Structural Cores and Problems of Vulnerability of Partially Overlapped Multilayer Networks

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Abstract. The concept of aggregate-network of multilayer network (MLN), which in many cases significantly simplifies the study of intersystem interactions is introduced, and the properties of its *k*-cores are investigated. The notion of *p*-cores is determined, with help of which the components of MLN that are directly involved in the implementation of intersystem interactions are distinguished. Methods of reducing the complexity of multilayer network models are investigated, which allow us to significantly decrease their dimensionality and better understand the processes that take place in intersystem interactions of different types. Effective scenarios of simultaneous group and system-wide targeted attacks on partially overlapped multilayer networks have been proposed, the main attention of which is focused on the transition points of MLN through which the intersystem interactions are actually implemented. It is shown that these scenarios can also be used to solve the inverse problem, namely, which elements of MLN should be blocked in the first place to prevent the acceleration of spread of dangerous infectious epidemics diseases, etc.

Keywords: Intersystem interactions · Complexity · Multilayer network · Aggregate-network · Network core · Vulnerability · Targeted attack · Epidemic spreading

1 Introduction

Any real system is open, i.e. it interacts with other systems [1]. Intersystem interactions of different kinds give rise to different types of interconnected network structures, which in the theory of complex networks (TCN) are called multilayer networks [2]. At present, a number of the most common types of MLN in the physical world, nature and human society have been identified. First of all these are multidimensional networks, each layer of which reflects a different type of intra- and inter-system interaction [3]. An example of multidimensional MLN is international cooperation, which includes the layers of political, economic, military, security, cultural, sports and other types of interactions between the nodes-countries of the Earth. Each city of the country is a node of several transport networks of different types, state and local government networks, economic

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and financial networks, etc. The city's life support system includes electricity, gas and water supply networks, telephone networks, cable television, Internet services, etc.

The models of intersystem interactions structures often have a huge dimensionality and complexity, associated with both a large number of MLN nodes and amount of various intra- and inter-layer interactions between them. This requires the development of methods to reduce the complexity of models of such formations, aimed at simplifying their understanding and cognition [4], as well as overcoming the problem of complexity of systems research in general [5, 6]. One way to solve this problem is the division of multiflow MLN into monoflow multilayer networks, i.e. those, that allow the movement of flows of one type by different media or systems-operators and the ability to move from one layer to another through the so-called transition points, i.e. nodes, that are part of several system-layers [7]. Transition points can have multiple connections to nodes of other layers (e-mail or telephone communication) or only to points with the same number from the total set of nodes of the multilayer network (common transport MLN). From a functional point of view, the latter case means that the corresponding node is an element of several system-layers and performs in them different functions or one function, but in different ways. For example, using this approach, it is advisable to divide the general transport system into two monoflow four-layer MLNs, each of which provides the movement of passenger or freight flows by road, rail, air or water (sea or river) transport, respectively. The next step to overcome the problem of complexity is to identify structurally or functionally most important components of monoflow MLN, i.e. its cores of different types [8, 9]. Identification of such cores allows us to develop effective scenarios of simultaneous group and system-wide targeted attacks on the process of intersystem interactions and appropriate means of their protection.

2 Structural Model of MLN

Usually, multilayer network structures are described as

$$G^M = \left(\bigcup_{m=1}^{M} G_m, \bigcup_{\substack{m,k=1 \\ m \neq k}}^{M} E_{mk} \right), \tag{1}$$

where $G_m = (V_m, E_m)$ is a description of structure of the m-th network layer of MLN; V_m is a set of nodes of the network G_m; E_m is a set of edges of the network G_m; E_{mk} is a set of edges between the nodes of sets V_m and V_k, $m \neq k$, $m, k = \overline{1, M}$, M is a number of layers of MLN. The set

$$V^M = \bigcup_{m=1}^{M} V_m$$

will be called the total set of MLN nodes, N^M is a number of nodes of V^M.

We represent the mathematical model of MLN in the form of adjacency matrix $\boldsymbol{A}^M = \{\boldsymbol{A}^{km}\}_{k,m=1}^{M}$. Blocks $\boldsymbol{A}^{km} = \{a_{ij}^{km}\}_{i,j=1}^{N^M}$ of this matrix are defined for the total set of MLN nodes, i.e. the problem of coordination of nodes numbers in the case of their independent numbering for each layer disappears. It is obvious that the diagonal blocks of matrix $\boldsymbol{A}^M$ describe the structure of intra-layer interactions, and the off-diagonal ones describe the structure of inter-layer interactions.Below, to simplify the presentation, we assume that the structure (1) and adjacency matrix $\boldsymbol{A}^M$ describe the undirected MLN of most general form.

From a structural point of view, the most common type of multilayer networks are the partially overlapped MLN, the intersection of sets of nodes V_m of which is not empty (Fig. 1) [10]. The boundary cases of partially overlapped (PO) multilayer networks are multiplexes, i.e. MLN, the sets of nodes V_m of which coincide [8], and multi-networks, i.e. MLN, the sets of nodes V_m of which do not intersect [7], $m = \overline{1, M}$.

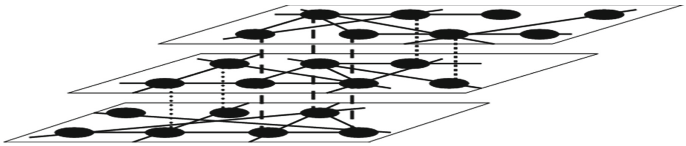

Fig. 1. Example of structure of partially overlapped multilayer network

3 Aggregate-Network of Partially Overlapped MLN

The local characteristic ε_{ij} of edge (n_i, n_j), in PO MLN, which will be called its aggregate-weight, is determined by formula.

$$\varepsilon_{ij} = \sum_{l,m=1}^{M} a_{ij}^{lm}, i, j = \overline{1, N^M}.$$

The aggregate-weight ε_{ii} of node n_i, $i = \overline{1, N^M}$, in a partially overlapped multilayer network is equal to the number of layers of which it is a part. A node that belongs to several layers of MLN and through which the flow can move from one layer to another will be called the transition point of multilayer network. For an arbitrary PO MLN the adjacency matrix $\boldsymbol{E} = \{\varepsilon_{ij}\}_{i,j=1}^{N^M}$ completely defines the weighted network, which will be called the aggregate-network of partially overlapped MLN. The elements of matrix **E** determine the integral structural characteristics of nodes and edges of such multilayer network (Fig. 2).

For multiflows (multidimensional) networks, the value of aggregate-weights $\varepsilon_{ij}, \quad i \neq j$, of weighted aggregate-network determines the number of interactions of different types between the nodes of such structures. The projection on weighted aggregate-network of multidimensional MLN loses some meaning, because the weight of each of its edge determines the total number of connections of different types. For monoflow MLN, this disadvantage is absent, because the weight of each edge reflects the number of possible carriers or systems-operators that can provide the movement of corresponding type of flow.

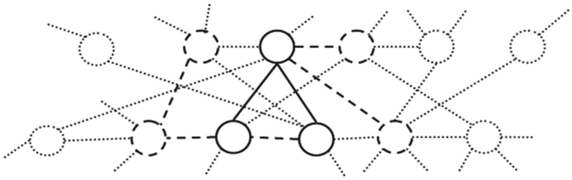

Fig. 2. Fragment of aggregate-network for the shown in Fig. 1 partially overlapped three-layer network (____—for $\varepsilon_{ij} = 3$, _ _ _—for $\varepsilon_{ij} = 2$, **.....**—for $\varepsilon_{ij} = 1$)

During the study of monoflow PO MLN there are many practically important problems that can be solved at least in the first approximation, using the concept of its aggregate-network. Such problems include:

1. Determination of the shortest paths through the MLN [11] (change of transport modes can significantly speed up time or reduce the cost of movement of passengers and cargos). The shortest path is built in aggregate-network, the structure of which is described by matrix $\boldsymbol{E}$. When such path is determined, it remains to choose the optimal by certain criteria carriers or system-operators on each edge of this path, the aggregate-weight of which exceeds 1.
2. Search for alternative paths of transit flows through different network layers during the isolation of certain zone in separate network layer (use of subway in large cities during traffic jams) [4].
3. Counteracting the spread of epidemics or computer viruses, which due to multi-layer interactions can spread much faster than in one layer [12]. Thus, the weighted aggregate-degree d_i^ε, of aggregate-network's node n_i calculated by formula

$$d_i^\varepsilon = \sum_{j=1}^{N^M} \varepsilon_{ji} = \sum_{j=1}^{N^M} \varepsilon_{ij}, \tag{2}$$

determines the number of nodes adjacent to n_i from which the threat of infection of this node may come or the number of nodes adjacent to n_i in which there may be a threat of infection from this node; the aggregate-weight ε_{ii} of node n_i determines the number of layers of MLN, in which this node can contribute to the spread of infection.
4. Finding the path from arbitrary node of one layer to arbitrary node of another layer, especially if they lie outside the intersection of sets of nodes of these layers. The ability to move a flow from one network layer to another and back through transition points expands access to nodes that are unreachable in separate network layers and allows communication between unconnected components of such layers (the movement of traffic flows across the oceans or to remote regions of separate countries—the northern regions of Canada, the central regions of Australia, etc.).

Using the aggregate-network to solve the above problems means reducing the dimensionality of source MLN's model or its quantitative complexity by M times.

4 Structural *k*- and *p*-cores of Aggregate-Network of Partially Overlapped MLN

In TCN, one of the ways to determine the structurally most important components of MLN and simplify the models of multilayer networks is the concept of **k**-core [13].

$$\boldsymbol{k} = \{k_1, k_2, ..., k_M\},$$

as a combination of k_m-cores of separate layers of MLN. The values k_m, $m = \overline{1, M}$, for different layers may differ, i.e. the **k**-core determines the elements that are structurally most important more for the MLN layers than for organization of interlayer interactions in it. To determine the structurally most important components of MLN at a whole, we can use the concept of k-cores of its weighted aggregate-network. Then under k_{ag}^{ε}-core of weighted aggregate-network we understand its largest subnetwork, the degree of nodes d_i^{ε} of which, calculated by means of formula (2), are not less than k_{ag}^{ε}. Adjacency matrices $\boldsymbol{E}(k_{ag}^{\varepsilon})$, which fully describe the structures of weighted k_{ag}^{ε}-cores of PO MLN aggregate-network, are obviously obtained from adjacency matrix $\boldsymbol{E}$ by excluding rows and columns for nodes whose values $d_i^{\varepsilon} < k_{ag}^{\varepsilon}$, $i = \overline{1, N^M}$. Indirectly, the weighted aggregate-degrees of nodes of PO MLN's aggregate-network determine the importance of this node in multilayer network, as duplication is usually subject to elements that implement the most important functions in the system. It should be noted that k_{ag}^{ε}-core of weighted aggregate-network of PO MLN provides much more important information for the study of intersystem interactions than **k**-core.

To solve the problem of determining the structurally most important components of intersystem interactions in MLN, we introduce the notion of p-core of partially overlapped multilayer network

$$\tilde{G}^p = (\tilde{V}^p, \tilde{E}^p),$$

as combination of those subnetworks of separate layers with connections between nodes of them that are part of at least p, $2 \le p \le M$, layers of MLN (it is obvious that $\tilde{G}^1$ is identical to the source multilayer network G^M defined by formula (1). In other words, the p-core of MLN is its multilayer subnet, each of the nodes of which has an aggregate-weight $\varepsilon_{ii} \ge p$. If

$$p_{\max} = \max_{i=\overline{1, N^M}} \{\varepsilon_{ii}\} = M,$$

that is, if the structural M-core of multilayer network is nonempty, then the studied MLN we will call the kernel (Fig. 3). Such structures are generated by the general transport system of the Earth and its separate continents, systems of maintenance of vital activity of the city, postal systems, systems of telephone communication, etc. It is obviously that the kernel $\tilde{G}^M$ has a multiplex structure. If the condition

$$p_{\max} < M,$$

is met then the studied MLN will call the non-kernel. Such MLN are generated by linguistic multilayer network systems (there is hardly at least one person who speaks all existing languages), social network systems, online services, mobile operators and provider of cable TV, etc. [7].

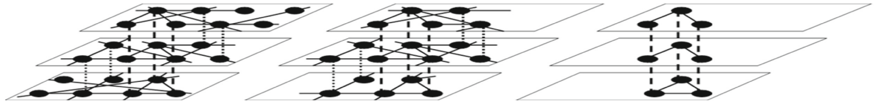

Fig. 3. Examples of reflected in Fig. 1 source three-layer partially overlapped kernel MLN (a) and its *2*- (b) and *3*-core (c)

Hereinafter, the p_{ag}-core of aggregate-network of MLN will be called such its weighted subnet, each node n_i of which has an aggregate weight ε_{ii} not less than p (Fig. 4).

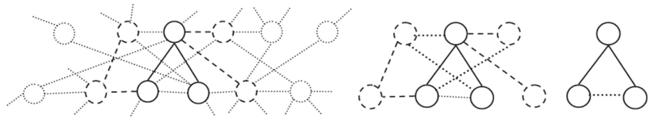

Fig. 4. Examples of weighted aggregate-network of reflected in Fig. 3 source three-layer partially overlapped kernel MLN (a) and its 2_{ag}- (b) and 3_{ag}-core (c, ____—element is part of three layers, **- - -**—element is part of two layers, **….**—element is part of one layer)

Denote by N^p the number of elements of the total set of nodes of the *p*-core of partially overlapped multilayer network. Let us determine the specific weight η^p of *p*-core transition points in this MLN by the ratio

$$\eta^p = N^p \Big/ N^M .$$

The *p*-cores determine the set of transition points that are directly involved in the organization of intersystem interactions at least of *p* layers of MLN. Therefore, the closer the value η^p to 1 for all $p = 1,2,\ldots, M$, the stronger the interaction of MLN layers. The specific weight of transition points of the layer determines its ability to participate in intersystem interactions with other layers of the multilayer network.

The value N^M only partially reflects the complexity of multilayer network. Much more adequate indicators are the characteristics of dimensional and connectional complexity of MLN. The dimensional complexity of partially overlapped MLN can be estimated through the diagonal elements of matrix **E** by means of parameter

$$\phi = \sum_{i=1}^{N^M} \varepsilon_{ii},$$

and its connectional complexity—using parameter

$$\varphi = \sum_{\substack{i,j=1 \\ i \neq j}}^{N^M} \varepsilon_{ij}.$$

To calculate the corresponding indicators of complexity of the *p*-core of MLN, we define the matrix

$$\boldsymbol{E}_p = \{\varepsilon_{ij}^p\}_{i,j=1}^{N^M}, \varepsilon_{ij}^p = \begin{cases} \varepsilon_{ij}, & \text{if } \varepsilon_{ii} \geq p, \\ 0, & \text{if } \varepsilon_{ii} < p, \end{cases}, i,j = \overline{1, N^M}, p = 2, 3, ..., M.$$

Then the dimensional complexity of *p*-core of PO MLN we determine by means of parameter

$$\phi_p = \sum_{i=1}^{N^M} \varepsilon_{ii}^p,$$

and its connectional complexity—using parameter

$$\varphi_p = \sum_{\substack{i,j=1 \\ i \neq j}}^{N^M} \varepsilon_{ij}^p, p = 2, 3, ..., M$$

Based on this, we can determine the reduction of dimensional complexity of the *p*-core model compared to the PO MLN model in general by the ratio

$$\eta_p = \phi_p / \phi,$$

and reduction of connectional complexity of the *p*-core model compared to the MF PO MLN model in general by the ratio

$$\mu_p = \varphi_p / \varphi, p = 2, 3, ..., M.$$

Note that k_{ag}^{ε}-cores are formed on the basis of values of the sum of nondiagonal elements of the rows or columns of matrix **E**. At the same time, *p*-cores are formed on the basis of values of the diagonal elements of matrix **E**. This means that there is no direct connection between the *k*- and *p*-cores. However, their properties can be combined as follows. To determine the sets of nodes with the largest values of aggregate-degree centrality, which at the same time take the greatest part in process of intersystem interactions, we introduce the notion of $p_{ag}(k_{ag}^{\varepsilon})$-core as a such subnet of source aggregate-network, nodes of which at first have aggregate-degree $d_i^{\varepsilon} \geq k_{ag}^{\varepsilon}$ and at second belong to not less than *p* layers of MLN. The adjacency matrix $\boldsymbol{E}(p_{ag}(k_{ag}^{\varepsilon}))$ of corresponding subnet of source MLN is obviously obtained from the adjacency matrix $\boldsymbol{E}(k_{ag}^{\varepsilon})$ by excluding rows and columns whose diagonal elements are smaller than p_{ag}. Conversely, to determine the sets of nodes that participate most in the process of intersystem interactions and among them have the greatest values of aggregate-degree centrality, we introduce the concept of $k_{ag}^{\varepsilon}(p_{ag})$-core as a subnet of source aggregate-network, nodes which at first are the part of at least p layers of MLN and at second have the aggregate-degree $d_i^{\varepsilon} \geq k_{ag}^{\varepsilon}$. The adjacency matrix $\boldsymbol{E}(k_{ag}^{\varepsilon}(p_{ag}))$ of corresponding subnet of the source MLN is obviously obtained from the adjacency matrix $\boldsymbol{E}_p$ by excluding rows and columns for which the nodes have the aggregate-degree $d_i^{\varepsilon} < k_{ag}^{\varepsilon}$. It is obvious that the structures of $p_{ag}(k_{ag}^{\varepsilon})$- and $k_{ag}^{\varepsilon}(p_{ag})$-cores, as well as the adjacency matrices $\boldsymbol{E}(p_{ag}(k_{ag}^{\varepsilon}))$ and $\boldsymbol{E}(k_{ag}^{\varepsilon}(p_{ag}))$ may differ, because the methods and purposes of construction of each of them are different. The use of k- and p-cores allows us at least partially to solve the problem of complexity of the study of real intersystem interactions, highlighting in them the most structurally important components.

5 Vulnerability of Intersystem Interactions

The study of stability of real systems and intersystem interactions of different types to targeted attacks is one of the most important problems of modern systems analysis. Let us that all layers of MLN are the free-scale networks [14], which are quite widespread and at the same time the most vulnerable to such attacks. The stability of separate system-layers can be determined using the scenarios proposed in [15, 16] etc. These scenarios are based on the defeat of network nodes with the highest degree or nodes with the highest betweenness centrality. Stability of MLN to targeted attacks is determined primarily by the vulnerability of transition points of the multilayer network, i.e. its p-cores with different values p, starting with the largest $p_{\max} \leq M$. We will build scenarios of attacks on intersystem interactions, using local and global characteristics of the elements of weighted aggregate-network, which determine the integrated indicators of their importance in PO MLN. Note that usually the removal of a particular node from the system structure leads to redistribution of traffic routes that passed through it by the network, and the establishment of new connections between the remaining nodes. This process reflects the reaction of system to changes in operating conditions and means a change of degrees and betweenness centralities of MLN nodes.

Before constructing scenarios of targeted attacks, it is necessary to determine the criteria for their success, i.e. the desired level of damage of system structure. Such criteria include the division of aggregate-network into unconnected components in which the MLS ceases to exist as a single supersystem formation, the cessation of intersystem interactions between layers, and so on. Let us build the first attack scenario on the basis of aggregate-weights and weighted aggregate-degrees of nodes of the weighted aggregate-network of MLN:

1. make a list of nodes of the aggregate-network of MLN in descending order of their aggregate-weights (values ε_{ii} of matrix $\mathbf{E}$);
2. in each group of nodes with the same values ε_{ii}, we arrange the nodes on the basis of decreasing values of their weighted aggregate degrees d_i^{ε};
3. delete the first node from the beginning of created list; if the selected criterion of attack success is fulfilled than algorithm is finished, otherwise go to the next point;

4. the removal of particular node usually leads to the establishment of new connections between the nodes that remain in MLN, i.e. the values d_i^{ε} of nodes may change; therefore, if the list of nodes with the current value ε_{ii} is not exhausted, we pass to point 2 of this scenario;
5. after processing the nodes of group with the current value $\varepsilon_{ii} > 1$ go to step 2 with the value ε_{ii}-1; if $\varepsilon_{ii} = 1, i = \overline{1, N^M}$, then the algorithm is finished.

The second scenario of targeted attack will be built on the use of aggregate-weights and betweenness centrality of nodes of the aggregate-network of MLN:

1. make a list of nodes of the aggregate-network of MLN in descending order of their aggregate weights (values ε_{ii} of matrix $\mathbf{E}$);
2. in each group of nodes with the same values ε_{ii}, we arrange the nodes on the basis of decreasing values of their betweenness centralities in aggregate-network of MLN;
3. delete the first node from the beginning of created list; if the selected criterion of attack success is fulfilled than algorithm is finished, otherwise go to the next point;
4. the removal of particular node usually leads to the establishment of new connections between the nodes that remain in MLN, i.e. the values of betweenness centrality of nodes may change; therefore, if the list of nodes with the current value ε_{ii} is not exhausted we pass to point 2 of this scenario;
5. after processing the nodes of group with the current value $\varepsilon_{ii} > 1$ go to step 2 with the value ε_{ii}-1; if $\varepsilon_{ii} = 1, i = \overline{1, N^M}$, then the algorithm is finished.

Usually, scenarios that list the characteristics of elements after removal of the next node and are based on the use of betweenness centrality, are more effective for achieving the goal of attack than scenarios that use degree centrality [17].

Along with described above, there is a reverse problem, which is to prevent the spread of epidemics of dangerous infectious diseases, computer viruses, invasion processes, etc. This problem is especially acute during Covid-19 and is the need to block those components of MLN that most "contribute" to the pandemic spreading [18]. The greatest risk of infection in settlements arises during direct contact of people in places of their mass concentration or constant communication. The spread of infection between settlements, regions and countries is due to large volumes of passenger traffic between them. These features are quite well correlated with the elements of MLN with large values of aggregate weights and weighted aggregate-degrees of nodes of its aggregate-network. Hence the ways to prevent the pandemic spreading, which are to block the nodes where crowds are possible, and to block the paths of flows motion to/from the nodes, which have a high level of infection. Obviously, to implement these ways, we can use the proposed above scenarios, in which the criterion for success of anti-epidemiological measures is to minimize access to the nodes with the highest level of infection.

In [15] was shown that the average performance of Internet is halved if only 1% of nodes-domains with the largest degrees fail, and the Internet becomes divided into unconnected components if 4% of such nodes fail. In Ukraine, the number of state-owned banks in the country's banking system does not exceed 0.7%. At the same time, the share of their assets in this system is 55.2%, and the share of deposits of individuals is 61.6% [19]. A successful attack on this group of banks will lead to the largest losses in financial system of the state. Simultaneous *DDoS*-attacks on January 14 and February 15, 2022 on the computer networks of more than 70 major state, security and financial structures of Ukraine became a serious threat to the public administration system [20]. This means that for critical destabilization or shutdown of the system operation, it is usually necessary to simultaneously block the operation of a certain group of nodes. Indeed, successive attacks on separate, even the most important nodes, often allow us to distribute their functions among other nodes of the system. This is duly taken into account in the above scenarios. However, to counteract the simultaneous successful attack on a group of the most important elements of MLN, and the main thing to overcome the consequences of such attack, is much more difficult. To defeat intersystem interactions, such group can be a certain set of the most important transition points. The notions of p_{ag}-, $p_{ag}(k_{ag}^{\varepsilon})$- and $k_{ag}^{\varepsilon}(p_{ag})$-cores defined in the previous paragraphs allow us to determine the most important for MLN groups of nodes, simultaneous targeted attacks on which will certainly cause the most damage or even lead to the cessation of intersystem interactions.

One of scenarios of such defeat, based on the use of notion of $k_{ag}^{\varepsilon}(p_{ag})$-core of aggregate-network of MLN, consists of sequential implementation of the following steps:

1. we accept equal $p_{ag} = p_{\max} \leq M$ and $k_{ag}^{\varepsilon} = \max\limits_{i=\overline{1,N^M}} d_i^{\varepsilon}$;
2. remove from structure of MLN nodes that belong to $k_{ag}^{\varepsilon}(p_{ag})$-core and its connections; if selected criterion for the attack success is met, the algorithm is finished, otherwise go to the next point;
3. reduce the value k_{ag}^{ε} by 1; if the list of nodes of $k_{ag}^{\varepsilon}(p_{ag})$-core is not exhausted, then go to point 2, otherwise go to the next point;
4. if the value $p_{ag} > 1$, then reduce it by 1 and go to point 2 with a value k_{ag}^{ε} equal to the maximum weighted aggregate-degree of remaining nodes of aggregate-network; otherwise the algorithm is finished.

To counteract the epidemic spreading, it is also advisable to use scenarios of blocking groups of nodes (settlements) in which the highest incidence is observed. Thus, the practice of preventing the spread of Covid-19 in Ukraine has shown that sequential blocking of the most infected nodes (settlements) gives a worse result than simultaneous blocking groups of nodes (regions of the country) from which the infection can potentially spread [21].

Usually, the spread of epidemics is stopped by the introduction of quarantine, i.e. the isolation of areas where carriers of infection are found. From a functional point of view, the isolation of certain subnet of the source network means the complete cessation or significant restriction of flows motion from (in, through) it. For many reasons, measures taken to combat the deployment of Covid-19 pandemic have transformed the world into network of isolated zones, movement of flows between which (especially human ones) reduced by dozens due to the cessation or significant restriction of rail, air and road services. Moreover, as a result of introduction of such restrictions, many states have also become networks of isolated communities or separate settlements. The self-isolation of majority of citizens, caused by traffic restrictions, large fines for non-compliance with quarantine conditions and the closure of enterprises or their operation in remote access, has significantly reduced not only external but also internal flows in such isolated zones. With the constant network structure there was a kind of "granulation" of the system, which was divided into a hierarchy of successively isolated subsystems in terms of limiting the interaction between them. Thus, a new type of "granular" networks has emerged, which has not been studied so far. In this case, the losses suffered by the system are not caused by blocking its separate component, but by restricting the movement of flows between all components as a whole. The possibility of transition of network system to the "granular" state, which is characterized by complete or partial isolation of all its components, is a separate kind of system-wide targeted attack. The collapse of USSR was a successful outcome of such attack, which was carried out in response to the war in Afghanistan, and led to its division (granulation) into separate independent states. As a result, the volume of flows within the former Soviet Union has declined significantly, and the total GDP of all these states has not yet reached the level of GDP of the former USSR. Such attacks can be implemented in the form of international sanctions against countries that threaten world security and so on. It should also be noted that much more effective scenarios of targeted attacks of various directions can be built on the basis of flow models of complex network systems and intersystem interactions [4, 22–24].

6 Conclusions

The study of real complex networks and intersystem interactions of different types allows us to understand many processes that take place in the physical world, nature and human society. The main obstacle that arises in this way is the problem of complexity, caused both by dimension of such systems and the number of heterogeneous interactions in them. One way to solve this problem is to highlight the most important from structural and functional point of view components of system that determine its behavior. To simplify the study of intra- and intersystem interactions, the article introduces the notion of aggregate-network, and its k- and p-cores, which determines the most structurally important components of MLN. It is shown how the notions introduced in the paper can be used to solve a number of practically important problems of the theory of complex networks, in particular the stability of intersystem interactions to various vulnerable factors of both artificial and natural origin. As the emergence of new threats, such as Covid-19, is not excluded, the development of scenarios for such defeats and means of timely counteraction to them is an extremely important applied problem, for the theoretical and practical solution of which a lot of effort will be spent.

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