Chapter 1
An Appetizer of Propositional Logic

Abstract Mathematics is based on the language of proposition logic: every statement is a combination of logical propositions, and theorems are simply true statements that can be deduced according to the rules of logic. We follow the first chapter of Mendelson (Introduction to mathematical logic. CRC Press, Boca Raton, 2015).

1.1 The Propositional Calculus

Sentences are just statements to which it is possible to attach a binary value: true (T) or false (F). For example, "Roses are flowers" is a sentence, "dogs have five legs" is another sentence. But "Any cat is" is not. Sometimes sentences depend on free variables, as in "The integer n is a prime", or "The real number x is irrational". The variable n and x are free in the sense that they may take any (admissible) value: compare with "For every integer n, $n + 1 > n$". In this sentence, the variable n is quantified by "For each", and is not a free variable. Another example is "There exists a positive real number r such that $r^2 = 2$".

Definition 1.1 (Negation) If A is a sentence, its negation $\neg A$ is the sentence governed by the following table:

A	$\neg A$
T	F
F	T

S. Secchi, *A Circle-Line Study of Mathematical Analysis*,
La Matematica per il 3+2 141, https://doi.org/10.1007/978-3-031-19738-3_1

Definition 1.2 (Conjunction) If A and B are sentences, then their conjunction $A \wedge B$ is the sentence governed by the following table:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.3 (Disjunction) If A and B are sentences, then their disjunction $A \vee B$ is the sentence governed by the following table:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Remark 1.1 Although the mathematical conjunction agrees with the use of "and" in everyday language, the mathematical disjunction reflects a use of "or" which may differ from the use in common language. To be explicit, we may formulate a golden rule: $A \vee B$ corresponds to "either A, or B, or both". In common language we tend to understand "either A or B, but not both."

Definition 1.4 (Implication) If A and B are sentences, the sentence $A \implies B$ is defined by the following table:

A	B	$A \implies B$
T	T	T
F	T	T
T	F	F
F	F	T

Remark 1.2 Logical implication may be written in different ways: $A \supset B$ was common in Logic textbooks a few years ago, but also $A \to B$ is often found. The symbol $A \implies B$ is pronounced "If A, then B", and we also call it a conditional.

Definition 1.5 (Logical Equivalence) If A and B are sentences, the sentence $A \iff B$ is the sentence governed by the following table:

A	B	$A \iff B$
T	T	T
F	T	F
T	F	F
F	F	T

Logical equivalence is also denoted by $A \equiv B$.

1.2 Quantifiers

As we said before, sentences may contain one or more free variables.

Example 1.1 The sentence $A(x, y)$ defined by "the real number x is strictly smaller than the real number y" is sentence with two free variables x and y. From a logical viewpoint, $A(x, y)$ is indistinguishable from $A(\alpha, \beta)$ or $A(\clubsuit, \spadesuit)$. Of course we cannot replace free variables with symbols that are already taken: $A(x, y)$ is not the same as $A(1, 4)$ or $A(\cos, \log)$. However $A(\pi, e)$ may be acceptable, provided that we do not understand π as the number 3.14159... and e as the Napier number 2.718281... As a stronger example, think of $A(i)$: is $i^2 = -1$ as in Complex Analysis, or is i a free variable?

The truth value of a sentence depending on free variables may depend on the choice of these variables. If $A(x, y)$ is defined by "the real number x is strictly smaller than the real number y", then $A(1, 2)$ is certainly true, while $A(4, 0)$ is false.

Definition 1.6 (Universal Quantifier) The universal quantifier $\forall$ means "for all", or "for each".

Definition 1.7 (Existential Quantifier) The existential quantifier $\exists$ means "there exists".

Important: $\exists$ vs. $\exists!$

In mathematics, "there exists" always means "there exists at least one". The sentence "there exists a solution $x \in \mathbb{R}$ to the equation $x^2 = 1$" is true, although we know that there exist *exactly* two real solutions to the equation $x^2 = 1$. Since existence and uniqueness is often important, the symbol $\exists!$ is reserved for the sentence "there exists a unique".

The syntax of sentences with quantifiers is not completely universal. The sentence “For each x the sentence $A(x)$ holds” can be written in different ways:

$$\forall x \ A(x)$$

$$(\forall x) A(x)$$

$$\forall x, A(x)$$

$$(x) \ A(x).$$

The last one is clearly the most economic, and the first one is affordable. Logicians tend to avoid brackets as far as they may, and also commas are seen as inessential objects. It is a matter of facts that most mathematicians use brackets freely on their blackboards, and commas are also ubiquitous.

Remark 1.3 The existential quantifier is not a primitive symbol, since the sentence $\exists x \ A(x)$ is logically equivalent to (and actually defined as) $\neg(\forall x \ \neg A(x))$. As a consequence, the negation of $\exists x \ A(x)$ is precisely $\forall x \ \neg A(x)$, and the negation of $\forall x \ A(x)$ is precisely $\exists x \ \neg A(x)$.

If we are given a sentence $A(x_1, \ldots, x_n)$ defined by n free variables and one or more of them is quantified by either $\forall$ or $\exists$, the quantified variables become bound variables. Bound variables essentially disappear from the arguments of the sentence.

Example 1.2 Suppose that $A(x_1, x_2)$ means “$x_1 - x_2 = 0$”. The sentence A contains two free variables, but the sentence $\exists x_2 \ A(x_1, x_2)$—which means “there exists x_2 such that $x_1 - x_2 = 0$”—contains one free variable. The sentence $\forall x_1 \exists x_2 \ A(x_1, x_2)$ does not contain any free variable, and means “For every x_1 there exists (at least one) x_2 such that $x_1 - x_2 = 0$”.

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