r_j(x_j(n_k)), \ n \in J_k, \ k \in \mathbb{Z}_1, \tag{13}$$

where $\mathcal{F}_i(y) = f_i(y - u_i^*) - f_i(u_i^*)$ and $r_i(y) = S_i(y - u_i^*) - S_i(u_i^*)$, $i = 1, 2, \ldots, N$, for $y \in \mathbb{R}$.

The stability behavior of the equilibrium of (11) is equivalent to the stability behavior of zero solution of (13).

The system (13) could be written in the matrix form (1) where

$$A = \begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_N \end{bmatrix}, \quad \mathcal{B}(n) = \begin{bmatrix} \Psi_{11}(n) & \Psi_{12}(n) & \Psi_{13}(n) & \ldots & \Psi_{1N}(n) \\ \Psi_{21}(n) & \Psi_{22}(n) & \Psi_{23}(n) & \ldots & \Psi_{2N}(n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \Psi_{N1}(n) & \Psi_{N2}(n) & \Psi_{N3}(n) & \ldots & \Psi_{NN}(n) \end{bmatrix},$$

$$\mathcal{F}(u) = \begin{bmatrix} \mathcal{F}_1(u_1) \\ \mathcal{F}_2(u_2) \\ \ldots \\ \mathcal{F}_N(u_N) \end{bmatrix}, \quad r(u) = \begin{bmatrix} r_1(u_1) \\ r_2(u_2) \\ \ldots \\ r_N(u_N) \end{bmatrix}, \quad \mathcal{M}_k = \begin{bmatrix} M_{1k} & 0 & 0 & \ldots & 0 \\ 0 & M_{2k} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & M_{Nk} \end{bmatrix},$$

$$\mathcal{G}_k(n) = \begin{bmatrix} \Phi_{11}^k(n) & \Phi_{12}^k(n) & \gamma_{13}^k(n) & \ldots & \Phi_{1N}^k(n) \\ \Phi_{21}^k(n) & \Phi_{22}^k(n) & \Phi_{23}^k(n) & \ldots & \Phi_{2N}^k(n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \Phi_{N1}^k(n) & \Phi_{N2}^k(n) & \Phi_{N3}^k(n) & \ldots & \Phi_{NN}^k(n) \end{bmatrix},$$

$u = (u_1, u_2, \ldots, u_N)$, $F = (F_1, F_2, \ldots, F_N)$, $F(n, u) = \mathcal{B}(n)\mathcal{F}(u)$, $P_k = (P_{k,1}, P_{k,2}, \ldots, P_{k,N})$, $P_k(n, u) = \mathcal{G}_k(n)r(u) + \mathcal{M}_k u^T$.

From assumption (A1) and the inequality $\left(\sum_{j=1}^{N} \gamma_j u_j \right)^2 \leq N \sum_{j=1}^{N} \left(\gamma_j u_j \right)^2$ we have

$|F(n, u)| \leq \sqrt{N \sum_{i=1}^{N} \max_j (L_j \beta_{ij})^2} \sqrt{\sum_{j=1}^{N} u_j^2}$, i.e. the condition 3 of Theorem 1 is satisfied with $K = N \sum_{i=1}^{N} \max_j (L_j \beta_{ij})^2$.

From assumption (A2) and the inequality $\left(\sum_{j=1}^{N} \gamma_j u_j \right)^2 \leq N \sum_{j=1}^{N} \left(\gamma_j u_j \right)^2$ we have

$|P_k(n, u)| \leq \sqrt{2 \left(\max_i M_{ik}^2 + \sum_{i=1}^{N} N \max_j \left(\gamma_{ij}^k K_j \right)^2 \right) \sum_{i=1}^{N} u_i^2}$, i.e. the condition 4 of Theorem 1 is satisfied with $M_k = 2 \left(\max_i M_{ik}^2 + \sum_{i=1}^{N} N \max_j \left(\gamma_{ij}^k K_j \right)^2 \right)$.

Theorem 2 *Let the conditions (A1)–(A3) be satisfied and:*

1. *The discrete model (11) has an equilibrium u^*.*
2. *The constants a_i, $M_{ik} \in \mathbb{R}$, β_{ij}, $\gamma_{ij}^k > 0$, G_i, $Q_{ik} \in \mathbb{R}$, i, $j \in \mathbb{Z}[1, N$ $k \in \mathbb{Z}_1$.*
3. *The inequalities $\max_i M_{ik}^2 + N \sum_{i=1}^{N} \max_j \left(\gamma_{ij}^k K_j \right)^2 < 1$ for $k \in \mathbb{Z}_1$, $\max_i a_i^2 + \max_i |a_i| + N(1 - \max_i |a_i|) \sum_{i=1}^{N} \max_j (L_j \beta_{ij})^2) < 1$, and $\max_i a_i^2 + \max_i |a_i| + K \left(1 + \max_i |a_i| \right) < 1$ hold.*

Then the equilibrium point of the difference neural network with non-instantaneous impulses (11) is exponentially stable with a rate $\alpha = {}^{2(h+1)}\!\sqrt{\Theta} < 1$. where $\Theta = \max \left\{ \Lambda, \max_i a_i^2 + \max_i |a_i| + K \left(1 + \max_i |a_i| \right) \right\} < 1$ with $\Lambda = \sup_{k \geq 1} \left(\max_i M_{ik}^2 + \sum_{i=1}^{N} N \max_j \left(\gamma_{ij}^k K_j \right)^2 \right)$.

Proof Let $H = E$, where $E \in \mathbb{M}_n$ is the unit matrix. Then (2) is satisfied with $C \in \mathbb{M}_n : c_{ii} = 1 - a_i^2$ and $c_{ij} = 0$ for $i \neq j$. Then $\lambda_{max}(H) = \lambda_{min}(H) = \phi(H) = 1$ and $\lambda_{min}(C) = 1 - \max_i a_i^2$, $|A| = \max_i |a_i|$. According to Theorem 1 the zero solution of (13) is exponentially stable.

5   Application

Consider a system with three agent with constant connection weights modeled by the following discrete model of neural network

$$
u_1(n+1) = \frac{1}{2}u_1(n) + \frac{1}{8}\sin\left(\max_{\xi \in \mathbb{Z}(n-3,n)} u_1(\xi)\right) - \frac{1}{4}\sin\left(\max_{\xi \in \mathbb{Z}(n-3,n)} u_2(\xi)\right)
$$
$$
+ \frac{1}{16}\max_{\xi \in \mathbb{Z}(n-3,n)} u_3(\xi) + 1
$$
$$
u_2(n+1) = \frac{1}{3}u_2(n) + \frac{1}{4}\sin\left(\max_{\xi \in \mathbb{Z}(n-3,n)} u_1(\xi)\right) + \frac{1}{8}\sin\left(\max_{\xi \in \mathbb{Z}(n-3,n)} u_2(\xi)\right) + 2
$$
$$
u_3(n+1) = \frac{1}{4}u_3(n) + \frac{1}{16}\sin\left(\max_{\xi \in \mathbb{Z}(n-3,n)} u_1(\xi)\right) - \frac{1}{8}\sin\left(\max_{\xi \in \mathbb{Z}(n-3,n)} u_2(\xi)\right)
$$
$$
+ \frac{1}{16}\max_{\xi \in \mathbb{Z}(n-3,n)} u_3(\xi)) + 1 \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,
$$

$$(14)$$

with non-instantaneous impulses for $n \in J_k$, $k \in \mathbb{Z}_1$

$$
u_1(n) = \frac{1}{2}u_1(n_k) + \frac{1}{8}\sin(u_1(n_k)) - \frac{1}{4}\sin(u_2(n_k)) + \frac{1}{16}u_3(n_k) + 1
$$
$$
u_2(n) = \frac{1}{3}u_2(n_k) + \frac{1}{4}\sin(u_1(n_k)) + \frac{1}{8}\sin(u_2(n_k)) + 2
$$
$$
u_3(n) = \frac{1}{4}u_3(n_k) + \frac{1}{16}\sin(u_1(n_k)) - \frac{1}{8}\sin(u_2(n_k)) + \frac{1}{16}u_3(n_k) + 1,
$$

$$(15)$$

and initial conditions

$$
u_i(n) = \phi_i(n), \quad n = -3, -2, -1, 0, \tag{16}
$$

where $n_0 = 0$, $d_0 = 0$, $n_1 = 4$, $d_1 = 6$, $n_2 = 18$, $d_2 = 5$, $n_3 = 33$, $d_3 = 7$, $n_4 = 45$. In this particular case $I_0 = \mathbb{Z}[0, 3]$, $I_1 = \mathbb{Z}[10, 17]$, $I_2 = \mathbb{Z}[23, 32]$, $I_3 = \mathbb{Z}[40, 44]$ $J_1 = \mathbb{Z}[5, 10]$, $J_2 = \mathbb{Z}[19, 23]$, $J_3 = \mathbb{Z}[34, 40]$. (The whole interval is $\mathbb{Z}[-3,45]$).

The point $u^* = (2.40568, 3.23436, 1.53241)$ is the equilibrium point of (14), (15). The conditions of Theorem 2 are reduced to $0.25 + 3(\frac{1}{16} + \frac{1}{16} + \frac{1}{64}) = 0.680556 < 1$, and $0.25 + 0.5 + 3(1 - 0.5)(\frac{1}{16} + \frac{1}{16} + \frac{1}{64}) = 0.965278 < 1$. Therefore, the equilibrium u^* is exponentially stable with a rate $\alpha = \sqrt[8]{0.965278} \approx$
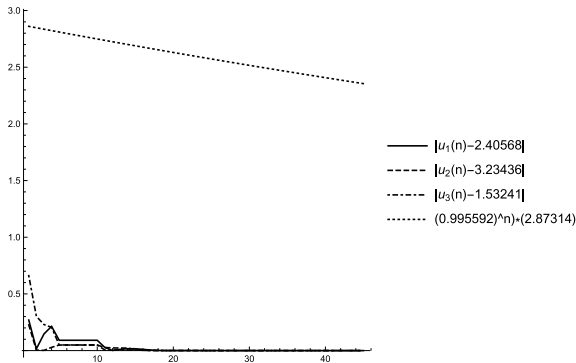
Fig. 1 Graphs of the bound $\alpha^n \|\phi - u^*\|_0 = 2.87314 * 0.995592^n$ and the differences $|u_1(n) - 2.40568|$, $|u_2(n) - 3.23436|$, $|u_3(n) - 1.53241|$ for $n \in [1, 45]$

Table 1 Values of the bound $\alpha^n \|\phi - u^*\|_0 = 2.87314 * 0.995592^n$ and the differences $|u_1(n) - 2.40568|$, $|u_2(n) - 3.23436|$, $|u_3(n) - 1.53241|$ for $n = 1, \ldots, 45$

n	$2.87314 * 0.995592^n$	$\|u_1(n) - 2.40568\|$	$\|u_2(n) - 3.23436\|$	$\|u_3(n) - 1.53241\|$
1	2.86048	0.269342	0.226707	0.660759
2	2.84787	0.0153586	0.00404188	0.301958
3	2.83531	0.146954	0.00197866	0.229919
4	2.82281	0.213725	0.0285321	0.204716
5	2.81037	0.0909878	0.0490802	0.0496394
…	…	…	…	…
38	2.42913	0.000020443	$1.71883 * 10^{-6}$	$3.53387 * 10^{-6}$
39	2.41842	0.000020443	$1.71883 * 10^{-6}$	$3.53387 * 10^{-6}$
40	2.40776	0.000020443	$1.71883 * 10^{-6}$	$3.53387 * 10^{-6}$
41	2.39714	$6.77539 * 10^{-6}$	$1.13363 * 10^{-7}$	$1.04167 * 10^{-6}$
42	2.38658	$4.65835 * 10^{-7}$	$2.03659 * 10^{-6}$	$2.0664 * 10^{-6}$
43	2.37606	$2.16839 * 10^{-6}$	$3.92243 * 10^{-6}$	$2.09433 * 10^{-6}$
44	2.36558	$3.24319 * 10^{-6}$	$5.03918 * 10^{-6}$	$1.98103 * 10^{-6}$
45	2.35516	$3.681 * 10^{-6}$	$5.61059 * 10^{-6}$	$1.90291 * 10^{-6}$

0.995592 and $L = \sqrt{\Phi(H)}$, $\Phi(H) = \frac{\lambda_{max}(H)}{\lambda_{min}(H)}$, $H = E$, where $E \in \mathbb{M}_n$ is the unit matrix.

Consider the solution $\tilde{u}(n)$ of the discrete model (14), (15), (16) with initial functions $\phi_1(n) = 3n + 2$, $\phi_2(n) = 2n + 2$, $\phi_3(n) = n + 4$, $n = -3, -2, -1, 0$. In this case $\|\phi - u^*\|_0 = \max_{\sigma \in [-h,0]} \|\phi(\sigma) - u^*\| = 2.87314$. The graphs of the differences $|u_1(n) - 2.40568|$, $|u_2(n) - 3.23436|$, $|u_3(n) - 1.53241|$ and the bound $\alpha^n \|\phi - u^*\|_0 = 2.87314 * 0.995592^n$ are given on Fig. 1 and Table 1. From both it could be seen the equilibrium point u^* is exponentially stable.

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