Fuzzy Soft Relations-Based Rough Soft Sets Classified by Overlaps of Successor Classes with Measurement Issues

Aiyared Iampan[1] and Rukchart Prasertpong[2(✉)]

[1] Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand

[2] Division of Mathematics and Statistics, Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Nakhon Sawan 60000, Thailand

rukchart.p@nsru.ac.th

Abstract. In this paper, a new class for data classification in rough set theory is defined as an overlap of successor classes. This kind of rough approximation is proposed via fuzzy soft relations. Depending on the class of fuzzy soft relation, this paper defines the upper and lower rough approximations of a soft set. At this point, the fundamental of rough soft sets and definable soft sets is proposed. Then, some related theories are proved. In the aftermath, the notion of accuracy and roughness measures of soft sets in terms of rough set theory is studied combined with the concept of distance measures.

Keywords: Fuzzy soft relation · Rough set · Soft set · Rough soft set

1 Introduction

For a given non-empty universal set V and an equivalence relation E on V, a pair (V, E) is denoted as a Pawlak's approximation space, and $[v]_E$ is denoted as an equivalence class of $v \in V$ induced by E. Now, let (V, E) be a given Pawlak's approximation space and let X be a subset of V. The set

$$\overline{E}(X) := \{v \in V : [v]_E \cap X \neq \emptyset\}$$

is said to be an *upper approximation* of X within (V, E). The set

$$\underline{E}(X) := \{v \in V : [v]_E \subseteq F(a)\}$$

is said to be a *lower approximation* of X within (V, E). A difference $\overline{E}(X) - \underline{E}(X)$ is said to be a *boundary region* of X within (V, E). Three sets are obtained the following interpretation.

- The upper approximation $\overline{E}(X)$ of X contains all objects which possibly belong to X. At this point, a complement of $\overline{E}(X)$ is said to be a *negative region* of X within (V, E).

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- The lower approximation $\underline{E}(X)$ of X consists of all objects which surely belong to X. In this way, such the set is said to be a *positive region* of X within (V, E).
- $\overline{E}(X) - \underline{E}(X)$ is a set of all objects, which can be classified neither as X nor as non-X using E.

In what follows, a pair $(\overline{E}(X), \underline{E}(X))$ is said to be a *rough set* of X within (V, E) if $\overline{E}(X) - \underline{E}(X)$ is a non-empty set. In this way, X is said to be a rough set. X is said to be a *definable* (or an *exact*) *set* within (V, E) if $\overline{E}(X) - \underline{E}(X)$ is an empty set.

As mentioned above, it is a classical theory proposed by Pawlak [1] in 1982. Observe that the notion of a Pawlak's rough set theory is classified by all equivalence classes via an equivalence relation. At this point, it has been extended to arbitrary binary relations and fuzzy relations. In 2019, Prasertpong and Siripitukdet [2] proposed the fundamental of rough sets induced by fuzzy serial relations. It is classified by overlaps of successor classes with respect to level in a closed unit interval under a fuzzy serial relation. This class is defined as follows. Let R be a fuzzy serial relation from V to W and $\alpha \in [0, 1]$ [0, 1]. For an element $v \in V$, the set

$$[v]_R^S := \{w \in W \ : \ R(v, w) \ \geq \ \alpha\}$$

is said to be a *successor class* of v with respect to α–level based on R. For $v \in V$,

$$[v]_R^{OS} := \{v' \in V \ : \ [v]_R^S \cap \left[v'\right]_R^S \neq \emptyset\}$$

is called an *overlap of the successor class* of v with respect to α–level based on R.

As an extension of Zadeh's fuzzy relations [3], the notion of Zhang's fuzzy soft relations [4] is a mathematical tool for dealing with uncertainty problems. Then, in this research, an overlap of the successor class is considered in terms of fuzzy soft relations to rough approximations. In Sect. 3, the contributions of the section are as follows.

- We extend the concept of fuzzy serial relations by the sense of fuzzy soft relations. That is, a fuzzy soft serial relation over two universes is proposed. An overlap of successor classes via fuzzy soft serial relations is defined. Some related properties are investigated.
- We propose the notion of upper and lower rough approximations of a soft set based on overlaps of successor classes. We introduce the concept of rough soft sets and definable soft sets, and a corresponding example is provided. The relationships between such the softs and fuzzy soft relations are verified.
- As a novel rough soft set theory of the section, we further study the argument to accuracy and roughness measures of soft sets in terms of Pawlak's rough set theory. The relationship between a roughness measure and a distance measure is discussed.

In the end, the work is summarized in Sect. 4.

2 Preliminaries

In this section, let us first review some basic concepts which wi/ll be necessary for subsequent sections. Throughout this paper, K, V, and W denote non-empty sets.

2.1 Some Basic Notions of Fuzzy Sets

Definition 2.1.1 [5]. f is said to be a *fuzzy subset (*or *fuzzy set)* of V if it is a function from V to the closed unit interval [0, 1]. In this way, $FP(V)$ is denoted as a collection of all fuzzy subsets of V.

Definition 2.1.2 [5]. Let f and g be fuzzy subsets of V. $f \prec g$ is denoted by meaning $f(v) \leq g(v)$ for all $v \in V$.

Definition 2.1.3 [3]. An element in $FP(V \times W)$ is said to be a *fuzzy relation* from V to W. An element in $FP(V \times V)$ is a fuzzy relation on V if W is replaced by V. Given a fuzzy relation R from V to W and $v \in V, w \in W$, the value $R(v, w)$ in [0, 1] is the *membership grade* of the relation between v and w based on R. If R is a given fuzzy relation from V to W, where $V = \{v_1, v_2, v_3, \ldots, v_p\}$ and $W = \{w_1, w_2, w_3, \ldots, w_q\}$, then every membership grade under R is represented by the $p \times q$ matrix form as

$$\begin{pmatrix} R(v_1, w_1) & R(v_1, w_2) & R(v_1, w_3) & \ldots & R(v_1, w_q) \\ R(v_2, w_1) & R(v_2, w_2) & R(v_2, w_3) & \ldots & R(v_2, w_q) \\ R(v_3, w_1) & R(v_3, w_2) & R(v_3, w_3) & \ldots & R(v_3, w_q) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ R(v_p, w_1) & R(v_p, w_2) & R(v_p, w_3) & \cdots & R(v_p, w_q) \end{pmatrix}.$$

Definition 2.1.4 [6]. Let $R \in FP(V \times W)$. R is said to be a *fuzzy serial relation* if for all $v \in V$, there exists $w \in W$ such that $R(v, w) = 1$.

Definition 2.1.5 [3]. Let $R \in FP(V \times V)$.

(1) R is said to be a *fuzzy reflexive relation* if $R(v, v) = 1$ for all $v \in V$.
(2) R is said to be a *fuzzy transitive relation* if it satisfies $R(v_1, v_2) = \sup_{v \in V}(R(v_1, v) \wedge R(v, v_2))$ for all $v_1, v_2 \in V$.
(3) R is said to be a *fuzzy symmetric relation* if $R(v_1, v_2) = R(v_2, v_1)$ for all $v_1, v_2 \in V$.
(4) R is said to be a *fuzzy equivalence relation* if it is a fuzzy reflexive relation, a fuzzy transitive relation, and a fuzzy symmetric relation.

Definition 2.1.6 [7]. Let $R \in FP(V \times V)$. R is said to be a *fuzzy antisymmetric relation* if for all $v_1, v_2 \in V$, $R(v_1, v_2) > 0$ and $R(v_2, v_1) > 0$ imply $v_1 = v_2$.

2.2 Some Basic Notions of Soft Sets and Fuzzy Soft Relations

Definition 2.2.1 [8]. Throughout this work, $P(V)$ denotes a power set of V. Let A be a non-empty subset of K. If F is a mapping from A to $P(V)$, then (F, A) is said to be a *soft set* over V with respect to A. As the understanding of the soft set, V is said to be a universe of all alternative objects of (F, A), and K is said to be a set of all parameters of (F, A), where the parameter is an attribute, a characteristic or a statement of the alternative object of V. For any element $a \in A$, $F(a)$ is considered as a set of a-approximate elements (or a-alternative objects) of (F, A).

Definition 2.2.2 [9]. Let A be a non-empty subset of K.

(1) A *relative null soft set* over V with respect to A is denoted by $\mathcal{N}_{\emptyset_A} := (\emptyset_A, A)$, where $\emptyset_A$ is a set valued-mapping given by $\emptyset_A(a) = \emptyset$ for all $a \in A$.
(2) For a soft set $\mathscr{F} := (F, A)$ over V with respect to A, a *support* of $\mathscr{F}$ is denoted by $Supp(\mathcal{F})$, where $Supp(\mathcal{F}) := \{a \in A : F(a) \neq \emptyset\}$.
(3) A *relative whole soft set* over V with respect to A is denoted by $\mathcal{V}_{V_A} := (V_A, A)$, where V_A is a set valued-mapping given by $V_A(a) = V$ for all $a \in A$.

Definition 2.2.3 [9]. Let $\mathscr{F} := (F, A)$ and $\mathcal{G} := (G, B)$ be two soft sets over a common alternative universe with respect to non-empty subsets A and B of K, respectively. $\mathscr{F}$ is a soft subset of $\mathcal{G}$ if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$. We denote by $\mathcal{F} \Subset \mathcal{G}$.

Definition 2.2.4 [9]. Let $\mathscr{F} := (F, A)$ and $\mathcal{G} := (G, B)$ be two soft sets over a common alternative universe with respect to non-empty subsets A and B of K, respectively.

(1) A *restricted intersection* of $\mathscr{F}$ and $\mathcal{G}$, denoted by $\mathscr{F} \cap_r \mathcal{G}$ is defined as a soft set (H, C), where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$.
(2) A *restricted union* of $\mathscr{F}$ and $\mathcal{G}$, denoted by $\mathscr{F} \cup_r \mathcal{G}$ is defined as a soft set (H, C), where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.
(3) An *extended intersection* of $\mathscr{F}$ and $\mathcal{G}$, denoted by $\mathscr{F} \cap_e \mathcal{G}$ is defined as a soft set (H, C), where $C = A \cup B$ and $H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \\ F(c) \cap G(c) & \text{if } c \in A \cap B \end{cases}$ for all $c \in C$.
(4) An *extended union* of $\mathscr{F}$ and $\mathcal{G}$, denoted by $\mathscr{F} \cup_e \mathcal{G}$ is defined as a soft set (H, C), where $C = A \cup B$ and $H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \\ F(c) \cup G(c) & \text{if } c \in A \cap B \end{cases}$ for all $c \in C$.
(5) A *restricted difference* of $\mathscr{F}$ and $\mathcal{G}$, denoted by $\mathscr{F} -_r \mathcal{G}$ is defined as a soft set (H, C), where $C = A \cap B$ and $H(c) = F(c) - G(c)$ for all $c \in C$.

Definition 2.2.5 [10]. Let A be a non-empty subset of K. If F is a mapping from A to $FP(V)$, then (F, A) is said to be a *fuzzy soft set* over V with respect to A.

Definition 2.2.6 [10]. Let $\mathscr{F} := (F, A)$ and $\mathscr{F} := (G, B)$ be two fuzzy soft sets over a common alternative universe with respect to non-empty subsets A and B of K, respectively. $\mathscr{F}$ is a fuzzy soft subset of $\mathcal{G}$ if $A \subseteq B$ and $F(a) \prec G(a)$ for all $a \in A$.

Definition 2.2.7 [4]. Let A be a non-empty subset of K. If F is a mapping from A to $FP(V \times W)$, then (F, A) is said to be a *fuzzy soft relation* over $V \times W$.

Definition 2.2.8 [4]. Let A be a non-empty subset of K, and let $\mathcal{R} := (R, A)$ be a fuzzy soft relation over $V \times V$.

(1) $\mathcal{R}$ is called a *fuzzy soft reflexive relation* if $R(a)$ is a fuzzy reflexive relation for all $a \in A$.
(2) $\mathcal{R}$ is called a *fuzzy soft transitive relation* if $R(a)$ is a fuzzy transitive relation for all $a \in A$.
(3) $\mathcal{R}$ is called a *fuzzy soft symmetric relation* if $R(a)$ is a fuzzy symmetric relation for all $a \in A$.
(4) $\mathcal{R}$ is called a *fuzzy soft equivalence relation* if it is a fuzzy soft reflexive relation, a fuzzy soft transitive relation, and a fuzzy soft symmetric relation.

3 Main Results

In this section, we propose the concept of a successor class and an overlap of the successor classes based on fuzzy soft relations. The related theories are verified. Then, upper and lower rough approximations of a soft set are proposed under the classification of all overlaps of the successor classes. Of course, the notion of rough soft set is defined. A corresponding example is provided. Furthermore, the measurement issue is discussed via rough set theory.

Throughout this section, the set A and B denote two non-empty subsets of V.

3.1 Overlaps of Successor Classes via Fuzzy Soft Relations

In this subsection, we construct a new class for rough approximations of a soft set. Such the class is called the overlaps of successor classes based on fuzzy soft relations.

Definition 3.1.1 Let $\mathcal{R} := (R, A)$ be a fuzzy soft relation over $V \times W$ and $\alpha \in [0, 1]$. For an element $v \in V$, the set

$$[v]^S_{\mathcal{R},\alpha} := \{w \in W \ : \ R(a)(v, w) \ \geq \ \alpha, \forall a \in A\}$$

is called a *successor class* of v with respect to α–level based on $\mathcal{R}$. We denote by $[V]^S_{\mathcal{R},\alpha}$ the collection of $[v]^S_{\mathcal{R},\alpha}$ for all $v \in V$.

Example 3.1.1 Let
$V = \{v_i \in \mathbb{R} : i \in \mathbb{N},\ 1 \leq i \leq 6\}$ and $W = \{w_i \in \mathbb{R} : i \in \mathbb{N}, 1 \leq i \leq 5\}$. We define a fuzzy relation $\theta \in FP(V \times W)$ by the matrix representation as

$$\begin{pmatrix} 0.6 & 0.3 & 0.7 & 0.9 & 0.4 \\ 0.2 & 0.4 & 0.8 & 0.1 & 0.3 \\ 0.7 & 0.4 & 0.2 & 0.2 & 0.4 \\ 0.4 & 0.8 & 0.2 & 0.1 & 0.3 \\ 0.2 & 0.9 & 0.1 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.4 & 0.4 & 0.6 \end{pmatrix}.$$

Suppose that $\mathcal{R} := (R, A)$ is a fuzzy soft relation over $V \times W$ defined by $R(a) = \theta$ for all $a \in A$. Then, the successor class of each element in V with respect to 0.5–level based on R is presented by

$$[v_1]^S_{\mathcal{R},0.5} = \{w_1, w_3, w_4\}, [v_2]^S_{\mathcal{R},0.5} = \{w_3\}, [v_3]^S_{\mathcal{R},0.5} = \{w_1\},$$

$[v_4]^S_{\mathcal{R},0.5} = \{w_2\}$, $[v_5]^S_{\mathcal{R},0.5} = \{w_2\}$, and $[v_6]^S_{\mathcal{R},0.5} = \{w_5\}$. This is a corresponding example of Definition 3.1.1.

Definition 3.1.2 Let $\mathcal{R} := (R, A)$ be a fuzzy soft relation over $V \times W$. $\mathcal{R}$ is called a *fuzzy soft serial relation* if $R(a)$ is a fuzzy serial relation for all $a \in A$.

Remark 3.1.1 A fuzzy soft serial relation over $V \times V$ is a generalization concept of a fuzzy soft reflexive relation over $V \times V$.

Proposition 3.1.1 If $\mathcal{R} := (R, A)$ is a fuzzy soft serial relation over $V \times W$ and $\alpha \in [0, 1]$, then $[v]^S_{\mathcal{R},\alpha} \neq \emptyset$ for all $v \in V$.

Proof. Assume that $\mathcal{R}$ is a fuzzy soft serial relation over $V \times W$ and $\alpha \in [0, 1]$. Now, we let $v \in V$. Then, there exists $w \in W$ such that

$$R(a)(v, w) = 1 \geq \alpha$$

for all $a \in A$. Therefore $w \in [v]^S_{\mathcal{R},\alpha}$. It follows that $[v]^S_{\mathcal{R},\alpha} \neq \emptyset$.

Definition 3.1.3 Let $\mathcal{R} := (R, A)$ be a fuzzy soft serial relation over $V \times W$ and $\alpha \in [0, 1]$. For an element $v \in V$, the set

$$[v]^{OS}_{\mathcal{R},\alpha} := \{v' \in V \ : \ [v]^S_{\mathcal{R},\alpha} \cap [v']^S_{\mathcal{R},\alpha} \neq \emptyset\}$$

is called an *overlap of successor class* of v with respect to α–level based on $\mathcal{R}$ We shall denote by $[V]^{OS}_{\mathcal{R},\alpha}$ the collection of $[v]^{OS}_{\mathcal{R},\alpha}$ for all $v \in V$.

Example 3.1.2 Based on Example 3.1.1, we observe that

$$[v_1]^{OS}_{\mathcal{R},0.5} = \{v_1, v_2, v_3\}, [v_2]^{OS}_{\mathcal{R},0.5} = \{v_1, v_2\}, [v_3]^{OS}_{\mathcal{R},0.5} = \{v_1, v_3\},$$

$[v_4]^{OS}_{\mathcal{R},0.5} = \{v_4, v_5\}$, $[v_5]^{OS}_{\mathcal{R},0.5} = \{v_4, v_5\}$, and $[v_6]^{OS}_{\mathcal{R},0.5} = \{v_6\}$. Observe that it is a corresponding example of Definition 3.1.3.

Proposition 3.1.2 If $\mathcal{R} := (R, A)$ is a fuzzy soft serial relation over $V \times W$ and $\alpha \in [0, 1]$, then $v \in [v]^{OS}_{\mathcal{R},\alpha}$ for all $v \in V$.

Proof. Suppose that $\mathcal{R}$ is a fuzzy soft serial relation over $V \times W$ and $\alpha \in [0, 1]$. Then, by Proposition 3.1.1, we have $[v]^S_{\mathcal{R},\alpha} \neq \emptyset$ for all $v \in V$. Let $v \in V$. Then $[v]^S_{\mathcal{R},\alpha} \cap [v]^S_{\mathcal{R}\alpha} \neq \emptyset$. This implies that $v \in [v]^{OS}_{\mathcal{R},\alpha}$.

Proposition 3.1.3 Let $\mathcal{R} := (R, A)$ be a fuzzy soft serial relation over $V \times V$ and $\alpha \in [0, 1]$. If $\mathcal{R}$ is a fuzzy soft reflexive relation, then $[v]^S_{\mathcal{R},\alpha} \subseteq [v]^{OS}_{\mathcal{R},\alpha}$ for all $v \in V$.

Proof. Suppose that $\mathcal{R}$ is a fuzzy soft reflexive relation and $v \in V$. Assume that $v' \in [v]^S_{\mathcal{R},\alpha}$. Then, we get that $v' \in V$. Thus $R(a)(v', v') = 1 \geq \alpha$ for all $a \in A$. Whence $v' \in [v']^S_{\mathcal{R},\alpha}$. Hence $[v]^S_{\mathcal{R},\alpha} \cap [v']^S_{\mathcal{R},\alpha} \neq \emptyset$. Then $v' \in [v]^{OS}_{\mathcal{R},\alpha}$. It follows that $[v]^S_{\mathcal{R},\alpha} \subseteq [v]^{OS}_{\mathcal{R},\alpha}$.

Proposition 3.1.4 If $\mathcal{R} := (R, A)$ is a fuzzy soft equivalence relation over $V \times V$ and $\alpha \in [0, 1]$, then $[v]^S_{\mathcal{R},\alpha}$ and $[v]^{OS}_{\mathcal{R},\alpha}$ are identical for all $v \in V$.

Proof. By Proposition 3.1.3, we obtain that $[v]^S_{\mathcal{R},\alpha}$ is a subset of $[v]^{OS}_{\mathcal{R},\alpha}$ for all $v \in V$. Let $v \in V$ be given. Suppose that $v' \in [v]^{OS}_{\mathcal{R},\alpha}$. Then $[v]^S_{R,\alpha} \cap [v']^S_{R,\alpha} \neq \emptyset$. Thus, there exists $v'' \in V$ such that $v'' \in [v]^S_{\mathcal{R},\alpha} \cap [v']^S_{\mathcal{R},\alpha}$. It is true that $R(a)(v, v'') \geq \alpha$ and $R(a)(v', v'') \geq \alpha$ for all $a \in A$. Since $\mathcal{R}$ is a fuzzy soft symmetric relation, we have $R(a)(v'', v') \geq \alpha$ for all $a \in A$. Since $\mathcal{R}$ is a fuzzy soft transitive relation, we get that

$$\begin{aligned} R(a)(v, v') &\geq \sup_{v''' v \in V} (R(a)(v, v''') \wedge R(a)(v''' v, v')) \\ &\geq R(a)(v, v'') \wedge R(a)(v'', v') \\ &\geq \alpha \wedge \alpha \\ &= \alpha \end{aligned}$$

for all $a \in A$. Hence $v' \in [v]^S_{\mathcal{R},\alpha}$. Thus $[v]^{OS}_{\mathcal{R},\alpha} \subseteq [v]^S_{\mathcal{R},\alpha}$. Hence $[v]^S_{\mathcal{R},\alpha} = [v]^{OS}_{\mathcal{R},\alpha}$.

Next, we shall introduce the notion of fuzzy soft antisymmetric relations in terms of fuzzy soft relations on a single universe.

Definition 3.1.4 Let $\mathcal{R} := (R, A)$ be a fuzzy soft relation over $V \times V$. $\mathcal{R}$ is called a *fuzzy soft antisymmetric relation* if $R(a)$ is a fuzzy antisymmetric relation for all $a \in A$.

Example 3.1.3 Let $V = \{v_i \in \mathbb{R} : i \in \mathbb{N} \text{ and } 1 \leq i \leq 6\}$. Suppose that $\theta \in FP(V \times V)$ is a fuzzy relation defined by the square matrix representation as.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Assume that $\mathcal{R} := (R, A)$ is a fuzzy soft relation over $V \times V$ defined by $R(a) = \theta$ for all $a \in A$. Then, it is easy to check that $\mathcal{R}$ is a fuzzy soft antisymmetric relation over $V \times V$. In fact, for all $a \in A$, $v, v' \in V$, $R(a)(v, v') > 0$ and $R(a)(v', v) > 0$ imply $v = v'$. This is a corresponding example of Definition 3.1.4.

Proposition 3.1.5 Let $\mathcal{R} := (R, A)$ be a fuzzy soft serial relation over $V \times V$ and $\alpha \in (0, 1]$. If $[V]^S_{R,\alpha}$ is the partition of V and $\mathcal{R}$ is a fuzzy soft reflexive relation and a fuzzy soft antisymmetric relation over $V \times V$, then the following statements are equivalent.

(1) $v = v'$ for all $v, v' \in V$.

(2) $[v]^{OS}_{\mathcal{R},\alpha} = \left[v'\right]^{OS}_{\mathcal{R},\alpha}$ for all $v, v' \in V$.
(3) $v \in \left[v'\right]^{OS}_{\mathcal{R},\alpha}$ for all $v, v' \in V$.

Proof. It is clear that (1) implies (2). According to Proposition 3.3, we obtain that (2) implies (3). In order to prove that (3) implies (1), we let $v, v' \in V$ be such that $v \in \left[v'\right]^{OS}_{\mathcal{R},\alpha}$. Then $[v]^{S}_{\mathcal{R},\alpha} \cap \left[v'\right]^{S}_{\mathcal{R},\alpha} \neq \emptyset$. Hence $[v]^{S}_{\mathcal{R},\alpha} = \left[v'\right]^{S}_{\mathcal{R},\alpha}$. Since $\mathcal{R}$ is a fuzzy soft reflexive relation, it is easy to prove that $v \in [v]^{S}_{\mathcal{R},\alpha}$ and $v' \in \left[v'\right]^{S}_{\mathcal{R},\alpha}$. Thus $v \in \left[v'\right]^{S}_{\mathcal{R},\alpha}$ and $v' \in [v]^{S}_{\mathcal{R},\alpha}$. Therefore $R(a)(v', v) \geq \alpha > 0$ and $R(a)(v, v') \geq \alpha > 0$ for all $a \in A$. As $\mathcal{R}$ is a fuzzy soft antisymmetric relation, we obtain that $v = v'$.

3.2 Rough Soft Sets Based on Overlaps of Successor Classes

As rough set theory based on a non-partition classification, in this subsection, the foundation of roughness of soft sets induced by all overlaps of successor classes is proposed via fuzzy soft relations. This concept is generated by upper and lower rough approximations. That is, it is constructed by two distinct classes classified by all overlaps of successor classes. Then, some related properties are investigated under the novel notion.

Definition 3.2.1 If $\alpha \in [0, 1]$ and $\mathcal{R} := (R, K)$ is a fuzzy soft relation over $V \times W$ related to $[V]^{OS}_{\mathcal{R},\alpha}$, then $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ is called an *approximation space* based on $[V]^{OS}_{\mathcal{R},\alpha}$.

Definition 3.2.2 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,\alpha})$ be a given approximation space, and let $\mathscr{F} := (F, A)$ be a soft set over V. An upper rough approximation of $\mathscr{F}$ within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ is denoted by $\overline{\mathscr{F}}\Big|^{OS}_{\mathcal{R},\alpha} := (\overline{F}\big|^{OS}_{\mathcal{R},\alpha}, A)$, where

$$\overline{F}\Big|^{OS}_{\mathcal{R},\alpha}(a) := \left\{v \in V : [v]^{OS}_{\mathcal{R},\alpha} \cap F(a) \neq \emptyset\right\}$$

for all $a \in A$. A lower rough approximation of $\mathscr{F}$ within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ denoted by $\underline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} := (\underline{F}\big|^{OS}_{\mathcal{R},\alpha}, A)$, where

$$\underline{F}\Big|^{OS}_{\mathcal{R},\alpha}(a) := \left\{v \in V : [v]^{OS}_{\mathcal{R},\alpha} \subseteq F(a)\right\}$$

for all $a \in A$. A boundary region of $\mathscr{F}$ within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ is denoted by $\mathscr{F}]^{OS}_{\mathcal{R},\alpha} := (F]^{OS}_{\mathcal{R},\alpha}, A)$, where

$$\mathscr{F}]^{OS}_{\mathcal{R},\alpha} = \overline{\mathscr{F}}\Big|^{OS}_{\mathcal{R},\alpha} -_r \underline{\mathscr{F}}\Big|^{OS}_{\mathcal{R},\alpha}.$$

As introduced above, such sets are obtained the following interpretation.

(1) $\overline{F}\big|^{OS}_{\mathcal{R},\alpha}(a)$ is a set of all elements, which can be possibly classified as $F(a)$ using $\mathcal{R}$ (are possibly in view of $\mathcal{R}$) for all $a \in A$. In this way, a complement of $\overline{F}\big|^{OS}_{\mathcal{R},\alpha}(a)$ is said to be a *negative region* of $F(a)$ within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ for all $a \in A$.

(2) $\underline{F}|^{OS}_{\mathcal{R},\alpha}(a)$ is a set of all elements, which can be certain classified as $F(a)$ using $\mathcal{R}$ (are certainly $F(a)$ in view of $\mathcal{R}$) for all $a \in A$. In this way, such the set is said to be a *positive region* of $F(a)$ within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ for all $a \in A$.

(3) $F]^{OS}_{\mathcal{R},\alpha}(a)$ is a set of all elements, which can be classified neither as $F(a)$ nor as non-$F(a)$ using $\mathcal{R}$ for all $a \in A$.

As introduced above, for all $a \in A$, if $F]^{OS}_{\mathcal{R},\alpha}(a) \neq \emptyset$, then $(\overline{F}|^{OS}_{\mathcal{R},\alpha}(a), \underline{F}|^{OS}_{\mathcal{R},\alpha}(a))$ is called a rough (or an inexact) set of $F(a)$ within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ and we call $F(a)$ a *rough set*. For all $a \in A$, if $F]^{OS}_{\mathcal{R},\alpha}(a) = \emptyset$, then $F(a)$ is called a *definable* (or an *exact*) *set* within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$. The soft set $\mathscr{F}$ is called a *definable soft set* within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$ if $\mathscr{F}]^{OS}_{\mathcal{R},\alpha} = \mathcal{N}_{\emptyset_A}$; otherwise $\mathscr{F}$ is called a *rough soft set* within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$.

In the following, we shall introduce a corresponding example of Definition 3.2.2.

Example 3.2.1 Define an approximation space $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,0.5})$ based on the data from Example 3.1.1, where $\mathcal{R} := (R, K) = (R, A)$. Suppose that $\mathscr{F} := (F, A)$ is a soft set over V defined by.

$$F(a) = \{v_1, v_3, v_5\}$$

for all $a \in A$. Then, by Example 3.1.2, it is true that

$$\overline{F}|^{OS}_{\mathcal{R},0.5}(a) = \{v_i : i \in \mathbb{N} \text{ and } 1 \leq i \leq 5\},$$

$$\underline{F}|^{OS}_{\mathcal{R},0.5}(a) = \{v_3\}, \text{ and}$$

$$F]^{OS}_{\mathcal{R},0.5}(a) = \{v_1, v_2, v_4, v_5\}$$

for all $a \in A$. Therefore $\mathscr{F}$ is a rough soft set within $(V, W, [V]^{OS}_{\mathcal{R},0.5})$. Observe that upper and lower approximations are necessary for approximated soft sets. In addition, negative and positive regions exist in $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,0.5})$.

Remark 3.2.1 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,\alpha})$ be a given approximation space, and let $\mathscr{F} := (F, A)$ be a soft set over V. Then, it is easy to see that $\underline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha} \Subset \mathcal{F} \Subset \overline{\mathscr{F}}\Big|^{\text{OS}}_{\mathcal{R},\alpha}$. This is a relationship between upper and lower approximations in general.

The following three results are a straightforward consequence of Definition 3.2.2.

Proposition 3.2.1 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,\alpha})$ be a given approximation space. If $\mathscr{F} := (F, A)$ is a soft set over V, then we have the following statements.

(1) If $\mathscr{F} = \mathcal{V}_{V_A}$, then $\mathscr{F}$ is equal to $\underline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha}$ and $\overline{\mathscr{F}}\Big|^{OS}_{\mathcal{R},\alpha}$. Moreover, $\mathscr{F}$ is a definable soft set within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$.

(2) If $\mathscr{F} = \mathcal{N}_{\emptyset_A}$, then $\mathscr{F}$ is equal to $\underline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha}$ and $\overline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha}$. Moreover, $\mathscr{F}$ is a definable soft set within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$.

Proposition 3.2.2 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,\alpha})$ be a given approximation space, and let $\mathscr{F} := (F, A)$ and $\mathcal{G} := (G, B)$ be soft sets over V. Then, we have the following statements.

(1) $\overline{\mathscr{F} \cup_r \mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} = \overline{\mathscr{F}}\big|^{OS}_{\mathscr{F},\alpha} \cup_r \overline{\mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha}$.
(2) $\overline{\mathscr{F} \cup_e \mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} = \overline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \cup_e \overline{\mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha}$.
(3) $\underline{\mathscr{F} \cap_r \mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} = \underline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \cap_r \underline{\mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha}$.
(4) $\underline{\mathscr{F} \cap_e \mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} = \underline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \cap_e \underline{\mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha}$.
(5) $\overline{\mathscr{F} \cap_r \mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} \Subset \overline{\mathscr{F}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha} \cap_{\mathrm{r}} \overline{\mathcal{G}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$.
(6) $\overline{\mathscr{F} \cap_e \mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} \Subset \overline{\mathscr{F}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha} \cap_{\mathrm{e}} \overline{\mathcal{G}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$.
(7) $\underline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \cup_r \underline{\mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} \Subset \underline{\mathscr{F} \cup_{\mathrm{r}} \mathrm{G}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$.
(8) $\underline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \cup_e \underline{\mathcal{G}}\big|^{OS}_{\mathcal{R},\alpha} \Subset \underline{\mathscr{F} \cup_{\mathrm{e}} \mathcal{G}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$.

Proposition 3.2.3 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,\alpha})$ be a given approximation space, and let $\mathscr{F} := (F, A)$ and $\mathcal{G} := (G, B)$ be soft sets over V. If $\mathcal{F} \Subset \mathcal{G}$, then $\underline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \Subset \underline{\mathcal{G}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$ and $\overline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \Subset \overline{\mathcal{G}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$.

Proposition 3.2.4 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\,\alpha})$ and let $(V, W, [V]^{OS}_{\mathcal{S}:=(S,K),\,\beta})$ be given two approximation spaces with property that $\mathcal{R} \Subset \mathcal{S}$ and that $\alpha \geq \beta$. If $\mathscr{F} := (F, A)$ is a soft set over V, then $\underline{\mathscr{F}}\big|^{OS}_{\mathcal{S},\beta} \Subset \underline{\mathscr{F}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$ and $\overline{\mathscr{F}}\big|^{OS}_{\mathcal{R},\alpha} \Subset \overline{\mathscr{F}}\big|^{\mathrm{OS}}_{\mathcal{S},\beta}$.

Proof. Suppose that $\mathscr{F}$ is a soft set over V. We shall show that $\underline{\mathscr{F}}\big|^{OS}_{\mathcal{S},\beta} \Subset \underline{\mathscr{F}}\big|^{\mathrm{OS}}_{\mathcal{R},\alpha}$. Let $a \in A$ be given. Suppose $v \in \underline{F}\big|^{OS}_{\mathcal{S},\beta}(a)$. Then $[v]^{OS}_{\mathcal{S},\beta} \subseteq F(a)$. Now, we must to prove that $[v]^{OS}_{\mathcal{R},\alpha} \subseteq [v]^{OS}_{\mathcal{S},\beta}$. Assume that $v' \in [v]^{OS}_{\mathcal{R},\alpha}$. Then $[v]^{S}_{\mathcal{R}\alpha} \cap \left[v'\right]^{S}_{\mathcal{R}\alpha} \neq \emptyset$. There exists $v'' \in V$ such that $v'' \in [v]^{S}_{\mathcal{R},\alpha} \cap \left[v'\right]^{S}_{\mathcal{R},\alpha}$. Whence $R(k)(v, v'') \geq \alpha$ and $R(k)(v', v'') \geq \alpha$ for all $k \in K$. From the hypothesis, we get that.

$$S(k)(v, v'') \geq R(k)(v, v'') \geq \alpha \geq \beta$$

and

$$S(k)(v', v'') \geq R(k)(v', v'') \geq \alpha \geq \beta$$

for all $k \in K$. It follows that $v'' \in [v]^S_{\mathcal{S},\beta} \cap [v']^S_{\mathcal{S},\beta}$. Then $[v]^S_{\mathcal{S},\beta} \cap [v']^S_{\mathcal{S},\beta} \neq \emptyset$. Thus $v' \in [v]^{OS}_{\mathcal{S},\beta}$. Hence $[v]^{OS}_{\mathcal{R},\alpha} \subseteq [v]^{OS}_{\mathcal{S},\beta} \subseteq F(a)$. Therefore $v \in \underline{F}|^{OS}_{\mathcal{R},\alpha}(a)$. Thus $\underline{F}|^{OS}_{\mathcal{S},\beta}(a) \subseteq \underline{F}|^{OS}_{\mathcal{R},\alpha}(a)$. This implies that $\underline{\mathscr{F}}|^{OS}_{\mathcal{S},\beta} \Subset \underline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha}$. The remain argument is straightforward, so we omit it.

Proposition 3.2.5 Let $(V, V, [V]^{OS}_{\mathcal{R}=(R,K),\alpha})$ be a given approximation space with property that $\alpha \in (0, 1]$, $[V]^S_{\mathcal{R},\alpha}$ is the partition of V, and $\mathcal{R}$ is a fuzzy soft reflexive relation and a fuzzy soft antisymmetric relation. If $\mathscr{F} := (F, A)$ is a soft set over V, then $\mathscr{F}$ is a definable soft set within $(V, W, [V]^{OS}_{\mathcal{R},\alpha})$.

Proof. Assume that $\mathscr{F}$ is a soft set over V. Then, by Remark 3.2.1, we obtain that $\underline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha} \Subset \overline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha}$. Let $a \in A$. Suppose that $v \in \overline{F}|^{OS}_{\mathcal{R},\alpha}(a)$. Then $[v]^{OS}_{\mathcal{R},\alpha} \cap F(a) \neq \emptyset$. Thus, there exists $v' \in V$ such that $v' \in [v]^{OS}_{\mathcal{R},\alpha}$ and $v' \in F(a)$. By Proposition 3.1.5, we have $v = v'$. We must prove that $[v]^{OS}_{\mathcal{R},\alpha} \subseteq F(a)$. Let $v'' \in [v]^{OS}_{\mathcal{R},\alpha}$. From Proposition 3.1.5, we have $v = v''$. Hence $v'' = v' \in F(a)$, which implies that $[v]^{OS}_{\mathcal{R},\alpha} \subseteq F(a)$. Therefore $v \in \underline{F}|^{OS}_{\mathcal{R},\alpha}(a)$. Thus, it is true that $\overline{F}|^{OS}_{\mathcal{R},\alpha}(a) \subseteq \underline{F}|^{OS}_{\mathcal{R},\alpha}(a)$. It follows that $\overline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha} \Subset \underline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha}$. Thus $\underline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha}$ is equal to $\overline{\mathscr{F}}|^{OS}_{\mathcal{R},\alpha}$. Consequently, $\mathscr{F}$ is a definable soft set within $(V, V, [V]^{OS}_{\mathcal{R},\alpha})$.

3.3 Measurement Issues

In this subsection, we shall study to accuracy and roughness measures of soft sets in terms of Pawlak's rough set theory [1]. Now, let X be a subset of V.

An *accuracy measure* of X based on (V, E), denoted by $X|_E$, is defined by

$$X|_E := \frac{card(\underline{E}(X))}{card(\overline{E}(X))}.$$

A *roughness measure* of X based on (V, E), denoted by $X||_E$, is defined by

$$X||_E := 1 - X|_E.$$

In what follows, the concept of rough set theory-based accuracy and roughness measurements via fuzzy soft relation is proposed below.

Definition 3.3.1 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\alpha})$ be an approximation space, and let $\mathscr{F} := (F, A)$ be a soft set over V. For $a \in A$, an *accuracy measure* of $F(a)$ based on $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\alpha})$, denoted by $F(a)|^{OS}_{\mathcal{R},\alpha}$, is defined by

$$F(a)|^{OS}_{\mathcal{R},\alpha} := \frac{card(\underline{F}|^{OS}_{\mathcal{R},\alpha}(a))}{card(\overline{F}|^{OS}_{\mathcal{R},\alpha}(a))}.$$

In generality, observe that $F(a)|^{OS}_{\mathcal{R},\alpha} \in [0, 1]$ for all $a \in A$. Then, for all $a \in A$, a *roughness measure* of $F(a)$ based on $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\alpha})$, denoted by $F(a)||^{OS}_{\mathcal{R},\alpha}$, is defined by

$$F(a)||^{OS}_{\mathcal{R},\alpha} := 1 - F(a)|^{OS}_{\mathcal{R},\alpha}.$$

Example 3.3.1 Based on Example 3.2.1, we compute that

$$F(a)|^{OS}_{\mathcal{R},0.5} := \frac{card(\underline{F}|^{OS}_{\mathcal{R},0.5}(a))}{card(\overline{F}|^{OS}_{\mathcal{R},0.5}(a))} = \frac{1}{5} = 0.2$$

and

$$F(a)||^{OS}_{\mathcal{R},0.5} := 1 - F(a)|^{OS}_{\mathcal{R},0.5} = 1 - 0.2 = 0.8$$

for all $a \in A$, i.e., the accuracy measure of $F(a)$ is 0.2 and the roughness measure of $F(a)$ is 0.8 for all $a \in A$.

Proposition 3.3.1 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\alpha})$ and Let $(V, V, [V]^{OS}_{\mathcal{S}:=(S,K),\beta})$ be given two approximation spaces with property that $\mathcal{R} \Subset \mathcal{S}$ and that $\alpha \geq \beta$. If $\mathscr{F} := (F, A)$ is a soft set over V, then $F(a)|^{OS}_{\mathcal{R},\alpha} \geq F(a)|^{OS}_{\mathcal{S},\beta}$ for all $a \in Supp(\mathscr{F})$.

Proof. Suppose that $\mathscr{F} := (F, A)$ is a soft set over V and $a \in Supp(\mathscr{F})$. Then, by Remark 3.2.1, we have $\overline{F}|^{OS}_{\mathcal{R},\alpha}(a) \neq \emptyset$. From Proposition 3.2.4, it follows that

$$card(\overline{F}|^{OS}_{\mathcal{R},\alpha}(a)) \leq card(\overline{F}|^{OS}_{\mathcal{S},\beta}(a)) \text{ and } card(\underline{F}|^{OS}_{\mathcal{R},\alpha}(a)) \geq card(\underline{F}|^{OS}_{\mathcal{S},\beta}(a)).$$

Now

$$F(a)|^{OS}_{\mathcal{S},\beta} := \frac{card(\underline{F}|^{OS}_{\mathcal{S},\beta}(a))}{card(\overline{F}|^{OS}_{\mathcal{S},\beta}(a))} \leq \frac{card(\underline{F}|^{OS}_{\mathcal{R},\alpha}(a))}{card(\overline{F}|^{OS}_{\mathcal{S},\beta}(a))} \leq \frac{card(\underline{F}|^{OS}_{\mathcal{R},\alpha}(a))}{card(\overline{F}|^{OS}_{\mathcal{R},\alpha}(a))} =: F(a)|^{OS}_{\mathcal{R},\alpha}.$$

Proposition 3.3.2 $(V, V, [V]^{OS}_{\mathcal{R}:=(R,K),\alpha})$ be a given approximation space with property that $\alpha \in (0, 1]$, $[V]^{S}_{\mathcal{R}\alpha}$ is the partition of V, and $\mathcal{R}$ is a fuzzy soft reflexive relation and a fuzzy soft antisymmetric relation. If $\mathscr{F} := (F, A)$ is a soft set over V, then $F(a)|^{OS}_{\mathcal{R},\alpha} = 1$ for all $a \in Supp(\mathscr{F})$. Furthermore, $F(a)||^{OS}_{\mathcal{R},\alpha} = 0$ for all $a \in Supp(\mathscr{F})$.

Proof. Suppose that $\mathscr{F} := (F, A)$ is a soft set over V and $a \in Supp(\mathscr{F})$. Then $\overline{F}|^{OS}_{\mathcal{R},\alpha}(a) \neq \emptyset$ due to Remark 3.2.1. From Proposition 3.2.5, it follows that $card(\underline{F}|^{OS}_{\mathcal{R},\alpha}(a)) = card(\overline{F}|^{OS}_{\mathcal{R},\alpha}(a))$. Thus $F(a)|^{OS}_{\mathcal{R},\alpha} := \frac{card(\underline{F}|^{OS}_{\mathcal{R},\alpha}(a))}{card(\overline{F}|^{OS}_{\mathcal{R},\alpha}(a))} = 1$. It is true that $F(a)||^{OS}_{\mathcal{R},\alpha} := 1 - F(a)|^{OS}_{\mathcal{R},\alpha} = 1 - 1 = 0$.

In the following, we further consider the fact related to the distance measurement of Marczewski and Steinhaus [11]. Let X and Y be given subsets of V. The distance measure of X and Y is defined as follows.

A *symmetric difference* between X and Y, denoted by $X \triangle Y$, is defined by

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

A *distance measure* of X and Y, denoted by $DM(X, Y)$, is defined by

$$DM(X, Y) = \begin{cases} \dfrac{card(X \triangle Y)}{card(X \cup Y)} & if \quad card(X \cup Y) > 0, \\ 0 & if \quad card(X \cup Y) = 0. \end{cases}$$

As mentioned above, we further study the argument under the relationship between distance measures and roughness measures as the following.

Proposition 3.3.3 Let $(V, W, [V]^{OS}_{\mathcal{R}:=(R,K),\alpha})$ be a given approximation space. If $\mathscr{F} := (F, A)$ is a soft set over V, then $DM(\overline{F}|^{OS}_{\mathcal{R},\alpha}(a), \underline{F}|^{OS}_{\mathcal{R},\alpha}(a)) = F(a)||^{OS}_{\mathcal{R},\alpha}$. For all $a \in Supp(\mathscr{F})$.

Proof. Suppose that $\mathscr{F} := (F, A)$ is a soft set over V and $a \in Supp(\mathcal{F})$. Then, by Remark 3.2.1, we have $\overline{F}|^{OS}_{\mathcal{R},\alpha}(a) \neq \emptyset$. Now

$$\begin{aligned} DM\left(\overline{F}|^{OS}_{R,\alpha}(a), \underline{F}|^{OS}_{R,\alpha}(a)\right) &:= \frac{card(\overline{F}|^{OS}_{R,\alpha}(a) \cup \underline{F}|^{OS}_{R,\alpha}(a))}{card(\overline{F}|^{OS}_{R,\alpha}(a) \cup \underline{F}|^{OS}_{R,\alpha}(a))} \\ &= \frac{card(\overline{F}|^{OS}_{R,\alpha}(a) \cup \underline{F}|^{OS}_{R,\alpha}(a))}{card(\overline{F}|^{OS}_{R,\alpha}(a) \cup \underline{F}|^{OS}_{R,\alpha}(a))} - \frac{card(\overline{F}|^{OS}_{R,\alpha}(a) \cap \underline{F}|^{OS}_{R,\alpha}(a))}{card(\overline{F}|^{OS}_{R,\alpha}(a) \cup \underline{F}|^{OS}_{R,\alpha}(a))} \\ &= 1 - \frac{card(\underline{F}|^{OS}_{R,\alpha}(a))}{card(\overline{F}|^{OS}_{R,\alpha}(a))} \\ &= 1 - F(a)|^{OS}_{R,\alpha} =: F(a)||^{OS}_{R,\alpha} \end{aligned}$$

as required.

Example 3.3.2 According to Example 3.2.1, we observe that

$$card(\underline{F}|^{OS}_{\mathcal{R},0.5}(a)) \cup card(\overline{F}|^{OS}_{\mathcal{R},0.5}(a)) = 5 \text{ and } card(\underline{F}|^{OS}_{\mathcal{R},0.5}(a)) \cap card(\overline{F}|^{OS}_{\mathcal{R},0.5}(a)) = 1.$$

Then, we compute that $DM(\overline{F}|^{OS}_{\mathcal{R},0.5}(a), \underline{F}|^{OS}_{\mathcal{R},0.5}(a)) = F(a)||^{OS}_{\mathcal{R},0.5}$. In fact,

$$DM(\overline{F}|^{OS}_{\mathcal{R},0.5}(a), \underline{F}|^{OS}_{\mathcal{R},0.5}(a)) = \frac{5-1}{5} = \frac{4}{5} = 0.8 = F(a)||^{OS}_{\mathcal{R},0.5}$$

due to Example 3.3.1.

As mentioned above, we observe that the notion of these measurements is related to the novel rough soft set model in the previous subsection. Moreover, we see that the distance measure of upper and lower approximations is a roughness measure. It is exhibited in Example 3.3.2.

4 Conclusions

The concept of fuzzy soft serial relations was defined. Then, an overlap of successor classes was proposed as a new class. We introduced a new rough soft set theory. That is, upper and lower rough approximations of a soft set were provided via all overlaps of successor classes under fuzzy soft serial relations. Then, we got that a soft set is definable if a fuzzy soft relation is reflexive and antisymmetric. Finally, we obtained that a roughness measure and a distance measure are identical.

As the novel rough soft set theory, in the future, we shall further study the notion to deal with decision-making problems.

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