Chapter 2
Special Relativity

2.1 Introduction

In this chapter we summarise some of the standard results of conventional special relativity theory that are needed to formulate the proposed extension of Newton's second law. The word special alludes to invariance under transformations relating constant relative velocity frames of reference, which is in contrast to general relativity which relates to invariance under arbitrary space-time coordinate transformations. The first section deals with the fundamental notion of Lorentz transformations and the importance of invariance with respect to frames that are moving with constant relative velocity. The following section highlights the Einstein addition of velocities law which is an immediate consequence of the notion of invariance under Lorentz transformations.

The next six sections provide a number of results dealing with Lorentz invariances, including the determination of Lorentz invariant velocity fields, a general framework for Lorentz invariances, solving a first order partial differential equation by Lagrange's method to determine their general form, a novel method of their validation and showing that the Jacobian for all Lorentz invariances necessarily vanishes. Two sections thereafter deal, respectively, with the space-time transformation $x' = ct$ and $t' = x/c$ and the de Broglie wave velocity $u' = c^2/u$. In the subsequent section, the fundamental rate-of-working equation (or work done equation) is formulated for the determination of the conventional physical energy of a particle moving under an applied force. The next two sections of the chapter deal with the Lorentz invariant energy-momentum relations and force invariance for two frames moving with constant relative velocity. The penultimate section of the chapter provides a specific example of particle motion in an invariant potential field, while the final section deals with a possible extension of the conventional Einstein variation of mass formula with a specific expression arising from a Lorentz invariant equation for the energy rate de/dp.

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We assume throughout the text that the Einstein energy-mass expression $e = e_0/(1-(u/c)^2)^{1/2}$ applies. In the final section of this chapter, assuming $e = mc^2$, $p = mu$, and a Lorentz invariant equation for the energy rate de/dp, which is motivated from the energy equation (5.8), we derive an extension of the conventional Einstein variation of the energy-mass formula. This is the simplest one-parameter Lorentz invariant extension of the Einstein mass-energy relation. Implicit in the new expression is space-time anisotropy such that the particle has different rest masses in the positive and negative x directions. The resulting energy-mass formula (2.59), involving an arbitrary constant κ, predicts in particular that the rest energies in the moving and reference frames, respectively, e_0 and E_0, are related by the equation

$$e_0 = E_0 \left(\frac{1+(v/c)}{1-(v/c)} \right)^{\kappa/2}, \tag{2.1}$$

which for $\kappa \neq 0$ therefore impinges on the basic assumption relating to the isotropy of space. If $e_0 = E_0$, then necessarily κ is zero, and for $\kappa \neq 0$ the rest mass values will vary with the direction of motion, namely, two different values are obtained for positive and negative velocities v. While numerous experiments have been undertaken aimed at testing such hypothesis, and all indicate the veracity of the assumed isotropy of space, nevertheless the validity or otherwise of (2.1) might only be properly tested in those situations for which both rest energies e_0 and E_0 are non-zero and the fraction $(1+v/c)/(1-v/c)$ significantly differs from unity. Further, since it is generally believed that black holes exist at the centres of galaxies, space must be intrinsically anisotropic in some sense.

The topic of special relativity has acquired the status of a standard subject in both physics and mechanics, and almost every text on physics or mechanics contains either a chapter on special relativity or at least some reference to special relativity. The older texts tend to be closer to the original motivating issues and the developments that gave birth to the subject. Both Dingle [25] and McCrea [76] are concise student texts, while more comprehensive accounts of special relativity can be found in Bohm [8], French [34] and Resnick [86]. Both Moller [78] and Tolman [102] are two texts which have become standard works of reference to many aspects of relativity. The reader may wish to also consult [83] which contains a collection of the original papers of Einstein, Lorentz, Minkowski and Weyl with additional notes by Arnold Sommerfeld.

2.2 Lorentz Transformations

At the very heart of special relativistic mechanics lies the notion of invariance with respect to frames moving with constant relative velocity, and particularly under those transformations of space and time leaving the wave equation unchanged, referred to as Lorentz transformations. We consider a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to

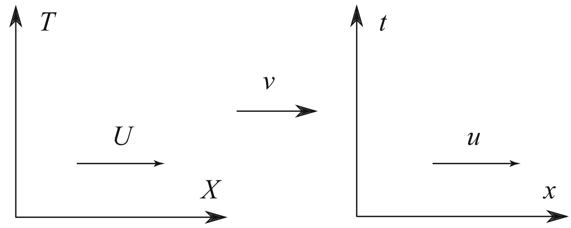

Fig. 2.1 Two inertial frames moving along x-axis with relative velocity v

the first frame, and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 2.1. We view the relative velocity v as a parameter measuring the departure of the current frame (x, y, z) from the rest frame (X, Y, Z), and for this purpose we adopt a notation employing lower case for variables associated with the moving (x, y, z) frame and upper case or capitals for those variables associated with the rest (X, Y, Z) frame. Accordingly, time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t. Following normal practice, we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

For $0 \leqslant v < c$, the standard Lorentz transformations are

$$X = \frac{x + vt}{[1 - (v/c)^2]^{1/2}}, \quad T = \frac{t + vx/c^2}{[1 - (v/c)^2]^{1/2}}, \tag{2.2}$$

with the inverse transformation characterised by $-v$; thus

$$x = \frac{X - vT}{[1 - (v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1 - (v/c)^2]^{1/2}}, \tag{2.3}$$

and various derivations of these equations can be found in many standard textbooks such as Feynmann et al. [33] and Landau and Lifshitz [66], and other novel derivations are given by Lee and Kalotas [67] and Levy-Leblond [68]. The above equations reflect, of course, that the two coordinate frames coincide when the relative velocity v is zero, namely, $x = X$, $t = T$, when $v = 0$.

Throughout the text we adopt the notation $\alpha = ct + x$ and $\beta = ct - x$ for the characteristic variables, and from either of the above equations, by direct substitution, we may readily deduce the relations

$$ct + x = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} (cT + X), \quad ct - x = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} (cT - X), \tag{2.4}$$

so that in particular we may confirm the simple Lorentz invariance

$$(ct)^2 - x^2 = (cT)^2 - X^2. \tag{2.5}$$

2.3 Einstein Addition of Velocities Law

The relative frame velocity v is assumed to be constant, so that with velocities $U = dX/dT$ and $u = dx/dt$, on taking the differentials of both equations in (2.3), thus

$$dx = \frac{dX - vdT}{[1-(v/c)^2]^{1/2}}, \quad dt = \frac{dT - vdX/c^2}{[1-(v/c)^2]^{1/2}},$$

and on dividing the first differential by the second yields the Einstein addition of velocity law

$$u = \frac{U - v}{\left(1 - Uv/c^2\right)}. \tag{2.6}$$

An immediate consequence of (2.6) is the identity

$$[1-(u/c)^2]^{1/2}(1 - Uv/c^2) = [1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2}, \tag{2.7}$$

which can be easily established by using (2.6) in the left-hand side of (2.7). This latter equation is fundamental to the development of special relativity, and in particular it is necessary to establish Lorentz invariance of certain quantities.

There are special relativity theories which apply for relative velocities greater than the speed of light and which are complementary to the Einstein special theory of relativity that applies to relative velocities less than the speed of light. Hill and Cox [54] derive Lorentz transformations corresponding to (2.3) for superluminal relative velocities and show that the Einstein addition of velocities law still applies. The two formulae (2.6) and (2.7), when expressed in the form (2.8), reveal that at least one of the velocities u, v or U must not exceed the speed of light, and in terms of taking square roots or logarithms, all need appropriate re-arrangement depending upon the particular values of the three velocities.

A formula arising from (2.6) that is not so well known is

$$\left(\frac{1+U/c}{1-U/c}\right) = \left(\frac{1+u/c}{1-u/c}\right)\left(\frac{1+v/c}{1-v/c}\right), \tag{2.8}$$

so that on introducing velocity variables $(\Theta, \theta, \epsilon)$ defined by

$$\Theta = \tanh^{-1}(U/c), \quad \theta = \tanh^{-1}(u/c), \quad \epsilon = \tanh^{-1}(v/c), \tag{2.9}$$

equation (2.8) becomes simply the translation $\Theta = \theta + \epsilon$ noting again that within the context of special relativity, v and therefore ϵ are both assumed to be constants. The angle θ assumes an important role and is the angle in which Lorentz invariance appears through a translational invariance, and so for completeness we note the elementary relations

$$\theta = \frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \tanh^{-1}(u/c), \qquad \left(\frac{1+u/c}{1-u/c}\right)^{1/2} = e^{\theta}. \tag{2.10}$$

Further, with $u/c = \tanh\theta$, we have from the usual formulae of special relativity for energy and momentum, namely, $e = mc^2$ and $p = mu$, respectively, where $m(u) = m_0[1-(u/c)^2]^{-1/2}$ and m_0 denotes the rest mass, the following additional relations:

$$m = m_0\cosh\theta, \qquad e = e_0\cosh\theta, \qquad pc = e_0\sinh\theta, \tag{2.11}$$

where $e_0 = m_0c^2$ denotes the rest mass energy. We comment that the formulation of Lorentz transformations as a one-parameter group of geometric transformations is due to Minkowski [77].

Superluminal de Broglie Waves de Broglie [17] showed that while the group velocity of the wave package coincides with the particle velocity u, the wave velocity is given by $w = c^2/u$. This means that if the particle velocity u is subluminal, then not only is the associated wave or phase velocity c^2/u necessarily superluminal but also the average velocity $(u+w)/2$, and this will be examined in a subsequent section in this chapter. Furthermore, if the relationship $uw = c^2$ holds in one Lorentz frame, then it holds in all Lorentz frames. This is most easily seen if we rearrange (2.6) to read

$$U = \frac{u+v}{\left(1+uv/c^2\right)}, \tag{2.12}$$

so that the corresponding respective velocities U and W as measured from the (X, T) rest frame and corresponding to u and w in the moving (x, t) frame are given by

$$U = \frac{u+v}{\left(1+uv/c^2\right)}, \qquad W = \frac{w+v}{\left(1+wv/c^2\right)},$$

and from the identity

$$UW - c^2 = \frac{(1-(v/c)^2)(uw-c^2)}{\left(1+uv/c^2\right)\left(1+wv/c^2\right)},$$

so that $UW = c^2$ if and only if $uw = c^2$.

As noted above, an outcome predicted in [54] is that the Einstein velocity addition formula (2.6) remains valid for the proposed superluminal extension of special relativity in [54], and in particular, in the limit $v \to \infty$ the de Broglie relation $uU = c^2$ emerges. It also formally emerges from the envelope [41], namely, by simultaneously solving (2.6) and

$$\frac{\partial u}{\partial v} = \frac{-1}{\left(1 - Uv/c^2\right)} + \frac{U(U-v)}{c^2\left(1 - Uv/c^2\right)^2} = \frac{-\left(1-(U/c)^2\right)}{\left(1 - Uv/c^2\right)^2} = 0,$$

which can only vanish in the limit $v \to \infty$.

2.4 Lorentz Invariances

A Lorentz invariant quantity is one which assumes an identical form under a Lorentz transformation, and we have previously noted the Lorentz invariant $(ct)^2 - x^2 = (cT)^2 - X^2$ given by (2.5). In this section we identify a further three important Lorentz invariances which can all be established by direct substitution using the above equations. In the following sections, we develop a general understanding of the origin of these invariances. By substitution of the Lorentz transformations (2.3) and the addition of velocities law in the form of (2.6) or (2.7) into the left-hand sides, we may verify directly the following interesting Lorentz invariances:

$$\frac{x-ut}{\left(1-(u/c)^2\right)^{1/2}} = \frac{X-UT}{\left(1-(U/c)^2\right)^{1/2}}, \quad \frac{t-ux/c^2}{\left(1-(u/c)^2\right)^{1/2}} = \frac{T-UX/c^2}{\left(1-(U/c)^2\right)^{1/2}}, \tag{2.13}$$

which are also verified independently and further discussed in a subsequent section in this chapter. For the first equality, on substitution of (2.3) and (2.6) into the left-hand side, we need to simplify

$$\frac{x-ut}{\left(1-(u/c)^2\right)^{1/2}} = \frac{\left[(X-vT)\left(1-Uv/c^2\right) - (U-v)\left(T-vX/c^2\right)\right]}{[1-(u/c)^2]^{1/2}[1-(v/c)^2]^{1/2}(1-Uv/c^2)},$$

which on simplification of the numerator and using (2.7) gives the required right-hand side. Similarly, for the second equality, on substitution of (2.3) and (2.6) into the left-hand side, we need to simplify

$$\frac{t-ux/c^2}{\left(1-(u/c)^2\right)^{1/2}} = \frac{\left[(T-vX/c^2)\left(1-Uv/c^2\right) - (U-v)\left(X-vT\right)\right]}{[1-(u/c)^2]^{1/2}[1-(v/c)^2]^{1/2}(1-Uv/c^2)},$$

which again on simplification of the numerator and using (2.7) gives the desired outcome.

Similarly, we may establish a third Lorentz invariant

$$\left(\frac{ct+x}{ct-x}\right)\left(\frac{1-u/c}{1+u/c}\right) = \left(\frac{cT+X}{cT-X}\right)\left(\frac{1-U/c}{1+U/c}\right), \tag{2.14}$$

since by division of the two Eqs. (2.4) we have

$$\left(\frac{ct+x}{ct-x}\right)\left(\frac{1+v/c}{1-v/c}\right)=\left(\frac{cT+X}{cT-X}\right),$$

and on using the Einstein addition of velocity law in the form of (2.8), this equation simplifies to give (2.14). Thus, from Eqs. (2.5) and (2.14), we have the two Lorentz invariants

$$\zeta=((ct)^2-x^2)^{1/2}, \qquad \tau=\left(\frac{ct+x}{ct-x}\right)^{1/2}\left(\frac{1-u/c}{1+u/c}\right)^{1/2}=\left(\frac{ct+x}{ct-x}\right)^{1/2}e^{-\theta}, \tag{2.15}$$

where the latter equality arises from (2.10) where $\theta=\tanh^{-1}(u/c)$. In terms of the characteristic coordinates $\alpha=ct+x$ and $\beta=ct-x$, we have

$$\zeta=(\alpha\beta)^{1/2}, \qquad \tau=\left(\frac{\alpha}{\beta}\right)^{1/2}e^{-\theta},$$

and from the partial derivatives

$$\frac{\partial\zeta}{\partial x}=-\frac{x}{(\alpha\beta)^{1/2}}, \qquad \frac{\partial\tau}{\partial x}=\left\{\frac{ct}{\alpha^{1/2}\beta^{3/2}}-\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{\partial\theta}{\partial x}\right\}e^{-\theta},$$

$$\frac{\partial\zeta}{\partial t}=\frac{c^2t}{(\alpha\beta)^{1/2}}, \qquad \frac{\partial\tau}{\partial t}=-\left\{\frac{cx}{\alpha^{1/2}\beta^{3/2}}+\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{\partial\theta}{\partial t}\right\}e^{-\theta},$$

we may deduce that the Jacobian of the transformation (2.15) becomes

$$\frac{\partial(\zeta,\tau)}{\partial(x,t)}=-\frac{e^{-\theta}}{\beta}\left(x\frac{\partial\theta}{\partial t}-c^2t\frac{\partial\theta}{\partial x}+c\right)=\frac{ce^{-\theta}}{\beta}\left(\alpha\frac{\partial\theta}{\partial\alpha}-\beta\frac{\partial\theta}{\partial\beta}-1\right). \tag{2.16}$$

We note that in terms of the velocity $u(\alpha,\beta)$, from the relations (2.10), the above Jacobian becomes

$$\frac{\partial(\zeta,\tau)}{\partial(x,t)}=\frac{1}{\beta(1-u/c)^{1/2}(1+u/c)^{3/2}}\left(\alpha\frac{\partial u}{\partial\alpha}-\beta\frac{\partial u}{\partial\beta}-c\left(1-(u/c)^2\right)\right). \tag{2.17}$$

Assuming that this Jacobian is non-zero, we may express other Lorentz invariants as functions of both ζ and τ. Thus, for example, for the two Lorentz invariants given by Eq. (2.13), we have

$$\frac{x-ut}{\left(1-(u/c)^2\right)^{1/2}}=\frac{((\alpha-\beta)-(\alpha+\beta)(u/c))}{2(1-(u/c)^2)^{1/2}} \tag{2.18}$$

$$=\frac{1}{2}\left\{\alpha\left(\frac{1-u/c}{1+u/c}\right)^{1/2}-\beta\left(\frac{1+u/c}{1-u/c}\right)^{1/2}\right\}$$

$$= \frac{\zeta}{2}\left(\tau - \frac{1}{\tau}\right),$$

and

$$\frac{t - ux/c^2}{\left(1-(u/c)^2\right)^{1/2}} = \frac{((\alpha+\beta)-(\alpha-\beta)(u/c))}{2c(1-(u/c)^2)^{1/2}} \tag{2.19}$$

$$= \frac{1}{2c}\left\{\alpha\left(\frac{1-u/c}{1+u/c}\right)^{1/2} + \beta\left(\frac{1+u/c}{1-u/c}\right)^{1/2}\right\}$$

$$= \frac{\zeta}{2c}\left(\tau + \frac{1}{\tau}\right),$$

on making use of the following two relations which may be deduced from (2.15), namely,

$$\alpha\left(\frac{1-u/c}{1+u/c}\right)^{1/2} = \alpha e^{-\theta} = \zeta\tau, \qquad \beta\left(\frac{1+u/c}{1-u/c}\right)^{1/2} = \beta e^{\theta} = \frac{\zeta}{\tau}.$$

The above invariants ζ and τ might also be useful in demonstrating that certain quantities are not Lorentz invariant. For example, for the exact wave-like solution examined in some detail in Chaps. 5 and 6, the assumed linear force equations

$$f(u) = f_0(1+\lambda u/c), \qquad cg(u) = f_0(\lambda + u/c).$$

where λ and f_0 denote arbitrary constants, are not Lorentz invariant, and it turns out that the Lorentz invariant quantities are $e(u)f(u)$ and $e(u)g(u)$. The force relations themselves are only partially Lorentz invariant in the sense that their functional form is preserved under a Lorentz transformation with different constants λ and f_0. This is most easily seen from the relations

$$f(u) + cg(u) = f_0(1+\lambda)(1+u/c), \qquad f(u) - cg(u) = f_0(1-\lambda)(1-u/c),$$

so that

$$e(u)\left(f(u)+cg(u)\right) = e_0 f_0(1+\lambda)\left(\frac{1+u/c}{1-u/c}\right)^{1/2},$$

$$e(u)\left(f(u)-cg(u)\right) = e_0 f_0(1-\lambda)\left(\frac{1-u/c}{1+u/c}\right)^{1/2},$$

which cannot be expressed solely in terms of the invariants ζ and τ. However, on using (2.9) and $\Theta = \theta + \epsilon$, these relations become

$$e(u)\,(f(u)+cg(u)) = e_0 f_0(1+\lambda)e^{\theta} = e_0 f_0(1+\lambda)e^{-\epsilon}e^{\Theta},$$
$$e(u)\,(f(u)-cg(u)) = e_0 f_0(1-\lambda)e^{-\theta} = e_0 f_0(1-\lambda)e^{\epsilon}e^{-\Theta},$$

demonstrating that the Lorentz transformed force relations preserve the functional dependence on velocity u (viz. on θ) but with changed force parameters arising through the translational velocity v (viz. through ϵ).

In the event that the Jacobian (2.16) is zero, then $\theta(\alpha, \beta)$ satisfies the first order linear partial differential equation

$$\alpha\frac{\partial\theta}{\partial\alpha} - \beta\frac{\partial\theta}{\partial\beta} = 1. \tag{2.20}$$

This partial differential equation may be solved using Lagrange's characteristic method which is formally to introduce a characteristic parameter s through the three equations

$$\frac{d\alpha}{ds} = \alpha, \qquad \frac{d\beta}{ds} = -\beta, \qquad \frac{d\theta}{ds} = 1,$$

and then the general solution is obtained by taking one integral of these equations to be an arbitrary function of a second independent integral. Thus, for example, by division of the first equation, we have

$$\frac{d\beta}{d\alpha} = -\frac{\beta}{\alpha}, \qquad \frac{d\theta}{d\alpha} = \frac{1}{\alpha},$$

which can both be integrated to yield $\alpha\beta = C_1$ and $\theta = \log\alpha + C_2$, where C_1 and C_2 denote arbitrary constants, and from which we might deduce that the general solution of (2.20) may be determined from $C_2 = \Phi(C_1)$ where Φ denotes an arbitrary function. Accordingly, in terms of $\zeta = (\alpha\beta)^{1/2}$, the general solution of (2.20) is given by $\theta(\alpha, \beta) = \log\alpha + \Psi(\zeta)$, where Ψ denotes an arbitrary function. In this case, with $\exp\theta = \alpha\phi(\zeta)$ where $\phi(\zeta) = \exp\Psi(\zeta)$, we have from Eqs. (2.10) and (2.11) that the velocity u, energy e and momentum p are given, respectively, by

$$\frac{u}{c} = \frac{(\alpha\phi)^2-1}{(\alpha\phi)^2+1}, \qquad e = \frac{e_0}{2}\left(\alpha\phi + \frac{1}{\alpha\phi}\right), \qquad pc = \frac{e_0}{2}\left(\alpha\phi - \frac{1}{\alpha\phi}\right).$$

We show in the following section that these velocity fields are those for which $dx/dt = u(x,t)$ remains invariant under Lorentz transformation.

2.5 Lorentz Invariant Velocity Fields $u(x, t)$

In this section we pose the question of determining the most general one-dimensional velocity fields $u(x, t)$ that remain invariant under the Lorentz transformation (2.3). Alternatively, an equivalent question is to determine those velocity fields $u(x, t)$ and $u(X, T)$ which are such that the two differential problems

$$\frac{dx}{dt} = u(x, t), \qquad \frac{dX}{dT} = u(X, T). \tag{2.21}$$

transform into each other under the Lorentz transformations (2.2) and (2.3). Since the Lorentz transformations form a one-parameter group of transformations in the frame velocity v, we need only expand either (2.2) or (2.3) and equate the corresponding infinitesimals for either differential problem. Thus the infinitesimal versions of (2.2) are $X \approx x + vt$ and $T \approx t + vx/c^2$, and therefore from $(2.21)_2$, we obtain

$$\frac{dx + vdt}{dt + (v/c^2)dx} = \frac{dx/dt + v}{1 + (v/c^2)(dx/dt)} \approx \left(\frac{dx}{dt} + v\right)\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right) \approx \frac{dx}{dt} + v\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)$$
$$= u\left(x + vt, t + \frac{vx}{c^2}\right) \approx u(x, t) + v\left(t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t}\right),$$

on using Taylor's theorem to expand the last term. Thus by equating infinitesimals, we may readily deduce the following first order partial differential equation for $u(x, t)$; thus

$$t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t} = 1 - \left(\frac{u}{c}\right)^2. \tag{2.22}$$

In terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, we have $x = (\alpha - \beta)/2$ and $t = (\alpha + \beta)/2c$, and the differential formulae

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \qquad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta},$$

and the first order partial differential equation (2.22) becomes

$$\alpha\frac{\partial u}{\partial \alpha} - \beta\frac{\partial u}{\partial \beta} = c\left(1 - \left(\frac{u}{c}\right)^2\right),$$

which is precisely the condition for the vanishing of the Jacobian (2.17), as might be anticipated since essentially we seek a functional relationship between the two invariants ζ and τ, namely, $\tau = \phi(\zeta)$. Using Lagrange's characteristic method described above, we may formally solve the three characteristic equations with parameter s; thus

$$\frac{d\alpha}{ds} = \alpha, \qquad \frac{d\beta}{ds} = -\beta, \qquad \frac{du}{ds} = c\left(1 - \left(\frac{u}{c}\right)^2\right),$$

to confirm that the general solution is indeed given by $\tau = \phi(\zeta)$. From this relation and (2.15), namely,

$$\zeta = ((ct)^2 - x^2)^{1/2} = (\alpha\beta)^{1/2}, \qquad \tau = \left(\frac{ct+x}{ct-x}\right)^{1/2}\left(\frac{1-u/c}{1+u/c}\right)^{1/2} = \left(\frac{\alpha}{\beta}\right)^{1/2}\left(\frac{1-u/c}{1+u/c}\right)^{1/2},$$

we might deduce

$$\frac{d\beta}{d\alpha} = \left(\frac{1-u/c}{1+u/c}\right) = \frac{\beta}{\alpha}\phi^2((\alpha\beta)^{1/2}) = \frac{(\alpha\beta)\phi^2((\alpha\beta)^{1/2})}{\alpha^2}.$$

On introducing a new arbitrary function $\Phi(\zeta)$ which is defined by $\Phi(\zeta) = 1/\zeta^2\phi^2(\zeta)$, we obtain the differential equation

$$\frac{d\alpha}{d\beta} = \alpha^2\Phi((\alpha\beta)^{1/2}),$$

which is readily integrable with the substitution $\zeta = (\alpha\beta)^{1/2}$; thus

$$\frac{d\zeta}{d\beta} = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2} + \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{1/2}\frac{d\alpha}{d\beta} = \frac{\zeta}{2\beta}\left(1 + \zeta^2\Phi(\zeta)\right),$$

or similarly

$$\frac{d\zeta}{d\alpha} = \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{1/2} + \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{d\beta}{d\alpha} = \frac{\zeta}{2\alpha}\left(1 + \frac{1}{\zeta^2\Phi(\zeta)}\right),$$

and both of which are evidently separable.

2.6 General Framework for Lorentz Invariances

In this section we develop a general understanding of the formal origin of Lorentz invariances, and in particular we provide an independent confirmation of those established in the previous section. As we have previously stated, in this text we are considering a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to the first frame, and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 2.1. The coordinate notation adopted here is different to that normally used in special relativity which tends to involve both primed and unprimed variables. We do this purposely here because we wish to view the relative velocity v as a parameter measuring the departure of the current frame (x, y, z) from the rest frame (X, Y, Z), and for this purpose the notation employed in non-linear continuum mechanics is preferable. Time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t, and we assume throughout that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

We emphasise that although v is a constant, the approach adopted here and in [54] is to assume that the current frame is defined by the two variables (x, t) and each of x and t are characterised by the three independent variables (X, T, v), so that explicitly we have $x = x(X, T, v)$ and $t = t(X, T, v)$. Our basic approach is not to think of the relative velocity v as a velocity as such, but rather to envisage v simply as the parameter which provides a measure of the departure of one inertial from the rest frame. We are then able to exploit the well-trodden formalism from non-linear continuum mechanics, treating v as the time-like variable for which we can formally differentiate to produce velocity-like quantities.

In this context, it is important for the reader to fully appreciate that this approach is motivated from and completely analogous to that in non-linear continuum mechanics for which in standard notation, the spatial locations defined by (x, y, z) are viewed as functions of the four independent variables (X, Y, Z, t), namely, $x = x(X, Y, Z, t)$ and so on. Further, in continuum mechanics we usually introduce velocities such as $u = dx/dt$ where the total derivative d/dt means partial differentiation with respect to time keeping (X, Y, Z) fixed, and there is the well-known connection between this derivative, the velocities (u, v, w) and the spatial time derivative $\partial/\partial t$, which means partial differentiation with respect to time keeping the three variables (x, y, z) fixed; thus for $\phi = \phi(x, y, z, t)$ we have

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z},$$

and there are many other such mathematical formalities that can be exploited in the context of relativity, treating the relative frame velocity v as a time-like variable.

We have in mind that the Lorentz transformation (2.3) is a one-parameter group of transformations, with the relative velocity v serving as the parameter, and the identity $x = X$ and $t = T$ arising from the value $v = 0$. By the "pseudo-velocity" equations, we refer to the derivatives dx/dv and dt/dv, subject to the initial data $x = X$ and $t = T$ for $v = 0$. We now view x and t defined by (2.3) as functions of v, and by straightforward differentiation and subsequent simplification, we may derive the following "pseudo-velocity" equations:

$$\frac{dx}{dv} = \frac{-t}{1-(v/c)^2}, \quad \frac{dt}{dv} = \frac{-x/c^2}{1-(v/c)^2}, \tag{2.23}$$

where d/dv denotes total differentiation with respect to v, keeping the initial variables (X, T) fixed. We may now proceed to show that by formally solving (2.23), Einstein's theory emerges from the initial data $x = X$ and $t = T$ for $v = 0$. On re-arrangement of (2.23), we obtain

$$\left[1-\left(\frac{v}{c}\right)^2\right]\frac{dx}{dv}=-t,\quad \left[1-\left(\frac{v}{c}\right)^2\right]\frac{dt}{dv}=-\frac{x}{c^2},$$

which becomes a fully autonomous system if we introduce a new parameter ϵ such that

$$\frac{d}{d\epsilon}=\left[1-\left(\frac{v}{c}\right)^2\right]\frac{d}{dv},$$

so that $dv/d\epsilon = 1-(v/c)^2$. On making the further substitution $v = c\sin\phi$, this equation becomes $d\phi/\cos\phi = d\epsilon/c$, which integrates to give

$$\frac{1}{2}\log\left(\frac{1+\sin\phi}{1-\sin\phi}\right)=\frac{\epsilon}{c}+\text{constant}.$$

Suppose we assign the value $v = v_0$ at $\epsilon = 0$, then from this equation we may deduce

$$\frac{1+v/c}{1-v/c}=\left(\frac{1+v_0/c}{1-v_0/c}\right)e^{2\epsilon/c}, \tag{2.24}$$

and the special theory of relativity arises from the initial data $x = X$ and $t = T$ for $v_0 = 0$.

With $v_0 = 0$ we may deduce from (2.24) $v = c\tanh(\epsilon/c)$ for $0 \leqslant v < c$, and the two ordinary differential equations (2.23) become

$$\frac{dx}{d\epsilon}=-t,\quad \frac{dt}{d\epsilon}=-\frac{x}{c^2},$$

so that on differentiating either with respect to ϵ, we may eventually deduce

$$x(\epsilon)=A\sinh(\epsilon/c)+B\cosh(\epsilon/c),\qquad t(\epsilon)=-[A\cosh(\epsilon/c)+B\sinh(\epsilon/c)]/c, \tag{2.25}$$

where A and B denote arbitrary constants of integration. From the initial data and (2.25), we may deduce that $A = -cT$ and $B = X$, giving rise to the well-known pseudo-Euclidean rotation of special relativity (see [66, p. 10])

$$x(\epsilon)=X\cosh(\epsilon/c)-cT\sinh(\epsilon/c),\qquad t(\epsilon)=T\cosh(\epsilon/c)-\frac{X}{c}\sinh(\epsilon/c), \tag{2.26}$$

noting that from $v = c\tanh(\epsilon/c)$, we have

$$\cosh(\epsilon/c)=\frac{1}{[1-(v/c)^2]^{1/2}},\quad \sinh(\epsilon/c)=\frac{v/c}{[1-(v/c)^2]^{1/2}}, \tag{2.27}$$

and together (2.26) and (2.27) yield (2.3), as might be expected. Of particular relevance in the above discussion are the expressions (2.23) for the infinitesimal vector of the Lorentz one-parameter group of transformations together with the initial data $x = X$ and $t = T$ for $v = 0$. In the following section, we demonstrate how we might characterise the integral invariants of the Lorentz group.

2.7 Integral Invariants of the Lorentz Group

In a previous section, we have established certain integral invariants of the Lorentz group such as (2.13) or (2.14) by direct substitution of either the Lorentz transformations (2.2) or their inverses (2.3). In this section we develop the machinery necessary to provide an alternative validation of the invariants of the Lorentz group, the details of which are presented in the following section. From the preceding section, it is clear that for any integral $I(x, t, v)$ for which $dI/dv = 0$, its value is determined by its value at $v = 0$; thus

$$\frac{dI}{dv} = 0, \qquad I(x, t, v) = I(X, T, 0), \qquad 0 \leqslant v < c.$$

Now on differentiating $v = v(x, t)$, we have from the chain rule for partial differentiation

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial t}dt,$$

so that

$$\frac{\partial v}{\partial x}\frac{dx}{dv} + \frac{\partial v}{\partial t}\frac{dt}{dv} = 1.$$

On using the above expressions (2.23) for the infinitesimal vector of the Lorentz one-parameter group of transformations, we may deduce the following first order partial differential equation:

$$t\frac{\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t} = \left(\frac{v}{c}\right)^2 - 1. \tag{2.28}$$

Again such first order partial differential equations are formally solved using Lagrange's characteristic method, which involves introducing a characteristic parameter s, and formulating the three ordinary differential relations

$$\frac{dx}{ds} = t, \quad \frac{dt}{ds} = \frac{x}{c^2}, \quad \frac{dv}{ds} = \left(\frac{v}{c}\right)^2 - 1,$$

and then if $I_1(x, t, v)$ and $I_2(x, t, v)$ denote any two independent integrals of the reduced system

$$\frac{dx}{dt} = \frac{c^2 t}{x}, \qquad \frac{dv}{dx} = \frac{(v/c)^2 - 1}{t}, \tag{2.29}$$

the general solution of the first order partial differential equation (2.28) is then given by $I_1(x, t, v) = \Phi(I_2(x, t, v))$, where Φ denotes an arbitrary function.

The first differential equation readily integrates to give $\zeta^2 = (ct)^2 - x^2 = constant$, while the second differential equation becomes

$$\frac{dv}{dx} = c\frac{(v/c)^2 - 1}{(x^2 + \zeta^2)^{1/2}},$$

and on treating ζ as a constant for the purposes of this integration, the integration may then be effected through the substitutions,

$$x = \zeta \sinh \psi, \qquad v = c \sin \phi,$$

to yield the differential relation $d\psi + d\phi/\cos\phi = 0$. This equation integrates to give

$$\psi + \frac{1}{2}\log\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right) = \text{constant},$$

and on using the elementary formula $\sinh^{-1}(z) = \log(z + (1 + z^2)^{1/2})$, we may deduce, from the above relations, the second independent integral

$$\rho = \left(\frac{ct + x}{ct - x}\right)\left(\frac{1 + v/c}{1 - v/c}\right) = \text{constant},$$

and the validity of ζ = constant and ρ = constant can be verified either directly from the characteristic relations (2.4) or using the above differential relations (2.23) to show that $d\zeta/dv = d\rho/dv = 0$ which we now proceed to describe.

In order to achieve this, we again need the two differential relations (2.23), namely,

$$\frac{dx}{dv} = \frac{-t}{1 - (v/c)^2}, \qquad \frac{dt}{dv} = \frac{-x/c^2}{1 - (v/c)^2}, \tag{2.30}$$

which we need to use in the evaluation of $d\zeta/dv$ and $d\rho/dv$; thus

$$\frac{d\zeta^2}{dv} = \frac{d}{dv}((ct)^2 - x^2) = \frac{2}{1 - (v/c)^2}\left(c^2 t(-x/c^2) - x(-t)\right) = 0,$$

$$\frac{d\rho}{dv} = \frac{d}{dv}\left[\left(\frac{ct+x}{ct-x}\right)\left(\frac{1+v/c}{1-v/c}\right)\right] = \left(\frac{1+v/c}{1-v/c}\right)\frac{d}{dv}\left(\frac{ct+x}{ct-x}\right) + \left(\frac{ct+x}{ct-x}\right)\frac{d}{dv}\left(\frac{1+v/c}{1-v/c}\right)$$
$$= -\frac{2}{c}\left(\frac{ct+x}{ct-x}\right)\frac{1}{(1-v/c)^2} + \frac{2}{c}\left(\frac{ct+x}{ct-x}\right)\frac{1}{(1-v/c)^2} = 0,$$

as required. Thus, two independent integrals of the system (2.29) are

$$\zeta^2 = (ct)^2 - x^2, \qquad \rho = \left(\frac{ct+x}{ct-x}\right)\left(\frac{1+v/c}{1-v/c}\right),$$

so that the general solution of the above first order partial differential equation is given by $\rho = \Phi(\zeta)$, where Φ denotes an arbitrary function.

This may be formally confirmed as follows: From $\rho = \Phi(\zeta)$ we may deduce that

$$\left(\frac{1+v/c}{1-v/c}\right) = \beta^2\Psi((\alpha\beta)^{1/2}), \tag{2.31}$$

where Ψ denotes a second arbitrary function and α and β denote the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$. From the immediately above equation, we obtain

$$\frac{v(\alpha,\beta)}{c} = \left(\frac{\beta^2\Psi((\alpha\beta)^{1/2}) - 1}{\beta^2\Psi((\alpha\beta)^{1/2}) + 1}\right), \tag{2.32}$$

and in terms of the characteristic variables α and β, the first order partial differential equation (2.28) becomes

$$\frac{1}{c}\left(\alpha\frac{\partial v}{\partial\alpha} - \beta\frac{\partial v}{\partial\beta}\right) = \left(\frac{v}{c}\right)^2 - 1, \tag{2.33}$$

which we observe, apart from a change of sign, coincides with the equation for $u(\alpha,\beta)$ that arises from the vanishing of the Jacobian given by (2.17), noting however that this is an important distinction, and that the two calculations are distinct, although evidently related. On evaluating the two partial derivatives $\partial v/\partial\alpha$ and $\partial v/\partial\beta$ using the above expression (2.32), we have

$$\frac{1}{c}\frac{\partial v}{\partial\alpha} = \frac{\beta^3\Psi'(\zeta)}{\zeta(\beta^2\Psi(\zeta)+1)^2}, \qquad \frac{1}{c}\frac{\partial v}{\partial\beta} = \frac{\beta(4\Psi(\zeta) + \zeta\Psi'(\zeta))}{(\beta^2\Psi(\zeta)+1)^2},$$

where $\zeta = (\alpha\beta)^{1/2}$, and we may now show that (2.32) constitutes a formal solution of (2.33) for all arbitrary functions $\Psi((\alpha\beta)^{1/2})$. Subsequently, we see that the Jacobian $\partial(X,T)/\partial(x,t)$ vanishes whenever $v(x,t)$ satisfies the above partial differential equation (2.28) or (2.33), implying that the integral invariants correspond to families of singular transformations.

2.8 Alternative Validation of Lorentz Invariants

We are now well placed to establish the integral invariants of the Lorentz group in a very interesting manner using the "pseudo-velocity" equations (2.23), namely,

$$\frac{dx}{dv} = \frac{-t}{1-(v/c)^2}, \quad \frac{dt}{dv} = \frac{-x/c^2}{1-(v/c)^2}. \tag{2.34}$$

In order to do this, we need an expression for the velocity $u(x, t)$ which we obtain by division of the differential relations (2.23) or (2.34) or directly from (2.29); thus

$$u = \frac{dx}{dt} = \frac{c^2 t}{x}.$$

As a first example, we consider the Einstein addition of velocities law (2.6) expressed in the form

$$U = \frac{u+v}{1+uv/c^2}, \quad 0 \leqslant v < c.$$

On replacement of u by $c^2 t/x$, we have

$$\frac{u+v}{1+uv/c^2} = \frac{xv+tc^2}{x+vt},$$

which we now differentiate with respect to v, and using the differential relations (2.30), we find that

$$\frac{d}{dv}\left(\frac{u+v}{1+uv/c^2}\right) = \frac{d}{dv}\left(\frac{xv+tc^2}{x+vt}\right) = \frac{1}{(x+vt)}\frac{d}{dv}(xv+tc^2) - \frac{(xv+tc^2)}{(x+vt)^2}\frac{d}{dv}(x+vt)$$
$$= \frac{x}{(x+vt)} - \frac{1}{(1-(v/c)^2)} - \left(\frac{xv+tc^2}{x+vt}\right)\frac{v}{c^2(1-(v/c)^2)} = 0,$$

as might be expected. Again, formally the value of the integral is fixed through the initial data which in this case is $u = U$ when $v = 0$, which produces the required result.

Similarly, for the Lorentz invariants ζ and τ defined by (2.15), we have

$$\frac{d\zeta^2}{dv} = \frac{d((ct)^2 - x^2)}{dv} = 2\left(c^2 t\frac{dt}{dv} - x\frac{dx}{dv}\right) = 0,$$
$$\frac{d\tau^2}{dv} = \frac{d}{dv}\left[\left(\frac{ct+x}{ct-x}\right)\left(\frac{1-u/c}{1+u/c}\right)\right] = \frac{d}{dv}\left[\left(\frac{ct+x}{ct-x}\right)\left(\frac{1-ct/x}{1+ct/x}\right)\right] = \frac{d1}{dv} = 0.$$

Other integral invariants previously given by (2.13) are as follows:

$$\frac{x-ut}{\left(1-(u/c)^2\right)^{1/2}} = \frac{X-UT}{\left(1-(U/c)^2\right)^{1/2}}, \quad \frac{t-ux/c^2}{\left(1-(u/c)^2\right)^{1/2}} = \frac{T-UX/c^2}{\left(1-(U/c)^2\right)^{1/2}},$$

which we have previously established by direct substitution of the Lorentz transformations into the left-hand sides with appropriate use of the addition of velocities law. Alternatively, we may establish these by differentiation using $u = c^2t/x$; thus

$$\frac{d}{dv}\left(\frac{x-ut}{\left(1-(u/c)^2\right)^{1/2}}\right) = \frac{d}{dv}\left(\frac{x-(ct)^2/x}{(1-(ct/x)^2)^{1/2}}\right) = \frac{d}{dv}\left(x^2-(ct)^2\right)^{1/2} = 0,$$

$$\frac{d}{dv}\left(\frac{t-ux/c^2}{\left(1-(u/c)^2\right)^{1/2}}\right) = \frac{d}{dv}\left(\frac{t-tx/x}{\left(1-(u/c)^2\right)^{1/2}}\right) = 0,$$

as required.

2.9 Jacobians of the Lorentz Transformations

In this section we deduce formulae for the Jacobians of the Lorentz transformations (2.2) and their inverses (2.3). We observe from Eq. (2.2) on carefully calculating the four partial derivatives $\partial X/\partial x$, $\partial X/\partial t$, $\partial T/\partial x$ and $\partial T/\partial t$ treating v as a function of x and t, so that, for example, we have

$$\frac{\partial X}{\partial x} = \frac{1}{(1-(v/c)^2)^{1/2}} + \frac{(xv+c^2t)}{c^2(1-(v/c)^2)^{3/2}}\frac{\partial v}{\partial x}, \tag{2.35}$$

$$\frac{\partial X}{\partial t} = \frac{v}{(1-(v/c)^2)^{1/2}} + \frac{(xv+c^2t)}{c^2(1-(v/c)^2)^{3/2}}\frac{\partial v}{\partial t},$$

$$\frac{\partial T}{\partial x} = \frac{v}{c^2(1-(v/c)^2)^{1/2}} + \frac{(x+tv)}{c^2(1-(v/c)^2)^{3/2}}\frac{\partial v}{\partial x},$$

$$\frac{\partial T}{\partial t} = \frac{1}{(1-(v/c)^2)^{1/2}} + \frac{(x+tv)}{c^2(1-(v/c)^2)^{3/2}}\frac{\partial v}{\partial t},$$

and from which we may eventually deduce

$$\frac{\partial(X,T)}{\partial(x,t)} = \left(1 + \frac{\frac{t\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t}}{\left(1-(v/c)^2\right)}\right), \tag{2.36}$$

and we see that the Jacobian vanishes whenever $v(x,t)$ satisfies the above partial differential equation (2.28), implying that the integral invariants correspond to families of singular transformations. Similarly, on treating the relative velocity v

as a function of X and T, from the inverse Lorentz relations (2.3), we may deduce the corresponding equation

$$\frac{\partial(x,t)}{\partial(X,T)} = \left(1 - \frac{\frac{T\partial v}{\partial X} + \frac{X}{c^2}\frac{\partial v}{\partial T}}{\left(1-(v/c)^2\right)}\right). \tag{2.37}$$

For convenience, we introduce the notation J, δ and Δ such that

$$1/J = \frac{\partial(X,T)}{\partial(x,t)} = (1+\delta), \qquad J = \frac{\partial(x,t)}{\partial(X,T)} = (1-\Delta),$$

where δ and Δ are defined, respectively, by

$$\delta = \frac{\frac{t\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t}}{\left(1-(v/c)^2\right)}, \qquad \Delta = \frac{\frac{T\partial v}{\partial X} + \frac{X}{c^2}\frac{\partial v}{\partial T}}{\left(1-(v/c)^2\right)},$$

and since the product of the two Jacobians is necessarily unity, there follows the necessary identity $(1/\Delta) = 1 + (1/\delta)$ which may be formally verified as follows:

$$\delta = \frac{\frac{t\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t}}{\left(1-(v/c)^2\right)} = \frac{t\frac{\partial(v,t)}{\partial(x,t)} - \frac{x}{c^2}\frac{\partial(v,x)}{\partial(x,t)}}{\left(1-(v/c)^2\right)},$$

and multiplication by J gives

$$\delta J\left(1-(v/c)^2\right) = t\frac{\partial(v,t)}{\partial(X,T)} - \frac{x}{c^2}\frac{\partial(v,x)}{\partial(X,T)} = \frac{\partial\left(v, t^2-(x/c)^2\right)}{2\partial(X,T)},$$

and on using the invariant $(ct)^2 - x^2 = (cT)^2 - X^2$, we have

$$\delta J\left(1-(v/c)^2\right) = \frac{\partial\left(v, T^2-(X/c)^2\right)}{2\partial(X,T)} = \frac{T\partial v}{\partial X} + \frac{X}{c^2}\frac{\partial v}{\partial T} = \left(1-(v/c)^2\right)\Delta,$$

and the identity $(1/\Delta) = 1 + (1/\delta)$ now follows from $J = 1 - \Delta$.

As previously mentioned, we view the relative velocity v as a time-like variable in a manner completely analogous to non-linear continuum mechanics, and by formal differentiation of $J = \partial(x,t)/\partial(X,T)$ with respect to v, recalling that (X, T, v) are treated as the independent variables, we may deduce

$$\frac{dJ}{dv} = \frac{\partial(\frac{dx}{dv},t)}{\partial(X,T)} + \frac{\partial(x,\frac{dt}{dv})}{\partial(X,T)} = \left(\frac{\partial(\frac{dx}{dv},t)}{\partial(x,t)} + \frac{\partial(x,\frac{dt}{dv})}{\partial(x,t)}\right)\frac{\partial(x,t)}{\partial(X,T)},$$

and on using (2.23) or (2.30), we may eventually deduce

$$\frac{dJ}{dv} = \frac{2v/c^2}{\left(1-(v/c)^2\right)}(J-1),$$

which may be integrated to yield

$$J = \frac{\partial(x,t)}{\partial(X,T)} = \left(1 - \frac{A(X,T)}{\left(1-(v/c)^2\right)}\right),$$

where $A(X, T)$ is at most a function of X and T, and this equation is entirely consistent with (2.37).

2.10 Space-Time Transformation $x' = ct$ and $t' = x/c$

The relation $uU = c^2$ formally arises from the underlying transformation $x = cT, t = X/c$. With a primed notation, the space-time transformation $x' = ct$ and $t' = x/c$, for which $u' = dx'/dt' = c^2 dt/dx = c^2/u$, has been widely used to connect the Galilean and Carroll transformations as significant limits of Lorentz invariant theories, for example, in electromagnetism. The transformations $x' = ct$ and $t' = x/c$ were originally introduced by Jean-Marc Levy-Leblond, and their origin and development is fully detailed by Rousseaux [89] and Houlrik and Rousseaux [61]. Here we observe that the Lorentz transformations (2.2) or (2.3) are left unchanged by this transformation, namely,

$$X' = cT, \qquad T' = \frac{X}{c}, \qquad x' = ct, \qquad t' = \frac{x}{c}, \tag{2.38}$$

assuming the same constant value for the frame velocity v. Further, on making this transformation for the equation describing Galilean invariance for particles, namely, $x = X - vT$ and $t = T$, we obtain the equations $x' = X'$ and $t' = T' - vX'/c^2$ describing Galilean invariance for waves, and for a fuller account, we refer the reader to Houlrik and Rousseaux [61].

We observe the seemingly curious property that under the space-time transformation (2.38) with $u' = c^2/u$, $U' = c^2/U$ and $v' = c^2/v$, changing any two means that the Einstein addition of velocities law (2.6) remains invariant, in the sense that the same value for the third is recorded as in the unprimed frame, so that, for example, for u we have from (2.6)

$$u = \frac{U - v}{\left(1 - Uv/c^2\right)} = \frac{\left(c^2/U' - c^2/v'\right)}{\left(1 - c^2/U'v'\right)} = \frac{U' - v'}{\left(1 - U'v'/c^2\right)}.$$

This property reflects the symmetry and parity of the Einstein addition of velocities law which is most apparent when written in the form of (2.8).

We now examine the changes to the wavelengths and frequencies in the unprimed and primed frames. We consider a simple wave of wavelength μ and frequency ν in the unprimed (x, t) frame with $\mu\nu = c^2/u$; thus

$$y(x, t) = A \exp\left\{2\pi i \left(\frac{x}{\mu} - t\nu\right)\right\}, \tag{2.39}$$

where A denotes the constant amplitude. A corresponding simple wave in the primed frame becomes

$$y(x', t') = A \exp\left\{2\pi i \left(\frac{x'}{\mu'} - t'\nu'\right)\right\} = A \exp\left\{-2\pi i \left(\frac{x\nu'}{c} - \frac{ct}{\mu'}\right)\right\}, \tag{2.40}$$

and therefore under the primed transformation $x' = ct$ and $t' = x/c$, the wavelength and frequency transform according to the formulae

$$\mu' = \frac{c}{\nu}, \qquad \nu' = \frac{c}{\mu}, \tag{2.41}$$

with the consequence that $\mu'\nu' = u$.

Remark We emphasise that the space-time transformation $x' = ct$ and $t' = x/c$ involves only one Cartesian spatial dimension, and reflects a simple invariance of the operator for the classical one-dimensional wave equation, so that the appropriate extension and interpretation to three spatial dimensions and other coordinates involving curvature is by no means obvious. However, while Guemez et al. [44] have proposed one extension of the de Broglie formula $uu' = c^2$ to three spatial dimensions (x, y, z), namely, $\mathbf{u}.\mathbf{u}' = c^2$, the most likely coordinate decomposition of this formula $\mathbf{r}' = ct\mathbf{r}/r$ and $t' = r/c$, where $\mathbf{r}$ and $\mathbf{r}'$ are the obvious position vectors and $r = (x^2 + y^2 + z^2)^{1/2}$, and with the identical inverse transformations $\mathbf{r} = ct'\mathbf{r}'/r'$ and $t = r'/c$ where $r' = (x'^2+y'^2+z'^2)^{1/2}$, remains essentially a single spatial dimension transformation. The coordinate transformation $\mathbf{r}' = ct\mathbf{r}/r$ is spatially spherically symmetric, and polar angles remain unchanged and $r' = ct$ and $t' = r/c$, so that the transformation $\mathbf{r}' = ct\mathbf{r}/r$ is essentially one dimensional. We further comment that although this coordinate decomposition is a natural extension of the one-dimensional coordinate transformation, it may not be unique. Indeed, the one-dimensional transformation itself $x' = ct$ and $t' = x/c$ for $uu' = c^2$ does not provide a unique decomposition of the equation $uu' = c^2$, since the negative transformation $x' = -ct$ and $t' = -x/c$ is equally effective.

2.11 The de Broglie Wave Velocity $u' = c^2/u$

All matter exhibits wave-like behaviour, and Louis de Broglie [17] first predicted light to display the dual characteristics both as a collection of particles, called photons, and in some respects as a wave. The particle velocity u is the group velocity of the wave, and if the particle velocity u is sub-luminal, then the associated wave or phase velocity c^2/u through the de Broglie relation is necessarily superluminal. At various times in his life, de Broglie held a concrete physical picture of the co-existence of both particle and its associated wave and refers to "the theory of the double solution", for which he formulated an equation which he called "the guidance formula" (see Eq. (1.2) and also [108]). Since $uu' = c^2$, from (2.6) their relative velocity is necessarily infinite. However, the average of the particle and wave velocities, denoted here by V, is given by

$$V = \frac{1}{2}(u + u') = \frac{1}{2}\left(u + \frac{c^2}{u}\right), \tag{2.42}$$

and from the elementary identity

$$\left(\frac{V}{c}\right)^2 - 1 = \frac{1}{4}\left[\left(\frac{u}{c}\right)^2 + \left(\frac{c}{u}\right)^2 + 2\right] - 1 = \frac{1}{4}\left(\frac{c}{u} - \frac{u}{c}\right)^2 \geq 0,$$

so that for sub-luminal particle velocity u, both the wave velocity u' and their average V are necessarily both superluminal. In addition, the three velocities u, u' and V satisfy the Einstein addition of velocity law in its various forms, but specifically in the form of (2.8), we have

$$\left(\frac{V/c + 1}{V/c - 1}\right) = \left(\frac{1 + u/c}{1 - u/c}\right)\left(\frac{u'/c + 1}{u'/c - 1}\right).$$

With θ as defined previously by (2.11), we have the following relations:

$$u = c\tanh\theta, \qquad u' = c\coth\theta, \qquad e = e_0\cosh\theta, \qquad pc = e_0\sinh\theta,$$

where as before $e_0 = m_0c^2$ denotes the rest mass energy, and it is not difficult to establish the formulae

$$V = c\coth 2\theta, \qquad \frac{1}{\left((V/c)^2 - 1\right)^{1/2}} = \sinh 2\theta,$$

which may be used to show that

$$\frac{e_0^2(V/c)}{\left((V/c)^2 - 1\right)^{1/2}} = e^2 + (pc)^2, \qquad \frac{e_0^2}{\left((V/c)^2 - 1\right)^{1/2}} = 2epc.$$

By addition and subtraction of these expressions, we might readily deduce

$$e + cp = e_0 \left(\frac{V/c + 1}{V/c - 1} \right)^{1/4}, \qquad e - cp = e_0 \left(\frac{V/c - 1}{V/c + 1} \right)^{1/4},$$

and therefore

$$e = \frac{e_0}{2} \left[\left(\frac{V/c + 1}{V/c - 1} \right)^{1/4} + \left(\frac{V/c - 1}{V/c + 1} \right)^{1/4} \right],$$
$$cp = \frac{e_0}{2} \left[\left(\frac{V/c + 1}{V/c - 1} \right)^{1/4} - \left(\frac{V/c - 1}{V/c + 1} \right)^{1/4} \right].$$

Now since the average velocity V is a symmetric function of u and u', we expect this symmetry to be reflected in the expression $V = c \coth 2\theta$, and in order to see this, we need to determine θ' through (2.10); thus formally we have

$$e^{\theta'} = \left(\frac{1 + u'/c}{1 - u'/c} \right)^{1/2} = \left(\frac{1 + u/c}{u/c - 1} \right)^{1/2} = \frac{1}{i} \left(\frac{1 + u/c}{1 - u/c} \right)^{1/2} = e^{\theta} e^{-i\pi/2},$$

and from which we might deduce the relation $\theta' = \theta - i\pi/2$. Using the standard expressions,

$$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$$
$$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y,$$

from which we obtain

$$V' = c \coth 2\theta' = c \frac{\cosh 2\theta'}{\sinh 2\theta'} == c \frac{\cosh(2\theta - i\pi)}{\sinh(2\theta - i\pi)} = c \frac{\cosh 2\theta}{\sinh 2\theta} = V,$$

as might be expected.

2.12 Force and Physical Energy Arising from Work Done

The basic notions of force, as rate of change of momenta, and physical energy, as the work done (viz. force times distance) arise in the two rate-of-working equations (or work done equations) for the physical energies E and e in the (X, T) and (x, t) frames, respectively, and these are as follows:

$$dE = F dX = \frac{dP}{dT} dX, \qquad de = f dx = \frac{dp}{dt} dx, \tag{2.43}$$

where $F = dP/dT$ and $f = dp/dt$ denote the physical force in the two frames and $P = MU$ and $p = mu$ the momenta where $U = dX/dT$ and $u = dx/dt$ are the respective particle velocities. Using these relations and the expressions $E = Mc^2$ and $e = mc^2$, the Eqs. (2.43) on multiplication by their respective masses may be rewritten as

$$E\frac{dE}{dT} = c^2 P\frac{dP}{dT}, \qquad e\frac{de}{dt} = c^2 p\frac{dp}{dt}.$$

These equations evidently integrate to yield the respective equations $E^2 = (Pc)^2 + constant$ and $e^2 = (pc)^2 + constant$. It might be important to appreciate that the arbitrary constants in these equations are generally fixed by taking the particle energy at $E = e = e_0 = m_0c^2$ at zero velocity, so that we have $E^2 - (Pc)^2 = e^2 - (pc)^2 = e_0^2$. However, this does not necessarily have to be the case and there may be other interpretations for the constants. Here we assume these arbitrary constants are as generally prescribed, and we assume the energy statements $E^2 = e_0{}^2 + (Pc)^2$ and $e^2 = e_0{}^2 + (pc)^2$, so that along with $E = Mc^2$ and $e = mc^2$, we might deduce the Einstein formulae for the variation of mass with velocity

$$M(U) = \frac{m_0}{[1-(U/c)^2]^{1/2}}, \qquad m(u) = \frac{m_0}{[1-(u/c)^2]^{1/2}}. \tag{2.44}$$

We comment that the formula $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ is only one of the many expressions showing a particular variation of mass with its velocity, and this expression has a long and extensive history involving many eminent scientists such as Abraham, Bücherer, Lorentz, Ehrenfest, Kaufmann and of course Einstein, who first grappled with the notion that the “transverse and longitudinal” masses may be distinct. The development of the Einstein expression is fully detailed by Weinstein [109].

In [56] and [57], the authors generalise the Einstein mass variation for both sub-luminal and superluminal velocities by supplementing the condition $f = F$ with the condition that the energy-mass rate is the same in both frames, namely, $de/dm = dE/dM$, which for the Einstein expression is simply constant and equal to c^2. Specifically, assuming the Lorentz transformations (2.3), the Lorentz invariant energy-momentum relations discussed in the following section and the two invariances

$$\frac{dp}{dt} = \frac{dP}{dT}, \quad \frac{dm}{dx} = \frac{dM}{dX},$$

which are known to apply in special relativity, new mass variation formulae involving two arbitrary constants are obtained, noting that the Einstein expression involves only the rest mass as a single arbitrary constant. For example, for sub-luminal velocities $0 < u < c$, and with θ and ϵ defined by Eqs. (2.9) and (2.10), we may deduce in [56] the mass variation expression

$$m(u) = \frac{C_2}{(1-(u/c)^2)^{1/2}}\left\{C_1 + \frac{1}{4\tanh(\epsilon/2)}\int_0^{\theta+\epsilon/2}\frac{d\rho}{(\sinh\rho)^{1/2}} + \frac{1}{2\,[\sinh(\theta+\epsilon/2)]^{1/2}}\right\},$$

where C_1 and C_2 denote arbitrary constants and the Einstein expression arises from the above equation in the limiting case $C_1 \rightarrow \infty$. The corresponding energy is found to be given by

$$e(u) = m(u)c^2 - \frac{C_2c^2}{\sinh(\epsilon/2)}\left\{[\sinh(\theta+\epsilon/2)]^{1/2} - [\sinh(\epsilon/2)]^{1/2}\right\},$$

noting that in this expression the energy has been normalised such that $e = m_0c^2$ when $u = 0$. Although to a certain extent this datum energy level is arbitrary, we observe that for this particular normalisation, the additive constant might become pure imaginary depending upon whether $v > 0$ or $v < 0$. This highlights the issue that the region of validity of the above expression for $m(u)$ might require careful consideration since even the seemingly natural choice of datum energy levels may not be applicable, and this aspect is fully examined in [56].

For superluminal velocities $c < u/c < \infty$, there are corresponding expressions derived in [57], essentially with the sinh replaced by cosh; thus

$$m(u) = \frac{C_2}{((u/c)^2-1)^{1/2}}\left\{C_1 + \frac{1}{4}\tanh(\epsilon/2)\int_0^{\theta+\epsilon/2}\frac{d\rho}{(\cosh\rho)^{1/2}} + \frac{1}{2\,[\cosh(\theta+\epsilon/2)]^{1/2}}\right\},$$

with corresponding energy formulae given by

$$e(u) = m(u)c^2 - \frac{C_2c^2}{\cosh(\epsilon/2)}\left\{\frac{\sinh(\theta+\epsilon/2)/2}{[\cosh(\theta+\gamma/2)]^{1/2}} - \frac{\sinh(\epsilon/2)}{[\cosh(\epsilon/2)]^{1/2}}\right\},$$

where again C_1 and C_2 denote arbitrary constants and in this case the energy is assumed to vanish for $u \longrightarrow \infty$. We note especially that in the case of superluminal velocities, from Eq. (2.8), it is clear that at least one of the (U, u, v) must be subluminal and the immediately above expressions are derived on the assumption that $u, v > c$ and that $U < c$, so that in this case Θ, θ and ϵ must be appropriately defined; thus

$$\Theta = \log\left(\frac{1+U/c}{1-U/c}\right), \qquad \theta = \log\left(\frac{u/c+1}{u/c-1}\right), \qquad \epsilon = \log\left(\frac{v/c+1}{v/c-1}\right),$$

for which again $\Theta = \theta + \epsilon$ and with inverses given by

$$U = c\tanh(\Theta/2), \qquad u = c\coth(\theta/2), \qquad v = c\coth(\epsilon/2).$$

Applications of the above formulae are given in [56] and [57] and we refer the reader to these papers for details.

2.13 Lorentz Invariant Energy-Momentum Relations

Lorentz Invariant Energy-Momentum Relations The Lorentz invariant energy-momentum relations given by Eq. (2.46) can both be deduced from (2.6) as follows: For sub-luminal velocities $v, u, U < c$, the Einstein formulae for mass and energy in both frames are summarised by the formulae

$$E = Mc^2, \qquad M = \frac{m_0}{[1-(U/c)^2]^{1/2}}, \qquad e = mc^2, \qquad m = \frac{m_0}{[1-(u/c)^2]^{1/2}}, \tag{2.45}$$

so that with momenta $P = MU$ and $p = mu$, we have on multiplication of (2.6) by $m_0\left[1-(u/c)^2\right]^{-1/2}$, and by using Eq. (2.7), we may readily deduce $(2.46)_1$; thus

$$\frac{um_0}{[1-(u/c)^2]^{1/2}} = \frac{m_0U - m_0v}{[1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2}},$$

while $(2.46)_2$ arises directly from $(2.45)_3$ and $(2.45)_4$, on using Eq. (2.7); thus

$$e = \frac{m_0c^2}{[1-(u/c)^2]^{1/2}} = \frac{m_0c^2(1-Uv/c^2)}{[1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2}},$$

so that altogether we obtain

$$p = \frac{P - Ev/c^2}{[1-(v/c)^2]^{1/2}}, \qquad e = \frac{E - Pv}{[1-(v/c)^2]^{1/2}}. \tag{2.46}$$

The inverse relations are given by

$$P = \frac{p + ev/c^2}{[1-(v/c)^2]^{1/2}}, \qquad E = \frac{e + pv}{[1-(v/c)^2]^{1/2}}, \tag{2.47}$$

and together these equations are referred to as the Lorentz invariant energy-momentum relations. Some authors [13] refer, respectively, to the above notions of mass m and momentum p as "temporal" and "spatial" momentum. We do not follow that distinction here and we refer to the relations (2.46) as the Lorentz invariant energy-momentum equations. From these relations, it is also clear that we have the Lorentz invariant $e^2 - (pc)^2 = E^2 - (Pc)^2 = e_0^2$, where $e_0 = m_0c^2$ denotes the rest mass energy.

Lorentz Invariants $\xi(x,t) = ex - c^2pt$ and $\eta(x,t) = px - et$ By direct substitution we may establish, using Eqs. (2.3) and (2.46), that $\xi(x,t)$ and $\eta(x,t)$ as defined by the equations

$$\xi = ex - c^2 pt, \qquad \eta = px - et, \tag{2.48}$$

constitute two Lorentz invariances of special relativity, which are readily verified as follows: On evaluating $\xi = ex - c^2 pt$ and $\eta = px - et$ using (2.3) and (2.46), we have

$$\frac{\big((E - Pv)(X - vT) - c^2(P - Ev/c^2)(T - vX/c^2)\big)}{\big(1 - (v/c)^2\big)} = EX - c^2 PT,$$

and

$$\frac{\big((P - Ev/c^2)(X - vT) - (E - Pv)(T - vX/c^2)\big)}{\big(1 - (v/c)^2\big)} = PX - ET,$$

as required. We further observe that with γ and δ defined, respectively, by

$$\gamma = \frac{1}{2c}(e + pc)(ct - x), \qquad \delta = \frac{1}{2c}(e - pc)(ct + x), \tag{2.49}$$

the two Lorentz invariants ξ and η given by (2.48) become

$$\xi = ex - c^2 pt = -c(\gamma - \delta), \qquad \eta = px - et = -(\gamma + \delta), \tag{2.50}$$

so that we have the expressions

$$\xi + c\eta = -(e + pc)(ct - x), \qquad \xi - c\eta = (e - pc)(ct + x), \tag{2.51}$$

and the Lorentz invariant $e^2 - (pc)^2 = e_0^2$ becomes $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$. From these relations we may deduce

$$e - pc = e_0^2\left(\frac{x - ct}{\xi + c\eta}\right) = \left(\frac{\xi - c\eta}{x + ct}\right), \qquad e + pc = e_0^2\left(\frac{x + ct}{\xi - c\eta}\right) = \left(\frac{\xi + c\eta}{x - ct}\right),$$

revealing that each of the conditions $e = \pm pc$ occurs for both $x = \pm ct$ and $\xi = \pm c\eta$. Further, from (2.51) we observe that

$$\frac{\xi - c\eta}{\xi + c\eta} = -\left(\frac{e - pc}{e + pc}\right)\left(\frac{ct + x}{ct - x}\right) = -\left(\frac{1 - u/c}{1 + u/c}\right)\left(\frac{ct + x}{ct - x}\right) = -\tau^2, \tag{2.52}$$

where τ is the second Lorentz invariant defined by (2.15), and from (2.52), we may deduce the following relationship between the three Lorentz invariants ξ, η and τ; thus

$$\frac{c\eta}{\xi} = \left(\frac{1+\tau^2}{1-\tau^2}\right).$$

Inverse Relations for Lorentz Invariants On writing the Lorentz invariants (2.48) as follows

$$x - ut = \frac{\xi(x,t)}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{1/2}, \qquad x\frac{u}{c} - ct = \frac{c\eta(x,t)}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{1/2},$$

which we may regard as two equations in the two unknowns x and t, and formally solve to obtain

$$x = \frac{\xi(x,t) - u\eta(x,t)}{e_0(1-(u/c)^2)^{1/2}}, \quad ct = \frac{\xi(x,t)u/c - c\eta(x,t)}{e_0(1-(u/c)^2)^{1/2}},$$

which are evidently reminiscent of the Lorentz transformations (2.3), and we make this connection more precise in the following:

Lorentz Invariants as a Coordinate Transformation Clearly we may express the immediately above equations in the form of a Lorentz transformation, namely,

$$x^* = \frac{x - ut}{[1-(u/c)^2]^{1/2}}, \quad t^* = \frac{t - ux/c^2}{[1-(u/c)^2]^{1/2}}, \tag{2.53}$$

where $\xi = e_0 x^*$ and $\eta = -e_0 t^*$, and from which we may deduce the following expression for the velocity $u^* = dx^*/dt^*$; thus

$$u^* = \frac{dx^*}{dt^*} = \frac{(dx - udt - tdu)(1-(u/c)^2) + (x-ut)udu/c^2}{(dt - udx/c^2 - xdu/c^2)(1-(u/c)^2) + (t - xu/c^2)udu/c^2}, \tag{2.54}$$

$$= \frac{(dx - udt)(1-(u/c)^2) - (t - xu/c^2)du}{(dt - udx/c^2)(1-(u/c)^2) - (x-ut)du/c^2},$$

$$= \frac{-(t - xu/c^2)du/dt}{(1-(u/c)^2)^2 - (x-ut)/c^2(du/dt)},$$

$$= \frac{-\frac{(t-xu/c^2)}{(1-(u/c)^2)^{3/2}}\frac{du}{dt}}{(1-(u/c)^2)^{1/2} - \frac{(x-ut)}{c^2(1-(u/c)^2)^{3/2}}\frac{du}{dt}},$$

on using $u(x,t) = dx/dt$. Subsequently, we show that the two fundamental total derivatives underpinning the model structure are the material or total time derivative following the particle and the spatial or total space derivative following the wave, which are defined below in terms of partial differential operators (see also (4.56)). It is therefore tempting to investigate the transformation properties under (2.53) of the two total differential operators

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \qquad \frac{d}{dx} = \frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t}.$$

We find that with the transformed velocity given by (2.54), the structure of the first is preserved while that of the second is not. We need the following expressions for the partial derivatives, which are essentially those arising from (2.35), with (X, T) replaced by (x^*, t^*) and v replaced by $-u$; thus

$$\frac{\partial x^*}{\partial x} = \frac{1}{(1-(u/c)^2)^{1/2}} - \frac{(t - xu/c^2)}{(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial x},$$

$$\frac{\partial x^*}{\partial t} = \frac{-u}{(1-(u/c)^2)^{1/2}} - \frac{(t - xu/c^2)}{(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial t},$$

$$\frac{\partial t^*}{\partial x} = \frac{-u}{c^2(1-(u/c)^2)^{1/2}} - \frac{(x - ut)}{c^2(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial x},$$

$$\frac{\partial t^*}{\partial t} = \frac{1}{(1-(u/c)^2)^{1/2}} - \frac{(x - ut)}{c^2(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial t}.$$

On making use of these expressions, we obtain

$$\begin{aligned}\frac{d}{dt} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},\\ &= \left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial t}\right) + u\left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial x} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\right),\\ &= \left\{(1-(u/c)^2)^{1/2} - \frac{(x-ut)}{c^2(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\right\}\left(\frac{\partial}{\partial t^*} + u^*\frac{\partial}{\partial x^*}\right),\end{aligned}$$

and similarly for d/dx we have

$$\begin{aligned}\frac{d}{dx} &= \frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t},\\ &= \left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial x} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\right) + \frac{u}{c^2}\left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial t}\right),\\ &= \left\{(1-(u/c)^2)^{1/2} - \frac{(t-ux/c^2)}{(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t}\right)\right\}\left(\frac{\partial}{\partial x^*} + \frac{w^*}{c^2}\frac{\partial}{\partial t^*}\right),\end{aligned}$$

where w^*/c is given by

$$\frac{w^*}{c} = \frac{-\frac{(x-ut)}{c(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t}\right)}{\left\{(1-(u/c)^2)^{1/2} - \frac{(t-ux/c^2)}{(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t}\right)\right\}},$$

which in general does not coincide with u^*/c as given by (2.54), namely,

$$\frac{u^*}{c} = \frac{-\frac{(t-xu/c^2)}{c(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)}{\left\{(1-(u/c)^2)^{1/2} - \frac{(x-ut)}{c^2(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\right\}}.$$

However, it is more than curious to observe that their product

$$\frac{u^*w^*}{c^2} = \frac{(x-ut)(t-\frac{xu}{c^2})\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t}\right)}{c^2\left(1-\left(\frac{u}{c}\right)^2\right)^3\left\{\left(1-\left(\frac{u}{c}\right)^2\right) - \left(t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t}\right)\right\} + (x-ut)(t-\frac{xu}{c^2})\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t}\right)},$$

becomes unity in the event that either $u(x, t) = \pm c$ or that $u(x, t)$ satisfies the following first order partial differential equation:

$$t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t} = 1 - \left(\frac{u}{c}\right)^2, \tag{2.55}$$

corresponding to (2.28), and for which we have previously noted is the condition for which the Jacobian of the transformation (2.53) becomes singular (see Eqs. (2.35) and (2.36)). Further in general, u^* and w^* are complementary velocities satisfying the de Broglie condition, contingent on the vanishing of the single parameter, Γ, which is defined by

$$\Gamma = \frac{c^2\left(1-\left(\frac{u}{c}\right)^2\right)^3\left\{\left(1-\left(\frac{u}{c}\right)^2\right) - \left(t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t}\right)\right\}}{(x-ut)(t-\frac{xu}{c^2})\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t}\right)},$$

since then the product u^*w^*/c^2 becomes simply $u^*w^*/c^2 = 1/(1+\Gamma)$. We comment that $u(x, t) = \pm c$ are also formally solutions of (2.55) and that this may be relevant in terms of the continuity and matching of the solutions of this equation.

2.14 Force Invariance for Constant Velocity Frames

Fundamental to special relativistic mechanics is the physically motivated assumption that force, as defined to be the rate of change of momentum, remains invariant in the direction of relative motion for frames moving with constant relative velocity. This assumption is related to Newton's first law that particles remain at rest or in uniform motion unless operated on by an external force. Thus, although force equality $f = F$ for non-accelerating frames is a basic physical hypothesis of special relativity, it nevertheless formally hinges on assuming the Einstein mass variation (2.44), and this can be established in at least two different ways.

Firstly, on taking the differentials of $(2.46)_1$ and $(2.3)_2$, and using $dE = U dP$ which arise from the rate-of-working equation $(2.43)_1$, we have

$$f = \frac{dp}{dt} = \frac{dP - v\,dE/c^2}{dT - v\,dX/c^2} = \frac{dP(1 - Uv/c^2)}{dT(1 - Uv/c^2)} = \frac{dP}{dT} = F.$$

Alternatively,

$$f = \frac{d}{dt}\left(\frac{m_0 u}{[1 - (u/c)^2]^{1/2}}\right) = \frac{m_0\,du/dt}{[1 - (u/c)^2]^{3/2}},$$

and on using the differential of $(2.3)_2$ and the velocity addition formula (2.6), we have

$$f = \frac{m_0\left[1 - (v/c)^2\right]^{3/2} dU}{\left[1 - (u/c)^2\right]^{3/2}\left[1 - vU/c^2\right]^3 dT} = \frac{m_0\,dU/dT}{\left[1 - (U/c)^2\right]^{3/2}},$$

where the final step follows from (2.7) and again gives $f = F$. This result is well known and can be found, for example, in Moller [78] (page 73). Thus although one might expect force invariance $f = F$ in the direction of relative motion for non-accelerating frames to be a fundamental physical hypothesis of special relativity, it is formally equivalent to the Einstein relations (2.45) and the Lorentz transformations, and their consequences.

The equation $dp/dt = dP/dT$ is the starting point for the model developed in [47–52] and described in subsequent chapters; that is, instead of simply a spatial force being Lorentz invariant, we investigate a model for which both a spatial physical force **f** and a new force g in the direction of time are defined by two Lorentz invariant equations. In the final two sections of this chapter, we present an illustration for particle motion in an invariant potential field, and we derive one possible extension of the conventional Einstein variation of mass formula with a specific expression arising from a Lorentz invariant equation for the energy rate de/dp.

2.15 Example: Motion in an Invariant Potential Field

In special relativity, the full integration of the equation corresponding to Newton's second law, along with appropriate boundary or initial data, may well constitute a non-trivial problem. As an illustration of this formal integration procedure, we consider the general problem of the determination of the motion a single particle that is moving under the action of a properly invariant potential field that might arise, for example, from electromagnetism, gravity or some other force-generating field. As usual, we view the particle from two frames of reference (X, T) and (x, t) with the latter moving with relative velocity v in the aligned x and X direction, and we assume that Hamilton's equations apply in both frames. Thus, for Hamiltonians $K(X, P, T)$ and $H(x, p, t)$ where $P = MU$ and $p = mu$, we assume that

$$\frac{dP}{dT} = -\frac{\partial K}{\partial X}, \qquad \frac{dX}{dT} = \frac{\partial K}{\partial P},$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \qquad \frac{dx}{dt} = \frac{\partial H}{\partial p},$$

and for a single particle, we assume that the Hamiltonians take the specific forms

$$K(X, P, T) = \left(e_0^2 + (Pc)^2\right)^{1/2} + V(X, T),$$
$$H(x, p, t) = \left(e_0^2 + (pc)^2\right)^{1/2} + V(x, t),$$

where the potential functions $V(X, T)$ and $V(x, t)$ are the same function of the variables (X, T) and (x, t), respectively. For conventional special relativity, we have from the previous section $dp/dt = dP/dT$, and therefore the partial derivative $\partial V(x, t)/\partial x$ must either be at most a constant or an invariant function. Now each of the second Hamiltonian equations is automatically satisfied, while the first equations yield simply

$$\frac{dP}{dT} = -\frac{\partial V(X, T)}{\partial X}, \qquad \frac{dp}{dt} = -\frac{\partial V(x, t)}{\partial x},$$

so that from $dP/dt = dp/dt$ we may conclude that

$$\frac{\partial V(x, t)}{\partial x} = \Phi(\zeta, \tau),$$

where $\Phi(\zeta, \tau)$ denotes any arbitrary function of ζ and τ, which are the Lorentz invariants defined by (2.15); thus

$$\zeta = ((ct)^2 - x^2)^{1/2} = (\alpha\beta)^{1/2},$$
$$\tau = \left(\frac{ct + x}{ct - x}\right)^{1/2} \left(\frac{1 - u/c}{1 + u/c}\right)^{1/2} = \left(\frac{\alpha}{\beta}\right)^{1/2} e^{-\theta},$$

and α and β denote the characteristic coordinates defined by $\alpha = ct + x$ and $\beta = ct - x$.

For purposes of illustration, we might consider the case when $\Phi(\zeta, \tau)$ depends only on ζ; thus $\Phi = \Phi(\zeta)$, so that we need to integrate the momentum equation

$$\frac{d(mu)}{dt} = -\Phi(\zeta),$$

and we assume that appropriate boundary or initial data is adequately prescribed. On using $d\zeta/dt = (c^2t - xu)/\zeta$ we may deduce the equation

$$(xu - c^2t)d(mu) = \zeta\Phi(\zeta)d\zeta,$$

which on integration by parts using $u = dx/dt$ yields

$$\begin{aligned}
&(xu - c^2t)mu - \int mu\left(udx + xdu - c^2dt\right)\\
&= (xu - c^2t)mu + \int \left\{mc^2\left(1 - \left(\frac{u}{c}\right)^2\right)dx - muxdu\right\}\\
&= \frac{m_0u(xu - c^2t)}{(1 - (u/c)^2)^{1/2}} + m_0c^2x(1 - (u/c)^2)^{1/2}\\
&= \int^{\zeta} \zeta\Phi(\zeta)d\zeta.
\end{aligned}$$

Subsequent simplification readily yields the formal first integral

$$\frac{x - ut}{\left(1 - (u/c)^2\right)^{1/2}} = \int^{\zeta} \frac{\zeta\Phi(\zeta)d\zeta}{m_0c^2} + C_1,$$

where C_1 denotes an arbitrary constant, and we note that the expression on the left-hand side is one of the invariants in Eq. (2.13). If we now introduce $\Psi(\zeta)$ defined by

$$\Psi(\zeta) = \int^{\zeta} \frac{\zeta\Phi(\zeta)d\zeta}{m_0c^2} + C_1,$$

then with $u = dx/dt$ we require to obtain a further integral of the first order ordinary differential equation for $x = x(t)$, namely,

$$\frac{x - tdx/dt}{\left(1 - \frac{1}{c^2}(\frac{dx}{dt})^2\right)^{1/2}} = \Psi(\zeta),$$

where $\zeta = \left((ct)^2 - x^2\right)^{1/2}$. This equation may be formally integrated by introducing the two angles (θ, ϕ) such that

$$ct = \zeta \cosh\theta, \qquad x = \zeta \sinh\theta,$$

and

$$\cosh\phi = \frac{1}{\left(1 - \frac{1}{c^2}(\frac{dx}{dt})^2\right)^{1/2}}, \qquad \sinh\phi = \frac{\frac{1}{c}\frac{dx}{dt}}{\left(1 - \frac{1}{c^2}(\frac{dx}{dt})^2\right)^{1/2}},$$

and we assume that any boundary or initial data, perhaps involving velocity, position and time, may be transferred to boundary or initial data for θ and ϕ. Using $dx/dt = c\tanh\phi$, we may deduce from the above relations

$$\sinh(\theta - \phi) = \frac{\Psi}{\zeta}, \qquad \coth(\theta - \phi) = -\frac{1}{\zeta}\frac{d\zeta}{d\theta},$$

from which we may deduce the second formal integral

$$\theta = \pm \int^{\zeta} \frac{\Psi(\zeta)d\zeta}{\zeta\left[\Psi(\zeta)^2 + \zeta^2\right]^{1/2}} + C_2,$$

where C_2 denotes a further arbitrary constant of integration.

As a specific example, given that $\partial V(x,t)/\partial x = \Phi(\zeta)$, it follows that

$$V(x,t) = \int^{x} \Phi(\zeta)dx + W(t),$$

where the integration is partial with respect to x and $W(t)$ denotes an arbitrary function of time. Thus, for the partial derivatives, we have

$$\frac{\partial V}{\partial x} = \Phi(\zeta), \qquad \frac{\partial^2 V}{\partial x^2} = -\frac{x\Phi(\zeta)'}{\zeta},$$
$$\frac{\partial V}{\partial t} = c^2 t \int^{x} \frac{\Phi(\zeta)'}{\zeta} dx + W(t)', \qquad \frac{\partial^2 V}{\partial t^2} = c^4 t^2 \int^{x} \left(\Phi(\zeta)'' - \frac{\Phi(\zeta)'}{\zeta}\right)\frac{dx}{\zeta^2} + W(t)'',$$

so that in particular we have

$$\frac{\partial^2 V}{\partial x^2} = -\frac{x\Phi(\zeta)'}{\zeta} = -\int^{x} \frac{\partial}{\partial x}\left(\frac{x\Phi(\zeta)'}{\zeta}\right)dx = \int^{x} \left\{\frac{x^2}{\zeta^2}\left(\Phi(\zeta)'' - \frac{\Phi(\zeta)'}{\zeta}\right) - \frac{\Phi(\zeta)'}{\zeta}\right\}dx,$$

and from which we may verify that

$$\frac{\partial^2 V}{\partial t^2} - c^2\frac{\partial^2 V}{\partial x^2} = -c^2 \int^{x} \Phi(\zeta)'' dx + W(t)'',$$

where primes denote differentiation with respect to the indicated argument. Thus for a specific example, we might assume that

$$\Phi(\zeta)'' = -k^2\Phi(\zeta),$$

for some constant k, which then implies that the assumed potential $V(x, t)$ satisfies a partial differential equation of the form

$$\frac{\partial^2 V}{\partial t^2} - c^2\frac{\partial^2 V}{\partial x^2} = (kc)^2 V + [W(t)'' - (kc)^2 W(t)].$$

Accordingly, an example might be $\Phi(\zeta) = A\sin(k\zeta)$ for certain constants A and k, so that $\Psi(\zeta)$ is given by

$$\Psi(\zeta) = \frac{A}{e_0}\int^{\zeta} \zeta\sin(k\zeta)d\zeta + C_1 = -\frac{A}{e_0 k^2}\{k\zeta\cos(k\zeta) - \sin(k\zeta)\} + C_1,$$

where $e_0 = m_0c^2$, and with the constant $C_1 = 0$ giving rise to the formal integral

$$\theta = \pm\int^{\zeta} \frac{\{k\zeta\cos(k\zeta) - \sin(k\zeta)\}\, d\zeta}{\zeta\left[\{k\zeta\cos(k\zeta) - \sin(k\zeta)\}^2 + (e_0k^2\zeta/A)^2\right]^{1/2}} + C_2.$$

With the substitution $z = k\zeta$ and the constant δ defined by $\delta = ke_0/A$ simplifies somewhat to give

$$\theta = \pm\int^{z} \frac{(z\cos z - \sin z)\, dz}{z\left[(z\cos z - \sin z)^2 + (\delta z)^2\right]^{1/2}} + C_2,$$

but no doubt would still require to be evaluated numerically.

2.16 Alternative Energy-Mass Velocity Variation

Throughout the text we assume that the Einstein energy-mass expression $e = e_0/(1 - (u/c)^2)^{1/2}$ applies, which is not an essential assumption other than being the simplest and most logical, and any other hypothesis would necessarily be more complicated and might involve additional implied consequences. The author [53] derives the simplest one-parameter Lorentz invariant extension of the Einstein mass-energy relation, and implicit in the new expression is space-time anisotropy such that the particle has different rest masses in the positive and negative x directions. This alternative energy-mass velocity variation formula arises from the following general Lorentz invariant equation:

$$\frac{de}{dp} = c\left(\frac{\kappa + u/c}{1 + \kappa u/c}\right), \tag{2.56}$$

which for $\kappa \neq 0$ implies a non-isotropy of space. This equation is motivated from Eq. (5.8) and involves an arbitrary constant κ for which the particle and wave velocities arise as two special cases corresponding, respectively, to the values $\kappa = 0$ and $\kappa = \pm\infty$; thus

$$\frac{de}{dp} = u, \qquad \frac{de}{dp} = \frac{c^2}{u}. \tag{2.57}$$

The case $\kappa = 0$ arises by re-writing the standard relations $m = m_0[1 - (u/c)^2]^{-1/2}$, $e = mc^2$ and $p = mu$ using momentum as the variable; thus

$$\frac{u}{c} = \frac{pc}{(e_0^2 + (pc)^2)^{1/2}}, \qquad e = (e_0^2 + (pc)^2)^{1/2}, \tag{2.58}$$

where $e_0 = m_0c^2$ denotes the rest mass energy. The relationship $(2.57)_1$ then arises immediately on differentiating $(2.58)_2$ with respect to p and then using $(2.58)_1$. The case $\kappa = \pm\infty$ corresponds to the de Broglie wave that is associated with a particle moving with velocity u and moving with the superluminal wave velocity $w = c^2/u$ (see de Broglie [17]).

Equation (2.56) is Lorentz invariant in the sense that for fixed relative frame velocities v, by division of the differentials of the inverse energy-momentum relations (2.47), namely,

$$dP = \frac{dp + vde/c^2}{[1 - (v/c)^2]^{1/2}}, \quad dE = \frac{de + vdp}{[1 - (v/c)^2]^{1/2}},$$

we may deduce the equation

$$\frac{dE}{dP} = \left(\frac{de}{dp} + v\right) / \left(1 + \frac{v}{c^2}\frac{de}{dp}\right),$$

and on substitution of (2.56) into this equation, we obtain

$$\frac{dE}{dP} = \left(\frac{(u + v) + c\kappa(1 + uv/c^2)}{(1 + uv/c^2) + \kappa(u + v)/c}\right) = c\left(\frac{\kappa + U/c}{1 + \kappa U/c}\right),$$

on using (2.12). Thus, there is the same velocity dependence in both the moving and reference frames, and therefore equation (2.56) is a Lorentz invariant equation.

Assuming the usual relations $e = mc^2$ and $p = mu$, we have $p = eu/c^2$, and therefore from (2.56) we obtain

$$\frac{dp}{de} = \frac{1}{c^2}\left(u + e\frac{du}{de}\right) = \frac{1}{c}\left(\frac{1+\kappa u/c}{\kappa + u/c}\right),$$

which simplifies to become

$$\frac{de}{e} = \frac{(\kappa + u/c)du}{c(1-(u/c)^2)} = \frac{1}{2c}\left(\frac{(\kappa+1)du}{(1-(u/c))} + \frac{(\kappa-1)du}{(1+(u/c))}\right),$$

and this integrates to give

$$e(u) = \frac{e_0}{(1-(u/c)^2)^{1/2}}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}, \tag{2.59}$$

where as usual e_0 denotes the rest energy, and evidently the Einstein variation arises from the special case $\kappa = 0$. In terms of the angle θ defined by (2.10) in which the Lorentz invariance appears through a translational invariance, we have the following alternative expressions on using Eq. (2.10):

$$e(u) = \frac{e_0}{(1-(u/c)^2)^{1/2}}e^{\kappa\theta} = e_0 \cosh\theta e^{\kappa\theta} = \frac{e_0}{2}\left(e^{(\kappa+1)\theta} + e^{(\kappa-1)\theta}\right), \tag{2.60}$$

and the following relations also apply:

$$e(u) - cp(u) = e_0 e^{(\kappa-1)\theta}, \qquad e(u) + cp(u) = e_0 e^{(\kappa+1)\theta},$$

so that in this case we have

$$e(u)^2 - (cp(u))^2 = e_0^2 e^{2\kappa\theta} = e_0^2\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa}.$$

With the angles $(\Theta, \theta, \epsilon)$ defined by (2.9) and the Lorentz invariance represented by the translation $\Theta = \theta + \epsilon$, it is clear from $(2.60)_1$ that the energy-momentum relations (2.47) remain properly Lorentz invariant, noting however that the rest mass as perceived from the reference frame becomes $E_0 = e_0 e^{-\kappa\epsilon}$.

Specifically, from the inverse energy-momentum relation $(2.47)_2$, we have

$$E = \frac{e + pv}{[1-(v/c)^2]^{1/2}} = \frac{e_0 e^{\kappa\theta}(1 + uv/c^2)}{(1-(u/c)^2)^{1/2}(1-(v/c)^2)^{1/2}},$$

and on using (2.6) to replace u in the denominator of this equation, we might deduce

$$E = \frac{e_0 e^{\kappa\theta}(1-(v/c)^2)}{(1-(u/c)^2)^{1/2}(1-(v/c)^2)^{1/2}(1-Uv/c^2)} = \frac{e_0 e^{-\kappa\epsilon}e^{\kappa\Theta}}{(1-(U/c)^2)^{1/2}}, \tag{2.61}$$

demonstrating that E has the same dependence on U as e has on u, except that the rest energy $E_0 = e_0 e^{-\kappa\epsilon}$ which is dependent on the constant relative frame velocity v through $\epsilon = \tanh^{-1}(v/c)$; thus

$$E_0 = e_0 e^{-\kappa\epsilon} = e_0 \left(\frac{1-(v/c)}{1+(v/c)}\right)^{\kappa/2}. \tag{2.62}$$

We further comment that the final line of (2.61) follows from Eq. (2.7) and that a similar calculation applies to the inverse energy-momentum relation $(2.47)_1$.

We observe that if we require $e(u) = e(-u)$, then necessarily $\kappa = 0$, but note that conventionally this requirement does not hold for light for which the de Broglie relations become $e = \pm pc$, dependent upon the direction. For $\kappa \neq 0$ Eqs. (2.59) and (2.62) impinge on one of the most basic postulates in special relativity relating to the assumed isotropy of space. These equations predict that for $\kappa \neq 0$ the rest mass values will vary with the direction of motion, namely, two different values are obtained for positive and negative velocities v. While numerous experiments have been undertaken aimed at testing such hypothesis, and all indicate the veracity of the assumed isotropy of space, nevertheless the validity or otherwise of (2.62) might only be properly tested in those situations for which both rest masses e_0 and E_0 are non-zero and the fraction $(1 + v/c)/(1 - v/c)$ significantly differs from unity. Accordingly, any test must involve speeds close to the speed of light but involving finite (non-zero) rest masses which therefore excludes those tests dealing with light such as the Michelson-Morley experiments.

It might also be worth noting that since it is generally believed that black holes exist at the centres of galaxies, then space-time must be intrinsically anisotropic in some sense. It is conceivable that space-time is anisotropic at galactic scales and possibly a massive black hole might cause particle rest mass to depend on the direction of particle velocity. This might be the case when the black hole is not at rest in the rest frame of the cosmic microwave background. The cosmic microwave background does have a sizeable dipole component, and its rest frame is measured to be travelling at 627 km/s relative to the centre of mass of our galaxy group (see the mini-review of cosmic microwave background by Scott and Smoot [92]).

Further, while the space-time of special relativity is assumed to be isotropic, this is not taken as an assumption in general relativity. The use of the isotropy assumption in cosmology to select the basic models tends to reflect known experimental outcomes rather than being a necessary part of the theory. We further comment that [96] provides a very general approach to mechanical anisotropy in relativistic mechanics which includes the simple model described here, although derived differently.

We note that for $\kappa = 1$ and $\kappa = -1$, we have, respectively, from (2.59) the following relations:

$$e(u) = \frac{e_0}{1-(u/c)}, \qquad p(u) = \frac{e_0(u/c)}{c(1-(u/c))}, \qquad e(u) - cp(u) = e_0,$$

$$e(u) = \frac{e_0}{1 + (u/c)}, \qquad p(u) = \frac{e_0(u/c)}{c(1 + (u/c))}, \qquad e(u) + cp(u) = e_0,$$

allowing the possibility of $e = \pm pc + e_0$ with non-zero rest energy e_0.

Finally, we comment that it is a very curious fact that both the conventional Einstein energy-mass expression $e = e_0/(1 - (u/c)^2)^{1/2}$ and the generalisation derived here (2.59) bear a relationship with certain singular integral equations associated with aerofoil problems, fluid mechanics and punch problems in elasticity and that this relationship is not some vague intangible connection but involves an exact correspondence. Linear singular equations arise in many areas of applied mathematics but particularly within fluid and solid mechanics. Specifically, in nondimensional variables, the formal solution of the singular integral equation of the second kind

$$\phi(x) + \frac{\lambda}{\pi} \int_{-1}^{1} \frac{\phi(y)dy}{(x - y)} = g(x), \tag{2.63}$$

is given by

$$\phi(x) = \frac{C\lambda}{\pi(1 + \lambda^2)^{1/2}(1 - x^2)^{1/2}} \left(\frac{1 + x}{1 - x}\right)^{\gamma} \tag{2.64}$$
$$+ \frac{g(x)}{(1 + \lambda^2)} + \frac{\lambda}{\pi(1 + \lambda^2)(1 - x^2)^{1/2}} \left(\frac{1 + x}{1 - x}\right)^{\gamma} \int_{-1}^{1} (1 - y^2)^{1/2} \left(\frac{1 - y}{1 + y}\right)^{\gamma} \frac{g(y)dy}{(y - x)},$$

where here λ and γ are related by $\lambda = \cot(\pi\gamma)$ and the constant C is defined by

$$C = \int_{-1}^{1} \phi(x)dx. \tag{2.65}$$

The singular integral appearing in (2.63) is sometimes referred to as the finite Hilbert transform. There are numerous standard results available such as (see for example [39, 60] or [63])

$$\frac{1}{\pi} \int_{-1}^{1} \frac{dy}{(1 - y^2)^{1/2}(x - y)} = 0, \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{(1 - y^2)^{1/2}dy}{(x - y)} = x, \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{y(1 - y^2)^{1/2}dy}{(x - y)} = x^2 - \frac{1}{2},$$
$$\frac{1}{\pi} \int_{-1}^{1} \frac{y^3(1 - y^2)^{1/2}dy}{(x - y)} = x^4 - \frac{x^2}{2} - \frac{1}{8}, \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{y^5(1 - y^2)^{1/2}dy}{(x - y)} = x^6 - \frac{x^4}{2} - \frac{x^2}{8} - \frac{1}{16},$$

and there are usually other constraints such as $\phi(\pm 1) = 0$, and the function $g(x)$ is assumed to be an odd function, but since the major issue here is simply the connection with the Einstein expression, we do not need concern ourselves with such details.

Strictly speaking, the Einstein expression arising from the case $\gamma = 0$ corresponds to $\lambda \longrightarrow \infty$ and accordingly arises from the integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(y)dy}{(x-y)} = f(x),$$

which has solution given by Tricomi [103] (pages 173–185)

$$\phi(x) = \frac{C}{\pi(1-x^2)^{1/2}} + \frac{1}{\pi(1-x^2)^{1/2}}\int_{-1}^{1}(1-y^2)^{1/2}\frac{f(y)dy}{(y-x)}, \tag{2.66}$$

with the constant C as previously given by (2.65), and since the function $f(x)$ is necessarily orthogonal to $(1-x^2)^{-1/2}$, thus

$$\int_{-1}^{1}\frac{f(x)dx}{(1-x^2)^{1/2}} = 0,$$

there are several equivalent expressions available for this solution (see [103], page 179). We comment that the expression (2.66) formally emerges from (2.64) with $g(x) = \lambda f(x)$ and in the limit $\lambda \longrightarrow \infty$ and $\gamma \longrightarrow 0$. Evidently with $x = u/c$, there is an exact correspondence with $e = e_0/(1-(u/c)^2)^{1/2}$ and Eq. (2.66), and with the new expression (2.59) and (2.64), and both of which arise as the solution of the homogeneous problem ($f(x) = g(x) = 0$).

The correspondence between Einstein's fundamental energy expression and areas of classical fluid and solid mechanics is very curious to say the least, and perhaps it is just one of those coincidences. However, perhaps also if these interconnections were properly understood, they might spark the onset of some fundamental revelations in particle physics. In particular, it is natural to pose the question as to what might be the physical meaning for the corresponding energy-mass expressions arising from the above singular integral equations with $f(x), g(x) \neq 0$? Apart from the Newtonian interpretation using the classical kinetic energy $e = m_0u^2/2$ in the final three sections of Chap. 3, in the remainder of the text, we deal exclusively with the Einstein energy-mass expression $e = e_0/(1-(u/c)^2)^{1/2}$ as applying to the particle energy.