Chapter 2
Sets, Relations, Functions in a Naïve Way

Abstract We start our journey with naïve set theory. In the second half of the book we will provide a rigorous foundation of these ideas.

We begin this book in the worst possible manner: we introduce a meaningless definition.

Definition 2.1 (Sets) A *set* is a collection of *elements*.

Important: Sets Remain Undefined

It should be clear in the reader's mind that the previous sentence is far from being a mathematical definition. A set is defined through the word "collection", but we do not provide any primitive definition of collections. In other words, we are assuming that the concept of set is already present in our minds. More formally, we can say that our set theory is based on two primitive objects: sets and elements.

We write $x \in X$ to mean that x is an element of the set X, and we say that x is an element of X, or that x belongs to X. We will avoid the reversed symbol $X \ni x$, since $\ni$ is sometimes used in mathematics with a different meaning.

The typical way of constructing a set is as follows:

$$X = \{x \mid \text{some proposition about } x\}.$$

The variable x is a dummy variable, in the sense that it can be replaced by any other symbol without affecting the validity of the definition of the set X.

Example 2.1 To clarify the use of dummy variables, consider

$$\{x \mid x \text{ is a cat}\} = \{C \mid C \text{ is a cat}\}.$$

S. Secchi, *A Circle-Line Study of Mathematical Analysis*,
La Matematica per il 3+2 141, https://doi.org/10.1007/978-3-031-19738-3_2

On both sides we are introducing the set of all cats, no matter how we name the generic cat.

By definition, $X = \{x \mid x \in X\}$. Two sets X and Y are equal when they share the same elements: $x \in X$ if and only if $x \in Y$.

Definition 2.2 (Empty Set) The *empty set* is

$$\emptyset = \{x \mid x \neq x\},$$

Exercise 2.1 Prove that and $\emptyset$ contains no element at all. *Hint:* for every x, the statement $x \neq x$ is false.

It should be remarked that the definition of the empty set is meaningful, in the sense that it does not rely on some intuitive knowledge. The empty set could be equally defined by means of any statement which is false, for instance

$$\begin{aligned}\emptyset &= \left\{x \in \mathbb{R} \;\middle|\; x^2 = -1\right\} \\ &= \{n \in \mathbb{N} \mid n \text{ is neither odd nor even}\} \\ &= \{f \mid f \text{ is a function which is both bounded and unbounded}\}\end{aligned}$$

Example 2.2 Why don't we define the *opposite* of the empty set, namely

$$\mathcal{U} = \{x \mid x = x\}?$$

This object would contain anything, since anything is equal to itself by definition of equality. It would be desirable to have such a "set". wouldn't it? Unfortunately $\mathcal{U}$ cannot be a set, as Russel showed in his celebrated *paradox*. Let us consider $R = \{x \mid x \notin x\}$, the set of all sets which do not belong to themselves. What can we say about the relation $R \in R$?

Well, if $R \in R$, then R is a set which does not belong to itself, so that $R \notin R$. Viceversa, if $R \notin R$, then R is not a set which does not belong to itself, hence $R \in R$. Formally, $R \in R$ if and only if $R \notin R$. The consequence of this logical equivalence is that sets cannot be described unrestrictedly, and the universe $\mathcal{U}$ cannot be a set in the naïve sense. We will see in the second part of this book that Axiomatic Set Theory can be used to speak of sets without facing Russel's paradox. But most mathematicians think of sets naïvely, and so will we do for the moment. The only recommendation is to avoid any use of the universe.

Definition 2.3 (Subsets) If A and B are sets, then A is a subset of B if and only if each element of A is an element of B: in symbols,

$$\forall x (x \in A \Rightarrow x \in B).$$

In this situation we write $A \subset B$ or $B \supset A$. A set A is a *proper* subset of B if $A \subset B$ and $A \neq B$. We remark that $A = B$ if and only if $(A \subset B) \wedge (B \subset A)$.

Important: Proper Inclusion

It must be observed that $A \subset B$ is compatible with $A = B$. Since many mathematicians do not like this occurrence, the notation $A \subseteq B$ is often found in the literature, so that $A \subset B$ means $A \subseteq B$ *and* $A \neq B$. In this book we will never understand $\subset$ in this restrictive sense.

Definition 2.4 (Union and Intersection) The *union* of two sets A and B is the set $A \cup B$ of all points that are element of either A or B (or both):

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$

The *intersection* of two sets A and B is the set $A \cap B$ of all points that are elements of both A and B:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$

Two sets A and B are *disjoint* if $A \cap B = \emptyset$.

Definition 2.5 (Complement) The *absolute complement* of a set A is the set $\complement A = \{x \mid x \notin A\}$. We remark that $\complement\complement A = A$. The *relative complement* of a set A with respect to a set X is $X \setminus A = X \cap \complement A$.

Figures 2.1, 2.2, and 2.3 describe visually the basic operations on sets.

Definition 2.6 (Singleton) The set that contains only the element x is denoted by $\{x\}$ and called *singleton* x.

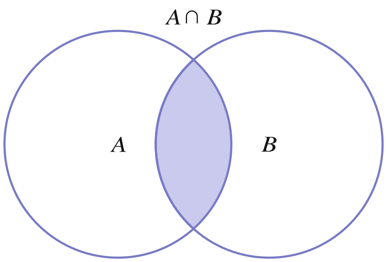

Fig. 2.1 Intersection of two sets

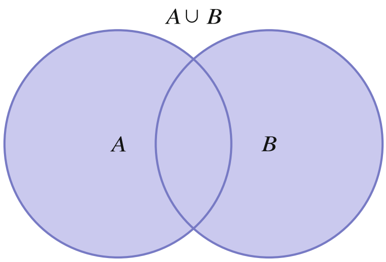

Fig. 2.2 Union of two sets

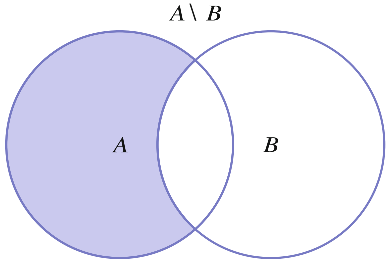

Fig. 2.3 Difference of two sets

Let us suppose that for each element α of a set A, which is called the *index set*, we are given a set X_α. We can extend our definition of union and intersection as follows:

$$\bigcup\{X_\alpha \mid \alpha \in A\} = \bigcup_{\alpha \in A} X_\alpha = \{x \mid \exists\alpha(\alpha \in A \wedge x \in X_\alpha)\} \tag{2.1}$$

$$\bigcap\{X_\alpha \mid \alpha \in A\} = \bigcap_{\alpha \in A} X_\alpha = \{x \mid \forall\alpha(\alpha \in A \wedge x \in X_\alpha)\}. \tag{2.2}$$

A particular case arises when the index set is a collection $\mathcal{A}$ of sets, and in this case we can write

$$\bigcup\{A \mid A \in \mathcal{A}\} = \{x \mid x \in A \text{ for some } A \in \mathcal{A}\}$$

and similarly

$$\bigcap\{A \mid A \in \mathcal{A}\} = \{x \mid x \in A \text{ for each } A \in \mathcal{A}\}.$$

Exercise 2.2 For each positive real numbers α and β, let $Q_{\alpha,\beta}$ be the rectangle $[0, \alpha] \times [0, \beta]$ in the plane. Describe the sets

$$\bigcap\{Q_{\alpha,\beta} \mid \alpha > 0,\ \beta > 0\}, \quad \bigcup\{Q_{\alpha,\beta} \mid \alpha > 0,\ \beta > 0\}.$$

Theorem 2.1 *Let A be an index set, and for each $\alpha \in A$ let X_α be a subset of a fixed set Y. Then*

(a) If B is a subset of A, then

$$\bigcup\{X_\beta \mid \beta \in B\} \subset \bigcup\{X_\alpha \mid \alpha \in A\},$$

and

$$\bigcap\{X_\beta \mid \beta \in B\} \supset \bigcap\{X_\alpha \mid \alpha \in A\}.$$

(b) $Y \setminus \bigcup\{X_\alpha \mid \alpha \in A\} = \bigcap\{Y \setminus X_\alpha \mid \alpha \in A\}$, *and* $Y \setminus \bigcap\{X_\alpha \mid \alpha \in A\} = \bigcup\{Y \setminus X_\alpha \mid \alpha \in A\}$.

Proof

(a) If $x \in \bigcup\{X_\beta \mid \beta \in B\}$ then there exists $\beta \in B$ such that $x \in X_\beta$. By assumption $\beta \in A$, and thus $x \in \bigcup\{X_\alpha \mid \alpha \in A\}$. If $x \in \bigcap\{X_\alpha \mid \alpha \in A\}$ then $x \in X_\alpha$ for each $\alpha \in A$, so that in particular $x \in X_\beta$ for each $\beta \in B$. Thus $x \in \bigcap\{X_\beta \mid \beta \in B\}$.

(b) If $x \in Y \setminus \bigcup\{X_\alpha \mid \alpha \in A\}$ then $x \in Y$ and x is not an element of any X_α, $\alpha \in A$. Hence x belongs to Y and for each $\alpha \in A$ there holds $x \notin X_\alpha$. This means that $x \in \bigcap\{Y \setminus X_\alpha \mid \alpha \in A\}$. Reversing this argument we prove the first identity. Now, if $x \in Y \setminus \bigcap\{X_\alpha \mid \alpha \in A\}$ then $x \in Y$ and there exists $\alpha \in A$ such that $x \notin X_\alpha$. Hence $x \in \bigcup\{Y \setminus X_\alpha \mid \alpha \in A\}$. Reversing this argument we prove the second identity.

□

An *ordered pair* is a new object (x, y) characterized by the following property: two ordered pairs (x, y) and (u, v) are equal if and only if $x = u$ and $y = v$. Actually an ordered pair may be defined in terms of sets as follows.

Definition 2.7 (Ordered Pair)

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Exercise 2.3 Prove that indeed $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$. *Hint:* by assumption $\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}$. Consider first the case $x = y$, then deal with the general case.

Definition 2.8 (Relations) A *relation* is a set of ordered pairs: a relation is therefore a set whose elements are ordered pairs.

If R is a relation, we usually write xRy instead of the more formal $(x, y) \in R$, and we say that x is related to y via R.

Definition 2.9 The *domain* of a relation R is the set $\{x \mid \exists y((x, y) \in R)\}$. The *range* of a relation R is the set $\{x \mid \exists x((x, y) \in R)\}$. The *field* of a relation R is the union of the domain and of the range of R.

One of the simplest relations is the set of ordered pairs (x, y) such that x is a member of a fixed set A, and y is a member of a fixed set B. This relation reduces

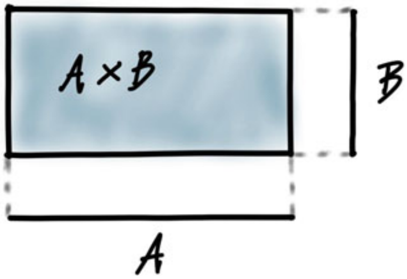

Fig. 2.4 A cartesian product

therefore to

$$A \times B = \{(x, y) \mid (x \in A) \wedge (y \in B)\},$$

and is called the *cartesian product* of A and B: see Fig. 2.4. It is clear that any relation is a subset of the cartesian product of its domain and its range.

Remark 2.1 The identification of sets and relations usually sounds strange to students. In this book we will never think of relations or functions like *black boxes* which transform elements of some set into elements of some other set.

The *inverse* of a relation R, denoted by R^{-1}, is the relation obtained by swapping each of the ordered pairs belonging to R. Formally,

$$R^{-1} = \{(y, x) \mid (x, y) \in R\},$$

or equivalently $y R^{-1} x$ if and only if $x R y$.

The *composition* of two relations R and S is

$$R \circ S = \{(x, z) \mid \exists y ((x, y) \in S \wedge (y, z) \in R)\}.$$

We remark that, roughly speaking, first comes S, then comes R, and not viceversa. The domain of $R \circ S$ is the domain of S, while the range of $R \circ S$ is the range of R. This will be of crucial importance when we introduce functions.

Definition 2.10 Suppose that R is a relation and X is the set of all points that are elements of either the domain or the range of R. We say that R is

- reflexive, if each element of X is in relation R with itself;
- symmetric, if $x R y$ whenever $y R x$;
- antisymmetric, if $x R y$ and $y R x$ imply $x = y$;
- transitive, if $x R y$ and $y R z$ imply $x R z$.

Definition 2.11 An equivalence relation is a reflexive, symmetric and transitive relation. An order relation is a reflexive, antisymmetric and transitive relation.

It is customary to use the symbol $\sim$ for equivalence relations, and $\leq$ for order relations.

A *function* is a relation such that no two distinct members have the same first coordinate. More explicitly, a relation f is a function if for each element x of its domain there exists a unique element y of its range such that $(x, y) \in f$, see Fig. 2.5. Uniqueness means that if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. For a function it is customary to abandon the general notation $(x, y) \in f$ (or xfy) in favor of $y = f(x)$. Then $f(x)$ is the *image* of the element x of the domain of f. In mathematical analysis a function $f \subset X \times Y$ is denoted by the (more complicated) symbol

$$f : X \to Y, \quad x \mapsto f(x).$$

A function $f : X \to Y$ is *injective* if distinct points of X have distinct images in Y. Equivalently, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. A function $f : X \to Y$ is *surjective* if the range of f coincides with Y. Equivalently, for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Finally, a function $f : X \to Y$ is *bijective* if it is both injective and surjective.

Exercise 2.4 Let X and Y be sets. Prove that the map $f : X \times Y \to Y \times X$ defined by $f(x, y) = (y, x)$ for each $(x, y) \in X \times Y$ is a bijection. In this sense, $X \times Y$ and $Y \times X$ are *essentially* the same object.

If A is a set and f is a function, the set

$$f(A) = \{y \mid \exists x(x \in A \wedge f(x) = y)\} = \{f(x) \mid x \in A\}$$

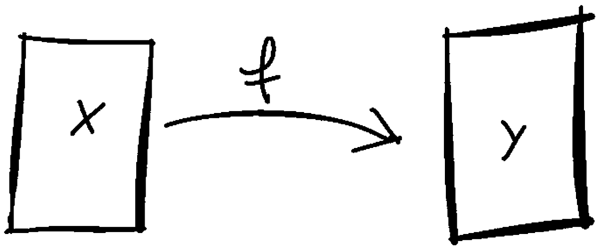

Fig. 2.5 Intuition of a function

is called the *image* of the set A under f. Similarly, if B is a set and f is a function,

$$f^{-1}(B) = \{x \mid \exists y(x \in B \wedge f(x) = y)\}$$

is called the *pre-image* of B under f. We notice that $f^{-1}(B)$ is just the image of the set B under the inverse relation f^{-1}. Clearly $f(A)$ is a subset of the range of f, while $f^{-1}(B)$ is a subset of the domain of f.

Theorem 2.2 *If f is a function and A and B are sets, then*

(a) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$*;*
(b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$*;*
(c) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$*.*

More generally, if we are given a set X_α for each member α of a non-empty index set C, then

(d) $f^{-1}(\bigcup\{X_\alpha \mid \alpha \in C\}) = \bigcup\{f^{-1}(X_\alpha) \mid \alpha \in C\}$*;*
(e) $f^{-1}(\bigcap\{X_\alpha \mid \alpha \in C\}) = \bigcap\{f^{-1}(X_\alpha) \mid \alpha \in C\}$*.*

Proof We prove part (e), leaving the rest of the proof as a simple exercise. A point x is an element of $f^{-1}(\bigcap\{X_\alpha \mid \alpha \in C\})$ if and only if $f(x)$ is an element of this intersection, in which case $f(x) \in X_\alpha$ for each $\alpha \in C$. But the latter condition is equivalent to $x \in f^{-1}(X_\alpha)$ for each $\alpha \in C$, i.e. $x \in \bigcup\{f^{-1}(X_\alpha) \mid \alpha \in C\}$. □

Remark 2.2 Any function f is invertible *as a relation*. However the inverse relation f^{-1} need not be again a function: this happens if and only if for each y there exists a unique x such that $yf^{-1}x$, i.e. $f(x) = y$. We have proved that the relation f^{-1} is a function if and only if f is a bijective function. It is customary to say that a function $f\colon X \to Y$ is *invertible* if it is bijective.

Remark 2.3 Any injective function $f\colon X \to Y$ can be somehow inverted, in the sense that we can define a function $g\colon f(X) \to X$ such that $g(y) = x$ if and only if $f(x) = y$. In general the domain of g is a proper subset of Y, but the rule which defines g is exactly the same rule which defines f^{-1}. Many mathematicians do not require surjectivity in order to define invertible functions. This is fairly reasonable, since $f(X)$ is the largest subset of Y on which we can define the inverse function of the injective function f.

Exercise 2.5 Let $f\colon X \to U$ and $g\colon Y \to V$ denote two functions. Prove that $(x, y) \mapsto (f(x), g(y))$ defines a function $f \times g\colon X \times Y \to U \times V$, which we call the Cartesian product of f and g. Prove the following statements:

(i) if f and g are injective, then so is $f \times g$;
(ii) if f and g are surjective, then so is $f \times g$.

Important: Sets or Subsets?

Most surveys of naïve set theory for mathematical analysis only deal with subsets of a given *universe*. We followed another route, and this may have been surprising. The use of a given universe is motivated by Russel's paradox, but for the moment this remains irrelevant to us. As we will see, denying the set of all sets is not the only escape from Russel's paradox.

2.1 Comments

We have presented a quick survey of Set Theory from a non-axiomatic viewpoint. Most textbook in Mathematical Analysis contain similar information, with only minor differences in the language. As an example, functions are typically defined as *rules* of assignment instead of special relations between two sets. A standard reference is [1], a book which goes however much beyond the level suggested by the title.

Before proceeding further, we should stop and think about notation. It is a matter of facts that most instructors discourage the abstract use of

$$\{x \mid P(x)\} \tag{2.3}$$

for the definition of a set. In this book we may seem to be lazy, since such a notation is allowed and even typical. Let us try to elaborate on this issue.

From a very abstract viewpoint, (2.3) contains the troublesome formula

$$\{x \mid x = x\},$$

which leads to the paradox of the *universe*. On the contrary, the more precise formula

$$\{x \mid x \in \mathcal{U} \wedge P(x)\},$$

often written as $\{x \in \mathcal{U} \mid P(x)\}$, is admissible, since it defines a subset of a (given) set $\mathcal{U}$. Nowadays, most introductory discussions about (naïve) Set Theory are based on axiomatic theories which discard arbitrarily large sets, like ZF (Zermelo-Fraenkel), and this accounts for the recommendation against the use of $\{x \mid P(x)\}$.

On the contrary, we will discuss a different Axiomatic Theory of Sets which allows large objects (called *classes*). In some sense, we should say that $\{x \mid x = x\}$ exists as a class, but not as a set. Since the algebra of classes is quite similar to the algebra of sets, at a first stage we forget the distinction and we allow a more relaxed notation.

Reference

1. P.R. Halmos, *Naive Set Theory* (Dover Publications, 2017)