Chapter 6
Series

Abstract Series are just a special type of sequences. The main feature of numerical series is that they lead us to finding convergence theorems which do not involve the value of the limit.

If $a = \{a_n\}_n$ is a sequence of real numbers, we use the symbol

$$\sum_{n=p}^{q} a_n$$

to denote the finite sum $a_p + a_{p+1} + \cdots + a_{q-1} + a_q$. We use the sequence a to construct a new sequence $s = \{s_n\}_n$ by means of the formula

$$s_n = \sum_{k=1}^{n} a_k.$$

The sequence s is called the *sequence of partial sums* of a. It is customary to introduce a different notation for the sequence s:

$$s = \sum_{n=1}^{\infty} a_n.$$

In a really formal world, a series should be defined as an ordered couple (a, s) such that a is a sequence, s is a sequence, and $s_n = \sum_{k=1}^{n} a_k$ for each $n \in \mathbb{N}$.

Remark 6.1 The language about series is very unprecise. In a completely rigorous world, we should probably remove the word *series* and continue to use the word

S. Secchi, *A Circle-Line Study of Mathematical Analysis*,
La Matematica per il 3+2 141, https://doi.org/10.1007/978-3-031-19738-3_6

sequence, as in

consider the sequence $\left\{\sum_{k=1}^{n} \frac{k}{k^2+1}\right\}_n$.

Furthermore, several mathematicians interpret $\sum_{n=1}^{\infty} a_n$ as $\lim_{N \to +\infty} \sum_{n=1}^{N} a_n$, which is either a real number of a symbol of infinity. Despite these difficulties, tradition rules, and in this chapter we will freely abuse of language and define a series with the symbol $\sum_n a_n$.

Definition 6.1 We say that the series $\sum_{n=1}^{\infty} a_n$ converges to s if $\lim_{n \to +\infty} s_n = s$. In this case, we will often say that s is the sum of the series.

Remark 6.2 It should be clear that sequences and series are the same object. Indeed, series are sequences by definition. Conversely, the sequence $\{a_n\}_n$ can be recovered from the sequence $\{s_n\}_n$ by writing $a_n = s_n - s_{n-1}$. Of course this logical equivalence is not a good reason to forget about numerical series at all.

We will often write $\sum_n a_n$ or even $\sum a_n$ to denote a series. We agree that the first index of the sum may also be different than 1, as in $\sum_{n=7}^{\infty} a_n$. Clearly, the convergence of a series does not depend on the first terms that we add or discard: remember that the character of a sequence is not altered by the modification of finitely many terms.

Example 6.1 Let us consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}.$$

Since

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n},$$

we see that

$$s_n = \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 - \frac{1}{n} \to 1$$

as $n \to +\infty$. Hence the series converges to the sum 1.

Example 6.2 The previous example can be easily generalized. Suppose that we are given the series

$$\sum_{n=1}^{\infty} (b_{n+1} - b_n),$$

where $\{b_n\}_n$ is a sequence such that $\lim_{n\to+\infty} b_n = b$. Then

$$\sum_{n=1}^{\infty}(b_n - b_{n+1}) = b_1 - b.$$

These are called *telescoping series*.

Since a closed formula for the partial sums of a sequence is usually unavailable, the whole theory of convergence must be based on some *indirect* approach. A very general one is the Cauchy characterization of convergence.

Theorem 6.1 (Cauchy for Series) *A series* $\sum a_n$ *converges if and only if for every* $\varepsilon > 0$ *there exists a positive integer* N *such that*

$$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$$

for any $m \geq n > N$.

Proof Since

$$\left|\sum_{k=n}^{m} a_k\right| = |s_m - s_{n-1}|,$$

the conclusion follows from Theorem 5.14. □

Corollary 6.1 (Necessary Condition for Convergence) *If* $\sum a_n$ *converges, then* $\lim_{n\to+\infty} a_n = 0$.

Proof We take $m = n$ in the previous theorem. □

Remark 6.3 We will see that this corollary cannot be reversed. For instance the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, although $1/n \to 0$ as $n \to +\infty$.

> The necessary condition for convergence confirms an intuitive fact: you cannot sum infinitely many numbers and obtain a finite result, unless the numbers you add get smaller and smaller. As usual, intuitive results in mathematics are weak results.

Theorem 6.2 *Suppose that* $a_n \geq 0$ *for each* n. *The series* $\sum a_n$ *converges if and only if its partial sums form a bounded sequence.*

Proof For a series of non-negative terms, we clearly have

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

for every n. In other words, the sequence of partial sums is increasing. The conclusion follows from Theorem 5.7. □

The most important test of convergence is based on comparison. We will see that actually all convergence tests are based on some comparison argument.

Theorem 6.3 (Comparison Test)

(a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed positive integer, and if $\sum c_n$ converges, then $\sum a_n$ converges as well.
(b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges as well.

Proof

(a) Given $\varepsilon > 0$, there exists a positive integer $n_0 \geq N_0$ such that $m \geq n > n_0$ implies $\sum_{k=n}^{m} c_k \leq \varepsilon$. Hence $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \varepsilon$.
(b) If $\sum a_n$ converges, by (a) $\sum d_n$ converges. Contradiction.

□

An important corollary is described in the next result.

Theorem 6.4 (Asymptotic Comparison Test) *Let $\sum a_n$ and $\sum b_n$ be series of positive terms, and suppose that*

$$\lim_{n \to +\infty} \frac{a_n}{b_n} = 1.$$

The series $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof Indeed, there exists a positive integer N_0 such that $1/2 < a_n/b_n < 3/2$ for every $n > N_0$. Hence $\frac{b_n}{2} < a_n < \frac{3}{2} b_n$ for $n > N_0$. The conclusion follows from the Comparison test. □

Example 6.3 The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Indeed,

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)}$$

for $n = 2, 3, \ldots$ We conclude by comparison with Example 6.1.

Remark 6.4 If $a_n = 1/n^2$ and $b_n = \frac{1}{n(n-1)}$, we have $\lim_{n \to +\infty} a_n/b_n = 1$. This shows a typical application of the asymptotic comparison test to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which often requires less care in checking the validity of the comparison.

The triangle inequality always ensures that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k|, \tag{6.1}$$

leading us to the following definition via the Cauchy condition for convergence.

Definition 6.2 (Absolute Convergence) We say that the series $\sum a_n$ converges absolutely, if the series $\sum |a_n|$ is convergent.

An easy but not trivial consequence of (6.1) is the next result.

Theorem 6.5 *Every absolutely convergent series is convergent.*

Proof Let $\sum a_n$ be an absolutely convergent series. By (6.1), the series $\sum a_n$ satisfies the Cauchy condition, and is therefore convergent. □

The converse is false, as Exercise 6.4 shows.

6.1 Convergence Tests for Positive Series

Theorem 6.2 says that series of positive terms are somehow easier to deal with, since no oscillation phenomenon can arise. In this section we develop several convergence tests for positive series, i.e. series of positive terms.

Important: Negative Series

Of course the very same tests can be applied to series of *negative* terms, just by changing signs to each term. For the sake of definiteness, we will always deal with positive series.

Let us start with a milestone of the theory.

Theorem 6.6 (Geometric Series) *If* $0 \leq x < 1$, *then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \geq 1$, *the series* $\sum_{n=0}^{\infty} x^n$ *diverges.*

Proof If $x = 1$, then $\sum_{k=0}^{n} 1^k = n + 1$, and the series diverges. Suppose $x \neq 1$, and compute

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x}.$$

Indeed

$$(1-x)(1+x+x^2+\cdots+x^n) = 1+x-x+x^2-x^2+\cdots+x^n-x^n-x^{n+1}$$
$$= 1-x^{n+1}.$$

The conclusion follows by letting $n \to +\infty$. □

Exercise 6.1 Prove the identity

$$(1-x)(1+x+x^2+\cdots+x^n) = 1-x^{n+1}$$

by induction.

The following test is usually a difficult one for students. It states a rather surprising fact: under a monotonicity assumption, only those terms of a very particular subsequence decide whether a series converges.

Theorem 6.7 (Condensation Test) *Suppose that* $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. *The series* $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if the series* $\sum_{k=0}^{\infty} 2^k a_{2^k}$ *is convergent.*

Proof It suffices to prove that the partial sums of the two series are simultaneously bounded from above. Set

$$s_n = a_1 + \cdots + a_n$$
$$t_k = 2^0 a_{2^0} + 2^1 a_{2^1} + \cdots + 2^k a_{2^k}.$$

We consider two cases. If $n < 2^k$, then

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k \end{aligned}$$

by the monotonicity of $\{a_n\}_n$. Notice that we have grouped terms in blocks that begin with a power of 2 and end one step before the subsequent power of 2. We deduce that $s_n \leq t_k$.

On the other hand, if $2^k < n$, we group terms in a different way:

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\ &= \frac{1}{2} t_k. \end{aligned}$$

In this case, $t_k \le 2s_n$. In any case the sequences $\{s_n\}_n$ and $\{t_k\}_k$ are both bounded or unbounded above, and the proof is complete. □

Example 6.4 As a fundamental application, we consider the *generalized harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is a fixed real number. Clearly $p \le 0$ implies divergence of the series, since the general term does not converge to zero. For $p > 0$ we use the condensation test, and look at the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

This is a geometric series, and we know that the latter series converges if and only if $2^{1-p} < 1$, i.e. $p > 1$.

We propose the following tests for historical reasons. They are based on a comparison with a geometric series, and we will comment on the weakness of these tests after the proof.

Theorem 6.8 (Root and Ratio Tests) *The series* $\sum a_n$

(a) converges, if $\limsup_{n\to+\infty} \sqrt[n]{|a_n|} < 1$*;*
(b) diverges, if $\limsup_{n\to+\infty} \sqrt[n]{|a_n|} > 1$*;*
(c) converges, if $\limsup_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$*;*
(d) diverges, if $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ *for each* $n \ge n_0$*, where* n_0 *is some fixed positive integer.*

Proof Put $\alpha = \limsup_{n\to+\infty} \sqrt[n]{|a_n|}$. If $\alpha < 1$, we can choose β such that $\alpha < \beta < 1$, and a positive integer N such that $\sqrt[n]{|a_n|} < \beta$ for each $n \ge N$. Hence $n \ge N$ implies $|a_n| < \beta^n$. Since $\beta < 1$, the comparison test leads to (a).

If $\alpha > 1$, then $\sqrt[n]{|a_n|} > 1$ for infinitely many indices n (otherwise 1 would be an eventual upper bound). This prevents a_n from converging to 0 as $n \to +\infty$, and the series $\sum a_n$ is divergent. This proves (b).

Suppose that $\limsup_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$: we can find $\beta < 1$ and a positive integer N such that $\left|\frac{a_{n+1}}{a_n}\right| < \beta$ for each $n \ge N$. In particular

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ &\vdots \\ |a_{N+p}| &< \beta^p |a_N| \end{aligned}$$

for each positive integer p. Writing $n = N + p$ we discover that

$$|a_n| < |a_N|\beta^{-N} \cdot \beta^n$$

for each $n \geq N$. Again (c) follows from the comparison theorem. Finally, if $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, then the condition $a_n \to 0$ fails, and the series $\sum a_n$ is divergent. □

The root and the ration tests are popular but *weak*. We know that the series $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges. The ratio and the root tests are both inconclusive, since the limsup equals 1.

Remark 6.5 It follows from Theorem 5.19 that the root test is stronger than the ratio test. In particular, if the root test is inconclusive, the ratio test must be inconclusive as well.

Example 6.5 Consider the series $\sum_n \frac{n}{n^2+3}$. If we put $a_n = \frac{n}{n^2+3}$, there results

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \frac{n^2+3}{n^2+2n+4}.$$

We deduce that $\lim_{n\to+\infty} |a_{n+1}/a_n| = 1$. Similarly, $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1$. The root test and the ratio test are inconclusive, although the series is divergent by comparison:

$$a_n \geq \frac{n}{n^2+3n^2} = \frac{1}{4n}.$$

Once more, we remark that a clever direct comparison is often preferable to a standard test.

Exercise 6.2 Prove that if a series $\sum_n a_n$ of nonnegative numbers converges, then the series $\sum_n a_n^p$ converges for every real number $p > 1$. *Hint:* the inequality $a_n < 1$ must hold eventually.

Exercise 6.3 Prove that $\sum_n a_n$ and $\sum_n b_n$ are convergent series of nonnegative numbers, then the series $\sum_n \sqrt{a_n b_n}$ converges. *Hint:* prove that $\sqrt{a_n b_n} \leq a_n + b_n$.

6.2 Euler's Number as the Sum of a Series

The typical Calculus approach to the definition of the number e is via the "fundamental limit"

$$\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n.$$

Unfortunately the existence of this limit is not straightforward. In the next theorem we propose a different approach.

Theorem 6.9 (The Euler Number) *The series* $\sum_{n=0}^{\infty} \frac{1}{n!}$ *converges to a limit that is denoted by e and called the Euler number. Furthermore,* $e = \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n$.

Proof Recall that $0! = 1$ and, for any positive integer n, the factorial of n is defined as $n! = 1 \cdot 2 \cdots (n-1)n$. Since

$$\begin{aligned} s_n &= 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdots n} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3, \end{aligned}$$

the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to a limit $e < 3$. To prove the second part, we introduce the sequences

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad t_n = \left(1+\frac{1}{n}\right)^n.$$

The binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

yields

$$\begin{aligned} t_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \\ + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \end{aligned}$$

Then $t_n \le s_n$ and $\limsup_{n\to+\infty} t_n \le e$. If $n \ge m$,

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right),$$

so that $s_m \le \liminf_{n\to+\infty} t_n$ for any m. Letting $m \to +\infty$, $e \le \liminf_{n\to+\infty} t_n$, and the proof is complete. □

The definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is rather flexible, and allows us to derive a theoretical property of the Euler number.

Theorem 6.10 *The number e is irrational.*

Proof We begin with an estimate of the convergence of the series $\sum 1/n!$ to e. Letting s_n denote the n-th partial sum of this series, we have

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) \\ &= \frac{1}{n!n}. \end{aligned}$$

Therefore $0 < e - s_n < \frac{1}{n!n}$ for each positive integer n. Now suppose that $e = p/q$ is a rational number, where p and q are positive integers. Then $0 < q!(e - s_q) < 1/q$. The number $q!e$ must be an integer, since e is rational. Also

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q}\right)$$

is an integer. Hence $q!(e - s_q)$ is an integer between 0 and 1: contradiction. The number e is therefore irrational. □

6.3 Alternating Series

The reader should suspect that a complete analysis of series whose terms do not have constant sign is out of reach. In this section we focus our attention on a particular class of series of variable sign. We begin with a general result which reminds us of the popular formula of integration by parts.

Proposition 6.1 (Summation by Parts) *Two sequences $\{a_n\}_n$ and $\{b_n\}_n$ are given. Put $A_{-1} = 0$ and $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$. For each positive integers $p \leq q$ we have*

$$\sum_{n=p}^{p} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Proof Since $a_n = A_n - A_{n-1}$, we write

$$\sum_{n=p}^{p} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1})\, b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}.$$

The last difference is equal to $\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$, and the proof is complete. □

Theorem 6.11 (Dirichlet's Test) *Suppose*

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;
(b) $b_0 \geq b_1 \geq b_2 \geq \ldots$;
(c) $\lim_{n\to+\infty} b_n = 0$.

Then the series $\sum a_n b_n$ is convergent.

Proof There exists $M > 0$ such that $|A_n| \leq M$ for each n. Let $\varepsilon > 0$, and pick a positive integer ν such that $b_\nu \leq \varepsilon/(2M)$. For $\nu \leq p \leq q$ we have by Proposition 6.1

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &\leq \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq M \left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| \\ &= 2M b_p \leq 2M b_\nu \leq \varepsilon.\end{aligned}$$

The series $\sum a_n b_n$ converges by the Cauchy theorem. □

Choosing $a_n = (-1)^{n+1}$ and $b_n = |c_n|$ in the previous theorem yields a popular test for alternating series.

Theorem 6.12 (Leibnitz Theorem for Alternating Series) *Suppose that*

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \ldots$
(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ for $m = 1, 2, 3, \ldots$
(c) $\lim_{n\to+\infty} c_n = 0$

Then the series $\sum c_n$ is convergent.

Exercise 6.4 Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but it does not converge absolutely. This fact often seems to be surprising, but we must remember that the factor $(-1)^n$ contributes to a huge balancing of the terms in the series.

6.3.1 Product of Series

Numerical series can be multiplied together. The definition is reminiscent of the product of two polynomials $p(x)$ and $q(x)$, in which terms are grouped according to the power of the unknown x.

Definition 6.3 (Cauchy Product of Two Series) The Cauchy product of the series $\sum a_n$ and $\sum b_n$ is the series $\sum c_n$ defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Remark 6.6 Properly speaking, the Cauchy product of two series is a discrete *convolution* product. Since we do not assume the reader to be familiar with integral convolutions, we will not use this language in the book.

The convergence of a product of two series is a delicate issue. Consider for example the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

Convergence follows from Theorem 6.12. Let us now multiply this series by itself, obtaining

$$\begin{aligned}\sum_{n=0}^{\infty} c_n &= 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}.\end{aligned}$$

But

$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2,$$

and

$$|c_n| \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}.$$

Since the necessary condition $c_n \to 0$ is violated, the series $\sum c_n$ must diverge. Here comes the basic convergence result about the product of convergent series.

Theorem 6.13 (Mertens) *Suppose that*

(a) $\sum_{n=0}^{\infty} a_n$ *converges absolutely*
(b) $\sum_{n=0}^{\infty} a_n = A$
(c) $\sum_{n=0}^{\infty} b_n = B$
(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Then $\sum_{n=0}^{\infty} c_n$ *converges.*

Proof We follow [1], and set

$$A_n = \sum_{k=0}^{n} a_k$$
$$B_n = \sum_{k=0}^{n} b_k$$
$$C_n = \sum_{k=0}^{n} c_k$$
$$\beta_n = B_n - B.$$

We compute

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0 (B + \beta_n) + \cdots + a_n (B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0. \end{aligned}$$

To conclude the proof, we must show that $\lim_{n \to +\infty} \gamma_n = 0$, where $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$. Let $\alpha = \sum_{n=0}^{\infty} |a_n|$. Notice that this is the first time we invoke assumption (a). Given any $\varepsilon > 0$, we can choose a positive integer ν such that $|\beta_n| \leq \varepsilon$ for each $n \geq \nu$. Thus

$$\begin{aligned} |\gamma_n| &\leq |\beta_0 a_n + \cdots + \beta_\nu a_{n-\nu}| + |\beta_{\nu+1} a_{n-\nu-1} + \cdots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \cdots + \beta_n a_{n-\nu}| + \varepsilon \alpha. \end{aligned}$$

Since $\limsup_{n \to +\infty} (\beta_0 a_n + \cdots + \beta_n a_{n-\nu}) = 0$, we find $\limsup_{n \to +\infty} |\gamma_n| \leq \varepsilon \alpha$, and the conclusion follows. □

6.4 Problems

6.1 Decide whether the series

$$\sum_{n=1}^{\infty} \sin(\alpha) \sin(2\alpha) \cdots \sin(n\alpha)$$

is convergent, for any fixed value of $\alpha \in \mathbb{R}$.

6.2 Let $\{a_n\}_n$ be a sequence with the property that there exists a real number $h < 1$ such that $|a_{n+1} - a_n| \leq h|a_n - a_{n-1}|$ for each n. Prove that the sequence converges.

6.3 Using the previous problem, show that the sequence defined by choosing any two real numbers a_1 and a_2, and defining

$$a_{n+1} = \frac{a_{n-1} + a_n}{2}$$

converges. Compute its limit.

6.4 Let $\{a_n\}_n$ be a sequence of positive real numbers. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

6.5 Starting from

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

and using Cauchy products, prove that

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

for each real number x with $|x| < 1$.

6.5 Comments

Once upon a time, the treatment of numerical series used to fill up long chapters in Calculus textbooks. As I have tried to show, the theory of series is indeed a long collection of sufficient conditions for the convergence of particular sequences of numbers. In recent years this awareness has become prevalent, and we no longer annoy our students with awful convergence tests. Last but not least, many of these tests are based on the algebraic properties of real numbers, and they do not extend to series of complex numbers, for instance.

Reference

1. W. Rudin, *Principles of Mathematical Analysis*. International Series in Pure and Applied Mathematics, 3rd edn. (McGraw-Hill Book Co., New York, 1976)