Chapter 11
Coordinate Transformations, Tensors and General Relativity

In this chapter, we provide some basic information on coordinate transformations, tensors, partial covariant differentiation, Christoffel symbols and Ricci and Einstein tensors, leading to general relativity. As previously stated, this is necessarily a limited progressive introduction, termed progressive, in the sense that the reader is invited to read on to later sections if more information and further detail are required. Einstein's general theory of relativity was published over a century ago, and up to this point in time provides the best description of a gravitation field, and is capable of describing a myriad of interesting phenomena in the universe, such as the bending of light through gravitational lensing, the slowing of clocks in gravitational fields and the recently detected ripples in space time due to cataclysmic astrophysical events, such as the coalescence of dense stellar objects. The next generation of GPS systems will require a detailed mapping of the earth's gravitational field combined with the development of accurate predictive mathematical models. Accordingly for such applications, a superficial understanding may not be sufficient for future space scientists, but rather some prior experience of the actual "gory" details of the discipline may be required.

The first two sections of the chapter deal with an introduction to Cartesian tensors and an alternative derivation of the basic identity (3.7), previously derived in Chap. 3. The two sections thereafter deal with general curvilinear coordinates and the important notion of partial covariant differentiation, including briefly mentioning the fundamental tensors of general relativity, which are the Riemann-Christoffel tensor, the covariant curvature tensor, the Ricci tensor, the curvature invariant and the Einstein tensor. Bianchi's identity is also briefly mentioned, and the subsequent section provides an illustrative example involving a single spatial Cartesian dimension.

Throughout we follow the approach to general relativity adopted in the excellent recent text of [15] and the older texts of [78, 100, 102], and we follow the development of tensor analysis as pursued by [97, 101] or the Appendix of [31]. While there exist a number of exact space-times for the Einstein field equations

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of general relativity, such as those listed in [42, 81, 98] and to a lesser extent in [84, 91, 107], in the search for new exact space-time solutions of general relativity, the major prohibiting factors include the shear complexity of the underlying equations and the lack of any strategic perspective to guide the analysis. In Sect. 11.6, we deduce some general formulae for a line element involving seven arbitrary metric tensor components, and each component is assumed to depend only on two spatial variables and the temporal variable. While the assumed line element is not completely general, it nevertheless includes many known exact space-time solutions of general relativity. Some general formulae are presented for the Ricci and Einstein tensors expressed in terms of six components of the covariant curvature tensor, and two well-known cosmological models are presented as illustrative examples of the formulation in the section thereafter. In the final section of the chapter, we examine a possible cosmological model involving logarithmic spirals and seven arbitrary constants. While spiral structures frequently occur in the universe, at present there appears to be no known formal solutions of the general relativistic field equations reflecting such structures. A particular case of the spiral model gives rise to an Einstein tensor, which is reminiscent of the tensor involving the cosmological constant.

11.1 Summation Convention and Cartesian Tensors

In this section, we present the background material required to give an alternative derivation of the identity (3.7) for the spatial physical force **f** using a suffix or index notation, for which we represent the Cartesian coordinates (x, y, z) as (x^1, x^2, x^3), and accordingly, we would write (x, y, z) as simply x^j for $j = 1, 2, 3$. In order to achieve this, there are four important ideas and results that we need to introduce. These ideas provide an elementary introduction to the general topic of tensor analysis, which deals with the transformations of curvilinear coordinates for which the issue regarding the level of the index, namely either upper of lower, is critical, and this is the case in the subsequent sects. 11.3 and 11.4. In this and the following section, however, we deal only with Cartesian coordinates for which the level of the index is not relevant. Also, we comment that we could if necessary deal with a four-dimensional space time with coordinates (ct, x, y, z) as (x^0, x^1, x^2, x^3) or x^j for $j = 1, 2, 3, 4$, but since the identity (3.7) involves only the three spatial dimensions (x, y, z), we restrict attention to x^j for $j = 1, 2, 3$.

Einstein Summation Convention We first need to mention the Einstein summation convention, which is an important accepted convention that any repeated index implies that a summation is taken over that index. Thus, for the two vectors $\mathbf{u} = (u_x, u_y, u_z) = (u^1, u^2, u^3)$ and $\mathbf{p} = (p_x, p_y, p_z) = (p^1, p^2, p^3)$, their scalar product **u**.**p** would be written as

$$\mathbf{u} \cdot \mathbf{p} = u_x p_x + u_y p_y + u_z p_z = u^1 p^1 + u^2 p^2 + u^3 p^3 = u^j p^j,$$

and the repeated j index implies that a summation must be made over $j = 1, 2, 3$.

Kronecker Delta δ^{ij} The second idea that is needed is the Kronecker delta symbol δ^{ij}, which is defined to have a unit value if $i = j$ and the value zero otherwise, thus

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

so as an example, combining the two notions of the Einstein summation convention and the Kronecker delta symbol δ^{ij}, we might write the above scalar vector product $\mathbf{u} \cdot \mathbf{p}$ as follows:

$$\mathbf{u} \cdot \mathbf{p} = \delta^{ij} u^i p^j = u^j p^j = u^1 p^1 + u^2 p^2 + u^3 p^3.$$

We would observe that repeated indices occur over both i and j, so that in reality the first equality reveals a summation of precisely nine terms, thus

$$\delta^{ij} u^i p^j = \delta^{11} u^1 p^1 + \delta^{12} u^1 p^2 + \delta^{13} u^1 p^3 + \delta^{21} u^2 p^1 + \delta^{22} u^2 p^2 + \delta^{23} u^2 p^3 \\ + \delta^{31} u^3 p^1 + \delta^{32} u^3 p^2 + \delta^{33} u^3 p^3,$$

but of course, only the three terms arising from δ^{11}, δ^{22} and δ^{33} provide a non-zero contribution to the summation.

Levi-Civita Symbol Denoted by ε^{ijk} The third notion that is required is the mathematical symbol, often referred to as the Levi-Civita symbol denoted by ε^{ijk}, which represents numbers arising from the sign of a permutation of the natural numbers 1, 2 and 3. They are also referred to as the permutation symbols, antisymmetric symbols or alternating symbols, referring to their antisymmetric property and their definition in terms of permutations. The Levi-Civita symbols ε^{ijk} are defined to be unity if ijk is a cyclic permutation of 1, 2 and 3, minus one if ijk is an anticyclic permutation of 1, 2 and 3 and zero otherwise, thus

$$\varepsilon^{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is cyclic permutation of 123} \\ -1 & \text{if } ijk \text{ is anticyclic permutation of 123} \\ 0 & \text{if } ijk \text{ is otherwise} \end{cases} \tag{11.1}$$

Vector Product $\mathbf{u} \wedge \mathbf{p}$ Armed with this symbol, we are now able to express a number of important vector relations and vector differential relations and, principally, those vector relations involving either the vector product or the differential operator, known as the curl of a vector. For example, the three components of the vector product $\mathbf{u} \wedge \mathbf{p}$, which has the formal determinant definition

$$\mathbf{u} \wedge \mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ p_x & p_y & p_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u^1 & u^2 & u^3 \\ p^1 & p^2 & p^3 \end{vmatrix},$$

are given by $(\mathbf{u} \wedge \mathbf{p})^i = \varepsilon^{ijk} u^j p^k$, where there are implied summations over both j and k, and the specific ordering of ijk is critical, where as usual $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ denote the unit vectors in the three Cartesian directions. Thus, for example

$$(\mathbf{u} \wedge \mathbf{p})^1 = \varepsilon^{1jk} u^j p^k = \varepsilon^{123} u^2 p^3 + \varepsilon^{132} u^3 p^2 = u^2 p^3 - u^3 p^2 = u_y p_z - u_z p_y,$$

since $\varepsilon^{123} = 1$ and $\varepsilon^{132} = -1$ are the only non-zero values of ε^{ijk} involving 1 in the indices.

Further, we may use these symbols to write the i^{th} component of the curl differential operator $\nabla \wedge \mathbf{p}$, which has the formal determinant definition

$$\nabla \wedge \mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p_x & p_y & p_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ p^1 & p^2 & p^3, \end{vmatrix},$$

thus

$$(\nabla \wedge \mathbf{p})^i = \varepsilon^{ijk} \frac{\partial p^k}{\partial x^j},$$

again noting the summations over both repeated indices j and k and their specific placement in the above expression.

Fundamental Identity Involving δ^{ij} and ε^{ijk} The fourth and final result that we need is the following fundamental identity involving both the Kronecker delta symbol δ^{ij} and the Levi-Civita symbol ε^{ijk}, thus

$$\varepsilon^{ijk} \varepsilon^{imn} = \delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}, \tag{11.2}$$

for which we observe the implied summation over the repeated index i and the specific association of the indices in the positive and negative contributions. This is the fundamental identity from which the vast majority of all other vector identities involving either the vector product or the curl differential operator can be quickly established.

11.2 Alternative Derivation of Basic Identity

We are now well placed to establish the identity (3.7) for the spatial physical force $\mathbf{f}$, and we start with the i^{th} component of the triple vector differential product $\mathbf{u} \wedge (\nabla \wedge \mathbf{p})$, thus

$$\begin{aligned}\varepsilon^{ijk} u^j (\nabla \wedge \mathbf{p})^k &= \varepsilon^{ijk} \varepsilon^{kmn} u^j \frac{\partial p^n}{\partial x^m} \\ &= \varepsilon^{kij} \varepsilon^{kmn} u^j \frac{\partial p^n}{\partial x^m} = (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}) u^j \frac{\partial p^n}{\partial x^m},\end{aligned}$$

on using the fundamental identity (11.2); the fact that the indices of ε^{kij} are a cyclic permutation of those of ε^{ijk}, therefore the two symbols have the same value. On applying the values of the Kronecker delta symbols, we finally have that the i^{th} component of the triple vector differential product $\mathbf{u} \wedge (\nabla \wedge \mathbf{p})$ is given by

$$(\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = \varepsilon^{ijk} u^j (\nabla \wedge \mathbf{p})^k = u^n \frac{\partial p^n}{\partial x^i} - u^m \frac{\partial p^i}{\partial x^m}. \tag{11.3}$$

In this equation, we recognise the term $u^n \partial p^n / \partial x^i$ as arising from the rate-of-working equation $de = \mathbf{dx} \cdot d\mathbf{p}/dt = \mathbf{u} \cdot \mathbf{dp}$, so that

$$\frac{\partial e}{\partial x^i} = u^n \frac{\partial p^n}{\partial x^i} = u^1 \frac{\partial p^1}{\partial x^i} + u^2 \frac{\partial p^2}{\partial x^i} + u^3 \frac{\partial p^3}{\partial x^i},$$

while the term $u^m \partial p^i / \partial x^m$ we recognise as that arising in the total or material derivative defined by (3.6), namely $(\mathbf{u} \cdot \nabla) p^i$. Thus, on rearrangement of (11.3), we obtain

$$u^n \frac{\partial p^n}{\partial x^i} = (\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i + (\mathbf{u} \cdot \nabla) p^i,$$

so that finally from (3.4), we may deduce

$$f^i = \frac{\partial p^i}{\partial t} + \frac{\partial e}{\partial x^i} = \frac{\partial p^i}{\partial t} + (\mathbf{u} \cdot \nabla) p^i + (\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i,$$

and the required result follows, thus

$$f^i = \frac{dp^i}{dt} + (\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i,$$

where the total or material derivative is defined by Eq. (3.6).

11.3 General Curvilinear Coordinates

In this section, we present the main details for the extended force formulation given by (3.4) but in terms of a general spatial curvilinear coordinate system x^j for $j = 1, 2, 3$, where now the index level shown here as upper is critical. We follow the development of tensor analysis as pursued in the standard text [97], and the reader might be referred to any of [15, 97, 101] or the Appendix of [31] for much of the necessary background detail. In this section, we utilise the notion of the partial covariant derivative without giving the full details. For the interested reader, these details are presented in a subsequent section along with the notion of Christoffel symbols. For a general curvilinear coordinate system (x^1, x^2, x^3), we assume a symmetric metric tensor with components g_{ij} for $i, j = 1, 2, 3$. The metric tensor g_{ij} may be expressed in terms of rectangular Cartesian coordinates $(z^1, z^2, z^3) = (x, y, z)$ and is defined by the following equation for the three-dimensional line element, thus

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = dz^k dz^k = g_{ij} dx^i dx^j, \tag{11.4}$$

so that formally the metric tensor is given by

$$g_{ij} = \frac{\partial z^k}{\partial x^i} \frac{\partial z^k}{\partial x^j},$$

noting the implied summation over $k = 1, 2, 3$, the apparent symmetry in i and j, and that both sides of this equation are lower in the indices i and j, and we say that g_{ij} is a covariant tensor of rank 2. Further, we remind the reader that the index level for Cartesian vectors and tensors is not relevant. We also remind the reader that we would refer to the general curvilinear coordinate system x^j as either a contravariant vector or as a contravariant tensor of rank 1, and we observe from Eq. (11.4) that, in such a coordinate system, the Einstein summation convention only applies over one raised index and one lower index and that it has no meaning if both indices are on the same level. Formally, the proper tensorial version of the Kronecker delta symbol δ^i_j would be defined, thus

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{11.5}$$

and we would refer to δ^i_j as a mixed tensor of rank 2, contravaraint in i and covariant in j.

Associated with the symmetric metric tensor with components g_{ij} is the conjugate symmetric metric tensor with components g^{ij} for $i, j = 1, 2, 3$, which is such that

$$g_{ij}g^{jk} = \delta_i^k, \tag{11.6}$$

so that viewing g_{ij} as a matrix, the conjugate metric tensor with components g^{ij} is simply the formal matrix, which is the inverse to g_{ij}. We would refer to the metric tensor g_{ij} as a symmetric covariant tensor of rank 2, while the conjugate metric tensor g^{ij} would be referred to as a symmetric contravariant tensor of rank 2. We further note that in this section, we use the symbol g to denote the formal determinant of the matrix, which has components g_{ij}, thus $g = |g_{ij}|$, and there should be no confusion with the same symbol g, which is used for the force in the direction of time.

The metric tensor and its conjugate are fundamental in terms of raising and lowering indices, thus for example, we have

$$A_{ij} = g_{ik}g_{jm}A^{km}, \qquad A^i_j = g^{ik}A_{kj}, \qquad A^{ij} = g^{ik}g^{jm}A_{km},$$

most importantly noting that in all cases the indices i and j appear on the same level on each side of the equations and that summation only occurs over two indices on different levels. In this manner, with the Kronecker delta δ^i_j defined by (11.5), we might give meaning to the covariant and contravaraint tensors of rank 2, namely δ_{ij} and δ^{ij}, respectively, thus

$$\delta_{ij} = g_{ik}\delta^k_j = g_{ij}, \qquad \delta^{ij} = g^{ik}\delta^j_k = g^{ij},$$

so that δ_{ij} is simply the covariant metric tensor g_{ij}, while δ^{ij} is the contravariant conjugate metric tensor g^{ij}.

Spherical Polar Coordinates (r, θ, ϕ) As a simple example, for the spherical polar coordinates (r, θ, ϕ) defined by the relations

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

from Eq. (11.4), we might deduce

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

so that the metric tensor g_{ij} and its conjugate g^{ij} are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\sin\theta)^2 \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (r\sin\theta)^{-2} \end{pmatrix}.$$

With these preliminary observations in tensor calculus, we may proceed to present the main details for the extended force formulation given by (3.4) in a general spatial curvilinear coordinate system (x^1, x^2, x^3). The velocity vector $\mathbf{u}$ has components $u^i = dx^i/dt$ with magnitude u arising from (11.4), thus

$$u^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = g_{ij}u^i u^j,$$

while the momentum vector $\mathbf{p}$ has components $p^i = mu^i$ and magnitude p, where $p^2 = g_{ij}p^i p^j$, and as usual the mass m is defined in terms of the magnitude of the velocity vector by $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, where m_0 denotes the rest mass.

On taking the total derivative of the usual energy equation in tensorial form

$$e^2 = e_0^2 + (pc)^2 = e_0^2 + c^2 g_{ij}p^i p^j,$$

where p denotes the magnitude of the momentum vector given explicitly by $p^2 = g_{ij}p^i p^j$, we might deduce the rate-of-working equation $ede = c^2 g_{ij}p^i dp^j$, which using $e = mc^2$ simplifies to give $de = g_{ij}u^i dp^j = g_{ij}dp^i u^j$ on using the symmetry of the metric tensor g_{ij}. Further, in tensor calculus, the notion of partial differentiation is generalised to become partial covariant differentiation, which is designated by a semicolon, thus ; and since here the energy e is a scalar quantity (namely a tensor of rank 0), the partial derivatives of e coincide with the partial covariant derivative, so that from $de = g_{ij}u^i dp^j$ we might deduce the important relation

$$\frac{\partial e}{\partial x^k} = e_{;k} = g_{ij}u^i p^j_{;k}, \tag{11.7}$$

where $p^j_{;k}$ refers to the partial covariant derivative with respect to x^k of the contravariant vector p^j. For further information relating to the partial covariant derivatives, we refer the reader to the following section or again to either [97] or the Appendix of [31]. Here, it suffices to mention that partial covariant derivatives satisfy the usual rules of differentiation, such as the product rule, and they possess the very curious properties that all partial covariant derivatives of both the metric tensor and its conjugate vanish, thus $g_{ij;k} = 0$ and $g^{ij}_{;k} = 0$, results which are used frequently below. Maybe it is worth noting that, for example, these two partial covariant derivatives with respect to x^k are formally quite distinct. The first is the partial covariant derivative of a covariant tensor of rank 2, while the second is the partial covariant derivative of a contravariant tensor of rank 2, which are different.

For $i = 1, 2, 3$, the tensorial version of (3.4) in a general curvilinear coordinate system (x^1, x^2, x^3) becomes

$$f^i = \frac{\partial p^i}{\partial t} + g^{ik}e_{;k} = \frac{\partial p^i}{\partial t} + g^{ik}\frac{\partial e}{\partial x^k}, \tag{11.8}$$

$$g^0 = \frac{1}{c^2}\frac{\partial e}{\partial t} + p^k_{;k} = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}p^k)}{\partial x^k},$$

where $p^k_{;k}$ is the divergence of the contravariant vector p^k, and in the last part of Eq. (11.8), we have used a standard formula of tensor calculus for the divergence of a vector $\mathbf{p}$ that is verified in the following section and given on page 32 of Spain [97], namely

$$div\mathbf{p} = p^k_{;k} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} p^k)}{\partial x^k}, \tag{11.9}$$

where we again remind the reader that g denotes the determinant of the metric tensor g_{ij}, thus $g = |g_{ij}|$, and that in this present section, we have momentarily added a zero to the force g in the direction of time, thus g^0, to ensure that there is no confusion when the two quantities are used in the same equation.

Work Done Or Energy Function $W(x^i, t)$ On assuming the existence of a work done or energy function $W(x^i, t)$ which is defined by

$$\begin{aligned} dW &= \mathbf{f} \cdot dx + gc^2 dt \\ &= \left(\frac{\partial p^i}{\partial t} + g^{ik} \frac{\partial e}{\partial x^k} \right) g_{im} dx^m + \left(\frac{1}{c^2} \frac{\partial e}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} p^k)}{\partial x^k} \right) dt, \end{aligned}$$

and using $g_{ij} g^{jk} = \delta^k_i$ and $\mathscr{E} = W - e$, we might deduce

$$d\mathscr{E} = g_{im} \frac{\partial p^i}{\partial t} dx^m + \frac{c^2}{\sqrt{g}} \frac{\partial(\sqrt{g} p^k)}{\partial x^k} dt,$$

for which the following partial differential relations evidently apply

$$\frac{\partial \mathscr{E}}{\partial x^m} = g_{im} \frac{\partial p^i}{\partial t}, \qquad \frac{\partial \mathscr{E}}{\partial t} = \frac{c^2}{\sqrt{g}} \frac{\partial(\sqrt{g} p^k)}{\partial x^k}. \tag{11.10}$$

Basic Identity (3.7) In order to provide a formal tensorial proof of the identity (3.7), we have from both (11.7) and $(11.8)_1$

$$f^i = \frac{\partial p^i}{\partial t} + g^{ik} g_{mj} u^m p^j_{;k} = \frac{dp^i}{dt} + g^{ik} g_{mj} u^m p^j_{;k} - u^m p^i_{;m}, \tag{11.11}$$

where d/dt denotes the total or material time derivative that is defined by

$$\frac{dp^i}{dt} = \frac{\partial p^i}{\partial t} + u^m p^i_{;m}.$$

Now on using (11.6) in the form $\delta^k_m = g^{kj} g_{mj}$, we have from (11.11)

$$f^i = \frac{dp^i}{dt} + g^{ik} g_{mj} u^m p^j_{;k} - u^m p^i_{;k} \delta^k_m \tag{11.12}$$
$$= \frac{dp^i}{dt} + g_{mj} u^m (g^{ik} p^j_{;k} - g^{jk} p^i_{;k}),$$

and the tensorial versions of the curl operator $(\nabla \wedge \mathbf{p})$ and the vector product $\mathbf{u} \wedge (\nabla \wedge \mathbf{p})$ are given by

$$(\nabla \wedge \mathbf{p})_i = \epsilon^*_{ijn} g^{jk} p^n_{;k}, \qquad ((\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = \epsilon^{*ijk} g_{jm} u^m (\nabla \wedge \mathbf{p})_k,$$

where the tensorial permutation symbols ϵ^*_{ijk} and ϵ^{*ijk} are defined in terms of the Cartesian permutation symbols ϵ^{ijk} (see Eq. (11.1)) by the following formulae that may be found in Eringen [31] (pages 441 and 442), thus

$$\epsilon^*_{ijk} = \sqrt{g}\epsilon^{ijk}, \qquad \epsilon^{*ijk} = \frac{\epsilon^{ijk}}{\sqrt{g}}.$$

The identity (3.7) now follows on using both

$$((\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = \epsilon^{*ijk} g_{jm} u^m \epsilon^*_{knr} g^{ns} p^r_{;s} = \epsilon^{*kij} \epsilon^*_{knr} g_{jm} u^m g^{ns} p^r_{;s},$$

and the tensorial version of the identity (11.2) becomes

$$\epsilon^{*kij} \epsilon^*_{knr} = \delta^i_n \delta^j_r - \delta^i_r \delta^j_n,$$

so that together we obtain

$$((\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = (\delta^i_n \delta^j_r - \delta^i_r \delta^j_n) g_{jm} u^m g^{ns} p^r_{;s} = g_{jm} u^m (g^{is} p^j_{;s} - g^{js} p^i_{;s}),$$

which coincides precisely with the term in $(11.12)_2$, and therefore, we have provided a formal tensorial proof of (3.7).

Now on using $g_{ij} g^{jk} = \delta^k_i$, Eq. $(11.10)_1$ may be rewritten as

$$\frac{\partial p^i}{\partial t} = g^{ik} \frac{\partial \mathscr{E}}{\partial x^k} = g^{ik} \mathscr{E}_{;k} = (g^{ik} \mathscr{E})_{;k},$$

and together the two relations (11.10) become

$$\frac{\partial p^i}{\partial t} = (g^{ik} \mathscr{E})_{;k}, \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2 p^k_{;k}. \tag{11.13}$$

From these relations, we may deduce that there exists a scalar function $\phi(x^j, t)$ such that

$$p^i = g^{ik}\frac{\partial \phi}{\partial x^k} = g^{ik}\phi_{;k} = (g^{ik}\phi)_{;k}, \qquad \mathscr{E} = \frac{\partial \phi}{\partial t},$$

and that $\phi(x^j, t)$ satisfies the equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 p^k_{;k} = c^2 g^{ik}\phi_{;ik} = c^2\nabla^2\phi,$$

which is the wave equation, where $\nabla^2\phi = g^{ik}\phi_{;ik}$ is the tensorial version of the Laplacian operator for which further details may be found in the following section or in [97] (page 32). The existence of the scalar function $\phi(x^j, t)$ ensures the vanishing of the curl of the vector $\mathbf{p}$, since from

$$(\nabla \wedge \mathbf{p})_i = \epsilon^*_{ijk} g^{jm} p^k_{;m} = \epsilon^*_{ijk} g^{jm}(g^{kn}\phi)_{;mn} = \epsilon^*_{ijk} g^{jm} g^{kn}\phi_{;mn},$$

we may, for example, deduce that in particular for $i = 1$, we have

$$(\nabla \wedge \mathbf{p})_1 = \sqrt{g}(\epsilon^{123} g^{2m} g^{3n}\phi_{;mn} + \epsilon^{132} g^{3m} g^{2n}\phi_{;mn}),$$

which evidently vanishes, since $\epsilon^{123} = 1$ and $\epsilon^{132} = -1$, the conjugate metric tensor g^{ij} is symmetric and assuming the partial derivatives coincide $\phi_{;mn} = \phi_{;nm}$.

Wave Energy $\mathscr{E}(x^j, t)$ Satisfies Wave Equation From Eq. (11.13), we observe that the wave energy $\mathscr{E}(x^j, t)$ satisfies the wave equation

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2\left(\frac{\partial p^k}{\partial t}\right)_{;k} = c^2\left((g^{km}\mathscr{E})_{;m}\right)_{;k} = c^2 g^{km}\mathscr{E}_{;km} = c^2\nabla^2\mathscr{E}, \tag{11.14}$$

while the momentum vector $p^i(x^j, t)$ satisfies

$$\frac{\partial^2 p^i}{\partial t^2} = c^2\left(g^{ik} p^m_{;m}\right)_{;k} = c^2 g^{ik}\left(p^m_{;m}\right)_{;k}. \tag{11.15}$$

We note that while the double covariant partial differentiations appear to be the same in both cases, there is a distinction arising from the fact that in the former case we are undertaking the double partial covariant derivative of the scalar quantity $\mathscr{E}(x^j, t)$, while in the second case we are undertaking the double partial covariant derivative of the momentum vector $p^m(x^j, t)$, the first leading to the divergence of the momentum vector and the second simply the partial derivative of a scalar. In the former case, the first covariant derivative is simply the partial derivative $\partial\mathscr{E}/\partial x^k$, while the second differentiation involves the partial covariant derivative of a covariant vector. This is in contrast to the two differentiations of the momentum vector $p^m(x^j, t)$, for which the first $p^m_{;m}$ is the partial covariant derivative of a contravariant vector, namely the divergence of the vector $\mathbf{p}$. Now the divergence of $\mathbf{p}$ is a scalar, and the second partial covariant derivative with respect x^k is that of differentiation of a scalar, namely the partial derivative, even though it arises from $p^m_{;n}$, which a mixed tensor of rank 2. This distinction corresponds to the distinction between the divergence of the gradient, which is the former case, and the gradient of the divergence, corresponding to the latter case.

The reader may find it instructive to compare the above Eqs. (11.14) and (11.15) directly with the corresponding general equations (3.22) and (3.19) and those for centrally symmetric systems, namely (9.4). For the general equations, we have for the de Broglie wave energy $\mathscr{E}(\mathbf{x}, t)$ and the momentum vector $\mathbf{p}(\mathbf{x}, t)$,

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \nabla^2 \mathscr{E} = c^2 \nabla \cdot (\nabla \mathscr{E}), \qquad \frac{\partial^2 \mathbf{p}}{\partial t^2} = c^2 \nabla (\nabla \cdot \mathbf{p}),$$

while for the centrally symmetric systems, the de Broglie wave energy $\mathscr{E}(r, t)$ and the radial component of momentum $p(r, t)$, satisfy

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \left(\frac{\partial^2 \mathscr{E}}{\partial r^2} + \frac{2}{r} \frac{\partial \mathscr{E}}{\partial r} \right), \qquad \frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial}{\partial r} \left(\frac{\partial p}{\partial r} + \frac{2p}{r} \right),$$

since under these circumstances the gradient $\nabla = \partial/\partial r$.

11.4 Partial Covariant Differentiation

In the previous section, we have utilised the notion of the partial covariant derivative without giving a full explanation and presenting all the details, and this can only be achieved by introducing the Christoffel symbols which is the major objective of this section. Again the reader is referred to any of [15, 97, 101] or the Appendix of [31] for a fuller explanation and much of the necessary background detail. Essentially, both the notions of partial covariant differentiation and Christoffel symbols arise in consequence that when we transform from rectangular Cartesian coordinates $(x, y, z) = (z^1, z^2, z^3)$ with corresponding fixed unit base vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ to a general curvilinear coordinate system (x^1, x^2, x^3), then the corresponding base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are no longer fixed in space, and the Christoffel symbols are the mechanism required to describe their variability. In addition, and as a secondary issue, we note that the base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are not necessarily unit vectors.

In rectangular Cartesian coordinates, the position vector $\mathbf{r}$ is defined by the equation

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = z^1\hat{\mathbf{i}} + z^2\hat{\mathbf{j}} + z^3\hat{\mathbf{k}},$$

and we might conceive that the three fixed base vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ are defined by the three equations

$$\hat{\mathbf{i}} = \frac{\partial \mathbf{r}}{\partial x}, \qquad \hat{\mathbf{j}} = \frac{\partial \mathbf{r}}{\partial y}, \qquad \hat{\mathbf{k}} = \frac{\partial \mathbf{r}}{\partial z}.$$

In fact, in general curvilinear coordinates (x^1, x^2, x^3), we adopt this definition as the defining equation for the base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, thus

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial x^1}, \qquad \mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial x^2}, \qquad \mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial x^3},$$

or equivalently

$$\mathbf{e}_j = \frac{\partial \mathbf{r}}{\partial x^j}, \tag{11.16}$$

for $j = 1, 2, 3$. So, for example, suppose that we transform to cylindrical polar coordinates (r, θ, z), then, with the usual relations $x = r\cos\theta$ and $y = r\sin\theta$, the position vector becomes

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} + z\hat{\mathbf{k}},$$

while the three base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are given by

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}, \quad \mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}}, \quad \mathbf{e}_z = \frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{k}},$$

and we notice that, in this particular case, the three base vectors are mutually orthogonal to one another and that $\mathbf{e}_r$ and $\mathbf{e}_z$ are both unit vectors, while $\mathbf{e}_{\boldsymbol{\theta}}$ is not a unit vector, since $\mathbf{e}_\theta . \mathbf{e}_\theta = r^2$.

In general curvilinear coordinates (x^1, x^2, x^3), we have from (11.4) that the line element becomes on using (11.16)

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i} dx^i \cdot \frac{\partial \mathbf{r}}{\partial x^j} dx^j = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} dx^i dx^j = g_{ij} dx^i dx^j,$$

and therefore we may make the important identification

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} = \mathbf{e}_i \cdot \mathbf{e}_j, \tag{11.17}$$

for $i, j = 1, 2, 3$. From this equation, it is apparent that if the base vectors are mutually orthogonal, then $g_{ij} = 0$ for $i \neq j$, and that for a particular i, the quantity g_{ii} provides the magnitude squared of the base vector $\mathbf{e}_i$; thus, we have $g_{ii} = \mathbf{e}_i \cdot \mathbf{e}_i$, with no implied summation over the i in this particular instance.

In an analogous manner, we might introduce a set of base vectors $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ of simply $\mathbf{e}^j$ for $j = 1, 2, 3$, so that for an arbitrary vector $\mathbf{u}$, we have the two representations

$$\mathbf{u} = u^j \mathbf{e}_j = u_k \mathbf{e}^k, \tag{11.18}$$

with implied summations over both j and k, noticing that each of these summations occurs over two levels, namely one index is an upper index and one index is a lower index. We refer to u^j as the contravariant components of the vector $\mathbf{u}$, while u_k are referred to as the covariant components of the vector $\mathbf{u}$. Thus, without being overly specific, when we consider a tensor of higher rank, say the mixed tensor of rank 4, $A^{jk}{}_{mn}$, then we have in mind that the contravariant indices jk somehow relate to the base vectors $\mathbf{e}_j$, while the covariant indices mn somehow relate to the base vectors $\mathbf{e}^j$. Further, in addition to (11.17) the complete set of relations connecting the scalar products of the two sets of base vectors $\mathbf{e}_j$ and $\mathbf{e}^j$ with the metric tensor g_{ij}, the conjugate metric tensor g^{ij} and the Kronecker delta δ^i_j are as follows:

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \qquad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j, \qquad \delta^i_j = \mathbf{e}^i \cdot \mathbf{e}_j.$$

Christoffel Symbols of the Second Kind Γ^k_{ij} The Christoffel symbols arise on taking the partial derivative of the vector $\mathbf{u}$, and from Eq. (11.18), we have

$$\frac{\partial \mathbf{u}}{\partial x^i} = \frac{\partial u^j}{\partial x^i}\mathbf{e}_j + u^j \frac{\partial \mathbf{e}_j}{\partial x^i} = \frac{\partial u_k}{\partial x^i}\mathbf{e}^k + u_k \frac{\partial \mathbf{e}^k}{\partial x^i}, \tag{11.19}$$

and the Christoffel symbols of the second kind Γ^k_{ij} arise as follows:

$$\frac{\partial \mathbf{e}_j}{\partial x^i} = \Gamma^k_{ij}\mathbf{e}_k, \qquad \frac{\partial \mathbf{e}^k}{\partial x^i} = -\Gamma^k_{ij}\mathbf{e}^j,$$

and, of course, with summation over repeated indices. On using these results, we obtain from (11.19) the formulae for the partial covariant derivatives of contravariant and covariant vectors, respectively, thus

$$u^j_{;i} = \frac{\partial u^j}{\partial x^i} + \Gamma^j_{ik}u^k, \qquad u_{j;i} = \frac{\partial u_j}{\partial x^i} - \Gamma^k_{ij}u_k, \tag{11.20}$$

and it is instructive to examine such formulae carefully, noticing that corresponding indices on one side of the equation are at the same level on the other side of the equation, and if an index is repeated on one side of the equation, one index is upper while the other is lower. If the reader keeps these two simple rules in mind, then in tensor analysis, it is very difficult to write down an equation that is incorrect. We note further that the partial covariant derivatives are themselves well-defined tensors, so that $u^j_{;i}$ is a mixed tensor of rank 2, while $u_{j;i}$ is a covariant tensor of rank 2. Also if we bear in mind the two formulae (11.20) with the plus sign for contravariant vectors and the minus sign for convariant vectors, then we may readily write down the partial covariant derivative of a tensor of any rank. For example, for the mixed tensor of rank 3 A^{ij}_k, which is contravariant in two indices and covariant in one, we have two plus signs and one minus sign, and the partial covariant derivative is given by the formula

$$A^{ij}_{k;m} = \frac{\partial A^{ij}_k}{\partial x^m} + \Gamma^j_{mn} A^{in}_k + \Gamma^i_{pm} A^{pj}_k - \Gamma^q_{mk} A^{ij}_q . \tag{11.21}$$

Christoffel Symbols of the First Kind $[ij, k]$ The Christoffel symbols of the second kind Γ^k_{ij} are defined in terms of the metric tensor g_{ij} and the conjugate metric tensor g^{ij} through the Christoffel symbols of the first kind $[ij, k]$, which are defined in terms the partial derivatives of the metric tensor, thus

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right), \tag{11.22}$$

and these are related to the Christoffel symbols of the second kind through the formulae

$$[ij, k] = g_{km}\Gamma^m_{ij}, \qquad \Gamma^k_{ij} = g^{kn}[ij, n],$$

and are such that both symbols are symmetric with respect to the paired indices, thus

$$[ij, k] = [ji, k], \qquad \Gamma^k_{ij} = \Gamma^k_{ji}.$$

Partial Covariant Derivatives of Metric Tensors Are Zero: $g_{ij;k} = 0$ **and** $g^{ij}_{;k} = 0$ The definition of the Christoffel symbols in terms of the metric tensor g_{ij} and the conjugate metric tensor g^{ij} is such that the partial covariant derivatives of all components of the metric tensor and its conjugate vanish, thus $g_{ij;k} = 0$ and $g^{ij}_{;k} = 0$. In order to formally prove these important results, we proceed as follows:

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^m_{jk} g_{im} - \Gamma^n_{ik} g_{jn} = \frac{\partial g_{ij}}{\partial x^k} - g^{mp}[jk, p] g_{im} - g^{nq}[ik, q] g_{jn},$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \delta^p_i [jk, p] - \delta^q_j [ik, q] = \frac{\partial g_{ij}}{\partial x^k} - [jk, i] - [ik, j],$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) - \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right),$$

$$= 0,$$

and similarly for the conjugate metric tensor, we have

$$g^{ij}_{;k} = \frac{\partial g^{ij}}{\partial x^k} + \Gamma^i_{mk} g^{jm} + \Gamma^j_{nk} g^{in} = \frac{\partial g^{ij}}{\partial x^k} + g^{ip}[mk, p] g^{jm} + g^{jq}[nk, q] g^{in},$$

$$= \frac{\partial g^{ij}}{\partial x^k} + \frac{g^{ip} g^{jm}}{2} \left(\frac{\partial g_{mp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^p} \right) + \frac{g^{jq} g^{in}}{2} \left(\frac{\partial g_{nq}}{\partial x^k} + \frac{\partial g_{kq}}{\partial x^n} - \frac{\partial g_{nk}}{\partial x^q} \right),$$

$$= \frac{\partial g^{ij}}{\partial x^k} + \frac{g^{ip} g^{jm}}{2} \left(\frac{\partial g_{mp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^p} \right) + \frac{g^{ip} g^{jm}}{2} \left(\frac{\partial g_{pm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^p} - \frac{\partial g_{pk}}{\partial x^m} \right),$$

$$= \frac{\partial g^{ij}}{\partial x^k} + g^{ip} g^{jm} \frac{\partial g_{mp}}{\partial x^k},$$

where in the final term of the second last equality we have changed the repeated indices $n \longrightarrow p$ and $q \longrightarrow m$. Now on taking the partial derivative with respect to x^k of Eq. (11.6), namely $g_{mp} g^{pi} = \delta^i_m$, the above equality becomes

$$g^{ij}_{;k} = \frac{\partial g^{ij}}{\partial x^k} - g_{mp} g^{jm} \frac{\partial g^{ip}}{\partial x^k} = \frac{\partial g^{ij}}{\partial x^k} - \delta^j_p \frac{\partial g^{ip}}{\partial x^k} = 0,$$

and the result is established.

In order to verify the formula (11.9) for the divergence of a contravariant vector p^j, we first recall that if g denotes the determinant of the metric tensor g_{ij}, thus $g = |g_{ij}|$; then, $g g^{ij}$ is the cofactor of the element g_{ij} in the determinant, and from the rule for differentiating determinants, we have

$$\frac{\partial g}{\partial x^k} = g g^{ij} \frac{\partial g_{ij}}{\partial x^k}.$$

Now the partial covariant derivative of the contravariant vector p^j is given by

$$p^j_{;i} = \frac{\partial p^j}{\partial x^i} + \Gamma^j_{ik} p^k,$$

and therefore summation over the repeated index j gives

$$p^j_{;j} = \frac{\partial p^j}{\partial x^j} + \Gamma^j_{jk} p^k = \frac{\partial p^j}{\partial x^j} + g^{jm}[jk, m] p^k,$$

$$= \frac{\partial p^j}{\partial x^j} + \frac{g^{jm}}{2} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right) p^k,$$

$$= \frac{\partial p^j}{\partial x^j} + \frac{g^{jm}}{2} \frac{\partial g_{jm}}{\partial x^k} p^k = \frac{\partial p^j}{\partial x^j} + \frac{1}{2g} \frac{\partial g}{\partial x^k} p^k,$$

$$= \frac{\partial p^j}{\partial x^j} + \frac{1}{2g} \frac{\partial g}{\partial x^j} p^j = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} p^j)}{\partial x^j},$$

as required. We comment that in this derivation, we have also established the formula

$$\Gamma^j_{jk} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^k}, \tag{11.23}$$

where g is replaced by $-g$ if $g < 0$, and this identity becomes important as a useful check on any expressions obtained for the Chistoffel symbols of the second kind.

Formula for the Laplacian of a Scalar We may exploit the above formalism to obtain the following useful formula for the Laplacian of a scalar function $\phi(x^i, t)$, thus

$$\nabla^2\phi = g^{ik}\phi_{;ik} = g^{ik}\left(\frac{\partial^2\phi}{\partial x^i \partial x^k} - \Gamma^j_{ik}\frac{\partial \phi}{\partial x^j}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{ik}\frac{\partial \phi}{\partial x^i}\right), \tag{11.24}$$

which follows, since on using the fact that the partial covariant derivatives of all metric tensors are zero, we have

$$\nabla^2\phi = g^{ik}\phi_{;ik} = \left(g^{ik}\frac{\partial \phi}{\partial x^i}\right)_{;k} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{ik}\frac{\partial \phi}{\partial x^i}\right),$$

where the final equality follows on using the above result for the divergence of a contravariant vector p^j, since the quantity $g^{ik}\partial\phi/\partial x^i$ operates as a contravariant vector. The expression (11.24) provides a convenient means to evaluate the Laplacian in a particular given coordinate system.

The full import of the notions of partial covariant differentiation and Christoffel symbols comes to fruition in the formulation of the four-dimensional theory of general relativity, and indeed these notions are fundamental to the development of the subject. For this reason, we briefly state some of the major results underpinning general relativity theory. Now we have previously established that the partial covariant derivatives of a covariant vector A_i and a covariant tensor of rank 2 B_{ij} are given by the formulae

$$A_{i;j} = \frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij}A_k, \qquad B_{ij;k} = \frac{\partial B_{ij}}{\partial x^k} - \Gamma^m_{kj}B_{im} - \Gamma^n_{ik}B_{jn},$$

and for both partial covariant derivatives, we may take a further partial covariant derivative $A_{i;jk}$ and $B_{ij;km}$ and pose the question as to the circumstances under which these partial covariant derivatives commute. A full investigation reveals the following relations:

$$A_{i;jk} - A_{i;kj} = R^m_{ijk}A_m, \qquad B_{ij;km} - B_{ij;mk} = R^n_{jkm}B_{in} + R^p_{ikm}B_{jp},$$

where R^m_{ijk} is a tensor of rank 4 covariant in three indices and contravariant in one, and can be shown to be given by (see, e.g. [97], page 53)

$$R^m_{ijk} = \frac{\partial \Gamma^m_{ik}}{\partial x^j} - \frac{\partial \Gamma^m_{ij}}{\partial x^k} + \Gamma^m_{jn}\Gamma^n_{ik} - \Gamma^m_{kp}\Gamma^p_{ij},$$

and altogether there are five major tensors leading to the development of general relativity as follows:

- Riemann-Christoffel tensor R^m_{ijk} defined above
- Covariant curvature tensor $R_{ijkm} = g_{in} R^n_{jkm}$
- Ricci tensor $R_{ij} = R^k_{ijk}$
- Curvature invariant $R = g^{ij} R_{ij}$
- Einstein tensor $G^i_j = g^{ik} R_{jk} - R\delta^i_j/2$

for which there are many known results and many formulae are known to exist such as

$$R_{ijkm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^m} - \frac{\partial^2 g_{jm}}{\partial x^i \partial x^k}\right) + g^{np}\left([jk, n][im, p] - [jm, n][ik, p]\right), \tag{11.25}$$

$$R_{ijkm} = \frac{\partial [jm, i]}{\partial x^k} - \frac{\partial [jk, i]}{\partial x^m} + \Gamma^n_{jk}[im, n] - \Gamma^n_{jm}[ik, n], \tag{11.26}$$

and Bianchi's identity

$$R_{ijkm;n} + R_{ijnk;m} + R_{ijmn;k} = 0. \tag{11.27}$$

Further, we have the relation

$$R_{ij} = \frac{\partial^2 \log \sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^m_{in}\Gamma^n_{jm} - \Gamma^p_{ij}\frac{\partial \log \sqrt{g}}{\partial x^p},$$

from which the symmetry $R_{ij} = R_{ji}$ is apparent. We further note that on careful inspection of (11.25), we may verify the following symmetries of the covariant curvature tensor R_{ijkm}, namely

$$R_{ijkm} = -R_{jikm}, \qquad R_{ijkm} = -R_{ijmk}, \qquad R_{ijkm} = R_{kmij}, \qquad R_{ijkm} + R_{ikmj} + R_{imjk} = 0. \tag{11.28}$$

For a proof of Bianchi's identity and further details on these symmetries, we refer the reader to either Spain [97] or to the Appendix of Eringen [31].

On making use of Bianchi's identity, by purposeful construction, the Einstein tensor is divergence free, namely $G^i_{j;i} = 0$ (also refer to Spain [97]). Briefly, this may be established using the fact that all partial covariant derivatives of all metric tensors vanish, and therefore in Bianchi's identity (11.27), we may raise and lower the symbols freely. Thus, from Bianchi's identity and the above symmetries, so that $R_{ijmn;k} = -R_{ijnm;k}$, we may deduce

$$R^i_{jkm;n} + R^i_{jnk;m} - R^i_{jnm;k} = 0,$$

which on contracting i and m produces

$$R_{jk;n} + R^i_{jnk;i} - R_{jn;k} = 0,$$

and these are referred to as the contracted Bianchi identity. A second contraction of this identity may be shown to yield $R_{;j} = 2R^k_{j;k}$, which are equivalent to the vanishing of the divergence of the Einstein tensor, namely $G^i_{j;i} = 0$. While the vanishing of the divergence of the Einstein tensor is a fundamental result in general relativity, the above abbreviated derivation belies the underlying complexities involved in demonstrating the result in a particular coordinate representation. Two illustrative examples are provided in Sect. 11.7, and the result is also verified for the spiral gravitating structures examined in the final section of the chapter.

In the verification of the equations $G^i_{j;i} = 0$, sometimes it is simpler to express the equations in terms of the covariant Einstein tensor G_{ij} through the relations $G^i_k = g^{ij} G_{jk}$, so that on taking the partial covariant derivative with respect to x^m, we have

$$G^i_{k;m} = g^{ij} G_{jk;m} = g^{ij} \left\{ \frac{\partial G_{jk}}{\partial x^m} - \Gamma^n_{mj} G_{nk} - \Gamma^p_{mk} G_{jp} \right\}, \tag{11.29}$$

noting again that all partial covariant derivatives of the metric tensor g_{ij} and its conjugate g^{ij} are zero. On contracting this equation, that is, setting $m = i$ and then summing over i, the equations $G^i_{j;i} = 0$ become

$$G^i_{k;i} = g^{ij} G_{jk;i} = g^{ij} \left\{ \frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij} G_{nk} - \Gamma^p_{ik} G_{jp} \right\} = 0, \tag{11.30}$$

and the latter equality is sometimes the most convenient formulation to use. We note that in a stress-free vacuum space, the Einstein tensor is assumed to vanish identically, namely $G_{ij} = 0$.

11.5 Illustration for Single Space Dimension

To provide an illustration of the machinery for the above tensors, in a four-dimensional space $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ where (x, y, z) denote rectangular Cartesian coordinates, we consider the particular four-dimensional line element, thus

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j \\ &= a(x,t)(cdt)^2 - b(x,t)(dx)^2 - 2f(x,t)cdtdx - (dy)^2 - (dz)^2, \end{aligned}$$

where $a(x, t)$, $b(x, t)$ and $f(x, t)$ denote three functions to be determined from $G^i_{j;i} = 0$, and the indices i, j run through $i, j = 0, 1, 2, 3$. The metric tensor g_{ij} has components given by

$$g_{ij} = \begin{pmatrix} a(x,t) & -f(x,t) & 0 & 0 \\ -f(x,t) & -b(x,t) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

while the conjugate metric tensor g^{ij} has components given by

$$g^{ij} = \begin{pmatrix} b(x,t)/\delta & -f(x,t)/\delta & 0 & 0 \\ -f(x,t)/\delta & -a(x,t)/\delta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{11.31}$$

where $\delta(x, t)$ is defined by $\delta = ab + f^2$ and the determinant $g = |g_{ij}| = -\delta$.

Using the formula (11.22) for the Christoffel symbols of the first kind $[ij, k]$, namely

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

we may deduce the following expressions for the non-zero Christoffel symbols of the first kind, thus

$$[01, 0] = \frac{1}{2}\frac{\partial a}{\partial x}, \quad [01, 1] = -\frac{1}{2c}\frac{\partial b}{\partial t}, \quad [00, 0] = \frac{1}{2c}\frac{\partial a}{\partial t}, \quad [11, 1] = -\frac{1}{2}\frac{\partial b}{\partial x},$$
$$[00, 1] = -\left(\frac{1}{2}\frac{\partial a}{\partial x} + \frac{1}{c}\frac{\partial f}{\partial t}\right), \qquad [11, 0] = \left(\frac{1}{2c}\frac{\partial b}{\partial t} - \frac{\partial f}{\partial x}\right),$$

noting that other non-zero components may be deduced from the symmetry $[ij, k] = [ji, k]$. From these relations and the formulae $\Gamma^k_{ij} = g^{kn}[ij, n]$, we may deduce that the only non-zero Christoffel symbols of the second kind are as follows:

$$\Gamma^0_{01} = \frac{1}{2\delta}\left(b\frac{\partial a}{\partial x} + \frac{f}{c}\frac{\partial b}{\partial t}\right), \qquad \Gamma^1_{01} = \frac{1}{2\delta}\left(\frac{a}{c}\frac{\partial b}{\partial t} - f\frac{\partial a}{\partial x}\right),$$

$$\Gamma^0_{00} = \frac{1}{2\delta}\left(f\frac{\partial a}{\partial x} + \frac{b}{c}\frac{\partial a}{\partial t} + \frac{2f}{c}\frac{\partial f}{\partial t}\right), \quad \Gamma^1_{00} = \frac{1}{2\delta}\left(a\frac{\partial a}{\partial x} - \frac{f}{c}\frac{\partial a}{\partial t} + \frac{2a}{c}\frac{\partial f}{\partial t}\right),$$

$$\Gamma^0_{11} = \frac{1}{2\delta}\left(f\frac{\partial b}{\partial x} + \frac{b}{c}\frac{\partial b}{\partial t} - 2b\frac{\partial f}{\partial x}\right), \quad \Gamma^1_{11} = \frac{1}{2\delta}\left(a\frac{\partial b}{\partial x} - \frac{f}{c}\frac{\partial b}{\partial t} + 2f\frac{\partial f}{\partial x}\right).$$

We may make use of (11.23) to check these formulae, thus

$$\begin{aligned}
\Gamma^j_{j0} &= \Gamma^0_{00} + \Gamma^1_{10} \\
&= \frac{1}{2\delta}\left(f\frac{\partial a}{\partial x} + \frac{b}{c}\frac{\partial a}{\partial t} + \frac{2f}{c}\frac{\partial f}{\partial t} + \frac{a}{c}\frac{\partial b}{\partial t} - f\frac{\partial a}{\partial x}\right) \\
&= \frac{1}{2c\delta}\frac{\partial \delta}{\partial t},
\end{aligned}$$

and

$$\begin{aligned}
\Gamma^j_{j1} &= \Gamma^0_{01} + \Gamma^1_{11} \\
&= \frac{1}{2\delta}\left(b\frac{\partial a}{\partial x} + \frac{f}{c}\frac{\partial b}{\partial t} + a\frac{\partial b}{\partial x} - \frac{f}{c}\frac{\partial b}{\partial t} + 2f\frac{\partial f}{\partial x}\right) \\
&= \frac{1}{2\delta}\frac{\partial \delta}{\partial x},
\end{aligned}$$

as required.

With $i = k = 1$ and $j = m = 0$, we may deduce from the symmetries of the covariant curvature tensor R_{ijkm} (see Eq. (11.28)) the following results $R_{1010} = -R_{0110} = R_{0101}$ and $R_{0011} = 0$, and since there is essentially only one non-zero component, we chose to adopt R_{0101} and express the other non-zero components in terms of R_{0101}. Again with $i = k = 1$ and $j = m = 0$, we may deduce from (11.25) the following expression for R_{0101} in terms of the metric tensor and the Christoffel symbols of the first kind, thus

$$\begin{aligned}
R_{0101} = &\frac{1}{2}\left(\frac{\partial^2 g_{01}}{\partial x^1 \partial x^0} + \frac{\partial^2 g_{10}}{\partial x^1 \partial x^0} - \frac{\partial^2 g_{00}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{11}}{\partial x^0 \partial x^0}\right) \\
&+ g^{np}\left([10, n][01, p] - [11, n][00, p]\right)
\end{aligned}$$

$$= \frac{1}{2}\left(2\frac{\partial^2 g_{01}}{\partial x^1 \partial x^0} - \frac{\partial^2 g_{00}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{11}}{\partial x^0 \partial x^0}\right)$$
$$+ g^{n0}\left([10,n][01,0] - [11,n][00,0]\right) + g^{n1}\left([10,n][01,1] - [11,n][00,1]\right)$$
$$= \frac{1}{2}\left(2\frac{\partial^2 g_{01}}{\partial x^1 \partial x^0} - \frac{\partial^2 g_{00}}{\partial x^{1^2}} - \frac{\partial^2 g_{11}}{\partial x^{0^2}}\right)$$
$$+ g^{00}\left([10,0][01,0] - [11,0][00,0]\right) + g^{10}\left([10,1][01,0] - [11,1][00,0]\right)$$
$$+ g^{01}\left([10,0][01,1] - [11,0][00,1]\right) + g^{11}\left([10,1][01,1] - [11,1][00,1]\right).$$

After much simplification and rearrangement, this expression can be shown to become

$$R_{0101} = \frac{1}{2}\left(\frac{1}{c^2}\frac{\partial^2 b}{\partial t^2} - \frac{2}{c}\frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 a}{\partial x^2}\right)$$
$$+ \frac{1}{4\delta}\left\{\left(\frac{\partial a}{\partial x} + \frac{1}{c}\frac{\partial f}{\partial t}\right)\frac{\partial \delta}{\partial x} - \left(\frac{1}{c}\frac{\partial b}{\partial t} - \frac{\partial f}{\partial x}\right)\frac{1}{c}\frac{\partial \delta}{\partial t}\right\}$$
$$+ \frac{1}{4c\delta}\left\{a\frac{\partial(f,b)}{\partial(t,x)} + b\frac{\partial(a,f)}{\partial(t,x)} + f\frac{\partial(b,a)}{\partial(t,x)}\right\},$$

where $\delta(x,t)$ is defined by $\delta = ab + f^2$ and closer inspection of the final term reveals the structure of the scalar triple product, thus

$$\frac{1}{4c\delta}\left\{a\frac{\partial(f,b)}{\partial(t,x)} + b\frac{\partial(a,f)}{\partial(t,x)} + f\frac{\partial(b,a)}{\partial(t,x)}\right\} = -\frac{1}{4c\delta}\begin{vmatrix} a & b & f \\ \frac{\partial a}{\partial t} & \frac{\partial b}{\partial t} & \frac{\partial f}{\partial t} \\ \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} & \frac{\partial f}{\partial x} \end{vmatrix}.$$

We use $R_{jk} = g^{im} R_{ijkm}$ to determine the components of the Ricci tensor $R_{ij} = R^k_{ijk}$, thus

$$R_{00} = g^{im} R_{i00m} = g^{i0} R_{i000} + g^{i1} R_{i001}$$
$$= g^{00} R_{0000} + g^{10} R_{1000} + g^{01} R_{0001} + g^{11} R_{1001}$$
$$= g^{11} R_{1001} = g^{11} R_{0110} = -g^{11} R_{0101},$$

on using the above symmetries of the covariant curvature tensor R_{ijkm} given by (11.28), since from Eq. (11.25) it is apparent that any component of the tensor R_{ijkm} with three indices coinciding must be zero. Proceeding further in this manner and making use of (11.31) for the components the conjugate metric tensor g^{ij}, we may show that the only non-zero components of the Ricci tensor R_{jk} are given by

$$R_{00} = aR_{0101}/\delta, \qquad R_{10} = R_{01} = -fR_{0101}/\delta, \qquad R_{11} = -bR_{0101}/\delta,$$

so that the curvature invariant $R = g^{ij}R_{ij}$ becomes

$$R = g^{ij}R_{ij} = g^{00}R_{00} + 2g^{01}R_{01} + g^{11}R_{11} = 2(ab + f^2)R_{0101}/\delta^2 = 2R_{0101}/\delta,$$

since by definition $\delta(x, t)$ is defined by $\delta = ab + f^2$. From the relation $G_{ij} = R_{ij} - Rg_{ij}/2$, it is apparent that the only non-zero components of the covariant Einstein tensor G_{ij} arising through the term $-Rg_{ij}/2$ are $G_{22} = G_{33} = R_{0101}/\delta$ with all others identically zero, so that the equations for the vanishing of the divergence of the Einstein tensor become simply $\partial G_{22}/\partial y = 0$ and $\partial G_{33}/\partial z = 0$, which are trivially satisfied. In the following section, we provide some general formulae for the Riemann and Einstein tensors applying to a particular metric of some generality.

11.6 Formulae for Ricci and Einstein Tensors

While there exist a number of exact space-times for the Einstein field equations of general relativity, such as those listed in [42, 81, 98] and to a lesser extent in [84, 91, 107], as previously stated, in the search for further exact space-time solutions of general relativity, the major prohibiting factors include the shear complexity of the underlying equations and the lack of any strategic perspective that might guide the analysis. For any given line element, the Riemann-Christoffel tensor itself has a complicated dependence on the components of the particular assumed metric tensor. The metric tensor for the most general four-dimensional line element involves ten arbitrary components, which are possibly dependent upon three spatial variables and the temporal variable.

We attempt to reduce the complexity of the problem in the determination of exact solutions by considering a line element involving only seven arbitrary metric tensor components, and each component is assumed to depend only on two of the spatial variables and the temporal variable. Of course, while not completely general, the assumed line element includes many existing exact space-time solutions of general relativity. We show that six components of the covariant curvature tensor act as a basis, and we present some general formulae for the Ricci and Einstein tensors expressed in terms of these six components of the covariant curvature tensor. In the next section, we illustrate this general formulation with two well-known cosmological models. Although similar general metrics have been studied in the past and it is known that the four-dimensional structure can be represented in terms of the corresponding three-dimensional tensors (see, e.g. [30] or [42]), the present approach differs in the sense that it involves the determination of explicit formulae.

In four-dimensional space (x^0, x^1, x^2, x^3), we consider the line element with the particular structure,

$$ds^2 = g_{ij}dx^i dx^j = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 \quad (11.32)$$
$$+ 2g_{01}dx^0dx^1 + 2g_{02}dx^0dx^2 + 2g_{12}dx^1dx^2,$$

where $x^0 = ct$, and we assume that the non-zero components of the metric tensor g_{ij} depend only on x^0, x^1 and x^2 and are independent of the x^3 coordinate. Specifically, we assume that the metric tensor has the structure

$$g_{ij} = \begin{pmatrix} g_{00}(x^0, x^1, x^2) & g_{01}(x^0, x^1, x^2) & g_{02}(x^0, x^1, x^2) & 0 \\ g_{01}(x^0, x^1, x^2) & g_{11}(x^0, x^1, x^2) & g_{12}(x^0, x^1, x^2) & 0 \\ g_{02}(x^0, x^1, x^2) & g_{12}(x^0, x^1, x^2) & g_{22}(x^0, x^1, x^2) & 0 \\ 0 & 0 & 0 & g_{33}(x^0, x^1, x^2) \end{pmatrix}. \quad (11.33)$$

This particular metric, although not completely general, includes many of the metrics for which exact space-time solutions have been determined (see, e.g. [81]).

Working Notation Now in order to undertake lengthy algebraic calculations with this tensor, we may facilitate these manipulations and without loss of generality, we may employ the notation (ct, x, y, z) for (x^0, x^1, x^2, x^3) and in place of the above equations we use a notation devoid of indices, thus

$$ds^2 = g_{ij}dx^i dx^j = a(x, y, t)(cdt)^2 - b(x, y, t)(dx)^2 - h(x, y, t)(dy)^2 \quad (11.34)$$
$$- 2f(x, y, t)cdtdx - 2j(x, y, t)cdtdy - 2k(x, y, t)dxdy - m(x, y, t)(dz)^2,$$

where $a(x, y, t)$, $b(x, y, t)$, $f(x, y, t)$, $h(x, y, t)$, $j(x, y, t)$, $k(x, y, t)$ and $m(x, y, t)$ all denote functions to be ultimately determined from the Einstein equations $G^i_{j;i} = 0$, and the indices i, j here run through $i, j = 0, 1, 2, 3$. Thus, the metric tensor g_{ij} has components given by

$$g_{ij} = \begin{pmatrix} a(x, y, t) & -f(x, y, t) & -j(x, y, t) & 0 \\ -f(x, y, t) & -b(x, y, t) & -k(x, y, t) & 0 \\ -j(x, y, t) & -k(x, y, t) & -h(x, y, t) & 0 \\ 0 & 0 & 0 & -m(x, y, t) \end{pmatrix}, \quad (11.35)$$

while the conjugate metric tensor g^{ij} has components given by

$$g^{ij} = \begin{pmatrix} m(k^2 - bh)/g^* & m(fh - jk))/g^* & m(bj - fk)/g^* & 0 \\ m(fh - jk))/g^* & m(ah + j^2)/g^* & -m(ak + jf)/g^* & 0 \\ m(bj - fk)/g^* & -m(ak + jf)/g^* & m(f^2 + ab)/g^* & 0 \\ 0 & 0 & 0 & -1/m \end{pmatrix},$$

where $g^*(x, y, t)$ is the determinant $g^* = |g_{ij}|$ given by

$$g^* = m(ak^2 + 2jkf - abh - bj^2 - hf^2),$$

and we emphasise that the specific line element (11.34) with the symbols (x, y, z) is for working purposes only and that they do not necessarily represent rectangular Cartesian coordinates and might represent any three-dimensional spatial coordinates (x^1, x^2, x^3).

The assumptions underlying the metric tensor g_{ij} mean that in many respects, and specifically in terms of the conjugate metric tensor g^{ij}, the three-dimensional and four-dimensional problems uncouple and the conjugate metric tensor becomes

$$g^{ij} = \begin{pmatrix} (bh - k^2)/g & (jk - fh))/g & (fk - bj)/g & 0 \\ (jk - fh))/g & -(ah + j^2)/g & (ak + jf)/g & 0 \\ (fk - bj)/g & (ak + jf)/g & -(f^2 + ab)/g & 0 \\ 0 & 0 & 0 & -1/m \end{pmatrix},$$

where $g(x, y, t)$ is the three-dimensional determinant $g = |g_{ij}|$ given by

$$g = abh + bj^2 + hf^2 - ak^2 - 2jkf,$$

and $g^* = g_{33}g$.

Christoffel Symbols of the First Kind $[ij, k]$ Using the standard formula for the Christoffel symbols of the first kind $[ij, k]$, namely

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

we may deduce the following expressions for the non-zero Christoffel symbols of the first kind in a straightforward manner, thus

$$[00, 0] = \frac{1}{2}\frac{\partial g_{00}}{\partial x^0}, \quad [00, 1] = \left(\frac{\partial g_{01}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^1}\right), \quad [00, 2] = \left(\frac{\partial g_{02}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^2}\right),$$

$$[11, 0] = \left(\frac{\partial g_{01}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^0}\right), \quad [11, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1}, \quad [11, 2] = \left(\frac{\partial g_{12}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^2}\right),$$

$$[22, 0] = \left(\frac{\partial g_{02}}{\partial x^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial x^0}\right), \quad [22, 1] = \left(\frac{\partial g_{12}}{\partial x^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial x^1}\right), \quad [22, 2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^2},$$

$$[01, 0] = \frac{1}{2}\frac{\partial g_{00}}{\partial x^1}, \quad [01, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^0}, \quad [01, 2] = \frac{1}{2}\left(\frac{\partial g_{02}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^2}\right),$$

$$[02, 0] = \frac{1}{2}\frac{\partial g_{00}}{\partial x^2}, \quad [02, 1] = \frac{1}{2}\left(\frac{\partial g_{01}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^0} - \frac{\partial g_{02}}{\partial x^1}\right), \quad [02, 2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^0},$$

$$[12, 0] = \frac{1}{2}\left(\frac{\partial g_{02}}{\partial x^1} + \frac{\partial g_{01}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^0}\right), \quad [12, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^2}, \quad [12, 2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1},$$

which in terms of the working symbols become

$$[00,0]=\frac{1}{2c}\frac{\partial a}{\partial t},\quad [00,1]=-\left(\frac{1}{2}\frac{\partial a}{\partial x}+\frac{1}{c}\frac{\partial f}{\partial t}\right),\quad [00,2]=-\left(\frac{1}{2}\frac{\partial a}{\partial y}+\frac{1}{c}\frac{\partial j}{\partial t}\right),$$

$$[11,0]=\left(\frac{1}{2c}\frac{\partial b}{\partial t}-\frac{\partial f}{\partial x}\right),\qquad [11,1]=-\frac{1}{2}\frac{\partial b}{\partial x},\qquad [11,2]=\left(\frac{1}{2}\frac{\partial b}{\partial y}-\frac{\partial k}{\partial x}\right),$$

$$[22,0]=\left(\frac{1}{2c}\frac{\partial h}{\partial t}-\frac{\partial j}{\partial y}\right),\qquad [22,1]=\left(\frac{1}{2}\frac{\partial h}{\partial x}-\frac{\partial k}{\partial y}\right),\qquad [22,2]=-\frac{1}{2}\frac{\partial h}{\partial y},$$

$$[01,0]=\frac{1}{2}\frac{\partial a}{\partial x},\qquad [01,1]=-\frac{1}{2c}\frac{\partial b}{\partial t},\qquad [01,2]=\frac{1}{2}\left(\frac{\partial f}{\partial y}-\frac{\partial j}{\partial x}-\frac{1}{c}\frac{\partial k}{\partial t}\right),$$

$$[02,0]=\frac{1}{2}\frac{\partial a}{\partial y},\qquad [02,1]=\frac{1}{2}\left(\frac{\partial j}{\partial x}-\frac{\partial f}{\partial y}-\frac{1}{c}\frac{\partial k}{\partial t}\right),\qquad [02,2]=-\frac{1}{2c}\frac{\partial h}{\partial t},$$

$$[12,0]=-\frac{1}{2}\left(\frac{\partial j}{\partial x}+\frac{\partial f}{\partial y}-\frac{1}{c}\frac{\partial k}{\partial t}\right),\qquad [12,1]=-\frac{1}{2}\frac{\partial b}{\partial y},\qquad [12,2]=-\frac{1}{2}\frac{\partial h}{\partial x},$$

noting that other non-zero components may be deduced from the symmetry $[ij,k]=[ji,k]$, and with the convention that i and j immediately below refer only to 0, 1, 2, we have

$$[ij,3]=[3i,j]=0,\qquad [3i,3]=-\frac{1}{2}\frac{\partial m}{\partial x^i},\qquad [33,i]=\frac{1}{2}\frac{\partial m}{\partial x^i}.$$

Christoffel Symbols of the Second Kind Γ^k_{ij} The Christoffel symbols of the second kind are more complicated, and we derive the following formulae as follows. From the above 4×4 matrix, we formulate the 3×3 matrix

$$g_{ij}=\begin{pmatrix} g_{00} & g_{01} & g_{02}\\ g_{01} & g_{11} & g_{12}\\ g_{02} & g_{12} & g_{22}\end{pmatrix}, \tag{11.36}$$

and we assign g to be the determinant of this matrix, so that $g^*=|g_{ij}|=gg_{33}$. Basically, in terms of the index-free notation, we need to individually calculate each of the Christoffel symbols of the second kind using the above formulae for the Christoffel symbols of the first kind along with the formulae $\Gamma^k_{ij}=g^{kn}[ij,n]$. Having undertaken these extensive algebraic calculations, it becomes apparent that we may express the final formulae for each of the non-zero Christoffel symbols of the second kind Γ^k_{ij} in terms of three 3×3 determinants, two of positive sign and one of negative sign, no doubt emanating from the corresponding signatures involved in the three terms in the Christoffel symbols of the first kind $[ij,k]$. With the designation of i, j and k as in the symbol Γ^k_{ij}, we proceed as follows:

- In each of the three determinants, we delete the k^{th} row.
- In one of the determinants of positive sign, we replace the k^{th} row by $(\partial g_{i0}/\partial x^j, \partial g_{i1}/\partial x^j, \partial g_{i2}/\partial x^j)$.
- In the other determinant of positive sign, we replace the k^{th} row by $(\partial g_{j0}/\partial x^i, \partial g_{j1}/\partial x^i, \partial g_{j2}/\partial x^i)$.
- In the determinant of negative sign, we replace the k^{th} row by $(\partial g_{ij}/\partial x^0, \partial g_{ij}/\partial x^1, \partial g_{ij}/\partial x^2)$.
- Each of the three determinants is weighted with a factor $1/2g$.
- In the event that $i = j$, the two determinants of positive sign coincide and merge to produce one determinant of positive sign of weight $1/g$.

The final expressions are as follows:

$$\Gamma^0_{00} = \frac{1}{g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{00} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{00} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{11} = \frac{1}{g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{12}}{\partial x^1} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{11} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{12}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{11} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{12}}{\partial x^1} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{22} = \frac{1}{g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^2} & \frac{\partial g_{21}}{\partial x^2} & \frac{\partial g_{22}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{22}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{22} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^2} & \frac{\partial g_{21}}{\partial x^2} & \frac{\partial g_{22}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{22}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{22} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^2} & \frac{\partial g_{21}}{\partial x^2} & \frac{\partial g_{22}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{22}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{01} = \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{01} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{01} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{02} = \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{02} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma_{02}^{2}=\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \end{vmatrix}+\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \end{vmatrix}-\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \end{vmatrix},$$

$$\Gamma_{12}^{0}=\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^1} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}+\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^2} & \frac{\partial g_{11}}{\partial x^2} & \frac{\partial g_{12}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}-\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma_{12}^{1}=\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}+\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^2} & \frac{\partial g_{11}}{\partial x^2} & \frac{\partial g_{12}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}-\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma_{12}^{2}=\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^1} \end{vmatrix}+\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^2} & \frac{\partial g_{11}}{\partial x^2} & \frac{\partial g_{12}}{\partial x^2} \end{vmatrix}-\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^2} \end{vmatrix}.$$

Since these expressions involve the scalar triple product, no doubt there exists any number of equivalent alternative expressions dependent upon one's own personal preferences. We may use the well-known formula

$$\Gamma_{ij}^{j}=\frac{1}{2g}\frac{\partial g}{\partial x^i},$$

to check the above expressions obtained for the Chistoffel symbols of the second kind. These calculations are not entirely trivial, and for purposes of illustration, the proof for the particular value $i=0$ is as follows. On writing the identity out in full for $i=0$, we have

$$\Gamma_{0j}^{j}=\Gamma_{00}^{0}+\Gamma_{01}^{1}+\Gamma_{02}^{2},$$

and from the above formulae, we obtain

$$\Gamma_{ij}^{j}$$
$$=\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}+\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}-\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}$$

$$+\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}$$

$$+\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \end{vmatrix},$$

where we have specifically written the first term of this equation as two identical terms, so that the sum of the first three terms on the left-hand side gives the desired result as the partial derivative of g with respect to x^0 divided by $2g$, and it is left to show that the remaining six terms are zero. This is most easily achieved using the permutation symbols ε^{ijk} and the summation convention. In this notation, the determinant g of the 3×3 metric tensor becomes $g = \varepsilon^{ijk} g_{0i} g_{1j} g_{2k}$, and the six determinants in the above expression that are required to be shown to be zero become

$$\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}$$

$$-\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \end{vmatrix}$$

$$= \frac{\varepsilon^{ijk}}{2g}\left(\frac{\partial g_{0i}}{\partial x^0} g_{1j} g_{2k} + g_{0i}\frac{\partial g_{0j}}{\partial x^1} g_{2k} + g_{0i} g_{1j}\frac{\partial g_{0k}}{\partial x^2}\right)$$

$$- \frac{\varepsilon^{ijk}}{2g}\left(\frac{\partial g_{00}}{\partial x^i} g_{1j} g_{2k} + g_{0i}\frac{\partial g_{10}}{\partial x^j} g_{2k} + g_{0i} g_{1j}\frac{\partial g_{20}}{\partial x^k}\right)$$

$$= \frac{\varepsilon^{ijk}}{2g}\left\{\left(\frac{\partial g_{0i}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^i}\right) g_{1j} g_{2k} + \left(\frac{\partial g_{0j}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^j}\right) g_{0i} g_{2k} + \left(\frac{\partial g_{0k}}{\partial x^2} - \frac{\partial g_{20}}{\partial x^k}\right) g_{0i} g_{1j}\right\}$$

$$= \frac{1}{2g}\left\{\left(\frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1}\right)(g_{12}g_{20} - g_{10}g_{22}) + \left(\frac{\partial g_{02}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^2}\right)(g_{10}g_{21} - g_{11}g_{20})\right\}$$

$$+ \frac{1}{2g}\left\{\left(\frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^0}\right)(g_{02}g_{21} - g_{01}g_{22}) + \left(\frac{\partial g_{02}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^2}\right)(g_{01}g_{20} - g_{00}g_{21})\right\}$$

$$+ \frac{1}{2g}\left\{\left(\frac{\partial g_{00}}{\partial x^2} - \frac{\partial g_{20}}{\partial x^0}\right)(g_{01}g_{12} - g_{02}g_{11}) + \left(\frac{\partial g_{01}}{\partial x^2} - \frac{\partial g_{20}}{\partial x^1}\right)(g_{02}g_{10} - g_{00}g_{12})\right\},$$

each term of which can be seen to have a corresponding term that cancels, and therefore, the sum total produces zero as required, and a similar proof can be presented for $i = 1$ and $i = 2$.

Christoffel Symbols of the Second Kind Γ^k_{ij} Involving the Index 3 For the Christoffel symbols of the second kind involving the index 3, we again assume that the indices i and j immediately below refer only to 0, 1, 2, and we have the following formulae:

$$\Gamma^3_{ij} = g^{3n}[ij, n] = g^{33}[ij, 3] = 0,$$

$$\Gamma^i_{33} = g^{in}[33, n] = g^{ij}[33, j] = -\frac{g^{ij}}{2}\frac{\partial g_{33}}{\partial x^j},$$

$$\Gamma^3_{3j} = g^{3n}[3j, n] = g^{33}[3j, 3] = \frac{g^{33}}{2}\frac{\partial g_{33}}{\partial x^j} = \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial x^j},$$

and we observe that these expressions are entirely consistent with the extended identity

$$\Gamma^j_{ij} = \Gamma^0_{i0} + \Gamma^1_{i1} + \Gamma^2_{i2} + \Gamma^3_{i3} = \frac{1}{2g}\frac{\partial g}{\partial x^i} + \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial x^i} = \frac{1}{2g^*}\frac{\partial g^*}{\partial x^i},$$

where g^* is the determinant of the 4×4 matrix and $g^* = |g_{ij}| = gg_{33}$, and the derived identity is as might be expected.

Covariant Curvature Tensor R_{ijkm} The metric tensor (11.32) (see also (11.34)) has been purposely chosen so that there is an uncoupling of certain aspects of the three-dimensional problem (x^0, x^1, x^2) and the four-dimensional problem (x^0, x^1, x^2, x^3). Thus, with i, j, k and m restricted to 0, 1 and 2, other than those connected through the symmetries (11.28), there are essentially only six non-zero components of the covariant curvature tensor R_{ijkm}, which here we adopt to be R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012}. As far as the author is aware, there is no immediate way to identify the trivial components of the covariant curvature tensor R_{ijkm}, other than close inspection of the assumed metric tensor and (11.25) or (11.27) along with the known symmetries (11.28), so for the sake of completeness, these components are listed below. From the assumed metric tensor and either Eqs. (11.25) or (11.27), we may verify that the following components of the following components of the covariant curvature tensor R_{ijkm} are all zero, thus

$$\begin{aligned} &R_{0000} = R_{0001} = R_{0002} = R_{0111} = R_{0222} = R_{0010} \\ &= R_{0011} = R_{0012} = R_{0020} = R_{0021} = R_{0022} = R_{0122} \\ &= R_{1000} = R_{1110} = R_{1111} = R_{1112} = R_{1222} = R_{1011} \\ &= R_{1022} = R_{1120} = R_{1121} = R_{1122} = R_{2000} = R_{2111} \\ &= R_{2220} = R_{2221} = R_{2222} = R_{2011} = R_{2022} = R_{2122} = 0, \end{aligned}$$

while through the symmetries (11.28), we have the results:

$$R_{0110} = R_{1001} = -R_{0101}, \quad R_{1010} = R_{0101},$$
$$R_{0220} = R_{2002} = -R_{0202}, \quad R_{2020} = R_{0202},$$
$$R_{1221} = R_{2112} = -R_{1212}, \quad R_{2121} = R_{1212},$$
$$R_{0121} = R_{1012} = -R_{0112}, \quad R_{1021} = R_{2110} = R_{0112},$$
$$R_{0102} = R_{2010} = -R_{0120}, \quad R_{1002} = R_{2001} = R_{0120},$$
$$R_{0212} = R_{2120} = -R_{2012}, \quad R_{2012} = R_{1220} = R_{2012}.$$

Now following similar lines, and with i, j, k and m still restricted to 0, 1 and 2, we may show that for the covariant curvature tensor R_{ijkm} involving the index 3, we have the following results

$$R_{3jkm} = R_{i3km} = R_{ij3m} = R_{ijk3} = R_{33km} = R_{ij33} = 0,$$
$$R_{j3k3} = R_{3j3k} = -R_{3jk3},$$

where R_{3jk3} is given by

$$R_{3jk3} = \frac{1}{2}\left(\frac{\partial^2 g_{33}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial g_{33}}{\partial x^p}\right) - \frac{g^{33}}{4}\frac{\partial g_{33}}{\partial x^j}\frac{\partial g_{33}}{\partial x^k},$$

and since $g^{33} = 1/g_{33}$, it is apparent that

$$R_{3jk3} = \sqrt{g_{33}}\left(\frac{\partial^2 \sqrt{g_{33}}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial \sqrt{g_{33}}}{\partial x^p}\right),$$

or

$$R_{3jk3} = -\sqrt{-g_{33}}\left(\frac{\partial^2 \sqrt{-g_{33}}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial \sqrt{-g_{33}}}{\partial x^p}\right),$$

in the event that $g^{33} < 0$.

The components R_{3jk3} involve another six non-zero components of the covariant curvature tensor R_{ijkm} arising from $j, k = 0, 1, 2$, which contribute to the Ricci tensor through $R_{jk} = R^n_{jkn} = g^{im}R_{ijkm} = g^{33}R_{3jk3}$, thus

$$R_{jk} = g^{33}R_{3jk3} = \frac{1}{\sqrt{g_{33}}}\left(\frac{\partial^2 \sqrt{g_{33}}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial \sqrt{g_{33}}}{\partial x^p}\right),$$

and to the curvature invariant $R = g^{jk}R_{jk} = g^{im}g^{jk}R_{ijkm}$ through the term R^* given by

$$R^* = g^{jk}g^{33}R_{3jk3} + g^{im}g^{33}R_{i33m} = 2g^{jk}g^{33}R_{3jk3},$$

which becomes

$$R^* = \frac{2g^{jk}}{\sqrt{g_{33}}}\left(\frac{\partial^2\sqrt{g_{33}}}{\partial x^j\partial x^k} - \Gamma^p_{jk}\frac{\partial\sqrt{g_{33}}}{\partial x^p}\right) = \frac{2\nabla^2\sqrt{g_{33}}}{\sqrt{g_{33}}},$$

where ∇^2 is the conventional generalised Laplacian associated with the three-dimensional coordinates (x^0, x^1, x^2), namely

$$\nabla^2 = g^{jk}\left(\frac{\partial^2}{\partial x^j\partial x^k} - \Gamma^p_{jk}\frac{\partial}{\partial x^p}\right).$$

In general, there are overall 12 essentially independent non-zero components of the covariant curvature tensor R_{ijkm}, which act as a basis in which we might express the Ricci tensor and Einstein tensor components R_{jk} and G_{jk} and the curvature invariant R, and we now proceed to do precisely this. As far as the author is aware, in order to deduce the final expressions, there is no approach other than by direct individual calculations, since each of the 12 curvature tensor components are not readily characterised in some general way. By way of illustration, we start with the calculation for R_{00}, thus

$$\begin{aligned} R_{00} &= g^{im}R_{i00m} \\ &= g^{11}R_{1001} + g^{12}R_{1002} + g^{21}R_{2001} + g^{22}R_{2002} + g^{33}R_{3003} \\ &= -g^{11}R_{0101} + g^{12}R_{0120} + g^{21}R_{0120} - g^{22}R_{0202} + g^{33}R_{3003} \\ &= -g^{11}R_{0101} + 2g^{12}R_{0120} - g^{22}R_{0202} + g^{33}R_{3003}, \end{aligned}$$

and proceeding similarly, we may deduce the following expressions:

$$\begin{aligned} R_{00} &= -g^{11}R_{0101} + 2g^{12}R_{0120} - g^{22}R_{0202} + g^{33}R_{3003}, \\ R_{11} &= -g^{00}R_{0101} + 2g^{02}R_{0112} - g^{22}R_{1212} + g^{33}R_{3113}, \\ R_{22} &= -g^{00}R_{0202} + 2g^{01}R_{2012} - g^{11}R_{1212} + g^{33}R_{3223}, \\ R_{01} &= g^{10}R_{0101} - g^{12}R_{0112} - g^{20}R_{0120} + g^{22}R_{2012} + g^{33}R_{3013}, \\ R_{02} &= -g^{10}R_{0120} + g^{11}R_{0112} + g^{20}R_{0202} - g^{21}R_{2012} + g^{33}R_{3023}, \\ R_{12} &= -g^{00}R_{0102} - g^{01}R_{0112} - g^{20}R_{2012} + g^{21}R_{1212} + g^{33}R_{3123}, \end{aligned} \tag{11.37}$$

and $R_{33} = g^{im}R_{i33m} = R^*g_{33}/2$. Now on forming the curvature invariant $R = g^{jk}R_{jk} = g^{im}g^{jk}R_{ijkm}$, the x^3 spatial direction contributes both through the terms R_{3jk3} in the above relations and through the term R_{33}, and we have

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + 2g^{01}R_{01} + 2g^{02}R_{02} + 2g^{12}R_{12} + R^*, \tag{11.38}$$

where $R^* = g^{33}g^{jk}R_{3jk3} + g^{im}g^{33}R_{i33m} = 2g^{jk}g^{33}R_{3jk3}$ is the total contribution to the curvature invariant arising from the x^3 spatial dimension and the summations over j and k in this expression are restricted to $0, 1, 2$. On using the above expressions for the Ricci tensor components R_{jk}, we find that

$$\begin{aligned} R &= R^* \\ &- 2[(g^{00}g^{11} - g^{01}g^{01})R_{0101} + (g^{00}g^{22} - g^{02}g^{02})R_{0202} + (g^{11}g^{22} - g^{12}g^{12})R_{1212} \\ &- 2(g^{00}g^{12} - g^{20}g^{10})R_{0120} - 2(g^{11}g^{02} - g^{01}g^{12})R_{0112} - 2(g^{22}g^{10} - g^{02}g^{21})R_{2012}]. \end{aligned}$$

Now by direct calculation or otherwise, we may deduce the formulae

$$g^{00}g^{11} - g^{01}g^{01} = \frac{g_{22}}{g}, \quad g^{00}g^{22} - g^{02}g^{02} = \frac{g_{11}}{g}, \quad g^{11}g^{22} - g^{12}g^{12} = \frac{g_{00}}{g},$$
$$g^{00}g^{12} - g^{20}g^{10} = -\frac{g_{12}}{g}, \quad g^{11}g^{20} - g^{01}g^{12} = -\frac{g_{02}}{g}, \quad g^{22}g^{10} - g^{02}g^{12} = -\frac{g_{01}}{g},$$

where g is the determinant of the 3×3 matrix (11.36), and from Eqs. (11.37) and (11.38), we may eventually deduce the expression for the curvature invariant R, namely

$$\begin{aligned} R &= 2(g^{mn}R_{3mn3})/g_{33} \\ &- 2(g_{00}R_{1212} + g_{11}R_{0202} + g_{22}R_{0101} + 2g_{12}R_{0120} + 2g_{02}R_{1021} + 2g_{01}R_{2012})/g, \end{aligned} \tag{11.39}$$

where $g^* = gg_{33}$ is the determinant of the 4×4 matrix given by (11.33), and the first term in this expression can be alternatively written as simply $R^* = 2g^{jk}g^{33}R_{3jk3}$, and R^* is the contribution to the curvature invariant R arising from the x^3 spatial direction.

We are now in a position to evaluate the Einstein tensor from $G_{ij} = R_{ij} - Rgij/2$, and to achieve this, we need the particular relations

$$\frac{g_{11}g_{22}}{g} - g^{00} = \frac{g_{12}g_{12}}{g}, \quad \frac{g_{00}g_{22}}{g} - g^{11} = \frac{g_{02}g_{02}}{g}, \quad \frac{g_{00}g_{11}}{g} - g^{22} = \frac{g_{01}g_{01}}{g},$$
$$\frac{g_{02}g_{12}}{g} - g^{01} = \frac{g_{01}g_{22}}{g}, \quad \frac{g_{01}g_{12}}{g} - g^{02} = \frac{g_{02}g_{11}}{g}, \quad \frac{g_{01}g_{02}}{g} - g^{12} = \frac{g_{00}g_{12}}{g}.$$

On combining these relations with the above Eqs. (11.37) and (11.38), we may deduce the following equations for the Einstein tensor $G_{ij} = R_{ij} - Rgij/2$, thus

$$G_{00} = \frac{g_{02}g_{02}}{g}R_{0101} + \frac{g_{01}g_{01}}{g}R_{0202} + \frac{g_{00}g_{00}}{g}R_{1212} + \frac{2g_{01}g_{02}}{g}R_{0120} \tag{11.40}$$

$$+\frac{2g_{00}g_{02}}{g}R_{0112}+\frac{2g_{00}g_{01}}{g}R_{2012}-\frac{R^*}{2}g_{00}+g^{33}R_{3003},$$

$$G_{11}=\frac{g_{12}g_{12}}{g}R_{0101}+\frac{g_{11}g_{11}}{g}R_{0202}+\frac{g_{01}g_{01}}{g}R_{1212}+\frac{2g_{01}g_{12}}{g}R_{0112}$$

$$+\frac{2g_{11}g_{12}}{g}R_{0120}+\frac{2g_{11}g_{01}}{g}R_{2012}-\frac{R^*}{2}g_{11}+g^{33}R_{3113},$$

$$G_{22}=\frac{g_{22}g_{22}}{g}R_{0101}+\frac{g_{12}g_{12}}{g}R_{0202}+\frac{g_{02}g_{02}}{g}R_{1212}+\frac{2g_{12}g_{22}}{g}R_{0120}$$

$$+\frac{2g_{22}g_{02}}{g}R_{0112}+\frac{2g_{12}g_{02}}{g}R_{2012}-\frac{R^*}{2}g_{22}+g^{33}R_{3223},$$

$$G_{01}=\frac{g_{02}g_{12}}{g}R_{0101}+\frac{g_{01}g_{11}}{g}R_{0202}+\frac{g_{00}g_{01}}{g}R_{1212}+\frac{(g_{00}g_{12}+g_{01}g_{02})}{g}R_{0112}$$

$$+\frac{(g_{11}g_{02}+g_{01}g_{12})}{g}R_{0120}+\frac{(g_{00}g_{11}+g_{01}g_{01})}{g}R_{2012}-\frac{R^*}{2}g_{01}+g^{33}R_{3013},$$

$$G_{02}=\frac{g_{02}g_{22}}{g}R_{0101}+\frac{g_{01}g_{12}}{g}R_{0202}+\frac{g_{00}g_{02}}{g}R_{1212}+\frac{(g_{00}g_{22}+g_{02}g_{02})}{g}R_{0112}$$

$$+\frac{(g_{01}g_{22}+g_{02}g_{12})}{g}R_{0120}+\frac{(g_{00}g_{12}+g_{01}g_{02})}{g}R_{2012}-\frac{R^*}{2}g_{02}+g^{33}R_{3023},$$

$$G_{12}=\frac{g_{12}g_{22}}{g}R_{0101}+\frac{g_{12}g_{11}}{g}R_{0202}+\frac{g_{01}g_{02}}{g}R_{1212}+\frac{(g_{01}g_{22}+g_{02}g_{12})}{g}R_{0112}$$

$$+\frac{(g_{11}g_{22}+g_{12}g_{12})}{g}R_{0120}+\frac{(g_{11}g_{02}+g_{01}g_{12})}{g}R_{2012}-\frac{R^*}{2}g_{12}+g^{33}R_{3123},$$

and $G_{33}=R_{33}-Rg_{33}/2=(R^*-R)g_{33}/2$ and all other components involving the index 3, namely G_{3j} for $j=0, 1, 2$ are zero.

Now from the expressions for the Einstein tensor $G^i_j = g^{ik}R_{jk}-R\delta^i_j/2$, it is immediately apparent that the trace $G=R-2R=-R$, and we can use this result to check the veracity of the above expressions for G_{ij} as follows. In the summation $G=g^{jk}G_{jk}$ and because of the essentially three-dimensional nature of the tensors involved, we first need to perform the summation over $j,k = 0,1,2$, and then we have to include the term involving G_{33}. Further, each of the above expressions involve three types of terms: the terms involving the six non-zero components of the covariant curvature tensor R_{ijkm}, which are R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012}, and then the two terms $-R^*g_{jk}/2$ and $g^{33}R_{3jk3}$.

First, we examine the contribution to $G=g^{ik}G_{jk}$ arising from the six terms R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012}, and for purposes of illustration, we examine the contribution from the term R_{0101}, which becomes

$$(g^{00}g_{02}g_{02}+g^{11}g_{12}g_{12}+g^{22}g_{22}g_{22}+2g^{01}g_{02}g_{12}$$

$$+ 2g^{02}g_{02}g_{22} + 2g^{12}g_{12}g_{22})\frac{R_{0101}}{g},$$

which we may reorganise as

$$[(g_{02}(g^{00}g_{02} + g^{01}g_{12} + g^{02}g_{22}) + g_{12}(g^{01}g_{02} + g^{11}g_{12} + g^{12}g_{22})$$
$$+ g_{22}(g^{02}g_{02} + g^{12}g_{12} + g^{22}g_{22})]\frac{R_{0101}}{g},$$

so that on using $g^{ik}g_{kj} = \delta^i_j$, this term becomes simply $g_{22}R_{0101}/g$. Similarly, we may show that the contribution arising from the six terms R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012} becomes

$$(g_{00}R_{1212} + g_{11}R_{0202} + g_{22}R_{0101} + 2g_{12}R_{0120} + 2g_{02}R_{1021} + 2g_{01}R_{2012})/g,$$

which from (11.39) we can identify as simply $(R^* - R)/2$. Thus, the total summation becomes

$$G = g^{jk}G_{jk} = \frac{(R^* - R)}{2} - \frac{R^*}{2}g^{jk}g_{jk} + g^{33}g^{jk}R_{3jk3} + g^{33}G_{33},$$
$$= \frac{(R^* - R)}{2} - \frac{3R^*}{2} + \frac{R^*}{2} + \frac{(R^* - R)}{2} = -R,$$

as required.

Although not particularly illuminating or insightful, with i, j and k all distinct and taking on only the values 0, 1 and 2, and with no summation over repeated indices, these formulae can be formally summarised by the following expressions:

$$G_{ii} = \frac{g_{ij}g_{ij}}{g}R_{kiki} + \frac{g_{ik}g_{ik}}{g}R_{ijij} + \frac{g_{ii}g_{ii}}{g}R_{kjkj} + \frac{2g_{ij}g_{ik}}{g}R_{kiij}$$
$$+ \frac{2g_{ii}g_{ij}}{g}R_{ikkj} + \frac{2g_{ii}g_{ik}}{g}R_{jkij} - \frac{R^*}{2}g_{ii} + g^{33}R_{3ii3},$$
$$G_{ij} = \frac{g_{ik}g_{jk}}{g}R_{ijij} + \frac{g_{ij}g_{ii}}{g}R_{kjkj} + \frac{g_{ij}g_{jj}}{g}R_{kiki} + \frac{(g_{ii}g_{jj} + g_{ij}g_{ij})}{g}R_{kijk}$$
$$+ \frac{(g_{ki}g_{jj} + g_{ij}g_{kj})}{g}R_{kiij} + \frac{(g_{kj}g_{ii} + g_{ij}g_{ki})}{g}R_{kjji} - \frac{R^*}{2}g_{ij} + g^{33}R_{3ij3}.$$

11.7 Two Illustrative Line Elements

Illustration (i) As an illustration of these results, we consider the simple wormhole geometry of [79], which is given by the four-dimensional line element, thus

$$ds^2 = (cdt)^2 - (d\rho)^2 - (b^2 + \rho^2)(d\theta)^2 - (b^2 + \rho^2)\sin^2\theta(d\phi)^2,$$

where b is the throat radius and (ρ, θ, ϕ) are the usual spherical polar coordinates, so that in this case, we have $(x^0, x^1, x^2, x^3) = (ct, \rho, \theta, \phi)$ and the metric and conjugate metric tensors are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -(b^2+\rho^2) & 0 \\ 0 & 0 & 0 & -(b^2+\rho^2)\sin^2\theta \end{pmatrix},$$

and

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{(b^2+\rho^2)} & 0 \\ 0 & 0 & 0 & -\frac{1}{(b^2+\rho^2)\sin^2\theta} \end{pmatrix},$$

where the determinants $g^* = -(b^2 + \rho^2)^2 \sin^2\theta$ and $g = (b^2 + \rho^2)$. The non-zero Christoffel symbols of the first kind are

$$[22, 1] = \rho, \quad [12, 2] = -\rho, \quad [33, 1] = \rho\sin^2\theta, \quad [31, 3] = -\rho\sin^2\theta,$$
$$[33, 2] = (b^2 + \rho^2)\sin\theta\cos\theta, \qquad [32, 3] = -(b^2 + \rho^2)\sin\theta\cos\theta,$$

and from these relations and Eq. (11.25), we may deduce the following non-zero components of the covariant curvature tensor R_{ijkm}, thus

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^{1\,2}} + g^{22}[12, 2][12, 2] = 1 - \frac{\rho^2}{(b^2+\rho^2)} = \frac{b^2}{(b^2+\rho^2)},$$
$$R_{3113} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{1\,2}} - g^{33}[13, 3][13, 3] = -\sin^2\theta + \frac{\rho^2\sin^2\theta}{(b^2+\rho^2)} = -\frac{b^2\sin^2\theta}{(b^2+\rho^2)},$$
$$R_{3223} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{2\,2}} + g^{11}[22, 1][33, 1] - g^{33}[23, 3][23, 3]$$
$$= (b^2 + \rho^2)(\sin^2\theta - \cos^2\theta) - \rho^2\sin^2\theta + (b^2 + \rho^2)\cos^2\theta = b^2\sin^2\theta.$$

The latter two components result in a net zero contribution to the curvature invariant arising from the x^3 spatial direction, since

$$R^* = 2g^{33}(g^{11}R_{3113} + g^{22}R_{3223}) = \frac{2}{(b^2+\rho^2)\sin^2\theta}\left(-\frac{b^2\sin^2\theta}{(b^2+\rho^2)} + \frac{b^2\sin^2\theta}{(b^2+\rho^2)}\right) = 0,$$

so that only the component R_{1212} contributes to the curvature invariant R through the term $-2g_{00}R_{1212}/g$, and we have simply $R = -2b^2/(b^2+\rho^2)^2$.

From the relations (11.40), we find that the only non-zero components of the Einstein tensor G_{ij} become

$$G_{00} = \frac{g_{00}g_{00}}{g}R_{1212} = \frac{b^2}{(b^2+\rho^2)^2}, \qquad G_{11} = g^{33}R_{3113} = \frac{b^2}{(b^2+\rho^2)^2},$$

$$G_{22} = g^{33}R_{3223} = -\frac{b^2}{(b^2+\rho^2)}, \qquad G_{33} = -\frac{Rg_{33}}{2} = -\frac{b^2\sin^2\theta}{(b^2+\rho^2)},$$

where G_{33} originates from the expression $G_{33} = R_{33} - Rg_{33}/2 = (R^* - R)g_{33}/2$ with $R^* = 0$ and noting that correctly $G = g^{ij}G_{ij} = -R$. On noting that the only non-zero Christoffel symbols of the second kind are

$$\Gamma^2_{12} = \frac{\rho}{(b^2+\rho^2)}, \qquad \Gamma^3_{13} = \frac{\rho}{(b^2+\rho^2)}, \qquad \Gamma^1_{22} = \rho,$$

$$\Gamma^1_{33} = -\rho\sin^2\theta, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3_{23} = \cot\theta,$$

the above expressions for the Einstein tensor can be shown to satisfy the divergence free equations $G^i_{k;i} = 0$ in the form of Eqs. (11.35); thus, we have

$$G^i_{k;i} = g^{ij}G_{jk;i} = g^{ij}\left\{\frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij}G_{nk} - \Gamma^p_{ik}G_{jp}\right\} = 0,$$

and for $k = 1$ we obtain

$$\begin{aligned}
G^i_{k;i} &= g^{ij}\left\{\frac{\partial G_{j1}}{\partial x^i} - \Gamma^n_{ij}G_{n1} - \Gamma^p_{i1}G_{jp}\right\} \\
&= \left\{-\frac{\partial G_{11}}{\partial \rho} - g^{ij}\Gamma^1_{ij}G_{11} - g^{ij}\Gamma^2_{i1}G_{j2} - g^{ij}\Gamma^3_{i1}G_{j3}\right\} \\
&= -\left\{\frac{\partial G_{11}}{\partial \rho} + g^{22}\Gamma^1_{22}G_{11} + g^{33}\Gamma^1_{33}G_{11} + g^{22}\Gamma^2_{21}G_{22} + g^{33}\Gamma^3_{31}G_{33}\right\} \\
&= -\left\{\frac{\partial G_{11}}{\partial \rho} + \frac{2\rho}{(b^2+\rho^2)}G_{11} - \frac{\rho}{(b^2+\rho^2)^2}\left(G_{22} + \frac{G_{33}}{\sin^2\theta}\right)\right\},
\end{aligned}$$

which on performing the differentiation can be shown to be identically zero, and similarly the three equations $G^i_{k;i} = 0$ arising from $k = 0, 2$ and 3 can be shown to be trivially satisfied. The reader may be comforted to know that these results are not completely apparent and are consequences of the particular diagonal structure of the tensor G_{ij} and the particular non-zero Christoffel symbols of the second kind and the reader may need to give each equation a close examination before being convinced.

Illustration (ii) As a second illustration of the results of the previous section, we consider the Einstein universe given by the four-dimensional line element, thus

$$ds^2 = (cdt)^2 - \frac{(dr)^2}{(1-(r/\mathscr{R})^2)} - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2,$$

where $\mathscr{R}$ is a constant and (r,θ,ϕ) are the usual spherical polar coordinates, so that in this case, we have $(x^0,x^1,x^2,x^3)=(ct,r,\theta,\phi)$ and the metric and conjugate metric tensors are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1}{(1-(r/\mathscr{R})^2)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},$$

and

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1-(r/\mathscr{R})^2) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix},$$

and for the determinants g^* and g, we have $g^* = -r^4\sin^2\theta/(1-(r/\mathscr{R})^2)$ and $g = r^2/(1-(r/\mathscr{R})^2)$. The non-zero Christoffel symbols of the first kind are

$$[11,1] = -\frac{r/\mathscr{R}^2}{(1-(r/\mathscr{R})^2)^2}, \quad [12,2] = -r, \quad [22,1] = r, \quad [33,1] = r\sin^2\theta,$$

$$[31,3] = -r\sin^2\theta, \quad [33,2] = r^2\sin\theta\cos\theta, \qquad [32,3] = -r^2\sin\theta\cos\theta,$$

and from these relations and Eq. (11.25), we may deduce the following non-zero components of the covariant curvature tensor R_{ijkm}, thus

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^{1^2}} + g^{22}[12,2][12,2] - g^{11}[22,1][11,1] = -\frac{(r/\mathscr{R})^2}{(1-(r/\mathscr{R})^2)},$$

$$R_{3113} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{1^2}} + g^{11}[33,1][11,1] - g^{33}[13,3][13,3] = \frac{(r/\mathscr{R})^2\sin^2\theta}{(1-(r/\mathscr{R})^2)},$$

$$R_{3223} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{2^2}} + g^{11}[22,1][33,1] - g^{33}[23,3][23,3] = (r/\mathscr{R})^2 r^2\sin^2\theta,$$

$$R_{3123} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^1\partial x^2} + g^{22}[12,2][33,2] - g^{33}[13,3][23,3] = 0.$$

The two components R_{3113} and R_{3223} both contribute to the curvature invariant R^* arising from the x^3 spatial direction, thus

$$R^* = 2g^{33}(g^{11}R_{3113} + g^{22}R_{3223}) = \frac{2}{r^2\sin^2\theta}\left((1-(r/\mathscr{R})^2)R_{3113} + \frac{R_{3223}}{r^2}\right)$$
$$= \frac{2}{r^2\sin^2\theta}\left((r/\mathscr{R})^2\sin^2\theta + (r/\mathscr{R})^2\sin^2\theta\right) = \frac{4}{\mathscr{R}^2},$$

so that altogether along with the contribution to the curvature invariant R from the component R_{1212} we have from Eq. (11.39) the following result:

$$R = \frac{4}{\mathscr{R}^2} + \frac{2(1-(r/\mathscr{R})^2)}{r^2}\frac{(r/\mathscr{R})^2}{(1-(r/\mathscr{R})^2)} = \frac{6}{\mathscr{R}^2},$$

which is well-known.

From Eq. (11.40), we may show that the only non-zero components of the Einstein tensor G_{ij} become

$$G_{00} = \frac{g_{00}g_{00}}{g}R_{1212} - \frac{R^*}{2}g_{00} = -\frac{3}{\mathscr{R}^2},$$
$$G_{11} = -\frac{R^*}{2}g_{11} + g^{33}R_{3113} = \frac{1}{\mathscr{R}^2(1-(r/\mathscr{R})^2)},$$
$$G_{22} = -\frac{R^*}{2}g_{22} + g^{33}R_{3223} = (r/\mathscr{R})^2,$$
$$G_{33} = \frac{(R^*-R)}{2}g_{33} = (r/\mathscr{R})^2\sin^2\theta,$$

where G_{33} originates from the expression $G_{33} = R_{33} - Rg_{33}/2 = (R^* - R)g_{33}/2$, and again as a check, we may confirm that the above expressions correctly satisfy $G = g^{ij}G_{ij} = -6/\mathscr{R}^2 = -R$. For this metric, the only non-zero Christoffel symbols of the second kind are

$$\Gamma^1_{11} = \frac{r/\mathscr{R}^2}{(1-(r/\mathscr{R})^2)}, \quad \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^1_{22} = -r(1-(r/\mathscr{R})^2),$$
$$\Gamma^1_{33} = -r\sin^2\theta(1-(r/\mathscr{R})^2), \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta,$$

and again the above expressions for the Einstein tensor can be shown to satisfy the divergence free equation $G^i_{k;i} = 0$ in the form of Eqs. (11.35); thus, we have

$$G^i_{k;i} = g^{ij}G_{jk;i} = g^{ij}\left\{\frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij}G_{nk} - \Gamma^p_{ik}G_{jp}\right\} = 0,$$

so that for $k = 1$, we have

$$G^i_{k;i} = g^{ij}\left\{\frac{\partial G_{j1}}{\partial x^i} - \Gamma^n_{ij}G_{n1} - \Gamma^p_{i1}G_{jp}\right\}$$

$$= \left\{g^{11}\frac{\partial G_{11}}{\partial x^1} - g^{ij}\Gamma^1_{ij}G_{11} - g^{ij}\Gamma^1_{i1}G_{j1} - g^{ij}\Gamma^2_{i1}G_{j2} - g^{ij}\Gamma^3_{i1}G_{j3}\right\}$$

$$= \left\{g^{11}\frac{\partial G_{11}}{\partial r} - (g^{11}\Gamma^1_{11} + g^{22}\Gamma^1_{22} + g^{33}\Gamma^1_{33})G_{11} - g^{11}\Gamma^1_{11}G_{11} - g^{22}\Gamma^2_{21}G_{22} - g^{33}\Gamma^3_{31}G_{33}\right\},$$

and again on performing the differentiation and using the above expressions for the conjugate metric tensor and the Christoffel symbols, this equation may be shown to be identically zero, and similarly the three equations $G^i_{k;i} = 0$ arising from $k = 0, 2$ and 3 can be shown to be trivially satisfied. Again, we remind the reader that none of these outcomes are immediately obvious, and in formulating each of the results, we constantly have in mind the particular diagonal structures of the tensors g^{ij} and G_{ij} and the particular non-zero Christoffel symbols of the second kind, and the reader should give each outcome a close examination to convince themselves of its veracity. In the final section of this chapter, we provide an example for which the details are far more complicated.

11.8 Spiral Gravitating Structures

In this section, we examine a possible cosmological model of general relativity involving logarithmic spirals and seven arbitrary constants. While there is considerable astronomical evidence that spiral structures commonly occur in the universe, there appear to be no known formal solutions of the general relativistic field equations reflecting such structures. The Schwarzschild solution relates to spatially spherically symmetric gravitational fields, while the stationary Kerr solution is axially symmetric. However, in nonlinear continuum mechanics, formal similarity solutions involving logarithmic spiral similarity invariants are well-known, and indeed the most general similarity forms for isotropic materials satisfying the principle of material indifference have been given by the author [46]. These formal similarity solutions essentially arise from rotational invariances and invariance under the basic length scale that is adopted. We generalise the particular Minkowski line element given by (11.42) to the line element given by Eq. (11.41), which involves the seven completely arbitrary constants A, B, C, D, E, F and λ. The motivating Minkowski line element (11.42) is a special case of (11.41) that arises from the particular values of the constants A, B, C, D, E, F and λ given by (11.43), which satisfy the three relations (11.44).

Below we show that solutions of the Einstein vacuum equations that might involve logarithmic spirals as a similarity variable arise from the essentially two-dimensional (spatially) metric given by

$$(ds)^2 = (1 - \lambda r^2)(cdt)^2 - A(dr)^2 - Br^2(d\theta)^2 - 2Crdrcdt \tag{11.41}$$
$$- 2Dr^2 d\theta cdt - 2Erdrd\theta - F(dz)^2,$$

where $(x^0, x^1, x^2, x^3) = (ct, r, \theta, z)$, (r, θ, z) denote the usual cylindrical polar coordinates and where A, B, C, D, E, F and λ denote seven arbitrary constants, in principle, to be determined from the general relativistic field equations. We provide a detailed verification below that the Einstein tensor for the metric (11.41) is indeed divergence free for all values of the seven constants without additional restrictions.

From a conventional general relativistic perspective, only five of these constants would be regarded as essential, since on face value the constants A and F can be scaled immediately to unity, even though the special case $A = 0$ would have to be considered as a separate case. From the author's perspective, setting $A = F = 1$, there is an apparent loss of symmetry and parity in the structure, and moreover as shown in [59], reducing and simplifying the metric in this manner often closes off opportunities to determine new solutions. We demonstrate below that the metric (11.41) provides a formal solution of the divergence free vacuum field equations for all values of the seven constants without additional restrictions. We comment that while the two illustrative examples presented above are both such that the Einstein tensor satisfies the divergence free vacuum condition, it is however non-vanishing, and therefore, these models are considered to represent some underlying intrinsic property of space-time that is additional to that generated from the energy-momentum tensor. It is important to point out that the model examined here also shares this characteristic, and formally here, we simply verify the vanishing of the divergence of the Einstein tensor.

The particular structure of this metric is motivated as follows. In the cylindrical polar coordinates (r, θ, z), a rotation around the $z-$axis is given simply by $\theta^{'} = \theta + \omega t$, and therefore in order generalise this to generate logarithmic spirals, we consider the invariant $\xi = \sigma\theta + \mu \log r + \omega t$ and the generalised rotational and stretching transformation which in rectangular Cartesian coordinates (x, y, z) is given by

$$x = \nu r \cos(\sigma\theta + \mu \log r + \omega t), \qquad y = \nu r \sin(\sigma\theta + \mu \log r + \omega t),$$

combined with a constant stretch along the $z-$axis, thus $z \longrightarrow \kappa z$, where σ, μ, ν, ω and κ all denote arbitrary constants. From the differentials

$$dx = \nu \cos\xi dr - \nu r \sin\xi \left(\sigma d\theta + \frac{\mu}{r} dr + \omega dt\right),$$
$$dy = \nu \sin\xi dr + \nu r \cos\xi \left(\sigma d\theta + \frac{\mu}{r} dr + \omega dt\right),$$

the Minkowski line element becomes

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{11.42}$$

$$= (1 - (\nu\omega r/c)^2)(cdt)^2 - \nu^2(1+\mu^2)(dr)^2 - (\nu\sigma r)^2(d\theta)^2 - 2(\mu\omega\nu^2/c)rdrcdt$$
$$- 2(\sigma\omega\nu^2/c)r^2 d\theta cdt - 2\mu\sigma\nu^2 rdrd\theta - \kappa^2(dz)^2,$$

which evidently assumes the form of the line element (11.41) with the seven arbitrary constants A, B, C, D, E, F and λ taking on the particular valiues

$$A = \nu^2(1+\mu^2), \quad B = (\nu\sigma)^2, \quad C = \mu\omega\nu^2/c, \tag{11.43}$$
$$D = \sigma\omega\nu^2/c, \quad E = \mu\sigma\nu^2, \quad F = \kappa^2, \quad \lambda = (\nu\omega/c)^2.$$

We emphasise that subsequently we regard the seven constants A, B, C, D, E, F and λ to be completely arbitrary, and we exploit the above particular values rather as reference values to guide the ensuring mathematical analysis. Specifically, we observe that these particular values satisfy the three relations,

$$BC - DE = 0, \qquad CD - \lambda E = 0, \qquad D^2 - \lambda B = 0, \tag{11.44}$$

and we subsequently observe these quantities emerging as factors in the analysis. We further comment that in proposing the metric (11.41) as a potentially interesting line element, we have in mind certain issues arising from nonlinear continuum mechanics. If we have in mind a spiral gravitating structure, then note that the invariant $\xi = \sigma\theta + \mu \log r + \omega t$ arises from both the two- and three-dimensional spatial rotational groups and that use of this invariant has two important features. Firstly, the constant $\sigma \neq 1$ embodies more than a strict rotation and might well on its own produce an interesting outcome. Secondly, the $\log r$ term is a natural amendment, since it behaves formally like θ in the sense that the factor $1/r$ precedes both $\partial/\partial\theta$ and $\partial/\partial r$. In finite elasticity, this characteristic is well-known and the invariant $\sigma\theta + \mu \log r$ forms the basis of an important exact solution in that discipline, and further references to this topic may be found in [46]. Accordingly, these two features in their own right might generate interesting and new outcomes, but here in proposing the line element (11.41), we are adding further levels of arbitrariness.

Again noting that (r, θ, z) are the usual cylindrical polar coordinates and $(x^0, x^1, x^2, x^3) = (ct, r, \theta, z)$, then corresponding to (11.41) the metric tensor g_{ij} and conjugate metric tensor g^{ij} are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1-\lambda r^2 & -Cr & -Dr^2 & 0 \\ -Cr & -A & -Er & 0 \\ -Dr^2 & -Er & -Br^2 & 0 \\ 0 & 0 & 0 & -F \end{pmatrix}, \tag{11.45}$$

and

$$g^{ij} = \begin{pmatrix} (AB - E^2)r^2/g & -(BC - DE)r^3/g & -(AD - CE)r^2/g & 0 \\ -(BC - DE)r^3/g & -[Br^2 + (D^2 - \lambda B)r^4]/g & [Er + (DC - \lambda E)r^3]/g & 0 \\ -(AD - CE)r^2/g & [Er + (DC - \lambda E)r^3]/g & -[A + (C^2 - \lambda A)r^2]/g & 0 \\ 0 & 0 & 0 & -1/F \end{pmatrix}, \tag{11.46}$$

where the determinant g of the three-dimensional matrix g_{ij} is given by

$$\begin{aligned} g &= (1 - \lambda r^2)(AB - E^2)r^2 + (BC^2 - 2CDE + AD^2)r^4 \\ &= (AB - E^2)r^2 + [(BC - DE)^2 + (AB - E^2)(D^2 - \lambda B)]r^4/B, \end{aligned} \tag{11.47}$$

while the determinant g^* of the four-dimensional matrix g_{ij} is simply given by $g^* = -Fg$.

Christoffel Symbols of First Kind $[ij, k]$ The only non-zero Christoffel symbols of the first kind for the metric (11.41) are

$$[00, 1] = \lambda r, \quad [11, 0] = -C, \quad [11, 2] = -E, \quad [22, 1] = Br, \quad [01, 0] = -\lambda r, \tag{11.48}$$

$$[01, 2] = -Dr, \quad [12, 2] = -Br, \quad [02, 1] = Dr, \quad [12, 0] = -Dr.$$

Covariant Curvature Tensor R_{ijkm} From the relations (11.48) and Eq. (11.25), we may deduce the following non-zero components of the covariant curvature tensor R_{ijkm}, thus

$$R_{0101} = -\frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^{1^2}} + g^{np}([01, n][01, p] - [11, n][00, p])$$

$$= \lambda + g^{0p}[01, 0][01, p] + g^{2p}[01, 2][01, p] - g^{n1}[11, n][00, 1]$$

$$\begin{aligned} &= \lambda + g^{00}[01, 0]^2 + 2g^{02}[01, 0][01, 2] + g^{22}[01, 2]^2 \\ &\quad - g^{01}[11, 0][00, 1] - g^{12}[11, 2][00, 1] \end{aligned}$$

$$= -\frac{r^2}{g}[A(D^2 - \lambda B) + (DC - \lambda E)^2 r^2],$$

$$R_{0202} = g^{np}([02, n][02, p] - [22, n][00, p])$$

$$= g^{1p}([02, 1][02, p] - [22, 1][00, p])$$

$$= -\frac{r^4}{g}[B + (D^2 - \lambda B)r^2](D^2 - \lambda B),$$

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^{1^2}} + g^{np}([12, n][12, p] - [22, n][11, p])$$

$$= B + g^{00}[12, 0]^2 + g^{22}[12, 2]^2 + 2g^{02}[12, 0][12, 2] \\ - g^{01}[22, 1][11, 0] - g^{12}[22, 1][11, 2]$$

$$= -\frac{r^4}{g}(BC - DE)^2,$$

$$R_{2012} = g^{np}([01, n][22, p] - [02, n][12, p])$$

$$= g^{n1}([01, n][22, 1] - g^{1p}[02, 1][12, p])$$

$$= g^{01}[01, 0][22, 1] + g^{21}[01, 2][22, 1] - g^{01}[02, 1][12, 0] - g^{12}[02, 1][12, 2]$$

$$= -\frac{r^5}{g}(BC - DE)(D^2 - \lambda B),$$

$$R_{0112} = \frac{1}{2}\frac{\partial^2 g_{02}}{\partial x^{1^2}} + g^{np}([11, n][02, p] - [12, n][01, p])$$

$$= -D + g^{10}[02, 1][11, 0] + g^{21}[02, 1][11, 2] - g^{00}[12, 0][01, 0] \\ - g^{20}[12, 2][01, 0] - g^{02}[12, 0][01, 2] - g^{22}[12, 2][01, 2]$$

$$= \frac{r^4}{g}(BC - DE)(DC - \lambda E),$$

$$R_{0120} = g^{np}([12, n][00, p] - [01, n][02, p])$$

$$= g^{n1}([12, n][00, 1] - [01, n][02, 1])$$

$$= g^{01}[12, 0][00, 1] + g^{21}[12, 2][00, 1] - g^{01}[01, 0][02, 1] - g^{21}[01, 2][02, 1]$$

$$= \frac{r^3}{g}[E + (DC - \lambda E)r^2](D^2 - \lambda B),$$

We now introduce five new constants α, β, γ, δ and ϵ defined by

$$\alpha = BC - DE, \qquad \beta = D^2 - \lambda B, \qquad \gamma = \lambda E - DC,$$
$$\delta = AB - E^2, \qquad \epsilon = AD - CE,$$

so that the above expressions for $R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120}$ and R_{2012}, simplify to become

$$R_{0101} = -\frac{r^2}{g}(\beta A + \gamma^2 r^2), \qquad R_{0202} = -\frac{r^4}{g}(\beta B + \beta^2 r^2), \qquad R_{1212} = -\frac{r^4}{g}\alpha^2, \tag{11.49}$$

$$R_{2012} = -\frac{r^5}{g}\alpha\beta, \qquad R_{0112} = -\frac{r^4}{g}\alpha\gamma, \qquad R_{0120} = -\frac{r^3}{g}(\beta\gamma r^2 - \beta E).$$

We may readily verify that the new constants satisfy

$$\alpha\lambda + \beta C + \gamma D = 0, \qquad \alpha D + \beta E + \gamma B = 0,$$
$$\alpha C + \beta A + \gamma E = AD^2 + BC^2 - 2CDE - \lambda(AB - E^2),$$

and it proves convenient to introduce yet another new constant Ω defined by

$$\Omega = \alpha C + \beta A + \gamma E,$$

so that we have the important relations

$$AD^2 + BC^2 - 2CDE = \Omega + \lambda\delta, \qquad \Omega = (\alpha^2 + \beta\delta)/B,$$

and, noting that in terms of these new constants, the determinant g given by Eq. (11.47) becomes

$$g = \delta r^2 + \Omega r^4 = \delta r^2 + (\alpha^2 + \beta\delta) r^4 / B.$$

Curvature Invariant R On using the immediately above relations together with Eq. (11.39) and the metric tensor (11.45), we may deduce the following expression for the curvature invariant R, thus

$$R = \frac{2r^4}{g^2}\left\{(\alpha^2 - 2\beta\delta) - r^2(\lambda\alpha^2 + A\beta^2 + B\gamma^2 + 2C\alpha\beta + 2D\alpha\gamma + 2E\beta\gamma)\right\}.$$

We may simplify the coefficient of r^2 in this expression as follows:

$$\begin{aligned} &\lambda\alpha^2 + A\beta^2 + B\gamma^2 + 2C\alpha\beta + 2D\alpha\gamma + 2E\beta\gamma \\ &= \alpha(\alpha\lambda + \beta C + \gamma D) + \beta(\alpha C + \beta A + \gamma E) + \gamma(\alpha D + \beta E + \gamma B) \\ &= \beta(\alpha C + \beta A + \gamma E) = \beta\Omega, \end{aligned}$$

and therefore, the above expression for the curvature invariant R becomes simply

$$R = \frac{2r^4}{g^2}\left\{(\alpha^2 - 2\beta\delta) - \beta\Omega r^2\right\}.$$

Einstein Tensor G_{ij} Using the expressions (11.49) for the non-zero components of the covariant curvature tensor R_{ijkm}, the components of the metric tensor (11.45), we may deduce directly from the relations (11.40) the following components of the Einstein tensor

$$G_{00} = -\frac{r^4}{g^2}\left\{\alpha^2 + \beta(AD^2 + BC^2 - 2CDE)r^2\right\} = -\frac{r^4}{g^2}\left\{\alpha^2 + \beta(\Omega + \lambda\delta)r^2\right\}, \tag{11.50}$$

$$G_{11} = -\frac{r^4}{g^2}\left\{\beta A(AB - E^2) + (\alpha C + \beta A + \gamma E)^2 r^2\right\} = -\frac{r^4}{g^2}(\beta\delta A + \Omega^2 r^2), \tag{11.51}$$

$$G_{22} = -\frac{r^6}{g^2}\left\{\beta B(AB - E^2) + (\alpha D + \beta E + \gamma B)^2 r^2\right\} = -\frac{r^6 \beta\delta B}{g^2}, \tag{11.52}$$

$$G_{01} = -\frac{r^5}{g^2}\left\{\beta C(AB - E^2) - \alpha(\alpha C + \beta A + \gamma E)\right\} = -\frac{r^5}{g^2}(\beta\delta C - \alpha\Omega), \tag{11.53}$$

$$G_{02} = -\frac{r^6}{g^2}\beta D(AB - E^2) = -\frac{r^6\beta\delta D}{g^2}, \quad G_{12} = -\frac{r^5}{g^2}\beta E(AB - E^2) = -\frac{r^5\beta\delta E}{g^2}, \tag{11.54}$$

noting that in this case $G_{33} = -(R/2)g_{33}$. We observe that these components can be alternatively expressed as

$$\begin{aligned}
G_{00} &= \frac{\beta\delta r^4}{g^2}g_{00} - \frac{r^4}{g^2}\left\{\alpha^2 + \beta(\delta + \Omega r^2)\right\}, \qquad G_{11} = \frac{\beta\delta r^4}{g^2}g_{11} - \frac{r^6}{g^2}\Omega^2,\\
G_{22} &= \frac{\beta\delta r^4}{g^2}g_{22}, \qquad G_{33} = \frac{\beta\delta r^4}{g^2}g_{33} - \frac{r^4}{g^2}\left\{\alpha^2 - \beta(\delta + \Omega r^2)\right\},\\
G_{01} &= \frac{\beta\delta r^4}{g^2}g_{01} + \frac{\alpha\Omega r^5}{g^2}, \qquad G_{02} = \frac{\beta\delta r^4}{g^2}g_{02}, \qquad G_{12} = \frac{\beta\delta r^4}{g^2}g_{12}.
\end{aligned}$$

We further observe that in the event $\Omega = 0$, then $\alpha^2 = -\beta\delta$ and these expressions simplify to become

$$G_{ij} = \frac{\beta}{\delta}g_{ij}, \quad (i, j = 0, 1, 2), \qquad G_{33} = \frac{3\beta}{\delta}g_{33},$$

and $R = -6\beta/\delta$, and we note that these equations are slightly reminiscent of Einstein's cosmological constant Λ for which $G_{ij} = -\Lambda g_{ij}$, but of course here the constant takes on two distinct values.

Check on Einstein Tensor We may apply a reasonably robust check on the veracity of these individual expressions by evaluating the trace of the Einstein tensor $G_k^k = g^{ij}G_{ij} + g^{33}G_{33}$ which should give $-R$. Using the above equations for G_{ij} and (11.46) for the components of the conjugate metric tensor, which in terms of the new constants α, β, γ, δ and ϵ become

$$g^{ij} = \begin{pmatrix} \delta r^2/g & -\alpha r^3/g & -\epsilon r^2/g & 0\\ -\alpha r^3/g & -(B + \beta r^2)r^2/g & -(\gamma r^2 - E)r/g & 0\\ -\epsilon r^2/g & -(\gamma r^2 - E)r/g & -[A + (C^2 - \lambda A)r^2]/g & 0\\ 0 & 0 & 0 & -1/F \end{pmatrix},$$

and after some algebra and simplification, the contribution $g^{ij}G_{ij}$ admits g as a factor and simplifies as follows:

$$
\begin{aligned}
g^{ij}G_{ij} &= \frac{r^6}{g^3}\left\{\delta(2\beta\delta-\alpha^2)+[B\Omega+2(\beta\delta-\alpha^2)]\Omega r^2+\beta\Omega^2 r^4\right\}\\
&= \frac{r^6}{g^3}\left\{\delta(2\beta\delta-\alpha^2)+(3\beta\delta-\alpha^2)\Omega r^2+\beta\Omega^2 r^4\right\}\\
&= \frac{r^6}{g^3}(\delta+\Omega r^2)(2\beta\delta-\alpha^2+\beta\Omega r^2)\\
&= \frac{r^4}{g^2}(2\beta\delta-\alpha^2+\beta\Omega r^2)=-\frac{R}{2},
\end{aligned}
$$

where we have used $B\Omega = \alpha^2 + \beta\delta$ and $g = r^2(\delta + \Omega r^2)$, and the required result follows since $G_{33} = -(R/2)g_{33}$.

Christoffel Symbols of First Kind Γ^k_{ij} The non-zero Christoffel symbols of the second kind are

$$
\begin{aligned}
&\Gamma^0_{00} = -\frac{\alpha\lambda r^4}{g}, \quad \Gamma^1_{00} = -\frac{\lambda r^3}{g}(\beta r^2 + B), \quad \Gamma^2_{00} = -\frac{\lambda r^2}{g}(\gamma r^2 - E),\\
&\Gamma^0_{11} = -\frac{\alpha A r^2}{g}, \quad \Gamma^1_{11} = \frac{r}{g}\left\{(\alpha C + \gamma E)r^2 - E^2\right\}, \quad \Gamma^2_{11} = -\frac{A}{g}(\gamma r^2 - E),\\
&\Gamma^0_{22} = -\frac{\alpha B r^4}{g}, \quad \Gamma^1_{22} = -\frac{B r^3}{g}(\beta r^2 + B), \quad \Gamma^2_{22} = -\frac{B r^2}{g}(\gamma r^2 - E),\\
&\Gamma^0_{01} = \frac{r^3}{g}(\beta A + \gamma E), \quad \Gamma^1_{01} = -\frac{r^2}{g}(\beta C r^2 + DE), \quad \Gamma^2_{01} = -\frac{r}{g}(\gamma C r^2 - AD),\\
&\Gamma^0_{02} = -\frac{\alpha D r^4}{g}, \quad \Gamma^1_{02} = -\frac{D r^3}{g}(\beta r^2 + B), \quad \Gamma^2_{02} = -\frac{D r^2}{g}(\gamma r^2 - E),\\
&\Gamma^0_{12} = -\frac{\alpha E r^3}{g}, \quad \Gamma^1_{12} = -\frac{E r^2}{g}(\beta r^2 + B), \quad \Gamma^2_{12} = \frac{r}{g}\left\{(\alpha C + \beta A)r^2 + AB\right\},
\end{aligned}
$$

and we might provide a limited check on these expressions with the well-known formula $\Gamma^i_{ki} = (1/2g)\partial g/\partial x^k$, and for $k = 0, 1, 2$, we obtain

$$
\begin{aligned}
\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} &= -\frac{r^4}{g}(\alpha\lambda + \beta C + \gamma D) = 0,\\
\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} &= \frac{r}{g}\left\{\delta + 2(\alpha C + \beta A + \gamma E)r^2\right\} = \frac{1}{2g}\frac{\partial g}{\partial r},\\
\Gamma^0_{20} + \Gamma^1_{21} + \Gamma^2_{22} &= -\frac{r^4}{g}(\alpha D + \beta E + \gamma B) = 0,
\end{aligned}
$$

as required.

Verification Einstein Tensor Is Divergence Free Although the Einstein tensor is purposely constructed to be divergence free, it is often a nontrivial matter to convince oneself that this is indeed the case, and the following provides a demonstration of the underlying complexities and detail that are often required. In order to verify that the metric (11.41) produces a divergence free Einstein tensor, that is, the equations $G^i_{j;i} = 0$ are indeed satisfied, we proceed as follows. Using the relations $G^i_k = g^{ij}G_{jk}$, we need the partial covariant derivative with respect to x^m, thus

$$G^i_{k;m} = g^{ij}G_{jk;m} = g^{ij}\left\{\frac{\partial G_{jk}}{\partial x^m} - \Gamma^n_{mj}G_{nk} - \Gamma^p_{mk}G_{jp}\right\}, \tag{11.55}$$

which we contract; that is, we set $m = i$ and then sum over i to obtain

$$G^i_{k;i} = g^{ij}G_{jk;i} = g^{ij}\left\{\frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij}G_{nk} - \Gamma^p_{ik}G_{jp}\right\} = 0. \tag{11.56}$$

It is clear from these latter relations that for the problem on hand, the summations in each equation generates a large number of terms, actually 57 separate terms in all for each of the equations arising from $k = 0, 1, 2$. We observe that while the scale of this problem may be greatly reduced with the observation that the above particular expressions for the Einstein tensor G_{ij} have the structure

$$G_{ij} = \beta\delta\psi^2 g_{ij} + h_{ij}, \tag{11.57}$$

where $\psi(r)$ is a function of r only that is defined by $\psi(r) = r^2/g$ and h_{ij} is the tensor defined by

$$h_{ij} = \begin{pmatrix} -\psi(\beta+\alpha^2\psi) & \alpha\psi^2\Omega r & 0 & 0 \\ \alpha\psi^2\Omega r & -(\psi\Omega r)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we do not however follow this approach, since there is an apparent lack of parity in the calculation, and it turns out to be more straightforward to deal with the full equations directly.

However, if we were to adopt this approach, then it would seem worthwhile noting that the field equations might be manipulated as follows. On observing again that all the partial covariant derivatives of the metric tensor g_{ij} and its conjugate g^{ij} are zero and that the partial covariant derivative of a scalar is simply the partial derivative, we have from (11.55) and (11.57)

$$G^i_{k;m} = \beta\delta g^{ij}g_{jk}\frac{\partial\psi^2}{\partial x^m} + g^{ij}\left\{\frac{\partial h_{jk}}{\partial x^m} - \Gamma^n_{mj}h_{nk} - \Gamma^p_{mk}h_{jp}\right\},$$

which becomes

$$G^i_{k;m} = \beta\delta\delta^i_k \frac{\partial \psi^2}{\partial x^m} + g^{ij}\left\{\frac{\partial h_{jk}}{\partial x^m} - \Gamma^n_{mj} h_{nk} - \Gamma^p_{mk} h_{jp}\right\}.$$

Further, since ψ is a function of $r = x^1$ only, on contraction the divergence free equations become

$$G^i_{k;i} = \beta\delta\delta^1_k \frac{\partial \psi^2}{\partial x^1} + g^{ij}\left\{\frac{\partial h_{jk}}{\partial x^i} - \Gamma^n_{ij} h_{nk} - \Gamma^p_{ik} h_{jp}\right\} = 0,$$

which are the alternative form for the three basic equations for $k = 0, 1, 2$ to be satisfied.

Proceeding directly, we observe that the three basic equations (11.56) for $k = 0, 1, 2$ become

$$g^{1j}\frac{\partial G_{jk}}{\partial x^1} = g^{ij}\Gamma^n_{ij} G_{nk} + g^{ij}\Gamma^p_{ik} G_{jp},$$

since all components G_{ij} are functions of $x^1 = r$ only, and so

$$\begin{aligned} & g^{10}\frac{\partial G_{0k}}{\partial x^1} + g^{11}\frac{\partial G_{1k}}{\partial x^1} + g^{12}\frac{\partial G_{2k}}{\partial x^1} \\ & = G_{0k} g^{ij}\Gamma^0_{ij} + G_{1k} g^{ij}\Gamma^1_{ij} + G_{2k} g^{ij}\Gamma^2_{ij} \\ & + g^{ij}\Gamma^0_{ik} G_{j0} + g^{ij}\Gamma^1_{ik} G_{j1} + g^{ij}\Gamma^2_{ik} G_{j2}, \end{aligned}$$

and the three basic equations (11.56) for $k = 0, 1, 2$ when written out in full are as given below.

Equation for $k = 0$ For $k = 0$, we have

$$\begin{aligned} & g^{10}\frac{\partial G_{00}}{\partial x^1} + g^{11}\frac{\partial G_{10}}{\partial x^1} + g^{12}\frac{\partial G_{20}}{\partial x^1} \\ & = G_{00} g^{ij}\Gamma^0_{ij} + G_{10} g^{ij}\Gamma^1_{ij} + G_{20} g^{ij}\Gamma^2_{ij} \\ & + g^{ij}\Gamma^0_{i0} G_{j0} + g^{ij}\Gamma^1_{i0} G_{j1} + g^{ij}\Gamma^2_{i0} G_{j2}. \end{aligned}$$

On writing this equation out in full, we have

$$\begin{aligned} & g^{10}\frac{\partial G_{00}}{\partial x^1} + g^{11}\frac{\partial G_{10}}{\partial x^1} + g^{12}\frac{\partial G_{20}}{\partial x^1} \qquad (11.58) \\ & = G_{00}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02}) + G_{00}(g^{01}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12}) \end{aligned}$$

$$
\begin{aligned}
&+ G_{00}(g^{02}\Gamma^0_{02} + g^{12}\Gamma^0_{12} + g^{22}\Gamma^0_{22}) + G_{10}(g^{00}\Gamma^1_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^2_{02}) \\
&+ G_{10}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{12}) + G_{10}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22}) \\
&+ G_{20}(g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02}) + G_{20}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12}) \\
&+ G_{20}(g^{02}\Gamma^2_{02} + g^{12}\Gamma^2_{12} + g^{22}\Gamma^2_{22}) + G_{00}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{10} + g^{02}\Gamma^0_{20}) \\
&+ G_{10}(g^{10}\Gamma^0_{00} + g^{11}\Gamma^0_{10} + g^{12}\Gamma^0_{20}) + G_{20}(g^{20}\Gamma^0_{00} + g^{21}\Gamma^0_{10} + g^{22}\Gamma^0_{20}) \\
&+ G_{01}(g^{00}\Gamma^1_{00} + g^{10}\Gamma^1_{10} + g^{20}\Gamma^1_{20}) + G_{11}(g^{01}\Gamma^1_{00} + g^{11}\Gamma^1_{10} + g^{21}\Gamma^1_{20}) \\
&+ G_{21}(g^{02}\Gamma^1_{00} + g^{12}\Gamma^1_{10} + g^{22}\Gamma^1_{20}) + G_{02}(g^{00}\Gamma^2_{00} + g^{10}\Gamma^2_{10} + g^{20}\Gamma^2_{20}) \\
&+ G_{12}(g^{01}\Gamma^2_{00} + g^{11}\Gamma^2_{10} + g^{12}\Gamma^2_{20}) + G_{22}(g^{02}\Gamma^2_{00} + g^{21}\Gamma^2_{10} + g^{22}\Gamma^2_{20}).
\end{aligned}
$$

Equation for $k = 1$ Similarly, the corresponding equation for $k = 1$ is

$$
\begin{aligned}
&g^{10}\frac{\partial G_{01}}{\partial x^1} + g^{11}\frac{\partial G_{11}}{\partial x^1} + g^{12}\frac{\partial G_{21}}{\partial x^1} \qquad (11.59) \\
&= G_{01}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02}) + G_{01}(g^{01}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12}) \\
&+ G_{01}(g^{02}\Gamma^0_{02} + g^{12}\Gamma^0_{12} + g^{22}\Gamma^0_{22}) + G_{11}(g^{00}\Gamma^1_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^2_{02}) \\
&+ G_{11}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{11}) + G_{11}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22}) \\
&+ G_{12}(g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02}) + G_{12}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12}) \\
&+ G_{12}(g^{02}\Gamma^2_{02} + g^{12}\Gamma^2_{12} + g^{22}\Gamma^2_{22}) + G_{00}(g^{00}\Gamma^0_{01} + g^{01}\Gamma^0_{11} + g^{02}\Gamma^0_{21}) \\
&+ G_{10}(g^{10}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{21}) + G_{20}(g^{20}\Gamma^0_{01} + g^{21}\Gamma^0_{11} + g^{22}\Gamma^0_{21}) \\
&+ G_{01}(g^{00}\Gamma^1_{01} + g^{10}\Gamma^1_{11} + g^{20}\Gamma^1_{21}) + G_{11}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{21}\Gamma^1_{21}) \\
&+ G_{21}(g^{02}\Gamma^1_{01} + g^{12}\Gamma^1_{11} + g^{22}\Gamma^1_{21}) + G_{02}(g^{00}\Gamma^2_{01} + g^{10}\Gamma^2_{11} + g^{20}\Gamma^2_{21}) \\
&+ G_{12}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{21}) + G_{22}(g^{02}\Gamma^2_{01} + g^{21}\Gamma^2_{11} + g^{22}\Gamma^2_{21}).
\end{aligned}
$$

Equation for $k = 2$ For $k = 2$, we have

$$
\begin{aligned}
&g^{10}\frac{\partial G_{02}}{\partial x^1} + g^{11}\frac{\partial G_{12}}{\partial x^1} + g^{12}\frac{\partial G_{22}}{\partial x^1} \qquad (11.60) \\
&= G_{02}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02}) + G_{02}(g^{01}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12}) \\
&+ G_{02}(g^{02}\Gamma^0_{02} + g^{12}\Gamma^0_{12} + g^{22}\Gamma^0_{22}) + G_{12}(g^{00}\Gamma^1_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^2_{02})
\end{aligned}
$$

$$+ G_{12}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{12}) + G_{12}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22})$$
$$+ G_{22}(g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02}) + G_{22}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12})$$
$$+ G_{22}(g^{02}\Gamma^2_{02} + g^{12}\Gamma^2_{12} + g^{22}\Gamma^2_{22}) + G_{00}(g^{00}\Gamma^0_{02} + g^{01}\Gamma^0_{12} + g^{02}\Gamma^0_{22})$$
$$+ G_{10}(g^{10}\Gamma^0_{02} + g^{11}\Gamma^0_{12} + g^{12}\Gamma^0_{22}) + G_{20}(g^{20}\Gamma^0_{02} + g^{21}\Gamma^0_{12} + g^{22}\Gamma^0_{22})$$
$$+ G_{01}(g^{00}\Gamma^1_{02} + g^{10}\Gamma^1_{12} + g^{20}\Gamma^1_{22}) + G_{11}(g^{01}\Gamma^1_{02} + g^{11}\Gamma^1_{12} + g^{21}\Gamma^1_{22})$$
$$+ G_{21}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22}) + G_{02}(g^{00}\Gamma^2_{02} + g^{10}\Gamma^2_{12} + g^{20}\Gamma^2_{22})$$
$$+ G_{12}(g^{01}\Gamma^2_{02} + g^{11}\Gamma^2_{12} + g^{12}\Gamma^2_{22}) + G_{22}(g^{02}\Gamma^2_{02} + g^{21}\Gamma^2_{12} + g^{22}\Gamma^2_{22}).$$

Christoffel Symbol Identities In order to confirm, the given expressions for the Einstein tensor G_{ij} given by Eqs. (11.50), (11.51), (11.52), (11.53) and (11.54) constitute a bonafide solution of these lengthy divergence equations by means of the following identities, which take the form $g^{ik}\Gamma^m_{ij}$ for fixed j, k and m and with summation over i, which are no doubt related to the particular case of this formula given by [15] (page 68), namely

$$g^{ij}\Gamma^k_{ij} = -\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}g^{km})}{\partial x^m}.$$

By direct individual calculations, we may confirm the following identities:

$$g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02} = 0, \qquad g^{00}\Gamma^1_{00} + g^{01}\Gamma^1_{01} + g^{02}\Gamma^1_{02} = \frac{\beta r^3}{g}, \qquad (11.61)$$
$$g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02} = \frac{\gamma r^2}{g}, \qquad g^{10}\Gamma^0_{10} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12} = \frac{\alpha\delta r^4}{g^2},$$
$$g^{10}\Gamma^1_{10} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{12} = -\frac{\alpha^2 r^5}{g^2}, \qquad g^{10}\Gamma^2_{10} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12} = -\frac{\alpha\epsilon r^4}{g^2},$$
$$g^{20}\Gamma^0_{20} + g^{21}\Gamma^0_{21} + g^{22}\Gamma^0_{22} = \frac{\alpha r^2}{g}, \qquad g^{20}\Gamma^1_{20} + g^{21}\Gamma^1_{21} + g^{22}\Gamma^1_{22} = (\beta r^2 + B)\frac{r}{g},$$
$$g^{20}\Gamma^2_{20} + g^{21}\Gamma^2_{21} + g^{22}\Gamma^2_{22} = 0, \qquad g^{01}\Gamma^0_{00} + g^{11}\Gamma^0_{10} + g^{21}\Gamma^0_{20} = -\frac{\beta r^3}{g},$$
$$g^{01}\Gamma^1_{00} + g^{11}\Gamma^1_{10} + g^{21}\Gamma^1_{20} = 0, \qquad g^{01}\Gamma^2_{00} + g^{11}\Gamma^2_{10} + g^{21}\Gamma^2_{20} = -\frac{Dr}{g},$$
$$g^{00}\Gamma^0_{01} + g^{01}\Gamma^0_{11} + g^{02}\Gamma^0_{21} = \frac{\Omega\delta r^5}{g^2}, \qquad g^{00}\Gamma^1_{01} + g^{01}\Gamma^1_{11} + g^{02}\Gamma^1_{21} = -\frac{\Omega\alpha r^6}{g^2},$$

$$g^{00}\Gamma^2_{01}+g^{01}\Gamma^2_{11}+g^{02}\Gamma^2_{21}=-\frac{\Omega\epsilon r^5}{g^2},\qquad g^{02}\Gamma^0_{00}+g^{12}\Gamma^0_{10}+g^{22}\Gamma^0_{20}=-\frac{\gamma r^2}{g},$$
$$g^{02}\Gamma^1_{00}+g^{12}\Gamma^1_{10}+g^{22}\Gamma^1_{20}=\frac{Dr}{g},\qquad g^{02}\Gamma^2_{00}+g^{12}\Gamma^2_{10}+g^{22}\Gamma^2_{20}=0,$$
$$g^{02}\Gamma^0_{01}+g^{12}\Gamma^0_{11}+g^{22}\Gamma^0_{21}=-\frac{\Omega\epsilon r^5}{g^2},$$
$$g^{02}\Gamma^1_{01}+g^{12}\Gamma^1_{11}+g^{22}\Gamma^1_{21}=(\delta+2\Omega r^2)\frac{Er^2}{g^2}-\frac{\Omega\gamma r^6}{g^2},$$
$$g^{02}\Gamma^2_{01}+g^{12}\Gamma^2_{11}+g^{22}\Gamma^2_{21}=-(\delta+2\Omega r^2)\frac{Ar}{g^2}-(C^2-\lambda A)\frac{\Omega r^5}{g^2},$$
$$g^{00}\Gamma^0_{02}+g^{10}\Gamma^0_{12}+g^{20}\Gamma^0_{22}=0,\qquad g^{00}\Gamma^1_{02}+g^{10}\Gamma^1_{12}+g^{20}\Gamma^1_{22}=0,$$
$$g^{00}\Gamma^2_{02}+g^{10}\Gamma^2_{12}+g^{20}\Gamma^2_{22}=-\frac{\alpha r^2}{g},\qquad g^{01}\Gamma^0_{02}+g^{11}\Gamma^0_{12}+g^{21}\Gamma^0_{22}=0,$$
$$g^{01}\Gamma^1_{02}+g^{11}\Gamma^1_{12}+g^{21}\Gamma^1_{22}=0,\qquad g^{01}\Gamma^2_{02}+g^{11}\Gamma^2_{12}+g^{21}\Gamma^2_{22}=-(\beta r^2+B)\frac{r}{g}.$$

In terms of $\psi(r)=r^2/g=1/(\delta+\Omega r^2)$, the components of the Einstein tensor become

$$G_{00}=-\psi^2\left\{\alpha^2+\beta(\Omega+\lambda\delta)r^2\right\},\qquad G_{11}=-\psi^2(\beta\delta A+\Omega^2r^2),\tag{11.62}$$
$$G_{22}=-\psi^2r^2\beta\delta B,\qquad G_{01}=-\psi^2 r(\beta\delta C-\alpha\Omega),$$
$$G_{02}=-\psi^2r^2\beta\delta D,\qquad G_{12}=-\psi^2 r\beta\delta E,$$

and we may use the immediately above identities given by (11.61) to show that the lengthy Eqs. (11.58), (11.59) and (11.60) simplify as follows. For $k=0$, we have

$$\alpha r\frac{dG_{00}}{dr}+(\beta r^2+B)\frac{dG_{10}}{dr}+\left(\gamma r-\frac{E}{r}\right)\frac{dG_{20}}{dr}\tag{11.63}$$
$$+\alpha(1+\delta\psi)G_{00}+\left(\left(\beta r+\frac{B}{r}\right)+\beta r-\alpha^2\psi r\right)G_{10}+(\gamma-\alpha\epsilon\psi)G_{20}=0,$$

while for $k=1$, we obtain

$$\alpha r\frac{dG_{01}}{dr}+(\beta r^2+B)\frac{dG_{11}}{dr}+\left(\gamma r-\frac{E}{r}\right)\frac{dG_{21}}{dr}\tag{11.64}$$
$$+\alpha(1+\delta\psi)G_{01}+\left(\left(\beta r+\frac{B}{r}\right)+\beta r-\alpha^2\psi r\right)G_{11}+(\gamma-\alpha\epsilon\psi)G_{21}$$
$$+\delta\Omega\psi rG_{00}+\alpha\delta\psi G_{01}-\epsilon\Omega\psi rG_{02}-\alpha\Omega\psi r^2G_{01}-\alpha^2\psi rG_{11}$$

$$+\left((\delta+2\Omega r^2)\frac{E}{r^2}-\gamma\Omega r^2\right)\psi G_{21}-\Omega\epsilon\psi r G_{02}-\alpha\epsilon\psi G_{12}$$
$$-\left((\delta+2\Omega r^2)\frac{A}{r^3}+(C^2-\lambda A)\Omega r\right)\psi G_{22}=0,$$

noting that evidently further grouping of terms is possible. The above form is that in which the terms arise immediately from Eq. (11.59) and in the verification below, it is useful to leave in this form. For $k=2$, we have

$$\alpha r\frac{dG_{02}}{dr}+(\beta r^2+B)\frac{dG_{12}}{dr}+\left(\gamma r-\frac{E}{r}\right)\frac{dG_{22}}{dr} \tag{11.65}$$
$$+\alpha(1+\delta\psi)G_{02}+\left(\left(\beta r+\frac{B}{r}\right)+\beta r-\alpha^2\psi r\right)G_{12}+(\gamma-\alpha\epsilon\psi)G_{22}=0.$$

We may verify that each of Eqs. (11.63), (11.64) and (11.65) is correctly satisfied by the expressions for G_{ij} given by (11.62) as follows. All three can be verified in a similar manner, and for $k=0$ and $k=2$, there are two important equalities, namely

$$\alpha G_{00}+\beta r G_{10}+\gamma G_{20}=-\alpha^3\psi^2,\quad \delta G_{00}-\alpha r G_{10}-\epsilon G_{20}=-\alpha^2\psi,\ (k=0)$$
$$\alpha G_{02}+\beta r G_{12}+\gamma G_{22}=0,\quad \delta G_{02}-\alpha r G_{12}-\epsilon G_{22}=0,\ (k=2)$$

while for $k=1$, there are three important equalities, which are as follows:

$$\alpha G_{01}+\beta r G_{11}+\gamma G_{21}=(\alpha^2\psi-\beta)\Omega\psi r,\quad \delta G_{01}-\alpha r G_{11}-\epsilon G_{21}=\alpha\Omega\psi r,$$
$$\epsilon G_{02}+\gamma r G_{12}+(C^2-\lambda A)G_{22}=-\beta\delta\Omega\psi^2 r^2.$$

In addition to these equalities, we also require the three subsidiary equalities for $k=0$, 1 and 2, respectively,

$$BrG_{01}-EG_{02}=\alpha^3\psi^2 r^2,\quad BrG_{11}-EG_{12}=-(\alpha^2\Omega\psi r^2+\beta\delta)\psi r,$$
$$BrG_{21}-EG_{22}=0.$$

The proofs for all three values of k follow a similar approach, so we only give the details for the case of $k=1$, which is the most complicated. In each case, we purposely rearrange each of Eqs. (11.63), (11.64) and (11.65) to exploit the above equalities. For $k=1$, Eq. (11.64) can be shown to become

$$r\frac{d}{dr}(\alpha G_{01}+\beta r G_{11}+\gamma G_{21})+\frac{1}{r}\frac{d}{dr}(BrG_{11}-EG_{12})$$
$$+(\alpha G_{01}+\beta r G_{11}+\gamma G_{21})+2\alpha\psi(\delta G_{01}-\alpha r G_{11}-\epsilon G_{21})$$
$$+\Omega\psi r(\delta G_{01}-\alpha r G_{11}-\epsilon G_{21})-\Omega\psi r(\epsilon G_{02}+\gamma r G_{12}+(C^2-\lambda A)G_{22})$$

$$+ (\delta + 2\Omega r^2)(ErG_{21} - AG_{22})\frac{\psi}{r^3} = 0,$$

and in this case, we need the additional equality that $ErG_{21} - AG_{22} = \beta(\delta\psi r)^2$. Throughout these proofs, we make frequent use of the relations arising from $g = r^2(\delta + \Omega r^2)$ and $\psi(r) = r^2/g = 1/(\delta + \Omega r^2)$, namely

$$\delta + \Omega r^2 = \frac{1}{\psi}, \qquad \frac{d\psi}{dr} = -2\Omega\psi^2 r,$$

along with the identity $B\Omega = \alpha^2 + \beta\delta$.