Chapter 11
Elementary Functions

Abstract Most of us make use of *elementary* functions in a formal way. We graph exponentials, logarithms, sines, cosines, we differentiate and integrate them. But Calculus does not teach us an acceptable definition of these functions. We accept their existence, and we keep using their properties. In this chapter we offer a more advanced description of the most important functions, and show that their definition is indeed far from being *elementary*.

11.1 Sequences and Series of Functions

A particular type of sequence is of fundamental importance for defining the elementary functions in a rigorous way: sequences (and series) of *functions*. Although these are nothing else than sequences in a *set* of functions, we follow here a classical approach which does not lean on General Topology.

Suppose that E is a set and that for every positive integer n we have a function f_n defined on E. We can say that $\{f_n\}_n$ is a sequence of functions on E.[1] When we speak of sequences, sooner or later we speak of limits.

Definition 11.1 Let $\{f_n\}_n$ be a sequence of functions on a set[2] E into $\mathbb{R}$. We say that this sequence converges pointwise to a function f, if the numerical sequence $\{f_n(x)\}_n$ converges for every $x \in E$. The function $f : E \to \mathbb{R}$ defined by

$$f(x) = \lim_{n \to +\infty} f_n(x)$$

is the (pointwise) limit of the sequence $\{f_n\}_n$.

[1] It should be remarked that E does not depend on the index n. From a theoretical view point we could consider sequences of functions $f_n : E_n \to \mathbb{R}$ in which every term is defined on a set E_n. For our purposes such a generality can be troublesome.

[2] The nature of E is not particularly relevant. In many concrete cases, E is a subset of $\mathbb{R}$ of $\mathbb{C}$.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022
S. Secchi, *A Circle-Line Study of Mathematical Analysis*,
La Matematica per il 3+2 141, https://doi.org/10.1007/978-3-031-19738-3_11

In a similar way—and indeed in an equivalent way—we can define *series* of functions. Indeed we can consider the numerical series $\sum_n f_n(x)$ for every $x \in E$: if this series converges, the (pointwise) limit of the series $\sum_n f_n$ is $f(x) = \sum_n f_n(x)$.

Example 11.1 For $m = 1, 2, 3, \dots$ we define

$$f_m(x) = \lim_{n \to +\infty} (\cos(m!\pi x))^{2n}.$$

When $m!x \in \mathbb{Z}$, we have $f_m(x) = 1$. Otherwise we have $f_m(x) = 0$. Let $f(x) = \lim_{m \to +\infty} f_m(x)$.

If x is irrational, then $f_m(x) = 0$ for every m, so $f(x) = 0$. For rational values of $x = p/q$, we see that $m!x$ is an integer for every $m \geq q$, and therefore $f(x) = 1$. To summarize,

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

This shows that the pointwise limit of very smooth functions may well be a very irregular function.

Exercise 11.1 Consider

$$f_n(x) = \frac{x^2}{(1 + x^2)^n}$$

for $x \in \mathbb{R}$ and $n = 0, 1, 2, 3, \dots$ Let $f(x) = \sum_{n=0}^{\infty} f_n(x)$. Show that f is defined for every real x, and that

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 + x^2 & \text{if } x \neq 0. \end{cases}$$

Deduce that the pointwise limit of a sequence of continuous functions is in general a discontinuous function.

Example 11.2 Let $f_n \colon [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = n^2 x(1 - x^2)^n$. Since $f_n(0) = 0$, we have that

$$f(0) = \lim_{n \to +\infty} f_n(0) = 0.$$

For $0 < x \leq 1$, we trivially have $\lim_{n \to +\infty} n^2 x(1 - x^2)^n = 0$. By the Fundamental Theorem of Calculus,

$$\int_0^1 x(1 - x^2)^n \, dx = \frac{1}{2n + 2},$$

so that

$$\lim_{n\to+\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n^2}{2n+2} = +\infty.$$

In particular $0 = \int_0^1 f(x)\,dx \neq \lim_{n\to+\infty}\int_0^1 f_n(x)\,dx$.

Exercise 11.2 Consider instead $f_n(x) = nx(1-x^2)^n$ for $x \in [0, 1]$. Show that $\lim_{n\to+\infty}\int_0^1 f_n(x)\,dx = 1/2$, while $f(x) = \lim_{n\to+\infty} f_n(x) = 0$ for every $x \in [0, 1]$.

To summarize: the pointwise limit of a sequence of function does not preserve continuity, differentiability, integrability. The natural question is whether we can replace our pointwise convergence with another convergence which preserves these properties.

11.2 Uniform Convergence

Definition 11.2 A sequence $\{f_n\}_n$ of functions defined on a set E converges uniformly to a limit f on E, if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $x \in E$ and $n \geq N$ imply

$$|f_n(x) - f(x)| < \varepsilon.$$

When dealing with series of functions, we say that $\sum_n f_n$ converges uniformly on E if the sequence $\{s_n\}_n$ of partial sums defined by $s_n(x) = \sum_{j=1}^n f_j(x)$ converges uniformly (to some limit).

Remark 11.1 It is easy to check that $\{f_n\}_n$ converges uniformly to f on E if and only if

$$\lim_{n\to+\infty} \sup\{|f_n(x) - f(x)| \mid x \in E\} = 0.$$

The quantity $\sup\{|f_n(x) - f(x)| \mid x \in E\}$ is often denoted by $\|f_n - f\|_{\infty,E}$.

Theorem 11.1 (Cauchy Criterion for Uniform Convergence) *A sequence $\{f_n\}_n$ of functions defined on a set E converges uniformly if and only if the Cauchy condition holds: for every $\varepsilon > 0$ there exists a positive integer N such that $m \geq N$, $n \geq N$, $x \in E$ imply*

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Proof Suppose that $\{f_n\}_n$ converges uniformly on E to a limit f, and let $\varepsilon > 0$ be fixed. By Definition 11.2 there exists a positive integer N such that $n \geq N$ and $x \in E$ imply $|f_n(x) - f(x)| < \varepsilon/2$. Thus

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon$$

for every $n \geq N$, $m \geq N$, $x \in E$.

On the contrary, suppose that the Cauchy condition holds. For every $x \in E$, the numerical sequence $\{f_n(x)\}_n$ is then a Cauchy sequence in $\mathbb{R}$, so it converges to some limit that we call $f(x)$. We need to prove that this convergence is uniform on E. Let $\varepsilon > 0$ be given, and choose a positive integer N such that $|f_n(x) - f_m(x)| < \varepsilon$ for every $m \geq N$, $n \geq N$, $x \in E$. Letting $m \to +\infty$, since $f_m(x) \to f(x)$, we deduce that

$$|f_n(x) - f(x)| \leq \varepsilon$$

for every $n \geq N$ and $x \in E$. Hence $\{f_n\}_n$ converges uniformly on E to the function f. □

The Cauchy condition immediately implies a useful test for the uniform convergence of series of functions.

Theorem 11.2 (Weierstrass' M-Test for Series) *Suppose that $\{f_n\}_n$ is a sequence of functions on a set E, and suppose that $|f_n(x)| \leq M_n$ for every $x \in E$ and every $n \in \mathbb{N}$. If $\sum_n M_n$ converges, then $\sum_n f_n$ converges uniformly on E.*

Proof For any $\varepsilon > 0$, let N be an integer such that $n \geq N$, $m \geq N$ and $m \geq n$ imply $\sum_{j=n}^{m} M_j < \varepsilon$. For every $x \in E$ we see that

$$\left| \sum_{j=n}^{m} f_j(x) \right| \leq \sum_{j=n}^{m} M_j \leq \varepsilon,$$

and the conclusion follows from Theorem 11.1. □

We will now present a few statements which relate uniform convergence with limits, derivatives and integrals.

Theorem 11.3 (Uniform Convergence and Continuity) *Suppose that $f_n \to f$ uniformly on a set E in a metric space. Let x be an accumulation point of E, and suppose that*

$$\lim_{t \to x} f_n(t) = A_n$$

for $n = 1, 2, 3, \ldots$ *Then* $\{A_n\}_n$ *converges, and*

$$\lim_{t \to x} f_n(t) = \lim_{n \to +\infty} A_n.$$

Proof Fix $\varepsilon > 0$. By uniform convergence there exists N such that $n \geq N$, $m \geq N$, $t \in E$ imply

$$|f_n(t) - f_m(t)| < \varepsilon.$$

As $t \to x$, $|A_n - A_m| \leq \varepsilon$. Hence $\{A_n\}_n$ is a Cauchy sequence in $\mathbb{R}$, and it converges to a limit A. Now

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

We choose $n_0 \geq N$ so that $|f(t) - f_{n_0}(t)| < \varepsilon/3$ for all $t \in E$, and $|A_{n_0} - A| < \varepsilon/3$. With this n_0 we choose a neighborhood V of x such that $t \in V \cap E$, $t \neq x$ imply $|f_{n_0}(t) - A_n| < \varepsilon/3$. It follows that $|f(t) - A| < \varepsilon$ for every $t \in V \cap E$, $t \neq x$. The proof is complete. □

Corollary 11.1 *If* $\{f_n\}_n$ *is a sequence of continuous functions on* E *that converges uniformly to a limit* f*, then* f *is a continuous function.*

Theorem 11.4 (Uniform Convergence and Differentiation) *Suppose that* $\{f_n\}_n$ *is a sequence of functions, differentiable on* $[a, b]$ *and such that there exists a point* $x_0 \in [a, b]$ *such that* $\{f_n(x_0)\}_n$ *converges. If* $\{f_n'\}_n$ *converges uniformly on* $[a, b]$*, then* $\{f_n\}_n$ *converges uniformly on* $[a, b]$ *to a limit* f*, and*

$$f'(x) = \lim_{n \to +\infty} f_n'(x)$$

for every $x \in [a, b]$.

In other words, uniform convergence of the derivatives a pointwise convergence of the functions at *some* point x_0 imply uniform convergence of the functions.

Proof We fix $\varepsilon > 0$. By assumption there exists a positive integer N such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$$

and

$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)} \tag{11.1}$$

for every $n \geq N, m \geq N$. Let us apply the mean value theorem to $f_n - f_m$: Eq. (11.1) yields

$$|f_n(x) - f_m(x) - (f_n(t) - f_m(t))| \leq \frac{|x-t|\varepsilon}{2(b-a)} \leq \frac{\varepsilon}{2} \tag{11.2}$$

for every $x \in [a,b]$, $t \in [a,b]$, $n \geq N$, $m \geq N$. Splitting

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

we deduce that $x \in [a,b]$, $n \geq N$, $m \geq N$ imply

$$|f_n(x) - f_m(x)| \leq \varepsilon.$$

This proves that $\{f_n\}_n$ converges uniformly to a limit that we call f. We fix a point $x \in [a,b]$ and introduce the incremental ratios

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}, \quad \phi(t) = \frac{f(t) - f(x)}{t-x}$$

for any $t \in [a,b] \setminus \{x_0\}$. Clearly $\lim_{t \to x} \phi_n(t) = f_n'(x)$. From (11.2) we see that $n \geq N$, $m \geq N$ imply

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b-a)}.$$

Hence $\{\phi_n\}_n$ converges uniformly on $[a,b] \setminus \{x\}$. But $\{f_n\}_n$ converges to f, hence $\lim_{n \to +\infty} \phi_n(t) = \phi(t)$ uniformly on $[a,b] \setminus \{x\}$. We conclude from Theorem 11.3 that $\lim_{t \to x} \phi(t) = \lim_{n \to +\infty} f_n'(x)$. The proof is complete. □

Theorem 11.5 (Uniform Convergence and Integration) *Suppose that each f_n is R-integrable on $[a,b]$, and suppose that $f_n \to f$ uniformly on $[a,b]$. Then f is R-integrable on $[a,b]$ and $\int_a^b f \, dx = \lim_{n \to +\infty} \int_a^b f_n \, dx$.*

Proof Let

$$\varepsilon_n = \sup \{|f_n(x) - f(x)| \mid x \in [a,b]\}.$$

Hence $f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n$, and this implies

$$\int_a^b (f_n - \varepsilon_n) \, dx \leq \underline{\int_a^b} f \, dx \leq \overline{\int_a^b} f \, dx \leq \int_a^b (f_n + \varepsilon_n) \, dx.$$

Hence

$$0 \le \overline{\int_a^b} f \, dx - \underline{\int_a^b} f \, dx \le 2\varepsilon_n (b-a).$$

Since $\varepsilon_n \to 0$ as $n \to +\infty$, we conclude that $\underline{\int_a^b} f \, dx = \overline{\int_a^b} f \, dx$, and f is R-integrable. As before,

$$\left| \int_a^b f \, dx - \int_a^b f_n \, dx \right| = \left| \int_a^b (f - f_n) \, dx \right| \le \varepsilon_n (b-a),$$

and it follows that $\int_a^b f_n \, dx \to \int_a^b f \, dx$ as $n \to +\infty$. □

Remark 11.2 The rigidity of the Riemann integral under passage to the limit is one of the reasons why it has been superseded by more flexible integrals. We will meet Lebesgue's generalization in Chap. 15.

11.3 The Exponential Function

It has been said that the exponential function is the most important function in Mathematical Analysis. We propose a definition which entails a lot of useful properties.

Definition 11.3 For each $z \in \mathbb{C}$, we define

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

By the Ratio Test, the series converges absolutely. The function

$$\exp \colon \mathbb{C} \to \mathbb{C}$$

is the exponential function.[3]

Proposition 11.1

1. *For every* $z \in \mathbb{C}$, $w \in \mathbb{C}$, *there results* $\exp(z+w) = \exp z \cdot \exp w$.
2. $\exp 0 = 1$ *and* $\exp 1 = \mathrm{e}$.
3. $\exp z \neq 0$ *for every* $z \in \mathbb{C}$.

[3] We refrain from writing e^z instead of $\exp z$. This is only a pedagogical choice, since we want to prevent the reader from believing that all properties of this function are trivial since we are deling with an ordinary power.

4. $\exp(-z) = 1/\exp z$ *for every* $z \in \mathbb{C}$.
5. exp *is a continuous function.*

Proof We form the Cauchy product according to Definition 6.3:

$$\begin{aligned}\exp z \cdot \exp w &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n = \exp(z+w).\end{aligned}$$

This proves 1. Part 2 is clear from the definitions. Using 1 and 2 we see that $\exp(-z) \exp z = 1$ for every $z \in \mathbb{C}$, and both 3 and 4 follow. To prove 5, we fix a point $z \in \mathbb{C}$ and $\varepsilon > 0$. Let

$$\delta = \min \left\{ 1, \frac{\varepsilon}{2|\exp z|} \right\}.$$

Hence $h \in \mathbb{C}$ and $|h| < \delta$ imply

$$\begin{aligned}|\exp(z+h) - \exp z| &= |\exp z| \cdot |\exp h - 1| \\ &\le \frac{\varepsilon}{2\delta} \left| \sum_{n=1}^{\infty} \frac{h^n}{n!} \right| \le \frac{\varepsilon}{2\delta} \sum_{n=1}^{\infty} \frac{|h|^n}{n!} \\ &< \frac{\varepsilon}{2} \sum_{n=1}^{\infty} \frac{1}{n!} = \frac{\varepsilon}{2}(\mathrm{e} - 1) < \varepsilon.\end{aligned}$$

The proof of 5 is complete. □

Theorem 11.6 *There results*

1. $\exp x > 0$ *for every* $x \in \mathbb{R}$.
2. exp *is strictly increasing on* $\mathbb{R}$.
3. $\lim_{x \to +\infty} \exp x = +\infty$.
4. $\lim_{x \to -\infty} \exp x = 0$.
5. $\exp(\mathbb{R}) = (0, +\infty)$.
6. $\lim_{x \to +\infty} x^{-n} \exp x = +\infty$ *for every integer* $n \ge 0$ *and every* $x \in \mathbb{R}$.

Proof We first notice that $\exp(\mathbb{R}) \subset \mathbb{R}$. Since $x > 0$ implies $\exp x > 1 + x$, 1 follows for $x \ge 0$. If $x < 0$, $\exp(-x) = 1/\exp x$, and 1 follows also in this case. To prove 2, we fix $x \in \mathbb{R}$ and $h > 0$. Then $\exp(x+h) = \exp x \cdot \exp h > \exp x$ since $\exp h > 1$. Similarly, $\lim_{x \to +\infty} \exp x \ge \lim_{x \to +\infty} (1 + x) = +\infty$, and 3 follows.

Recalling again that

$$\lim_{x\to-\infty} \exp x = \lim_{x\to+\infty} \exp(-x) = \lim_{x\to+\infty} \frac{1}{\exp x},$$

we see that 4 follows from 3. The proof of 6 goes as follows: $x > 0$ implies

$$x^{-n} \exp x > x^{-n} \frac{x^{n+1}}{(n+1)!} = \frac{x}{(n+1)!}$$

To conclude, 5 follows from 1, 3, 4 and the fact that exp is injective on $\mathbb{R}$. □

Theorem 11.7 *The restriction of* exp *to* $\mathbb{R}$ *is differentiable at every point, and there results* $\exp' = \exp$.

Proof Just differentiate the series of function that defines the exponential, and use Theorem 11.4. □

One an exponential function has been introduced, the logarithm comes into play as its inverse.

Definition 11.4 The (real) logarithm is defined as

$$\log = \left(\exp_{|\mathbb{R}}\right)^{-1}.$$

As a function, $\log\colon (0, +\infty) \to \mathbb{R}$.

Theorem 11.8 *There results*

1. $\log((0, +\infty)) = \mathbb{R}$.
2. log *is strictly increasing on* $(0, +\infty)$.
3. log *is continuous on* $(0, +\infty)$.
4. $\lim_{x\to+\infty} \log x = +\infty$.
5. $\lim_{x\to0+} \log x = -\infty$.
6. $\log 1 = 0$ *and* $\log \mathrm{e} = 1$.
7. $\log(ab) = \log a + \log b$ *for every* $a > 0$, $b > 0$.
8. $\log(a^n) = n \log a$ *for every* $a > 0$ *and every* $n \in \mathbb{Z}$.
9. $\log(a^{1/n}) = (1/n) \log a$ *for every* $a > 0$ *and* $n \in \mathbb{N}$.
10. $\lim_{x\to+\infty} \frac{\log x}{\sqrt[n]{x}} = 0$ *for every* $n \in \mathbb{N}$.

Proof Properties from 1 to 6 follow from the analogous properties of the exponential. For $a > 0$ and $b > 0$ we have

$$\exp(\log a + \log b) = \exp(\log a) \cdot \exp \log b = ab,$$

and this proves 7. Properties 8 and 9 are left as a simple exercise about induction. To prove 10 we fix $\varepsilon > 0$ and $n \in \mathbb{N}$. By Theorem 11.6 we can choose $\alpha > 1$ such that $y > \alpha$ implies $y^{-n} \exp y > \varepsilon^{-n}$. By property 4 there exists $\beta > 1$ such that

$x > \beta$ implies $\log x > \alpha$. Therefore $x > \beta$ implies

$$(\log x)^{-n}\, x > \varepsilon^{-n},$$

or

$$0 < \frac{\log x}{\sqrt[n]{x}} < \varepsilon.$$

□

Theorem 11.9 *The function* log *is differentiable at every point of* $(0, +\infty)$*, and there results* $(\log)'(y) = 1/y$ *for every* $y > 0$*.*

Proof Exercise on the derivative of the inverse function! □

Remark 11.3 Another approach is possible, which avoids the use of series. The first idea is to define a function $\log \colon (0, +\infty) \to \mathbb{R}$ by

$$\log x = \int_1^x \frac{dt}{t}.$$

Here we are using the convention $\int_a^b = -\int_b^a$. The main properties of the real logarithm follows at once, in particular the fact that log has a continuous inverse. We call the inverse the real exponential function.

11.4 Sine and Cosine

Definition 11.5 The functions sin and cos are defined on $\mathbb{C}$ by

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

for every $z \in \mathbb{C}$. The convention $0^0 = 1$ is used in these formulas.

Theorem 11.10 (Euler) *For every* $z \in \mathbb{C}$ *we have*

1. $\exp(\mathrm{i}z) = \cos z + \mathrm{i} \sin z$.
2. $\exp(-\mathrm{i}z) = \cos z - \mathrm{i} \sin z$.
3. $\sin z = \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}$.
4. $\cos z = \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2}$.

Proof Observe that $(-1)^n z^{2n} = (\mathrm{i}z)^{2n}$ and $\mathrm{i}(-1)^n z^{2n+1} = (\mathrm{i}z)^{2n+1}$. Then 1 follows from the definition of sin and cos. Property 2 follows by replacing z by $-z$ in 1. If we solve the system

$$\begin{cases} \exp(\mathrm{i}z) = \cos z + \mathrm{i} \sin z \\ \exp(-\mathrm{i}z) = \cos z - \mathrm{i} \sin z \end{cases}$$

with respect to $\sin z$ and $\cos z$, we find 3 and 4. □

The next result may also be taken as a *definition* of π.

Theorem 11.11 *There exists one and only one real number* π *such that*

(i) $\pi > 0$*;*
(ii) $\cos(\pi/2) = 0$*;*
(iii) $0 < x < \pi/2$ *implies* $\cos x > 0$.

Furthermore, $\pi < 4$.

Proof If π and π' are different numbers which satisfy (i)–(iii) and $\pi < \pi'$, then $\cos(\pi/2) > 0$. This contradiction shows the uniqueness of π.

To prove the existence, we reason as follows. For every $n > 1$ we have

$$\frac{2^{2n}}{(2n)!} > \frac{2^{2n+2}}{(2n+2)!},$$

hence

$$\begin{aligned} \cos 2 &= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \cdots \\ &= -1 + \frac{16}{24} - \cdots < -1 + \frac{2}{3} < 0. \end{aligned}$$

Since $x \mapsto \cos x$ is a continuous real-valued function on $[0, 2]$ and $\cos 0 = 1 > 0$, it follows that the set

$$A = \{x \in [0, 2] \mid \cos x = 0\}$$

is non-empty. We define $\pi/2 = \inf A$. As an accumulation point of the closed set A, we have $\pi/2 \in A$, and $\pi/2 > 0$ since $\cos 0 = 1$. As a consequence π satisfies (i) and (ii). Suppose (iii) was false. Then another application of the Intermediate Value Theorem would produce an element of A which would be smaller than $\pi/2$. Finally, since $\cos 2 < -1 + 2/3$, we find $\pi/2 < 2$, and the proof is complete. □

From Euler's Theorem we can recover all the properties of the trigonometric functions. The interested reader can expand the details and prove the main results. It is however remarkable that it would be impossible to postpone the use of these functions until they can be rigorously defined. Calculus is built around elementary functions, but it does not provide sufficient tools to define them without any reference to geometric or intuitive facts.

Remark 11.4 The approach to trigonometric functions without power series is slightly involved. A possible approach is to begin with $\arctan\colon \mathbb{R} \to (-\pi/2, \pi/2)$ in terms of a definite R-integral:

$$\arctan x = \int_0^x \frac{\mathrm{d}t}{1+t^2}.$$

This function is invertible, so that tan is defined on $(-\pi/2, \pi/2)$ as $\arctan^{-1}$. Then sin and cos are recovered as suggested by the identity $\sin^2 + \cos^2 = 1$, i.e. $\tan^2 + 1 = 1/\cos^2$.

We end this section with a discussion about periodic functions.

Definition 11.6 A function $f : \mathbb{R} \to \mathbb{R}$ is periodic if and only if there exists a real number $T \neq 0$ such that $f(x+T) = f(x)$ for every $x \in \mathbb{R}$.

Exercise 11.3 Let $P = \{T \in \mathbb{R} \setminus \{0\} \mid T \text{ is a period of } f\}$ be the set of all periods of a given function f. Prove that the sum and the difference of two elements of P are elements of P. We can summarize this by saying that P is an additive subgroup of $\mathbb{R}$.

We now investigate additive subgroups of $\mathbb{R}$.

Theorem 11.12 *If H is an additive subgroup of $\mathbb{R}$, then either $\overline{H} = \mathbb{R}$ or there exists $T^* \neq 0$ such that $H = \{mT^* \mid m \in \mathbb{Z}\}$.*[4]

Proof Since H is an additive subgroup, $H \cap [0, +\infty) \neq \emptyset$. We define

$$\eta = \inf(H \cap [0, +\infty)).$$

Two cases are possible. If $\eta > 0$, then we pick $h \in H$ and $m \in \mathbb{Z}$ such that

$$m\eta \leq |h| < (m+1)\eta.$$

Since $|h| - m\eta \in H$ and $0 \leq |h| - m\eta < (m+1)\eta - m\eta = \eta$, the definition of η implies $|h| - m\eta = 0$, or $h = \pm m\eta$. Hence H consists of all integer multiples of η.

If $\eta = 0$, we must prove that H is dense in $\mathbb{R}$. Let $r \in \mathbb{R}$ and let $\varepsilon > 0$. Since $\eta = 0$, there exists $h_\varepsilon \in H \cap [0, \varepsilon]$. We may suppose $r \geq 0$, the case $r < 0$ being similar. By the Archimedean property, there exists $k \in \mathbb{N}$ such that $kh \leq r < (k+1)h$. Since $hk \in H$ and $0 \leq r - kh < (k+1)h - kh = h \leq \varepsilon$, we see that $|r - kh| \leq \varepsilon$, which shows that H is dense in $\mathbb{R}$. The proof is complete. □

[4] Algebraists say that H is a cyclic group.

Theorem 11.13 *Let $f: \mathbb{R} \to \mathbb{R}$ be a periodic continuous function. If f is non-constant, then there exists a smallest positive period of f.*

Proof Let P be the set of all periods of f. We are going to rule out the possibility that P is dense in $\mathbb{R}$. Indeed, if this were true, then f would be constant on the dense subset P, and thus f would be globally constant by continuity. Since this contradicts the assumption, we conclude that all periods are integer multiples of some period $T^* \neq 0$. Since $-T^*$ is also a period, we may assume that $T^* > 0$, and by Theorem 11.12 T^* is the smallest positive period of f. The proof is complete. □

Important: Periodic Functions Without a Smallest Period

Many Calculus students believe that any periodic function possesses *the* period, i.e. a unique number like π for the sine or cosine. Although some textbooks restrict the definition of periodicity to functions which do have a smallest period, in this book we will always think of constant functions as periodic functions. We invite the reader to elaborate on the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It is clear that no smallest positive period exists, although f is surely periodic in the sense of Definition 11.6.

11.5 Polynomial Approximation

Polynomials are the most elementary functions of mathematical analysis. They are built on arithmetic operations, and they turn out to be a flexible class of infinitely differentiable functions. Of course not all functions are polynomial.

Exercise 11.4 Prove rigorously that not all functions are polynomials. *Hint:* if P is a non-constant polynomial, either $\lim_{x\to\pm\infty} |P(x)| = +\infty$.

Nevertheless, polynomials do *approximate* continuous functions in a strong way.

Theorem 11.14 (Weierstrass Approximation Theorem) *If $f: [a, b] \to \mathbb{R}$ is a continuous function and $\varepsilon > 0$, there exists a polynomial P such that* $\sup_{x\in[a,b]} |f(x) - P(x)| < \varepsilon$.

Proof We will prove an equivalent statement: there exists a sequence $\{P_n\}_n$ of polynomials such that $P_n \to f$ uniformly on $[a, b]$. Considering an affine change of variable, we may assume that $[a, b] = [0, 1]$. Furthermore, we may also assume that $f(0) = f(1) = 0$. Indeed, the function f may be replaced by the function

$$x \mapsto f(x) - f(0) - x(f(1) - f(0)).$$

This function differs from f by a polynomial (of degree ≤ 1), so that a uniform approximation of this function by means of polynomials implies that f is approximated by polynomials.

Finally, for technical reasons, we define $f(x) = 0$ for each $x \in \mathbb{R} \setminus [0, 1]$. Hence f is defined on the whole real line. For each $n = 1, 2, 3, \ldots$ define

$$c_n = \frac{1}{\int_{-1}^{1} (1 - x^2)^n \, \mathrm{d}x}$$

and

$$Q_n(x) = c_n (1 - x^2)^n.$$

Trivially, $\int_{-1}^{1} Q_n(x) \, \mathrm{d}x = 1$ for each n. Furthermore,

$$\begin{aligned}
\int_{-1}^{1} (1 - x^2)^n \, \mathrm{d}x &= 2 \int_0^1 (1 - x^2)^n \, \mathrm{d}x \\
&\geq 2 \int_0^{1/\sqrt{n}} (1 - x^2)^n \, \mathrm{d}x \\
&\geq 2 \int_0^{1/\sqrt{n}} (1 - nx^2) \, \mathrm{d}x \\
&= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.
\end{aligned}$$

As a consequence, $c_n < \sqrt{n}$ for each n. Now, fix any $\delta > 0$, and observe that $\delta \leq |x| \leq 1$ implies

$$Q_n(x) \leq \sqrt{n}(1 - \delta^2)^n.$$

The right-hand side converges to zero, hence Q_n converges to zero uniformly in the region $\delta \leq |x| \leq 1$, i.e. away from zero.

We introduce the sequence of functions

$$P_n(x) = \int_{-1}^{1} f(x + t) Q_n(t) \, \mathrm{d}t,$$

defined for $0 \le x \le 1$. A change of variable shows that

$$P_n(x) = \int_0^1 f(t) Q_n(t - x)\, \mathrm{d}t,$$

which is a polynomial function. We claim that $\{P_n\}_n$ is the approximating sequence of polynomials we are looking for. Indeed, given $\varepsilon > 0$ we choose $\delta > 0$ so that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon/2$. Here we are exploiting the uniform continuity of f on $[0, 1]$. Call $M = \sup\{|f(x)| \mid 0 \le x \le 1\}$. Recalling that $Q_n \ge 0$ we see that for each $0 \le x \le 1$ we have

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^1 (f(x+t) - f(x))\, Q_n(t)\, \mathrm{d}t \right| \\
&\le \int_{-1}^1 |f(x+t) - f(x)|\, Q_n(t)\, \mathrm{d}t \\
&\le 2M \int_{-1}^{-\delta} Q_n(t)\, \mathrm{d}t + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\, \mathrm{d}t + 2M \int_{\delta}^1 Q_n(t)\, \mathrm{d}t \\
&\le 4M\sqrt{n}(1 - \delta^2)^n + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

provided that n is big enough. □

Example 11.3 Consider the function $f \colon [-1, 1] \to \mathbb{R}$ defined by $f(x) = |x|$. By the previous result, a sequence $\{\tilde{P}_n\}_n$ of polynomials exists which converges uniformly to f on $[-1, 1]$. Setting $P_n(x) = \tilde{P}_n(x) - \tilde{P}_n(0)$, we see that $\{P_n\}_n$ is a sequence of polynomials that converges to f on $[-1, 1]$ and such that $P_n(0) = 0$ for every n. Notice that f is not differentiable at $x = 0$: in some sense, the sequence $\{P_n\}$ is a smooth uniform approximation of f.

11.6 A Continuous Non-differentiable Function

Every student learns that a continuous function may fail to be differentiable at all points, and the simplest example is usually the absolute value $x \mapsto |x|$. Karl Weierstrass proved a much stronger result about a continuous function exists which is *nowhere* differentiable. Intuitively, such a function cannot be represented by a simple formula. In this section we propose a reasonable construction.

Theorem 11.15 *There exists a continuous function on the real line* $\mathbb{R}$ *which is nowhere differentiable .*

Proof Let us start with $\varphi(x) = |x|$, for each $x \in [-1, 1]$. Then we extend it by periodicity to $\mathbb{R}$, i.e. $\varphi(x+2) = \varphi(x)$ for each $x \in \mathbb{R}$. It is clear that

$$|\varphi(s) - \varphi(t)| \leq |s - t| \tag{11.3}$$

for each s, t in $\mathbb{R}$. The function $\varphi \colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

Now we define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi\left(4^n x\right).$$

The M-test 11.2 shows that f is defined by a series that converges uniformly on $\mathbb{R}$, so that f is a continuous function. We claim that f is differentiable at *no* point of $\mathbb{R}$. To prove this claim, we pick any $x \in \mathbb{R}$ and any positive integer m. Let

$$\delta_m = \pm \frac{1}{2} \cdot 4^{-m},$$

where the sign is chosen so that the interval $[4^m x, 4^m(x + \delta_m)]$ contains no integer. Since $4^m |\delta_m| = 1/2$, this is indeed possible. Consider now the incremental ratio

$$\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}.$$

If $n > m$, the number $4^n \delta_m$ is an even integer, so that $\gamma_n = 0$. If $0 \leq n \leq m$, it follows from (11.3) that $|\gamma_n| \leq 4^n$. Recalling that $|\gamma_m| = 4^m$, we see that

$$\begin{aligned}
\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| &= \left| \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \gamma_n \right| \\
&\geq 3^m - \sum_{n=0}^{m-1} 3^n \\
&= \frac{1}{2}\left(3^m + 1\right).
\end{aligned}$$

Letting $m \to +\infty$ we get $\delta_m \to 0$. Hence f is not differentiable at x, and the proof is complete. ☐

Remark 11.5 Such a function is a typical example of a *fractal* curve, whose graph is essentially impossible to sketch. Our function f is based on the function φ, which is already irregular at countably many points. However, Weierstrass constructed a more complicated example of the form

$$f(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right),$$

where $0 < a < 1$ and b is a positive odd integer such that

$$ab > 1 + \frac{3}{2}\pi.$$

The function f is then a *trigonometric* series, and each term of the infinite sum is a smooth function. Once more we see that a limit of regular functions may be a very irregular function.

11.7 Asymptotic Estimates for the Factorial Function

Asymptotic estimates are a major tool in several fields of mathematics. We want to present a couple of results which describe the behavior of the factorial $n!$ as n gets larger and larger. Since the factorial has been introduced as a discrete function, our road is not really straight.

Definition 11.7 (Double Factorial) We define inductively on $n \in \mathbb{N}$,

$$(-1)!! = 1, \quad 0!! = 1, \quad (n+1)!! = (n+1) \cdot (n-1)!!$$

Exercise 11.5 Prove that $n!!$ is the product of all odd numbers $m \le n$ when n is odd, and it is the product of all even numbers $m \le n$ when n is even.

Theorem 11.16 (Wallis Integrals) *If $n \in \mathbb{N}$, then*

$$\int_0^{\pi/2} (\sin x)^{2n+1}\, dx = \frac{(2n)!!}{(2n+1)!!} \tag{11.4}$$

$$\int_0^{\pi/2} (\sin x)^{2n}\, dx = \frac{\pi}{2} \frac{(2n-1)!!}{(2n)!!} \tag{11.5}$$

$$\int_0^{\pi/2} (\sin x)^{2n+1}\, dx \le \int_0^{\pi/2} (\sin x)^{2n}\, dx \le \int_0^{\pi/2} (\sin x)^{2n-1}\, dx. \tag{11.6}$$

Proof For every $x \in [0, \pi/2]$ we have $0 \le \sin x \le 1$. Hence, for any such x, the sequence $n \mapsto (\sin x)^n$ is decreasing. In particular (11.6) follows at once from the monotonicity properties of the Riemann integral. We prove (11.4) and (11.5) by induction on n. They clearly hold true when $n = 0$, and we integrate by parts as follows:

$$\begin{aligned}\int_0^{\pi/2} (\sin x)^m \, \mathrm{d}x &= \int_0^{\pi/2} (\sin x)^{m-2} (1-\cos^2 x) \, \mathrm{d}x \\ &= \int_0^{\pi/2} (\sin x)^{m-2} \, \mathrm{d}x - \int_0^{\pi/2} \cos x \cdot (\sin x)^{m-2} x \cos x \, \mathrm{d}x \\ &= \int_0^{\pi/2} (\sin x)^{m-2} \, \mathrm{d}x - \frac{1}{m-1} \int_0^{\pi/2} (\sin x)^m \, \mathrm{d}x ,\end{aligned}$$

deducing the identity

$$\int_0^{\pi/2} (\sin x)^m \, \mathrm{d}x = \frac{m-1}{m} \int_0^{\pi/2} (\sin x)^{m-2} \, \mathrm{d}x$$

for every $m \geq 2$. The induction step is now easy. □

Theorem 11.17 (Wallis Formulas) *As* $n \to +\infty$,

$$\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{n\pi} \tag{11.7}$$

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}} \tag{11.8}$$

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots (2n-1) \cdot (2n-1)} = \frac{\pi}{2} + o(1). \tag{11.9}$$

Proof We set

$$q = \frac{\sqrt{n\pi}}{\frac{(2n)!!}{(2n-1)!!}}$$

and we see from (11.4) and (11.5) that

$$\begin{aligned} q &= \frac{2n}{\sqrt{n\pi}} \int_0^{\pi/2} (\sin x)^{2n} \, \mathrm{d}x \geq \frac{2n}{\sqrt{n\pi}} \int_0^{\pi/2} (\sin x)^{2n+1} \, \mathrm{d}x \\ &= \frac{2n}{\sqrt{n\pi}} \frac{(2n)!!}{(2n-1)!!} \frac{1}{2n+1} = \frac{1}{q} \frac{2n}{2n+1} \end{aligned}$$

and

$$q = \frac{2n}{\sqrt{n\pi}} \int_0^{\pi/2} (\sin x)^{2n} \, dx \le \frac{2n}{\sqrt{n\pi}} \int_0^{\pi/2} (\sin x)^{2n-1} \, dx$$
$$= \frac{2n}{\sqrt{n\pi}} \frac{(2n)!!}{(2n-1)!!} \frac{1}{2n} = \frac{1}{q}.$$

We have thus proved that

$$\frac{2n}{2n+1} \le \left(\frac{\sqrt{n\pi}}{\frac{(2n)!!}{(2n-1)!!}} \right)^2 \le 1$$

for every $n \ge 1$. Now (11.7) follows easily. Since

$$\binom{2n}{n} = \frac{(2n)!!}{(n!)^2} = \frac{2^{2n}(2n-1)!!}{(2n)!!} \sim \frac{2^{2n}}{\sqrt{n\pi}},$$

also (11.8) follows. To conclude, we observe that

$$\frac{2 \cdot 2 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdots (2n-1) \cdot (2n-1)} \sim \frac{(2n)!!(2n)!!}{(2n-1)!!(2n-1)!!(2n+1)} \sim \frac{n\pi}{2n+1} \sim \frac{\pi}{2}.$$

□

Theorem 11.18 (Stirling) *As $n \to +\infty$,*

$$n! \sim n^n e^{-n} \sqrt{2n\pi}.$$

Proof We define the sequence

$$x_n = n! e^n n^{-n-1/2}.$$

A direct calculation shows that

$$\frac{x_{n+1}}{x_n} = e^{-\left(n+\frac{1}{2}\right) \log \frac{n+1}{n} + 1}.$$

By taking logarithms we see that

$$\log x_{n+1} - \log x_n = \log \frac{x_{n+1}}{x_n} = -\left(n + \frac{1}{2}\right) \log \frac{n+1}{n} + 1 \sim -\frac{1}{12n^2}$$

as $n \to +\infty$. As a consequence, the series $\sum_n \log x_{n+1} - \log x_n$ converges, and

$$k = \lim_{n \to +\infty} \log x_n$$

exists in $\mathbb{R}$. Thus $n! \sim n^n e^{-n} \sqrt{n} e^k$. Inserting this into (11.8) we see that

$$\frac{2^{2n}}{\sqrt{n\pi}} \sim \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{(2n)^{2n} e^{-2n} \sqrt{2n} e^k}{n^{2n} e^{-2n} n e^{2k}} = \frac{2^{2n}\sqrt{2}}{\sqrt{n} e^k} = \frac{2^{2n}}{\sqrt{n\pi}} \frac{\sqrt{2\pi}}{e^k},$$

and necessarily $e^k = \sqrt{2\pi}$. The proof is complete. □

11.8 Problems

11.1 Consider the series of functions

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

For what values of x does the series converge absolutely? In what intervals does the series converge uniformly? If the series converges, is f a continuous function?

11.2 For each $n = 1, 2, 3, \ldots$ let

$$f_n(x) = \begin{cases} 0 & \text{if } x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n}. \end{cases}$$

Prove that the sequence $\{f_n\}_n$ converges pointwise to a continuous function, but the converges is not uniform.

11.3 For each real number x, let $\{x\} = x - [x]$, where $[x]$ denotes the integer part of x. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2}.$$

Find all discontinuity points of f, and prove that these points form a dense, countable subset of $\mathbb{R}$. Prove also that f is R-integrable on each bounded interval.

11.4 Let f be a continuous function on $[0, 1]$, and suppose that

$$\int_0^1 f(x)x^n \, \mathrm{d}x = 0 \quad \text{for } n = 0, 1, 2, 3, \ldots$$

Prove that $f(x) = 0$ for each $x \in [0, 1]$.

11.5 Suppose that a sequence $\{f_n\}_n$ converges pointwise to f on a compact set K, and suppose moreover that $f_n(x) \leq f_{n+1}(x)$ for each $x \in K$ and each $n \in \mathbb{N}$.

1. By setting $g = f - f_n$, reduce to the case of a sequence of function which converges pointwise to zero in a decreasing way.
2. Assume that f and each f_n are continuous functions on K. Fix any $\varepsilon > 0$ and define for each n the set $K_n = \{x \in K \mid g_n(x) \geq \varepsilon\}$. Prove that

$$K_1 \supset K_2 \supset K_3 \supset \ldots,$$

and conclude that $f_n \to f$ uniformly on K.