Periodic Systems of Coatings on an Elastic Half-Space

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Abstract. Periodic mixed boundary value problems are considered for linear systems of rigid flexible coatings cohesive with an elastic half-space. The half-space boundary is rigidly fixed over a half-plane. The half-plane boundary is parallel to the axis of the periodic system. Two problems are analyzed. In the first one, the system of coatings is shifted in the perpendicular direction to the half-plane boundary. In the second problem, it is shifted parallel to this line. In fact, both contact problems have an extra line of changing boundary conditions which allows us to derive correct equations. By using the method of Fourier and Kontorovich–Lebedev integral transformations and taking the periodicity into account, the problems are reduced to integral equations with respect to tangential contact stresses over only one coating. For elliptic coatings, the regular asymptotic method is used to construct analytical solutions of the integral equations. The contact characteristics are calculated for different values of dimensionless geometric parameters.

Keywords: Periodic contact problems · Elastic half-space · Integral equations · Regular asymptotic method

1 Introduction

Investigating contact problems allow us to estimate distributions of contact stresses over contact domains [1, 2]. One can see contact interactions in everyday life, e.g. contact of a finger with a smartphone screen [3], mechanical palpation tomography [4].

Periodicity usually arises in contact for rough wavy surfaces. Most articles deal with plane periodic contact problems [5, 6]. Evolution of the periodic two-dimensional contact area was studied numerically [7]. Effect of adhesion in periodic contact was analyzed by many authors [5, 8, 9]. Integral equations in periodic contact problems are connected with those in periodic crack problems [10–12]. An elastic or viscoelastic half-space is the simplest spatial model of deformable solid [13]. The normal periodic contact under normal forces, including the three-dimensional contact for non-classical elastic solids, was investigating in [14, 15]. The presented paper focuses on the three-dimensional tangential periodic contact under tangential forces. Such contacts along straight line on a half-space with free boundary outside the contact zone will not be mathematically correct because the corresponding integral equations contain kernels with divergent series. To regularize the problems, it is sufficient to fix a part of the

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half-space boundary over a half-plane whose boundary is parallel the axis of the contact system. Then the series in the kernels of the integral equations converge. The tangential forces can be directed perpendicular or parallel to the contact axis. Like in [14], the problems include two dimensionless geometric parameters. One of them describes the relative distance between the neighboring contact zones and the other corresponds to the relative distance from the contact axis to the half-plane boundary. It is presupposed that the two parameters are linearly connected. Then asymptotic solutions can be constructed as power series in only one parameter.

2 Formulation of Problems

Let us consider a half-space in cylindrical coordinates, $0 \le r < \infty, 0 \le \varphi \le \pi, -\infty < z < \infty$, with elasticity parameters G (shear modulus) and ν (Poisson's ratio). For simplicity, we will take the value $\nu = 0.5$ (incompressible material). Suppose the half-plane $\varphi = \pi$ is fixed while the face $\varphi = 0$ contacts over domain Ω with a periodic system of rigid flexible coatings (thin plates) situated along z-axis. The period of the system is equal to $2l$. This is a tangential cohesive contact, the plates can be shifted along r-axis (problem A) or along z-axis (problem B, Fig. 1).

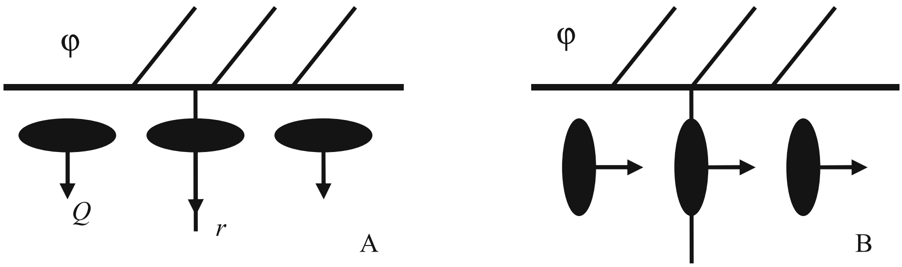

Fig. 1. Systems of plates on a half-space shifted along r-axis (problem A) or z-axis (problem B)

The displacement of the coatings is equal to δ under the action of tangential forces Q. The plates are elongated perpendicular to the shift direction so that one can take only one component of the tangential contact stresses into account.

The boundary conditions for the differential equations of elastic equilibrium have the following form:

$$\text{A)}\ \phi = 0: \quad u_r = \delta, \ \ (r, z \in \Omega), \ \ \tau_{r\phi} = 0 \ \ (r, z \notin \Omega), \quad \sigma_\phi = \tau_{\phi z} = 0 \tag{1}$$

$$\text{B)}\ \phi = 0: \quad u_z = \delta, \ \ (r, z \in \Omega), \ \ \tau_{\phi z} = 0 \ \ (r, z \notin \Omega), \quad \sigma_\phi = \tau_{r\phi} = 0 \tag{2}$$

$$\phi = \pi : \ u_r = u_\phi = u_z = 0 \tag{3}$$

Let the plates have elliptic shape and the central ellipse be Ω0 = {(r − c)2/a2 + z2/b2 ≤ 1}, c > a, where l > b ≥ a in problem A and l > b, a ≥ b in problem B. For given values of G, δ, l, a, b and c, one should determine the contact stresses $\tau_{r\phi}$ (problem A) and $\tau_{\phi z}$ (problem B) as well as the force Q.

3 Integral Equations

To reduce problems A and B to integral equations with respect to the contact stresses, one should solve auxiliary boundary value problems A* and B* on concentrated tangential forces T acted on the half-space boundary $\varphi = 0$ with the fixed face $\varphi = \pi$. Namely,

$$\text{A*)}\ \phi = 0: \quad \tau_{r\phi} = T\delta\,(r - x, z - y), \quad \sigma_\phi = \tau_{\phi z} = 0 \tag{4}$$

$$\text{B*)}\ \phi = 0: \quad \tau_{\phi z} = T\delta(r - x, z - y), \quad \sigma_\phi = \tau_{r\phi} = 0, \tag{5}$$

where $\delta(x)$ is Dirac δ-function and the boundary conditions for $\varphi = \pi$ have the form (3).

The fundamental solutions of problems (3)–(5) can be determined by using the method of Fourier and Kontorovich– Lebedev integral transformations [14]. Then, integrating these solutions over Ω, satisfying boundary conditions (1), (2) with δ and taking the periodicity into account, we derive the governing integral equations ($n = 1, 2$)

$$\iint\limits_{\Omega_0} \tau_n(x, y) K_n(x, y, r, z) dx dy = 4\pi G\delta, \quad (r, z) \in \Omega_0 \tag{6}$$

$$K_1(x, y, r, z) = \sum_{k=-\infty}^{\infty} \{(\frac{1}{R_k} + \frac{(r - x)^2}{R_k^3})(1 - \frac{2}{\pi}\arctan\frac{R_k}{2\sqrt{xr}}) - \frac{4\sqrt{xr} z_k^2}{\pi R_k^2[(r + x)^2 + z_k^2]}\} \tag{7}$$

$$K_2(x, y, r, z) = \sum_{k=-\infty}^{\infty} \{(\frac{1}{R_k} + \frac{z_k^2}{R_k^3})(1 - \frac{2}{\pi}\arctan\frac{R_k}{2\sqrt{xr}}) + \frac{4\sqrt{xr} z_k^2}{\pi R_k^2[(r + x)^2 + z_k^2]}\} \tag{8}$$

$$R_k = \sqrt{(r - x)^2 + z_k^2}, \ z_k = z - y + 2kl$$

Here, $n = 1$, $\tau_1(r, z) = \tau_{r\phi}(r, z)$ for problem A and $n = 2$, $\tau_2(r, z) = \tau_{\phi z}(r, z)$ for problem B.

The kernels (7) and (8) can also be rewritten in the equivalent differential form

$$K_1(x, y, r, z) = \sum_{k=-\infty}^{\infty} [\frac{1}{R_k} + \frac{(r - x)^2}{R_k^3} + (x\frac{\partial}{\partial x} + r\frac{\partial}{\partial r} - 1)\frac{2}{\pi R_k} \arctan \frac{R_k}{2\sqrt{xr}}] \tag{9}$$

$$K_2(x, y, r, z) = \sum_{k=-\infty}^{\infty} [\frac{1}{R_k} + \frac{z_k^2}{R_k^3} - (x\frac{\partial}{\partial x} + r\frac{\partial}{\partial r} + 2)\frac{2}{\pi R_k} \arctan \frac{R_k}{2\sqrt{xr}}] \tag{10}$$

Note that for the case of stress-free face $\varphi = \pi$, the corresponding integral equations would have the divergent kernels

$$K_1(x, y, r, z) = \sum_{k=-\infty}^{\infty} (\frac{1}{R_k} + \frac{(r - x)^2}{R_k^3}), K_2(x, y, r, z) = \sum_{k=-\infty}^{\infty} (\frac{1}{R_k} + \frac{z_k^2}{R_k^3})$$

It means that fixation of the face $\varphi = \pi$ allows us to regularize the periodic contact problems because the series in formulas (7)–(10) converge.

4 Asymptotic Solutions

To derive analytical solutions of the integral Eqs. (6)–(10), we apply the regular asymptotic method [14]. We introduce the dimensionless notation

$$\begin{aligned} &\text{A)}\ r' = \frac{r-c}{b},\ z' = \frac{z}{b},\ \delta' = \frac{\delta}{b},\ \varepsilon = \frac{a}{b},\ \lambda = \frac{c}{b},\ \mu = \frac{l}{b},\ \tau_1' = \frac{\tau_1}{2G},\\ &Q' = \frac{Q}{2Gb^2},\ \Omega_0 \to \Omega_0' \end{aligned} \tag{11}$$

$$\begin{aligned} &\text{B)}\ r' = \frac{r-c}{a},\ z' = \frac{z}{a},\ \delta' = \frac{\delta}{b},\ \varepsilon = \frac{b}{a},\ \lambda = \frac{c}{a},\ \mu = \frac{l}{a},\ \tau_2' = \frac{\tau_2}{2G},\\ &Q' = \frac{Q}{2Ga^2},\ \Omega_0 \to \Omega_0' \end{aligned} \tag{12}$$

(for x and y similarly) and omit the primes in what follows.

The notation (11), (12) includes two principal geometric parameters, λ and μ. The first one takes care about the relative distance between the periodic system axis and the interface (z-axis) while the second one serves as the relative distance between the neighboring coatings. Let us suppose that the two parameters are linearly related as

$$\mu = \gamma\lambda,\ \gamma = \text{const} \tag{13}$$

to simplify the applicability of the regular asymptotic method.

Equations (6), (9) and (10) in notation (11)–(13) take the form ($R = R0$)

$$\iint\limits_{\Omega_0} \tau_1(x,y)[\frac{1}{R} + \frac{(r-x)^2}{R^3} + F_1(x+\lambda, y, r+\lambda, z)]dxdy = 2\pi\delta,\quad (r,z) \in \Omega_0 \tag{14}$$

$$\iint\limits_{\Omega_0} \tau_2(x,y)[\frac{1}{R} + \frac{(z-y)^2}{R^3} + F_2(x+\lambda, y, r+\lambda, z)]dxdy = 2\pi\delta,\quad (r,z) \in \Omega_0 \tag{15}$$

$$\begin{aligned} F_1(x,y,r,z) = {} & (x\frac{\partial}{\partial x} + r\frac{\partial}{\partial r} - 1)\frac{2}{\pi R}\arctan\frac{R}{2\sqrt{xr}}\\ & + \sum_{k=-\infty,k\neq 0}^{\infty}[\frac{1}{R_k} + \frac{(r-x)^2}{R_k^3} + (x\frac{\partial}{\partial x} + r\frac{\partial}{\partial r} - 1)\frac{2}{\pi R_k}\arctan\frac{R_k}{2\sqrt{xr}}] \end{aligned} \tag{16}$$

$$\begin{aligned} F_2(x,y,r,z) = {} & -(x\frac{\partial}{\partial x} + r\frac{\partial}{\partial r} + 2)\frac{2}{\pi R}\arctan\frac{R}{2\sqrt{xr}}\\ & + \sum_{k=-\infty,k\neq 0}^{\infty}[\frac{1}{R_k} + \frac{z_k^2}{R_k^3} - (x\frac{\partial}{\partial x} + r\frac{\partial}{\partial r} + 2)\frac{2}{\pi R_k}\arctan\frac{R_k}{2\sqrt{xr}}] \end{aligned} \tag{17}$$

The principal parts are separated out in the kernels of Eqs. (14) and (15). The smooth kernels parts (16) and (17) can be expanded in power series of λ with the help of the well-known expansions

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots\quad (|z| \leq 1) \tag{18}$$

$$\arctan z = \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} - \frac{1}{5z^5} + \ldots \quad (|z| > 1) \tag{19}$$

$$(1+z)^\alpha = 1 + \alpha z + \alpha(\alpha - 1)\frac{z^2}{2!} + \ldots \quad (|z| < 1)$$

Formulas (18) and (19) are respectively needed for $k = 0$ and $k \neq 0$ in (16) and (17).

As a result, we get the power expansions

$$F_1(x+\lambda, y, r+\lambda, z) = \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + O(\frac{1}{\lambda^3}),\ \lambda \to \infty \tag{20}$$

$$a_1 = -\frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^m(2m-2)}{(k\gamma)^{2m}(2m-1)},\quad a_2 = \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^m(m-1)}{(k\gamma)^{2m}} \tag{21}$$

$$F_2(x+\lambda, y, r+\lambda, z) = \frac{b_1}{\lambda} + \frac{b_2}{\lambda^2} + O(\frac{1}{\lambda^3}),\ \lambda \to \infty \tag{22}$$

$$b_1 = -\frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(2m+1)}{(k\gamma)^{2m}(2m-1)},$$
$$b_2 = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(2m+1)}{(k\gamma)^{2m}} \tag{23}$$

The power series (20) converges uniformly in $(x, y), (r, z) \in \Omega_0$ as

$$\lambda > \max(1, \sqrt{2}\varepsilon),\quad \gamma > \max(1 + \frac{1+\varepsilon}{\lambda}, \frac{1+\sqrt{2+\varepsilon^2}}{\lambda}) \tag{24}$$

while the series (22) converges as

$$\lambda > \max(1+\varepsilon, \sqrt{2+\varepsilon^2}),\quad \gamma > 1 + \frac{1+\varepsilon}{\lambda} \tag{25}$$

Inequalities (24) and (25) restrict the frames of applicability of the regular asymptotic method in problems A and B, respectively.

The coefficients (21) and (23) are presented in Table 1 for some values of γ.

We will seek the solutions of Eqs. (14) and (15) as asymptotic expansions ($n = 1, 2$)

$$\tau_n(x, y) = \tau_{n0}(x, y) + \frac{\tau_{n1}(x, y)}{\lambda} + \frac{\tau_{n2}(x, y)}{\lambda^2} + O(\frac{1}{\lambda^3}),\ \lambda \to \infty \tag{26}$$

Substituting the expressions (20), (22) and (26) into Eqs. (14) and (15) and equating terms of equal powers of λ, we come to sequences of the integral equations with respect to $\tau_{nk}(x, y)$ (k = 1,2,...)

$$\iint_{\Omega_0} \tau_{1k}(x, y)[\frac{1}{R} + \frac{(r-x)^2}{R^3}]dxdy = 2\pi P_{1k}(r, z),\ (r, z) \in \Omega_0 \tag{27}$$

Table 1. The coefficients (21) and (23).

γ	1.5	2	2.5	3	3.5	4	∞
$-a_1$	0.577	0.614	0.627	0.632	0.634	0.635	0.637
a_2	0.387	0.347	0.332	0.325	0.322	0.321	0.318
b_1	0.904	0.405	0.157	0.0167	–0.0697	–0.127	–0.318
b_2	0.641	0.470	0.373	0.315	0.277	0.251	0.159

$$\iint\limits_{\Omega_0} \tau_{2k}(x,y)[\frac{1}{R}+\frac{(z-y)^2}{R^3}]dxdy = 2\pi P_{2k}(r,z),\ (r,z)\in\Omega_0 \tag{28}$$

with determined polynomial right-hand sides.

Since Eqs. (27) and (28) have exact solutions, we finally arrive at the asymptotics

$$\tau_1(r,z)=\frac{\delta}{\varepsilon B}(1+\frac{T_{11}}{\lambda}+\frac{T_{12}+T_{13}\,r}{\lambda^2}+O(\frac{1}{\lambda^3}))[1-\frac{r^2}{\varepsilon^2}-z^2]^{-1/2},\ \lambda\to\infty \tag{29}$$

$$\tau_2(r,z)=\frac{\delta}{\varepsilon B}(1+\frac{T_{21}}{\lambda}+\frac{T_{22}+T_{23}\,r}{\lambda^2}+O(\frac{1}{\lambda^3}))[1-r^2-\frac{z^2}{\varepsilon^2}]^{-1/2},\ \lambda\to\infty \tag{30}$$

$$T_{11}=-\frac{a_1}{B},\ T_{12}=\frac{a_1^2}{B^2},\ T_{13}=-\frac{a_2}{\varepsilon^2(2\varepsilon^2S_{02}-S_{11})},\ B=S_{00}+\varepsilon^2S_{01} \tag{31}$$

$$T_{21}=-\frac{b_1}{B},\ T_{22}=\frac{b_1^2}{B^2},\ T_{23}=-\frac{b_2}{S_{10}+3\varepsilon^2S_{11}} \tag{32}$$

$$S_{km}=\int\limits_0^{\pi/2}\frac{\cos^{2k}t\sin^{2m}t}{(1-e^2\sin^2t)^{k+m+1/2}}dt,\ e^2=1-\varepsilon^2$$

$$S_{00}=K,\ S_{01}=\frac{E-(1-e^2)K}{e^2(1-e^2)},\ S_{10}=\frac{K-E}{e^2}$$

$$S_{11}=\frac{(2-e^2)E-2(1-e^2)K}{3e^4(1-e^2)},\ S_{02}=\frac{-2(1-2e^2)E+(1-e^2)(2-3e^2)K}{3e^4(1-e^2)^2},$$

where $K=K(e)$ and $E=E(e)$ are the complete elliptic integrals.

On the basis of formulas (29)–(32), we can derive the integral characteristic

$$Q=\iint\limits_{\Omega_0}\tau_n(x,y)dxdy=\frac{2\pi\delta}{B}Q_\bullet,\ Q_\bullet=1+\frac{T_{n1}}{\lambda}+\frac{T_{n2}}{\lambda^2}+O(\frac{1}{\lambda^3}),\ \lambda\to\infty \tag{33}$$

Table 2. The integral characteristics (33).

ε	0.2	0.4	0.6	0.8	1
Problem A					
$\gamma = 1.5$	1.030	1.036	1.042	1.047	1.051
$\gamma = 2$	1.032	1.039	1.045	1.050	1.055
$\gamma = \infty$	1.033	1.040	1.046	1.051	1.057
Problem B					
$\gamma = 1.5$	0.957	0.948	0.941	0.935	0.929
$\gamma = 2$	0.980	0.976	0.973	0.970	0.967
$\gamma = \infty$	1.016	1.020	1.023	1.025	1.028

5 Analysis and Conclusion

The values of the integral characteristics $Q_\bullet$ are presented in Table 2 calculated for $\lambda = 5$ for both problems.

Note that the value $\gamma = \infty$ corresponds to one coating on the half-space with the fixed half-plane while the value $\lambda = \infty$ is related to the case of one coating on a free half-space outside the contact zone. We have $Q_\bullet = 1$ for $\lambda = \infty$, see formula (33). As γ decreases (the period 2l diminishes), the tangential force Q decreases too. For the circular coatings ($\varepsilon = 1$), the parameters λ and μ do not depend on the type of problem. As one can see in Table 2, the periodic system of circular plates can be more easily shifted along the system axis. In problem A, the integral characteristics for elongated coatings are smaller than that for the circular coatings. It is vice versa in problem B, where the force Q for elongated coatings is bigger than that for the circular ones. The interaction between coatings in the periodic system is apparently greater in problem B than that in problem A.

The asymptotics obtained above can be recommended for comparative analysis with direct numerical solutions.

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