Chapter 4
Elementary Cardinality

Abstract What does it mean that two sets have the same number of elements? This may appear clear if we can write down all the members in a finite list. The answer becomes complicated if the sets contain infinitely many elements. In this chapter we propose a definition of cardinality in an elementary fashion.

4.1 Countable and Uncountable Sets

Definition 4.1 (Sequences) A sequence is any function whose domain is of the form $\mathbb{N} \setminus F$, for some finite subset F of $\mathbb{N}$. If X is a set, a sequence in X is any function which takes values in X and whose domain is of the form $\mathbb{N} \setminus F$, for some finite subset F of $\mathbb{N}$.

If s is a sequence, it is customary to abridge the notation $s(n)$ to s_n. Hence we will also write $\{s_n\}_n$ for a sequence, but we remark that n is a dummy variable: $\{s_n\}_n = \{s_j\}_j = \{s_k\}_k = \dots$

Important: Notation for Sequences

Since a sequence is a function, one might wonder why we make so many efforts to avoid the natural use of functional notation. This sounds as a reasonable question, because historical habit remains the only answer. Sequence are often denoted by $(s_n)_n$ or $\langle s_n \rangle_n$, to distinguish the sequence from the *set* of its values.

We try to illustrate our definition of sequences.

Theorem 4.1 *Let N be a subset of $\mathbb{N}$. The following statements are equivalent:*

(a) $N = \mathbb{N} \setminus F$ for some finite subset F of $\mathbb{N}$;
(b) N contains an interval of the form $\mathbb{N} \cap [n_0, +\infty)$ for some $n_0 \in \mathbb{N}$.

Proof If (a) holds, we call $n_0 - 1$ the largest positive integer which does not belong to N. Then (b) holds. Conversely, we suppose that (b) holds and we consider the

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finite set $\{1, 2, \ldots, n_0 - 1\}$. Thus at most finitely many positive integers do not belong to N, and (a) holds. □

In other words, our sequences may be considered as functions from an unbounded interval $\mathbb{N} \cap [n_0, +\infty)$ for some $n_0 \in \mathbb{N}$. In the Comments at the end of the chapter we will discuss again our definition.

Definition 4.2 (Subsequences) Let s be a sequence, and let $k \colon \mathbb{N} \to \mathbb{N}$ a sequence of positive integers with the property that $k_n < k_{n+1}$ for each $n \in \mathbb{N}$. Then the composition $s \circ k$ is called a subsequence of s. Explicitly, $s \circ k = \{s_{k_n}\}_n$.

Remark 4.1 In a subsequent chapter we will see that a weaker condition on the sequence k could be assumed in order to define subsequences. The strong monotonicity $k_n < k_{n+1}$ is however more popular in the literature.

Definition 4.3 (Equal Cardinality) Two sets A and B are equinumerous (or have the same cardinality), if there exists a bijective function $F \colon A \to B$. In this case we will write $A \sim B$, or even $\#A = \#B$.

It is an easy exercise in set theory to check that $\sim$ is actually an equivalence relation between sets. We will use this fact in the rest of the chapter.

Definition 4.4 We say that a set A has cardinality n, if $A \sim \{1, 2, \ldots, n\}$. By extension, the cardinality of the empty set is zero. A set A is finite, if there exists a positive integer n such that A has cardinality n. Otherwise it is called infinite. A set A is countably infinite if $A \sim \mathbb{N}$, and it is countable if it is either finite or countably infinite. If A is not countable, we say that A is uncountable.

Important: Finite or Countable?

The use of the adjective "countable" is not completely universal. Several mathematicians actually think of countable sets as countably infinite sets. Hence they would not say that $\{5, 7, 11, 23\}$ is a countable set. In my opinion, such an agreement is popular among analysts, who seldom work with finite structures. For this reason, it may happen that in this book the word countable can be used instead of countably infinite. The reader should not have any trouble in recognizing such an abuse of language.

Exercise 4.1 Prove that the Cartesian product of two finite sets is a finite set. *Hint:* this is essentially a "matrix" proof. If X has n members and Y has m members, you can write down $X \times Y$ as a matrix of n rows and m columns. Then just... count the entries of this matrix.

A countably infinite set S can always be described as $S = \{s_1, s_2, \ldots\}$, where s is the bijective function that describes the fact that $A \sim \mathbb{N}$. In this sense, a countably infinite set can be seen as a *labeled* list of points.

Theorem 4.2 *Every subset of a countable set is countable.*

Proof Let S be a countable set, and let $A \subset S$. If A is finite, there is nothing to prove. We may therefore assume that A is infinite, and S is infinite as well. We select a sequence $s = \{s_n\}_n$ of distinct points such that $S = \{s_1, s_2, \ldots\}$. We define a function as follows: let k_1 be the smallest positive integer such that $s_{k_1} \in A$. If $k_2, k_3, \ldots, k_{n-1}$ have been selected, we choose k_n as the smallest positive integer $> k_{n-1}$ such that $s_{k_n} \in A$. It is evident that $k_n < k_{n+1}$ for each n. The composition $s \circ k$ is defined on $\mathbb{N}$ and its range is A. Since $s_{k_n} = s_{k_m}$ implies $k_n = k_m$ (because the points $s_1, s_2, \ldots$ are distinct) and this implies $n = m$, we see that $s \circ k$ is injective. The proof is complete. □

Theorem 4.3 *The cartesian product* $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

Proof For each $(m, n) \in \mathbb{N} \times \mathbb{N}$ we set $f(m, n) = 2^m 3^n$. This is an injective function whose range is contained in $\mathbb{N}$. Since this range is countable by the previous theorem and $\mathbb{N} \times \mathbb{N}$ is clearly infinite, the proof is complete. □

What about the cardinality of $\mathbb{Q}$? To answer this question we need some preliminary result about unions of countable sets.

We say that a family F of sets is a collection of disjoint sets, if any two elements of F are disjoint.

Theorem 4.4 *If F is a countable collection of disjoint sets, say $F = \{A_1, A_2, \ldots\}$, such that each A_n is countable, then $\bigcup F = \bigcup_{n=1}^{\infty} A_n$ is also countable.*

Proof For each n, let $A_n = \{a_{1,n}, a_{2,n}, a_{3,n}, \ldots\}$. Call $S = \bigcup_{n=1}^{\infty} A_n$. Every element x of S must lie in some A_n, thus $x = a_{m,n}$ for some pair of integers (m, n). This pair is uniquely determined, since F is a collection of disjoint sets. This defines a function $f \colon S \to \mathbb{N} \times \mathbb{N}$ via $f(x) = a_{m,n}$. We have just seen that f is injective, so its range is countable. We conclude that S is also countable. □

We want to remove the assumption that F should be a collection of disjoint sets. This is possible, but it requires some attention.

Theorem 4.5 *If F is a countable collection of countable sets, then the union of all the members of F is also countable.*

Proof We need to reduce to the case of a collection of disjoint sets. A standard way to achieve this result is as follows: put $B_1 = A_1$, and, for $n > 1$,

$$B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k.$$

Clearly $G = \{B_1, B_2, B_3, \ldots\}$ is a disjoint collection. Setting $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, we show that $A = B$. If $x \in A$, then $x \in A_k$ for some k. Let n be the smallest k with this property, so that $x \notin A_k$ for $k < n$. This implies $x \in B_n$, and in turn $x \in B$. Viceversa, if $x \in B$, then $x \in B_n$ for some n, and in particular $x \in A_n$ for the same n. The proof is complete. □

Corollary 4.1 *The set $\mathbb{Q}$ of rational numbers is countably infinite.*

Proof We call A_n the set of all positive rational numbers whose denominator is n. The set $\mathbb{Q}$ is therefore equal to $\bigcup_{n=1}^{\infty} A_n$, a union of countable sets. The result follows from the previous theorem and the trivial remark that $\mathbb{Q}$ is an infinite set. □

We already know that $\mathbb{R} \neq \mathbb{Q}$ as sets. We can now show that $\mathbb{R}$ has actually more elements than $\mathbb{Q}$.

Theorem 4.6 *The set $\mathbb{R}$ is uncountable.*

Proof Since the interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is a subset of $\mathbb{R}$, it suffices to show that $(0, 1)$ is uncountable. Suppose not, so that there exists a sequence $s = \{s_n\}_n$ whose range is $(0, 1)$. We show that this is impossible by constructing a real number in $(0, 1)$ which is not a term of the sequence s. As a starting point, we assume that each real number can be uniquely written as an infinite decimal, and in particular $s_n = 0.u_{n,1}u_{n,2}u_{n,3}\ldots$ Each $u_{n,i}$ is a digit, i.e. an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Consider the number $y = 0.v_1v_2v_3\ldots$ where

$$v_n = \begin{cases} 1 & \text{if } u_{n,n} \neq 1 \\ 2 & \text{if } u_{n,n} = 1. \end{cases}$$

We claim that no term of the sequence $\{s_n\}_n$ can equal y. Indeed y differs from s_1 in the first digit, differs from s_2 in the second digit, and in general differs from s_n in the n-th digit. But $0 < y < 1$ by construction, and this contradicts the assumption that $(0, 1)$ is countable. □

Example 4.1 Every open subset (a, b) of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$. Indeed, we choose a number $c \in (a, b)$ and we define $f\colon (a, b) \to \mathbb{R}$ as

$$f(x) = \begin{cases} \frac{x-c}{b-x} & \text{if } c \leq x < b \\ \frac{x-c}{x-a} & \text{if } a < x \leq c. \end{cases}$$

It is easy to check that f is a bijective map.

Exercise 4.2 Let P be the set of all positive real numbers. Prove that $(0, 1)$ and P have the same cardinality by using the function $f\colon (0, 1) \to P$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 1/2 \\ \frac{1}{4(1-x)} & \text{if } 1/2 < x < 1. \end{cases}$$

Exercise 4.3 Prove that any infinite set contains a countably infinite subset. *Hint:* let X be an infinite set. Pick any $x_1 \in X$. Since X is infinite, there exists $x_2 \in$

$X \setminus \{x_1\}$. For the same reason, there exists $x_3 \in X \setminus \{x_1, x_2\}$, and so on. In this way we construct a subset $\{x_j \mid j \in \mathbb{N}\}$ of X which is clearly countably infinite.

Let us call $\mathfrak{c}$ the cardinality of $\mathbb{R}$ and $\aleph_0$ for the cardinality of $\mathbb{N}$. From our discussion it is clear that

$$\aleph_0 < \mathfrak{c},$$

in the sense that there exists an injective function from $\mathbb{N}$ into $\mathbb{R}$, but there cannot exist a bijection between these two sets.

Important: Question

Is there any set whose cardinality is strictly larger than $\aleph_0$ and strictly smaller than $\mathfrak{c}$?

The answer is more than difficult: it is actually impossible! To be more precise, let us state the following

Continuum Hypothesis There exists no set whose cardinality κ satisfies $\aleph_0 < \kappa < \mathfrak{c}$.

Although David Hilbert proposed a proof that the continuum hypothesis was actually true, it soon turned out that his proof was incorrect. Some years later, Gödel showed that the continuum hypothesis cannot be disproved in the framework of any consistent theory of sets. The debate was closed in 1963 by Paul Cohen, who showed that the continuum hypothesis cannot be proved in the framework of any consistent theory of sets, either. Roughly speaking, and since we always assume to have a consistent Set Theory at our disposal, the continuum hypothesis remains independent: it is a matter of taste whether we want to include it among our axioms. Luckily enough, it is rather hard to single out a milestone of Mathematical Analysis which depends on the continuum hypothesis. For this reason, we will not pursue further this topic in the book.

4.2 The Schröder-Bernstein Theorem

We have decided that two sets have the same cardinality if a bijective map exists which takes one set onto the other. A celebrated result by Schröder and Bernstein simplifies our task.

Theorem 4.7 (Schröder-Bernstein) *If there is a one-to-one function on a set A to a subset of a set B and there is also a one-to-one function on B to a subset of A, then A and B have the same cardinality.*

Proof Suppose that $f: A \to B$ and $g: B \to A$ are two injective maps. We may assume without loss of generality that $A \cap B = \emptyset$. We say that a point x of either A or B is an ancestor of a point y if and only if y can be obtained from x by successive application of f and g, or of g and f. Now we split A into three subsets: A_E consisting of all points of A which have an even number of ancestors, A_O consisting of all points of A which have an odd number of ancestors, and A_I consisting of all points of A which have infinitely many ancestors. The set B can be split in the same way. We finally define $F: A \to B$ as follows:

$$F = \begin{cases} f & \text{on } A_E \cup A_I \\ g^{-1} & \text{on } A_O \end{cases}$$

is a bijective map. □

Remark 4.2 How do we interpret the previous proof? We have actually constructed the map F by an inductive process:

$$\begin{aligned} E_0 &= A \setminus g(B) \\ E_1 &= g(f(E_0)) \\ E_2 &= g(f(E_1)) \\ &\dots \\ E_{n+1} &= g(f(E_n)), \end{aligned}$$

and so on. Then we set $E = \bigcup_n E_n$. The function F is constructed in such a way that $F = f$ on A, and $F = g^{-1}$ on $A \setminus E$.

We present a second proof of this important result in Set Theory. We need a preliminary tool.

Lemma 4.1 *Let $\mathfrak{X}$ be an ordered set such that every non-empty subset has a greatest lower bound. If $\mathfrak{f}: \mathfrak{X} \to \mathfrak{X}$ is such that*

1. *there exists $x \in \mathfrak{X}$ such that $\mathfrak{f}(x) \leq x$;*
2. *for every $x \in \mathfrak{X}$, $y \in \mathfrak{X}$, $x \leq y$ implies $\mathfrak{f}(x) \leq \mathfrak{f}(y)$,*

then $\mathfrak{f}$ has a fixed point, i.e. there exists $a \in \mathfrak{X}$ such that $\mathfrak{f}(a) = a$.

Proof The set

$$A = \{x \in \mathfrak{X} \mid \mathfrak{f}(x) \leq x\}$$

is non-empty, hence there exists a greatest lower bound $a \in \mathcal{X}$ for A. If $x \in A$, then $a \leq x$, hence assumption 2 implies $\mathfrak{f}(a) \leq \mathfrak{f}(x) \leq x$. Thus $\mathfrak{f}(a) \leq a$, since $a = \inf A$. Using again 2, we see that $\mathfrak{f}(\mathfrak{f}(a)) \leq \mathfrak{f}(a)$, hence $\mathfrak{f}(a) \in A$ and so $a \leq \mathfrak{f}(a)$. The proof is complete. □

***Proof** (of Theorem 4.7)* Let $f: X \to Y$ and $g: Y \to X$ be injective functions. We claim that there exists a subset A of X such that $g(Y \setminus f(A)) = X \setminus A$. Once this claim is proved, the construction of a bijective application of X onto Y is easy.

Let us define $F: 2^X \to 2^X$ such that

$$A \mapsto X \setminus g(Y \setminus f(A)).$$

Lemma 4.1 can be applied with $\mathcal{X} = 2^X$, ordered by inclusion $\subset$, and $\mathfrak{f} = F$, since F satisfies condition 2. Condition 1 is also satisfied, since 2^X contains a largest element. Thus $F(A) = A$ for some $A \subset X$, and the proof follows. □

A remarkable fact is that given a set A, one can always construct another set whose cardinality is different than the cardinality of A. We call $\mathcal{P}(A)$ the set of all subsets of A.

Theorem 4.8 (Cantor) *If $A \neq \emptyset$, then there exists no surjective map $f: A \to \mathcal{P}(A)$. In particular, A and $\mathcal{P}(A)$ do not have the same cardinality.*

Proof Let $f: A \to \mathcal{P}(A)$; we will prove that the set

$$S = \{x \in A \mid x \notin f(x)\}$$

does not belong to the image of f. Suppose that $S \in f(A)$, so that $S = f(s)$ for some member $s \in A$. If $s \in S$, then $s \notin f(s) = S$; if $s \notin S$, then $s \in f(s) = S$. In any case we reach a contradiction. □

Exercise 4.4 Suppose that $A = \{x\}$. What is the cardinality of $\mathcal{P}(A)$? Think carefully!

4.3 Problems

4.1 A complex number z is an algebraic number if there exist integers $a_0, \dots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* given $N \in \mathbb{N}$, there exist only finitely many equations with $n + |a_0| + \cdots + |a_n| = N$.

4.2 Is the set $\mathbb{R} \setminus \mathbb{Q}$ countable?

4.3 Prove that a set E is infinite if and only if E has the same cardinality of a proper subset of E. *Hint:* one direction is Exercise 4.3. Conversely, if $f: E \to E$ is an injective function and $a \in E \setminus f(E)$, define recursively $a_1 = f(a)$, $a_{n+1} = f(a_n)$.

4.4 Comments

The rigorous definition of sequences is more problematic than we might suspect. Most textbooks propose to call sequence in a set X any function from $\mathbb{N}$ to X. But a problem immediately arises: with this definition the function $n \mapsto \sqrt{n^2 - 9}$ should not be termed sequence. Our definition clearly absorbs the previous one.

A more refined definition appears in [1]: a sequence in a set X is any function defined on an infinite subset of $\mathbb{N}$, taking values in X. It is easy to check that infinite subsets of $\mathbb{N}$ are characterized as follows.

Theorem 4.9 *Let N be a subset of $\mathbb{N}$. The following statements are equivalent:*

(a) N is an infinite set;
(b) for every $n \in \mathbb{N}$ there exists $p \in N$ such that $p \geq n$.

We will see later that (b) is actually the characterizing property of *nets*, a generalization of sequences.

Comparing sets by counting their elements obviously leads to a rather rough classification. However, this is the first appearance of the concept of *infinity*, which students consider from a philosophical viewpoint. We have proposed a standard approach to elementary cardinality of sets, and in particular we have avoided any explicit reference to the complicated issue of *choosing* elements from non-empty sets. This immediately leads to the Axiom of Choice and to the exhausting discussions about the necessity of using it.

Luckily, I have never found a student who needed an axiom to label the elements of a countable collection of countable sets, although such an operation requires some flavor of the Axiom of Choice. To clarify this point, we should always compare the sentences

1. A is a countable set;
2. let $\{s_1, s_2, s_3, \ldots\}$ be the elements of the countable set A.

The first statement is intrinsic, and we understand that an enumeration of the elements of A *exists*. The second statement already contains the *choice* of an enumeration of A, since the same countable set can be enumerated in infinitely many different ways. To summarize, the Axiom of Choice is not needed to define countable sets, but it comes into play as soon as we want to write down an enumeration of a countable set.

The Schröder-Bernstein Theorem is a useful result which can be proved in several ways. The first proof appears in [2] (but the author attributes it to G. Birkhoff and S. Mac Lane), while the second in based on the *fixed point* Lemma 4.1. I believe that both proofs are elegant and readable at an early stage.

References

1. S. Dolecki, F. Mynard, *Convergence Foundations of Topology* (World Scientific, 2016)
2. J.L. Kelley, *General Topology*. Graduate Texts in Mathematics, No. 27 (Springer, New York, 1975). Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.]