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Mathematics of Particle-Wave Mechanical Systems

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To Jessica, for so much.

Author’s Foreword

The present text attempts to develop a combined theory for the two distinct and well-defined primitive notions of particles and waves. The intended contents are most aptly characterised in the words of the eminent Irish mathematical physicist John Lighton Synge, who in describing the contents of his book, *Geometrical Mechanics and de Broglie Waves*, writes in the introduction, ‘This book is intended as a contribution to “mathematical” physics rather than to “naturalistic” physics’ [100]. By this he means, ‘the creation of a coherent mathematical theory, with clearly set forth assumptions being the primary object, with physical interpretation in terms of experiment being relegated to a secondary position’ [99]. In the engineering community when theory and observation no longer coincide, the notion of a ‘fudge factor’ becomes paramount to reconcile the two positions. In the physics community, however, the same predicament forces the introduction of new factors which are coined mysterious because they can no longer be explained through conventional thinking, and the word ‘dark’ becomes attached to these mysterious phenomena. It is clear to the writer that the dark issues of astrophysics are in consequence of incorrect mechanical accounting, namely omitting insignificant issues at a local level which become increasingly significant at larger scales. Accordingly, there is a need to re-assess fundamental mechanical accounting practice while incorporating the essential aspects of the successful well-established areas of physics. This book describes one such formulation and attempts to develop the key implications of the proposal.

As the theory of electro-magnetism developed in the late nineteenth century, in attempting to formulate the critical electric and magnetic concepts, pioneering scientists such as Faraday and Maxwell sought inspiration from well-established mechanical analogues from both solid and fluid mechanics. Subsequently there has always been a strong interplay between ideas and concepts in electro-magnetism and mechanics, to the extent that the entire mechanical theory of special relativity springs directly from Lorentz invariance of the equations of electro-magnetism. One of the key lessons in electro-magnetism is the more fundamental importance of the vector and scalar potentials $(\mathbf{A}, V)$ as compared to the fields $(\mathbf{E}, \mathbf{B})$. The vector potential $\mathbf{A}$ corresponds to the electromagnetic momentum, and is the analogue of mechanical momentum, and referred to by both Faraday and Maxwell as the electro-tonic intensity, while the potential V is the analogue of the velocity potential in fluid mechanics. This book adopts the perspective that a similar situation might apply in special relativistic mechanics, and it is not just the momentum vector $\mathbf{p}$ that contributes to the work done W but rather the particle energy e itself also plays an important role. We develop here a Lorentz invariant modification of Newton's second law applying when particle energy itself is of comparable magnitude to the potential energy of the applied external field.

While the original idea came from electro-magnetism, much later I was able to reflect on an often-repeated remark of a colleague, who would say that general relativity specialists do not care which one in particular of their four variables corresponds to time in their four-dimensional description of space-time. Coming from continuum mechanics, this struck me as an interesting comment, but if it were true, then we certainly might be willing to put space and time on equal footings, and the notion of a force g in the direction of time might not be such a radical thought. Indeed, at the very heart of the theory of general relativity lies the notion of an ideal test particle, the presence of which is assumed not to affect the gravitational field. The theory expounded here attempts to make some accommodation for this energy omission within the context of the theory of special relativity.

After toying with the replacement formulation (3.4) for Newton's second law, the idea seemed to have some merit. I was especially encouraged that the force relations Eqs. (3.4) are Lorentz invariant, and that Eq. (5.8) admits the two suggestive limits $d\mathscr{E}/dp = u$ and $d\mathscr{E}/dp = c^2/u$, where $\mathscr{E}$ is wave energy and p is particle momentum. These two limits are the particle and wave limits, which also arise from Eq. (3.10) as purely spatial ($dt = 0$) or purely temporal ($d\mathbf{x} = \mathbf{0}$) extreme limits of the proposed theory. Also immediate from Eq. (5.8) are the relations $\mathscr{E} = \pm cp$, termed here the de Broglie relations. The positive case is equivalent to the known relations for light, namely $p = h/\mu$ and $\mathscr{E} = h\nu$, which together imply $\mathscr{E} = cp$ where h is the Planck constant and μ is the wave length ($c = \mu\nu$). These initial results provided sufficient incentive to contemplate other allowable outcomes.

Also important is that the structure of the standard operator relations of quantum mechanics $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$ is immediately apparent from the proposed theory, assuming only that the external forces f and g are generated from a potential $V(x,t)$ such that $f = -\partial V/\partial x$ and $gc^2 = -\partial V/\partial t$. Specifically, from (8.1), we have $p = \partial\psi/\partial x$ and $\mathscr{E} = \partial\psi/\partial t$, and conservation of energy $e + \mathscr{E} + V = constant$ immediately gives $e = -(\partial\psi/\partial t + V)$, which is precisely the structure of the standard operator relations of quantum mechanics. That is, the established structure of conventional quantum mechanics is immediately inherent in the present approach and arises from conservation of energy.

This highlights the fact that particle energy e and wave energy $\mathscr{E}$ must be accommodated separately. Conventionally, energy is just energy, and in quantum mechanics, particle-like and wave-like states are compromised by restricting the particle-like properties so that they do not permit the description of an exact space-time motion (Heisenberg uncertainty) achieved through the introduction of probability waves. Here, we deal with particle energy e and wave energy $\mathscr{E}$ as strictly different entities, and for the above-mentioned example of light, we would say $e = 0$ and $\mathscr{E} = cp$. The notion of energy commonly referred to as particle energy is an acquired characteristic arising from the properties of space, while wave energy is an accumulation of energy arising from time.

Also noteworthy is the fact that a simple and logical analysis of the Einstein energy statement $e^2 = {e_0}^2 + (pc)^2$ immediately gives rise to the prospect of four distinct types of matter, since either the rest energy e_0 is zero or it is non-zero, and for each of these two alternatives, we may adopt a plus or minus sign. This simple and logical explanation of at least four distinct types of matter is only meaningful because it is interpreted within the present proposed extension of special relativity, and such an interpretation is simply not available within the narrower confines of traditional special relativity because it lacks the notion of a force in the direction of time.

Here, we speculate that dark matter and dark energy arise as special or privileged states occurring for particular alignments of the magnitude of the spatial physical force f and the force g in the direction of time, which in a single spatial dimension become simply $f = \pm gc$. The particle and wave energies for these alignments are such that $e = \pm\mathscr{E}$, with dark matter an essentially backward wave and dark energy an essentially forward wave, both propagating at the speed of light. In a real circumstance, we might expect a situation comparable to a 'fuzzy region' where the key equalities are constantly switching on and off dependent upon a varying local environment. The case $e = -\mathscr{E}$ is especially interesting since it may operate under zero potential $V = 0$, which might account for the abundance of dark energy and dark matter in the universe.

This text presents an elementary and much expanded introduction to the basic mathematics and mechanics of the ideas formulated in [47–52] involving a Lorentz invariant alternative to Newton's second law. The Lorentz invariant modification of Newton's second law extends conventional special relativity theory by developing a dual particle-wave formulation accommodating both particle and wave energies, and allowing exceptions to the law that matter cannot be created or destroyed. de Broglie was first to propose a concrete physical picture of the co-existence of both particle and its associated wave, and [47–52] make a distinction between particle and wave energies e and $\mathscr{E}$, respectively, such that the total work done by the particle $W = e + \mathscr{E}$ accumulates from both a spatial physical force $\mathbf{f}$ and a force g in the direction of time. Generally, experiments are undertaken with the classical notions of either particles or waves in mind, so that in an experiment either particles or waves are reported, and only one of e or $\mathscr{E}$ is measured. Since nature tends to adopt the least energy structure, for the present theory, we propose that particles appear for $e < \mathscr{E}$ and waves for $\mathscr{E} < e$, but in either event, both a measurable and an unmeasurable energy exists.

Our specific purpose here is to provide an introduction to the mathematical and mechanical framework underpinning these ideas with a level of mathematical detail not included in the original papers, as might be comprehensible to an undergraduate student in either physics or applied mathematics. However, it is not a definitive account of a successful completed story, but rather a single proposal and a log for some of the mathematical results pertaining to the proposed model. The applied mathematical and mathematical modelling communities are used to the idea that there often exist a range of perfectly sensible and meaningful mathematical models predicting comparable numerical outcomes. While the writer is open to the idea that the particular model structure advanced here may not be immaculately correct, he is nevertheless firmly of the view that the ultimate resolution of the dark issues of mechanics will require a comparable re-thinking of Newton's second law, and that the dark issues will emerge as essentially artefacts of the mechanical accounting. The author is grateful to Professor David Steigmann for his constant support and encouragement of this work, and to Professor Phil Broadbridge, Dr Barry Cox, Professor Sam Drake, Dr Joe O'Leary and Mr Arthur Rorris, for numerous enjoyable discussions over coffee.

Contents

Chapter 1
Introduction

1.1 Introduction

In this chapter we provide an overall background introduction and motivation to the contents of the book. The following section deals with some of the background issues which is followed by two brief sections on special relativity and quantum mechanics that emphasise the importance that any new theories might sensibly be cognisant of the considerable successes of existing theories. The section thereafter introduces the general idea of de Broglie particle-wave mechanical systems which are mechanical systems incorporating both particle and wave energies. The chapter closes with a brief plan of the text and two tables of the major symbols and basic equations.

1.2 General Introduction

The latest Planck data [2] gives the relative fractions for dark energy, dark matter and ordinary or baryonic matter as comprising, respectively, 68.3, 26.8 and 4.9 per cent of all matter in the universe, and it is only for ordinary matter that existing mechanical and physical theories apply. There are fundamental problems with mechanical accounting at the astrophysical scale, and the word dark implies a mysterious aspect that is not properly understood. The dark issues of cosmological mechanics imply that our accounting for mass and energy at this scale is incorrect, and in the search to understand the dark issues of astrophysics, a fundamental re-examination of mechanical theory is necessary, and the most basic of all mechanics is Newton's second law involving force, mass and acceleration.

On the other hand, existing theory not only accounts for atomic physics, but does so to a very high degree of accuracy, and our current knowledge and understanding of the physical universe, combined with the great achievements of Newtonian

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mechanics and modern physics, are so overwhelming that as time passes, it becomes increasingly difficult to attempt the fundamental problems with a fresh mind and an open disposition. If we steadfastly maintain all the fundamental pillars of modern physics principles, then nothing new can be produced within such a rigid framework. The major problems can only be solved by venturing outside the pre-confined paradigm, to gain new insights and new perspectives. It seems to the writer that we need to start at the beginning while being as inclusive as is possible of well-established theory, and the mathematical construct described here and in [47–52] is an attempt to satisfy these two not entirely consistent objectives.

Current astrophysical theory is tending more towards increasing complexity, and many novel and creative ideas have been proposed to explain the origins of dark energy and dark matter, including negative and imaginary masses. Many proposals are compounded with other theories or defined in terms of other equally complex scientific concepts. Scientific measurement and data are also ambiguous, complex and shrouded in sophisticated statistical analysis, as is immediately clear from the briefest examination of the technical summaries of the data arising out of the Planck mission [1–3] or the data associated with measurement of the Hubble parameter (see for example [87]). In such a prevailing complex scientific environment that is characterised by a lack of clarity, and in the absence of the correct mechanical framework to interpret data, this is not the time for further elaborate theory; rather there is a need for fundamental questioning of basic mechanical principles, and the most basic of all mechanical principles is Newton's second law relating force, mass and acceleration. This is not to suggest that existing mechanical theory will need revisiting, but rather Newton's second law is the place where the fundamental thinking is required, and that understanding can then be transferred to more sophisticated gravitational theories.

Both special relativity and quantum mechanics provide accurate mathematical models that are locally reliable, and therefore the first challenge is to enhance these theories to explain "dark" mechanics but in a manner that embraces their successful features. Special relativity is especially relevant as a preliminary testing ground, since firstly the necessary adjustments to this theory will demonstrate how more general gravitational theories, such as general relativity, might be modified. Secondly, within special relativity the criterion of Lorentz invariance is there to establish the veracity or otherwise of any proposal. In other words, a good proposition in special relativity must satisfy the criterion of Lorentz invariance, which is a non-trivial constraint. Our aim is to modify special relativistic mechanics in a manner that is inclusive of its successful features. It turns out that in this reappraisal of basic mechanics, unintentionally other doors are opened, including the role of quantum mechanics and the "long searched for" relationship with special relativity.

A Lorentz invariant extension of Newton's second law is proposed within the context of special relativity that simultaneously produces all the successful results of classical mechanics, special relativity and quantum mechanics in the form of Schrödinger's second order wave equation, and provides a common basis for special relativity and quantum mechanics, with a single theory. The major achievements of these theories are preserved, and the formulation identifies underlying reasons for our present incorrect accounting of energy at the cosmological scale. It presupposes that along with the usual spatial physical force $\mathbf{f}$, there is a force g in the "direction of time", such that when the external forces are derivable from a potential, the particle energy itself acts as a boost to the potential. If the external forces $\mathbf{f}$ and g are generated from an underlying potential field, then when the spatial physical force is switched off, the force in the direction of time is also switched off, and the particle velocity is zero, indicating that there is nothing that is non-physical concerning the proposed theory.

In Bohr's theory of the hydrogen atom (see [35], pages 29–61 for a general historical account), de Broglie [17] showed that the group velocity of the wave package coincides with the particle velocity u, and the wave velocity is given by $w = c^2/u$. Accordingly, if the particle velocity u is sub-luminal, then the associated wave or phase velocity c^2/u through the de Broglie relation is necessarily superluminal. This is "believed" not to contradict the fact that information cannot be carried faster than the speed of light c because "supposedly" the wave phase does not carry energy. However, the superluminal phase velocity may well be physically significant, and as suggested in [47–49, 51], dark matter or dark energy may well exist as a consequence that either the particle energy or the de Broglie wave energy is neglected. If the wave energy through the superluminal wave speed c^2/u is accommodated, then it is clear that there will be interesting outcomes for slowing particle speeds u tending to zero.

The singularity arising from $u = 0$ and the consequent logarithmic singularities that can be produced even for slowly moving mechanical structures are at odds with our mechanical experience. Traditional mechanical theory and thinking presuppose smooth and sensible physical behaviour and an absence of singularities. In fact, expectations of decreasing energy for slowing systems and avoiding singularities lie at the very heart of mechanical thinking of natural systems. We may accommodate a singularity at the speed of light because it is believed to constitute an unattainable physical barrier. The formalism developed here hints at the prospect of physical singularities, supported by large energies, for possibly slowly moving mechanical structures, that display characteristics of unstoppable mechanical systems.

Our purpose is to progressively build a picture that a distinction must be made between particle energy $e = mc^2$ and the de Broglie wave energy $\mathscr{E}$ such that the total work done by the particle is $W = e + \mathscr{E}$ which accumulates from both a spatial physical force $\mathbf{f}$ and a force g in the direction of time. Since in any experiment, either particles or de Broglie waves are reported, so that only one of e or $\mathscr{E}$ is physically measured, and particles appear for $e < \mathscr{E}$ and de Broglie waves occur for $\mathscr{E} < e$, in either event, both a measurable and an unmeasurable energy exist. The question arises as to whether there are other reasons why it is important to include both particle and wave energies and for which there is no simple physical response. However, a simple mathematical response might be as follows: Formally, known particle energy obtained from $e = mc^2$ is an even function of velocity, while known wave energy obtained from the de Broglie formula $\mathscr{E} = pc$ is an odd function of velocity. Now it is well known that every function can be expressed as the sum of both an even and odd function (namely, $f(v) = [f(v) + f(-v)]/2 + [f(v) - f(-v)]/2$) so that both even and odd functions of velocity might be needed to correctly represent some arbitrary function of velocity.

We propose that dark matter and dark energy might arise as special states, termed here de Broglie states that are characterised by the relations $\mathscr{E} = \pm pc$, where p denotes the magnitude of the momentum vector, namely, $p^2 = \mathbf{p}.\mathbf{p}$. Further, we propose that the conventional particle energy equation $e^2 = {e_0}^2 + (pc)^2$, where $e_0 = m_0c^2$ denotes the particle rest mass energy, logically admits four distinct types of matter. The rest mass energy e_0 is either zero or non-zero and so gives rise to precisely four distinct types of matter: (I) $e = ({e_0}^2 + (pc)^2)^{1/2}$, $e_0 \neq 0$; (II) $e = -({e_0}^2 + (pc)^2)^{1/2}$, $e_0 \neq 0$; (III) $e = pc$, $e_0 = 0$; and (IV) $e = -pc$, $e_0 = 0$. Here we identify these (I) as baryonic matter; (II) as some form of antimatter; (III) as dark matter; and (IV) as dark energy, and no doubt using this particular characterisation, there are numerous elementary particles belonging to each category.

This straightforward identification of the distinct types of matter is only meaningful because it is interpreted within the proposed extension of special relativity examined in [47–52]. Such an interpretation is simply not available within the narrower confines of traditional special relativity because it lacks the notion of a force in the direction of time. We speculate that dark matter and dark energy arise from a particular alignment of the spatial physical force $\mathbf{f}$ and the force g in the direction of time, namely, $f = \pm gc$, where f denotes the magnitude $f^2 = \mathbf{f}.\mathbf{f}$. This alignment is such that the particle and wave energies coincide, namely, $e = \mathscr{E}$, and a consistent mathematical framework supports this proposal. The proposed model also admits $e = -\mathscr{E}$, with the important outcome that such a privileged state might be sustained under a zero or at least constant potential. In a real circumstance, we might expect a situation comparable to a "fuzzy region" where the key equalities are constantly switching on and off dependent upon a varying local environment. We identify dark matter as a backward wave and dark energy as a forward wave, in consequence that it is known that there is more dark energy in the universe than dark matter, which is equivalent to saying that time past is less than future time.

1.3 Special Relativity

While Einstein formulae $e = mc^2$ and $m = m_0[1 - (u/c)^2]^{-1/2}$, for the variation of mass m with its velocity u, where m_0 denotes the rest mass, have been overwhelmingly verified in our own local environment, it is clear that on a cosmological or astrophysical scale, our understanding of energy and matter is not so successful and issues such as dark energy and dark matter remain improperly understood. In our local environment, the rest mass m_0 is deemed to be the sole critical parameter, and yet the mysteries associated with dark energy and dark matter indicate that matter itself may adopt other forms or possess other defining characteristics (see for example [90]).

The notion of invariance refers to formulations of physical laws or quantities that are left unchanged with respect to changing frames of reference, and the particular notion of Lorentz invariance refers to invariance of Maxwell's equations of electromagnetism with regard to those frames of reference that are moving with constant relative velocity with respect to each other. The notion of Lorentz invariance with respect to non-accelerating frames is fundamental to special relativity, and the Einstein formula for mass m as a function of velocity u, namely, $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, is both a necessary and sufficient condition for force invariance in two non-accelerating frames.

The underlying philosophy here is firstly the recognition of the importance of special relativity and secondly to extend the theory in a manner that embraces the essential features of the existing theory. Now given the veracity of the special theory, it may not be too unreasonable to expect that somewhere embodied within the theory are clues as to the notions which have been termed dark matter and dark energy. However, since the special theory deals only with non-accelerating frames, we certainly would not expect any such extension to tell the complete story, but we might expect some definite pointers as to how a more complete picture may be subsequently developed.

The formulae $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ are fundamental to the development of special relativity, and these expressions together with the Lorentz transformations and the law for the addition of velocities are fundamental to the derivation of other results, including the Lorentz invariant energy-momentum relations. Here, assuming the Lorentz transformations and their consequences, and the formulae $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, we seek to extend the existing formalism of special relativity. The proposed formulation is a natural extension of special relativity that retains the major features and indicates that dark matter and dark energy arise from neglecting the work done in the direction of time.

1.4 Quantum Mechanics

In 1980, Bernard Cohen in his book on the Newtonian revolution [12], page 147 writes,"I believe that the outlook of post-Newtonian scientists, using a system based on the action of a universal force that they could not understand, was not wholly unlike that of present day physicists with respect to quantum field theory", and he quotes Murray Gell-Mann, who said in 1977, "All of modern physics is governed by that magnificent and thoroughly confusing discipline called quantum mechanics, invented more than fifty years ago. It has survived all tests and there is no reason to believe that there is any flaw in it. We suppose that it is exactly correct. Nobody understands it, but we all know how to use it and how to apply it to problems; and so we have learned to live with the fact that nobody can understand it".

The phenomenon of quantum entanglement indicates that systems of particles exist which individually display certain characteristics while collectively the same characteristic is absent simply because it has cancelled out between individual particles. To understand such complex physical issues, it may be irrelevant whether or not a particular long held conservation law applies, as long as there is a framework to include exceptions to the rule. It might be advantageous to formulate a framework in which there are exceptions to the rule that matter cannot be created or destroyed, without necessarily believing that it is either absolutely true or not. In [47–52] a modified special relativistic framework for mechanics is proposed to properly accommodate both particle and wave energies while still retaining the major achievements in mechanics and atomic physics.

In conventional quantum mechanics, Schrödinger's second order wave equation is typically motivated from the classical wave equation. Conventional quantum mechanics postulates that the equation be linear, so that different solutions may be superimposed, and that it involves only fundamental constants, rather than parameters associated with a particular motion of the particle such as momentum, energy, frequency or propagation number, and the classical wave equation emerges as the most likely candidate. It is therefore important to emphasise that within the theory proposed here, the classical wave equation is not a matter of speculation, and it is not difficult to envisage the Schrödinger wave equation arising in the present context as a formal consequence, following the numerous ad hoc derivations of the Schrödinger wave equation presented in several texts (such as [71], pages 18–19, or [93], pages 218–220), and we refer the reader to [49] for further details.

The proposed theory extends special relativistic mechanics by invoking the concept of a "force g in the direction of time". The model is inclusive of Newtonian mechanics and of quantum mechanics, in the form of Schrödinger's second order wave equation, and therefore inclusive of much existing theory of atomic physics. In this approach the wave equation emerges as a consequence of the theory, and the open question arises as to whether quantised energy levels can be deduced directly from the wave equation without forcing an eigenvalue problem.

Moreover, the operator relations of quantum mechanics $\mathbf{p} \longrightarrow -i\hbar\nabla$ and $e \longrightarrow i\hbar\partial/\partial t$, where $\hbar = h/2\pi$ and h is Planck's constant, are immediately apparent from the proposed theory if the external forces $\mathbf{f}$ and g are generated from a potential $V(\mathbf{x}, t)$ so that $\mathbf{f} = -\nabla V$ and $gc^2 = -\partial V/\partial t$, and therefore if $\mathbf{p} \longrightarrow -i\hbar\nabla$, then the theory suggests that the operator relation $\mathscr{E} \longrightarrow -i\hbar\partial/\partial t$ applies to the de Broglie wave energy. From these two operator relations and the conservation of energy $e + \mathscr{E} + V = constant$, we obtain the conventional operator relation of quantum mechanics that applies to the particle energy; thus $e \longrightarrow i\hbar\partial/\partial t - V(\mathbf{x}, t)$. The emergence of the operator relations of quantum mechanics from the theory highlights the fact that the particle energy e and the wave energy $\mathscr{E}$ might be accommodated separately.

In conventional quantum mechanics, particle-like and wave-like states are compromised by restricting the particle-like properties so that they do not permit the description of an exact space-time motion. Of course through the Hilbert space formalism of quantum mechanics, the mathematical theory of the particle-wave duality has come a long way since the rudimentary ideas of de Broglie and is quite capable of dealing with states in which there is neither pure particle behaviour (characterised by a well-defined position) nor pure wave behaviour (characterised by well-defined momentum) with both having significant variances of measurement. Here we deal with particle energy e and wave energy $\mathscr{E}$ as strictly different entities, and we examine the implications of the conventional quantum mechanical operators for the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$ of special relativity to identify the "long searched for" formal connection between special relativity and quantum mechanics, such that η associates with the particle or sub-luminal world, while ξ associates with the wave or superluminal world.

1.5 de Broglie Particle-Wave Mechanics

Writing in 1970, Louis de Broglie [23] makes clear that at two distinct stages of his life, his interpretation of the dual particle-wave nature of matter involved a concrete physical picture of the co-existence of both particle and the associated wave, which he referred to as "the theory of the double solution", and for which he formulated an equation which he called "the guidance formula" (see also de Broglie [21, 24]). At other times he followed the then current thinking in quantum mechanics, in which particle-like and wave-like states are compromised by restricting the particle-like properties so that they do not permit the description of an exact space-time motion (Heisenberg uncertainty), and this is achieved through the introduction of probability waves.

Roughly speaking, de Broglie's guidance equation arises from a combination of special relativistic and quantum mechanical ideas as follows: The momentum **p** and particle energy e are assumed to simultaneously admit the two representations

$$\mathbf{p} = m\mathbf{u} = -\nabla\psi, \qquad e = mc^2 = \frac{\partial\psi}{\partial t}, \tag{1.1}$$

where m is assumed to be given by the relativistic expression $m = m_0/[1-(u/c)^2]^{1/2}$ and therefore the velocity **u** is given by

$$\mathbf{u} = -c^2\frac{\nabla\psi}{(\partial\psi/\partial t)}, \tag{1.2}$$

and de Broglie refers to this formula, which determines the motion of the particle at each point of its trajectory in the wave, as the "the guidance formula of the particle by its wave". Notice that in the approach suggested here, the two Eqs. (3.4) combined with de Broglie relations (1.1) indicate that $\mathbf{f} = \mathbf{0}$ while

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = g. \tag{1.3}$$

For the special case of a single spatial dimension, an extended analysis of the above three relations is provided in Chap. 4.

In 1923 Louis de Broglie [16, 17] (see also de Broglie [18–20] and Weinberger [108]) first predicted light to display the dual characteristics both as a collection of particles, called photons, and as a wave. He predicted that all matter may, under appropriate circumstances, exhibit either particle-like or wave-like behaviour. He envisaged that an electron orbiting a hydrogen atom is accompanied by a mysterious pilot wave (now known as a de Broglie wave) extending the circumferential length of the orbit, and he speculated that the length of the orbit circumference comprised an integer number of wavelengths, from which he deduced $p = h/\mu$ and $\mathscr{E} = h\nu$, where p is momentum, $\mathscr{E}$ is energy, h is Planck's constant, μ is the wave length, and ν is the frequency and from which on using $\mu\nu = c$, we may deduce the de Broglie relation $\mathscr{E} = cp$ (see [35]). In this text we reserve the symbol λ for the nondimensional parameter in the exact solution given by Eq. (5.1).

Recently, de Broglie's quantum mechanical pilot-wave ideas have received something of a boost and a revival following the considerable attention focussed on the experiments of Yves Couder and colleagues [14], who observed droplets walking on the surface of a vibrating fluid bath and exhibiting phenomena previously thought to be exclusive to the microscopic quantum realm. As detailed by John W. M. Bush and co-workers (see for example [10, 45, 82]), "the walking-drop system displays comparable effects such as single and double-slit diffraction, tunneling, orbital quantisation, level-splitting and wave-like statistics in confined geometries. The walking drop is propelled through resonant interaction with its own wave field, and represents the first macroscopic realisation of a double-wave pilot-wave quantum mechanical system envisaged by de Broglie" [45].

John Lighton Synge's book on "geometrical mechanics and de Broglie waves" [100] develops Hamilton's formal methods of geometrical optics in the space-time of Minkowski, making proper allowance for the four-dimensionality of special relativity, to formulate a topic which might be termed "relativistic geometric mechanics". Synge explains that in optics, Hamilton's theory of geometrical optics and Maxwell's equations of electromagnetism form the two complementary coherent mathematical theories of optics, while in mechanics the Newtonian theory of particle motion is the analogue of geometrical optics and Schrödinger's wave mechanics is the analogue of Maxwell's theory, all providing coherent mathematical theories. Within this framework the topic developed here might be best described as a contribution to "Newtonian theory of particle motion within the space-time of Minkowski", noting that the distinction between the theory here and the existing subject of this name is as follows: At present the relativistic theory of Newtonian particle motion deals only with a spatially Lorentz invariant Newton's second law. In this text we develop a coherent mathematical theory describing Lorentz invariant Newton's second law for both space and time.

The theory proposed here might be viewed as intermediate between special and general relativity, in the sense that for an energy function to exist, any external forces $\mathbf{f}$ and g must satisfy a compatibility condition (3.11). In the event that the forces are generated from a potential $V(\mathbf{x}, t)$ through Eq. (5.8), then unlike conventional theory, the potential itself cannot be arbitrarily assigned but is determined as part of the solution procedure. This feature is reminiscent of general relativity for which the field equations determine the metric tensor and the gravitational nature of the field itself.

In the present theory, we make a distinction between particle energy $e = mc^2$ and the de Broglie wave energy $\mathscr{E}$, which are such that the total work done by the particle $W = e + \mathscr{E}$ accumulates from both the spatial physical force $\mathbf{f}$ and the force g in the direction of time. We propose that in an experiment, particles appear for $e < \mathscr{E}$ and waves for $\mathscr{E} < e$, but in either event, both a measurable and an unmeasurable energy exist, and in making this assertion, we are motivated by the fact that, generally speaking, nature tends to prefer to adopt minimum energy structures. We further speculate that dark matter and dark energy arise from particular alignments of the magnitude of the spatial physical force $\mathbf{f}$ and the force g in the direction of time, such that the particle and wave energies coincide, namely, $e = \mathscr{E}$.

Although the theory proposed in [47–52] might be referred to as a "toy theory" in the sense that it does not involve the full gravitational nature of general relativity, we might expect that it provides the correct conceptual outcomes which at a later time might be refined through more sophisticated theory. The mere fact of the existence of the dark issues of mechanics implies that there is something fundamentally flawed with mechanical accounting at the cosmological scale, but now may not be the time for further elaborate theory; rather there is a need for fundamental questioning of basic mechanical principles, and the most basic of all mechanical principles is Newton's second law.

The proposed mathematical theory indicates that dark matter might arise from an essentially backward wave (time past) while dark energy might arise from an essentially forward wave (future time), and the conventional wisdom of more dark matter than dark energy in the universe arises as a consequence that future time exceeds time past. If the spatial physical force $\mathbf{f}$ and the force g in the direction of time are generated as the gradient of a potential function, then the total particle energy is necessarily conserved in the conventional manner, and the formulation predicts the conventional operator structure of quantum mechanics leading to the Schrödinger wave equation. The present approach links two invariances of the Lorentz group of special relativity with the corresponding Lorentz invariant differential operators arising in quantum mechanics and the de Broglie particle and wave duality and giving rise to the Klein-Gordon equation of relativistic quantum mechanics.

1.6 Plan of Text

In the following chapter, we present a brief summary of the basic equations of conventional special relativity including the Lorentz transformations, the addition of velocities law, several sections on Lorentz invariances, force invariance and the Lorentz invariant mass-momentum relations. Throughout the text we assume the Einstein variation of mass with velocity formulae of special relativity. However, in the final section of Chap. 2, we include a Lorentz invariant alternative, which indicates that any departure from the Einstein expressions inevitably introduces additional complexities to the theory.

We begin Chap. 3 with brief biographies of de Broglie and Maxwell, mentioning some of their considerable achievements. We then detail the general formulation of the basic mathematical and mechanical model formulated in [47–52], including the four types of matter, a discussion of the likely rest mass energy profile, the proposed extension of Newton's second law, the introduction of the work done W and the de Broglie wave energy $\mathscr{E}$, the conservation of energy principle for forces $\mathbf{f}$ and g derivable from a potential $V(\mathbf{x}, t)$ and a formal correspondence with Maxwell's equations. As an illustrative example, the basic equations corresponding to purely radial motion in a spherically symmetric environment with no angular effect are examined. The model proposed here is primarily intended to be interpreted as an extension of special relativity. However, by way of an introduction to the new theory, the final three sections of Chap. 3 examine a Newtonian interpretation of the theory that is based upon the classical interpretations with particle energy $e = m_0u^2/2$ and particle momentum $\mathbf{p} = m_0\mathbf{u}$, where the mass m_0 is assumed to be constant.

In Chap. 4 we detail special results for the model that apply to one Cartesian space dimension. In this circumstance, the formal equations take on their simplest form, and extending some of these results may not be entirely straightforward. In this chapter, we present the basic equations, a number of reformulations of these equations and an identity involving the particle energy. The next five sections deal with various formulations of the basic equations using the Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$ and culminating in de Broglie's guidance equation leading to an insightful formulation in terms of Clairaut's differential equation with parameter u. The final two sections of the chapter briefly discuss the Hamiltonian and Lagrangian for a single dimension.

The next two chapters deal with the derivations and formulae for a specific one-dimensional exact wave-like solution (5.1) of the basic equations (3.4). In Chap. 5 we discuss the solution and its relationship to a known solution in relativity and its application to derive an analytical expression for the Hubble parameter. In Chap. 6 we present a formal derivation of this solution and some technical details for the evaluation of expressions for the corresponding de Broglie wave energy $\mathscr{E}$.

The Lorentz and associated invariances of the underlying one space dimension model and of the exact solution are examined in detail in Chap. 7. The general equations are fully Lorentz invariant, so that the model allows the force in all Lorentz frames to be the same. However, the condition for the existence of a work done or energy function is not Lorentz invariant, so that not all aspects of the solutions of the model will be fully Lorentz invariant. This is the case for the exact wave-like solution examined in Chap. 5, which might be termed partially Lorentz invariant. The functional forms of the assumed linear forces are preserved under Lorentz transformation, with new constants dependent upon the Lorentz translational velocity v, while the actual Lorentz invariants turn out to involve the products of force and energy. The final two sections of this chapter deal with force invariance under superluminal Lorentz frames, and using the space-time transformation $x' = ct$ and $t' = x/c$, relating sub-luminal particle motion and superluminal waves, to identify particle, wave and momentum energies.

In Chap. 8 some further results for a single space dimension are developed. The general solution in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$ is examined in Chap. 8, and a generalisation of the wave-like solution is presented there. In two subsequent sections, the notion of solving the basic equations for non-constant or variable rest mass is examined. In the final section of the chapter, a number of results are developed based on the assumption that the momentum $p(x, t)$ and wave energy $\mathscr{E}(x, t)$ can be treated as independent variables which essentially is equivalent to the assumption that the applied forces f and g are such that $f \neq \pm cg$.

The basic equations and results for centrally or spherically symmetric gravitating environments are developed in Chap. 9, and it is noted that the special relativistic estimate of the Schwarzschild radius is precisely one-half of the conventional Schwarzschild radius obtained from Newtonian mechanics. The four types of matter are discussed and four partial differential equations are determined for the four potentials $V(r, t)$ corresponding to each of the predicted distinct types of matter, and some of the simpler stretching similarity potentials are determined. The final section of the chapter deals with de Broglie's centrally symmetric guidance formula.

In Chap. 10, we examine the relationship of the model presented here with conventional quantum mechanics in the form of Schrödinger's second order wave equation, and we show that the two invariants $\xi = ex - c^2pt$ and $\eta = px - et$ of special relativity (see Eq. (2.48)) provide the formal connection between special relativity and quantum mechanics that has long attracted many eminent researchers. This formal connection has always eluded researchers since in conventional quantum mechanics, there is no distinction between particle energy e and wave energy $\mathscr{E}$. On adopting the standard operator relations of quantum mechanics, ξ and η give rise to two Lorentz invariant operators leading to the Klein-Gordon partial differential equation arising as the operator equivalent of the algebraic identity $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$. By incorporating a potential function $V(x, t)$ into the operator relations, we also deduce the modified second order Klein-Gordon equation (10.21) which is an entirely new partial differential equation displaying characteristics of both the Klein-Gordon equation and of Schrödinger's second order wave equation.

As an essential perspective for any reader, Chap. 11 of the book provides an introduction to some of the necessary technical aspects leading to general relativity, which, although an important topic, is admittedly by any measure a technically difficult one. At the heart of general relativity theory lies the notion of an ideal test particle that does not influence the gravitational field in any manner. The present text attempts to develop a special relativistic theory that applies when the particle energy itself is comparable to the external potential energy generating the motion, and the next challenge might be to incorporate a similar feature in general relativity. The question also arises as to whether the present proposal is already embodied in the general theory, and if not, how might the general theory be modified to accommodate particle energy? With this in mind, and given the practical importance of the subject, a progressive introduction to general relativity theory is provided in Chap. 11. This chapter presents some notes on general coordinate transformations, tensors and an introduction to general relativity, which is necessarily a limited introduction to the topic and termed progressive in the sense that the reader can simply read on to further sections if more information and further detail are required.

Beginning with Cartesian tensors and the Einstein summation convention and progressing through tensors arising from general curvilinear coordinates, the notions of partial covariant differentiation and the Christoffel symbols are developed leading to the tensorial equations of general relativity. To provide some insight into the mathematical and technical detail involved in the discipline, some general formulae are presented for the Ricci and Einstein tensors expressed in terms of six components of the covariant curvature tensor, and two well-known cosmological models are presented as illustrative examples of the formulation. In the final section of Chap. 11, some calculations involving logarithmic spirals and demonstrating more complicated detail are presented. Chapter 12 provides some overall conclusions, a dot point summary of some of the key results with reference to a single spatial dimension and some reflective commentary on the text.

1.7 Tables of Major Symbols and Basic Equations

In this section, for convenience, we present Tables 1.1 and 1.2 for the major symbols and basic equations that are used throughout the text.

Table 1.1 Major symbols and equation number of first appearance

Quantity	Symbol	Number
Mass	$m = m_0/(1 - (u/c)^2)^{1/2}$	(2.11)
Rest mass	m_0	(2.11)
Particle energy	$e = mc^2$	(2.11)
Rest particle energy	$e_0 = m_0c^2$	(2.11)
Wave energy	$\mathscr{E}$	(3.10)
Position vector	$\mathbf{x}$	(3.2)
Time	t	(2.2)
Velocity vector	$\mathbf{u} = d\mathbf{x}/dt$	(3.6)
Momentum vector	$\mathbf{p} = m\mathbf{u}$	(3.2)
One-dimensional position	x	(2.2)
One-dimensional velocity	$u = dx/dt$	(2.2)
Also magnitude of $\mathbf{u}(\mathbf{x}, t)$	$u = (\mathbf{u} \cdot \mathbf{u})^{1/2}$	(3.33)
Also centrally symmetric radial velocity	$u = dr/dt$	(9.13)
One-dimensional momentum	$p = mu$	(2.11)
Also magnitude of $\mathbf{p}(\mathbf{x}, t)$	$p = (\mathbf{p} \cdot \mathbf{p})^{1/2}$	(3.3)
Relative frame velocity	v	(2.2)
Position from fixed frame	X	(2.2)
Time from fixed frame	T	(2.2)

(continued)

Velocity from fixed frame	$U = dX/dT$	(2.6)
Average particle and wave velocity	$V = (u + c^2/u)/2$	(2.42)
Also force potential $V(\mathbf{x}, t)$	$\mathbf{f} = -\nabla V,\ gc^2 = -\partial V/\partial t$	(3.25)
Lorentz invariant	$\xi(x,t) = ex - c^2 pt$	(2.48)
Lorentz invariant	$\eta(x,t) = px - et$	(2.48)
Lorentz invariance angle	$\theta(x,t) = \tanh^{-1}(u/c)$	(2.10)
Velocity angle	$\phi(x,t) = \sin^{-1}(u/c)$	(5.11)
Gauge invariance $\psi(\mathbf{x}, t)$	$\mathbf{p}' = \mathbf{p} + \nabla\psi,\ e' = e - \partial\psi/\partial t$	(3.5)
Also potential for Lorentz invariants	$\eta - tV = \partial\psi/\partial x,\ \xi + xV = -\partial\psi/\partial t$	(4.38)
Characteristic variables	$\alpha = ct + x,\ \beta = ct - x$	(8.37)
Arbitrary functions	$F(\alpha),\ G(\beta)$	(4.42)
Extended Newton's 2nd law	$\mathbf{f} = \partial\mathbf{p}/\partial t + \nabla e,\ g = (1/c^2)\partial e/\partial t + \nabla\cdot\mathbf{p},$	(3.4)

Table 1.2 Basic equations and first appearance (* denotes assuming force potential)

Name	Equation	Number
Total time derivative	$d/dt = \partial/\partial t + (\mathbf{u}\cdot\nabla),$	(3.6)
Identity for spatial force $\mathbf{f}(\mathbf{x}, t)$	$\mathbf{f} = d\mathbf{p}/dt + \mathbf{u}\wedge(\nabla\wedge\mathbf{p}),$	(3.7)
Incremental wave energy $\mathcal{E}(\mathbf{x}, t)$	$d\mathcal{E} = \partial\mathbf{p}/\partial t\cdot d\mathbf{x} + c^2(\nabla\cdot\mathbf{p})dt,$	(3.10)
Force potential $V(\mathbf{x}, t)$	$\mathbf{f} = -\nabla V,\ gc^2 = -\partial V/\partial t,$	(3.25)*
Conservation of energy	$e + \mathcal{E} + V = constant,$	(3.26)*
Force formulae	$\mathbf{f} = d\mathbf{p}/dt,\ gc^2 = d\mathcal{E}/dt,$	(3.15)*
Wave equation for $\mathbf{p}(\mathbf{x}, t)$	$\partial^2\mathbf{p}/\partial t^2 = c^2\nabla^2\mathbf{p},$	(3.21)*
Wave equation for $\mathcal{E}(\mathbf{x}, t)$	$\partial^2\mathcal{E}/\partial t^2 = c^2\nabla^2\mathcal{E},$	(3.22)*
General solution for $\mathbf{p}$ and $\mathcal{E}$	$\mathbf{p} = \nabla\phi,\ \mathcal{E} = \partial\phi/\partial t,\ \partial^2\phi/\partial t^2 = c^2\nabla^2\phi,$	(3.23)*
One-dimensional version	$f = \partial p/\partial t + \partial e/\partial x,\ gc^2 = \partial e/\partial t + c^2\partial p/\partial x,$	(7.1)
Characteristic variables	$\alpha = ct + x,\ \beta = ct - x,$	(2.31)
Characteristic variable α	$f + cg = 2\partial(e + cp)/\partial\alpha,$	(8.37)
Characteristic variable β	$f - cg = -2\partial(e - cp)/\partial\beta,$	(8.37)
Force potential $V(x, t)$	$f = -\partial V/\partial x,\ gc^2 = -\partial V/\partial t,$	(8.40)*
Equations for p and $\mathcal{E}$	$\partial\mathcal{E}/\partial x = \partial p/\partial t,\ \partial\mathcal{E}/\partial t = c^2\partial p/\partial x,$	(8.44)*
General solution for $p(x, t)$	$p(x,t) = F(\alpha) + G(\beta),$	(4.43)*
General solution for $\mathcal{E}(x, t)$	$\mathcal{E}(x,t) = c\,(F(\alpha) - G(\beta)),$	(4.43)*
General solution for $e(x, t)$	$e(x,t) = -V(x,t) - c(F(\alpha) - G(\beta)),$	(4.44)*
Lorentz invariant $\xi(x, t)$	$\xi(x,t) = -xV(x,t) - c(\alpha F(\alpha) + \beta G(\beta)),$	(4.43)*
Lorentz invariant $\eta(x, t)$	$\eta(x,t) = tV(x,t) + (\alpha F(\alpha) - \beta G(\beta)),$	(4.43)*
Working variable $\gamma(x, t)$	$\gamma(x,t) = (e + pc)(ct - x)/2c,$	(2.49)
Working variable $\delta(x, t)$	$\delta(x,t) = (e - pc)(ct + x)/2c,$	(2.49)
Centrally symmetric systems	$r = (x^2 + y^2 + z^2)^{1/2},$	(3.28)
Momentum	$\mathbf{p} = p(r,t)\hat{\mathbf{r}} = p(r,t)(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})/r,$	(3.28)
Wave energy $\mathcal{E}(r, t)$	$\partial\mathcal{E}/\partial r = \partial p/\partial t,\ \partial\mathcal{E}/\partial t = c^2\,(\partial p/\partial r + 2p/r),$	(9.3)*
General solution $p(r, t)$	$p(r,t) = (\partial\psi/\partial r)/r - \psi/r^2 = \partial\,(\psi/r)/\partial r,$	(9.5)*
General solution $\mathcal{E}(r, t)$	$\mathcal{E}(r,t) = (\partial\psi/\partial t)/r = \partial\,(\psi/r)/\partial t,$	(9.5)*
Function $\psi(r, t)$	$\partial^2\psi/\partial t^2 = c^2\partial^2\psi/\partial r^2,$	(9.6)*

Chapter 2
Special Relativity

2.1 Introduction

In this chapter we summarise some of the standard results of conventional special relativity theory that are needed to formulate the proposed extension of Newton's second law. The word special alludes to invariance under transformations relating constant relative velocity frames of reference, which is in contrast to general relativity which relates to invariance under arbitrary space-time coordinate transformations. The first section deals with the fundamental notion of Lorentz transformations and the importance of invariance with respect to frames that are moving with constant relative velocity. The following section highlights the Einstein addition of velocities law which is an immediate consequence of the notion of invariance under Lorentz transformations.

The next six sections provide a number of results dealing with Lorentz invariances, including the determination of Lorentz invariant velocity fields, a general framework for Lorentz invariances, solving a first order partial differential equation by Lagrange's method to determine their general form, a novel method of their validation and showing that the Jacobian for all Lorentz invariances necessarily vanishes. Two sections thereafter deal, respectively, with the space-time transformation $x' = ct$ and $t' = x/c$ and the de Broglie wave velocity $u' = c^2/u$. In the subsequent section, the fundamental rate-of-working equation (or work done equation) is formulated for the determination of the conventional physical energy of a particle moving under an applied force. The next two sections of the chapter deal with the Lorentz invariant energy-momentum relations and force invariance for two frames moving with constant relative velocity. The penultimate section of the chapter provides a specific example of particle motion in an invariant potential field, while the final section deals with a possible extension of the conventional Einstein variation of mass formula with a specific expression arising from a Lorentz invariant equation for the energy rate de/dp.

We assume throughout the text that the Einstein energy-mass expression $e = e_0/(1-(u/c)^2)^{1/2}$ applies. In the final section of this chapter, assuming $e = mc^2$,

J. M. Hill, *Mathematics of Particle-Wave Mechanical Systems*,
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$p = mu$, and a Lorentz invariant equation for the energy rate de/dp, which is motivated from the energy equation (5.8), we derive an extension of the conventional Einstein variation of the energy-mass formula. This is the simplest one-parameter Lorentz invariant extension of the Einstein mass-energy relation. Implicit in the new expression is space-time anisotropy such that the particle has different rest masses in the positive and negative x directions. The resulting energy-mass formula (2.59), involving an arbitrary constant κ, predicts in particular that the rest energies in the moving and reference frames, respectively, e_0 and E_0, are related by the equation

$$e_0 = E_0 \left(\frac{1 + (v/c)}{1 - (v/c)} \right)^{\kappa/2}, \tag{2.1}$$

which for $\kappa \neq 0$ therefore impinges on the basic assumption relating to the isotropy of space. If $e_0 = E_0$, then necessarily κ is zero, and for $\kappa \neq 0$ the rest mass values will vary with the direction of motion, namely, two different values are obtained for positive and negative velocities v. While numerous experiments have been undertaken aimed at testing such hypothesis, and all indicate the veracity of the assumed isotropy of space, nevertheless the validity or otherwise of (2.1) might only be properly tested in those situations for which both rest energies e_0 and E_0 are non-zero and the fraction $(1 + v/c)/(1 - v/c)$ significantly differs from unity. Further, since it is generally believed that black holes exist at the centres of galaxies, space must be intrinsically anisotropic in some sense.

The topic of special relativity has acquired the status of a standard subject in both physics and mechanics, and almost every text on physics or mechanics contains either a chapter on special relativity or at least some reference to special relativity. The older texts tend to be closer to the original motivating issues and the developments that gave birth to the subject. Both Dingle [25] and McCrea [76] are concise student texts, while more comprehensive accounts of special relativity can be found in Bohm [8], French [34] and Resnick [86]. Both Moller [78] and Tolman [102] are two texts which have become standard works of reference to many aspects of relativity. The reader may wish to also consult [83] which contains a collection of the original papers of Einstein, Lorentz, Minkowski and Weyl with additional notes by Arnold Sommerfeld.

2.2 Lorentz Transformations

At the very heart of special relativistic mechanics lies the notion of invariance with respect to frames moving with constant relative velocity, and particularly under those transformations of space and time leaving the wave equation unchanged, referred to as Lorentz transformations. We consider a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to the first frame, and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 2.1. We view the relative velocity v as a parameter measuring

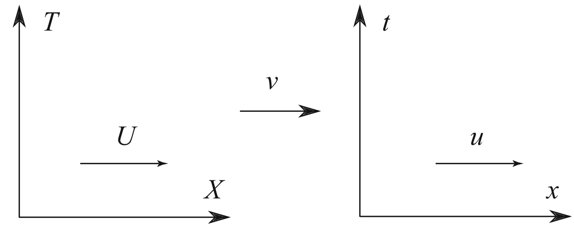

Fig. 2.1 Two inertial frames moving along x-axis with relative velocity v

the departure of the current frame (x, y, z) from the rest frame (X, Y, Z), and for this purpose we adopt a notation employing lower case for variables associated with the moving (x, y, z) frame and upper case or capitals for those variables associated with the rest (X, Y, Z) frame. Accordingly, time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t. Following normal practice, we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

For $0 \leqslant v < c$, the standard Lorentz transformations are

$$X = \frac{x + vt}{[1 - (v/c)^2]^{1/2}}, \quad T = \frac{t + vx/c^2}{[1 - (v/c)^2]^{1/2}}, \tag{2.2}$$

with the inverse transformation characterised by $-v$; thus

$$x = \frac{X - vT}{[1 - (v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1 - (v/c)^2]^{1/2}}, \tag{2.3}$$

and various derivations of these equations can be found in many standard textbooks such as Feynmann et al. [33] and Landau and Lifshitz [66], and other novel derivations are given by Lee and Kalotas [67] and Levy-Leblond [68]. The above equations reflect, of course, that the two coordinate frames coincide when the relative velocity v is zero, namely, $x = X$, $t = T$, when $v = 0$.

Throughout the text we adopt the notation $\alpha = ct + x$ and $\beta = ct - x$ for the characteristic variables, and from either of the above equations, by direct substitution, we may readily deduce the relations

$$ct + x = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} (cT + X), \quad ct - x = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} (cT - X), \tag{2.4}$$

so that in particular we may confirm the simple Lorentz invariance

$$(ct)^2 - x^2 = (cT)^2 - X^2. \tag{2.5}$$

2.3 Einstein Addition of Velocities Law

The relative frame velocity v is assumed to be constant, so that with velocities $U = dX/dT$ and $u = dx/dt$, on taking the differentials of both equations in (2.3), thus

$$dx = \frac{dX - vdT}{[1-(v/c)^2]^{1/2}}, \quad dt = \frac{dT - vdX/c^2}{[1-(v/c)^2]^{1/2}},$$

and on dividing the first differential by the second yields the Einstein addition of velocity law

$$u = \frac{U - v}{\left(1 - Uv/c^2\right)}. \tag{2.6}$$

An immediate consequence of (2.6) is the identity

$$[1-(u/c)^2]^{1/2}(1 - Uv/c^2) = [1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2}, \tag{2.7}$$

which can be easily established by using (2.6) in the left-hand side of (2.7). This latter equation is fundamental to the development of special relativity, and in particular it is necessary to establish Lorentz invariance of certain quantities.

There are special relativity theories which apply for relative velocities greater than the speed of light and which are complementary to the Einstein special theory of relativity that applies to relative velocities less than the speed of light. Hill and Cox [54] derive Lorentz transformations corresponding to (2.3) for superluminal relative velocities and show that the Einstein addition of velocities law still applies. The two formulae (2.6) and (2.7), when expressed in the form (2.8), reveal that at least one of the velocities u, v or U must not exceed the speed of light, and in terms of taking square roots or logarithms, all need appropriate re-arrangement depending upon the particular values of the three velocities.

A formula arising from (2.6) that is not so well known is

$$\left(\frac{1 + U/c}{1 - U/c}\right) = \left(\frac{1 + u/c}{1 - u/c}\right)\left(\frac{1 + v/c}{1 - v/c}\right), \tag{2.8}$$

so that on introducing velocity variables $(\Theta, \theta, \epsilon)$ defined by

$$\Theta = \tanh^{-1}(U/c), \quad \theta = \tanh^{-1}(u/c), \quad \epsilon = \tanh^{-1}(v/c), \tag{2.9}$$

equation (2.8) becomes simply the translation $\Theta = \theta + \epsilon$ noting again that within the context of special relativity, v and therefore ϵ are both assumed to be constants. The angle θ assumes an important role and is the angle in which Lorentz invariance appears through a translational invariance, and so for completeness we note the elementary relations

$$\theta = \frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \tanh^{-1}(u/c), \qquad \left(\frac{1+u/c}{1-u/c}\right)^{1/2} = e^{\theta}. \tag{2.10}$$

Further, with $u/c = \tanh\theta$, we have from the usual formulae of special relativity for energy and momentum, namely, $e = mc^2$ and $p = mu$, respectively, where $m(u) = m_0[1-(u/c)^2]^{-1/2}$ and m_0 denotes the rest mass, the following additional relations:

$$m = m_0\cosh\theta, \qquad e = e_0\cosh\theta, \qquad pc = e_0\sinh\theta, \tag{2.11}$$

where $e_0 = m_0c^2$ denotes the rest mass energy. We comment that the formulation of Lorentz transformations as a one-parameter group of geometric transformations is due to Minkowski [77].

Superluminal de Broglie Waves de Broglie [17] showed that while the group velocity of the wave package coincides with the particle velocity u, the wave velocity is given by $w = c^2/u$. This means that if the particle velocity u is subluminal, then not only is the associated wave or phase velocity c^2/u necessarily superluminal but also the average velocity $(u+w)/2$, and this will be examined in a subsequent section in this chapter. Furthermore, if the relationship $uw = c^2$ holds in one Lorentz frame, then it holds in all Lorentz frames. This is most easily seen if we rearrange (2.6) to read

$$U = \frac{u+v}{\left(1+uv/c^2\right)}, \tag{2.12}$$

so that the corresponding respective velocities U and W as measured from the (X,T) rest frame and corresponding to u and w in the moving (x,t) frame are given by

$$U = \frac{u+v}{\left(1+uv/c^2\right)}, \qquad W = \frac{w+v}{\left(1+wv/c^2\right)},$$

and from the identity

$$UW - c^2 = \frac{(1-(v/c)^2)(uw-c^2)}{\left(1+uv/c^2\right)\left(1+wv/c^2\right)},$$

so that $UW = c^2$ if and only if $uw = c^2$.

As noted above, an outcome predicted in [54] is that the Einstein velocity addition formula (2.6) remains valid for the proposed superluminal extension of special relativity in [54], and in particular, in the limit $v \to \infty$ the de Broglie relation $uU = c^2$ emerges. It also formally emerges from the envelope [41], namely, by simultaneously solving (2.6) and

$$\frac{\partial u}{\partial v} = \frac{-1}{\left(1 - Uv/c^2\right)} + \frac{U(U - v)}{c^2\left(1 - Uv/c^2\right)^2} = \frac{-\left(1 - (U/c)^2\right)}{\left(1 - Uv/c^2\right)^2} = 0,$$

which can only vanish in the limit $v \to \infty$.

2.4 Lorentz Invariances

A Lorentz invariant quantity is one which assumes an identical form under a Lorentz transformation, and we have previously noted the Lorentz invariant $(ct)^2 - x^2 = (cT)^2 - X^2$ given by (2.5). In this section we identify a further three important Lorentz invariances which can all be established by direct substitution using the above equations. In the following sections, we develop a general understanding of the origin of these invariances. By substitution of the Lorentz transformations (2.3) and the addition of velocities law in the form of (2.6) or (2.7) into the left-hand sides, we may verify directly the following interesting Lorentz invariances:

$$\frac{x - ut}{\left(1 - (u/c)^2\right)^{1/2}} = \frac{X - UT}{\left(1 - (U/c)^2\right)^{1/2}}, \quad \frac{t - ux/c^2}{\left(1 - (u/c)^2\right)^{1/2}} = \frac{T - UX/c^2}{\left(1 - (U/c)^2\right)^{1/2}}, \tag{2.13}$$

which are also verified independently and further discussed in a subsequent section in this chapter. For the first equality, on substitution of (2.3) and (2.6) into the left-hand side, we need to simplify

$$\frac{x - ut}{\left(1 - (u/c)^2\right)^{1/2}} = \frac{\left[(X - vT)\left(1 - Uv/c^2\right) - (U - v)\left(T - vX/c^2\right)\right]}{[1 - (u/c)^2]^{1/2}[1 - (v/c)^2]^{1/2}(1 - Uv/c^2)},$$

which on simplification of the numerator and using (2.7) gives the required right-hand side. Similarly, for the second equality, on substitution of (2.3) and (2.6) into the left-hand side, we need to simplify

$$\frac{t - ux/c^2}{\left(1 - (u/c)^2\right)^{1/2}} = \frac{\left[(T - vX/c^2)\left(1 - Uv/c^2\right) - (U - v)(X - vT)\right]}{[1 - (u/c)^2]^{1/2}[1 - (v/c)^2]^{1/2}(1 - Uv/c^2)},$$

which again on simplification of the numerator and using (2.7) gives the desired outcome.

Similarly, we may establish a third Lorentz invariant

$$\left(\frac{ct + x}{ct - x}\right)\left(\frac{1 - u/c}{1 + u/c}\right) = \left(\frac{cT + X}{cT - X}\right)\left(\frac{1 - U/c}{1 + U/c}\right), \tag{2.14}$$

since by division of the two Eqs. (2.4) we have

$$\left(\frac{ct+x}{ct-x}\right)\left(\frac{1+v/c}{1-v/c}\right)=\left(\frac{cT+X}{cT-X}\right),$$

and on using the Einstein addition of velocity law in the form of (2.8), this equation simplifies to give (2.14). Thus, from Eqs. (2.5) and (2.14), we have the two Lorentz invariants

$$\zeta=((ct)^2-x^2)^{1/2}, \qquad \tau=\left(\frac{ct+x}{ct-x}\right)^{1/2}\left(\frac{1-u/c}{1+u/c}\right)^{1/2}=\left(\frac{ct+x}{ct-x}\right)^{1/2}e^{-\theta}, \tag{2.15}$$

where the latter equality arises from (2.10) where $\theta=\tanh^{-1}(u/c)$. In terms of the characteristic coordinates $\alpha=ct+x$ and $\beta=ct-x$, we have

$$\zeta=(\alpha\beta)^{1/2}, \qquad \tau=\left(\frac{\alpha}{\beta}\right)^{1/2}e^{-\theta},$$

and from the partial derivatives

$$\frac{\partial\zeta}{\partial x}=-\frac{x}{(\alpha\beta)^{1/2}}, \qquad \frac{\partial\tau}{\partial x}=\left\{\frac{ct}{\alpha^{1/2}\beta^{3/2}}-\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{\partial\theta}{\partial x}\right\}e^{-\theta},$$

$$\frac{\partial\zeta}{\partial t}=\frac{c^2t}{(\alpha\beta)^{1/2}}, \qquad \frac{\partial\tau}{\partial t}=-\left\{\frac{cx}{\alpha^{1/2}\beta^{3/2}}+\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{\partial\theta}{\partial t}\right\}e^{-\theta},$$

we may deduce that the Jacobian of the transformation (2.15) becomes

$$\frac{\partial(\zeta,\tau)}{\partial(x,t)}=-\frac{e^{-\theta}}{\beta}\left(x\frac{\partial\theta}{\partial t}-c^2t\frac{\partial\theta}{\partial x}+c\right)=\frac{ce^{-\theta}}{\beta}\left(\alpha\frac{\partial\theta}{\partial\alpha}-\beta\frac{\partial\theta}{\partial\beta}-1\right). \tag{2.16}$$

We note that in terms of the velocity $u(\alpha,\beta)$, from the relations (2.10), the above Jacobian becomes

$$\frac{\partial(\zeta,\tau)}{\partial(x,t)}=\frac{1}{\beta(1-u/c)^{1/2}(1+u/c)^{3/2}}\left(\alpha\frac{\partial u}{\partial\alpha}-\beta\frac{\partial u}{\partial\beta}-c\left(1-(u/c)^2\right)\right). \tag{2.17}$$

Assuming that this Jacobian is non-zero, we may express other Lorentz invariants as functions of both ζ and τ. Thus, for example, for the two Lorentz invariants given by Eq. (2.13), we have

$$\frac{x-ut}{\left(1-(u/c)^2\right)^{1/2}}=\frac{((\alpha-\beta)-(\alpha+\beta)(u/c))}{2(1-(u/c)^2)^{1/2}} \tag{2.18}$$

$$=\frac{1}{2}\left\{\alpha\left(\frac{1-u/c}{1+u/c}\right)^{1/2}-\beta\left(\frac{1+u/c}{1-u/c}\right)^{1/2}\right\}$$

$$= \frac{\zeta}{2}\left(\tau - \frac{1}{\tau}\right),$$

and

$$\frac{t - ux/c^2}{\left(1-(u/c)^2\right)^{1/2}} = \frac{((\alpha+\beta)-(\alpha-\beta)(u/c))}{2c(1-(u/c)^2)^{1/2}} \tag{2.19}$$

$$= \frac{1}{2c}\left\{\alpha\left(\frac{1-u/c}{1+u/c}\right)^{1/2} + \beta\left(\frac{1+u/c}{1-u/c}\right)^{1/2}\right\}$$

$$= \frac{\zeta}{2c}\left(\tau + \frac{1}{\tau}\right),$$

on making use of the following two relations which may be deduced from (2.15), namely,

$$\alpha\left(\frac{1-u/c}{1+u/c}\right)^{1/2} = \alpha e^{-\theta} = \zeta\tau, \qquad \beta\left(\frac{1+u/c}{1-u/c}\right)^{1/2} = \beta e^{\theta} = \frac{\zeta}{\tau}.$$

The above invariants ζ and τ might also be useful in demonstrating that certain quantities are not Lorentz invariant. For example, for the exact wave-like solution examined in some detail in Chaps. 5 and 6, the assumed linear force equations

$$f(u) = f_0(1+\lambda u/c), \qquad cg(u) = f_0(\lambda + u/c).$$

where λ and f_0 denote arbitrary constants, are not Lorentz invariant, and it turns out that the Lorentz invariant quantities are $e(u)f(u)$ and $e(u)g(u)$. The force relations themselves are only partially Lorentz invariant in the sense that their functional form is preserved under a Lorentz transformation with different constants λ and f_0. This is most easily seen from the relations

$$f(u) + cg(u) = f_0(1+\lambda)(1+u/c), \qquad f(u) - cg(u) = f_0(1-\lambda)(1-u/c),$$

so that

$$e(u)\left(f(u)+cg(u)\right) = e_0 f_0(1+\lambda)\left(\frac{1+u/c}{1-u/c}\right)^{1/2},$$

$$e(u)\left(f(u)-cg(u)\right) = e_0 f_0(1-\lambda)\left(\frac{1-u/c}{1+u/c}\right)^{1/2},$$

which cannot be expressed solely in terms of the invariants ζ and τ. However, on using (2.9) and $\Theta = \theta + \epsilon$, these relations become

$$e(u)\,(f(u)+cg(u)) = e_0 f_0(1+\lambda)e^{\theta} = e_0 f_0(1+\lambda)e^{-\epsilon}e^{\Theta},$$
$$e(u)\,(f(u)-cg(u)) = e_0 f_0(1-\lambda)e^{-\theta} = e_0 f_0(1-\lambda)e^{\epsilon}e^{-\Theta},$$

demonstrating that the Lorentz transformed force relations preserve the functional dependence on velocity u (viz. on θ) but with changed force parameters arising through the translational velocity v (viz. through ϵ).

In the event that the Jacobian (2.16) is zero, then $\theta(\alpha, \beta)$ satisfies the first order linear partial differential equation

$$\alpha\frac{\partial\theta}{\partial\alpha} - \beta\frac{\partial\theta}{\partial\beta} = 1. \tag{2.20}$$

This partial differential equation may be solved using Lagrange's characteristic method which is formally to introduce a characteristic parameter s through the three equations

$$\frac{d\alpha}{ds} = \alpha, \qquad \frac{d\beta}{ds} = -\beta, \qquad \frac{d\theta}{ds} = 1,$$

and then the general solution is obtained by taking one integral of these equations to be an arbitrary function of a second independent integral. Thus, for example, by division of the first equation, we have

$$\frac{d\beta}{d\alpha} = -\frac{\beta}{\alpha}, \qquad \frac{d\theta}{d\alpha} = \frac{1}{\alpha},$$

which can both be integrated to yield $\alpha\beta = C_1$ and $\theta = \log\alpha + C_2$, where C_1 and C_2 denote arbitrary constants, and from which we might deduce that the general solution of (2.20) may be determined from $C_2 = \Phi(C_1)$ where Φ denotes an arbitrary function. Accordingly, in terms of $\zeta = (\alpha\beta)^{1/2}$, the general solution of (2.20) is given by $\theta(\alpha, \beta) = \log\alpha + \Psi(\zeta)$, where Ψ denotes an arbitrary function. In this case, with $\exp\theta = \alpha\phi(\zeta)$ where $\phi(\zeta) = \exp\Psi(\zeta)$, we have from Eqs. (2.10) and (2.11) that the velocity u, energy e and momentum p are given, respectively, by

$$\frac{u}{c} = \frac{(\alpha\phi)^2 - 1}{(\alpha\phi)^2 + 1}, \qquad e = \frac{e_0}{2}\left(\alpha\phi + \frac{1}{\alpha\phi}\right), \qquad pc = \frac{e_0}{2}\left(\alpha\phi - \frac{1}{\alpha\phi}\right).$$

We show in the following section that these velocity fields are those for which $dx/dt = u(x,t)$ remains invariant under Lorentz transformation.

2.5 Lorentz Invariant Velocity Fields $u(x, t)$

In this section we pose the question of determining the most general one-dimensional velocity fields $u(x, t)$ that remain invariant under the Lorentz transformation (2.3). Alternatively, an equivalent question is to determine those velocity fields $u(x, t)$ and $u(X, T)$ which are such that the two differential problems

$$\frac{dx}{dt} = u(x, t), \qquad \frac{dX}{dT} = u(X, T). \tag{2.21}$$

transform into each other under the Lorentz transformations (2.2) and (2.3). Since the Lorentz transformations form a one-parameter group of transformations in the frame velocity v, we need only expand either (2.2) or (2.3) and equate the corresponding infinitesimals for either differential problem. Thus the infinitesimal versions of (2.2) are $X \approx x + vt$ and $T \approx t + vx/c^2$, and therefore from $(2.21)_2$, we obtain

$$\frac{dx + vdt}{dt + (v/c^2)dx} = \frac{dx/dt + v}{1 + (v/c^2)(dx/dt)} \approx \left(\frac{dx}{dt} + v\right)\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right) \approx \frac{dx}{dt} + v\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)$$
$$= u\left(x + vt, t + \frac{vx}{c^2}\right) \approx u(x, t) + v\left(t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t}\right),$$

on using Taylor's theorem to expand the last term. Thus by equating infinitesimals, we may readily deduce the following first order partial differential equation for $u(x, t)$; thus

$$t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t} = 1 - \left(\frac{u}{c}\right)^2. \tag{2.22}$$

In terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, we have $x = (\alpha - \beta)/2$ and $t = (\alpha + \beta)/2c$, and the differential formulae

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \qquad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta},$$

and the first order partial differential equation (2.22) becomes

$$\alpha\frac{\partial u}{\partial \alpha} - \beta\frac{\partial u}{\partial \beta} = c\left(1 - \left(\frac{u}{c}\right)^2\right),$$

which is precisely the condition for the vanishing of the Jacobian (2.17), as might be anticipated since essentially we seek a functional relationship between the two invariants ζ and τ, namely, $\tau = \phi(\zeta)$. Using Lagrange's characteristic method described above, we may formally solve the three characteristic equations with parameter s; thus

$$\frac{d\alpha}{ds} = \alpha, \qquad \frac{d\beta}{ds} = -\beta, \qquad \frac{du}{ds} = c\left(1 - \left(\frac{u}{c}\right)^2\right),$$

to confirm that the general solution is indeed given by $\tau = \phi(\zeta)$. From this relation and (2.15), namely,

$$\zeta = ((ct)^2 - x^2)^{1/2} = (\alpha\beta)^{1/2}, \qquad \tau = \left(\frac{ct+x}{ct-x}\right)^{1/2}\left(\frac{1-u/c}{1+u/c}\right)^{1/2} = \left(\frac{\alpha}{\beta}\right)^{1/2}\left(\frac{1-u/c}{1+u/c}\right)^{1/2},$$

we might deduce

$$\frac{d\beta}{d\alpha} = \left(\frac{1-u/c}{1+u/c}\right) = \frac{\beta}{\alpha}\phi^2((\alpha\beta)^{1/2}) = \frac{(\alpha\beta)\phi^2((\alpha\beta)^{1/2})}{\alpha^2}.$$

On introducing a new arbitrary function $\Phi(\zeta)$ which is defined by $\Phi(\zeta) = 1/\zeta^2\phi^2(\zeta)$, we obtain the differential equation

$$\frac{d\alpha}{d\beta} = \alpha^2\Phi((\alpha\beta)^{1/2}),$$

which is readily integrable with the substitution $\zeta = (\alpha\beta)^{1/2}$; thus

$$\frac{d\zeta}{d\beta} = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2} + \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{1/2}\frac{d\alpha}{d\beta} = \frac{\zeta}{2\beta}\left(1 + \zeta^2\Phi(\zeta)\right),$$

or similarly

$$\frac{d\zeta}{d\alpha} = \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{1/2} + \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{d\beta}{d\alpha} = \frac{\zeta}{2\alpha}\left(1 + \frac{1}{\zeta^2\Phi(\zeta)}\right),$$

and both of which are evidently separable.

2.6 General Framework for Lorentz Invariances

In this section we develop a general understanding of the formal origin of Lorentz invariances, and in particular we provide an independent confirmation of those established in the previous section. As we have previously stated, in this text we are considering a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to the first frame, and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 2.1. The coordinate notation adopted here is different to that normally used in special relativity which tends to involve both primed and unprimed variables. We do this purposely here because we wish to view the relative velocity v as a parameter measuring the departure of the current frame (x, y, z) from the rest frame (X, Y, Z), and for this purpose the notation employed in non-linear continuum mechanics is preferable. Time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t, and we assume throughout that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

We emphasise that although v is a constant, the approach adopted here and in [54] is to assume that the current frame is defined by the two variables (x, t) and each of x and t are characterised by the three independent variables (X, T, v), so that explicitly we have $x = x(X, T, v)$ and $t = t(X, T, v)$. Our basic approach is not to think of the relative velocity v as a velocity as such, but rather to envisage v simply as the parameter which provides a measure of the departure of one inertial from the rest frame. We are then able to exploit the well-trodden formalism from non-linear continuum mechanics, treating v as the time-like variable for which we can formally differentiate to produce velocity-like quantities.

In this context, it is important for the reader to fully appreciate that this approach is motivated from and completely analogous to that in non-linear continuum mechanics for which in standard notation, the spatial locations defined by (x, y, z) are viewed as functions of the four independent variables (X, Y, Z, t), namely, $x = x(X, Y, Z, t)$ and so on. Further, in continuum mechanics we usually introduce velocities such as $u = dx/dt$ where the total derivative d/dt means partial differentiation with respect to time keeping (X, Y, Z) fixed, and there is the well-known connection between this derivative, the velocities (u, v, w) and the spatial time derivative $\partial/\partial t$, which means partial differentiation with respect to time keeping the three variables (x, y, z) fixed; thus for $\phi = \phi(x, y, z, t)$ we have

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z},$$

and there are many other such mathematical formalities that can be exploited in the context of relativity, treating the relative frame velocity v as a time-like variable.

We have in mind that the Lorentz transformation (2.3) is a one-parameter group of transformations, with the relative velocity v serving as the parameter, and the identity $x = X$ and $t = T$ arising from the value $v = 0$. By the "pseudo-velocity" equations, we refer to the derivatives dx/dv and dt/dv, subject to the initial data $x = X$ and $t = T$ for $v = 0$. We now view x and t defined by (2.3) as functions of v, and by straightforward differentiation and subsequent simplification, we may derive the following "pseudo-velocity" equations:

$$\frac{dx}{dv} = \frac{-t}{1-(v/c)^2}, \quad \frac{dt}{dv} = \frac{-x/c^2}{1-(v/c)^2}, \tag{2.23}$$

where d/dv denotes total differentiation with respect to v, keeping the initial variables (X, T) fixed. We may now proceed to show that by formally solving (2.23), Einstein's theory emerges from the initial data $x = X$ and $t = T$ for $v = 0$. On re-arrangement of (2.23), we obtain

$$\left[1-\left(\frac{v}{c}\right)^2\right]\frac{dx}{dv}=-t, \quad \left[1-\left(\frac{v}{c}\right)^2\right]\frac{dt}{dv}=-\frac{x}{c^2},$$

which becomes a fully autonomous system if we introduce a new parameter ϵ such that

$$\frac{d}{d\epsilon}=\left[1-\left(\frac{v}{c}\right)^2\right]\frac{d}{dv},$$

so that $dv/d\epsilon = 1-(v/c)^2$. On making the further substitution $v = c\sin\phi$, this equation becomes $d\phi/\cos\phi = d\epsilon/c$, which integrates to give

$$\frac{1}{2}\log\left(\frac{1+\sin\phi}{1-\sin\phi}\right)=\frac{\epsilon}{c}+\text{constant}.$$

Suppose we assign the value $v = v_0$ at $\epsilon = 0$, then from this equation we may deduce

$$\frac{1+v/c}{1-v/c}=\left(\frac{1+v_0/c}{1-v_0/c}\right)e^{2\epsilon/c}, \tag{2.24}$$

and the special theory of relativity arises from the initial data $x = X$ and $t = T$ for $v_0 = 0$.

With $v_0 = 0$ we may deduce from (2.24) $v = c\tanh(\epsilon/c)$ for $0 \leqslant v < c$, and the two ordinary differential equations (2.23) become

$$\frac{dx}{d\epsilon}=-t, \quad \frac{dt}{d\epsilon}=-\frac{x}{c^2},$$

so that on differentiating either with respect to ϵ, we may eventually deduce

$$x(\epsilon)=A\sinh(\epsilon/c)+B\cosh(\epsilon/c), \qquad t(\epsilon)=-[A\cosh(\epsilon/c)+B\sinh(\epsilon/c)]/c, \tag{2.25}$$

where A and B denote arbitrary constants of integration. From the initial data and (2.25), we may deduce that $A = -cT$ and $B = X$, giving rise to the well-known pseudo-Euclidean rotation of special relativity (see [66, p. 10])

$$x(\epsilon)=X\cosh(\epsilon/c)-cT\sinh(\epsilon/c), \qquad t(\epsilon)=T\cosh(\epsilon/c)-\frac{X}{c}\sinh(\epsilon/c), \tag{2.26}$$

noting that from $v = c\tanh(\epsilon/c)$, we have

$$\cosh(\epsilon/c)=\frac{1}{[1-(v/c)^2]^{1/2}}, \quad \sinh(\epsilon/c)=\frac{v/c}{[1-(v/c)^2]^{1/2}}, \tag{2.27}$$

and together (2.26) and (2.27) yield (2.3), as might be expected. Of particular relevance in the above discussion are the expressions (2.23) for the infinitesimal vector of the Lorentz one-parameter group of transformations together with the initial data $x = X$ and $t = T$ for $v = 0$. In the following section, we demonstrate how we might characterise the integral invariants of the Lorentz group.

2.7 Integral Invariants of the Lorentz Group

In a previous section, we have established certain integral invariants of the Lorentz group such as (2.13) or (2.14) by direct substitution of either the Lorentz transformations (2.2) or their inverses (2.3). In this section we develop the machinery necessary to provide an alternative validation of the invariants of the Lorentz group, the details of which are presented in the following section. From the preceding section, it is clear that for any integral $I(x, t, v)$ for which $dI/dv = 0$, its value is determined by its value at $v = 0$; thus

$$\frac{dI}{dv} = 0, \qquad I(x, t, v) = I(X, T, 0), \qquad 0 \leqslant v < c.$$

Now on differentiating $v = v(x, t)$, we have from the chain rule for partial differentiation

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial t} dt,$$

so that

$$\frac{\partial v}{\partial x}\frac{dx}{dv} + \frac{\partial v}{\partial t}\frac{dt}{dv} = 1.$$

On using the above expressions (2.23) for the infinitesimal vector of the Lorentz one-parameter group of transformations, we may deduce the following first order partial differential equation:

$$t\frac{\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t} = \left(\frac{v}{c}\right)^2 - 1. \tag{2.28}$$

Again such first order partial differential equations are formally solved using Lagrange's characteristic method, which involves introducing a characteristic parameter s, and formulating the three ordinary differential relations

$$\frac{dx}{ds} = t, \quad \frac{dt}{ds} = \frac{x}{c^2}, \quad \frac{dv}{ds} = \left(\frac{v}{c}\right)^2 - 1,$$

and then if $I_1(x, t, v)$ and $I_2(x, t, v)$ denote any two independent integrals of the reduced system

$$\frac{dx}{dt} = \frac{c^2 t}{x}, \quad \frac{dv}{dx} = \frac{(v/c)^2 - 1}{t}, \tag{2.29}$$

the general solution of the first order partial differential equation (2.28) is then given by $I_1(x, t, v) = \Phi(I_2(x, t, v))$, where Φ denotes an arbitrary function.

The first differential equation readily integrates to give $\zeta^2 = (ct)^2 - x^2 = constant$, while the second differential equation becomes

$$\frac{dv}{dx} = c\frac{(v/c)^2 - 1}{(x^2 + \zeta^2)^{1/2}},$$

and on treating ζ as a constant for the purposes of this integration, the integration may then be effected through the substitutions,

$$x = \zeta \sinh \psi, \quad v = c \sin \phi,$$

to yield the differential relation $d\psi + d\phi/\cos\phi = 0$. This equation integrates to give

$$\psi + \frac{1}{2}\log\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right) = \text{constant},$$

and on using the elementary formula $\sinh^{-1}(z) = \log(z + (1 + z^2)^{1/2})$, we may deduce, from the above relations, the second independent integral

$$\rho = \left(\frac{ct + x}{ct - x}\right)\left(\frac{1 + v/c}{1 - v/c}\right) = \text{constant},$$

and the validity of ζ = constant and ρ = constant can be verified either directly from the characteristic relations (2.4) or using the above differential relations (2.23) to show that $d\zeta/dv = d\rho/dv = 0$ which we now proceed to describe.

In order to achieve this, we again need the two differential relations (2.23), namely,

$$\frac{dx}{dv} = \frac{-t}{1 - (v/c)^2}, \quad \frac{dt}{dv} = \frac{-x/c^2}{1 - (v/c)^2}, \tag{2.30}$$

which we need to use in the evaluation of $d\zeta/dv$ and $d\rho/dv$; thus

$$\frac{d\zeta^2}{dv} = \frac{d}{dv}((ct)^2 - x^2) = \frac{2}{1 - (v/c)^2}\left(c^2 t(-x/c^2) - x(-t)\right) = 0,$$

$$\frac{d\rho}{dv} = \frac{d}{dv}\left[\left(\frac{ct+x}{ct-x}\right)\left(\frac{1+v/c}{1-v/c}\right)\right] = \left(\frac{1+v/c}{1-v/c}\right)\frac{d}{dv}\left(\frac{ct+x}{ct-x}\right) + \left(\frac{ct+x}{ct-x}\right)\frac{d}{dv}\left(\frac{1+v/c}{1-v/c}\right)$$
$$= -\frac{2}{c}\left(\frac{ct+x}{ct-x}\right)\frac{1}{(1-v/c)^2} + \frac{2}{c}\left(\frac{ct+x}{ct-x}\right)\frac{1}{(1-v/c)^2} = 0,$$

as required. Thus, two independent integrals of the system (2.29) are

$$\zeta^2 = (ct)^2 - x^2, \qquad \rho = \left(\frac{ct+x}{ct-x}\right)\left(\frac{1+v/c}{1-v/c}\right),$$

so that the general solution of the above first order partial differential equation is given by $\rho = \Phi(\zeta)$, where Φ denotes an arbitrary function.

This may be formally confirmed as follows: From $\rho = \Phi(\zeta)$ we may deduce that

$$\left(\frac{1+v/c}{1-v/c}\right) = \beta^2\Psi((\alpha\beta)^{1/2}), \tag{2.31}$$

where Ψ denotes a second arbitrary function and α and β denote the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$. From the immediately above equation, we obtain

$$\frac{v(\alpha,\beta)}{c} = \left(\frac{\beta^2\Psi((\alpha\beta)^{1/2}) - 1}{\beta^2\Psi((\alpha\beta)^{1/2}) + 1}\right), \tag{2.32}$$

and in terms of the characteristic variables α and β, the first order partial differential equation (2.28) becomes

$$\frac{1}{c}\left(\alpha\frac{\partial v}{\partial\alpha} - \beta\frac{\partial v}{\partial\beta}\right) = \left(\frac{v}{c}\right)^2 - 1, \tag{2.33}$$

which we observe, apart from a change of sign, coincides with the equation for $u(\alpha,\beta)$ that arises from the vanishing of the Jacobian given by (2.17), noting however that this is an important distinction, and that the two calculations are distinct, although evidently related. On evaluating the two partial derivatives $\partial v/\partial\alpha$ and $\partial v/\partial\beta$ using the above expression (2.32), we have

$$\frac{1}{c}\frac{\partial v}{\partial\alpha} = \frac{\beta^3\Psi'(\zeta)}{\zeta(\beta^2\Psi(\zeta)+1)^2}, \qquad \frac{1}{c}\frac{\partial v}{\partial\beta} = \frac{\beta(4\Psi(\zeta) + \zeta\Psi'(\zeta))}{(\beta^2\Psi(\zeta)+1)^2},$$

where $\zeta = (\alpha\beta)^{1/2}$, and we may now show that (2.32) constitutes a formal solution of (2.33) for all arbitrary functions $\Psi((\alpha\beta)^{1/2})$. Subsequently, we see that the Jacobian $\partial(X,T)/\partial(x,t)$ vanishes whenever $v(x,t)$ satisfies the above partial differential equation (2.28) or (2.33), implying that the integral invariants correspond to families of singular transformations.

2.8 Alternative Validation of Lorentz Invariants

We are now well placed to establish the integral invariants of the Lorentz group in a very interesting manner using the "pseudo-velocity" equations (2.23), namely,

$$\frac{dx}{dv} = \frac{-t}{1-(v/c)^2}, \quad \frac{dt}{dv} = \frac{-x/c^2}{1-(v/c)^2}. \tag{2.34}$$

In order to do this, we need an expression for the velocity $u(x, t)$ which we obtain by division of the differential relations (2.23) or (2.34) or directly from (2.29); thus

$$u = \frac{dx}{dt} = \frac{c^2 t}{x}.$$

As a first example, we consider the Einstein addition of velocities law (2.6) expressed in the form

$$U = \frac{u+v}{1+uv/c^2}, \quad 0 \leqslant v < c.$$

On replacement of u by c^2t/x, we have

$$\frac{u+v}{1+uv/c^2} = \frac{xv+tc^2}{x+vt},$$

which we now differentiate with respect to v, and using the differential relations (2.30), we find that

$$\frac{d}{dv}\left(\frac{u+v}{1+uv/c^2}\right) = \frac{d}{dv}\left(\frac{xv+tc^2}{x+vt}\right) = \frac{1}{(x+vt)}\frac{d}{dv}(xv+tc^2) - \frac{(xv+tc^2)}{(x+vt)^2}\frac{d}{dv}(x+vt)$$
$$= \frac{x}{(x+vt)} - \frac{1}{(1-(v/c)^2)} - \left(\frac{xv+tc^2}{x+vt}\right)\frac{v}{c^2(1-(v/c)^2)} = 0,$$

as might be expected. Again, formally the value of the integral is fixed through the initial data which in this case is $u = U$ when $v = 0$, which produces the required result.

Similarly, for the Lorentz invariants ζ and τ defined by (2.15), we have

$$\frac{d\zeta^2}{dv} = \frac{d((ct)^2 - x^2)}{dv} = 2\left(c^2 t\frac{dt}{dv} - x\frac{dx}{dv}\right) = 0,$$
$$\frac{d\tau^2}{dv} = \frac{d}{dv}\left[\left(\frac{ct+x}{ct-x}\right)\left(\frac{1-u/c}{1+u/c}\right)\right] = \frac{d}{dv}\left[\left(\frac{ct+x}{ct-x}\right)\left(\frac{1-ct/x}{1+ct/x}\right)\right] = \frac{d1}{dv} = 0.$$

Other integral invariants previously given by (2.13) are as follows:

$$\frac{x - ut}{\left(1 - (u/c)^2\right)^{1/2}} = \frac{X - UT}{\left(1 - (U/c)^2\right)^{1/2}}, \quad \frac{t - ux/c^2}{\left(1 - (u/c)^2\right)^{1/2}} = \frac{T - UX/c^2}{\left(1 - (U/c)^2\right)^{1/2}},$$

which we have previously established by direct substitution of the Lorentz transformations into the left-hand sides with appropriate use of the addition of velocities law. Alternatively, we may establish these by differentiation using $u = c^2t/x$; thus

$$\frac{d}{dv}\left(\frac{x - ut}{\left(1 - (u/c)^2\right)^{1/2}}\right) = \frac{d}{dv}\left(\frac{x - (ct)^2/x}{(1 - (ct/x)^2)^{1/2}}\right) = \frac{d}{dv}\left(x^2 - (ct)^2\right)^{1/2} = 0,$$

$$\frac{d}{dv}\left(\frac{t - ux/c^2}{\left(1 - (u/c)^2\right)^{1/2}}\right) = \frac{d}{dv}\left(\frac{t - tx/x}{\left(1 - (u/c)^2\right)^{1/2}}\right) = 0,$$

as required.

2.9 Jacobians of the Lorentz Transformations

In this section we deduce formulae for the Jacobians of the Lorentz transformations (2.2) and their inverses (2.3). We observe from Eq. (2.2) on carefully calculating the four partial derivatives $\partial X/\partial x$, $\partial X/\partial t$, $\partial T/\partial x$ and $\partial T/\partial t$ treating v as a function of x and t, so that, for example, we have

$$\frac{\partial X}{\partial x} = \frac{1}{(1 - (v/c)^2)^{1/2}} + \frac{(xv + c^2t)}{c^2(1 - (v/c)^2)^{3/2}}\frac{\partial v}{\partial x}, \tag{2.35}$$

$$\frac{\partial X}{\partial t} = \frac{v}{(1 - (v/c)^2)^{1/2}} + \frac{(xv + c^2t)}{c^2(1 - (v/c)^2)^{3/2}}\frac{\partial v}{\partial t},$$

$$\frac{\partial T}{\partial x} = \frac{v}{c^2(1 - (v/c)^2)^{1/2}} + \frac{(x + tv)}{c^2(1 - (v/c)^2)^{3/2}}\frac{\partial v}{\partial x},$$

$$\frac{\partial T}{\partial t} = \frac{1}{(1 - (v/c)^2)^{1/2}} + \frac{(x + tv)}{c^2(1 - (v/c)^2)^{3/2}}\frac{\partial v}{\partial t},$$

and from which we may eventually deduce

$$\frac{\partial(X, T)}{\partial(x, t)} = \left(1 + \frac{\frac{t\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t}}{\left(1 - (v/c)^2\right)}\right), \tag{2.36}$$

and we see that the Jacobian vanishes whenever $v(x, t)$ satisfies the above partial differential equation (2.28), implying that the integral invariants correspond to families of singular transformations. Similarly, on treating the relative velocity v

as a function of X and T, from the inverse Lorentz relations (2.3), we may deduce the corresponding equation

$$\frac{\partial(x,t)}{\partial(X,T)} = \left(1 - \frac{\frac{T\partial v}{\partial X} + \frac{X}{c^2}\frac{\partial v}{\partial T}}{\left(1-(v/c)^2\right)}\right). \tag{2.37}$$

For convenience, we introduce the notation J, δ and Δ such that

$$1/J = \frac{\partial(X,T)}{\partial(x,t)} = (1+\delta), \qquad J = \frac{\partial(x,t)}{\partial(X,T)} = (1-\Delta),$$

where δ and Δ are defined, respectively, by

$$\delta = \frac{\frac{t\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t}}{\left(1-(v/c)^2\right)}, \qquad \Delta = \frac{\frac{T\partial v}{\partial X} + \frac{X}{c^2}\frac{\partial v}{\partial T}}{\left(1-(v/c)^2\right)},$$

and since the product of the two Jacobians is necessarily unity, there follows the necessary identity $(1/\Delta) = 1 + (1/\delta)$ which may be formally verified as follows:

$$\delta = \frac{\frac{t\partial v}{\partial x} + \frac{x}{c^2}\frac{\partial v}{\partial t}}{\left(1-(v/c)^2\right)} = \frac{t\frac{\partial(v,t)}{\partial(x,t)} - \frac{x}{c^2}\frac{\partial(v,x)}{\partial(x,t)}}{\left(1-(v/c)^2\right)},$$

and multiplication by J gives

$$\delta J\left(1-(v/c)^2\right) = t\frac{\partial(v,t)}{\partial(X,T)} - \frac{x}{c^2}\frac{\partial(v,x)}{\partial(X,T)} = \frac{\partial\left(v, t^2-(x/c)^2\right)}{2\partial(X,T)},$$

and on using the invariant $(ct)^2 - x^2 = (cT)^2 - X^2$, we have

$$\delta J\left(1-(v/c)^2\right) = \frac{\partial\left(v, T^2-(X/c)^2\right)}{2\partial(X,T)} = \frac{T\partial v}{\partial X} + \frac{X}{c^2}\frac{\partial v}{\partial T} = \left(1-(v/c)^2\right)\Delta,$$

and the identity $(1/\Delta) = 1 + (1/\delta)$ now follows from $J = 1 - \Delta$.

As previously mentioned, we view the relative velocity v as a time-like variable in a manner completely analogous to non-linear continuum mechanics, and by formal differentiation of $J = \partial(x,t)/\partial(X,T)$ with respect to v, recalling that (X, T, v) are treated as the independent variables, we may deduce

$$\frac{dJ}{dv} = \frac{\partial(\frac{dx}{dv},t)}{\partial(X,T)} + \frac{\partial(x,\frac{dt}{dv})}{\partial(X,T)} = \left(\frac{\partial(\frac{dx}{dv},t)}{\partial(x,t)} + \frac{\partial(x,\frac{dt}{dv})}{\partial(x,t)}\right)\frac{\partial(x,t)}{\partial(X,T)},$$

and on using (2.23) or (2.30), we may eventually deduce

$$\frac{dJ}{dv} = \frac{2v/c^2}{\left(1-(v/c)^2\right)}(J-1),$$

which may be integrated to yield

$$J = \frac{\partial(x,t)}{\partial(X,T)} = \left(1 - \frac{A(X,T)}{\left(1-(v/c)^2\right)}\right),$$

where $A(X, T)$ is at most a function of X and T, and this equation is entirely consistent with (2.37).

2.10 Space-Time Transformation $x' = ct$ and $t' = x/c$

The relation $uU = c^2$ formally arises from the underlying transformation $x = cT, t = X/c$. With a primed notation, the space-time transformation $x' = ct$ and $t' = x/c$, for which $u' = dx'/dt' = c^2 dt/dx = c^2/u$, has been widely used to connect the Galilean and Carroll transformations as significant limits of Lorentz invariant theories, for example, in electromagnetism. The transformations $x' = ct$ and $t' = x/c$ were originally introduced by Jean-Marc Levy-Leblond, and their origin and development is fully detailed by Rousseaux [89] and Houlrik and Rousseaux [61]. Here we observe that the Lorentz transformations (2.2) or (2.3) are left unchanged by this transformation, namely,

$$X' = cT, \qquad T' = \frac{X}{c}, \qquad x' = ct, \qquad t' = \frac{x}{c}, \tag{2.38}$$

assuming the same constant value for the frame velocity v. Further, on making this transformation for the equation describing Galilean invariance for particles, namely, $x = X - vT$ and $t = T$, we obtain the equations $x' = X'$ and $t' = T' - vX'/c^2$ describing Galilean invariance for waves, and for a fuller account, we refer the reader to Houlrik and Rousseaux [61].

We observe the seemingly curious property that under the space-time transformation (2.38) with $u' = c^2/u$, $U' = c^2/U$ and $v' = c^2/v$, changing any two means that the Einstein addition of velocities law (2.6) remains invariant, in the sense that the same value for the third is recorded as in the unprimed frame, so that, for example, for u we have from (2.6)

$$u = \frac{U - v}{\left(1 - Uv/c^2\right)} = \frac{\left(c^2/U' - c^2/v'\right)}{\left(1 - c^2/U'v'\right)} = \frac{U' - v'}{\left(1 - U'v'/c^2\right)}.$$

This property reflects the symmetry and parity of the Einstein addition of velocities law which is most apparent when written in the form of (2.8).

We now examine the changes to the wavelengths and frequencies in the unprimed and primed frames. We consider a simple wave of wavelength μ and frequency ν in the unprimed (x, t) frame with $\mu\nu = c^2/u$; thus

$$y(x, t) = A \exp\left\{2\pi i \left(\frac{x}{\mu} - t\nu\right)\right\}, \tag{2.39}$$

where A denotes the constant amplitude. A corresponding simple wave in the primed frame becomes

$$y(x', t') = A \exp\left\{2\pi i \left(\frac{x'}{\mu'} - t'\nu'\right)\right\} = A \exp\left\{-2\pi i \left(\frac{x\nu'}{c} - \frac{ct}{\mu'}\right)\right\}, \tag{2.40}$$

and therefore under the primed transformation $x' = ct$ and $t' = x/c$, the wavelength and frequency transform according to the formulae

$$\mu' = \frac{c}{\nu}, \qquad \nu' = \frac{c}{\mu}, \tag{2.41}$$

with the consequence that $\mu'\nu' = u$.

Remark We emphasise that the space-time transformation $x' = ct$ and $t' = x/c$ involves only one Cartesian spatial dimension, and reflects a simple invariance of the operator for the classical one-dimensional wave equation, so that the appropriate extension and interpretation to three spatial dimensions and other coordinates involving curvature is by no means obvious. However, while Guemez et al. [44] have proposed one extension of the de Broglie formula $uu' = c^2$ to three spatial dimensions (x, y, z), namely, $\mathbf{u}.\mathbf{u}' = c^2$, the most likely coordinate decomposition of this formula $\mathbf{r}' = ct\mathbf{r}/r$ and $t' = r/c$, where $\mathbf{r}$ and $\mathbf{r}'$ are the obvious position vectors and $r = (x^2 + y^2 + z^2)^{1/2}$, and with the identical inverse transformations $\mathbf{r} = ct'\mathbf{r}'/r'$ and $t = r'/c$ where $r' = (x'^2 + y'^2 + z'^2)^{1/2}$, remains essentially a single spatial dimension transformation. The coordinate transformation $\mathbf{r}' = ct\mathbf{r}/r$ is spatially spherically symmetric, and polar angles remain unchanged and $r' = ct$ and $t' = r/c$, so that the transformation $\mathbf{r}' = ct\mathbf{r}/r$ is essentially one dimensional. We further comment that although this coordinate decomposition is a natural extension of the one-dimensional coordinate transformation, it may not be unique. Indeed, the one-dimensional transformation itself $x' = ct$ and $t' = x/c$ for $uu' = c^2$ does not provide a unique decomposition of the equation $uu' = c^2$, since the negative transformation $x' = -ct$ and $t' = -x/c$ is equally effective.

2.11 The de Broglie Wave Velocity $u' = c^2/u$

All matter exhibits wave-like behaviour, and Louis de Broglie [17] first predicted light to display the dual characteristics both as a collection of particles, called photons, and in some respects as a wave. The particle velocity u is the group velocity of the wave, and if the particle velocity u is sub-luminal, then the associated wave or phase velocity c^2/u through the de Broglie relation is necessarily superluminal. At various times in his life, de Broglie held a concrete physical picture of the co-existence of both particle and its associated wave and refers to "the theory of the double solution", for which he formulated an equation which he called "the guidance formula" (see Eq. (1.2) and also [108]). Since $uu' = c^2$, from (2.6) their relative velocity is necessarily infinite. However, the average of the particle and wave velocities, denoted here by V, is given by

$$V = \frac{1}{2}(u + u') = \frac{1}{2}\left(u + \frac{c^2}{u}\right), \tag{2.42}$$

and from the elementary identity

$$\left(\frac{V}{c}\right)^2 - 1 = \frac{1}{4}\left[\left(\frac{u}{c}\right)^2 + \left(\frac{c}{u}\right)^2 + 2\right] - 1 = \frac{1}{4}\left(\frac{c}{u} - \frac{u}{c}\right)^2 \geq 0,$$

so that for sub-luminal particle velocity u, both the wave velocity u' and their average V are necessarily both superluminal. In addition, the three velocities u, u' and V satisfy the Einstein addition of velocity law in its various forms, but specifically in the form of (2.8), we have

$$\left(\frac{V/c+1}{V/c-1}\right) = \left(\frac{1+u/c}{1-u/c}\right)\left(\frac{u'/c+1}{u'/c-1}\right).$$

With θ as defined previously by (2.11), we have the following relations:

$$u = c\tanh\theta, \qquad u' = c\coth\theta, \qquad e = e_0\cosh\theta, \qquad pc = e_0\sinh\theta,$$

where as before $e_0 = m_0c^2$ denotes the rest mass energy, and it is not difficult to establish the formulae

$$V = c\coth 2\theta, \qquad \frac{1}{\left((V/c)^2 - 1\right)^{1/2}} = \sinh 2\theta,$$

which may be used to show that

$$\frac{e_0^2(V/c)}{\left((V/c)^2 - 1\right)^{1/2}} = e^2 + (pc)^2, \qquad \frac{e_0^2}{\left((V/c)^2 - 1\right)^{1/2}} = 2epc.$$

By addition and subtraction of these expressions, we might readily deduce

$$e + cp = e_0 \left(\frac{V/c + 1}{V/c - 1} \right)^{1/4}, \qquad e - cp = e_0 \left(\frac{V/c - 1}{V/c + 1} \right)^{1/4},$$

and therefore

$$e = \frac{e_0}{2} \left[\left(\frac{V/c + 1}{V/c - 1} \right)^{1/4} + \left(\frac{V/c - 1}{V/c + 1} \right)^{1/4} \right],$$

$$cp = \frac{e_0}{2} \left[\left(\frac{V/c + 1}{V/c - 1} \right)^{1/4} - \left(\frac{V/c - 1}{V/c + 1} \right)^{1/4} \right].$$

Now since the average velocity V is a symmetric function of u and u', we expect this symmetry to be reflected in the expression $V = c \coth 2\theta$, and in order to see this, we need to determine θ' through (2.10); thus formally we have

$$e^{\theta'} = \left(\frac{1 + u'/c}{1 - u'/c} \right)^{1/2} = \left(\frac{1 + u/c}{u/c - 1} \right)^{1/2} = \frac{1}{i} \left(\frac{1 + u/c}{1 - u/c} \right)^{1/2} = e^{\theta} e^{-i\pi/2},$$

and from which we might deduce the relation $\theta' = \theta - i\pi/2$. Using the standard expressions,

$$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$$

$$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y,$$

from which we obtain

$$V' = c \coth 2\theta' = c \frac{\cosh 2\theta'}{\sinh 2\theta'} == c \frac{\cosh(2\theta - i\pi)}{\sinh(2\theta - i\pi)} = c \frac{\cosh 2\theta}{\sinh 2\theta} = V,$$

as might be expected.

2.12 Force and Physical Energy Arising from Work Done

The basic notions of force, as rate of change of momenta, and physical energy, as the work done (viz. force times distance) arise in the two rate-of-working equations (or work done equations) for the physical energies E and e in the (X, T) and (x, t) frames, respectively, and these are as follows:

$$dE = F dX = \frac{dP}{dT} dX, \qquad de = f dx = \frac{dp}{dt} dx, \tag{2.43}$$

where $F = dP/dT$ and $f = dp/dt$ denote the physical force in the two frames and $P = MU$ and $p = mu$ the momenta where $U = dX/dT$ and $u = dx/dt$ are the respective particle velocities. Using these relations and the expressions $E = Mc^2$ and $e = mc^2$, the Eqs. (2.43) on multiplication by their respective masses may be rewritten as

$$E\frac{dE}{dT} = c^2 P\frac{dP}{dT}, \qquad e\frac{de}{dt} = c^2 p\frac{dp}{dt}.$$

These equations evidently integrate to yield the respective equations $E^2 = (Pc)^2 + constant$ and $e^2 = (pc)^2 + constant$. It might be important to appreciate that the arbitrary constants in these equations are generally fixed by taking the particle energy at $E = e = e_0 = m_0c^2$ at zero velocity, so that we have $E^2 - (Pc)^2 = e^2 - (pc)^2 = e_0^2$. However, this does not necessarily have to be the case and there may be other interpretations for the constants. Here we assume these arbitrary constants are as generally prescribed, and we assume the energy statements $E^2 = {e_0}^2 + (Pc)^2$ and $e^2 = {e_0}^2 + (pc)^2$, so that along with $E = Mc^2$ and $e = mc^2$, we might deduce the Einstein formulae for the variation of mass with velocity

$$M(U) = \frac{m_0}{[1 - (U/c)^2]^{1/2}}, \qquad m(u) = \frac{m_0}{[1 - (u/c)^2]^{1/2}}. \tag{2.44}$$

We comment that the formula $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ is only one of the many expressions showing a particular variation of mass with its velocity, and this expression has a long and extensive history involving many eminent scientists such as Abraham, Bücherer, Lorentz, Ehrenfest, Kaufmann and of course Einstein, who first grappled with the notion that the "transverse and longitudinal" masses may be distinct. The development of the Einstein expression is fully detailed by Weinstein [109].

In [56] and [57], the authors generalise the Einstein mass variation for both sub-luminal and superluminal velocities by supplementing the condition $f = F$ with the condition that the energy-mass rate is the same in both frames, namely, $de/dm = dE/dM$, which for the Einstein expression is simply constant and equal to c^2. Specifically, assuming the Lorentz transformations (2.3), the Lorentz invariant energy-momentum relations discussed in the following section and the two invariances

$$\frac{dp}{dt} = \frac{dP}{dT}, \quad \frac{dm}{dx} = \frac{dM}{dX},$$

which are known to apply in special relativity, new mass variation formulae involving two arbitrary constants are obtained, noting that the Einstein expression involves only the rest mass as a single arbitrary constant. For example, for sub-luminal velocities $0 < u < c$, and with θ and ϵ defined by Eqs. (2.9) and (2.10), we may deduce in [56] the mass variation expression

$$m(u) = \frac{C_2}{(1-(u/c)^2)^{1/2}} \left\{ C_1 + \frac{1}{4\tanh(\epsilon/2)} \int_0^{\theta+\epsilon/2} \frac{d\rho}{(\sinh\rho)^{1/2}} + \frac{1}{2\,[\sinh(\theta+\epsilon/2)]^{1/2}} \right\},$$

where C_1 and C_2 denote arbitrary constants and the Einstein expression arises from the above equation in the limiting case $C_1 \to \infty$. The corresponding energy is found to be given by

$$e(u) = m(u)c^2 - \frac{C_2 c^2}{\sinh(\epsilon/2)} \left\{ [\sinh(\theta+\epsilon/2)]^{1/2} - [\sinh(\epsilon/2)]^{1/2} \right\},$$

noting that in this expression the energy has been normalised such that $e = m_0 c^2$ when $u = 0$. Although to a certain extent this datum energy level is arbitrary, we observe that for this particular normalisation, the additive constant might become pure imaginary depending upon whether $v > 0$ or $v < 0$. This highlights the issue that the region of validity of the above expression for $m(u)$ might require careful consideration since even the seemingly natural choice of datum energy levels may not be applicable, and this aspect is fully examined in [56].

For superluminal velocities $c < u/c < \infty$, there are corresponding expressions derived in [57], essentially with the sinh replaced by cosh; thus

$$m(u) = \frac{C_2}{((u/c)^2-1)^{1/2}} \left\{ C_1 + \frac{1}{4}\tanh(\epsilon/2) \int_0^{\theta+\epsilon/2} \frac{d\rho}{(\cosh\rho)^{1/2}} + \frac{1}{2\,[\cosh(\theta+\epsilon/2)]^{1/2}} \right\},$$

with corresponding energy formulae given by

$$e(u) = m(u)c^2 - \frac{C_2 c^2}{\cosh(\epsilon/2)} \left\{ \frac{\sinh(\theta+\epsilon/2)/2}{[\cosh(\theta+\gamma/2)]^{1/2}} - \frac{\sinh(\epsilon/2)}{[\cosh(\epsilon/2)]^{1/2}} \right\},$$

where again C_1 and C_2 denote arbitrary constants and in this case the energy is assumed to vanish for $u \longrightarrow \infty$. We note especially that in the case of superluminal velocities, from Eq. (2.8), it is clear that at least one of the (U, u, v) must be subluminal and the immediately above expressions are derived on the assumption that $u, v > c$ and that $U < c$, so that in this case Θ, θ and ϵ must be appropriately defined; thus

$$\Theta = \log\left(\frac{1+U/c}{1-U/c}\right), \qquad \theta = \log\left(\frac{u/c+1}{u/c-1}\right), \qquad \epsilon = \log\left(\frac{v/c+1}{v/c-1}\right),$$

for which again $\Theta = \theta + \epsilon$ and with inverses given by

$$U = c\tanh(\Theta/2), \qquad u = c\coth(\theta/2), \qquad v = c\coth(\epsilon/2).$$

Applications of the above formulae are given in [56] and [57] and we refer the reader to these papers for details.

2.13 Lorentz Invariant Energy-Momentum Relations

Lorentz Invariant Energy-Momentum Relations The Lorentz invariant energy-momentum relations given by Eq. (2.46) can both be deduced from (2.6) as follows: For sub-luminal velocities $v, u, U < c$, the Einstein formulae for mass and energy in both frames are summarised by the formulae

$$E = Mc^2, \qquad M = \frac{m_0}{[1-(U/c)^2]^{1/2}}, \qquad e = mc^2, \qquad m = \frac{m_0}{[1-(u/c)^2]^{1/2}}, \tag{2.45}$$

so that with momenta $P = MU$ and $p = mu$, we have on multiplication of (2.6) by $m_0\left[1-(u/c)^2\right]^{-1/2}$, and by using Eq. (2.7), we may readily deduce $(2.46)_1$; thus

$$\frac{um_0}{[1-(u/c)^2]^{1/2}} = \frac{m_0U - m_0v}{[1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2}},$$

while $(2.46)_2$ arises directly from $(2.45)_3$ and $(2.45)_4$, on using Eq. (2.7); thus

$$e = \frac{m_0c^2}{[1-(u/c)^2]^{1/2}} = \frac{m_0c^2(1-Uv/c^2)}{[1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2}},$$

so that altogether we obtain

$$p = \frac{P - Ev/c^2}{[1-(v/c)^2]^{1/2}}, \qquad e = \frac{E - Pv}{[1-(v/c)^2]^{1/2}}. \tag{2.46}$$

The inverse relations are given by

$$P = \frac{p + ev/c^2}{[1-(v/c)^2]^{1/2}}, \qquad E = \frac{e + pv}{[1-(v/c)^2]^{1/2}}, \tag{2.47}$$

and together these equations are referred to as the Lorentz invariant energy-momentum relations. Some authors [13] refer, respectively, to the above notions of mass m and momentum p as "temporal" and "spatial" momentum. We do not follow that distinction here and we refer to the relations (2.46) as the Lorentz invariant energy-momentum equations. From these relations, it is also clear that we have the Lorentz invariant $e^2 - (pc)^2 = E^2 - (Pc)^2 = e_0^2$, where $e_0 = m_0c^2$ denotes the rest mass energy.

Lorentz Invariants $\xi(x,t) = ex - c^2pt$ and $\eta(x,t) = px - et$ By direct substitution we may establish, using Eqs. (2.3) and (2.46), that $\xi(x,t)$ and $\eta(x,t)$ as defined by the equations

$$\xi = ex - c^2 pt, \qquad \eta = px - et, \tag{2.48}$$

constitute two Lorentz invariances of special relativity, which are readily verified as follows: On evaluating $\xi = ex - c^2 pt$ and $\eta = px - et$ using (2.3) and (2.46), we have

$$\frac{\big((E - Pv)(X - vT) - c^2(P - Ev/c^2)(T - vX/c^2)\big)}{\big(1 - (v/c)^2\big)} = EX - c^2 PT,$$

and

$$\frac{\big((P - Ev/c^2)(X - vT) - (E - Pv)(T - vX/c^2)\big)}{\big(1 - (v/c)^2\big)} = PX - ET,$$

as required. We further observe that with γ and δ defined, respectively, by

$$\gamma = \frac{1}{2c}(e + pc)(ct - x), \qquad \delta = \frac{1}{2c}(e - pc)(ct + x), \tag{2.49}$$

the two Lorentz invariants ξ and η given by (2.48) become

$$\xi = ex - c^2 pt = -c(\gamma - \delta), \qquad \eta = px - et = -(\gamma + \delta), \tag{2.50}$$

so that we have the expressions

$$\xi + c\eta = -(e + pc)(ct - x), \qquad \xi - c\eta = (e - pc)(ct + x), \tag{2.51}$$

and the Lorentz invariant $e^2 - (pc)^2 = e_0^2$ becomes $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$. From these relations we may deduce

$$e - pc = e_0^2\left(\frac{x - ct}{\xi + c\eta}\right) = \left(\frac{\xi - c\eta}{x + ct}\right), \qquad e + pc = e_0^2\left(\frac{x + ct}{\xi - c\eta}\right) = \left(\frac{\xi + c\eta}{x - ct}\right),$$

revealing that each of the conditions $e = \pm pc$ occurs for both $x = \pm ct$ and $\xi = \pm c\eta$. Further, from (2.51) we observe that

$$\frac{\xi - c\eta}{\xi + c\eta} = -\left(\frac{e - pc}{e + pc}\right)\left(\frac{ct + x}{ct - x}\right) = -\left(\frac{1 - u/c}{1 + u/c}\right)\left(\frac{ct + x}{ct - x}\right) = -\tau^2, \tag{2.52}$$

where τ is the second Lorentz invariant defined by (2.15), and from (2.52), we may deduce the following relationship between the three Lorentz invariants ξ, η and τ; thus

$$\frac{c\eta}{\xi} = \left(\frac{1+\tau^2}{1-\tau^2}\right).$$

Inverse Relations for Lorentz Invariants On writing the Lorentz invariants (2.48) as follows

$$x - ut = \frac{\xi(x,t)}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{1/2}, \qquad x\frac{u}{c} - ct = \frac{c\eta(x,t)}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{1/2},$$

which we may regard as two equations in the two unknowns x and t, and formally solve to obtain

$$x = \frac{\xi(x,t) - u\eta(x,t)}{e_0(1-(u/c)^2)^{1/2}}, \quad ct = \frac{\xi(x,t)u/c - c\eta(x,t)}{e_0(1-(u/c)^2)^{1/2}},$$

which are evidently reminiscent of the Lorentz transformations (2.3), and we make this connection more precise in the following:

Lorentz Invariants as a Coordinate Transformation Clearly we may express the immediately above equations in the form of a Lorentz transformation, namely,

$$x^* = \frac{x - ut}{[1-(u/c)^2]^{1/2}}, \quad t^* = \frac{t - ux/c^2}{[1-(u/c)^2]^{1/2}}, \tag{2.53}$$

where $\xi = e_0 x^*$ and $\eta = -e_0 t^*$, and from which we may deduce the following expression for the velocity $u^* = dx^*/dt^*$; thus

$$u^* = \frac{dx^*}{dt^*} = \frac{(dx - udt - tdu)(1-(u/c)^2) + (x-ut)udu/c^2}{(dt - udx/c^2 - xdu/c^2)(1-(u/c)^2) + (t - xu/c^2)udu/c^2}, \tag{2.54}$$

$$= \frac{(dx - udt)(1-(u/c)^2) - (t - xu/c^2)du}{(dt - udx/c^2)(1-(u/c)^2) - (x-ut)du/c^2},$$

$$= \frac{-(t - xu/c^2)du/dt}{(1-(u/c)^2)^2 - (x-ut)/c^2(du/dt)},$$

$$= \frac{-\frac{(t-xu/c^2)}{(1-(u/c)^2)^{3/2}}\frac{du}{dt}}{(1-(u/c)^2)^{1/2} - \frac{(x-ut)}{c^2(1-(u/c)^2)^{3/2}}\frac{du}{dt}},$$

on using $u(x,t) = dx/dt$. Subsequently, we show that the two fundamental total derivatives underpinning the model structure are the material or total time derivative following the particle and the spatial or total space derivative following the wave, which are defined below in terms of partial differential operators (see also (4.56)). It is therefore tempting to investigate the transformation properties under (2.53) of the two total differential operators

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \qquad \frac{d}{dx} = \frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t}.$$

We find that with the transformed velocity given by (2.54), the structure of the first is preserved while that of the second is not. We need the following expressions for the partial derivatives, which are essentially those arising from (2.35), with (X, T) replaced by (x^*, t^*) and v replaced by $-u$; thus

$$\frac{\partial x^*}{\partial x} = \frac{1}{(1-(u/c)^2)^{1/2}} - \frac{(t - xu/c^2)}{(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial x},$$

$$\frac{\partial x^*}{\partial t} = \frac{-u}{(1-(u/c)^2)^{1/2}} - \frac{(t - xu/c^2)}{(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial t},$$

$$\frac{\partial t^*}{\partial x} = \frac{-u}{c^2(1-(u/c)^2)^{1/2}} - \frac{(x - ut)}{c^2(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial x},$$

$$\frac{\partial t^*}{\partial t} = \frac{1}{(1-(u/c)^2)^{1/2}} - \frac{(x - ut)}{c^2(1-(u/c)^2)^{3/2}}\frac{\partial u}{\partial t}.$$

On making use of these expressions, we obtain

$$\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},\\
&= \left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial t}\right) + u\left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial x} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\right),\\
&= \left\{(1-(u/c)^2)^{1/2} - \frac{(x-ut)}{c^2(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\right\}\left(\frac{\partial}{\partial t^*} + u^*\frac{\partial}{\partial x^*}\right),
\end{aligned}$$

and similarly for d/dx we have

$$\begin{aligned}
\frac{d}{dx} &= \frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t},\\
&= \left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial x} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\right) + \frac{u}{c^2}\left(\frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} + \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial t}\right),\\
&= \left\{(1-(u/c)^2)^{1/2} - \frac{(t-ux/c^2)}{(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t}\right)\right\}\left(\frac{\partial}{\partial x^*} + \frac{w^*}{c^2}\frac{\partial}{\partial t^*}\right),
\end{aligned}$$

where w^*/c is given by

$$\frac{w^*}{c} = \frac{-\frac{(x-ut)}{c(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial x}+\frac{u}{c^2}\frac{\partial u}{\partial t}\right)}{\left\{(1-(u/c)^2)^{1/2}-\frac{(t-ux/c^2)}{(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial x}+\frac{u}{c^2}\frac{\partial u}{\partial t}\right)\right\}},$$

which in general does not coincide with u^*/c as given by (2.54), namely,

$$\frac{u^*}{c} = \frac{-\frac{(t-xu/c^2)}{c(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)}{\left\{(1-(u/c)^2)^{1/2}-\frac{(x-ut)}{c^2(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)\right\}}.$$

However, it is more than curious to observe that their product

$$\frac{u^*w^*}{c^2} = \frac{(x-ut)(t-\frac{xu}{c^2})\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}+\frac{u}{c^2}\frac{\partial u}{\partial t}\right)}{c^2\left(1-\left(\frac{u}{c}\right)^2\right)^3\left\{\left(1-\left(\frac{u}{c}\right)^2\right)-\left(t\frac{\partial u}{\partial x}+\frac{x}{c^2}\frac{\partial u}{\partial t}\right)\right\}+(x-ut)(t-\frac{xu}{c^2})\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}+\frac{u}{c^2}\frac{\partial u}{\partial t}\right)},$$

becomes unity in the event that either $u(x,t) = \pm c$ or that $u(x,t)$ satisfies the following first order partial differential equation:

$$t\frac{\partial u}{\partial x}+\frac{x}{c^2}\frac{\partial u}{\partial t} = 1-\left(\frac{u}{c}\right)^2, \tag{2.55}$$

corresponding to (2.28), and for which we have previously noted is the condition for which the Jacobian of the transformation (2.53) becomes singular (see Eqs. (2.35) and (2.36)). Further in general, u^* and w^* are complementary velocities satisfying the de Broglie condition, contingent on the vanishing of the single parameter, Γ, which is defined by

$$\Gamma = \frac{c^2\left(1-\left(\frac{u}{c}\right)^2\right)^3\left\{\left(1-\left(\frac{u}{c}\right)^2\right)-\left(t\frac{\partial u}{\partial x}+\frac{x}{c^2}\frac{\partial u}{\partial t}\right)\right\}}{(x-ut)(t-\frac{xu}{c^2})\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}+\frac{u}{c^2}\frac{\partial u}{\partial t}\right)},$$

since then the product u^*w^*/c^2 becomes simply $u^*w^*/c^2 = 1/(1+\Gamma)$. We comment that $u(x,t) = \pm c$ are also formally solutions of (2.55) and that this may be relevant in terms of the continuity and matching of the solutions of this equation.

2.14 Force Invariance for Constant Velocity Frames

Fundamental to special relativistic mechanics is the physically motivated assumption that force, as defined to be the rate of change of momentum, remains invariant in the direction of relative motion for frames moving with constant relative velocity. This assumption is related to Newton's first law that particles remain at rest or in uniform motion unless operated on by an external force. Thus, although force equality $f = F$ for non-accelerating frames is a basic physical hypothesis of special relativity, it nevertheless formally hinges on assuming the Einstein mass variation (2.44), and this can be established in at least two different ways.

Firstly, on taking the differentials of $(2.46)_1$ and $(2.3)_2$, and using $dE = UdP$ which arise from the rate-of-working equation $(2.43)_1$, we have

$$f = \frac{dp}{dt} = \frac{dP - v\,dE/c^2}{dT - v\,dX/c^2} = \frac{dP(1 - Uv/c^2)}{dT(1 - Uv/c^2)} = \frac{dP}{dT} = F.$$

Alternatively,

$$f = \frac{d}{dt}\left(\frac{m_0 u}{[1 - (u/c)^2]^{1/2}}\right) = \frac{m_0\,du/dt}{[1 - (u/c)^2]^{3/2}},$$

and on using the differential of $(2.3)_2$ and the velocity addition formula (2.6), we have

$$f = \frac{m_0\left[1 - (v/c)^2\right]^{3/2} dU}{\left[1 - (u/c)^2\right]^{3/2}\left[1 - vU/c^2\right]^3 dT} = \frac{m_0\,dU/dT}{\left[1 - (U/c)^2\right]^{3/2}},$$

where the final step follows from (2.7) and again gives $f = F$. This result is well known and can be found, for example, in Moller [78] (page 73). Thus although one might expect force invariance $f = F$ in the direction of relative motion for non-accelerating frames to be a fundamental physical hypothesis of special relativity, it is formally equivalent to the Einstein relations (2.45) and the Lorentz transformations, and their consequences.

The equation $dp/dt = dP/dT$ is the starting point for the model developed in [47–52] and described in subsequent chapters; that is, instead of simply a spatial force being Lorentz invariant, we investigate a model for which both a spatial physical force **f** and a new force g in the direction of time are defined by two Lorentz invariant equations. In the final two sections of this chapter, we present an illustration for particle motion in an invariant potential field, and we derive one possible extension of the conventional Einstein variation of mass formula with a specific expression arising from a Lorentz invariant equation for the energy rate de/dp.

2.15 Example: Motion in an Invariant Potential Field

In special relativity, the full integration of the equation corresponding to Newton's second law, along with appropriate boundary or initial data, may well constitute a non-trivial problem. As an illustration of this formal integration procedure, we consider the general problem of the determination of the motion a single particle that is moving under the action of a properly invariant potential field that might arise, for example, from electromagnetism, gravity or some other force-generating field. As usual, we view the particle from two frames of reference (X, T) and (x, t) with the latter moving with relative velocity v in the aligned x and X direction, and we assume that Hamilton's equations apply in both frames. Thus, for Hamiltonians $K(X, P, T)$ and $H(x, p, t)$ where $P = MU$ and $p = mu$, we assume that

$$\frac{dP}{dT} = -\frac{\partial K}{\partial X}, \qquad \frac{dX}{dT} = \frac{\partial K}{\partial P},$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \qquad \frac{dx}{dt} = \frac{\partial H}{\partial p},$$

and for a single particle, we assume that the Hamiltonians take the specific forms

$$K(X, P, T) = \left(e_0^2 + (Pc)^2\right)^{1/2} + V(X, T),$$
$$H(x, p, t) = \left(e_0^2 + (pc)^2\right)^{1/2} + V(x, t),$$

where the potential functions $V(X, T)$ and $V(x, t)$ are the same function of the variables (X, T) and (x, t), respectively. For conventional special relativity, we have from the previous section $dp/dt = dP/dT$, and therefore the partial derivative $\partial V(x, t)/\partial x$ must either be at most a constant or an invariant function. Now each of the second Hamiltonian equations is automatically satisfied, while the first equations yield simply

$$\frac{dP}{dT} = -\frac{\partial V(X, T)}{\partial X}, \qquad \frac{dp}{dt} = -\frac{\partial V(x, t)}{\partial x},$$

so that from $dP/dt = dp/dt$ we may conclude that

$$\frac{\partial V(x, t)}{\partial x} = \Phi(\zeta, \tau),$$

where $\Phi(\zeta, \tau)$ denotes any arbitrary function of ζ and τ, which are the Lorentz invariants defined by (2.15); thus

$$\zeta = ((ct)^2 - x^2)^{1/2} = (\alpha\beta)^{1/2},$$
$$\tau = \left(\frac{ct + x}{ct - x}\right)^{1/2} \left(\frac{1 - u/c}{1 + u/c}\right)^{1/2} = \left(\frac{\alpha}{\beta}\right)^{1/2} e^{-\theta},$$

and α and β denote the characteristic coordinates defined by $\alpha = ct + x$ and $\beta = ct - x$.

For purposes of illustration, we might consider the case when $\Phi(\zeta, \tau)$ depends only on ζ; thus $\Phi = \Phi(\zeta)$, so that we need to integrate the momentum equation

$$\frac{d(mu)}{dt} = -\Phi(\zeta),$$

and we assume that appropriate boundary or initial data is adequately prescribed. On using $d\zeta/dt = (c^2t - xu)/\zeta$ we may deduce the equation

$$(xu - c^2t)d(mu) = \zeta\Phi(\zeta)d\zeta,$$

which on integration by parts using $u = dx/dt$ yields

$$\begin{aligned}
&(xu - c^2t)mu - \int mu\left(udx + xdu - c^2dt\right) \\
&= (xu - c^2t)mu + \int\left\{mc^2\left(1 - \left(\frac{u}{c}\right)^2\right)dx - muxdu\right\} \\
&= \frac{m_0u(xu - c^2t)}{(1 - (u/c)^2)^{1/2}} + m_0c^2x(1 - (u/c)^2)^{1/2} \\
&= \int^\zeta \zeta\Phi(\zeta)d\zeta.
\end{aligned}$$

Subsequent simplification readily yields the formal first integral

$$\frac{x - ut}{\left(1 - (u/c)^2\right)^{1/2}} = \int^\zeta \frac{\zeta\Phi(\zeta)d\zeta}{m_0c^2} + C_1,$$

where C_1 denotes an arbitrary constant, and we note that the expression on the left-hand side is one of the invariants in Eq. (2.13). If we now introduce $\Psi(\zeta)$ defined by

$$\Psi(\zeta) = \int^\zeta \frac{\zeta\Phi(\zeta)d\zeta}{m_0c^2} + C_1,$$

then with $u = dx/dt$ we require to obtain a further integral of the first order ordinary differential equation for $x = x(t)$, namely,

$$\frac{x - tdx/dt}{\left(1 - \frac{1}{c^2}(\frac{dx}{dt})^2\right)^{1/2}} = \Psi(\zeta),$$

where $\zeta = \left((ct)^2 - x^2\right)^{1/2}$. This equation may be formally integrated by introducing the two angles (θ, ϕ) such that

$$ct = \zeta \cosh\theta, \qquad x = \zeta \sinh\theta,$$

and

$$\cosh\phi = \frac{1}{\left(1 - \frac{1}{c^2}(\frac{dx}{dt})^2\right)^{1/2}}, \qquad \sinh\phi = \frac{\frac{1}{c}\frac{dx}{dt}}{\left(1 - \frac{1}{c^2}(\frac{dx}{dt})^2\right)^{1/2}},$$

and we assume that any boundary or initial data, perhaps involving velocity, position and time, may be transferred to boundary or initial data for θ and ϕ. Using $dx/dt = c\tanh\phi$, we may deduce from the above relations

$$\sinh(\theta - \phi) = \frac{\Psi}{\zeta}, \qquad \coth(\theta - \phi) = -\frac{1}{\zeta}\frac{d\zeta}{d\theta},$$

from which we may deduce the second formal integral

$$\theta = \pm \int^{\zeta} \frac{\Psi(\zeta)d\zeta}{\zeta\left[\Psi(\zeta)^2 + \zeta^2\right]^{1/2}} + C_2,$$

where C_2 denotes a further arbitrary constant of integration.

As a specific example, given that $\partial V(x,t)/\partial x = \Phi(\zeta)$, it follows that

$$V(x,t) = \int^{x} \Phi(\zeta)dx + W(t),$$

where the integration is partial with respect to x and $W(t)$ denotes an arbitrary function of time. Thus, for the partial derivatives, we have

$$\frac{\partial V}{\partial x} = \Phi(\zeta), \qquad \frac{\partial^2 V}{\partial x^2} = -\frac{x\Phi(\zeta)'}{\zeta},$$

$$\frac{\partial V}{\partial t} = c^2 t \int^{x} \frac{\Phi(\zeta)'}{\zeta} dx + W(t)', \qquad \frac{\partial^2 V}{\partial t^2} = c^4 t^2 \int^{x} \left(\Phi(\zeta)'' - \frac{\Phi(\zeta)'}{\zeta}\right)\frac{dx}{\zeta^2} + W(t)'',$$

so that in particular we have

$$\frac{\partial^2 V}{\partial x^2} = -\frac{x\Phi(\zeta)'}{\zeta} = -\int^{x} \frac{\partial}{\partial x}\left(\frac{x\Phi(\zeta)'}{\zeta}\right)dx = \int^{x}\left\{\frac{x^2}{\zeta^2}\left(\Phi(\zeta)'' - \frac{\Phi(\zeta)'}{\zeta}\right) - \frac{\Phi(\zeta)'}{\zeta}\right\}dx,$$

and from which we may verify that

$$\frac{\partial^2 V}{\partial t^2} - c^2\frac{\partial^2 V}{\partial x^2} = -c^2\int^{x} \Phi(\zeta)'' dx + W(t)'',$$

where primes denote differentiation with respect to the indicated argument. Thus for a specific example, we might assume that

$$\Phi(\zeta)^{''} = -k^2\Phi(\zeta),$$

for some constant k, which then implies that the assumed potential $V(x, t)$ satisfies a partial differential equation of the form

$$\frac{\partial^2 V}{\partial t^2} - c^2\frac{\partial^2 V}{\partial x^2} = (kc)^2 V + [W(t)^{''} - (kc)^2 W(t)].$$

Accordingly, an example might be $\Phi(\zeta) = A\sin(k\zeta)$ for certain constants A and k, so that $\Psi(\zeta)$ is given by

$$\Psi(\zeta) = \frac{A}{e_0}\int^{\zeta} \zeta \sin(k\zeta)d\zeta + C_1 = -\frac{A}{e_0 k^2}\{k\zeta\cos(k\zeta) - \sin(k\zeta)\} + C_1,$$

where $e_0 = m_0c^2$, and with the constant $C_1 = 0$ giving rise to the formal integral

$$\theta = \pm\int^{\zeta} \frac{\{k\zeta\cos(k\zeta) - \sin(k\zeta)\}\, d\zeta}{\zeta\left[\{k\zeta\cos(k\zeta) - \sin(k\zeta)\}^2 + (e_0k^2\zeta/A)^2\right]^{1/2}} + C_2.$$

With the substitution $z = k\zeta$ and the constant δ defined by $\delta = ke_0/A$ simplifies somewhat to give

$$\theta = \pm\int^{z} \frac{(z\cos z - \sin z)\, dz}{z\left[(z\cos z - \sin z)^2 + (\delta z)^2\right]^{1/2}} + C_2,$$

but no doubt would still require to be evaluated numerically.

2.16 Alternative Energy-Mass Velocity Variation

Throughout the text we assume that the Einstein energy-mass expression $e = e_0/(1 - (u/c)^2)^{1/2}$ applies, which is not an essential assumption other than being the simplest and most logical, and any other hypothesis would necessarily be more complicated and might involve additional implied consequences. The author [53] derives the simplest one-parameter Lorentz invariant extension of the Einstein mass-energy relation, and implicit in the new expression is space-time anisotropy such that the particle has different rest masses in the positive and negative x directions. This alternative energy-mass velocity variation formula arises from the following general Lorentz invariant equation:

$$\frac{de}{dp} = c\left(\frac{\kappa + u/c}{1 + \kappa u/c}\right), \tag{2.56}$$

which for $\kappa \neq 0$ implies a non-isotropy of space. This equation is motivated from Eq. (5.8) and involves an arbitrary constant κ for which the particle and wave velocities arise as two special cases corresponding, respectively, to the values $\kappa = 0$ and $\kappa = \pm\infty$; thus

$$\frac{de}{dp} = u, \qquad \frac{de}{dp} = \frac{c^2}{u}. \tag{2.57}$$

The case $\kappa = 0$ arises by re-writing the standard relations $m = m_0[1 - (u/c)^2]^{-1/2}$, $e = mc^2$ and $p = mu$ using momentum as the variable; thus

$$\frac{u}{c} = \frac{pc}{(e_0^2 + (pc)^2)^{1/2}}, \qquad e = (e_0^2 + (pc)^2)^{1/2}, \tag{2.58}$$

where $e_0 = m_0c^2$ denotes the rest mass energy. The relationship $(2.57)_1$ then arises immediately on differentiating $(2.58)_2$ with respect to p and then using $(2.58)_1$. The case $\kappa = \pm\infty$ corresponds to the de Broglie wave that is associated with a particle moving with velocity u and moving with the superluminal wave velocity $w = c^2/u$ (see de Broglie [17]).

Equation (2.56) is Lorentz invariant in the sense that for fixed relative frame velocities v, by division of the differentials of the inverse energy-momentum relations (2.47), namely,

$$dP = \frac{dp + vde/c^2}{[1 - (v/c)^2]^{1/2}}, \quad dE = \frac{de + vdp}{[1 - (v/c)^2]^{1/2}},$$

we may deduce the equation

$$\frac{dE}{dP} = \left(\frac{de}{dp} + v\right) / \left(1 + \frac{v}{c^2}\frac{de}{dp}\right),$$

and on substitution of (2.56) into this equation, we obtain

$$\frac{dE}{dP} = \left(\frac{(u + v) + c\kappa(1 + uv/c^2)}{(1 + uv/c^2) + \kappa(u + v)/c}\right) = c\left(\frac{\kappa + U/c}{1 + \kappa U/c}\right),$$

on using (2.12). Thus, there is the same velocity dependence in both the moving and reference frames, and therefore equation (2.56) is a Lorentz invariant equation.

Assuming the usual relations $e = mc^2$ and $p = mu$, we have $p = eu/c^2$, and therefore from (2.56) we obtain

$$\frac{dp}{de} = \frac{1}{c^2}\left(u + e\frac{du}{de}\right) = \frac{1}{c}\left(\frac{1+\kappa u/c}{\kappa + u/c}\right),$$

which simplifies to become

$$\frac{de}{e} = \frac{(\kappa + u/c)du}{c(1-(u/c)^2)} = \frac{1}{2c}\left(\frac{(\kappa+1)du}{(1-(u/c))} + \frac{(\kappa-1)du}{(1+(u/c))}\right),$$

and this integrates to give

$$e(u) = \frac{e_0}{(1-(u/c)^2)^{1/2}}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2}, \tag{2.59}$$

where as usual e_0 denotes the rest energy, and evidently the Einstein variation arises from the special case $\kappa = 0$. In terms of the angle θ defined by (2.10) in which the Lorentz invariance appears through a translational invariance, we have the following alternative expressions on using Eq. (2.10):

$$e(u) = \frac{e_0}{(1-(u/c)^2)^{1/2}}e^{\kappa\theta} = e_0\cosh\theta e^{\kappa\theta} = \frac{e_0}{2}\left(e^{(\kappa+1)\theta} + e^{(\kappa-1)\theta}\right), \tag{2.60}$$

and the following relations also apply:

$$e(u) - cp(u) = e_0e^{(\kappa-1)\theta}, \qquad e(u) + cp(u) = e_0e^{(\kappa+1)\theta},$$

so that in this case we have

$$e(u)^2 - (cp(u))^2 = e_0^2e^{2\kappa\theta} = e_0^2\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa}.$$

With the angles $(\Theta, \theta, \epsilon)$ defined by (2.9) and the Lorentz invariance represented by the translation $\Theta = \theta + \epsilon$, it is clear from $(2.60)_1$ that the energy-momentum relations (2.47) remain properly Lorentz invariant, noting however that the rest mass as perceived from the reference frame becomes $E_0 = e_0e^{-\kappa\epsilon}$.

Specifically, from the inverse energy-momentum relation $(2.47)_2$, we have

$$E = \frac{e + pv}{[1-(v/c)^2]^{1/2}} = \frac{e_0e^{\kappa\theta}(1+uv/c^2)}{(1-(u/c)^2)^{1/2}(1-(v/c)^2)^{1/2}},$$

and on using (2.6) to replace u in the denominator of this equation, we might deduce

$$E = \frac{e_0e^{\kappa\theta}(1-(v/c)^2)}{(1-(u/c)^2)^{1/2}(1-(v/c)^2)^{1/2}(1-Uv/c^2)} = \frac{e_0e^{-\kappa\epsilon}e^{\kappa\Theta}}{(1-(U/c)^2)^{1/2}}, \tag{2.61}$$

demonstrating that E has the same dependence on U as e has on u, except that the rest energy $E_0 = e_0 e^{-\kappa\epsilon}$ which is dependent on the constant relative frame velocity v through $\epsilon = \tanh^{-1}(v/c)$; thus

$$E_0 = e_0 e^{-\kappa\epsilon} = e_0 \left(\frac{1-(v/c)}{1+(v/c)} \right)^{\kappa/2}. \tag{2.62}$$

We further comment that the final line of (2.61) follows from Eq. (2.7) and that a similar calculation applies to the inverse energy-momentum relation $(2.47)_1$.

We observe that if we require $e(u) = e(-u)$, then necessarily $\kappa = 0$, but note that conventionally this requirement does not hold for light for which the de Broglie relations become $e = \pm pc$, dependent upon the direction. For $\kappa \neq 0$ Eqs. (2.59) and (2.62) impinge on one of the most basic postulates in special relativity relating to the assumed isotropy of space. These equations predict that for $\kappa \neq 0$ the rest mass values will vary with the direction of motion, namely, two different values are obtained for positive and negative velocities v. While numerous experiments have been undertaken aimed at testing such hypothesis, and all indicate the veracity of the assumed isotropy of space, nevertheless the validity or otherwise of (2.62) might only be properly tested in those situations for which both rest masses e_0 and E_0 are non-zero and the fraction $(1 + v/c)/(1 - v/c)$ significantly differs from unity. Accordingly, any test must involve speeds close to the speed of light but involving finite (non-zero) rest masses which therefore excludes those tests dealing with light such as the Michelson-Morley experiments.

It might also be worth noting that since it is generally believed that black holes exist at the centres of galaxies, then space-time must be intrinsically anisotropic in some sense. It is conceivable that space-time is anisotropic at galactic scales and possibly a massive black hole might cause particle rest mass to depend on the direction of particle velocity. This might be the case when the black hole is not at rest in the rest frame of the cosmic microwave background. The cosmic microwave background does have a sizeable dipole component, and its rest frame is measured to be travelling at 627 km/s relative to the centre of mass of our galaxy group (see the mini-review of cosmic microwave background by Scott and Smoot [92]).

Further, while the space-time of special relativity is assumed to be isotropic, this is not taken as an assumption in general relativity. The use of the isotropy assumption in cosmology to select the basic models tends to reflect known experimental outcomes rather than being a necessary part of the theory. We further comment that [96] provides a very general approach to mechanical anisotropy in relativistic mechanics which includes the simple model described here, although derived differently.

We note that for $\kappa = 1$ and $\kappa = -1$, we have, respectively, from (2.59) the following relations:

$$e(u) = \frac{e_0}{1-(u/c)}, \qquad p(u) = \frac{e_0(u/c)}{c(1-(u/c))}, \qquad e(u) - cp(u) = e_0,$$

$$e(u) = \frac{e_0}{1 + (u/c)}, \qquad p(u) = \frac{e_0(u/c)}{c(1 + (u/c))}, \qquad e(u) + cp(u) = e_0,$$

allowing the possibility of $e = \pm pc + e_0$ with non-zero rest energy e_0.

Finally, we comment that it is a very curious fact that both the conventional Einstein energy-mass expression $e = e_0/(1 - (u/c)^2)^{1/2}$ and the generalisation derived here (2.59) bear a relationship with certain singular integral equations associated with aerofoil problems, fluid mechanics and punch problems in elasticity and that this relationship is not some vague intangible connection but involves an exact correspondence. Linear singular equations arise in many areas of applied mathematics but particularly within fluid and solid mechanics. Specifically, in nondimensional variables, the formal solution of the singular integral equation of the second kind

$$\phi(x) + \frac{\lambda}{\pi} \int_{-1}^{1} \frac{\phi(y)dy}{(x - y)} = g(x), \tag{2.63}$$

is given by

$$\phi(x) = \frac{C\lambda}{\pi(1 + \lambda^2)^{1/2}(1 - x^2)^{1/2}} \left(\frac{1 + x}{1 - x}\right)^{\gamma} \tag{2.64}$$
$$+ \frac{g(x)}{(1 + \lambda^2)} + \frac{\lambda}{\pi(1 + \lambda^2)(1 - x^2)^{1/2}} \left(\frac{1 + x}{1 - x}\right)^{\gamma} \int_{-1}^{1} (1 - y^2)^{1/2} \left(\frac{1 - y}{1 + y}\right)^{\gamma} \frac{g(y)dy}{(y - x)},$$

where here λ and γ are related by $\lambda = \cot(\pi\gamma)$ and the constant C is defined by

$$C = \int_{-1}^{1} \phi(x)dx. \tag{2.65}$$

The singular integral appearing in (2.63) is sometimes referred to as the finite Hilbert transform. There are numerous standard results available such as (see for example [39, 60] or [63])

$$\frac{1}{\pi} \int_{-1}^{1} \frac{dy}{(1 - y^2)^{1/2}(x - y)} = 0, \quad \frac{1}{\pi} \int_{-1}^{1} \frac{(1 - y^2)^{1/2}dy}{(x - y)} = x, \quad \frac{1}{\pi} \int_{-1}^{1} \frac{y(1 - y^2)^{1/2}dy}{(x - y)} = x^2 - \frac{1}{2},$$
$$\frac{1}{\pi} \int_{-1}^{1} \frac{y^3(1 - y^2)^{1/2}dy}{(x - y)} = x^4 - \frac{x^2}{2} - \frac{1}{8}, \quad \frac{1}{\pi} \int_{-1}^{1} \frac{y^5(1 - y^2)^{1/2}dy}{(x - y)} = x^6 - \frac{x^4}{2} - \frac{x^2}{8} - \frac{1}{16},$$

and there are usually other constraints such as $\phi(\pm 1) = 0$, and the function $g(x)$ is assumed to be an odd function, but since the major issue here is simply the connection with the Einstein expression, we do not need concern ourselves with such details.

Strictly speaking, the Einstein expression arising from the case $\gamma = 0$ corresponds to $\lambda \longrightarrow \infty$ and accordingly arises from the integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(y)dy}{(x-y)}=f(x),$$

which has solution given by Tricomi [103] (pages 173–185)

$$\phi(x)=\frac{C}{\pi(1-x^2)^{1/2}}+\frac{1}{\pi(1-x^2)^{1/2}}\int_{-1}^{1}(1-y^2)^{1/2}\frac{f(y)dy}{(y-x)}, \tag{2.66}$$

with the constant C as previously given by (2.65), and since the function $f(x)$ is necessarily orthogonal to $(1-x^2)^{-1/2}$, thus

$$\int_{-1}^{1}\frac{f(x)dx}{(1-x^2)^{1/2}}=0,$$

there are several equivalent expressions available for this solution (see [103], page 179). We comment that the expression (2.66) formally emerges from (2.64) with $g(x)=\lambda f(x)$ and in the limit $\lambda\longrightarrow\infty$ and $\gamma\longrightarrow 0$. Evidently with $x=u/c$, there is an exact correspondence with $e=e_0/(1-(u/c)^2)^{1/2}$ and Eq. (2.66), and with the new expression (2.59) and (2.64), and both of which arise as the solution of the homogeneous problem ($f(x)=g(x)=0$).

The correspondence between Einstein's fundamental energy expression and areas of classical fluid and solid mechanics is very curious to say the least, and perhaps it is just one of those coincidences. However, perhaps also if these interconnections were properly understood, they might spark the onset of some fundamental revelations in particle physics. In particular, it is natural to pose the question as to what might be the physical meaning for the corresponding energy-mass expressions arising from the above singular integral equations with $f(x), g(x)\neq 0$? Apart from the Newtonian interpretation using the classical kinetic energy $e=m_0u^2/2$ in the final three sections of Chap. 3, in the remainder of the text, we deal exclusively with the Einstein energy-mass expression $e=e_0/(1-(u/c)^2)^{1/2}$ as applying to the particle energy.

Chapter 3
General Formulation and Basic Equations

3.1 Introduction

In this chapter we present the proposed reformulation of Newton's second law and a number of general results relating to this extension. Since the origin and motivation for the proposed model emanate from both the particle-wave duality of de Broglie and Maxwell's formulation of electromagnetism, in the following two sections, we mention some of their considerable achievements. These two sections provide some of the background motivation leading to the model developed here. In the subsequent section, we show that a straightforward analysis of the integrated rate-of-working equation indicates four distinct states of matter. In five sections thereafter, we progressively state the proposed extension of Newton's second law and then develop a number of general outcomes relating to this proposal, including an important identity for the spatial physical force. In addition, assuming the existence of either a work done function or if the external forces are generated from an applied external potential, then in either case a conservation of energy statement applies. In either of these cases or if we assume the Einstein energy-mass variation with velocity, then Newton's second law as the total rate of change of momentum emerges from the proposed reformulation.

The model presented here is analogous to the use of potentials in Maxwell's equations of electromagnetism, and there is an interesting connection between the two formulations which sheds much light on the approach adopted here, and this is examined in the next section of the chapter. The following section illustrates the general equations for centrally or spherically symmetric systems applicable to a central body generating a spherically symmetric environment such that particle motion occurs in a radial direction only with no angular contribution or dependence. In the final three sections of the chapter, we briefly examine the model that is generated assuming the conventional Newtonian expressions for kinetic energy and momentum and assuming that the mass remains constant. The Newtonian interpretation of the proposed model although mathematically simple is nevertheless

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instructive since it embodies the essential solution types of the full theory. For a single spatial dimension, we show that the Newtonian version of this model admits a simple wave-like solution for which the corresponding full relativistic solution is examined in considerable detail in subsequent chapters. Assuming the existence of a work done function defined by (3.8), we make explicit the allowable solution options exhibited by the linearised Newtonian theory.

3.2 Louis Victor de Broglie

Early in the twentieth century, the long-standing debate about the nature of light as to whether it was particle-like or wave-like was finally concluding as scientists began to accept that light could assume both characteristics. The possibility that the same duality might also apply to all other forms of matter was first proposed in 1923 by the French physicist Louis de Broglie [16–20] (Fig. 3.1), for whom much has been written and a detailed biography can be found in [37] or [65].

Louis Victor de Broglie FRS (15 August 1892–19 March 1987) was born into a French aristocratic family with a long history of both political and military service. Born Louis Victor Pierre Raymond de Broglie in Dieppe, Seine-Maritime, he was destined to become the seventh Duc de Broglie in 1960 following the death of his elder brother Maurice, the sixth Duc de Broglie. As the youngest child in the family, Louis grew up in relative loneliness, read extensively and was fond of history,

Fig. 3.1 Louis Victor de Broglie FRS (15 August 1892–19 March 1987)

especially politically related subjects. He had a good memory and could always provide a complete listing of the ministers of the Third Republic (1870–1940) to the extent that in his youth, he considered a career as a diplomat but later turned to science and pursued theoretical physics. His brother, Maurice, who had also decided to become a physicist and made many advances in the study of X-rays, was a considerable influence on his younger brother and indeed was the person to introduce him to the work of both Planck and Einstein.

Louis's first degree was in history, but afterwards, he turned his attention towards mathematics and physics and received a second degree in physics in 1913. With the outbreak of the First World War in 1914, he offered his services to the army and joined the engineering forces to undergo compulsory service as a sapper. However, on the initiative of his brother, he was soon seconded to the Wireless Communications Service to work on the radio transmitter at the Eiffel Tower, and during the war he was stationed there as part of the Army's Wireless Telegraphy Subdivision. He remained in military service throughout the First World War, dealing with purely technical issues, and with Leon Brillouin and brother Maurice established wireless communications with submarines. He was demobilised in August 1919 with the rank of adjutant. Later in life he would regret that period of time spent away from his real passion for the fundamental problems of science.

During his war service, when not occupied with official duties, de Broglie spent many hours thinking about science, and this helped him to continue his studies in 1920 after the war had ended. His first research in the early 1920s was undertaken in the laboratory of his older brother Maurice and dealt with the photoelectric effect and properties of X-rays. The brothers examined the absorption of X-rays and described the phenomenon using Bohr's theory, applied quantum principles to the interpretation of photoelectron spectra, and gave a systematic classification of X-ray spectra. Optical spectra are used to determine the structure of the outer electron orbits, and their studies of the X-ray spectra were important for elucidating the structure of the internal electron orbits of the atoms. The results of de Broglie's experiments with Dauvillier revealed the shortcomings of existing schemes for the distribution of electrons in atoms. Other research work at this time involved the elucidation of the insufficiency of the Sommerfeld formula for determining the position of the lines in X-ray spectra, and this discrepancy was resolved after the discovery of the electron spin.

Upon de Broglie's return to Paris, his research focus shifted to mathematical physics and his doctoral thesis consisted of research on quantum theory. Studying the nature of X-ray radiation and discussing its properties with his brother Maurice, who considered these rays to be some combination of waves and particles, gave de Broglie an awareness of the need to build a theory linking particle and wave representations. He was also familiar with the work of Marcel Brillouin, who proposed a hydrodynamic model of an atom and attempted to relate it to Bohr's theory. In his first article on this subject, de Broglie's starting point was an idea of Einstein about the quanta of light, where he considered blackbody radiation as a gas of light quanta, and using classical statistical mechanics, he derived the Wien radiation law. In his second publication, he interpreted light quanta as relativistic particles of very small mass to reconcile the phenomena of interference and diffraction, with the conclusion that it was necessary to associate a certain periodicity with light quanta.

It remained to extend the wave considerations to all matter, and in the summer of 1923, de Broglie outlined his ideas in a short note [16], "Ondes et quanta" ("Waves and quanta") presented at a meeting of the Paris Academy of Sciences on 10 September 1923, an event which marked the beginning of quantum wave mechanics. In this paper, de Broglie suggested that a moving particle with energy e and velocity v is characterised by some "internal periodic process" with a frequency e/h, where h is Planck's constant. To reconcile these quantum considerations with the ideas of special relativity, de Broglie was forced to postulate a "fictitious wave" associated with a moving body, which propagates with the velocity c^2/v. Such a wave, later to be referred to as a de Broglie wave, was speculated to occur in the process of body movement and remain in phase with the "internal periodic process". On examining the motion of an electron in a closed orbit, he showed that phase matching led to the Bohr-Sommerfeld quantisation condition for angular momentum. In his next two notes reported at the Paris Academy of Sciences meetings on 24 September and 8 October 1923, de Broglie concluded that the particle velocity is equal to the group velocity of phase waves and that the particle moves along the normal to surfaces of equal phase. In general, the trajectories can be determined using Fermat's principle for waves or the principle of least action for particles, indicating a connection between geometrical optics and classical mechanics.

In his 1924 PhD thesis [17], "Recherches sur la theorie des quanta" ("Research on the theory of the quanta"), de Broglie conjectured that the electron has an internal clock that constitutes part of the mechanism by which a pilot wave guides a particle. While attempts at verifying the internal clock hypothesis and measuring clock frequency are so far not conclusive, experimental data is at least compatible with de Broglie's conjecture. In his PhD thesis, he introduced his theory of electron waves, and he postulated that just as light has both wave-like and particle-like properties, electrons also adopt these dual characteristics. Moreover, based on the work of Planck and Einstein on light, he extended the particle-wave duality theory of matter by suggesting that all matter might display wave properties. This research culminated in the de Broglie hypothesis stating that any moving particle or object had an associated wave, and he thus created the new field in physics of wave mechanics, uniting the physics of energy (waves) and the physics of matter (particles).

In the second part of his PhD thesis, de Broglie used the equivalence of the mechanical principle of least action with Fermat's optical principle, stating that "Fermat's principle applied to phase waves is identical to Maupertuis' principle applied to the moving body; the possible dynamic trajectories of the moving body are identical to the possible rays of the wave". This equivalence had been pointed out by the Irish mathematician, Hamilton, almost a century earlier, and published by him around 1830, in an era where atomic phenomena were not directly related to the fundamental principles of physics.

Within de Broglie's PhD thesis, many of his most profound ideas were proposed, including his ground-breaking theory of electron waves. He had previously published work on electron waves in scientific journals, but these articles were given little attention. It was not until Einstein read a copy of the thesis that the revolutionary nature of his ideas began to be appreciated. Due to Einstein's attention, other physicists became familiar with de Broglie's wave theory and utilised it in shaping their own work.

Most notably, de Broglie's theory served as the foundation upon which Schrödinger and others developed quantum mechanics. Indeed, his concept now known as the de Broglie hypothesis is an example of particle-wave duality and has become a central part of quantum mechanics. In his PhD thesis, de Broglie rearranged the momentum equation to obtain a relationship between the wavelength λ, associated with an electron and its momentum p, through the Planck's constant h; thus $\lambda = h/p$. This relationship has since been shown to hold for all types of matter; that is, all matter exhibits properties of both particles and waves, and he subsequently wrote:

> When I conceived the first basic ideas of wave mechanics in 1923–1924, I was guided by the aim to perform a real physical synthesis, valid for all particles, of the co-existence of the wave and of the corpuscular aspects that Einstein had introduced for photons in his theory of light quanta in 1905.

Although his ideas were subsequently extended by the work of Schrödinger, Schrödinger's generalisation was probabilistic in nature, and de Broglie did not approve, saying:

> that the particle must be the seat of an internal periodic movement and that it must move in a wave in order to remain in phase with it, was ignored by the physicists (who are) wrong to consider a wave propagation without localisation of the particle, quite contrary to my original ideas.

Far from claiming to make "the particle-wave contradiction disappear" which Born thought could be achieved with a probabilistic approach, de Broglie extended the particle-wave duality to all particles and to those crystals which revealed the effects of diffraction and extended the principle of duality to the laws of nature. In his Nature article on waves and quanta written from Paris, in 12 September 1923 (Nature, no. 2815, Vol. 112, October 13, 1923, page 540), de Broglie writes:

> The quantum relation, energy $= h\times$ frequency, leads one to associate a periodical phenomena with any isolated portion of matter or energy. An observer bound to the portion of matter will associate with it a frequency determined by its internal energy, namely, by its "matter at rest". An observer for whom the portion of matter is in steady motion with velocity βc, will see this frequency lower in consequence of the Lorentz-Einstein time transformation. I have been able to show (Comptes Rendus, September 10th and 24th, of the Paris Academy of Sciences) that the fixed observer will constantly see the internal periodical phenomenon in phase with a wave the frequency of which is $\nu = m_0c^2/h(1-\beta^2)^{1/2}$ is determined by the quantum relation using the whole energy of the moving body, provided that it is assumed that the wave spreads with the velocity c/β. This wave, the velocity of which is greater than c, cannot carry energy.

A radiation of frequency ν has to be considered as divided into atoms of light of very small internal mass ($\approx 10^{-10}$ gm.) which move with a velocity very nearly equal to that given by $h\nu = m_0c^2/(1-\beta^2)^{1/2}$. The atom of light slides slowly upon the non-material wave, the frequency of which is ν and velocity c/β, very little higher that c.

The "phase wave" has a very great importance in determining the motion of any moving body, and I have been able to show that the stability conditions in Bohr's atom express that the wave is tuned with the length of the closed path.

The path of a luminous atom is no longer straight when this atom crosses a narrow opening; that is, diffraction. It is then necessary to give up the inertia principle, and we must suppose that any moving body follows always the ray of its "phase wave"; its path will then bend by passing through a sufficiently small aperture. Dynamics must undergo the same evolution that optics has undergone when undulations took the place of purely geometrical optics. Hypotheses based upon those of the wave theory allowed us to explain interferences and diffraction fringes. By means of these new ideas, it will probably be possible to reconcile also diffusion and dispersion with the discontinuity of light, and to solve almost all the problems brought up by the quanta.

At first there was no experimental data to support de Broglie's theory that electrons behaved like waves as well as particles. However, the theory did help to explain many previously unaccountable phenomena. For example, experimental evidence had shown that electrons must move around the nucleus of an atom with certain restrictions on the motion. de Broglie's consideration of the electron as a wave suggested a plausible explanation for the constrained *nature* of the motion. In a closed loop like that for an electron moving around a nucleus, the undulations of a wave must stretch evenly around the entirety of the loop and consist of a whole number of wavelengths. If they do not, then the wave is cancelled. These conditions on a closed-loop wave are consistent with the evidence that an electron in an atom has only a number of possible energy configurations. Also, the de Broglie view of electrons may be used to rationalise how subatomic particles may materialise in unexpected locations, since the waves are believed to travel through obstacles.

A few years after de Broglie's theory was published, in 1927 it was confirmed experimentally, by the electron diffraction experiments of the American physicists Thomson, Davisson and Germer, that electrons do indeed exhibit wave-like characteristics when scattered from the surface of a solid crystal. Additional support for the wave-like behaviour of electrons was provided around the same time by the British scientist George Paget Thomson, and soon after by the German physicist Otto Stern, who carried out experiments on helium atoms and hydrogen molecules that supported de Broglie's additional claim that complex particles, and not just electrons, also exhibit properties similar to waves. With this experimental evidence, and the support of Einstein, de Broglie's particle-wave duality theory of matter gained widespread acceptance as the framework of wave mechanics, and de Broglie was recognised with the Nobel Prize for Physics in 1929.

Previously, in 1925 and 1926, the Leningrad physicist Orest Khvolson had nominated the de Broglie brothers for the Nobel Prize for their work in the field of X-rays. The 1925 pilot-wave model and the wave-like behaviour of particles were subsequently used by Schrödinger in his 1926 formulation of wave mechanics.

Schrödinger published an equation describing how a matter wave should evolve, which is the matter wave analogue of Maxwell's equations, and he used it to derive the energy spectrum of hydrogen. Following Schrödinger's success, the pilot-wave model and de Broglie's interpretation were abandoned in favour of the quantum formalism. In his later career, de Broglie worked to develop a causal explanation of wave mechanics, in opposition to the wholly probabilistic models which now dominate quantum mechanical theory. de Broglie's approach was rediscovered and refined by David Bohm in the 1950 and is now referred to as the de Broglie-Bohm theory.

Originally, de Broglie had thought that a real wave was associated with the particles, that is, a wave with a direct physical interpretation. In fact, the wave aspect of matter was formalised by the wave function introduced by Schrödinger's equation, which is a purely mathematical entity requiring a probabilistic interpretation and without support of any real physical elements. That is, Schrödinger's wave function gives the appearance of the wave behaviour of matter, without the appearance of any real physical waves. The de Broglie-Bohm theory is today the only interpretation giving real status to matter waves in representing the predictions of quantum theory.

Between 1930 and 1950, de Broglie worked on various aspects of wave mechanics, including Dirac's electron theory, the new theory of light, the general theory of spin particles and applications of wave mechanics to nuclear physics. During the remainder of his long and illustrious career, he published numerous notes and several papers on these subjects and is the author of more than 25 scientific texts. In addition to his scientific work, he also taught theoretical physics at the Sorbonne in Paris and composed several books exploring the relationship between physics and philosophy. He thought and wrote about the philosophy of science, including the value of modern scientific discoveries. From a philosophical viewpoint, his dual particle-wave theory has contributed greatly to the ruin of the atomism of the past. According to de Broglie, the neutrino and the photon have non-zero rest masses, although necessarily extremely small to preserve the coherence of his theory. His rejection of the hypothesis of the massless photon enabled him to doubt the hypothesis of the expansion of the universe. In addition, he believed that the true mass of particles is not constant, but variable, and that each particle can be represented as a thermodynamic machine equivalent to a cyclic integral of action.

de Broglie's final idea [22] was the hidden thermodynamics of isolated particles, and he attempted to bring together the three distinct principles of physics, namely, those of Fermat, Maupertuis and Carnot. In this work, action becomes opposed to entropy, through an equation relating the two universal dimensions of the form $action/h = entropy/k$, where k is thermal conductivity. One consequence is that this theory brings back the uncertainty principle to distances around the extrema of action, distances that correspond to reductions in entropy. He became the physicist who most sought that dimension of action which Planck, at the beginning of the twentieth century, had shown to be the only universal unity with his own dimension of entropy. de Broglie's last work made a single system of laws from the two large systems of thermodynamics and of mechanics, saying that:

> When Boltzmann and his continuators developed their statistical interpretation of Thermodynamics, one could have considered Thermodynamics to be a complicated branch of Dynamics. But, with my actual ideas, it's Dynamics that appear to be a simplified branch of Thermodynamics. I think that, of all the ideas that I've introduced in quantum theory in these past years, it's that idea that is, by far, the most important and the most profound.

> That idea seems to match the continuous-discontinuous duality, since its dynamics could be the limit of its thermodynamics when transitions to continuous limits are postulated. It is also close to that of Leibniz, who posited the necessity of "architectonic principles" to complete the system of mechanical laws.

However, according to de Broglie, there is less duality, in the sense of opposition, than synthesis since one is the limit of the other, and the effort of synthesis is constant according to him, like in his first formula $mc^2 = h\nu$ in which the first symbol relates to mechanics and the second to optics. His neutrino theory of light dates from 1934 and introduces the idea that the photon is equivalent to the fusion of two Dirac neutrinos. It shows that the movement of the centre of gravity of these two particles obeys Maxwell's equations, which implies that the neutrino and the photon both have rest masses that are non-zero, although very small.

Since 1951, together with young colleagues, de Broglie [21, 23, 24] resumed the study of an attempt which he made in 1927 under the name of the "theory of the double solution" to give a causal interpretation to wave mechanics in the classical terms of space and time, an attempt which he had then abandoned in the face of almost universal adherence of physicists to the purely probabilistic interpretation of Born, Bohr and Heisenberg. Back again in his former field of research, he obtained several new and encouraging results published in notes to Comptes Rendus de L'Academie des Sciences and elsewhere.

In 1929 the Academie des Sciences awarded him the first Henri Poincare Medal and then, in 1932, the Albert I of Monaco Prize. In 1929 the Swedish Academy of Sciences conferred on him the Nobel Prize for Physics "for his discovery of the wave nature of electrons". de Broglie became a member of the French Academy of Sciences in 1933, and he served as the Perpetual Secretary of the French Academy of Sciences from 1942. He was asked to join Le Conseil de L'Union Catholique des Scientifiques Francaise but declined because he was nonreligious. In 1944, he was the 16th member elected to occupy the First Seat of the Académie Française, replacing the mathematician Emile Picard. However, because of the deaths and imprisonments of academy members during the German occupation, the academy was unable to gather a quorum of 20 members for his election. Nevertheless, despite the exceptional circumstances, his unanimous election was accepted by the 17 members present. In an event unique in the history of the Académie des Sciences, he was received as a member by his own brother Maurice, who had been elected in 1934.

In addition to the Nobel Prize, de Broglie received a large number of other honours, including a number of honorary doctorate degrees and an appointment as an adviser to the French Atomic Energy Commissariat. He held honorary doctorates from the Universities of Warsaw, Bucharest, Athens, Lausanne, Quebec and Brussels and was a member of 18 foreign academies in Europe, India and the USA. He was a member of the Bureau des Longitudes since 1944, and he made major contributions to the fostering of international scientific co-operation. He was the first eminent scientist to call for the establishment of a multi-national laboratory, a proposal that ultimately led to the European Organisation for Nuclear Research (CERN). de Broglie was awarded a post as the Consular to the French High Commission of Atomic Energy in 1945 for his efforts to bring industry and science closer together. He established a centre for applied mechanics at the Henri Poincare Institute, where research into optics, cybernetics and atomic energy were carried out. He inspired the formation of the International Academy of Quantum Molecular Science and was an early member.

In 1952 he was awarded the first Kalinga Prize by the United Nations Economic and Social Council (UNESCO) for his contributions to popularising scientific knowledge and his efforts to explain modern physics to the layman. He was elected a Foreign Member of the Royal Society on 23 April 1953 and awarded the gold medal of the French National Scientific Research Centre in 1956. He is an Officer of the Order of Leopold of Belgium, and in 1960 he became the seventh Duc de Broglie, following the death without heir of his elder brother Maurice, the sixth Duc de Broglie. In 1961 he received the title of Knight of the Grand Cross of the Legion d'Honneur. He never married, and when he died in Louveciennes, he was succeeded as the duke by his distant cousin, Victor-Francois, the eighth Duc de Broglie. His funeral was held on 23 March 1987 at the Church of Saint-Pierre-de-Neuilly.

3.3 James Clerk Maxwell

James Clerk Maxwell FRS FRSE (13 June 1831–5 November 1879) (Fig. 3.2) was a Scottish scientist in the field of mathematical physics, for whom an extensive and detailed biography can be found in [38]. His most notable achievement was to formulate the classical theory of electromagnetic radiation, bringing together for the first time electricity, magnetism and light as different manifestations of the same phenomenon. Maxwell's equations for electromagnetism have been called the "second great unification in physics" after the first in mechanics achieved by Sir Isaac Newton. From the long view of the history of mankind, Maxwell may well be viewed as the greatest theoretical physicist of the nineteenth century, and there can be little doubt that the most significant event of the nineteenth century will be his discovery of the laws of electrodynamics. His discoveries helped usher in the era of modern physics, laying the foundation for such fields as special relativity and quantum mechanics. He also helped develop the Maxwell-Boltzmann distribution, a statistical means of describing aspects of the kinetic theory of gases. He is also known for presenting the first durable colour photograph in 1861 and for his foundational work on analysing the rigidity of rod-and-joint frameworks (trusses) like those in many bridges. His tutor at Cambridge, William Hopkins, wrote, "it appears impossible for Maxwell to think incorrectly on physical subjects". Maxwell

Fig. 3.2 James Clerk Maxwell FRS FRSE (13 June 1831–5 November 1879)

has said, “it is of great advantage to the student of any subject to read the original memoirs on that subject, for science is almost completely assimilated when it is in the nascent state”.

Maxwell had studied and commented on electricity and magnetism as early as 1855 when his paper titled “On Faraday’s Lines of Force” was read to the Cambridge Philosophical Society [72]. The paper presented a simplified model of Faraday’s work and how electricity and magnetism are related. He reduced all of the current knowledge into a linked set of differential equations with 20 equations in 20 variables, and the work was later published as “On Physical Lines of Force” in March 1861 in the *Philosophical Magazine* [73].

Around 1862, while lecturing at King’s College London, Maxwell calculated that the speed of propagation of an electromagnetic field is approximately that of the speed of light. He obtained a velocity of 310,740,000 metres per second and considered this to be more than just a coincidence. In his 1865 paper [74] appearing in the *Philosophical Transactions of the Royal Society* of London and titled “A dynamical theory of the electromagnetic field”, he comments that “The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws” and “We can scarcely avoid the conclusion that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena”. In this paper Maxwell derives an electromagnetic wave equation with a wave velocity in close agreement with that for light. He deduces that light is an electromagnetic wave and that light is an undulation in the same medium that is the cause of electric and magnetic phenomena. Subsequently Maxwell was proved to be correct, and his quantitative connection between light and electromagnetism is considered to be one of the great accomplishments of nineteenth-century mathematical physics that led to the prediction of the existence of radio waves.

The origins of his 1865 paper lay in his earlier papers of 1855–1856, in which he began the mathematical elaboration of Faraday's researches into electromagnetism, and his papers during the period 1861–1862, in which the displacement current was introduced. Of particular interest to the present text is that much of this earlier work was motivated from mechanical analogies, while in the 1865 paper, the focus shifts to the role of the fields themselves as descriptions of electromagnetic phenomena, and the mechanical models by which he had arrived at the field equations a few years earlier were stripped away. Maxwell's introduction of the concept of fields to explain physical phenomena provided the essential link between the mechanical world of Newtonian physics and the theory of fields, as elaborated by Einstein and others, which lies at the heart of twentieth- and twenty-first-century physics. The 1865 paper provided a new theoretical framework for the subject, based on experiment and a few general dynamical principles, from which the propagation of electromagnetic waves through space followed without any special assumptions.

In the treatise (*A Treatise on Electricity and Magnetism*) [75], Maxwell presents electromagnetic theory and advanced ideas and thinking that were to become essential for modern physics, including his landmark hypothesis that light and electricity correspond in their ultimate nature. His treatise did for electromagnetism what Newton's *Principia* had done for classical mechanics. It provided a mathematical basis for the investigation and representation of the whole electromagnetic theory and altered the framework of both theoretical and experimental physics. It was this work that finally displaced action-at-a-distance physics and substituted the physics of the field.

His famous 20 equations first appear in their modern form of four partial differential equations in the treatise [75], and most of this work was done at Glenlair during the period between holding his King's College London post and taking up the Cavendish Chair at Cambridge. He expressed electromagnetism in the algebra of quaternions and made the electromagnetic potential the centrepiece of his theory. Subsequently, in 1881 Oliver Heaviside reduced the complexity of his theory to four differential equations, now collectively known as Maxwell's equations of electromagnetism, and the electromagnetic potential field was replaced by force fields as the centrepiece of electromagnetic theory which later became the cause of some debate concerning the relative merits of vector analysis and quaternions. The end result was the realisation that there was no need for the greater physical insights provided by quaternions if the theory was purely local, and the use of scalar and vector potentials is now standard in the solution of Maxwell's equations.

His treatise on electromagnetism extended the dynamical formalism through a more rigorous application of Lagrange's equations than he had attempted in 1865 and coincided with a general movement in Britain and Europe towards a wider use of the methods of analytical dynamics in physical problems, by expressing the Lagrangian for an electromagnetic system in its most general form. George Green and others had developed similar arguments to study the dynamics of the luminiferous aether. However, Maxwell's use of Lagrange's techniques was a new to physics, and many years passed before other physicists adopted these ideas. In 1865, and again in the treatise, Maxwell's next step after completing the dynamical analogy was to develop eight equations describing the electromagnetic field. Their underlying principle is that electromagnetic processes are transmitted by the separate and independent action of each charge (or magnetised body) on the surrounding space rather than by direct action at a distance. Force formulae between moving charged bodies may be derived from Maxwell's equations, but the action is not along the line joining them and can only be reconciled with dynamical principles by taking into account the exchange of momentum with the field.

Maxwell once remarked that the aim of the treatise was not to expound the final view of his electromagnetic theory, which he had developed in a series of five major papers between 1855 and 1868; rather it was to educate himself by presenting a view of the stage that he had reached in his thinking. Accordingly, the work is loosely organised on historical and experimental rather than systematically deductive lines. It extends Maxwell's ideas beyond the scope of his earlier work in many directions, demonstrating the special importance of electricity to physics as a whole. He began the investigation of moving frames of reference, which in Einstein's hands were to revolutionise physics; gave proofs of the existence of electromagnetic waves that paved the way for Hertz's discovery of radio waves; worked out connections between electrical and optical qualities of bodies that would lead to modern solid-state physics; and applied Tait's quaternion formulae to the field equations, from which Heaviside and Gibbs would later develop. Viewing electromagnetism as a mathematical model, Hertz would comment that Maxwell's equations seemed to contain more than their constituent parts.

Maxwell also introduced the concept of the electromagnetic field in comparison to force lines that Faraday described. By understanding the propagation of electromagnetism as a field emitted by active particles, Maxwell could advance his work on light. At that time, Maxwell believed that the propagation of light required a medium for the waves, dubbed the luminiferous aether or space-time fabric. Over time, the existence of such a medium, permeating all space and yet apparently undetectable by mechanical means, proved impossible to reconcile with experiments such as the Michelson and Morley experiment. Moreover, it seemed to require an absolute frame of reference in which the equations were valid, and in consequence the equations might change their form for a moving observer. These difficulties inspired Einstein to formulate the theory of special relativity and to dispense with the requirement of a stationary luminiferous aether.

Einstein's work on relativity was founded directly upon Maxwell's electromagnetic theory, and it was this that led him to equate Faraday with Galileo and Maxwell with Newton. On the occasion of the centenary of Maxwell's birth in 1931, Einstein summed up Maxwell's achievement: "We may say that, before Maxwell, physical reality, in so far as it was to represent the process of nature, was thought of as consisting in material particles, whose variations consist only in movements governed by ordinary differential equations. Since Maxwell's time, physical reality has been thought of as represented by continuous fields, governed by partial differential equations, and not capable of any mechanical interpretation. This change in the conception of reality is the most profound and the most fruitful that physics has experienced since the time of Newton". Many physicists regard Maxwell as the nineteenth-century scientist having the greatest influence on twentieth-century physics. His contributions to science are considered by many to be of the same magnitude as those of Isaac Newton and Albert Einstein. In a survey of 100 of the most prominent physicists, Maxwell was voted as the third greatest physicist of all time, behind only Newton and Einstein. On the centenary of Maxwell's birthday, Einstein described Maxwell's work as the "most profound and the most fruitful that physics has experienced since the time of Newton".

The key idea formulated in [47–52] and expounded below originally arose from a series of five lectures by Professor Germain Rousseaux presented at the University of Poitiers on five successive Tuesdays, 9 May–6 June 2017. Professor Rousseaux, as a passionate and lifelong student of the life and work of James Clerk Maxwell, presented five remarkably insightful lectures, principally dealing with the life of Maxwell and his numerous outstanding contributions to electromagnetism that included three important themes. First is that much of our present understanding of electromagnetism developed from mechanical analogues from both solid and fluid mechanics. Second is the recognition as to the importance as to which variables constitute the force and which identify the associated flux, and the assumed linear connection between them, such as in standard notation in electromagnetic theory $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{J} = \sigma\mathbf{E}$. The third and perhaps the most important theme that permeated throughout the five lectures was the more fundamental importance of the vector and scalar potentials $(\mathbf{A}, V)$ as compared to the fields $(\mathbf{E}, B)$ that are related by the formulae

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V. \tag{3.1}$$

The vector potential $\mathbf{A}$ corresponds to the electromagnetic momentum and is the analogue of mechanical momentum and referred to by both Faraday and Maxwell as the electro-tonic intensity, while the potential V is the analogue of the velocity potential in fluid mechanics.

While the recognition of the relative importance of the potentials $(\mathbf{A}, V)$ has waxed and waned over the past 150 years, it is clear that mechanical analogues and models have been critical to our present advanced understanding of electromagnetism. Maxwell has remarked, "a physical analogue is that partial similarity between the results of one science and those of another which makes each of them illustrate the other", and later in relation to the electro-tonic state, "using the mechanical illustrations to assist the imagination, but not to account for the phenomena". Writing on the physical lines of force in 1861, Maxwell, in extending Thomson's idea that magnetism represents the alignment of the axes of vortices, commented, "it appears the phenomena of induced currents are part of a communication of the rotatory velocity of the vortices from one part of the fluid to another", and he developed the idea that the vortices might be thought of as elastic cells or balls and wrote, "the electric field exerts a force to the fixed charge balls, they tend to distort the vortices, and the charges are displaced from their stationary location by a distance proportional to the electric force. In this way the electric force is transferred through the medium because of the elasticity", and later described "electric deformation as a kind of elastic yielding".

Regarding the interplay between developments in mechanics and those in electromagnetism and on the electro-tonic intensity, Maxwell wrote, "what I have called the electromagnetic momentum is the same quantity which is called by Faraday the electro-tonic state of the circuit, every change of which involves the action of an electromotive force". Later he wrote, "the electro-tonic intensity is the fundamental quantity in the theory of electromagnetism. By a careful study of the laws of elastic solids and of the motions of viscous fluids, I hope to discover a method of forming a mechanical conception of this electro-tonic state". And later he wrote, "Faraday was led to recognise the existence of something which we now know to be a mathematical quantity and which may even be called the fundamental quantity in the theory of electromagnetism".

These lengthy preliminary comments are designed to convince the reader that progress in mechanics and electromagnetism did not occur in isolation of each other. Indeed, after Maxwell had fully settled the fundamentals of electromagnetism, its importance and the importance of Lorentz invariance were fundamental to the development of the theory of special relativity. After attending Professor Rousseaux's lectures, it occurred to the author that if the four potentials $(\mathbf{A}, V)$ play such an important role in electromagnetism, might there not also be a corresponding set of potentials that might play a similar role in mechanics? And the question posed here is that, "if we accept the proposition that the potentials $(\mathbf{A}, V)$ are indeed fundamental to the theory of electromagnetism, does this result hold any implications for mechanics fundamentals, such as the validity or otherwise of Newton's second law?" Given that momentum and mass conservation are necessarily partnered in a special relativistic four-vector sense, and the requirement of Lorentz invariant energy-momentum relations, the only available option is the adoption of momentum $\mathbf{p} = m\mathbf{u}$, particle energy $e = mc^2$ and necessarily the Einstein mass variation. This choice appears to be fortuitous, since the proposed work done expression automatically makes accommodation for the de Broglie wave energy.

3.4 Four Types of Matter and Variable Rest Mass

Assuming all quantities are position $\mathbf{x}$ and time t dependent along with the usual formulae of special relativity, namely, $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ where m_0 denotes rest mass and u is the magnitude of the particle velocity, thus $u^2 = \mathbf{u} \cdot \mathbf{u}$; the conventional rate-of-working Eq. (2.43) for the physical energy e of a particle is as follows:

$$de = \mathbf{f} \cdot \mathbf{dx} = \frac{d\mathbf{p}}{dt} \cdot \mathbf{dx}, \tag{3.2}$$

where $\mathbf{f} = d\mathbf{p}/dt$ denotes the physical force and $\mathbf{p} = m\mathbf{u}$ is the momentum where $\mathbf{u} = d\mathbf{x}/dt$ is the particle velocity. Using these relations, Eq. (3.2) may be rewritten

$$e\frac{de}{dt} = c^2\mathbf{p} \cdot \frac{d\mathbf{p}}{dt} = c^2 p\frac{dp}{dt}, \tag{3.3}$$

which evidently integrates to yield $e^2 = (pc)^2 + constant$, where p denotes the magnitude of the momentum vector, namely, $p^2 = \mathbf{p} \cdot \mathbf{p}$. Again we comment that this constant is generally fixed by taking the particle energy at $e = e_0 = m_0c^2$ at zero velocity, so that $e^2 = e_0{}^2 + (pc)^2$, but this does not necessarily have to be the case, and there might be other interpretations for the constant, although here again we assume that this arbitrary constant is as generally prescribed. In the analysis of the energy statement $e^2 = e_0{}^2 + (pc)^2$, either e_0 is zero or it is non-zero, and therefore four distinct alternatives arise:

- (I) $e = (e_0{}^2 + (pc)^2)^{1/2}, \quad e_0 \neq 0$
- (II) $e = -(e_0{}^2 + (pc)^2)^{1/2}, \quad e_0 \neq 0$
- (III) $e = pc, \quad e_0 = 0,$
- (IV) $e = -pc, \quad e_0 = 0,$

and while it is straightforward to assign distinct types of matter to each of the four distinct alternatives, other than (I) as baryonic matter, at this point in time, we may only speculate to correlate the others with the various known distinct types, such as dark energy and dark matter. Here we identify (III) and (IV) with dark matter and dark energy, respectively, while we propose (II) corresponds perhaps to a form of antimatter. This seemingly naive identification of the distinct types of matter is subsequently discussed at length and is only meaningful because it is interpreted within the extension of special relativity examined here. Such an interpretation is simply not available within the narrower confines of traditional special relativity because it lacks the notion of a force in the direction of time. We further note that [32] puts forwards the interesting notion of negative mass as one explanation of dark matter and dark energy.

In the following extension of Newton's second law, we assume throughout the text the Einstein formulae of special relativity and in particular that the rest energy $e_0 = m_0c^2$ is a constant. However, it is generally believed that for baryonic matter the rest mass is constant for speeds $u < c$ and zero at the speed of light $u = c$. Further, de Broglie believed that the rest mass of particles is not constant, but variable, and that both the neutrino and the photon have non-zero rest masses. Following de Broglie, the superluminal world $u > c$ is in all probability the world of waves, and by inverting the Einstein relation $pc = e_0(u/c)/(1 - (u/c)^2)^{1/2}$ for velocity and momentum magnitudes $u = (\mathbf{u} \cdot \mathbf{u})^{1/2}$ and $p = (\mathbf{p} \cdot \mathbf{p})^{1/2}$ respectively, we obtain the simple but important relation

$$\frac{u}{c} = \frac{pc}{(e_0^2 + (pc)^2)^{1/2}},$$

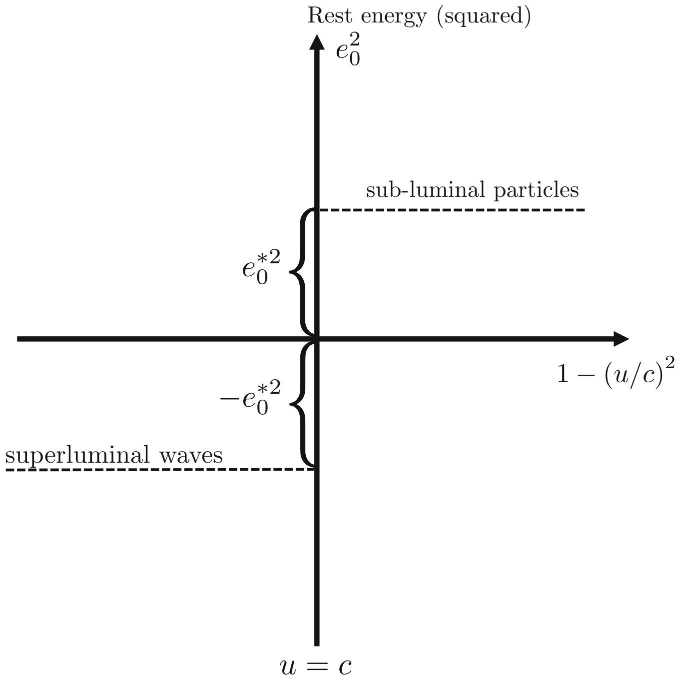

Fig. 3.3 Variable rest mass energy e_0^2 in terms of the Heaviside unit step function

which reveals the inescapable consequences that $e_0^2 > 0$ for $u < c$, $e_0^2 = 0$ for $u = c$ and $e_0^2 < 0$ for $u > c$, and therefore the square of the rest mass energy e_0^2 must adopt a profile something like that represented in Fig. 3.3. This means, at the very least, that we need to entertain the possibility of a variable rest mass energy $e_0 = e_0(\mathbf{x}, t)$ and that the square of a variable rest mass energy $e_0(\mathbf{x}, t)^2$ might involve generalised functions and in particular the Heaviside unit step function.

For example, the profile of the square of the rest energy $e_0(\mathbf{x}, t)^2$ shown in Fig. 3.3 might be represented as follows:

$$e_0(\mathbf{x}, t)^2 = e_0^{*2} \begin{cases} H(1-(u/c)^2), & \text{if } u < c \text{ sub-luminal,} \\ 0, & \text{if } u = c \text{ speed of light,} \\ -H((u/c)^2-1), & \text{if } u > c \text{ superluminal,} \end{cases}$$

and given explicitly by the equation

$$e_0(\mathbf{x}, t)^2 = e_0^{*2} \left(H\left(1-\left(\frac{u}{c}\right)^2\right) - H\left(\left(\frac{u}{c}\right)^2 - 1\right)\right),$$

where e_0^* is assumed to be a positive real number and H denotes the Heaviside unit step function. Equally well, the profile shown in Fig. 3.3 might be represented alternatively as

$$e_0(\mathbf{x},t)^2 = e_0^{*2}\begin{cases} H((ct)^2 - x^2), & \text{if } u < c \text{ sub-luminal,} \\ 0, & \text{if } u = c \text{ speed of light,} \\ -H(x^2 - (ct)^2), & \text{if } u > c \text{ superluminal,} \end{cases}$$

and given explicitly by the equation

$$e_0(\mathbf{x},t)^2 = e_0^{*2}(H((ct)^2 - x^2) - H(x^2 - (ct)^2)),$$

where x here denotes the magnitude of the position vector, namely, $x = (\mathbf{x}\cdot\mathbf{x})^{1/2}$, and for both representations any derivatives of $e_0(\mathbf{x},t)$ will involve the Dirac delta function, either $\delta(1-(u/c)^2)$ or $\delta((ct)^2 - x^2)$.

Some of the basic properties of both the Dirac delta function and the Heaviside unit step function are discussed briefly in Chap. 9, along with the basic references to these important generalised functions. Two sections of Chap. 8 deal with non-constant rest energy $e_0 = e_0(\mathbf{x},t)$, and at least for a single space dimension, the formal equations for the extension of Newton's second law that are presented in the following section appear to be perfectly well-defined and meaningful for variable rest energy. Although we assume throughout the text that the rest energy e_0 remains constant, Eqs. (3.4) in terms of momentum $\mathbf{p} = m\mathbf{u}$ and energy $e = mc^2$ appear to be equally sensible in the absence of this assumption, that is, $e^2 - (pc)^2 = e_0(\mathbf{x},t)^2$, and this seems to apply without further restriction on $e_0(\mathbf{x},t)$, including, perhaps, generalised functions.

3.5 Modified Newton's Laws of Motion

We assume that all quantities are both position $\mathbf{x}$ and time t dependent, and assuming the usual formulae of special relativity, so that $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ where m_0 is the rest mass and u is the magnitude of the particle velocity, thus $u^2 = \mathbf{u}\cdot\mathbf{u}$. Specifically, in view of the above considerations, for sub-luminal particle velocity u with an associated superluminal wave velocity w, where $uw = c^2$, we have in mind developing a mechanical framework within which the following might apply:

$$\begin{cases} e_{part} = \dfrac{e_0}{(1-(u/c)^2)^{1/2}}, \\ e_{light} = cp_{light}, \\ e_{wave} = \dfrac{e_0}{((w/c)^2 - 1)^{1/2}}, \end{cases} \qquad \begin{cases} p_{part} = \dfrac{e_0 u}{c^2(1-(u/c)^2)^{1/2}}, & 0 \leqslant u/c < 1, \\ cp_{light} = e_{light}, & u/c = 1, \\ p_{wave} = \dfrac{e_0 w}{c^2((w/c)^2 - 1)^{1/2}}, & 1 < w/c < \infty, \end{cases}$$

so that the combined or total energies $e_{total} = e_{part} + e_{wave}$ and momenta $p_{total} = p_{part} + p_{wave}$ are given by

$$e_{total} = cp_{total} = e_0 \left(\frac{1 + u/c}{1 - u/c} \right)^{1/2} = e^{\theta},$$

where the angle θ which is previously defined by (2.10) is the angle in which a Lorentz invariance appears as a translational invariance, so that the energy totals involve only a multiplicative factor under Lorentz transformation. Further, at the speed of light, privileged or singular states exist which are generalisations of the known relations for light, namely, $p_{light} = h/\mu$ and $e_{light} = h\nu$, which together imply $e_{light} = cp_{light}$ where h is the Planck's constant, μ is the wave length, and ν is the frequency ($c = \mu\nu$), and we refer to such privileged states as de Broglie states. It is important to note that throughout we are making the implicit hypothesis that any superluminal motion is only possible within the energy interpretation $e = e_0/(1 - (u/c)^2)^{1/2}$, provided either that $e_0^2 < 0$ or that for $u/c > 1$ the formulae $e = e_0/((u/c)^2 - 1)^{1/2}$ and $p = e_0 u/c^2((u/c)^2 - 1)^{1/2}$ apply.

In order to formulate the proposed model, we need to make a distinction between the particle energy $e = mc^2$ and the actual work done by the particle W, and it is not just the momentum vector $\mathbf{p} = m\mathbf{u}$ that contributes to the work done W, but also the intrinsic particle energy e itself plays an important role through the combined potentials $(\mathbf{p}, e/c)$ as a well-defined four vector within special relativity. Specifically, we propose the following formal extension of Newton's second law, such that the force $\mathbf{f}$ and energy-mass production g are given by

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{p}, \tag{3.4}$$

noting that this formulation assumes that space and time are on an equal footing so that all derivatives in (3.4) are assumed to be partial. This is an important point since unlike conventional dynamics the equation $\mathbf{f} = d\mathbf{p}/dt = \mathbf{0}$ does not necessarily imply that $\mathbf{p}$ is necessarily a constant vector, since the total time derivatives

$$\frac{d\mathbf{p}}{dt} = \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{p} = \mathbf{0},$$

constitute connections between partial differentials which can be satisfied without the individual partial derivatives of $\mathbf{p}$ necessarily all vanishing. For example, adopting de Broglie's guidance formula (see Eqs. (1.1), (1.2) and (1.3)) provides non-trivial illustrations of solutions for which $\mathbf{f} = \mathbf{0}$, and yet both $\mathbf{p}$ and e are non-constant with $e^2 - (pc)^2 = e_0^2$ where $p^2 = \mathbf{p} \cdot \mathbf{p}$, and specific examples are given in Chaps. 4 and 9 for the cases of a single spatial dimension and centrally symmetric mechanical systems, respectively. Accordingly for $g \neq 0$, the above proposal also has implications for Newton's first law, giving rise to the possibility of motion in the absence of any spatial force.

Clearly the proposed formulation (3.4) is by no means novel and is well rooted in conventional continuum mechanics. The second equation of (3.4) is merely the generally accepted mass continuity equation but including an energy-mass production term g and might be viewed as Newton's second law in the direction of time. We observe that these relations are left unchanged by the gauge transformation

$$\mathbf{p}' = \mathbf{p} + \nabla\psi, \qquad e' = e - \frac{\partial\psi}{\partial t}, \tag{3.5}$$

where $\psi(\mathbf{x}, t)$ satisfies the wave equation

$$\frac{\partial^2\psi}{\partial t^2} = c^2\nabla^2\psi.$$

In Chap. 7 we show that the expressions for both $\mathbf{f}$ and g are Lorentz invariant so that $(\mathbf{f}, gc)$ is a well-defined four vector.

We further comment that adopting the Einstein mass variation is necessitated by the requirement that the Lorentz invariant energy-momentum relations (2.46) are satisfied. If the (x, t) frame is moving with velocity v with respect to the (X, T) frame, then without any assumptions as to the mass variation, but with $p = mu$, $e = mc^2$ in the (x, t) frame and $P = MU$, $E = Mc^2$ in the (X, T) frame, where $u = (U - v)/(1 - Uv/c^2)$, the requirement of Lorentz invariant energy-momentum relations yields the two consistent equations

$$m(u + v) = MU\left(1 - (v/c)^2\right)^{1/2}, \qquad M(U - v) = mu\left(1 - (v/c)^2\right)^{1/2}.$$

On locating a particle at the origin of the (x, t) frame, we have $U = v$, $u = 0$ and $m = m_0$ and therefore necessarily $M(v) = m_0[1 - (v/c)^2]^{-1/2}$.

3.6 Identity for Spatial Physical Force f

Throughout most of the book, we assume the Einstein mass variation so that the energy e and momentum $\mathbf{p}$ satisfy the relation $e^2 = e_0^2 + c^2\mathbf{p}\cdot\mathbf{p}$, where $e_0 = m_0c^2$ and m_0 is the rest mass. We also frequently assume that the forces $\mathbf{f}$ and g are generated from a potential function $V(\mathbf{x}, t)$ through the relations (3.25). In both situations we are able to establish the conventional formula for Newton's second law, namely, $\mathbf{f} = d\mathbf{p}/dt$, where $\mathbf{p} = m\mathbf{u}$ is the momentum vector and d/dt denotes the material or total time derivative which is defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla). \tag{3.6}$$

However, in general without restriction on the forces $\mathbf{f}$ and g, the spatial physical force $\mathbf{f}$ admits the interesting formula

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} + \mathbf{u} \wedge (\boldsymbol{\nabla} \wedge \mathbf{p}), \tag{3.7}$$

and this equation may be established in component form as follows: Using the subscript notation for Cartesian components $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{p} = (p_x, p_y, p_z)$, we have from the rate-of-working equation $de = \mathbf{dx} \cdot d\mathbf{p}/dt = \mathbf{u} \cdot \mathbf{dp}$, so that for the x-component, we have from $(3.4)_1$

$$f_x = \frac{\partial p_x}{\partial t} + \frac{\partial e}{\partial x} = \frac{\partial p_x}{\partial t} + u_x \frac{\partial p_x}{\partial x} + u_y \frac{\partial p_y}{\partial x} + u_z \frac{\partial p_z}{\partial x},$$

which with some re-arrangement becomes

$$f_x = \frac{\partial p_x}{\partial t} + u_x \frac{\partial p_x}{\partial x} + u_y \frac{\partial p_x}{\partial y} + u_z \frac{\partial p_x}{\partial z} + u_y \left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y} \right) + u_z \left(\frac{\partial p_z}{\partial x} - \frac{\partial p_x}{\partial z} \right).$$

This equation can be seen to further simplify to give

$$f_x = \frac{dp_x}{dt} + u_y (\boldsymbol{\nabla} \wedge \mathbf{p})_z - u_z (\boldsymbol{\nabla} \wedge \mathbf{p})_y,$$

where $\boldsymbol{\nabla} \wedge \mathbf{p}$ has the formal determinant definition

$$\boldsymbol{\nabla} \wedge \mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p_x & p_y & p_z \end{vmatrix},$$

where as usual $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ denote the unit vectors in the three Cartesian directions, and (3.7) now follows where $\mathbf{u} \wedge (\boldsymbol{\nabla} \wedge \mathbf{p})$ is formally given by the determinant

$$\mathbf{u} \wedge (\boldsymbol{\nabla} \wedge \mathbf{p}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ (\boldsymbol{\nabla} \wedge \mathbf{p})_x & (\boldsymbol{\nabla} \wedge \mathbf{p})_y & (\boldsymbol{\nabla} \wedge \mathbf{p})_z \end{vmatrix}.$$

We observe that Eq. (3.7) implies that for general forces $\mathbf{f}$ and g, the scalar product, $\mathbf{f} \cdot \mathbf{u}$, becomes $\mathbf{f} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{dp}/dt = de/dt$. Further, in the following section, assuming the existence of an energy or work done function $W(\mathbf{x}, t)$ implies that the curl of the momentum vector is the zero vector, namely, $\boldsymbol{\nabla} \wedge \mathbf{p} = \mathbf{0}$, which we prove subsequently. We further comment that an alternative derivation of (3.7) is presented in Chap. 12 using Cartesian tensors.

3.7 Assumed Existence of Work Done Function $W(\mathbf{x}, t)$

To extend the conventional notion of work done, say dW from the accepted notion of force times distance, we propose that the incremental work done dW arises as the scalar product of the two four vectors $(\mathbf{f}, gc)$ and $(\mathbf{dx}, cdt)$; thus

$$dW = \mathbf{f} \cdot \mathbf{dx} + gc^2 dt = \left(\frac{\partial \mathbf{p}}{\partial t} + \nabla e \right) \cdot \mathbf{dx} + \left(\frac{\partial e}{\partial t} + c^2 \nabla \cdot \mathbf{p} \right) dt, \tag{3.8}$$

which may be simplified to yield

$$d(W - e) = \frac{\partial \mathbf{p}}{\partial t} \cdot \mathbf{dx} + c^2 (\nabla \cdot \mathbf{p}) dt, \tag{3.9}$$

and therefore we propose that (3.8) generalises the conventional work done equation $de = (d\mathbf{p}/dt) \cdot \mathbf{dx}$, with the additional equation for the wave energy; thus

$$d\mathscr{E} = \frac{\partial \mathbf{p}}{\partial t} \cdot \mathbf{dx} + c^2 (\nabla \cdot \mathbf{p}) dt, \tag{3.10}$$

and we adopt this statement as the defining equation for the determination of the incremental de Broglie wave energy $d\mathscr{E}(\mathbf{x}, t)$. The term $c^2(\nabla \cdot \mathbf{p})dt$ does not appear in conventional special relativistic mechanics, and automatically accommodates the de Broglie wave energy.

Assuming the existence of either W or $\mathscr{E}$ imposes certain constraints, and it is apparent from (3.8) that $\mathbf{f}$ and g must satisfy the compatibility condition

$$\frac{\partial \mathbf{f}}{\partial t} = c^2 \nabla g, \tag{3.11}$$

in order that either (3.8) or (3.10) represents well-defined differential relations for W and $\mathscr{E}$, respectively. At this juncture, it is important to observe that while the proposed force equations (3.4) are fully Lorentz invariant, the above compatibility equation (3.11) is not automatically Lorentz invariant without further restriction, and this is discussed at length subsequently in Chap. 7. This fact gives rise to the notion of partial Lorentz invariance which is exhibited by the exact wave-like solution discussed in Chaps. 5 and 6, for which the assumed linear force expressions satisfy another Lorentz invariance involving the product of force times energy, or in other words, force times mass.

Einstein Mass Variation $e^2 = e_0^2 + c^2 \mathbf{p} \cdot \mathbf{p}$ From the Einstein mass variation, the energy e and momentum $\mathbf{p}$ satisfy the relation $e^2 = e_0^2 + c^2 \mathbf{p} \cdot \mathbf{p}$, where $e_0 = m_0 c^2$ and m_0 is the rest mass, so the following relations are known to apply:

$$\frac{\partial e}{\partial t} = \mathbf{u} \cdot \frac{\partial \mathbf{p}}{\partial t}, \qquad \nabla e = (\mathbf{u} \cdot \nabla)\mathbf{p}, \qquad \frac{de}{dt} = \mathbf{u} \cdot \frac{d\mathbf{p}}{dt}, \tag{3.12}$$

which we may exploit to establish the identities:

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e = \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{p} = \frac{d\mathbf{p}}{dt},$$

so that in this case, the force $\mathbf{f}$ coincides precisely with the usual notion as the total time derivative of momentum. This is an important point since it means that all the great achievements of Newtonian mechanics remain essentially unaltered. Further from (3.10) on division by dt, we have

$$\frac{d\mathscr{E}}{dt} = \mathbf{u} \cdot \frac{\partial \mathbf{p}}{\partial t} + c^2(\nabla \cdot \mathbf{p}),$$

which on using (3.12) can be shown to become

$$g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{p} = \frac{1}{c^2}\left(\frac{\partial e}{\partial t} + \nabla \cdot (e\mathbf{u})\right) = \frac{1}{c^2}\left(\frac{de}{dt} + e(\nabla \cdot \mathbf{u})\right), \tag{3.13}$$

so that from (3.10) we may view the equation

$$d\mathscr{E} = \frac{\partial \mathbf{p}}{\partial t} \cdot \mathbf{dx} + c^2(\nabla \cdot \mathbf{p})dt, \tag{3.14}$$

as the defining statement for the de Broglie wave energy $\mathscr{E}(\mathbf{x}, t)$, extending the conventional notion of work done $de = (d\mathbf{p}/dt) \cdot \mathbf{dx}$. Thus, there are two energies: the conventional particle energy $e(\mathbf{x}, t)$ arising from $de = (d\mathbf{p}/dt) \cdot \mathbf{dx}$ and the de Broglie wave energy $\mathscr{E}(\mathbf{x}, t)$, for which we may deduce the companion formulae

$$\mathbf{f} = \frac{d\mathbf{p}}{dt}, \qquad g = \frac{1}{c^2}\frac{d\mathscr{E}}{dt}, \tag{3.15}$$

the latter equation also following directly from Eqs. $(3.4)_2$ and $(3.12)_1$, so that

$$g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{p} = \frac{\mathbf{u}}{c^2} \cdot \frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot \mathbf{p} = \frac{1}{c^2}\frac{d\mathscr{E}}{dt},$$

where the final equality arises from (3.14).

The equation $W = e + \mathscr{E}$ takes into account both the particle energy $e(\mathbf{x}, t)$ and the de Broglie wave energy $\mathscr{E}(\mathbf{x}, t)$, and it is the contribution arising from the de Broglie wave energy $\mathscr{E}(\mathbf{x}, t)$ that is not accommodated in traditional mechanical thinking. From (3.13) and (3.15), we may deduce the important rate equation connecting the particle and wave energies; thus

$$\frac{d\mathscr{E}}{dt} = \frac{de}{dt} + e(\nabla \cdot \mathbf{u}), \tag{3.16}$$

where the divergence of the velocity field $\nabla \cdot \mathbf{u}$ can be related to the Jacobian of the transformation between material coordinates $\mathbf{X} = (X, Y, Z)$ and spatial coordinates $\mathbf{x} = (x, y, z)$; thus

$$\nabla \cdot \mathbf{u} = \frac{1}{J}\frac{dJ}{dt}, \qquad J = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}, \tag{3.17}$$

the proof of which may be found in any textbook on non-linear continuum mechanics (see for example [31], page 71). The existence of this relation means that we may formulate the following incremental equation connecting the particle energies with the Jacobian of the transformation between material and spatial coordinates; thus

$$d\mathscr{E} = de + \frac{e}{J}dJ. \tag{3.18}$$

While the particle energy $e = mc^2$ can be calculated for any sub-luminal velocity field, the de Broglie wave energy $\mathscr{E}(\mathbf{x}, t)$ is only generated from velocity fields $\mathbf{u}$ for which the corresponding momentum $\mathbf{p}(\mathbf{x}, t)$ satisfies the wave-like equation

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} = c^2 \nabla(\nabla \cdot \mathbf{p}), \tag{3.19}$$

arising from the compatibility of either differential relations (3.8). This equation may be further simplified using the standard vector identity

$$\nabla_\wedge(\nabla_\wedge \mathbf{p}) = \nabla(\nabla \cdot \mathbf{p}) - \nabla^2 \mathbf{p}, \tag{3.20}$$

and since assuming the existence of $V(\mathbf{x}, t)$ or $W(\mathbf{x}, t)$ implies $\nabla \wedge \mathbf{p} = \mathbf{0}$, Eq. (3.19) becomes simply

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} = c^2 \nabla^2 \mathbf{p}, \tag{3.21}$$

so that both the momentum vector $\mathbf{p}$ and the wave energy $\mathscr{E}(\mathbf{x}, t)$ satisfy the wave equation,

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \nabla^2 \mathscr{E}. \tag{3.22}$$

If $\phi(\mathbf{x}, t)$ satisfies the wave equation, we may formally solve equations (3.21) and (3.22) through the relations:

$$\mathbf{p} = \nabla\phi, \qquad \mathscr{E} = \frac{\partial\phi}{\partial t}, \qquad \frac{\partial^2\phi}{\partial t^2} = c^2\nabla^2\phi, \tag{3.23}$$

and $\nabla \wedge \mathbf{p} = \mathbf{0}$ which is apparent immediately from the well-known vector identity curl grad ϕ is zero for all $\phi(\mathbf{x}, t)$, namely, $\nabla \wedge (\nabla\phi) = \mathbf{0}$.

Conservation Equation for Energy Density $\mathscr{W}(\mathbf{x}, t)$ and Instantaneous Power $\mathbf{Q}(\mathbf{x}, t)$ On using the standard vector identity

$$\nabla \cdot (\mathscr{E}\mathbf{p}) = \mathscr{E}\nabla \cdot \mathbf{p} + (\mathbf{p} \cdot \nabla)\mathscr{E},$$

the above force relations may be expressed in the form of a conservation equation in the following manner: From Eq. (3.10) we may deduce four equations

$$\frac{\partial \mathbf{p}}{\partial t} = \nabla\mathscr{E}, \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2(\nabla \cdot \mathbf{p}).$$

On taking the scalar product of the first of these with $c^2\mathbf{p}$, by multiplying the second by $\mathscr{E}$ and then by addition we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\mathscr{E}^2 + c^2\mathbf{p} \cdot \mathbf{p}\right) = c^2\nabla \cdot (\mathscr{E}\mathbf{p}).$$

In conservation form, this equation becomes

$$\frac{\partial \mathscr{W}}{\partial t} + \nabla \cdot \mathbf{Q} = 0, \tag{3.24}$$

where the energy density $\mathscr{W}(\mathbf{x}, t)$ and energy flow or instantaneous power $\mathbf{Q}(\mathbf{x}, t)$ are defined by

$$\mathscr{W} = \frac{1}{2}\left(\mathscr{E}^2 + c^2\mathbf{p} \cdot \mathbf{p}\right), \qquad \mathbf{Q} = -c^2\mathscr{E}\mathbf{p},$$

and Eq. (3.24) relates the time rate of increase (decrease) of the energy density $\mathscr{W}(\mathbf{x}, t)$ that is balanced by a decrease (increase) in the instantaneous power $\mathbf{Q}(\mathbf{x}, t)$. Subsequently, we examine the form of the conservation equation (3.24) for the two special cases of a single spatial dimension and for centrally symmetric mechanical systems for which motion is in the radial direction only with no angular effect.

Remark We comment that in principle, we may postulate the existence of any energy or work done function $\mathscr{E}$ or W, and in doing so, we are inevitably making certain assumptions and inheriting certain implied consequences. In formulating the particular four-vector scalar product (3.8), we assume the spatial and temporal coordinates $\mathbf{x}$ and ct have equal footings, so that for this purpose we are not adopting the conventional signature $(1, 1, 1, -1)$ of special relativity. The latter approach would generate a minus sign in the above Eqs. (3.8) and (3.10) which may well lead to physically interesting outcomes in its own right. The wave equation is also associated with special relativity, and we have adopted the positive sign in the above equations for the following four reasons given below. The question as to whether there is a positive or negative sign in (3.8) is reminiscent of the positive or negative signs appearing in the Hamiltonian and the Lagrangian for conservative systems in classical mechanics, namely, $\mathscr{H} = T + V$ and $\mathscr{L} = T - V$, with the usual notation for kinetic energy T and potential energy V. Here we are dealing with energy, and so the positive sign is chosen, and we provide the following detailed reasoning in support of this choice.

Firstly, the positive sign means that the terms ∇e and $\partial e/\partial t$ reinforce each other to produce the total differential de, which secondly leads to Eq. (3.10) involving the momentum $\mathbf{p}$ only on the right-hand side, and which satisfies the linear wave equation, rather than some more complicated coupled equation involving both $\mathbf{p}$ and e and their derivatives. Thirdly, and perhaps most importantly, as noted below the two terms of (3.10) produce two extreme limits, one involving the particle velocity u and the other the de Broglie wave velocity c^2/u, and specifically it is $+c^2/u$ that is required if the de Broglie wave energy is to be correctly incorporated in the theory. Finally, the positive sign always generates the classical Einstein formula as the leading term (see the general Eqs. (6.13) and (6.14)), and indeed even in the extreme de Broglie limit (see Eq. (5.5)), the Einstein expression emerges as the leading term. These four important outcomes do not occur if the minus sign is adopted in (3.8).

3.8 Forces f and g Derivable from a Potential $V(\mathbf{x}, t)$

In the present formulation, the compatibility condition (3.11) represents a new equation and an important constraint that is not present in conventional theory. For example, if the particle is exposed to some external field such as gravity, then (3.11) implies that on extending the conventional approach, we might propose that $(\mathbf{f}, gc)$ are generated as external forces from a scalar potential $V(\mathbf{x}, t)$ such that

$$\mathbf{f} = -\nabla V, \qquad gc^2 = -\frac{\partial V}{\partial t}, \tag{3.25}$$

and such mechanical systems are conservative in the sense that a conservation of energy principle applies. If these relations apply, then (3.4) becomes

$$\frac{\partial \mathbf{p}}{\partial t} + \nabla(e + V) = 0, \qquad \frac{1}{c^2}\frac{\partial(e + V)}{\partial t} + \nabla \cdot \mathbf{p} = 0,$$

so that by taking the divergence of the first equation and the partial time derivative of the second, it implies that $e + V$ satisfies the classical wave equation; thus

$$\frac{\partial^2(e+V)}{\partial t^2} = c^2\nabla^2(e+V).$$

Further, from the definition of $\mathscr{E}$, Eq. (3.10), we find that

$$d\mathscr{E} = \frac{\partial \mathbf{p}}{\partial t}\cdot \mathbf{dx} + c^2(\nabla\cdot\mathbf{p})dt = -\nabla(e+V)\cdot\mathbf{dx} - \frac{\partial(e+V)}{\partial t}dt = -d(e+V), \tag{3.26}$$

and therefore $d(e+\mathscr{E}+V)=0$, and a conventional type of conservation of energy applies, namely, $W+V=e+\mathscr{E}+V=constant$, or in words $Particle\ Energy + Wave\ Energy + Potential\ Energy = constant.$

$\nabla\wedge\mathbf{p}=\mathbf{0}$ for Conservative Mechanical Systems Assuming the existence of an energy potential $V(\mathbf{x},t)$ or equivalently a work done function $W(\mathbf{x},t)$ implies that the curl of the momentum vector is the zero vector, namely, $\nabla\wedge\mathbf{p}=\mathbf{0}$, since the three spatial force equations become

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e = -\nabla V,$$

so that in component form for the momentum vector $\mathbf{p}=(p_x, p_y, p_z)$, we have

$$\frac{\partial p_x}{\partial t} = \frac{\partial \mathscr{E}}{\partial x}, \qquad \frac{\partial p_y}{\partial t} = \frac{\partial \mathscr{E}}{\partial y}, \qquad \frac{\partial p_z}{\partial t} = \frac{\partial \mathscr{E}}{\partial z}.$$

On examination of each of these three equations in turn, we may progressively deduce that there exists a function $\phi(x,y,z,t)$ such that

$$p_x = \frac{\partial\phi}{\partial x}, \qquad p_y = \frac{\partial\phi}{\partial y}, \qquad p_z = \frac{\partial\phi}{\partial z}, \qquad \mathscr{E} = \frac{\partial\phi}{\partial t},$$

and therefore trivially we have $\nabla\wedge\mathbf{p}=\mathbf{0}$, since, for example, $\partial p_x/\partial y = \partial p_y/\partial x$ and therefore each component of $\nabla\wedge\mathbf{p}$ vanishes.

3.9 Correspondence with Maxwell's Equations

In view of the fact that the model developed here springs from electromagnetic theory, it will come as no surprise that there is a correspondence between the two sets of equations, and the relationship sheds much light on the notion of a force in the direction of time. Specifically, within this correspondence the "electric field" $\mathbf{E}=\mathbf{f}$, the "magnetic induction" $\mathbf{B}=\nabla\wedge\mathbf{p}$, and the gradient ∇g of the force g in the direction of time is equivalent to the "current" $\mathbf{j}$; that is, if there is no force in the direction of time, there is no "current".

In this section we examine this formal connection between Eqs. (3.4) and Maxwell's equations of electromagnetism. Maxwell's equations might appear in a variety of forms depending upon the particular electrical and magnetic units that are adopted. For our purposes, the relationship to be established is most apparent if we do not specify units and we examine Maxwell's equations in a form with unspecified units and involving four arbitrary constants, as given by Wachter and Hoeber [106] (page 124). Specifically, we establish a formal connection between Eqs. (3.4), (3.7) and (3.11), namely,

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e = \frac{d\mathbf{p}}{dt} + \mathbf{u} \wedge (\nabla \wedge \mathbf{p}), \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{p}, \qquad \frac{\partial \mathbf{f}}{\partial t} = c^2 \nabla g,$$

and the general Maxwell's equations of electromagnetism as given by Wachter and Hoeber [106] (page 124); thus

$$\nabla \cdot \mathbf{E} = 4\pi k_1 \rho, \qquad \nabla \wedge \mathbf{E} + k_2 \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \wedge \mathbf{B} - k_3 \frac{\partial \mathbf{E}}{\partial t} = 4\pi k_4 \mathbf{j},$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction, ρ is the charge, $\mathbf{j}$ is the current, and k_1, k_2, k_3 and k_4 are four constants which take on a variety of values dependent upon the units adopted but in all cases are required to satisfy the two constraints $k_1 k_3 = k_4$ and $k_2 k_3 = 1/c^2$.

If we attempt the correspondence

$$\mathbf{E} = \mathbf{f}, \qquad \mathbf{B} = (\nabla \wedge \mathbf{p}),$$

then the above equations become

$$\rho = \frac{\nabla \cdot \mathbf{f}}{4\pi k_1}, \qquad \nabla \wedge \mathbf{f} = -k_2 \frac{\partial (\nabla \wedge \mathbf{p})}{\partial t}, \tag{3.27}$$

$$\nabla \wedge (\nabla \wedge \mathbf{p}) - k_3 \frac{\partial \mathbf{f}}{\partial t} = 4\pi k_4 \mathbf{j},$$

and on substituting $\mathbf{f} = \partial \mathbf{p}/\partial t + \nabla e$ into the second of these equations, it is immediately apparent that $k_2 = -1$, and therefore from the constraints $k_1 k_3 = k_4$ and $k_2 k_3 = 1/c^2$, we have $k_3 = -1/c^2$ and $k_4 = -k_1/c^2$. Thus the third of these equations becomes

$$\mathbf{j} = -\frac{c^2}{4\pi k_1} \left\{ \nabla \wedge (\nabla \wedge \mathbf{p}) + \frac{1}{c^2} \frac{\partial \mathbf{f}}{\partial t} \right\},$$

$$= -\frac{c^2}{4\pi k_1}\left\{\nabla(\nabla\cdot\mathbf{p}) - \nabla^2\mathbf{p} + \frac{1}{c^2}\left(\frac{\partial^2\mathbf{p}}{\partial t^2} + \nabla\frac{\partial e}{\partial t}\right)\right\},$$
$$= -\frac{c^2}{4\pi k_1}\left\{\nabla\left(g - \frac{1}{c^2}\frac{\partial e}{\partial t}\right) + \frac{1}{c^2}\nabla\frac{\partial e}{\partial t}\right\},$$
$$= -\frac{c^2}{4\pi k_1}\nabla g,$$

on using $g = \partial e/c^2\partial t + \nabla\cdot\mathbf{p}$ and assuming that $\mathbf{p}$ satisfies the wave equation (3.21), so that charge conservation becomes

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = \frac{1}{4\pi k_1}\left\{\frac{\partial\nabla\cdot\mathbf{f}}{\partial t} - c^2\nabla^2 g\right\},$$
$$= \frac{1}{4\pi k_1}\nabla\left\{\frac{\partial\mathbf{f}}{\partial t} - c^2\nabla g\right\},$$
$$= 0,$$

and therefore for all choices of the constant k_1, the two sets of equations correspond provided that the forces $\mathbf{f}$ and g satisfy (3.11) and are consequently derivable from a potential function $V(\mathbf{x}, t)$ through Eqs. (3.25).

In summary, if the spatial force $\mathbf{f}$ and the force g in the direction of time satisfy equation (3.11), then for all values of the constant k_1, we have established the following correspondence with Maxwell's equations:

$$\mathbf{E} = \mathbf{f}, \qquad \mathbf{B} = (\nabla\wedge\mathbf{p}), \qquad \rho = \frac{\nabla\cdot\mathbf{f}}{4\pi k_1}, \qquad \mathbf{j} = -\frac{c^2}{4\pi k_1}\nabla g.$$

In this event, Maxwell's equations (3.27) become simply

$$\nabla\wedge\mathbf{f} = \frac{\partial(\nabla\wedge\mathbf{p})}{\partial t}, \qquad \nabla\wedge(\nabla\wedge\mathbf{p}) = \mathbf{0},$$

so that both $\nabla\wedge(\nabla\wedge\mathbf{f}) = \nabla\wedge(\nabla\wedge\mathbf{p}) = \mathbf{0}$. In fact if the forces $\mathbf{f}$ and g are derived from a potential function $V(\mathbf{x}, t)$ through Eqs. (3.25), then both $\nabla\wedge\mathbf{f} = \nabla\wedge\mathbf{p} = \mathbf{0}$, since $\mathbf{f} = -\nabla V$ and from (3.23) $\mathbf{p} = \nabla\phi$, so that the curl of the "electric field" $\nabla\wedge\mathbf{E}$ and the curl of the "magnetic induction" $\mathbf{B}$ both vanish. However, of most interest in this correspondence is that the gradient ∇g of the force in the direction of time corresponds to the notion of "current", and is such that if the force in the direction of time does not exist, then there is no "current".

3.10 Centrally or Spherically Symmetric Systems

Here for purposes of illustration, we have in mind a central body generating a spherically symmetric environment, such that the motion generated is in the radial direction only with no angular contribution or dependence, and is such that all variables are functions of (r, t) only, where r is the spatial radius taken from the centre of the gravitating body and defined by $r = (x^2 + y^2 + z^2)^{1/2}$. This topic is examined in detail in Chap. 9 which provides an illustration of the curious equality $\nabla(\nabla \cdot \mathbf{p}) = \nabla^2\mathbf{p}$ arising from (3.20) in the event that curl of the momentum vector $\mathbf{p}$ vanishes, namely, $\nabla \wedge (\nabla \wedge \mathbf{p}) = \mathbf{0}$. The left-hand side of the equality $\nabla(\nabla \cdot \mathbf{p}) = \nabla^2\mathbf{p}$ involves the gradient of the divergence of the momentum vector $\mathbf{p}$, while the right-hand side involves the scalar operator ∇^2, namely, the divergence of the gradient, operating on the momentum vector $\mathbf{p}$. Accordingly, we have a situation where the divergence and gradient appear to commute, and since it is not immediately obvious why this should occur, it is instructive to examine the details for what this really means in this particular case.

In this case the momentum vector $\mathbf{p}$ becomes

$$\mathbf{p} = p(r,t)\hat{\mathbf{r}} = \frac{p(r,t)}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}), \tag{3.28}$$

where $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ denote the usual unit vectors in the (x, y, z) directions, respectively, and $\hat{\mathbf{r}}$ is a unit vector in the spherical radial direction. Thus, explicitly we have the three relations

$$p_x = p(r,t)\frac{x}{r}, \qquad p_y = p(r,t)\frac{y}{r}, \qquad p_z = p(r,t)\frac{z}{r},$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, and for which by partially differentiating these expressions, it is a simple matter to verify directly that, for example, $\partial p_x/\partial y - \partial p_y/\partial x = 0$, and that all three components of $\nabla \wedge \mathbf{p}$ vanish.

For $\mathbf{p}$ given by Eq. (3.28), the divergence of the momentum vector is given by

$$(\nabla \cdot \mathbf{p}) = \frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z} = \frac{3p}{r} + \left(\frac{1}{r}\frac{\partial p}{\partial r} - \frac{p}{r^2}\right)\frac{(x^2+y^2+z^2)}{r} = \left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right),$$

and on taking the gradient of this scalar quantity, we have

$$\nabla\left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right)\hat{\mathbf{r}} = \frac{\partial}{\partial r}\left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right)(\frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}}),$$

and from this equation we may deduce

$$\nabla(\nabla \cdot \mathbf{p}) = \left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{2p}{r^2}\right)(\frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}}). \tag{3.29}$$

We now need to evaluate the scalar operator ∇^2 operating on the momentum vector $\mathbf{p}$, so that we need to evaluate $\nabla^2 p_x$ and similarly for the other two components. Now with $p_x = p(r,t)x/r$, we require the following partial derivatives:

$$\frac{\partial p_x}{\partial x} = \frac{x^2}{r^2}\frac{\partial p}{\partial r} + (y^2+z^2)\frac{p}{r^3}, \quad \frac{\partial^2 p_x}{\partial x^2} = \frac{x}{r}\left\{\frac{x^2}{r^2}\frac{\partial^2 p}{\partial r^2} + \frac{3}{r^3}(y^2+z^2)\left(\frac{\partial p}{\partial r} - \frac{p}{r}\right)\right\},$$

along with

$$\frac{\partial p_x}{\partial y} = \frac{xy}{r^2}\left(\frac{\partial p}{\partial r} - \frac{p}{r}\right), \qquad \frac{\partial^2 p_x}{\partial y^2} = \frac{x}{r}\left\{\frac{y^2}{r^2}\frac{\partial^2 p}{\partial r^2} + \frac{(r^2-3y^2)}{r^3}\left(\frac{\partial p}{\partial r} - \frac{p}{r}\right)\right\},$$

and

$$\frac{\partial p_x}{\partial z} = \frac{xz}{r^2}\left(\frac{\partial p}{\partial r} - \frac{p}{r}\right), \qquad \frac{\partial^2 p_x}{\partial z^2} = \frac{x}{r}\left\{\frac{z^2}{r^2}\frac{\partial^2 p}{\partial r^2} + \frac{(r^2-3z^2)}{r^3}\left(\frac{\partial p}{\partial r} - \frac{p}{r}\right)\right\}.$$

By addition of the second order partial derivatives, we might deduce

$$\nabla^2 p_x = \frac{\partial^2 p_x}{\partial x^2} + \frac{\partial^2 p_x}{\partial y^2} + \frac{\partial^2 p_x}{\partial z^2} = \frac{x}{r}\left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{2p}{r^2}\right),$$

which is entirely in agreement with the first component of Eq. (3.29), and there are corresponding calculations for the other two components, establishing that $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{p}) = \nabla^2\mathbf{p}$ holds in this particular case.

Another way of understanding this result is to use spherical polar coordinates (r,θ,ϕ) defined by the relations

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

so that the momentum vector $\mathbf{p}$ becomes

$$\begin{aligned}\mathbf{p} = p(r,t)\hat{\mathbf{r}} &= \frac{p(r,t)}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \\ &= p(r,t)(\sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}),\end{aligned} \tag{3.30}$$

and in terms of spherical polar coordinates, the Laplacian ∇^2, which is a scalar operator, becomes

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right). \tag{3.31}$$

On performing the partial differentiations, we may show from Eqs. (3.30) and (3.31)

$$\nabla^2 \mathbf{p} = \nabla^2 \left\{ p(r,t)(\sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}) \right\}$$
$$= \left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{2p}{r^2} \right) (\sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}),$$

in complete agreement with the previously obtained result.

3.11 Newtonian Kinetic Energy and Momentum

In the final three sections of this chapter, we provide a Newtonian interpretation of the proposed extension of Newton's second law represented by Eqs. (3.4), thus

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{p}, \tag{3.32}$$

and with a work done function defined by (3.8), namely,

$$dW = \mathbf{f} \cdot \mathbf{dx} + gc^2 dt.$$

These equations are primarily intended to be interpreted in the context of the special relativistic expressions for both $e = mc^2$ and $\mathbf{p} = m\mathbf{u}$ with $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ where u is the magnitude of the velocity defined by $u^2 = \mathbf{u} \cdot \mathbf{u}$ and m_0 denotes the rest mass. However, it would seem worthwhile mentioning that a Newtonian version of (3.32) is also meaningful that is based upon the classical interpretations $e = m_0 u^2/2$ and $\mathbf{p} = m_0 \mathbf{u}$, for which we might readily deduce

$$\mathbf{f} = m_0 \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right), \qquad g = m_0 \left(\frac{u}{c^2}\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{u} \right), \tag{3.33}$$

and for which the first expression can be shown to be simply the conventional Newton's second law, as might be expected; thus

$$\mathbf{f} = m_0 \frac{d\mathbf{u}}{dt},$$

where d/dt denotes the total or material time derivative. The second equation of (3.33) provides a new element to Newton's second law, and while the two constitutive equations (3.33) are themselves non-linear, the model itself may be referred to as a linearised model in the sense that the governing equation is linear, namely, the classical linear wave equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = c^2 \nabla^2 \mathbf{u}, \tag{3.34}$$

which, assuming the existence of a work done function, is most easily seen from Eq. (3.21) with the momentum vector $\mathbf{p} = m_0\mathbf{u}$. A linearised model often has the distinct advantages of simplicity and tractability, and the various contributing effects not only become clearer and more apparent but often may be superimposed.

For a single spatial dimension x with particle velocity $u = u(x, t)$, Eqs. (3.33) become quite simply

$$\frac{f}{m_0} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \qquad \frac{g}{m_0} = \frac{u}{c^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}. \tag{3.35}$$

We observe that on using the primed notation for the space-time transformation $x' = ct$ and $t' = x/c$, we have $dx' = cdt$ and $dt' = dx/c$, and $u' = c^2/u$ so that these equations become

$$\frac{f'}{m_0} = \frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} = -\left(\frac{c}{u}\right)^3\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\left(\frac{c}{u}\right)^3\frac{f}{m_0},$$
$$\frac{g'}{m_0} = \frac{u'}{c^2}\frac{\partial u'}{\partial t'} + \frac{\partial u'}{\partial x'} = -\left(\frac{c}{u}\right)^3\left(\frac{u}{c^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) = -\left(\frac{c}{u}\right)^3\frac{g}{m_0},$$

from which the following force transformation relations are apparent

$$\frac{f'}{(u'/c)^{3/2}} = -\frac{f}{(u/c)^{3/2}}, \qquad \frac{g'}{(u'/c)^{3/2}} = -\frac{g}{(u/c)^{3/2}}. \tag{3.36}$$

3.12 Newtonian Wave-Like Solution

In Chaps. 5 and 6, we examine in some detail an exact relativistic wave-like solution of Eqs. (3.4), and we seek a formal solution for which the forces f and g and particle energy e are all functions of the particle velocity u only. In this section we present the corresponding solution details assuming Newtonian kinetic energy and momentum. Accordingly, we now look for forces $f(u)$ and $g(u)$ dependent only upon the velocity $u(x, t)$, and we employ the notation $A(u) = f(u)/m_0$ and $B(u) = g(u)/m_0$, so that Eqs. (3.35) become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = A(u), \qquad \frac{u}{c^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = B(u), \tag{3.37}$$

which we solve as two equations in the two unknowns $\partial u/\partial x$ and $\partial u/\partial t$ to obtain

$$\frac{\partial u}{\partial x} = \frac{B(u) - A(u)u/c^2}{(1 - (u/c)^2)}, \qquad \frac{\partial u}{\partial t} = \frac{A(u) - B(u)u}{(1 - (u/c)^2)}. \tag{3.38}$$

On partially differentiating the first of these with respect to t and the second with respect to x, equating the two expressions and using the equations themselves, we may eventually deduce the intriguingly simple result

$$(1-(u/c)^2)\left(B\frac{dA}{du}-A\frac{dB}{du}\right)=(B^2-(A/c)^2),$$

which may be rearranged as

$$\frac{d(A/B)}{(1-(A/Bc)^2)}=\frac{du}{(1-(u/c)^2)},$$

and integrated to yield

$$\left(\frac{1+A/Bc}{1-A/Bc}\right)=C\left(\frac{1+u/c}{1-u/c}\right),$$

where C denotes the arbitrary constant of integration. This equation may be rearranged to give

$$cB(u)=A(u)\left(\frac{u/c+\lambda}{1+\lambda u/c}\right),$$

or more directly in terms of the forces $f(u)$ and $g(u)$

$$cg(u)=f(u)\left(\frac{u/c+\lambda}{1+\lambda u/c}\right),$$

where λ denotes another arbitrary constant related to C through $\lambda=(C+1)/(C-1)$, and we observe that this equation coincides with the corresponding equation obtained for the full theory (see Eq. (6.4)). With this latter relation, the two Eqs. (3.38) now become

$$\frac{\partial u}{\partial x}=\frac{\lambda f(u)}{cm_0(1+\lambda u/c)},\qquad \frac{\partial u}{\partial t}=\frac{f(u)}{m_0(1+\lambda u/c)},\tag{3.39}$$

and from which it is clear that the particle velocity $u(x,t)$ satisfies the partial differential equation

$$c\frac{\partial u}{\partial x}=\lambda\frac{\partial u}{\partial t},$$

and therefore the particle velocity is of the form $u(x,t)=u(\xi)$ where $\xi=\lambda x+ct$, and both Eqs. (3.39) give rise to the ordinary differential equation

$$\frac{du}{d\xi} = \frac{f(u)}{cm_0(1+\lambda u/c)} = \frac{g(u)}{m_0(\lambda + u/c)}. \tag{3.40}$$

This outcome is a direct consequence of the search for consistent solutions such that both forces $f(u)$ and $g(u)$ depend only upon the velocity $u(x,t)$.

3.13 Newtonian Work Done $W(u,\lambda)$ from $\partial f/\partial t = c^2 \partial g/\partial x$

Now from the two Eqs. (3.37) and the functional dependence $u(x,t) = u(\xi)$ where $\xi = \lambda x + ct$, or directly from (3.40), we may deduce

$$f(u) = m_0 c \left(1 + \frac{\lambda u}{c}\right) u'(\xi), \qquad g(u) = m_0 \left(\lambda + \frac{u}{c}\right) u'(\xi),$$

where the prime here denotes differentiation with respect to ξ. In order that an energy function W exists, we have the constraint $\partial f/\partial t = c^2 \partial g/\partial x$, and from which we obtain the condition $(1-\lambda^2)u''(\xi) = 0$, so that either $\lambda = \pm 1$ or $u''(\xi) = 0$. Alternatively, if $\partial f/\partial t = c^2 \partial g/\partial x$, then there exists a potential function $V(x,t)$ such that

$$f(u) = m_0 c \left(1 + \frac{\lambda u}{c}\right) u'(\xi) = -\frac{\partial V}{\partial x}, \quad g(u)c^2 = m_0 c^2 \left(\lambda + \frac{u}{c}\right) u'(\xi) = -\frac{\partial V}{\partial t},$$

and the condition $(1-\lambda^2)u''(\xi) = 0$ arises from equating expressions for the mixed partial derivative $\partial^2 V/\partial x \partial t$. Assuming that this condition is satisfied, on using $d\xi/dt = \lambda u + c$, we might formally integrate these two equations as follows:

$$\begin{aligned}
-dV &= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial t} dt, \\
&= m_0 c \left(1 + \frac{\lambda u}{c}\right) u'(\xi) dx + m_0 c^2 \left(\lambda + \frac{u}{c}\right) u'(\xi) dt, \\
&= \frac{m_0 c}{\lambda} \left(1 + \frac{\lambda u}{c}\right) u'(\xi) d\xi + m_0 c^2 \left(\lambda - \frac{1}{\lambda}\right) u'(\xi) dt, \\
&= \frac{m_0 c}{\lambda} \left(1 + \frac{\lambda u}{c}\right) u'(\xi) d\xi + m_0 c^2 \frac{\left(\lambda - \frac{1}{\lambda}\right) u'(\xi)}{(\lambda u + c)} d\xi,
\end{aligned}$$

and a straightforward integration yields

$$V(x,t) = -\frac{m_0}{2} u^2 - \frac{m_0 c u}{\lambda} - m_0 c^2 \left(1 - \frac{1}{\lambda^2}\right) \log\left(1 + \frac{\lambda u}{c}\right) + V_0,$$

where V_0 denotes an arbitrary constant. We comment that the first two terms explicitly demonstrate contributions from both the conventional kinetic energy and a momentum term corresponding to the de Broglie wave energy, while the term involving the logarithm, if it appears, is new. We recall that in this derivation we are assuming that the condition $(1 - \lambda^2)u''(\xi) = 0$ is satisfied, so that either $\lambda = \pm 1$ or $u''(\xi) = 0$.

Of course, the condition $(1 - \lambda^2)u''(\xi) = 0$ also follows immediately from the classical wave equation (3.34) for a single spatial dimension. We note that subsequently we identify $\lambda = 1$ as corresponding to a backward dark matter wave, while the case $\lambda = -1$ corresponds to a forward dark energy wave. The solutions arising from $u''(\xi) = 0$ are the linearised versions of the explicit wave-like solutions representing both forward and backward waves that the corresponding full relativistic solutions are discussed in some considerable detail in two subsequent chapters.

In terms of the work done function $W(x, t) = -V(x, t)$, and again assuming that either $\lambda = \pm 1$ or that $u''(\xi) = 0$, we have

$$dW = f dx + gc^2 dt = m_0 c \left(1 + \frac{\lambda u}{c}\right) u'(\xi) dx + m_0 c^2 \left(\lambda + \frac{u}{c}\right) u'(\xi) dt,$$

and on using $d\xi/dt = \lambda u + c$, the same integration yields the energy expression

$$W(u, \lambda) = \frac{m_0}{2} u^2 + \frac{m_0 c u}{\lambda} + m_0 c^2 \left(1 - \frac{1}{\lambda^2}\right) \log\left(1 + \frac{\lambda u}{c}\right) + m_0 c^2,$$

and here for the sake of definiteness, the constant of integration is chosen so that the Einstein rest mass energy $e_0 = m_0 c^2$ is recovered when $u = 0$. In the event that $\lambda = \pm 1$, the following expressions for the total work done by the particle W apply, where $u = F(ct + x)$ for $\lambda = 1$ and $u = G(ct - x)$ for $\lambda = -1$ and where F and G denote arbitrary functions. Thus for $\lambda = 1$ and $\lambda = -1$, we have, respectively,

$$W(u, 1) = \frac{m_0}{2} u^2 + m_0 c u + m_0 c^2, \qquad W(u, -1) = \frac{m_0}{2} u^2 - m_0 c u + m_0 c^2.$$

Again the first two terms of each show explicitly the contributions from the kinetic energy and the de Broglie wave energy, and we have

$$W(u, 1) + W(u, -1) = e_0 \left[2 + \left(\frac{u}{c}\right)^2\right],$$

where $e_0 = m_0 c^2$.

Chapter 4
Special Results for One Space Dimension

4.1 Introduction

The purpose of this chapter is to illustrate the formal extension of Newton's second law given by (3.4) in their simplest conceivable context. In this chapter we present a number of additional general results which apply for de Broglie particle-wave mechanical systems for a single Cartesian spatial dimension. We need to emphasise that these results might be termed special or privileged in the sense that they are only applicable to a single Cartesian spatial dimension and in the absence of any curvature effects. In some instances their generalisation may be straightforward while, in others, perhaps an entirely non-trivial matter. Nevertheless, even with this in mind, special exact results and consequences are always important and helpful in providing insight.

In the first section we present the basic equations applying to a single spatial dimension and an illustrative similarity solution. In the following section, we detail a number of general reformulations of the basic equations, and in the subsequent section, we establish an important insightful formal identity which shows that if $f = \pm cg$, then the physical energy $e(x, t)$ is necessarily either identically zero, or it satisfies the wave equation. If it satisfies the wave equation, then the rest energy $e_0 = 0$, and this gives rise to the prospects of dark matter $e = cp$ and dark energy $e = -cp$, and for both the particle and wave energies coincide; thus $e = \mathscr{E}$. In the subsequent two sections, we formulate the basic equations in terms of two invariants of the Lorentz group of transformations and which are later shown to be important in the context of quantum mechanics.

We have previously noted de Broglie's guidance formula of the particle by its wave, as given by Eqs. (1.1) and (1.2), and that in the context of the present theory, these equations suggest that $f = 0$ while g is determined from (1.3). In the next two sections of the chapter, for a single spatial dimension x, we extend the analysis of these equations first for the case $f = 0$ and $g \neq 0$, which essentially applies to quantum mechanics, and then we present the corresponding analysis for the case

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$f \neq 0$ and $g = 0$, which essentially applies to special relativistic mechanics. In these two sections, we present exact solutions arising by adopting constant values for the Lorentz invariants, and we make the point that the approach presented here no doubt also applies to more general situations. Adopting at least one of the Lorentz invariants to be constant means that we may present explicit formulae and the various partial derivatives can be easily evaluated. The analysis is suggestive of the importance of Clairaut's differential equation $x - ut = U_0(u)$ with parameter u, and this constitutes the topic of the next section of the chapter. In the final two sections of the chapter, we briefly outline, respectively, the Hamiltonian and Lagrangian formulations for particle-wave mechanics with a single Cartesian spatial dimension, with the conclusion that since $dH/dt \neq 0$, the system is nonconservative in the conventional sense with energy integral $H + \mathscr{E} = constant$.

4.2 Basic Equations

For a single spatial dimension, we propose that every particle moving with velocity u acquires a conventional momentum $p = mu$ and energy $e = mc^2$ where $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ such that p satisfies the wave equation. We propose that associated with this particle motion is a de Broglie wave moving with a wave velocity c^2/u for which there is an associated wave energy $\mathscr{E}$ such that the total energy of the particle W is given by $W = e + \mathscr{E}$. For a given momentum $p(x, t)$, we have

$$u(x, t) = \frac{pc^2}{(e_0^2 + (pc)^2)^{1/2}}, \qquad e(x, t) = (e_0^2 + (pc)^2)^{1/2}, \tag{4.1}$$

from the one-dimensional versions of (3.10) or (3.14), namely,

$$d\mathscr{E} = \frac{\partial \mathscr{E}}{\partial x}dx + \frac{\partial \mathscr{E}}{\partial t}dt = \frac{\partial p}{\partial t}dx + c^2\frac{\partial p}{\partial x}dt, \tag{4.2}$$

we may deduce

$$\frac{\partial \mathscr{E}}{\partial t} = c^2\frac{\partial p}{\partial x}, \qquad \frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t}, \tag{4.3}$$

and therefore the momentum $p = mu$ and the de Broglie wave energy $\mathscr{E}$ both satisfy the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2\frac{\partial^2 p}{\partial x^2}, \qquad \frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2\frac{\partial^2 \mathscr{E}}{\partial x^2}. \tag{4.4}$$

Thus for a single spatial dimension, both the wave energy $\mathscr{E}(x,t)$ and the momentum $p(x,t)$ always satisfy the wave equation. In terms of velocity, using the expression $p(x,t) = m_0 u/(1-(u/c)^2)^{1/2}$, the velocity $u(x,t)$ can be shown to satisfy the non-linear partial differential equation

$$\left\{1-\left(\frac{u}{c}\right)^2\right\}\left(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}\right) + \frac{3u}{c^2}\left\{\left(\frac{\partial u}{\partial t}\right)^2 - c^2\left(\frac{\partial u}{\partial x}\right)^2\right\} = 0, \tag{4.5}$$

noting especially the induced singular behaviour at the speed of light $u = c$. Since the formal general solution of the wave equation $(4.4)_1$ is given by $p(x,t) = F(\alpha) + G(\beta)$, where $F(\alpha)$ and $G(\beta)$ denote arbitrary functions and $\alpha = ct + x$ and $\beta = ct - x$ denote the characteristic coordinates, from $(4.1)_1$ it is clear that the general solution of the highly non-linear partial differential equation (4.5) becomes

$$\frac{u(x,t)}{c} = \frac{pc}{(e_0^2 + (pc)^2)^{1/2}} = \frac{c(F(\alpha)+G(\beta))}{(e_0^2 + c^2(F(\alpha)+G(\beta))^2)^{1/2}}.$$

Illustrative Similarity Stretching Solutions of (4.4) **and** (4.5) The one-dimensional classical wave equations (4.4) are well known to remain invariant under the stretching group of transformations

$$x^* = \lambda x, \quad t^* = \lambda t, \quad p^* = \lambda^m p, \quad \mathscr{E}^* = \lambda^m \mathscr{E}, \tag{4.6}$$

where λ denotes any positive arbitrary parameter and m denotes any arbitrary exponent. From these invariances, we observe the particular invariant quantities that are independent of the parameter λ, and thus

$$\frac{x^*}{t^*} = \frac{x}{t}, \qquad \frac{p^*}{t^{*m}} = \frac{p}{t^m}, \qquad \frac{\mathscr{E}^*}{t^{*m}} = \frac{\mathscr{E}}{t^m},$$

and the partial differential equations (4.4) are reduced to ordinary differential equations by adopting one invariant as a function of another; thus

$$\frac{p}{t^m} = F\left(\frac{x}{t}\right), \qquad \frac{\mathscr{E}}{t^m} = G\left(\frac{x}{t}\right),$$

where F and G denote functions to be determined by substitution into the partial differential equations, which in this case are simply the classical one-dimensional wave equation. For convenience we use the variable $\xi = x/ct$, and invariance under the group of transformations (4.6) implies that the classical wave equations (4.4) admit similarity solutions of the form

$$p(x,t) = t^m \phi(\xi), \qquad \mathscr{E}(x,t) = t^m \psi(\xi), \tag{4.7}$$

where $\phi(\xi)$ and $\psi(\xi)$ denote functions of ξ only. We show below that $p(x, t)$ and $\mathscr{E}(x, t)$ as solutions of the classical wave equation are given by expressions of the form $p(x, t) = C_1(ct + x)^m + C_2(ct - x)^m$, where C_1 and C_2 denote arbitrary constants. This outcome might well be anticipated, nevertheless, it is interesting that it can be formally established, and in the context of centrally symmetric mechanical systems, a corresponding analysis is provided in Chap. 9.

For example, for the momentum $p(x, t)$, from $(4.7)_1$, we may determine the four partial derivatives

$$\frac{\partial p}{\partial t} = t^{m-1}(m\phi - \xi\phi'), \qquad \frac{\partial p}{\partial x} = t^{m-1}\frac{\phi'}{c},$$
$$\frac{\partial^2 p}{\partial t^2} = t^{m-2}\left[-\xi(m\phi - \xi\phi')' + (m-1)(m\phi - \xi\phi')\right], \qquad \frac{\partial^2 p}{\partial x^2} = t^{m-2}\frac{\phi''}{c^2},$$

and therefore we find from $(4.4)_1$

$$(1-\xi^2)\frac{d^2\phi}{d\xi^2} + 2(m-1)\xi\frac{d\phi}{d\xi} - m(m-1)\phi = 0,$$

and with the further substitution $\eta = (1+\xi)/2$, we obtain

$$\eta(1-\eta)\frac{d^2\phi}{d\eta^2} + [2(m-1)\eta - (m-1)]\frac{d\phi}{d\eta} - m(m-1)\phi = 0.$$

This is the hypergeometric equation which is usually denoted by $F(-m, 1-m; 1-m; \eta)$, and we may readily verify the exact solution $\phi(\eta) = (1-\eta)^m$. On making the further substitution $\phi(\eta) = (1-\eta)^m\Phi(\eta)$, we obtain

$$\eta(1-\eta)\frac{d^2\Phi}{d\eta^2} = (2\eta + m - 1)\frac{d\Phi}{d\eta},$$

which may be integrated to give altogether

$$\phi(\eta) = (1-\eta)^m\left(C_1^* + C_2^*\int^\eta z^{m-1}(1-z)^{-m-1}dz\right),$$

where C_1^* and C_2^* denote two arbitrary constants. On making the substitution $\rho = z/(1-z)$ and noting that $d\rho = dz/(1-z)^2$, the above integral becomes

$$\int^\eta \left(\frac{z}{1-z}\right)^{m-1}\frac{dz}{(1-z)^2} = \int \rho^{m-1}d\rho = \frac{\rho^m}{m} = \frac{1}{m}\left(\frac{\eta}{1-\eta}\right)^m,$$

and therefore we have $\Phi(\eta) = C_1^*(1-\eta)^m + C_2^*\eta^m/m$. On retracing the substitutions $\eta = (1+\xi)/2$, $\xi = x/ct$ and $p(x, t) = t^m\phi(\xi)$, we may readily deduce the solution

$p(x,t) = C_1(ct+x)^m + C_2(ct-x)^m$, on introducing modified arbitrary constants C_1 and C_2. We note however the singular case arising from $m = 0$ for which we have

$$p(x,t) = C_1 + C_2 \log\left(\frac{ct-x}{ct+x}\right). \tag{4.8}$$

From the relations $p = mu$ and $e = mc^2$, we may deduce

$$\frac{u(x,t)}{c} = \frac{pc}{e} = \frac{pc}{(e_0^2 + (pc)^2)^{1/2}}, \tag{4.9}$$

and therefore the velocity field $u(x,t)$ corresponding to the most general stretching similarity solution for the momentum $p(x,t)$ may be determined from the expression

$$\frac{u(x,t)}{c} = \frac{(C_1(ct+x)^m + C_2(ct-x)^m)}{[(e_0/c)^2 + (C_1(ct+x)^m + C_2(ct-x)^m)^2]^{1/2}}.$$

We observe that the existence of such solutions to the partial differential equation (4.5) is not entirely obvious since on face value Eq. (4.5) only remains invariant under the restricted one-parameter group of stretching transformations

$$x^* = \lambda x, \qquad t^* = \lambda t, \qquad u^* = u,$$

corresponding to the group given by (4.6) with the index parameter $m = 0$. In this case the similarity solutions of (4.5) have the functional form $u(x,t) = c\Phi(\xi)$, where again $\xi = x/ct$. On substitution of the four partial derivatives

$$\frac{\partial u}{\partial t} = -\frac{c\xi\Phi'}{t}, \qquad \frac{\partial u}{\partial x} = \frac{\Phi'}{t}, \qquad \frac{\partial^2 u}{\partial t^2} = \frac{c}{t^2}\left[\xi(\xi\Phi')' + \xi\Phi'\right], \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\Phi''}{ct^2},$$

the partial differential equation (4.5) reduces to the ordinary differential equation

$$(1-\Phi^2)((1-\xi^2)\Phi'' - 2\xi\Phi') + 3\Phi(1-\xi^2)\Phi'^2 = 0,$$

and this equation may be reformulated to give

$$\frac{((1-\xi^2)\Phi')'}{(1-\xi^2)\Phi'} = -3\frac{\Phi\Phi'}{(1-\Phi^2)},$$

which upon integration yields $(1-\xi^2)\Phi' = -2C_2(1-\Phi^2)^{3/2}$, where C_2 denotes the constant of integration. Thus we require to integrate

$$\frac{d\Phi}{(1-\Phi^2)^{3/2}} = -C_2\left(\frac{1}{(1-\xi)} + \frac{1}{(1+\xi)}\right),$$

and the substitution $\Phi = \sin\Theta$ eventually produces

$$\tan\Theta = \frac{\Phi}{(1-\Phi^2)^{1/2}} = \frac{1}{(1/\Phi^2-1)^{1/2}} = C_1 + C_2\log\left(\frac{1-\xi}{1+\xi}\right),$$

where C_1 denotes a further constant of integration. Upon re-arrangement and using $u(x,t) = c\Phi(\xi)$, we may finally obtain

$$\frac{u(x,t)}{c} = \frac{\left[C_1 + C_2\log\left(\frac{1-\xi}{1+\xi}\right)\right]}{\left(1+\left[C_1 + C_2\log\left(\frac{1-\xi}{1+\xi}\right)\right]^2\right)^{1/2}},$$

which, with slightly re-defined arbitrary constants C_1 and C_2, can be seen to coincide with the above solution arising from (4.9) and (4.8).

4.3 General Reformulations of Basic Equations

In this section we detail a number of general reformulations of the basic equations for a single space dimension given in the previous section.

Energy-Momentum Relations From the basic equations (3.4) and the companion formulae (3.15), for one space dimension, these expressions become

$$f = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{d\mathscr{E}}{dt} = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{4.10}$$

so that from $p = m_0u/(1-(u/c)^2)^{1/2}$ and using Eqs. (4.3), the formulae (4.10) become

$$f = \frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = \frac{m_0}{(1-(u/c)^2)^{3/2}}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \frac{e_0}{c^2(1-(u/c)^2)^{3/2}}\left(\frac{du}{dt}\right)_{part},$$

$$gc^2 = \frac{d\mathscr{E}}{dt} = c^2\frac{\partial p}{\partial x} + u\frac{\partial p}{\partial t} = \frac{m_0}{(1-(u/c)^2)^{3/2}}\left(c^2\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial t}\right) = \frac{e_0}{(1-(u/c)^2)^{3/2}}\left(\frac{du}{dx}\right)_{wave}.$$

On division of these two equations, we may deduce the expression

$$\frac{d\mathscr{E}}{dp} = \frac{\left(c^2\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial t}\right)}{\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)} = c^2\frac{(du/dx)_{wave}}{(du/dt)_{part}},$$

and using this we may deduce the interesting equation that

$$\frac{d(\mathscr{E}+cp)}{d(\mathscr{E}-cp)} = \frac{(d\mathscr{E}/dp)+c}{(d\mathscr{E}/dp)-c} = \left(\frac{1+u/c}{1-u/c}\right)\left(\frac{c\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}}{c\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}}\right),$$

which might also be deduced by division of the following two equations:

$$\frac{d(\mathscr{E}+cp)}{dt} = (c+u)\left(\frac{\partial p}{\partial t}+c\frac{\partial p}{\partial x}\right) = \frac{m_0 c(1+u/c)}{(1-(u/c)^2)^{3/2}}\left(c\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}\right),$$

$$\frac{d(\mathscr{E}-cp)}{dt} = -(c-u)\left(\frac{\partial p}{\partial t}-c\frac{\partial p}{\partial x}\right) = \frac{m_0 c(1-u/c)}{(1-(u/c)^2)^{3/2}}\left(c\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}\right).$$

These relations reveal that $\mathscr{E} = \pm cp$ either for waves travelling at the speed of light, namely, $u(x,t) = \pm c$, or for momentum waves travelling in the opposite direction; thus $p(x,t) = p(ct \pm x)$.

In particular, if $\mathscr{E} = \pm cp$, then either $u = \pm c$ and we have

$$e + \mathscr{E} = (e_0^2 + (pc)^2)^{1/2} \pm pc = e_0 \exp(\sinh^{-1}(\pm cp/e_0)) = e_0 e^{\pm\theta},$$

on making use of the angle $\theta = \sinh^{-1}(cp/e_0)$ defined by (2.10) and the elementary relation $\sinh^{-1}(z) = \log(z + (1+z^2)^{1/2})$. Alternatively, if $\mathscr{E} = \pm cp$, then from the above or Eqs. (4.3), $p(x,t)$ might satisfy $\partial p/\partial t = \pm c\partial p/\partial x$, and therefore $p(x,t) = p(ct \pm x)$.

Equation in Conservation Form On multiplication of the first equation of (4.3) by $\mathscr{E}$ and the second by $c^2 p$ and then by addition, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\mathscr{E}^2 + (pc)^2\right) = c^2\frac{\partial}{\partial x}(\mathscr{E}p),$$

which in conservation form becomes

$$\frac{\partial \mathscr{W}}{\partial t} + \frac{\partial \mathscr{Q}}{\partial x} = 0, \tag{4.11}$$

where the energy density $\mathscr{W}(x,t)$ and energy flow or instantaneous power $\mathscr{Q}(x,t)$ are defined by

$$\mathscr{W} = \frac{1}{2}(\mathscr{E}^2 + (pc)^2) = \frac{1}{4}\left((\mathscr{E}-cp)^2 + (\mathscr{E}+cp)^2\right),$$

$$\mathscr{Q} = -c^2(\mathscr{E}p) = \frac{1}{4}\left((\mathscr{E}-cp)^2 - (\mathscr{E}+cp)^2\right),$$

and Eq. (4.11) relates the time rate of increase (decrease) of the energy density $\mathscr{W}(x,t)$ that is balanced by a decrease (increase) in the instantaneous power $\mathscr{Q}(x,t)$.

Partial Differential Equation for Wave Energy-Momentum Ratio $\mathscr{E}(x,t)/cp(x,t)$ We comment that if we make the assumption that $\mathscr{E}(x,t) = \omega(x,t)cp(x,t)$ where $\omega(x,t)$ is to be determined, then from (4.3) we obtain the two equations

$$p\frac{\partial \omega}{\partial t} + \omega\frac{\partial p}{\partial t} = c\frac{\partial p}{\partial x}, \qquad cp\frac{\partial \omega}{\partial x} + \omega c\frac{\partial p}{\partial x} = \frac{\partial p}{\partial t},$$

which may be solved to give

$$(1-\omega^2)\frac{\partial p}{\partial t} = \left(c\frac{\partial \omega}{\partial x} + \omega\frac{\partial \omega}{\partial t}\right)p, \qquad c(1-\omega^2)\frac{\partial p}{\partial x} = \left(\frac{\partial \omega}{\partial t} + c\omega\frac{\partial \omega}{\partial x}\right)p,$$

which with some re-arrangement become

$$(1-\omega^2)\frac{\partial q}{\partial t} = c\frac{\partial \omega}{\partial x}, \qquad c(1-\omega^2)\frac{\partial q}{\partial x} = \frac{\partial \omega}{\partial t},$$

where $q = \log\left(p(1-\omega^2)^{1/2}\right)$. From these equations, we may deduce that both $q(x,t)$ and $\Omega(x,t)$ defined by the relations

$$\Omega = \frac{1}{2}\log\left(\frac{1+\omega}{1-\omega}\right) = \tanh^{-1}(\omega), \qquad \left(\frac{1+\omega}{1-\omega}\right)^{1/2} = e^{\Omega}. \tag{4.12}$$

satisfy the classical one-dimensional wave equation, namely,

$$\frac{\partial^2 \Omega}{\partial t^2} = c^2\frac{\partial^2 \Omega}{\partial x^2}. \tag{4.13}$$

Thus, if $\Omega(x,t) = C(\alpha) + D(\beta)$, then $q(x,t) = C(\alpha) - D(\beta)$, where $C(\alpha)$ and $D(\beta)$ denote arbitrary functions, and we may make the identification $2F(\alpha) = \exp(2C(\alpha))$ and $2G(\beta) = \exp(-2D(\beta))$ where $F(\alpha)$ and $G(\beta)$ are the previously introduced arbitrary functions. From these expressions we might deduce the following three curious expressions for $p(x,t)$; thus

$$p(x,t) = \frac{2F(\alpha)}{(1+\omega)}, \qquad p(x,t) = \frac{2G(\beta)}{(1-\omega)}, \qquad p(x,t) = \frac{2(F(\alpha)G(\beta))^{1/2}}{(1-\omega^2)^{1/2}},$$

results which are also immediately apparent from the general formulae (4.43), noting that $\omega(x,t) = (F(\alpha) - G(\beta))\,/\,(F(\alpha) + G(\beta))$ and noting in particular the special relativistic-like dependence of $p(x,t)$ on $(1-\omega^2)^{1/2}$ in the latter relation. As with much of the analysis of this present chapter, the immediately above analysis is very much dependent upon the assumption of a single Cartesian spatial variable and in the absence of any effects due to curvature. For centrally symmetric mechanical systems, a corresponding analysis is presented in Chap. 9, which follows similar lines to the above but is more complicated due to the presence of the $1/r$ terms.

Partial Derivatives Following Both Particle and Wave It may also be worthwhile observing that the above Eqs. (4.3) take on an interesting form if we evaluate the total space and time derivatives following both the particle and the wave, rather than the partial derivatives; thus

$$\left(\frac{d\mathscr{E}}{dt}\right)_{part} = \frac{\partial \mathscr{E}}{\partial t} + u\frac{\partial \mathscr{E}}{\partial x} = c^2\frac{\partial p}{\partial x} + u\frac{\partial p}{\partial t} = c^2\left(\frac{\partial p}{\partial x} + \frac{u}{c^2}\frac{\partial p}{\partial t}\right) = c^2\left(\frac{dp}{dx}\right)_{wave},$$

$$\left(\frac{d\mathscr{E}}{dx}\right)_{part} = \frac{\partial \mathscr{E}}{\partial x} + \frac{1}{u}\frac{\partial \mathscr{E}}{\partial t} = \frac{\partial p}{\partial t} + \frac{c^2}{u}\frac{\partial p}{\partial x} = \left(\frac{dp}{dt}\right)_{wave},$$

so that together we have the equations

$$\left(\frac{d\mathscr{E}}{dt}\right)_{part} = c^2\left(\frac{dp}{dx}\right)_{wave}, \qquad \left(\frac{d\mathscr{E}}{dx}\right)_{part} = \left(\frac{dp}{dt}\right)_{wave},$$

relating the total spatial and time rates of the energy exchanges between the particle and the wave, namely, between the energies $cp(x,t)$ and $\mathscr{E}(x,t)$.

Comparable Equations for Forces f and cg For a single spatial dimension, we derive here an alternative expression for the spatial force f in a form for which there is a comparable form for the force cg except that the velocity of light c is replaced by the particle velocity u. We first make the observation that from the basic rate-of-working relation $de = f dx = u dp$, we have

$$f = \frac{dp}{dt} = \frac{de}{dx} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = \frac{\partial e}{\partial x} + \frac{1}{u}\frac{\partial e}{\partial t},$$

so that by addition of these expressions we may deduce

$$2f = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \frac{\partial e}{\partial x} + \frac{1}{u}\frac{\partial e}{\partial t},$$

and therefore on using $(4.10)_1$ we might deduce the interesting and comparable equations

$$f = \frac{1}{u}\frac{\partial e}{\partial t} + u\frac{\partial p}{\partial x}, \qquad cg = \frac{1}{c}\frac{\partial e}{\partial t} + c\frac{\partial p}{\partial x},$$

indicating that at least for one space dimension, the two forces f and cg assume similar forms except that the velocity u is replaced by the velocity of light c in the second equation.

Net Forces as Line Integrals In order to express the one spatial dimension resultant or net forces as line integrals, we first need the standard two-dimensional Green's theorem for line and surface integrals. For two functions $P(x, y)$ and $Q(x, y)$ which are assumed to be both finite and continuous inside and on the closed boundary C of the region R of the $(x, y)-$plane, we have

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

noting that the line integral around the contour C is taken to be in the counter-clockwise direction. If we now take C to be a closed contour surrounding a region R of $(x, t)-$space, and we assume that the momentum $p(x, t)$ and energy $e(x, t)$ are both finite and continuous inside and on C, then on making use of the one spatial dimension equations (4.10) and Green's theorem, we have

$$\oint_C (p\,dx - e\,dt) = -\iint_R \left(\frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} \right) dxdt = -\iint_R f dxdt,$$

$$\oint_C (e\,dx - c^2 p\,dt) = -c^2 \iint_R \left(\frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} \right) dxdt = -c^2 \iint_R g dxdt,$$

so that for a single space dimension the resultant or net forces are given by the respective line integrals

$$\iint_R f dxdt = \oint_C (e\,dt - p\,dx), \qquad \iint_R cg dxdt = \oint_C (cp\,dt - e/c\,dx).$$

4.4 Important Identity

From the above equations and $e = \left(e_0^2 + (pc)^2\right)^{1/2}$, we may establish the important equation

$$f^2 - (cg)^2 = \frac{e}{c^2} \left(\frac{\partial^2 e}{\partial t^2} - c^2 \frac{\partial^2 e}{\partial x^2} \right). \tag{4.14}$$

as follows: The relationship $e^2 = e_0^2 + (pc)^2$ implies that e and p are functionally related, and therefore the Jacobian $\partial(e, p)/\partial(x, t)$ vanishes, namely,

$$\frac{\partial(e, p)}{\partial(x, t)} = \left(\frac{\partial e}{\partial x}\frac{\partial p}{\partial t} - \frac{\partial e}{\partial t}\frac{\partial p}{\partial x}\right) = 0,$$

so that either from (3.4) or the one-dimensional version (7.1), we have

$$f^2 - (cg)^2 = \left(\frac{\partial p}{\partial t}\right)^2 + \left(\frac{\partial e}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial e}{\partial t}\right)^2 - c^2\left(\frac{\partial p}{\partial x}\right)^2. \tag{4.15}$$

Now on using the relations

$$e\frac{\partial e}{\partial t} = c^2 p\frac{\partial p}{\partial t}, \qquad e\frac{\partial e}{\partial x} = c^2 p\frac{\partial p}{\partial x},$$

which are obtained by differentiating $e^2 = e_0^2 + (pc)^2$, and then from the derivatives of these equations, we might obtain

$$e\frac{\partial^2 e}{\partial t^2} + \left(\frac{\partial e}{\partial t}\right)^2 = c^2 p\frac{\partial^2 p}{\partial t^2} + c^2\left(\frac{\partial p}{\partial t}\right)^2,$$

and

$$e\frac{\partial^2 e}{\partial x^2} + \left(\frac{\partial e}{\partial x}\right)^2 = c^2 p\frac{\partial^2 p}{\partial x^2} + c^2\left(\frac{\partial p}{\partial x}\right)^2.$$

If we use these relations to simplify Eq. (4.15), then on using the fact that $p(x, t)$ satisfies the wave equation, we might deduce (4.14). This important insightful equation indicates that if either condition $f = \pm cg$ is satisfied, then either the physical energy $e(x, t)$ is necessarily identically zero, or it satisfies the wave equation. If it is non-zero and satisfies the wave equation, then necessarily $e_0 = 0$ and $e = \pm cp$, and giving rise to the prospects of dark matter $e = cp$ and dark energy $e = -cp$, and for both, we have the condition that particle and wave energies coincide; thus $e = \mathscr{E}$.

We also observe from Eq. (4.14) that for a conservative mechanical system, the external applied forces f and g are generated from a potential function $V(x, t)$ such that

$$f = -\frac{\partial V}{\partial x}, \qquad gc^2 = -\frac{\partial V}{\partial t}, \tag{4.16}$$

then the identity becomes

$$\frac{e}{c^2}\left(\frac{\partial^2 e}{\partial t^2} - c^2\frac{\partial^2 e}{\partial x^2}\right) + \frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2 - \left(\frac{\partial V}{\partial x}\right)^2 = 0, \tag{4.17}$$

and since the wave energy $\mathscr{E}$ always satisfies the wave equation, we may deduce from (4.17) and conservation of energy $e + \mathscr{E} = -V$ the following interesting equation:

$$e\left(\frac{\partial^2 V}{\partial t^2} - c^2\frac{\partial^2 V}{\partial x^2}\right) = \left(\frac{\partial V}{\partial t}\right)^2 - c^2\left(\frac{\partial V}{\partial x}\right)^2, \tag{4.18}$$

which coincides with (8.42) obtained using the characteristic coordinates $\alpha = ct+x$ and $\beta = ct - x$. This curious equation has the appearance of a singular perturbation problem for which the potential $V(x, t)$ loses its highest derivatives in the limit of the particle energy e tending to zero and would therefore be governed by a lower order partial differential equation. However, in this present situation, there is no limit of the particle energy e tending to zero except for zero rest mass and zero velocity, so that when the governing equation is the lower order equation

$$\left(\frac{\partial V}{\partial t}\right)^2 - c^2\left(\frac{\partial V}{\partial x}\right)^2 = 0,$$

it is then balanced by the vanishing of the other factor term, namely,

$$\frac{\partial^2 V}{\partial t^2} - c^2\frac{\partial^2 V}{\partial x^2} = 0,$$

which is the case for dark energy as a forward wave $V(x, t) = V(ct - x)$ and dark matter as a backward wave $V(x, t) = V(ct + x)$.

4.5 Formulation in Terms of Lorentz Invariants

Lorentz Invariants $\xi = ex - c^2pt$ and $\eta = px - et$ Here for a single spatial dimension x, we examine the formulation in terms of the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$ which are defined previously by (2.48) and shown to be invariants of the full Lorentz group. First, we examine the immediate algebraic relations arising from these definitions, and it is not difficult to establish the following formulae:

$$\xi + c\eta = (e + cp)(x - ct), \qquad \xi - c\eta = (e - cp)(x + ct),$$

and therefore we have

$$\xi^2 - (c\eta)^2 = (e^2 - (cp)^2)(x^2 - (ct)^2) = e_0^2(x^2 - (ct)^2), \tag{4.19}$$

on using $e^2 - (cp)^2 = e_0^2$ where $e_0 = m_0c^2$. Further, on using the relations

$$e + cp = e_0 \left(\frac{1 + u/c}{1 - u/c}\right)^{1/2} = e_0 e^{\theta}, \qquad e - cp = e_0 \left(\frac{1 - u/c}{1 + u/c}\right)^{1/2} = e_0 e^{-\theta},$$

where θ is the angle defined by $(2.9)_2$ and satisfying the relations (2.10), we may deduce

$$\xi + c\eta = e_0(x - ct)e^{\theta}, \qquad \xi - c\eta = e_0(x + ct)e^{-\theta},$$

and from which we may readily obtain

$$\xi = e_0 x \cosh\theta - e_0 ct \sinh\theta, \qquad c\eta = e_0 x \sinh\theta - e_0 ct \cosh\theta. \tag{4.20}$$

Now the relations $\xi = ex - c^2pt$ and $\eta = px - et$ may be inverted to yield

$$e = \left(\frac{x\xi + c^2t\eta}{x^2 - (ct)^2}\right), \qquad p = \left(\frac{t\xi + x\eta}{x^2 - (ct)^2}\right), \tag{4.21}$$

and under the space-time transformation $x' = ct$ and $t' = x/c$ with $u' = c^2/u$, we have $e' = cp$ and $cp' = e$, and Eqs. (4.10) are left invariant under this symmetry with $f' = f$ and $g' = g$ since from these general Lorentz invariant equations for the three energies $e(x, t)$, $cp(x, t)$ and $\mathscr{E}(x, t)$ for a single spatial dimension, we have

$$f = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{d\mathscr{E}}{dt} = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{4.22}$$

where $f(x, t)$ and $cg(x, t)$ denote the applied external forces in the spatial and time directions, respectively. The primed version of the above Eqs. (4.10) becomes

$$f' = \frac{dp'}{dt'} = \frac{\partial p'}{\partial t'} + \frac{\partial e'}{\partial x'}, \qquad g' = \frac{1}{c^2}\frac{d\mathscr{E}'}{dt'} = \frac{1}{c^2}\frac{\partial e'}{\partial t'} + \frac{\partial p'}{\partial x'},$$

which under the space-time transformation $x' = ct$ and $t' = x/c$ become, respectively,

$$f' = \frac{de}{dx} = \frac{1}{u}\frac{de}{dt} = \frac{dp}{dt} = \frac{\partial e}{\partial x} + \frac{\partial p}{\partial t},$$
$$g' = \frac{d\mathscr{P}}{dx} = \frac{1}{u}\frac{d\mathscr{P}}{dt} = \frac{1}{c^2}\frac{d\mathscr{E}}{dt} = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x},$$

where $\mathscr{E}' = c\mathscr{P}$, so therefore $f' = f$ and $g' = g$, and these details are more fully discussed in the final section of Chap. 7. Either directly from the relations (4.21) or the equations corresponding to (4.22) when expressed in terms of $\xi(x,t)$ and $\eta(x,t)$, namely,

$$xf - c^2 tg = \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x}, \qquad xg - tf = \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x},$$

we may deduce that under the transformation we have that $\xi' = -\xi$ and $\eta' = -\eta$. These particular transformation rules are important in understanding the transformations of the differential relations for ξ and η which are subsequently derived in this chapter.

Quadratic Equation for the Determination of $u(x,t)$ If we assume for the time being that the two Lorentz invariants $\xi(x,t)$ and $\eta(x,t)$ are known, then from their definition we have

$$x - ut = \frac{\xi(x,t)}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{1/2}, \qquad x\frac{u}{c} - ct = \frac{c\eta(x,t)}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{1/2}, \tag{4.23}$$

and on squaring and re-arranging these equations, we obtain two quadratic equations for the determination of $u(x,t)$; thus

$$\left(\frac{u}{c}\right)^2\left\{(ct)^2 + \left(\frac{\xi}{e_0}\right)^2\right\} - 2xct\left(\frac{u}{c}\right) + \left\{x^2 - \left(\frac{\xi}{e_0}\right)^2\right\} = 0,$$
$$\left(\frac{u}{c}\right)^2\left\{x^2 + \left(\frac{c\eta}{e_0}\right)^2\right\} - 2xct\left(\frac{u}{c}\right) + \left\{(ct)^2 - \left(\frac{c\eta}{e_0}\right)^2\right\} = 0.$$

In view of the relationship $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, we have in particular

$$(ct)^2 + \left(\frac{\xi}{e_0}\right)^2 = x^2 + \left(\frac{c\eta}{e_0}\right)^2, \qquad (ct)^2 - \left(\frac{c\eta}{e_0}\right)^2 = x^2 - \left(\frac{\xi}{e_0}\right)^2,$$

leading to the important conclusion that these two quadratic equations coincide assuming that the condition $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$ is satisfied.

On solving, say the first quadratic equation, we have

$$\frac{u(x,t)}{c} = \frac{xct \pm (\xi/e_0)\left\{(ct)^2 - x^2 + (\xi/e_0)^2\right\}^{1/2}}{\left\{(ct)^2 + (\xi/e_0)^2\right\}}, \tag{4.24}$$

which in view of the above relations can be expressed in a number of alternative forms including

$$\frac{u(x,t)}{c} = \frac{xct \pm c\xi\eta/e_0^2}{\left\{(ct)^2 + (\xi/e_0)^2\right\}} = \frac{xct \pm c\xi\eta/e_0^2}{\left\{x^2 + (c\eta/e_0)^2\right\}}, \tag{4.25}$$

as well as the following expression which is symmetrical in x and ct as well as in ξ and $c\eta$; thus

$$\frac{u(x,t)}{c} = \frac{xct \pm c\xi\eta/e_0^2}{\left\{(ct)^2 + (\xi/e_0)^2\right\}^{1/2}\left\{x^2 + (c\eta/e_0)^2\right\}^{1/2}}, \tag{4.26}$$

Complementary Velocities $u(x,t)$ and $u^*(x,t)$ Such That $uu^* = c^2$ For a known velocity $u(x,t)$, of particular importance is the identification of the complementary velocities $u^*(x,t)$ such that $uu^* = c^2$. Here we derive the condition on the Lorentz invariants $\xi(x,t)$ and $\eta(x,t)$ and $\xi^*(x,t)$ and $\eta^*(x,t)$ that must be satisfied in order that the associated velocities $u(x,t)$ and $u^*(x,t)$ satisfy $uu^* = c^2$. By division of the two equations (4.23), we may deduce the following expression for $u(x,t)$ in terms of the two Lorentz invariants $\xi(x,t)$ and $\eta(x,t)$; thus

$$\frac{u}{c} = \frac{c(x\eta + t\xi)}{(x\xi + c^2t\eta)}.$$

Accordingly, a necessary condition for $uu^* = c^2$ is that

$$\frac{c(x\eta + t\xi)}{(x\xi + c^2t\eta)}\,\frac{c(x\eta^* + t\xi^*)}{(x\xi^* + c^2t\eta^*)} = 1,$$

which, assuming that $x \neq \pm ct$, simplifies considerably to become simply $\xi\xi^* = c^2\eta\eta^*$ as the required condition necessary to ensure that $uu^* = c^2$.

Two Complementary Velocities $u^*(x,t)$ Such That $uu^* = c^2$ The two complementary velocities $u^*(x,t)$ such that $uu^* = c^2$ corresponding to the positive and negative cases given in (4.25), namely,

$$\frac{u(x,t)}{c} = \frac{xct + c\xi\eta/e_0^2}{\left\{(ct)^2 + (\xi/e_0)^2\right\}}, \qquad \frac{u(x,t)}{c} = \frac{xct - c\xi\eta/e_0^2}{\left\{(ct)^2 + (\xi/e_0)^2\right\}}, \tag{4.27}$$

are given, respectively, by

$$\frac{u^*(x,t)}{c} = \frac{xct - c\xi\eta/e_0^2}{\left\{x^2 - (\xi/e_0)^2\right\}}, \qquad \frac{u^*(x,t)}{c} = \frac{xct + c\xi\eta/e_0^2}{\left\{x^2 - (\xi/e_0)^2\right\}}, \tag{4.28}$$

since in both cases we have

$$\frac{uu^*}{c^2} = \frac{(xct)^2 - (c\xi\eta/e_0^2)^2}{\left\{(ct)^2 + (\xi/e_0)^2\right\}\left\{x^2 - (\xi/e_0)^2\right\}} = \frac{(xct)^2 - (c\xi\eta/e_0^2)^2}{(xct)^2 - (\xi/e_0)^2\left\{(\xi/e_0)^2 + (ct)^2 - x^2\right\}} = 1,$$

in view of the relation $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$. Thus for prescribed Lorentz invariants $\xi(x,t)$ and $\eta(x,t)$, we may determine the two associated velocities $u(x,t)$ through (4.29), while their complementary velocities $u^*(x,t)$ are determined from (4.28).

Furthermore, we may show independently that under reasonable assumptions, the two velocities given by (4.29) are such that $u/c < 1$ while those given by (4.28) are such that $u/c > 1$. Firstly for $u(x,t)$, we introduce the working variables α, β and γ defined by

$$\alpha = \frac{x}{\left\{(ct)^2 + (\xi/e_0)^2\right\}^{1/2}} = \frac{x}{\left\{x^2 + (c\eta/e_0)^2\right\}^{1/2}} < 1, \tag{4.29}$$

$$\beta = \frac{ct}{\left\{(ct)^2 + (\xi/e_0)^2\right\}^{1/2}} < 1, \qquad \gamma = \frac{(\xi/e_0)}{\left\{(ct)^2 + (\xi/e_0)^2\right\}^{1/2}} < 1,$$

then evidently $\beta^2 + \gamma^2 = 1$, and from $u(x,t)$ expressed in the form of (4.24), we have

$$\begin{aligned}\frac{u(x,t)}{c} &= \frac{xct \pm (\xi/e_0)\left\{(ct)^2 - x^2 + (\xi/e_0)^2\right\}^{1/2}}{\left\{(ct)^2 + (\xi/e_0)^2\right\}} \\ &= \alpha\beta \pm \gamma\left\{\gamma^2 + \beta^2 - \alpha^2\right\}^{1/2} = \alpha\beta \pm \gamma\left\{1 - \alpha^2\right\}^{1/2} \\ &= \alpha\beta \pm \left\{1 - \beta^2\right\}^{1/2}\left\{1 - \alpha^2\right\}^{1/2} < 1,\end{aligned}$$

since evidently $\pm\left\{1 - \beta^2\right\}^{1/2}\left\{1 - \alpha^2\right\}^{1/2} < 1 - \alpha\beta$. The negative case is obvious since $1 - \alpha\beta > 0$, while the positive case follows by squaring $\left\{1 - \beta^2\right\}^{1/2}\left\{1 - \alpha^2\right\}^{1/2} < 1 - \alpha\beta$ to obtain $-(\alpha - \beta)^2 < 0$, which is certainly true. The final equation reveals the critical structure of the expression, since with $\alpha = \sin\theta$ and $\beta = \sin\phi$, then the positive case gives rise to $u/c = \cos(\theta - \phi)$, while the negative case results in $u/c = -\cos(\theta + \phi)$, which are useful expressions in terms of simplifying $(1 - (u/c)^2)^{1/2}$ involved in the denominator of both the energy e and momentum p.

For the two complementary velocities defined by (4.28), a similar analysis applies, and we might introduce the working variables a, b and δ defined by

$$a = \frac{x}{\left\{(x^2 - (\xi/e_0)^2\right\}^{1/2}} > 1, \qquad \delta = \frac{(\xi/e_0)}{\left\{x^2 - (\xi/e_0)^2\right\}^{1/2}},$$

$$b = \frac{ct}{\left\{x^2 - (\xi/e_0)^2\right\}^{1/2}} = \frac{ct}{\left\{(ct)^2 - (c\eta/e_0)^2\right\}^{1/2}} > 1,$$

along with $a^2 - \delta^2 = 1$, and the two complementary velocities (4.28) become

$$\begin{aligned}\frac{u^*(x,t)}{c} &= \frac{xct \pm (\xi/e_0)\left\{(ct)^2 - x^2 + (\xi/e_0)^2\right\}^{1/2}}{\left\{x^2 - (\xi/e_0)^2\right\}} \\ &= ab \pm \delta\left\{\delta^2 + b^2 - a^2\right\}^{1/2} = ab \pm \delta\left\{b^2 - 1\right\}^{1/2} \\ &= ab \pm \left\{a^2 - 1\right\}^{1/2}\left\{b^2 - 1\right\}^{1/2} > 1,\end{aligned}$$

since $\pm\left\{a^2 - 1\right\}^{1/2}\left\{b^2 - 1\right\}^{1/2} > 1 - ab$. The positive case is obvious, while for the negative case, both sides of the inequality are negative, and therefore when we square both sides, we need to reverse the inequality leading to the requirement $-(a-b)^2 < 0$ which is certainly true. Again, with $a = \cosh\theta$ and $b = \cosh\phi$, the positive case leads to $u^*/c = \cosh(\theta + \phi)$, while the negative case gives rise to $u^*/c = \cosh(\theta - \phi)$, and again these expressions are useful in the simplification of the energy e and the momentum p.

Single Space Equations in Terms of ξ and η We now examine the particular consequences of the one spatial dimension equations arising from (3.4), namely,

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{4.30}$$

in terms of the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$, and by a straightforward evaluation of the right-hand sides of the following equations, we may readily verify that

$$xf - c^2tg = \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x}, \qquad xg - tf = \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x}, \tag{4.31}$$

which in comparison with (4.30) seems to indicate a certain correspondence between η and p and between ξ and e.

By partial differentiation of the expressions (4.20) with respect to both x and t, it is not difficult to show that

$$xf - c^2tg = \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x} = c\eta\frac{\partial \theta}{\partial x} + \frac{\xi}{c}\frac{\partial \theta}{\partial t}, \qquad xg - tf = \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x} = \frac{\xi}{c}\frac{\partial \theta}{\partial x} + \frac{\eta}{c}\frac{\partial \theta}{\partial t},$$

and on solving these two equations for $\partial\theta/\partial x$ and $\partial\theta/\partial t$ and using the relation $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, we may deduce

$$\frac{\partial \theta}{\partial x} = \frac{c}{e_0^2}\left(\frac{\xi \mathscr{G} - \eta \mathscr{F}}{x^2 - (ct)^2}\right), \qquad \frac{\partial \theta}{\partial t} = \frac{c}{e_0^2}\left(\frac{\xi \mathscr{F} - c^2 \eta \mathscr{G}}{x^2 - (ct)^2}\right), \tag{4.32}$$

where here for convenience we use the abbreviation $\mathscr{F} = xf - c^2 tg$ and $\mathscr{G} = xg - tf$. On using the expressions $\xi = ex - c^2 pt$ and $\eta = px - et$, we find that

$$\xi \mathscr{G} - \eta \mathscr{F} = (x^2 - (ct)^2)(eg - pf), \qquad \xi \mathscr{F} - c^2 \eta \mathscr{G} = (x^2 - (ct)^2)(ef - c^2 pg),$$

so that from (4.32) we obtain

$$\frac{\partial \theta}{\partial x} = \frac{c}{e_0^2}(eg - pf), \qquad \frac{\partial \theta}{\partial t} = \frac{c}{e_0^2}(ef - c^2 pg). \tag{4.33}$$

Now with arbitrarily assigned external forces (f, gc), these equations must be well-defined and compatible in the sense that $\partial/\partial t(\partial\theta/\partial x) = \partial/\partial x(\partial\theta/\partial t)$. However, with f and g given by the expressions (4.30), these equations are automatically well-defined and meaningful. In order to see this, on using the relations (4.30), performing the partial differentiations and equating the two expressions for the second order partial derivative $\partial^2\theta/\partial x \partial t$, we might eventually deduce the interesting equation

$$f^2 - (cg)^2 + e\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial t}\right) + p\left(\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\right) = 0. \tag{4.34}$$

We observe that on using $p = eu/c^2$, this equation takes on the alternative interesting form

$$f^2 + e\left(\frac{df}{dx}\right)_{wave} = (cg)^2 + e\left(\frac{dg}{dt}\right)_{part}.$$

It is important to emphasise that this equation must be satisfied, and might be viewed as a consistency condition imposed upon any allowable velocity fields arising from the applied external forces (f, gc), so that f and g may not be entirely arbitrarily assigned. Subsequently, we see that this equation corresponds to the important identity (4.14) discussed in the previous section. On using the relations (4.30), Eq. (4.34) becomes

$$\frac{\partial}{\partial x}\left(e\frac{\partial e}{\partial x}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(e\frac{\partial e}{\partial t}\right) + \frac{\partial}{\partial t}\left(p\frac{\partial p}{\partial t}\right) - c^2\frac{\partial}{\partial x}\left(p\frac{\partial p}{\partial x}\right) + 2\frac{\partial(e, p)}{\partial(x, t)} = 0,$$

which evidently can be rewritten as simply

$$\frac{\partial^2(e^2 - (cp)^2)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2(e^2 - (cp)^2)}{\partial t^2} + 4\frac{\partial(e, p)}{\partial(x, t)} = 0,$$

which is trivially satisfied since $e^2 - (cp)^2 = e_0^2$, so that e and p are functionally related and therefore the Jacobian also vanishes. We comment that throughout this text we are assuming that the rest energy $e_0 = m_0 c^2$ remains constant and all the equations are derived accordingly. However, non-constant rest mass is thought to be a real physical possibility, and in two sections of Chap. 8, a specific solution for non-constant rest mass is derived, along with the equations corresponding to those derived immediately above. The corresponding equations derived in Chap. 8 place the above results in a more reasonable context.

Thus with external forces (f, gc) satisfying (4.30), Eqs. (4.33) constitute two well-defined relations for the determination of $\theta(x, t)$, which also becomes clear on closer examination, since from (4.33) on using (4.30), we have

$$d\theta = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial t}dt = \frac{c}{e_0^2}\Big((eg - pf)dx + (ef - c^2 pg)dt\Big) = \frac{c}{e_0^2}(edp - pde),$$

and on noting the relations $e = e_0 \cosh\theta$ and $pc = e_0 \sinh\theta$, obtained from (2.11), this equation is evidently satisfied.

Characteristic Coordinates $\alpha = ct + x$ **and** $\beta = ct - x$ From (4.31), we have

$$\mathscr{F} = xf - c^2 tg = \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x}, \qquad \mathscr{G} = xg - tf = \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x},$$

and from these two equations, in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$ and the differential formulae

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \qquad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}, \tag{4.35}$$

we may deduce

$$\mathscr{F} + c\mathscr{G} = 2\frac{\partial(\xi + c\eta)}{\partial \alpha}, \qquad \mathscr{F} - c\mathscr{G} = -2\frac{\partial(\xi - c\eta)}{\partial \beta}. \tag{4.36}$$

From the definitions of $\mathscr{F}$ and $\mathscr{G}$, we may deduce

$$\mathscr{F} + c\mathscr{G} = -(f + cg)(ct - x), \qquad \mathscr{F} - c\mathscr{G} = (f - cg)(ct + x),$$

and therefore Eqs. (4.36) become

$$f + cg = -\frac{2}{\beta}\frac{\partial(\xi + c\eta)}{\partial \alpha}, \qquad f - cg = -\frac{2}{\alpha}\frac{\partial(\xi - c\eta)}{\partial \beta}.$$

and we have $(\mathscr{F}^2 - (c\mathscr{G})^2) = (f^2 - (cg)^2)(x^2 - (ct)^2) = -(f^2 - (cg)^2)\alpha\beta$. Again, on noting the relation $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2) = -\alpha\beta e_0^2$, we may deduce the curious equation

$$f^2 - (cg)^2 = \frac{-4e_0^2}{(\xi^2 - (c\eta)^2)} \frac{\partial(\xi + c\eta)}{\partial\alpha} \frac{\partial(\xi - c\eta)}{\partial\beta} = -4e_0^2 \frac{\partial(\log(\xi + c\eta))}{\partial\alpha} \frac{(\log \partial(\xi - c\eta))}{\partial\beta}.$$

We now examine in some detail the case when the external forces (f, gc) are generated from a single potential function $V(x, t)$ as given by (4.16). For completeness we also include below some of the details arising from the case when the external forces (f, gc) are generated from two scalar functions $V(x, t)$ and $W(x, t)$, including the structure of Eq. (4.34). In the latter case, for non-trivial $W(x, t)$, the system is generally nonconservative.

Forces (f, gc) Generated from Single Potential Function $V(x, t)$ If we now assume that the external forces (f, gc) are generated as external forces from the scalar field such that

$$f = -\frac{\partial V}{\partial x}, \qquad gc^2 = -\frac{\partial V}{\partial t}, \tag{4.37}$$

for some potential function $V(x, t)$, then it is not difficult to show that the above Eqs. (4.31) become simply

$$\frac{\partial(\eta - tV)}{\partial t} + \frac{\partial(\xi + xV)}{\partial x} = 0, \qquad \frac{1}{c^2}\frac{\partial(\xi + xV)}{\partial t} + \frac{\partial(\eta - tV)}{\partial x} = 0,$$

and therefore there exists $\psi(x, t)$ such that $\eta - tV = \partial\psi/\partial x$ and $\xi + xV = -\partial\psi/\partial t$, where $\psi(x, t)$ satisfies the classical wave equation

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}. \tag{4.38}$$

and we have the following explicit formulae:

$$\xi(x, t) = -\left(\frac{\partial\psi}{\partial t} + xV(x, t)\right), \qquad \eta(x, t) = \left(\frac{\partial\psi}{\partial x} + tV(x, t)\right). \tag{4.39}$$

Now the relations $\xi = ex - c^2pt$ and $\eta = px - et$ may be inverted to yield

$$e = \left(\frac{x\xi + c^2t\eta}{x^2 - (ct)^2}\right), \qquad p = \left(\frac{t\xi + x\eta}{x^2 - (ct)^2}\right),$$

and from these expressions and (4.39), we might deduce the following for the momentum p and energy e; thus

$$p = \left(\frac{x\frac{\partial \psi}{\partial x} - t\frac{\partial \psi}{\partial t}}{x^2 - (ct)^2} \right), \qquad e = \left(\frac{c^2 t\frac{\partial \psi}{\partial x} - x\frac{\partial \psi}{\partial t}}{x^2 - (ct)^2} \right) - V, \tag{4.40}$$

In terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$ and the differential formulae (4.35), the force relations (4.37) become

$$f + cg = -2\frac{\partial V}{\partial \alpha}, \qquad f - gc = 2\frac{\partial V}{\partial \beta}.$$

The general solution of (4.38) is given by $\psi(x, t) = A(\alpha) + B(\beta)$, where both A and B denote arbitrary functions of the indicated arguments, while from the above equations, (4.47) we have

$$p = \frac{1}{\alpha\beta}\left(\alpha\frac{\partial \psi}{\partial \beta} + \beta\frac{\partial \psi}{\partial \alpha} \right), \qquad \mathscr{E} = \frac{c}{\alpha\beta}\left(\beta\frac{\partial \psi}{\partial \alpha} - \alpha\frac{\partial \psi}{\partial \beta} \right), \tag{4.41}$$

where in the second equality we have used the conservation of energy $e + \mathscr{E} + V = constant$. Subsequently, we present the simple general solutions of the wave equation for the momentum $p(x, t) = F(\alpha) + G(\beta)$ and wave energy $\mathscr{E}(x, t) = c\,(F(\alpha) - G(\beta))$ where $\alpha = ct + x$ and $\beta = ct - x$, and where both F and G denote arbitrary functions of their arguments, and from the above expressions (4.41), we may make the following straightforward identifications $F(\alpha) = A'(\alpha)/\alpha$ and $G(\beta) = B'(\beta)/\beta$, which on using (4.39) altogether give rise to the following expressions for $\xi(x, t)$ and $\eta(x, t)$; thus

$$\xi(x, t) = -xV(x, t) - c(\alpha F(\alpha) + \beta G(\beta)), \tag{4.42}$$
$$\eta(x, t) = tV(x, t) + (\alpha F(\alpha) - \beta G(\beta)).$$

Summary for Forces (f, gc) **Generated from Single Potential Function** $V(x, t)$
For a single Cartesian spatial dimension, with forces f and g derivable from a potential $V(x, t)$ as given by (4.37), the complete general solution of (4.30)

$$\frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = -\frac{\partial V}{\partial x}, \qquad \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = -\frac{1}{c^2}\frac{\partial V}{\partial t},$$

is given by

$$\psi(x, t) = A(\alpha) + B(\beta), \qquad A'(\alpha) = \alpha F(\alpha), \qquad B'(\beta) = \beta G(\beta), \tag{4.43}$$
$$p(x, t) = F(\alpha) + G(\beta), \qquad \mathscr{E}(x, t) = c\left(F(\alpha) - G(\beta)\right),$$
$$\xi(x, t) = -xV(x, t) - c(\alpha F(\alpha) + \beta G(\beta)),$$
$$\eta(x, t) = tV(x, t) + (\alpha F(\alpha) - \beta G(\beta)),$$

and since the particle energy $e(x,t) = (e_0^2 + (pc)^2)^{1/2}$, from the conservation of energy $e + \mathscr{E} + V =$ constant, formally the potential energy $V(x,t)$ is given by

$$V(x,t) = -(e_0^2 + c^2(F(\alpha) + G(\beta))^2)^{1/2} - c(F(\alpha) - G(\beta)), \tag{4.44}$$

where $\alpha = ct + x$ and $\beta = ct - x$. In the laboratory we are accustomed to imposing whatever electric or magnetic field that our equipment permits. However, this is not the case here, and this equation reveals explicitly that unlike conventional theory, the potential for the proposed model is not completely arbitrary and must satisfy certain other requirements. This is similar to the gravitational field in general relativity being determined as part of the solution. We note that in consequence of the above general formulae (4.43), we may deduce in particular the important relations

$$\mathscr{E} + pc = 2cF(\alpha), \qquad \mathscr{E} - pc = -2cG(\beta),$$

results that will appear again throughout the text.

Forces (f, gc) Generated from Two Potential Functions $V(x,t)$ and $W(x,t)$
More generally we might assume that the external forces (f, gc) are generated from two potential functions $V(x,t)$ and $W(x,t)$, namely,

$$f = \frac{1}{c}\frac{\partial W}{\partial t} - \frac{\partial V}{\partial x}, \qquad gc = \frac{\partial W}{\partial x} - \frac{1}{c}\frac{\partial V}{\partial t}, \tag{4.45}$$

and in terms of $\alpha = ct + x$ and $\beta = ct - x$, the force relations (4.45) become

$$f + cg = 2\frac{\partial(W - V)}{\partial \alpha}, \qquad f - gc = 2\frac{\partial(W + V)}{\partial \beta}.$$

In this case the basic one-dimensional equations (4.30) become

$$\frac{\partial}{\partial t}(p - W/c) + \frac{\partial}{\partial x}(e + V) = 0, \qquad \frac{1}{c^2}\frac{\partial}{\partial t}(e + V) + \frac{\partial}{\partial x}(p - W/c) = 0,$$

and therefore from the basic defining equation for the de Broglie wave energy (3.10), we have $d\mathscr{E} = (\partial p/\partial t)dx + c^2(\partial p/\partial x)dt$, and we may deduce

$$d(e + \mathscr{E} + V) = \frac{1}{c}\frac{\partial W}{\partial t}dx + c\frac{\partial W}{\partial x}dt. \tag{4.46}$$

From this equation it is clear that unless $W(x,t)$ is a solution of the classical wave equation, then mechanical systems for which (4.45) applies will generally be nonconservative. We comment that in the event $W(x,t)$ is a solution of the classical wave equation, so that $W(x,t) = C(\alpha) + D(\beta)$; then from (4.46), we might deduce the conservation principle $e + \mathscr{E} + V = C(\alpha) - D(\beta) + constant$, noting that under certain circumstances conventional conservation of energy may take place along the characteristics.

Further, it is not difficult to show that the above Eqs. (4.31) become

$$\frac{\partial}{\partial t}(\eta - tV - xW/c) + \frac{\partial}{\partial x}(\xi + xV + ctW) = 0,$$
$$\frac{1}{c^2}\frac{\partial}{\partial t}(\xi + xV + ctW) + \frac{\partial}{\partial x}(\eta - tV - xW/c) = 0,$$

and therefore there exists $\psi(x,t)$ such that $\eta - tV - xW/c = \partial\psi/\partial x$ and $\xi + xV + ctW = -\partial\psi/\partial t$, where again $\psi(x,t)$ satisfies the classical wave equation and we have

$$\xi(x,t) = -\left(\frac{\partial\psi}{\partial t} + xV(x,t) + ctW(x,t)\right), \qquad \eta(x,t) = \left(\frac{\partial\psi}{\partial x} + tV(x,t) + \frac{xW(x,t)}{c}\right).$$

The relations $\xi = ex - c^2pt$ and $\eta = px - et$ may be inverted to yield the following expressions for the momentum p and energy e; thus

$$p = \left(\frac{x\frac{\partial\psi}{\partial x} - t\frac{\partial\psi}{\partial t}}{x^2 - (ct)^2}\right) + \frac{W}{c}, \qquad e = \left(\frac{c^2t\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial t}}{x^2 - (ct)^2}\right) - V. \tag{4.47}$$

Finally, we comment that in this case, Eq. (4.34) becomes

$$f^2 - (cg)^2 + \frac{e}{c^2}\left(\frac{\partial^2 V}{\partial t^2} - c^2\frac{\partial^2 V}{\partial x^2}\right) + \frac{p}{c}\left(\frac{\partial^2 W}{\partial t^2} - c^2\frac{\partial^2 W}{\partial x^2}\right) = 0.$$

4.6 Differential Relations for Invariants ξ and η

In this section we formulate certain differential relations for the Lorentz invariants ξ and η defined by (2.48). On taking the total time derivative d/dt of the two invariants $\xi = ex - c^2pt$ and $\eta = px - et$ and making use of $de/dt = udp/dt$, we obtain

$$e_0\frac{d\xi}{ds} = fc\eta, \qquad e_0c\frac{d\eta}{ds} = f\xi - e_0^2, \tag{4.48}$$

where $f = dp/dt$ is the force and ds denotes the infinitesimal line element $ds = c\left(1 - (u/c)^2\right)^{1/2} dt$ arising from $(ds)^2 = (cdt)^2 - (dx)^2$. Accordingly, ds is related to the proper time τ through the relation $ds = cd\tau$. We may confirm the relations (4.48) since on differentiating equation (4.19), namely, $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, totally with respect to time, we may deduce

$$\xi \frac{d\xi}{dt} - c^2 \eta \frac{d\eta}{dt} = e_0^2 (xu - c^2 t),$$

and this equation simplifies to give

$$\xi \frac{d\xi}{ds} - c^2 \eta \frac{d\eta}{ds} = e_0 c \eta,$$

and Eqs. (4.48) can be readily seen to be consistent with this result.

On introducing the function $\sigma(x, t)$ defined by the differential relation $d\sigma = f ds = fc\left(1 - (u/c)^2\right)^{1/2} dt$, from $f = dp/dt$ we may readily deduce $d\sigma = m_0 c du/\left(1 - (u/c)^2\right)$ so that on integration the function σ is given by

$$\sigma = \frac{m_0 c^2}{2} \log\left(\frac{1 + u/c}{1 - u/c}\right) = e_0 \tanh^{-1}\left(\frac{u}{c}\right).$$

Accordingly, in terms of the angle θ defined by (2.9), we have simply $\sigma = e_0\theta$, and the two equations (4.48) become

$$\frac{d\xi}{d\theta} = c\eta, \qquad c\frac{d\eta}{d\theta} = \xi - \frac{e_0^2}{f}, \tag{4.49}$$

and from which on differentiation with respect to θ, we may readily deduce

$$\frac{d^2\xi}{d\theta^2} - \xi = -\frac{e_0^2}{f}, \qquad \frac{d^2\eta}{d\theta^2} - \eta = -\frac{1}{c}\frac{d(e_0^2/f)}{d\theta}. \tag{4.50}$$

Equations (4.49) and (4.50) can appear in a variety of forms. For example, on noting the relations for the force and the line element, namely, $f = dp/dt$ and $ds = c\left(1 - (u/c)^2\right)^{1/2} dt$, we may deduce the equation $e_0/f = ds/d\theta$, and in place of Eqs. (4.49) and (4.50), we have

$$\frac{d\xi}{d\theta} = c\eta, \qquad c\frac{d\eta}{d\theta} = \xi - e_0\frac{ds}{d\theta}, \tag{4.51}$$

and from which we may readily deduce

$$\frac{d^2\xi}{d\theta^2} - \xi = -e_0\frac{ds}{d\theta}, \qquad c\frac{d^2\eta}{d\theta^2} - c\eta = -e_0\frac{d^2 s}{d\theta^2}, \tag{4.52}$$

and we can provide an independent derivation of the latter Eqs. (4.51 and 4.52) as follows:

On taking the total time derivative d/dt of the two Lorentz invariants $\xi = ex - c^2 pt$ and $\eta = px - et$, and making use of $de/dt = udp/dt$, we obtain

$$\frac{d\xi}{dt} = \frac{\eta}{\left(1-(u/c)^2\right)}\frac{du}{dt}, \qquad c\frac{d\eta}{dt} = \frac{\xi}{c\left(1-(u/c)^2\right)}\frac{du}{dt} - e_0 c\left(1-(u/c)^2\right)^{1/2},$$

on using $dp/dt = m_0\left(1-(u/c)^2\right)^{-3/2} du/dt$. Assuming that $du/dt \neq 0$, these two equations simplify to give

$$\frac{d\xi}{du} = \frac{\eta}{\left(1-(u/c)^2\right)}, \qquad c\frac{d\eta}{du} = \frac{\xi}{c\left(1-(u/c)^2\right)} - e_0\frac{ds}{du},$$

so that on introducing the substitution $u = c\sin\phi$, we have

$$\cos\phi\frac{d\xi}{d\phi} = c\eta, \qquad c\cos\phi\frac{d\eta}{d\phi} = \xi - e_0\cos\phi\frac{ds}{d\phi}.$$

Now on introducing χ defined by

$$d\chi = \frac{d\phi}{\cos\phi} = \frac{d\phi}{\left(\cos^2(\phi/2)-\sin^2(\phi/2)\right)} = \frac{\sec^2(\phi/2)d\phi}{\left(1-\tan^2(\phi/2)\right)},$$

and from which, we may readily deduce that

$$\chi = \log\left(\frac{1+\tan(\phi/2)}{1-\tan(\phi/2)}\right) = \frac{1}{2}\log\left(\frac{1+\sin\phi}{1-\sin\phi}\right) = \frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \theta,$$

where θ is again as previously defined in Eqs. (2.9), (4.51), and (4.52) follow immediately.

Further, we comment that by direct differentiation of (4.20) totally with respect to time, and using $u = dx/dt = c\tanh\theta$, we have

$$\frac{d\xi}{dt} = e_0\left(x\sinh\theta - ct\cosh\theta\right)\frac{d\theta}{dt}, \quad c\frac{d\eta}{dt} = e_0\left(x\cosh\theta - ct\sinh\theta\right)\frac{d\theta}{dt} - e_0\operatorname{sech}\theta, \tag{4.53}$$

which coincide with (4.49) and (4.51), since on using $u = c\tanh\theta$, we have $ds/dt = c\operatorname{sech}\theta$ and (4.51) follows immediately from (4.53).

Summary and Differential Relations in Symmetric Form To recapitulate, for a single spatial dimension x, in terms of the arbitrary applied forces $f(x,t)$ and $g(x,t)$ and the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$, the two postulated basic Eqs. (4.30), thus

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{4.54}$$

can be shown to become

$$xf - c^2tg = \frac{\partial\eta}{\partial t} + \frac{\partial\xi}{\partial x}, \qquad xg - tf = \frac{1}{c^2}\frac{\partial\xi}{\partial t} + \frac{\partial\eta}{\partial x},$$

which might be verified simply by performing the partial differentiations on the left-hand side and making use of (4.54).

Now by taking the material or total time derivative d/dt, the two Lorentz invariants can be shown to satisfy the differential relations (4.48) which can be expressed as

$$e\frac{d\xi}{dt} = fc^2\eta, \qquad e\frac{d\eta}{dt} = f\xi - e_0^2, \tag{4.55}$$

together with the integral $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$. We observe especially that the differential relations do not involve the force $g(x,t)$ in the direction of time. In order to deduce a comparable set of differential relations involving $g(x,t)$, we need to determine a corresponding total derivative of the Lorentz invariants. Now, after trialling a number of potential candidates, it eventuates that the companion total derivative is a spatial derivative, but not that following the particle but rather that following the wave, so that the fundamental time and space total derivatives underpinning the structure of the proposed model are defined by

$$\frac{d}{dt} = \left(\frac{d}{dt}\right)_{part} = \frac{\partial}{\partial t} + \left(\frac{dx}{dt}\right)_{part}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \tag{4.56}$$
$$\frac{d}{dx} = \left(\frac{d}{dx}\right)_{wave} = \frac{\partial}{\partial x} + \left(\frac{dt}{dx}\right)_{wave}\frac{\partial}{\partial t} = \frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t},$$

which are not Lorentz invariant, but, under Lorentz transformation, transform in the same manner (see Eqs. (4.59) and (4.60)). In terms of the total spatial derivative following the wave d/dx, we show below that the relations corresponding to (4.55) are

$$e\frac{d\xi}{dx} = c^2g\eta + e_0^2, \qquad e\frac{d\eta}{dx} = g\xi, \tag{4.57}$$

involving only the force gc in the direction of time.

Thus, for the first Lorentz invariant $\xi = ex - c^2pt$, we have

$$\begin{aligned} e\frac{d\xi}{dx} &= e\left(\frac{\partial\xi}{\partial x} + \frac{u}{c^2}\frac{\partial\xi}{\partial t}\right), \\ &= e\left\{(xf - c^2tg) - \frac{\partial\eta}{\partial t} + u(xg - tf) - u\frac{\partial\eta}{\partial x}\right\}, \\ &= e\left\{(x - ut)f + (xu - c^2t)g - \frac{d\eta}{dt}\right\}, \end{aligned}$$

$$= \left\{ f\xi + c^2 g\eta - (f\xi - e_0^2) \right\} = c^2 g\eta + e_0^2,$$

while for the second Lorentz invariant $\eta = px - et$, we have

$$\begin{aligned}
e\frac{d\eta}{dx} &= e\left(\frac{\partial \eta}{\partial x} + \frac{u}{c^2}\frac{\partial \eta}{\partial t}\right), \\
&= e\left\{(xg - tf) - \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{u}{c^2}(xf - c^2 tg) - \frac{u}{c^2}\frac{\partial \xi}{\partial x}\right\}, \\
&= e\left\{(x - ut)g + (xu - c^2 t)\frac{f}{c^2} - \frac{1}{c^2}\frac{d\xi}{dt}\right\}, \\
&= g\xi + f\eta - f\eta = g\xi,
\end{aligned}$$

so that altogether the comparable differential relations to (4.55) involving only the force $g(x, t)$ become (4.57), which together are important relations in revealing the immediate consequences for the two cases examined below, namely, $f = 0$ and $g \neq 0$ suggested by de Broglie's guidance equation, and the complementary case $f \neq 0$ and $g = 0$ arising in special relativistic mechanics.

The symmetry of Eqs. (4.55) and (4.57) arises because under a Lorentz transformation (2.3), the two total derivatives

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \qquad \frac{d}{dx} = \frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t}, \tag{4.58}$$

are not Lorentz invariant, but rather transform in the same manner since from the differential relations

$$\frac{\partial}{\partial x} = \frac{1}{(1 - (v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\}, \quad \frac{\partial}{\partial t} = \frac{1}{(1 - (v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\},$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \\
&= \frac{1}{(1 - (v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\} \\
&+ \frac{(U - v)}{(1 - Uv/c^2)(1 - (v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\}, \\
&= \frac{(1 - (v/c)^2)^{1/2}}{(1 - Uv/c^2)}\left\{\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right\},
\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}&\frac{\partial}{\partial x}+\frac{u}{c^2}\frac{\partial}{\partial t},\\&=\frac{1}{(1-(v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial X}+\frac{v}{c^2}\frac{\partial}{\partial T}\right\}\\&+\frac{(U-v)}{c^2(1-Uv/c^2)(1-(v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial T}+v\frac{\partial}{\partial X}\right\},\\&=\frac{(1-(v/c)^2)^{1/2}}{(1-Uv/c^2)}\left\{\frac{\partial}{\partial X}+\frac{U}{c^2}\frac{\partial}{\partial T}\right\},\end{aligned}\tag{4.60}$$

demonstrating that the two total derivatives defined by (4.58) transform under a Lorentz transformation in an identical manner.

In order to understand the transformation properties of Eqs. (4.55) and (4.57) under the space-time transformation $x' = ct$ and $t' = x/c$, as we have noted previously, we have $u' = c^2/u$, $e' = cp$, $cp' = e$, $f' = f$, $g' = g$, $\xi' = -\xi$, $\eta' = -\eta$ and that under this transformation the two total derivatives d/dt and d/dx given by (4.58) transform in the following manner:

$$\frac{d}{dt'}=\frac{\partial}{\partial t'}+u'\frac{\partial}{\partial x'}=\frac{c}{u}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)=\frac{c}{u}\frac{d}{dt},$$

$$\frac{d}{dx'}=\frac{\partial}{\partial x'}+\frac{u'}{c^2}\frac{\partial}{\partial t'}=\frac{c}{u}\left(\frac{\partial}{\partial x}+\frac{u}{c^2}\frac{\partial}{\partial t}\right)=\frac{c}{u}\frac{d}{dx},$$

so that we have

$$e'\frac{d}{dt'}=e\frac{d}{dt},\qquad e'\frac{d}{dx'}=e\frac{d}{dx}.$$

We also note that under this transformation the two total derivatives du/dt and du/dx transform in the following manner:

$$\frac{du'}{dt'}=\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}=-\left(\frac{c}{u}\right)^3\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\left(\frac{c}{u}\right)^3\frac{du}{dt},$$

$$\frac{du'}{dx'}=\frac{\partial u'}{\partial x'}+\frac{u'}{c^2}\frac{\partial u'}{\partial t'}=-\left(\frac{c}{u}\right)^3\left(\frac{\partial u}{\partial x}+\frac{u}{c^2}\frac{\partial u}{\partial t}\right)=-\left(\frac{c}{u}\right)^3\frac{du}{dx}.$$

4.7 de Broglie's Guidance Equation

In Chap. 1 we have noted de Broglie's general guidance formula of the particle by its wave, as given by Eqs. (1.1) and (1.2), and we have further noted that in the context of the present theory, these equations suggest that $f = 0$ while g is determined from (1.3). In this section, for a single spatial dimension x, we present an extended analysis to determine $\psi(x, t)$ for the relations

$$p = mu = -\frac{\partial \psi}{\partial x}, \qquad e = mc^2 = \frac{\partial \psi}{\partial t}, \tag{4.61}$$

$$\left(\frac{\partial \psi}{\partial t}\right)^2 - c^2\left(\frac{\partial \psi}{\partial x}\right)^2 = e_0^2, \qquad \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = g, \tag{4.62}$$

where $e_0 = m_0c^2$, $(4.62)_1$ arises from the relativistic expression $m = m_0/[1 - (u/c)^2]^{1/2}$ in the form $e^2 - (pc)^2 = e_0^2$, and the velocity $u(x, t)$ is assumed to be given by

$$\frac{u}{c} = \frac{pc}{e} = -c\frac{(\partial \psi/\partial x)}{(\partial \psi/\partial t)}.$$

From the relation $(4.62)_1$, we may without loss of generality suppose that there exists a function $\phi(x, t)$ such that

$$\frac{\partial \psi}{\partial t} = e_0 \cosh \phi, \qquad c\frac{\partial \psi}{\partial x} = e_0 \sinh \phi, \tag{4.63}$$

so that the velocity $u(x, t)$ becomes

$$\frac{u}{c} = \frac{pc}{e} = -c\frac{(\partial \psi/\partial x)}{(\partial \psi/\partial t)} = -\tanh \phi. \tag{4.64}$$

On equating the two expressions for the second derivative, $\partial^2\psi/\partial t\partial x$ gives $(\partial\phi/\partial t) = c \tanh \phi(\partial\phi/\partial x)$, while a second equation is obtained from $(4.62)_2$, which when combined yield

$$\frac{\partial \phi}{\partial x} = -\frac{cg}{e_0}\cosh \phi, \qquad \frac{\partial \phi}{\partial t} = -\frac{c^2 g}{e_0}\sinh \phi. \tag{4.65}$$

We observe that these relations imply that $d\phi/dt = 0$ since we have

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = -\frac{c^2 g}{e_0}\sinh \phi + c \tanh \phi \frac{cg}{e_0}\cosh \phi = 0,$$

and therefore from $u = -c \tanh \phi$, we may conclude that we also have $du/dt = 0$. We note especially that in reality this condition constitutes a first order partial differential equation, the solution of which involves an arbitrary function, and therefore the condition might involve more than simply $u(x, t) = \textit{constant}$. Further, we find that the two equations (4.65) are compatible provided that $g(x, t)$ is a solution of the first order partial differential equation

$$c \sinh \phi \frac{\partial g}{\partial x} - \cosh \phi \frac{\partial g}{\partial t} = \frac{c^2 g^2}{e_0}, \tag{4.66}$$

which we may formally solve using Lagrange's characteristic method.

This method involves introducing a characteristic parameter s and formally solving the three ordinary differential equations

$$\frac{dx}{ds} = c \sinh \phi, \qquad \frac{dt}{ds} = -\cosh \phi, \qquad \frac{dg}{ds} = \frac{c^2 g^2}{e_0}.$$

From the first two equations, we have the relation which is entirely consistent with (4.64), namely,

$$\frac{dx}{dt} = -c \tanh \phi = -\frac{(\partial \phi / \partial t)}{(\partial \phi / \partial x)},$$

and therefore one integral is $\phi(x, t) = constant$, while from the first and the third, we obtain

$$\frac{dg}{dx} = \frac{cg^2}{e_0 \sinh \phi}.$$

On noting that, for the purposes of this integration, $\phi(x, t) = constant$, a second integral is found to be $1/g + cx/e_0 \sinh \phi = constant$, so that the general solution of (4.66) which is obtained by equating one integral to be an arbitrary function of the second integral may be expressed in the form

$$g(x, t) = -\frac{e_0 \sinh \phi}{c(x - x_0(\phi))}, \tag{4.67}$$

where $x_0(\phi)$ denotes an arbitrary function of ϕ. On substitution of this expression into (4.66), we may readily confirm that (4.67) constitutes a solution of the equation without further restriction on the arbitrary function $x_0(\phi)$.

From Eqs. (4.65) and (4.67), we find that we are required to integrate

$$\frac{\partial \phi}{\partial x} = \frac{\sinh \phi \cosh \phi}{x - x_0(\phi)}, \qquad \frac{\partial \phi}{\partial t} = \frac{c \sinh^2 \phi}{x - x_0(\phi)}, \tag{4.68}$$

which at first sight appears to be a non-trivial problem. However, if we introduce $\zeta = x - x_0(\phi)$ as a working variable and change the independent variables from (x, t) to (ζ, t), then from the relations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi^*}{\partial\zeta}\left(1 - x_0^{'}(\phi)\frac{\partial\phi}{\partial x}\right), \qquad \frac{\partial\phi}{\partial t} = \frac{\partial\phi^*}{\partial t} + \frac{\partial\phi^*}{\partial\zeta}\left(-x_0^{'}(\phi)\frac{\partial\phi}{\partial t}\right),$$

where we are using an asterisk to make a distinction between the partial derivatives, so that ϕ^* designates that the partial differentiations are taking place with respect to (ζ, t) as the independent variables, namely, $\phi(x, t) = \phi^*(\zeta, t)$. From these relations we may deduce

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi^*/\partial\zeta}{\left(1 + x_0^{'}(\phi)(\partial\phi^*/\partial\zeta)\right)} = \frac{\sinh\phi\cosh\phi}{\zeta},$$
$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi^*/\partial t}{\left(1 + x_0^{'}(\phi)(\partial\phi^*/\partial\zeta)\right)} = \frac{c\sinh^2\phi}{\zeta},$$

which give rise to the expressions

$$\frac{\partial\phi^*}{\partial\zeta} = \frac{1}{\left(\zeta/(\sinh\phi\cosh\phi) - x_0^{'}(\phi)\right)}, \qquad \frac{\partial\phi^*}{\partial t} = \frac{c\tanh\phi}{\left(\zeta/(\sinh\phi\cosh\phi) - x_0^{'}(\phi)\right)}. \tag{4.69}$$

Now the two expressions (4.69) result in the first order partial differential equation

$$\frac{\partial\phi^*}{\partial t} - c\tanh\phi\frac{\partial\phi^*}{\partial\zeta} = 0,$$

which can be readily solved using Lagrange's characteristic method to deduce $\zeta + ct\tanh\phi = F(\phi)$, where $F(\phi)$ denotes an arbitrary function and on substitution of this relation into either of Eqs. (4.69) yields the following relation between the arbitrary function $F(\phi)$ and $x_0(\phi)$, namely,

$$\frac{dF(\phi)}{d\phi} - \frac{F(\phi)}{\sinh\phi\cosh\phi} = -\frac{dx_0(\phi)}{d\phi}.$$

Integration of this equation yields

$$F(\phi) = -x_0(\phi) - \tanh\phi\int\frac{x_0(\phi)d\phi}{\sinh^2\phi},$$

as the formal connection between the two arbitrary functions $F(\phi)$ and $x_0(\phi)$, so that from $\zeta + ct\tanh\phi = F(\phi)$, we have finally the formal integral for Eqs. (4.68) becomes

$$x + ct\tanh\phi = -\tanh\phi\int\frac{x_0(\phi)d\phi}{\sinh^2\phi}, \tag{4.70}$$

which on using $u = -c \tanh \phi$ from (4.64) can be alternatively written as

$$x - ut = \tanh \phi \int x_0(\phi) d(\coth \phi) = -u \int \frac{x_0^*(u)du}{u^2}, \tag{4.71}$$

where $x_0^*(u)$ denotes $x_0(\phi)$ with ϕ replaced by $-\tanh^{-1}(u/c)$. Of course, since $x_0(\phi)$ refers to an arbitrary function, at one level these details might be considered to be irrelevant, except that we do need to know the relationship between the two arbitrary functions $F(\phi)$ and $x_0(\phi)$. We also observe that the formal integral (4.70) or (4.71) has the structure of the first Lorentz invariant $\xi = ex - c^2 pt = e(x - ut)$ previously defined in this chapter.

The structure of (4.71) leads to the topic of Clairaut's differential equation with parameter u which is examined subsequently in a separate section. We also comment that we may undertake a corresponding calculation to that given in this section for centrally symmetric mechanical systems which is presented in the final section of Chap. 9.

Special Case Arising from $x_0(\phi) = 0$ For $x_0(\phi) \neq 0$, Eq. (4.70) generally involves implicit functions for $u(x, t)$. The simplest solution in this family of solutions which can be given explicitly arises from the special case $x_0(\phi) = 0$. Necessarily $u = -c \tanh \phi = x/t$, and we have

$$e = \frac{e_0 ct}{((ct)^2 - x^2)^{1/2}}, \qquad p = \frac{e_0 x}{c((ct)^2 - x^2)^{1/2}}, \qquad \frac{c^2 p}{e} = \frac{x}{t} = u, \tag{4.72}$$

with particle paths arising from $dx/dt = x/t$, namely, $x/t = constant$. We comment that while the velocity $u(x, t) = x/t$ is perfectly well-defined, it is however an extremely special case but nevertheless sufficiently simple and tractable to calculate explicit formulae and to demonstrate certain important characteristics. Evidently this solution allows both sub-luminal ($x < ct$) and superluminal ($x > ct$) motion such that at some fixed point in space, say $x = a$, we have

$$\frac{u(a, t)}{c} = \begin{cases} a/ct > 1 & \text{if } ct < a \text{ superluminal,} \\ a/ct < 1 & \text{if } ct > a \text{ sub-luminal,} \end{cases}$$

which as far as an observer is concerned means that shortly after time $t = a/c$, a particle suddenly appears moving at a velocity just below that of light. Further, we may readily confirm that the total or material time derivative du/dt is zero, since we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{x}{t^2} + \frac{u}{t} = -\frac{x}{t^2} + \frac{x}{t^2} = 0,$$

and since p is a function of u only, this ensures that $f = dp/dt = 0$.

For this special case, we may deduce simple expressions for all the various quantities as follows: For $u = -c \tanh \phi = x/t$, we may readily show that

$$\frac{\partial \psi}{\partial t} = e_0 \cosh \phi = \frac{e_0 ct}{((ct)^2 - x^2)^{1/2}}, \qquad c\frac{\partial \psi}{\partial x} = e_0 \sinh \phi = -\frac{e_0 x}{((ct)^2 - x^2)^{1/2}},$$

and therefore together with $u(x,t) = x/t$, we have

$$\psi(x,t) = \frac{e_0}{c}((ct)^2 - x^2)^{1/2}, \quad g(x,t) = \frac{e_0}{c((ct)^2 - x^2)^{1/2}}.$$

From these expressions and (4.61), we may confirm $e(x,t)$ and $p(x,t)$ are as given by (4.72) and these expressions provide particular illustrations of Eqs. (3.12) in the case of a single spatial dimension, namely,

$$\frac{\partial e}{\partial t} = u\frac{\partial p}{\partial t}, \qquad \frac{\partial e}{\partial x} = u\frac{\partial p}{\partial x}, \qquad \frac{de}{dt} = u\frac{dp}{dt}.$$

Further, from $(4.72)_2$ we may readily confirm that $f = dp/dt = 0$ since we have

$$\begin{aligned}\frac{dp}{dt} &= \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x},\\ &= \frac{e_0}{c}\left\{\frac{-xc^2 t}{((ct)^2 - x^2)^{3/2}} + \frac{x}{t}\left(\frac{1}{((ct)^2 - x^2)^{1/2}} + \frac{x^2}{((ct)^2 - x^2)^{3/2}}\right)\right\},\\ &= 0,\end{aligned}$$

and providing a specific illustration of non-constant momentum $p(x,t)$ and yet $f = dp/dt = 0$. With $f = 0$ the work done $W(x,t)$ as defined by (3.8) is given by the relation $dW = gc^2 dt$, from which we may deduce

$$W(x,t) = \int \frac{e_0 c dt}{((ct)^2 - x^2)^{1/2}} = e_0 \cosh^{-1}\left(\frac{ct}{x}\right),$$

on using the substitution $ct = x \cosh \theta$ and ignoring any arbitrary constants of integration. From the relation $\cosh^{-1} z = \log(z + (z^2 - 1)^{1/2})$ and $u(x,t) = x/t$, we might deduce

$$W(x,t) = e_0 \log\left(\frac{1 + \left(1 - (u/c)^2\right)^{1/2}}{u/c}\right) = -e_0 \log\left(\frac{u/c}{1 + \left(1 - (u/c)^2\right)^{1/2}}\right).$$

This expression happens to coincide with one of the terms arising from the general expression (6.14) and the special case $\lambda \to -\infty$ for the exact solution (5.1) which is examined in some detail in the next two chapters.

Special Case Arising from $x_0(\phi) = (\xi_0/e_0)\cosh\phi$ Another special case giving rise to an explicit solution is obtained by taking $x_0(\phi) = (\xi_0/e_0)\cosh\phi$ where ξ_0 denotes an arbitrary constant. In this case the integral in (4.70) may be evaluated immediately to obtain

$$x - ut = -\frac{\xi_0 \tanh\phi}{e_0}\int\frac{\cosh\phi d\phi}{\sinh^2\phi} = \frac{\xi_0}{e_0\cosh\phi} = \frac{\xi_0}{e_0}\left\{1-\left(\frac{u}{c}\right)^2\right\}^{1/2} = \frac{\xi_0}{e}, \tag{4.73}$$

on neglecting a non-essential arbitrary additive constant. This case corresponds to assuming that the first Lorentz invariant is a constant, namely, $\xi(x,t) = ex - c^2pt = e(x-ut) = \xi_0$. Here, following the analysis leading to (4.24), (4.25) and (4.26), we note again that by squaring (4.73) we may deduce the quadratic equation

$$\left(\frac{u}{c}\right)^2\left\{(ct)^2+\left(\frac{\xi_0}{e_0}\right)^2\right\} - 2xct\left(\frac{u}{c}\right) + \left\{x^2-\left(\frac{\xi_0}{e_0}\right)^2\right\} = 0,$$

which may be readily solved to give

$$\frac{u(x,t)}{c} = \frac{xct \pm (\xi_0/e_0)\left\{(ct)^2 - x^2 + (\xi_0/e_0)^2\right\}^{1/2}}{\left\{(ct)^2 + (\xi_0/e_0)^2\right\}},$$

noting that the two limiting cases $\xi_0 \longrightarrow 0$ give the above special solution $u(x,t) = x/t$ arising from $x_0(\phi) = 0$, while $\xi_0 \longrightarrow \infty$ corresponds to $u = \pm c$. In view of the constraint $\xi^2 - (c\eta)^2 = e_0^2(x^2-(ct)^2)$, these solutions take on a number of alternative forms including

$$\frac{u(x,t)}{c} = \frac{xct \pm (c\xi_0/e_0^2)\eta_0(x,t)}{\left\{x^2 + (c\eta_0(x,t)/e_0)^2\right\}},$$

where $\eta_0(x,t)$ is assumed to be defined by the relation $c\eta_0(x,t)/e_0 = \pm\left\{(ct)^2 - x^2 + (\xi_0/e_0)\right\}^2$.

In the following analysis, it proves convenient to introduce the working variables

$$\lambda = \frac{\xi_0}{e_0}, \qquad \mu = \frac{c\eta_0(x,t)}{e_0},$$

$$\alpha = \frac{x}{((ct)^2+\lambda^2)^{1/2}} = \frac{x}{(x^2+\mu^2)^{1/2}} = \sin\theta^*,$$

$$\beta = \frac{ct}{((ct)^2+\lambda^2)^{1/2}} = \sin\phi^*,$$

giving rise to the important relations

$$(ct)^2 + \lambda^2 = x^2 + \mu^2, \qquad \mu = \pm((ct)^2 + \lambda^2 - x^2)^{1/2},$$

$$\frac{u(x,t)}{c} = \alpha\beta \pm (1-\alpha^2)^{1/2}(1-\beta^2)^{1/2} = \cos(\theta^* - \phi^*) = \cos\psi^*,$$

where $\psi^* = \theta^* - \phi^*$, and to clarify matters we have adopted the positive sign in the expression for the velocity. If we adopt the negative sign, then $u(x,t)/c = -\cos(\theta^* + \phi^*)$ and a similar analysis applies. For the positive case, we have

$$e(x,t) = \frac{e_0}{(1-(u/c)^2)^{1/2}} = \frac{e_0}{\sin(\theta^* - \phi^*)} = \frac{e_0}{\sin\psi^*},$$

$$p(x,t) = \frac{e_0 u}{c^2(1-(u/c)^2)^{1/2}} = \frac{e_0}{c\tan(\theta^* - \phi^*)} = \frac{e_0}{c\tan\psi^*},$$

where the angle $\psi^* = \theta^* - \phi^*$ is given explicitly by the expression

$$\psi^*(x,t) = \sin^{-1}\left(\frac{x}{((ct)^2 + \lambda^2)^{1/2}}\right) - \sin^{-1}\left(\frac{ct}{((ct)^2 + \lambda^2)^{1/2}}\right),$$

so that we may evaluate the forces f and cg from the general expressions; thus

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = -\frac{e_0}{c\sin^2\psi^*}\left(\frac{\partial \psi^*}{\partial t} + c\cos\psi^*\frac{\partial \psi^*}{\partial x}\right) = 0,$$

$$g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = -\frac{e_0}{c^2\sin^2\psi^*}\left(\cos\psi^*\frac{\partial \psi^*}{\partial t} + c\frac{\partial \psi^*}{\partial x}\right) = -\frac{e_0}{c}\frac{\partial \psi^*}{\partial x} = -\frac{e_0}{c\mu},$$

on making use of the formulae

$$\frac{\partial \psi^*}{\partial t} = -\frac{c(xct + \lambda\mu)}{\mu((ct)^2 + \lambda^2)}, \qquad \frac{\partial \psi^*}{\partial x} = \frac{1}{\mu},$$

$$\cos\psi^* = \cos(\theta^* - \phi^*) = \cos\theta^*\cos\phi^* + \sin\theta^*\sin\phi^* = \frac{xct + \lambda\mu}{(ct)^2 + \lambda^2},$$

and the expression for g is most easily obtained using the result for $f = 0$. The analysis is greatly facilitated by both the assumption that $\xi(x,t) = constant$ and the judicious use of the identity $(ct)^2 + \lambda^2 = x^2 + \mu^2$ to facilitate the explicit evaluation of the partial derivatives. We also observe that the approach adopted for the above analysis for the case of $\xi(x,t) = constant$ no doubt applies to more general situations.

Alternative Analysis of (4.66) We may provide an equivalent analysis to the above as follows: Instead of using Lagrange's characteristic method for the first order partial differential equation (4.66), we may assume that $g(x,t) = g(\phi, \psi)$ which is a valid assumption since from Eqs. (4.63) and (4.65), the Jacobian becomes

$$\frac{\partial(\phi,\psi)}{\partial(x,t)}=\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial t}-\frac{\partial\phi}{\partial t}\frac{\partial\psi}{\partial x}=-cg,$$

and we assume that g is nonvanishing. With the assumption $g(x,t)=g(\phi,\psi)$, Eq. (4.66) becomes

$$c\sinh\phi\left(\frac{\partial g}{\partial\phi}\frac{\partial\phi}{\partial x}+\frac{\partial g}{\partial\psi}\frac{\partial\psi}{\partial x}\right)-\cosh\phi\left(\frac{\partial g}{\partial\phi}\frac{\partial\phi}{\partial t}+\frac{\partial g}{\partial\psi}\frac{\partial\psi}{\partial t}\right)=\frac{c^2g^2}{e_0},$$

and again on using (4.63) and (4.65), this equation simplifies to give simply

$$\frac{\partial g}{\partial\psi}=-\left(\frac{cg}{e_0}\right)^2,$$

which may be readily integrated to yield

$$g(\phi,\psi)=\frac{e_0^2}{c^2(\psi-\psi_0(\phi))},\tag{4.74}$$

where $\psi_0(\phi)$ denotes an arbitrary function of ϕ. From this expression and (4.65), we have

$$\frac{\partial\phi}{\partial x}=-\frac{e_0\cosh\phi}{c(\psi-\psi_0(\phi))},\qquad\frac{\partial\phi}{\partial t}=-\frac{e_0\sinh\phi}{(\psi-\psi_0(\phi))},$$

which we may readily confirm are well-defined in the sense that the second order mixed partial derivatives $\frac{\partial^2\phi}{\partial x\partial t}$ and $\frac{\partial^2\phi}{\partial t\partial x}$ automatically coincide for all arbitrary functions $\psi_0(\phi)$. Further, from $u=-c\tanh\phi$ and (4.63), these relations may be used to confirm that

$$\frac{d\phi}{dt}=\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}=0,\qquad\frac{d\psi}{dt}=\frac{\partial\psi}{\partial t}+u\frac{\partial\psi}{\partial x}=\frac{e_0}{\cosh\phi}.$$

Integration of the latter equation yields

$$\psi=\frac{e_0(t-t_0(\phi))}{\cosh\phi}.$$

where $t_0(\phi)$ denotes some arbitrary function of ϕ, and on combining with (4.74) and equating the alternative expression (4.67) for $g(\phi,\psi)$, we obtain

$$g(\phi,\psi)=\frac{e_0\cosh\phi}{c^2(t-t_0(\phi)-\cosh\phi\,\psi_0(\phi)/e_0)}=-\frac{e_0\sinh\phi}{c(x-x_0(\phi))},$$

and on simplification again it leads to an equation of the form $x - ut = u_0(\phi)$ where $u_0(\phi)$ is essentially an arbitrary function of ϕ and specifically in terms of the other introduced arbitrary functions is given by $u_0(\phi) = x_0(\phi) + c \tanh\phi t_0(\phi) + c \sinh\phi\psi_0(\phi)/e_0$. Since $u = -c\tanh\phi$, the equation $x - ut = u_0(\phi)$ constitutes Clairaut's differential equation with parameter u, and this is the subject of a subsequent section of this chapter.

4.8 Vanishing of Force *g* in Direction of Time

In the previous section, we have presented a detailed analysis for the case $f = 0$ and $g \neq 0$ which is motivated from de Broglie's guidance equation. In this section we present the corresponding analysis for the complementary case $f \neq 0$ and $g = 0$, with the understanding that the product of the two associated velocities for the two problems is unity. We comment that while the calculation details of this section evidently follow similar lines to those of the previous section, in places there are critical departures from this rule. Specifically, in this section for one space dimension, we solve the two equations

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{4.75}$$

where e and p are given, respectively, by

$$e = \frac{m_0}{(1-(u/c)^2)^{1/2}}, \qquad p = \frac{m_0 u}{(1-(u/c)^2)^{1/2}},$$

for the case $f \neq 0$ and $g = 0$. From the condition that $g = 0$, there exists $\Psi(x, t)$ such that

$$p = \frac{1}{c^2}\frac{\partial \Psi}{\partial t}, \qquad e = -\frac{\partial \Psi}{\partial x},$$

so that from the energy relationship $e^2 - (pc)^2 = e_0^2$ and $(4.75)_1$, we have

$$\left(\frac{\partial \Psi}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial \Psi}{\partial t}\right)^2 = e_0^2, \qquad \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} = f, \tag{4.76}$$

where $e_0 = m_0 c^2$, and the velocity $u(x, t)$ is assumed to be given by

$$\frac{u}{c} = \frac{pc}{e} = -\frac{1}{c}\frac{(\partial \Psi/\partial t)}{(\partial \Psi/\partial x)}.$$

From the relation $(4.76)_1$, we can without loss of generality introduce a function $\Phi(x, t)$ such that

$$\frac{\partial \Psi}{\partial x} = e_0 \cosh \Phi, \qquad \frac{1}{c}\frac{\partial \Psi}{\partial t} = e_0 \sinh \Phi, \tag{4.77}$$

so that the velocity $u(x, t)$ becomes

$$\frac{u}{c} = \frac{pc}{e} = -\frac{1}{c}\frac{(\partial \Psi/\partial t)}{(\partial \Psi/\partial x)} = -\tanh \Phi, \tag{4.78}$$

and together these relations imply that $d\Psi/dt = 0$ since we have

$$\frac{d\Psi}{dt} = \frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial x} = e_0 c \sinh \Phi - e_0 c \tanh \Phi \cosh \Phi = 0.$$

From the two expressions for the second derivative $\partial^2\Psi/\partial t \partial x$, we obtain $c\partial\Phi/\partial x = \tanh \Phi(\partial\Phi/\partial t)$, while a second equation is obtained from $(4.76)_2$, which when combined yield

$$\frac{\partial \Phi}{\partial x} = \frac{f}{e_0} \sinh \Phi, \qquad \frac{\partial \Phi}{\partial t} = \frac{cf}{e_0} \cosh \Phi, \tag{4.79}$$

which are well-defined provided that $f(x, t)$ is a solution of the first order partial differential equation

$$\sinh \Phi \frac{\partial f}{\partial t} - c \cosh \Phi \frac{\partial f}{\partial x} = -\frac{cf^2}{e_0}, \tag{4.80}$$

which again we may formally solve using Lagrange's characteristic method. From (4.79) we have

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x} = \frac{cf}{e_0 \cosh \Phi}.$$

Again, in order to solve (4.80), we introduce a characteristic parameter s and the three ordinary differential equations

$$\frac{dx}{ds} = -c \cosh \Phi, \qquad \frac{dt}{ds} = \sinh \Phi, \qquad \frac{df}{ds} = -\frac{cf^2}{e_0}. \tag{4.81}$$

so that on this occasion

$$\frac{dx}{dt} = -c \coth \Phi = -\frac{(\partial \Phi/\partial t)}{(\partial \Phi/\partial x)}. \tag{4.82}$$

We note especially that for this present case $f \neq 0$ and $g = 0$, from the two equations (4.78 and 4.82), dx/dt does not coincide with the velocity u, but rather satisfies the de Broglie relationship such that their product equals c^2, and therefore corresponds to the complementary velocity.

From (4.82), one integral is $\Phi(x, t) = constant$, and by division of the first and the third differential equations (4.81), we obtain

$$\frac{df}{dx} = \frac{f^2}{e_0 \cosh \Phi}.$$

and since for the purposes of this integration, $\Phi(x, t) = constant$, a second integral is found to be $1/f + x/e_0 \cosh \Phi = constant$, so that the general solution of (4.80), which is obtained by equating one integral to be an arbitrary function of the second integral, may be expressed in the form

$$f(x, t) = -\frac{e_0 \cosh \Phi}{x - x_0(\Phi)}, \tag{4.83}$$

where $x_0(\Phi)$ denotes an arbitrary function of Φ. On substitution of this expression into (4.80), we may readily confirm that (4.83) constitutes a solution of the equation without further restriction on the arbitrary function $x_0(\Phi)$.

The details for the determination of $\Phi(x, t)$, although virtually identical to those for (4.68) for the determination of $\phi(x, t)$, are nevertheless sufficiently different to warrant inclusion. From Eqs. (4.79) and (4.83), we are required to integrate

$$\frac{\partial \Phi}{\partial x} = -\frac{\sinh \Phi \cosh \Phi}{x - x_0(\Phi)}, \qquad \frac{\partial \Phi}{\partial t} = -\frac{c \cosh^2 \Phi}{x - x_0(\Phi)}, \tag{4.84}$$

so as before we introduce $\zeta = x - x_0(\Phi)$ as the working variable and change the independent variables from (x, t) to (ζ, t), then from the relations

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi^*}{\partial \zeta}\left(1 - x_0'(\Phi)\frac{\partial \Phi}{\partial x}\right), \qquad \frac{\partial \Phi}{\partial t} = \frac{\partial \Phi^*}{\partial t} + \frac{\partial \Phi^*}{\partial \zeta}\left(-x_0'(\Phi)\frac{\partial \Phi}{\partial t}\right),$$

where we again use an asterisk to make a distinction between the partial derivatives, so that Φ^* designates partial derivatives with respect to (ζ, t) as the independent variables, namely, $\Phi(x, t) = \Phi^*(\zeta, t)$. From these relations we may deduce

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi^*/\partial \zeta}{\left(1 + x_0'(\Phi)(\partial \Phi^*/\partial \zeta)\right)} = -\frac{\sinh \Phi \cosh \Phi}{\zeta},$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi^*/\partial t}{\left(1 + x_0'(\Phi)(\partial \Phi^*/\partial \zeta)\right)} = -\frac{c \cosh^2 \Phi}{\zeta},$$

which give rise to the expressions

$$\frac{\partial \Phi^*}{\partial \zeta} = \frac{-1}{\left(\zeta/(\sinh\Phi\cosh\Phi) + x_0'(\Phi)\right)}, \qquad \frac{\partial \Phi^*}{\partial t} = \frac{-c\coth\Phi}{\left(\zeta/(\sinh\Phi\cosh\Phi) + x_0'(\Phi)\right)}. \tag{4.85}$$

Now the two expressions (4.85) result in the first order partial differential equation

$$\frac{\partial \Phi^*}{\partial t} - c\coth\Phi\frac{\partial \Phi^*}{\partial \zeta} = 0,$$

which can be readily solved using Lagrange's characteristic method to deduce $\zeta + ct\coth\Phi = G(\Phi)$, where $G(\Phi)$ denotes an arbitrary function and on substitution of this relation into either of Eqs. (4.85) yields the following relation between the arbitrary functions $G(\Phi)$ and $x_0(\Phi)$, namely,

$$\frac{dG(\Phi)}{d\Phi} + \frac{G(\Phi)}{\sinh\Phi\cosh\Phi} = -\frac{dx_0(\Phi)}{d\Phi}.$$

Integration of this equation yields

$$G(\Phi) = -x_0(\Phi) + \coth\Phi\int\frac{x_0(\Phi)d\Phi}{\cosh^2\Phi},$$

as the formal connection between the two arbitrary functions $G(\Phi)$ and $x_0(\Phi)$, so that from $\zeta + ct\coth\Phi = G(\Phi)$, we have finally the formal integral for Eqs. (4.84) becomes

$$x + ct\coth\Phi = \coth\Phi\int\frac{x_0(\Phi)d\Phi}{\cosh^2\Phi}.$$

On re-arrangement we obtain

$$x\tanh\Phi + ct = \int\frac{x_0(\Phi)d\Phi}{\cosh^2\Phi}, \tag{4.86}$$

which on using $u = -c\tanh\Phi$ from (4.78) can be alternatively written as

$$x\frac{u}{c} - ct = -\int x_0(\Phi)d(\tanh\Phi) = \frac{1}{c}\int x_0^*(u)du, \tag{4.87}$$

where $x_0^*(u)$ denotes $x_0(\Phi)$ with Φ replaced by $-\tanh^{-1}(u/c)$, and we note that the integral (4.87) has the structure of the second Lorentz invariant $\eta = px - et = e(xu - c^2t)/c^2$ previously defined in this chapter.

Special Case Arising from $x_0(\Phi) = 0$ The special case $x_0(\Phi) \neq 0$ is the complementary solution of the special case $u(x,t) = x/t$ arising in the previous section. Necessarily $u = -c\tanh\Phi = c^2t/x$, and again this solution allows both sub-luminal ($x > ct$) and superluminal ($x < ct$) motion, and we have

$$e = \frac{e_0 x}{(x^2 - (ct)^2)^{1/2}}, \qquad p = \frac{e_0 t}{(x^2 - (ct)^2)^{1/2}}, \qquad \frac{c^2 p}{e} = \frac{c^2 t}{x} = u,$$

with particle paths arising from $dx/dt = c^2t/x$, namely, $x^2 - (ct)^2 = constant$, and the total or material time derivative du/dt is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{c^2}{x}\left\{1 - \left(\frac{ct}{x}\right)^2\right\} = 0.$$

From these explicit expressions we may readily verify $f \neq 0$ and $g = 0$; thus

$$\begin{aligned} f &= \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} \\ &= e_0\left\{\frac{1}{(x^2-(ct)^2)^{1/2}} + \frac{(ct)^2}{(x^2-(ct)^2)^{3/2}}\right\} + e_0\left\{\frac{1}{(x^2-(ct)^2)^{1/2}} - \frac{x^2}{(x^2-(ct)^2)^{3/2}}\right\}, \\ &= \frac{e_0}{(x^2-(ct)^2)^{1/2}}, \\ g &= \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = e_0\left\{\frac{xt}{(x^2-(ct)^2)^{3/2}} - \frac{xt}{(x^2-(ct)^2)^{3/2}}\right\} = 0. \end{aligned}$$

In this special case, the second Lorentz invariant $\eta(x,t) = 0$, while the first Lorentz invariant $\xi(x,t)$ is given by

$$\xi(x,t) = e(x - ut) = \frac{e_0 x}{(x^2-(ct)^2)^{1/2}}\left\{x - \frac{(ct)^2}{x}\right\} = e_0(x^2 - (ct)^2)^{1/2},$$

which is completely in agreement with the relation $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$.

Again while the velocity $u(x,t) = c^2t/x$ is an extremely special case, it is sufficiently simple and tractable to calculate explicit formulae and to demonstrate certain important characteristics. In this case, at some fixed point in space, say $x = a$, we have

$$\frac{u(a,t)}{c} = \begin{cases} a/ct > 1 & \text{if } ct < a \text{ sub-luminal,} \\ a/ct < 1 & \text{if } ct > a \text{ superluminal,} \end{cases}$$

which as far as an observer is concerned means that shortly after time $t = a/c$, a particle moving at a velocity just below that of light suddenly disappears.

Special Case Arising from $x_0(\Phi) = (c\eta_0/e_0)\sinh\Phi$ The special case $x_0(\Phi) = (c\eta_0/e_0)\sinh\Phi$ also gives rise to an explicit solution, where η_0 denotes an arbitrary constant. In this case the integral in (4.86) may be evaluated immediately to obtain

$$x\frac{u}{c} - ct = -\frac{c\eta_0}{e_0}\int \frac{\sinh \Phi d\phi}{\cosh^2 \Phi} = \frac{c\eta_0}{e_0 \cosh \Phi} = \frac{c\eta_0}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{1/2}, \tag{4.88}$$

on neglecting a non-essential arbitrary additive constant. This case corresponds to assuming that the second Lorentz invariant is a constant, namely, $\eta(x,t) = px - et = e(xu/c - ct)/c = \eta_0$. Here, following the analysis leading to (4.24), (4.25) and (4.26), we note again that by squaring (4.88) we may deduce the quadratic equation

$$\left(\frac{u}{c}\right)^2\left\{x^2 + \left(\frac{c\eta_0}{e_0}\right)^2\right\} - 2xct\left(\frac{u}{c}\right) + \left\{(ct)^2 - \left(\frac{c\eta_0}{e_0}\right)^2\right\} = 0,$$

which may be readily solved to give

$$\frac{u(x,t)}{c} = \frac{xct \pm (c\eta_0/e_0)\left\{x^2 - (ct)^2 + (c\eta_0/e_0)^2\right\}^{1/2}}{\left\{x^2 + (c\eta_0/e_0)^2\right\}},$$

noting that the two limiting cases $\eta_0 \longrightarrow 0$ give the above special solution $u(x,t) = c^2t/x$ arising from $x_0(\Phi) = 0$, while $\eta_0 \longrightarrow \infty$ corresponds to $u = \pm c$. In view of the constraint $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, these solutions take on a number of alternative forms including

$$\frac{u(x,t)}{c} = \frac{xct \pm (c\eta_0/e_0^2)\xi_0(x,t)}{\left\{x^2 + (c\eta_0/e_0)^2\right\}},$$

where $\xi_0(x,t)$ is assumed to be defined explicitly by the relation $\xi_0(x,t)/e_0 = \pm\left\{x^2 - (ct)^2 + (c\eta_0/e_0)\right\}^2$.

Again, it proves convenient to introduce the working variables

$$\lambda = \frac{\xi_0(x,t)}{e_0}, \qquad \mu = \frac{c\eta_0}{e_0},$$

$$a = \frac{x}{(x^2 - \lambda^2)^{1/2}} = \frac{x}{((ct)^2 - \mu^2)^{1/2}} = \cosh\theta^*, \qquad \frac{\lambda}{(x^2 - \lambda^2)^{1/2}} = \sinh\theta^*,$$

$$b = \frac{ct}{((ct)^2 - \mu^2)^{1/2}} = \cosh\phi^*, \qquad \frac{\mu}{((ct)^2 - \mu^2)^{1/2}} = \sinh\phi^*,$$

noting again that here μ is a constant and giving rise to the important relations

$$x^2 - \lambda^2 = (ct)^2 - \mu^2, \qquad \lambda = \pm(x^2 + \mu^2 - (ct)^2)^{1/2},$$

$$\frac{u(x,t)}{c} = ab \pm (a^2 - 1)^{1/2}(b^2 - 1)^{1/2} = \cosh(\theta^* + \phi^*) = \cosh\psi^*,$$

where $\psi^* = \theta^* + \phi^*$, and to clarify matters we have adopted the positive sign in the expression for the velocity. If we adopt the negative sign, then $u(x,t)/c = -\cosh(\theta^* - \phi^*)$ and a similar analysis applies. Evidently, in both cases $u/c > 1$, and it is important to note that in this analysis, we are making an implicit hypothesis that any superluminal motion is only possible within the energy interpretation $e = e_0/(1-(u/c)^2)^{1/2}$, provided that $e_0^2 < 0$, and formally we assume this to be the case. Alternatively, we might achieve the same outcome simply by adopting the formulae $e = e_0/((u/c)^2 - 1)^{1/2}$ and $p = e_0 u/c^2((u/c)^2 - 1)^{1/2}$ for $u/c > 1$.

For the positive case, we have

$$e(x,t) = \frac{e_0}{((u/c)^2 - 1)^{1/2}} = \frac{e_0}{\sinh(\theta^* + \phi^*)} = \frac{e_0}{\sinh \psi^*},$$
$$p(x,t) = \frac{e_0 u}{c^2((u/c)^2 - 1)^{1/2}} = \frac{e_0}{c \tanh(\theta^* + \phi^*)} = \frac{e_0}{c \tanh \psi^*},$$

where the angle $\psi^* = \theta^* + \phi^*$ is given explicitly by the expression

$$\psi^*(x,t) = \sinh^{-1}\left(\frac{\lambda}{(x^2 - \lambda^2)^{1/2}}\right) + \sinh^{-1}\left(\frac{\mu}{((ct)^2 - \mu^2)^{1/2}}\right),$$

so that we may evaluate the forces f and cg from the general expressions; thus

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = -\frac{e_0}{c \sinh^2 \psi^*}\left(\frac{\partial \psi^*}{\partial t} + c \cosh \psi^* \frac{\partial \psi^*}{\partial x}\right) = 0,$$
$$g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = -\frac{e_0}{c^2 \sinh^2 \psi^*}\left(\cosh \psi^* \frac{\partial \psi^*}{\partial t} + c\frac{\partial \psi^*}{\partial x}\right) = -\frac{e_0}{c^2 \cosh \psi^*}\frac{\partial \psi^*}{\partial t} = -\frac{e_0}{c\lambda},$$

on making use of the formulae

$$\frac{\partial \psi^*}{\partial t} = -\frac{c(xct + \lambda\mu)}{\lambda((ct)^2 - \mu^2)}, \qquad \frac{\partial \psi^*}{\partial x} = \frac{1}{\lambda},$$
$$\cosh \psi^* = \cosh(\theta^* + \phi^*) = \cosh \theta^* \cosh \phi^* + \sinh \theta^* \sinh \phi^* = \frac{xct + \lambda\mu}{(ct)^2 - \mu^2},$$

and again the expression for g is most easily obtained using the result for $f = 0$. Again, the present analysis is greatly facilitated by both the assumption that $\eta(x,t) = constant$ and the judicious use of the identity $(ct)^2 - \lambda^2 = x^2 - \mu^2$ to evaluate the partial derivatives. We again observe that the approach adopted in this and the previous sections for the two cases $\eta(x,t) = constant$ and $\xi(x,t) = constant$ might also apply to more general situations.

From Eqs. (4.77) and (4.79), we may readily show that the Jacobian becomes

$$\frac{\partial(\Phi, \Psi)}{\partial(x,t)} = \frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial t} - \frac{\partial \Phi}{\partial t}\frac{\partial \Psi}{\partial x} = -cf,$$

and here we assume that f is nonvanishing. As in the previous section, this equation allows an alternative analysis to the Lagrange's characteristic method by assuming that $f(x,t) = f(\Phi, \Psi)$, so that Eq. (4.80) becomes

$$\sinh \Phi \left(\frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial x} + \frac{\partial f}{\partial \Psi} \frac{\partial \Psi}{\partial x} \right) - c \cosh \Phi \left(\frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial t} + \frac{\partial f}{\partial \Psi} \frac{\partial \Psi}{\partial t} \right) = -\frac{cf^2}{e_0},$$

which on using (4.77) and (4.79) this equation simplifies to give simply

$$\frac{\partial f}{\partial \Psi} = \left(\frac{f}{e_0} \right)^2,$$

which may be readily integrated to yield

$$f(\Phi, \Psi) = \frac{e_0^2}{\Psi_0(\Phi) - \Psi},$$

where $\Psi_0(\Phi)$ denotes an arbitrary function of Φ.

Further, to summarise the various relationships for the two distinct problems, we have for $f = 0$ and $g \neq 0$

$$\frac{u}{c} = -c \frac{(\partial \psi / \partial x)}{(\partial \psi / \partial t)} = -\tanh \phi = -\frac{1}{c} \frac{(\partial \phi / \partial t)}{(\partial \phi / \partial x)},$$

with particle paths given by $\phi(x,t) = constant$, while in this present section for $f \neq 0$ and $g = 0$, we have

$$\frac{u}{c} = -\frac{1}{c} \frac{(\partial \Psi / \partial t)}{(\partial \Psi / \partial x)} = -\tanh \Phi = -c \frac{(\partial \Phi / \partial x)}{(\partial \Phi / \partial t)},$$

with particle paths given by $\Psi(x,t) = constant$, and a number of simple relations now follow.

Firstly, simply by equating the two alternative expressions for the same velocity $u(x,t)$, it is clear that for the two problems that we have,

$$\frac{\partial \phi}{\partial t} \frac{\partial \psi}{\partial t} = c^2 \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x}, \qquad \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial t} = c^2 \frac{\partial \Phi}{\partial x} \frac{\partial \Psi}{\partial x}, \tag{4.89}$$

and secondly, if we impose the requirement that the two distinct velocities correspond to the two solutions of the same physical problem, namely, the wave and particulate solutions, so that we impose two versions of the same de Broglie condition, then we may verify the Jacobians

$$\frac{\partial(\phi, \Phi)}{\partial(x,t)} = \frac{\partial \phi}{\partial x} \frac{\partial \Phi}{\partial t} - \frac{\partial \phi}{\partial t} \frac{\partial \Phi}{\partial x} = 0, \qquad \frac{\partial(\psi, \Psi)}{\partial(x,t)} = \frac{\partial \psi}{\partial x} \frac{\partial \Psi}{\partial t} - \frac{\partial \psi}{\partial t} \frac{\partial \Psi}{\partial x} = 0,$$

and from which we may deduce that $\Phi = F(\phi)$ and $\Psi = G(\psi)$, where F and G denote arbitrary functions of the indicated argument. Thus, the requirement that the two problems are complementary is equivalent to the condition that the (ϕ, ψ) and (Φ, Ψ) constant curves coincide, noting from (4.89) that these networks are non-orthogonal.

4.9 Clairaut's Differential Equation with Parameter *u*

In this section, we make the assumption that the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$ are functions of the velocity u only, and we frame the particle-wave formulation in terms of Clairaut's differential equation using the velocity u as the parameter. In order to present this formulation, we need previously stated equations from this present chapter which for convenience we list here. Firstly, from (4.31) the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$, satisfy $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$ and the following equations:

$$xf - c^2tg = \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x}, \qquad xg - tf = \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x}, \tag{4.90}$$

where $f(x, t)$ and $g(x, t)$ denote arbitrary applied forces and where, in this instance, $f = dp/dt$. By eliminating the mixed partial derivatives $\frac{\partial^2 \xi}{\partial x \partial t}$ and $\frac{\partial^2 \eta}{\partial x \partial t}$, we might deduce from (4.90) the following two important equations:

$$\left(x\frac{\partial g}{\partial t} + c^2t\frac{\partial g}{\partial x}\right) - \left(x\frac{\partial f}{\partial x} + t\frac{\partial f}{\partial t} + 2f\right) = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2}, \tag{4.91}$$

$$\frac{1}{c^2}\left(x\frac{\partial f}{\partial t} + c^2t\frac{\partial f}{\partial x}\right) - \left(x\frac{\partial g}{\partial x} + t\frac{\partial g}{\partial t} + 2g\right) = \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2},$$

noting the evident symmetries with regard to both space and time. Bearing in mind that the two Lorentz invariants ξ and η involve the variable $(ct)^2 - x^2$ through the relation $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, it may be worthwhile noting that the two force functional forms $f(x, t)$ for which either of

$$x\frac{\partial f}{\partial t} + c^2t\frac{\partial f}{\partial x} = 0, \qquad x\frac{\partial f}{\partial x} + t\frac{\partial f}{\partial t} + 2f = 0,$$

are, respectively, $f(x, t) = f((ct)^2 - x^2)$ and $f(x, t) = f(x/t)/x^2$, which are connected since a wave velocity $w = x/t$ has a corresponding particle velocity $u = c^2/w = c^2t/x$ for which the integral curves arising from $dx/dt = c^2t/x$ are $(ct)^2 - x^2 = constant$.

Secondly, from (4.48) by taking the total time derivative d/dt of the two invariants, we have obtained the following differential relations:

$$e\frac{d\xi}{dt} = fc^2\eta, \qquad e\frac{d\eta}{dt} = f\xi - e_0^2. \tag{4.92}$$

The three equations (4.90), (4.91), and (4.92) constitute results necessary to provide a complete description of the particle-wave formulation in terms of Clairaut's differential equation using the velocity u as the parameter.

We first determine expressions for the two partial derivatives $\partial u/\partial t$ and $\partial u/\partial x$. From the two basic one-dimensional equations (4.30) and the two expressions for the momentum and particle energy, namely, $p(x,t) = e_0 u/c^2(1-(u/c)^2)^{1/2}$ and $e(x,t) = e_0/(1-(u/c)^2)^{1/2}$, where e_0 denotes the rest energy, we may deduce

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{fc^2}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{3/2},$$

$$\frac{\partial u}{\partial x} + \frac{u}{c^2}\frac{\partial u}{\partial t} = \frac{gc^2}{e_0}\left\{1 - \left(\frac{u}{c}\right)^2\right\}^{3/2}.$$

On solving these equations as two equations in the two unknowns $\partial u/\partial x$ and $\partial u/\partial t$, we obtain the following two expressions:

$$\frac{\partial u}{\partial x} = \frac{gc^2 - fu}{e}, \qquad \frac{\partial u}{\partial t} = \frac{c^2(f - gu)}{e}, \tag{4.93}$$

which we adopt to determine

$$\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 = \frac{c^2}{e^2}(f^2 - (gc)^2)\left\{1 - \left(\frac{u}{c}\right)^2\right\}. \tag{4.94}$$

In order that (4.93) constitute two well-defined expressions for the first order partial derivatives, the second order mixed partial derivatives $\frac{\partial^2 u}{\partial x \partial t}$ and $\frac{\partial^2 u}{\partial t \partial x}$ must coincide, and from this condition and (4.93), we obtain

$$c^2\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right) - u\left(\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\right) = \frac{c^2}{e}(f^2 - (gc)^2),$$

which we use as a means of expressing the first term in terms of the other two terms; thus

$$\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right) = \frac{u}{c^2}\left(\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\right) + \frac{1}{e}(f^2 - (gc)^2). \tag{4.95}$$

We now use the relations (4.93) to determine an expression for the quantity involving the second order derivatives, namely, $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}$, so we partially differentiate $(4.93)_1$ with respect to x and $(4.93)_2$ with respect to t and make use of the expressions themselves to eventually deduce

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{e}\left(c^2\frac{\partial g}{\partial x} - u\frac{\partial f}{\partial x}\right) - \frac{(gc^2 - fu)}{(ec)^2(1-(u/c)^2)}\left(fc^2\left\{1-\left(\frac{u}{c}\right)^2\right\} + u(gc^2 - fu)\right),$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{e}\left(\frac{\partial f}{\partial t} - u\frac{\partial g}{\partial t}\right) - \frac{(f - gu)}{e^2(1-(u/c)^2)}\left(gc^2\left\{1-\left(\frac{u}{c}\right)^2\right\} + u(f - gu)\right).$$

On combining these two expressions and making use of (4.95), we obtain finally

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{1}{e}\left(\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\right)\left\{1-\left(\frac{u}{c}\right)^2\right\} - \frac{3u}{e^2}(f^2 - (gc)^2), \qquad (4.96)$$

which we observe is in complete agreement with (4.5) for which the velocity $u(x,t)$ satisfies the non-linear partial differential equation

$$\left\{1-\left(\frac{u}{c}\right)^2\right\}\left(\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{3u}{c^2}\left\{\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right\} = 0, \qquad (4.97)$$

which arises from the fact that if $\frac{\partial f}{\partial t} = c^2\frac{\partial g}{\partial x}$, then the momentum $p(x,t)$ satisfies the wave equation. Thus, in this case we have from (4.94) and (4.97) that

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\frac{3u}{e^2}(f^2 - (gc)^2).$$

which is entirely consistent with the general result (4.96).

We note in passing that from the expressions (4.93), the total time derivative of $u(x,t)$ does not depend explicitly on $g(x,t)$ and simplifies to yield

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{fc^2}{e}\left\{1-\left(\frac{u}{c}\right)^2\right\}, \qquad (4.98)$$

which is entirely consistent with $f = dp/dt$ and $p = mu = eu/c^2$. We may wish to examine Eq. (4.98) rewritten as follows:

$$e\frac{du}{dt} = c^2 f\left\{1-\left(\frac{u}{c}\right)^2\right\},$$

because this version reveals the full options that are available if one side of the equation vanishes. For example, if $du/dt = 0$, then either $f = 0$ or $u = \pm c$ or both of these conditions hold. Similarly, if $u = \pm c$, then either $e = 0$ (viz. $e_0 = 0$) or $du/dt = 0$ or both.

We now make the assumption that the Lorentz invariants $\xi(x,t) = e(x - ut)$ and $\eta(x,t) = e(ux - c^2 t)/c^2$ are functions of the velocity u only, namely, $\xi(x,t) = \xi(u)$ and $\eta(x,t) = \eta(u)$, so that we have

$$x - ut = \frac{\xi(u)}{e_0}(1-(u/c)^2)^{1/2} = A(u), \tag{4.99}$$

$$xu - c^2t = \frac{c^2\eta(u)}{e_0}(1-(u/c)^2)^{1/2} = B(u),$$

where $A(u)$ and $B(u)$ denote the shown functions of u only. In the following development, we may either use $\xi(u)$ and $\eta(u)$ or $A(u)$ and $B(u)$ as entirely equivalent formulations. We make the comment in passing that since in the development below, we use the velocity u as a parametric variable with both x and t defined by certain functions of u (see Eqs. (4.100)), the extent of the assumption $\xi(x,t) = \xi(u)$ and $\eta(x,t) = \eta(u)$ is not entirely clear and is something of a technical question.

The first equation (4.99) is in the form of Clairaut's differential equation $x-ut = A(u)$, which on taking the total time derivative yields

$$\frac{du}{dt}\left(t + A'(u)\right) = 0,$$

so that either $du/dt = 0$, corresponding to the constancy of the velocity of light, or $t + A'(u) = 0$, in which case

$$t = -A'(u), \qquad x = A(u) - uA'(u), \tag{4.100}$$

and we may check that these parametric equations provide the desired expression for the velocity; thus

$$\frac{dx}{dt} = \frac{A'(u) - uA''(u) - A'(u)}{-A''(u)} = u,$$

as required. In addition, on taking the total time derivative of $t = -A'(u)$, we have the simple relationship

$$\frac{du}{dt} = \frac{-1}{A''(u)}. \tag{4.101}$$

It is perhaps important to point out that the option $du/dt = 0$ as well as implying the constancy of the velocity of light also embraces non-constant velocities such as $u(x,t) = x/t$ considered in the previous section.

If we now turn to the second equation of (4.99), namely, $xu - c^2t = B(u)$, then on using the parametric relations (4.100), we have the basic connection between the two functions $A(u)$ and $B(u)$; thus

$$B(u) = c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\}A'(u) + uA(u), \tag{4.102}$$

and it is not difficult to show that this equation coincides with $(4.92)_1$, since on using

$$f = \frac{dp}{dt} = \frac{e_0}{c^2(1-(u/c)^2)^{3/2}}\frac{du}{dt},$$

Equations (4.92) become, respectively,

$$\xi'(u) = \frac{\eta(u)}{(1-(u/c)^2)}, \qquad \eta'(u) = \frac{\xi(u)}{c^2(1-(u/c)^2)} - \frac{e_0(1-(u/c)^2)^{1/2}}{du/dt}, \tag{4.103}$$

noting especially that both relations are derived on the assumption that $du/dt \neq 0$, which for the time being we assume to be the case.

Illustration of the Differential Conditions (4.103) On taking the total time derivative of $(c\eta)^2 - \xi^2 = e_0^2((ct)^2 - x^2)$, we may deduce

$$\left(c^2\eta(u)\eta'(u) - \xi(u)\xi'(u)\right)\frac{du}{dt} = e_0^2(c^2t - xu) = -e_0c^2\eta(u)(1-(u/c)^2)^{1/2}, \tag{4.104}$$

on using $(4.99)_2$. Now on adopting the differential conditions (4.103), Eq. (4.104) becomes

$$\left(c^2\eta(u)\left\{\frac{\xi(u)}{c^2(1-(u/c)^2)} - \frac{e_0(1-(u/c)^2)^{1/2}}{du/dt}\right\} - \frac{\xi(u)\eta(u)}{(1-(u/c)^2)}\right)\frac{du}{dt}$$
$$= -e_0c^2\eta(u)(1-(u/c)^2)^{1/2},$$

which is evidently properly satisfied, so that the differential conditions (4.103) are entirely consistent with the integral $(c\eta)^2 - \xi^2 = e_0^2((ct)^2 - x^2)$, on the assumption that the total time derivative $du/dt \neq 0$.

From the basic relations $A(u) = \xi(u)(1-(u/c)^2)^{1/2}/e_0$ and $B(u) = c^2\eta(u)(1-(u/c)^2)^{1/2}/e_0$, we may show that Eq. (4.102) coincides with $(4.103)_1$. Further, on taking the total time derivative of $xu - c^2t = B(u)$, we obtain

$$\frac{du}{dt}\left(x - B'(u)\right) = c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\},$$

and on using $x = A(u) - uA'(u)$ and (4.101), this equation becomes

$$B'(u) = c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\}A''(u) - uA'(u) + A(u), \tag{4.105}$$

which is evidently entirely consistent with the derivative with respect to u of (4.102).

First Illustration of Clairaut's Differential Equation In the special case that the first Lorentz invariant is constant, $\xi(u) = \xi_0$, and the second is zero, $\eta(u) = 0$, we have $A(u) = \xi_0(1-(u/c)^2)^{1/2}/e_0$ and $B(u) = 0$, and the latter condition yields

$$u = \frac{c^2 t}{x} = \frac{c^2 A'(u)}{uA'(u) - A(u)}, \tag{4.106}$$

which on re-arrangement is simply (4.102) with $B(u) = 0$. Further, $\xi(u) = \xi_0$, $\eta(u) = 0$ and $A(u) = \xi_0(1-(u/c)^2)^{1/2}/e_0$ properly satisfy both of (4.103). We also observe that for the associated wave velocity $u(x,t) = x/t$ for which $du/dt = 0$, formally in place of (4.106), we have

$$u = \frac{x}{t} = \frac{uA'(u) - A(u)}{A'(u)},$$

resulting in $A(u) = B(u) = 0$ and therefore, of course, $\xi(u) = \eta(u) = 0$.

Second Illustration of Clairaut's Differential Equation To provide a second simple analytical example, we pose $(4.103)_2$ in the following form:

$$\frac{\eta'(u)}{(1-(u/c)^2)^{1/2}} - \frac{\xi(u)}{u}\left(\frac{1}{(1-(u/c)^2)^{1/2}}\right)' = e_0 A''(u),$$

and we choose $\xi(u)$ such that this equation becomes an exact differential, namely, we adopt $\xi(u)/u = -\eta(u)$ so that the above equation becomes

$$\frac{\eta(u)}{(1-(u/c)^2)^{1/2}} = e_0 A'(u) + C_1,$$

where C_1 denotes the arbitrary constant of integration. From $\eta(u) = -\xi(u)/u$ and $A(u) = \xi(u)(1-(u/c)^2)^{1/2}/e_0$, this differential relation becomes

$$\xi'(u) + \frac{\xi(u)}{u} = -\frac{C_1}{(1-(u/c)^2)^{1/2}},$$

which upon integration yields

$$\xi(u) = \frac{C_1 c^2}{u}(1-(u/c)^2)^{1/2} + \frac{C_2}{u}, \qquad \eta(u) = -\frac{C_1 c^2}{u^2}(1-(u/c)^2)^{1/2} - \frac{C_2}{u^2},$$

where C_2 denotes a second arbitrary constant of integration. However, from the relation $\xi'(u) = \eta(u)/(1-(u/c)^2)$, we may readily establish that we require $C_2 = 0$ and therefore two simple analytical expressions satisfying both of the relations (4.103) become

$$\xi(u) = \frac{C_1 c^2}{u}(1-(u/c)^2)^{1/2}, \qquad \eta(u) = -\frac{C_1 c^2}{u^2}(1-(u/c)^2)^{1/2}, \tag{4.107}$$

where C_1 denotes an arbitrary constant. In terms of $A(u)$ and $B(u)$, these relations become

$$A(u) = \frac{C_1 c}{e_0}\left(\frac{c}{u} - \frac{u}{c}\right), \qquad B(u) = \frac{C_1 c^2}{e_0}\left\{1 - \left(\frac{c}{u}\right)^2\right\},$$

and from which we may confirm the relation (4.102). From the expressions (4.107) and $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, we may readily deduce that the particle velocity $u(x,t)$ adopts one of the four allowable alternatives

$$u(x,t) = \frac{\pm c}{\left(1 \pm e_0((ct)^2 - x^2)^{1/2}/C_1 c\right)^{1/2}}.$$

Summary for Clairaut's Differential Equation Using Parameter u The assumption that the Lorentz invariants are functions of u only gives rise to two entirely consistent differential conditions $x - ut = A(u)$ and $xu - c^2 t = B(u)$, where $A(u)$ and $B(u)$ are defined in terms of the Lorentz invariants by $A(u) = \xi(u)(1 - (u/c)^2)^{1/2}/e_0$ and $B(u) = c^2\eta(u)(1-(u/c)^2)^{1/2}/e_0$. The important parametric relations are

$$t = -A'(u), \qquad x = A(u) - uA'(u), \qquad \frac{du}{dt} = \frac{-1}{A''(u)}, \tag{4.108}$$

and it eventuates that the key relationships between $\xi(u)$ and $\eta(u)$ or between $A(u)$ and $B(u)$ are either of the equivalent relations

$$\xi'(u) = \frac{\eta(u)}{(1-(u/c)^2)}, \qquad B(u) = c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\} A'(u) + uA(u). \tag{4.109}$$

For example, the parametric relations (4.108) along with $(4.109)_1$ automatically ensure the validity of both $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$ and the relation $(4.103)_2$ which can be demonstrated by direct substitution. In the former case, we have

$$\begin{aligned}(c\eta)^2 - \xi^2 &= c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\}^2 \xi'^2 - \xi^2 = e_0^2((ct)^2 - x^2) = e_0^2(ct - x)(ct + x),\\ &= e_0^2\left\{c\left(1 - \frac{u}{c}\right)A'(u) + A(u)\right\}\left\{c\left(1 + \frac{u}{c}\right)A'(u) - A(u)\right\},\\ &= e_0^2\left[c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\}^2 A'(u)^2 + 2uA(u)A'(u) - A(u)^2\right],\end{aligned}$$

which on using $A(u) = \xi(u)(1-(u/c)^2)^{1/2}/e_0$ and its derivative can be shown to be identically satisfied. Similarly for the differential relation $(4.103)_2$, on using $(4.109)_1$, we have

$$\eta'(u) = \left\{1-\left(\frac{u}{c}\right)^2\right\}\xi''(u) - \frac{2u}{c^2}\xi'(u) = \frac{\xi(u)}{c^2(1-(u/c)^2)} - \frac{e_0(1-(u/c)^2)^{1/2}}{du/dt},$$
$$= \frac{\xi(u)}{c^2(1-(u/c)^2)} + e_0(1-(u/c)^2)^{1/2}A''(u),$$

which is equivalent to

$$e_0A''(u) = (1-(u/c)^2)^{1/2}\xi''(u) - \frac{2u\xi'(u)}{c^2(1-(u/c)^2)^{1/2}} - \frac{\xi(u)}{c^2(1-(u/c)^2)^{3/2}},$$

and this is simply the second derivative of the relation $A(u) = \xi(u)(1-(u/c)^2)^{1/2}/e_0$. Thus to recapitulate, the key relations comprise the three parametric equations (4.108) coupled with either of the equivalent equations (4.109).

Up to this point, we have only examined what might be termed the "kinematics" of the motion, without serious reference to the mechanical aspects embodied in either of Eqs. (4.90) or (4.91) which are formulated in terms of arbitrary applied forces $f(x,t)$ and $g(x,t)$. As shown below, there are no "universal" allowable Lorentz invariants $\xi(u)$ and $\eta(u)$ applying to all applied forces, and in order to proceed further, we need to make a more precise prescription of $f(x,t)$ and $g(x,t)$. The above analysis is useful since it provides insight into the particle-wave duality, in the sense that either under Lorentz transformation or under the space-time transformation $x' = ct$ and $t' = x/c$, the formulation in terms of Clairaut's differential equation is preserved. In the former case, this is obvious, while in the latter case, the velocity transforms as $u' = c^2/u$, and the Lorentz invariants $\xi = e(x-ut)$ and $\eta = e(ux-c^2t)/c^2$ become

$$\xi' = e'(x'-u't') = \frac{e_0(ct-(cx/u))}{((c/u)^2-1)^{1/2}} = -\frac{e_0(x-ut)}{(1-(u/c)^2)^{1/2}} = -\xi,$$
$$\eta' = \frac{e'}{c^2}(u'x'-c^2t') = \frac{e_0((c^3t/u)-cx)}{((c/u)^2-1)^{1/2}} = -\frac{e_0(ux-c^2t)}{(1-(u/c)^2)^{1/2}} = -\eta,$$

which is based on the assumption of a formal interpretation for the energy e' for superluminal velocities u', namely, $e' = e_0/((u'/c)^2-1)^{1/2} = e_0/((c/u)^2-1)^{1/2} = eu/c$.

The Lorentz invariants are connected to the applied forces $f(x,t)$ and $g(x,t)$ through either of Eqs. (4.90) or (4.91) with the former the stronger version, since the latter equations involve partial derivatives of the former equations. In the following we examine both sets of equations.

Arbitrary Applied Forces $f(x,t)$ and $g(x,t)$ with (4.90) From the two equations (4.90) and using the expressions (4.93) for the partial derivatives $\partial u/\partial x$ and $\partial u/\partial t$ along with the expressions (4.103) for $\xi'(u)$ and $\eta'(u)$, we obtain

$$xf - c^2tg = \left(\frac{\xi(u)}{(1-(u/c)^2)} + e_0c^2\left\{1-\left(\frac{u}{c}\right)^2\right\}^{1/2}A''(u)\right)\frac{(f-gu)}{e} + \frac{\eta(u)(gc^2-fu)}{e(1-(u/c)^2)}, \tag{4.110}$$

$$xg - tf = \frac{\eta(u)(f-gu)}{e(1-(u/c)^2)} + \left(\frac{\xi(u)}{c^2(1-(u/c)^2)} + e_0\left\{1-\left(\frac{u}{c}\right)^2\right\}^{1/2}A''(u)\right)\frac{(gc^2-fu)}{e}.$$

On collecting the coefficients of the linear terms in f and g, using the basic definitions of $\xi(u)$ and $\eta(u)$ (Eqs. (4.99)) in terms of $A(u)$ and $B(u)$ and the parametric relationships (4.100) for x and t, and then using the relationship (4.102) to eliminate $B(u)$, we may ultimately simplify (4.110) to become, respectively,

$$fX + gc^2Y = 0, \qquad fY + gX = 0,$$

where the quantities X and Y are given by

$$X = -c^2\left\{1-\left(\frac{u}{c}\right)^2\right\}A''(u), \qquad Y = u\left\{1-\left(\frac{u}{c}\right)^2\right\}A''(u).$$

Assuming for the time being that $u \neq \pm c$ and that $A''(u) \neq 0$, these equations become simply

$$-f + gu = 0, \qquad fu - gc^2 = 0,$$

which only coincide if $f = \pm gc$ and in which case necessarily $u = \pm c$, noting, however, that from (4.101), namely, $du/dt = -1/A''(u)$, so that within this parametric formulation, $A''(u) = 0$ is not an option unless $u = \pm c$ and the rest energy e_0 also vanishes in such a manner as to be physically meaningful. However, clearly the intent of this parametric formulation is to exploit u as a variable parameter, and therefore the approach is not sensible with $u = \pm c$.

Prescribed Applied Forces $f(u) = f_0(1 + \lambda u/c)$ and $cg(u) = f_0(\lambda + u/c)$ with (4.90) In the following chapters, we examine in some detail an exact solution arising from assuming the particular prescribed applied forces $f(u) = f_0(1+\lambda u/c)$ and $cg(u) = f_0(\lambda + u/c)$ where f_0 and λ denote arbitrary constants. Here as an illustrative example, we adopt the same applied forces in conjunction with the two equations (4.90). The eventual outcome is unchanged from the above outcome for arbitrary applied forces $f(x,t)$ and $g(x,t)$, except that the details are more transparent resulting in a more insightful calculation. From the force relations $f(u) = f_0(1 + \lambda u/c)$, $cg(u) = f_0(\lambda + u/c)$ and (4.90), the left-hand sides of this equation become

$$xf - c^2tg = f_0\left(A(u) + \frac{\lambda B(u)}{c}\right), \qquad xg - tf = \frac{f_0}{c^2}\left(B(u) + \lambda c A(u)\right),$$

while from (4.93) we obtain the following expressions for the partial derivatives $\partial u/\partial x$ and $\partial u/\partial t$; thus

$$\frac{\partial u}{\partial x} = \frac{cf_0\lambda}{e}\left\{1 - \left(\frac{u}{c}\right)^2\right\}, \qquad \frac{\partial u}{\partial t} = \frac{c^2 f_0}{e}\left\{1 - \left(\frac{u}{c}\right)^2\right\},$$

Using these expressions and (4.103) for the derivatives $\xi'(u)$ and $\eta'(u)$, we find that the right-hand sides of (4.90) simplify to give

$$\frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x} = f_0\left(c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\}A''(u) + A(u)\right) + \frac{f_0\lambda B(u)}{c}, \tag{4.111}$$

$$\frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x} = \frac{f_0 B(u)}{c^2} + \frac{f_0\lambda}{c}\left(c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\}A''(u) + A(u)\right),$$

so that on comparing the left- and right-hand sides of (4.90), we again require that the term involving $A''(u)$ must vanish, and accordingly bearing in mind the assumption that $u(x,t) \neq \pm c$, there are again no solutions of these mechanical equations with Lorentz invariants that are functions of u only.

We observe that in the case when the first Lorentz invariant is constant, $\xi(u) = \xi_0$, so that $A(u) = \xi_0(1 - (u/c)^2)^{1/2}/e_0$, and then we have

$$c^2\left\{1 - \left(\frac{u}{c}\right)^2\right\}A''(u) + A(u) = -\frac{\xi_0(u/c)^2}{e_0(1 - (u/c)^2)^{1/2}}, \qquad B(u) = -\frac{\xi_0 u(u/c)^2}{e_0(1 - (u/c)^2)^{1/2}},$$

so that Eqs. (4.111) become

$$\frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x} = -\frac{f_0\xi_0(u/c)^2}{e_0(1 - (u/c)^2)^{1/2}}\left(1 + \frac{\lambda u}{c}\right) = -\frac{\xi_0(u/c)^2 f(u)}{e_0(1 - (u/c)^2)^{1/2}},$$

$$\frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x} = -\frac{f_0\xi_0(u/c)^2}{ce_0(1 - (u/c)^2)^{1/2}}\left(\lambda + \frac{u}{c}\right) = -\frac{\xi_0(u/c)^2 g(u)}{e_0(1 - (u/c)^2)^{1/2}}.$$

Arbitrary Applied Forces $f(x,t)$ and $g(x,t)$ with (4.91) On using the results that

$$\frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} = \xi'(u)\left(\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) + \xi''(u)\left(\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right),$$

$$\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial x^2} = \eta'(u)\left(\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) + \eta''(u)\left(\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right),$$

and (4.95) along with the two expressions (4.94) and (4.96), we may eventually show that Eqs. (4.91) become

$$-2f = \frac{c^2(f^2-(gc)^2)}{e}\left\{1-\left(\frac{u}{c}\right)^2\right\}A''(u),$$

$$-2g = \frac{c^2(f^2-(gc)^2)}{e}\left(\left\{1-\left(\frac{u}{c}\right)^2\right\}^2 A''(u)\right)' + \left(\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\right)\left\{1-\left(\frac{u}{c}\right)^2\right\}^2 A''(u),$$

where we have frequently used the basic relations and their derivatives (4.99) in the form

$$\xi(u) = \frac{e_0 A(u)}{(1-(u/c)^2)^{1/2}}, \qquad \eta(u) = \frac{e_0 B(u)}{c^2(1-(u/c)^2)^{1/2}},$$

and the two equations (4.102) and (4.108) connecting the functions $A(u)$ and $B(u)$.

4.10 Hamiltonian for One Space Dimension

In both classical and quantum mechanics, the variational principles underlying Lagrange's equations and Hamilton's equations constitute both important general mechanical principles and correspondingly important general methods for the solution of problems. In this and the following section, we briefly discuss the Hamiltonian and Lagrangian approaches for dual particle-wave mechanics. The Hamiltonian formulation for dual particle-wave mechanics is most easily introduced for a single spatial dimension x. In this section we need to be alive to the fact that there are two distinct partial derivatives: one referring to independent variables (x, t) and one referring to independent variables (x, p, t). Accordingly, in this section we adopt the convention that the full symbols $\partial/\partial x$ and $\partial/\partial t$ refer to independent variables (x, t), while subscripts refer to independent variables (x, p, t).

Assuming the existence of a Hamiltonian function $H = H(x, p, t)$, and then on taking the total derivative of this function with respect to time, we obtain

$$\frac{dH}{dt} = H_t + H_x\frac{dx}{dt} + H_p\frac{dp}{dt},$$

and on using $u = dx/dt$ with the relations arising from (3.15) and assuming the Hamiltonian relations, thus

$$u = \frac{dx}{dt} = H_p, \qquad f = \frac{dp}{dt} = -H_x, \qquad gc^2 = \frac{d\mathscr{E}}{dt} = -H_t, \tag{4.112}$$

we may deduce

$$\frac{dH}{dt} = H_t + uH_x + fH_p = H_t = -\frac{d\mathscr{E}}{dt}, \tag{4.113}$$

and since $dH/dt \neq 0$, the system is nonconservative in the conventional sense with energy integral $H + \mathscr{E} = constant$. We note that in the derivation of Eq. (4.113), we are putting space and time on equal footings and assuming that the force g in the direction of time is derivable from the Hamiltonian in the same manner as the spatial force f, that is, we are assuming the relation $gc^2 = -H_t$. Thus in summary, from (4.113) we may conclude that $H + \mathscr{E} = constant$, and we have the two relations,

$$f = -H_x, \qquad gc^2 = -H_t,$$

which are evidently formally equivalent to the existence of a work done function $W(x,t)$ (Eq. (3.8)) or a potential energy function $V(x,t)$ (Eq. (3.25)). Accordingly, we now assume the compatibility constraint $\partial f/\partial t = c^2 \partial g/\partial x$ for the existence of the work function $W(x,t)$ is satisfied, or that there exists a potential $V(x,t)$ such that

$$f = -\frac{\partial V}{\partial x}, \qquad gc^2 = -\frac{\partial V}{\partial t},$$

and therefore on combining these equations, we obtain

$$\frac{\partial V}{\partial x} = H_x, \qquad \frac{\partial V}{\partial t} = H_t.$$

On evaluating the partial derivatives $\partial H/\partial x$ and $\partial H/\partial t$, we have, respectively,

$$\begin{aligned} \frac{\partial H}{\partial x} &= H_x + H_p \frac{\partial p}{\partial x} = \frac{\partial V}{\partial x} + H_p \frac{\partial p}{\partial x}, \\ \frac{\partial H}{\partial t} &= H_t + H_p \frac{\partial p}{\partial t} = \frac{\partial V}{\partial t} + H_p \frac{\partial p}{\partial t}, \end{aligned} \tag{4.114}$$

so that on equating expressions for the mixed partial derivative $\partial^2 H/\partial x \partial t$, we might readily deduce that the following Jacobian vanishes; thus

$$\frac{\partial(H_p, p)}{\partial(x,t)} = \frac{\partial H_p}{\partial x}\frac{\partial p}{\partial t} - \frac{\partial H_p}{\partial t}\frac{\partial p}{\partial x} = 0,$$

and therefore the compatibility condition $\partial f/\partial t = c^2 \partial g/\partial x$ is equivalent to the condition $H_p = \Phi'(p)$ where Φ' denotes an arbitrary function of p. In this case

Eqs. (4.114) partially integrate immediately to give the Hamiltonian $H = V + \Phi(p)$, which is entirely consistent with the Hamiltonian $H = e + V$ where e is the conventional particle energy considered here, namely, $e = (e_0^2 + (pc)^2)^{1/2}$, as might be anticipated.

4.11 Lagrangian for One Space Dimension

In this section we provide the Lagrangian formulation for the dual particle-wave mechanics in a single spatial dimension. Assuming the existence of a Lagrangian function $L(x, p, t)$ and with the convention for partial derivatives described in the previous section, the standard relations connecting the Lagrangian and Hamiltonian functions become

$$p = L_u, \qquad H = uL_u - L = pH_p - L,$$

and therefore the Lagrangian function $L(x, p, t)$ for a single spatial dimension is defined in terms of the Hamiltonian function $H = H(x, p, t)$ by the equation

$$L = pH_p - H = pu - H.$$

On taking the total time derivative of this equation and making use of the basic relations (4.112) for the Hamiltonian, we might deduce

$$\frac{dL}{dt} = \frac{d(pu)}{dt} - \frac{dH}{dt} = \frac{dp}{dt}u + p\frac{du}{dt} + \frac{d\mathscr{E}}{dt} = \frac{d(pu + \mathscr{E})}{dt} = \frac{d(mu^2 + \mathscr{E})}{dt},$$

and therefore apart from an arbitrary constant, the Lagrangian function $L(x, p, t)$ for dual particle-wave mechanics is given by

$$L = mu^2 + \mathscr{E} = mu^2 - e - V = mu^2 - mc^2 - V = -\frac{m_0c^2(1 - (u/c)^2)}{(1 - (u/c)^2)^{1/2}} - V, \tag{4.115}$$

on assuming a potential energy $V(x, t)$ giving rise to an energy integral $e + \mathscr{E} + V =$ *constant*. From this latter equation, we might readily deduce

$$L = -e_0(1 - (u/c)^2)^{1/2} - V, \tag{4.116}$$

where $e_0 = m_0c^2$ denotes the rest mass energy, and the given expression for the Lagrangian function coincides precisely with the conventional expression for special relativistic mechanics (see for example [66], page 26). On using the well-known identity $e = pu + e_0(1 - (u/c)^2)^{1/2}$, Eq. (4.116) becomes simply $L = pu + \mathscr{E}$, in complete accord with $(4.115)_1$.

Chapter 5
Exact Wave-Like Solution

5.1 Introduction

In this and the following two chapters, we consider in some detail an exact relativistic wave-like solution for the proposed model that is defined by Eqs. (3.4). We seek a formal mathematical solution of (3.4) for which the forces f and g and particle energy e are all functions of the particle velocity u only. In this chapter we present the major solution details and some implications of the solution. In the following chapter, we focus on the mathematical formalities and present the solution derivation and the evaluation of various integrals for the de Broglie wave energy. The chapter thereafter deals with Lorentz invariances and functional dependence of the linear forces f and g.

In the following section of this chapter, we simply state the wave-like solution and its major characteristics. In the subsequent section, assuming the linear force relations (5.2), we present a detailed derivation of the work done function $W(x, t)$ (potential energy $V(x, t) = -W(x, t)$) which is basically the sum of the two energies e and $\mathscr{E}$ arising from energy conservation. In the section thereafter, we provide an elementary validation of the solution, and in the next section, we examine its relation to a well-known solution of special relativity. In the following two sections of the chapter, we show that the exact solution gives rise to an expression for the Hubble parameter, and we evaluate certain integrals required for the Hubble parameter formula. In the final section of the chapter, we put forward a tentative proposal for dark matter and dark energy as de Broglie states characterised by force relations $f = \pm cg$ and $e_0 = 0$, with particle and wave energies either coinciding or in balance under a zero potential; thus $e = \pm\mathscr{E}$.

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5.2 Wave-Like Solution

In order to deduce a solution of (3.4) that corresponds to Einstein's simple formula $e = mc^2$, namely, one for which both momentum and energy are functions of velocity u only, we show in the subsequent section that only wave-like solutions are possible. Specifically, we show that for a single spatial dimension x, the only solution for which both momentum and energy are functions of u has velocity $u(x, t)$ given explicitly by

$$u(x,t) = c\left\{\frac{\lambda x + ct}{((e_0/f_0)^2 + (\lambda x + ct)^2)^{1/2}}\right\}, \tag{5.1}$$

where again $e_0 = m_0c^2$ is the rest mass energy and λ and f_0 denote arbitrary constants appearing in the assumed linear force equations

$$f(u) = f_0(1 + \lambda u/c), \qquad cg(u) = f_0(\lambda + u/c). \tag{5.2}$$

We emphasise that with the assumption the momentum $p = mu$ and the particle energy $e = mc^2$ are functions of velocity u only, the present formulation only permits forces $f(u)$ and energy-mass production $g(u)$ that are the above linear functions in the particle velocity u. We observe that from the form of the solution (5.1) for $\lambda \to \pm\infty$, the velocity $u \to \pm c$.

From the above two equations (5.2), it is apparent that the following identity holds:

$$f^2 - (cg)^2 = f_0^2(1 - \lambda^2)(1 - (u/c)^2), \tag{5.3}$$

and to fix ideas, we suppose that $f(u) > 0$ and $g(u) \geq 0$, since we have in mind that particle-like behaviour might occur for $f(u) > cg(u) \geq 0$, so that under the circumstance $f(u) > 0$ and $g(u) \geq 0$, the following allowable options are as follows:

$$\begin{cases} f^2 - (cg)^2 > 0 & \text{if } f - cg > 0 \text{ and } f + cg > 0, \\ f^2 - (cg)^2 < 0 & \text{if } f - cg < 0 \text{ and } f + cg > 0. \end{cases}$$

More generally, Eq. (5.3) gives rise to the following four possibilities:

$$\begin{cases} f^2 - (cg)^2 > 0 & \text{if } \lambda^2 < 1 \text{ and } u/c < 1, \text{ particle-like dynamics}, \\ f^2 - (cg)^2 > 0 & \text{if } \lambda^2 > 1 \text{ and } u/c > 1, \text{ wave-like dynamics}, \\ f^2 - (cg)^2 < 0 & \text{if } \lambda^2 < 1 \text{ and } u/c > 1, \text{ wave-like dynamics}, \\ f^2 - (cg)^2 < 0 & \text{if } \lambda^2 > 1 \text{ and } u/c < 1, \text{ particle-like dynamics}, \end{cases}$$

and of course light-like dynamics occurs for either $\lambda^2 = 1$ or $u/c = \pm 1$ in which case $f = \pm cg$. The above situation suggests that both particle- and wave-like behaviour might occur for all values of $\lambda \neq 1$. However, throughout the book, we principally have in mind that the velocity u represents either a sub-luminal particle velocity with a corresponding superluminal wave velocity c^2/u or a sub-luminal wave velocity with a corresponding superluminal particle velocity c^2/u, so that in either case the substitution $u = c\sin\phi$ is meaningful with ϕ real. In this situation it is clear that both particle-like and wave-like dynamics may occur for both $\lambda^2 < 1$ and for $\lambda^2 > 1$, with light-like dynamics for $\lambda^2 = 1$. The three cases give rise to different expressions for the de Broglie wave energy which are discussed at length in the following chapters. We note that from Eq. (5.3) we may deduce

$$(ef)^2 - (ceg)^2 = (e_0 f_0)^2(1-\lambda^2),$$

where as usual e denotes the particle energy, and this equation connects the two Lorentz invariances ef and eg subsequently examined in Chap. 7.

The two terms of (3.10) produce two extreme limits of the proposed theory, $de = \partial\mathbf{p}/\partial t \cdot \mathbf{dx}$ and $d\mathscr{E} = c^2(\nabla\cdot\mathbf{p})dt$, and correspond to either purely spatial or purely temporal and also arise, respectively, from the above solution for the two limiting parameter values $\lambda = 0$ and $\lambda = \pm\infty$. For a single spatial dimension, these extreme limits become $de = (dp/dt)dx$ and $d\mathscr{E} = c^2(dp/dx)dt$ and with $u = dx/dt$ give rise to the two suggestive equations, $de/dp = u$ and $d\mathscr{E}/dp = c^2/u$, namely, the particle velocity and the de Broglie wave velocity, respectively, and subsequently generate the two expressions

$$e(u) = \frac{e_0}{\left(1-(u/c)^2\right)^{1/2}}, \tag{5.4}$$

and

$$\mathscr{E}(u) = e_0\left\{\frac{1}{\left(1-(u/c)^2\right)^{1/2}} + \log\left(\frac{u/c}{1+\left(1-(u/c)^2\right)^{1/2}}\right)\right\}. \tag{5.5}$$

The first equation is the Einstein expression, and the second is singular at both $u = 0$ and $u = \pm c$, and it is clear that for $0 < u/c < 1$ the logarithm is negative and therefore $\mathscr{E}(u) < e(u)$. This inequality says that it is energetically more favourable to travel solely through time than solely through space, if travelling solely through time were technically feasible. From (5.5) it is not difficult to envisage that extremely large negative energies might be generated for relatively slowly moving bodies. For all other values of λ, we see in the following chapter that there are two distinct energy expressions corresponding to $\pm\lambda$, and it is only at the extremes that the two energy expressions coincide.

The two extreme limits (5.4) and (5.5) arise, respectively, from the differential relations

$$\frac{de}{dp} = u, \qquad \frac{d\mathscr{E}}{dp} = \frac{c^2}{u}, \tag{5.6}$$

so that formally by division we have

$$\frac{d\mathscr{E}}{de} = \left(\frac{c}{u}\right)^2 = \frac{1}{1-(e_0/e)^2} = 1 + \frac{e_0}{2}\left(\frac{1}{e-e_0} - \frac{1}{e+e_0}\right),$$

and on integration we may express $\mathscr{E}$ as a function of e; thus

$$\mathscr{E}(e) = e + \frac{e_0}{2}\log\left(\frac{e-e_0}{e+e_0}\right) + \mathscr{E}_0, \tag{5.7}$$

where $\mathscr{E}_0$ denotes an arbitrary constant. Note that this formal relation only connects the two extreme limiting cases for $\lambda = 0$ and $\lambda = \pm\infty$, and if $\mathscr{E}_0 = 0$, then for $e_0 > 0$ it is clear that $\mathscr{E} < e$, and in this circumstance we would anticipate the appearance of waves rather than particles.

We comment that the expression (5.5) can be alternatively derived directly from

$$d\mathscr{E} = \frac{c^2dp}{u} = \frac{e_0du}{u\left(1-(u/c)^2\right)^{3/2}} = \frac{e_0d\phi}{\sin\phi\cos^2\phi},$$

after making the substitution $u = c\sin\phi$, and this expression readily integrates to yield

$$\mathscr{E}(\phi) = e_0\left\{\frac{1}{\cos\phi} + \log\left(\tan\frac{\phi}{2}\right)\right\} + \mathscr{E}_0,$$

where again $\mathscr{E}_0$ denotes an arbitrary constant. Also making use of the relation $(pc)^2 = e^2 - e_0^2$, the same expression in the form of Eq. (5.7) admits some re-arrangement; thus

$$\mathscr{E}(e) = e + e_0\log\left(\frac{pc}{e+e_0}\right) + \mathscr{E}_0, \quad \mathscr{E}(e) = e - e_0\log\left(\frac{pc}{e-e_0}\right) + \mathscr{E}_0,$$

reflecting an identical dependence on $\pm e_0$.

5.3 Work Done $W(u,\lambda)$ from $\partial f/\partial t = c^2\partial g/\partial x$

The above two limits given by (5.6) arise as special cases of the more general Lorentz invariant equation

$$\frac{d\mathscr{E}}{dp} = c\left(\frac{\lambda + u/c}{1+\lambda u/c}\right), \tag{5.8}$$

involving the arbitrary constant λ for which the above two special cases correspond, respectively, to the values $\lambda = 0$ and $\lambda = \pm\infty$. We observe that Eq. (5.8) can be alternatively derived quite simply by division of the two total derivatives of the two solutions of the wave equation, namely, $p(x, t) = f_0(\lambda x + ct)/c$ and $\mathscr{E}(x, t) = f_0(x + c\lambda t)$; thus

$$d\mathscr{E} = f_0(dx + c\lambda dt), \qquad dp = f_0(\lambda dx + cdt)/c,$$

and therefore with $u = dx/dt$, on division we obtain Eq. (5.8). This approach is described later with reference to a general solution of the wave equation. Also since the operator appearing in (3.10) is Lorentz invariant, and as a result, the work done transfer rate $d\mathscr{E}/dp$ is also Lorentz invariant, and this is also discussed subsequently.

We further comment that from the exact solution $cp(x, t) = f_0(\lambda x + ct)$ and $\mathscr{E}(x, t) = f_0(x + c\lambda t)$, we may deduce the following relations:

$$\begin{cases} \mathscr{E} - cp = f_0(1-\lambda)(x - ct), & \mathscr{E} + cp = f_0(1+\lambda)(x + ct), \\ \mathscr{E}^2 - (cp)^2 = f_0^2(1-\lambda^2)(x^2 - (ct)^2), & \mathscr{E}^2 - e^2 = f_0^2(1-\lambda^2)(x^2 - (ct)^2) - e_0^2. \end{cases}$$

Assuming that the forces f and g given by (5.2) are generated by a potential $V(x, t)$ and given by (3.11), then we have

$$f_0\left(1 + \frac{\lambda u}{c}\right) = -\frac{\partial V}{\partial x}, \qquad f_0 c\left(\lambda + \frac{u}{c}\right) = -\frac{\partial V}{\partial t},$$

and for $u = u(\xi)$ where $\xi = \lambda x + ct$, these two equations are consistent in the sense that the same expression is obtained for the mixed partial derivative $\partial^2 V/\partial x \partial t$. The two equations may then be integrated as follows:

$$\begin{aligned}
-dV(x, t) &= -\frac{\partial V}{\partial x}dx - \frac{\partial V}{\partial t}dt, \\
&= f_0\left(1 + \frac{\lambda u}{c}\right)dx + f_0 c\left(\lambda + \frac{u}{c}\right)dt, \\
&= f_0\left[\left(1 + \frac{\lambda u}{c}\right)\frac{u}{c} + \left(\lambda + \frac{u}{c}\right)\right]cdt, \\
&= f_0\frac{\left[\lambda\,(u/c)^2 + 2(u/c) + \lambda\right]d\xi}{(1 + \lambda u/c)}, \\
&= f_0\frac{\left[(1 + \lambda u/c)\,(\lambda + (u/c)) - (\lambda^2 - 1)(u/c)\right]d\xi}{(1 + \lambda u/c)}, \\
&= f_0\left[\left(\lambda + \frac{u}{c}\right)d\xi - \frac{(\lambda^2 - 1)(u/c)d\xi}{(1 + \lambda u/c)}\right],
\end{aligned}$$

$$= f_0\left[\lambda d\xi + \frac{\xi d\xi}{(\xi_0^2+\xi^2)^{1/2}} - (\lambda^2-1)udt\right],$$

where $\xi_0 = e_0/f_0$, which integrates to give

$$V(x,t) = -f_0\left(\lambda\xi + (\xi_0^2+\xi^2)^{1/2} - (\lambda^2-1)x\right) + V_0,$$

where V_0 denotes an arbitrary constant, and this equation finally simplifies to yield

$$V(x,t) = -f_0\left(x + \lambda ct + (\xi_0^2+\xi^2)^{1/2}\right) + V_0.$$

Since $\mathscr{E}(x,t) = f_0(x + c\lambda t)$ and $e = f_0(\xi_0^2+\xi^2)^{1/2}$, this equation is merely a statement of the conservation of energy $e + \mathscr{E} + V = \text{constant}$. We comment that even though, modulo a constant, the de Broglie wave energy is given simply by $\mathscr{E}(x,t) = f_0(x + c\lambda t)$, the determination of the wave energy as a function of velocity, namely, $\mathscr{E}(u)$, is far more complicated, and this aspect is examined in some detail in the following chapter.

5.4 Simple Derivation of Wave-Like Solution

In the following chapter, we provide a formal derivation (5.1) and (5.2). Here we merely assume the particular expressions (5.2) for the forces f and g and derive the wave-like solution (5.1) from Eq. (7.1); thus

$$\frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = f_0(1+\lambda u/c), \qquad \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = \frac{f_0}{c}(\lambda + u/c).$$

From the two expressions for the momentum and particle energy, $p(x,t) = m_0u/(1-(u/c)^2)^{1/2}$ and $e(x,t) = m_0c^2/(1-(u/c)^2)^{1/2}$ where m_0 denotes the rest mass, we may deduce

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = w_0(1+\lambda u/c)(1-(u/c)^2)^{3/2},$$
$$c^2\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial t} = w_0c(\lambda + u/c)(1-(u/c)^2)^{3/2},$$

where $w_0 = f_0/m_0$. On solving these equations as two equations in the two unknowns $\partial u/\partial x$ and $\partial u/\partial t$, we may deduce the following two expressions:

$$\frac{\partial u}{\partial x} = \frac{\lambda w_0}{c}(1-(u/c)^2)^{3/2}, \qquad \frac{\partial u}{\partial t} = w_0(1-(u/c)^2)^{3/2},$$

which by partial integration can be shown to give $u/(1-(u/c)^2)^{1/2} = w_0(\lambda x+ct)/c$ and from which we may deduce the wave-like solution (5.1). Further, a minor rearrangement of $u/(1-(u/c)^2)^{1/2} = w_0(\lambda x+ct)/c$ gives $p(x,t) = f_0(\lambda x+ct)/c$ which corresponds to $\mathscr{E}(x,t) = f_0(x + c\lambda t)$ where f_0 denotes the arbitrary force constant appearing in the expressions (5.2).

5.5 Relation to Solution of Special Relativity

Equation (5.1) reduces to $u = c$ in the event that the rest mass is zero, namely, $m_0 = 0$, and apparently extends a well-known formula of conventional special relativity that appears in many standard texts on the subject (see for example either [66], page 24, or [26], page 37) arising from assuming a Lorentz invariant constant acceleration w_0. In other words, at each instant of time, in any co-moving frame moving at the same velocity, the particle has the same constant value acceleration w_0, giving rise to the equation

$$u(t) = \frac{w_0 t}{(1+(w_0 t/c)^2)^{1/2}}, \tag{5.9}$$

which is formally derived from the differential relation $\left(1-(u/c)^2\right)^{-3/2} du/dt = w_0$, a relationship applying in any Lorentz frame. On setting $\lambda = 0$ in Eq. (5.1), we may recover (5.9) with the interesting identification of the force and acceleration constants f_0 and w_0, namely, $f_0 = m_0 w_0$. Similarly, with the same identification of constants, we might alternatively deduce (5.1) from the differential relation $m_0\left(1-(u/c)^2\right)^{-3/2} du/dt = f_0(1+\lambda u/c)$, which becomes $dp/dt = f_0(1+\lambda u/c) = f_0 d(t+\lambda x/c)/dt$. On integration and omitting the constant of integration, we might deduce $p(x,t) = f_0(\lambda x+ct)/c$, and (5.1) follows immediately from $u(x,t) = cp(x,t)/\left((m_0 c)^2 + p(x,t)^2\right)^{1/2}$.

5.6 Relation to Hubble Parameter

We may also observe that for sufficiently large values e_0/f_0, the above Eq. (5.1) bears a remarkable resemblance to the Hubble law for receding galaxies. Specifically, for $f_0(\lambda x+ct)/e_0 \ll 1$, Eq. (5.1) becomes simply $u(x,t) \approx f_0 c(\lambda x+ct)/e_0$, and from which we might deduce $H_0 \approx f_0\lambda/m_0 c$ as an approximate expression for the Hubble constant H_0 in terms of the constants f_0 and λ and the mass m_0 of the receding galaxy.

Since the Hubble constant is known to vary over large distances, we have more precisely, on using Eq. (5.1), a one-dimensional Hubble parameter $H(t)$ (see for example [26], page 311) which might be defined by

$$H(t) = \frac{1}{x(t)}\frac{dx}{dt} = c(1-\lambda^2)\left\{\frac{\xi/((e_0/f_0)^2+\xi^2)^{1/2}}{\mathscr{E}(\xi)/f_0 - \lambda\xi + C_1}\right\},$$

where C_1 denotes an arbitrary constant and $\xi = \lambda x + ct$. We are using the fact that the integral of (5.1) is given by

$$x(t) = \frac{\mathscr{E}(\xi)/f_0 - \lambda\xi + C_1}{(1-\lambda^2)}, \tag{5.10}$$

where $\mathscr{E}(\xi)$ denotes the de Broglie wave energy for which explicit expressions, depending on whether $\lambda^2 < 1$ or $\lambda^2 > 1$, are given in the following chapter (see Eqs. (6.13), (6.14), (6.21), and (6.22)). The formula (5.10) is formally derived in the following section, but it can also be deduced from the previously noted relation $\mathscr{E}(x,t) = f_0(x + c\lambda t)$ on elimination of t using $\xi = \lambda x + ct$; thus

$$\mathscr{E}(\xi) = f_0(x + \lambda(\xi - \lambda x) - C_1) = f_0((1-\lambda^2)x + \lambda\xi - C_1),$$

and since in terms of the angle ϕ defined by $u = c\sin\phi$, we have $\xi = (e_0/f_0)\tan\phi$, and the following expression may be deduced for the Hubble parameter:

$$H(t) = \frac{1}{x(t)}\frac{dx}{dt} = cf_0(1-\lambda^2)\left\{\frac{\sin\phi}{\mathscr{E}(\phi) - \lambda e_0\tan\phi + C_2}\right\}, \tag{5.11}$$

where C_2 denotes a different arbitrary constant ($C_1 f_0$), and explicit expressions for $\mathscr{E}(\phi)$ are given above for the two cases $\lambda^2 < 1$ and $\lambda^2 > 1$ (see Eqs. (6.13) and (6.14)). For $\lambda^2 \approx 1$ we may deduce from the above approximation for $\mathscr{E}(\phi)$ given by (6.19) the simple approximate expression for the Hubble parameter

$$H(t) \approx \frac{cf_0}{e_0}\frac{\cos\phi(\lambda+\sin\phi)}{(1+C_3(\lambda+\sin\phi)\cos\phi)} = \frac{cf_0}{e_0}\frac{\left(1-(u/c)^2\right)^{1/2}(\lambda+(u/c))}{\left(1+C_3(\lambda+(u/c))\left(1-(u/c)^2\right)^{1/2}\right)},$$

valid for $\lambda^2 \approx 1$ and where C_3 denotes another arbitrary constant ($C_2/(1-\lambda^2)e_0$), and noting that the value $H_0 \approx f_0\lambda/m_0c$ arises in the limit u tending to zero with the constant C_3 zero, or alternatively we have the value $H_1 \approx c\lambda/\xi_0(1+\lambda C_3)$ for C_3 non-zero.

5.7 Derivation of Integral for Hubble Formula

Here we give a formal derivation of the integral (5.10) arising from Eq. (5.1) that is required for the formula (5.11) for the Hubble parameter. With $u(x,t) = dx/dt$ and $\xi = \lambda x + ct$, we find that (5.1) becomes

$$\frac{d\xi}{dt} = \frac{c\lambda\xi}{((e_0/f_0)^2 + \xi^2)^{1/2}} + c = c\left(\frac{\lambda\xi + ((e_0/f_0)^2 + \xi^2)^{1/2}}{((e_0/f_0)^2 + \xi^2)^{1/2}}\right),$$

and with the substitution $\xi = (e_0/f_0)\tan\phi$ yields

$$e_0 \int \left(\frac{\sec^2\phi}{1 + \lambda\sin\phi}\right) d\phi = cf_0 t + constant. \tag{5.12}$$

Now from the following chapter, various expressions for the de Broglie wave energy $\mathscr{E}(\phi)$ are evaluated for the integral

$$\mathscr{E}(\phi) = e_0 \int \left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right) \sec^2\phi d\phi, \tag{5.13}$$

and therefore we write (5.12)

$$e_0 \int \left(\frac{\lambda + \sin\phi - \sin\phi}{1 + \lambda\sin\phi}\right) \sec^2\phi d\phi = c\lambda f_0 t + constant.$$

With the integral $I(\phi)$ defined by

$$I(\phi) = e_0 \int \left(\frac{\lambda\sec^2\phi}{1 + \lambda\sin\phi}\right) d\phi,$$

we may deduce

$$I(\phi) = e_0 \int \left(\frac{\lambda + \sin\phi - \sin\phi}{1 + \lambda\sin\phi}\right) \sec^2\phi d\phi,$$

and therefore

$$I(\phi) = \mathscr{E}(\phi) - \frac{e_0}{\lambda} \int \left(\frac{1 + \lambda\sin\phi - 1}{1 + \lambda\sin\phi}\right) \sec^2\phi d\phi.$$

This equation simplifies to yield

$$I(\phi) = \mathscr{E}(\phi) - \frac{e_0}{\lambda}\tan\phi + \frac{1}{\lambda^2} I(\phi) + constant,$$

and from this equation and (5.12), we can deduce

$$\frac{(\mathscr{E}(\xi) - f_0\xi/\lambda)}{(1 - 1/\lambda^2)} = cf_0 t + constant = \lambda f_0(\xi - \lambda x) + constant,$$

which can be simplified to yield (5.10).

5.8 Dark Matter and Dark Energy as de Broglie States

The Lorentz invariant extension of Newton's second law given by Eqs. (3.4) naturally admits two singular or privileged states, so that it does not require too much of a leap in one's imagination to envisage that possibly these two singular states may well correspond to the two states which are commonly referred as dark energy and dark matter. In this section, we speculate that this indeed might be the case, but at this point in time, it is necessarily a matter of speculation, and much more future work is required. Nevertheless given the preponderance of dark energy and dark matter in the universe, it is likely that any ultimate resolution will require a comparable rethinking of Newton's second law and that the dark issues of mechanics will emerge as essentially artefacts of the mechanical accounting.

We have previously noted that Einstein's particle energy statement $e^2 = {e_0}^2 + (pc)^2$, where $e_0 = m_0c^2$ denotes the particle rest mass energy, logically admits four distinct types of matter. Either the rest mass energy e_0 is zero or non-zero and so gives rise to precisely four distinct types of matter:

$$e = \begin{cases} ({e_0}^2 + (pc)^2)^{1/2} & \text{if } e_0 \neq 0 \text{ baryonic matter,} \\ pc & \text{if } e_0 = 0 \text{ dark or invisible matter,} \\ -pc & \text{if } e_0 = 0 \text{ dark energy,} \\ -({e_0}^2 + (pc)^2)^{1/2} & \text{if } e_0 \neq 0 \text{ anti-matter,} \end{cases} \tag{5.14}$$

with the evident inequalities (Fig. 5.1)

$$-({e_0}^2 + (pc)^2)^{1/2} \leqslant -pc \leqslant pc \leqslant ({e_0}^2 + (pc)^2)^{1/2},$$

and this interpretation constitutes the underlying basis for the work described in [47–52]. We view baryonic matter as the most energetic form of matter which is consistent with the Einstein picture that we should envisage baryonic matter energetically as a form of an energy-battery. From the above inequalities, this picture is also consistent with the cited percentages of matter in the universe and the fact that each of the higher energy states has a ready access to the lower energy states in the context of the allowable occupancies indicated in Table 5.1. The table shows, for example, that a baryonic particle has the potential to be in any of the four states. Of course however, depending upon local conditions, all energy states are allowable, but in a natural environment, we might expect the lowest energy state to be the most occupied.

The question arises as to whether this approach sheds any light on the particular structure of the latest Planck data, either as a three-state universe or as in the above interpretation as a four-state universe. The approximate known fractions given in the table are based upon the latest Planck data [2] for our universe provided by the Planck space craft observations between 2009 and 2015, which gives the relative fractions for a three-state universe, namely, dark energy, dark matter and ordinary

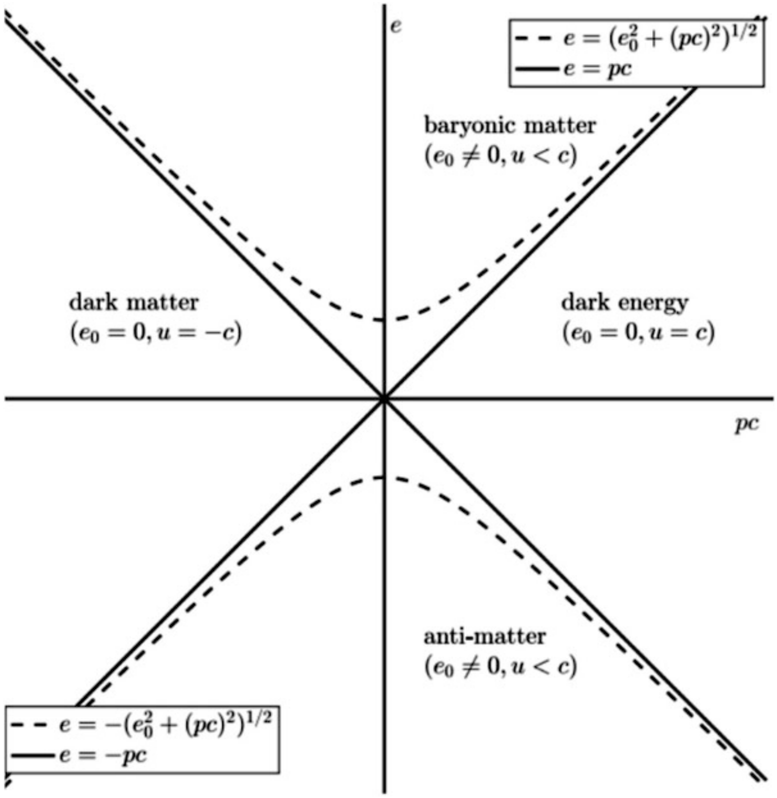

Fig. 5.1 Energy-momentum identification of the four types arising from $-(e_0{}^2 + (pc)^2)^{1/2} \leqslant -pc \leqslant pc \leqslant (e_0{}^2 + (pc)^2)^{1/2}$

or baryonic matter comprise, respectively, 68.3, 26.8 and 4.9% of all matter in the universe, and this data has been modified for a four-state universe [52].

Specifically, the Planck data [2] gives the relative fractions for baryonic matter $\alpha^* = 0.049$ per unit, dark matter $\beta^* = 0.268$ per unit and dark energy $\gamma^* = 0.683$ per unit, where α^*, β^* and γ^* are such that $\alpha^* + \beta^* + \gamma^* = 1$. As noted in [52], it seems to have gone unmentioned that this new data very nearly satisfies the relationship $2\alpha^*\gamma^* \approx \beta^{*2}$, namely, $2\alpha^*\gamma^* = 0.067$, as compared to $\beta^{*2} = 0.072$. It also seems to have gone unnoticed that this relationship still holds true if there were a fourth component of matter, say antimatter, such that $\alpha + \beta + \gamma + \delta = 1$, since then the ratios $\alpha^* = \alpha/(1 - \delta)$, $\beta^* = \beta/(1 - \delta)$ and $\gamma^* = \gamma/(1 - \delta)$ are such that $\alpha^* + \beta^* + \gamma^* = 1$ and moreover still satisfy the same formal approximate relationship $2\alpha^*\gamma^* \approx \beta^{*2}$. This opens up the prospect of having under consideration four fundamental components of matter, namely, baryonic matter; dark matter sometimes referred to as invisible matter; dark energy; and antimatter.

Table 5.1 Allowable occupancies for four energy states

Energy state	Fractions	Allowable	Predicted
Baryonic matter	≈0.0262	1 0 0 0	0.1
Dark matter	≈0.1433	1 1 0 0	0.2
Dark energy	≈0.3652	1 1 1 0	0.3
Antimatter	≈0.4653	1 1 1 1	0.4

Thus with baryonic fraction α, dark matter β, dark energy γ and antimatter δ, where $\alpha + \beta + \gamma + \delta = 1$, if we suppose that $2\alpha\gamma \approx \beta^2$, then, since the four energy states mirror each other in sign, it seems perfectly reasonable to suggest that by symmetry we also have the approximate relation $2\delta\beta \approx \gamma^2$. We now make use of the following relations and the Planck data

$$\alpha^* = \frac{\alpha}{1-\delta} = 0.049, \qquad \beta^* = \frac{\beta}{1-\delta} = 0.268, \qquad \gamma^* = \frac{\gamma}{1-\delta} = 0.683,$$

and use the relation $2\delta\beta \approx \gamma^2$ to determine the value of δ; thus $2\delta\beta^* \approx \gamma^{*2}(1-\delta)$, and from which we may deduce the determining equation for the fraction δ; thus

$$\delta \approx \frac{\gamma^{*2}}{\gamma^{*2} + 2\beta^*} = \frac{1}{1 + 2\beta^*/\gamma^{*2}}.$$

On adopting the above values for β^* and γ^*, we may determine an approximate value of δ from this equation and then determine the remaining as

$$\alpha \approx 0.0262, \qquad \beta \approx 0.1433, \qquad \gamma \approx 0.3652, \qquad \delta \approx 0.4653,$$

which are more or less in accord with the predicted values arising from the allowable occupancies, namely,

$$\alpha = 0.1, \qquad \beta = 0.2, \qquad \gamma = 0.3, \qquad \delta = 0.4.$$

We observe from Eq. (5.8) that de Broglie relation $\mathscr{E} = cp$ for the electron and the companion relation $\mathscr{E} = -cp$ arise immediately from (5.8) with $\lambda = \pm 1$, since from (5.8) with $\lambda = \pm 1$, we may readily deduce $d\mathscr{E} = \pm c dp$ and $\mathscr{E} = \pm cp$ follows immediately. Further, it is also apparent from (5.2) that these two de Broglie states are characterised by the conditions $f = \pm gc$, arising from the identity (5.3), namely,

$$f^2 - (cg)^2 = f_0^2(1-\lambda^2)\left(1 - (u/c)^2\right),$$

indicating that $f = \pm cg$ if either $\lambda = \pm 1$ or $u = \pm c$.

de Broglie states also arise from the above general solution $p(x,t) = F(ct + x) + G(ct - x)$ and $\mathscr{E}(x,t) = c\,(F(ct+x) - G(ct-x))$ for arbitrary functions F

and G, by assuming only one family of characteristics. Assuming only negatively inclined characteristics, we have $p(x,t) = F(ct+x)$ and $\mathscr{E}(x,t) = cF(ct+x)$, while assuming only positively inclined characteristics yields $p(x,t) = G(ct-x)$ and $\mathscr{E}(x,t) = -cG(ct-x)$, from which the relations $\mathscr{E} = \pm cp$ are apparent.

The special states also arise from Eq. (8.37) assuming either of the two force relations $f = cg$ or $f = -cg$, since if $f = cg$, then $e = cp + A(\alpha)$, and if $f = -cg$, then $e = -cp + B(\beta)$ where $A(\alpha)$ and $B(\beta)$ denote, respectively, arbitrary functions of $\alpha = ct + x$ and $\beta = ct - x$. This is because from the functional relation $e^2 = e_0^2 + (pc)^2$, we may conclude that in both cases both $p(x,t)$ depend upon a single characteristic only, and formally in the two cases, we may deduce

$$p(x,t) = \frac{e_0^2 - A(ct+x)^2}{2cA(ct+x)}, \qquad p(x,t) = \frac{B(ct-x)^2 - e_0^2}{2cB(ct-x)},$$

indicating explicitly that in both cases that $p(x,t)$ is a function of a single characteristic only, so that each is equivalent to one of the de Broglie relations $\mathscr{E} = \pm cp$.

Consolidating these results we propose the following consistent mathematical picture motivated from the results of this section. Since it is believed that there is more dark energy in the universe than dark matter, we propose that dark matter as a positive gravity force is characterised by $\mathscr{E} = cp$ and arises as an essentially backward wave accruing from time past. On the other hand, we propose that dark energy as an antigravity force is characterised by $\mathscr{E} = -cp$ arising as an essentially forward wave occurring in consequence of future time. We propose that dark energy and dark matter arise when there is a particular alignment of the physical force f with the force g in the direction of time, so that the particle energy e and wave energy $\mathscr{E}$ coincide; thus $e = \mathscr{E}$. In a real circumstance, we might expect a situation comparable to a "fuzzy region" where the key equalities are constantly switching on and off dependent upon a varying local environment.

Specifically, we propose dark matter to be a backward wave occurring whenever $f = cg$ for which

$$e(x,t) = cp(x,t), \qquad \mathscr{E}(x,t) = cp(x,t), \qquad p(x,t) = F(\alpha),$$

where $\alpha = ct + x$ and $F(\alpha)$ denotes an arbitrary function. Similarly, we propose that dark energy is a forward wave occurring whenever $f = -cg$ for which

$$e(x,t) = -cp(x,t), \qquad \mathscr{E}(x,t) = -cp(x,t), \qquad p(x,t) = G(\beta),$$

where $\beta = ct - x$ and $G(\beta)$ denotes an arbitrary function, further noting that in both cases the rest mass energy e_0 vanishes and that since in both cases we have $e = \mathscr{E}$, we are assuming that the underlying potential $V(x,t)$ is generated from the conservation of energy statement, namely, $V(x,t) = -2e(x,t) + constant$. Also as noted above, we have the more general results that $f = cg$ when $e = pc + A(\alpha)$, and $f = -cg$, when $e = -pc + B(\beta)$, where $A(\alpha)$ and $B(\beta)$ denote arbitrary functions of their arguments.

We provide the following reasoning in support of the above identification of dark matter and dark energy:

- The overall picture is suggested from the particular mathematical framework under investigation.
- The identification of dark matter and dark energy arises from special or singular states that are permitted within the theory.
- The identification of dark matter as a backward wave and dark energy as a forward wave arises in consequence that there is believed to be more dark energy in the universe than dark matter, which is equivalent to saying that time past is less than future time.
- The particular identification of dark matter by the equation $e = pc$ is motivated from our present understanding of dark matter within conventional theory that it is characterised by a positive gravity effect.
- On the other hand, the particular identification of dark energy by the equation $e = -pc$ is motivated from our present understanding that dark energy is an antigravity force within conventional theory and that it is characterised by a peculiar equation of state involving a minus sign (private communication).
- The combined identification of dark energy and dark matter by the respective equations $e = -pc$ and $e = pc$ secures dark energy as the lower energy state, and therefore it is energetically more accessible.

Of course, in the absence of hard numerical evidence, the above proposal must be regarded as necessarily speculative. It does, however, provide a model capable of direct testing, and while it may not be completely in accord with such findings, it is highly likely that dark matter and dark energy arise in consequence of an artefact of the mechanical equations and that their formal origin lies in the current mechanical models neglecting the wave energy. In the proposed model it is suggested that the most likely explanation of dark energy and dark matter is that they arise from a balancing of particle and wave energies $e = \mathscr{E}$ which are supported by a potential $V = -2e$. However, it is also apparent that another possibility allowed in the present model is $e = -\mathscr{E}$ operating under zero potential $V = 0$, which might well explain the perceived preponderance of dark energy and dark matter in the universe.

Chapter 6
Derivations and Formulae

6.1 Introduction

In this chapter we present the formal derivation and mathematical details of the relativistic wave-like solution given by Eq. (5.1) discussed in the previous chapter. We first derive the solution and then present various details relating to the integrals and formulae for the de Broglie wave energy. As we have previously mentioned, even though the de Broglie wave energy is given simply by $\mathscr{E}(x, t) = f_0(x + c\lambda t)$, the determination of the wave energy as a function of velocity $\mathscr{E}(u)$ is far more complicated, and there appears to be no simple single expression such as that for the particle energy, namely, $e = e_0[1 - (u/c)^2]^{-1/2}$. In the final section of this chapter, for the sake of completeness, we present an alternative derivation that is based on the hyperbolic representation $u = c \tanh \theta$ rather than the trigonometric representation $u = c \sin \phi$. The following chapter deals with Lorentz invariances and functional dependence of the assumed underlying linear forces f and g given by (5.2).

Now since the de Broglie wave energy $\mathscr{E}(\phi)$ involves both an arbitrary additive constant and the inverse tangent function admits many curious features, the bulk of this chapter relates to the various seemingly differing expressions for $\mathscr{E}(\phi)$ that are obtained from the integral (5.13) by performing the integration in different ways. In each case we provide an independent demonstration that the various expressions do indeed coincide. As noted above we employ both the trigonometric representation $u = c \sin \phi$ and the hyperbolic representation $u = c \tanh \theta$. However, for clarity of presentation, we present here a summary of the major results obtained in this chapter with all formulae expressed in terms of the angle ϕ for which it may be useful to have in mind the following elementary results:

$$\tan\left(\frac{\phi}{2}\right) = \frac{\sin \phi}{(1 + \cos \phi)} = \frac{(1 - \cos \phi)}{\sin \phi},$$

J. M. Hill, *Mathematics of Particle-Wave Mechanical Systems*,
https://doi.org/10.1007/978-3-031-19793-2_6

and

$$\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^{1/2} = \frac{\cos\phi}{1-\sin\phi} = \frac{(1+\sin\phi)}{\cos\phi}.$$

For $\lambda^2 < 1$ the three major formulae obtained are as follows:

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\frac{(\lambda+\sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi} \right) \right\},$$

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{2\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\frac{\lambda+\tan(\phi/2)}{(1-\lambda^2)^{1/2}} \right) \right\},$$

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{2\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\left(\frac{(1+\lambda)(1+\sin\phi)}{(1-\lambda)(1-\sin\phi)} \right)^{1/2} \right) \right\}.$$

For $\lambda^2 > 1$ we obtain the following three expressions:

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log\left(\frac{(\lambda+\sin\phi) - (\lambda^2-1)^{1/2}\cos\phi}{(\lambda+\sin\phi) + (\lambda^2-1)^{1/2}\cos\phi} \right)^{1/2} \right\},$$

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log\left(\frac{\lambda+\tan(\phi/2) - (\lambda^2-1)^{1/2}}{\lambda+\tan(\phi/2) + (\lambda^2-1)^{1/2}} \right) \right\},$$

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log\left(\frac{\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^{1/2} - \left(\frac{\lambda-1}{\lambda+1}\right)^{1/2}}{\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^{1/2} + \left(\frac{\lambda-1}{\lambda+1}\right)^{1/2}} \right) \right\},$$

where in each case we have omitted an arbitrary additive constant and again noting that $u = c\sin\phi$ and that, contrary to appearances, all expressions are well-defined for the important special case $\lambda = \pm 1$. Further, we have not included the modulus signs in the logarithms, and we are assuming throughout that all logarithms are appropriately well-defined.

Since the leading term for all of the above expressions is the Einstein expression $e(\phi)$, it is apparent that $e(\phi) \le \mathscr{E}(\phi)$ or $\mathscr{E}(\phi) \le e(\phi)$ depending on whether the second term is positive or negative. In order to make a meaningful comparison, we need to prescribe a common datum energy level, say $\mathscr{E}(0) = e_0$. However, it is clear that a number of particular cases arise depending upon the signs of both u/c and λ. For example, for $0 < u/c < 1$ and $0 < \lambda < 1$, with $\mathscr{E}(0) = e_0$ it is apparent from the above three relevant expressions involving the inverse tangent that all the second terms are necessarily positive and therefore $\mathscr{E}(\phi) > e(\phi)$. On the other hand, if $0 < u/c < 1$ and $1 < \lambda < \infty$, then all three of the second terms involving the logarithm are necessarily negative and therefore $\mathscr{E}(\phi) < e(\phi)$, and a number of other special situations arise when either or both of u/c and λ are negative.

6.2 Derivation of Wave-Like Solution

In this section, in attempting to mimic the Einstein formulae for which both momentum $p = mu$ and particle energy $e = mc^2$ are functions of velocity u only, we seek corresponding solutions arising from (3.4) for which the forces f and cg are functions of velocity u only and that we might subsequently use to determine expressions for the de Broglie wave energy $\mathscr{E}$. Here we assume the two basic equations (4.10), and on introducing the angle $\phi(x,t)$ such that $u = c\sin\phi$, we have from $e^2 - (pc)^2 = e_0^2$ the following relations:

$$u = c\sin\phi, \quad m = m_0\sec\phi, \quad e = e_0\sec\phi, \quad pc = e_0\tan\phi, \tag{6.1}$$

where again $e_0 = m_0c^2$ is the rest mass energy. On substitution of these relations into the two basic equations (4.10), we may readily deduce two equations for the determination of the partial derivatives ϕ_x and ϕ_t; thus

$$\phi_t + c\sin\phi\phi_x = a(\phi)\cos^2\phi, \qquad \sin\phi\phi_t + c\phi_x = b(\phi)\cos^2\phi,$$

where we have introduced $a(\phi) = cf(\phi)/e_0$ and $b(\phi) = c^2g(\phi)/e_0$. On solving these equations as two equations in the two unknowns ϕ_x and ϕ_t, we find

$$c\phi_x = b(\phi) - a(\phi)\sin\phi, \qquad \phi_t = a(\phi) - b(\phi)\sin\phi. \tag{6.2}$$

On cross differentiation of these equations and equating two expressions for ϕ_{xt}, we may deduce the following simple equation:

$$\frac{d(b/a)}{d\phi} = \left(1 - (b/a)^2\right)\sec\phi,$$

which may be readily integrated, and further simplification yields

$$\frac{b(\phi)}{a(\phi)} = \left(\frac{\sin\phi + \lambda}{1 + \lambda\sin\phi}\right), \tag{6.3}$$

where λ denotes a nondimensional constant of integration. In terms of the force $f(u)$ and the energy-mass production $g(u)$, we have the implied relation

$$cg(u) = f(u)\left(\frac{u/c+\lambda}{1+\lambda u/c}\right), \tag{6.4}$$

noting that the case $\lambda = 0$ gives $g(u) = f(u)u/c^2$ while the case $\lambda = v/c$, where v denotes the relative frame velocity, produces $g(u) = f(u)U/c^2$ where $U = (u + v)/(1 + uv/c^2)$ arising from (2.6).

On substitution of (6.3) into (6.2) to eliminate $a(\phi)$, we obtain

$$c\phi_x = \frac{b(\phi)\lambda\cos^2\phi}{(\sin\phi+\lambda)}, \qquad \phi_t = \frac{b(\phi)\cos^2\phi}{(\sin\phi+\lambda)}, \tag{6.5}$$

and from these two equations, it is apparent that $c\phi_x = \lambda\phi_t$, so that $\phi(x, t) = \phi(\xi)$ where $\xi = \lambda x + ct$, and indicating that with the assumption that the momentum $p = mu$ and the particle energy $e = mc^2$ are functions of velocity u only, then only wave-like solutions are permitted by this formulation. Further substitution of this expression for $\phi(x, t)$ into either of Eqs. (6.5) yields

$$c\frac{d\phi}{d\xi} = \frac{b(\phi)\cos^2\phi}{(\sin\phi+\lambda)}. \tag{6.6}$$

Now from the one-dimensional version of the compatibility condition Eq. (3.11), namely,

$$\frac{\partial f}{\partial t} = c^2\frac{\partial g}{\partial x}, \tag{6.7}$$

we may deduce $a_t = cb_x$, and from $\phi(x, t) = \phi(\xi)$, we find

$$\frac{da(\phi)}{d\phi} = \lambda\frac{db(\phi)}{d\phi}, \tag{6.8}$$

which together with (6.3) constitutes a second equation for the determination of $a(\phi)$ and $b(\phi)$. On elimination of one of $a(\phi)$ and $b(\phi)$ and integration of the resulting first order ordinary differential equation, or more simply by direct integration of (6.8), we may deduce

$$a(\phi) = a_0(1+\lambda\sin\phi), \qquad b(\phi) = a_0(\sin\phi+\lambda), \tag{6.9}$$

where a_0 denotes the constant of integration. In terms of the force $f(u)$ and the energy-mass production $g(u)$, these equations yield (5.2), that is,

$$f(u) = f_0(1+\lambda u/c), \qquad cg(u) = f_0(\lambda + u/c),$$

where f_0 is a re-defined constant given by $f_0 = e_0a_0/c$, and it is clear from these results that with the assumption that the momentum $p = mu$ and the particle energy

$e = mc^2$ are functions of velocity u only, the formulation only permits forces $f(u)$ and energy-mass production $g(u)$ that are linear functions in the particle velocity u.

From Eqs. (6.6) and (6.9), we may readily deduce $cd\phi/d\xi = a_0 \cos^2\phi$ which readily integrates to yield $c\tan\phi = a_0(\xi - \xi_0)$, where ξ_0 denotes the constant of integration, reflecting the arbitrary choice of coordinate and time origins. On taking $\xi_0 = 0$ the equation for ϕ simplifies to give the above explicit formula (5.1) for the particle velocity $u(x, t)$, namely,

$$u(x,t) = c\left\{\frac{\lambda x + ct}{((e_0/f_0)^2 + (\lambda x + ct)^2)^{1/2}}\right\}, \tag{6.10}$$

where again $e_0 = m_0c^2$ is the rest mass energy and f_0 is the arbitrary constant appearing in Eq. (5.2).

6.3 Expressions for de Broglie Wave Energy

In this section we utilise the solution of the previous section to determine expressions for the de Broglie wave energy $\mathscr{E}$. For one-dimensional motion, we have from (3.10) that the incremental work done $d\mathscr{E} = (\partial p/\partial t)dx + c^2(\partial p/\partial x)dt$ becomes on using $p = mu$

$$\frac{d\mathscr{E}}{dt} = u\frac{\partial p}{\partial t} + c^2\frac{\partial p}{\partial x} = m_0\left\{\frac{u\frac{\partial u}{\partial t} + c^2\frac{\partial u}{\partial x}}{(1-(u/c)^2)^{3/2}}\right\},$$

and on using $u = c\sin\phi$ and $cd\phi/d\xi = a_0\cos^2\phi$, we might deduce

$$\frac{d\mathscr{E}}{dt} = e_0\left\{\frac{\lambda + \sin\phi}{\cos^2\phi}\right\}c\frac{d\phi}{d\xi} = cf_0(\lambda + \sin\phi), \tag{6.11}$$

noting the relation $cf_0 = e_0a_0$. Now on using the relation $d\xi/dt = \lambda u + c$, Eq. (6.11) becomes

$$d\mathscr{E} = f_0\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right)d\xi = e_0\sec^2\phi\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right)d\phi, \tag{6.12}$$

and the formal Lorentz invariance of the transfer rates $d\mathscr{E}/dp$ or $d\mathscr{E}/d\xi$ specifically for $d\mathscr{E}$ arising from (6.12) is examined subsequently. The integral, although elementary, is lengthy, and evaluation involves the substitution $z = \tan\phi$ to yield

$$\int\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right)\sec^2\phi d\phi = \int\left(\frac{\lambda + (1-\lambda^2)z(1+z^2)^{1/2}}{1 + (1-\lambda^2)z^2}\right)dz,$$

and there are two cases to consider. If $\lambda^2 < 1$, we make the substitution $(1 - \lambda^2)^{1/2} z = \tan \Phi$, while if $\lambda^2 > 1$, we make the substitution $(\lambda^2 - 1)^{1/2} z = \sin \Psi$, and the final results are as follows: For $\lambda^2 < 1$ we find

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\frac{(\lambda + \sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi} \right) \right\} + \mathscr{E}_0, \tag{6.13}$$

while if $\lambda^2 > 1$, we obtain

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log\left(\frac{(\lambda + \sin\phi) - (\lambda^2-1)^{1/2}\cos\phi}{(\lambda + \sin\phi) + (\lambda^2-1)^{1/2}\cos\phi} \right)^{1/2} \right\} + \mathscr{E}_0, \tag{6.14}$$

where again $u = c \sin\phi$ and in both cases $\mathscr{E}_0$ denotes a suitable constant. We observe that the underlying symmetry in both (6.13) and (6.14) is $\mathscr{E}(\phi, \lambda) = \mathscr{E}(-\phi, -\lambda)$; that is, the de Broglie wave energy $\mathscr{E}(\phi)$ remains invariant under simultaneous changing of both ϕ and λ to $-\phi$ and $-\lambda$, respectively. We further comment that the two expressions involving the inverse tangent and the logarithm are formally connected through the elementary relation

$$\tan^{-1}(z) = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz} \right),$$

and the logarithm relation valid for $\lambda^2 > 1$ can be formally obtained from that for the inverse tangent valid for $\lambda^2 < 1$ using $(1-\lambda^2)^{1/2} = i(\lambda^2-1)^{1/2}$ and the above elementary relation.

Again, since the inverse tangent function admits many curious features, and in particular satisfies the relation $\tan^{-1}(z) + \tan^{-1}(1/z) = \pi/2$, and the fact that the wave energy is determined modulo an arbitrary constant, there arise differing expressions for $\mathscr{E}(\phi)$ which at first sight might only be reconciled after some thought. However, we may verify by direct differentiation that the above expressions for the de Broglie wave energy are indeed correct as follows: For $\lambda^2 < 1$ we have

$$\begin{aligned}
&\frac{d}{d\phi} \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\frac{(\lambda + \sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi} \right) \right\} \\
&= \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda}{(1-\lambda^2)^{1/2}} \frac{\left(\frac{1}{(1-\lambda^2)^{1/2}} + \frac{(\lambda+\sin\phi)\sin\phi}{(1-\lambda^2)^{1/2}\cos^2\phi} \right)}{\left(1 + \frac{(\lambda+\sin\phi)^2}{(1-\lambda^2)\cos^2\phi} \right)} \\
&= \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda\left(\cos^2\phi + (\lambda + \sin\phi)\sin\phi\right)}{(1-\lambda^2)\cos^2\phi + (\lambda + \sin\phi)^2} \\
&= \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda(1 + \lambda\sin\phi)}{(1 + \lambda\sin\phi)^2} = \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda}{(1 + \lambda\sin\phi)}
\end{aligned}$$

$$= \frac{\sin\phi(1+\lambda\sin\phi)+\lambda\cos^2\phi}{\cos^2\phi(1+\lambda\sin\phi)} = \frac{(\lambda+\sin\phi)}{\cos^2\phi(1+\lambda\sin\phi)},$$

as required. For $\lambda^2 > 1$ we have

$$\frac{d}{d\phi}\left\{\frac{1}{\cos\phi} + \frac{\lambda}{(\lambda^2-1)^{1/2}}\log\left(\frac{\lambda+\sin\phi-(\lambda^2-1)^{1/2}\cos\phi}{\lambda+\sin\phi+(\lambda^2-1)^{1/2}\cos\phi}\right)^{1/2}\right\} = \frac{\sin\phi}{\cos^2\phi}$$

$$+\frac{\lambda}{2(\lambda^2-1)^{1/2}}\left(\frac{\cos\phi+(\lambda^2-1)^{1/2}\sin\phi}{\lambda+\sin\phi-(\lambda^2-1)^{1/2}\cos\phi} - \frac{\cos\phi-(\lambda^2-1)^{1/2}\sin\phi}{\lambda+\sin\phi+(\lambda^2-1)^{1/2}\cos\phi}\right),$$

which after considerable algebra and some simplification becomes exactly as for $\lambda^2 < 1$; thus

$$\frac{\sin\phi}{\cos^2\phi} + \frac{\lambda}{(1+\lambda\sin\phi)} = \frac{\sin\phi(1+\lambda\sin\phi)+\lambda\cos^2\phi}{\cos^2\phi(1+\lambda\sin\phi)}$$
$$= \frac{(\lambda+\sin\phi)}{\cos^2\phi(1+\lambda\sin\phi)},$$

which again is the required expression.

In terms of $e = e_0(1-(u/c)^2)^{-1/2}$, we have the relations

$$\frac{u}{c} = \sin\phi = \frac{(e^2-e_0^2)^{1/2}}{e}, \qquad \left(1-\left(\frac{u}{c}\right)^2\right)^{1/2} = \cos\phi = \frac{e_0}{e},$$

and the following formulae are immediately apparent from Eqs. (6.13) and (6.14). For $\lambda^2 < 1$ we find

$$\mathscr{E}(\phi) = e + \frac{\lambda e_0}{(1-\lambda^2)^{1/2}}\tan^{-1}\left(\frac{\lambda e+(e^2-e_0^2)^{1/2}}{(1-\lambda^2)^{1/2}e_0}\right) + \mathscr{E}_0, \tag{6.15}$$

while if $\lambda^2 > 1$, we obtain

$$\mathscr{E}(\phi) = e + \frac{\lambda e_0}{2(\lambda^2-1)^{1/2}}\log\left(\frac{\lambda e+(e^2-e_0^2)^{1/2}-(\lambda^2-1)^{1/2}e_0}{\lambda e+(e^2-e_0^2)^{1/2}+(\lambda^2-1)^{1/2}e_0}\right) + \mathscr{E}_0, \tag{6.16}$$

where $e = e_0(1-(u/c)^2)^{-1/2}$ and in both cases $\mathscr{E}_0$ denotes a suitable constant. We observe that Eq. (5.7) emerges from Eq. (6.16) in the limit $\lambda \to \pm\infty$, and again we note that we are assuming throughout that the logarithms are always appropriately well-defined, namely, the modulus signs have been omitted from the logarithms.

We observe that from either (6.15) or (6.16), we may deduce

$$\frac{d\mathscr{E}}{dt} = \frac{de}{dt} + \frac{\lambda e_0^2 \frac{de}{dt}}{(e^2 - e_0^2)^{1/2}(e + \lambda(e^2 - e_0^2)^{1/2})},$$

so that from the general equations (3.13) and (3.15), we have

$$gc^2 = \frac{d\mathscr{E}}{dt} = \frac{\partial e}{\partial t} + \nabla \cdot (e\mathbf{u}) = \frac{de}{dt} + e(\nabla \cdot \mathbf{u}),$$

and from $u/c = (e^2 - e_0^2)^{1/2}/e$, for a single spatial dimension x, we might deduce

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} = \frac{e_0^2}{e^2(e^2 - e_0^2)^{1/2}} \frac{\partial e}{\partial x}.$$

Altogether the above three equations become

$$\lambda e \frac{de}{dt} = c(e + \lambda(e^2 - e_0^2)^{1/2}) \frac{\partial e}{\partial x},$$

which further simplifies to give

$$\lambda e \left(\frac{\partial e}{\partial t} + c \frac{(e^2 - e_0^2)^{1/2}}{e} \frac{\partial e}{\partial x} \right) = c(e + \lambda(e^2 - e_0^2)^{1/2}) \frac{\partial e}{\partial x},$$

and this equation finally yields

$$\lambda \frac{\partial e}{\partial t} = c \frac{\partial e}{\partial x}.$$

Thus, the same formulae (6.15) or (6.16) still apply provided that $e(x, t)$ and therefore $p(x, t)$ are functions of the variable $\xi = \lambda x + ct$. Since $p(x, t) = p(\xi)$ satisfies the classical one-dimensional wave equation, this means that $(\lambda^2 - 1)p''(\xi) = 0$, and therefore there are no further new solutions valid for all λ since $p(\xi) = A\xi + B$, where A and B denote arbitrary constants, generates the wave-like solution already examined in detail. However, for the special cases $\lambda = \pm 1$, we have $u = \pm c$, $e = \mathscr{E} = cp(\xi)$ with $p(\xi)$ denoting an arbitrary function of $\xi = ct \pm x$.

The new terms in the work done equation arising from the de Broglie wave energy are identified for the two cases $\lambda^2 < 1$ and $\lambda^2 > 1$, and given explicitly above by (6.13) and (6.14). For both $\lambda^2 < 1$ and $\lambda^2 > 1$, the given expressions become singular, and we use an asterisk to designate the values at the singularity; thus x^*, t^*, p^* and u^*. For $\lambda^2 < 1$ the inverse tangent function becomes unbounded whenever

$$\frac{(\lambda + \sin\phi^*)}{(1 - \lambda^2)^{1/2}\cos\phi^*} = \pm\frac{\pi}{2},$$

which we might square to deduce a quadratic equation in $\sin\phi^*$ which has the following two roots:

$$\sin\phi^* = \frac{(1+(2/\pi)^2)^{1/2}\lambda - 1}{\lambda - (1+(2/\pi)^2)^{1/2}}, \qquad \sin\phi^* = \frac{(1+(2/\pi)^2)^{1/2}\lambda + 1}{\lambda + (1+(2/\pi)^2)^{1/2}},$$

with corresponding values of $\tan\phi^*$ given by

$$\tan\phi^* = \frac{(1+(\pi/2)^2)^{1/2}\lambda - (\pi/2)}{(1-\lambda^2)^{1/2}}, \qquad \tan\phi^* = \frac{(1+(\pi/2)^2)^{1/2}\lambda + (\pi/2)}{(1-\lambda^2)^{1/2}}.$$

Thus the inverse tangent function in (6.13) becomes singular at the values

$$cp^* = f_0\left(\lambda x^* + ct^*\right) = e_0\left(\frac{(1+(\pi/2)^2)^{1/2}\lambda \pm (\pi/2)}{(1-\lambda^2)^{1/2}}\right),$$
$$\frac{u^*}{c} = \frac{(1+(2/\pi)^2)^{1/2}\lambda \pm 1}{\lambda \pm (1+(2/\pi)^2)^{1/2}}.$$

For $\lambda^2 > 1$ the new terms appearing in the work done equation involve the log function, possibly giving rise to massive energies and becoming singular whenever

$$cp^* = f_0\left(\lambda x^* + ct^*\right) = \pm\frac{e_0}{c(\lambda^2-1)^{1/2}}, \qquad \frac{u^*}{c} = \frac{\pm 1}{\lambda},$$

where f_0 is the constant appearing in the force equation (5.2) given above and x^*, t^*, p^* and u^* designate the values at the singularity. The equation $u^* = \pm c/\lambda$ infers that at the singularity the particle velocity u^* coincides with the wave velocity $-c/\lambda$, which means that at this critical juncture, the particles move with the wave and are not left behind.

6.4 de Broglie Wave Energy for Particular λ

We observe that clearly the first terms in both (6.13) and (6.14) correspond to the Einstein contribution and formally arise in the limit $\lambda \to 0$ and in this limit we have

$$\mathscr{E}(\phi) \approx e_0\left(\frac{1}{\cos\phi} + \lambda\phi\right) + \mathscr{E}_0.$$

For $\lambda \to \pm\infty$ we have from (6.14) the following limiting expression:

$$\mathscr{E}(\phi) \approx e_0 \left\{ \frac{1}{\cos\phi} \pm \log\left(\frac{\sin\phi}{1+\cos\phi} \right) \right\} + \mathscr{E}_0 = e_0 \left\{ \frac{1}{\cos\phi} \pm \log\left(\tan\frac{\phi}{2} \right) \right\} + \mathscr{E}_0, \tag{6.17}$$

which for $\lambda \to \infty$ gives rise to Eq. (5.5) and also formally arises directly from (3.10) assuming that all variables are spatially dependent only, so that (3.10) becomes

$$d\mathscr{E} = c^2 \frac{dp}{dx} dt = \frac{e_0 du}{u\left(1-(u/c)^2\right)^{3/2}} = \frac{e_0 d\phi}{\sin\phi\cos^2\phi},$$

after making the substitution $u = c\sin\phi$, and this expression readily integrates to yield (6.17).

Again, on recalling that the inverse tangent function admits a number of curious relations, including $\tan^{-1}(z) + \tan^{-1}(1/z) = \pi/2$, we may if necessary adopt the reciprocal of the argument of the inverse tangent function, together with a change of sign and a redefinition of the arbitrary constant $\mathscr{E}_0$. By this means we observe that in both cases, the additional terms are well-defined for $\lambda = 1$, which is the case of energy-mass waves travelling at the speed of light, and both (6.13) and (6.14) simplify to give

$$\mathscr{E}(\phi) = e_0 \left(\frac{1}{\cos\phi} - \frac{\cos\phi}{1+\sin\phi} \right) + \mathscr{E}_0 = e_0 \tan\phi + \mathscr{E}_0 = e_0 + pc, \tag{6.18}$$

with an appropriate choice for $\mathscr{E}_0$. Similarly, for energy-mass waves travelling at the speed of light in the opposite direction, $\lambda = -1$, and we may deduce

$$\mathscr{E}(\phi) = e_0 \left(\frac{1}{\cos\phi} - \frac{\cos\phi}{1-\sin\phi} \right) + \mathscr{E}_0 = -e_0 \tan\phi + \mathscr{E}_0 = e_0 - pc,$$

and we observe that both formulae are in complete accord with the well-established relations for photons and light, namely, $p = h\nu/c$ and $\mathscr{E} = h\nu$ where h is Planck's constant and ν denotes the frequency, which together yield $\mathscr{E} = pc$ and noting that e_0 is generally adopted to be zero for photons.

Further, assuming that $\lambda^2 \approx 1$, expanding both the inverse tangent and the logarithm and retaining in each case only the leading term in each expansion give rise to the following simple approximate expression:

$$\mathscr{E}(\phi) \approx e_0 \frac{(1+\lambda\sin\phi)\tan\phi}{(\lambda+\sin\phi)} + \mathscr{E}_0, \tag{6.19}$$

valid for both $\lambda^2 < 1$ and $\lambda^2 > 1$, and this expression is exact when $\lambda = 1$ and $\lambda = -1$ giving precisely (6.18) and (10.11), respectively. At the speed of light, we anticipate that both $f = \pm gc$ and $e = \mathscr{E} = \pm pc$, and Eq. (6.19) allows a more precise statement, and there are two possible approaches.

Firstly, if we write this equation as

$$\mathscr{E}(p) \approx e_0 \frac{(1+\lambda(u/c))\tan\phi}{(\lambda+(u/c))} + \mathscr{E}_0,$$

then from $(6.1)_4$ and (5.8), this equation might be rearranged to yield $\mathscr{E} \approx c^2 p(dp/d\mathscr{E}) + \mathscr{E}_0$ which on integration becomes

$$(\mathscr{E}-\mathscr{E}_0)^2 \approx (cp)^2 + \mathscr{E}_1^2, \tag{6.20}$$

where $\mathscr{E}_1$ denotes a further arbitrary constant and the equation is valid for $\lambda^2 \approx 1$. With appropriately chosen constants $\mathscr{E}_0$ and $\mathscr{E}_1$ and in comparison with $e^2 = e_0^2 + (pc)^2$, it is clear that the above Eq. (6.20) admits the possibility that close to the speed of light $\mathscr{E} \approx e$.

Secondly, from Eq. (6.19) and the assumed force relations (5.2)

$$\mathscr{E} \approx \frac{f(u)p}{g(u)} + \mathscr{E}_0 \approx \frac{f(u)e}{cg(u)} + \mathscr{E}_0,$$

since close to the speed of light $e \approx pc$. Thus on taking $\mathscr{E}_0 = 0$, we have the interesting possibility that close to the speed of light, the forces and energies exhibit a balance such that $f(u)e \approx cg(u)\mathscr{E}$.

6.5 Alternative Approach to Evaluation of Integrals

In this section we provide an alternative approach to the evaluation of the energy integrals which also serves to provide an independent confirmation of the de Broglie energy expressions given by Eqs. (6.13) and (6.14). In this section we make use of the variable $\xi = \lambda x + ct$ for which

$$p(x,t) = f_0\xi, \qquad \mathscr{E}(x,t) = f_0(x + c\lambda t), \qquad e(x,t) = f_0(\xi_0^2 + \xi^2)^{1/2},$$

where $\xi_0 = e_0/f_0$, and which give rise to the symmetrical expressions

$$\frac{u}{c} = \frac{\xi}{(\xi_0^2+\xi^2)^{1/2}} = \sin\phi, \qquad \left(1 - \left(\frac{u}{c}\right)^2\right)^{1/2} = \frac{\xi_0}{(\xi_0^2+\xi^2)^{1/2}} = \cos\phi.$$

In terms of the velocity u, the two integrals given above for the combined energy $W = e + \mathscr{E}$ become for $\lambda^2 < 1$

$$W = e_0 \left\{ \frac{2}{(1-(u/c)^2)^{1/2}} + \frac{\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\frac{\lambda + u/c}{(1-\lambda^2)^{1/2}(1-(u/c)^2)^{1/2}} \right) \right\} + \mathscr{E}_0, \tag{6.21}$$

while for $\lambda^2 > 1$ we have

$$W = e_0 \left\{ \frac{2}{(1-(u/c)^2)^{1/2}} + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log \left(\frac{\frac{\lambda+u/c}{(\lambda^2-1)^{1/2}(1-(u/c)^2)^{1/2}} - 1}{\frac{\lambda+u/c}{(\lambda^2-1)^{1/2}(1-(u/c)^2)^{1/2}} + 1} \right)^{1/2} \right\} + \mathscr{E}_0, \tag{6.22}$$

noting that in both cases $\mathscr{E}_0$ denotes a suitable constant and in particular that the arguments of both the inverse tangent and the log functions involve the Lorentz invariant $(\lambda + u/c)/(1-\lambda^2)^{1/2}(1-(u/c)^2)^{1/2}$ which is noted below (see Eq. (7.9)).

We may provide an alternative approach to these integral evaluations through integration using the variable $\xi = \lambda x + ct$. From the basic energy equation (3.8) and the force expressions (5.2) for the wave-like solution, we have

$$dW = f\,dx + gc^2 dt = f_0 \left[\left(1 + \frac{\lambda u}{c}\right) dx + \left(\lambda + \frac{u}{c}\right) c\,dt \right],$$

so that on using $u = dx/dt$ and $d\xi/dt = c + \lambda u$, we may deduce

$$\frac{dW}{d\xi} = f_0 \left(\frac{u}{c} + \frac{\lambda + u/c}{1 + \lambda u/c} \right).$$

From this equation and the relation $u/c = \xi/(\xi_0^2 + \xi^2)^{1/2}$, we have

$$\frac{dW}{d\xi} = f_0 \left(\frac{\xi}{(\xi_0^2 + \xi^2)^{1/2}} + \frac{\xi + \lambda(\xi_0^2 + \xi^2)^{1/2}}{\lambda\xi + (\xi_0^2 + \xi^2)^{1/2}} \right). \tag{6.23}$$

In terms of the variable ξ, the above two energy expressions become for $\lambda^2 < 1$

$$W = e_0 \left\{ \frac{2(\xi_0^2 + \xi^2)^{1/2}}{\xi_0} + \frac{\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1} \left(\frac{\xi + \lambda(\xi_0^2 + \xi^2)^{1/2}}{(1-\lambda^2)^{1/2}\xi_0} \right) \right\} + \mathscr{E}_0,$$

while for $\lambda^2 > 1$ we have

$$W = e_0 \left\{ \frac{2(\xi_0^2 + \xi^2)^{1/2}}{\xi_0} + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log \left(\frac{\frac{\xi + \lambda(\xi_0^2+\xi^2)^{1/2}}{(\lambda^2-1)^{1/2}\xi_0} - 1}{\frac{\xi + \lambda(\xi_0^2+\xi^2)^{1/2}}{(\lambda^2-1)^{1/2}\xi_0} + 1} \right)^{1/2} \right\} + \mathscr{E}_0,$$

and by differentiation of these expressions, it is a straightforward matter to confirm that (6.23) is satisfied in both cases, noting that in the latter case $\xi + \lambda(\xi_0^2 + \xi^2)^{1/2} > (\lambda^2 - 1)^{1/2}\xi_0$ since in terms of ϕ this inequality is equivalent to $(\lambda + \sin\phi) > (\lambda^2 - 1)^{1/2}\cos\phi$, which on simplification is equivalent to $(\lambda \sin\phi)^2 + 2\lambda\sin\phi + 1 =$

$(\lambda \sin\phi + 1)^2 > 0$, which evidently always holds. Both formulae may be readily verified using the following respective results: For $\lambda^2 < 1$ we have

$$\frac{d}{d\xi}\left\{\tan^{-1}\left(\frac{\xi + \lambda(\xi_0^2 + \xi^2)^{1/2}}{(1-\lambda^2)^{1/2}\xi_0}\right)\right\} = \frac{\xi_0(1-\lambda^2)^{1/2}}{\left(\lambda\xi + (\xi_0^2 + \xi^2)^{1/2}\right)(\xi_0^2 + \xi^2)^{1/2}},$$

while for $\lambda^2 > 1$ we have

$$\frac{d}{d\xi}\left\{\log\left(\frac{\frac{\xi+\lambda(\xi_0^2+\xi^2)^{1/2}}{(\lambda^2-1)^{1/2}\xi_0} - 1}{\frac{\xi+\lambda(\xi_0^2+\xi^2)^{1/2}}{(\lambda^2-1)^{1/2}\xi_0} + 1}\right)^{1/2}\right\} = \frac{\xi_0(\lambda^2-1)^{1/2}}{\left(\lambda\xi + (\xi_0^2 + \xi^2)^{1/2}\right)(\xi_0^2 + \xi^2)^{1/2}}.$$

6.6 Alternative Derivation for Wave Energy

In this section we provide yet another method to evaluate the integrals required to determine the de Broglie wave energy and which result in alternative expressions. Starting with the integral (6.12), we perform one integration by parts, to produce

$$\begin{aligned}
&\int \sec^2\phi\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right)d\phi \\
&= \tan\phi\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right) - (1-\lambda^2)\int\frac{\sin\phi}{(1+\lambda\sin\phi)^2}d\phi \\
&= \tan\phi\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right) + (1-\lambda^2)\frac{\partial I(\phi,\lambda)}{\partial\lambda},
\end{aligned}$$

where the integral $I(\phi, \lambda)$ is given by

$$I(\phi,\lambda) = \int\frac{d\phi}{1+\lambda\sin\phi} = 2\int\frac{dt}{(t+\lambda)^2 + (1-\lambda^2)},$$

where $t = \tan(\phi/2)$ is used as a working variable for the half-tangent substitution, and again there are two cases to consider. For $\lambda^2 < 1$ we may make the substitution $t + \lambda = (1-\lambda^2)^{1/2}\tan\psi$ to determine the following expressions:

$$I(\phi,\lambda) = \frac{2}{(1-\lambda^2)^{1/2}}\tan^{-1}\left(\frac{\lambda + \tan(\phi/2)}{(1-\lambda^2)^{1/2}}\right), \tag{6.24}$$

$$\begin{aligned}
\frac{\partial I(\phi,\lambda)}{\partial\lambda} &= \frac{2\lambda}{(1-\lambda^2)^{3/2}}\tan^{-1}\left(\frac{\lambda + \tan(\phi/2)}{(1-\lambda^2)^{1/2}}\right) \\
&+ \frac{\cos\phi}{(1-\lambda^2)(1+\lambda\sin\phi)} + \frac{1}{(1-\lambda^2)},
\end{aligned}$$

while for $\lambda^2 > 1$, on using partial fractions, we find

$$I(\phi, \lambda) = \frac{1}{(\lambda^2 - 1)^{1/2}} \log \left(\frac{\lambda + \tan(\phi/2) - (\lambda^2 - 1)^{1/2}}{\lambda + \tan(\phi/2) + (\lambda^2 - 1)^{1/2}} \right),$$

$$\frac{\partial I(\phi, \lambda)}{\partial \lambda} = -\frac{\lambda}{(\lambda^2 - 1)^{3/2}} \log \left(\frac{\lambda + \tan(\phi/2) - (\lambda^2 - 1)^{1/2}}{\lambda + \tan(\phi/2) + (\lambda^2 - 1)^{1/2}} \right)$$
$$- \frac{\cos\phi}{(\lambda^2 - 1)(1 + \lambda \sin\phi)} - \frac{1}{(\lambda^2 - 1)}.$$

Altogether, from these expressions we may deduce the following formulae for the de Broglie wave energy. For $\lambda^2 < 1$ we obtain

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{2\lambda}{(1 - \lambda^2)^{1/2}} \tan^{-1} \left(\frac{\lambda + \tan(\phi/2)}{(1 - \lambda^2)^{1/2}} \right) \right\} + \mathscr{E}_0, \tag{6.25}$$

while for $\lambda^2 > 1$ we have

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{(\lambda^2 - 1)^{1/2}} \log \left(\frac{\lambda + \tan(\phi/2) - (\lambda^2 - 1)^{1/2}}{\lambda + \tan(\phi/2) + (\lambda^2 - 1)^{1/2}} \right) \right\} + \mathscr{E}_0,$$

where in both cases $u = c \sin\phi$ and $\mathscr{E}_0$ denotes an arbitrary constant.

The equality of these expressions with Eqs. (6.13) and (6.14) is not immediately obvious, and their veracity is most easily established by direct differentiation. For $\lambda^2 < 1$ we have

$$\frac{d}{d\phi} \left\{ \frac{1}{\cos\phi} + \frac{2\lambda}{(1 - \lambda^2)^{1/2}} \tan^{-1} \left(\frac{\lambda + \tan(\phi/2)}{(1 - \lambda^2)^{1/2}} \right) \right\}$$
$$= \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda \sec^2(\phi/2)}{(1 - \lambda^2) \left(1 + \frac{(\lambda + \tan(\phi/2))^2}{(1 - \lambda^2)} \right)}$$
$$= \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda \sec^2(\phi/2)}{(1 - \lambda^2) + (\lambda + \tan(\phi/2))^2}$$
$$= \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda(1 + \tan^2(\phi/2))}{(1 + \tan(\phi/2)^2 + 2\lambda \tan(\phi/2))}$$
$$= \frac{\sin\phi}{\cos^2\phi} + \frac{\lambda}{(1 + \lambda \sin\phi)} = \frac{\sin\phi(1 + \lambda \sin\phi) + \lambda \cos^2\phi}{\cos^2\phi(1 + \lambda \sin\phi)}$$
$$= \frac{(\lambda + \sin\phi)}{\cos^2\phi(1 + \lambda \sin\phi)},$$

as required. Similarly for $\lambda^2 > 1$ we have

$$\frac{d}{d\phi}\left\{\frac{1}{\cos\phi}+\frac{\lambda}{(\lambda^2-1)^{1/2}}\log\left(\frac{\lambda+\tan(\phi/2)-(\lambda^2-1)^{1/2}}{\lambda+\tan(\phi/2)+(\lambda^2-1)^{1/2}}\right)\right\}=\frac{\sin\phi}{\cos^2\phi}$$

$$+\frac{\lambda\sec^2(\phi/2)}{(\lambda^2-1)^{1/2}}\left(\frac{1}{\lambda+\tan(\phi/2)-(\lambda^2-1)^{1/2}}-\frac{1}{\lambda+\tan(\phi/2)+(\lambda^2-1)^{1/2}}\right)$$

$$=\frac{\sin\phi}{\cos^2\phi}+\frac{\lambda\sec^2(\phi/2)}{(1-\lambda^2)+(\lambda+\tan(\phi/2))^2}=\frac{(\lambda+\sin\phi)}{\cos^2\phi(1+\lambda\sin\phi)},$$

following precisely the immediately above calculation for $\lambda^2 < 1$.

For $\lambda^2 < 1$, in order to provide an independent proof that Eq. (6.25) coincides with the expression (6.13), we need to establish that the difference

$$\tan^{-1}\left(\frac{(\lambda+\sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi}\right)-2\tan^{-1}\left(\frac{\lambda+\tan(\phi/2)}{(1-\lambda^2)^{1/2}}\right),$$

is at most a constant, and we do this in two stages. We first examine the difference

$$\tan^{-1}\left(\frac{(\lambda+\sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi}\right)-\tan^{-1}\left(\frac{\lambda+\tan(\phi/2)}{(1-\lambda^2)^{1/2}}\right).$$

On using $\tan(\phi/2) = (1-\cos\phi)/\sin\phi$ and taking the tan of this expression, after some lengthy algebra, we may eventually deduce

$$\tan\left\{\tan^{-1}\left(\frac{(\lambda+\sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi}\right)-\tan^{-1}\left(\frac{\lambda+\tan(\phi/2)}{(1-\lambda^2)^{1/2}}\right)\right\}$$

$$=\frac{(1-\lambda^2)^{1/2}}{\lambda+\cot(\phi/2)}=\frac{(1-\lambda^2)^{1/2}\tan(\phi/2)}{1+\lambda\tan(\phi/2)},$$

so that now we are led to examine

$$\tan\left\{\tan^{-1}\left(\frac{(1-\lambda^2)^{1/2}\tan(\phi/2)}{1+\lambda\tan(\phi/2)}\right)-\tan^{-1}\left(\frac{\lambda+\tan(\phi/2)}{(1-\lambda^2)^{1/2}}\right)\right\}$$

$$=\frac{-\lambda}{(1-\lambda^2)^{1/2}},$$

and therefore altogether we obtain

$$\tan^{-1}\left(\frac{(\lambda+\sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi}\right)-2\tan^{-1}\left(\frac{\lambda+\tan(\phi/2)}{(1-\lambda^2)^{1/2}}\right)=\tan^{-1}\left(\frac{-\lambda}{(1-\lambda^2)^{1/2}}\right),$$

as required and a result that is confirmed by setting $\phi = 0$. (The above proof of this result is due to Dr Barry Cox, personal communication.)

For $\lambda^2 > 1$ we may also formally establish the equality of (6.24) with Eq. (6.14), but the proof is quite subtle and not at all immediately obvious. The proof hinges on a subtle use of the identity

$$(\lambda + (\lambda^2 - 1)^{1/2})(\lambda - (\lambda^2 - 1)^{1/2}) = 1,$$

and the fact that the wave energy is determined only up to an arbitrary constant, so that the log of any constants may be ignored. We start by writing the above expression as

$$\mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \frac{\lambda}{2(\lambda^2 - 1)^{1/2}} \log \left(\frac{\lambda + \tan(\phi/2) - (\lambda^2 - 1)^{1/2}}{\lambda + \tan(\phi/2) + (\lambda^2 - 1)^{1/2}} \right)^2 \right\} + \mathscr{E}_0,$$

and then for the argument of the log, modulo any multiplicative constants, we proceed as follows:

$$\begin{aligned}
&\left(\frac{\lambda + \tan(\phi/2) - (\lambda^2 - 1)^{1/2}}{\lambda + \tan(\phi/2) + (\lambda^2 - 1)^{1/2}} \right)^2 \\
&= \left(\frac{\tan(\phi/2) + \lambda - (\lambda^2 - 1)^{1/2}}{\tan(\phi/2) + \lambda + (\lambda^2 - 1)^{1/2}} \right) \left(\frac{\tan(\phi/2) + \lambda - (\lambda^2 - 1)^{1/2}}{\tan(\phi/2) + \lambda + (\lambda^2 - 1)^{1/2}} \right) \\
&= \left(\frac{\tan(\phi/2) + \lambda - (\lambda^2 - 1)^{1/2}}{\tan(\phi/2) + \lambda + (\lambda^2 - 1)^{1/2}} \right) \left(\frac{(\lambda + (\lambda^2 - 1)^{1/2}) \tan(\phi/2) + 1}{(\lambda - (\lambda^2 - 1)^{1/2}) \tan(\phi/2) + 1)} \right) \\
&= \frac{(\lambda + (\lambda^2 - 1)^{1/2}) \tan^2(\phi/2) + 2\tan(\phi/2) + (\lambda - (\lambda^2 - 1)^{1/2})}{(\lambda - (\lambda^2 - 1)^{1/2}) \tan^2(\phi/2) + 2\tan(\phi/2) + (\lambda + (\lambda^2 - 1)^{1/2})} \\
&= \frac{\lambda(1 + \tan^2(\phi/2)) + 2\tan(\phi/2) - (\lambda^2 - 1)^{1/2}(1 - \tan^2(\phi/2))}{\lambda(1 + \tan^2(\phi/2)) + 2\tan(\phi/2) + (\lambda^2 - 1)^{1/2}(1 - \tan^2(\phi/2))} \\
&= \frac{(\lambda + \sin\phi) - (\lambda^2 - 1)^{1/2} \cos\phi}{(\lambda + \sin\phi) + (\lambda^2 - 1)^{1/2} \cos\phi},
\end{aligned}$$

as required, and again we emphasise that in the above calculation, we have simply ignored any multiplicative constants since these only affect the arbitrary additive constant involved in the energy.

6.7 Alternative Derivation of Exact Solution

In this section we present an alternative derivation of the relativistic wave-like solution given by (5.1) which is derived on the basis that the forces $f(u)$ and $g(u)$ are assumed to be functions of velocity only and given by the linear expressions

Eq. (5.2). Here we again assume the two basic equations (4.10) and $e^2-(pc)^2=e_0^2$, and on introducing the angle $\theta(x,t)$ such that $u(x,t) = c\tanh\theta$, we have from (2.11) the relations

$$e(x,t)=e_0\cosh\theta, \quad cp(x,t)=e_0\sinh\theta, \tag{6.26}$$

where again $e_0 = m_0c^2$ is the rest mass energy. On substitution of these relations into the two basic equations (4.10), we may readily deduce two equations for the determination of the partial derivatives $\partial\theta/\partial t$ and $\partial\theta/\partial x$; thus

$$\frac{\partial\theta}{\partial t} = a(\theta)\cosh\theta - b(\theta)\sinh\theta, \quad c\frac{\partial\theta}{\partial x} = b(\theta)\cosh\theta - a(\theta)\sinh\theta, \tag{6.27}$$

where we have introduced $a(\theta) = cf(\theta)/e_0$ and $b(\theta) = c^2g(\theta)/e_0$. On cross differentiation these two equations, using the equations themselves and equating the two expressions for $\partial^2\theta/\partial t\partial x$, we may deduce the following simple ordinary differential equation:

$$\frac{d(a/b)}{d\theta} = \left(1-(a/b)^2\right),$$

which may be readily integrated, and further simplification yields simply $a(\theta) = b(\theta)\tanh(\theta-\theta_0)$ where θ_0 denotes the constant of integration. With this relation the two equations (6.27) simplify to become

$$\frac{\partial\theta}{\partial t} = -\frac{b(\theta)\sinh\theta_0}{\cosh(\theta-\theta_0)}, \qquad c\frac{\partial\theta}{\partial x} = \frac{b(\theta)\cosh\theta_0}{\cosh(\theta-\theta_0)},$$

so that evidently $\theta(x,t)$ satisfies the simple partial differential equation

$$\frac{\partial\theta}{\partial t} + c\tanh\theta_0\frac{\partial\theta}{\partial x} = 0,$$

Again we may deduce $\theta(x,t)=\theta(\xi)$ where $\xi = \lambda x + ct$, we make the identification $\lambda = -\coth\theta_0$, and the two partial differential equations both reduce to the same ordinary differential equation

$$\frac{d\theta}{d\xi} = -\frac{b(\theta)\sinh\theta_0}{c\cosh(\theta-\theta_0)} = \frac{cg(\theta)}{e_0(\sinh\theta+\lambda\cosh\theta)},$$

noting again that here $u = c\tanh\theta$. Now from the condition $\partial f/\partial t = c^2\partial g/\partial x$ and the functional relationship $\theta(x,t)=\theta(\xi)$, we may deduce $df/d\theta = c\lambda dg/d\theta$, and therefore on integration we have $f(\theta)-\lambda cg(\theta) = f_0(\lambda^2-1)$ where the constant of integration is specifically chosen to agree with the previously used constants.

From this relation, together with the integral $f(\theta) = cg(\theta)\tanh(\theta - \theta_0)$, we may eventually deduce as before $f(u) = f_0(1 + \lambda(u/c))$ and $cg(u) = f_0(\lambda + (u/c))$.

6.8 Yet Another Approach to Evaluation of Integrals

It would seem worthwhile noting that yet another approach to the evaluation of the integrals involving the angle θ arises directly from Eq. (5.8) by simply multiplying both numerator and denominator of this equation by $e = mc^2 = (e_0^2 + (pc)^2)^{1/2}$; thus

$$\frac{d\mathscr{E}}{dp} = c\left(\frac{\lambda + u/c}{1 + \lambda u/c}\right) = c\left(\frac{\lambda e + pc}{e + \lambda pc}\right) = c\left(\frac{\lambda(e_0^2 + (pc)^2)^{1/2} + pc}{(e_0^2 + (pc)^2)^{1/2} + \lambda pc}\right),$$

and on using ϕ through the trigonometric expressions (6.1), this equation can be seen to coincide precisely with (6.12). Here however, we use the angle $\theta = \tanh^{-1}(u/c)$ and the hyperbolic relations given by either (2.11) or (6.26) so that the immediately above equation becomes

$$\frac{d\mathscr{E}}{d\theta} = e_0 \cosh\theta \left(\frac{\lambda\cosh\theta + \sinh\theta}{\cosh\theta + \lambda\sinh\theta}\right),$$

which with some re-arrangement becomes

$$\frac{d\mathscr{E}}{d\theta} = e_0\left(\sinh\theta + \frac{\lambda}{\cosh\theta + \lambda\sinh\theta}\right).$$

On using the basic definitions of cosh and sinh, we obtain on integration

$$\mathscr{E}(\theta) = e_0\left(\cosh\theta + \frac{2\lambda}{(\lambda+1)}\int\frac{e^{\theta}\,d\theta}{e^{2\theta} - (\lambda-1)/(\lambda+1)}\right) + \mathscr{E}_0,$$

and again $\mathscr{E}_0$ denotes a suitable constant, and there are two cases to consider. If $\lambda^2 < 1$ and $\gamma^2 = (1-\lambda)/(1+\lambda)$ and making the substitution $\omega = e^{\theta}$, we have

$$\mathscr{E}(\theta) = e_0\left(\cosh\theta + \frac{2\lambda}{(\lambda+1)}\int\frac{d\omega}{\omega^2 + \gamma^2}\right) + \mathscr{E}_0,$$

and on integration we have

$$\mathscr{E}(\theta) = e_0\left\{\cosh\theta + \frac{2\lambda}{(1-\lambda^2)^{1/2}}\tan^{-1}\left(\frac{\omega}{\gamma}\right)\right\} + \mathscr{E}_0.$$

If $\lambda^2 > 1$ then with $\delta^2 = (\lambda-1)/(\lambda+1)$, and again making the substitution $\omega = e^\theta$, we have

$$\mathscr{E}(\theta) = e_0 \left(\cosh\theta + \frac{2\lambda}{(\lambda+1)} \int \frac{d\omega}{\omega^2 - \delta^2} \right) + \mathscr{E}_0,$$

which becomes

$$\mathscr{E}(\theta) = e_0 \left\{ \cosh\theta + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log\left(\frac{\omega-\delta}{\omega+\delta} \right) \right\} + \mathscr{E}_0.$$

Thus in summary, in terms of the angle θ for which $u = c\tanh\theta$ and the hyperbolic relations given by either (2.11) or (6.26), for $\lambda^2 < 1$ we have

$$\mathscr{E}(\theta) = e_0 \left\{ \cosh\theta + \frac{2\lambda}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\frac{e^\theta}{\gamma} \right) \right\} + \mathscr{E}_0, \tag{6.28}$$

where $\gamma^2 = (1-\lambda)/(1+\lambda)$, while if $\lambda^2 > 1$, we have

$$\mathscr{E}(\theta) = e_0 \left\{ \cosh\theta + \frac{\lambda}{(\lambda^2-1)^{1/2}} \log\left(\frac{e^\theta-\delta}{e^\theta+\delta} \right) \right\} + \mathscr{E}_0, \tag{6.29}$$

where $\delta^2 = (\lambda-1)/(\lambda+1)$ and again noting that in both cases $\mathscr{E}_0$ denotes a suitable constant.

In order to see that the above Eqs. (6.28) and (6.29) coincide, respectively, with both Eqs. (6.13) and (6.14) and with both Eqs. (6.21) and (6.22), we need to use the previously given identity (2.10), namely,

$$e^\theta = \left(\frac{1+u/c}{1-u/c} \right)^{1/2}.$$

To confirm that Eq. (6.28) agrees with Eq. (6.21), we again recall the identity $\tan^{-1}(z) + \tan^{-1}(1/z) = \pi/2$, and we write the term involving the inverse tangent as follows:

$$2\tan^{-1}\left(\frac{e^\theta}{\gamma} \right) = \frac{\pi}{2} - \left(\tan^{-1}\left(\frac{\gamma}{e^\theta} \right) - \tan^{-1}\left(\frac{e^\theta}{\gamma} \right) \right),$$

so that on taking the tangent of this expression, we have

$$\tan\left(2\tan^{-1}\left(\frac{e^\theta}{\gamma} \right) \right) = \cot\left(\tan^{-1}\left(\frac{\gamma}{e^\theta} \right) - \tan^{-1}\left(\frac{e^\theta}{\gamma} \right) \right)$$

$$= \frac{1}{\tan\left(\tan^{-1}\left(\frac{\gamma}{e^\theta}\right) - \tan^{-1}\left(\frac{e^\theta}{\gamma}\right)\right)} = \frac{1}{\frac{1}{2}\left(\frac{\gamma}{e^\theta} - \frac{e^\theta}{\gamma}\right)}$$

$$= \frac{-2\gamma e^\theta}{e^{2\theta} - \gamma^2} = -\frac{(1-\lambda^2)^{1/2}(1-(u/c)^2)^{1/2}}{\lambda + u/c},$$

on simplification of the latter expression using the formulae

$$e^\theta = \left(\frac{1+u/c}{1-u/c}\right)^{1/2}, \qquad \gamma = \left(\frac{1-\lambda}{1+\lambda}\right)^{1/2},$$

and the desired result, modulo an arbitrary constant, follows on again noting the relation $\tan^{-1}(z) + \tan^{-1}(1/z) = \pi/2$. Similarly, in order to confirm that Eq. (6.29) coincides with (6.22), we write the logarithm as

$$\frac{1}{2}\log\left(\frac{e^\theta - \delta}{e^\theta + \delta}\right)^2 = \frac{1}{2}\log\left(\frac{e^{2\theta} - 2e^\theta\delta + \delta^2}{e^{2\theta} + 2e^\theta\delta + \delta^2}\right),$$

and again the desired result follows on simplification of this equation using

$$e^\theta = \left(\frac{1+u/c}{1-u/c}\right)^{1/2}, \qquad \delta = \left(\frac{\lambda-1}{\lambda+1}\right)^{1/2}.$$

Chapter 7
Lorentz and Other Invariances

7.1 Introduction

In special relativity the criterion of Lorentz invariance is there to establish the veracity or otherwise of any proposal, or in other words, a good proposition in special relativity must satisfy the criterion of Lorentz invariance, and this is a nontrivial constraint. In this chapter we examine the Lorentz invariances of the basic equations themselves and the associated Lorentz invariances implicit in the previously derived exact wave-like solution. We have previously stated that a Lorentz invariant quantity is one that assumes an identical form under a Lorentz transformation, and we establish here that the general force equations (3.4) are fully Lorentz invariant. However, this does not imply that all aspects of the solutions of (3.4) are Lorentz invariant, and there will be solutions that are not fully Lorentz invariant, and the exact wave-like solution provides an example.

In the following section, assuming the Lorentz invariant energy-momentum relations (2.46) and (2.47), we show that the proposed general one-dimensional force expressions given by (7.1) remain invariant under Lorentz transformations. This is a particularly important and fundamental outcome, which means that the model allows the same force values to be recorded in all frames moving with constant relative velocity. In this section we also show that the operator arising in (3.9) and (3.10) remains invariant under a Lorentz transformation.

In three subsequent sections of this chapter, with specific reference to the wave-like solution given by (5.1), we investigate in some detail the formal Lorentz invariance of the derivative $d\mathscr{E}/dp$ and the Lorentz invariances and the functional dependence of the linear force expressions $f(u) = f_0(1 + \lambda u/c)$ and $cg(u) = f_0(\lambda + u/c)$ arising in the wave-like solution. In the following section we examine the variable $\xi = \lambda x + ct$ under a Lorentz transformation to determine the corresponding parameter μ that corresponds to the parameter λ and the associated invariances. We comment that the parameters λ and f_0 are not themselves Lorentz invariants since their transformations are dependent on the frame velocity v (see

J. M. Hill, *Mathematics of Particle-Wave Mechanical Systems*,
https://doi.org/10.1007/978-3-031-19793-2_7

Eq. (7.8) below). The exact wave-like solution might be termed partially Lorentz invariant in the sense that, under a Lorentz transformation, the assumed linear force expressions $f(u) = f_0(1+\lambda u/c)$ and $cg(u) = f_0(\lambda+u/c)$ transform to linear force expressions involving $F(U) = F_0(1+\mu U/c)$ and $cG(U) = F_0(\mu+U/c)$ with new force constants F_0 and μ dependent upon the translational velocity v arising in the Lorentz transformation, but the actual Lorentz invariances turn out to involve the products of force and energy.

In the next two sections, for the wave-like solution, we examine explicitly the Lorentz invariance and functional dependence of the forces $f(u)$ and $cg(u)$. We confirm that, with the assumption of the Lorentz invariant energy-momentum relations (2.46) and (2.47), the basic equations (7.1) generate the same force values in all Lorentz frames. We conclude with the curious outcome that the actual Lorentz invariants involve mass times force or energy times force. That is, for forces and energy $f(u)$, $g(u)$ and $e(u)$ in one frame and $F(U)$, $G(U)$ and $E(U)$ in another, we formally establish the interesting result that $e(u)f(u) = E(U)F(U)$ and $e(u)g(u) = E(U)G(U)$.

In the final three sections of this chapter, we examine the implications of the space-time transformation $x' = ct$ and $t' = x/c$, involving only one Cartesian spatial dimension and reflecting a simple invariance of the classical one-dimensional wave equation. Formally, the de Broglie hypothesis arises from this transformation, which essentially interchanges the roles of space and time and, for sub-luminal particle velocities $u = dx/dt$, produces superluminal wave velocities $w = u' = dx'/dt' = c^2 dt/dx = c^2/u$. Thus, any discussion of de Broglie waves necessitates at the very least an appreciation of superluminal waves, and therefore, some understanding of superluminal motion in general is necessary. Under the transformation $x' = ct$ and $t' = x/c$, there are two important and distinct restricted energy interchanges that we need to accommodate. Firstly, in the first of the final three sections, we show that the two energies $\mathscr{E}(x,t)$ and momentum $cp(x,t)$ may be interchanged, and we also show that the operator arising in (3.9) and (3.10) remains invariant under the space-time transformation $x' = ct$ and $t' = x/c$. In the penultimate section of the chapter, and following the superluminal Lorentz transformations derived in [54], we establish that the proposed basic force equations (3.4) also remain unchanged under Lorentz frames moving with constant velocities v that are in excess of the speed of light. In the final section, since particle and wave velocities in the unprimed (x,t) frame are perceived as, respectively, wave and particle velocities in the primed (x',t') frame, the particle energy $e(x,t)$ in the unprimed (x,t) frame may be interchanged with the wave energy $\mathscr{E}(x',t')$ in the primed (x',t') frame, and vice versa. We also show that when the two distinct energy interchanges are combined, an invariant symmetry (7.26) of the basic equations (3.4) of the model is produced.

7.2 Force Invariance Under Lorentz Transformations

For one spatial dimension x, the proposed equations (3.4) become simply

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{7.1}$$

and it is not difficult to show that these equations remain invariant under the Lorentz group (2.3) and the Lorentz invariant energy-momentum relations (2.46); in other words the following relations hold:

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = \frac{\partial P}{\partial T} + \frac{\partial E}{\partial X}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial E}{\partial T} + \frac{\partial P}{\partial X}, \tag{7.2}$$

which we establish as follows. From Eq. (2.3) we have the differential relations

$$\frac{\partial}{\partial x} = \frac{1}{(1-(v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\}, \quad \frac{\partial}{\partial t} = \frac{1}{(1-(v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\}, \tag{7.3}$$

so that on using (2.46) and employing the subscript notation to denote partial derivatives, for the spatial physical force f, we have

$$\begin{aligned} p_t + e_x &= \\ &\frac{1}{(1-(v/c)^2)}\left\{(P_T - \frac{v}{c^2}E_T) + v(P_X - \frac{v}{c^2}E_X) + (E_X - vP_X) + \frac{v}{c^2}(E_T - vP_T)\right\} \\ &= P_T + E_X, \end{aligned}$$

and similarly for the force in the direction of time g, we have

$$\begin{aligned} \frac{1}{c^2}e_t + p_x &= \\ &\frac{1}{(1-(v/c)^2)}\left\{\frac{1}{c^2}(E_T - vP_T) + \frac{v}{c^2}(E_X - vP_X) + (P_X - \frac{v}{c^2}E_X) + \frac{v}{c^2}(P_T - \frac{v}{c^2}E_T)\right\} \\ &= \frac{1}{c^2}E_T + P_X. \end{aligned}$$

This important invariance property, derived under the assumption of the Lorentz invariant energy-momentum relations (2.46) and (2.47), ensures that the same force values are observed as perceived from any Lorentz frame. This key outcome indicates, at the very least, that the two Eqs. (3.4) are well formulated. However, it does not mean that all aspects of the solutions of the Eqs. (7.1) will be fully Lorentz invariant. Specifically, there will be aspects of solutions of these equations that are not fully Lorentz invariant, and the exact wave-like solution presented in Chap. 5 is one such example. We show in a subsequent section of this chapter that the Lorentz invariants for this particular solution are force times energy, thus fe and cge.

This partial Lorentz invariance arises from the fact that while the general force expressions (7.1) are fully Lorentz invariant, the corresponding condition for the existence of a work-done function, namely

$$\frac{\partial f}{\partial t} = c^2 \frac{\partial g}{\partial x}, \tag{7.4}$$

is not Lorentz invariant. In terms of the (X, T) variables, from the formulae (7.3), Eq. (7.5) becomes

$$\frac{\partial f}{\partial T} + v\frac{\partial f}{\partial X} = c^2 \frac{\partial g}{\partial X} + v\frac{\partial g}{\partial T},$$

so that for full Lorentz invariance $f(x, t) = F(X, T)$ and $g(x, t) = G(X, T)$, and therefore from the above equation it follows that

$$\frac{\partial F}{\partial T} = c^2 \frac{\partial G}{\partial X}, \qquad \frac{\partial F}{\partial X} = \frac{\partial G}{\partial T}.$$

Accordingly, for full Lorentz invariance, we need to supplement (7.5) with the additional constraint

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}, \tag{7.5}$$

and in terms of a potential function $V(x, t)$ defined by (4.16), namely

$$f = -\frac{\partial V}{\partial x}, \qquad gc^2 = -\frac{\partial V}{\partial t},$$

it is clear that $V(x, t)$ satisfies the classical wave equation

$$\frac{\partial^2 V}{\partial t^2} - c^2 \frac{\partial^2 V}{\partial x^2} = 0.$$

Thus, the only force fields (f, cg) providing full Lorentz invariance are those generated from potentials $V(x, t)$, which satisfy the classical wave equation.

We comment that a more familiar example exhibiting related phenomena might be the partial differential equation describing the one-dimensional temperature for the movement of heat or for the concentration of diffusing particles; thus,

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2}, \tag{7.6}$$

for $C = C(x, t)$. It is well known that this partial differential equation remains invariant under the one-parameter group of stretching transformations

$$x^* = e^{\epsilon} x, \qquad t^* = e^{2\epsilon} t, \qquad C^* = e^{n\epsilon} C,$$

where ϵ is the one parameter and n denotes an arbitrary constant. The existence of this one-parameter stretching group of transformations means that there are similarity solutions of the partial differential equation (7.6) with the functional form $C(x, t) = t^{n/2}\Phi(x/t^{1/2})$ for some function Φ, and the partial differential equation (7.6) may be reduced to a second-order ordinary differential equation. However, it does not mean that every solution of (7.6) has this structure, and as is well known, there are certainly solutions of (7.6) that are not similarity solutions. A similar situation applies to (7.1), which will admit solutions and embody consequences that are not fully Lorentz invariant.

We may also confirm that the operator appearing in (3.9) and (3.10) is Lorentz invariant, namely for any three-dimensional spatial vector $\mathbf{q}$, we have

$$\frac{\partial \mathbf{q}}{\partial t} \cdot \mathbf{dx} + c^2(\nabla \cdot \mathbf{q})dt = \frac{\partial \mathbf{q}}{\partial T} \cdot \mathbf{dX} + c^2(\nabla^* \cdot \mathbf{q})dT,$$

where ∇^* denotes the del operator with respect to the $\mathbf{X}$ variables. This equation follows since for one dimension and relative frame velocity in the x-direction, we have from (2.3) and the above differential relations

$$\begin{aligned} &q_t dx + c^2 q_x dt = \\ &\frac{1}{(1-(v/c)^2)}\left\{(q_T + vq_X)(dX - vdT) + c^2\left(q_X + \frac{v}{c^2}q_T\right)\left(dT - \frac{v}{c^2}dX\right)\right\} \\ &= q_T dX + c^2 q_X dT, \end{aligned}$$

as required and demonstrating that the operator in (3.9) and (3.10) is Lorentz invariant.

7.3 Lorentz Invariance of $d\mathscr{E}/dp$ or $d\mathscr{E}/d\xi$

In this section we make a Lorentz transformation of the exact wave-like solution examined in the two prior chapters, and we show the formal Lorentz invariance of $d\mathscr{E}/dp$ or $d\mathscr{E}/d\xi$ for the specific $d\mathscr{E}/dp$ given by (5.8). From $\xi = \lambda x + ct$ and the Lorentz transformations (2.3)

$$x = \frac{X - vT}{[1-(v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1-(v/c)^2]^{1/2}}, \tag{7.7}$$

we have

$$\xi = \frac{\lambda(X - vT) + c(T - vX/c^2)}{[1 - (v/c)^2]^{1/2}} = \frac{(\lambda - v/c)\, X + (1 - \lambda v/c)\, cT}{[1 - (v/c)^2]^{1/2}},$$

and therefore with $\eta = \mu X + cT$, we have $\xi = \sigma\eta$ where μ is the essential constant in the (X, T) variables that corresponds to λ in the (x, t) variable and μ and σ are given by

$$\mu = \left(\frac{\lambda - v/c}{1 - \lambda v/c}\right), \qquad \sigma = \frac{(1 - \lambda v/c)}{[1 - (v/c)^2]^{1/2}}, \tag{7.8}$$

and if F_0 is the constant in the (X, T) variables corresponding to the constant f_0 in the (x, t) variables, then $F_0 = \sigma f_0$. With this notation, in either coordinates the momentum $p(x, t)$ and the wave energy $\mathscr{E}(x, t)$ for the wave-like solution become

$$p(x, t) = f_0(\lambda x + ct)/c = F_0(\mu X + cT)/c = p^*(X, T),$$
$$\mathscr{E}(x, t) = f_0(x + c\lambda t) = F_0(X + c\mu T) = \mathscr{E}^*(X, T).$$

Under this Lorentz transformation and with the usual Einstein formula for velocities, we may verify by direct substitution of λ and u into the left-hand side, namely

$$u = \frac{U - v}{\left(1 - Uv/c^2\right)}, \qquad \lambda = \left(\frac{\mu + v/c}{1 + \mu v/c}\right),$$

and subsequent simplification, the important relations

$$\frac{\lambda + u/c}{1 + \lambda u/c} = \frac{\mu + U/c}{1 + \mu U/c}, \qquad \frac{1 - \lambda^2}{1 - \lambda v/c} = \frac{1 - \mu^2}{1 + \mu v/c}. \tag{7.9}$$

By a similar process, we may verify the following Lorentz invariants:

$$\frac{1 + \lambda u/c}{(1 - \lambda^2)^{1/2}(1 - (u/c)^2)^{1/2}} = \frac{1 + \mu U/c}{(1 - \mu^2)^{1/2}(1 - (U/c)^2)^{1/2}},$$
$$\frac{\lambda + u/c}{(1 - \lambda^2)^{1/2}(1 - (u/c)^2)^{1/2}} = \frac{\mu + U/c}{(1 - \mu^2)^{1/2}(1 - (U/c)^2)^{1/2}},$$

noting especially that the left-hand side of the latter invariant is precisely the argument of the inverse tangent function in the expression (6.13) for the wave energy. Further, on making use of $(7.9)_2$ and the symmetric identity (7.15), through the identity

$$\left(\frac{1 - \lambda^2}{1 - \mu^2}\right)^{1/2} = \left(\frac{1 - \lambda v/c}{1 + \mu v/c}\right)^{1/2} = \frac{(1 - \lambda v/c)}{(1 - (v/c)^2)^{1/2}} = \sigma,$$

the immediately above Lorentz invariants might be seen to coincide with the two invariants $ef = EF$ and $eg = EG$ subsequently discussed.

From Eq. (7.9)$_1$ we find that Eq. (5.8) becomes

$$\frac{d\mathscr{E}}{dp} = c\left(\frac{\lambda + u/c}{1 + \lambda u/c}\right) = c\left(\frac{\mu + U/c}{1 + \mu U/c}\right) = \frac{d\mathscr{E}^*}{dp^*},$$

where $\mathscr{E}^*(X, T)$ and $p^*(X, T)$ designate, respectively, $\mathscr{E}(x, t)$ and $p(x, t)$ in terms of the (X, T) variables. The Lorentz invariance $d\mathscr{E}/d\xi = d\mathscr{E}^*/d\eta$ now follows from $d\mathscr{E}/dp = d\mathscr{E}^*/dp^*$ on noting the relations $pc = f_0\xi$ and $p^*c = F_0\eta$ and where $\mathscr{E}^*(X, T) = F_0(X + c\mu T)$ denotes the wave energy with respect to the (X, T) variables. The constant f_0 and the corresponding quantity in the (X, T) variables F_0 are related by the equation $F_0 = \sigma f_0$, where σ is defined by the latter relation of (7.8).

We observe from (7.9) the privileged role played by the two special cases $\lambda = \pm 1$. The values $\lambda = \pm 1$ correspond to energy-mass waves travelling at the speed of light and from the de Broglie wave energy expressions given in the two previous chapters, arising as well-defined limiting cases that are consistent with established results for photons and light, namely $p = h\nu/c$ and $\mathscr{E} = h\nu$, where h is Planck's constant and ν denotes the frequency, which together yield $\mathscr{E} = pc$. The two cases $\lambda = 1$ and $\lambda = -1$ correspond, respectively, to $\mu = 1$ and $\mu = -1$, and as noted previously in Chap. 2, Eq. (2.4), the following important relations apply for the characteristic variables $\alpha = ct + x$ and $\beta = ct - x$, also given by Hill and Cox [54], thus

$$ct + x = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2}(cT + X), \qquad ct - x = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}(cT - X),$$

which can also be seen to arise directly from $\xi = \sigma\eta$ and (2.3)$_2$ with $\lambda = \pm 1$. These relations imply that the special cases $\lambda = \pm 1$ possess certain privileged properties under Lorentz transformation that are not shared by other values of λ. For example, the particle energy corresponding to Eq. (5.1) is simply $e(x, t) = f_0\left((e_0/f_0)^2 + \xi^2\right)^{1/2}$, where $\xi = \lambda x + ct$, and $e(x, t)$ only satisfies the wave equation in the two special cases $\lambda = \pm 1$.

7.4 Lorentz Invariance of Forces

In this section, it is instructive to examine explicitly the Lorentz invariance for the particular exact wave-like solution given by either (5.1) or (6.10), and we show explicitly that the forces f and cg as determined through the basic equations (7.2) are indeed independent of the Lorentz reference frame. However, as we establish in the following section, this does not necessarily imply that the forces $f(u)$ and $cg(u)$

coincide, respectively, with $F(U)$ and $cG(U)$ in every Lorentz reference frame, and we show in the next section that the Lorentz invariants turn out to be $ef = EF$ and $eg = EG$.

Here, we show explicitly that under the Lorentz transformation given by (2.3) or immediately above by (7.7), and the transformations which are inverse to the energy-momentum relations (2.46), namely

$$P = \frac{p + ev/c^2}{[1-(v/c)^2]^{1/2}}, \qquad E = \frac{e + pv}{[1-(v/c)^2]^{1/2}}, \tag{7.10}$$

the following general relations verified above hold for the particular solution, thus

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = \frac{\partial P}{\partial T} + \frac{\partial E}{\partial X}, \quad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial E}{\partial T} + \frac{\partial P}{\partial X}. \tag{7.11}$$

Summarising the formulae from the previous section, we have

$$\xi = \lambda x + ct, \qquad \eta = \mu X + cT, \tag{7.12}$$
$$f_0 \xi = F_0 \eta, \qquad F_0 = \sigma f_0, \qquad \xi = \sigma \eta,$$

where λ and μ are related through the inverse formulae

$$\mu = \left(\frac{\lambda - v/c}{1 - \lambda v/c}\right), \qquad \lambda = \left(\frac{\mu + v/c}{1 + \mu v/c}\right), \tag{7.13}$$

while σ is given by either of the expressions

$$\sigma = \frac{(1 - \lambda v/c)}{[1-(v/c)^2]^{1/2}}, \qquad \sigma = \frac{[1-(v/c)^2]^{1/2}}{(1 + \mu v/c)}, \tag{7.14}$$

and from the latter relations, we may deduce the useful symmetric equality

$$(1 - \lambda v/c)\,(1 + \mu v/c) = [1 - (v/c)^2]. \tag{7.15}$$

Now, in terms of $\xi = \lambda x + ct$, the wave-like solution (5.1) has momentum $p(x,t)$ and energy $e(x,t)$ given by

$$p(x,t) = f_0 \xi / c, \qquad e(x,t) = \left(e_0^2 + (f_0 \xi)^2\right)^{1/2},$$

while from the above inverse energy-momentum relations (7.10) and $(7.12)_3$ we may deduce

$$P = \frac{F_0 \eta + \left(e_0^2 + (F_0 \eta)^2\right)^{1/2}(v/c)}{c[1-(v/c)^2]^{1/2}}, \qquad E = \frac{\left(e_0^2 + (F_0 \eta)^2\right)^{1/2} + F_0 \eta (v/c)}{[1-(v/c)^2]^{1/2}}.$$

Thus, from these equations with $\xi = \lambda x + ct$ and $\eta = \mu X + cT$, we may perform the partial differentiations in (7.11) to demonstrate explicitly that

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = f_0\left(1 + \frac{\lambda f_0 \xi}{\left(e_0^2 + (f_0\xi)^2\right)^{1/2}}\right),$$

$$g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = \frac{f_0}{c}\left(\frac{f_0\xi}{\left(e_0^2 + (f_0\xi)^2\right)^{1/2}} + \lambda\right),$$

while on simplifying the partial derivatives, we obtain

$$\frac{\partial P}{\partial T} + \frac{\partial E}{\partial X} = \frac{F_0}{[1-(v/c)^2]^{1/2}}\left(1 + \frac{\mu v}{c} + \frac{F_0\eta(\mu + v/c)}{\left(e_0^2 + (F_0\eta)^2\right)^{1/2}}\right),$$

$$\frac{1}{c^2}\frac{\partial E}{\partial T} + \frac{\partial P}{\partial X} = \frac{F_0}{c[1-(v/c)^2]^{1/2}}\left(\frac{F_0\eta(1 + \mu v/c)}{\left(e_0^2 + (F_0\eta)^2\right)^{1/2}} + \mu + \frac{v}{c}\right),$$

and on writing $F_0 = \sigma f_0$ in the latter expressions, using the above relations (7.13) and making the recognition that $f_0\xi = F_0\eta$, we may conclude that the two expressions for the forces coincide, as indeed they must.

7.5 Functional Dependence of Forces

In a prior section we have established that the general force equations (7.1) for f and cg are Lorentz invariant, and in the previous section, we have verified explicitly this Lorentz invariance for the exact wave-like solution. In this section we investigate the functional dependence under Lorentz transformation of the linear force expressions (5.2) arising from the wave-like solution, namely the forces $f(u)$ and $cg(u)$ are given by

$$f(u) = f_0(1 + \lambda u/c), \qquad cg(u) = f_0(\lambda + u/c). \tag{7.16}$$

Making use of the relations (7.12), (7.13) and (7.14)

$$f(u) = f_0(1 + \lambda u/c) = \frac{F_0\,(1 + \mu v/c)}{[1-(v/c)^2]^{1/2}}\left(1 + \frac{(\mu + v/c)(U/c - v/c)}{(1 + \mu v/c)(1 - Uv/c^2)}\right),$$

and

$$cg(u) = f_0(\lambda + u/c) = \frac{F_0\,(1 + \mu v/c)}{[1-(v/c)^2]^{1/2}}\left(\frac{(\mu + v/c)}{(1 + \mu v/c)} + \frac{(U/c - v/c)}{(1 - Uv/c^2)}\right),$$

which upon simplification become

$$f(u)=f_0\left(1+\frac{\lambda u}{c}\right)=F_0\frac{[1-(v/c)^2]^{1/2}}{(1-Uv/c^2)}\left(1+\frac{\mu U}{c}\right),\tag{7.17}$$

$$cg(u)=f_0\left(\lambda+\frac{u}{c}\right)=F_0\frac{[1-(v/c)^2]^{1/2}}{(1-Uv/c^2)}\left(\mu+\frac{U}{c}\right),$$

which indicate a different functional dependence on velocity in each of the Lorentz frames. However, on using the identity for velocity addition (2.7), namely

$$[1-(u/c)^2]^{1/2}(1-Uv/c^2)=[1-(v/c)^2]^{1/2}[1-(U/c)^2]^{1/2},$$

we may deduce the rather intriguing Lorentz invariant expressions

$$\frac{f_0\,(1+\lambda u/c)}{\left(1-(u/c)^2\right)^{1/2}}=\frac{F_0\,(1+\mu U/c)}{\left(1-(U/c)^2\right)^{1/2}},$$

$$\frac{f_0(\lambda+u/c)}{\left(1-(u/c)^2\right)^{1/2}}=\frac{F_0(\mu+U/c)}{\left(1-(U/c)^2\right)^{1/2}}.$$

If we identify forces F and G in the Lorentz frame, thus

$$F(U)=F_0\left(1+\frac{\mu U}{c}\right),\qquad cG(U)=F_0\left(\mu+\frac{U}{c}\right),$$

then the immediately above formulae indicate that the Lorentz invariants are mass times force or energy times force; thus, $ef=EF$ and $eg=EG$. This may be independently confirmed from the force relations (7.16) by multiplication of the energy e and making use of the previously derived Lorentz invariant energy-momentum relations (2.46), which have inverse relations given by (7.10), thus

$$p=\frac{P-Ev/c^2}{[1-(v/c)^2]^{1/2}},\quad e=\frac{E-Pv}{[1-(v/c)^2]^{1/2}}.\tag{7.18}$$

Directly from $(7.16)_1$ and (7.18) we have

$$f(u)e=f_0(e+\lambda pc)=\frac{f_0}{(1-(v/c)^2)^{1/2}}\left(\left(1-\frac{\lambda v}{c}\right)E+\left(\lambda-\frac{v}{c}\right)Pc\right)$$
$$=\frac{f_0(1-\lambda v/c)}{(1-(v/c)^2)^{1/2}}(E+\mu Pc)=\sigma f_0(E+\mu Pc)=F_0(E+\mu Pc),$$

and similarly, directly from $(7.16)_2$ we have

$$cg(u)e = f_0(\lambda e + pc) = \frac{f_0}{(1-(v/c)^2)^{1/2}}\left(\left(\lambda - \frac{v}{c}\right)E + \left(1 - \frac{\lambda v}{c}\right)Pc\right)$$
$$= \frac{f_0(1-\lambda v/c)}{(1-(v/c)^2)^{1/2}}(Pc + \mu E) = \sigma f_0(Pc + \mu E) = F_0(Pc + \mu E),$$

where $F_0 = f_0\sigma$ and μ and σ are as previously defined by (7.8), that is,

$$\mu = \left(\frac{\lambda - v/c}{1 - \lambda v/c}\right), \qquad \sigma = \frac{(1-\lambda v/c)}{[1-(v/c)^2]^{1/2}}.$$

In a prior section we have established that the operator arising in the energy equations (3.10) and (3.9) is Lorentz invariant for any three-dimensional spatial vector $\mathbf{q}$, and for a single dimension we verified directly that, under a Lorentz transformation, we have

$$\frac{\partial q}{\partial t}dx + c^2\frac{\partial q}{\partial x}dt = \frac{\partial q}{\partial T}dX + c^2\frac{\partial q}{\partial X}dT, \tag{7.19}$$

which applies for an arbitrary scalar q. In terms of the wave-like solution, this invariance characteristic is revealed in the following explicit manner. For a single spatial dimension, the total energy as defined by (3.8) becomes

$$dW = f\,dx + gc^2dt$$
$$= f_0\left[\left(1 + \frac{\lambda u}{c}\right)dx + \left(\lambda + \frac{u}{c}\right)cdt\right]$$
$$= F_0\left[\left(1 + \frac{\mu u}{c}\right)dX + \left(\mu + \frac{u}{c}\right)cdT\right],$$

noting especially that, in the latter equation, while both the parameters F_0 and μ and the variables (X, T) all refer to the Lorentz frame, the velocity u is that which applies in the (x, t) frame, that is, it is lower case u and not upper case U, reflecting the invariant nature of the operator (7.19). This result follows immediately from the above relations (7.17) and the Lorentz transformations, either equation (2.3) or above Eq. (7.7), thus

$$f_0\left[\left(1 + \frac{\lambda u}{c}\right)dx + \left(\lambda + \frac{u}{c}\right)cdt\right]$$
$$= F_0\frac{[1-(v/c)^2]^{1/2}}{(1-Uv/c^2)}\left[\left(1 + \frac{\mu U}{c}\right)\frac{(dX - vdT)}{[1-(v/c)^2]^{1/2}} + \left(\mu + \frac{U}{c}\right)\frac{(cdT - vdX/c)}{[1-(v/c)^2]^{1/2}}\right]$$
$$= \frac{F_0}{(1-Uv/c^2)}\left(\left(1 - \frac{Uv}{c^2}\right) + \mu\left(\frac{U}{c} - \frac{v}{c}\right)\right)dX$$
$$+ \frac{F_0}{(1-Uv/c^2)}\left(\mu\left(1 - \frac{Uv}{c^2}\right) + \left(\frac{U}{c} - \frac{v}{c}\right)\right)cdT,$$

and the desired result follows upon simplification and using the Einstein addition of velocity law (2.6).

7.6 Transformation $x' = ct$ and $t' = x/c$

Formally, de Broglie's particle-wave velocity relation $w = c^2/u$ arises from the underlying space-time transformation $x' = ct$ and $t' = x/c$, for which $w = u' = dx'/dt' = c^2 dt/dx = c^2/u$ and which we have previously noted in Chapt. 2 that it has been used to connect the Galilean and Carroll transformations as significant limits of Lorentz invariant theories. Under this transformation, there are two important and distinct restricted energy interchanges that may be accommodated. In this section, we examine the interchange of the two energies $\mathscr{E}(x,t)$ and momentum $cp(x',t')$, and we also show that the operator arising in (3.9) and (3.10) remains invariant. In the final section of the chapter, we examine the interchange of the particle and wave energies $e(x,t)$ and $\mathscr{E}(x',t')$, and then we combine the two distinct energy interchanges to produce an invariant transformation of the basic equations (3.4).

In this section, we first show that the one space dimension equations (4.3), namely

$$\frac{\partial \mathscr{E}}{\partial t} = c^2 \frac{\partial p}{\partial x}, \qquad \frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t}, \tag{7.20}$$

admit the relations $\mathscr{E}' = \mathscr{E}(x',t') = cp(x,t)$ and $cp' = cp(x',t') = \mathscr{E}(x,t)$. This restricted symmetry might be motivated from the simple exact solution of these equations that underlies the previously discussed exact wave-like solution and which is given by

$$\mathscr{E}(x,t) = f_0(x + c\lambda t), \qquad cp(x,t) = f_0(\lambda x + ct), \tag{7.21}$$

where f_0 and λ denote arbitrary constants. Under the space-time transformation $x' = ct$ and $t' = x/c$, the above solution (7.21) becomes

$$\mathscr{E}(x,t) = f_0(x + c\lambda t) = f_0(\lambda x' + ct') = cp(x',t'),$$
$$cp(x,t) = f_0(\lambda x + ct) = f_0(x' + c\lambda t') = \mathscr{E}(x',t').$$

Thus, under certain circumstances, the wave-energy $\mathscr{E}(x,t)$ and the momentum $cp(x,t)$ interchange under the space-time transformation $x' = ct$ and $t' = x/c$, that is, we have $\mathscr{E}(x',t') = cp(x,t)$ and $cp(x',t') = \mathscr{E}(x,t)$. This might be verified directly from (4.3) or (7.20), since first on changing x and t these equations become

$$\frac{\partial \mathscr{E}}{\partial x'} = \frac{\partial p}{\partial t'}, \qquad \frac{\partial \mathscr{E}}{\partial t'} = c^2 \frac{\partial p}{\partial x'},$$

and then a further change $\mathscr{E} = cp' = cp(x', t')$ and $cp = \mathscr{E}' = \mathscr{E}(x', t')$ produces

$$\frac{\partial \mathscr{E}'}{\partial x'} = \frac{\partial p'}{\partial t'}, \qquad \frac{\partial \mathscr{E}'}{\partial t'} = c^2 \frac{\partial p'}{\partial x'},$$

showing that Eqs. (4.3) allow the transformation property $\mathscr{E}' = \mathscr{E}(x', t') = cp(x, t)$ and $cp = cp(x', t') = \mathscr{E}(x, t)$. Of course this outcome is not entirely unexpected, since the system of Eqs. (4.3) is equivalent to the classical one spatial dimension wave equation, which is left invariant by the transformation $x' = ct$ and $t' = x/c$. It is important to emphasise again that this transformation involves only one Cartesian spatial dimension, and that the appropriate extension and interpretation to other coordinates involving curvature, is by no means obvious. Even with this in mind, the simple exact solution (7.21) is still important because it is indicative of a significant outcome.

On using the relations $x' = ct$, $t' = x/c$, $\mathscr{E}' = \mathscr{E}(x', t') = cp(x, t)$ and $cp' = cp(x', t') = \mathscr{E}(x, t)$, the primed version of Eq. (8.45), namely

$$d\mathscr{E}' = \frac{\partial \mathscr{E}'}{\partial x'} dx' + \frac{\partial \mathscr{E}'}{\partial t'} dt' = \frac{\partial p'}{\partial t'} dx' + c^2 \frac{\partial p'}{\partial x'} dt',$$

becomes precisely equation (8.46). Further, the primed versions of Eqs. (7.1) become

$$f' = \frac{\partial p'}{\partial t'} + \frac{\partial e'}{\partial x'} = \frac{\partial \mathscr{E}}{\partial x} + \frac{1}{c}\frac{\partial e'}{\partial t} = \frac{\partial p}{\partial t} + \frac{1}{c}\frac{\partial e'}{\partial t} = \frac{\partial \left(p + e'/c\right)}{\partial t},$$

$$c^2 g' = \frac{\partial e'}{\partial t'} + c^2 \frac{\partial p'}{\partial x'} = c\frac{\partial e'}{\partial x} + \frac{\partial \mathscr{E}}{\partial t} = c\frac{\partial e'}{\partial x} + c^2 \frac{\partial p}{\partial x} = c^2 \frac{\partial \left(p + e'/c\right)}{\partial x}.$$

Thus, if the forces f and g are generated through a potential $V(x, t)$ through the equations $f = -\partial V/\partial x$ and $gc^2 = -\partial V/\partial t$, then the primed versions of these equations become

$$f' = -\frac{\partial V'}{\partial x'} = -\frac{1}{c}\frac{\partial V'}{\partial t} = \frac{\partial \left(p + e'/c\right)}{\partial t},$$

$$c^2 g' = -\frac{\partial V'}{\partial t'} = -c\frac{\partial V'}{\partial x} = c^2 \frac{\partial \left(p + e'/c\right)}{\partial x},$$

which may be integrated to yield $V'(x', t') = -(e' + pc) + constant$, which is simply the statement of conservation of energy in the primed variables, namely $e + \mathscr{E} + V(x, t) = constant$, noting that $V' = V(x', t') = V(ct, x/c)$.

Further, for a single spatial dimension, the two basic equations (3.4) or (7.1) become simply

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = \frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x},$$
$$c^2 g = \frac{\partial e}{\partial t} + c^2\frac{\partial p}{\partial x} = \frac{d\mathscr{E}}{dt} = \frac{\partial \mathscr{E}}{\partial t} + u\frac{\partial \mathscr{E}}{\partial x},$$

so that the primed version of the latter equalities becomes

$$f' = \frac{\partial p'}{\partial t'} + u'\frac{\partial p'}{\partial x'} = \frac{1}{u}\left(\frac{\partial \mathscr{E}}{\partial t} + u\frac{\partial \mathscr{E}}{\partial x}\right) = \frac{1}{u}\frac{d\mathscr{E}}{dt} = \frac{c^2 g}{u},$$
$$c^2 g' = \frac{\partial \mathscr{E}'}{\partial t'} + u'\frac{\partial \mathscr{E}'}{\partial x'} = \frac{c^2}{u}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) = \frac{c^2}{u}\frac{dp}{dt} = \frac{c^2 f}{u}.$$

These equations give rise to the following interesting force transformation relations:

$$\frac{f'}{(u'/c)^{1/2}} = \frac{cg}{(u/c)^{1/2}}, \qquad \frac{cg'}{(u'/c)^{1/2}} = \frac{f}{(u/c)^{1/2}},$$

as compared to the corresponding relations arising from a Newtonian interpretation of the proposed model, which are given by Eq. (3.36). Clearly, these are the primed versions of each other, and on multiplication of the first equation by dx' and the second equation by dt', we might deduce

$$\frac{f'dx'}{(u'/c)^{1/2}} = \frac{c^2 g dt}{(u/c)^{1/2}}, \qquad \frac{c^2 g' dt'}{(u'/c)^{1/2}} = \frac{f dx}{(u/c)^{1/2}},$$

which simplify to become

$$\frac{de'}{(u'/c)^{1/2}} = \frac{d\mathscr{E}}{(u/c)^{1/2}}, \qquad \frac{d\mathscr{E}'}{(u'/c)^{1/2}} = \frac{de}{(u/c)^{1/2}}.$$

Noting that $\mathscr{E}'(x', t') = cp(x, t)$, the second relation is merely a restatement of the rate-of-working equation $de/dp = u$, and the first relation is the primed version of this equation.

Finally, the relations $\mathscr{E}' = \mathscr{E}(x', t') = cp(x, t)$ and $cp' = cp(x', t') = \mathscr{E}(x, t)$ are consistent with the exact wave-like solution (7.21) discussed in previous chapters, for which $d\mathscr{E}/dp$ is given by (5.8), thus

$$\frac{d\mathscr{E}}{dp} = c\left(\frac{\lambda + u/c}{1 + \lambda u/c}\right), \tag{7.22}$$

arising from the specific forces f and g given by Eq. (5.2), namely

$$f(u) = f_0(1 + \lambda u/c), \qquad cg(u) = f_0(\lambda + u/c), \tag{7.23}$$

where f_0 denotes an arbitrary constant. In the primed variables equation (7.22) becomes

$$\frac{d\mathscr{E}'}{dp'} = c\left(\frac{\lambda + u'/c}{1 + \lambda u'/c}\right) = c\left(\frac{1 + \lambda u/c}{\lambda + u/c}\right) = c^2\frac{dp}{d\mathscr{E}},$$

from which on rearrangement yields the unprimed version (7.22).

7.7 Force Invariance Under Superluminal Lorentz Frames

In this section we establish that the proposed basic force equations (3.4) also remain unchanged under Lorentz frames moving with constant velocities v that are in excess of the speed of light. In de Broglie's Nature article on waves and quanta, previously quoted in Chap. 3 (Nature, no. 2815, Vol. 112, October 13, 1923, page 540), he writes:

> The quantum energy-frequency relation $e = h\nu$, leads one to associate a periodical phenomena with any isolated portion of matter or energy. An observer bound to the portion of matter will associate with it a frequency ν_0 determined by its internal energy, namely, by its "matter at rest" ($e_0 = h\nu_0$ where $e_0 = m_0c^2$). An observer for whom the portion of matter is in steady motion with velocity u, will see this frequency lower in consequence of the Lorentz-Einstein time transformation. I have been able to show (Comptes Rendus, September 10th and 24th, of the Paris Academy of Sciences) that the fixed observer will constantly see the internal periodical phenomenon in phase with a wave the frequency given by $h\nu = e_0/(1-(u/c)^2)^{1/2}$ which is determined by the quantum relation using the whole energy of the moving body, provided that it is assumed that the wave spreads with the velocity c^2/u. This wave, the velocity of which is greater than c, cannot carry energy.

> A radiation of frequency ν has to be considered as divided into atoms of light of very small internal mass ($\approx 10^{-10}gm.$) which move with a velocity u very nearly equal to that determined by the relation $h\nu = e_0/(1-(u/c)^2)^{1/2}$. The atom of light slides slowly upon the non-material wave, the frequency of which is ν and velocity c^2/u, very little higher that c.

It is clear from this commentary that the de Broglie hypothesis inevitably involves at least an appreciation of superluminal waves, and therefore some understanding of superluminal motion in general. These matters are naturally controversial and are examined at length in Hill and Cox [54], including a new derivation of Lorentz space-time transformations applicable at superluminal velocities. Subsequent to the publication of [54], the work of Vieira [104] appeared at about the same time, who proposed two alternative derivations of the same extended Lorentz transformations, one algebraic in character and the other geometric. Accordingly, there is some commonality of agreement in the extended Lorentz transformations for superluminal motion and in the basic equations underlying superluminal motion. After the publication of [54], Andreka et al [4] provided another derivation of the same extended Lorentz transformations, and [62] point out that these extended Lorentz transformations are not entirely new and have a prior history (see also [70]). In these earlier contributions, the transformations for $v > c$ are presented but not formally derived as such, and the main original contribution of [54] is the mode of their derivation as arising from the tangent vector for special relativity, combined with an initial condition for infinite relative velocity. In summary then, superluminal Lorentz transformations do exist, and while the topic as a whole remains controversial, there is some agreement on some of the basic equations, which are needed in any discussion on de Broglie's particle-wave duality.

We now establish that the force equations (3.4) remain invariant under Lorentz frames moving with constant superluminal velocities v. For superluminal velocities v such that $c \leqslant v < \infty$, Hill and Cox [54] show that the extended Lorentz transformations become

$$X = \frac{-\epsilon(x + vt)}{((v/c)^2 - 1)^{1/2}}, \quad T = \frac{-\epsilon(t + vx/c^2)}{((v/c)^2 - 1)^{1/2}}, \tag{7.24}$$

with the inverse transformation characterised by $-v$, thus

$$x = \frac{\epsilon(X - vT)}{((v/c)^2 - 1)^{1/2}}, \quad t = \frac{\epsilon(T - vX/c^2)}{((v/c)^2 - 1)^{1/2}},$$

where $\epsilon^2 = 1$ and $\epsilon = \pm 1$ according to the assumed behaviour at $v = \infty$, namely

$$x = -\epsilon c T, \quad t = -\epsilon X/c.$$

For these superluminal Lorentz transformations, a key feature is that the Einstein addition of velocities in its various forms, namely Eqs. (2.6), (2.7) and (2.12), are left unchanged, and with the implication that at least one of the three velocities u, v and U must be less than the speed of light. From (2.7) it is apparent that if the frame velocity v exceeds the speed of light, then one, and only one, of u or U will be superluminal and the other sub-luminal. Further, for superluminal Lorentz frames, the Lorentz invariant energy-momentum relations become

$$P = \frac{-\epsilon(p + ev/c^2)}{((v/c)^2 - 1)^{1/2}}, \quad E = \frac{-\epsilon(e + pv)}{((v/c)^2 - 1)^{1/2}},$$

along with the inverse relations

$$p = \frac{\epsilon(P - Ev/c^2)}{((v/c)^2 - 1)^{1/2}}, \quad e = \frac{\epsilon(E - Pv)}{((v/c)^2 - 1)^{1/2}}. \tag{7.25}$$

From the above relations (7.24) and (7.25), it is now a straightforward matter to show that for one spatial dimension x, the proposed Eqs. (3.4), namely

$$f=\frac{\partial p}{\partial t}+\frac{\partial e}{\partial x},\qquad g=\frac{1}{c^2}\frac{\partial e}{\partial t}+\frac{\partial p}{\partial x},$$

are also invariant under the above superluminal Lorentz group. More precisely, for both values of ϵ, the two Eqs. (3.4) have the same form in all superluminal Lorentz frames, namely

$$f=\frac{\partial p}{\partial t}+\frac{\partial e}{\partial x}=\frac{\partial P}{\partial T}+\frac{\partial E}{\partial X},\qquad g=\frac{1}{c^2}\frac{\partial e}{\partial t}+\frac{\partial p}{\partial x}=\frac{1}{c^2}\frac{\partial E}{\partial T}+\frac{\partial P}{\partial X},$$

which we establish as follows. From Eq. (7.24) we have the differential relations

$$\frac{\partial}{\partial x}=\frac{-\epsilon}{((v/c)^2-1)^{1/2}}\left\{\frac{\partial}{\partial X}+\frac{v}{c^2}\frac{\partial}{\partial T}\right\},\quad \frac{\partial}{\partial t}=\frac{-\epsilon}{((v/c)^2-1)^{1/2}}\left\{\frac{\partial}{\partial T}+v\frac{\partial}{\partial X}\right\},$$

so that on using (7.25) and employing the subscript notation to denote partial derivatives, for the spatial physical force f, we have

$$\begin{aligned}
&p_t+e_x=\\
&\frac{-1}{((v/c)^2-1)}\left\{(P_T-\frac{v}{c^2}E_T)+v(P_X-\frac{v}{c^2}E_X)+(E_X-vP_X)+\frac{v}{c^2}(E_T-vP_T)\right\}\\
&=\ P_T+E_X,
\end{aligned}$$

and similarly for the force in the direction of time g, we have

$$\begin{aligned}
&\frac{1}{c^2}e_t+p_x=\\
&\frac{-1}{((v/c)^2-1)}\left\{\frac{1}{c^2}(E_T-vP_T)+\frac{v}{c^2}(E_X-vP_X)+(P_X-\frac{v}{c^2}E_X)+\frac{v}{c^2}(P_T-\frac{v}{c^2}E_T)\right\}\\
&=\ \frac{1}{c^2}E_T+P_X.
\end{aligned}$$

Again this is an important key outcome, which tends to reinforce the belief that the two Eqs. (3.4) are well formulated.

7.8 Particle and Wave Energies and Momenta

In this section, we detail expressions for the particle and wave energies and momenta denoted, respectively, by e, $\mathscr{E}$, pc and $\mathscr{P}$. We deal with particle energy $e(x,t)$, particle momentum $p(x,t)$, wave energy $\mathscr{E}(x,t)$ and wave momentum $\mathscr{P}(x,t)$, and we frequently loosely refer to the energy pc as the momentum, since e, pc and $\mathscr{E}$ constitute the three basic energies. For sub-luminal particle velocities $0\leqslant u/c<1$, the particle momentum p couples the particle and wave energies e and $\mathscr{E}$, while for superluminal particle velocities $1 < u/c < \infty$, it is the wave momentum $\mathscr{P}$ that acts to couple the wave and particle energies $\mathscr{E}$ and e, respectively.

The space-time transformation $x' = ct$ and $t' = x/c$ formally arises from the superluminal Lorentz transformations given in [54], in the limit of the frame velocity $v \longrightarrow \infty$, and moreover, the developments of [54] are based on the key premise that for any two inertial frames separated by an infinite relative velocity, the product of the two perceived velocities of the same physical incident equals the velocity of light squared. Based upon this assumption, the Einstein addition law for velocities still holds for superluminal motions and leads to the important physical notion of two worlds separated by the speed of light c. In a subsequent publication, Hill and Cox [55] infer the coexistence of two worlds comprising, with respect to some fixed inertial frame, all the sub-luminal Lorentz frames and all the superluminal Lorentz frames. From the perspective of special relativity, an observer in either world would be unable to detect in which of the two worlds they existed, since from either perspective both appear identical. This leads to the idea that not only that the speed of light is relative to all observers in frames, which are sub-luminal to some reference frame, but also there is a separate and symmetric reciprocal set of frames of reference, which have an identical world view, in the sense that observers in the reciprocal frames consider their velocities to be sub-luminal and those from the original set of frames to be superluminal.

As described in Hill and Cox [55], this duality is very suggestive of a theory of relativity, which transcends the usual notion of uniform measurement of the speed of light. In this meta-relativity, the speed of light is not only constant for all observers, but it also provides a boundary that separates two distinct but equivalent sets of frames. Each frame from one set is associated with a single frame from the other set by the reciprocal relationship, a relationship that holds independent of the specific frame from which the measurement of velocities is made. Furthermore, each frame from both sets measures the relative velocities as sub-luminal for all the frames in their own set and superluminal for all the frames in the reciprocal set. The only velocity that lacks a distinct reciprocal is the speed of light itself, which is its own reciprocal and is not properly a member of either set. The implications to causality are that either set of frames can interact with all others in their set without violating the conventional notion of causality, since all velocities are less than c for an observer at rest in any of the frames in a single set.

These considerations inevitably lead to the conclusion that the superluminal world comprises the world of superluminal waves and that the notions of either sub-luminal particles or superluminal waves are interchangeable and are themselves relative notions dependent upon the frame of reference. Accordingly, particle and wave velocities in the unprimed (x, t) frame are perceived as, respectively, wave and particle velocities in the primed (x', t') frame, with the implication that particle energy $e(x, t)$ in the unprimed (x, t) frame "coincides" with the wave energy $\mathscr{E}(x', t')$ in the primed (x', t') frame. Similarly, wave energy $\mathscr{E}(x, t)$ in the unprimed (x, t) frame "coincides" with the particle energy $e(x', t')$ in the primed (x', t') frame. To be more precise, in the formulae given below, there are two separate effects that have been accommodated. Firstly, the perceived particle and wave energies in the primed frame become, respectively, $e' = e(x', t') = \mathscr{E}(x, t)$ and $\mathscr{E}' = \mathscr{E}(x', t') = e(x, t)$, and secondly, a further transformation $\mathscr{E}' = \mathscr{E}(x', t') = cp(x, t)$ and $cp' = cp(x', t') = \mathscr{E}(x, t)$ has been applied, which is necessary to bring the quantities back to the unprimed frame.

The Lorentz invariant alternative to Newton's second law developed by the author in [47–52] is based upon the following structure. In the (x, t) frame, a particle is assumed to be moving with velocity $u = dx/dt$, where $0 \leqslant u/c < 1$, with an associated de Broglie wave, which is moving with a superluminal wave velocity $w = c^2/u$. Assuming the standard Einstein formulae $e = mc^2$ and $m(u) = m_0(1 - (u/c)^2)^{-1/2}$, where m_0 denotes the rest mass, so that with $e_0 = m_0c^2$, the particle energy $e = e(x, t)$ and particle momentum $p = p(x, t)$ are given by

$$e = \frac{e_0}{(1 - (u/c)^2)^{1/2}}, \quad pc = \frac{e_0 u}{c(1 - (u/c)^2)^{1/2}}, \quad \frac{\partial \mathscr{E}}{\partial t} = c^2 \frac{\partial p}{\partial x}, \quad \frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t},$$

where $\mathscr{E} = \mathscr{E}(x, t)$ is the associated wave energy, which is determined from the stated coupled partial differential equations through the momentum p. Note that $\mathscr{E} = \mathscr{E}(x, t)$ satisfies the classical wave equation and that the coupled partial differential equations remain unchanged by the transformation $\mathscr{E}' = \mathscr{E}(x', t') = cp(x, t)$ and $cp' = cp(x', t') = \mathscr{E}(x, t)$.

To summarise the above discussion, the unprimed (x, t) frame and primed (x', t') frame are characterised as follows. Particle and wave velocities in the unprimed (x, t) frame become, respectively, wave and particle velocities in the primed (x', t') frame, and in both frames the "perceived" particles are assumed to follow the Einstein formulation with respective particle energy $e = mc^2$ or $e' = m'c^2$ and particle momentum $p = mu$ or $p' = m'u'$, so that altogether we have:

- In the unprimed (x, t) frame, a particle is moving with velocity $u = dx/dt$, particle energy $e = e(x, t)$, and there is an associated wave with wave velocity $w = c^2/u$ and wave energy $\mathscr{E} = \mathscr{E}(x, t)$, which is determined from the classical wave equation.
- In the primed (x', t') frame, a particle is moving with velocity $u' = dx'/dt' = c^2/u = w$, particle energy $e' = e(x', t') = \mathscr{E}(x, t)$, and there is an associated wave with wave velocity $u = c^2/w$ and wave energy $\mathscr{E}' = \mathscr{E}(x', t') = e(x, t)$, which is determined from the classical wave equation.
- The interchange of particle and wave energies is then followed by the application of a further transformation, namely $\mathscr{E}' = \mathscr{E}(x', t') = cp(x, t)$ and $cp' = cp(x', t') = \mathscr{E}(x, t)$.
- These two steps when combined produce the following transformation, which is subsequently shown to be a symmetry of the basic model given by (3.4), thus

$$e' \longrightarrow cp, \qquad cp' \longrightarrow e, \qquad \mathscr{E}' \longrightarrow c\mathscr{P}. \tag{7.26}$$

On adopting the standard Einstein formulae $e = mc^2$ and $m(u) = m_0(1 - (u/c)^2)^{-1/2}$, where m_0 denotes the rest mass, then arising from the superluminal extension provided in [54], the key relations for particle and wave energies and momenta may be summarised as follows:

$$\begin{cases} e = \dfrac{e_0}{(1-(u/c)^2)^{1/2}}, \\ e = \mathscr{E} = pc = \mathscr{P}c, \\ \mathscr{E} = \dfrac{\mathscr{E}_\infty (u/c)}{((u/c)^2-1)^{1/2}}, \end{cases} \qquad \begin{cases} pc = \dfrac{e_0 u}{c(1-(u/c)^2)^{1/2}}, & 0 \leqslant u/c < 1, \\ pc = \mathscr{P}c = e = \mathscr{E}, & u/c = 1, \\ \mathscr{P}c = \dfrac{\mathscr{E}_\infty}{((u/c)^2-1)^{1/2}}, & 1 < u/c < \infty, \end{cases} \tag{7.27}$$

with the complimentary energy, either $\mathscr{E}(x,t)$ or $e(x,t)$, is assumed to be determined from coupled first-order partial differential equations through the corresponding momenta $p(x,t)$ or $\mathscr{P}(x,t)$, or from the classical wave equation, thus

$$\begin{cases} \dfrac{\partial \mathscr{E}}{\partial t} = c^2 \dfrac{\partial p}{\partial x}, \quad \dfrac{\partial \mathscr{E}}{\partial x} = \dfrac{\partial p}{\partial t}, \\ \quad e = \mathscr{E} = pc = \mathscr{P}c, \\ \dfrac{\partial e}{\partial t} = c^2 \dfrac{\partial \mathscr{P}}{\partial x}, \quad \dfrac{\partial e}{\partial x} = \dfrac{\partial \mathscr{P}}{\partial t}, \end{cases} \qquad \begin{cases} \dfrac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \dfrac{\partial^2 \mathscr{E}}{\partial x^2}, & 0 \leqslant u/c < 1, \\ e = \mathscr{E} = pc = \mathscr{P}c, & u/c = 1, \\ \dfrac{\partial^2 e}{\partial t^2} = c^2 \dfrac{\partial^2 e}{\partial x^2}, & 1 < u/c < \infty, \end{cases} \tag{7.28}$$

where $e_0 = m_0 c^2$ and $\mathscr{E}_\infty$ denote the limiting values of the particle and wave energies in the limits u tending to zero and infinity, respectively, and in general $\mathscr{E}_\infty \neq e_0$. However, if we require consistency for the determination of the de Broglie internal frequency, namely the two sets of equations should coincide,

$$\lambda \nu = \frac{c^2}{u}, \qquad h\nu = e(x,t) = \frac{e_0}{(1-(u/c)^2)^{1/2}},$$

and

$$\lambda' \nu' = \frac{c^2}{w}, \qquad h\nu' = e(x',t') = c\mathscr{P}(x,t) = \frac{\mathscr{E}_\infty}{((w/c)^2-1)^{1/2}},$$

then we require $\mathscr{E}_\infty = e_0$, noting that for parity in the second set of equations we are using $w = c^2/u = u'$ for the perceived particle velocity in the primed frame, and in the final equation we have utilised the equivalent version of the transformation $\mathscr{E}' = \mathscr{E}(x',t') = cp(x,t)$ and $cp' = cp(x',t') = \mathscr{E}(x,t)$, but with e in place of $\mathscr{E}$ and $\mathscr{P}$ in place of p, respectively. We also have in mind that for a simple wave of wavelength λ and frequency ν in the unprimed (x,t) frame with $\lambda\nu = c^2/u$, as noted previously in Chap. 2 (see Eqs. (2.39), (2.40) and (2.41)), the wavelength and frequency transform according to the formulae $\lambda' = c/\nu$ and $\nu' = c/\lambda$ so that $\lambda'\nu' = u$. In this formulation, other than generating insight into the role interchange of particles and waves, there can be no material advantage in adopting either the sub-luminal or superluminal set of frames, since any perceived advantage would imply a loss of complete parity.

We also note that the above relations (7.27) satisfy the basic rate-of-working equations and energy statements

$$\begin{cases} de = u dp, \\ d\mathscr{E} = \dfrac{c^2}{u} d\mathscr{P}, \end{cases} \qquad \begin{cases} e^2 = (pc)^2 + e_0^2, & 0 \leqslant u/c < 1, \\ \mathscr{E}^2 = (c\mathscr{P})^2 + \mathscr{E}_\infty^2, & 1 < u/c < \infty, \end{cases}$$

with $e_0 = \mathscr{E}_\infty = 0$ when $u/c = 1$.

We now show that the space-time transformation $x' = ct$ and $t' = x/c$ with $u' = c^2/u$ and the symmetry (7.26) involving the interchange of the energy and momentum expressions across the page, namely

$$e' \longrightarrow cp, \qquad cp' \longrightarrow e, \qquad \mathscr{E}' \longrightarrow c\mathscr{P}, \tag{7.29}$$

may be confirmed independently as follows. The general Lorentz invariant equations for the three energies $e(x, t)$, $cp(x, t)$ and $\mathscr{E}(x, t)$ for a single spatial dimension become simply

$$f = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{d\mathscr{E}}{dt} = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{7.30}$$

where $f(x, t)$ and $cg(x, t)$ denote the applied external forces in the spatial and time directions, respectively. Now the primed version of the above Eqs. (7.30), namely

$$f' = \frac{dp'}{dt'} = \frac{\partial p'}{\partial t'} + \frac{\partial e'}{\partial x'}, \qquad g' = \frac{1}{c^2}\frac{d\mathscr{E}'}{dt'} = \frac{1}{c^2}\frac{\partial e'}{\partial t'} + \frac{\partial p'}{\partial x'},$$

under the symmetry (7.29), becomes, respectively,

$$f' = \frac{de}{dx} = \frac{1}{u}\frac{de}{dt} = \frac{dp}{dt} = \frac{\partial e}{\partial x} + \frac{\partial p}{\partial t},$$
$$g' = \frac{d\mathscr{P}}{dx} = \frac{1}{u}\frac{d\mathscr{P}}{dt} = \frac{1}{c^2}\frac{d\mathscr{E}}{dt} = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x},$$

and therefore Eqs. (7.30) are left invariant under this symmetry, and moreover, $f' = f$ and $g' = g$.

As previously described, we may view the symmetry (7.29) as arising from the combination of two transformations. Firstly, in the primed frame there is an interchange of the particle and wave energies e and $\mathscr{E}$, thus $e' = \mathscr{E}(x', t')$ and $\mathscr{E}' = e(x', t')$. This interchange is then followed by an application of a limited symmetry of the Eqs. (7.20), namely $\mathscr{E}' = \mathscr{E}(x', t') = cp(x, t)$ and $cp' = cp(x', t') = \mathscr{E}(x, t)$, and we use the term limited to convey the fact that this is a symmetry of (7.20) but not of (7.30). These two transformations then combined produce the symmetry (7.29).

For the velocity u $(0 < u/c < \infty)$, in terms of the momenta pc and $c\mathscr{P}$, we find from (7.27) and (7.28) that the full set of transformations for the particle velocities u/c, along with the particle and wave energies e and $\mathscr{E}$, become

$$u/c = \begin{cases} \dfrac{pc}{((pc)^2 + e_0^2)^{1/2}}, \\ 1, \\ \dfrac{((\mathscr{P}c)^2 + \mathscr{E}_\infty^2)^{1/2}}{\mathscr{P}c}, \end{cases} \qquad \begin{cases} e = ((pc)^2 + e_0^2)^{1/2}, & 0 \leqslant u/c < 1, \\ e = \mathscr{E} = pc = \mathscr{P}c, & u/c = 1, \\ \mathscr{E} = ((\mathscr{P}c)^2 + \mathscr{E}_\infty^2)^{1/2}, & 1 < u/c < \infty, \end{cases}$$

while the corresponding expressions for the wave velocities w/c and particle and wave energies e and $\mathscr{E}$ become

$$w/c = \begin{cases} \dfrac{((pc)^2 + e_0^2)^{1/2}}{pc}, \\ 1, \\ \dfrac{\mathscr{P}c}{((\mathscr{P}c)^2 + \mathscr{E}_\infty^2)^{1/2}}, \end{cases} \qquad \begin{cases} e = ((pc)^2 + e_0^2)^{1/2}, & 0 \leqslant u/c < 1, \\ e = \mathscr{E} = pc = \mathscr{P}c, & u/c = 1, \\ \mathscr{E} = ((\mathscr{P}c)^2 + \mathscr{E}_\infty^2)^{1/2}, & 1 < u/c < \infty, \end{cases}$$

noting again that $uw = c^2$.

The major known fields, such as electricity, magnetism and gravitation, are believed to propagate as waves travelling at the speed of light, and the immediately above relations would seem to indicate that a necessary and sufficient condition for $u/c = 1$ is that both the rest energy and the wave energy at infinite velocity are zero, namely $e_0 = \mathscr{E}_\infty = 0$. However, these relations also imply that for sufficiently large momenta with $e_0 \neq 0$ and $\mathscr{E}_\infty \neq 0$, a propagating field might well be travelling sufficiently close to the speed of light so as to be perceived to be travelling at the speed of light. It is also apparent that a wave travelling at the speed of light can only materialise as a particle if there is a slowing to a sub-luminal speed, which hints at a possible origin of the perceived probabilistic nature of quantum mechanics. That is, the above relations indicate that the particle or wave nature of either light or electrons occurs when the particular physical conditions are such that their speeds drift into either the sub-luminal or superluminal ranges to give the appearance of a random phenomenon. It is also apparent from this analysis that the phenomenon of quantum entanglement might simply be explained by the two separate cases arising from the positive and negative values of either the rest energy e_0 or the wave energy at infinite velocity $\mathscr{E}_\infty$.

Chapter 8
Further Results for One Space Dimension

8.1 Introduction

In this chapter, we present further results applying to a single spatial dimension, for which the key feature is that the momentum $p(x, t)$ satisfies the conventional wave equation, for which there exists a simple general solution. In the following section, we detail the well-known general solution of the classical one-dimensional wave equation. In the subsequent two sections, we are interested in the existence or otherwise of solutions in the event that the force $f(x, t)$ is switched off. If there exists a work done function, namely $\partial f/\partial t = c^2 \partial g/\partial x$, then only the trivial solution exists if the spatial physical force $f(x, t)$ is set to zero, as might be anticipated. If however $f(x, t) = 0$ and there is no other underlying constraint on the forces, then it is possible to generate an interesting class of general nontrivial solutions for zero spatial force f. The section thereafter deals with a particular generalisation of the relativistic wave-like solution studied in prior chapters, and the subsequent section provides another generalisation, which is an example of a solution with a variable rest mass. The penultimate section of the chapter deals with the formulation of the basic equations in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$. The final section of the chapter develops a number of results assuming that the momentum $p(x, t)$ and wave energy $\mathscr{E}(x, t)$ can be treated as independent variables which corresponds to the assumption that the applied forces f and g are such that $f \neq \pm cg$.

8.2 Wave Equation General Solution

For a single spatial dimension, a particle moving with velocity u has a conventional momentum $p = mu$ and energy $e = mc^2$ where $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, such that p satisfies the wave equation. In addition, associated with the particle is a de

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Broglie wave moving with a wave velocity c^2/u, for which there is an associated wave energy $\mathscr{E}$, such that the total energy of the particle W is given by $W = e + \mathscr{E}$. For a given momentum $p(x, t)$, we have

$$u(x, t) = \frac{pc^2}{(e_0^2 + (pc)^2)^{1/2}}, \qquad e(x, t) = (e_0^2 + (pc)^2)^{1/2},$$

so that from Eq. (3.10), we obtain

$$d\mathscr{E} = \frac{\partial p}{\partial t} dx + c^2 \frac{\partial p}{\partial x} dt,$$

and we may deduce the basic relations

$$\frac{\partial \mathscr{E}}{\partial t} = c^2 \frac{\partial p}{\partial x}, \qquad \frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t}.$$

Accordingly, both the momentum $p = mu$ and the wave energy $\mathscr{E}$ satisfy the wave equations,

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}, \qquad \frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \frac{\partial^2 \mathscr{E}}{\partial x^2}.$$

We may exploit the simple general solutions of the wave equation $p(x, t) = F(\alpha) + G(\beta)$ and $\mathscr{E}(x, t) = c\,(F(\alpha) - G(\beta))$, where $\alpha = ct + x$ and $\beta = ct - x$ and where both F and G denote arbitrary functions of their arguments.

Thus, if $p(x, t) = F(ct + x) + G(ct - x)$ for arbitrary functions F and G, then the wave energy $\mathscr{E}(x, t)$ is given by $\mathscr{E}(x, t) = c\,(F(ct + x) - G(ct - x))$, and by total differentiation of this expression with respect to time, it is not difficult to show that the four expressions

$$\begin{aligned}
p(x, t) &= F(ct + x) + G(ct - x), \\
\mathscr{E}(x, t) &= c\,(F(ct + x) - G(ct - x))\,, \\
u(x, t) &= \frac{c^2\,(F(ct + x) + G(ct - x))}{(e_0^2 + c^2\,(F(ct + x) + G(ct - x))^2)^{1/2}}, \\
e(x, t) &= (e_0^2 + c^2\,(F(ct + x) + G(ct - x))^2)^{1/2},
\end{aligned}$$

automatically satisfy the equation

$$\frac{d\mathscr{E}}{dt} = \frac{de}{dt} + e\frac{\partial u}{\partial x},$$

for all functions $F(ct + x)$ and $G(ct - x)$, which therefore constitutes the formal general solution in a single spatial dimension.

The explicit solution given above by (5.1), which is valid for $f(u)$ and $g(u)$ given by (5.2), arises from the immediately above analysis by taking $F(\alpha) = f_0 \alpha A/c$ and $G(\beta) = f_0 \beta B/c$, where A and B are constants, such that $A + B = 1$ and the constant λ in (5.8) is given by $A - B = \lambda$, so that $A = (1 + \lambda)/2$, $B = (1 - \lambda)/2$, $F(\alpha) = f_0(1 + \lambda)\alpha/2c$ and $G(\beta) = f_0(1 - \lambda)\beta/2c$. Thus, in this particular case, we have $p(x, t) = f_0(\lambda x + ct)/c$ and $\mathscr{E}(x, t) = f_0(x + c\lambda t)$, where f_0 denotes the arbitrary force constant appearing in the expressions (5.2), and these expressions are in agreement with the previously given results.

From Eq. $(4.3)_2$, there certainly exists a function $\psi(x, t)$ such that

$$p = \frac{\partial \psi}{\partial x}, \qquad \mathscr{E} = \frac{\partial \psi}{\partial t}. \tag{8.1}$$

and satisfying the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0. \tag{8.2}$$

We also observe that the one-dimensional version of (3.10), namely

$$d\mathscr{E} = udp + eu_x dt,$$

uses a subscript notation for partial derivatives, and this equation can be shown to become

$$\psi_{tx} dx + \psi_{tt} dt = u(\psi_{xx} dx + \psi_{xt} dt) + eu_x dt.$$

On using $u = dx/dt$, together with the wave equation (8.2) to eliminate ψ_{tt}, the resulting equation for ψ_{xx} becomes $\psi_{xx}(1 - (u/c)^2) = mu_x$ or $\psi_{xx} = m_0 u_x/(1 - (u/c)^2)^{3/2}$, which can be seen to be merely the partial derivative with respect to x of the relation $p = mu = \partial \psi/\partial x$.

8.3 Trivial Solution Only for Zero Spatial Force

For any physically reasonable theory, we might expect that only the trivial solution arises if the spatial physical force is switched off. This is certainly the case for Eqs. (7.1) arising from the basic model (3.4), if we assume the existence of a work done function $W(x, t)$. Without this assumption, we show in the subsequent section that it is not necessarily the case.

If for the above relations (8.1) and (8.2) we make the additional assumption that $f = p_t + e_x = 0$, then apart from an arbitrary function of time, in addition to

$p = \partial\psi/\partial x$, we have the further relation $e = -\partial\psi/\partial t$, so that in the particular case that there exists a work done function $W(x,t)$, we have $g = 0$ and therefore $W = e + \mathscr{E} = 0$, as might be anticipated. Further, within the present theory, it is a simple matter to show that there are no nontrivial solutions for $\psi(x,t)$ for which $f = 0$. In terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, the equation $e^2 - (pc)^2 = e_0^2$ becomes $\psi_\alpha \psi_\beta = (e_0/2c)^2$, and it is not difficult to show that the only solutions for e and p arising from $\psi_{\alpha\beta} = 0$ and $\psi_\alpha \psi_\beta = (e_0/2c)^2$ are constant state solutions emanating from $\psi(x,t) = C_1\alpha + C_2\beta$, where C_1 and C_2 denote constants such $C_1 C_2 = (e_0/2c)^2$, and for which we have

$$e = -c(C_1 + C_2), \qquad p = (C_1 - C_2), \qquad \mathscr{E} = c(C_1 + C_2),$$

corresponding to the constant velocity $u/c = (C_2 - C_1)/(C_1 + C_2)$. However, since e is a constant, we have $(C_1 + C_2) = -e_0/c$, and together with $C_1 C_2 = (e_0/2c)^2$, we might readily conclude $C_1 = C_2 = -e_0/2c$, and therefore only the trivial solution $p = u = 0$ applies, which confirms that if the work done function $W(x,t)$ exists, then zero external force f leads to only the trivial solution.

8.4 Nontrivial Solutions for Zero Spatial Force

If there is no work done function $W(x,t)$ and the spatial force f is zero, that is, we are not assuming the relation $f_t = c^2 g_x$, then it is easy to demonstrate that indeed nontrivial solutions do exist. If $f = p_t + e_x = 0$, then from the relations (8.1) we have altogether

$$p = \frac{\partial\psi}{\partial x}, \qquad e = -\frac{\partial\psi}{\partial t}, \qquad \mathscr{E} = \frac{\partial\psi}{\partial t},$$

where $\psi(x,t)$ satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = g. \tag{8.3}$$

and where the force in the direction of time $g(x,t)$ is an assumed prescribed function. From the equation $e^2 - (pc)^2 = e_0^2$, we have $(\partial\psi/\partial t)^2 - c^2(\partial\psi/\partial x)^2 = e_0^2$, and therefore we might deduce

$$\frac{\partial\psi}{\partial t} = e_0 \cosh\theta, \qquad c\frac{\partial\psi}{\partial x} = e_0 \sinh\theta, \tag{8.4}$$

for some angle $\theta(x,t)$. On equating the two expressions for $\partial^2\psi/\partial t\partial x$, we obtain $\partial\theta/\partial t = c\tanh\theta\,\partial\theta/\partial x$, which together with (8.3) we obtain

$$\frac{\partial \theta}{\partial x} = \frac{cg}{e_0} \cosh \theta, \qquad \frac{\partial \theta}{\partial t} = \frac{c^2 g}{e_0} \sinh \theta, \tag{8.5}$$

and the compatibility equation obtained by equating the two expressions for $\partial^2 \theta / \partial t \partial x$ becomes

$$c \sinh \theta \frac{\partial g}{\partial x} + \frac{(cg)^2}{e_0} = \cosh \theta \frac{\partial g}{\partial t}. \tag{8.6}$$

Using the basic definitions of $\cosh \theta$ and $\sinh \theta$, this equation may be reformulated as a quadratic equation for e^{θ} which on solving yields

$$e^{\theta} = \frac{\mu g^2 + \left(\mu^2 g^4 - (\partial g/\partial t)^2 + c^2 (\partial g/\partial x)^2\right)^{1/2}}{(\partial g/\partial t - c \partial g/\partial x)}, \tag{8.7}$$

$$e^{-\theta} = \frac{\mu g^2 - \left(\mu^2 g^4 - (\partial g/\partial t)^2 + c^2 (\partial g/\partial x)^2\right)^{1/2}}{(\partial g/\partial t + c \partial g/\partial x)},$$

where $\mu = 1/m_0$, and from these relations, we find that

$$\cosh \theta = \frac{\mu g^2 \partial g/\partial t + \left(\mu^2 g^4 - (\partial g/\partial t)^2 + c^2 (\partial g/\partial x)^2\right)^{1/2} c \partial g/\partial x}{((\partial g/\partial t)^2 - c^2 (\partial g/\partial x)^2)}, \tag{8.8}$$

$$\sinh \theta = \frac{\mu g^2 c \partial g/\partial x + \left(\mu^2 g^4 - (\partial g/\partial t)^2 + c^2 (\partial g/\partial x)^2\right)^{1/2} \partial g/\partial t}{((\partial g/\partial t)^2 - c^2 (\partial g/\partial x)^2)}.$$

It happens that a simple but very general family of solutions exist for which $(\partial g/\partial t)^2 - c^2 (\partial g/\partial x)^2 = \mu^2 g^4$, so that there exists some function $\phi(x,t)$ such that

$$\frac{\partial g}{\partial t} = \mu g^2 \cosh \phi, \qquad c \frac{\partial g}{\partial x} = \mu g^2 \sinh \phi,$$

but immediately from (8.8), in this particular case, we have simply

$$\frac{\partial g}{\partial t} = \mu g^2 \cosh \theta, \qquad c \frac{\partial g}{\partial x} = \mu g^2 \sinh \theta,$$

and therefore, we may conclude that $\phi(x,t) = \theta(x,t)$. These equations together with (8.5) yield

$$\frac{\partial \theta}{\partial x} = \frac{1}{cg} \frac{\partial g}{\partial t}, \qquad \frac{\partial \theta}{\partial t} = \frac{c}{g} \frac{\partial g}{\partial x},$$

and therefore, both $\theta(x,t)$ and $\log g(x,t)$ satisfy the wave equation, so that with $\alpha = ct + x$ and $\beta = ct - x$, we have $\log g(x,t) = F(\alpha) + G(\beta)$ and $\theta(x,t) =$

$F(\alpha) - G(\beta)$, where $F(\alpha)$ and $G(\beta)$ denote arbitrary functions, and we obtain

$$g(x, t) = e^{F(\alpha)+G(\beta)} = A(\alpha)B(\beta), \tag{8.9}$$

where $A(\alpha)$ and $B(\beta)$ also denote arbitrary functions given by $A(\alpha) = e^{F(\alpha)}$ and $B(\beta) = e^{G(\beta)}$.

Further, from the relations $pc = c\partial\psi/\partial x = e_0 \sinh\theta$ and $e = -\partial\psi/\partial t = -e_0 \cosh\theta$ and therefore from the equation

$$\frac{u}{c} = \frac{pc}{e} = -\frac{\sinh\theta}{\cosh\theta} = \frac{B(\beta)/A(\alpha) - A(\alpha)/B(\beta)}{B(\beta)/A(\alpha) + A(\alpha)/B(\beta)},$$

we find that u/c is given by

$$\frac{u}{c} = \frac{B(\beta)^2 - A(\alpha)^2}{B(\beta)^2 + A(\alpha)^2}, \tag{8.10}$$

noting that $u/c < 1$. Also, in terms of $\alpha = ct + x$ and $\beta = ct - x$, Eq. (8.3) becomes simply $\partial^2\psi/\partial\alpha\partial\beta = -A(\alpha)B(\beta)/4$ and therefore integrates immediately to give the general solution arising from the assumed $g(x, t)$ given by (8.9), thus

$$\psi(x, t) = -\frac{1}{4}\int A(\alpha)d\alpha \int B(\beta)d\beta + C(\alpha) + D(\beta),$$

where $A(\alpha)$, $B(\beta)$, $C(\alpha)$ and $D(\beta)$ denote four arbitrary functions. Thus, even though the spatial physical force vanishes, for a wide range of forces in the direction of time $g(x, t)$ given by (8.9), we can generate velocities less than the speed of light and given explicitly by (8.10).

Finally, in this section, we comment that the above solution happens to be one particularly simple solution of many solutions that exist for arbitrary $g(x, t)$. Further, we comment that on using $h(x, t) = 1/g(x, t)$, the above Eqs. (8.7) and (8.8) simplify somewhat to become, respectively,

$$e^{\theta} = \frac{\mu + \left(\mu^2 - (\partial h/\partial t)^2 + c^2(\partial h/\partial x)^2\right)^{1/2}}{(c\partial h/\partial x - \partial h/\partial t)},$$

$$e^{-\theta} = \frac{\mu - \left(\mu^2 - (\partial h/\partial t)^2 + c^2(\partial h/\partial x)^2\right)^{1/2}}{(c\partial h/\partial x + \partial h/\partial t)},$$

and

$$\cosh\theta = \frac{\mu\partial h/\partial t + \left(\mu^2 - (\partial h/\partial t)^2 + c^2(\partial h/\partial x)^2\right)^{1/2} c\partial h/\partial x}{(c^2(\partial h/\partial x)^2 - (\partial h/\partial t)^2)},$$

$$\sinh\theta = \frac{\mu c \partial h/\partial x + \left(\mu^2 - (\partial h/\partial t)^2 + c^2(\partial h/\partial x)^2\right)^{1/2} \partial h/\partial t}{(c^2(\partial h/\partial x)^2 - (\partial h/\partial t)^2)}.$$

where again $\mu = 1/m_0$. In terms of the characteristic variables $\alpha = ct + x$ and $\beta = ct - x$, the three Eqs. (8.5), (8.6) and (8.7) become, respectively,

$$\frac{\partial\theta}{\partial\alpha} = \frac{\mu}{2ch}e^{\theta}, \qquad \frac{\partial\theta}{\partial\beta} = -\frac{\mu}{2ch}e^{\theta}, \tag{8.11}$$

while the compatibility equation obtained by equating the two expressions for $\partial^2\theta/\partial\alpha\partial\beta$ becomes

$$e^{-\theta}\frac{\partial h}{\partial\alpha} + e^{\theta}\frac{\partial h}{\partial\beta} = -\frac{\mu}{c},$$

and this may be solved as a quadratic equation for e^{θ} to yield

$$e^{\theta} = -\frac{\left(\mu + \left(\mu^2 - 4c^2\partial h/\partial\alpha\partial h/\partial\beta\right)^{1/2}\right)}{2c\partial h/\partial\beta},$$

$$e^{-\theta} = -\frac{\left(\mu - \left(\mu^2 - 4c^2\partial h/\partial\alpha\partial h/\partial\beta\right)^{1/2}\right)}{2c\partial h/\partial\alpha}.$$

These equations mean that in principle, for any prescribed force $g(x, t)$ in the direction of time, namely any $h(\alpha, \beta)$, we may generate $\theta(\alpha, \beta)$ satisfying (8.11), and therefore through (8.4), we might generate a solution of (8.3).

8.5 Generalisation of Wave-Like Solution

For a single spatial dimension, the above wave-like solution (5.1) can be generalised in a variety of ways, and in this section, we provide a particular generalised version of (5.1) and (5.8). We start with $p(x, t) = F(\alpha) + G(\beta)$ and Eq. (4.2) to deduce the two differential relations

$$\frac{dp}{dt} = u\left(F'(\alpha) - G'(\beta)\right) + c\left(F'(\alpha) + G'(\beta)\right),$$

and

$$\frac{d\mathscr{E}}{dt} = uc\left(F'(\alpha) + G'(\beta)\right) + c^2\left(F'(\alpha) - G'(\beta)\right),$$

and by division, we may deduce an equation, which constitutes the generalised version of (5.8) thus

$$\frac{d\mathscr{E}}{dp} = c\left(\frac{\omega + u/c}{1 + \omega u/c}\right), \tag{8.12}$$

where ω is defined by the ratio $\omega = \big(F'(\alpha) - G'(\beta)\big) / \big(F'(\alpha) + G'(\beta)\big)$.

For example, for the special case $F(\alpha) = \gamma \log \alpha$ and $G(\beta) = \gamma \log \beta$ where γ is a constant, we find that $\omega = -x/ct$ and (8.12) becomes simply

$$\frac{d\mathscr{E}}{dp} = \frac{u - x/t}{1 - xu/c^2 t}.$$

From $pc = e_0(u/c)/(1-(u/c)^2)^{1/2}$, we can deduce that $u/c = pc/(e_0^2+(pc)^2)^{1/2}$, and therefore from $p(x,t) = \gamma \log((ct)^2 - x^2)$, the particle velocity $u(x,t)$ is given explicitly by

$$u(x,t) = c\left\{\frac{\log\big((ct)^2 - x^2\big)}{\Big[(e_0/\gamma c)^2 + \big(\log\big((ct)^2 - x^2\big)\big)^2\Big]^{1/2}}\right\}.$$

For the general solution $p(x,t) = F(\alpha) + G(\beta)$ and $\mathscr{E}(x,t) = c\,(F(\alpha) - G(\beta))$, the corresponding general expressions for the velocity $u(x,t)$ and the required force functions $f(x,t)$ and $g(x,t)$ might be evaluated as follows. From $p(x,t) = F(\alpha) + G(\beta)$ and

$$u(x,t) = c\left\{\frac{cp(x,t)}{(e_0^2 + (cp(x,t))^2)^{1/2}}\right\},$$

therefore the velocity profile $u(x,t)$ corresponding to the general solution is given by

$$u(x,t) = c\left\{\frac{c(F(\alpha) + G(\beta))}{(e_0^2 + c^2(F(\alpha) + G(\beta))^2)^{1/2}}\right\},$$

where again F and G denote arbitrary functions of $\alpha = ct + x$ and $\beta = ct - x$, respectively.

We now seek the most general $f(x,t)$ and $g(x,t)$ satisfying (6.7), such that (7.1) admit nontrivial solutions. Again on introducing ϕ through $u = c \sin\phi$, there follows the relations (6.1), and we might readily deduce

$$c\phi_x = b(x,t) - a(x,t)\sin\phi, \qquad \phi_t = a(x,t) - b(x,t)\sin\phi, \tag{8.13}$$

where $a(x, t)$ and $b(x, t)$ are given by $a(x, t) = cf(x, t)/e_0$ and $b(x, t) = c^2 g(x, t)/e_0$ and $e_0 = m_0 c^2$. In terms of $a(x, t)$ and $b(x, t)$, Eq. (6.7) and the equation obtained by equating expressions for ϕ_{xt} become

$$a_t = cb_x, \qquad b_t = ca_x + (a^2 - b^2)\cos\phi,$$

which not only constitutes two equations for $a(x, t)$ and $b(x, t)$ but also involves the function $\phi(x, t)$. In order to obtain an equation in the single variable $\phi(x, t)$, we use $a_t = cb_x$ to introduce the function $\psi(x, t)$ such that $a = c\psi_x$ and $b = \psi_t$, and Eqs. (8.13) become

$$c\phi_x = \psi_t - c\psi_x \sin\phi, \qquad \phi_t = c\psi_x - \psi_t \sin\phi,$$

and from which, we may deduce

$$\psi_t = (c\phi_x + \phi_t \sin\phi)\sec^2\phi, \qquad c\psi_x = (\phi_t + c\phi_x \sin\phi)\sec^2\phi,$$

and on elimination of $\psi(x, t)$ by equating expressions for the derivative ψ_{xt}, we obtain

$$\phi_{tt} + 2\tan\phi{\phi_t}^2 = c^2(\phi_{xx} + 2\tan\phi{\phi_x}^2),$$

which simplifies to give

$$\frac{\partial^2(\tan\phi)}{\partial t^2} = c^2\frac{\partial^2(\tan\phi)}{\partial x^2}.$$

This equation is of course merely $(4.4)_1$ with $pc = e_0\tan\phi$. This means that the most general $f(x, t)$ and $g(x, t)$ satisfying (6.7) are determined from (9.55) through the relations

$$a(x, t) = (\tan\phi)_t + c\sin\phi(\tan\phi)_x, \qquad b(x, t) = c(\tan\phi)_x + \sin\phi(\tan\phi)_t,$$

along with $\tan\phi = c(F(\zeta) + G(\eta))/e_0$. The final relations for $f(x, t)$ and $g(x, t)$ become

$$f(x, t) = c\left\{(F'(\zeta) - G'(\eta)) + \sin\phi(F'(\zeta) + G'(\eta))\right\},$$
$$g(x, t) = \left\{(F'(\zeta) + G'(\eta)) + \sin\phi(F'(\zeta) - G'(\eta))\right\},$$

where the primes denote differentiation with respect to the argument indicated and with $\sin\phi = u(x, t)/c$ given explicitly by

$$\sin\phi = \frac{u(x, t)}{c} = \left\{\frac{(F(\zeta) + G(\eta))}{((e_0/c)^2 + (F(\zeta) + G(\eta))^2)^{1/2}}\right\}.$$

8.6 Solutions with Non-constant Rest Mass

Baryonic matter appears to be characterised by a finite rest mass, and yet miraculously this finite mass necessarily plunges to zero if the particle attains the speed of light. As we have previously cited, according to de Broglie, he believed that the rest mass of particles is not constant, but variable, and that both the neutrino and the photon have nonzero rest masses, although necessarily extremely small. This means that we need to entertain the possibility of a variable rest mass energy $e_0 = e_0(x, t)$, which might involve generalised functions, and in particular the representation shown in Fig. 3.3, involving the Heaviside unit step function. Accordingly, the question arises as to whether the present theory permits a variable rest mass, which is finite at $u = 0$ and zero at $u = c$. In this section, we provide one example of this, which is yet another generalisation of the wave-like solution (5.1) for a single spatial dimension and with a variable rest mass. In the following section, we show that the formal equations for the extension of Newton's second law given by Eqs. (3.4), at least for a single space dimension, are indeed well-defined for variable rest energy.

We follow the derivation given in the previous chapter, and we assume the two basic equations either (4.30) or (7.1), and following (6.1), we again introduce an angle $\phi(x, t)$ such that $u = c \sin\phi$ through the relations

$$e = e_0(x, t) \sec\phi, \qquad pc = e_0(x, t) \tan\phi, \tag{8.14}$$

so that we have correctly $e^2 - (pc)^2 = e_0^2$, but here we suppose that the quantity $e_0(x, t)$ is itself also varying with position and time. Subsequently, in order to facilitate the analysis, we make the additional assumption that $e_0(x, t) = e_0(\phi)$, and ultimately, we determine constants in the solution that we may identify as the rest mass energy constant, say e_0^*, or as the rest mass constant, say m_0^*, where $e_0^* = m_0^* c^2$.

On substitution of the above two relations into the two basic equations given by either (7.1) or (4.30), we may obtain two equations for the determination of the two partial derivatives ϕ_x and ϕ_t; thus,

$$\phi_t + \psi_t \sin\phi \cos\phi + c \sin\phi \phi_x + c\psi_x \cos\phi = a(x, t) \cos^2\phi,$$

$$\sin\phi \phi_t + \psi_t \cos\phi + c\phi_x + c\psi_x \sin\phi \cos\phi = b(x, t) \cos^2\phi,$$

where $\psi(x, t) = \log e_0(x, t)$, and we have again introduced $a(x, t) = cf(x, t)/e_0(x, t)$ and $b(x, t) = c^2 g(x, t)/e_0(x, t)$. On solving these equations as two equations in the two unknowns ϕ_x and ϕ_t, we find

$$c\phi_x = b(x, t) - a(x, t) \sin\phi - \psi_t \cos\phi, \qquad \phi_t = a(x, t) - b(x, t) \sin\phi - c\psi_x \cos\phi.$$

In order to proceed further, we need to cross differentiate these equations and equate the two expressions for ϕ_{xt}. There results a complicated second-order partial differential equation for $\psi(x, t)$, for which further progress appears difficult in the absence of additional restrictions on the assumed form of $\psi(x, t)$.

Accordingly, we return to the case when the forces f and g depend only on velocity $u = c \sin \phi$, and we make the additional assumption that $e_0(x, t) = e_0(\phi)$ so that $a(x, t)$, $b(x, t)$ and $\psi(x, t)$ are all functions of ϕ only; thus, $a(x, t) = a(\phi)$, $b(x, t) = b(\phi)$ and $\psi(x, t) = \psi(\phi)$ and the above equations become

$$c\phi_x + \psi'(\phi) \cos \phi \phi_t = b(\phi) - a(\phi) \sin \phi, \quad \phi_t + c\psi'(\phi) \cos \phi \phi_x = a(\phi) - b(\phi) \sin \phi,$$

which on solving as two equations for the determination of the two partial derivatives ϕ_x and ϕ_t yields

$$c\phi_x = \frac{b(\phi) - a(\phi) \sin \phi}{(1 - F^2)} - \frac{(a(\phi) - b(\phi) \sin \phi)F}{(1 - F^2)}, \tag{8.15}$$
$$\phi_t = \frac{a(\phi) - b(\phi) \sin \phi}{(1 - F^2)} - \frac{(b(\phi) - a(\phi) \sin \phi)F}{(1 - F^2)},$$

where for convenience we have introduced the function $F(\phi) = \psi'(\phi) \cos \phi$. Further, on introducing the two functions $A(\phi)$ and $B(\phi)$ defined by

$$A(\phi) = \frac{a(\phi) - b(\phi) \sin \phi}{(1 - F^2)}, \qquad B(\phi) = \frac{b(\phi) - a(\phi) \sin \phi}{(1 - F^2)},$$

the two Eqs. (8.15) become simply

$$c\phi_x = B(\phi) - A(\phi)F(\phi), \qquad \phi_t = A(\phi) - B(\phi)F(\phi).$$

We observe that these latter equations enjoy exactly the same mathematical structure as (6.2), except that $\sin \phi$ has been replaced by $F(\phi)$. Effectively, this means that the solution procedure follows much the same lines as that for (6.2). On cross differentiation of these equations, equating two expressions for ϕ_{xt} and repeatedly using the equations themselves, we may deduce the following simple equation:

$$\frac{d(B/A)}{d\phi} = \left(1 - (B/A)^2\right) \frac{dF}{d\phi}, \tag{8.16}$$

which may be readily integrated, and further simplification yields

$$\log \left(\frac{1 + B/A}{1 - B/A} \right) = 2(F - F_0),$$

where F_0 denotes the constant of integration. With some rearrangement, this equation becomes simply $B(\phi) = A(\phi) \tanh(F - F_0)$, which in terms of $a(\phi)$ and $b(\phi)$ becomes

$$\frac{b(\phi)}{a(\phi)} = \left(\frac{\sin\phi + \tanh(F - F_0)}{1 + \sin\phi\tanh(F - F_0)}\right), \tag{8.17}$$

while in terms of the force $f(u)$ and the energy-mass production $g(u)$, we have the implied relation

$$cg(u) = f(u)\left(\frac{(u/c) + \tanh(F - F_0)}{1 + (u/c)\tanh(F - F_0)}\right), \tag{8.18}$$

where $F(\phi)$ is given by

$$F(\phi) = \frac{\cos\phi}{e_0(\phi)}\frac{de_0(\phi)}{d\phi}. \tag{8.19}$$

We observe that the case $e_0 = \text{constant}$ gives $F(\phi) = 0$ and that the two Eqs. (8.17) and (8.18) recover precisely (6.3) and (6.4), respectively, with the constant $\lambda = -\tanh F_0$ as might be expected.

With the relation (8.17), the two Eqs. (8.15) for the partial derivatives ϕ_x and ϕ_t simplify dramatically to give

$$c\phi_x = \frac{a(\phi)\cos^2\phi(\tanh(F - F_0) - F)}{(1 - F^2)(1 + \sin\phi\tanh(F - F_0))}, \tag{8.20}$$
$$\phi_t = \frac{a(\phi)\cos^2\phi(1 - F\tanh(F - F_0))}{(1 - F^2)(1 + \sin\phi\tanh(F - F_0))},$$

which correspond to Eqs. (6.5) and for which it is clear that $\phi(x, t)$ satisfies the interesting nonlinear partial differential equation

$$c\phi_x = \left(\frac{\tanh(F - F_0) - F}{1 - F\tanh(F - F_0)}\right)\phi_t,$$

where $F(\phi)$ is defined in terms of $e_0(\phi)$ by Eq. (8.19). At this point in the mathematical analysis, we may determine further solutions in any number of ways, depending upon particular assumptions.

As an illustration of the process, we assume that the forces $f(u)$ and $g(u)$ are given by the previously introduced linear force equations (5.2), so that

$$f(\phi) = f_0(1 + \lambda\sin\phi), \qquad cg(\phi) = f_0(\lambda + \sin\phi). \tag{8.21}$$

where f_0 and λ are constants, and in the above analysis, we make the identification $\lambda = \tanh(F - F_0)$ so that the function $F(\phi)$ is assumed to be a constant, say $F(\phi) = \kappa$ for $\kappa \neq \pm 1$, and for which we have the relation, $F_0 + \tanh^{-1}\lambda = \kappa$. We comment that the terminology for the constant κ is intentionally chosen so as to be consistent with the constant κ used in the final section of Chap. 2. We also comment that with the forces f and cg defined by (8.21), $B/A = \lambda$ and therefore with $F = \kappa$, Eq. (8.16) is satisfied trivially.

The function $e_0(\phi)$ is determined from the condition $F(\phi) = \kappa$, and from (8.19), we require to integrate

$$\frac{1}{e_0(\phi)}\frac{de_0(\phi)}{d\phi} = \frac{\kappa}{\cos\phi}, \tag{8.22}$$

and we close this section with an entirely independent derivation of this condition arising from (2.56). In order to integrate equation (8.22), we introduce the working variable $t = \tan(\phi/2)$ to deduce

$$\log e_0(\phi) = \kappa \log\left(\frac{1+t}{1-t}\right) + \log e_0^*,$$

where e_0^* is a constant that can be identified with the rest mass energy. From this equation, we may progressively deduce an expression for $e_0(\phi)$, thus

$$e_0(\phi) = e_0^*\left(\frac{1+t}{1-t}\right)^\kappa = e_0^*\left(\frac{\cos(\phi/2)+\sin(\phi/2)}{\cos(\phi/2)-\sin(\phi/2)}\right)^\kappa = e_0^*\left(\frac{1+\sin\phi}{\cos\phi}\right)^\kappa,$$

to eventually determine

$$e_0(\phi) = e_0^*\left(\frac{1+u/c}{(1-(u/c)^2)^{1/2}}\right)^\kappa = e_0^*\left(\frac{1+u/c}{1-u/c}\right)^{\kappa/2}, \tag{8.23}$$

which for both $u/c > 0$ and $\kappa < 0$ provides an example of a rest mass, which is finite at $u = 0$ and zero at $u = c$. However, (8.23) is not a symmetric function of u/c, and for $u/c < 0$, we require $\kappa > 0$. Despite this, the expression for $e_0(\phi)$ is particularly illuminating, since from Eq. (2.10), the angle θ that is defined by

$$\theta = \frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \tanh^{-1}(u/c), \qquad \left(\frac{1+u/c}{1-u/c}\right)^{1/2} = e^\theta,$$

is the angle for which a Lorentz invariance appears as a translational invariance, and therefore, we have the expression $e_0(\phi) = e_0^* e^{\kappa\theta}$. Further, the resulting energy expression $e(u)$ coincides exactly with the energy expression (2.59), arising from (2.56).

Now from the force assumptions (8.21) and the assumption that $F(\phi)$ is a constant, we may deduce that the two Eqs. (8.20) become simply

$$\phi_x = \frac{f_0(\lambda-\kappa)\cos^2\phi}{e_0(\phi)(1-\kappa^2)}, \qquad \phi_t = \frac{cf_0(1-\kappa\lambda)\cos^2\phi}{e_0(\phi)(1-\kappa^2)}, \tag{8.24}$$

where $e_0(\phi)$ is assumed to be given by (8.23). From these two equations, it is apparent that $c(1-\kappa\lambda)\phi_x = (\lambda-\kappa)\phi_t$, so that $\phi(x,t) = \phi(\zeta)$ where $\zeta = (\lambda-\kappa)x + c(1-\kappa\lambda)t$, and both Eqs. (8.24) reduce to the ordinary differential equation

$$\frac{d\phi}{d\zeta} = \frac{f_0 \cos^2\phi}{e_0(\phi)(1-\kappa^2)}. \tag{8.25}$$

Again with the working variable $t = \tan(\phi/2)$, this equation becomes

$$\left(\frac{1+t}{1-t}\right)^{\kappa} \frac{d\tan\phi}{d\zeta} = \frac{f_0}{e_0^*(1-\kappa^2)}, \tag{8.26}$$

so that we make the substitution

$$z = \left(\frac{1+t}{1-t}\right) = \left(\frac{\cos(\phi/2)+\sin(\phi/2)}{\cos(\phi/2)-\sin(\phi/2)}\right) = \left(\frac{1+\sin\phi}{\cos\phi}\right) = (\tan\phi + \sec\phi)\,,$$

and after rearrangement and squaring the equation $z = \tan\phi + (1+\tan^2\phi)^{1/2}$, we obtain

$$\tan\phi = \frac{1}{2}\left(z - \frac{1}{z}\right),$$

and the integration (8.26) becomes merely

$$z^{\kappa}\left(1+\frac{1}{z^2}\right)dz = \frac{2f_0}{e_0^*(1-\kappa^2)}d\xi.$$

Thus, assuming that the constant $\kappa \neq \pm 1$, this differential equation readily integrates to yield

$$\frac{z^{1+\kappa}}{(1+\kappa)} - \frac{z^{-1+\kappa}}{(1-\kappa)} = \frac{2f_0(\zeta-\zeta_0)}{e_0^*(1-\kappa^2)},$$

where ζ_0 denotes the constant of integration. This equation may be alternatively written in a variety of ways, including

$$\zeta = \zeta_0 + \frac{e_0^*}{2f_0}\left((1-\kappa)z^{1+\kappa} - (1+\kappa)z^{-1+\kappa}\right) = \zeta_0 + \frac{e_0^* e^{\kappa\theta}}{f_0}(\sinh\theta - \kappa\cosh\theta)\,,$$

where $\zeta = (\lambda-\kappa)x + c(1-\kappa\lambda)t$, $z = \tan\phi + (1+\tan^2\phi)^{1/2}$ and ζ_0, f_0, e_0^*, λ and $\kappa \neq \pm 1$ all denote arbitrary constants, where f_0 and λ are the two constants appearing in the assumed linear force relations (8.21) and e_0^* denotes the rest mass energy.

Further, θ is the angle, in which the Lorentz invariance appears as a translational invariance and defined by (2.10).

Independent Derivation of the Condition (8.22) In this note, we show that we may provide an independent derivation of the condition (8.22) by assuming (2.56) for the incremental energy-momentum raio de/dp, namely

$$\frac{de}{dp} = c\left(\frac{\kappa + u/c}{1 + \kappa u/c}\right),$$

since from the relations (8.14), namely $e = e_0(x,t)\sec\phi$, $pc = e_0(x,t)\tan\phi$ and $u = c\sin\phi$, this equation becomes

$$\left(\frac{\sin\phi + \rho(\phi)\cos\phi}{1 + \rho(\phi)\sin\phi\cos\phi}\right) = \left(\frac{\kappa + \sin\phi}{1 + \kappa\sin\phi}\right), \tag{8.27}$$

where $\rho(\phi)$ is used here to denote $(de_0(\phi)/d\phi)/e_0(\phi)$. Equation (8.27) readily simplifies to give $\rho(\phi) = \kappa\sec\phi$, which is precisely the assumed condition (8.22).

8.7 Formulation for Variable Rest Mass

In this text, we assume throughout the Einstein relations of special relativity and that the rest energy e_0 remains constant. In the prior section, we have obtained a particular solution applying for non-constant rest energy $e_0 = e_0(\mathbf{x}, t)$. In this section, we show that at least for a single space dimension, the formal equations for the extension of Newton's second law given by Eqs. (3.4) appear to be perfectly well-defined and meaningful for variable rest energy. Thus, the basic equations (3.4), in terms of the momentum $\mathbf{p} = m\mathbf{u}$ and the energy $e = mc^2$, would appear to be equally sensible for variable rest energy, that is, $e^2 - (pc)^2 = e_0(\mathbf{x}, t)^2$, where $p = (\mathbf{p}\cdot\mathbf{p})^{1/2}$ is the momentum magnitude, and this validity seems to apply without further restriction on $e_0(\mathbf{x}, t)$.

In this section, for a single space dimension x, we present the formulation of the basic equations in terms of the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$, assuming that the rest energy is non-constant, thus $e_0 = e_0(x, t)$ and $e^2 - (cp)^2 = e_0^2(x, t)$, and we show after a complicated calculation that the governing partial differential equations are well-defined for all $e_0(x, t)$. This calculation gives some perspective to the corresponding equations derived in Chap. 4 for constant rest energy, as well as an independent verification of the special solution derived in the previous section. Unless partial differentiation of the rest energy is involved, the corresponding equations derived in Chap. 4 remain unchanged.

Accordingly, these are simply listed below, except that we make explicit the assumed non-constant nature of the rest energy, thus

$$\xi + c\eta = (e + cp)(x - ct), \qquad \xi - c\eta = (e - cp)(x + ct),$$

$$\xi^2 - (c\eta)^2 = (e^2 - (cp)^2)(x^2 - (ct)^2) = e_0^2(x,t)(x^2 - (ct)^2),$$

$$e + cp = e_0(x,t)\left(\frac{1 + u/c}{1 - u/c}\right)^{1/2} = e_0(x,t)e^{\theta}, \qquad e - cp = e_0(x,t)\left(\frac{1 - u/c}{1 + u/c}\right)^{1/2} = e_0(x,t)e^{-\theta},$$

$$\xi + c\eta = e_0(x,t)(x - ct)e^{\theta}, \qquad \xi - c\eta = e_0(x,t)(x + ct)e^{-\theta},$$

$$\xi = e_0(x,t)(x\cosh\theta - ct\sinh\theta), \qquad c\eta = e_0(x,t)(x\sinh\theta - ct\cosh\theta), \tag{8.28}$$

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{8.29}$$

$$xf - c^2 tg = \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x}, \qquad xg - tf = \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x},$$

where θ is the angle defined by $(2.9)_2$ and satisfying the relations (2.10).

By partial differentiation of the expressions (8.28) with respect to both x and t, we may show that

$$xf - c^2 tg = c\eta\frac{\partial\theta}{\partial x} + \frac{\xi}{c}\frac{\partial\theta}{\partial t} + \xi\frac{\partial\psi}{\partial x} + \eta\frac{\partial\psi}{\partial t} = \xi\left(\frac{1}{c}\frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial x}\right) + \eta\left(c\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial t}\right),$$

$$xg - tf = \frac{\xi}{c}\frac{\partial\theta}{\partial x} + \frac{\eta}{c}\frac{\partial\theta}{\partial t} + \frac{\xi}{c^2}\frac{\partial\psi}{\partial t} + \eta\frac{\partial\psi}{\partial x} = \frac{\xi}{c^2}\left(c\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial t}\right) + \eta\left(\frac{1}{c}\frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial x}\right),$$

where $\psi(x,t) = \log e_0(x,t)$, and with the abbreviation $F = xf - c^2 tg$ and $G = xg - tf$, we solve these as two equations for the two unknowns A and B defined by

$$A = \left(\frac{1}{c}\frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial x}\right), \qquad B = \left(c\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial t}\right),$$

namely, $F = \xi A + \eta B$ and $G = \xi B/c^2 + \eta A$ to obtain

$$A = \left(\frac{\xi F - c^2\eta G}{\xi^2 - (c\eta)^2}\right), \qquad B = c^2\left(\frac{\xi G - \eta F}{\xi^2 - (c\eta)^2}\right).$$

On using the expressions $\xi = ex - c^2pt$ and $\eta = px - et$ and the relation $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$,we obtain

$$\xi G - \eta F = (x^2 - (ct)^2)(eg - pf), \qquad \xi F - c^2\eta G = (x^2 - (ct)^2)(ef - c^2pg),$$

and from which we may eventually deduce

$$\frac{\partial \theta}{\partial t} = \frac{c}{e_0^2}(ef - c^2pg) - c\frac{\partial \psi}{\partial x}, \qquad \frac{\partial \theta}{\partial x} = \frac{c}{e_0^2}(eg - pf) - \frac{1}{c}\frac{\partial \psi}{\partial t}, \tag{8.30}$$

as the basic underlying coupled partial differential relations for the case of non-constant rest energy, where θ is the angle defined by (2.9)$_2$ and $\psi(x, t) = \log e_0(x, t)$.

Again, we comment that with arbitrarily assigned external forces (f, gc), these equations must be well-defined and compatible in the sense that $\partial/\partial x(\partial\theta/\partial t) = \partial/\partial t(\partial\theta/\partial x)$. Again, however, with f and g defined by the expressions (8.29), these equations are surprisingly automatically well-defined and meaningful, although for variable rest energy, this is not immediately apparent, and the calculation is slightly more complicated than that previously given.

In order to see this, on using the relations (8.29), performing the partial differentiations in (8.30) and equating the two expressions for the second order partial derivative $\partial^2\theta/\partial x\partial t$, we might deduce the generalisation of (4.34), namely

$$f^2 - (cg)^2 + e\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial t}\right) + p\left(\frac{\partial f}{\partial t} - c^2\frac{\partial g}{\partial x}\right) + e_0^2\left(\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2}\right) \tag{8.31}$$

$$= 2(ef - c^2pg)\frac{\partial \psi}{\partial x} - 2(eg - pf)\frac{\partial \psi}{\partial t},$$

which again might be viewed as a consistency condition imposed upon any allowable velocity fields arising from the applied external forces (f, gc), indicating that f and g may not be entirely arbitrarily assigned. With f and g given by (8.29), these equations can be shown to simplify to give

$$\frac{1}{2}\left(\frac{\partial^2 e_0^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 e_0^2}{\partial t^2}\right) + e_0^2\left(\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2}\right) + 2\frac{\partial(e, p)}{\partial(x, t)}$$

$$= \left\{\frac{\partial e_0^2}{\partial x} + 2\left(e\frac{\partial p}{\partial t} - p\frac{\partial e}{\partial t}\right)\right\}\frac{\partial \psi}{\partial x} - \left\{\frac{1}{c^2}\frac{\partial e_0^2}{\partial t} + 2\left(e\frac{\partial p}{\partial x} - p\frac{\partial e}{\partial x}\right)\right\}\frac{\partial \psi}{\partial t},$$

and on using $\psi(x, t) = \log e_0(x, t)$, we might finally deduce the apparently cyclicly interchangeable constraint

$$p\frac{\partial(e,e_0)}{\partial(x,t)}+e\frac{\partial(e_0,p)}{\partial(x,t)}+e_0\frac{\partial(p,e)}{\partial(x,t)}=0.$$

By direct calculation, this seemingly nontrivial condition can be shown to be identically satisfied using the relations $e = e_0(x,t)\sec\phi$ and $pc = e_0(x,t)\tan\phi$. Thus, with external forces (f, gc) satisfying (8.29), Eqs. (8.30) constitute two well-defined relations for the determination of $\theta(x,t)$, which also becomes clear on closer examination, since from (8.30) on using (8.29), we have

$$\frac{\partial\theta}{\partial t}=\frac{c}{e_0^2}\left\{e\left(\frac{\partial p}{\partial t}+\frac{\partial e}{\partial x}\right)-p\left(\frac{\partial e}{\partial t}+c^2\frac{\partial p}{\partial x}\right)\right\}-\frac{c}{e_0}\frac{\partial e_0}{\partial x}=\frac{c}{e_0^2}\left(e\frac{\partial p}{\partial t}-p\frac{\partial e}{\partial t}\right), \tag{8.32}$$

$$\frac{\partial\theta}{\partial x}=\frac{c}{e_0^2}\left\{e\left(\frac{1}{c^2}\frac{\partial e}{\partial t}+\frac{\partial p}{\partial x}\right)-p\left(\frac{\partial p}{\partial t}+\frac{\partial e}{\partial x}\right)\right\}-\frac{1}{ce_0}\frac{\partial e_0}{\partial t}=\frac{c}{e_0^2}\left(e\frac{\partial p}{\partial x}-p\frac{\partial e}{\partial x}\right).$$

Thus, as for the case of constant rest energy mass, we obtain the seemingly simple differential relation

$$d\theta=\frac{\partial\theta}{\partial x}dx+\frac{\partial\theta}{\partial t}dt=\frac{c}{e_0^2}(edp-pde), \tag{8.33}$$

applying to all variable rest energies $e_0 = e_0(x,t)$. Now using the relations $e = e_0(x,t)\sec\phi$ and $pc = e_0(x,t)\tan\phi$, Eq. (8.33) becomes

$$d\theta=\frac{1}{e_0^2}\{e_0\sec\phi(e_0\sec^2\phi d\phi+de_0\tan\phi)-e_0\tan\phi(e_0\sec\phi\tan\phi d\phi+de_0\sec\phi)\}=\sec\phi d\phi,$$

and, apart from an arbitrary additive constant, the simple relation $d\theta = \sec\phi d\phi$ integrates to give $\theta = \log((1+\sin\phi)/\cos\phi)$, and with $u = c\sin\phi$, this is precisely the relation $(2.10)_1$, where θ is the angle in which the Lorentz invariance appears as a translational invariance.

The upshot of this long calculation, which applies to any rest energy $e_0 = e_0(x,t)$, is that we may view Eq. (8.31) as a constraint that must be satisfied by the applied external forces (f, gc) and allowable momentum-energy fields (pc, e). If, however, the applied external forces (f, gc) are generated from (8.29) from the (pc, e) momentum-energy field, then the above equations are all automatically satisfied and the coupled relations such as (8.30) or (8.32) are well-defined and meaningful as two coupled partial differential relations for $\theta(x,t)$.

Independent Verification for Solution of Previous Section Given the above formalism, it is a relatively easy matter to verify the particular solution derived in the previous section. The solution derived there applies to rest energies $e_0(\phi)$ given by (8.23), namely

$$e_0(\phi) = e_0^* \left(\frac{1 + u/c}{1 - u/c} \right)^{\kappa/2},$$

which is derived from the condition (8.22)

$$\frac{1}{e_0(\phi)} \frac{de_0(\phi)}{d\phi} = \frac{\kappa}{\cos\phi}, \tag{8.34}$$

where κ is a constant. Further, the angle $\phi(\zeta)$ where $u = c\sin\phi$ is assumed to be determined from the differential relation (8.25)

$$\frac{d\phi}{d\zeta} = \frac{f_0 \cos^2\phi}{e_0(\phi)(1-\kappa^2)}, \tag{8.35}$$

where $\zeta = (\lambda - \kappa)x + c(1 - \kappa\lambda)t$, and λ is an arbitrary constant appearing in the assumed linear force relations (8.21), thus

$$f(\phi) = f_0(1 + \lambda \sin\phi), \qquad cg(\phi) = f_0(\lambda + \sin\phi),$$

where f_0 is a further arbitrary force constant. Using these equations, it is now a simple matter to independently verify the solution given in the previous section. From the relations $e = e_0(\phi)\sec\phi$, $pc = e_0(\phi)\tan\phi$ and $\psi(x,t) = \log e_0(\phi)$, the basic underlying coupled partial differential equations for the case of non-constant rest energy equations (8.30) become

$$\begin{aligned}
\frac{\partial\theta}{\partial t} &= \frac{c}{e_0^2}(ef - c^2 pg) - c\frac{\partial\psi}{\partial x}, \\
&= \frac{cf_0}{e_0}\{\sec\phi(1 + \lambda\sin\phi) - \tan\phi(\lambda + \sin\phi)\} - \frac{ce_0'(\phi)}{e_0(\phi)}\frac{\partial\phi}{\partial x}, \\
&= \frac{cf_0}{e_0}\cos\phi - c\kappa\sec\phi\frac{\partial\phi}{\partial x},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\theta}{\partial x} &= \frac{c}{e_0^2}(eg - pf) - \frac{1}{c}\frac{\partial\psi}{\partial t}, \\
&= \frac{f_0}{e_0}\{\sec\phi(\lambda + \sin\phi) - \tan\phi(1 + \lambda\sin\phi)\} - \frac{e_0'(\phi)}{ce_0(\phi)}\frac{\partial\phi}{\partial t}, \\
&= \frac{f_0}{e_0}\lambda\cos\phi - \frac{\kappa\sec\phi}{c}\frac{\partial\phi}{\partial t},
\end{aligned}$$

on using the condition (8.34), where θ is the angle defined by $(2.9)_2$. Now on using $\phi(x,t) = \phi(\zeta)$ where $\zeta = (\lambda - \kappa)x + c(1 - \kappa\lambda)t$, these two equations become

$$\frac{\partial\theta}{\partial t} = \frac{cf_0}{e_0}\cos\phi - c\kappa(\lambda-\kappa)\sec\phi\frac{d\phi}{d\zeta} = \frac{cf_0}{e_0}\cos\phi\left(1 - \frac{\kappa(\lambda-\kappa)}{(1-\kappa^2)}\right) = \frac{cf_0}{e_0}\cos\phi\left(\frac{1-\kappa\lambda}{1-\kappa^2}\right),$$

and

$$\frac{\partial\theta}{\partial x} = \frac{f_0}{e_0}\lambda\cos\phi - \kappa\sec\phi(1-\kappa\lambda)\frac{d\phi}{d\zeta} = \frac{f_0}{e_0}\cos\phi\left(\lambda - \frac{\kappa(1-\kappa\lambda)}{(1-\kappa^2)}\right) = \frac{f_0}{e_0}\cos\phi\left(\frac{\lambda-\kappa}{1-\kappa^2}\right),$$

on using the condition (8.35). From both equations, it is clear that the assumption $\theta(x,t) = \theta(\zeta)$ leads to the single condition

$$\frac{d\theta}{d\zeta} = \frac{f_0\cos\phi}{e_0(\phi)(1-\kappa^2)} = \sec\phi\frac{d\phi}{d\zeta},$$

on using the relation (8.35), and therefore, as expected, we obtain the correct outcome $d\theta = \sec\phi d\phi$, and we have provided an independent check on the validity of the solution given in the previous section.

8.8 Characteristics $\alpha = ct + x$ and $\beta = ct - x$

In terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$ and the differential relations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}, \qquad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}, \tag{8.36}$$

Equations (7.1) may be alternatively written as

$$f - cg = -2\frac{\partial(e-cp)}{\partial\beta}, \qquad f + cg = 2\frac{\partial(e+cp)}{\partial\alpha}, \tag{8.37}$$

indicating that if $f = cg$, then $e = cp + A(\alpha)$, and if $f = -cg$, then $e = -cp + B(\beta)$, where $A(\alpha)$ and $B(\beta)$ denote arbitrary functions of the indicated argument. These results give rise to the prospect of identifying dark matter with the equation $e = cp$ and identifying dark energy with the equation $e = -cp$ and with the arbitrary functions $A(\alpha)$ and $B(\beta)$, taken to be zero.

On recalling that $e^2 - (cp)^2 = e_0^2$, we may use the relations

$$e - cp = \frac{e_0^2}{(e+cp)}, \qquad e + cp = \frac{e_0^2}{(e-cp)}, \tag{8.38}$$

to show that the above Eqs. (8.37) become

$$\frac{\partial p}{\partial \alpha} = \frac{e}{2ce_0^2}(e - cp)(f + cg), \qquad \frac{\partial p}{\partial \beta} = \frac{e}{2ce_0^2}(e + cp)(f - cg). \tag{8.39}$$

Further, if f and g are generated from a potential function $V(x, t)$ such that

$$f = -\frac{\partial V}{\partial x}, \qquad gc^2 = -\frac{\partial V}{\partial t}, \tag{8.40}$$

then it is not difficult to show that the above Eqs. (8.37) become simply

$$\frac{\partial V}{\partial \beta} = -\frac{\partial (e - pc)}{\partial \beta}, \qquad \frac{\partial V}{\partial \alpha} = -\frac{\partial (e + pc)}{\partial \alpha}, \tag{8.41}$$

so that we have the two relations $V = -(e - pc) - 2cF(\alpha)$ and $V = -(e + pc) + 2cG(\beta)$, where $F(\alpha)$ and $G(\beta)$ coincide with the arbitrary functions previously introduced. On using $f + cg = -2\partial V/\partial \alpha$ and $f - cg = 2\partial V/\partial \beta$, Eqs. (8.39) can be shown to become

$$\frac{\partial p}{\partial \alpha} = -\frac{e}{ce_0^2}(e - cp)\frac{\partial V}{\partial \alpha}, \qquad \frac{\partial p}{\partial \beta} = \frac{e}{ce_0^2}(e + cp)\frac{\partial V}{\partial \beta}.$$

On differentiating the first with respect to β, the second with respect to α, and using the equations themselves to eliminate the partial derivatives of p with respect to both α and β, the two equations yield

$$\frac{\partial^2 p}{\partial \alpha \partial \beta} = -\frac{(e - cp)}{ce_0^2}\left(e\frac{\partial^2 V}{\partial \alpha \partial \beta} - \frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial \beta}\right) = \frac{(e + cp)}{ce_0^2}\left(e\frac{\partial^2 V}{\partial \alpha \partial \beta} - \frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial \beta}\right),$$

demonstrating that necessarily the potential $V(\alpha, \beta)$ and the particle energy $e(\alpha, \beta)$ must be such that the following partial differential equation is satisfied

$$e\frac{\partial^2 V}{\partial \alpha \partial \beta} = \frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial \beta}, \tag{8.42}$$

in order that the momentum $p(\alpha, \beta)$ satisfies the wave equation $\partial^2 p/\partial \alpha \partial \beta = 0$. We observe that this equation is in complete agreement with the same Eq. (4.18), expressed in terms of (x, t) coordinates, and again reflects a singular limit for particle energy e tending to zero.

With $\mathscr{E} = -(e + V)$, the two basic equations (7.1) become simply

$$\frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t}, \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2\frac{\partial p}{\partial x}, \tag{8.43}$$

which in terms of the characteristic variables $\alpha = ct + x$ and $\beta = ct - x$ can be alternatively expressed as

$$\frac{\partial \mathscr{E}}{\partial \alpha} = c\frac{\partial p}{\partial \alpha}, \qquad \frac{\partial \mathscr{E}}{\partial \beta} = -c\frac{\partial p}{\partial \beta}. \tag{8.44}$$

Alternatively, for a single spatial dimension x, the one-dimensional equations (4.3) or (8.44) above are obtained from either (3.9) or (3.14) by equating the expressions, thus

$$d\mathscr{E} = \frac{\partial \mathscr{E}}{\partial x}dx + \frac{\partial \mathscr{E}}{\partial t}dt = \frac{\partial p}{\partial t}dx + c^2\frac{\partial p}{\partial x}dt. \tag{8.45}$$

The corresponding equation for $p(x, t)$ becomes

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial t}dt = \frac{1}{c^2}\frac{\partial \mathscr{E}}{\partial t}dx + \frac{\partial \mathscr{E}}{\partial x}dt, \tag{8.46}$$

so that on using $(8.45)_2$ together with $(8.46)_2$, we might deduce the relations

$$d(\mathscr{E} - pc) = -\frac{1}{c}\left(\frac{\partial(\mathscr{E} - pc)}{\partial t}dx + c^2\frac{\partial(\mathscr{E} - pc)}{\partial x}dt\right) = \frac{\partial(\mathscr{E} - pc)}{\partial x}dx + \frac{\partial(\mathscr{E} - pc)}{\partial t}dt,$$

$$d(\mathscr{E} + pc) = -\frac{1}{c}\left(\frac{\partial(\mathscr{E} + pc)}{\partial t}dx + c^2\frac{\partial(\mathscr{E} + pc)}{\partial x}dt\right) = \frac{\partial(\mathscr{E} + pc)}{\partial x}dx + \frac{\partial(\mathscr{E} + pc)}{\partial t}dt.$$

On equating coefficients of the differentials, using the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$ and the differential relations (8.36), the above equations can be shown to yield

$$\frac{\partial(\mathscr{E} - pc)}{\partial \alpha} = 0, \qquad \frac{\partial(\mathscr{E} + pc)}{\partial \beta} = 0,$$

from which we might deduce

$$\mathscr{E} + pc = 2cF(\alpha), \qquad \mathscr{E} - pc = -2cG(\beta), \tag{8.47}$$

where $F(\alpha)$ and $G(\beta)$ refer to the same arbitrary functions that are used throughout.

We note that for the particularly simple exact solution of Eqs. (4.3) underlying the previously discussed exact wave-light solution, which is given by

$$\mathscr{E}(x, t) = f_0(x + c\lambda t), \qquad cp(x, t) = f_0(\lambda x + ct),$$

where f_0 and λ denote arbitrary constants, the functions $F(\alpha)$ and $G(\beta)$ are simply

$$F(\alpha) = \frac{f_0}{2c}(1 + \lambda)\alpha, \qquad G(\beta) = \frac{f_0}{2c}(1 - \lambda)\beta.$$

Finally, in this section, we determine two families of solutions of the partial differential equations (8.41) or (8.42), both of which involve a general solution of the wave equation.

Solution (i) With $\rho = e + pc$, we may deduce, directly from Eqs. (8.38) and (8.41), the two equations

$$\frac{\partial V}{\partial \beta} = \frac{e_0^2}{\rho^2}\frac{\partial \rho}{\partial \beta}, \qquad \frac{\partial V}{\partial \alpha} = -\frac{\partial \rho}{\partial \alpha}, \tag{8.48}$$

and therefore, on eliminating $V(\alpha, \beta)$ and introducing $\sigma = \rho/e_0$, we may deduce the nonlinear partial differential equation

$$\left(1 + \frac{1}{\sigma^2}\right)\frac{\partial^2 \sigma}{\partial \alpha \partial \beta} = \frac{2}{\sigma^3}\frac{\partial \sigma}{\partial \alpha}\frac{\partial \sigma}{\partial \beta}. \tag{8.49}$$

We may deduce simple solutions of this equation with the assumption $\sigma = f(\theta)$, where $\theta(\alpha, \beta)$ is assumed to be a solution of the wave equation, that is, $\partial^2\theta/\partial\alpha\partial\beta = 0$. We have the partial derivatives

$$\frac{\partial \sigma}{\partial \alpha} = f'(\theta)\frac{\partial \theta}{\partial \alpha}, \qquad \frac{\partial \sigma}{\partial \beta} = f'(\theta)\frac{\partial \theta}{\partial \beta}, \qquad \frac{\partial^2 \sigma}{\partial \alpha \partial \beta} = f'(\theta)\frac{\partial^2 \theta}{\partial \alpha \partial \beta} + f''(\theta)\frac{\partial \theta}{\partial \alpha}\frac{\partial \theta}{\partial \beta},$$

and from which we may deduce the ordinary differential equation

$$\left(1 + \frac{1}{f^2}\right) f'' = \frac{2f'^2}{f^3},$$

assuming that both $\partial\theta/\partial\alpha \neq 0$ and $\partial\theta/\partial\beta \neq 0$. On rearrangement, this equation becomes $f''/f' = 2f'/f(1 + f^2)$, which integrates to give $f' = C_1 f^2/(1 + f^2)$, and a further integration gives

$$f - \frac{1}{f} = C_1(\theta - \theta_0),$$

where C_1 and θ_0 denote two arbitrary constants. This final equation may be solved as a quadratic equation to yield $f(\theta)$ as an explicit function of θ and a general family of solutions for (8.38) and the two partial differential equations (8.41), thus

$$f(\theta) = \frac{1}{2}\left(C_1(\theta - \theta_0) \pm \left[C_1^2(\theta - \theta_0)^2 + 4\right]^{1/2}\right),$$

where $f(\theta) = (e + pc)/e_0$ and where $\theta(\alpha, \beta)$ is assumed to be a solution of the wave equation, that is, $\partial^2\theta/\partial\alpha\partial\beta = 0$. With the notation $z = C_1(\theta - \theta_0)/2$, altogether we have

$$\frac{e+pc}{e_0} = \left(\frac{1+u/c}{1-u/c}\right)^{1/2} = z \pm (z^2+1)^{1/2},$$

so that for the positive case, we obtain

$$\frac{e+pc}{e_0} = \left(\frac{1+u/c}{1-u/c}\right)^{1/2} = e^{\tanh^{-1}(u/c)} = z + (z^2+1)^{1/2} = e^{\sinh^{-1}(z)},$$

and from the relation $\tanh^{-1}(u/c) = \sinh^{-1}(z)$, we may deduce that

$$\frac{u}{c} = \tanh(\sinh^{-1}(z)) = \frac{\sinh(\sinh^{-1}(z))}{\cosh(\sinh^{-1}(z))} = \frac{z}{(z^2+1)^{1/2}}.$$

On rearrangement of this latter equation, we may identify the quantity $z = C_1(\theta - \theta_0)/2$ with essentially the momentum p, which we know for a single spatial dimension satisfies the wave equation, and this ties in with the fact that $\theta(\alpha, \beta)$ is assumed to be a solution of the wave equation.

Alternatively, with $\rho = e - pc$ and then directly from Eqs. (8.38) and (8.41), in place of (8.48) we have

$$\frac{\partial V}{\partial \alpha} = \frac{e_0^2}{\rho^2}\frac{\partial \rho}{\partial \alpha}, \qquad \frac{\partial V}{\partial \beta} = -\frac{\partial \rho}{\partial \beta}, \tag{8.50}$$

so that again on eliminating $V(\alpha, \beta)$ and introducing $\sigma = \rho/e_0$, we may deduce that $\sigma(\alpha, \beta)$ satisfies exactly the same nonlinear partial differential equation as (8.49), and we may proceed as above.

Further, if in the above two solutions we identify the wave energy $\mathscr{E} = \pm pc$, then because of the conservation of energy $e + \mathscr{E} + V = constant$, only one of Eqs. (8.48) or (8.50) applies, since one is trivially satisfied. This ties in directly with the expressions $e = e_0/(1-(u/c)^2)^{1/2}$, $pc = e_0 u/c(1-(u/c)^2)^{1/2}$, and the rate equation (3.18), which becomes $d\mathscr{E}/dt = de/dt + e\partial u/\partial x$ and which for $\mathscr{E} = \pm pc$ simplifies considerably to give

$$\frac{\partial u}{\partial t} = \pm c\frac{\partial u}{\partial x},$$

and the two solutions reduce to either forward or backward waves only, which connects with the remaining nontrivial equation from either (8.48) or (8.50).

Solution (ii) The wave energy $\mathscr{E}(\alpha, \beta)$ satisfies the wave equation $\partial^2\mathscr{E}/\partial\alpha\partial\beta = 0$, and in the terminology of (4.43), we have $\mathscr{E}(\alpha, \beta) = c\,(F(\alpha) - G(\beta))$, where F and G denote the arbitrary functions used in (4.43). From the conservation of energy, we have $e = -(\mathscr{E} + V)$, modulo a constant, and therefore Eq. (8.42) becomes the nonlinear partial differential equation

$$(V + \mathscr{E})\frac{\partial^2 V}{\partial\alpha\partial\beta} + \frac{\partial V}{\partial\alpha}\frac{\partial V}{\partial\beta} = 0, \tag{8.51}$$

for which some limited solutions may be determined from the assumption $V(\alpha, \beta) = V(\mathscr{E})$. In this case, we have

$$\frac{\partial V}{\partial\alpha} = V'(\mathscr{E})\frac{\partial\mathscr{E}}{\partial\alpha}, \quad \frac{\partial V}{\partial\beta} = V'(\mathscr{E})\frac{\partial\mathscr{E}}{\partial\beta}, \quad \frac{\partial^2 V}{\partial\alpha\partial\beta} = V'(\mathscr{E})\frac{\partial^2\mathscr{E}}{\partial\alpha\partial\beta} + V''(\mathscr{E})\frac{\partial\mathscr{E}}{\partial\alpha}\frac{\partial\mathscr{E}}{\partial\beta} = V''(\mathscr{E})\frac{\partial\mathscr{E}}{\partial\alpha}\frac{\partial\mathscr{E}}{\partial\beta},$$

since $\partial^2\mathscr{E}/\partial\alpha\partial\beta = 0$. With the further assumptions that both $\partial\mathscr{E}/\partial\alpha \neq 0$ and $\partial\mathscr{E}/\partial\beta \neq 0$, the above partial differential equation (8.51) may be reduced to the following ordinary differential equation, thus

$$(V + \mathscr{E})V''(\mathscr{E}) + V'(\mathscr{E})^2 = 0,$$

where primes of course denote differentiation with respect to $\mathscr{E}$. The first integral may be readily obtained, thus

$$(V + \mathscr{E})V'(\mathscr{E}) = V + V_0, \tag{8.52}$$

where V_0 denotes the constant of integration. A further integral of this equation may be deduced by introducing the working variables $y = V + V_0$ and $x = \mathscr{E} - V_0$, so that the differential equation becomes simply $dy/dx = y/(y + x)$, which may be integrated with the standard substitution $v = y/x$, to give the final result

$$\mathscr{E} = V_0 + (V + V_0)\left(\log(V + V_0) + V_1\right), \tag{8.53}$$

where V_1 denotes the second constant of integration, and notice that $V(\mathscr{E})$ is defined only as an implicit function. We observe that in the case $V_1 = -1$, the particle energy e is essentially $(V + V_0)\log(V + V_0)$, and note the curiosity that this function happens to coincide with the entropy function arising in Shannon's information theory, namely the function $p \log p$ (see, e.g. [80], page 124).

We may extend this result somewhat by making use of the one-dimensional version of Eq. (3.17). For a single space dimension x, this result becomes simply $J\partial u/\partial x = dJ/dt$, where J is the magnitude of the derivative, thus $J = |\partial x/\partial X|$, the validity of which follows since $J\partial u/\partial x = \partial u/\partial X = dJ/dt$, and this latter equality follows from the fact that the total or material time derivative d/dt and the material spatial derivative $\partial/\partial X$ are interchangeable. With the working notation $W = V + V_0$, Eq. (8.53) becomes

$$\mathscr{E} + V = W + W\left(\log W + V_1\right) = W \log(W/\kappa),$$

where κ is another constant defined by $\log\kappa = -(1 + V_1)$, so that from $e = -(\mathscr{E} + V)$, we have $e = -W\log(W/\kappa)$ and the incremental relation (3.18) yields

$$\frac{2}{e}\frac{de}{dW} + \frac{1}{J}\frac{dJ}{dW} = \frac{-1}{e} = \frac{1}{W\log(W/\kappa)},$$

on using $dW = dV$. This equation integrates immediately to give

$$\log Je^2 = \log(\log(W/\kappa)) + \log J_0,$$

where J_0 denotes a further arbitrary constant, and from this latter relation, we may deduce $J = J_0/W^2\log(W/\kappa))$, so that altogether the Jacobian of the transformation between material and spatial coordinates is given explicitly by

$$J = \left|\tfrac{\partial x}{\partial X}\right| = \frac{J_0}{(V+V_0)^2\log((V+V_0)/\kappa)},$$

where J_0, V_0 and $\kappa = -(1+V_1)$ all denote arbitrary constants.

8.9 $p(x,t)$ and $\mathscr{E}(x,t)$ Assumed Independent Variables

In this section, we assume that the Jacobian $\Delta = \partial(p,\mathscr{E})/\partial(x,t)$ is non-zero, so that we may regard $p(x,t)$ and $\mathscr{E}(x,t)$ as independent variables, and we observe that this Jacobian is given by

$$\Delta = \frac{\partial(p,\mathscr{E})}{\partial(x,t)} = \frac{\partial p}{\partial x}\frac{\partial \mathscr{E}}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial \mathscr{E}}{\partial x} = c^2\left(\frac{\partial p}{\partial x}\right)^2 - \left(\frac{\partial p}{\partial t}\right)^2 = -4c^2\frac{\partial p}{\partial \alpha}\frac{\partial p}{\partial \beta}$$
$$= -\left(\frac{e}{e_0}\right)^2(f^2-(cg)^2) = 4\left(\frac{e}{e_0}\right)^2\frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial \beta},$$

so that here we are explicitly assuming that the forces f and g are such that $f \neq \pm cg$. Now with the working notation $\mathscr{F} = f + cg$ and $\mathscr{G} = f - cg$, Eq. (8.39) becomes

$$\frac{\partial p}{\partial \alpha} = \frac{e}{2ce_0^2}(e-cp)\mathscr{F}, \qquad \frac{\partial p}{\partial \beta} = \frac{e}{2ce_0^2}(e+cp)\mathscr{G}.$$

Again on differentiating the first equation with respect to β, the second with respect to α, and using the chain rule and the equations themselves to eliminate the partial derivatives of p with respect to both α and β, except that on this occasion we are treating both $\mathscr{F}$ and $\mathscr{G}$ as functions of the independent variables p and $\mathscr{E}$, thus $\mathscr{F} = \mathscr{F}(p,\mathscr{E})$ and $\mathscr{G} = \mathscr{G}(p,\mathscr{E})$, the two equations can be shown to eventually yield

$$\frac{\partial^2 p}{\partial\alpha\partial\beta} = \frac{\mathscr{G}e^2}{4c^2}\left(\frac{\partial\mathscr{F}}{\partial p} - c\frac{\partial\mathscr{F}}{\partial\mathscr{E}} - \frac{c(e-pc)}{e^2}\mathscr{F}\right), \tag{8.54}$$
$$= \frac{\mathscr{F}e^2}{4c^2}\left(\frac{\partial\mathscr{G}}{\partial p} + c\frac{\partial\mathscr{G}}{\partial\mathscr{E}} + \frac{c(e+pc)}{e^2}\mathscr{G}\right).$$

Now since the momentum p satisfies the classical wave equation, both expressions must vanish independently. If however we simply equate the two expressions for $\partial^2 p/\partial\alpha\partial\beta$, then we might deduce the following equation:

$$\left(\mathscr{G}\frac{\partial\mathscr{F}}{\partial p} - \mathscr{F}\frac{\partial\mathscr{G}}{\partial p}\right) - c\left(\mathscr{G}\frac{\partial\mathscr{F}}{\partial\mathscr{E}} + \mathscr{F}\frac{\partial\mathscr{G}}{\partial\mathscr{E}}\right) = \frac{2c\mathscr{F}\mathscr{G}}{e}. \tag{8.55}$$

We comment that the wave-like solution, exhaustively examined in subsequent chapters, also arises from this equation, assuming that both $\mathscr{F}$ and $\mathscr{G}$ are functions of the variable p only, so that the following relation holds

$$\left(\mathscr{G}\frac{d\mathscr{F}}{dp} - \mathscr{F}\frac{d\mathscr{G}}{dp}\right) = \frac{2c\mathscr{F}\mathscr{G}}{e}.$$

This equation simplifies to become

$$\frac{d\,(\mathscr{F}/\mathscr{G})}{dp} = \frac{2c}{(e_0^2 + (pc)^2)^{1/2}}\,(\mathscr{F}/\mathscr{G}),$$

so that on integration using the substitution $pc = e_0 \tan\phi$ yields

$$\frac{1}{2}\log\left(\frac{\mathscr{F}}{\mathscr{G}}\right) = \frac{1}{2}\log\left(\frac{1+\sin\phi}{1-\sin\phi}\right) + constant.$$

This equation can be shown to be equivalent to Eq. (6.4) arising in a different derivation of the wave-like solution. Specifically, from the above equation, we have

$$\frac{\mathscr{F}}{\mathscr{G}} = C\left(\frac{1+\sin\phi}{1-\sin\phi}\right),$$

where C is a constant, which may be reconciled with the constant λ appearing in Eq. (6.4) through the relation $\lambda = (C-1)/(C+1)$.

More generally, on solving the two partial differential equations arising from (8.54), namely

$$\frac{\partial\mathscr{F}}{\partial p} - c\frac{\partial\mathscr{F}}{\partial\mathscr{E}} - \frac{c(e-pc)}{e^2}\mathscr{F} = 0, \qquad \frac{\partial\mathscr{G}}{\partial p} + c\frac{\partial\mathscr{G}}{\partial\mathscr{E}} + \frac{c(e+pc)}{e^2}\mathscr{G} = 0,$$

by Lagrange's characteristic method, we may deduce the two general solutions

$$\mathscr{F}(p, \mathscr{E}) = \frac{(e + pc)}{e}\Phi(\mathscr{E} + pc), \qquad \mathscr{G}(p, \mathscr{E}) = \frac{(e - pc)}{e}\Psi(\mathscr{E} - pc),$$

where both Φ and Ψ denote arbitrary functions of the indicated argument. By direct substitution and performing the partial differentiations, and perhaps contrary to expectations, these general solutions can be shown to satisty equation (8.55) for all arbitrary functions $\Phi(\mathscr{E} + pc)$ and $\Psi(\mathscr{E} - pc)$. From these general solutions, we may deduce the following explicit expressions for the forces f and g as functions of p and $\mathscr{E}$, thus

$$f(p, \mathscr{E}) = \frac{1}{2}\left(\Phi(\mathscr{E} + pc) + \Psi(\mathscr{E} - pc)\right) + \frac{pc}{2e}\left(\Phi(\mathscr{E} + pc) - \Psi(\mathscr{E} - pc)\right),$$
$$cg(p, \mathscr{E}) = \frac{1}{2}\left(\Phi(\mathscr{E} + pc) - \Psi(\mathscr{E} - pc)\right) + \frac{pc}{2e}\left(\Phi(\mathscr{E} + pc) + \Psi(\mathscr{E} - pc)\right),$$

noting again the underlying assumption that the Jacobian $\Delta = \partial(p, \mathscr{E})/\partial(x, t)$ is non-zero, so that $p(x, t)$ and $\mathscr{E}(x, t)$ can be treated as independent variables, which corresponds to the assumption that $f \neq \pm cg$. We also note that since $pc/e = u/c$, these equations have the same essential structure of the assumed forces f and g underlying the exact wave-like solution and given by either (5.2) or (7.23), thus

$$f(p, \mathscr{E}) = \frac{1}{2}\left(\Phi(\mathscr{E} + pc) + \Psi(\mathscr{E} - pc)\right) + \frac{u}{2c}\left(\Phi(\mathscr{E} + pc) - \Psi(\mathscr{E} - pc)\right),$$
$$cg(p, \mathscr{E}) = \frac{1}{2}\left(\Phi(\mathscr{E} + pc) - \Psi(\mathscr{E} - pc)\right) + \frac{u}{2c}\left(\Phi(\mathscr{E} + pc) + \Psi(\mathscr{E} - pc)\right),$$

giving rise to the simple expressions

$$f(p, \mathscr{E}) + cg(p, \mathscr{E}) = \Phi(\mathscr{E} + pc)\left(1 + \frac{u}{c}\right), \quad f(p, \mathscr{E}) - cg(p, \mathscr{E}) = \Psi(\mathscr{E} - pc)\left(1 - \frac{u}{c}\right), \tag{8.56}$$

and therefore

$$f(p, \mathscr{E})^2 - c^2 g(p, \mathscr{E})^2 = \Phi(\mathscr{E} + pc)\Psi(\mathscr{E} - pc)\left(1 - \left(\frac{u}{c}\right)^2\right).$$

We note that on using the formulae (3.15), the relations (8.56) can be shown to be equivalent to

$$\frac{d(\mathscr{E} + pc)}{\Phi(\mathscr{E} + pc)} = d\alpha, \qquad \frac{d(\mathscr{E} - pc)}{\Psi(\mathscr{E} - pc)} = d\beta,$$

where as usual α and β denote the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, and after formal integration, the latter relations are entirely consistent with (8.47).

Chapter 9
Centrally Symmetric Mechanical Systems

9.1 Introduction

In this chapter, our specific purpose is to examine the consequences of the model in a centrally or spherically symmetric gravitating environment to determine the physically admissible or allowable potentials $V(r,t)$. We have in mind a central gravitating body generating a spherically symmetric environment, such that the motion generated is in the radial direction only with no angular contribution or dependence and is such that all variables are functions of (r,t) only, where r is the spatial radius taken from the centre of the gravitating body and defined by $r = (x^2 + y^2 + z^2)^{1/2}$. The potentials $V(r,t)$ are shown to arise as the solutions of four distinct partial differential equations, and the mathematical framework, corresponding to each of the four states of matter, is briefly described and, where possible, some of the simpler and physically more interesting solutions are noted. Further, as in Chap. 5, we speculate that dark matter and dark energy arise when there is a particular alignment of f and g, such that $e = \mathscr{E}$ and for which a special solution becomes available. In this chapter, however, the curvature is involved in the particular alignment of f and g. These proposals are supported by an entirely consistent mechanical and mathematical framework, for which there remains much future work, including the solution of specific problems and a stability analysis of the various states of matter.

The following four sections relate the basic equations, their general solution, the statement of energy conservation and the fundamental identity involving the external forces f and cg. In the section thereafter, we pose the question as to whether there exist solutions, for which both e_0 is non-zero and one of the constraints $f = \pm c(g - 2p/r)$ is satisfied. The next three sections deal with the Schwarzschild radius $r_0 = GM/c^2$ used here, which is one-half of the conventional value; the pseudo-Newtonian gravitational potential $V(r,t) = -e(r,t)r_0/r$; and the proposal for dark energy and dark matter. The subsequent four sections deal with the determination of the gravitational potentials $V(r,t)$, corresponding to each of the

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designated four types of matter. The following three sections present some technical details relating to similarity solutions, and there is one section containing a number of illustrative examples of spherically symmetric pulse waves, all the examples involving the Dirac delta function. The final section of the chapter deals with de Broglie's guidance equation in a centrally symmetric environment.

9.2 Basic Equations with Spherical Symmetry

Here, we assume a spherically symmetric environment, so that for the momentum vector $\mathbf{p} = (p_x, p_y, p_z)$, we may deduce from the differential relation (3.9), namely

$$d\mathscr{E} = \frac{\partial \mathbf{p}}{\partial t}.\mathbf{dx} + c^2(\nabla.\mathbf{p})dt,$$

the following four equations

$$\frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p_x}{\partial t}, \quad \frac{\partial \mathscr{E}}{\partial y} = \frac{\partial p_y}{\partial t}, \quad \frac{\partial \mathscr{E}}{\partial z} = \frac{\partial p_z}{\partial t}, \quad \frac{\partial \mathscr{E}}{\partial t} = c^2\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z}\right). \tag{9.1}$$

Accordingly, for the case of spherical symmetry, for which we have

$$p = p(r, t), \qquad \mathscr{E} = \mathscr{E}(r, t), \tag{9.2}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, we may deduce

$$\mathbf{p} = p(r, t)\hat{\mathbf{r}} = \frac{p(r, t)}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}),$$

where $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ denote the usual unit vectors in the (x, y, z) directions, respectively, and $\hat{\mathbf{r}}$ is a unit vector in the spherical radial direction. From this equation and (9.2), we may deduce that the four Eqs. (9.1) become

$$\frac{\partial \mathscr{E}}{\partial r} = \frac{\partial p}{\partial t}, \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2\left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right), \tag{9.3}$$

and from these two equations, we may readily obtain

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2\left(\frac{\partial^2 \mathscr{E}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathscr{E}}{\partial r}\right), \qquad \frac{\partial^2 p}{\partial t^2} = c^2\left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{2p}{r^2}\right), \tag{9.4}$$

again noting that the wave energy $\mathscr{E}(r, t)$ satisfies the classical wave equation, while the momentum $p(r, t)$ satisfies a wave-like equation. Notice also that the equation for the wave energy involves only derivatives of $\mathscr{E}(r,t)$, reflecting the fact that the wave energy is determined modulo an arbitrary additive constant.

Both Eqs. (9.4) are standard linear partial differential equations, for which many analytical solutions may be generated. Indeed, we may readily confirm that the immediately above equations are formally satisfied by

$$\mathscr{E}(r,t) = \frac{1}{r}\frac{\partial \psi}{\partial t} = \frac{\partial\,(\psi/r)}{\partial t}, \qquad p(r,t) = \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} = \frac{\partial\,(\psi/r)}{\partial r}, \tag{9.5}$$

where the potential $\psi(r,t)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial r^2}. \tag{9.6}$$

Further, if $\mathscr{E}(r,t)$ is any solution of the spherically symmetric wave equation $(9.4)_1$, then $\mathscr{E}^* = \partial\mathscr{E}/\partial t$ is a further solution, while $p^* = \partial\mathscr{E}/\partial r$ is a solution of $(9.4)_2$, and it is not difficult to show that $\mathscr{E}^*(r,t)$ and $p^*(r,t)$ constitute a matching or corresponding pair of the coupled equations (9.3), since from

$$\frac{\partial^2 \mathscr{E}}{\partial r \partial t} = \frac{\partial}{\partial r}\left(\frac{\partial \mathscr{E}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{E}}{\partial r}\right),$$

we have immediately

$$\frac{\partial \mathscr{E}^*}{\partial r} = \frac{\partial p^*}{\partial t},$$

and therefore, the first equation of (9.3) is satisfied. Further, on writing the second equation of (9.3) in the form

$$\frac{\partial \mathscr{E}}{\partial t} = \left(\frac{c}{r}\right)^2 \frac{\partial}{\partial r}(r^2 p),$$

and on partially differentiating this equation with respect to time, we obtain

$$\frac{\partial \mathscr{E}^*}{\partial t} = \left(\frac{c}{r}\right)^2 \frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial t}\right) = \left(\frac{c}{r}\right)^2 \frac{\partial}{\partial r}\left(r^2 \frac{\partial \mathscr{E}}{\partial r}\right) = \left(\frac{c}{r}\right)^2 \frac{\partial}{\partial r}(r^2 p^*) = c^2\left(\frac{\partial p^*}{\partial r} + \frac{2p^*}{r}\right).$$

Thus, if $\mathscr{E}(r,t)$ is any solution of the wave equation $(9.4)_1$, then

$$\mathscr{E}^*(r,t) = \frac{\partial \mathscr{E}}{\partial t}, \qquad p^*(r,t) = \frac{\partial \mathscr{E}}{\partial r},$$

constitute a matching or corresponding pair of the coupled equations (9.3). This result can be a useful mechanism to identify corresponding pairs $(\mathscr{E}, p)$ of the coupled equations (9.3). It is also implicit in the general solution (9.5), since if

$\psi(r,t)$ is any solution of the one-dimensional wave equation (9.6), then $\mathscr{E}^{**}(r,t) = \psi(r,t)/r$ satisfies the spherically symmetric wave equation $(9.4)_1$, as can be readily verified by substituting $\psi(r,t) = r\mathscr{E}^{**}(r,t)$ into Eq. (9.6).

It may also be worth noting that if $p(r,t) = q(r,t)/r^2$ satisfies $(9.4)_2$ then $q(r,t)$ satisfies the deceptively similar partial differential equation to $(9.4)_1$ but with an important sign difference, thus

$$\frac{\partial^2 q}{\partial t^2} = c^2 \left(\frac{\partial^2 q}{\partial r^2} - \frac{2}{r}\frac{\partial q}{\partial r} \right).$$

9.3 General Solutions for $\mathscr{E}(r,t)$ and $p(r,t)$

Equation (9.6) is well-known to admit the general solution $\psi(r,t) = F(ct+r) + G(ct-r)$, where F and G denote arbitrary functions, so that

$$\mathscr{E}(r,t) = \frac{c}{r}(F'(\alpha) + G'(\beta)), \quad p(r,t) = \frac{1}{r}(F'(\alpha) - G'(\beta)) - \frac{1}{r^2}(F(\alpha) + G(\beta)), \tag{9.7}$$

where $\alpha = ct + r$ and $\beta = ct - r$. In [49, 51], for a single spatial dimension, the notion of generalised de Broglie states are characterised by the condition $\mathscr{E}(r,t) = \pm cp$, being immediate generalisations of the two de Broglie relations $p = h/\mu$ and $\mathscr{E} = h\nu$. For one spatial dimension, these states may be characterised in a variety of ways, including as waves involving only a single family of characteristics. For more than one spatial dimension involving curvature, this characterisation is far more complicated, since it is clear from the immediately above equation that $\mathscr{E}(r,t) \mp cp$ necessarily involve both families of characteristics. Specifically, in terms of the characteristic coordinates, we have $r = (\alpha - \beta)/2$ and $t = (\alpha + \beta)/2c$, and the differential relations

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \qquad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta},$$

we have on using Eqs. (9.5) and (9.7) the following results:

$$\mathscr{E}(r,t) - cp(r,t) = 2c\frac{\partial}{\partial \beta}\left(\frac{\psi}{r}\right) = 2c\frac{\partial}{\partial \beta}\left(\frac{(F(\alpha) + G(\beta))}{r}\right) = \frac{2cG'(\beta)}{r} + \frac{c}{r^2}(F(\alpha) + G(\beta)),$$

$$\mathscr{E}(r,t) + cp(r,t) = 2c\frac{\partial}{\partial \alpha}\left(\frac{\psi}{r}\right) = 2c\frac{\partial}{\partial \alpha}\left(\frac{(F(\alpha) + G(\beta))}{r}\right) = \frac{2cF'(\alpha)}{r} - \frac{c}{r^2}(F(\alpha) + G(\beta)),$$

and therefore, for backward waves $G(\beta) = 0$ and forward waves $F(\alpha) = 0$, we have, respectively,

$$\mathscr{E}(r,t) - cp(r,t) = \frac{cF(\alpha)}{r^2}, \qquad \mathscr{E}(r,t) + cp(r,t) = -\frac{cG(\beta)}{r^2}.$$

Alternatively, with the definitions

$$E^-(r,t) = \mathscr{E}(r,t) - cp(r,t), \qquad E^+(r,t) = \mathscr{E}(r,t) + cp(r,t),$$

we may show using (9.3) and (9.4) that $E^-(r,t)$ and $E^+(r,t)$ satisfy, respectively, the following two partial differential equations:

$$\frac{\partial^2 E^-}{\partial t^2} = c^2 \left\{ \frac{\partial^2 E^-}{\partial r^2} + \frac{2}{r}\frac{\partial E^-}{\partial r} + \frac{1}{r}\left(\frac{\partial E^-}{\partial r} + \frac{1}{c}\frac{\partial E^-}{\partial t} \right) \right\}, \tag{9.8}$$

and

$$\frac{\partial^2 E^+}{\partial t^2} = c^2 \left\{ \frac{\partial^2 E^+}{\partial r^2} + \frac{2}{r}\frac{\partial E^+}{\partial r} + \frac{1}{r}\left(\frac{\partial E^+}{\partial r} - \frac{1}{c}\frac{\partial E^+}{\partial t} \right) \right\}. \tag{9.9}$$

In terms of the characteristic coordinates, these latter two equations become, respectively,

$$\frac{\partial^2 E^-}{\partial\alpha\partial\beta} - \frac{1}{r}\frac{\partial E^-}{\partial\alpha} + \frac{1}{2r}\frac{\partial E^-}{\partial\beta} = 0, \qquad \frac{\partial^2 E^+}{\partial\alpha\partial\beta} + \frac{1}{r}\frac{\partial E^+}{\partial\beta} - \frac{1}{2r}\frac{\partial E^+}{\partial\alpha} = 0. \tag{9.10}$$

With the respective substitutions

$$E^-(r,t) = \mathscr{E} - cp = 2c\frac{\partial}{\partial\beta}\left(\frac{\psi}{r}\right), \qquad E^+(r,t) = \mathscr{E} + cp = 2c\frac{\partial}{\partial\alpha}\left(\frac{\psi}{r}\right), \tag{9.11}$$

the two Eqs. (9.10) become, respectively,

$$\frac{\partial^3\psi}{\partial\alpha\partial\beta^2} = 0, \qquad \frac{\partial^3\psi}{\partial\alpha^2\partial\beta} = 0,$$

which from (9.6), namely $\partial^2\psi/\partial\alpha\partial\beta = 0$, are guaranteed to be satisfied. Equations (9.8) and (9.9) are important partial differential equations, which subsequently arise as Eqs. (9.53) and (9.45), respectively, and for which the major solution types are discussed in Sects. 9.12 and 9.13. Here, we merely comment that $e = pc$ associates with $E^+(r,t)$ (see Eq. (9.45)) while $e = -pc$ associates with $E^-(r,t)$ (see Eq. (9.53)).

The principal technical interest here is the determination of the allowable potentials $V(r,t)$ introduced subsequently in (9.12), and we may show that for the special cases $e = pc$ and $e = -pc$, the corresponding potentials $V(r,t)$ satisfy

Eqs. (9.9) and (9.8), respectively. However, the determination of a simple partial differential equation for $V(r,t)$ for the general cases $e = \pm({e_0}^2 + (pc)^2)^{1/2}$ with $e_0 \neq 0$ is more complicated. Throughout this chapter, we illustrate the theory with one or two of the major solution types. Since similarity stretching solutions are often physically the most interesting, we show in a subsequent section that the only similarity stretching solutions of the two wave equations (9.4) are those generated from $\psi(r,t) = C_1(ct+r)^n + C_2(ct-r)^n$, where C_1 and C_2 denote arbitrary constants and n is an arbitrary exponent. This result might well be anticipated, but it is not entirely obvious, and it is interesting that it can be formally established.

9.4 Conservation of Energy $e + \mathscr{E} + V$ = Constant

Now assuming that both f and g are spherically symmetric, so that $f = f(r,t)$ and $g = g(r,t)$ and that both are generated from a potential $V(r,t)$, then from the basic expressions (3.4), we have

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial r} = -\frac{\partial V}{\partial r}, \quad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right) = -\frac{1}{c^2}\frac{\partial V}{\partial t}, \tag{9.12}$$

which simplify to become

$$\frac{\partial p}{\partial t} = -\frac{\partial(e+V)}{\partial r}, \qquad \frac{\partial p}{\partial r} + \frac{2p}{r} = -\frac{1}{c^2}\frac{\partial(e+V)}{\partial t},$$

and therefore $(e+V)$ satisfies the classical wave equation. Further, from the immediately above relations and (9.3), we have

$$\frac{d\mathscr{E}}{dt} = \frac{\partial \mathscr{E}}{\partial t} + u\frac{\partial \mathscr{E}}{\partial r} = -\frac{\partial(e+V)}{\partial t} - u\frac{\partial(e+V)}{\partial r} = -\frac{d(e+V)}{dt}, \tag{9.13}$$

where $u = dr/dt$ is the particle velocity, and we confirm $e + \mathscr{E} + V =$ constant.

Equation in Conservation Form Following the corresponding result for a single spatial dimension (see Eq. (4.11)), Eqs. (9.3) can be expressed in conservation form as follows. On multiplication of the first equation of (9.3) by $\mathscr{E}$, the second by $c^2 p$ and then by addition, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\mathscr{E}^2 + (pc)^2\right) = c^2\left(\frac{\partial}{\partial r}(\mathscr{E}p) + \frac{2(\mathscr{E}p)}{r}\right),$$

which in conservation form becomes

$$\frac{\partial \mathscr{W}}{\partial t} + \frac{\partial \mathscr{Q}}{\partial r} + \frac{2\mathscr{Q}}{r} = 0, \tag{9.14}$$

where the energy density $\mathscr{W}(r,t)$ and energy flow or instantaneous power $\mathscr{Q}(r,t)$ are defined by

$$\mathscr{W} = \frac{1}{2}(\mathscr{E}^2 + (pc)^2) = \frac{1}{4}\left((\mathscr{E} - cp)^2 + (\mathscr{E} + cp)^2\right),$$
$$\mathscr{Q} = -c^2(\mathscr{E}p) = \frac{1}{4}\left((\mathscr{E} - cp)^2 - (\mathscr{E} + cp)^2\right),$$

and again Eq. (9.14) relates the time rate of increase (decrease) of the energy density $\mathscr{W}(r,t)$ that is balanced by a decrease (increase) in the instantaneous power $\mathscr{Q}(r,t)$.

Partial Differential Equation for Wave Energy-Momentum Ratio $\mathscr{E}(r,t)/cp(r,t)$ Again, following the corresponding result for a single spatial dimension (see Eqs. (4.12) and (4.13)), we may make the assumption that $\mathscr{E}(r,t) = \omega(r,t)cp(r,t)$, where $\omega(r,t)$ is to be determined, then from the above general solution (9.7), we might deduce the expression

$$\omega(r,t) = \left[\left(\frac{F'(\alpha) - G'(\beta)}{F'(\alpha) + G'(\beta)}\right) - \frac{1}{r}\left(\frac{F(\alpha) + G(\beta)}{F'(\alpha) + G'(\beta)}\right)\right]^{-1}. \tag{9.15}$$

Alternatively, in order to deduce the partial differential equation giving rise to this expression, from $\mathscr{E}(r,t) = \omega(r,t)cp(r,t)$ and the two first-order partial differential equations (9.3), we may deduce the two equations

$$p\frac{\partial \omega}{\partial t} + \omega\frac{\partial p}{\partial t} = c\left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right), \qquad cp\frac{\partial \omega}{\partial r} + \omega c\frac{\partial p}{\partial r} = \frac{\partial p}{\partial t},$$

which may be solved to give

$$(1-\omega^2)\frac{\partial p}{\partial t} = \left(c\frac{\partial \omega}{\partial r} + \omega\frac{\partial \omega}{\partial t} - \frac{2c\omega}{r}\right)p, \qquad c(1-\omega^2)\frac{\partial p}{\partial r} = \left(\frac{\partial \omega}{\partial t} + c\omega\frac{\partial \omega}{\partial r} - \frac{2c}{r}\right)p,$$

which with some rearrangement become

$$(1-\omega^2)\frac{\partial q}{\partial t} = c\left(\frac{\partial \omega}{\partial r} - \frac{2\omega}{r}\right), \qquad c(1-\omega^2)\frac{\partial q}{\partial r} = \frac{\partial \omega}{\partial t} - \frac{2c}{r}, \tag{9.16}$$

where $q = \log\left(p(1-\omega^2)^{1/2}\right)$. Following the corresponding analysis in Chap. 4, on introducing $\Omega(r,t)$ defined by the relations

$$\Omega = \frac{1}{2}\log\left(\frac{1+\omega}{1-\omega}\right) = \tanh^{-1}(\omega), \quad \left(\frac{1+\omega}{1-\omega}\right)^{1/2} = e^{\Omega}, \tag{9.17}$$

the two Eqs. (9.16) simplify to become

$$\frac{\partial q}{\partial t} = c\frac{\partial \Omega}{\partial r} - \frac{c}{r}\left(\frac{1}{(1-\omega)} - \frac{1}{(1+\omega)}\right), \qquad c\frac{\partial q}{\partial r} = \frac{\partial \Omega}{\partial t} - \frac{c}{r}\left(\frac{1}{(1-\omega)} + \frac{1}{(1+\omega)}\right).$$

In terms of the characteristic variables $\alpha = ct + r$ and $\beta = ct - r$, these two equations can be shown to become

$$\frac{\partial q}{\partial \alpha} = \frac{\partial \Omega}{\partial \alpha} - \frac{(1+e^{2\Omega})}{(\alpha-\beta)}, \qquad \frac{\partial q}{\partial \beta} = -\frac{\partial \Omega}{\partial \beta} + \frac{(1+e^{-2\Omega})}{(\alpha-\beta)},$$

which gives rise to the second order partial differential equation

$$\frac{\partial^2 \Omega}{\partial \alpha \partial \beta} = \frac{1}{(\alpha-\beta)}\left(e^{2\Omega}\frac{\partial \Omega}{\partial \beta} - e^{-2\Omega}\frac{\partial \Omega}{\partial \alpha}\right) + \frac{\sinh 2\Omega}{(\alpha-\beta)^2}, \tag{9.18}$$

for which Eq. (9.15), together with the relations (9.17), constitutes the formal general solution in terms of the characteristic variables $\alpha = ct + r$ and $\beta = ct - r$ and two arbitrary functions $F(\alpha)$ and $G(\beta)$.

We observe that in terms of the (r, t) variables, the highly nonlinear partial differential equation (9.18) simplifies eventually to become

$$\frac{1}{c^2}\frac{\partial^2 \Omega}{\partial t^2} - \frac{\partial^2 \Omega}{\partial r^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\cosh 2\Omega}{r}\right) - \frac{\partial}{\partial r}\left(\frac{\sinh 2\Omega}{r}\right),$$

which can be written alternatively as follows:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \Omega}{\partial t} - \frac{\cosh 2\Omega}{r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \Omega}{\partial r} - \frac{\sinh 2\Omega}{r}\right).$$

9.5 Fundamental Identity for *f* and *g*

From the functional relation $e^2 = {e_0}^2 + (pc)^2$ and the basic relations (9.12)

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial r}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right), \tag{9.19}$$

we find

$$f^2 - c^2\left(g - \frac{2p}{r}\right)^2 = \left(\frac{\partial p}{\partial t}\right)^2 + \left(\frac{\partial e}{\partial r}\right)^2 - c^2\left(\frac{\partial p}{\partial r}\right)^2 - \frac{1}{c^2}\left(\frac{\partial e}{\partial t}\right)^2, \tag{9.20}$$

which vanishes for the two special cases $e = \pm pc$. From the derived relations

$$e\frac{\partial e}{\partial t} = c^2 p\frac{\partial p}{\partial t}, \qquad e\frac{\partial e}{\partial r} = c^2 p\frac{\partial p}{\partial r} \tag{9.21}$$

we find that (9.20) can be rearranged to give

$$f^2 - c^2\left(g - \frac{2p}{r}\right)^2 = e\left(\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} - \frac{\partial^2 e}{\partial r^2}\right) - pc^2\left(\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial r^2}\right), \qquad (9.22)$$

using the wave-like equation (9.4) for $p(r,t)$ along with the relation $e^2 = {e_0}^2 + (pc)^2$ and the derived formulae (9.21), so that

$$\begin{aligned} & f^2 - c^2\left(g - \frac{2p}{r}\right)^2 \qquad (9.23)\\ &= e\left(\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} - \frac{\partial^2 e}{\partial r^2}\right) - pc^2\left(\frac{2}{r}\frac{\partial p}{\partial r} - \frac{2p}{r^2}\right)\\ &= e\left\{\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} - \left(\frac{\partial^2 e}{\partial r^2} + \frac{2}{r}\frac{\partial e}{\partial r} - \frac{2e}{r^2}\right)\right\} - \frac{2(e^2 - (cp)^2)}{r^2}, \end{aligned}$$

to eventually deduce the interesting and important formal identity

$$f^2 - c^2\left(g - \frac{2p}{r}\right)^2 = e\left\{\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} - \left(\frac{\partial^2 e}{\partial r^2} + \frac{2}{r}\frac{\partial e}{\partial r} - \frac{2e}{r^2}\right)\right\} - \frac{2{e_0}^2}{r^2}, \qquad (9.24)$$

demonstrating explicitly that $f = \pm c(g - 2p/r)$ when $e = \pm pc$ and $e_0 = 0$, and then $e(r,t)$ satisfies the same wave-like equation (9.4) as that for the momentum $p(r,t)$, as indeed it must. Using the relations (9.21) and their derivatives, in terms of the momentum $p(r,t)$, this identity becomes

$$f^2 - c^2\left(g - \frac{2p}{r}\right)^2 + p\left\{\frac{\partial^2 p}{\partial t^2} - c^2\left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{2p}{r^2}\right)\right\} = 0.$$

First Alternative Version The above identity (9.24) can be expressed in a number of ways. Firstly, from Eqs. (9.20) and (9.22), we have the formal identity

$$\begin{aligned} & \left(\frac{\partial p}{\partial t}\right)^2 + \left(\frac{\partial e}{\partial r}\right)^2 - c^2\left(\frac{\partial p}{\partial r}\right)^2 - \frac{1}{c^2}\left(\frac{\partial e}{\partial t}\right)^2\\ & - e\left(\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} - \frac{\partial^2 e}{\partial r^2}\right) + pc^2\left(\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial r^2}\right)\\ & = 0, \end{aligned}$$

and this equation can be simplified to give

$$\frac{\partial}{\partial r}\left(e\frac{\partial e}{\partial r}\right)-\frac{1}{c^2}\frac{\partial}{\partial t}\left(e\frac{\partial e}{\partial t}\right)+\frac{\partial}{\partial t}\left(p\frac{\partial p}{\partial t}\right)-c^2\frac{\partial}{\partial r}\left(p\frac{\partial p}{\partial r}\right)+2\frac{\partial(e,p)}{\partial(r,t)}=0,$$

which evidently can be rewritten as simply

$$\frac{\partial^2(e^2-(cp)^2)}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2(e^2-(cp)^2)}{\partial t^2}+4\frac{\partial(e,p)}{\partial(r,t)}=0.$$

This equation is of course trivially satisfied, since $e^2-(cp)^2=e_0^2$ and therefore e and p are functionally related and the Jacobian also vanishes. In terms of the basic assumed force relations

$$f=\frac{\partial p}{\partial t}+\frac{\partial e}{\partial r},\qquad g=\frac{1}{c^2}\frac{\partial e}{\partial t}+\left(\frac{\partial p}{\partial r}+\frac{2p}{r}\right),$$

the formal identity (9.23) becomes

$$f^2-c^2\left(g-\frac{2p}{r}\right)^2+e\left\{\frac{\partial f}{\partial r}-\frac{\partial}{\partial t}\left(g-\frac{2p}{r}\right)\right\}+p\left\{\frac{\partial f}{\partial t}-c^2\frac{\partial}{\partial r}\left(g-\frac{2p}{r}\right)\right\}=0,$$

which can be expressed in the alternative form

$$f^2+e\left(\frac{df}{dr}\right)_{wave}=c^2\left(g-\frac{2p}{r}\right)^2+e\left(\frac{d(g-2p/r)}{dt}\right)_{part},$$

where the time and spatial total derivatives follow the particle and the wave, respectively, thus

$$\left(\frac{d}{dt}\right)_{part}=\frac{\partial}{\partial t}+\left(\frac{dr}{dt}\right)_{part}\frac{\partial}{\partial r}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial r},$$

$$\left(\frac{d}{dr}\right)_{wave}=\frac{\partial}{\partial r}+\left(\frac{dt}{dr}\right)_{wave}\frac{\partial}{\partial t}=\frac{\partial}{\partial r}+\frac{u}{c^2}\frac{\partial}{\partial t}.$$

Second Alternative Version Secondly, we may make these relations more precise, since directly from (9.19) using the characteristic coordinates of the wave equation, namely $\alpha=ct+r$ and $\beta=ct-r$ and the differential formulae

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial \alpha}-\frac{\partial}{\partial \beta},\qquad \frac{1}{c}\frac{\partial}{\partial t}=\frac{\partial}{\partial \alpha}+\frac{\partial}{\partial \beta},$$

we may deduce

$$f - c\left(g - \frac{2p}{r}\right) = -2\frac{\partial(e-pc)}{\partial\beta}, \qquad f + c\left(g - \frac{2p}{r}\right) = 2\frac{\partial(e+pc)}{\partial\alpha}, \tag{9.25}$$

indicating explicitly that $f = c(g - 2p/r)$, when $e = pc$, and $f = -c(g - 2p/r)$, when $e = -pc$, and more generally we have $f = c(g-2p/r)$, when $e = pc+A(\alpha)$, and $f = -c(g - 2p/r)$, when $e = -pc + B(\beta)$, where $A(\alpha)$ and $B(\beta)$ denote arbitrary functions of their arguments.

Third Alternative Version in Terms of a Force Potential $V(r,t)$ Thirdly, we observe from the identity (9.24) that in the event that the applied forces f and g are generated from a potential $V(r,t)$ such that

$$f = -\frac{\partial V}{\partial r}, \quad g = -\frac{1}{c^2}\frac{\partial V}{\partial t},$$

then Eq. (9.24) becomes

$$\begin{aligned} &e\left\{\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} - \left(\frac{\partial^2 e}{\partial r^2} + \frac{2}{r}\frac{\partial e}{\partial r}\right)\right\} + \frac{2(e^2 - {e_0}^2)}{r^2} \\ &= f^2 - c^2\left(g - \frac{2p}{r}\right)^2 \\ &= \left(\frac{\partial V}{\partial r}\right)^2 - \left(\frac{1}{c}\frac{\partial V}{\partial t} + \frac{2pc}{r}\right)^2, \end{aligned}$$

and since the wave energy $\mathscr{E}(r,t)$ always satisfies the wave equation, we have from conservation of energy $e + \mathscr{E} = -V$ the following interesting equation:

$$\begin{aligned} &e\left\{\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}\right)\right\} \\ &= \left(\frac{1}{c}\frac{\partial V}{\partial t} + \frac{2pc}{r}\right)^2 - \left(\frac{\partial V}{\partial r}\right)^2 + \frac{2(e^2 - {e_0}^2)}{r^2} \\ &= \left(\frac{1}{c}\frac{\partial V}{\partial t}\right)^2 - \left(\frac{\partial V}{\partial r}\right)^2 + \frac{2pc}{r}\left(\frac{2}{c}\frac{\partial V}{\partial t} + \frac{3pc}{r}\right). \end{aligned}$$

Dark matter is an essentially backward wave occurring whenever $f = c(g-2p/r)$, while dark energy is an essentially forward wave occurring whenever $f = -c(g - 2p/r)$, and in both cases $e_0 = 0$ and the above equation simplify to become

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}\right) = \frac{1}{2}\left(\frac{\partial V}{\partial r} \pm \frac{1}{c}\frac{\partial V}{\partial t}\right)^2 = \frac{2(pc)^2}{er^2} = \frac{2e}{r^2}, \tag{9.26}$$

since $2pc/r = \pm\partial V/\partial r - (1/c)\partial V/\partial t$, noting that $e = \pm pc$ when $e_0 = 0$. These two states are discussed at length subsequently in this chapter. We note that from $V = -(\mathscr{E} + e)$ and using the fact that $\mathscr{E}(r, t)$ satisfies the wave equation $(9.4)_1$, we might deduce from (9.26) the equation arising from (9.27) in the case when $e_0 = 0$, which we now examine in the following section.

9.6 $f = \pm c(g - 2p/r)$ Implies e_0 Is Zero

Equation (9.23) would seem to indicate that the relations $f = \pm c(g - 2p/r)$ apply for $e_0 \neq 0$, provided that $e(r, t)$ is a solution of the nonlinear partial differential equation

$$\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} - \left(\frac{\partial^2 e}{\partial r^2} + \frac{2}{r}\frac{\partial e}{\partial r} - \frac{2e}{r^2}\right) = \frac{2{e_0}^2}{r^2 e}. \tag{9.27}$$

Noting that while $e = e_0$ constitutes a formal solution (9.27) with $e_0 \neq 0$, we show later in this section that although the differential equation evidently admits nontrivial solutions with $e_0 \neq 0$, it turns out that, when the additional constraints are taken into account that for such solutions to exist, e_0 is necessarily zero. However, in the absence of the other constraints it would still be interesting to determine any simple analytical solutions of this equation with $e_0 \neq 0$. There are certainly nontrivial solutions of the single equation (9.27) with $e_0 \neq 0$, and we mention here two such examples.

Time-Independent Solutions $e = e(r)$ Firstly, there are the time-independent solutions $e = e(r)$, for which the partial differential equation (9.27) reduces to the ordinary differential equation

$$\frac{d^2 e}{dr^2} + \frac{2}{r}\frac{de}{dr} - \frac{2e}{r^2} = -\frac{2{e_0}^2}{r^2 e}.$$

This equation remains invariant under the simple one-parameter stretching group of transformations, $r^* = \lambda r$ and $e^* = \lambda e$, and therefore may be reduced to a first-order ordinary differential equation by first making the substitution $\tau = \log r$ to give

$$\frac{d^2 e}{d\tau^2} + \frac{de}{d\tau} = 2\left(e - \frac{{e_0}^2}{e}\right),$$

followed by the substitution $z = de/d\tau$ to obtain the following first-order ordinary differential equation

$$z\left(\frac{dz}{de} + 1\right) = 2\left(e - \frac{{e_0}^2}{e}\right), \tag{9.28}$$

and it would be interesting to proceed further with this equation to determine any specific simple solutions. With a very small change of variables ($z = -y$ and $e = x$), Eq. (9.28) becomes an Abel equation of the second kind in standard form, for which [85] (first order 0.1.6–1 and 1.3.1 Tables 5–8) provides many special solvable cases. Unfortunately, however, Eq. (9.28) appears not to be included there.

Similarity Solutions $e = e(r/ct)$ Secondly, there are the similarity solutions $e = e(\xi)$, where $\xi = r/ct$ arising from the invariance of the partial differential equation (9.27) under the one-parameter stretching group of transformations, $r^* = \lambda r$, $t^* = \lambda t$ and $e^* = \lambda e$, and therefore on using the formulae

$$\frac{\partial e}{\partial t} = -\frac{\xi}{t}e', \quad \frac{\partial e}{\partial r} = \frac{e'}{ct}, \quad \frac{\partial^2 e}{\partial t^2} = \frac{\xi}{t^2}\left(\xi e'' + 2e'\right), \quad \frac{\partial^2 e}{\partial r^2} = \frac{e''}{(ct)^2},$$

where primes denote differentiation with respect to ξ, the partial differential equation reduces to the ordinary differential equation

$$(1-\xi^2)(\xi^2 e')' = 2\left(e - \frac{{e_0}^2}{e}\right).$$

This equation may be further simplified with the substitution $\eta = 1/\xi$ to obtain

$$(\eta^2 - 1)\frac{d^2 e}{d\eta^2} = 2\left(e - \frac{{e_0}^2}{e}\right),$$

and the transformations $x = (\eta+1)/2$ and $e = e_0 y$ simplify this equation somewhat to give

$$x(1-x)\frac{d^2 y}{dx^2} + y = \frac{1}{y}. \tag{9.29}$$

Again, it would be interesting to determine any specific simple solutions, but we do not examine either (9.28) or (9.29) further.

Thus, while there are certainly nontrivial solutions $e(r,t)$ of (9.27) for which $e_0 \neq 0$, the critical question is whether or not the momentum $p(r,t)$, as generated from the expression $pc = (e^2 - {e_0}^2)^{1/2}$, satisfies the Eq. $(9.4)_2$. Making use of the additional constraint $f = \pm c(g - 2p/r)$, so that either $e = pc + A(\alpha)$ or $e = -pc + B(\beta)$, then a straightforward analysis would seem to indicate that there are no such solutions of (9.27), unless $e_0 = 0$. For example, if $f = c(g - 2p/r)$, then $e = pc + A(\alpha)$, where $\alpha = ct + r$, and from Eq. (9.22), we may deduce

$$(pc + A(\alpha))\left(\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial r^2}\right) = pc\left(\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial r^2}\right),$$

and therefore,

$$A(\alpha)\left(\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}-\frac{\partial^2 p}{\partial r^2}\right)=0.$$

This implies that either $A(\alpha) = 0$, in which case e_0 is necessarily zero from the expression $pc = (e^2 - {e_0}^2)^{1/2}$, or $p(r, t)$ is a solution of the classical one-dimensional wave equation, which from Eq. (9.4)$_2$ implies that $\partial(p/r)/\partial r = 0$ and therefore $p(r, t) = rP(t)$, where $P(t)$ is a function of time only. Thus, we may conclude that $P''(t) = 0$ and therefore

$$p(r,t)=r(C_1t+C_2),\qquad e(r,t)=cr(C_1t+C_2)+A(\alpha),$$

where C_1 and C_2 denote arbitrary constants, and from $pc = (e^2 - {e_0}^2)^{1/2}$, it is clearly not possible to determine a meaningful nontrivial solution for $A(\alpha)$ from the resulting quadratic equation $A(\alpha)^2 + 2cr(C_1t + C_2)A(\alpha) - e_0^2 = 0$, and a similar proof applies when $f = -c(g - 2p/r)$ and $e = -pc + B(\beta)$, where $\beta = ct - r$.

For the time-independent solutions $e = e(r)$ and the similarity solutions $e = e(r/ct)$, this outcome is most apparent by direct examination of (9.20), since in the former case this equation becomes $(de/dr)^2 = c^2(dp/dr)^2$, but from (9.21)$_2$, we have $e(de/dr) = c^2p(dp/dr)$ and therefore necessarily $e_0 = 0$. In the latter case, a similar argument using (9.20) and (9.21) yields $e'^2 = c^2p'^2$ and $ee' = c^2pp'$, and again from which we might deduce that e_0 is necessarily zero. Thus, even though the proposed model appears to allow nontrivial solutions for $e(r, t)$ satisfying Eq. (9.27) for $e_0 \neq 0$, we have established the important outcome that whenever the relations $f = \pm c(g - 2p/r)$ apply, the rest energy e_0 is necessarily zero.

9.7 Newtonian Gravitation and Schwarzschild Radius

The Newtonian gravitational field $V(r, t) = -GMm_0/r$, where as usual G is the universal gravitational constant and M is the mass of the larger gravitating body, can be seen to arise from the above formulation in a variety of asymptotic limits. For example, the simplest solution of (9.6) $\psi(r, t) = GMm_0t + C_1r + C_2$, where C_1 and C_2 denote arbitrary constants, gives $\mathcal{E}(r, t) = r_0e_0/r$ and $p(r, t) = -r_0e_0(t - t_0)/r^2$, where for convenience we have adopted $C_2 = -r_0e_0t_0$ for some arbitrary time t_0, and $r_0 = GM/c^2$ denotes one-half of the conventional Schwarzschild radius. From the conservation of energy $e + \mathcal{E} + V =$ constant and modulo an arbitrary constant, we find that the potential $V(r, t)$ is given by

$$V(r,t)=-\frac{r_0e_0}{r}-e_0\left\{1+\left(\frac{r_0c(t-t_0)}{r^2}\right)^2\right\}^{1/2},\tag{9.30}$$

which on expansion in powers of $1/r^4$ gives the Newtonian potential correct to order $(r_0c(t-t_0))^2/r^4$, and given that r_0 is extremely small and typically r is large, we might expect that such a potential would be numerically indistinguishable from the conventional expression. It is clear from such approximations that the major contribution of the potential $V(r, t)$ arises from the wave energy $\mathscr{E}(r, t)$, which always satisfies the classical wave equation, and this is entirely consistent with the post-Newtonian approximation (see, e.g. [95]).

We comment that the escape velocity is defined to be the velocity necessary to escape a gravitational field, so that for classical Newtonian gravitation, from the usual conservation of energy kinetic energy + potential energy = constant, we have, assuming that $v = v_0$ at $r = a$,

$$\frac{mv^2}{2} - \frac{GMm}{r} = \frac{mv_0^2}{2} - \frac{GMm}{a},$$

so that in order that $v = 0$ at $r = \infty$, then classical Newtonian mechanics predicts that the escape velocity is given by $v_0^2 = 2GM/a$. If the escape velocity is prescribed to be the speed of light, namely $v_0 = c$, then $a = 2GM/c^2$, and this is conventionally referred to as the Schwarzschild radius. However, if we undertake the same calculation from the perspective of special relativistic mechanics, then we obtain a radius r_0, which is precisely one-half of the Schwarzschild radius, since in this case the corresponding conservation of energy principle gives

$$\frac{m_0c^2}{(1-(v/c)^2)^{1/2}} - \frac{GMm}{r} = \frac{m_0c^2}{(1-(v_0/c)^2)^{1/2}} - \frac{GMm_0}{a(1-(v_0/c)^2)^{1/2}},$$

so that again in order that $v = 0$ at $r = \infty$, we have $(1-(v_0/c)^2)^{1/2} = 1-GM/ac^2$, and therefore if the escape velocity is the speed of light, then we obtain $a = r_0 = GM/c^2$, namely precisely one-half of the conventional Schwarzschild radius.

9.8 Pseudo-Newtonian Gravitational Potential

It seems natural to ask the question as to whether or not the model admits the Newtonian gravitational potential $V(r, t) = -GMm/r$ as an exact consequence, where as usual m denotes the mass given by $m = m_0[1 - (u/c)^2]^{-1/2}$. That is, assuming that M remains fixed, does the proposed formulation admit the gravitational potential $V(r, t)$ given by

$$V(r, t) = -\frac{r_0}{r}e(r, t), \tag{9.31}$$

where $e(r, t) = mc^2$ and again $r_0 = GM/c^2$ is one-half of the conventional Schwarzschild radius. In making the hypothesis (9.31), we are imposing a definite constraint involving both $V(r, t)$ and $e(r, t)$, and while we might fully determine the general mathematical solution assuming this constraint, it is not possible to determine $p(r,t)$ and $e(r,t)$, which also satisfy the nonlinear relation $e^2 - (pc)^2 = e_0^2$. Nevertheless, the formal details of the general mathematical solution arising from (9.31) are not without interest.

From (9.31) and conservation of energy $e + \mathscr{E} + V = \text{constant}$, we might deduce

$$e(r,t) = -\frac{r}{r_0}V(r,t), \qquad \mathscr{E}(r,t) = -(e+V) = -\left(1 - \frac{r}{r_0}\right)V(r,t),$$

and since we know that $\mathscr{E}(r,t)$ satisfies the wave equation, we might readily obtain from $(9.4)_1$

$$\frac{\partial^2 V}{\partial t^2} = c^2\left\{\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{2}{(r-r_0)}\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right)\right\}, \tag{9.32}$$

and a general solution to this equation may be found as follows. On making the substitution $V(r,t) = U(r,t)/r$, Eq. (9.32) becomes

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{(r-r_0)}\frac{\partial U}{\partial r}\right),$$

so that on using $x = r - r_0$ as a working variable, this equation becomes

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{2}{x}\frac{\partial U}{\partial x}\right),$$

which with the substitution $U(r,t) = W(x,t)/x$ simplifies to give the classical one-dimensional wave equation, thus

$$\frac{\partial^2 W}{\partial t^2} = c^2\frac{\partial^2 W}{\partial x^2}. \tag{9.33}$$

From the general solution of (9.33), namely $W(x,t) = F(\alpha) + G(\beta)$, $F(\alpha)$ and $G(\beta)$ denote arbitrary functions, and in this section, α and β are defined by

$$\alpha = ct + x = ct + r - r_0, \qquad \beta = ct - x = ct - r + r_0.$$

On retracing the above substitutions, we might readily verify that the following results constitute the formal general mathematical solution arising from the assumption (9.31), thus

$$V(r,t) = \frac{F(\alpha)+G(\beta)}{r(r-r_0)}, \quad e(r,t) = -\frac{(F(\alpha)+G(\beta))}{r_0(r-r_0)}, \quad \mathscr{E}(r,t) = \frac{F(\alpha)+G(\beta)}{rr_0}, \tag{9.34}$$

and it is a straightforward matter to confirm that energy conservation $e + \mathscr{E} + V =$ constant is correctly satisfied. We observe that on face value the gravitational potential $V(r, t)$ has singularities at both $r = 0$ and $r = r_0$, while the particle and wave energies each acquire only one of these singularities. Further, from Eqs. (9.3), we can perform two partial integrations to show that $p(r, t)$ becomes

$$cp(r,t) = \frac{F(\alpha) - G(\beta)}{rr_0} - \frac{1}{r_0 r^2}\left(\int F(\alpha)d\alpha + \int G(\beta)d\beta + C_1\right), \tag{9.35}$$

where C_1 denotes an arbitrary constant. It is clear from the above expressions that even with $F(\alpha)$ and $G(\beta)$ as arbitrary functions, the nonlinear condition $e^2 - (pc)^2 = e_0^2$ presents a formidable constraint, and the very best that might be expected is to satisfy this latter constraint in some approximate manner. Taking into account the different singularities in the above expressions, we might readily verify the following relations:

$$cp(r,t) - \left(1 - \frac{r_0}{r}\right)e(r,t) = \frac{2F(\alpha)}{rr_0} - \frac{1}{r_0 r^2}\left(\int F(\alpha)d\alpha + \int G(\beta)d\beta + C_1\right),$$

$$cp(r,t) + \left(1 - \frac{r_0}{r}\right)e(r,t) = -\frac{2G(\beta)}{rr_0} - \frac{1}{r_0 r^2}\left(\int F(\alpha)d\alpha + \int G(\beta)d\beta + C_1\right),$$

and noting that since $\mathscr{E}(r, t) = -(1 - r_0/r)\, e(r, t)$, the same equations become

$$\mathscr{E}(r,t) + cp(r,t) = \frac{2F(\alpha)}{rr_0} - \frac{1}{r_0 r^2}\left(\int F(\alpha)d\alpha + \int G(\beta)d\beta + C_1\right),$$

$$\mathscr{E}(r,t) - cp(r,t) = \frac{2G(\beta)}{rr_0} + \frac{1}{r_0 r^2}\left(\int F(\alpha)d\alpha + \int G(\beta)d\beta + C_1\right).$$

As a simple illustrative example, we consider the case $F(\alpha) = \mu\alpha^2$ and $G(\beta) = -\mu\beta^2$ for some constant μ, so that from (9.34), we have

$$V(r,t) = \frac{4\mu ct}{r}, \qquad e(r,t) = -\frac{4\mu ct}{r_0}, \qquad \mathscr{E}(r,t) = 4\mu ct\left(\frac{1}{r_0} - \frac{1}{r}\right),$$

while from (9.35), we obtain

$$cp(r,t) = 2\mu\left[\left(\frac{ct}{r}\right)^2 + \frac{1}{3}\left(1 - \frac{r_0}{r}\right)^2\left(1 + \frac{2r}{r_0}\right) + C_2\right],$$

where C_2 denotes an arbitrary constant, and clearly these expressions do not satisfy the nonlinear relationship $e^2 - (pc)^2 = e_0^2$.

9.9 Dark Matter-Dark Energy and Four Types of Matter

Consolidating the results of this chapter, and since it is believed that there is more dark energy in the universe than dark matter, we propose that dark matter arises as an essentially backward wave and accrues from time past, while dark energy is essentially a forward wave arising in consequence of future time. The following is a consistent mathematical picture motivated from the results of this chapter. We propose that dark energy and dark matter arise when there is a particular alignment of the physical force f with the force g in the direction of time, so that the particle energy e and wave energy $\mathscr{E}$ coincide, thus $e = \mathscr{E}$. In a single spatial dimension, this alignment is exact, namely $f = \pm cg$ and $e = \mathscr{E}$, but here the problem is three-dimensional, and therefore there are curvature effects arising through the $1/r$ term, and precise equality occurs only for large values of r. In a real circumstance, we might expect a situation comparable to a "fuzzy region", where the key equalities are constantly switching on and off dependent upon a varying local environment.

Specifically, we propose dark matter to be an essentially backward wave occurring whenever $f = c(g - 2p/r)$, which in terms of the potential $V(r, t)$ becomes Eq. (9.44), namely $p(r, t) = r(\partial V/\partial r - \partial V/c\partial t)/2c$, and for which

$$e(r, t) = cp(r, t), \qquad \mathscr{E}(r, t) = cp(r, t) + \frac{cF(\alpha)}{r^2},$$

$$p(r, t) = \frac{F'(\alpha)}{r} - \frac{F(\alpha)}{r^2}, \qquad V(r, t) = -\frac{2cF'(\alpha)}{r} + \frac{cF(\alpha)}{r^2},$$

where $F(\alpha)$ denotes an arbitrary function. Similarly, we propose that dark energy is an essentially forward wave occurring whenever $f = -c(g - 2p/r)$, which in terms of the potential $V(r, t)$ becomes Eq. (9.52), namely $p(r, t) = -r(\partial V/\partial r + \partial V/c\partial t)/2c$, and for which

$$e(r, t) = -cp(r, t), \qquad \mathscr{E}(r, t) = -cp(r, t) - \frac{cG(\beta)}{r^2},$$

$$p(r, t) = \frac{-G'(\beta)}{r} - \frac{G(\beta)}{r^2}, \qquad V(r, t) = -\frac{2cG'(\beta)}{r} - \frac{cG(\beta)}{r^2},$$

where $G(\beta)$ denotes an arbitrary function. As noted above, we also have the more general results that $f = c(g-2p/r)$, when $e = pc+A(\alpha)$, and $f = -c(g-2p/r)$, when $e = -pc + B(\beta)$, where $A(\alpha)$ and $B(\beta)$ denote arbitrary functions of their arguments. In the above definition of states (III) and (IV) presented in Sect. 9.1, the arbitrary functions $A(\alpha)$ and $B(\beta)$ are taken to be zero. However, we note that the more general approach reinforces the picture of potentials $V(r, t)$ involving ascending powers of $1/r$ with coefficients that are either forward or backward waves.

In the remainder of this chapter, we are particularly concerned to determine the allowable potentials $V(r, t)$ corresponding to each of the four types. In one sense, by virtue of (9.5) and (9.6), the formal solutions of any problems for $\mathscr{E}(r, t)$ and $p(r, t)$ are fully determined, so that in principle, the potential is prescribed through the equation $V(r, t) = -\mathscr{E}(r, t) - (e_0{}^2 + (cp(r, t))^2)^{1/2}$, arising from conservation of energy. However, because of the square root, the determination of a partial differential equation for the allowable potentials is not obvious for types (I) and (II). For dark matter and dark energy, the situation is more straightforward.

9.10 Positive Energy (I) $e = (e_0{}^2 + (pc)^2)^{1/2}$, $e_0 \neq 0$

In this section, we assume that $e_0 \neq 0$ and that conventional or baryonic matter is described by the relation $e = (e_0{}^2 + (pc)^2)^{1/2}$, so that we introduce the angle $\phi(r, t)$ such that

$$pc = e_0 \tan\phi, \qquad e = e_0 \sec\phi, \tag{9.36}$$

and the relation $pc = e\sin\phi$ sheds insight on the subsequent analysis, since the cases $pc = \pm e$ correspond to the maximum and minimum values of $\sin\phi$, namely $\phi = \pm\pi/2$, which are examined subsequently. Using the fact that the momentum $p(r, t)$ satisfies the wave-like equation (9.4), we might readily deduce

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \left(\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}\right) + 2\tan\phi\left\{\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\cos^2\phi}{r^2}\right\},$$

as the governing equation for $\phi(r, t)$. Although nonlinear, this equation is merely a restatement of the wave-like equation (9.4) for the momentum $pc = e_0\tan\phi$, and for small angles ϕ, on neglecting the nonlinear terms and approximating the term $2\sin\phi\cos\phi = \sin 2\phi \approx 2\phi$, we again recover precisely the wave-like equation (9.4) for $\phi(r, t)$ as for the momentum $p(r, t)$.

Alternatively, on using the relations (9.36) together with the Eqs. (9.12), we may deduce the coupled partial differential equations

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \sin\phi\frac{\partial\phi}{\partial r} = \cos^2\phi\frac{\partial U}{\partial r}, \qquad \frac{\sin\phi}{c}\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial r} + \frac{2}{r}\sin\phi\cos\phi = \frac{\cos^2\phi}{c}\frac{\partial U}{\partial t},$$

where for working convenience we have introduced $U(r, t) = -V(r, t)/e_0$. These two equations may be reformulated to read

$$\frac{\partial \phi}{\partial r} = \left(\frac{1}{c}\frac{\partial U}{\partial t} - \frac{2}{r}\tan\phi \right) - \sin\phi \frac{\partial U}{\partial r}, \quad \frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{\partial U}{\partial r} - \sin\phi \left(\frac{1}{c}\frac{\partial U}{\partial t} - \frac{2}{r}\tan\phi \right), \tag{9.37}$$

and on noting the particular grouping identified in (9.37) and cross-differentiating the left-hand sides of the two equations, we may eventually deduce

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} \right) \tag{9.38}$$
$$+ \cos\phi \left\{ \left(\frac{\partial U}{\partial r} \right)^2 - \left(\frac{1}{c}\frac{\partial U}{\partial t} - \frac{2}{r}\tan\phi \right)^2 - \frac{2}{r^2}\tan^2\phi \right\},$$

Further, from (9.37), we have the identity

$$\cos^2\phi \left\{ \left(\frac{\partial U}{\partial r} \right)^2 - \left(\frac{1}{c}\frac{\partial U}{\partial t} - \frac{2}{r}\tan\phi \right)^2 \right\} = \frac{1}{c^2}\left(\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial r} \right)^2,$$

and from which we may deduce the alternative form for (9.38), namely

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} \right) + \sec\phi \left\{ \left(\frac{1}{c}\frac{\partial \phi}{\partial t} \right)^2 - \left(\frac{\partial \phi}{\partial r} \right)^2 - \frac{2}{r^2}\sin^2\phi \right\}. \tag{9.39}$$

In terms of the potential $V(r,t)$, Eqs. (9.38) and (9.39) become, respectively,

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} \right) + \frac{\cos\phi}{e_0}\left(\frac{1}{c}\frac{\partial V}{\partial t} + \frac{2e_0}{r}\tan\phi \right)^2$$
$$- \frac{\cos\phi}{e_0}\left\{ \left(\frac{\partial V}{\partial r} \right)^2 - \frac{2e_0^2}{r^2}\tan^2\phi \right\},$$

and

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} \right) + e_0\sec\phi \left\{ \left(\frac{\partial \phi}{\partial r} \right)^2 - \left(\frac{1}{c}\frac{\partial \phi}{\partial t} \right)^2 + \frac{2}{r^2}\sin^2\phi \right\},$$

from which it is apparent that the limiting situation $e_0 \to 0$ and $\phi \to \pm\pi/2$ is tricky and not entirely straightforward, and in both equations, e_0 and ϕ appear through the combination $e_0 \sec\phi$.

The two key points here are that unlike classical theory, the determination of the potential $V(r,t)$ arises as part of the solution procedure and cannot be arbitrarily assigned, so that the situation is more akin to general relativity, where the gravitational field is determined from solving the field equations. The second point is that while the potential $V(r,t)$ almost satisfies the classical wave equation, the correspondence is not exact, and the actual equation for $V(r,t)$ is far more complicated, noting that post-Newtonian theory predicts that $V(r,t)$ satisfies precisely the classical wave equation (see, e.g. [95]). As previously noted, this means that we cannot simply merge a Newtonian gravitational field $V(r,t) = -GMm_0/r$ (using the usual symbols) in a consistent manner, but we can frequently see such a potential emerging in a variety of asymptotic limits.

9.11 Negative Energy (II) $e = -({e_0}^2 + (pc)^2)^{1/2}$, $e_0 \neq 0$

In this section, we assume that $e_0 \neq 0$ and the relation $e = -({e_0}^2 + (pc)^2)^{1/2}$, so that in terms of the angle $\phi(r,t)$, we have

$$pc = e_0 \tan\phi, \qquad e = -e_0 \sec\phi, \tag{9.40}$$

and we may relate the governing equations of this section to those of the previous section through the transformation $\phi \to \phi - \pi$, noting that both $\sin\phi$ and $\cos\phi$ change sign while $\tan\phi$ remains unchanged by this transformation. Accordingly, the momentum wave-like equation (9.4) again yields the same equation as for case (I) for (9.40), while in place of (9.38) and (9.39), we may readily deduce from

$$\frac{\partial\phi}{\partial r} = \left(\frac{1}{c}\frac{\partial U}{\partial t} - \frac{2}{r}\tan\phi\right) + \sin\phi\frac{\partial U}{\partial r}, \qquad \frac{1}{c}\frac{\partial\phi}{\partial t} = \frac{\partial U}{\partial r} + \sin\phi\left(\frac{1}{c}\frac{\partial U}{\partial t} - \frac{2}{r}\tan\phi\right),$$

either of the following two equations:

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} &= \left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right) \\ &- \cos\phi\left\{\left(\frac{\partial U}{\partial r}\right)^2 - \left(\frac{1}{c}\frac{\partial U}{\partial t} - \frac{2}{r}\tan\phi\right)^2 - \frac{2}{r^2}\tan^2\phi\right\},\end{aligned} \tag{9.41}$$

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right) - \sec\phi\left\{\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2}{r^2}\sin^2\phi\right\}, \tag{9.42}$$

where again for working convenience $U(r,t) = -V(r,t)/e_0$. In terms of the potential $V(r,t)$, Eqs. (9.41) and (9.42)

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}\right)$$
$$+\frac{\cos\phi}{e_0}\left\{\left(\frac{\partial V}{\partial r}\right)^2 - \left(\frac{1}{c}\frac{\partial V}{\partial t} + \frac{2e_0}{r}\tan\phi\right)^2 - \frac{2e_0^2}{r^2}\tan^2\phi\right\},$$

and

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \left(\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}\right) + e_0\sec\phi\left\{\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{2}{r^2}\sin^2\phi\right\}.$$

9.12 Positive Energy (III) $e = pc$, $e_0 = 0$

In this section, we assume that the rest mass is zero, so that $e_0 = 0$ and that $e = pc$, which we speculate is the equation of state applying to dark matter. From this assumption and (9.12), we may deduce the two coupled partial differential equations:

$$\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial r} = -\frac{\partial V}{\partial r}, \qquad \frac{1}{c}\frac{\partial p}{\partial t} + \left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right) = -\frac{1}{c^2}\frac{\partial V}{\partial t}, \tag{9.43}$$

which are only consistent, provided that the momentum $p(r,t)$ is given explicitly by the expression

$$p(r,t) = \frac{r}{2c}\left(\frac{\partial V}{\partial r} - \frac{1}{c}\frac{\partial V}{\partial t}\right) = \frac{r}{2c}\left(cg - f\right), \tag{9.44}$$

and we comment that this equality is merely the equation $f = c(g - 2p/r)$ valid for $e = pc$ and $e_0 = 0$ and arising from Eq. (9.25). Substitution of this expression into either of (9.43) produces the resulting wave-like partial differential equation for the potential $V(r,t)$, thus

$$\frac{\partial^2 V}{\partial t^2} = c^2\left\{\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r}\left(\frac{\partial V}{\partial r} - \frac{1}{c}\frac{\partial V}{\partial t}\right)\right\}, \tag{9.45}$$

and indeed the assumed potential $V(r,t)$ must satisfy this equation in order that the assumption $e = pc$ is meaningful. We comment first that (9.45) is entirely consistent with (9.9) with $e = pc$, since in this event $E^+(r,t) = \mathscr{E}(r,t) + cp(r,t) = \mathscr{E}(r,t) + e(r,t) = -V(r,t) + constant$ and therefore $E^+(r,t)$ and $V(r,t)$ satisfy the same equation. We further comment, as a check on the consistency of the formulae, that if we differentiate (9.44) with respect to t and use both the first equation of (9.3) and (9.45), we obtain an expression for $\partial\mathscr{E}/\partial r$, which may be formally integrated with respect to r to give

$$\mathscr{E}(r,t) = \frac{r}{2}\left\{\frac{1}{c}\frac{\partial V}{\partial t} - \left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right)\right\},$$

which on using $(9.44)_1$ and $e = pc$ is again simply a restatement of the energy integral.

General Solution We may determine the general solution of Eq. (9.45) by first making the substitution $V(r,t) = U(r,t)/r^2$ so that (9.45) becomes

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial r^2} - \frac{1}{r}\frac{\partial U}{\partial r} - \frac{1}{cr}\frac{\partial U}{\partial t}\right),$$

which we may solve using the characteristic coordinates of the wave equation, namely $\alpha = ct + r$ and $\beta = ct - r$, and the differential formulae

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \qquad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta},$$

so that (9.45) becomes simply

$$(\alpha - \beta)\frac{\partial^2 U}{\partial \alpha \partial \beta} + \frac{\partial U}{\partial \alpha} = 0.$$

This equation may be integrated immediately to give $\partial U/\partial \alpha = F'(\alpha)(\alpha - \beta)$, where $F'(\alpha)$ denotes an arbitrary function of α, and a further integration with respect to α yields the general solution of (9.45), thus

$$U(\alpha,\beta) = (\alpha - \beta)F(\alpha) + G(\beta) - \int F(\alpha)d\alpha,$$

where $G(\beta)$ denotes a further arbitrary function of the variable β. Altogether, we have that the general solution of (9.45) becomes

$$V(r,t) = \frac{2}{r}F(ct + r) + \frac{1}{r^2}\left(G(ct - r) - \int^{ct+r} F(\alpha)d\alpha\right).$$

Separable Solutions Equation (9.45) also admits simple separable solutions of the form

$$V(r,t) = e^{-\lambda t}v(\lambda r) = e^{-\lambda t}v(x), \tag{9.46}$$

where λ is a constant, and here for convenience, we use $x = \lambda r$. From (9.45), we may deduce the second-order ordinary differential equation for $v(x)$

$$x\frac{d^2v}{dx^2} + 3\frac{dv}{dx} + (1 - x)v = 0, \tag{9.47}$$

which from [85] (Section 2.1.2.68, page 139) has solutions

$$v(x) = C_1^*\left(\frac{2}{x} + \frac{1}{x^2}\right)e^{-x} + \frac{C_2^*}{x^2}e^{x}, \tag{9.48}$$

where again C_1^* and C_2^* denote arbitrary constants. From these separable solutions and with a minor rearrangement of the arbitrary constants, we have from (9.46) and (9.48) that a particular solution of (9.45) becomes

$$V(r, t) = C_1\left(\frac{2\lambda}{r} + \frac{1}{r^2}\right)e^{-\lambda(ct+r)} + \frac{C_2}{r^2}e^{-\lambda(ct-r)}, \tag{9.49}$$

where again C_1 and C_2 denote re-defined arbitrary constants. This is another potential that would predict similar numerical outcomes to the standard Newtonian gravitational potential $V(r, t) = -GMm_0/r$, but as previously mentioned, this latter potential is certainly not an exact consequence of the present theory. However, from (9.49), with appropriate arbitrary constants and in the limit $\lambda \to \infty$, it is clear that the $1/r$ term might dominate the solution and give numerical outcomes comparable to those of the Newtonian potential.

We note that we might use (9.49) as the basic solution to generate other more complicated solutions. For example, by setting $\lambda = n\mu$, where n is an integer, $\rho = e^{-\mu(ct+r)}$ and $\sigma = e^{-\mu(ct-r)}$ so that

$$V(r, t) = C_1\left(\frac{2\mu n}{r} + \frac{1}{r^2}\right)\rho^n + \frac{C_2}{r^2}\sigma^n, \tag{9.50}$$

And by summation over n using the elementary results for both the geometric series and the derived geometric series, namely $\sum_{n=0}^{\infty} \rho^n = (1 - \rho)^{-1}$ and $\sum_{n=1}^{\infty} n\rho^n = \rho(1 - \rho)^{-2}$, we might eventually deduce solutions of (9.45) of the form

$$V(r, t) = \frac{A}{r^2} + \frac{\mu(A + B)}{r(\cosh(\mu(ct + r)) - 1)} + \frac{A\sinh(\mu ct) - B\sinh(\mu r)}{r^2(\cosh(\mu ct) - \cosh(\mu r))},$$

where A and B denote arbitrary constants related to the arbitrary constants C_1 and C_2 in (9.50) by the formulae $C_1 = A + B$ and $C_2 = A - B$.

Similarity Solutions Subsequently, in the final section of this chapter, we report details for similarity solutions of (9.45) of the form

$$V(r,t) = r^m w(r/ct) = r^m w(\xi),$$

where here $\xi = r/ct$, m is a constant and $w(\xi)$ is a function of ξ only. Generally, these solutions are quite complicated to determine analytical expressions. However, simple analytical expressions can be readily found for the four special cases corresponding to $m = 0$, $m = -1$, $m = -2$ and $m = -3$, and we derive the following explicit simple solutions:

$$V(r,t) = C_1 \left\{ \frac{2ct}{r} + \left(\frac{ct}{r} \right)^2 \right\} + C_2, \qquad (m = 0),$$

$$V(r,t) = \frac{C_1}{r^2} \left\{ ct + \frac{(ct+r)}{2} \log \left(\frac{ct+r}{ct-r} \right) \right\} + \frac{C_2}{r^2}(ct+r), \qquad (m = -1),$$

$$V(r,t) = \frac{C_1}{r^2} \left\{ \frac{ct}{(ct-r)} + \frac{1}{2} \log \left(\frac{ct-r}{ct+r} \right) \right\} + \frac{C_2}{r^2}, \qquad (m = -2),$$

$$V(r,t) = \frac{C_1}{r^2(ct+r)} \left\{ 1 - \left(\frac{r}{ct-r} \right)^2 \right\} + \frac{C_2}{(ct-r)^2(ct+r)}, \qquad (m = -3),$$

where in each case C_1 and C_2 denote re-defined arbitrary constants, slightly different from those used in the subsequent derivation. We note that the potentials for $m = 0$ and $m = -1$ both exhibit $1/r$ dependence as the leading term.

9.13 Negative Energy (IV) $e = -pc, e_0 = 0$

The results of this section may be deduced in an analogous manner to those given in the previous section. Here, we again assume that $e_0 = 0$ and that $e = -pc$, which we speculate is the equation of state applying to dark energy. Instead of (9.43), (9.44) and (9.45), we obtain, respectively,

$$\frac{\partial p}{\partial t} - c\frac{\partial p}{\partial r} = -\frac{\partial V}{\partial r}, \qquad \frac{\partial p}{\partial r} - \frac{1}{c}\frac{\partial p}{\partial t} + \frac{2p}{r} = -\frac{1}{c^2}\frac{\partial V}{\partial t}, \tag{9.51}$$

which are consistent only if the momentum $p(r,t)$ is given by the expression

$$p(r,t) = -\frac{r}{2c}\left(\frac{\partial V}{\partial r} + \frac{1}{c}\frac{\partial V}{\partial t} \right) = \frac{r}{2c}\left(f + cg \right), \tag{9.52}$$

and again we comment that this equation coincides with $f = -c(g - 2p/r)$ valid for $e = -pc$ and $e_0 = 0$ and arising from Eq. (9.25). Substitution of (9.52) into either of (9.51) produces the resulting wave-like partial differential equation for the determination of the potential $V(r, t)$, thus

$$\frac{\partial^2 V}{\partial t^2} = c^2 \left\{ \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r}\left(\frac{\partial V}{\partial r} + \frac{1}{c}\frac{\partial V}{\partial t} \right) \right\}, \tag{9.53}$$

and again the assumed potential $V(r, t)$ must satisfy this equation in order that the assumption $e = -pc$ is meaningful. We again comment that (9.53) is entirely consistent with (9.8) with $e = -pc$, since in this event $E^-(r, t) = \mathscr{E}(r, t) - cp(r, t) = \mathscr{E}(r, t) + e(r, t) = -V(r, t) + constant$ and therefore $E^-(r, t)$ and $V(r, t)$ must satisfy the same equation. We further comment that we may check the consistency of these formulae, by differentiating (9.52) with respect to t and using both the first equation of (9.3) and (9.53), to obtain an expression for $\partial\mathscr{E}/\partial r$, which may be formally integrated with respect to r to again produce the energy integral on using $e = -pc$.

General Solution These equations are identical to those of the previous section, except the sense of time is reversed, and the general solution of (9.53) is much the same as (9.45). After the transformation $V(r, t) = U(r, t)/r^2$, we obtain

$$\frac{\partial^2 U}{\partial t^2} = c^2 \left(\frac{\partial^2 U}{\partial r^2} - \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{cr}\frac{\partial U}{\partial t} \right),$$

which in terms of the variables $\alpha = ct + r$ and $\beta = ct - r$ becomes

$$(\alpha - \beta)\frac{\partial^2 U}{\partial\alpha\partial\beta} - \frac{\partial U}{\partial\beta} = 0,$$

and this has the general solution

$$U(\alpha, \beta) = (\alpha - \beta)G(\beta) + F(\alpha) + \int G(\beta)d\beta,$$

where again $F(\alpha)$ and $G(\beta)$ denote arbitrary functions of the variables $\alpha = ct + r$ and $\beta = ct - r$, and the general solution of (9.53) becomes

$$V(r, t) = \frac{2}{r}G(ct - r) + \frac{1}{r^2}\left(F(ct + r) + \int^{ct-r} G(\beta)d\beta \right).$$

Separable Solutions As might be expected, Eq. (9.53) admits separable solutions of the form

$$V(r,t) = e^{\lambda t} v(\lambda r) = e^{\lambda t} v(x),$$

where again λ is a constant and $x = \lambda r$. Precisely as before, $v(x)$ satisfies (9.47) and is again given explicitly by (9.48), so that similar results to those of the previous section might be obtained.

Similarity Solutions In the final section of the chapter, we present the details for similarity solutions of (9.53) of the form

$$V(r,t) = r^m w(r/ct) = r^m w(\xi),$$

where here $\xi = -r/ct$, m is a constant and $w(\xi)$ is a function of ξ only. We obtain the following simple solutions for the four special cases corresponding to $m = 0$, $m = -1$, $m = -2$ and $m = -3$:

$$V(r,t) = C_1 \left\{ \frac{2ct}{r} - \left(\frac{ct}{r}\right)^2 \right\} + C_2, \qquad (m = 0),$$

$$V(r,t) = \frac{C_1}{r^2} \left\{ ct - \frac{(ct - r)}{2} \log\left(\frac{ct + r}{ct - r}\right) \right\} + \frac{C_2}{r^2}(ct - r), \qquad (m = -1),$$

$$V(r,t) = \frac{C_1}{r^2} \left\{ \frac{ct}{(ct + r)} + \frac{1}{2} \log\left(\frac{ct + r}{ct - r}\right) \right\} + \frac{C_2}{r^2}, \qquad (m = -2),$$

$$V(r,t) = \frac{C_1}{r^2(ct - r)} \left\{ 1 - \left(\frac{r}{ct + r}\right)^2 \right\} + \frac{C_2}{(ct + r)^2(ct - r)}, \qquad (m = -3),$$

where in each case C_1 and C_2 denote re-defined arbitrary constants, which are slightly different from those used in the following derivation.

9.14 Similarity Stretching Solutions of Wave Equation

Here, we formally establish that the only solutions of (9.4) that remain invariant under the stretching group of transformations

$$r^* = \lambda r, \quad t^* = \lambda t, \quad p^* = \lambda^{n-2} p, \quad \mathscr{E}^* = \lambda^{n-2} \mathscr{E}, \tag{9.54}$$

where λ denotes a positive arbitrary parameter and n denotes an arbitrary exponent, are generated from the expressions given in Eq. (9.5), where the solution $\psi(r,t)$ of the classical wave equation (9.6) is given by $\psi(r,t) = C_1(ct+r)^n + C_2(ct-r)^n$ and where C_1 and C_2 denote arbitrary constants. Essentially the same calculation has been presented in Chap. 4 in the context of one-dimensional motion with (x,t) variables, and for the reader's convenience, the calculation is reproduced here in the notation of this chapter. As mentioned previously, although this outcome might be expected, it is not entirely obvious, and it is interesting that it can be formally established in the following manner.

Following invariance under the group of transformations (9.54), we look for similarity solutions of the classical wave equation (9.6) of the form $\psi(r,t) = t^n\Psi(x)$, where here $x = r/ct$, and we find

$$(1-x^2)\frac{d^2\Psi}{dx^2} + 2(n-1)x\frac{d\Psi}{dx} - n(n-1)\Psi = 0,$$

which with the further substitution $y = (1+x)/2$ gives

$$y(1-y)\frac{d^2\Psi}{dy^2} + [2(n-1)y - (n-1)]\frac{d\Psi}{dy} - n(n-1)\Psi = 0.$$

This is the hypergeometric equation $F(-n, 1-n; 1-n; y)$ and with the exact solution $\Psi(y) = (1-y)^n$ as can be readily verified. Accordingly, on making the substitution $\Psi(y) = (1-y)^n\Phi(y)$, we obtain

$$y(1-y)\frac{d^2\Phi}{dy^2} = (2y+n-1)\frac{d\Phi}{dy},$$

and this may be integrated to yield

$$\Psi(y) = (1-y)^n\left(C_1^* + C_2^*\int^y z^{n-1}(1-z)^{-n-1}dz\right),$$

where C_1^* and C_2^* denote two arbitrary constants. On making the substitution $\rho = z/(1-z)$ and noting that $d\rho = dz/(1-z)^2$, the above integral becomes

$$\int^y\left(\frac{z}{1-z}\right)^{n-1}\frac{dz}{(1-z)^2} = \int\rho^{n-1}d\rho = \frac{\rho^n}{n} = \frac{1}{n}\left(\frac{y}{1-y}\right)^n,$$

and therefore, we have $\Psi(y) = C_1^*(1-y)^n + C_2^*y^n/n$. On retracing the substitutions $y = (1+x)/2$, $x = r/ct$ and $\psi(r,t) = t^n\Psi(x)$, we may readily deduce the solution $\psi(r,t) = C_1(ct+r)^n + C_2(ct-r)^n$, on introducing modified arbitrary constants C_1 and C_2. We note however the singular case arising from $n = 0$ for which we have

$$\psi(r,t) = C_1 + C_2 \log\left(\frac{ct-r}{ct+r}\right). \tag{9.55}$$

Thus, in summary, the complete similarity solutions of (9.4) arising from $\psi(r,t) = C_1(ct+r)^n + C_2(ct-r)^n$ for n arbitrary are given by

$$p(r,t) = \frac{n}{r}\left[C_1(ct+r)^{n-1} - C_2(ct-r)^{n-1}\right] - \frac{1}{r^2}\left[C_1(ct+r)^n + C_2(ct-r)^n\right], \tag{9.56}$$

$$\mathscr{E}(r,t) = \frac{nc}{r}\left[C_1(ct+r)^{n-1} + C_2(ct-r)^{n-1}\right],$$

with the particle energy $e(r,t)$ and potential $V(r,t)$ determined, respectively, from the relations

$$e(r,t) = \left(e_0^2 + (cp(r,t))^2\right)^{1/2},$$
$$V(r,t) = -e(r,t) - \frac{nc}{r}\left[C_1(ct+r)^{n-1} + C_2(ct-r)^{n-1}\right] + V_0,$$

where V_0 denotes an arbitrary constant, noting that the above potential $V(r,t)$ given by (9.30) arises from the case $n = 1$.

9.15 Some Examples Involving the Dirac Delta Function

In this section, we consider some simple illustrative examples all involving the Dirac delta function. The Dirac delta function, which occurs throughout applied mathematics and mathematical physics, is usually denoted by the symbol δ and is referred to as a generalised function in contrast to an ordinary function, such as the familiar and well-defined functions, for example, x^3 and $\cosh x$. Roughly speaking, the Dirac delta function is zero for $x \neq 0$ and non-zero and very large and positive for $x = 0$. Its proper definition relies on a space of test functions $\psi(x)$, which are assumed to be well-behaved functions vanishing at infinity, and the Dirac delta function $\delta(x)$ is defined to be that function for which

$$\int_{-\infty}^{\infty} \psi(x)\delta(x)dx = \psi(0),$$

for all test functions $\psi(x)$. More generally, equality between generalised functions refers to two functions having an identical effect on a space of test functions. Also, the notion of delta convergent sequences is an important one, that is, a sequence of real functions which tend to a generalised function in some limiting process. There are many generalist textbooks containing excellent brief summaries of these issues and of the other important aspects of generalised functions, such as [5, 58] and [94], while both textbooks [36] and [69] are devoted to the topic and accordingly provide more comprehensive accounts.

Formally, for the Dirac delta function $\delta(x)$, we have the elementary properties

$$\delta(-x) = \delta(x), \qquad \delta(\lambda x) = \frac{\delta(x)}{\lambda}, \qquad \delta(x) = \frac{dH(x)}{dx},$$

where λ is assumed to be a positive scalar, and $H(x)$ is the Heaviside unit function defined by

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{9.57}$$

In addition, for $a > 0$, we have

$$\delta(a^2 - x^2) = \frac{1}{2a} \{\delta(a - x) + \delta(a + x)\}, \tag{9.58}$$

and if we use $\xi(x)$ to designate some arbitrary function of x, then on repeated differentiation of the formal identity $\xi\delta(\xi) = 0$, we may deduce the following successive relations between the derivatives of the delta function $\delta(\xi)$, thus

$$\xi\delta'(\xi) + \delta(\xi) = 0, \qquad \xi\delta''(\xi) + 2\delta'(\xi) = 0, \qquad \xi\delta'''(\xi) + 3\delta''(\xi) = 0, \qquad \xi\delta^{iv}(\xi) + 4\delta'''(\xi) = 0, \tag{9.59}$$

and so on, and further details of these interesting relations can be found in [36] (pages 212 and 233).

In the first example, the notion of a spherically symmetric delta function $\delta(r)$ in a three-dimensional space is required. In terms of rectangular Cartesian coordinates (x, y, z) and with a fully three-dimensional delta function $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$, we have the relationship $\delta(\mathbf{r}) = \delta(r)/4\pi$. The factor $1/4\pi$ arises from a normalisation requirement, since in terms of the usual spherical polar coordinates (r, θ, ϕ), the spherically symmetric delta function $\delta(r)$ requires a normalisation in order that $\delta(r)$ represents a source of unit magnitude, thus $\delta(\mathbf{r}) = \delta(r)/4\pi$, where the factor $1/4\pi$ is determined from the integral $\int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi = 4\pi$.

Impulsive Spherical Expansion in a Medium at Rest As an illustration of the similarity profiles, we consider the problem of a medium, which is initially at rest and set in motion by an impulsive spherical expansion. We assume that at time $t = 0$, the medium is subjected to an impulsive spherical expansion

$$p(\mathbf{r}, 0) = p_0^* \delta(\mathbf{r}),$$

where $\delta(\mathbf{r})$ is the three-dimensional delta function, and p_0^* denotes a constant prescribing the strength of the initial impulsive motion. In terms of the one-dimensional spherically symmetric delta function $\delta(r)$, this initial condition becomes

$$p(r,0) = \frac{p_0}{r^2}\delta(r),$$

where $p_0 = p_0^*/4\pi$ and from the stretching property of the one-dimensional delta function, namely $\delta(\lambda r) = \delta(r)/\lambda$, we have

$$p(\lambda r,0) = \frac{p_0}{\lambda^2 r^2}\delta(\lambda r) = \frac{p_0\delta(r)}{\lambda^3 r^2},$$

and therefore, we require a similarity stretching solution with $n = -1$. From (9.56) with $n = -1$, we may deduce

$$p(r,t) = -\frac{C_1(ct+2r)}{r^2(ct+r)^2} - \frac{C_2(ct-2r)}{r^2(ct-r)^2}, \tag{9.60}$$

where C_1 and C_2 denote two arbitrary constants to be determined by the continuity of both $p(r,t)$ and $\mathscr{E}(r,t)$ across the moving boundary $r = R(t)$ and the conservation of momentum, namely

$$4\pi\int_0^{R(t)} p(r,t)r^2 dr = 4\pi p_0.$$

With $\xi = r/ct$, we find from the two immediately above equations

$$\int_0^{R(t)} p(r,t)r^2 dr = -\int_0^{\xi_0}\left(\frac{C_1(1+2\xi)}{(1+\xi)^2} + \frac{C_2(1-2\xi)}{(1-\xi)^2}\right)d\xi = p_0, \tag{9.61}$$

where ξ_0 is the constant value of $\xi = r/ct$ on the moving boundary, which is necessarily defined by $r = R(t) = \xi_0 ct$.

On evaluation of the integral in (9.61), we obtain the onstraint

$$-C_1\log(1+\xi_0)^2 + C_2\log(1-\xi_0)^2 + \frac{C_1\xi_0}{(1+\xi_0)} + \frac{C_2\xi_0}{(1-\xi_0)} = p_0, \tag{9.62}$$

While from (9.60) continuity of $p(r,t)$ across the moving boundary, namely $p(\xi_0 ct, t) = 0$, gives

$$C_1(1-\xi_0)^2(1+2\xi_0) + C_2(1+\xi_0)^2(1-2\xi_0) = 0. \tag{9.63}$$

The third constraint comes from the continuity of the wave energy $\mathscr{E}(r,t)$ across the moving boundary. From (9.56) with $n = -1$, we have

$$\mathscr{E}(r,t) = -\frac{c}{r}\left(\frac{C_1}{(ct+r)^2} + \frac{C_2}{(ct-r)^2}\right),$$

and $\mathscr{E}(\xi_0 ct, t) = 0$ gives

$$C_1(1-\xi_0)^2 + C_2(1+\xi_0)^2 = 0. \tag{9.64}$$

Thus, immediately from (9.63), we might deduce the relation

$$C_1(1-\xi_0)^2 = C_2(1+\xi_0)^2,$$

which together with (9.64) implies that

$$C_1(1-\xi_0)^2 = C_2(1+\xi_0)^2 = 0.$$

These conditions give rise to the only physically allowable option (since $r > 0$) $\xi_0 = 1$ and $C_2 = 0$, so that the moving boundary is given by $R(t) = ct$ and

$$p(r,t) = -\frac{C_1(ct+2r)}{r^2(ct+r)^2}, \qquad \mathscr{E}(r,t) = -\frac{cC_1}{r(ct+r)^2},$$

where the constant C_1 is determined from (9.62), thus $C_1 = p_0/(1/2 - 2\log 2)$, and any such impulsive motion must necessarily propagate at the speed of light.

Spherically Symmetric Pulse Waves The following four examples originate from the existence of a well-known delta function solution $\mathscr{E}(r,t) = \delta((ct)^2 - r^2)$ of the spherically symmetric wave equation, namely equation $(9.4)_1$, and the result hinges upon the identity $(9.59)_2$ (see, e.g. [36], page 234). For these examples, we use ξ as the working variable, which is defined by

$$\xi = (ct)^2 - r^2,$$

and we note that for $r > 0$ and $t > 0$, strictly speaking, only the first term arises in the expansion

$$\delta((ct)^2 - r^2) = \frac{1}{2r}\left\{\delta(ct-r) + \delta(ct+r)\right\},$$

since evidently $ct + r > 0$. Here, however, we do not necessarily assume that this is the case, and we leave open the possibility that in certain scenarios time itself might be adopted to take on negative values.

From $\mathscr{E}(r,t) = \delta((ct)^2 - r^2)$, we may calculate the partial derivatives

$$\frac{\partial \mathscr{E}}{\partial t} = 2c^2 t\delta'(\xi), \qquad \frac{\partial \mathscr{E}}{\partial r} = -2r\delta'(\xi),$$

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = 2c^2\delta'(\xi) + 4c^4t^2\delta''(\xi), \qquad \frac{\partial^2 \mathscr{E}}{\partial r^2} = -2\delta'(\xi) + 4r^2\delta''(\xi),$$

so that from the spherically symmetric wave equation $(9.4)_1$,

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2\left(\frac{\partial^2 \mathscr{E}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathscr{E}}{\partial r}\right), \tag{9.65}$$

we might deduce

$$\xi\delta''(\xi) + 2\delta'(\xi) = 0, \tag{9.66}$$

which is the second relation of (9.59). We make two important points. Firstly, clearly the solution is modulo both an arbitrary scale factor and an arbitrary additive constant. Secondly, we have in mind that $\delta(\xi)$ refers to the Dirac delta function, but it could also refer to an ordinary function, which is the solution of the second-order ordinary differential equation (9.66), namely $\delta(\xi) = C_1/\xi^2 + C_2$, where C_1 and C_2 denote the two constants of integration.

Another solution of the spherically symmetric wave equation (9.65), which is not so well-known, is $\mathscr{E}(r,t) = H((ct)^2 - r^2)/r$, where $H(x)$ is the Heaviside unit function defined by (9.57). For this solution, we may calculate the following partial derivatives

$$\frac{\partial \mathscr{E}}{\partial t} = \frac{2c^2t}{r}\delta(\xi), \qquad \frac{\partial \mathscr{E}}{\partial r} = -2\delta(\xi) - \frac{H(\xi)}{r^2},$$

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = \frac{2c^2}{r}\delta(\xi) + \frac{4c^4t^2}{r}\delta'(\xi), \qquad \frac{\partial^2 \mathscr{E}}{\partial r^2} = 4r\delta'(\xi) + \frac{2H(\xi)}{r^3} + \frac{2\delta(\xi)}{r},$$

so that from the spherically symmetric wave equation (9.65), we might deduce

$$\xi\delta'(\xi) + \delta(\xi) = 0,$$

which in this case is the first relation of (9.59).

Thus, we have two pulse solutions of the spherically symmetric wave equation (9.65), namely

$$\mathscr{E}(r,t) = \delta((ct)^2 - r^2), \qquad \mathscr{E}(r,t) = \frac{H((ct)^2 - r^2)}{r}, \tag{9.67}$$

and corresponding to each of these, we may calculate $p(r,t) = \partial\mathscr{E}/\partial r$ to obtain a solution of $(9.4)_2$, namely

$$\frac{\partial^2 p}{\partial t^2} = c^2\left(\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{2p}{r^2}\right). \tag{9.68}$$

Accordingly, from (9.67), we might generate two solutions of (9.68), thus

$$p(r,t) = r\delta'((ct)^2 - r^2), \qquad p(r,t) = 2\delta((ct)^2 - r^2) + \frac{H((ct)^2 - r^2)}{r^2}, \tag{9.69}$$

and as previously noted, these solutions do not constitute a matching pair of the coupled equations (9.3). Arising from each of the four solutions (9.67) and (9.69), we determine the matching pairs of the coupled equations (9.3) in the four examples immediately below. The calculations involve varying degrees of difficulty, with the first example being by far the most challenging.

(i) Determination of $p(r,t)$ **corresponding to** $\mathscr{E}(r,t) = \delta((ct)^2 - r^2)$**:** In this example, we determine the function $p(r,t)$ from the coupled partial differential equations (9.3), namely

$$\frac{\partial \mathscr{E}}{\partial r} = \frac{\partial p}{\partial t}, \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2\left(\frac{\partial p}{\partial r} + \frac{2p}{r}\right), \tag{9.70}$$

and corresponding to $\mathscr{E}(r,t) = \delta((ct)^2 - r^2)$. From these equations and $\mathscr{E}(r,t) = \delta((ct)^2 - r^2)$, we may deduce

$$\frac{\partial p}{\partial t} = -2r\delta'(\xi), \qquad \frac{\partial p}{\partial r} + \frac{2p}{r} = 2t\delta'(\xi),$$

where $\xi = (ct)^2 - r^2$. On making the substitution $p(r,t) = q(r,t)/r^2$, these equations become

$$\frac{\partial q}{\partial t} = -2r^3\delta'(\xi), \qquad \frac{\partial q}{\partial r} = 2tr^2\delta'(\xi), \tag{9.71}$$

and we may confirm that this is a well-defined problem for the determination of $q(r,t)$ by equating the two expressions for the mixed partial derivative $\partial^2 q/\partial r\partial t$, which result in the second relation of (9.59). From the two Eqs. (9.71), we may deduce the first-order partial differential equation

$$c^2 t\frac{\partial q}{\partial r} + r\frac{\partial q}{\partial t} = 2r^2\xi\delta'(\xi) = -2r^2\delta(\xi),$$

where we have used the first relation of (9.59). On making the substitution $q(r,t) = \phi(r,t)\delta(\xi)$, this equation becomes

$$c^2 t\frac{\partial \phi}{\partial r} + r\frac{\partial \phi}{\partial t} = -2r^2, \tag{9.72}$$

and the general solution of this equation may be determined using Lagrange's characteristic method. We introduce a characteristic parameter s such that

$$\frac{dr}{ds} = c^2 t, \qquad \frac{dt}{ds} = r, \qquad \frac{d\phi}{ds} = -2r^2,$$

and then the general solution for $\phi(r, t)$ may be determined from any two integrals of the reduced system

$$\frac{dr}{dt} = \frac{c^2 t}{r}, \qquad \frac{d\phi}{dr} = -\frac{2r^2}{c^2 t}.$$

From the first equation, we see that one integral is $\xi = (ct)^2 - r^2$, which we may exploit as constant in the integration of the second equation, thus $ct = (\xi + r^2)^{1/2}$, and we have

$$\frac{d\phi}{dr} = -\frac{2r^2}{c(\xi + r^2)^{1/2}},$$

which integrates to give $\phi = -(r/c)(\xi + r^2)^{1/2} + (\xi/c)\log(r + (\xi + r^2)^{1/2})$, and therefore the general solution of the first-order partial differential equation (9.72) is given by

$$\phi(r, t) = -rt + \frac{\xi}{c}\log(ct + r) + \Phi(\xi), \tag{9.73}$$

where $\Phi(\xi)$ denotes an arbitrary function to be determined, such that either partial differential equation (9.71), involving the derivative of the delta function $\delta'(\xi)$, is properly satisfied by $q(r, t) = \phi(r, t)\delta(\xi)$. We find that either equation is properly satisfied and reduces to the first relation of (9.59), provided that $\Phi(\xi)$ satisfies the first-order ordinary differential equation

$$\xi\Phi'(\xi) - \Phi(\xi) = -\frac{\xi}{2c},$$

from which we may deduce $\Phi(\xi) = -(\xi/2c)\log\xi + C_1\xi$, where C_1 denotes the constant of integration. From Eq. (9.73), we obtain

$$\phi(r, t) = -rt + \frac{\xi}{2c}\log\left(\frac{ct + r}{ct - r}\right) + C_1\xi,$$

so that altogether the momentum $p(r, t)$ corresponding to the wave energy $\mathscr{E}(r, t) = \delta((ct)^2 - r^2)$ is given by

$$p(r, t) = \left\{-\frac{t}{r} + \frac{((ct)^2 - r^2)}{2cr^2}\log\left(\frac{ct + r}{ct - r}\right) + \frac{C_1}{r^2}((ct)^2 - r^2)\right\}\delta((ct)^2 - r^2).$$

We observe that while the term involving the arbitrary constant does not contribute due to the identity $\xi\delta(\xi) = 0$, on the other hand, the term involving the logarithm is essential, since it is singular and its behaviour is far more subtle. Thus, the final corresponding expressions for $\mathscr{E}(r,t)$ and $p(r,t)$ become

$$\mathscr{E}(r,t) = \delta((ct)^2 - r^2), \qquad p(r,t) = \left\{ -\frac{t}{r} + \frac{((ct)^2 - r^2)}{2cr^2} \log\left(\frac{ct+r}{ct-r}\right) \right\} \delta((ct)^2 - r^2).$$

(ii) Determination of $p(r,t)$ corresponding to $\mathscr{E}(r,t) = H((ct)^2 - r^2)/r$: The remaining three examples are far more straightforward than the previous example. From the coupled partial differential equations (9.3) or (9.70) and $\mathscr{E}(r,t) = H((ct)^2 - r^2)/r$, we may deduce

$$\frac{\partial p}{\partial t} = -2\delta(\xi) - \frac{H(\xi)}{r^2}, \qquad \frac{\partial p}{\partial r} + \frac{2p}{r} = \frac{2t}{r}\delta(\xi),$$

and on making the same substitution as in the previous example, namely $p(r,t) = q(r,t)/r^2$, these equations become

$$\frac{\partial q}{\partial t} = -2r^2\delta(\xi) - H(\xi), \qquad \frac{\partial q}{\partial r} = 2tr\delta(\xi). \tag{9.74}$$

On integrating the second equation with respect to r, we find $q(r,t) = -tH(\xi) + \Psi(t)$, where $\Psi(t)$ denotes an arbitrary function of time, and on substitution into the first of Eqs. (9.74), we obtain $\Psi'(t) = 2\xi\delta(\xi)$. Since $\xi\delta(\xi) = 0$, we have therefore $\Psi'(t) = 0$, and altogether the final corresponding expressions for $\mathscr{E}(r,t)$ and $p(r,t)$ become

$$\mathscr{E}(r,t) = \frac{H((ct)^2 - r^2)}{r}, \qquad p(r,t) = -\frac{tH((ct)^2 - r^2)}{r^2}.$$

(iii) Determination of $\mathscr{E}(r,t)$ corresponding to $p(r,t) = r\delta'((ct)^2 - r^2)$: In this case, from the coupled partial differential equations (9.70) and $p(r,t) = r\delta'((ct)^2 - r^2)$, we have

$$\frac{\partial \mathscr{E}}{\partial r} = 2c^2tr\delta''(\xi), \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2\left(3\delta'(\xi) - 2r^2\delta''(\xi)\right), \tag{9.75}$$

and again we may confirm that this is a well-defined problem for the determination of $\mathscr{E}(r,t)$ by equating the two expressions for the mixed partial derivative $\partial^2\mathscr{E}/\partial r\partial t$ which result in the third relation of (9.59). From the first of the Eqs. (9.75), it is not difficult to conclude that $\mathscr{E}(r,t) = -c^2t\delta'(\xi)$, and the second equation of (9.70) reduces to the second relation of (9.59). Thus, we have the corresponding or matching solutions of (9.70):

$$\mathscr{E}(r,t) = -c^2t\delta'(\xi), \qquad p(r,t) = r\delta'(\xi),$$

which can be seen to arise from the general solution (9.5) or (9.11), namely

$$\mathscr{E}(r,t) = \frac{\partial\,(\psi/r)}{\partial t}, \qquad p(r,t) = \frac{\partial\,(\psi/r)}{\partial r},$$

with $\psi(r,t)/r = -\delta(\xi)/2$, noting that formally $\psi(r,t)$ does indeed have the required structure $\psi(r,t) = F(\alpha) + G(\beta)$, since from the delta function identity (9.58), we have

$$\delta(\xi) = \delta((ct)^2 - r^2) = \frac{1}{2r}\{\delta(ct+r) + \delta(ct-r)\} = \frac{1}{2r}\{\delta(\alpha) + \delta(\beta)\}.$$

(iv) Determination of $\mathscr{E}(r,t)$ corresponding to $p(r,t) = 2\delta((ct)^2 - r^2) + H((ct)^2 - r^2)/r^2$: In this case from $p(r,t) = 2\delta(\xi) + H(\xi)/r^2$ and the two Eqs. (9.70), we have

$$\frac{\partial \mathscr{E}}{\partial r} = 4c^2 t\delta'(\xi) + \frac{2c^2 t\delta(\xi)}{r^2}, \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2\left(-4r\delta'(\xi) + \frac{2\delta(\xi)}{r}\right), \tag{9.76}$$

and again we may confirm that this is a well-defined problem for the determination of $\mathscr{E}(r,t)$ by equating the two expressions for the mixed partial derivative $\partial^2\mathscr{E}/\partial r\partial t$ which result in a condition involving a combination of the first and second relations of (9.59), specifically

$$2(\xi\delta''(\xi) + 2\delta'(\xi)) + \frac{(\xi\delta'(\xi) + \delta(\xi))}{r^2} = 0.$$

The first equation of (9.76) may be readily integrated to yield $\mathscr{E}(r,t) = -2c^2 t\delta(\xi)/r$, and the second equation of (9.76) becomes the identity $\xi\delta'(\xi) + \delta(\xi)$, so that altogether the corresponding or matching solutions of (9.70) become

$$\mathscr{E}(r,t) = -\frac{2c^2 t}{r}\delta((ct)^2 - r^2), \qquad p(r,t) = 2\delta((ct)^2 - r^2) + \frac{H((ct)^2 - r^2)}{r^2}.$$

9.16 Calculation Details for Similarity Solutions

Here, we present the calculation details for similarity solutions of both (9.45) and (9.53) of the form

$$V(r,t) = r^m w(\xi),$$

where $\xi = r/ct$ for (9.45) and $\xi = -r/ct$ for (9.53), and for both we have

$$\frac{\partial V}{\partial r} = r^{m-1}(\xi w' + mw),$$

while for $\xi = r/ct$ and for $\xi = -r/ct$ we have, respectively,

$$\frac{\partial V}{\partial t} = -cr^{m-1}\xi^2 w', \qquad \frac{\partial V}{\partial t} = cr^{m-1}\xi^2 w',$$

where primes denote differentiation with respect to ξ. On substitution of the above expressions into (9.45) and (9.53), both equations give rise to the same second-order ordinary differential equation

$$\xi^2(\xi^2 w')' + (\xi^2 w') = \xi(\xi w' + mw)' + (m-1)(\xi w' + mw) + 3(\xi w' + mw),$$

which may be simplified to give

$$\xi^2(\xi^2 w'' + 2\xi w' + w') = \xi^2 w'' + (2m+3)\xi w' + m(m+2)w. \tag{9.77}$$

It turns out that after making the change of variable $\xi = 1/x$, this differential equation reduces to a special case of the hypergeometric differential equation given by (9.85), where $\eta = (1+x)/2$. However, this end result is by no means obvious when initially faced with solving equation (9.77), and rather than simply stating the necessary change of variable, it may be more instructive to follow some of the thought processes, which ultimately prove effective. We first comment that while this equation is clearly linear, it nevertheless appears to be complicated, since it involves three terms of distinct character, namely $\xi^2(\xi^2 w')'$, $\xi^2 w'$, and the remaining terms. On face value, this means that any series solutions are generally more complicated than the generalised hypergeometric series, which arise from differential equations involving only two terms of distinct character (see, e.g. [6] volume 1, page 182). The first priority might be to undertake the most simple analysis to determine the accessible analytical solutions applying to particular values of m, and it turns out that the special cases $m = 0$, $m = -1$, $m = -2$ and $m = -3$ can be determined by elementary means.

We now proceed to derive some simple analytical expressions for Eq. (9.77) by elementary means for the four values of m, namely $m = 0$, $m = -1$, $m = -2$ and $m = -3$. For the two cases $m = 0$ and $m = -2$, Eq. (9.77) simplifies to give

$$(1-\xi^2)W' + ((2m+1)/\xi - 1)W = 0, \tag{9.78}$$

where $W = \xi^2 w'$. For $m = 0$, this equation simplifies further to become

$$\frac{W'}{W} + \frac{1}{\xi(1+\xi)} = 0,$$

which may be integrated once to give $W(\xi) = A(1+\xi)/\xi$, and a further straightforward integration yields

$$w(\xi) = -\frac{A}{2}\left(\frac{1}{\xi^2} + \frac{2}{\xi}\right) + B,$$

where A and B denote the arbitrary constants. For $m = -2$, (9.78) becomes

$$\frac{W'}{W} = \frac{3}{\xi(1-\xi^2)} + \frac{1}{(1-\xi^2)} = 3\left(\frac{1}{\xi} + \frac{\xi}{(1-\xi^2)}\right) + \frac{1}{2}\left(\frac{1}{(1-\xi)} + \frac{1}{(1+\xi)}\right),$$

which on integration becomes $W(\xi) = A\xi^3/(1-\xi)^2(1+\xi)$, so that we have

$$w'(\xi) = \frac{A\xi}{(1-\xi)^2(1+\xi)} = \frac{A}{4}\left(\frac{2}{(1-\xi)^2} - \frac{1}{(1-\xi)} - \frac{1}{(1+\xi)}\right),$$

and a further integration gives

$$w(\xi) = \frac{A}{2}\left\{\frac{1}{(1-\xi)} + \frac{1}{2}\log\left(\frac{1-\xi)}{1+\xi)}\right)\right\} + B,$$

where A and B denote the arbitrary constants.

Two further special cases of Eq. (9.77) can be readily integrated with the observation that

$$(\xi^2 w' + w)' = w'' + (2m+3)w'/\xi + m(m+2)w/\xi^2,$$

which might be immediately integrated, provided that $m(m+2) = -(2m+3)$, in which case, we have either $m = -1$ or $m = -3$, and the integral

$$\xi^2 w' + w = w' + (2m+3)w'/\xi + A, \tag{9.79}$$

where A denotes the constant of integration. For $m = -1$, we have

$$w' + \frac{w}{\xi(1+\xi)} = \frac{A}{(1-\xi^2)},$$

to obtain the further integral

$$w(\xi) = \frac{A}{2\xi}\left\{1 + \frac{(1+\xi)}{2}\log\left(\frac{1+\xi)}{1-\xi)}\right)\right\} + \frac{B(1+\xi)}{\xi},$$

where B denotes the second constant of integration. For the case $m = -3$, we have from (9.79)

$$w' - \frac{(3+\xi)w}{\xi(1-\xi^2)} = \frac{A}{(1-\xi^2)},$$

which may be integrated to give

$$w(\xi) = -\frac{A\xi(1-2\xi)}{2(1-\xi)^2(1+\xi)} + \frac{B\xi^3}{(1-\xi)^2(1+\xi)},$$

and A and B denote the constants of integration. The simple analytical results derived here by elementary means are summarised in Sects. 9.12 and 9.13 in terms of the potential $V(r, t)$.

We now proceed to attempt to identify the general solution of Eq. (9.77) in the following manner. We first note that Eq. (9.77) can be progressively transformed into an Heun differential equation (for further information, refer to any of the three comprehensive texts; [85] linear second order, 2.1.2, item 204, [88] page 7 or [6] volume 3, page 57). If we attempt the change of variable $w(\xi) = \xi^n y(\xi)$, then it becomes apparent that the case $n = -1$ is an important special case providing many major simplifications. Accordingly, we are led to make the transformation $w(\xi) = y(\xi)/\xi$, and (9.77) becomes

$$\xi^2(1-\xi^2)y'' + \xi(2m+1-\xi)y' + (m^2-1+\xi)y = 0, \tag{9.80}$$

which can be alternatively written as

$$y'' + \left(\frac{2m+1}{\xi} + \frac{m}{1-\xi} - \frac{(m+1)}{1+\xi}\right)y' + \left(\frac{(m^2-1)}{\xi} + 1\right)\frac{y}{\xi(1-\xi)(1+\xi)} = 0,$$

which on face value has regular singular points at $\xi = 0$ and $\xi = \pm 1$, except that later it becomes apparent that the singularity corresponding to $\xi = 0$ is an induced singularity as a result of the transformation $w(\xi) = y(\xi)/\xi$. Further, since with the differential equation in standard notation $y'' + p(\xi)y' + q(\xi)y = 0$, we have

$$p(\xi) = \frac{(2m+1-\xi)}{\xi(1-\xi^2)}, \qquad q(\xi) = \frac{(m^2-1+\xi)}{\xi^2(1-\xi^2)},$$

And therefore, the two quantities $2 - \xi p(\xi)$ and $\xi^2 q(\xi)$ are given by

$$2 - \xi p(\xi) = \frac{((1-2m)+\xi-2\xi^2)}{(1-\xi^2)}, \qquad \xi^2 q(\xi) = \frac{(m^2-1+\xi)}{(1-\xi^2)},$$

which are both are well-defined and analytic at $\xi = \infty$, and therefore the point $\xi = \infty$ is also a regular singular point. On face value, the existence of the four regular singular points for Eq. (9.80) indicate that the equation might be transformed into a Heun differential equation, which in standard notation takes the form

$$x(x-1)(x-a)\frac{d^2y}{dx^2}+\Big[(\alpha+\beta+1)x^2-(\alpha+\beta+1+a(\gamma+\delta)-\delta)x+a\gamma\Big]\frac{dy}{dx}+(\alpha\beta x-q)y=0, \tag{9.81}$$

where a, α, β, γ, δ and q denote six arbitrary parameters. On changing the independent variable $\xi = 1/x$ in Eq. (9.80), we may deduce

$$x(x-1)(x+1)\frac{d^2y}{dx^2}+\Big[(1-2m)x^2+x-2\Big]\frac{dy}{dx}+((m^2-1)x+1)y=0,$$

which is the Heun differential equation (9.81) with parameter values given by

$$a=-1,\quad \alpha=1-m,\quad \beta=-(1+m),\quad \gamma=2,\quad \delta=-m,\quad q=-1. \tag{9.82}$$

In this case, the parameter γ is an integer, and therefore, the standard Heun power series solutions are not available to provide the general solution (see [85], linear second order, 2.1.2, item 168). However, much information is known regarding Heun's equation, and especially integrable results for particular parameter values which can be found in the three previously cited references.

In particular, the Heun functions corresponding to the parameter values (9.82) can be shown to be expressed in terms of Jacobi polynomials $P_n^{\alpha,\beta}(x)$ (see, e.g. [85], linear second order, 2.1.2, item 168, [6] volume 3, page 247), which for integer n are defined by

$$P_n^{\alpha,\beta}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right],$$

which in standard notation satisfy the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+[\beta-\alpha-(\alpha+\beta+2)x]\frac{dy}{dx}+n(n+\alpha+\beta+1)y=0, \tag{9.83}$$

This is most easily seen by applying the change of variable $\xi = 1/x$ directly to Eq. (9.77), that is, to the equation

$$\xi^2(1-\xi^2)w''+\xi(2m+3-\xi-2\xi^2)w'+m(m+2)w=0,$$

and from which we may deduce

$$(1-x^2)\frac{d^2w}{dx^2}+[(2m+1)x-1]\frac{dw}{dx}-m(m+2)w=0, \tag{9.84}$$

and the parameter values in the general equation for Jacobi polynomials $P_n^{\alpha,\beta}(x)$ (9.84) giving rise to Eq. (9.84) are as follows:

$$\alpha = -(m+1), \qquad \beta = -(m+2), \qquad n = m, \qquad n = m+2.$$

Finally, we observe that on making the change of variable $\eta = (1+x)/2$, the general equation for Jacobi polynomials (9.84) becomes the hypergeometric equation

$$\eta(1-\eta)\frac{d^2y}{d\eta^2} + [\beta + 1 - (\alpha+\beta+2)\eta]\,\frac{dy}{d\eta} + n(n+\alpha+\beta+1)y = 0,$$

which in the standard notation for the hypergeometric differential equation becomes ([85] linear second order, 2.1.2, item 171, [6] volume 1, page 57)

$$\eta(1-\eta)\frac{d^2y}{d\eta^2} + [c - (a+b+1)\eta]\frac{dy}{d\eta} - aby = 0,$$

which has formal series solutions denoted by $F(a, b; c; \eta)$, with the hypergeometric parameter values a, b and c given by

$$a = -n, \qquad b = n + \alpha + \beta + 1, \qquad c = \beta + 1,$$

and therefore, the two sets of hypergeometric parameters a, b and c corresponding to Eq. (9.77) and arising from $n = m$ and $n = m + 2$ are

$$a = -m, \qquad b = -(m+2), \qquad c = -(m+1),$$
$$a = -(m+2), \qquad b = -m, \qquad c = -(m+1),$$

revealing the anticipated symmetry in the two parameters a and b.

In summary, the two linearly independent solutions of the second-order differential equation (9.77) may be determined from those of the hypergeometric equation, which arises from (9.84) and the substitution $\eta = (1 + x)/2$, thus

$$\eta(1-\eta)\frac{d^2w}{d\eta^2} + [(2m+1)\eta - (m+1)]\,\frac{dw}{d\eta} - m(m+2)w = 0, \tag{9.85}$$

which in standard notation has solutions given by $F(-m, -(m+2); -(m+1); \eta)$, and therefore, the general solutions of (9.77) are given explicitly by

$$w(\xi) = F\left(-m, -(m+2); -(m+1); \frac{1+\xi}{2\xi}\right).$$

There are many simple analytical solutions of the hypergeometric equation corresponding to particular parameter values a, b and c, and a table of the numerous special cases can be found in [85] (linear second order, 2.1.2, Tables 16 and 17, after item 176) and [6] (volume 1, page 89). In the present case $a = -m$, $b = -(m+2)$ and $c = -(m+1)$ and in the event that either a or b is a negative integer, the standard hypergeometric series terminates after a finite number of terms, resulting in a simple analytical expression; thus, with $a = -m$, we have

$$F(a, b; c; \eta) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{\eta^k}{k!}$$

$$= \sum_{k=0}^{m} \frac{(a)(a+1)(a+2)\ldots\ldots.(a+k-1)(b)(b+1)(b+2)\ldots\ldots.(b+k-1)}{(c)(c+1)(c+2)\ldots\ldots.(c+k-1)} \frac{\eta^k}{k!}$$

$$= \sum_{k=0}^{m} \frac{(-m)(-m+1)(-m+2)\ldots\ldots.(-m+k-1)(-m-2)(-m-1)(-m)\ldots\ldots.(-m+k-3)}{(-m-1)(-m)(-m+1)\ldots\ldots.(-m+k-2)} \frac{\eta^k}{k!}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (m)(m-1)(m-2)\ldots\ldots.(m-(k-1))(m+2)}{(m+2-k)} \frac{\eta^k}{k!}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k m!(m+2)}{(m-k)!(m+2-k)} \frac{\eta^k}{k!}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k (m+2)!(m+1-k)}{(m+2-k)!(m+1)} \frac{\eta^k}{k!}$$

$$= \sum_{k=0}^{m} (-1)^k \left(1 - \frac{k}{(m+1)}\right) \frac{(m+2)!}{(m+2-k)!k!} \eta^k$$

$$= \sum_{k=0}^{m} (-1)^k \left(1 - \frac{k}{(m+1)}\right) \binom{m+2}{k} \eta^k,$$

where $(a)_k$ denotes the Pochhammer symbols which are defined by

$$(a)_k = (a)(a+1)(a+2)\ldots\ldots.(a+k-1),$$

with $(a)_0 = 1$. Thus, if m is a positive integer, one solution of (9.77) is given explicitly by

$$w_m(\xi) = F\left(-m, -(m+2); -(m+1); \frac{1+\xi}{2\xi}\right) = \sum_{k=0}^{m} \left(1 - \frac{k}{(m+1)}\right) \binom{m+2}{k} \left(\frac{-(1+\xi)}{2\xi}\right)^k,$$

so that, for example, for $m = 1$, $m = 2$ and $m = 3$, we have, respectively,

$$w_1(\xi) = \frac{1}{4}\left(1 - \frac{3}{\xi}\right), \qquad w_2(\xi) = \frac{1}{6}\left(1 - \frac{2}{\xi} + \frac{3}{\xi^2}\right), \qquad w_3(\xi) = \frac{1}{16}\left(1 - \frac{5}{\xi} + \frac{5}{\xi^2} - \frac{5}{\xi^3}\right),$$

and we may verify by direct substitution that all three are indeed solutions of (9.77) for the appropriate values of m, and there are no doubt numerous other special values of m giving rise to equally simple analytical expressions.

Given one solution $w_m(\xi)$, a second linearly independent solution might be formally determined by making the substitution $w(\xi) = w_m(\xi)W(\xi)$ and then solving the resulting first-order differential equation for $W'(\xi)$, thus

$$\frac{W''(\xi)}{W'(\xi)} + 2\frac{w_m'(\xi)}{w_m(\xi)} = \frac{(2\xi^2 + \xi - (2m+3))}{\xi(1-\xi)(1+\xi)} = \frac{(m+1)}{(1+\xi)} - \frac{m}{(1-\xi)} - \frac{(2m+3)}{\xi},$$

which integrates finally to give the general solution

$$w(\xi) = C_1 w_m(\xi) \int_0^{\xi} \frac{(1-\zeta)^m (1+\zeta)^{1+m} d\zeta}{\zeta^{2m+3} w_m(\zeta)^2} + C_2 w_m(\xi),$$

where C_1 and C_2 denote two arbitrary constants.

9.17 de Broglie's Centrally Symmetric Guidance Formula

In Chap. 4, we have presented an analysis for de Broglie's guidance formula of the particle by its wave, as given by Eqs. (1.1) and (1.2) for a single spatial dimension x. For centrally symmetric mechanical systems, a similar analysis applies, which we present here. For the function $\psi(r, t)$, the particle momentum and energy are assumed to be defined by

$$p = mu = -\frac{\partial \psi}{\partial r}, \qquad e = mc^2 = \frac{\partial \psi}{\partial t},$$

where $\psi(r, t)$ satisfies

$$\left(\frac{\partial \psi}{\partial t}\right)^2 - c^2 \left(\frac{\partial \psi}{\partial r}\right)^2 = e_0^2, \qquad \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r}\right) = g, \tag{9.86}$$

where $e_0 = m_0 c^2$, (9.86) arises from the relativistic expression $m = m_0/[1 - (u/c)^2]^{1/2}$ in the form $e^2 - (pc)^2 = e_0^2$. From the relation $(9.86)_1$, there exists a function $\phi(r, t)$ such that

$$\frac{\partial \psi}{\partial t} = e_0 \cosh \phi, \qquad c\frac{\partial \psi}{\partial r} = e_0 \sinh \phi,$$

so that the velocity $u(r, t)$ becomes

$$\frac{u}{c} = \frac{pc}{e} = -c\frac{(\partial \psi/\partial r)}{(\partial \psi/\partial t)} = -\tanh \phi. \tag{9.87}$$

On equating the two expressions for the second derivative, $\partial^2\psi/\partial t\partial r$ gives $(\partial\phi/\partial t) = c\tanh\phi(\partial\phi/\partial r)$, while the second equation is obtained from $(9.86)_2$, which when combined yield

$$\frac{\partial\phi}{\partial r} = -\left(\frac{cg}{e_0} + \frac{2\sinh\phi}{r}\right)\cosh\phi, \qquad \frac{\partial\phi}{\partial t} = -c\left(\frac{cg}{e_0} + \frac{2\sinh\phi}{r}\right)\sinh\phi.$$

We now introduce $\omega = cg/e_0 + 2\sinh\phi/r$ so that the immediately above two equations become simply

$$\frac{\partial\phi}{\partial r} = -\omega\cosh\phi, \qquad \frac{\partial\phi}{\partial t} = -c\omega\sinh\phi, \tag{9.88}$$

and compatibility is satisfied, provided that $\omega(r,t)$ is a solution of the first-order partial differential equation

$$c\sinh\phi\frac{\partial\omega}{\partial r} - \cosh\phi\frac{\partial\omega}{\partial t} = c\omega^2, \tag{9.89}$$

which we again solve using Lagrange's characteristic method.

The three characteristic differential equations become

$$\frac{dr}{ds} = c\sinh\phi, \qquad \frac{dt}{ds} = -\cosh\phi, \qquad \frac{d\omega}{ds} = c\omega^2,$$

and from the first two equations, we have

$$\frac{dr}{dt} = -c\tanh\phi = -\frac{(\partial\phi/\partial t)}{(\partial\phi/\partial r)},$$

and therefore, one integral is $\phi(r,t) = constant$, while from the first and the third, we obtain

$$\frac{d\omega}{dr} = \frac{\omega^2}{\sinh\phi}.$$

On integration, the second integral is found to be $1/\omega + r/\sinh\phi = constant$, and the general solution of (9.89), which is obtained by equating one integral to be an arbitrary function of the second integral, may be expressed in the form

$$\omega(r,t) = -\frac{\sinh\phi}{r - r_0(\phi)}, \tag{9.90}$$

where $r_0(\phi)$ denotes an arbitrary function of ϕ. On substitution of this expression into (9.89), we may readily confirm that (9.90) constitutes a solution of the equation without further restriction on the arbitrary function $r_0(\phi)$. The function $g(r,t)$ is now given by

$$g(r,t) = -\frac{e_0}{c}\sinh\phi\left(\frac{2}{r} + \frac{1}{r - r_0(\phi)}\right), \tag{9.91}$$

while $\phi(r,t)$ is obtained by integration of the coupled relations (9.88) which now become

$$\frac{\partial\phi}{\partial r} = \frac{\sinh\phi\cosh\phi}{r - r_0(\phi)}, \qquad \frac{\partial\phi}{\partial t} = \frac{c\sinh^2\phi}{r - r_0(\phi)}.$$

The integration of (9.91) is precisely analogous to (4.68), and therefore, we may use that analysis to state that the general solution for $\phi(r,t)$ becomes $r - r_0(\phi) + ct\tanh\phi = G(\phi)$, where $G(\phi)$ denotes a second arbitrary function which is related to $r_0(\phi)$ through the differential relation

$$\frac{dG(\phi)}{d\phi} - \frac{G(\phi)}{\sinh\phi\cosh\phi} = -\frac{dr_0(\phi)}{d\phi}.$$

On integration of this equation, we have

$$G(\phi) = -r_0(\phi) - \tanh\phi\int\frac{r_0(\phi)d\phi}{\sinh^2\phi},$$

as the formal connection between the two arbitrary functions $G(\phi)$ and $r_0(\phi)$, and from $r - r_0(\phi) + ct\tanh\phi = G(\phi)$, we obtain

$$r + ct\tanh\phi = -\tanh\phi\int\frac{r_0(\phi)d\phi}{\sinh^2\phi}. \tag{9.92}$$

On using $u = -c\tanh\phi$ from (9.87), this equation may be alternatively written as

$$r - ut = \tanh\phi\int r_0(\phi)d(\coth\phi) = -u\int\frac{r_0^*(u)du}{u^2},$$

where $r_0^*(u)$ denotes $r_0(\phi)$ with ϕ replaced by $-\tanh^{-1}(u/c)$.

Again, for the sake of completeness, we consider the most basic solution in this family of solutions arising from the special case $r_0(\phi) = 0$. For $r_0(\phi) \neq 0$, Eq. (9.92) generally involves transcendental functions for $u(r,t)$, and the only solution in this family of solutions, which can be given explicitly, arises from the special case $r_0(\phi) = 0$. This solution corresponds to the wave velocity (see table in Chap. 10) with $u = -c\tanh\phi = r/t$, and therefore,

$$e = \frac{e_0 ct}{((ct)^2 - r^2)^{1/2}}, \qquad p = \frac{e_0 r}{c((ct)^2 - r^2)^{1/2}}, \qquad \frac{c^2 p}{e} = \frac{r}{t} = u, \tag{9.93}$$

and with particle paths arising from $dr/dt = c^2 t/r$, so that $r^2 - (ct)^2 = constant$. Further, the total or material time derivative du/dt is zero, arising from

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{r}{t^2} + \frac{u}{t} = -\frac{r}{t^2} + \frac{r}{t^2} = 0.$$

In this case, physically from $u(r, t) = r/t$, we have

$$\frac{u(a, t)}{c} = \begin{cases} a/ct > 1 & \text{if } ct < a \text{ superluminal,} \\ a/ct < 1 & \text{if } ct > a \text{ sub-luminal,} \end{cases}$$

which means that around the fixed sphere in space $r = a$, shortly after time $t = a/c$, there is the sudden appearance of particles moving outwards at a velocity just below that of light.

The details for this special case are virtually identical to those given in Chap. 4 for a single spatial dimension. Accordingly, we simply list the major results. We have $u(r, t) = -c \tanh\phi = r/t$, and $\psi(r, t)$ is determined from the relations

$$\frac{\partial \psi}{\partial t} = e_0 \cosh\phi = \frac{e_0 ct}{((ct)^2 - r^2)^{1/2}}, \qquad c\frac{\partial \psi}{\partial r} = e_0 \sinh\phi = -\frac{e_0 r}{((ct)^2 - r^2)^{1/2}},$$

so that, together from (9.91) with $u(r, t) = r/t$, we have

$$\psi(r, t) = \frac{e_0}{c}((ct)^2 - r^2)^{1/2}, \quad g(r, t) = \frac{3e_0}{c((ct)^2 - r^2)^{1/2}},$$

with $e(r, t)$ and $p(r, t)$ given by (9.93). Again, we may confirm these expressions as providing particular illustrations of the Eqs. (3.12) for centrally symmetric mechanical systems, namely

$$\frac{\partial e}{\partial t} = u\frac{\partial p}{\partial t}, \qquad \frac{\partial e}{\partial r} = u\frac{\partial p}{\partial r}, \qquad \frac{de}{dt} = u\frac{dp}{dt}.$$

Further, from $(9.93)_2$, we may again verify that $f = dp/dt = 0$, since we have

$$\begin{aligned} \frac{dp}{dt} &= \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r}, \\ &= \frac{e_0}{c}\left\{\frac{-rc^2 t}{((ct)^2 - r^2)^{3/2}} + \frac{r}{t}\left(\frac{1}{((ct)^2 - r^2)^{1/2}} + \frac{r^2}{((ct)^2 - r^2)^{3/2}}\right)\right\}, \\ &= 0, \end{aligned}$$

providing a specific illustration of non-constant momentum $p(r, t)$ and yet $f = dp/dt = 0$. With $f = 0$, the work done $W(r, t)$ as defined by (3.8) is given by the relation $dW = gc^2 dt$, from which we may deduce

$$W(r, t) = \int \frac{3e_0 c dt}{((ct)^2 - r^2)^{1/2}} = 3e_0 \cosh^{-1}\left(\frac{ct}{r}\right),$$

on using the substitution $ct = r \cosh\theta$ and ignoring any arbitrary constants of integration. From the relation $\cosh^{-1} z = \log(z + (z^2 - 1)^{1/2})$ and $u(r, t) = r/t$, we might deduce

$$W(r, t) = 3e_0 \log\left(\frac{1 + \left(1 - (u/c)^2\right)^{1/2}}{u/c}\right) = -3e_0 \log\left(\frac{u/c}{1 + \left(1 - (u/c)^2\right)^{1/2}}\right).$$

Thus, apart from the factor of 3, this expression coincides with that given for the case of a single spatial dimension and with one of the terms arising from the general expression (6.14) for the special case $\lambda \to -\infty$ for the exact solution (5.1) examined in some detail in Chaps. 5 and 6.

Chapter 10
Relation with Quantum Mechanics

10.1 Introduction

As we have previously mentioned, the operator structure of quantum mechanics, $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, is intrinsically inherent in the proposed theory, assuming conservative external forces $f = -\partial V/\partial x$ and $gc^2 = -\partial V/\partial t$. From the relations (10.4), we have $p = \partial\psi/\partial x$ and $\mathscr{E} = \partial\psi/\partial t$ and conservation of energy $e + \mathscr{E} + V = constant$ immediately gives $e = -(\partial\psi/\partial t + V)$, and this is precisely the structure of the standard operator relations of quantum mechanics. That is, the established structure of the conventional operators of quantum mechanics is embodied in the present approach and arises from conservation of energy.

The purpose of this chapter is to formulate the role of the invariants $\xi(x, t)$ and $\eta(x, t)$ defined by (2.48), namely $\xi(x, t) = ex - c^2 pt$ and $\eta(x, t) = px - et$, relating to de Broglie waves and quantum mechanics. The determination of a formal connection between special relativity and quantum mechanics has long attracted the interest of many eminent researchers, including Einstein himself, culminating in the highly successful Dirac and Klein–Gordon equations, and the literature leading to these developments and their numerous consequences is now extensive; see, for example, Bjorken and Drell [7], Dirac [27], Dirac [28] and Gross [43] to name only four of many substantial texts on this topic. In this chapter, we examine the relationship of the ideas developed here with quantum mechanics and, in particular, the relationship of the two invariants of $\xi(x, t)$ and $\eta(x, t)$ of special relativity.

In the following section, we make some general comments on quantum mechanics and the Schrödinger wave equation, and from which we might propose two open questions. Firstly, are there any alternative interpretations for the wave function $\psi(x, t)$ other than the probabilistic interpretation of quantum mechanics, and secondly is it possible to deduce discrete energy states directly from the wave equation without forcing an eigenvalue problem? In the subsequent section, we present the conventional approach to the notions of group velocities and de Broglie waves. In the section thereafter, we examine the role of the invariants $\xi(x, t)$

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and $\eta(x,t)$ as defined by Eq. (2.48) relating to de Broglie waves and quantum mechanics. We first show that the simple harmonic waves $e^{2\pi i\eta/h}$ and $e^{2\pi i\xi/hc}$ are connected to particles moving with velocities u and c^2/u, respectively, and moreover with corresponding de Broglie wave velocities c^2/u and u, respectively. Further, on adopting the standard operator relations of quantum mechanics $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, as usual $\hbar = h/2\pi$, ξ and η give rise to two Lorentz invariant operators $(10.10)_1$ and $(10.10)_2$, leading to the Klein–Gordon partial differential equation (10.13) arising as the operator equivalent of the algebraic identity $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$.

In the next section, we deduce a modified second-order Klein–Gordon equation (10.21), which arises from incorporating a potential function $V(x,t)$ into the operator relations, which become $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t + V(x,t)$, and in place of Eq. (10.13), we deduce the entirely new partial differential equation (10.21), which enjoys characteristics of both the Klein–Gordon equation and of Schrödinger's second-order wave equation. While the general solution of the classical wave equation is well-known, in the context of the de Broglie wave energy, the question of the essential general structure of both the amplitude and the phase of solutions of the wave equation might be of considerable interest, and this is examined in the penultimate section of this chapter. In the final section of the chapter, and for the sake of completeness, some details for the time-dependent Dirac equation for a free particle are included. The Klein–Gordon equation is a fundamental equation of relativistic quantum mechanics and is second order in both space and time coordinates. On the other hand, Dirac's Lorentz invariant relativistic equation is first order in both space and time coordinates and is purposely constructed so that the probability density function remains non-negative.

10.2 Quantum Mechanics and Schrödinger Wave Equation

In quantum mechanics, it is well-established that the variables become operators and wave functions involve a probability density. For a single spatial dimension, the standard quantum mechanical operator relations for momentum p and energy e are, respectively, $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, namely

$$p = -i\hbar\frac{\partial\Psi}{\partial x}, \qquad e = i\hbar\frac{\partial\Psi}{\partial t}, \tag{10.1}$$

for some function $\Psi(x,t)$. To a certain extent, the probabilistic interpretation in quantum mechanics arises in consequence of identifying variables as complex (pure imaginary) operators in order to adopt analogous energy integrals from classical mechanics.

From a special relativistic perspective, the conventional signatures of these operators are entirely meaningful in the sense of being precisely what is required to produce the correct Lorentz invariances. Firstly, if we adopt $P \longrightarrow -i\hbar\partial/\partial X$

and $E \longrightarrow i\hbar\partial/\partial T$ and we apply these operator relations to the Lorentz invariant energy-momentum relations (2.46) and then from $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, we obtain precisely the correct differential transformation formulae (7.3). Furthermore, the usual signatures of the quantum mechanical operators are precisely that required to ensure the Lorentz invariances of the operators arising from the Lorentz invariants ξ and η defined by (2.48) and established below.

In conventional quantum mechanics, the second-order Schrödinger equation is usually motivated as arising from the classical wave equation. The usual requirements being that the equation be linear, so that different solutions may be superimposed, and that it involves only fundamental constants, rather than parameters associated with a particular motion of the particle, such as momentum, energy, frequency or propagation number. It is therefore important to emphasise that within the theory proposed here, the classical wave equation is not a matter of speculation, but rather a consequence, and it is not difficult to envisage the second-order Schrödinger wave equation arising in the present context as a formal consequence, following the numerous ad hoc derivations of the Schrödinger wave equation presented in several texts (such as [71], pages 18–19 or [93], pages 218–220).

For example, for a single non-relativistic particle, Semat [93] starts with the wave equation

$$\frac{\partial^2 \Psi}{\partial t^2} = w^2 \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right), \tag{10.2}$$

where w denotes the wave speed, assumed constant, and $\Psi(x, y, z, t)$ denotes a wave function. On using the three relations $p = h/\mu$, $w = \mu\nu$ and $E = h\nu$, [93] makes use of conservation of energy $E = mu^2/2 + V$, where V denotes potential energy, to deduce $p = mu = (2m(E - V))^{1/2}$ and therefore $w = h\nu/(2m(E - V))^{1/2}$. On looking for solutions of (10.2) of the form $\Psi(x, y, z, t) = \psi(x, y, z)\exp(2\pi i \nu t)$, we might readily deduce Schrödinger's wave equation for a single particle, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{8\pi^2 m}{h^2}(E - V)\psi, \tag{10.3}$$

and the question arises as to whether or not there are other derivations of Schrödinger's equation arising from alternative formulations.

Both the standard operator relations of quantum mechanics and the Schrödinger wave equation are immediately apparent from the present formulation. For a single spatial dimension, within the present proposed model, the following equations apply for the momentum $p(x, t)$ and the wave energy $\mathscr{E}(x, t)$,

$$\frac{\partial \mathscr{E}}{\partial t} = c^2 \frac{\partial p}{\partial x}, \qquad \frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t},$$

so that there certainly exists a function $\psi(x, t)$ such that

$$p = \frac{\partial \psi}{\partial x}, \qquad \mathscr{E} = \frac{\partial \psi}{\partial t}, \tag{10.4}$$

satisfying the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0. \tag{10.5}$$

From (10.5) we may readily deduce Schrödinger's wave equation following Eq. (10.3) above, and which indicates that there may be alternative interpretations for $\psi(x, t)$ other than the probabilistic interpretation of quantum mechanics? We may derive Schrödinger's wave equation from the present formulation and for a single spatial dimension from the above wave equation (10.5), without the need to introduce complex operators. It is therefore natural to pose a second question as to whether it is possible to deduce discrete energy states directly from the wave equation without forcing an eigenvalue problem.

Remark We comment that on adopting the usual operator relations of quantum mechanics $\mathbf{p} \longrightarrow -i\hbar\nabla$ and $e \longrightarrow i\hbar\partial/\partial t$, where as usual $\hbar = h/2\pi$ and h is Planck's constant, so that

$$\mathbf{p} = -i\hbar\nabla\psi, \qquad e = i\hbar\frac{\partial \psi}{\partial t}, \tag{10.6}$$

for some function $\psi = \psi(\mathbf{x}, t)$, we see that on face value (3.4) gives $\mathbf{f} = 0$ and

$$g = \frac{i\hbar}{c^2}\left(\frac{\partial^2 \psi}{\partial t^2} - c^2\nabla^2\psi\right), \tag{10.7}$$

for which we make three observations. Firstly, as previously noted, conventional quantum mechanics is a compromised theory allowing both particle and wave states, with no distinction made between physical particle energy e and wave energy $\mathscr{E}$. In the formalism developed here, simply replacing $\mathbf{p}$ and e by (10.6), conventional quantum mechanics would appear to operate under circumstances for which the spatial physical force $\mathbf{f}$ vanishes. In a subsequent section, we examine the case when the external forces $\mathbf{f}$ and g are generated from a potential. Secondly, the three-dimensional Klein–Gordon equation (10.45) can be seen to emerge under the linearity assumption $g(\psi) \approx -i(e_0^2/\hbar c^2)\psi$ where $e_0 = m_0c^2$, so that the Klein–Gordon equation might be only a first approximation in a nonlinear setting. The third observation is that Schrödinger's wave equation might equally well be deduced as arising from the wave equation operator appearing in (10.7).

10.3 Group Velocity and de Broglie Waves

Energy transmission by a stream of particles involves mass transport through a flow of matter. However, wave motion, as the passage of a local disturbance in a medium, may transmit energy without any accompanying flow of matter. There are numerous types of wave motion, such as ocean waves, in which the disturbance is an oscillation of the sea water along the path of the wave; sound waves, in which the disturbance is a deformation of the medium (perhaps elastic or viscoelastic) along the direction of the medium; electromagnetic waves, in which the disturbance consists of oscillations of the electric and magnetic fields; and so on. Since it is generally believed that even the most complicated wave structure can be built up from a linear combination of simple harmonic waves, attention is accordingly focussed on simple harmonic waves of the form $e^{2\pi i(x/\mu - t/T)}$, where μ is the wavelength and T is the period of oscillation.

On introducing, the wave number $k = 2\pi/\mu$ and angular frequency $\omega = 2\pi/T$, in standard notation the simple harmonic wave becomes $e^{i(kx-\omega(k)t)}$, noting that typically the angular frequency ω depends upon the wave number k, namely the wavelength μ (see, e.g. [71], pages 13–16). In this notation, the wave velocity w as measured by the movement of one wave peak (crest) to the next, namely "wavelength/ period", becomes simply $w = \omega(k)/k$. Now in any dispersive medium, that is, one in which the wave velocity depends upon the wavelength, waves of different wavelengths are propagated through the medium as a group with a velocity $u = d\omega/dk$, which is generally different from $w = \omega/k$ and the relationship $u = d\omega/dk = d(kw)/dk = w + kdw/dk$ holds (see, e.g. [93], pages 208–213).

de Broglie's idea [17] of an accompanying pilot wave originates as follows. In the Bohr theory of the hydrogen atom (see, e.g. [35], pages 29–61 for a general historical account), the n^{th} orbit of the electron has radius r given by

$$r = \left(\frac{nh}{2\pi}\right)^2 \frac{1}{me*^2}, \tag{10.8}$$

where m and $e*$ denote, respectively, the mass and charge of the electron. Mechanical and electrical force balance gives

$$mr\omega^2 = \frac{e*^2}{r^2},$$

where ω denotes angular velocity. By substitution of $e*^2$ from this equation into Eq. (10.8) and taking the square root, we may deduce de Broglie's relation $2\pi r = nh/p$. On noting that the momentum $p = mu$ and the electron velocity $u = r\omega$, he makes the critical observation that $2\pi r$ is an integer multiple of the wavelength μ if $p = h/\mu$, and speculates that this results applies to all elementary particles.

The present conventional approach for such an elementary particle with wave velocity $w = \mu\nu$, where $\nu = 1/T$ is the frequency, and with the analogous result for energy $e = h\nu$, is to apply the special relativistic energy relation $e^2 - (pc)^2 = e_0^2$, where $e_0 = m_0c^2$ and m_0 is the rest mass. Thus, from $e = hw/\mu$, together with $p = h/\mu$, we may deduce that the wave velocity w is given by

$$w = c\left(1 + \left(\frac{e_0\mu}{hc}\right)^2\right)^{1/2}.$$

Now on using the abovementioned relations $k = 2\pi/\mu$, we have, after differentiation and simplification,

$$u = w + k\frac{dw}{dk} = w - \mu\frac{dw}{d\mu} = c\left(1 + \left(\frac{e_0\mu}{hc}\right)^2\right)^{-1/2},$$

from which we may deduce $uw = c^2$, connecting the group velocity u with the wave velocity w. Further, with the usual relations $e = mc^2$ and $p = mu$, we have $e/p = c^2/u = w = \mu\nu$, and it is not difficult to show that the group velocity u of the wave package coincides with the particle velocity as defined by that velocity occurring in the expression $p = mu$ for momentum.

Thus, the group velocity of the wave u coincides with the particle velocity, and if the particle velocity u is sub-luminal, then the associated wave or phase velocity c^2/u through the de Broglie relation is necessarily superluminal. This is "believed" not to contradict the fact that information cannot be carried faster than the speed of light c, because "supposedly" the wave phase does not carry energy. However, the superluminal phase velocity may well be physically significant, and dark energy may well exist as a consequence that the associated de Broglie wave energy is neglected. If the wave energy through the superluminal wave speed c^2/u is accommodated, then it is not difficult to envisage interesting outcomes for slowing particle speeds u tending to zero.

10.4 Lorentz Invariants $\xi = ex - c^2pt$ and $\eta = px - et$

In this section, assuming the usual quantum mechanical relations (10.1), we investigate the formal implications arising from the two Lorentz invariants $\xi(x,t) = ex - c^2pt$ and $\eta(x,t) = px - et$ of special relativity and previously defined by (2.48). The formal connection between special relativity and quantum mechanics has alluded researchers, since in quantum mechanics, there is no distinction between particle energy e and wave energy $\mathscr{E}$. Once this distinction is made clear and the two invariants $\xi = ex - c^2pt$ and $\eta = px - et$ of special relativity are identified, the connection between the two topics becomes apparent. We first show that the simple harmonic waves $e^{2\pi i\eta/h}$ and $e^{2\pi i\xi/hc}$ are connected to particles

Table 10.1 Wave and particle (group) velocities for Lorentz invariants ξ and η

Waves	Wave velocity	Particle (group) velocity
$\exp[i(kx - \omega(k)t)]$	$w = \omega(k)/k$	$u = d\omega/dk$
$\exp(i\eta/\hbar) = \exp[i(px - et)/\hbar]$	$e/p = c^2/u$	$de/dp = c^2p/e = u$
$\exp(i\xi/c\hbar) = \exp[i(ex - c^2pt)/c\hbar]$	$c^2p/e = u$	$c^2dp/de = e/p = c^2/u$

moving with velocities u and c^2/u, respectively, and moreover with corresponding de Broglie wave velocities c^2/u and u, respectively. Subsequently, following the usual replacement with operators in quantum mechanics $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, where as usual $\hbar = h/2\pi$, we show that ξ and η give rise to two commuting Lorentz invariant operators.

Waves of the Form $e^{2\pi i\eta/h}$ and $e^{2\pi i\xi/hc}$ With the relations $p = h/\mu$ and $e = h\nu$ in mind, on comparison of the expressions $e^{2\pi i(ex-c^2pt)/hc}$ and $e^{2\pi i(px-et)/h}$ with the previously noted standard simple harmonic wave $e^{i(kx-\omega(k)t)} = e^{2\pi i(x/\mu-\nu t)}$, we see an immediate correspondence in the latter case, while in the former case for the invariant ξ, we have $e^{2\pi i(ex-c^2pt)/hc} = e^{2\pi i(wx/\mu-\nu ut)/hc}$ involving both the wave and group velocities w and u, respectively. It is apparent that the invariant η is intimately connected to a particle moving with velocity u with an accompanying de Broglie wave speed c^2/u, since waves of the form $e^{2\pi i\eta/h} = e^{2\pi i(px-et)/h}$ have a particle velocity $de/dp = u$ and a wave velocity $e/p = c^2/u$. On the other hand, it is apparent that the invariant ξ is intimately connected to a particle moving with velocity c^2/u with an accompanying de Broglie wave speed u, since waves of the form $e^{2\pi i\xi/hc} = e^{2\pi i(ex-c^2pt)/hc}$ have a particle velocity $c^2dp/de = c^2/u$ and a wave velocity $c^2p/e = u$. We comment that in each case the de Broglie relation $uw = c^2$ arises from (2.43) and that we might anticipate that the invariant η associates with the sub-luminal or particle world, while the invariant ξ associates with the superluminal or wave world. This information is summarised in tabular form in Table 10.1 and most easily deduced, making use of $e = mc^2$, $p = mu$ and the differential relation $ede = c^2pdp$.

Waves of the Form $A\exp(i\xi/c\hbar) + B\exp(i\eta/\hbar)$ and $A\exp(i\xi/c\hbar) + B\exp(-i\eta/\hbar)$ With γ and δ defined by (2.49), the two Lorentz invariants ξ and η are given by (2.50), thus

$$\gamma = \frac{1}{2c}(e + pc)(ct - x) = -\frac{1}{2c}(\xi + c\eta), \qquad \delta = \frac{1}{2c}(e - pc)(ct + x) = \frac{1}{2c}(\xi - c\eta),$$
$$\xi = ex - c^2pt = -c(\gamma - \delta), \qquad \eta = px - et = -(\gamma + \delta),$$

and therefore, waves of the general form $y(x,t) = A\exp(i\xi/c\hbar) + B\exp(i\eta/\hbar)$ become

$$y(x,t) = A\exp[-i(\gamma - \delta)/\hbar] + B\exp[-i(\gamma + \delta)/\hbar]$$

$$= \left[A\exp(i\delta/\hbar) + B\exp-(i\delta/\hbar)\right]\exp(-i\gamma/\hbar)$$
$$= [(A+B)\cos(\delta/\hbar) + i(A-B)\sin(\delta/\hbar)]\exp(-i\gamma/\hbar),$$

and correspondingly, waves of the general form $y(x,t) = A\exp(i\xi/c\hbar) + B\exp(-i\eta/\hbar)$ become

$$y(x,t) = A\exp[-i(\gamma-\delta)/\hbar] + B\exp[i(\gamma+\delta)/\hbar]$$
$$= \left[A\exp(-i\gamma/\hbar) + B\exp(i\gamma/\hbar)\right]\exp(i\delta/\hbar)$$
$$= [(A+B)\cos(\gamma/\hbar) + i(B-A)\sin(\gamma/\hbar)]\exp(i\delta/\hbar),$$

where A and B are used here to designate arbitrary amplitudes. Thus, in summary, and in terms of the original variables, we have the companion formulae

$$A\exp\left(\frac{i\xi}{c\hbar}\right) + B\exp\left(\frac{i\eta}{\hbar}\right)$$
$$= \left\{(A+B)\cos\left(\frac{(e-pc)(ct+x)}{2c\hbar}\right) + i(A-B)\sin\left(\frac{(e-pc)(ct+x)}{2c\hbar}\right)\right\}\exp\left(-\frac{i(e+pc)(ct-x)}{2c\hbar}\right),$$
$$A\exp\left(\frac{i\xi}{c\hbar}\right) + B\exp\left(-\frac{i\eta}{\hbar}\right)$$
$$= \left\{(A+B)\cos\left(\frac{(e+pc)(ct-x)}{2c\hbar}\right) + i(B-A)\sin\left(\frac{(e+pc)(ct-x)}{2c\hbar}\right)\right\}\exp\left(\frac{i(e-pc)(ct+x)}{2c\hbar}\right).$$

Lorentz Invariant Quantum Mechanical Operators It proves convenient to adopt the convention that the operator corresponding to a given variable is subscripted, so that, for example, in this notation, the two standard operators arising from momentum p and energy e become

$$L_p = -i\hbar\frac{\partial}{\partial x}, \qquad L_e = i\hbar\frac{\partial}{\partial t}, \tag{10.9}$$

so that directly from Eq. (2.48), namely $\xi = ex - c^2pt$ and $\eta = px - et$, we might introduce operators L_ξ and L_η that are defined by

$$L_\xi = i\hbar\left(x\frac{\partial}{\partial t} + c^2t\frac{\partial}{\partial x}\right), \qquad L_\eta = -i\hbar\left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\right), \tag{10.10}$$

both of which are fully Lorentz invariant operators, as can be verified from Eqs. (2.3) and (7.3), thus

$$L_\xi = \frac{i\hbar(X-vT)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\} + \frac{i\hbar c^2(T-vX/c^2)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\},$$

and

$$L_\eta = -\frac{i\hbar(X - vT)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\} - \frac{i\hbar(T - vX/c^2)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\},$$

and from which we may deduce

$$L_\xi = i\hbar\left(X\frac{\partial}{\partial T} + c^2 T\frac{\partial}{\partial X}\right), \qquad L_\eta = -i\hbar\left(X\frac{\partial}{\partial X} + T\frac{\partial}{\partial T}\right),$$

and therefore, the operators L_ξ and L_η are Lorentz invariant. Furthermore, by direct computation, it is not difficult to show that the two operators commute, namely $L_\xi L_\eta = L_\eta L_\xi$. In conventional quantum mechanical thinking, this means that the corresponding observables ξ and η are simultaneously measurable (compatible), and the two operators share the same eigenfunctions (see, e.g. [11], page 101). We further observe that the operator L_ξ is essentially the Lorentz operator L_v arising from the one-parameter group of Lorentz transformations (2.3), which is given by

$$L_v = -\left(T\frac{\partial}{\partial X} + \frac{X}{c^2}\frac{\partial}{\partial T}\right),$$

which therefore also commutes with L_η. The full implications of this intriguing correspondence are not immediately apparent but further underscore the formal connection established here between special relativity and quantum mechanics. Starting with the full invariant $\xi = ex - c^2 pt$ of the Lorentz group, we formulate the quantum mechanical operator L_ξ given by (10.10), which turns out to be the Lorentz operator formed from the one-parameter group of Lorentz transformations (2.3).

10.5 Klein–Gordon Partial Differential Equation

In this section, we show that the Lorentz invariant operators (10.10) give rise to the Klein–Gordon partial differential equation (10.13), arising as the operator equivalent of the algebraic identity $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$. The various properties of the two operators L_ξ and L_η are most apparent in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$. Using the differential formulae

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}, \qquad \frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right),$$

we may deduce

$$L_\xi = i\hbar c\left(\alpha\frac{\partial}{\partial \alpha} - \beta\frac{\partial}{\partial \beta}\right), \qquad L_\eta = -i\hbar\left(\alpha\frac{\partial}{\partial \alpha} + \beta\frac{\partial}{\partial \beta}\right), \tag{10.11}$$

and from which there arises the intriguingly simple formulae

$$L_{\xi-c\eta} = 2i\hbar c\alpha \frac{\partial}{\partial\alpha}, \qquad L_{\xi+c\eta} = -2i\hbar c\beta \frac{\partial}{\partial\beta}. \tag{10.12}$$

From these two expressions, the formal operator equation corresponding to the identity (6.2), namely $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, gives rise immediately to the Klein–Gordon equation (see, e.g. [11], page 312)

$$L_{\xi-c\eta}L_{\xi+c\eta}\Psi = 4(\hbar c)^2\alpha\beta\frac{\partial^2\Psi}{\partial\alpha\partial\beta} = -e_0^2\alpha\beta\Psi,$$

for some function $\Psi(x, t)$ and from which we have

$$4\frac{\partial^2\Psi}{\partial\alpha\partial\beta} = -\left(\frac{e_0}{\hbar c}\right)^2\Psi,$$

or alternatively, in terms of the conventional (x, t) wave equation operator, we have the more usual form of the Klein–Gordon equation ([11], page 313)

$$\frac{\partial^2\Psi}{\partial t^2} - c^2\frac{\partial^2\Psi}{\partial x^2} = -\left(\frac{e_0}{\hbar}\right)^2\Psi. \tag{10.13}$$

Einstein Energy Relation $e^2 = e_0^2 + (pc)^2$ It is interesting to note that on assuming the usual de Broglie relations, $e = h\nu$ and $p = h/\mu$, where ν is the frequency, μ is the wavelength and $\nu\mu = c$, the Einstein relation $e^2 = e_0^2 + (pc)^2$ emerges from the so-called dispersion relation, arising from the Klein–Gordon equation (10.13) for the simple wave

$$\Psi(x, t) = \Psi_0 \exp(i(kx - \omega t)), \tag{10.14}$$

where Ψ_0 is a constant, $k = 2\pi/\mu$ and $\omega = 2\pi\nu$ in the following manner. From (10.14), we have

$$\frac{\partial^2\Psi}{\partial t^2} = -\omega^2\Psi, \qquad \frac{\partial^2\Psi}{\partial x^2} = -k^2\Psi,$$

so that the dispersion relation for the Klein–Gordon equation (10.13) becomes simply $\omega^2 - (kc)^2 = (e_0/\hbar)^2$, which on using $\omega = e/\hbar$ and $k = p/\hbar$ produces the desired relation $e^2 = e_0^2 + (pc)^2$.

Remark We observe that on introducing the new variables ζ and ρ defined by

$$\zeta = ((ct)^2 - x^2)^{1/2}, \qquad \rho = \frac{1}{2}\log\left(\frac{ct + x}{ct - x}\right),$$

which in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, become

$$\zeta = ((ct)^2 - x^2)^{1/2} = (\alpha\beta)^{1/2}, \qquad \rho = \frac{1}{2}\log\left(\frac{ct+x}{ct-x}\right) = \frac{1}{2}\log\left(\frac{\alpha}{\beta}\right),$$

and from (10.11), we may readily deduce the formulae

$$L_\xi(\zeta) = 0, \qquad L_\eta(\zeta) = -i\hbar\zeta, \qquad L_\xi(\rho) = i\hbar c, \qquad L_\eta(\rho) = 0,$$

along with

$$L_\xi L_\eta \Psi = \hbar^2 c\left(\alpha^2 \frac{\partial^2\Psi}{\partial\alpha^2} + \alpha\frac{\partial\Psi}{\partial\alpha} - \beta^2\frac{\partial^2\Psi}{\partial\beta^2} - \beta\frac{\partial\Psi}{\partial\beta}\right),$$

so that on making the Euler transformations $\gamma = \log\alpha$ and $\delta = \log\beta$, we obtain

$$L_\xi L_\eta \Psi = \hbar^2 c\left(\frac{\partial^2\Psi}{\partial\gamma^2} - \frac{\partial^2\Psi}{\partial\delta^2}\right).$$

We also observe that for the variables ζ and ρ, the Jacobian and wave equation for the momentum become

$$\frac{\partial(\zeta,\rho)}{\partial(x,t)} = \frac{c}{\zeta}, \qquad \frac{\partial^2 p}{\partial t^2} - c^2\frac{\partial^2 p}{\partial x^2} = 4c^2\frac{\partial^2 p}{\partial\alpha\partial\beta} = 0, \tag{10.15}$$

and the latter equation becomes

$$\zeta^2\frac{\partial^2 p}{\partial\zeta^2} + \zeta\frac{\partial p}{\partial\zeta} - \frac{\partial^2 p}{\partial\rho^2} = 0, \qquad \frac{\partial^2 p}{\partial\tau^2} - \frac{\partial^2 p}{\partial\rho^2} = 0, \tag{10.16}$$

where we have made another Euler transformation $\tau = \log\zeta$. On noting the relations $\gamma = \log\alpha = \tau + \rho$ and $\delta = \log\beta = \tau - \rho$, the well-known general solution of the wave equation $p = F(\alpha) + G(\beta)$ is apparent from the latter equations of either (10.15) or (10.16).

10.6 Alternative Klein–Gordon–Schrödinger Equation

In this section, we deduce the modified second-order Klein–Gordon equation (10.21), arising from incorporating a potential function $V(x,t)$ into the operator relations. In terms of the present formulation, from Eq. (8.1) and assuming that the external forces are generated from a potential function $V(x,t)$, then

$$p = \frac{\partial \psi}{\partial x}, \qquad \mathscr{E} = \frac{\partial \psi}{\partial t}, \qquad e = -\left(\frac{\partial \psi}{\partial t} + V(x,t)\right), \tag{10.17}$$

and it is important to note that the conventional quantum mechanical operator structure $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$ is immediately inherent in the present approach. Also, we observe that if the $\psi(x,t)$ satisfies the wave equation (8.2), then (10.17) immediately satisfies (3.4), assuming that the external forces f and g are generated from $f = -\partial V/\partial x$ and $gc^2 = -\partial V/\partial t$.

On incorporating a potential function $V(x,t)$, the operator relations now become $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t + V(x,t)$, so that in place of Eq. (10.9), we have

$$L_p = -i\hbar\frac{\partial}{\partial x}, \qquad L_e = i\hbar\frac{\partial}{\partial t} + V(x,t), \tag{10.18}$$

while Eqs. (10.12) become

$$L_{\xi-c\eta} = 2i\hbar c\alpha\frac{\partial}{\partial \alpha} + \alpha V, \qquad L_{\xi+c\eta} = -2i\hbar c\beta\frac{\partial}{\partial \beta} - \beta V, \tag{10.19}$$

and the structure of (10.18), and $(10.18)_2$ in particular, is not immediately apparent from the corresponding structure of (10.17), and $(10.18)_2$ is selected on the basis that the dispersion relation arising from (10.21) and (10.14) is sensible, noting that this point was not picked up in [52].

Now when we seek to define the operator equation corresponding to the algebraic identity $\xi^2 - (c\eta)^2 = (\xi + c\eta)(\xi - c\eta) = -e_0^2\alpha\beta$, we find that the two operators $L_{\xi-c\eta}$ and $L_{\xi+c\eta}$ are now non-commuting operators, and therefore, we need to define a product operator in the most obvious manner. Accordingly, we adopt the defintion

$$L_{\xi^2-(c\eta)^2} = \left(L_{\xi-c\eta}L_{\xi+c\eta} + L_{\xi+c\eta}L_{\xi-c\eta}\right)/2, \tag{10.20}$$

and in place of (10.13), we may eventually deduce the modified second-order Klein–Gordon equation

$$\frac{\partial^2 \Psi}{\partial t^2} - c^2\frac{\partial^2 \Psi}{\partial x^2} - \frac{2iV}{\hbar}\frac{\partial \Psi}{\partial t} - \frac{i\Psi}{\hbar}\frac{\partial V}{\partial t} - \left(\frac{V}{\hbar}\right)^2\Psi = -\left(\frac{e_0}{\hbar}\right)^2\Psi, \tag{10.21}$$

which is an entirely new partial differential equation that enjoys characteristics of both the Klein–Gordon equation and Schrödinger's second-order wave equation.

Now the dispersion relation arising from (10.21) and (10.14) can be readily shown to become

$$(\hbar\omega + V)^2 + i\hbar\frac{\partial V}{\partial t} = e_0^2 + (\hbar kc)^2,$$

which in terms of $\omega = e/\hbar$ and $k = p/\hbar$ becomes

$$(e+V)^2 + i\hbar\frac{\partial V}{\partial t} = e_0^2 + (pc)^2.$$

While we may readily find a physical interpretation for the term involving the sum of the two energies $(e+V)^2$ as $\mathscr{E}^2$, the physical meaning of the imaginary term $i\hbar\partial V/\partial t$ is not immediately apparent, noting however that in the approach adopted here, we might be more inclined to interpret $h\nu$ as $\mathscr{E}$ rather than as e, but either way the sum of the two energies, either $e+V$ or $\mathscr{E}+V$, is a physically sensible combination.

Remark Clearly Eq. (10.21) is a linear partial differential equation and therefore admits many analytical solutions. For example, we might make the assumption that the potential $V(x,t)$ satisfies the partial differential relation

$$i\hbar\frac{\partial V}{\partial t} = e_0^2 - V^2, \tag{10.22}$$

in which case the equation simplifies considerably to give

$$\frac{\partial^2\Psi}{\partial t^2} - c^2\frac{\partial^2\Psi}{\partial x^2} - \frac{2iV}{\hbar}\frac{\partial\Psi}{\partial t} = 0. \tag{10.23}$$

By partial integration of (10.22), we deduce that

$$\log\left(\frac{e_0+V}{e_0-V}\right) = -2ic\epsilon(t-t_0(x)),$$

where $\epsilon = e_0/c\hbar$ and $t_0(x)$ denotes an arbitrary function of x. From this relation, we obtain

$$iV(x,t) = e_0\tan[c\epsilon(t-t_0(x))], \tag{10.24}$$

and Eq. (10.23) becomes

$$\frac{\partial^2\Psi}{\partial t^2} - c^2\frac{\partial^2\Psi}{\partial x^2} - 2c\epsilon\tan[c\epsilon(t-t_0(x))]\frac{\partial\Psi}{\partial t} = 0. \tag{10.25}$$

In the case when t_0 is a constant, then it is easy to show that $\cos[c\epsilon(t-t_0)]\Psi(x,t)$ satisfies a Klein–Gordon equation. In the event that $t_0(x) = \gamma x/c$, where γ denotes an arbitrary constant, then Eq. (10.25) becomes

$$\frac{\partial^2\Psi}{\partial t^2} - c^2\frac{\partial^2\Psi}{\partial x^2} - 2c\epsilon\tan(\epsilon\xi)\frac{\partial\Psi}{\partial t} = 0,$$

where $\xi = ct - \gamma x$, and this equation admits solutions of the form $\Psi = \exp(-\omega t)\Phi(\xi)$, where ω is a constant and $\Phi(\xi)$ denotes a function of ξ only. These solutions simplify considerably in the two particular cases $\gamma = \pm 1$.

10.7 General Wave Structure of Solutions of Wave Equation

Of course, although the general solution of the classical wave equation is well-known, in the context of the de Broglie wave energy, it would be of some considerable interest to have an understanding of the essential general structure of both the amplitude and the phase of solutions of the wave equation. If we look for solutions of the classical wave equation of the form

$$\mathscr{E}(x,t) = a(x,t)\exp(i\eta(x,t)/\hbar),$$

where $a(x,t)$ denotes an arbitrary amplitude to be determined and we have in mind that $\eta(x,t)$ might be one of the Lorentz invariants $\eta(x,t) = px - et$, then we know that on solving the necessary partial differential equations for $a(x,t)$ and $\eta(x,t)$ and reconstituting the wave energy $\mathscr{E}(x,t) = a(x,t)\exp(i\eta(x,t)/\hbar)$, ultimately we must obtain the general solution of the wave equation, namely $\mathscr{E}(x,t) = c\,(F(\alpha) - G(\beta))$, where $F(\alpha)$ and $G(\beta)$ denote arbitrary functions of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, respectively. Accordingly, we know the end result; the real issue is to determine how the functions $a(x,t)$ and $\eta(x,t)$ are individually structured to ensure this outcome.

Now from the classical wave equation, we may deduce

$$\frac{\partial^2 a}{\partial t^2} - \frac{a}{\hbar^2}\left(\frac{\partial \eta}{\partial t}\right)^2 + \frac{i}{\hbar}\left(a\frac{\partial^2 \eta}{\partial t^2} + 2\frac{\partial a}{\partial t}\frac{\partial \eta}{\partial t}\right) = c^2\left\{\frac{\partial^2 a}{\partial x^2} - \frac{a}{\hbar^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{i}{\hbar}\left(a\frac{\partial^2 \eta}{\partial x^2} + 2\frac{\partial a}{\partial x}\frac{\partial \eta}{\partial x}\right)\right\},$$

so that by equating real and imaginary parts, we have

$$\frac{\partial^2 a}{\partial t^2} - c^2\frac{\partial^2 a}{\partial x^2} = \frac{a}{\hbar^2}\left\{\left(\frac{\partial \eta}{\partial t}\right)^2 - c^2\left(\frac{\partial \eta}{\partial x}\right)^2\right\}, \tag{10.26}$$

$$a\left(\frac{\partial^2 \eta}{\partial t^2} - c^2\frac{\partial^2 \eta}{\partial x^2}\right) + 2\left(\frac{\partial a}{\partial t}\frac{\partial \eta}{\partial t} - c^2\frac{\partial a}{\partial x}\frac{\partial \eta}{\partial x}\right) = 0.$$

From these two coupled partial differential equation,s we may deduce the interesting result that both $a(x,t)$ and the product $a(x,t)\eta(x,t)$ satisfy the same Klein–Gordon partial differential equation, thus

$$\frac{\partial^2 a}{\partial t^2} - c^2\frac{\partial^2 a}{\partial x^2} = \frac{a}{\hbar^2}\left\{\left(\frac{\partial \eta}{\partial t}\right)^2 - c^2\left(\frac{\partial \eta}{\partial x}\right)^2\right\}, \tag{10.27}$$

$$\frac{\partial^2(a\eta)}{\partial t^2} - c^2\frac{\partial^2(a\eta)}{\partial x^2} = \frac{a\eta}{\hbar^2}\left\{\left(\frac{\partial\eta}{\partial t}\right)^2 - c^2\left(\frac{\partial\eta}{\partial x}\right)^2\right\}.$$

In terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, the two equations (10.27) become

$$\frac{\partial^2 a}{\partial\alpha\partial\beta} = \frac{a}{\hbar^2}\frac{\partial\eta}{\partial\alpha}\frac{\partial\eta}{\partial\beta}, \qquad \frac{\partial^2(a\eta)}{\partial\alpha\partial\beta} = \frac{a\eta}{\hbar^2}\frac{\partial\eta}{\partial\alpha}\frac{\partial\eta}{\partial\beta}, \tag{10.28}$$

so that with $b = a\eta$, we may readily deduce $b\partial^2a/\partial\alpha\partial\beta - a\partial^2b/\partial\alpha\partial\beta = 0$, and therefore, $b\partial a/\partial\alpha - a\partial b/\partial\alpha = C'(\alpha)$ and $b\partial a/\partial\beta - a\partial b/\partial\beta = D'(\beta)$, where $C(\alpha)$ and $D(\beta)$ denote arbitrary functions of the indicated arguments. On solving these latter two equations as two equations for the two unknown functions a and b, we obtain

$$a = \frac{C'(\alpha)\frac{\partial a}{\partial\beta} - D'(\beta)\frac{\partial a}{\partial\alpha}}{\frac{\partial(a,b)}{\partial(\alpha,\beta)}}, \qquad b = \frac{C'(\alpha)\frac{\partial b}{\partial\beta} - D'(\beta)\frac{\partial b}{\partial\alpha}}{\frac{\partial(a,b)}{\partial(\alpha,\beta)}},$$

and therefore from $b = a\eta$, we have

$$a = \frac{C'(\alpha)\frac{\partial a}{\partial\beta} - D'(\beta)\frac{\partial a}{\partial\alpha}}{a\frac{\partial(a,\eta)}{\partial(\alpha,\beta)}}, \qquad a\eta = \frac{C'(\alpha)\left(\eta\frac{\partial a}{\partial\beta} + a\frac{\partial\eta}{\partial\beta}\right) - D'(\beta)\left(\eta\frac{\partial a}{\partial\alpha} + a\frac{\partial\eta}{\partial\alpha}\right)}{a\frac{\partial(a,\eta)}{\partial(\alpha,\beta)}}.$$

On using the first equation, the second simplifies to give simply $D'(\beta)\partial\eta/\partial\alpha - C'(\alpha)\partial\eta/\partial\beta = 0$ as a linear first-order partial differential equation for the determination of $\eta(\alpha,\beta)$, the general solution of which is $\eta(\alpha,\beta) = \phi(C(\alpha) + D(\beta))$, where ϕ denotes an arbitrary function of the indicated argument. Now on using this general solution in the first equation, the equation admits a factor, and we obtain

$$\left(1 + a^2\frac{d\phi}{d\zeta}\right)\left(D'(\beta)\frac{\partial a}{\partial\alpha} - C'(\alpha)\frac{\partial a}{\partial\beta}\right) = 0, \tag{10.29}$$

where here we are using $\zeta = C(\alpha) + D(\beta)$ as a working variable. Assuming for the time being that the first factor is non-zero, then $D'(\beta)\partial a/\partial\alpha - C'(\alpha)\partial a/\partial\beta = 0$, with general solution $a(\alpha,\beta) = \psi(C(\alpha) + D(\beta))$, where again ψ denotes an arbitrary function of $\zeta = C(\alpha) + D(\beta)$.

In terms of $\eta(\alpha,\beta) = \phi(\zeta)$ and $a(\alpha,\beta) = \psi(\zeta)$, the two partial differential equations (10.28) reduce to the two ordinary differential equations

$$\frac{d^2\psi}{d\zeta^2} = \epsilon^2\psi\left(\frac{d\phi}{d\zeta}\right)^2, \qquad \frac{d^2(\phi\psi)}{d\zeta^2} = \epsilon^2(\phi\psi)\left(\frac{d\phi}{d\zeta}\right)^2, \tag{10.30}$$

where for the moment we are using $\epsilon = 1/\hbar$. Now on adopting ψ as a function of ϕ, namely $\psi = \psi(\phi)$ and using

$$\frac{d^2\phi}{d\zeta^2}\Big/\left(\frac{d\phi}{d\zeta}\right)^2 = \frac{d(d\phi/d\zeta)}{d\phi}\Big/\left(\frac{d\phi}{d\zeta}\right) = \frac{d\log(d\phi/d\zeta)}{d\phi}, \tag{10.31}$$

so that with $\rho(\phi) = d\phi/d\zeta$, the two equations (10.30) become

$$\rho\frac{d^2\psi}{d\phi^2} + \frac{d\rho}{d\phi}\frac{d\psi}{d\phi} = \epsilon^2\rho\psi, \qquad \rho\frac{d^2(\phi\psi)}{d\phi^2} + \frac{d\rho}{d\phi}\frac{d(\phi\psi)}{d\phi} = \epsilon^2\rho\phi\psi.$$

Again, on using the first equation, the second simplifies to give simply $2\rho d\psi/d\phi + \psi d\rho/d\phi = 0$, and therefore, $\rho\psi^2 = C_1$ or $\rho = C_1/\psi^2$, where C_1 denotes an arbitrary constant, and noting that the assumed non-zero factor in (10.29) corresponds to the special case of $\rho = C_1/\psi^2$ with $C_1 = -1$. Substitution of $\rho = C_1/\psi^2$ into the first equation gives the simple equation $d^2\omega/d\phi^2 + \epsilon^2\omega = 0$, where ω denotes $1/\psi$. The general solution of this equation may be expressed as

$$\omega = \frac{1}{\psi} = C_2\cos\{\epsilon(\phi - \phi_0)\}, \tag{10.32}$$

where C_2 and ϕ_0 denote further arbitrary constants. Now from

$$\rho = \frac{d\phi}{d\zeta} = \frac{C_1}{\psi^2} = C_1C_2^2\cos^2\{\epsilon(\phi - \phi_0)\},$$

we are required to integrate $\sec^2\{\epsilon(\phi - \phi_0)\}d\phi = C_1C_2^2 d\zeta$, and therefore, we may finally deduce

$$\tan\{\epsilon(\phi - \phi_0)\} = \epsilon C_1C_2^2(\zeta - \zeta_0),$$

where ζ_0 denotes another arbitrary constant, and altogether we have

$$a(\alpha, \beta) = \psi(\zeta) = \frac{1}{C_2}\sec\{\epsilon(\phi - \phi_0)\} = \frac{1}{C_2}\left(1 + \{\epsilon C_1C_2^2(\zeta - \zeta_0)\}^2\right)^{1/2},$$
$$\eta(\alpha, \beta) = \phi(\zeta) = \phi_0 + \hbar\tan^{-1}\{\epsilon C_1C_2^2(\zeta - \zeta_0)\},$$

where $\zeta = C(\alpha) + D(\beta)$.

Now since $C(\alpha)$ and $D(\beta)$ are assumed to be arbitrary, we may include the two constants $\epsilon C_1C_2^2$ and ϕ_0 in these functions, and therefore, the essential structure of $a(\alpha, \beta)$ and $\eta(\alpha, \beta)$ becomes

$$a(\alpha, \beta) = a_0\left(1 + \zeta^2\right)^{1/2}, \qquad \eta(\alpha, \beta) = \eta_0 + \hbar\tan^{-1}(\zeta), \tag{10.33}$$

where a_0 and η_0 denote arbitrary constants, $\zeta = C(\alpha) + D(\beta)$ and $C(\alpha)$ and $D(\beta)$ are arbitrary functions of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$, respectively. We may check by direct substitution that these expressions indeed satisfy the two ordinary differential equations (10.30). Thus, on reconstituting the wave energy $\mathscr{E}(\alpha, \beta)$, these expressions give

$$\mathscr{E}(\alpha, \beta) = a(\alpha, \beta) \exp(i\eta(\alpha, \beta)/\hbar) = a_0 \exp(i\eta_0/\hbar) \left(1 + \zeta^2\right)^{1/2} \exp(i \tan^{-1} \zeta)$$

$$= a_0 \exp(i\eta_0/\hbar) \left(1 + \zeta^2\right)^{1/2} \left(\cos(\tan^{-1} \zeta) + i \sin(\tan^{-1} \zeta)\right) = a_0 \exp(i\eta_0/\hbar)(1 + i\zeta),$$

and therefore, as expected, $\mathscr{E}(\alpha, \beta)$ has the structure of the general solution of the wave equation. The question is, of course, how knowledge of this structure improves our understanding of the Lorentz invariant energy profiles arising from the classical wave equation, and we first consider the Einstein energy relation $e = e_0(1 - (u/c)^2)^{-1/2}$.

Einstein Energy Relation $e = e_0(1 - (u/c)^2)^{-1/2}$ On making a comparison of $(10.33)_1$ with $e = e_0(1 - (u/c)^2)^{-1/2}$, clearly we have the factor $\left(1 + \zeta^2\right)^{1/2}$ instead of $(1 - (u/c)^2)^{-1/2}$, where $\zeta = C(\alpha)+D(\beta)$ denotes the general solution of the classical one space dimension wave equation. With $\sigma = u/c$, the transformation

$$\sigma = \frac{\zeta}{(1 + \zeta^2)^{1/2}}, \qquad \zeta = \frac{\sigma}{(1 - \sigma^2)^{1/2}}, \tag{10.34}$$

takes the factor $\left(1 + \zeta^2\right)^{1/2}$ into the factor $(1 - \sigma^2)^{-1/2}$, as required, and with $\sigma = u/c$ we might examine this transformation in more detail, thus

$$\zeta = \frac{\sigma}{(1 - \sigma^2)^{1/2}} = \frac{u/c}{(1 - (u/c)^2)^{1/2}} = \frac{m_0 u}{m_0 c(1 - (u/c)^2)^{1/2}} = \frac{p}{m_0 c} = \frac{pc}{e_0},$$

where $p = mu$ denotes the momentum. Now for a single space dimension, the momentum p does indeed satisfy the classical wave equation (see Eq. $(4.4)_1$), and therefore, the above analysis together with the transformation (10.34) makes considerable sense.

General Lorentz Invariant Energy Relation (2.59) The most general Lorentz invariant energy expression that we have is that given by (2.59) for particle energy, namely

$$e(u) = \frac{e_0}{(1 - (u/c)^2)^{1/2}} \left(\frac{1 + (u/c)}{1 - (u/c)}\right)^{\kappa/2}, \tag{10.35}$$

where $e_0 = m_0 c^2$ is the rest energy and κ denotes an arbitrary constant, and the conventional relativistic energy expression $e(u) = e_0/(1 - (u/c)^2)^{1/2}$ arising from the case $\kappa = 0$. For $\kappa \neq 0$, we need the relation (see Eq. (2.10))

$$\frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \tanh^{-1}(u/c),$$

so that (10.35) becomes

$$\begin{aligned} e(u) &= \frac{e_0}{(1-(u/c)^2)^{1/2}}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2} \\ &= \frac{e_0}{(1-(u/c)^2)^{1/2}} e^{\kappa \tanh^{-1}(u/c)}, \end{aligned}$$

and from elementary relations, we have

$$\tanh^{-1}(u/c) = \tanh^{-1}\sigma = \tanh^{-1}(\zeta/(1+\zeta^2)^{1/2}) = \sinh^{-1}(\zeta),$$

so that we have

$$e(u) = \frac{e_0}{(1-(u/c)^2)^{1/2}} e^{\kappa \tanh^{-1}(u/c)} = e_0(1+\zeta^2)^{1/2} e^{\kappa \sinh^{-1}(\zeta)},$$

in comparison to that arising from (10.33). We also note that on using $\sinh^{-1}\zeta = \log(\zeta + (1+\zeta^2)^{1/2})$, we have the alternative expression

$$e(u) = \frac{e_0}{(1-(u/c)^2)^{1/2}}\left(\frac{1+(u/c)}{1-(u/c)}\right)^{\kappa/2} = e_0(1+\zeta^2)^{1/2}(\zeta + (1+\zeta^2)^{1/2})^{\kappa}.$$

10.8 Wave Solutions of Klein–Gordon Equation

Much of the analysis of the previous section also applies to the Klein–Gordon equation (10.13) in place of the classical wave equation, namely

$$\frac{\partial^2 \Psi}{\partial t^2} - c^2 \frac{\partial^2 \Psi}{\partial x^2} = -\left(\frac{e_0}{\hbar}\right)^2 \Psi,$$

so that on investigating solutions of the form $\Psi(x,t) = a(x,t)\exp(i\eta(x,t)/\hbar)$ to determine their possible amplitudes $a(x,t)$ and phases $\eta(x,t)/\hbar$, in place of (10.26), we have

$$\frac{\partial^2 a}{\partial t^2} - c^2 \frac{\partial^2 a}{\partial x^2} = \frac{a}{\hbar^2}\left\{\left(\frac{\partial \eta}{\partial t}\right)^2 - c^2\left(\frac{\partial \eta}{\partial x}\right)^2 - e_0^2\right\}, \tag{10.36}$$

$$a\left(\frac{\partial^2\eta}{\partial t^2}-c^2\frac{\partial^2\eta}{\partial x^2}\right)+2\left(\frac{\partial a}{\partial t}\frac{\partial \eta}{\partial t}-c^2\frac{\partial a}{\partial x}\frac{\partial \eta}{\partial x}\right)=0,$$

while in terms of the characteristic coordinates $\alpha = ct + x$ and $\beta = ct - x$ and in place of (10.28), we obtain

$$\frac{\partial^2 a}{\partial\alpha\partial\beta}=\frac{a}{\hbar^2}\left\{\frac{\partial\eta}{\partial\alpha}\frac{\partial\eta}{\partial\beta}-\left(\frac{e_0}{2c}\right)^2\right\},\qquad \frac{\partial^2(a\eta)}{\partial\alpha\partial\beta}=\frac{a\eta}{\hbar^2}\left\{\frac{\partial\eta}{\partial\alpha}\frac{\partial\eta}{\partial\beta}-\left(\frac{e_0}{2c}\right)^2\right\}. \tag{10.37}$$

Further, all the above analysis pertaining to a and $b = a\eta$ and leading to $\eta(\alpha,\beta) = \phi(C(\alpha) + D(\beta))$ and $a(\alpha,\beta) = \psi(C(\alpha) + D(\beta))$ also applies, where ϕ and ψ denote arbitrary functions of $\zeta = C(\alpha) + D(\beta)$ and $C(\alpha)$ and $D(\beta)$ are also arbitrary functions of the indicated argument. However, at this point further extension of the above analysis is not possible for arbitrary $C(\alpha)$ and $D(\beta)$, and in order to reduce (10.37) to ordinary differential equations, we need to assume that $\zeta = \lambda x + ct$, where λ denotes an arbitrary constant.

Accordingly, in the following we solve (10.36) or (10.37), assuming that $\eta(x,t) = \phi(\zeta)$ and $a(x,t) = \psi(\zeta)$, where $\zeta = \lambda x + ct$ and λ denotes an arbitrary constant. On substitution of these expressions for $\eta(x,t)$ and $a(x,t)$ into (10.36), the two partial differential equations reduce to the two ordinary differential equations

$$\frac{d^2\psi}{d\zeta^2}=\epsilon^2\psi\left\{\left(\frac{d\phi}{d\zeta}\right)^2-k^2\right\},\qquad \frac{d^2(\phi\psi)}{d\zeta^2}=\epsilon^2(\phi\psi)\left\{\left(\frac{d\phi}{d\zeta}\right)^2-k^2\right\}, \tag{10.38}$$

where $\epsilon = 1/\hbar$ and $k = e_0/c(1-\lambda^2)^{1/2}$, noting that k is pure imaginary for $\lambda^2 > 1$. On adopting ψ as a function of ϕ, namely $\psi = \psi(\phi)$ and using (10.31) with $\rho(\phi) = d\phi/d\zeta$, the two equations (10.38) become

$$\rho\frac{d^2\psi}{d\phi^2}+\frac{d\rho}{d\phi}\frac{d\psi}{d\phi}=\epsilon^2\rho\psi\left\{1-\left(\frac{k}{\rho}\right)^2\right\},\quad \rho\frac{d^2(\phi\psi)}{d\phi^2}+\frac{d\rho}{d\phi}\frac{d(\phi\psi)}{d\phi}=\epsilon^2\rho\phi\psi\left\{1-\left(\frac{k}{\rho}\right)^2\right\}.$$

Using the first equation, the second simplifies to give simply $2\rho d\psi/d\phi + \psi d\rho/d\phi = 0$, and therefore, again we obtain $\rho\psi^2 = C_1$ or $\rho = C_1/\psi^2$, where C_1 denotes an arbitrary constant. Substitution of $\rho = C_1/\psi^2$ into the first equation gives $d^2\omega/d\phi^2 + \epsilon^2\omega = \delta^2/\omega^3$, where ω denotes $1/\psi$ and δ is another constant defined by

$$\delta=\frac{k\epsilon}{C_1}=\frac{e_0\epsilon}{C_1c(1-\lambda^2)^{1/2}}=\frac{e_0}{C_1c\hbar(1-\lambda^2)^{1/2}}.$$

The second-order differential equation $d^2\omega/d\phi^2 + \epsilon^2\omega = \delta^2/\omega^3$ may be readily integrated to obtain

$$\left(\frac{d\omega}{d\phi}\right)^2 + (\epsilon\omega)^2 + \left(\frac{\delta}{\omega}\right)^2 = 2\epsilon C_2,$$

where C_2 denotes an arbitrary constant, and this equation may be rearranged to give

$$\frac{\omega d\omega}{(2\epsilon C_2\omega^2 - \epsilon^2\omega^4 - \delta^2)^{1/2}} = \pm d\phi,$$

which may be further integrated to yield

$$\frac{\epsilon}{\psi^2} = \epsilon\omega^2 = C_2 + \left(C_2^2 - \delta^2\right)^{1/2} \cos 2\epsilon(\phi - \phi_0), \tag{10.39}$$

where ϕ_0 denotes a second integration constant, and noting that in the limit $\delta \longrightarrow 0$, this expression agrees with Eq. (10.32) with a suitably adjusted arbitrary constant. Now from

$$\epsilon\frac{d\phi}{d\zeta} = \epsilon\rho = \epsilon\frac{C_1}{\psi^2} = \epsilon C_1\omega^2 = C_1\left\{C_2 + \left(C_2^2 - \delta^2\right)^{1/2} \cos 2\epsilon(\phi - \phi_0)\right\},$$

we are required to integrate

$$\frac{d(\tan\Phi)}{\left\{[C_2 + (C_2^2 - \delta^2)^{1/2}] + [C_2 - (C_2^2 - \delta^2)^{1/2}]\tan^2\Phi\right\}} = C_1 d\zeta,$$

where $\Phi = \epsilon(\phi - \phi_0)$, and this equation integrates to give

$$\left(\frac{C_2 - (C_2^2 - \delta^2)^{1/2}}{C_2 + (C_2^2 - \delta^2)^{1/2}}\right)^{1/2} \tan[\epsilon(\phi - \phi_0)] = \tan[C_1\delta(\zeta - \zeta_0)], \tag{10.40}$$

where ζ_0 denotes a further arbitrary constant. This equation may be rearranged to provide an expression for $\phi(\zeta)$, while an expression for $\psi(\phi)$ can be obtained from (10.39).

From the above expressions, it is clear that there are two critical combinations of the constants, which we denote by δ_1 and δ_2, thus

$$\delta_1 = C_1\delta = \frac{e_0}{c\hbar(1 - \lambda^2)^{1/2}}, \qquad \delta_2 = \frac{\delta}{C_2} = \frac{e_0}{C_1 C_2 c\hbar(1 - \lambda^2)^{1/2}},$$

and in terms of these constants, we have from (10.40)

$$\phi(\zeta) = \phi_0 + \hbar\tan^{-1}\left\{\left(\frac{1 + (1 - \delta_2^2)^{1/2}}{1 - (1 - \delta_2^2)^{1/2}}\right)^{1/2} \tan[\delta_1(\zeta - \zeta_0)]\right\}, \tag{10.41}$$

while from (10.39), we may deduce

$$\frac{1}{\psi^2} = \hbar C_2 \left\{1 + \left(1 - \delta_2^2\right)^{1/2} \cos 2\Phi\right\} = \hbar C_2 \frac{\left\{(1 + \left(1 - \delta_2^2\right)^{1/2}) + (1 - \left(1 - \delta_2^2\right)^{1/2}) \tan^2 \Phi\right\}}{(1 + \tan^2 \Phi)},$$

where $\Phi = \epsilon(\phi - \phi_0)$. On using the following expression for $\tan \Phi$ which is obtained from (10.41), namely

$$\tan \Phi = \left(\frac{1 + (1 - \delta_2^2)^{1/2}}{1 - (1 - \delta_2^2)^{1/2}}\right)^{1/2} \tan[\delta_1(\zeta - \zeta_0)],$$

we obtain

$$\frac{1}{\psi^2} = \hbar C_2 \delta_2^2 \frac{(1 + \tan^2[\delta_1(\zeta - \zeta_0)])}{\left\{(1 - \left(1 - \delta_2^2\right)^{1/2}) + (1 + \left(1 - \delta_2^2\right)^{1/2}) \tan^2[\delta_1(\zeta - \zeta_0)]\right\}},$$

and therefore, $\psi(\zeta)$ is given by

$$\psi(\zeta) = \frac{\cos[\delta_1(\zeta - \zeta_0)]}{(\hbar C_2)^{1/2} \delta_2} \left\{(1 - \left(1 - \delta_2^2\right)^{1/2}) + (1 + \left(1 - \delta_2^2\right)^{1/2}) \tan^2[\delta_1(\zeta - \zeta_0)]\right\}^{1/2}.$$

On reconstituting $\Psi(x,t) = a(x,t)\exp(i\eta(x,t)/\hbar) = \psi(\zeta)\exp(i\phi(\zeta)/\hbar)$, we obtain

$$\begin{aligned}
\Psi(\zeta) &= \frac{e^{i\phi_0/\hbar} \cos[\delta_1(\zeta - \zeta_0)]}{(\hbar C_2)^{1/2} \delta_2} \left\{(1 - \left(1 - \delta_2^2\right)^{1/2}) + (1 + \left(1 - \delta_2^2\right)^{1/2}) \tan^2[\delta_1(\zeta - \zeta_0)]\right\}^{1/2} \\
&\quad \times \exp i \tan^{-1} \left\{\left(\frac{1 + (1 - \delta_2^2)^{1/2}}{1 - (1 - \delta_2^2)^{1/2}}\right)^{1/2} \tan[\delta_1(\zeta - \zeta_0)]\right\}, \\
&= \frac{e^{i\phi_0/\hbar} \cos[\delta_1(\zeta - \zeta_0)]}{(\hbar C_2)^{1/2} \delta_2} \left\{(1 - \left(1 - \delta_2^2\right)^{1/2}) + (1 + \left(1 - \delta_2^2\right)^{1/2}) \tan^2[\delta_1(\zeta - \zeta_0)]\right\}^{1/2} \\
&\quad \times \left\{\frac{(1 - \left(1 - \delta_2^2\right)^{1/2})^{1/2} + i(1 + \left(1 - \delta_2^2\right)^{1/2})^{1/2} \tan[\delta_1(\zeta - \zeta_0)]}{\left\{(1 - \left(1 - \delta_2^2\right)^{1/2}) + (1 + \left(1 - \delta_2^2\right)^{1/2}) \tan^2[\delta_1(\zeta - \zeta_0)]\right\}^{1/2}}\right\}, \\
&= \frac{e^{i\phi_0/\hbar} \cos[\delta_1(\zeta - \zeta_0)]}{(\hbar C_2)^{1/2} \delta_2} \left\{(1 - \left(1 - \delta_2^2\right)^{1/2})^{1/2} + i(1 + \left(1 - \delta_2^2\right)^{1/2})^{1/2} \tan[\delta_1(\zeta - \zeta_0)]\right\}, \\
&= \frac{e^{i\phi_0/\hbar}}{(\hbar C_2)^{1/2} \delta_2} \left\{(1 - \left(1 - \delta_2^2\right)^{1/2})^{1/2} \cos[\delta_1(\zeta - \zeta_0)] + i(1 + \left(1 - \delta_2^2\right)^{1/2})^{1/2} \sin[\delta_1(\zeta - \zeta_0)]\right\},
\end{aligned}$$

which as might be expected comprises simply two linearly independent solutions of $\Psi''(\zeta) + \delta_1^2 \Psi(\zeta) = 0$, as indeed it must. However, it is important to appreciate that the amplitude $(a(x,t))$ and phase $(\eta(x,t)/\hbar)$ enjoy a certain structure, which in the absence of the nonessential constants (ζ_0 and ϕ_0) simplify somewhat to become

$$a(\zeta) = \psi(\zeta) = \frac{1}{(\hbar C_2)^{1/2}\delta_2}\left\{(1-\left(1-\delta_2^2\right)^{1/2})\cos^2(\delta_1\zeta) + (1+\left(1-\delta_2^2\right)^{1/2})\sin^2(\delta_1\zeta)\right\}^{1/2},$$

$$\frac{\eta(\zeta)}{\hbar} = \frac{\phi(\zeta)}{\hbar} = \tan^{-1}\left\{\left(\frac{1+(1-\delta_2^2)^{1/2}}{1-(1-\delta_2^2)^{1/2}}\right)^{1/2}\tan(\delta_1\zeta)\right\},$$

where $\zeta = \lambda x + ct$, and for a prescribed value of λ, the constant $\delta_1 = e_0/c\hbar(1-\lambda^2)^{1/2}$ is assumed to be known, while the constants δ_2 and C_2 are essentially completely arbitrary, provided only that the above expressions are meaningful. If we introduce the angle χ such that $\delta_2 = \sin\chi$, then the amplitude and phase are given, respectively, by

$$a(\zeta) = \frac{1}{(2\hbar C_2)^{1/2}}\left\{\left(\frac{\sin(\delta_1\zeta)}{\sin(\chi/2)}\right)^2 + \left(\frac{\cos(\delta_1\zeta)}{\cos(\chi/2)}\right)^2\right\}^{1/2}, \quad \frac{\eta(\zeta)}{\hbar} = \tan^{-1}\left\{\frac{\tan(\delta_1\zeta)}{\tan(\chi/2)}\right\}.$$

We also observe that $\lambda = \pm 1$ constitutes critical values of λ and that a transition from $\lambda^2 < 1$ to $\lambda^2 > 1$ and vice versa might well be achievable, if in the limit $\lambda \longrightarrow 1$ occurs in such a manner that the constant δ_1 always remains well-defined.

10.9 Time-Dependent Dirac Equation for Free Particle

Finally in this chapter and for the sake of completeness, some details for the time-dependent Dirac equation for a free particle are summarised in this section. The Klein–Gordon equation is second order in both space and time coordinates, and it is a fundamental equation of relativistic quantum mechanics. Dirac's Lorentz invariant relativistic equation, which is first order in both space and time coordinates, is purposely constructed so that the probability density remains non-negative, which is not a feature of the Klein–Gordon equation (see, e.g. [11], page 317). The Dirac equation itself for three spatial dimensions (x, y, z) becomes

$$i\hbar\frac{\partial \boldsymbol{\Psi}}{\partial t} + ic\hbar(\mathbf{A}.\nabla)\Psi = e_0\mathbf{B}\Psi,$$

where $\boldsymbol{\Psi}$ is given by

$$\boldsymbol{\Psi} = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix},$$

where the four matrices $\mathbf{A_x}$, $\mathbf{A_y}$, $\mathbf{A_z}$ and $\mathbf{B}$ are as defined in the following subsection. For three spatial dimensions, each component of $\boldsymbol{\Psi}$, namely $\psi_j = \psi_j(x, y, z, t)$ for

$j = 1, 2, 3, 4$, may be shown to satisfy the three spatial-dimensional Klein–Gordon equation (10.45).

Again for the sake of completeness, we present the following details for the time-dependent Dirac equation for a free particle and confirm that each component of Ψ satisfies the Klein–Gordon equation. The matrices $\mathbf{A_x}$, $\mathbf{A_y}$, $\mathbf{A_z}$ and $\mathbf{B}$ appearing in the time-dependent Dirac equation for a free particle are given, respectively, by

$$\mathbf{A_x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A_y} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A_z} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and the Dirac equation becomes

$$\begin{pmatrix} \frac{\partial}{\partial t} & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial t} & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial t} & 0 \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} -if_0\psi_1 \\ -if_0\psi_2 \\ if_0\psi_3 \\ if_0\psi_4 \end{pmatrix},$$

where $f_0 = e_0/\hbar$. With a subscript notation for partial derivatives, we have in component form

$$\begin{aligned} \psi_{1t} + c\psi_{3z} + c\psi_{4x} - ic\psi_{4y} &= -if_0\psi_1, \\ \psi_{2t} + c\psi_{3x} + ic\psi_{3y} - c\psi_{4z} &= -if_0\psi_2, \\ \psi_{3t} + c\psi_{1z} + c\psi_{2x} - ic\psi_{2y} &= if_0\psi_3, \\ \psi_{4t} + c\psi_{1x} + ic\psi_{1y} - c\psi_{2z} &= if_0\psi_4. \end{aligned} \tag{10.42}$$

By an examination of the particular case of one spatial dimension, so that $\psi_j = \psi_j(x, t)$ for $j = 1, 2, 3, 4$, it is not difficult to show the first equation couples with the last, while the second and third are coupled together, thus

$$\psi_{1t} + c\psi_{4x} = -if_0\psi_1, \qquad \psi_{4t} + c\psi_{1x} = if_0\psi_4,$$
$$\psi_{2t} + c\psi_{3x} = -if_0\psi_2, \qquad \psi_{3t} + c\psi_{2x} = if_0\psi_3,$$

and from which we may show that each component $\psi_j = \psi_j(x, t)$ for $j = 1, 2, 3, 4$ satisfies the one spatial-dimensional Klein–Gordon equation

$$\frac{\partial^2 \psi_j}{\partial t^2} - c^2 \frac{\partial^2 \psi_j}{\partial x^2} = -\left(\frac{e_0}{\hbar}\right)^2 \psi_j.$$

For three spatial dimensions with $\psi_j = \psi_j(x, y, z, t)$ for $j = 1, 2, 3, 4$, we may exploit the above coupling and write the first and last equations of (10.42)

$$\begin{aligned} \psi_{1t} + if_0\psi_1 + c\psi_{4x} - ic\psi_{4y} &= -c\psi_{3z}, \\ \psi_{4t} - if_0\psi_4 + c\psi_{1x} + ic\psi_{1y} &= c\psi_{2z} \end{aligned} \tag{10.43}$$

and the second and third as

$$\begin{aligned} \psi_{2t} + if_0\psi_2 + c\psi_{3x} + ic\psi_{3y} &= c\psi_{4z}, \\ \psi_{3t} - if_0\psi_3 + c\psi_{2x} - ic\psi_{2y} &= -c\psi_{1z}. \end{aligned} \tag{10.44}$$

On repeatedly using the expressions for two of the derivatives, say ψ_{2z} and ψ_{3z} as obtained from (10.43), and substituting into the differentiated with respect to z versions of (10.44), we may, after changing the order of the mixed partial derivatives, deduce that both ψ_2 and ψ_3 satisfy the Klein–Gordon equation for three spatial dimensions, thus

$$\frac{\partial^2 \psi_j}{\partial t^2} - c^2 \nabla^2 \psi_j = -\left(\frac{e_0}{\hbar}\right)^2 \psi_j. \tag{10.45}$$

for $j = 2, 3$. Similarly, by repeating this process with the two expressions obtained from (10.44) for ψ_{1z} and ψ_{4z} and substitution into the derived versions of (10.43), we may conclude that indeed both ψ_1 and ψ_4 also satisfy the three spatial-dimensional Klein–Gordon equation (10.45).

Chapter 11
Coordinate Transformations, Tensors and General Relativity

In this chapter, we provide some basic information on coordinate transformations, tensors, partial covariant differentiation, Christoffel symbols and Ricci and Einstein tensors, leading to general relativity. As previously stated, this is necessarily a limited progressive introduction, termed progressive, in the sense that the reader is invited to read on to later sections if more information and further detail are required. Einstein's general theory of relativity was published over a century ago, and up to this point in time provides the best description of a gravitation field, and is capable of describing a myriad of interesting phenomena in the universe, such as the bending of light through gravitational lensing, the slowing of clocks in gravitational fields and the recently detected ripples in space time due to cataclysmic astrophysical events, such as the coalescence of dense stellar objects. The next generation of GPS systems will require a detailed mapping of the earth's gravitational field combined with the development of accurate predictive mathematical models. Accordingly for such applications, a superficial understanding may not be sufficient for future space scientists, but rather some prior experience of the actual "gory" details of the discipline may be required.

The first two sections of the chapter deal with an introduction to Cartesian tensors and an alternative derivation of the basic identity (3.7), previously derived in Chap. 3. The two sections thereafter deal with general curvilinear coordinates and the important notion of partial covariant differentiation, including briefly mentioning the fundamental tensors of general relativity, which are the Riemann-Christoffel tensor, the covariant curvature tensor, the Ricci tensor, the curvature invariant and the Einstein tensor. Bianchi's identity is also briefly mentioned, and the subsequent section provides an illustrative example involving a single spatial Cartesian dimension.

Throughout we follow the approach to general relativity adopted in the excellent recent text of [15] and the older texts of [78, 100, 102], and we follow the development of tensor analysis as pursued by [97, 101] or the Appendix of [31]. While there exist a number of exact space-times for the Einstein field equations

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of general relativity, such as those listed in [42, 81, 98] and to a lesser extent in [84, 91, 107], in the search for new exact space-time solutions of general relativity, the major prohibiting factors include the shear complexity of the underlying equations and the lack of any strategic perspective to guide the analysis. In Sect. 11.6, we deduce some general formulae for a line element involving seven arbitrary metric tensor components, and each component is assumed to depend only on two spatial variables and the temporal variable. While the assumed line element is not completely general, it nevertheless includes many known exact space-time solutions of general relativity. Some general formulae are presented for the Ricci and Einstein tensors expressed in terms of six components of the covariant curvature tensor, and two well-known cosmological models are presented as illustrative examples of the formulation in the section thereafter. In the final section of the chapter, we examine a possible cosmological model involving logarithmic spirals and seven arbitrary constants. While spiral structures frequently occur in the universe, at present there appears to be no known formal solutions of the general relativistic field equations reflecting such structures. A particular case of the spiral model gives rise to an Einstein tensor, which is reminiscent of the tensor involving the cosmological constant.

11.1 Summation Convention and Cartesian Tensors

In this section, we present the background material required to give an alternative derivation of the identity (3.7) for the spatial physical force **f** using a suffix or index notation, for which we represent the Cartesian coordinates (x, y, z) as (x^1, x^2, x^3), and accordingly, we would write (x, y, z) as simply x^j for $j = 1, 2, 3$. In order to achieve this, there are four important ideas and results that we need to introduce. These ideas provide an elementary introduction to the general topic of tensor analysis, which deals with the transformations of curvilinear coordinates for which the issue regarding the level of the index, namely either upper of lower, is critical, and this is the case in the subsequent sects. 11.3 and 11.4. In this and the following section, however, we deal only with Cartesian coordinates for which the level of the index is not relevant. Also, we comment that we could if necessary deal with a four-dimensional space time with coordinates (ct, x, y, z) as (x^0, x^1, x^2, x^3) or x^j for $j = 1, 2, 3, 4$, but since the identity (3.7) involves only the three spatial dimensions (x, y, z), we restrict attention to x^j for $j = 1, 2, 3$.

Einstein Summation Convention We first need to mention the Einstein summation convention, which is an important accepted convention that any repeated index implies that a summation is taken over that index. Thus, for the two vectors $\mathbf{u} = (u_x, u_y, u_z) = (u^1, u^2, u^3)$ and $\mathbf{p} = (p_x, p_y, p_z) = (p^1, p^2, p^3)$, their scalar product $\mathbf{u}.\mathbf{p}$ would be written as

$$\mathbf{u} \cdot \mathbf{p} = u_x p_x + u_y p_y + u_z p_z = u^1 p^1 + u^2 p^2 + u^3 p^3 = u^j p^j,$$

and the repeated j index implies that a summation must be made over $j = 1, 2, 3$.

Kronecker Delta δ^{ij} The second idea that is needed is the Kronecker delta symbol δ^{ij}, which is defined to have a unit value if $i = j$ and the value zero otherwise, thus

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

so as an example, combining the two notions of the Einstein summation convention and the Kronecker delta symbol δ^{ij}, we might write the above scalar vector product $\mathbf{u} \cdot \mathbf{p}$ as follows:

$$\mathbf{u} \cdot \mathbf{p} = \delta^{ij} u^i p^j = u^j p^j = u^1 p^1 + u^2 p^2 + u^3 p^3.$$

We would observe that repeated indices occur over both i and j, so that in reality the first equality reveals a summation of precisely nine terms, thus

$$\delta^{ij} u^i p^j = \delta^{11} u^1 p^1 + \delta^{12} u^1 p^2 + \delta^{13} u^1 p^3 + \delta^{21} u^2 p^1 + \delta^{22} u^2 p^2 + \delta^{23} u^2 p^3 \\ + \delta^{31} u^3 p^1 + \delta^{32} u^3 p^2 + \delta^{33} u^3 p^3,$$

but of course, only the three terms arising from δ^{11}, δ^{22} and δ^{33} provide a non-zero contribution to the summation.

Levi-Civita Symbol Denoted by ε^{ijk} The third notion that is required is the mathematical symbol, often referred to as the Levi-Civita symbol denoted by ε^{ijk}, which represents numbers arising from the sign of a permutation of the natural numbers 1, 2 and 3. They are also referred to as the permutation symbols, antisymmetric symbols or alternating symbols, referring to their antisymmetric property and their definition in terms of permutations. The Levi-Civita symbols ε^{ijk} are defined to be unity if ijk is a cyclic permutation of 1, 2 and 3, minus one if ijk is an anticyclic permutation of 1, 2 and 3 and zero otherwise, thus

$$\varepsilon^{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is cyclic permutation of } 123 \\ -1 & \text{if } ijk \text{ is anticyclic permutation of } 123 \\ 0 & \text{if } ijk \text{ is otherwise} \end{cases} \tag{11.1}$$

Vector Product $\mathbf{u} \wedge \mathbf{p}$ Armed with this symbol, we are now able to express a number of important vector relations and vector differential relations and, principally, those vector relations involving either the vector product or the differential operator, known as the curl of a vector. For example, the three components of the vector product $\mathbf{u} \wedge \mathbf{p}$, which has the formal determinant definition

$$\mathbf{u} \wedge \mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ p_x & p_y & p_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u^1 & u^2 & u^3 \\ p^1 & p^2 & p^3 \end{vmatrix},$$

are given by $(\mathbf{u} \wedge \mathbf{p})^i = \varepsilon^{ijk} u^j p^k$, where there are implied summations over both j and k, and the specific ordering of ijk is critical, where as usual $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ denote the unit vectors in the three Cartesian directions. Thus, for example

$$(\mathbf{u} \wedge \mathbf{p})^1 = \varepsilon^{1jk} u^j p^k = \varepsilon^{123} u^2 p^3 + \varepsilon^{132} u^3 p^2 = u^2 p^3 - u^3 p^2 = u_y p_z - u_z p_y,$$

since $\varepsilon^{123} = 1$ and $\varepsilon^{132} = -1$ are the only non-zero values of ε^{ijk} involving 1 in the indices.

Further, we may use these symbols to write the i^{th} component of the curl differential operator $\nabla \wedge \mathbf{p}$, which has the formal determinant definition

$$\nabla \wedge \mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p_x & p_y & p_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ p^1 & p^2 & p^3, \end{vmatrix},$$

thus

$$(\nabla \wedge \mathbf{p})^i = \varepsilon^{ijk} \frac{\partial p^k}{\partial x^j},$$

again noting the summations over both repeated indices j and k and their specific placement in the above expression.

Fundamental Identity Involving δ^{ij} and ε^{ijk} The fourth and final result that we need is the following fundamental identity involving both the Kronecker delta symbol δ^{ij} and the Levi-Civita symbol ε^{ijk}, thus

$$\varepsilon^{ijk} \varepsilon^{imn} = \delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}, \tag{11.2}$$

for which we observe the implied summation over the repeated index i and the specific association of the indices in the positive and negative contributions. This is the fundamental identity from which the vast majority of all other vector identities involving either the vector product or the curl differential operator can be quickly established.

11.2 Alternative Derivation of Basic Identity

We are now well placed to establish the identity (3.7) for the spatial physical force **f**, and we start with the i^{th} component of the triple vector differential product $\mathbf{u} \wedge (\nabla \wedge \mathbf{p})$, thus

$$\begin{aligned}\varepsilon^{ijk} u^j (\nabla \wedge \mathbf{p})^k &= \varepsilon^{ijk} \varepsilon^{kmn} u^j \frac{\partial p^n}{\partial x^m} \\ &= \varepsilon^{kij} \varepsilon^{kmn} u^j \frac{\partial p^n}{\partial x^m} = (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}) u^j \frac{\partial p^n}{\partial x^m},\end{aligned}$$

on using the fundamental identity (11.2); the fact that the indices of ε^{kij} are a cyclic permutation of those of ε^{ijk}, therefore the two symbols have the same value. On applying the values of the Kronecker delta symbols, we finally have that the i^{th} component of the triple vector differential product $\mathbf{u} \wedge (\nabla \wedge \mathbf{p})$ is given by

$$(\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = \varepsilon^{ijk} u^j (\nabla \wedge \mathbf{p})^k = u^n \frac{\partial p^n}{\partial x^i} - u^m \frac{\partial p^i}{\partial x^m}. \tag{11.3}$$

In this equation, we recognise the term $u^n \partial p^n / \partial x^i$ as arising from the rate-of-working equation $de = \mathbf{dx} \cdot d\mathbf{p}/dt = \mathbf{u} \cdot \mathbf{dp}$, so that

$$\frac{\partial e}{\partial x^i} = u^n \frac{\partial p^n}{\partial x^i} = u^1 \frac{\partial p^1}{\partial x^i} + u^2 \frac{\partial p^2}{\partial x^i} + u^3 \frac{\partial p^3}{\partial x^i},$$

while the term $u^m \partial p^i / \partial x^m$ we recognise as that arising in the total or material derivative defined by (3.6), namely $(\mathbf{u} \cdot \nabla) p^i$. Thus, on rearrangement of (11.3), we obtain

$$u^n \frac{\partial p^n}{\partial x^i} = (\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i + (\mathbf{u} \cdot \nabla) p^i,$$

so that finally from (3.4), we may deduce

$$f^i = \frac{\partial p^i}{\partial t} + \frac{\partial e}{\partial x^i} = \frac{\partial p^i}{\partial t} + (\mathbf{u} \cdot \nabla) p^i + (\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i,$$

and the required result follows, thus

$$f^i = \frac{dp^i}{dt} + (\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i,$$

where the total or material derivative is defined by Eq. (3.6).

11.3 General Curvilinear Coordinates

In this section, we present the main details for the extended force formulation given by (3.4) but in terms of a general spatial curvilinear coordinate system x^j for $j = 1, 2, 3$, where now the index level shown here as upper is critical. We follow the development of tensor analysis as pursued in the standard text [97], and the reader might be referred to any of [15, 97, 101] or the Appendix of [31] for much of the necessary background detail. In this section, we utilise the notion of the partial covariant derivative without giving the full details. For the interested reader, these details are presented in a subsequent section along with the notion of Christoffel symbols. For a general curvilinear coordinate system (x^1, x^2, x^3), we assume a symmetric metric tensor with components g_{ij} for $i, j = 1, 2, 3$. The metric tensor g_{ij} may be expressed in terms of rectangular Cartesian coordinates $(z^1, z^2, z^3) = (x, y, z)$ and is defined by the following equation for the three-dimensional line element, thus

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = dz^k dz^k = g_{ij} dx^i dx^j, \tag{11.4}$$

so that formally the metric tensor is given by

$$g_{ij} = \frac{\partial z^k}{\partial x^i} \frac{\partial z^k}{\partial x^j},$$

noting the implied summation over $k = 1, 2, 3$, the apparent symmetry in i and j, and that both sides of this equation are lower in the indices i and j, and we say that g_{ij} is a covariant tensor of rank 2. Further, we remind the reader that the index level for Cartesian vectors and tensors is not relevant. We also remind the reader that we would refer to the general curvilinear coordinate system x^j as either a contravariant vector or as a contravariant tensor of rank 1, and we observe from Eq. (11.4) that, in such a coordinate system, the Einstein summation convention only applies over one raised index and one lower index and that it has no meaning if both indices are on the same level. Formally, the proper tensorial version of the Kronecker delta symbol δ^i_j would be defined, thus

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{11.5}$$

and we would refer to δ^i_j as a mixed tensor of rank 2, contravaraint in i and covariant in j.

Associated with the symmetric metric tensor with components g_{ij} is the conjugate symmetric metric tensor with components g^{ij} for $i, j = 1, 2, 3$, which is such that

$$g_{ij}g^{jk} = \delta_i^k, \tag{11.6}$$

so that viewing g_{ij} as a matrix, the conjugate metric tensor with components g^{ij} is simply the formal matrix, which is the inverse to g_{ij}. We would refer to the metric tensor g_{ij} as a symmetric covariant tensor of rank 2, while the conjugate metric tensor g^{ij} would be referred to as a symmetric contravariant tensor of rank 2. We further note that in this section, we use the symbol g to denote the formal determinant of the matrix, which has components g_{ij}, thus $g = |g_{ij}|$, and there should be no confusion with the same symbol g, which is used for the force in the direction of time.

The metric tensor and its conjugate are fundamental in terms of raising and lowering indices, thus for example, we have

$$A_{ij} = g_{ik}g_{jm}A^{km}, \qquad A^i_j = g^{ik}A_{kj}, \qquad A^{ij} = g^{ik}g^{jm}A_{km},$$

most importantly noting that in all cases the indices i and j appear on the same level on each side of the equations and that summation only occurs over two indices on different levels. In this manner, with the Kronecker delta δ^i_j defined by (11.5), we might give meaning to the covariant and contravaraint tensors of rank 2, namely δ_{ij} and δ^{ij}, respectively, thus

$$\delta_{ij} = g_{ik}\delta^k_j = g_{ij}, \qquad \delta^{ij} = g^{ik}\delta^j_k = g^{ij},$$

so that δ_{ij} is simply the covariant metric tensor g_{ij}, while δ^{ij} is the contravariant conjugate metric tensor g^{ij}.

Spherical Polar Coordinates (r, θ, ϕ) As a simple example, for the spherical polar coordinates (r, θ, ϕ) defined by the relations

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

from Eq. (11.4), we might deduce

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

so that the metric tensor g_{ij} and its conjugate g^{ij} are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\sin\theta)^2 \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (r\sin\theta)^{-2} \end{pmatrix}.$$

With these preliminary observations in tensor calculus, we may proceed to present the main details for the extended force formulation given by (3.4) in a general spatial curvilinear coordinate system (x^1, x^2, x^3). The velocity vector $\mathbf{u}$ has components $u^i = dx^i/dt$ with magnitude u arising from (11.4), thus

$$u^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = g_{ij}u^i u^j,$$

while the momentum vector **p** has components $p^i = mu^i$ and magnitude p, where $p^2 = g_{ij}p^i p^j$, and as usual the mass m is defined in terms of the magnitude of the velocity vector by $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, where m_0 denotes the rest mass.

On taking the total derivative of the usual energy equation in tensorial form

$$e^2 = e_0^2 + (pc)^2 = e_0^2 + c^2 g_{ij}p^i p^j,$$

where p denotes the magnitude of the momentum vector given explicitly by $p^2 = g_{ij}p^i p^j$, we might deduce the rate-of-working equation $ede = c^2 g_{ij}p^i dp^j$, which using $e = mc^2$ simplifies to give $de = g_{ij}u^i dp^j = g_{ij}dp^i u^j$ on using the symmetry of the metric tensor g_{ij}. Further, in tensor calculus, the notion of partial differentiation is generalised to become partial covariant differentiation, which is designated by a semicolon, thus ; and since here the energy e is a scalar quantity (namely a tensor of rank 0), the partial derivatives of e coincide with the partial covariant derivative, so that from $de = g_{ij}u^i dp^j$ we might deduce the important relation

$$\frac{\partial e}{\partial x^k} = e_{;k} = g_{ij}u^i p^j_{;k}, \tag{11.7}$$

where $p^j_{;k}$ refers to the partial covariant derivative with respect to x^k of the contravariant vector p^j. For further information relating to the partial covariant derivatives, we refer the reader to the following section or again to either [97] or the Appendix of [31]. Here, it suffices to mention that partial covariant derivatives satisfy the usual rules of differentiation, such as the product rule, and they possess the very curious properties that all partial covariant derivatives of both the metric tensor and its conjugate vanish, thus $g_{ij;k} = 0$ and $g^{ij}_{;k} = 0$, results which are used frequently below. Maybe it is worth noting that, for example, these two partial covariant derivatives with respect to x^k are formally quite distinct. The first is the partial covariant derivative of a covariant tensor of rank 2, while the second is the partial covariant derivative of a contravariant tensor of rank 2, which are different.

For $i = 1, 2, 3$, the tensorial version of (3.4) in a general curvilinear coordinate system (x^1, x^2, x^3) becomes

$$f^i = \frac{\partial p^i}{\partial t} + g^{ik}e_{;k} = \frac{\partial p^i}{\partial t} + g^{ik}\frac{\partial e}{\partial x^k}, \tag{11.8}$$

$$g^0 = \frac{1}{c^2}\frac{\partial e}{\partial t} + p^k_{;k} = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}p^k)}{\partial x^k},$$

where $p^k_{;k}$ is the divergence of the contravariant vector p^k, and in the last part of Eq. (11.8), we have used a standard formula of tensor calculus for the divergence of a vector $\mathbf{p}$ that is verified in the following section and given on page 32 of Spain [97], namely

$$div\mathbf{p} = p^k_{;k} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}p^k)}{\partial x^k}, \tag{11.9}$$

where we again remind the reader that g denotes the determinant of the metric tensor g_{ij}, thus $g = |g_{ij}|$, and that in this present section, we have momentarily added a zero to the force g in the direction of time, thus g^0, to ensure that there is no confusion when the two quantities are used in the same equation.

Work Done Or Energy Function $W(x^i, t)$ On assuming the existence of a work done or energy function $W(x^i, t)$ which is defined by

$$\begin{aligned} dW &= \mathbf{f}\cdot dx + gc^2 dt \\ &= \left(\frac{\partial p^i}{\partial t} + g^{ik}\frac{\partial e}{\partial x^k}\right) g_{im} dx^m + \left(\frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}p^k)}{\partial x^k}\right) dt, \end{aligned}$$

and using $g_{ij}g^{jk} = \delta_i^k$ and $\mathscr{E} = W - e$, we might deduce

$$d\mathscr{E} = g_{im}\frac{\partial p^i}{\partial t}dx^m + \frac{c^2}{\sqrt{g}}\frac{\partial(\sqrt{g}p^k)}{\partial x^k}dt,$$

for which the following partial differential relations evidently apply

$$\frac{\partial \mathscr{E}}{\partial x^m} = g_{im}\frac{\partial p^i}{\partial t}, \qquad \frac{\partial \mathscr{E}}{\partial t} = \frac{c^2}{\sqrt{g}}\frac{\partial(\sqrt{g}p^k)}{\partial x^k}. \tag{11.10}$$

Basic Identity (3.7) In order to provide a formal tensorial proof of the identity (3.7), we have from both (11.7) and $(11.8)_1$

$$f^i = \frac{\partial p^i}{\partial t} + g^{ik}g_{mj}u^m p^j_{;k} = \frac{dp^i}{dt} + g^{ik}g_{mj}u^m p^j_{;k} - u^m p^i_{;m}, \tag{11.11}$$

where d/dt denotes the total or material time derivative that is defined by

$$\frac{dp^i}{dt} = \frac{\partial p^i}{\partial t} + u^m p^i_{;m}.$$

Now on using (11.6) in the form $\delta^k_m = g^{kj}g_{mj}$, we have from (11.11)

$$f^i = \frac{dp^i}{dt} + g^{ik} g_{mj} u^m p^j_{;k} - u^m p^i_{;k} \delta^k_m \tag{11.12}$$
$$= \frac{dp^i}{dt} + g_{mj} u^m (g^{ik} p^j_{;k} - g^{jk} p^i_{;k}),$$

and the tensorial versions of the curl operator $(\nabla \wedge \mathbf{p})$ and the vector product $\mathbf{u} \wedge (\nabla \wedge \mathbf{p})$ are given by

$$(\nabla \wedge \mathbf{p})_i = \epsilon^*_{ijn} g^{jk} p^n_{;k}, \qquad ((\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = \epsilon^{*ijk} g_{jm} u^m (\nabla \wedge \mathbf{p})_k,$$

where the tensorial permutation symbols ϵ^*_{ijk} and ϵ^{*ijk} are defined in terms of the Cartesian permutation symbols ϵ^{ijk} (see Eq. (11.1)) by the following formulae that may be found in Eringen [31] (pages 441 and 442), thus

$$\epsilon^*_{ijk} = \sqrt{g}\epsilon^{ijk}, \qquad \epsilon^{*ijk} = \frac{\epsilon^{ijk}}{\sqrt{g}}.$$

The identity (3.7) now follows on using both

$$((\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = \epsilon^{*ijk} g_{jm} u^m \epsilon^*_{knr} g^{ns} p^r_{;s} = \epsilon^{*kij} \epsilon^*_{knr} g_{jm} u^m g^{ns} p^r_{;s},$$

and the tensorial version of the identity (11.2) becomes

$$\epsilon^{*kij} \epsilon^*_{knr} = \delta^i_n \delta^j_r - \delta^i_r \delta^j_n,$$

so that together we obtain

$$((\mathbf{u} \wedge (\nabla \wedge \mathbf{p}))^i = (\delta^i_n \delta^j_r - \delta^i_r \delta^j_n) g_{jm} u^m g^{ns} p^r_{;s} = g_{jm} u^m (g^{is} p^j_{;s} - g^{js} p^i_{;s}),$$

which coincides precisely with the term in $(11.12)_2$, and therefore, we have provided a formal tensorial proof of (3.7).

Now on using $g_{ij} g^{jk} = \delta^k_i$, Eq. $(11.10)_1$ may be rewritten as

$$\frac{\partial p^i}{\partial t} = g^{ik} \frac{\partial \mathscr{E}}{\partial x^k} = g^{ik} \mathscr{E}_{;k} = (g^{ik} \mathscr{E})_{;k},$$

and together the two relations (11.10) become

$$\frac{\partial p^i}{\partial t} = (g^{ik} \mathscr{E})_{;k}, \qquad \frac{\partial \mathscr{E}}{\partial t} = c^2 p^k_{;k}. \tag{11.13}$$

From these relations, we may deduce that there exists a scalar function $\phi(x^j, t)$ such that

$$p^i = g^{ik}\frac{\partial \phi}{\partial x^k} = g^{ik}\phi_{;k} = (g^{ik}\phi)_{;k}, \qquad \mathscr{E} = \frac{\partial \phi}{\partial t},$$

and that $\phi(x^j, t)$ satisfies the equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 p^k_{;k} = c^2 g^{ik}\phi_{;ik} = c^2 \nabla^2 \phi,$$

which is the wave equation, where $\nabla^2\phi = g^{ik}\phi_{;ik}$ is the tensorial version of the Laplacian operator for which further details may be found in the following section or in [97] (page 32). The existence of the scalar function $\phi(x^j, t)$ ensures the vanishing of the curl of the vector **p**, since from

$$(\nabla \wedge \mathbf{p})_i = \epsilon^*_{ijk} g^{jm} p^k_{;m} = \epsilon^*_{ijk} g^{jm}(g^{kn}\phi)_{;mn} = \epsilon^*_{ijk} g^{jm} g^{kn}\phi_{;mn},$$

we may, for example, deduce that in particular for $i = 1$, we have

$$(\nabla \wedge \mathbf{p})_1 = \sqrt{g}(\epsilon^{123} g^{2m} g^{3n}\phi_{;mn} + \epsilon^{132} g^{3m} g^{2n}\phi_{;mn}),$$

which evidently vanishes, since $\epsilon^{123} = 1$ and $\epsilon^{132} = -1$, the conjugate metric tensor g^{ij} is symmetric and assuming the partial derivatives coincide $\phi_{;mn} = \phi_{;nm}$.

Wave Energy $\mathscr{E}(x^j, t)$ Satisfies Wave Equation From Eq. (11.13), we observe that the wave energy $\mathscr{E}(x^j, t)$ satisfies the wave equation

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \left(\frac{\partial p^k}{\partial t}\right)_{;k} = c^2 \left((g^{km}\mathscr{E})_{;m}\right)_{;k} = c^2 g^{km}\mathscr{E}_{;km} = c^2 \nabla^2 \mathscr{E}, \tag{11.14}$$

while the momentum vector $p^i(x^j, t)$ satisfies

$$\frac{\partial^2 p^i}{\partial t^2} = c^2 \left(g^{ik} p^m_{;m}\right)_{;k} = c^2 g^{ik} \left(p^m_{;m}\right)_{;k}. \tag{11.15}$$

We note that while the double covariant partial differentiations appear to be the same in both cases, there is a distinction arising from the fact that in the former case we are undertaking the double partial covariant derivative of the scalar quantity $\mathscr{E}(x^j, t)$, while in the second case we are undertaking the double partial covariant derivative of the momentum vector $p^m(x^j, t)$, the first leading to the divergence of the momentum vector and the second simply the partial derivative of a scalar. In the former case, the first covariant derivative is simply the partial derivative $\partial \mathscr{E}/\partial x^k$, while the second differentiation involves the partial covariant derivative of a covariant vector. This is in contrast to the two differentiations of the momentum vector $p^m(x^j, t)$, for which the first $p^m_{;m}$ is the partial covariant derivative of a contravariant vector, namely the divergence of the vector **p**. Now the divergence of $\mathbf{p}$ is a scalar, and the second partial covariant derivative with respect x^k is that of differentiation of a scalar, namely the partial derivative, even though it arises from $p^m_{;n}$, which a mixed tensor of rank 2. This distinction corresponds to the distinction between the divergence of the gradient, which is the former case, and the gradient of the divergence, corresponding to the latter case.

The reader may find it instructive to compare the above Eqs. (11.14) and (11.15) directly with the corresponding general equations (3.22) and (3.19) and those for centrally symmetric systems, namely (9.4). For the general equations, we have for the de Broglie wave energy $\mathscr{E}(\mathbf{x}, t)$ and the momentum vector $\mathbf{p}(\mathbf{x}, t)$,

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \nabla^2 \mathscr{E} = c^2 \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \mathscr{E}), \qquad \frac{\partial^2 \mathbf{p}}{\partial t^2} = c^2 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{p}),$$

while for the centrally symmetric systems, the de Broglie wave energy $\mathscr{E}(r, t)$ and the radial component of momentum $p(r, t)$, satisfy

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2 \left(\frac{\partial^2 \mathscr{E}}{\partial r^2} + \frac{2}{r} \frac{\partial \mathscr{E}}{\partial r} \right), \qquad \frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial}{\partial r} \left(\frac{\partial p}{\partial r} + \frac{2p}{r} \right),$$

since under these circumstances the gradient $\boldsymbol{\nabla} = \partial / \partial r$.

11.4 Partial Covariant Differentiation

In the previous section, we have utilised the notion of the partial covariant derivative without giving a full explanation and presenting all the details, and this can only be achieved by introducing the Christoffel symbols which is the major objective of this section. Again the reader is referred to any of [15, 97, 101] or the Appendix of [31] for a fuller explanation and much of the necessary background detail. Essentially, both the notions of partial covariant differentiation and Christoffel symbols arise in consequence that when we transform from rectangular Cartesian coordinates $(x, y, z) = (z^1, z^2, z^3)$ with corresponding fixed unit base vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ to a general curvilinear coordinate system (x^1, x^2, x^3), then the corresponding base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are no longer fixed in space, and the Christoffel symbols are the mechanism required to describe their variability. In addition, and as a secondary issue, we note that the base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are not necessarily unit vectors.

In rectangular Cartesian coordinates, the position vector $\mathbf{r}$ is defined by the equation

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = z^1\hat{\mathbf{i}} + z^2\hat{\mathbf{j}} + z^3\hat{\mathbf{k}},$$

and we might conceive that the three fixed base vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ are defined by the three equations

$$\hat{\mathbf{i}} = \frac{\partial \mathbf{r}}{\partial x}, \qquad \hat{\mathbf{j}} = \frac{\partial \mathbf{r}}{\partial y}, \qquad \hat{\mathbf{k}} = \frac{\partial \mathbf{r}}{\partial z}.$$

In fact, in general curvilinear coordinates (x^1, x^2, x^3), we adopt this definition as the defining equation for the base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, thus

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial x^1}, \qquad \mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial x^2}, \qquad \mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial x^3},$$

or equivalently

$$\mathbf{e}_j = \frac{\partial \mathbf{r}}{\partial x^j}, \tag{11.16}$$

for $j = 1, 2, 3$. So, for example, suppose that we transform to cylindrical polar coordinates (r, θ, z), then, with the usual relations $x = r\cos\theta$ and $y = r\sin\theta$, the position vector becomes

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} + z\hat{\mathbf{k}},$$

while the three base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are given by

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}, \quad \mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}}, \quad \mathbf{e}_z = \frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{k}},$$

and we notice that, in this particular case, the three base vectors are mutually orthogonal to one another and that $\mathbf{e}_r$ and $\mathbf{e}_z$ are both unit vectors, while $\mathbf{e}_{\boldsymbol{\theta}}$ is not a unit vector, since $\mathbf{e}_\theta.\mathbf{e}_\theta = r^2$.

In general curvilinear coordinates (x^1, x^2, x^3), we have from (11.4) that the line element becomes on using (11.16)

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i}dx^i \cdot \frac{\partial \mathbf{r}}{\partial x^j}dx^j = \frac{\partial \mathbf{r}}{\partial x^i}\cdot\frac{\partial \mathbf{r}}{\partial x^j}dx^i dx^j = g_{ij}dx^i dx^j,$$

and therefore we may make the important identification

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial x^i}\cdot\frac{\partial \mathbf{r}}{\partial x^j} = \mathbf{e}_i \cdot \mathbf{e}_j, \tag{11.17}$$

for $i, j = 1, 2, 3$. From this equation, it is apparent that if the base vectors are mutually orthogonal, then $g_{ij} = 0$ for $i \neq j$, and that for a particular i, the quantity g_{ii} provides the magnitude squared of the base vector $\mathbf{e}_i$; thus, we have $g_{ii} = \mathbf{e}_i \cdot \mathbf{e}_i$, with no implied summation over the i in this particular instance.

In an analogous manner, we might introduce a set of base vectors $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$ of simply $\mathbf{e}^j$ for $j = 1, 2, 3$, so that for an arbitrary vector $\mathbf{u}$, we have the two representations

$$\mathbf{u} = u^j \mathbf{e}_j = u_k \mathbf{e}^k, \tag{11.18}$$

with implied summations over both j and k, noticing that each of these summations occurs over two levels, namely one index is an upper index and one index is a lower index. We refer to u^j as the contravariant components of the vector **u**, while u_k are referred to as the covariant components of the vector **u**. Thus, without being overly specific, when we consider a tensor of higher rank, say the mixed tensor of rank 4, $A^{jk}{}_{mn}$, then we have in mind that the contravariant indices jk somehow relate to the base vectors $\mathbf{e}_j$, while the covariant indices mn somehow relate to the base vectors $\mathbf{e}^j$. Further, in addition to (11.17) the complete set of relations connecting the scalar products of the two sets of base vectors $\mathbf{e}_j$ and $\mathbf{e}^j$ with the metric tensor g_{ij}, the conjugate metric tensor g^{ij} and the Kronecker delta δ^i_j are as follows:

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \qquad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j, \qquad \delta^i_j = \mathbf{e}^i \cdot \mathbf{e}_j.$$

Christoffel Symbols of the Second Kind Γ^k_{ij} The Christoffel symbols arise on taking the partial derivative of the vector **u**, and from Eq. (11.18), we have

$$\frac{\partial \mathbf{u}}{\partial x^i} = \frac{\partial u^j}{\partial x^i}\mathbf{e}_j + u^j \frac{\partial \mathbf{e}_j}{\partial x^i} = \frac{\partial u_k}{\partial x^i}\mathbf{e}^k + u_k \frac{\partial \mathbf{e}^k}{\partial x^i}, \tag{11.19}$$

and the Christoffel symbols of the second kind Γ^k_{ij} arise as follows:

$$\frac{\partial \mathbf{e}_j}{\partial x^i} = \Gamma^k_{ij}\mathbf{e}_k, \qquad \frac{\partial \mathbf{e}^k}{\partial x^i} = -\Gamma^k_{ij}\mathbf{e}^j,$$

and, of course, with summation over repeated indices. On using these results, we obtain from (11.19) the formulae for the partial covariant derivatives of contravariant and covariant vectors, respectively, thus

$$u^j_{;i} = \frac{\partial u^j}{\partial x^i} + \Gamma^j_{ik}u^k, \qquad u_{j;i} = \frac{\partial u_j}{\partial x^i} - \Gamma^k_{ij}u_k, \tag{11.20}$$

and it is instructive to examine such formulae carefully, noticing that corresponding indices on one side of the equation are at the same level on the other side of the equation, and if an index is repeated on one side of the equation, one index is upper while the other is lower. If the reader keeps these two simple rules in mind, then in tensor analysis, it is very difficult to write down an equation that is incorrect. We note further that the partial covariant derivatives are themselves well-defined tensors, so that $u^j_{;i}$ is a mixed tensor of rank 2, while $u_{j;i}$ is a covariant tensor of rank 2. Also if we bear in mind the two formulae (11.20) with the plus sign for contravariant vectors and the minus sign for convariant vectors, then we may readily write down the partial covariant derivative of a tensor of any rank. For example, for the mixed tensor of rank 3 A^{ij}_k, which is contravariant in two indices and covariant in one, we have two plus signs and one minus sign, and the partial covariant derivative is given by the formula

$$A^{ij}_{k;m} = \frac{\partial A^{ij}_k}{\partial x^m} + \Gamma^j_{mn} A^{in}_k + \Gamma^i_{pm} A^{pj}_k - \Gamma^q_{mk} A^{ij}_q . \qquad (11.21)$$

Christoffel Symbols of the First Kind $[ij, k]$ The Christoffel symbols of the second kind Γ^k_{ij} are defined in terms of the metric tensor g_{ij} and the conjugate metric tensor g^{ij} through the Christoffel symbols of the first kind $[ij, k]$, which are defined in terms the partial derivatives of the metric tensor, thus

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right), \qquad (11.22)$$

and these are related to the Christoffel symbols of the second kind through the formulae

$$[ij, k] = g_{km}\Gamma^m_{ij}, \qquad \Gamma^k_{ij} = g^{kn}[ij, n],$$

and are such that both symbols are symmetric with respect to the paired indices, thus

$$[ij, k] = [ji, k], \qquad \Gamma^k_{ij} = \Gamma^k_{ji}.$$

Partial Covariant Derivatives of Metric Tensors Are Zero: $g_{ij;k} = 0$ **and** $g^{ij}_{;k} = 0$ The definition of the Christoffel symbols in terms of the metric tensor g_{ij} and the conjugate metric tensor g^{ij} is such that the partial covariant derivatives of all components of the metric tensor and its conjugate vanish, thus $g_{ij;k} = 0$ and $g^{ij}_{;k} = 0$. In order to formally prove these important results, we proceed as follows:

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^m_{jk} g_{im} - \Gamma^n_{ik} g_{jn} = \frac{\partial g_{ij}}{\partial x^k} - g^{mp}[jk, p] g_{im} - g^{nq}[ik, q] g_{jn},$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \delta^p_i [jk, p] - \delta^q_j [ik, q] = \frac{\partial g_{ij}}{\partial x^k} - [jk, i] - [ik, j],$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) - \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right),$$

$$= 0,$$

and similarly for the conjugate metric tensor, we have

$$g^{ij}_{;k} = \frac{\partial g^{ij}}{\partial x^k} + \Gamma^i_{mk} g^{jm} + \Gamma^j_{nk} g^{in} = \frac{\partial g^{ij}}{\partial x^k} + g^{ip}[mk, p]g^{jm} + g^{jq}[nk, q]g^{in},$$
$$= \frac{\partial g^{ij}}{\partial x^k} + \frac{g^{ip}g^{jm}}{2}\left(\frac{\partial g_{mp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^p}\right) + \frac{g^{jq}g^{in}}{2}\left(\frac{\partial g_{nq}}{\partial x^k} + \frac{\partial g_{kq}}{\partial x^n} - \frac{\partial g_{nk}}{\partial x^q}\right),$$
$$= \frac{\partial g^{ij}}{\partial x^k} + \frac{g^{ip}g^{jm}}{2}\left(\frac{\partial g_{mp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^p}\right) + \frac{g^{ip}g^{jm}}{2}\left(\frac{\partial g_{pm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^p} - \frac{\partial g_{pk}}{\partial x^m}\right),$$
$$= \frac{\partial g^{ij}}{\partial x^k} + g^{ip}g^{jm}\frac{\partial g_{mp}}{\partial x^k},$$

where in the final term of the second last equality we have changed the repeated indices $n \longrightarrow p$ and $q \longrightarrow m$. Now on taking the partial derivative with respect to x^k of Eq. (11.6), namely $g_{mp}g^{pi} = \delta^i_m$, the above equality becomes

$$g^{ij}_{;k} = \frac{\partial g^{ij}}{\partial x^k} - g_{mp}g^{jm}\frac{\partial g^{ip}}{\partial x^k} = \frac{\partial g^{ij}}{\partial x^k} - \delta^j_p\frac{\partial g^{ip}}{\partial x^k} = 0,$$

and the result is established.

In order to verify the formula (11.9) for the divergence of a contravariant vector p^j, we first recall that if g denotes the determinant of the metric tensor g_{ij}, thus $g = |g_{ij}|$; then, gg^{ij} is the cofactor of the element g_{ij} in the determinant, and from the rule for differentiating determinants, we have

$$\frac{\partial g}{\partial x^k} = gg^{ij}\frac{\partial g_{ij}}{\partial x^k}.$$

Now the partial covariant derivative of the contravariant vector p^j is given by

$$p^j_{;i} = \frac{\partial p^j}{\partial x^i} + \Gamma^j_{ik}p^k,$$

and therefore summation over the repeated index j gives

$$p^j_{;j} = \frac{\partial p^j}{\partial x^j} + \Gamma^j_{jk}p^k = \frac{\partial p^j}{\partial x^j} + g^{jm}[jk, m]p^k,$$
$$= \frac{\partial p^j}{\partial x^j} + \frac{g^{jm}}{2}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right)p^k,$$
$$= \frac{\partial p^j}{\partial x^j} + \frac{g^{jm}}{2}\frac{\partial g_{jm}}{\partial x^k}p^k = \frac{\partial p^j}{\partial x^j} + \frac{1}{2g}\frac{\partial g}{\partial x^k}p^k,$$
$$= \frac{\partial p^j}{\partial x^j} + \frac{1}{2g}\frac{\partial g}{\partial x^j}p^j = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}p^j)}{\partial x^j},$$

as required. We comment that in this derivation, we have also established the formula

$$\Gamma^{j}_{jk} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^k}, \tag{11.23}$$

where g is replaced by $-g$ if $g < 0$, and this identity becomes important as a useful check on any expressions obtained for the Chistoffel symbols of the second kind.

Formula for the Laplacian of a Scalar We may exploit the above formalism to obtain the following useful formula for the Laplacian of a scalar function $\phi(x^i, t)$, thus

$$\nabla^2\phi = g^{ik}\phi_{;ik} = g^{ik}\left(\frac{\partial^2\phi}{\partial x^i \partial x^k} - \Gamma^{j}_{ik}\frac{\partial \phi}{\partial x^j}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{ik}\frac{\partial \phi}{\partial x^i}\right), \tag{11.24}$$

which follows, since on using the fact that the partial covariant derivatives of all metric tensors are zero, we have

$$\nabla^2\phi = g^{ik}\phi_{;ik} = \left(g^{ik}\frac{\partial \phi}{\partial x^i}\right)_{;k} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{ik}\frac{\partial \phi}{\partial x^i}\right),$$

where the final equality follows on using the above result for the divergence of a contravariant vector p^j, since the quantity $g^{ik}\partial\phi/\partial x^i$ operates as a contravariant vector. The expression (11.24) provides a convenient means to evaluate the Laplacian in a particular given coordinate system.

The full import of the notions of partial covariant differentiation and Christoffel symbols comes to fruition in the formulation of the four-dimensional theory of general relativity, and indeed these notions are fundamental to the development of the subject. For this reason, we briefly state some of the major results underpinning general relativity theory. Now we have previously established that the partial covariant derivatives of a covariant vector A_i and a covariant tensor of rank 2 B_{ij} are given by the formulae

$$A_{i;j} = \frac{\partial A_i}{\partial x^j} - \Gamma^{k}_{ij}A_k, \qquad B_{ij;k} = \frac{\partial B_{ij}}{\partial x^k} - \Gamma^{m}_{kj}B_{im} - \Gamma^{n}_{ik}B_{jn},$$

and for both partial covariant derivatives, we may take a further partial covariant derivative $A_{i;jk}$ and $B_{ij;km}$ and pose the question as to the circumstances under which these partial covariant derivatives commute. A full investigation reveals the following relations:

$$A_{i;jk} - A_{i;kj} = R^{m}_{ijk}A_m, \qquad B_{ij;km} - B_{ij;mk} = R^{n}_{jkm}B_{in} + R^{p}_{ikm}B_{jp},$$

where R^m_{ijk} is a tensor of rank 4 covariant in three indices and contravariant in one, and can be shown to be given by (see, e.g. [97], page 53)

$$R^m_{ijk} = \frac{\partial \Gamma^m_{ik}}{\partial x^j} - \frac{\partial \Gamma^m_{ij}}{\partial x^k} + \Gamma^m_{jn}\Gamma^n_{ik} - \Gamma^m_{kp}\Gamma^p_{ij},$$

and altogether there are five major tensors leading to the development of general relativity as follows:

- Riemann-Christoffel tensor R^m_{ijk} defined above
- Covariant curvature tensor $R_{ijkm} = g_{in}R^n_{jkm}$
- Ricci tensor $R_{ij} = R^k_{ijk}$
- Curvature invariant $R = g^{ij}R_{ij}$
- Einstein tensor $G^i_j = g^{ik}R_{jk} - R\delta^i_j/2$

for which there are many known results and many formulae are known to exist such as

$$\begin{aligned} R_{ijkm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^m} - \frac{\partial^2 g_{jm}}{\partial x^i \partial x^k}\right) \\ + g^{np}\left([jk, n][im, p] - [jm, n][ik, p]\right), \end{aligned} \tag{11.25}$$

$$R_{ijkm} = \frac{\partial [jm, i]}{\partial x^k} - \frac{\partial [jk, i]}{\partial x^m} + \Gamma^n_{jk}[im, n] - \Gamma^n_{jm}[ik, n], \tag{11.26}$$

and Bianchi's identity

$$R_{ijkm;n} + R_{ijnk;m} + R_{ijmn;k} = 0. \tag{11.27}$$

Further, we have the relation

$$R_{ij} = \frac{\partial^2 \log\sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^m_{in}\Gamma^n_{jm} - \Gamma^p_{ij}\frac{\partial \log\sqrt{g}}{\partial x^p},$$

from which the symmetry $R_{ij} = R_{ji}$ is apparent. We further note that on careful inspection of (11.25), we may verify the following symmetries of the covariant curvature tensor R_{ijkm}, namely

$$\begin{aligned} R_{ijkm} = -R_{jikm}, \qquad R_{ijkm} = -R_{ijmk}, \qquad R_{ijkm} = R_{kmij}, \\ R_{ijkm} + R_{ikmj} + R_{imjk} = 0. \end{aligned} \tag{11.28}$$

For a proof of Bianchi's identity and further details on these symmetries, we refer the reader to either Spain [97] or to the Appendix of Eringen [31].

On making use of Bianchi's identity, by purposeful construction, the Einstein tensor is divergence free, namely $G^i_{j;i} = 0$ (also refer to Spain [97]). Briefly, this may be established using the fact that all partial covariant derivatives of all metric tensors vanish, and therefore in Bianchi's identity (11.27), we may raise and lower the symbols freely. Thus, from Bianchi's identity and the above symmetries, so that $R_{ijmn;k} = -R_{ijnm;k}$, we may deduce

$$R^i_{jkm;n} + R^i_{jnk;m} - R^i_{jnm;k} = 0,$$

which on contracting i and m produces

$$R_{jk;n} + R^i_{jnk;i} - R_{jn;k} = 0,$$

and these are referred to as the contracted Bianchi identity. A second contraction of this identity may be shown to yield $R_{;j} = 2R^k_{j;k}$, which are equivalent to the vanishing of the divergence of the Einstein tensor, namely $G^i_{j;i} = 0$. While the vanishing of the divergence of the Einstein tensor is a fundamental result in general relativity, the above abbreviated derivation belies the underlying complexities involved in demonstrating the result in a particular coordinate representation. Two illustrative examples are provided in Sect. 11.7, and the result is also verified for the spiral gravitating structures examined in the final section of the chapter.

In the verification of the equations $G^i_{j;i} = 0$, sometimes it is simpler to express the equations in terms of the covariant Einstein tensor G_{ij} through the relations $G^i_k = g^{ij} G_{jk}$, so that on taking the partial covariant derivative with respect to x^m, we have

$$G^i_{k;m} = g^{ij} G_{jk;m} = g^{ij} \left\{ \frac{\partial G_{jk}}{\partial x^m} - \Gamma^n_{mj} G_{nk} - \Gamma^p_{mk} G_{jp} \right\}, \tag{11.29}$$

noting again that all partial covariant derivatives of the metric tensor g_{ij} and its conjugate g^{ij} are zero. On contracting this equation, that is, setting $m = i$ and then summing over i, the equations $G^i_{j;i} = 0$ become

$$G^i_{k;i} = g^{ij} G_{jk;i} = g^{ij} \left\{ \frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij} G_{nk} - \Gamma^p_{ik} G_{jp} \right\} = 0, \tag{11.30}$$

and the latter equality is sometimes the most convenient formulation to use. We note that in a stress-free vacuum space, the Einstein tensor is assumed to vanish identically, namely $G_{ij} = 0$.

11.5 Illustration for Single Space Dimension

To provide an illustration of the machinery for the above tensors, in a four-dimensional space $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ where (x, y, z) denote rectangular Cartesian coordinates, we consider the particular four-dimensional line element, thus

$$\begin{aligned} ds^2 &= g_{ij}dx^i dx^j \\ &= a(x,t)(cdt)^2 - b(x,t)(dx)^2 - 2f(x,t)cdtdx - (dy)^2 - (dz)^2, \end{aligned}$$

where $a(x, t)$, $b(x, t)$ and $f(x, t)$ denote three functions to be determined from $G^i_{j;i} = 0$, and the indices i, j run through $i, j = 0, 1, 2, 3$. The metric tensor g_{ij} has components given by

$$g_{ij} = \begin{pmatrix} a(x,t) & -f(x,t) & 0 & 0 \\ -f(x,t) & -b(x,t) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

while the conjugate metric tensor g^{ij} has components given by

$$g^{ij} = \begin{pmatrix} b(x,t)/\delta & -f(x,t)/\delta & 0 & 0 \\ -f(x,t)/\delta & -a(x,t)/\delta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (11.31)$$

where $\delta(x, t)$ is defined by $\delta = ab + f^2$ and the determinant $g = |g_{ij}| = -\delta$.

Using the formula (11.22) for the Christoffel symbols of the first kind $[ij, k]$, namely

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

we may deduce the following expressions for the non-zero Christoffel symbols of the first kind, thus

$$[01,0] = \frac{1}{2}\frac{\partial a}{\partial x}, \quad [01,1] = -\frac{1}{2c}\frac{\partial b}{\partial t}, \quad [00,0] = \frac{1}{2c}\frac{\partial a}{\partial t}, \quad [11,1] = -\frac{1}{2}\frac{\partial b}{\partial x},$$
$$[00,1] = -\left(\frac{1}{2}\frac{\partial a}{\partial x} + \frac{1}{c}\frac{\partial f}{\partial t}\right), \qquad [11,0] = \left(\frac{1}{2c}\frac{\partial b}{\partial t} - \frac{\partial f}{\partial x}\right),$$

noting that other non-zero components may be deduced from the symmetry $[ij, k] = [ji, k]$. From these relations and the formulae $\Gamma^k_{ij} = g^{kn}[ij, n]$, we may deduce that the only non-zero Christoffel symbols of the second kind are as follows:

$$\Gamma^0_{01} = \frac{1}{2\delta}\left(b\frac{\partial a}{\partial x} + \frac{f}{c}\frac{\partial b}{\partial t}\right), \qquad \Gamma^1_{01} = \frac{1}{2\delta}\left(\frac{a}{c}\frac{\partial b}{\partial t} - f\frac{\partial a}{\partial x}\right),$$

$$\Gamma^0_{00} = \frac{1}{2\delta}\left(f\frac{\partial a}{\partial x} + \frac{b}{c}\frac{\partial a}{\partial t} + \frac{2f}{c}\frac{\partial f}{\partial t}\right), \quad \Gamma^1_{00} = \frac{1}{2\delta}\left(a\frac{\partial a}{\partial x} - \frac{f}{c}\frac{\partial a}{\partial t} + \frac{2a}{c}\frac{\partial f}{\partial t}\right),$$

$$\Gamma^0_{11} = \frac{1}{2\delta}\left(f\frac{\partial b}{\partial x} + \frac{b}{c}\frac{\partial b}{\partial t} - 2b\frac{\partial f}{\partial x}\right), \quad \Gamma^1_{11} = \frac{1}{2\delta}\left(a\frac{\partial b}{\partial x} - \frac{f}{c}\frac{\partial b}{\partial t} + 2f\frac{\partial f}{\partial x}\right).$$

We may make use of (11.23) to check these formulae, thus

$$\begin{aligned}\Gamma^j_{j0} &= \Gamma^0_{00} + \Gamma^1_{10} \\ &= \frac{1}{2\delta}\left(f\frac{\partial a}{\partial x} + \frac{b}{c}\frac{\partial a}{\partial t} + \frac{2f}{c}\frac{\partial f}{\partial t} + \frac{a}{c}\frac{\partial b}{\partial t} - f\frac{\partial a}{\partial x}\right) \\ &= \frac{1}{2c\delta}\frac{\partial \delta}{\partial t},\end{aligned}$$

and

$$\begin{aligned}\Gamma^j_{j1} &= \Gamma^0_{01} + \Gamma^1_{11} \\ &= \frac{1}{2\delta}\left(b\frac{\partial a}{\partial x} + \frac{f}{c}\frac{\partial b}{\partial t} + a\frac{\partial b}{\partial x} - \frac{f}{c}\frac{\partial b}{\partial t} + 2f\frac{\partial f}{\partial x}\right) \\ &= \frac{1}{2\delta}\frac{\partial \delta}{\partial x},\end{aligned}$$

as required.

With $i = k = 1$ and $j = m = 0$, we may deduce from the symmetries of the covariant curvature tensor R_{ijkm} (see Eq. (11.28)) the following results $R_{1010} = -R_{0110} = R_{0101}$ and $R_{0011} = 0$, and since there is essentially only one non-zero component, we chose to adopt R_{0101} and express the other non-zero components in terms of R_{0101}. Again with $i = k = 1$ and $j = m = 0$, we may deduce from (11.25) the following expression for R_{0101} in terms of the metric tensor and the Christoffel symbols of the first kind, thus

$$\begin{aligned}R_{0101} = &\frac{1}{2}\left(\frac{\partial^2 g_{01}}{\partial x^1 \partial x^0} + \frac{\partial^2 g_{10}}{\partial x^1 \partial x^0} - \frac{\partial^2 g_{00}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{11}}{\partial x^0 \partial x^0}\right) \\ &+ g^{np}\left([10, n][01, p] - [11, n][00, p]\right)\end{aligned}$$

$$= \frac{1}{2}\left(2\frac{\partial^2 g_{01}}{\partial x^1 \partial x^0} - \frac{\partial^2 g_{00}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{11}}{\partial x^0 \partial x^0}\right)$$

$$+ g^{n0}\left([10, n][01, 0] - [11, n][00, 0]\right) + g^{n1}\left([10, n][01, 1] - [11, n][00, 1]\right)$$

$$= \frac{1}{2}\left(2\frac{\partial^2 g_{01}}{\partial x^1 \partial x^0} - \frac{\partial^2 g_{00}}{\partial x^{1^2}} - \frac{\partial^2 g_{11}}{\partial x^{0^2}}\right)$$

$$+ g^{00}\left([10, 0][01, 0] - [11, 0][00, 0]\right) + g^{10}\left([10, 1][01, 0] - [11, 1][00, 0]\right)$$

$$+ g^{01}\left([10, 0][01, 1] - [11, 0][00, 1]\right) + g^{11}\left([10, 1][01, 1] - [11, 1][00, 1]\right).$$

After much simplification and rearrangement, this expression can be shown to become

$$R_{0101} = \frac{1}{2}\left(\frac{1}{c^2}\frac{\partial^2 b}{\partial t^2} - \frac{2}{c}\frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 a}{\partial x^2}\right)$$

$$+ \frac{1}{4\delta}\left\{\left(\frac{\partial a}{\partial x} + \frac{1}{c}\frac{\partial f}{\partial t}\right)\frac{\partial \delta}{\partial x} - \left(\frac{1}{c}\frac{\partial b}{\partial t} - \frac{\partial f}{\partial x}\right)\frac{1}{c}\frac{\partial \delta}{\partial t}\right\}$$

$$+ \frac{1}{4c\delta}\left\{a\frac{\partial(f, b)}{\partial(t, x)} + b\frac{\partial(a, f)}{\partial(t, x)} + f\frac{\partial(b, a)}{\partial(t, x)}\right\},$$

where $\delta(x, t)$ is defined by $\delta = ab + f^2$ and closer inspection of the final term reveals the structure of the scalar triple product, thus

$$\frac{1}{4c\delta}\left\{a\frac{\partial(f, b)}{\partial(t, x)} + b\frac{\partial(a, f)}{\partial(t, x)} + f\frac{\partial(b, a)}{\partial(t, x)}\right\} = -\frac{1}{4c\delta}\begin{vmatrix} a & b & f \\ \frac{\partial a}{\partial t} & \frac{\partial b}{\partial t} & \frac{\partial f}{\partial t} \\ \frac{\partial a}{\partial x} & \frac{\partial b}{\partial x} & \frac{\partial f}{\partial x} \end{vmatrix}.$$

We use $R_{jk} = g^{im} R_{ijkm}$ to determine the components of the Ricci tensor $R_{ij} = R^k_{ijk}$, thus

$$R_{00} = g^{im} R_{i00m} = g^{i0} R_{i000} + g^{i1} R_{i001}$$

$$= g^{00} R_{0000} + g^{10} R_{1000} + g^{01} R_{0001} + g^{11} R_{1001}$$

$$= g^{11} R_{1001} = g^{11} R_{0110} = -g^{11} R_{0101},$$

on using the above symmetries of the covariant curvature tensor R_{ijkm} given by (11.28), since from Eq. (11.25) it is apparent that any component of the tensor R_{ijkm} with three indices coinciding must be zero. Proceeding further in this manner and making use of (11.31) for the components the conjugate metric tensor g^{ij}, we may show that the only non-zero components of the Ricci tensor R_{jk} are given by

$$R_{00} = aR_{0101}/\delta, \qquad R_{10} = R_{01} = -fR_{0101}/\delta, \qquad R_{11} = -bR_{0101}/\delta,$$

so that the curvature invariant $R = g^{ij}R_{ij}$ becomes

$$R = g^{ij}R_{ij} = g^{00}R_{00} + 2g^{01}R_{01} + g^{11}R_{11} = 2(ab + f^2)R_{0101}/\delta^2 = 2R_{0101}/\delta,$$

since by definition $\delta(x, t)$ is defined by $\delta = ab + f^2$. From the relation $G_{ij} = R_{ij} - Rg_{ij}/2$, it is apparent that the only non-zero components of the covariant Einstein tensor G_{ij} arising through the term $-Rg_{ij}/2$ are $G_{22} = G_{33} = R_{0101}/\delta$ with all others identically zero, so that the equations for the vanishing of the divergence of the Einstein tensor become simply $\partial G_{22}/\partial y = 0$ and $\partial G_{33}/\partial z = 0$, which are trivially satisfied. In the following section, we provide some general formulae for the Riemann and Einstein tensors applying to a particular metric of some generality.

11.6 Formulae for Ricci and Einstein Tensors

While there exist a number of exact space-times for the Einstein field equations of general relativity, such as those listed in [42, 81, 98] and to a lesser extent in [84, 91, 107], as previously stated, in the search for further exact space-time solutions of general relativity, the major prohibiting factors include the shear complexity of the underlying equations and the lack of any strategic perspective that might guide the analysis. For any given line element, the Riemann-Christoffel tensor itself has a complicated dependence on the components of the particular assumed metric tensor. The metric tensor for the most general four-dimensional line element involves ten arbitrary components, which are possibly dependent upon three spatial variables and the temporal variable.

We attempt to reduce the complexity of the problem in the determination of exact solutions by considering a line element involving only seven arbitrary metric tensor components, and each component is assumed to depend only on two of the spatial variables and the temporal variable. Of course, while not completely general, the assumed line element includes many existing exact space-time solutions of general relativity. We show that six components of the covariant curvature tensor act as a basis, and we present some general formulae for the Ricci and Einstein tensors expressed in terms of these six components of the covariant curvature tensor. In the next section, we illustrate this general formulation with two well-known cosmological models. Although similar general metrics have been studied in the past and it is known that the four-dimensional structure can be represented in terms of the corresponding three-dimensional tensors (see, e.g. [30] or [42]), the present approach differs in the sense that it involves the determination of explicit formulae.

In four-dimensional space (x^0, x^1, x^2, x^3), we consider the line element with the particular structure,

$$ds^2 = g_{ij}dx^i dx^j = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 \quad (11.32)$$
$$+ 2g_{01}dx^0dx^1 + 2g_{02}dx^0dx^2 + 2g_{12}dx^1dx^2,$$

where $x^0 = ct$, and we assume that the non-zero components of the metric tensor g_{ij} depend only on x^0, x^1 and x^2 and are independent of the x^3 coordinate. Specifically, we assume that the metric tensor has the structure

$$g_{ij} = \begin{pmatrix} g_{00}(x^0, x^1, x^2) & g_{01}(x^0, x^1, x^2) & g_{02}(x^0, x^1, x^2) & 0 \\ g_{01}(x^0, x^1, x^2) & g_{11}(x^0, x^1, x^2) & g_{12}(x^0, x^1, x^2) & 0 \\ g_{02}(x^0, x^1, x^2) & g_{12}(x^0, x^1, x^2) & g_{22}(x^0, x^1, x^2) & 0 \\ 0 & 0 & 0 & g_{33}(x^0, x^1, x^2) \end{pmatrix}. \quad (11.33)$$

This particular metric, although not completely general, includes many of the metrics for which exact space-time solutions have been determined (see, e.g. [81]).

Working Notation Now in order to undertake lengthy algebraic calculations with this tensor, we may facilitate these manipulations and without loss of generality, we may employ the notation (ct, x, y, z) for (x^0, x^1, x^2, x^3) and in place of the above equations we use a notation devoid of indices, thus

$$ds^2 = g_{ij}dx^i dx^j = a(x, y, t)(cdt)^2 - b(x, y, t)(dx)^2 - h(x, y, t)(dy)^2 \quad (11.34)$$
$$- 2f(x, y, t)cdtdx - 2j(x, y, t)cdtdy - 2k(x, y, t)dxdy - m(x, y, t)(dz)^2,$$

where $a(x, y, t)$, $b(x, y, t)$, $f(x, y, t)$, $h(x, y, t)$, $j(x, y, t)$, $k(x, y, t)$ and $m(x, y, t)$ all denote functions to be ultimately determined from the Einstein equations $G^i_{j;i} = 0$, and the indices i, j here run through $i, j = 0, 1, 2, 3$. Thus, the metric tensor g_{ij} has components given by

$$g_{ij} = \begin{pmatrix} a(x, y, t) & -f(x, y, t) & -j(x, y, t) & 0 \\ -f(x, y, t) & -b(x, y, t) & -k(x, y, t) & 0 \\ -j(x, y, t) & -k(x, y, t) & -h(x, y, t) & 0 \\ 0 & 0 & 0 & -m(x, y, t) \end{pmatrix}, \quad (11.35)$$

while the conjugate metric tensor g^{ij} has components given by

$$g^{ij} = \begin{pmatrix} m(k^2 - bh)/g^* & m(fh - jk))/g^* & m(bj - fk)/g^* & 0 \\ m(fh - jk))/g^* & m(ah + j^2)/g^* & -m(ak + jf)/g^* & 0 \\ m(bj - fk)/g^* & -m(ak + jf)/g^* & m(f^2 + ab)/g^* & 0 \\ 0 & 0 & 0 & -1/m \end{pmatrix},$$

where $g^*(x, y, t)$ is the determinant $g^* = |g_{ij}|$ given by

$$g^* = m(ak^2 + 2jkf - abh - bj^2 - hf^2),$$

and we emphasise that the specific line element (11.34) with the symbols (x, y, z) is for working purposes only and that they do not necessarily represent rectangular Cartesian coordinates and might represent any three-dimensional spatial coordinates (x^1, x^2, x^3).

The assumptions underlying the metric tensor g_{ij} mean that in many respects, and specifically in terms of the conjugate metric tensor g^{ij}, the three-dimensional and four-dimensional problems uncouple and the conjugate metric tensor becomes

$$g^{ij} = \begin{pmatrix} (bh - k^2)/g & (jk - fh))/g & (fk - bj)/g & 0 \\ (jk - fh))/g & -(ah + j^2)/g & (ak + jf)/g & 0 \\ (fk - bj)/g & (ak + jf)/g & -(f^2 + ab)/g & 0 \\ 0 & 0 & 0 & -1/m \end{pmatrix},$$

where $g(x, y, t)$ is the three-dimensional determinant $g = |g_{ij}|$ given by

$$g = abh + bj^2 + hf^2 - ak^2 - 2jkf,$$

and $g^* = g_{33}g$.

Christoffel Symbols of the First Kind $[ij, k]$ Using the standard formula for the Christoffel symbols of the first kind $[ij, k]$, namely

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

we may deduce the following expressions for the non-zero Christoffel symbols of the first kind in a straightforward manner, thus

$$[00, 0] = \frac{1}{2}\frac{\partial g_{00}}{\partial x^0}, \quad [00, 1] = \left(\frac{\partial g_{01}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^1}\right), \quad [00, 2] = \left(\frac{\partial g_{02}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^2}\right),$$

$$[11, 0] = \left(\frac{\partial g_{01}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^0}\right), \quad [11, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1}, \quad [11, 2] = \left(\frac{\partial g_{12}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^2}\right),$$

$$[22, 0] = \left(\frac{\partial g_{02}}{\partial x^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial x^0}\right), \quad [22, 1] = \left(\frac{\partial g_{12}}{\partial x^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial x^1}\right), \quad [22, 2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^2},$$

$$[01, 0] = \frac{1}{2}\frac{\partial g_{00}}{\partial x^1}, \quad [01, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^0}, \quad [01, 2] = \frac{1}{2}\left(\frac{\partial g_{02}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^2}\right),$$

$$[02, 0] = \frac{1}{2}\frac{\partial g_{00}}{\partial x^2}, \quad [02, 1] = \frac{1}{2}\left(\frac{\partial g_{01}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^0} - \frac{\partial g_{02}}{\partial x^1}\right), \quad [02, 2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^0},$$

$$[12, 0] = \frac{1}{2}\left(\frac{\partial g_{02}}{\partial x^1} + \frac{\partial g_{01}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^0}\right), \quad [12, 1] = \frac{1}{2}\frac{\partial g_{11}}{\partial x^2}, \quad [12, 2] = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1},$$

which in terms of the working symbols become

$$[00,0]=\frac{1}{2c}\frac{\partial a}{\partial t},\quad [00,1]=-\left(\frac{1}{2}\frac{\partial a}{\partial x}+\frac{1}{c}\frac{\partial f}{\partial t}\right),\quad [00,2]=-\left(\frac{1}{2}\frac{\partial a}{\partial y}+\frac{1}{c}\frac{\partial j}{\partial t}\right),$$

$$[11,0]=\left(\frac{1}{2c}\frac{\partial b}{\partial t}-\frac{\partial f}{\partial x}\right),\qquad [11,1]=-\frac{1}{2}\frac{\partial b}{\partial x},\qquad [11,2]=\left(\frac{1}{2}\frac{\partial b}{\partial y}-\frac{\partial k}{\partial x}\right),$$

$$[22,0]=\left(\frac{1}{2c}\frac{\partial h}{\partial t}-\frac{\partial j}{\partial y}\right),\qquad [22,1]=\left(\frac{1}{2}\frac{\partial h}{\partial x}-\frac{\partial k}{\partial y}\right),\qquad [22,2]=-\frac{1}{2}\frac{\partial h}{\partial y},$$

$$[01,0]=\frac{1}{2}\frac{\partial a}{\partial x},\qquad [01,1]=-\frac{1}{2c}\frac{\partial b}{\partial t},\qquad [01,2]=\frac{1}{2}\left(\frac{\partial f}{\partial y}-\frac{\partial j}{\partial x}-\frac{1}{c}\frac{\partial k}{\partial t}\right),$$

$$[02,0]=\frac{1}{2}\frac{\partial a}{\partial y},\qquad [02,1]=\frac{1}{2}\left(\frac{\partial j}{\partial x}-\frac{\partial f}{\partial y}-\frac{1}{c}\frac{\partial k}{\partial t}\right),\qquad [02,2]=-\frac{1}{2c}\frac{\partial h}{\partial t},$$

$$[12,0]=-\frac{1}{2}\left(\frac{\partial j}{\partial x}+\frac{\partial f}{\partial y}-\frac{1}{c}\frac{\partial k}{\partial t}\right),\qquad [12,1]=-\frac{1}{2}\frac{\partial b}{\partial y},\qquad [12,2]=-\frac{1}{2}\frac{\partial h}{\partial x},$$

noting that other non-zero components may be deduced from the symmetry $[ij,k]=[ji,k]$, and with the convention that i and j immediately below refer only to 0, 1, 2, we have

$$[ij,3]=[3i,j]=0,\qquad [3i,3]=-\frac{1}{2}\frac{\partial m}{\partial x^i},\qquad [33,i]=\frac{1}{2}\frac{\partial m}{\partial x^i}.$$

Christoffel Symbols of the Second Kind Γ^k_{ij} The Christoffel symbols of the second kind are more complicated, and we derive the following formulae as follows. From the above 4×4 matrix, we formulate the 3×3 matrix

$$g_{ij}=\begin{pmatrix} g_{00} & g_{01} & g_{02}\\ g_{01} & g_{11} & g_{12}\\ g_{02} & g_{12} & g_{22}\end{pmatrix},\tag{11.36}$$

and we assign g to be the determinant of this matrix, so that $g^*=|g_{ij}|=gg_{33}$. Basically, in terms of the index-free notation, we need to individually calculate each of the Christoffel symbols of the second kind using the above formulae for the Christoffel symbols of the first kind along with the formulae $\Gamma^k_{ij}=g^{kn}[ij,n]$. Having undertaken these extensive algebraic calculations, it becomes apparent that we may express the final formulae for each of the non-zero Christoffel symbols of the second kind Γ^k_{ij} in terms of three 3×3 determinants, two of positive sign and one of negative sign, no doubt emanating from the corresponding signatures involved in the three terms in the Christoffel symbols of the first kind $[ij,k]$. With the designation of i, j and k as in the symbol Γ^k_{ij}, we proceed as follows:

- In each of the three determinants, we delete the k^{th} row.
- In one of the determinants of positive sign, we replace the k^{th} row by $(\partial g_{i0}/\partial x^j, \partial g_{i1}/\partial x^j, \partial g_{i2}/\partial x^j)$.
- In the other determinant of positive sign, we replace the k^{th} row by $(\partial g_{j0}/\partial x^i, \partial g_{j1}/\partial x^i, \partial g_{j2}/\partial x^i)$.
- In the determinant of negative sign, we replace the k^{th} row by $(\partial g_{ij}/\partial x^0, \partial g_{ij}/\partial x^1, \partial g_{ij}/\partial x^2)$.
- Each of the three determinants is weighted with a factor $1/2g$.
- In the event that $i = j$, the two determinants of positive sign coincide and merge to produce one determinant of positive sign of weight $1/g$.

The final expressions are as follows:

$$\Gamma^0_{00} = \frac{1}{g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{00} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{00} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{11} = \frac{1}{g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{12}}{\partial x^1} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{11} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{12}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{11} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{12}}{\partial x^1} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^1} & \frac{\partial g_{11}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{22} = \frac{1}{g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^2} & \frac{\partial g_{21}}{\partial x^2} & \frac{\partial g_{22}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{22}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{22} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^2} & \frac{\partial g_{21}}{\partial x^2} & \frac{\partial g_{22}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{22}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{22} = \frac{1}{g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^2} & \frac{\partial g_{21}}{\partial x^2} & \frac{\partial g_{22}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{22}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{01} = \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{01} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{01} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{02} = \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{02} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{02} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \end{vmatrix},$$

$$\Gamma^0_{12} = \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^1} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{10}}{\partial x^2} & \frac{\partial g_{11}}{\partial x^2} & \frac{\partial g_{12}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^1_{12} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^2} & \frac{\partial g_{11}}{\partial x^2} & \frac{\partial g_{12}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix},$$

$$\Gamma^2_{12} = \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{22}}{\partial x^1} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{10}}{\partial x^2} & \frac{\partial g_{11}}{\partial x^2} & \frac{\partial g_{12}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^1} & \frac{\partial g_{21}}{\partial x^2} \end{vmatrix}.$$

Since these expressions involve the scalar triple product, no doubt there exists any number of equivalent alternative expressions dependent upon one's own personal preferences. We may use the well-known formula

$$\Gamma^j_{ij} = \frac{1}{2g}\frac{\partial g}{\partial x^i},$$

to check the above expressions obtained for the Chistoffel symbols of the second kind. These calculations are not entirely trivial, and for purposes of illustration, the proof for the particular value $i = 0$ is as follows. On writing the identity out in full for $i = 0$, we have

$$\Gamma^j_{0j} = \Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02},$$

and from the above formulae, we obtain

$$\Gamma^j_{ij}$$
$$= \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}$$

$$+\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{11}}{\partial x^0} & \frac{\partial g_{12}}{\partial x^0} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}$$

$$+\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{21}}{\partial x^0} & \frac{\partial g_{22}}{\partial x^0} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \end{vmatrix},$$

where we have specifically written the first term of this equation as two identical terms, so that the sum of the first three terms on the left-hand side gives the desired result as the partial derivative of g with respect to x^0 divided by $2g$, and it is left to show that the remaining six terms are zero. This is most easily achieved using the permutation symbols ε^{ijk} and the summation convention. In this notation, the determinant g of the 3×3 metric tensor becomes $g = \varepsilon^{ijk} g_{0i} g_{1j} g_{2k}$, and the six determinants in the above expression that are required to be shown to be zero become

$$\frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{01}}{\partial x^0} & \frac{\partial g_{02}}{\partial x^0} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} \frac{\partial g_{00}}{\partial x^0} & \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{00}}{\partial x^2} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{00}}{\partial x^1} & \frac{\partial g_{01}}{\partial x^1} & \frac{\partial g_{02}}{\partial x^1} \\ g_{20} & g_{21} & g_{22} \end{vmatrix}$$

$$-\frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ \frac{\partial g_{10}}{\partial x^0} & \frac{\partial g_{10}}{\partial x^1} & \frac{\partial g_{10}}{\partial x^2} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} + \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{00}}{\partial x^2} & \frac{\partial g_{01}}{\partial x^2} & \frac{\partial g_{02}}{\partial x^2} \end{vmatrix} - \frac{1}{2g}\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ \frac{\partial g_{20}}{\partial x^0} & \frac{\partial g_{20}}{\partial x^1} & \frac{\partial g_{20}}{\partial x^2} \end{vmatrix}$$

$$= \frac{\varepsilon^{ijk}}{2g}\left(\frac{\partial g_{0i}}{\partial x^0} g_{1j} g_{2k} + g_{0i}\frac{\partial g_{0j}}{\partial x^1} g_{2k} + g_{0i} g_{1j}\frac{\partial g_{0k}}{\partial x^2}\right)$$

$$- \frac{\varepsilon^{ijk}}{2g}\left(\frac{\partial g_{00}}{\partial x^i} g_{1j} g_{2k} + g_{0i}\frac{\partial g_{10}}{\partial x^j} g_{2k} + g_{0i} g_{1j}\frac{\partial g_{20}}{\partial x^k}\right)$$

$$= \frac{\varepsilon^{ijk}}{2g}\left\{\left(\frac{\partial g_{0i}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^i}\right) g_{1j} g_{2k} + \left(\frac{\partial g_{0j}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^j}\right) g_{0i} g_{2k} + \left(\frac{\partial g_{0k}}{\partial x^2} - \frac{\partial g_{20}}{\partial x^k}\right) g_{0i} g_{1j}\right\}$$

$$= \frac{1}{2g}\left\{\left(\frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1}\right)(g_{12}g_{20} - g_{10}g_{22}) + \left(\frac{\partial g_{02}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^2}\right)(g_{10}g_{21} - g_{11}g_{20})\right\}$$

$$+ \frac{1}{2g}\left\{\left(\frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^0}\right)(g_{02}g_{21} - g_{01}g_{22}) + \left(\frac{\partial g_{02}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^2}\right)(g_{01}g_{20} - g_{00}g_{21})\right\}$$

$$+ \frac{1}{2g}\left\{\left(\frac{\partial g_{00}}{\partial x^2} - \frac{\partial g_{20}}{\partial x^0}\right)(g_{01}g_{12} - g_{02}g_{11}) + \left(\frac{\partial g_{01}}{\partial x^2} - \frac{\partial g_{20}}{\partial x^1}\right)(g_{02}g_{10} - g_{00}g_{12})\right\},$$

each term of which can be seen to have a corresponding term that cancels, and therefore, the sum total produces zero as required, and a similar proof can be presented for $i = 1$ and $i = 2$.

Christoffel Symbols of the Second Kind Γ^k_{ij} Involving the Index 3 For the Christoffel symbols of the second kind involving the index 3, we again assume that the indices i and j immediately below refer only to 0, 1, 2, and we have the following formulae:

$$\Gamma^3_{ij} = g^{3n}[ij, n] = g^{33}[ij, 3] = 0,$$

$$\Gamma^i_{33} = g^{in}[33, n] = g^{ij}[33, j] = -\frac{g^{ij}}{2}\frac{\partial g_{33}}{\partial x^j},$$

$$\Gamma^3_{3j} = g^{3n}[3j, n] = g^{33}[3j, 3] = \frac{g^{33}}{2}\frac{\partial g_{33}}{\partial x^j} = \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial x^j},$$

and we observe that these expressions are entirely consistent with the extended identity

$$\Gamma^j_{ij} = \Gamma^0_{i0} + \Gamma^1_{i1} + \Gamma^2_{i2} + \Gamma^3_{i3} = \frac{1}{2g}\frac{\partial g}{\partial x^i} + \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial x^i} = \frac{1}{2g^*}\frac{\partial g^*}{\partial x^i},$$

where g^* is the determinant of the 4×4 matrix and $g^* = |g_{ij}| = gg_{33}$, and the derived identity is as might be expected.

Covariant Curvature Tensor R_{ijkm} The metric tensor (11.32) (see also (11.34)) has been purposely chosen so that there is an uncoupling of certain aspects of the three-dimensional problem (x^0, x^1, x^2) and the four-dimensional problem (x^0, x^1, x^2, x^3). Thus, with i, j, k and m restricted to 0, 1 and 2, other than those connected through the symmetries (11.28), there are essentially only six non-zero components of the covariant curvature tensor R_{ijkm}, which here we adopt to be R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012}. As far as the author is aware, there is no immediate way to identify the trivial components of the covariant curvature tensor R_{ijkm}, other than close inspection of the assumed metric tensor and (11.25) or (11.27) along with the known symmetries (11.28), so for the sake of completeness, these components are listed below. From the assumed metric tensor and either Eqs. (11.25) or (11.27), we may verify that the following components of the following components of the covariant curvature tensor R_{ijkm} are all zero, thus

$$\begin{aligned} R_{0000} &= R_{0001} = R_{0002} = R_{0111} = R_{0222} = R_{0010} \\ &= R_{0011} = R_{0012} = R_{0020} = R_{0021} = R_{0022} = R_{0122} \\ &= R_{1000} = R_{1110} = R_{1111} = R_{1112} = R_{1222} = R_{1011} \\ &= R_{1022} = R_{1120} = R_{1121} = R_{1122} = R_{2000} = R_{2111} \\ &= R_{2220} = R_{2221} = R_{2222} = R_{2011} = R_{2022} = R_{2122} = 0, \end{aligned}$$

while through the symmetries (11.28), we have the results:

$$R_{0110} = R_{1001} = -R_{0101}, \quad R_{1010} = R_{0101},$$
$$R_{0220} = R_{2002} = -R_{0202}, \quad R_{2020} = R_{0202},$$
$$R_{1221} = R_{2112} = -R_{1212}, \quad R_{2121} = R_{1212},$$
$$R_{0121} = R_{1012} = -R_{0112}, \quad R_{1021} = R_{2110} = R_{0112},$$
$$R_{0102} = R_{2010} = -R_{0120}, \quad R_{1002} = R_{2001} = R_{0120},$$
$$R_{0212} = R_{2120} = -R_{2012}, \quad R_{2012} = R_{1220} = R_{2012}.$$

Now following similar lines, and with i, j, k and m still restricted to 0, 1 and 2, we may show that for the covariant curvature tensor R_{ijkm} involving the index 3, we have the following results

$$R_{3jkm} = R_{i3km} = R_{ij3m} = R_{ijk3} = R_{33km} = R_{ij33} = 0,$$
$$R_{j3k3} = R_{3j3k} = -R_{3jk3},$$

where R_{3jk3} is given by

$$R_{3jk3} = \frac{1}{2}\left(\frac{\partial^2 g_{33}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial g_{33}}{\partial x^p}\right) - \frac{g^{33}}{4}\frac{\partial g_{33}}{\partial x^j}\frac{\partial g_{33}}{\partial x^k},$$

and since $g^{33} = 1/g_{33}$, it is apparent that

$$R_{3jk3} = \sqrt{g_{33}}\left(\frac{\partial^2 \sqrt{g_{33}}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial \sqrt{g_{33}}}{\partial x^p}\right),$$

or

$$R_{3jk3} = -\sqrt{-g_{33}}\left(\frac{\partial^2 \sqrt{-g_{33}}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial \sqrt{-g_{33}}}{\partial x^p}\right),$$

in the event that $g^{33} < 0$.

The components R_{3jk3} involve another six non-zero components of the covariant curvature tensor R_{ijkm} arising from $j, k = 0, 1, 2$, which contribute to the Ricci tensor through $R_{jk} = R^n_{jkn} = g^{im} R_{ijkm} = g^{33} R_{3jk3}$, thus

$$R_{jk} = g^{33} R_{3jk3} = \frac{1}{\sqrt{g_{33}}}\left(\frac{\partial^2 \sqrt{g_{33}}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial \sqrt{g_{33}}}{\partial x^p}\right),$$

and to the curvature invariant $R = g^{jk} R_{jk} = g^{im} g^{jk} R_{ijkm}$ through the term R^* given by

$$R^* = g^{jk}g^{33}R_{3jk3} + g^{im}g^{33}R_{i33m} = 2g^{jk}g^{33}R_{3jk3},$$

which becomes

$$R^* = \frac{2g^{jk}}{\sqrt{g_{33}}}\left(\frac{\partial^2\sqrt{g_{33}}}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial\sqrt{g_{33}}}{\partial x^p}\right) = \frac{2\nabla^2\sqrt{g_{33}}}{\sqrt{g_{33}}},$$

where ∇^2 is the conventional generalised Laplacian associated with the three-dimensional coordinates (x^0, x^1, x^2), namely

$$\nabla^2 = g^{jk}\left(\frac{\partial^2}{\partial x^j \partial x^k} - \Gamma^p_{jk}\frac{\partial}{\partial x^p}\right).$$

In general, there are overall 12 essentially independent non-zero components of the covariant curvature tensor R_{ijkm}, which act as a basis in which we might express the Ricci tensor and Einstein tensor components R_{jk} and G_{jk} and the curvature invariant R, and we now proceed to do precisely this. As far as the author is aware, in order to deduce the final expressions, there is no approach other than by direct individual calculations, since each of the 12 curvature tensor components are not readily characterised in some general way. By way of illustration, we start with the calculation for R_{00}, thus

$$\begin{aligned}
R_{00} &= g^{im}R_{i00m} \\
&= g^{11}R_{1001} + g^{12}R_{1002} + g^{21}R_{2001} + g^{22}R_{2002} + g^{33}R_{3003} \\
&= -g^{11}R_{0101} + g^{12}R_{0120} + g^{21}R_{0120} - g^{22}R_{0202} + g^{33}R_{3003} \\
&= -g^{11}R_{0101} + 2g^{12}R_{0120} - g^{22}R_{0202} + g^{33}R_{3003},
\end{aligned}$$

and proceeding similarly, we may deduce the following expressions:

$$\begin{aligned}
R_{00} &= -g^{11}R_{0101} + 2g^{12}R_{0120} - g^{22}R_{0202} + g^{33}R_{3003}, \qquad (11.37)\\
R_{11} &= -g^{00}R_{0101} + 2g^{02}R_{0112} - g^{22}R_{1212} + g^{33}R_{3113}, \\
R_{22} &= -g^{00}R_{0202} + 2g^{01}R_{2012} - g^{11}R_{1212} + g^{33}R_{3223}, \\
R_{01} &= g^{10}R_{0101} - g^{12}R_{0112} - g^{20}R_{0120} + g^{22}R_{2012} + g^{33}R_{3013}, \\
R_{02} &= -g^{10}R_{0120} + g^{11}R_{0112} + g^{20}R_{0202} - g^{21}R_{2012} + g^{33}R_{3023}, \\
R_{12} &= -g^{00}R_{0102} - g^{01}R_{0112} - g^{20}R_{2012} + g^{21}R_{1212} + g^{33}R_{3123},
\end{aligned}$$

and $R_{33} = g^{im}R_{i33m} = R^*g_{33}/2$. Now on forming the curvature invariant $R = g^{jk}R_{jk} = g^{im}g^{jk}R_{ijkm}$, the x^3 spatial direction contributes both through the terms R_{3jk3} in the above relations and through the term R_{33}, and we have

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + 2g^{01}R_{01} + 2g^{02}R_{02} + 2g^{12}R_{12} + R^*, \tag{11.38}$$

where $R^* = g^{33}g^{jk}R_{3jk3} + g^{im}g^{33}R_{i33m} = 2g^{jk}g^{33}R_{3jk3}$ is the total contribution to the curvature invariant arising from the x^3 spatial dimension and the summations over j and k in this expression are restricted to $0, 1, 2$. On using the above expressions for the Ricci tensor components R_{jk}, we find that

$$\begin{aligned} R &= R^* \\ &- 2[(g^{00}g^{11} - g^{01}g^{01})R_{0101} + (g^{00}g^{22} - g^{02}g^{02})R_{0202} + (g^{11}g^{22} - g^{12}g^{12})R_{1212} \\ &- 2(g^{00}g^{12} - g^{20}g^{10})R_{0120} - 2(g^{11}g^{02} - g^{01}g^{12})R_{0112} - 2(g^{22}g^{10} - g^{02}g^{21})R_{2012}]. \end{aligned}$$

Now by direct calculation or otherwise, we may deduce the formulae

$$g^{00}g^{11} - g^{01}g^{01} = \frac{g_{22}}{g}, \quad g^{00}g^{22} - g^{02}g^{02} = \frac{g_{11}}{g}, \quad g^{11}g^{22} - g^{12}g^{12} = \frac{g_{00}}{g},$$
$$g^{00}g^{12} - g^{20}g^{10} = -\frac{g_{12}}{g}, \quad g^{11}g^{20} - g^{01}g^{12} = -\frac{g_{02}}{g}, \quad g^{22}g^{10} - g^{02}g^{12} = -\frac{g_{01}}{g},$$

where g is the determinant of the 3×3 matrix (11.36), and from Eqs. (11.37) and (11.38), we may eventually deduce the expression for the curvature invariant R, namely

$$\begin{aligned} R &= 2(g^{mn}R_{3mn3})/g_{33} \\ &- 2(g_{00}R_{1212} + g_{11}R_{0202} + g_{22}R_{0101} + 2g_{12}R_{0120} + 2g_{02}R_{1021} + 2g_{01}R_{2012})/g, \end{aligned} \tag{11.39}$$

where $g^* = gg_{33}$ is the determinant of the 4×4 matrix given by (11.33), and the first term in this expression can be alternatively written as simply $R^* = 2g^{jk}g^{33}R_{3jk3}$, and R^* is the contribution to the curvature invariant R arising from the x^3 spatial direction.

We are now in a position to evaluate the Einstein tensor from $G_{ij} = R_{ij} - Rgij/2$, and to achieve this, we need the particular relations

$$\frac{g_{11}g_{22}}{g} - g^{00} = \frac{g_{12}g_{12}}{g}, \quad \frac{g_{00}g_{22}}{g} - g^{11} = \frac{g_{02}g_{02}}{g}, \quad \frac{g_{00}g_{11}}{g} - g^{22} = \frac{g_{01}g_{01}}{g},$$
$$\frac{g_{02}g_{12}}{g} - g^{01} = \frac{g_{01}g_{22}}{g}, \quad \frac{g_{01}g_{12}}{g} - g^{02} = \frac{g_{02}g_{11}}{g}, \quad \frac{g_{01}g_{02}}{g} - g^{12} = \frac{g_{00}g_{12}}{g}.$$

On combining these relations with the above Eqs. (11.37) and (11.38), we may deduce the following equations for the Einstein tensor $G_{ij} = R_{ij} - Rgij/2$, thus

$$G_{00} = \frac{g_{02}g_{02}}{g}R_{0101} + \frac{g_{01}g_{01}}{g}R_{0202} + \frac{g_{00}g_{00}}{g}R_{1212} + \frac{2g_{01}g_{02}}{g}R_{0120} \tag{11.40}$$

$$+\frac{2g_{00}g_{02}}{g}R_{0112}+\frac{2g_{00}g_{01}}{g}R_{2012}-\frac{R^*}{2}g_{00}+g^{33}R_{3003},$$

$$G_{11}=\frac{g_{12}g_{12}}{g}R_{0101}+\frac{g_{11}g_{11}}{g}R_{0202}+\frac{g_{01}g_{01}}{g}R_{1212}+\frac{2g_{01}g_{12}}{g}R_{0112}$$

$$+\frac{2g_{11}g_{12}}{g}R_{0120}+\frac{2g_{11}g_{01}}{g}R_{2012}-\frac{R^*}{2}g_{11}+g^{33}R_{3113},$$

$$G_{22}=\frac{g_{22}g_{22}}{g}R_{0101}+\frac{g_{12}g_{12}}{g}R_{0202}+\frac{g_{02}g_{02}}{g}R_{1212}+\frac{2g_{12}g_{22}}{g}R_{0120}$$

$$+\frac{2g_{22}g_{02}}{g}R_{0112}+\frac{2g_{12}g_{02}}{g}R_{2012}-\frac{R^*}{2}g_{22}+g^{33}R_{3223},$$

$$G_{01}=\frac{g_{02}g_{12}}{g}R_{0101}+\frac{g_{01}g_{11}}{g}R_{0202}+\frac{g_{00}g_{01}}{g}R_{1212}+\frac{(g_{00}g_{12}+g_{01}g_{02})}{g}R_{0112}$$

$$+\frac{(g_{11}g_{02}+g_{01}g_{12})}{g}R_{0120}+\frac{(g_{00}g_{11}+g_{01}g_{01})}{g}R_{2012}-\frac{R^*}{2}g_{01}+g^{33}R_{3013},$$

$$G_{02}=\frac{g_{02}g_{22}}{g}R_{0101}+\frac{g_{01}g_{12}}{g}R_{0202}+\frac{g_{00}g_{02}}{g}R_{1212}+\frac{(g_{00}g_{22}+g_{02}g_{02})}{g}R_{0112}$$

$$+\frac{(g_{01}g_{22}+g_{02}g_{12})}{g}R_{0120}+\frac{(g_{00}g_{12}+g_{01}g_{02})}{g}R_{2012}-\frac{R^*}{2}g_{02}+g^{33}R_{3023},$$

$$G_{12}=\frac{g_{12}g_{22}}{g}R_{0101}+\frac{g_{12}g_{11}}{g}R_{0202}+\frac{g_{01}g_{02}}{g}R_{1212}+\frac{(g_{01}g_{22}+g_{02}g_{12})}{g}R_{0112}$$

$$+\frac{(g_{11}g_{22}+g_{12}g_{12})}{g}R_{0120}+\frac{(g_{11}g_{02}+g_{01}g_{12})}{g}R_{2012}-\frac{R^*}{2}g_{12}+g^{33}R_{3123},$$

and $G_{33}=R_{33}-Rg_{33}/2=(R^*-R)g_{33}/2$ and all other components involving the index 3, namely G_{3j} for $j=0,1,2$ are zero.

Now from the expressions for the Einstein tensor $G^i_j=g^{ik}R_{jk}-R\delta^i_j/2$, it is immediately apparent that the trace $G=R-2R=-R$, and we can use this result to check the veracity of the above expressions for G_{ij} as follows. In the summation $G=g^{jk}G_{jk}$ and because of the essentially three-dimensional nature of the tensors involved, we first need to perform the summation over $j,k=0,1,2$, and then we have to include the term involving G_{33}. Further, each of the above expressions involve three types of terms: the terms involving the six non-zero components of the covariant curvature tensor R_{ijkm}, which are R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012}, and then the two terms $-R^*g_{jk}/2$ and $g^{33}R_{3jk3}$.

First, we examine the contribution to $G=g^{ik}G_{jk}$ arising from the six terms R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012}, and for purposes of illustration, we examine the contribution from the term R_{0101}, which becomes

$$(g^{00}g_{02}g_{02}+g^{11}g_{12}g_{12}+g^{22}g_{22}g_{22}+2g^{01}g_{02}g_{12}$$

$$+ 2g^{02}g_{02}g_{22} + 2g^{12}g_{12}g_{22})\frac{R_{0101}}{g},$$

which we may reorganise as

$$[(g_{02}(g^{00}g_{02} + g^{01}g_{12} + g^{02}g_{22}) + g_{12}(g^{01}g_{02} + g^{11}g_{12} + g^{12}g_{22})$$
$$+ g_{22}(g^{02}g_{02} + g^{12}g_{12} + g^{22}g_{22})]\frac{R_{0101}}{g},$$

so that on using $g^{ik}g_{kj} = \delta^i_j$, this term becomes simply $g_{22}R_{0101}/g$. Similarly, we may show that the contribution arising from the six terms R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120} and R_{2012} becomes

$$(g_{00}R_{1212} + g_{11}R_{0202} + g_{22}R_{0101} + 2g_{12}R_{0120} + 2g_{02}R_{1021} + 2g_{01}R_{2012})/g,$$

which from (11.39) we can identify as simply $(R^* - R)/2$. Thus, the total summation becomes

$$G = g^{jk}G_{jk} = \frac{(R^* - R)}{2} - \frac{R^*}{2}g^{jk}g_{jk} + g^{33}g^{jk}R_{3jk3} + g^{33}G_{33},$$
$$= \frac{(R^* - R)}{2} - \frac{3R^*}{2} + \frac{R^*}{2} + \frac{(R^* - R)}{2} = -R,$$

as required.

Although not particularly illuminating or insightful, with i, j and k all distinct and taking on only the values 0, 1 and 2, and with no summation over repeated indices, these formulae can be formally summarised by the following expressions:

$$G_{ii} = \frac{g_{ij}g_{ij}}{g}R_{kiki} + \frac{g_{ik}g_{ik}}{g}R_{ijij} + \frac{g_{ii}g_{ii}}{g}R_{kjkj} + \frac{2g_{ij}g_{ik}}{g}R_{kiij}$$
$$+ \frac{2g_{ii}g_{ij}}{g}R_{ikkj} + \frac{2g_{ii}g_{ik}}{g}R_{jkij} - \frac{R^*}{2}g_{ii} + g^{33}R_{3ii3},$$
$$G_{ij} = \frac{g_{ik}g_{jk}}{g}R_{ijij} + \frac{g_{ij}g_{ii}}{g}R_{kjkj} + \frac{g_{ij}g_{jj}}{g}R_{kiki} + \frac{(g_{ii}g_{jj} + g_{ij}g_{ij})}{g}R_{kijk}$$
$$+ \frac{(g_{ki}g_{jj} + g_{ij}g_{kj})}{g}R_{kiij} + \frac{(g_{kj}g_{ii} + g_{ij}g_{ki})}{g}R_{kjji} - \frac{R^*}{2}g_{ij} + g^{33}R_{3ij3}.$$

11.7 Two Illustrative Line Elements

Illustration (i) As an illustration of these results, we consider the simple wormhole geometry of [79], which is given by the four-dimensional line element, thus

$$ds^2 = (cdt)^2 - (d\rho)^2 - (b^2 + \rho^2)(d\theta)^2 - (b^2 + \rho^2)\sin^2\theta(d\phi)^2,$$

where b is the throat radius and (ρ, θ, ϕ) are the usual spherical polar coordinates, so that in this case, we have $(x^0, x^1, x^2, x^3) = (ct, \rho, \theta, \phi)$ and the metric and conjugate metric tensors are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -(b^2 + \rho^2) & 0 \\ 0 & 0 & 0 & -(b^2 + \rho^2)\sin^2\theta \end{pmatrix},$$

and

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{(b^2+\rho^2)} & 0 \\ 0 & 0 & 0 & -\frac{1}{(b^2+\rho^2)\sin^2\theta} \end{pmatrix},$$

where the determinants $g^* = -(b^2 + \rho^2)^2 \sin^2\theta$ and $g = (b^2 + \rho^2)$. The non-zero Christoffel symbols of the first kind are

$$[22, 1] = \rho, \quad [12, 2] = -\rho, \quad [33, 1] = \rho\sin^2\theta, \quad [31, 3] = -\rho\sin^2\theta,$$
$$[33, 2] = (b^2 + \rho^2)\sin\theta\cos\theta, \qquad [32, 3] = -(b^2 + \rho^2)\sin\theta\cos\theta,$$

and from these relations and Eq. (11.25), we may deduce the following non-zero components of the covariant curvature tensor R_{ijkm}, thus

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^{1^2}} + g^{22}[12, 2][12, 2] = 1 - \frac{\rho^2}{(b^2 + \rho^2)} = \frac{b^2}{(b^2 + \rho^2)},$$

$$R_{3113} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{1^2}} - g^{33}[13, 3][13, 3] = -\sin^2\theta + \frac{\rho^2\sin^2\theta}{(b^2 + \rho^2)} = -\frac{b^2\sin^2\theta}{(b^2 + \rho^2)},$$

$$R_{3223} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{2^2}} + g^{11}[22, 1][33, 1] - g^{33}[23, 3][23, 3]$$
$$= (b^2 + \rho^2)(\sin^2\theta - \cos^2\theta) - \rho^2\sin^2\theta + (b^2 + \rho^2)\cos^2\theta = b^2\sin^2\theta.$$

The latter two components result in a net zero contribution to the curvature invariant arising from the x^3 spatial direction, since

$$R^* = 2g^{33}(g^{11}R_{3113} + g^{22}R_{3223}) = \frac{2}{(b^2 + \rho^2)\sin^2\theta}\left(-\frac{b^2\sin^2\theta}{(b^2 + \rho^2)} + \frac{b^2\sin^2\theta}{(b^2 + \rho^2)}\right) = 0,$$

so that only the component R_{1212} contributes to the curvature invariant R through the term $-2g_{00}R_{1212}/g$, and we have simply $R = -2b^2/(b^2+\rho^2)^2$.

From the relations (11.40), we find that the only non-zero components of the Einstein tensor G_{ij} become

$$G_{00} = \frac{g_{00}g_{00}}{g}R_{1212} = \frac{b^2}{(b^2+\rho^2)^2}, \qquad G_{11} = g^{33}R_{3113} = \frac{b^2}{(b^2+\rho^2)^2},$$

$$G_{22} = g^{33}R_{3223} = -\frac{b^2}{(b^2+\rho^2)}, \qquad G_{33} = -\frac{Rg_{33}}{2} = -\frac{b^2\sin^2\theta}{(b^2+\rho^2)},$$

where G_{33} originates from the expression $G_{33} = R_{33} - Rg_{33}/2 = (R^* - R)g_{33}/2$ with $R^* = 0$ and noting that correctly $G = g^{ij}G_{ij} = -R$. On noting that the only non-zero Christoffel symbols of the second kind are

$$\Gamma^2_{12} = \frac{\rho}{(b^2+\rho^2)}, \qquad \Gamma^3_{13} = \frac{\rho}{(b^2+\rho^2)}, \qquad \Gamma^1_{22} = \rho,$$

$$\Gamma^1_{33} = -\rho\sin^2\theta, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3_{23} = \cot\theta,$$

the above expressions for the Einstein tensor can be shown to satisfy the divergence free equations $G^i_{k;i} = 0$ in the form of Eqs. (11.35); thus, we have

$$G^i_{k;i} = g^{ij}G_{jk;i} = g^{ij}\left\{\frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij}G_{nk} - \Gamma^p_{ik}G_{jp}\right\} = 0,$$

and for $k = 1$ we obtain

$$\begin{aligned} G^i_{k;i} &= g^{ij}\left\{\frac{\partial G_{j1}}{\partial x^i} - \Gamma^n_{ij}G_{n1} - \Gamma^p_{i1}G_{jp}\right\} \\ &= \left\{-\frac{\partial G_{11}}{\partial \rho} - g^{ij}\Gamma^1_{ij}G_{11} - g^{ij}\Gamma^2_{i1}G_{j2} - g^{ij}\Gamma^3_{i1}G_{j3}\right\} \\ &= -\left\{\frac{\partial G_{11}}{\partial \rho} + g^{22}\Gamma^1_{22}G_{11} + g^{33}\Gamma^1_{33}G_{11} + g^{22}\Gamma^2_{21}G_{22} + g^{33}\Gamma^3_{31}G_{33}\right\} \\ &= -\left\{\frac{\partial G_{11}}{\partial \rho} + \frac{2\rho}{(b^2+\rho^2)}G_{11} - \frac{\rho}{(b^2+\rho^2)^2}\left(G_{22} + \frac{G_{33}}{\sin^2\theta}\right)\right\}, \end{aligned}$$

which on performing the differentiation can be shown to be identically zero, and similarly the three equations $G^i_{k;i} = 0$ arising from $k = 0, 2$ and 3 can be shown to be trivially satisfied. The reader may be comforted to know that these results are not completely apparent and are consequences of the particular diagonal structure of the tensor G_{ij} and the particular non-zero Christoffel symbols of the second kind and the reader may need to give each equation a close examination before being convinced.

Illustration (ii) As a second illustration of the results of the previous section, we consider the Einstein universe given by the four-dimensional line element, thus

$$ds^2 = (cdt)^2 - \frac{(dr)^2}{(1-(r/\mathscr{R})^2)} - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2,$$

where $\mathscr{R}$ is a constant and (r, θ, ϕ) are the usual spherical polar coordinates, so that in this case, we have $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$ and the metric and conjugate metric tensors are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1}{(1-(r/\mathscr{R})^2)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},$$

and

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1-(r/\mathscr{R})^2) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix},$$

and for the determinants g^* and g, we have $g^* = -r^4\sin^2\theta/(1-(r/\mathscr{R})^2)$ and $g = r^2/(1-(r/\mathscr{R})^2)$. The non-zero Christoffel symbols of the first kind are

$$[11, 1] = -\frac{r/\mathscr{R}^2}{(1-(r/\mathscr{R})^2)^2}, \quad [12, 2] = -r, \quad [22, 1] = r, \quad [33, 1] = r\sin^2\theta,$$

$$[31, 3] = -r\sin^2\theta, \quad [33, 2] = r^2\sin\theta\cos\theta, \qquad [32, 3] = -r^2\sin\theta\cos\theta,$$

and from these relations and Eq. (11.25), we may deduce the following non-zero components of the covariant curvature tensor R_{ijkm}, thus

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^{1^2}} + g^{22}[12, 2][12, 2] - g^{11}[22, 1][11, 1] = -\frac{(r/\mathscr{R})^2}{(1-(r/\mathscr{R})^2)},$$

$$R_{3113} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{1^2}} + g^{11}[33, 1][11, 1] - g^{33}[13, 3][13, 3] = \frac{(r/\mathscr{R})^2\sin^2\theta}{(1-(r/\mathscr{R})^2)},$$

$$R_{3223} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^{2^2}} + g^{11}[22, 1][33, 1] - g^{33}[23, 3][23, 3] = (r/\mathscr{R})^2 r^2\sin^2\theta,$$

$$R_{3123} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x^1\partial x^2} + g^{22}[12, 2][33, 2] - g^{33}[13, 3][23, 3] = 0.$$

The two components R_{3113} and R_{3223} both contribute to the curvature invariant R^* arising from the x^3 spatial direction, thus

$$R^* = 2g^{33}(g^{11}R_{3113} + g^{22}R_{3223}) = \frac{2}{r^2\sin^2\theta}\left((1-(r/\mathscr{R})^2)R_{3113} + \frac{R_{3223}}{r^2}\right)$$
$$= \frac{2}{r^2\sin^2\theta}\Big((r/\mathscr{R})^2\sin^2\theta + (r/\mathscr{R})^2\sin^2\theta\Big) = \frac{4}{\mathscr{R}^2},$$

so that altogether along with the contribution to the curvature invariant R from the component R_{1212} we have from Eq. (11.39) the following result:

$$R = \frac{4}{\mathscr{R}^2} + \frac{2(1-(r/\mathscr{R})^2)}{r^2}\frac{(r/\mathscr{R})^2}{(1-(r/\mathscr{R})^2)} = \frac{6}{\mathscr{R}^2},$$

which is well-known.

From Eq. (11.40), we may show that the only non-zero components of the Einstein tensor G_{ij} become

$$G_{00} = \frac{g_{00}g_{00}}{g}R_{1212} - \frac{R^*}{2}g_{00} = -\frac{3}{\mathscr{R}^2},$$
$$G_{11} = -\frac{R^*}{2}g_{11} + g^{33}R_{3113} = \frac{1}{\mathscr{R}^2(1-(r/\mathscr{R})^2)},$$
$$G_{22} = -\frac{R^*}{2}g_{22} + g^{33}R_{3223} = (r/\mathscr{R})^2,$$
$$G_{33} = \frac{(R^*-R)}{2}g_{33} = (r/\mathscr{R})^2\sin^2\theta,$$

where G_{33} originates from the expression $G_{33} = R_{33} - Rg_{33}/2 = (R^* - R)g_{33}/2$, and again as a check, we may confirm that the above expressions correctly satisfy $G = g^{ij}G_{ij} = -6/\mathscr{R}^2 = -R$. For this metric, the only non-zero Christoffel symbols of the second kind are

$$\Gamma^1_{11} = \frac{r/\mathscr{R}^2}{(1-(r/\mathscr{R})^2)}, \quad \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^1_{22} = -r(1-(r/\mathscr{R})^2),$$
$$\Gamma^1_{33} = -r\sin^2\theta(1-(r/\mathscr{R})^2), \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta,$$

and again the above expressions for the Einstein tensor can be shown to satisfy the divergence free equation $G^i_{k;i} = 0$ in the form of Eqs. (11.35); thus, we have

$$G^i_{k;i} = g^{ij}G_{jk;i} = g^{ij}\left\{\frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij}G_{nk} - \Gamma^p_{ik}G_{jp}\right\} = 0,$$

so that for $k = 1$, we have

$$G^i_{k;i} = g^{ij}\left\{\frac{\partial G_{j1}}{\partial x^i} - \Gamma^n_{ij}G_{n1} - \Gamma^p_{i1}G_{jp}\right\}$$

$$= \left\{g^{11}\frac{\partial G_{11}}{\partial x^1} - g^{ij}\Gamma^1_{ij}G_{11} - g^{ij}\Gamma^1_{i1}G_{j1} - g^{ij}\Gamma^2_{i1}G_{j2} - g^{ij}\Gamma^3_{i1}G_{j3}\right\}$$

$$= \left\{g^{11}\frac{\partial G_{11}}{\partial r} - (g^{11}\Gamma^1_{11} + g^{22}\Gamma^1_{22} + g^{33}\Gamma^1_{33})G_{11} - g^{11}\Gamma^1_{11}G_{11} - g^{22}\Gamma^2_{21}G_{22} - g^{33}\Gamma^3_{31}G_{33}\right\},$$

and again on performing the differentiation and using the above expressions for the conjugate metric tensor and the Christoffel symbols, this equation may be shown to be identically zero, and similarly the three equations $G^i_{k;i} = 0$ arising from $k = 0, 2$ and 3 can be shown to be trivially satisfied. Again, we remind the reader that none of these outcomes are immediately obvious, and in formulating each of the results, we constantly have in mind the particular diagonal structures of the tensors g^{ij} and G_{ij} and the particular non-zero Christoffel symbols of the second kind, and the reader should give each outcome a close examination to convince themselves of its veracity. In the final section of this chapter, we provide an example for which the details are far more complicated.

11.8 Spiral Gravitating Structures

In this section, we examine a possible cosmological model of general relativity involving logarithmic spirals and seven arbitrary constants. While there is considerable astronomical evidence that spiral structures commonly occur in the universe, there appear to be no known formal solutions of the general relativistic field equations reflecting such structures. The Schwarzschild solution relates to spatially spherically symmetric gravitational fields, while the stationary Kerr solution is axially symmetric. However, in nonlinear continuum mechanics, formal similarity solutions involving logarithmic spiral similarity invariants are well-known, and indeed the most general similarity forms for isotropic materials satisfying the principle of material indifference have been given by the author [46]. These formal similarity solutions essentially arise from rotational invariances and invariance under the basic length scale that is adopted. We generalise the particular Minkowski line element given by (11.42) to the line element given by Eq. (11.41), which involves the seven completely arbitrary constants A, B, C, D, E, F and λ. The motivating Minkowski line element (11.42) is a special case of (11.41) that arises from the particular values of the constants A, B, C, D, E, F and λ given by (11.43), which satisfy the three relations (11.44).

Below we show that solutions of the Einstein vacuum equations that might involve logarithmic spirals as a similarity variable arise from the essentially two-dimensional (spatially) metric given by

$$(ds)^2 = (1-\lambda r^2)(cdt)^2 - A(dr)^2 - Br^2(d\theta)^2 - 2Crdrcdt \\ - 2Dr^2 d\theta cdt - 2Erdrd\theta - F(dz)^2, \tag{11.41}$$

where $(x^0, x^1, x^2, x^3) = (ct, r, \theta, z)$, (r, θ, z) denote the usual cylindrical polar coordinates and where A, B, C, D, E, F and λ denote seven arbitrary constants, in principle, to be determined from the general relativistic field equations. We provide a detailed verification below that the Einstein tensor for the metric (11.41) is indeed divergence free for all values of the seven constants without additional restrictions.

From a conventional general relativistic perspective, only five of these constants would be regarded as essential, since on face value the constants A and F can be scaled immediately to unity, even though the special case $A = 0$ would have to be considered as a separate case. From the author's perspective, setting $A = F = 1$, there is an apparent loss of symmetry and parity in the structure, and moreover as shown in [59], reducing and simplifying the metric in this manner often closes off opportunities to determine new solutions. We demonstrate below that the metric (11.41) provides a formal solution of the divergence free vacuum field equations for all values of the seven constants without additional restrictions. We comment that while the two illustrative examples presented above are both such that the Einstein tensor satisfies the divergence free vacuum condition, it is however non-vanishing, and therefore, these models are considered to represent some underlying intrinsic property of space-time that is additional to that generated from the energy-momentum tensor. It is important to point out that the model examined here also shares this characteristic, and formally here, we simply verify the vanishing of the divergence of the Einstein tensor.

The particular structure of this metric is motivated as follows. In the cylindrical polar coordinates (r, θ, z), a rotation around the $z-$axis is given simply by $\theta' = \theta + \omega t$, and therefore in order generalise this to generate logarithmic spirals, we consider the invariant $\xi = \sigma\theta + \mu \log r + \omega t$ and the generalised rotational and stretching transformation which in rectangular Cartesian coordinates (x, y, z) is given by

$$x = \nu r \cos(\sigma\theta + \mu \log r + \omega t), \qquad y = \nu r \sin(\sigma\theta + \mu \log r + \omega t),$$

combined with a constant stretch along the $z-$axis, thus $z \longrightarrow \kappa z$, where σ, μ, ν, ω and κ all denote arbitrary constants. From the differentials

$$dx = \nu \cos\xi dr - \nu r \sin\xi \left(\sigma d\theta + \frac{\mu}{r} dr + \omega dt\right),$$
$$dy = \nu \sin\xi dr + \nu r \cos\xi \left(\sigma d\theta + \frac{\mu}{r} dr + \omega dt\right),$$

the Minkowski line element becomes

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{11.42}$$

$$= (1 - (\nu\omega r/c)^2)(cdt)^2 - \nu^2(1+\mu^2)(dr)^2 - (\nu\sigma r)^2(d\theta)^2 - 2(\mu\omega\nu^2/c)rdrcdt$$
$$- 2(\sigma\omega\nu^2/c)r^2 d\theta cdt - 2\mu\sigma\nu^2 rdrd\theta - \kappa^2(dz)^2,$$

which evidently assumes the form of the line element (11.41) with the seven arbitrary constants A, B, C, D, E, F and λ taking on the particular valiues

$$A = \nu^2(1+\mu^2), \quad B = (\nu\sigma)^2, \quad C = \mu\omega\nu^2/c, \qquad (11.43)$$
$$D = \sigma\omega\nu^2/c, \quad E = \mu\sigma\nu^2, \quad F = \kappa^2, \quad \lambda = (\nu\omega/c)^2.$$

We emphasise that subsequently we regard the seven constants A, B, C, D, E, F and λ to be completely arbitrary, and we exploit the above particular values rather as reference values to guide the ensuring mathematical analysis. Specifically, we observe that these particular values satisfy the three relations,

$$BC - DE = 0, \qquad CD - \lambda E = 0, \qquad D^2 - \lambda B = 0, \qquad (11.44)$$

and we subsequently observe these quantities emerging as factors in the analysis. We further comment that in proposing the metric (11.41) as a potentially interesting line element, we have in mind certain issues arising from nonlinear continuum mechanics. If we have in mind a spiral gravitating structure, then note that the invariant $\xi = \sigma\theta + \mu \log r + \omega t$ arises from both the two- and three-dimensional spatial rotational groups and that use of this invariant has two important features. Firstly, the constant $\sigma \neq 1$ embodies more than a strict rotation and might well on its own produce an interesting outcome. Secondly, the $\log r$ term is a natural amendment, since it behaves formally like θ in the sense that the factor $1/r$ precedes both $\partial/\partial\theta$ and $\partial/\partial r$. In finite elasticity, this characteristic is well-known and the invariant $\sigma\theta + \mu \log r$ forms the basis of an important exact solution in that discipline, and further references to this topic may be found in [46]. Accordingly, these two features in their own right might generate interesting and new outcomes, but here in proposing the line element (11.41), we are adding further levels of arbitrariness.

Again noting that (r, θ, z) are the usual cylindrical polar coordinates and $(x^0, x^1, x^2, x^3) = (ct, r, \theta, z)$, then corresponding to (11.41) the metric tensor g_{ij} and conjugate metric tensor g^{ij} are given, respectively, by

$$g_{ij} = \begin{pmatrix} 1-\lambda r^2 & -Cr & -Dr^2 & 0 \\ -Cr & -A & -Er & 0 \\ -Dr^2 & -Er & -Br^2 & 0 \\ 0 & 0 & 0 & -F \end{pmatrix}, \qquad (11.45)$$

and

$$g^{ij} = \begin{pmatrix} (AB-E^2)r^2/g & -(BC-DE)r^3/g & -(AD-CE)r^2/g & 0 \\ -(BC-DE)r^3/g & -[Br^2+(D^2-\lambda B)r^4]/g & [Er+(DC-\lambda E)r^3]/g & 0 \\ -(AD-CE)r^2/g & [Er+(DC-\lambda E)r^3]/g & -[A+(C^2-\lambda A)r^2]/g & 0 \\ 0 & 0 & 0 & -1/F \end{pmatrix}, \tag{11.46}$$

where the determinant g of the three-dimensional matrix g_{ij} is given by

$$\begin{aligned} g &= (1-\lambda r^2)(AB-E^2)r^2 + (BC^2 - 2CDE + AD^2)r^4 \\ &= (AB-E^2)r^2 + [(BC-DE)^2 + (AB-E^2)(D^2-\lambda B)]r^4/B, \end{aligned} \tag{11.47}$$

while the determinant g^* of the four-dimensional matrix g_{ij} is simply given by $g^* = -Fg$.

Christoffel Symbols of First Kind $[ij, k]$ The only non-zero Christoffel symbols of the first kind for the metric (11.41) are

$$[00,1] = \lambda r, \quad [11,0] = -C, \quad [11,2] = -E, \quad [22,1] = Br, \quad [01,0] = -\lambda r, \tag{11.48}$$

$$[01,2] = -Dr, \quad [12,2] = -Br, \quad [02,1] = Dr, \quad [12,0] = -Dr.$$

Covariant Curvature Tensor R_{ijkm} From the relations (11.48) and Eq. (11.25), we may deduce the following non-zero components of the covariant curvature tensor R_{ijkm}, thus

$$R_{0101} = -\frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^{1^2}} + g^{np}([01,n][01,p] - [11,n][00,p])$$

$$= \lambda + g^{0p}[01,0][01,p] + g^{2p}[01,2][01,p] - g^{n1}[11,n][00,1]$$

$$\begin{aligned} &= \lambda + g^{00}[01,0]^2 + 2g^{02}[01,0][01,2] + g^{22}[01,2]^2 \\ &\quad - g^{01}[11,0][00,1] - g^{12}[11,2][00,1] \end{aligned}$$

$$= -\frac{r^2}{g}[A(D^2-\lambda B) + (DC-\lambda E)^2 r^2],$$

$$R_{0202} = g^{np}([02, n][02, p] - [22, n][00, p])$$

$$= g^{1p}([02, 1][02, p] - [22, 1][00, p])$$

$$= -\frac{r^4}{g}[B + (D^2 - \lambda B)r^2](D^2 - \lambda B),$$

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^{1^2}} + g^{np}([12, n][12, p] - [22, n][11, p])$$

$$= B + g^{00}[12, 0]^2 + g^{22}[12, 2]^2 + 2g^{02}[12, 0][12, 2]$$
$$- g^{01}[22, 1][11, 0] - g^{12}[22, 1][11, 2]$$

$$= -\frac{r^4}{g}(BC - DE)^2,$$

$$R_{2012} = g^{np}([01, n][22, p] - [02, n][12, p])$$

$$= g^{n1}([01, n][22, 1] - g^{1p}[02, 1][12, p])$$

$$= g^{01}[01, 0][22, 1] + g^{21}[01, 2][22, 1] - g^{01}[02, 1][12, 0] - g^{12}[02, 1][12, 2]$$

$$= -\frac{r^5}{g}(BC - DE)(D^2 - \lambda B),$$

$$R_{0112} = \frac{1}{2}\frac{\partial^2 g_{02}}{\partial x^{1^2}} + g^{np}([11, n][02, p] - [12, n][01, p])$$

$$= -D + g^{10}[02, 1][11, 0] + g^{21}[02, 1][11, 2] - g^{00}[12, 0][01, 0]$$
$$- g^{20}[12, 2][01, 0] - g^{02}[12, 0][01, 2] - g^{22}[12, 2][01, 2]$$

$$= \frac{r^4}{g}(BC - DE)(DC - \lambda E),$$

$$R_{0120} = g^{np}([12, n][00, p] - [01, n][02, p])$$

$$= g^{n1}([12, n][00, 1] - [01, n][02, 1])$$

$$= g^{01}[12, 0][00, 1] + g^{21}[12, 2][00, 1] - g^{01}[01, 0][02, 1] - g^{21}[01, 2][02, 1]$$

$$= \frac{r^3}{g}[E + (DC - \lambda E)r^2](D^2 - \lambda B),$$

We now introduce five new constants α, β, γ, δ and ϵ defined by

$$\alpha = BC - DE, \qquad \beta = D^2 - \lambda B, \qquad \gamma = \lambda E - DC,$$
$$\delta = AB - E^2, \qquad \epsilon = AD - CE,$$

so that the above expressions for $R_{0101}, R_{0202}, R_{1212}, R_{0112}, R_{0120}$ and R_{2012}, simplify to become

$$R_{0101} = -\frac{r^2}{g}(\beta A + \gamma^2 r^2), \qquad R_{0202} = -\frac{r^4}{g}(\beta B + \beta^2 r^2), \qquad R_{1212} = -\frac{r^4}{g}\alpha^2, \tag{11.49}$$

$$R_{2012} = -\frac{r^5}{g}\alpha\beta, \qquad R_{0112} = -\frac{r^4}{g}\alpha\gamma, \qquad R_{0120} = -\frac{r^3}{g}(\beta\gamma r^2 - \beta E).$$

We may readily verify that the new constants satisfy

$$\alpha\lambda + \beta C + \gamma D = 0, \qquad \alpha D + \beta E + \gamma B = 0,$$
$$\alpha C + \beta A + \gamma E = AD^2 + BC^2 - 2CDE - \lambda(AB - E^2),$$

and it proves convenient to introduce yet another new constant Ω defined by

$$\Omega = \alpha C + \beta A + \gamma E,$$

so that we have the important relations

$$AD^2 + BC^2 - 2CDE = \Omega + \lambda\delta, \qquad \Omega = (\alpha^2 + \beta\delta)/B,$$

and, noting that in terms of these new constants, the determinant g given by Eq. (11.47) becomes

$$g = \delta r^2 + \Omega r^4 = \delta r^2 + (\alpha^2 + \beta\delta)r^4/B.$$

Curvature Invariant R On using the immediately above relations together with Eq. (11.39) and the metric tensor (11.45), we may deduce the following expression for the curvature invariant R, thus

$$R = \frac{2r^4}{g^2}\left\{(\alpha^2 - 2\beta\delta) - r^2(\lambda\alpha^2 + A\beta^2 + B\gamma^2 + 2C\alpha\beta + 2D\alpha\gamma + 2E\beta\gamma)\right\}.$$

We may simplify the coefficient of r^2 in this expression as follows:

$$\begin{aligned}&\lambda\alpha^2 + A\beta^2 + B\gamma^2 + 2C\alpha\beta + 2D\alpha\gamma + 2E\beta\gamma\\ &= \alpha(\alpha\lambda + \beta C + \gamma D) + \beta(\alpha C + \beta A + \gamma E) + \gamma(\alpha D + \beta E + \gamma B)\\ &= \beta(\alpha C + \beta A + \gamma E) = \beta\Omega,\end{aligned}$$

and therefore, the above expression for the curvature invariant R becomes simply

$$R = \frac{2r^4}{g^2}\left\{(\alpha^2 - 2\beta\delta) - \beta\Omega r^2\right\}.$$

Einstein Tensor G_{ij} Using the expressions (11.49) for the non-zero components of the covariant curvature tensor R_{ijkm}, the components of the metric tensor (11.45), we may deduce directly from the relations (11.40) the following components of the Einstein tensor

$$G_{00} = -\frac{r^4}{g^2}\left\{\alpha^2 + \beta(AD^2 + BC^2 - 2CDE)r^2\right\} = -\frac{r^4}{g^2}\left\{\alpha^2 + \beta(\Omega + \lambda\delta)r^2\right\}, \tag{11.50}$$

$$G_{11} = -\frac{r^4}{g^2}\left\{\beta A(AB - E^2) + (\alpha C + \beta A + \gamma E)^2 r^2\right\} = -\frac{r^4}{g^2}(\beta\delta A + \Omega^2 r^2), \tag{11.51}$$

$$G_{22} = -\frac{r^6}{g^2}\left\{\beta B(AB - E^2) + (\alpha D + \beta E + \gamma B)^2 r^2\right\} = -\frac{r^6\beta\delta B}{g^2}, \tag{11.52}$$

$$G_{01} = -\frac{r^5}{g^2}\left\{\beta C(AB - E^2) - \alpha(\alpha C + \beta A + \gamma E)\right\} = -\frac{r^5}{g^2}(\beta\delta C - \alpha\Omega), \tag{11.53}$$

$$G_{02} = -\frac{r^6}{g^2}\beta D(AB - E^2) = -\frac{r^6\beta\delta D}{g^2}, \quad G_{12} = -\frac{r^5}{g^2}\beta E(AB - E^2) = -\frac{r^5\beta\delta E}{g^2}, \tag{11.54}$$

noting that in this case $G_{33} = -(R/2)g_{33}$. We observe that these components can be alternatively expressed as

$$\begin{aligned}
G_{00} &= \frac{\beta\delta r^4}{g^2}g_{00} - \frac{r^4}{g^2}\left\{\alpha^2 + \beta(\delta + \Omega r^2)\right\}, \qquad G_{11} = \frac{\beta\delta r^4}{g^2}g_{11} - \frac{r^6}{g^2}\Omega^2,\\
G_{22} &= \frac{\beta\delta r^4}{g^2}g_{22}, \qquad G_{33} = \frac{\beta\delta r^4}{g^2}g_{33} - \frac{r^4}{g^2}\left\{\alpha^2 - \beta(\delta + \Omega r^2)\right\},\\
G_{01} &= \frac{\beta\delta r^4}{g^2}g_{01} + \frac{\alpha\Omega r^5}{g^2}, \qquad G_{02} = \frac{\beta\delta r^4}{g^2}g_{02}, \qquad G_{12} = \frac{\beta\delta r^4}{g^2}g_{12}.
\end{aligned}$$

We further observe that in the event $\Omega = 0$, then $\alpha^2 = -\beta\delta$ and these expressions simplify to become

$$G_{ij} = \frac{\beta}{\delta}g_{ij}, \quad (i, j = 0, 1, 2), \qquad G_{33} = \frac{3\beta}{\delta}g_{33},$$

and $R = -6\beta/\delta$, and we note that these equations are slightly reminiscent of Einstein's cosmological constant Λ for which $G_{ij} = -\Lambda g_{ij}$, but of course here the constant takes on two distinct values.

Check on Einstein Tensor We may apply a reasonably robust check on the veracity of these individual expressions by evaluating the trace of the Einstein tensor $G_k^k = g^{ij}G_{ij} + g^{33}G_{33}$ which should give $-R$. Using the above equations for G_{ij} and (11.46) for the components of the conjugate metric tensor, which in terms of the new constants α, β, γ, δ and ϵ become

$$g^{ij} = \begin{pmatrix} \delta r^2/g & -\alpha r^3/g & -\epsilon r^2/g & 0\\ -\alpha r^3/g & -(B + \beta r^2)r^2/g & -(\gamma r^2 - E)r/g & 0\\ -\epsilon r^2/g & -(\gamma r^2 - E)r/g & -[A + (C^2 - \lambda A)r^2]/g & 0\\ 0 & 0 & 0 & -1/F \end{pmatrix},$$

and after some algebra and simplification, the contribution $g^{ij}G_{ij}$ admits g as a factor and simplifies as follows:

$$
\begin{aligned}
g^{ij}G_{ij} &= \frac{r^6}{g^3}\left\{\delta(2\beta\delta-\alpha^2)+[B\Omega+2(\beta\delta-\alpha^2)]\Omega r^2+\beta\Omega^2r^4\right\} \\
&= \frac{r^6}{g^3}\left\{\delta(2\beta\delta-\alpha^2)+(3\beta\delta-\alpha^2)\Omega r^2+\beta\Omega^2r^4\right\} \\
&= \frac{r^6}{g^3}(\delta+\Omega r^2)(2\beta\delta-\alpha^2+\beta\Omega r^2) \\
&= \frac{r^4}{g^2}(2\beta\delta-\alpha^2+\beta\Omega r^2) = -\frac{R}{2},
\end{aligned}
$$

where we have used $B\Omega = \alpha^2+\beta\delta$ and $g = r^2(\delta+\Omega r^2)$, and the required result follows since $G_{33} = -(R/2)g_{33}$.

Christoffel Symbols of First Kind Γ^k_{ij} The non-zero Christoffel symbols of the second kind are

$$
\begin{aligned}
&\Gamma^0_{00} = -\frac{\alpha\lambda r^4}{g}, \quad \Gamma^1_{00} = -\frac{\lambda r^3}{g}(\beta r^2+B), \quad \Gamma^2_{00} = -\frac{\lambda r^2}{g}(\gamma r^2-E), \\
&\Gamma^0_{11} = -\frac{\alpha A r^2}{g}, \quad \Gamma^1_{11} = \frac{r}{g}\left\{(\alpha C+\gamma E)r^2-E^2\right\}, \quad \Gamma^2_{11} = -\frac{A}{g}(\gamma r^2-E), \\
&\Gamma^0_{22} = -\frac{\alpha B r^4}{g}, \quad \Gamma^1_{22} = -\frac{B r^3}{g}(\beta r^2+B), \quad \Gamma^2_{22} = -\frac{B r^2}{g}(\gamma r^2-E), \\
&\Gamma^0_{01} = \frac{r^3}{g}(\beta A+\gamma E), \quad \Gamma^1_{01} = -\frac{r^2}{g}(\beta C r^2+DE), \quad \Gamma^2_{01} = -\frac{r}{g}(\gamma C r^2-AD), \\
&\Gamma^0_{02} = -\frac{\alpha D r^4}{g}, \quad \Gamma^1_{02} = -\frac{D r^3}{g}(\beta r^2+B), \quad \Gamma^2_{02} = -\frac{D r^2}{g}(\gamma r^2-E), \\
&\Gamma^0_{12} = -\frac{\alpha E r^3}{g}, \quad \Gamma^1_{12} = -\frac{E r^2}{g}(\beta r^2+B), \quad \Gamma^2_{12} = \frac{r}{g}\left\{(\alpha C+\beta A)r^2+AB\right\},
\end{aligned}
$$

and we might provide a limited check on these expressions with the well-known formula $\Gamma^i_{ki} = (1/2g)\partial g/\partial x^k$, and for $k = 0, 1, 2$, we obtain

$$
\begin{aligned}
\Gamma^0_{00}+\Gamma^1_{01}+\Gamma^2_{02} &= -\frac{r^4}{g}(\alpha\lambda+\beta C+\gamma D) = 0, \\
\Gamma^0_{10}+\Gamma^1_{11}+\Gamma^2_{12} &= \frac{r}{g}\left\{\delta+2(\alpha C+\beta A+\gamma E)r^2\right\} = \frac{1}{2g}\frac{\partial g}{\partial r}, \\
\Gamma^0_{20}+\Gamma^1_{21}+\Gamma^2_{22} &= -\frac{r^4}{g}(\alpha D+\beta E+\gamma B) = 0,
\end{aligned}
$$

as required.

Verification Einstein Tensor Is Divergence Free Although the Einstein tensor is purposely constructed to be divergence free, it is often a nontrivial matter to convince oneself that this is indeed the case, and the following provides a demonstration of the underlying complexities and detail that are often required. In order to verify that the metric (11.41) produces a divergence free Einstein tensor, that is, the equations $G^i_{j;i} = 0$ are indeed satisfied, we proceed as follows. Using the relations $G^i_k = g^{ij} G_{jk}$, we need the partial covariant derivative with respect to x^m, thus

$$G^i_{k;m} = g^{ij} G_{jk;m} = g^{ij} \left\{ \frac{\partial G_{jk}}{\partial x^m} - \Gamma^n_{mj} G_{nk} - \Gamma^p_{mk} G_{jp} \right\}, \tag{11.55}$$

which we contract; that is, we set $m = i$ and then sum over i to obtain

$$G^i_{k;i} = g^{ij} G_{jk;i} = g^{ij} \left\{ \frac{\partial G_{jk}}{\partial x^i} - \Gamma^n_{ij} G_{nk} - \Gamma^p_{ik} G_{jp} \right\} = 0. \tag{11.56}$$

It is clear from these latter relations that for the problem on hand, the summations in each equation generates a large number of terms, actually 57 separate terms in all for each of the equations arising from $k = 0, 1, 2$. We observe that while the scale of this problem may be greatly reduced with the observation that the above particular expressions for the Einstein tensor G_{ij} have the structure

$$G_{ij} = \beta\delta\psi^2 g_{ij} + h_{ij}, \tag{11.57}$$

where $\psi(r)$ is a function of r only that is defined by $\psi(r) = r^2/g$ and h_{ij} is the tensor defined by

$$h_{ij} = \begin{pmatrix} -\psi(\beta + \alpha^2\psi) & \alpha\psi^2\Omega r & 0 & 0 \\ \alpha\psi^2\Omega r & -(\psi\Omega r)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we do not however follow this approach, since there is an apparent lack of parity in the calculation, and it turns out to be more straightforward to deal with the full equations directly.

However, if we were to adopt this approach, then it would seem worthwhile noting that the field equations might be manipulated as follows. On observing again that all the partial covariant derivatives of the metric tensor g_{ij} and its conjugate g^{ij} are zero and that the partial covariant derivative of a scalar is simply the partial derivative, we have from (11.55) and (11.57)

$$G^i_{k;m} = \beta\delta g^{ij} g_{jk} \frac{\partial \psi^2}{\partial x^m} + g^{ij} \left\{ \frac{\partial h_{jk}}{\partial x^m} - \Gamma^n_{mj} h_{nk} - \Gamma^p_{mk} h_{jp} \right\},$$

which becomes

$$G^i_{k;m} = \beta\delta\delta^i_k \frac{\partial \psi^2}{\partial x^m} + g^{ij}\left\{\frac{\partial h_{jk}}{\partial x^m} - \Gamma^n_{mj}h_{nk} - \Gamma^p_{mk}h_{jp}\right\}.$$

Further, since ψ is a function of $r = x^1$ only, on contraction the divergence free equations become

$$G^i_{k;i} = \beta\delta\delta^1_k \frac{\partial \psi^2}{\partial x^1} + g^{ij}\left\{\frac{\partial h_{jk}}{\partial x^i} - \Gamma^n_{ij}h_{nk} - \Gamma^p_{ik}h_{jp}\right\} = 0,$$

which are the alternative form for the three basic equations for $k = 0, 1, 2$ to be satisfied.

Proceeding directly, we observe that the three basic equations (11.56) for $k = 0, 1, 2$ become

$$g^{1j}\frac{\partial G_{jk}}{\partial x^1} = g^{ij}\Gamma^n_{ij}G_{nk} + g^{ij}\Gamma^p_{ik}G_{jp},$$

since all components G_{ij} are functions of $x^1 = r$ only, and so

$$\begin{aligned}
&g^{10}\frac{\partial G_{0k}}{\partial x^1} + g^{11}\frac{\partial G_{1k}}{\partial x^1} + g^{12}\frac{\partial G_{2k}}{\partial x^1}\\
&= G_{0k}g^{ij}\Gamma^0_{ij} + G_{1k}g^{ij}\Gamma^1_{ij} + G_{2k}g^{ij}\Gamma^2_{ij}\\
&+ g^{ij}\Gamma^0_{ik}G_{j0} + g^{ij}\Gamma^1_{ik}G_{j1} + g^{ij}\Gamma^2_{ik}G_{j2},
\end{aligned}$$

and the three basic equations (11.56) for $k = 0, 1, 2$ when written out in full are as given below.

Equation for $k = 0$ For $k = 0$, we have

$$\begin{aligned}
&g^{10}\frac{\partial G_{00}}{\partial x^1} + g^{11}\frac{\partial G_{10}}{\partial x^1} + g^{12}\frac{\partial G_{20}}{\partial x^1}\\
&= G_{00}g^{ij}\Gamma^0_{ij} + G_{10}g^{ij}\Gamma^1_{ij} + G_{20}g^{ij}\Gamma^2_{ij}\\
&+ g^{ij}\Gamma^0_{i0}G_{j0} + g^{ij}\Gamma^1_{i0}G_{j1} + g^{ij}\Gamma^2_{i0}G_{j2}.
\end{aligned}$$

On writing this equation out in full, we have

$$\begin{aligned}
&g^{10}\frac{\partial G_{00}}{\partial x^1} + g^{11}\frac{\partial G_{10}}{\partial x^1} + g^{12}\frac{\partial G_{20}}{\partial x^1} \qquad (11.58)\\
&= G_{00}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02}) + G_{00}(g^{01}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12})
\end{aligned}$$

$$
\begin{aligned}
&+ G_{00}(g^{02}\Gamma^0_{02} + g^{12}\Gamma^0_{12} + g^{22}\Gamma^0_{22}) + G_{10}(g^{00}\Gamma^1_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^2_{02}) \\
&+ G_{10}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{12}) + G_{10}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22}) \\
&+ G_{20}(g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02}) + G_{20}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12}) \\
&+ G_{20}(g^{02}\Gamma^2_{02} + g^{12}\Gamma^2_{12} + g^{22}\Gamma^2_{22}) + G_{00}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{10} + g^{02}\Gamma^0_{20}) \\
&+ G_{10}(g^{10}\Gamma^0_{00} + g^{11}\Gamma^0_{10} + g^{12}\Gamma^0_{20}) + G_{20}(g^{20}\Gamma^0_{00} + g^{21}\Gamma^0_{10} + g^{22}\Gamma^0_{20}) \\
&+ G_{01}(g^{00}\Gamma^1_{00} + g^{10}\Gamma^1_{10} + g^{20}\Gamma^1_{20}) + G_{11}(g^{01}\Gamma^1_{00} + g^{11}\Gamma^1_{10} + g^{21}\Gamma^1_{20}) \\
&+ G_{21}(g^{02}\Gamma^1_{00} + g^{12}\Gamma^1_{10} + g^{22}\Gamma^1_{20}) + G_{02}(g^{00}\Gamma^2_{00} + g^{10}\Gamma^2_{10} + g^{20}\Gamma^2_{20}) \\
&+ G_{12}(g^{01}\Gamma^2_{00} + g^{11}\Gamma^2_{10} + g^{12}\Gamma^2_{20}) + G_{22}(g^{02}\Gamma^2_{00} + g^{21}\Gamma^2_{10} + g^{22}\Gamma^2_{20}).
\end{aligned}
$$

Equation for $k = 1$ Similarly, the corresponding equation for $k = 1$ is

$$
\begin{aligned}
&g^{10}\frac{\partial G_{01}}{\partial x^1} + g^{11}\frac{\partial G_{11}}{\partial x^1} + g^{12}\frac{\partial G_{21}}{\partial x^1} \qquad (11.59) \\
&= G_{01}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02}) + G_{01}(g^{01}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12}) \\
&+ G_{01}(g^{02}\Gamma^0_{02} + g^{12}\Gamma^0_{12} + g^{22}\Gamma^0_{22}) + G_{11}(g^{00}\Gamma^1_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^2_{02}) \\
&+ G_{11}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{11}) + G_{11}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22}) \\
&+ G_{12}(g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02}) + G_{12}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12}) \\
&+ G_{12}(g^{02}\Gamma^2_{02} + g^{12}\Gamma^2_{12} + g^{22}\Gamma^2_{22}) + G_{00}(g^{00}\Gamma^0_{01} + g^{01}\Gamma^0_{11} + g^{02}\Gamma^0_{21}) \\
&+ G_{10}(g^{10}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{21}) + G_{20}(g^{20}\Gamma^0_{01} + g^{21}\Gamma^0_{11} + g^{22}\Gamma^0_{21}) \\
&+ G_{01}(g^{00}\Gamma^1_{01} + g^{10}\Gamma^1_{11} + g^{20}\Gamma^1_{21}) + G_{11}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{21}\Gamma^1_{21}) \\
&+ G_{21}(g^{02}\Gamma^1_{01} + g^{12}\Gamma^1_{11} + g^{22}\Gamma^1_{21}) + G_{02}(g^{00}\Gamma^2_{01} + g^{10}\Gamma^2_{11} + g^{20}\Gamma^2_{21}) \\
&+ G_{12}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{21}) + G_{22}(g^{02}\Gamma^2_{01} + g^{21}\Gamma^2_{11} + g^{22}\Gamma^2_{21}).
\end{aligned}
$$

Equation for $k = 2$ For $k = 2$, we have

$$
\begin{aligned}
&g^{10}\frac{\partial G_{02}}{\partial x^1} + g^{11}\frac{\partial G_{12}}{\partial x^1} + g^{12}\frac{\partial G_{22}}{\partial x^1} \qquad (11.60) \\
&= G_{02}(g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02}) + G_{02}(g^{01}\Gamma^0_{01} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12}) \\
&+ G_{02}(g^{02}\Gamma^0_{02} + g^{12}\Gamma^0_{12} + g^{22}\Gamma^0_{22}) + G_{12}(g^{00}\Gamma^1_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^2_{02})
\end{aligned}
$$

$$+ G_{12}(g^{01}\Gamma^1_{01} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{12}) + G_{12}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22})$$
$$+ G_{22}(g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02}) + G_{22}(g^{01}\Gamma^2_{01} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12})$$
$$+ G_{22}(g^{02}\Gamma^2_{02} + g^{12}\Gamma^2_{12} + g^{22}\Gamma^2_{22}) + G_{00}(g^{00}\Gamma^0_{02} + g^{01}\Gamma^0_{12} + g^{02}\Gamma^0_{22})$$
$$+ G_{10}(g^{10}\Gamma^0_{02} + g^{11}\Gamma^0_{12} + g^{12}\Gamma^0_{22}) + G_{20}(g^{20}\Gamma^0_{02} + g^{21}\Gamma^0_{12} + g^{22}\Gamma^0_{22})$$
$$+ G_{01}(g^{00}\Gamma^1_{02} + g^{10}\Gamma^1_{12} + g^{20}\Gamma^1_{22}) + G_{11}(g^{01}\Gamma^1_{02} + g^{11}\Gamma^1_{12} + g^{21}\Gamma^1_{22})$$
$$+ G_{21}(g^{02}\Gamma^1_{02} + g^{12}\Gamma^1_{12} + g^{22}\Gamma^1_{22}) + G_{02}(g^{00}\Gamma^2_{02} + g^{10}\Gamma^2_{12} + g^{20}\Gamma^2_{22})$$
$$+ G_{12}(g^{01}\Gamma^2_{02} + g^{11}\Gamma^2_{12} + g^{12}\Gamma^2_{22}) + G_{22}(g^{02}\Gamma^2_{02} + g^{21}\Gamma^2_{12} + g^{22}\Gamma^2_{22}).$$

Christoffel Symbol Identities In order to confirm, the given expressions for the Einstein tensor G_{ij} given by Eqs. (11.50), (11.51), (11.52), (11.53) and (11.54) constitute a bonafide solution of these lengthy divergence equations by means of the following identities, which take the form $g^{ik}\Gamma^m_{ij}$ for fixed j, k and m and with summation over i, which are no doubt related to the particular case of this formula given by [15] (page 68), namely

$$g^{ij}\Gamma^k_{ij} = -\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}g^{km})}{\partial x^m}.$$

By direct individual calculations, we may confirm the following identities:

$$g^{00}\Gamma^0_{00} + g^{01}\Gamma^0_{01} + g^{02}\Gamma^0_{02} = 0, \qquad g^{00}\Gamma^1_{00} + g^{01}\Gamma^1_{01} + g^{02}\Gamma^1_{02} = \frac{\beta r^3}{g}, \qquad (11.61)$$
$$g^{00}\Gamma^2_{00} + g^{01}\Gamma^2_{01} + g^{02}\Gamma^2_{02} = \frac{\gamma r^2}{g}, \qquad g^{10}\Gamma^0_{10} + g^{11}\Gamma^0_{11} + g^{12}\Gamma^0_{12} = \frac{\alpha\delta r^4}{g^2},$$
$$g^{10}\Gamma^1_{10} + g^{11}\Gamma^1_{11} + g^{12}\Gamma^1_{12} = -\frac{\alpha^2 r^5}{g^2}, \qquad g^{10}\Gamma^2_{10} + g^{11}\Gamma^2_{11} + g^{12}\Gamma^2_{12} = -\frac{\alpha\epsilon r^4}{g^2},$$
$$g^{20}\Gamma^0_{20} + g^{21}\Gamma^0_{21} + g^{22}\Gamma^0_{22} = \frac{\alpha r^2}{g}, \qquad g^{20}\Gamma^1_{20} + g^{21}\Gamma^1_{21} + g^{22}\Gamma^1_{22} = (\beta r^2 + B)\frac{r}{g},$$
$$g^{20}\Gamma^2_{20} + g^{21}\Gamma^2_{21} + g^{22}\Gamma^2_{22} = 0, \qquad g^{01}\Gamma^0_{00} + g^{11}\Gamma^0_{10} + g^{21}\Gamma^0_{20} = -\frac{\beta r^3}{g},$$
$$g^{01}\Gamma^1_{00} + g^{11}\Gamma^1_{10} + g^{21}\Gamma^1_{20} = 0, \qquad g^{01}\Gamma^2_{00} + g^{11}\Gamma^2_{10} + g^{21}\Gamma^2_{20} = -\frac{Dr}{g},$$
$$g^{00}\Gamma^0_{01} + g^{01}\Gamma^0_{11} + g^{02}\Gamma^0_{21} = \frac{\Omega\delta r^5}{g^2}, \qquad g^{00}\Gamma^1_{01} + g^{01}\Gamma^1_{11} + g^{02}\Gamma^1_{21} = -\frac{\Omega\alpha r^6}{g^2},$$

$$g^{00}\Gamma^2_{01} + g^{01}\Gamma^2_{11} + g^{02}\Gamma^2_{21} = -\frac{\Omega\epsilon r^5}{g^2}, \qquad g^{02}\Gamma^0_{00} + g^{12}\Gamma^0_{10} + g^{22}\Gamma^0_{20} = -\frac{\gamma r^2}{g},$$
$$g^{02}\Gamma^1_{00} + g^{12}\Gamma^1_{10} + g^{22}\Gamma^1_{20} = \frac{Dr}{g}, \qquad g^{02}\Gamma^2_{00} + g^{12}\Gamma^2_{10} + g^{22}\Gamma^2_{20} = 0,$$
$$g^{02}\Gamma^0_{01} + g^{12}\Gamma^0_{11} + g^{22}\Gamma^0_{21} = -\frac{\Omega\epsilon r^5}{g^2},$$
$$g^{02}\Gamma^1_{01} + g^{12}\Gamma^1_{11} + g^{22}\Gamma^1_{21} = (\delta + 2\Omega r^2)\frac{Er^2}{g^2} - \frac{\Omega\gamma r^6}{g^2},$$
$$g^{02}\Gamma^2_{01} + g^{12}\Gamma^2_{11} + g^{22}\Gamma^2_{21} = -(\delta + 2\Omega r^2)\frac{Ar}{g^2} - (C^2 - \lambda A)\frac{\Omega r^5}{g^2},$$
$$g^{00}\Gamma^0_{02} + g^{10}\Gamma^0_{12} + g^{20}\Gamma^0_{22} = 0, \qquad g^{00}\Gamma^1_{02} + g^{10}\Gamma^1_{12} + g^{20}\Gamma^1_{22} = 0,$$
$$g^{00}\Gamma^2_{02} + g^{10}\Gamma^2_{12} + g^{20}\Gamma^2_{22} = -\frac{\alpha r^2}{g}, \qquad g^{01}\Gamma^0_{02} + g^{11}\Gamma^0_{12} + g^{21}\Gamma^0_{22} = 0,$$
$$g^{01}\Gamma^1_{02} + g^{11}\Gamma^1_{12} + g^{21}\Gamma^1_{22} = 0, \qquad g^{01}\Gamma^2_{02} + g^{11}\Gamma^2_{12} + g^{21}\Gamma^2_{22} = -(\beta r^2 + B)\frac{r}{g}.$$

In terms of $\psi(r) = r^2/g = 1/(\delta + \Omega r^2)$, the components of the Einstein tensor become

$$G_{00} = -\psi^2\left\{\alpha^2 + \beta(\Omega + \lambda\delta)r^2\right\}, \qquad G_{11} = -\psi^2(\beta\delta A + \Omega^2 r^2), \tag{11.62}$$
$$G_{22} = -\psi^2 r^2\beta\delta B, \qquad G_{01} = -\psi^2 r(\beta\delta C - \alpha\Omega),$$
$$G_{02} = -\psi^2 r^2\beta\delta D, \qquad G_{12} = -\psi^2 r\beta\delta E,$$

and we may use the immediately above identities given by (11.61) to show that the lengthy Eqs. (11.58), (11.59) and (11.60) simplify as follows. For $k = 0$, we have

$$\alpha r\frac{dG_{00}}{dr} + (\beta r^2 + B)\frac{dG_{10}}{dr} + \left(\gamma r - \frac{E}{r}\right)\frac{dG_{20}}{dr} \tag{11.63}$$
$$+\,\alpha(1 + \delta\psi)G_{00} + \left(\left(\beta r + \frac{B}{r}\right) + \beta r - \alpha^2\psi r\right)G_{10} + (\gamma - \alpha\epsilon\psi)G_{20} = 0,$$

while for $k = 1$, we obtain

$$\alpha r\frac{dG_{01}}{dr} + (\beta r^2 + B)\frac{dG_{11}}{dr} + \left(\gamma r - \frac{E}{r}\right)\frac{dG_{21}}{dr} \tag{11.64}$$
$$+\,\alpha(1 + \delta\psi)G_{01} + \left(\left(\beta r + \frac{B}{r}\right) + \beta r - \alpha^2\psi r\right)G_{11} + (\gamma - \alpha\epsilon\psi)G_{21}$$
$$+\,\delta\Omega\psi r G_{00} + \alpha\delta\psi G_{01} - \epsilon\Omega\psi r G_{02} - \alpha\Omega\psi r^2 G_{01} - \alpha^2\psi r G_{11}$$

$$+\left((\delta+2\Omega r^2)\frac{E}{r^2}-\gamma\Omega r^2\right)\psi G_{21}-\Omega\epsilon\psi r G_{02}-\alpha\epsilon\psi G_{12}$$
$$-\left((\delta+2\Omega r^2)\frac{A}{r^3}+(C^2-\lambda A)\Omega r\right)\psi G_{22}=0,$$

noting that evidently further grouping of terms is possible. The above form is that in which the terms arise immediately from Eq. (11.59) and in the verification below, it is useful to leave in this form. For $k=2$, we have

$$\alpha r\frac{dG_{02}}{dr}+(\beta r^2+B)\frac{dG_{12}}{dr}+\left(\gamma r-\frac{E}{r}\right)\frac{dG_{22}}{dr} \tag{11.65}$$
$$+\alpha(1+\delta\psi)G_{02}+\left(\left(\beta r+\frac{B}{r}\right)+\beta r-\alpha^2\psi r\right)G_{12}+(\gamma-\alpha\epsilon\psi)G_{22}=0.$$

We may verify that each of Eqs. (11.63), (11.64) and (11.65) is correctly satisfied by the expressions for G_{ij} given by (11.62) as follows. All three can be verified in a similar manner, and for $k=0$ and $k=2$, there are two important equalities, namely

$$\alpha G_{00}+\beta r G_{10}+\gamma G_{20}=-\alpha^3\psi^2,\quad \delta G_{00}-\alpha r G_{10}-\epsilon G_{20}=-\alpha^2\psi,\ (k=0)$$
$$\alpha G_{02}+\beta r G_{12}+\gamma G_{22}=0,\quad \delta G_{02}-\alpha r G_{12}-\epsilon G_{22}=0,\ (k=2)$$

while for $k=1$, there are three important equalities, which are as follows:

$$\alpha G_{01}+\beta r G_{11}+\gamma G_{21}=(\alpha^2\psi-\beta)\Omega\psi r,\quad \delta G_{01}-\alpha r G_{11}-\epsilon G_{21}=\alpha\Omega\psi r,$$
$$\epsilon G_{02}+\gamma r G_{12}+(C^2-\lambda A)G_{22}=-\beta\delta\Omega\psi^2 r^2.$$

In addition to these equalities, we also require the three subsidiary equalities for $k=0$, 1 and 2, respectively,

$$BrG_{01}-EG_{02}=\alpha^3\psi^2 r^2,\quad BrG_{11}-EG_{12}=-(\alpha^2\Omega\psi r^2+\beta\delta)\psi r,$$
$$BrG_{21}-EG_{22}=0.$$

The proofs for all three values of k follow a similar approach, so we only give the details for the case of $k=1$, which is the most complicated. In each case, we purposely rearrange each of Eqs. (11.63), (11.64) and (11.65) to exploit the above equalities. For $k=1$, Eq. (11.64) can be shown to become

$$r\frac{d}{dr}(\alpha G_{01}+\beta r G_{11}+\gamma G_{21})+\frac{1}{r}\frac{d}{dr}(BrG_{11}-EG_{12})$$
$$+(\alpha G_{01}+\beta r G_{11}+\gamma G_{21})+2\alpha\psi(\delta G_{01}-\alpha r G_{11}-\epsilon G_{21})$$
$$+\Omega\psi r(\delta G_{01}-\alpha r G_{11}-\epsilon G_{21})-\Omega\psi r(\epsilon G_{02}+\gamma r G_{12}+(C^2-\lambda A)G_{22})$$

$$+ (\delta + 2\Omega r^2)(ErG_{21} - AG_{22})\frac{\psi}{r^3} = 0,$$

and in this case, we need the additional equality that $ErG_{21} - AG_{22} = \beta(\delta\psi r)^2$. Throughout these proofs, we make frequent use of the relations arising from $g = r^2(\delta + \Omega r^2)$ and $\psi(r) = r^2/g = 1/(\delta + \Omega r^2)$, namely

$$\delta + \Omega r^2 = \frac{1}{\psi}, \qquad \frac{d\psi}{dr} = -2\Omega\psi^2 r,$$

along with the identity $B\Omega = \alpha^2 + \beta\delta$.

Chapter 12
Conclusions, Summary and Postscript

12.1 Introduction

In the search to understand the dark issues of astrophysics, a fundamental reexamination of mechanical accounting is necessary, and the most basic of all mechanics is Newton's second law involving force, mass and acceleration. In this chapter, we briefly summarise some of the major ideas and outcomes presented for the proposed dual particle-wave mechanical model given by Eqs. (3.4) or (12.3). The mathematics underpinning the model is motivated from the use of potentials in electromagnetism, while the physical idea originates from de Broglie's idea of the simultaneous existence of both particle and wave. Neither are sufficient alone, since even the best ideas require the correct theoretical framework for their successful implementation. de Broglie's [23] interpretation of the dual particle-wave nature of matter involved a concrete physical picture of the coexistence of both particle and the associated wave, and he proposed "the theory of the double solution", for which he formulated an equation which he called "the guidance formula" (see also de Broglie [21, 24]). In de Broglie's own words, previously quoted in Chap. 3:

> When I conceived the first basic ideas of wave mechanics in 1923–1924, I was guided by the aim to perform a real physical synthesis, valid for all particles, of the coexistence of the wave and of the corpuscular aspects that Einstein had introduced for photons in his theory of light quanta in 1905.
>
> that the particle must be the seat of an internal periodic movement ($e_0 = m_0c^2 = h\nu_0$ and $e = mc^2 = h\nu$) that must move in a wave in order to remain in phase with it, was ignored by the actual physicists (who are) wrong to consider a wave propagation without localisation of the particle, which was quite contrary to my original ideas.

de Broglie's guidance equation involves a combination of special relativistic and quantum mechanical ideas, such that the momentum **p** and the particle energy e simultaneously admit the two representations

J. M. Hill, *Mathematics of Particle-Wave Mechanical Systems*,
https://doi.org/10.1007/978-3-031-19793-2_12

$$\mathbf{p} = m\mathbf{u} = -\nabla\psi, \qquad e = mc^2 = \frac{\partial\psi}{\partial t}, \tag{12.1}$$

where m is assumed to be given by the relativistic expression $m = m_0/[1 - (u/c)^2]^{1/2}$ and therefore the velocity $\mathbf{u}$ is given by

$$\mathbf{u} = -c^2 \frac{\nabla\psi}{(\partial\psi/\partial t)}, \tag{12.2}$$

and de Broglie refers to this formula, which determines the motion of the particle at each point of its trajectory in the wave, as the "the guidance formula of the particle by its wave".

Given the attention focussed on the experiments of Yves Couder and colleagues [14], involving droplets walking on the surface of a vibrating fluid bath, and the theoretical work and commentrary of John W. M. Bush and coworkers [10, 45, 82] that "the walking-drop system displays comparable effects such as single and double-slit diffraction, tunnelling, orbital quantisation, level-splitting and wave-like statistics in confined geometries. The walking drop is propelled through resonant interaction with its own wave field, and represents the first macroscopic realisation of a double-wave pilot-wave quantum mechanical system envisaged by de Broglie" [45], de Broglie's quantum mechanical pilot-wave ideas, previously believed exclusive to the microscopic quantum realm, are likely to assume a new importance in numerous macroscopic particle-fluid systems.

In the following section, we present some of the major conclusions and outcomes, and in the subsequent section, we summarise some of the results that apply to a single space dimension x. In the final section of the chapter, we attempt to present some insight through the perspective of others on the hurdles encountered by those trying to develop new ideas in mechanics and physics.

12.2 Conclusions

In the mathematical modelling of various phenomena, there is often more than one model generating comparable numerical outcomes, and one of the major messages of applied mathematical modelling is to start with the simplest models first, with the objectives of both simplicity and tractability. Special relativity arising from Lorentz invariance is known to be successful and physically meaningful at least at certain scales, and the question arises as to whether or not these notions might be meaningfully extended to larger scales. Our purpose here is to produce an extended model of special relativity, which is both simple and tractable and capable of refinement and generalisation at a later time.

In [47–52], the author has proposed a modified version of special relativistic mechanics, which extends conventional theory, in such a way as to be inclusive of traditional Newtonian mechanics and of quantum mechanics, in the form of Schrödinger's second-order wave equation, and therefore inclusive of many of the great achievements of atomic physics. In conventional special relativity, Einstein's variation of mass formula is both a necessary and sufficient condition for the invariance of Newton's second law under Lorentz transformations. The modified Newton's second law proposed here involves the Lorentz invariance of two equations, one representing the spatial accumulation of energy and the other representing the temporal accumulation of energy. This Lorentz invariant extension of Newton's second law necessitates the notion of a force g in the direction of time, which in the language of continuum mechanics is merely the energy-mass production term.

Assuming the usual formulae of special relativity, namely energy $e = mc^2$ and momentum $\mathbf{p} = m\mathbf{u}$, with mass $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, we propose the following modified Newton's second law:

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{p}, \tag{12.3}$$

where $\mathbf{f}$ and g denote, respectively, the applied force and energy-mass production, and all derivatives in (12.3) are partial. For $g \neq 0$, this proposal also has implications for Newton's first law, giving rise to the possibility of motion in the absence of any spatial force. If the two equations (12.3) are combined with de Broglie's guidance formulae relations (12.1), then $\mathbf{f} = \mathbf{0}$, while

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = g.$$

For both the single spatial dimension and the centrally symmetric mechanical systems, examined in Chaps. 4 and 9, respectively, de Broglie's guidance formulae (12.1) and (12.2) provide nontrivial illustrations of solutions, for which $\mathbf{f} = \mathbf{0}$, and yet both $\mathbf{p}$ and e are non-constant with $e^2 - c^2\mathbf{p} \cdot \mathbf{p} = e_0^2$, and no doubt there are many such examples.

The proposed general force relations, Eqs. (3.4) or (12.3), constitute a modest extension of the special relativistic version of Newton's second law. In a special relativistic context, these equations attempt to elevate Newton's second law to a model, for which time and space are on an equal footing. Of course, this has always been the case for most of special relativity, but not for Newton's second law. For the spatial force $\mathbf{f}$ given by $(12.3)_1$, the following identity, reminiscent of the Lorentz force expression, holds

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} + \mathbf{u} \wedge (\nabla \wedge \mathbf{p}),$$

and in Chap. 3, we have described a correspondence of the proposed model with Maxwell's equations of electromagnetism. In this correspondence, the gradient ∇g of the force in the direction of time is analagous to the notion of "current", so that if the force in the direction of time does not exist, then there is no "current".

If the Einstein relativistic energy equation $e^2 - c^2\mathbf{p}\cdot\mathbf{p} = e_0^2$ is assumed, or if $(\mathbf{f}, gc)$ are generated as external forces from a scalar potential $V(\mathbf{x}, t)$ such that

$$\mathbf{f} = -\nabla V, \qquad gc^2 = -\frac{\partial V}{\partial t}, \tag{12.4}$$

then $\nabla \wedge \mathbf{p} = \mathbf{0}$ and the conventional special relativistic Newton second law applies, namely $\mathbf{f} = d\mathbf{p}/dt$. In the latter situation, a conventional conservation of energy applies, namely $e + \mathscr{E} + V = constant$, where $\mathscr{E}$ is identified as the de Broglie wave energy. In the case when the forces are generated from a potential $V(\mathbf{x}, t)$ through Eq. (3.25) or (12.4), then the potential itself cannot be arbitrarily assigned and must satisfy certain requirements that are determined as part of the solution procedure. This is in contrast to conventional thinking that in the laboratory we are free to impose whatever fields our equipment allows, and the feature is reminiscent of general relativity, for which the field equations determine the metric tensor and the gravitational nature of the field itself.

The modified theory applies when the particle and wave energies become of comparable magnitude to the potential energy generating the motion and might be viewed as intermediate theory between special and general relativity, in the sense that for an energy function $\mathscr{E}(\mathbf{x}, t)$ to exist, any external forces $\mathbf{f}$ and g must satisfy a compatibility condition (3.11), namely

$$\frac{\partial \mathbf{f}}{\partial t} = c^2 \nabla g. \tag{12.5}$$

and in these circumstances, the following companion formulae apply

$$\mathbf{f} = \frac{d\mathbf{p}}{dt}, \qquad gc^2 = \frac{d\mathscr{E}}{dt} = \frac{de}{dt} + e(\nabla \cdot \mathbf{u}), \tag{12.6}$$

so that a key inequality might be determined from the relation

$$\frac{d(\mathscr{E} - e)}{dt} = e(\nabla \cdot \mathbf{u}). \tag{12.7}$$

In Chaps. 5 and 6, based on the assumption that both momentum and energy are functions of velocity u only, we examine in some detail the exact wave-like solution of (12.3), namely

$$u(x, t) = c\left\{\frac{\lambda x + ct}{((e_0/f_0)^2 + (\lambda x + ct)^2)^{1/2}}\right\}, \tag{12.8}$$

where $e_0 = m_0c^2$ is the rest mass energy and λ and f_0 denote arbitrary constants, appearing in the assumed linear force equations

$$f(u) = f_0(1 + \lambda u/c), \qquad cg(u) = f_0(\lambda + u/c). \tag{12.9}$$

It is important to observe that while the proposed force equations (12.3) are fully Lorentz invariant, the above compatibility equation (12.5) is not automatically Lorentz invariant without further restriction, and this fact gives rise to the notion of partial Lorentz invariance, which is exhibited by the exact wave-like solution (12.8) that is discussed in depth in Chaps. 5 and 6. In Chap. 7, we show that the above assumed linear force expressions (12.9) satisfy another Lorentz invariance involving the product of force times energy, or in other words, force times mass.

In the modified theory, we make a distinction between particle energy $e = mc^2$ and the de Broglie wave energy $\mathscr{E}$, which are such that the total work done by the particle $W = e + \mathscr{E}$ accumulates from both the spatial physical force $\mathbf{f}$ and the force g in the direction of time. In the belief that nature tends to prefer to adopt minimum energy structures, we propose that in an experiment, particles appear for $e < \mathscr{E}$ and waves for $\mathscr{E} < e$, but in either event, both a measurable and an unmeasurable energy exists. In conventional quantum theory, energy is just energy; only one energy is measured, and the two energies become blurred into a single entity.

We show that within this modified theory, a simple but logical analysis of the conventional integrated rate-of-working equation might give rise to the prospect of four distinct types of matter: positive and negative energies, with non-zero rest mass, and positive and negative energies, with zero rest mass. This identification of the four types of matter is meaningful only because it is interpreted within the proposed modified theory. Within this new theory, dark energy and dark matter emerge as singular or privileged states, occurring when there are particular alignments of the forces $(\mathbf{f}, g)$, and these possibilities do not arise within conventional special relativistic mechanics, since there is no notion of "force in the direction of time".

The most likely explanation of dark energy and dark matter is that they arise from a balancing of particle and wave energies $e = \mathscr{E}$, which are supported by a potential $V = -2e$. However, another possibility permitted in the present model is $e = -\mathscr{E}$, which then operates under a zero potential $V = 0$, and this might well be the underlying reason for the perceived huge amounts of dark energy and dark matter in the universe. This necessarily speculative proposal provides a model capable of testing, and while it may not be completely in accord with such findings, it is probable that dark energy and dark matter are singular states, arising as artefacts of the mechanical accounting, and that their formal origin lies in the current mechanical models, neglecting either the particle or the wave energy.

Specifically, the proposed theory allows the following interpretation of Einstein's particle energy statement $e^2 = {e_0}^2 + (pc)^2$, where $e_0 = m_0c^2$ denotes the particle rest mass energy, which within the context of the present theory logically admits four distinct types of matter. Either the rest mass energy e_0 is zero or non-zero and so gives rise to precisely four distinct types of matter:

$$e = \begin{cases} ({e_0}^2 + (pc)^2)^{1/2} & \text{if } e_0 \neq 0 \text{ baryonic matter,} \\ pc & \text{if } e_0 = 0 \text{ dark or invisible matter,} \\ -pc & \text{if } e_0 = 0 \text{ dark energy,} \\ -({e_0}^2 + (pc)^2)^{1/2} & \text{if } e_0 \neq 0 \text{ anti-matter,} \end{cases}$$

with the evident inequalities

$$-({e_0}^2 + (pc)^2)^{1/2} \leqslant -pc \leqslant pc \leqslant ({e_0}^2 + (pc)^2)^{1/2},$$

and we accordingly view baryonic matter as the most energetic form of matter. This is consistent with the Einstein picture that we should envisage baryonic matter energetically as a form of an energy-battery. It is also consistent with the fact that each of the higher energy states has a ready access to the lower energy states, and the values are sensible in the context of the allowable occupancies indicated in Table 5.1. The table shows, for example, that a baryonic particle has the potential to be in any of the four states. Of course, however, depending upon local conditions, all energy states are allowable, but in a natural environment, we might expect the lowest energy state to be the most occupied.

12.3 Summary

In this section, for purposes of illustration, we summarise in dot point form some of the particular formulae applying to the case of one space dimension x.

- The model presented here and in [47–52] is based upon the idea that, along with a spatial force f, there exists a force g in the direction of time and that particle energy $e(x,t)$ and wave energy $\mathscr{E}(x,t)$ need to be considered distinct and accounted for separately. We propose that the particle energy accrues from the standard work-done relation $de = f dx$, while the wave energy accrues from the corresponding incremental relation in the direction of time, namely $d\mathscr{E}(x,t) = gc^2 dt$. The later elementary differential relation emerges from the general three-dimensional formulation (see either Eqs. (3.15) or (12.6)).
- In conventional special relativity, a necessary and sufficient condition for the Lorentz invariance of Newton's second law as a single equation is that Einstein's mass variation with velocity applies (see Sect. 2.13). As a system of two equations, the one-dimensional version (4.10) of the proposed force relations (3.4) is properly invariant under the full one-parameter group of both sub-luminal and superluminal Lorentz transformations. Further, if both forces f and g are derivable as the gradient of a potential function $V(x,t)$, namely $f = -\partial V/\partial x$ and $gc^2 = -\partial V/\partial t$, then the total particle energy is necessarily conserved in the conventional manner, namely $e + \mathscr{E} + V = constant$, and the usual law that the

spatial force equals the total time rate of change of momentum still holds (see Eq. (3.15)$_1$).

- In Chaps. 5, 6 and 7, we have presented an exhaustive account of an exact wave-like solution given by Eq. (5.1), which involves an arbitrary parameter λ, for which light-like behaviour occurs for $\lambda^2 = 1$, and both particle-like behaviour and wave-like behaviour occur for the two distinct cases $\lambda^2 < 1$ for $\lambda^2 > 1$. The exact wave-like solution is an illustrative example, exhibiting explicitly some of the major characteristics of the model and extending a well-known solution of relativity as well as giving rise to a formula for the Hubble parameter. Further, the well-known formulae for light, namely $p = h/\lambda$ and $\mathscr{E} = h\nu$, so that together $\mathscr{E} = cp$, termed here the de Broglie relation, emerges as an exact consequence from the wave-like solution (5.1) for the particular parameter value $\lambda = 1$.
- In the proposed extended special relativistic mechanics, there are two singular states termed here de Broglie states, which are characterised by particles with zero rest mass energy e_0 and by the relations $\mathscr{E} = \pm pc$, arising from particular force alignments $f = \pm cg$, and for which the particle and wave energies might satisfy $e = \pm\mathscr{E}$. It is therefore natural to propose that these two states might correspond to dark matter and dark energy. Specifically, we speculate that dark matter as a positive gravity contributor is characterised by the relations $f = cg$ and $\mathscr{E} = pc$, while dark energy as a negative gravity contributor is characterised by the relations $f = -cg$ and $\mathscr{E} = -pc$. The model also admits the possibility that such singular states might be supported by zero potential with $e = -\mathscr{E}$, giving rise to the notion that these singular states might occur as artefacts of the mechanical accounting.
- For a single spatial dimension x, with arbitrary applied forces $f(x, t)$ and $g(x, t)$, the two basic equations (12.3) are

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{12.10}$$

  and in terms of the two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$, these equations can be shown to become

$$xf - c^2tg = \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial x}, \qquad xg - tf = \frac{1}{c^2}\frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial x},$$

  which can be verified by undertaking the partial differentiations on the left-hand side and making use of (12.10).
- In traditional continuum mechanics, we automatically inherit the total or material time derivative d/dt as part of a well-established framework. However, within the present dual particle-wave context, the determination of a matching total derivative is not an entirely straightforward matter. Only after some trial and error does it become apparent that the corresponding total derivative is a spatial derivative d/dx, but following the wave rather than that following the particle. The two Lorentz invariants $\xi = ex - c^2pt$ and $\eta = px - et$ satisfy the basic rate

equations

$$e\frac{d\xi}{dt} = fc^2\eta, \qquad e\frac{d\eta}{dt} = f\xi - e_0^2,$$

and

$$e\frac{d\xi}{dx} = c^2 g\eta + e_0^2, \qquad e\frac{d\eta}{dx} = g\xi,$$

where d/dt and d/dx are the fundamental time and space total derivatives underpinning the structure of the proposed model. These time and spatial total derivatives follow the particle and the wave, respectively, thus

$$\frac{d}{dt} = \left(\frac{d}{dt}\right)_{part} = \frac{\partial}{\partial t} + \left(\frac{dx}{dt}\right)_{part}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},$$
$$\frac{d}{dx} = \left(\frac{d}{dx}\right)_{wave} = \frac{\partial}{\partial x} + \left(\frac{dt}{dx}\right)_{wave}\frac{\partial}{\partial t} = \frac{\partial}{\partial x} + \frac{u}{c^2}\frac{\partial}{\partial t},$$

which are not Lorentz invariant, but under Lorentz transformation, transforming in the same manner (see Eqs. (4.59) and (4.60)).

- The Schrödinger wave equation in quantum mechanics is usually motivated as arising from the classical wave equation. Within the present extended theory, the wave equation is not a matter of speculation, but rather a consequence of the theory, and the two Lorentz invariants of special relativity $\xi = ex - c^2pt$ and $\eta = px - et$ provide the formal mechanism to connect special relativity and quantum mechanics. These two Lorentz invariants may be expressed in terms of two other Lorentz invariants ζ and τ defined by (2.15) (see the relations (2.18) and (2.19)).
- On incorporating a potential function $V(x, t)$ into the usual operator relations in quantum mechanics, thus $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t + V(x, t)$, where $\hbar = h/2\pi$, the Klein–Gordon equation (10.13) emerges as the quantum mechanical equation corresponding to the algebraic identity $e^2 - (pc)^2 = e_0^2$, or $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$ in the case $V(x, t) = 0$. If $V(x, t)$ is assumed to be non-zero, the two quantum mechanical operators $L_{\xi - c\eta}$ and $L_{\xi + c\eta}$ defined by (10.19) are now non-commuting, and with the product definition (10.20), a modified Klein–Gordon equation (10.21) emerges as the quantum mechanical equation corresponding to the algebraic identity $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$.

12.4 Postscript

From the long view of the history of mankind, Maxwell may well be viewed as the greatest theoretical physicist of the nineteenth century, and there can be little doubt that the most significant event of the nineteenth century will be his discovery of the laws of electrodynamics. His discoveries helped usher in the era of modern physics, laying the foundation for such fields as special relativity and quantum mechanics. Existing special relativity and quantum mechanics not only account for atomic physics, but they do so to a very high degree of accuracy. However, this close agreement understandably makes it difficult to propose any alternative mechanical theories even though intended for the astrophysical scale. There is a problem with mechanical accounting at the astrophysical scale, and the question arises as to how best this might be resolved.

The intention of the theory proposed here is to extend special relativity, in such a way that both conventional special relativity and quantum theory in the form of Schrödinger's second-order wave equation are included, and therefore, the major achievements in atomic physics remain unchanged. Proposing new physical theory has never been easy or gladly received by the scientific community.

Thomas Kuhn in his book on the structure of scientific revolutions [64] argues that there are two types of science: normal science and revolutionary science. Normal science is conducted within an existing set of well-established rules that are accepted by all scientists working in that particular field. However, occasionally an unexpected discovery occurs that is inconsistent or lies outside the accepted rules, concepts and procedures. This causes a tense situation among the scientists, gradually increasing in intensity until a scientific revolution is achieved, after which a new paradigm emerges and normal science is resumed.

As early as 1612, Sir Francis Bacon wrote, "For when propositions are denied, there is an end of them, but if they be allowed, it requireth a new worke", which is taken from the Essaies of Sir Francis Bacon, London, 1612, and quoted by Sidney Goldstein in the first volume of "Modern developments in fluid dynamics" [40].

Writing in 1982, Bohm and Hiley [9], in relation to the then formal structural-driven approach to physics, comment that "For one can see that concepts that do not give immediate new experimental predictions may still be valuable, in that they permit new insight and understanding (from which new predictions may ultimately emerge). An approach that discourages this kind of insight will thus tend to prevent creative new perceptions, such as those of Einstein, de Broglie, etc. The pilot wave theory of de Broglie was indeed a significant and fruitful example of imaginative concepts that help lead to new insights". These authors conclude, "the early work of de Broglie played a key part in making possible the development of the mathematical form of the quantum theory itself. Unfortunately, his physical intuition and imaginative insights were not generally taken up, and therefore did not have a widespread effect on what was subsequently done". And later they write, "some are still working with extensions or developments of de Broglie ideas aimed at a better understanding of the underlying physical reality than is treated by the mathematical formalism alone".

Along similar lines, Weinberger [108] in his 2006 account of de Broglie's 1924 paper, "Revisiting Louis de Broglie's famous 1924 paper in the Philosophical Magazine", writes, "There is of course one final observation to be made: in terms of the present politics of publishing scientific papers; de Broglie's contribution could never have been published because it only essentially contains speculations. However, one can just as well say that this paper proves that speculations are an essential part of physics; without them no new ideas and theories are born. Quantum mechanics has to be regarded as a true rupture in the history of physics, as a revolution in the philosophy of science, a revolution that desperately needed speculations and deviations beyond well-accepted ways of thinking".

In 1887, Woldemar Voigt [105] published a paper on Doppler's principle, in which he proposed that in inertial reference frames, the speed of light remains constant and the classical wave equation remains invariant, and he obtained a set of space-time transformations different from the Lorentz transformations. Although not explicitly stated, Voigt anticipated special relativity arising out of the wave equation, and writing in relation to Voigt's 1887 paper on relativity, Freeman Dyson [29] said, "When the great innovation appears, it will almost certainly be in a muddled, incomplete and confusing form. To the discoverer himself it will only be half-understood; to everybody else it will be a complete mystery. For any speculation which does not at first glance look crazy, there is no hope". He further explained that "The reason why new concepts in any branch of science are hard to grasp is always the same; contemporary scientists try to picture the new concept in terms of ideas which existed before".

Bernard Cohen in his book on the Newtonian revolution comments, "Above all, Newton set forth a style of science that showed how mathematical principles might be applied to physics and astronomy (that is to natural philosophy) in a particularly fruitful way, and this may have been even more influential in the long run than his system of the world based on universal gravity" ([12], page 149). In a lecture in 1933, Albert Einstein emphasised "the free inventions of the human intellect", declared that reason and not empirical data must be the basis for any scientific system and said, "I am convinced that we can discover, by means of purely mathematical constructions, those concepts and those lawful connections between them which furnish the key to the understanding of natural phenomena. Experience may suggest the appropriate mathematical concepts, but they most certainly cannot be deduced from it. Experience remains, of course, the sole criterion of physical utility of a mathematical construction. In a certain sense, therefore, I hold it true that pure thought can grasp reality, as the ancients dreamed" (also quoted in [12], pages 152–153).

Our current knowledge and understanding of the physical universe, combined with the great achievements of Newtonian mechanics and modern physics, are so overwhelming, that as time passes, it becomes increasingly difficult to attempt the fundamental problems with a fresh mind and an open disposition. It seems to the author that we need to start at the beginning while being as inclusive as is possible of well-established theory. The mathematical construct presented here and in [47–52] is an attempt to satisfy these two not entirely consistent objectives.

Bibliography

1. R. Adam, P. A. R. Ade, and N. Aghanim et al. Planck 2015 results. i. Ooverview of products and scientific results. *Astronomy and Astrophysics*, 594: A1, 2016.
2. P. A. R. Ade, N. Aghanim, and M. Maraud et al. Planck 2015 results. xiii. Ccosmological parameters. *Astronomy and Astrophysics*, 594: A13, 2016.
3. N. Aghanim, Y. Karamu, and M. Ashton et al. Planck 2018 results. VI. Cosmological parameters. *Astronomy and Astrophysics*, 2019.
4. H. Andreka, J. X. Madarasz, I. Nemeti, and G. Szekely. A note on 'Einstein's special relativity beyond the speed of light by James M. Hill and Barry J. Cox'. *Proc. R. Soc. A*, 469:20120672, 2013.
5. G. Barton. *Elements of Green's functions and propagation*. Oxford Science Publications, Clarendon Press, Oxford, 1989.
6. H. Bateman and A. Erdelyi. *Higher transcendental functions*. McGraw-Hill, New York, 1955.
7. J. D. Bjorken and S. D. Drell. *Relativistic quantum fields (Pure and Applied Physics)*. McGraw-Hill, New York, 1965.
8. D. Bohm. *The special theory of relativity*. W. A. Benjamin, Inc., New York, 1965.
9. D. J. Bohm and B. J. Hiley. The de Broglie pilot wave theory and the further development of new insights arising out of it. *Foundations of Physics*, 12:1001–1016, 1982.
10. J. W. M. Bush. Pilot-wave hydrodynamics. *Annual Review of Fluid Mechanics*, 47:269–292, 2015.
11. H. Clark. *A first course in quantum mechanics*. Van Nostrand Reinhold (UK) Co. Ltd., 1982.
12. I. B. Cohen. *The Newtonian revolution*. Cambridge University Press, Cambridge, UK, 1980.
13. B. Coleman. Special relativity dynamics without *a priori* momentum conservation. *Eur. J. Phys.*, 26:647–650, 2005.
14. Y. Couder, S. Protiere, E. Fort, and A. Boudaoud. Walking and orbiting droplets. *Nature*, 437:208, 2005.
15. M. Dalarsson and N. Dalarsson. *Tensors, relativity and cosmology*. Academic Press, Elsevier, 2015.
16. L. de Broglie. Ondes et quanta. *Comptes Rendus*, 177:507–510, 1923.
17. L. de Broglie. Recherches sur la theorie des quanta. *PhD Thesis, Sorbonne University of Paris, France*, 1924.
18. L. de Broglie. *Ondes et mouvements*. Memorial des sciences physiques, Gauthier Villars et Cie, Paris, 1926.
19. L. de Broglie. *La mecanique ondulatoire*. Memorial des sciences physiques, Gauthier Villars et Cie, Paris, 1928.

J. M. Hill, *Mathematics of Particle-Wave Mechanical Systems*,
https://doi.org/10.1007/978-3-031-19793-2

20. L. de Broglie. *An introduction to the study of wave mechanics*. Methuen and Co London, 1930.
21. L. de Broglie. *Une interpretation causale et nonlinearie de la mechanique ondulatoire et la theorie de la double solution*. Memorial des sciences physiques, Gauthier Villars et Cie, Paris, 1956.
22. L. de Broglie. La thermodynamique cachee des particules. *Annales Institute Henri Poincare*, 1:1–19, 1964.
23. L. de Broglie. The reinterpretation of wave mechanics. *Foundations of Physics*, 1:5–15, 1970.
24. L. de Broglie. Interpretation of quantum mechanics by the double solution theory. *Annales de la Fondation Louis de Broglie*, 12:1–23, 1987.
25. H. Dingle. *The special theory of relativity*. Methuen and Co.Ltd., London, 1961.
26. R. d'Inverno. *Introducing Einstein's relativity*. Clarendon Press, Oxford, 1992.
27. P. A. M. Dirac. *Lectures on quantum mechanics*. Courier Dover Publications, New York, 1964.
28. P. A. M. Dirac. *Principles of quantum mechanics*. Clarendon Press, Oxford, 1981.
29. F. J. Dyson. Innovation in physics. *Scientific American*, 199:74–83, 1958.
30. J. Ehlers and W. Kundt. *Exact solutions of the gravitational field equations, in Gravitation: An introduction to current research*. Witten, L. (ed.) John Wiley, New York, 1962.
31. A. C. Eringen. *Nonlinear theory of continuous media*. McGraw-Hill Book Co., New York, 1962.
32. J. S. Farnes. A unifying theory of dark energy and dark matter and matter creation within a modified ΛCDM framework. *Astronomy and Astrophysics*, A92:1–20, 2018.
33. R. P. Feynman, R. B. Leighton, and M. Sands. *The Feynman Lectures on Physics*, volume 1. Addison-Wesley, 1964.
34. A. P. French. *Special relativity*. Thomas Nelson and Sons Ltd., London, 1968.
35. G. Gamow. *Thirty years that shook physics: The story of quantum theory*. Dover Publications, New York, 2014.
36. I. M. Gel'fand and G. E. Shilov. *Generalised functions, Vol. 1, Properties and operations*. Academic Press, New York, 1964.
37. C. C. Gillispie. *Dictionary of Scientific Biography*, volume 2. Charles Scribner's Sons, New York, 1970.
38. C. C. Gillispie. *Dictionary of Scientific Biography*, volume 9. Charles Scribner's Sons, New York, 1974.
39. G. M. L. Gladwell. *Contact problems in the classical theory of elasticity*. Sijthoff and Noordhoof, Leyden, 1980.
40. S. Goldstein. *Modern developments in fluid mechanics*, volume 1. Clarendon Press, Oxford, 1950.
41. E. Goursat. *A course in mathematical analysis*, volume 1. Dover Publications, New York, 1959.
42. J. B. Griffiths and J. Podolsky. *Exact space-times in Einstein's general relativity*. Cambridge University Press, Cambridge Monographs on Mathematical Physics, 2009.
43. F. Gross. *Relativistic quantum mechanics and field theory*. John Wiley and Sons, New York, 2008.
44. J. Guemez, M. Fiolhais, and L. A. Fernandez. The principle of relativity and the de Broglie relation. *Am. J Phys.*, 84:443–447, 2016.
45. D. M. Harris and J. W. M. Bush. The pilot-wave dynamics of walking droplets. *Physics of fluids*, 25:091112, 2013.
46. J. M. Hill. On static similarity deformations for isotropic materials. *Quarterly of Applied Mathematics*, 40:287–292, 1982.
47. J. M. Hill. On the formal origin of dark energy. *Zeitschrift fur angewandte Mathematik und Physik*, 69:133–145, 2018.
48. J. M. Hill. Some further comments on special relativity and dark energy. *Zeitschrift fur angewandte Mathematik und Physik*, 70:5–14, 2019.

49. J. M. Hill. Special relativity, de Broglie waves, dark energy and quantum mechanics. *Zeitschrift fur angewandte Mathematik und Physik*, 70:131–153, 2019.
50. J. M. Hill. Four states of matter and centrally symmetric de Broglie particle-wave mechanical systems. *Mathematics and Mechanics of Solids*, 26:263–284, 2020.
51. J. M. Hill. A review of de Broglie particle-wave mechanical systems. *Mathematics and Mechanics of Solids*, 25:1763–1777, 2020.
52. J. M. Hill. A mechanical model for dark matter and dark energy. *Zeitschrift fur angewandte Mathematik und Physik*, 72:56:14pp, 2021.
53. J. M. Hill. Einstein's energy and space isotropy. *Zeitschrift fur angewandte Mathematik und Physik*, 73:65:9pp, 2022.
54. J. M. Hill and B. J. Cox. Einstein's special relativity beyond the speed of light. *Proc. R. Soc. A*, 468:4174–4192, 2012.
55. J. M. Hill and B. J. Cox. Dual universe and hyperboloidal relative velocity surface arising from extended special relativity. *Zeitschrift fur angewandte Mathematik und Physik*, 65:1251–1260, 2014.
56. J. M. Hill and B. J. Cox. Generalised Einstein mass variation formulae: I Sub-luminal relative frame velocities. *Results in Physics*, 6:112–121, 2016.
57. J. M. Hill and B. J. Cox. Generalised Einstein mass variation formulae: II Superluminal relative frame velocities. *Results in Physics*, 6:122–130, 2016.
58. J. M. Hill and Dewynne J. N. *Heat Conduction*. Blackwell Science Publications, Oxford, 1987.
59. J. M. Hill and J. O'Leary. Generalized transformations and coordinates for static spherically symmetric general relativity. *Royal Society Open Science*, 5:171109 (9pp), 2018.
60. J. M. Hill and A. Tordesillas. The pressure distribution for symmetrical contact of circular elastic cylinders. *Quarterly Journal of Mechanics and Applied Mathematics*, 42:581–604, 1989.
61. J. M. Houlik and G. Rousseaux. "Non-relativistic" kinematics: Particles or waves. *Unpublished, available from Germain Rousseaux website University of Poitiers*, page 4pp., 2010.
62. C. Jin and M. Lazar. A note on Lorentz-like transformations and superluminal motion. *Z. Angew. Math. Mech. (ZAMM)*, 95:690–694, 2015.
63. K. L. Johnson. *Contact mechanics*. University Press, Cambridge, 1984.
64. T. S. Kuhn. *The structure of scientific revolutions*. Chicago University Press, Chicago, 1970.
65. J. Lacki. *New Dictionary of Scientific Biography*, volume 1. Charles Scribner's Sons, New York, 2008.
66. L. D. Landau and E. M. Lifshitz. *Course of Theoretical Physics*, volume 2. Addison-Wesley, 1951.
67. A. R. Lee and T. M. Kalotas. Lorentz transformations from the first postulate. *Amer. J. Phys.*, 43:434–437, 1975.
68. J.-M. Levy-Leblond. One more derivation of the Lorentz transformation. *Amer. J. Phys.*, 44:271–277, 1976.
69. M. J. Lighthill. *An introduction to Fourier analysis and generalised functions*. Cambridge University Press, Cambridge, 1989.
70. M. Mantegna. Revisiting Barry Cox and James Hill's theory of superluminal motion: a possible solution to the problem of spinless tachyon localisation. *Proc. R. Soc. A*, 471:20140541, 2015.
71. H. Massey. *Atomic and molecular collisions*. Taylor and Francis Ltd., London, 1979.
72. J. C. Maxwell. On Faraday's lines of force. *Transactions of the Cambridge Philosophical Society*, 10:155–229, 1855.
73. J. C. Maxwell. On physical lines of force. *Philosophical Magazine*, 90:11–23, 1861.
74. J. C. Maxwell. A dynamical theory of the electromagnetic field. *Philosophical Transactions of the Royal Society of London*, 155:459–512, 1865.
75. J. C. Maxwell. *A treatise on electricity and magnetism*. Clarendon Press, Oxford, 1873.
76. W. H. McCrea. *Relativity physics*. Methuen and Co.Ltd., London, 1947.

77. H. Minkowski. *Space and time, in "The Principle of Relativity", by H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl*. Dover, New York, 1952.
78. C. Moller. *The theory of relativity*. Clarendon Press, Oxford, 1966.
79. M. S. Morris and K. S. Thorne. Wormholes in space-time and their use for interstellar travel: A tool for teaching general relativity. *American Journal of Physics*, 56:395–412, 1988.
80. I. Müller. *A history of thermodynamics; The doctrine of energy and entropy*. Springer, 2007.
81. T. Müller and F. Grave. *Catalogue of spacetimes*. arXiv: 0904.4184v3 [gr-qc] 4 Nov 2010, 2010.
82. A. U. Oza, O. Wind Willassen, D. M. Harris, R. R. Rosales, and J. W. M. Bush. Pilot-wave hydrodynamics in a rotating frame: Exotic orbits. *Physics of fluids*, 26:082101, 2014.
83. W. Perrett and Jeffery G. B. *The principle of relativity*. Dover Publications Inc., 2017.
84. E. Poisson and C. M Will. *Gravity: Newtonian, post-newtonian, relativistic*. Cambridge University Press, 2014.
85. A. D. Polyanin and V. F. Zaitse. *Handbook of exact solutions for ordinary differential equations*. 2nd Edition, CRC Press, New York, 2003.
86. R. Resnick. *Introduction to special relativity*. John Wiley and Sons Inc., New York, 1968.
87. A. G. Riess, S. Casertano, W. Yuan, L. M. Macri, and D. Scolnic. Large magellanic cloud cepheid standards provide a one percent foundation for the determination of the Hubble constant and stronger evidence for physics beyond ΛCDM. *Astrophysical Journal ApJ*, 876:85–98, 2019.
88. A. Ronveaux. *Heun's differential equations*. Oxford University Press, Oxford, 1995.
89. G. Rousseaux. Forty years of Galilean Electromagnetism (1973–2013). *Eur. Phys. J. Plus*, 128:81–94, 2013.
90. D. G. Saari. Mathematics and the "dark matter" puzzle. *Amer. Math. Monthly*, 122:407–427, 2015.
91. B. Schutz. *A first course in general relativity*. Cambridge University Press, 2009.
92. D. Scott and G. F. Smoot. Cosmic microwave background mini-review. *P.A. Zyla et al. Particle Data Group, Review of Particle Physics, Prog. Theor. Exp. Physics*, 8:083C01, 2020.
93. H. Semat. *Introduction to atomic and nuclear physics*. Chapman and Hall Ltd., London, 1964.
94. I. N. Sneddon. *Special functions of mathematical physics and chemistry*. Oliver and Boyd Ltd., Interscience Publishers, New York, 1961.
95. Sergei M. Kopeikin, Ed. Soffel. M. *Frontiers in relativistic celestial mechanics*, volume de Gruyter Studies in Mathematical Physics Book 21. Walter de Gruyter, 2014.
96. S. Sonego and M. Pin. Foundations of anisotropic relativistic mechanics. *Journal of Mathematical Physics*, 50:042902, 2009.
97. B. Spain. *Tensor calculus*. Oliver and Boyd, Edinburgh and London, 1960.
98. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt. *Exact solutions of Einstein's field equations*. Cambridge University Press, 2009.
99. J. L. Synge. Orbits and rays in the gravitational field of a finite sphere according to the theory of A. N. Whitehead. *Proc. R. Soc. A*, 211:303 – 319, 1952.
100. J. L. Synge. *Geometrical mechanics and de Broglie waves*. Cambridge University Press, Cambridge, 1954.
101. J. L. Synge and A. Schild. *Tensor calculus*. University of Toronto Press, Toronto, 1949.
102. R. C. Tolman. *Relativity, thermodynamics and cosmology*. Clarendon Press, Oxford, 1946.
103. F. G. Tricomi. *Integral equations*. Interscience, New York, 1957.
104. R. S. Vieira. An introduction to the theory of tachyons. *Revista Brasileira de Ensino de Física*, 34:3306, 2012.
105. W. Voigt. On the principle of Doppler. *Nachr. Ges. Wiss. Gottingen*, 8:41–51, 1887.
106. A. Wachter and H. Hoeber. *Compendium of theoretical physics*. Springer, 2006.
107. R. M. Wald. *General relativity*. University of Chicago Press, 2010.
108. P. Weinberger. Revisiting Louis de Broglie's famous 1924 paper in the Philosophical Magazine. *Philosophical Magazine Letters*, 86:405–410, 2006.
109. G. Weinstein. Variation of mass with velocity: "kugeltheorie" or "relativtheorie". *arXiv:1205.5951 [physics.hist-ph]*, 2012.

Index

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