Chapter 6
On Equalities of Central Automorphism Group with Various Automorphism Groups

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6.1 Introduction

Throughout the chapter, p denotes a prime number. For group G, we denote by $G^{'}$, $Z(G)$, $cl(G)$, $d(G)$, $\Phi(G)$, and $\mathrm{Aut}(G)$, respectively, the commutator subgroup, the center, the nilpotency class, the rank, the Frattini subgroup, and the automorphism group of G. An automorphism σ of group G is called central if σ commutes with every automorphism in $\mathrm{Inn}(G)$, the group of inner automorphisms of G, (equivalently, if $g^{-1}\sigma(g)$ lies in the center $Z(G)$ of G, for all g in G.)

The central automorphisms of G fix the commutator subgroup of G elementwise and form a normal subgroup of the full automorphism group $\mathrm{Aut}(G)$; we denote this subgroup by $\mathrm{Aut_z}(G)$ in this paper. For groups G having $\mathrm{Aut}(G)$ abelian, it is necessarily the case that $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$. The non-abelian groups G with $\mathrm{Aut}(G)$ abelian are called as Miller groups (see [19]). However, several people constructed various groups G for which $\mathrm{Aut}(G)$ is non-abelian and $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$ (see [7, 11, 15, 18]). In 2001, Curran and McCaughan [6] considered the case where the central automorphisms are just the inner automorphisms of G, that is, $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$; one can also see [4, 23]. Continuing in this direction, in 2004, Curran [8], for group G, derived the equality $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$,; the same is derived in [1, 12, 22]. Let $\mathrm{Aut_z^z}(G)$ be the set of all central automorphisms of a group G which fixes the center $Z(G)$ of G elementwise. In 2007, Attar [2] characterized finite p-groups for which $\mathrm{Aut_z^z}(G) = \mathrm{Inn}(G)$ holds. In 2009, Yadav [25] characterized p-groups of nilpotency class 2 for which $\mathrm{Aut_z}(G) = \mathrm{Aut_z^z}(G)$ (for the same equality, also see [14]).

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An automorphism ϕ of a group G is called class preserving if $\phi(x)$ is conjugate to x for all $x \in G$. The set $\mathrm{Aut_c}(\mathrm{G})$ of all class-preserving automorphisms of G forms a normal subgroup of $\mathrm{Aut}(G)$ and contains $\mathrm{Inn}(G)$. In 2013, Yadav [26] characterized finite p-groups, and Kalra and Gumber [16] characterized all finite p-groups of order $\leq p^6$ (for any prime p) and $\leq p^5$ (for odd prime p) for which the set of all central automorphisms is equal to the set of all class-preserving automorphisms, that is, $\mathrm{Aut_z}(G) = \mathrm{Aut_c}(\mathrm{G})$; the same equality is derived in [10].

An automorphism σ of a group G is called IA-automorphism if it induces the identity automorphism on the abelian quotient $G/G^{'}$. Let $\mathrm{IA_z}(\mathrm{G})$ be the group of those IA automorphisms which fix the center of G elementwise. In 2014, Rai [21] characterized finite p-groups in which $\mathrm{Aut_z}(G) = \mathrm{IA_z}(\mathrm{G})$ if and only if $\gamma_2(G) = Z(G)$. In 2016, Kalra and Gumber [17], characterized finite non-abelian p-groups G for which $\mathrm{Aut_z}(G) = \mathrm{IA_z}(\mathrm{G})$ if and only if $G^{'} = Z(G)$.

Hegarty [13] defined the notions of absolute center and autocommutator of a group G (analogous to $Z(G)$ and G^* as follows:

$$L(G) = \{g \in G|\, \alpha(g) = g \,\forall \alpha \in \mathrm{Aut}(G)\}$$

$$G^* = \langle g^{-1}\alpha(g)|g \in G,\ \alpha \in \mathrm{Aut}(G)\rangle$$

These are clearly characteristic subgroups of G. Also, $Z(G) \supset L(G)$ and $G^{'} \subset G^*$. Hegarty [13] also defined absolute central automorphism of G as follows: an automorphism γ of a group G is called an absolute central automorphism if it induces identity automorphism on $G/L(G)$. The set $\mathrm{Aut_l}(G)$ of all absolute central automorphisms of G forms a normal subgroup of $\mathrm{Aut}(G)$; it is also a subgroup of $\mathrm{Aut_z}(G)$. Let $\mathrm{Aut_l^z}(G)$ denote the group of absolute central automorphisms of G which fix $Z(G)$ elementwise.

In 2020, Singh and Gumber [24] gave a necessary and sufficient condition on finite p-group G for which $\mathrm{Aut_z}(G) = \mathrm{Aut_l}(G)$ and also for which $\mathrm{Aut_z}(G) = \mathrm{Aut_l^z}(G)$.

6.2 Equalities of Central Automorphisms

6.2.1 Equalities with Group of All Automorphisms

Definition 6.1 Following Earnley, a non-abelian group with abelian automorphism group is called Miller group.

If $\mathrm{Aut}(G)$ is abelian, then it is clear that all the automorphisms are central, i.e., $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$. The obvious examples of groups with abelian automorphism group are the cyclic groups. There are non-abelian groups with abelian automorphism groups; these are called Miller groups ($see Earnley$ [9]). Several researchers constructed various examples of groups for which $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$, even if $\mathrm{Aut}(G)$ is non-abelian. Curran, in 1982, found first such example. He constructed a group of order 2^7 for which $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian.

Theorem 6.1 ([7], Proposition, p. 394) *There exists a non-abelian group G of order 2^7 which has a non-abelian automorphism group of order 2^{12} in which every automorphism is central, that is,* $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$.

Example of such group is given below:

Let M be the Miller group of order 2^6, and let

$$G = M \times Z_2 = \langle a, b, c, d \mid a^8 = b^4 = c^2 = d^2 = 1,\ a^b = a^5,\ b^c = b^{-1},$$

$$[a, c] = [a, d] = [b, d] = [c, d] = 1\rangle$$

This result of Curran leads the motivation to p-groups for p an odd prime in which $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian. In 1984, Malone proved the following result:

Theorem 6.2 ([18], Proposition, p. 36) *For each odd prime p, there exists a non-abelian p-group with a non-abelian automorphism group in which each automorphism is central, that is,* $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$.

For each odd prime p, we consider the group

$$F = \langle a_1,\ a_2,\ a_3,\ a_4,\ \mid (a_i,\ a_j,\ a_k) = 1\, and\, a_i^{p^2} = 1\ \ for$$

$$1 \leq i, j, k \leq 4;\ (a_1, a_2) = a_1^p;\ (a_1, a_3) = a_3^p;\ (a_1, a_4) = a_4^p$$

$$(a_2, a_3) = a_2^p;\ (a_2, a_4) = 1\,;\, (a_3, a_4) = a_3^p\rangle$$

$\mathrm{Aut}(F)$ is abelian group. We set $B = \langle\, b \mid b^p = 1\,\rangle$. Group $G = F \times B$ is non-abelian group which has $\mathrm{Aut}(G)$ non-abelian in which each automorphism is central.

Curran in [7] and Malone in [18] derived the examples of groups with direct factors for which $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian. The question was left if there is a group G with no direct factors for which $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian. Continuing in this direction, in 1986, Glasby produced an infinite family of 2-groups having no direct factors and which have a non-abelian automorphism group in which all automorphisms are central.

Definition 6.2 Define G_n to be the group generated by $x_1, \ldots, x_n$ $x_i^{2^i} = 1$ $(1 \leq i \leq n)$ $[x_i, x_{i+1}] = x_{i+1}^{2^i},\ \ (1 \leq i < n)$. $[x_i, x_j] = 1,\ \ (1 < i + 1 < j \leq n)$

Theorem 6.3 ([11], Theorem, p. 234) *For $n \geq 3$, G_n has no direct factors, and* $\mathrm{Aut}(G_n)$ *is non-abelian of order* $2^{p(n)}$, *where* $p(n) = (n-1)(2n^2 - n = 6/6)(n \geq 4)$*, in which every* automorphism *is central.*

In 2012, Jain and Yadav [15] constructed the following family of groups G_n with no direct factor, for which $\mathrm{Aut}_z(G_n) = \mathrm{Aut}(G_n)$.

Definition 6.3 Let n be a natural number greater than 2 and p an odd prime. Define G_n to be the group generated by x_1, x_2, x_3, x_4

$$x_1^{p^n} = x_2^{p^3} = x_3^{P^2} = x_4^{p^2} = 1,$$

$$[x_1, x_2] = x_2^{p^2}, \quad [x_1, x_3] = x_3^p$$

$$[x_1, x_4] = x_4^p, \quad [x_2, x_3] = x_1^{p^{n-1}}$$

$$[x_2, x_4] = x_2^{p^2}, \quad [x_3, x_4] = x_4^p.$$

This group G is a regular p-group of nilpotency class 2 having order p^{n+7} and exponent p^n. Further, $Z(G) = \Phi(G)$ and therefore G is purely non-abelian.

Theorem 6.4 ([15], Theorem A, p. 228) *Let $m = n + 7$ and p be an odd prime, where n is a positive integer greater than or equal to* 3*. Then there exists a group G of order p^m, exponent p^n, and with no nontrivial abelian direct factor such that* $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$ *is non-abelian.*

6.2.2 Equalities with Group of $\mathbf{Inn}(G)$ *and* $\mathbf{Z(Inn(G))}$

In 2001, Curran and McCaughan [6] characterized finite p-groups in which central automorphisms are precisely the inner automorphisms.

Theorem 6.5 ([6], Theorem, p. 2081) *If G is a finite p-group, then* $\mathrm{Aut}_z(G) = \mathrm{Inn}(G)$ *if and only if $G^{'} = Z(G)$ and $Z(G)$ is cyclic.*

Definition 6.4 A group G, whose only element of finite order is the identity, is called torsion-free group.

Definition 6.5 A non-abelian group G is purely non-abelian if G has no nontrivial abelian direct factor.

In 2016, Azhdari characterized all finitely generated groups G for which the equality $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ holds. He proved the following:

Theorem 6.6 ([4], Theorem 2, p. 4134) *Let G be a finitely generated group. Then* $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ *if and only if one of the following assertion holds:*

- *G is purely non-abelian and $Z(G) = G^{'}$ is cyclic.*
- *$G \cong C_2 \times N$ where N is purely non-abelian with $|Z(N)|$ odd and $Z(N) = N^{'}$ is cyclic (or $Z(G) = C_2 \times G^{'}$ is cyclic).*
- *G is torsion-free with $Z(G) = G^{'}$ is cyclic and $det(M_G) = 1$ where M_G is skew-symmetric matrix corresponding to G.*

In 2018, Sharma et al. [23] verified the equality $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ for the finite p-groups of order up to p^7 as follows:

Theorem 6.7 ([23], Theorem 2.1, p. 3) *There is no* p-group *G of order up to p^6 satisfying* $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$.

Theorem 6.8 ([23], Theorem 2.2, p. 3) *A* p-group *G of order p^7 satisfies* $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ *if and only if $Z(G) \cong C_p^2$, $|G^{'}| = p^4$ and $cl(G) = 4$.*

In 2004, Curran [8] considered the case where the central automorphism group is as small as possible. Clearly, $\mathrm{Z(Inn(G))} \leq \mathrm{Aut_z}(G)$, for any group G. When G is arbitrary, $\mathrm{Aut_z}(G)$ and $\mathrm{Z(Inn(G))}$ may coincide because both these subgroups of $\mathrm{Aut}(G)$ can be trivial. However, the situation becomes interesting if G is a p-group, since both subgroups are nontrivial.

Theorem 6.9 ([8], Theorem 1.1, p. 223) *Let G be a finite non-abelian p-group. If* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$*, then $Z(G) \leq G^{'}$, and furthermore,* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $\mathrm{Hom}(G/G, Z(G)) \approx Z(G/Z(G))$.

In 2013, Sharma and Gumber [22] characterized p-groups of order $\leq p^5$(for any prime p) and of order p^6(for p odd), for which $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$.

Theorem 6.10 ([22], Theorem 3.2, p. 3) *Let G be p-group of order p^5 and $cl(G) = 3$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if $d(G) = 2$ and $Z(G) \cong C_p$.*

Theorem 6.11 ([22], Theorem 3.3, p. 3) *Let G be a p-group of order p^6, for an odd prime p, and $cl(G) = 3\,0r4$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if $d(G) = 2$ and $Z(G) \cong C_p$.*

Continuing the study of Curran [8] of minimum order of $\mathrm{Aut_z}(G)$, Gumber and Kalra [12] obtained the following:

Let $G/G^{'} \cong C_{p^{r_1}} \times \ldots C_{p^{r_n}}$ $(r_1 \geq \cdots \geq r_n \geq 1)$ and $Z_2(G)/Z(G) \cong C_{p^{s_1}} \times \ldots C_{p^{s_m}}$ $(s_1 \geq \cdots \geq s_m \geq 1)$.

Theorem 6.12 ([12], Theorem 2.1, p. 1803) *Let G be a finite p-group with $Z(G) \cong C_{p^{b_1}}$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if either $G/G^{'} \cong Z_2(G)/Z(G)$ or $d(G) = d(Z_2(G)/Z(G))$, $s_i = b_1$ for $1 \leq i \leq c$, and $s_i = r_i$ for $c + 1 \leq i \leq n$, where c, $1 \leq c \leq n$ is the largest such that $r_c \geq b_1$.*

Definition 6.6 The coclass of a finite p-group G of order p^n is $n-c$, where c is the class of the group.

Corollary 6.1 ([12], Corollary 2.2, p. 1804) *Let G be a finite p-group of coclass* 2. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2$.

Corollary 6.2 ([12], Corollary 2.3, p. 1804) *Let G be a finite p-group of coclass* 3. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2,\ 3$ *or* $Z(G) \cong C_{p^2}$ *and* $Z_2(G)/Z(G) \cong G/G^{'}$.

Corollary 6.3 ([12], Corollary 2.4, p. 1804) *Let G be a finite p-group of coclass* 4. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if one of the following conditions holds:*

(a) $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2, 3, 4$.
(b) $Z(G) \cong C_{p^2}$ *and either* $Z_2(G)/Z(G) \cong G/G^{'}$ *or* $Z_2(G)/Z(G) \cong C_{p^2} \times C_p$ *and* $G/G^{'} \cong C_{p^3} \times C_p$ *or* $Z_2(G)/Z(G) \cong C_{p^2} \times C_p$ *and* $G/G^{'} \cong C_{p^4} \times C_p$.
(c) $Z(G) \cong C_{p^3}$ *and* $Z_2(G)/Z(G) \cong G/G^{'}$.

Gumber and Kalra also generalized the results of Sharma and Gumber [22] as follows:

Theorem 6.13 ([12], Theorem 3.1, p. 1804) *Let G be p-group of order* $= p^5$ *and* $cl(G) = 3$. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2$.

Theorem 6.14 ([12], Theorem 3.2, p. 1805) *Let G be a finite* p-group *such that* $cl(G) = 3$ *or* 4. *Then,* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2$.

Also, Gumber and Kalra obtained the result for $|G| = p^7$ *as in [22]; it was up to* p^6.

Theorem 6.15 ([12], Theorem 3.3, p. 1805) *Let G be a p-group of order* p^7. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if one of the following holds:*

$cl(G) = 3,\ Z(G) \simeq C_p$ *and* $rank(G) = rank(Z_2(G)/Z(G)) = 2,\ 3,\ 4$.
$cl(G) = 4$ *and either* $Z(G) \simeq C_p$ *and* $rank(G) = rank(Z_2(G)/Z(G)) = 2,\ 3$ *or* $Z(G)$ *is cyclic group of order* p^2 *and* $Z_2(G)/Z(G) \simeq G/G^{'}$.
$cl(G) = 5,\ Z(G) \simeq C_p$ *and* $rank(G) = rank(Z_2(G)/Z(G)) = 2$.

Let G be a non-abelian p-group G. Let $G/G^{'} \cong C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_r}}\ \ (c_1 \geq \cdots \geq c_r \geq 1)$ and $Z_2G/Z(G) \cong C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_s}}\ (d_1 \geq d_2 \geq \ldots d_s \geq 1)$, where $C_{p^{a_i}}$ is a cyclic group of order p^{a_i}.

In 2020, Attar [1] characterized the finite p-groups in some special cases, including p-groups G with $C_G(Z(\Phi(G)) \neq \Phi(G)$, p-groups with an abelian maximal subgroup, metacyclic p-groups with $p \geq 2$, p-groups of order p^n and exponent p^{n-2}, and Camina p-groups, for which $\mathrm{Aut_z}(G)$ is of minimal order, as follows:

Theorem 6.16 ([1], Theorem 3.1, p. 4) *Let* G *be a finite* p-group *such that* $C_G(Z(\Phi(G)) \neq \Phi(G)$. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G)$ *is cyclic and one of the following is true:*

- $G/G^{'} \cong Z_2(G)/Z(G)$.
- $r = s$, $d_i = h$ *for* $1 \leq i \leq t$, $d_i = c_i$ *for* $t+1 \leq i \leq r$, *where* $p^h = exp(Z(G))$ *and* t *is the largest integer between* 1 *and* s *such that* $c_t > h$.

Corollary 6.4 ([1], Corollary 3.2, p. 5) *Let* G *be a non-abelian finite* p*-group with an abelian maximal subgroup. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $G^{'} = Z(G)$ *and* $Z(G)$ *is cyclic.*

Theorem 6.17 ([1], Theorem 3.3, p. 6) *Let* G *be a non-abelian metacyclic finite* p*-group with* $p > 2$. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \leq G^{'}$.

Corollary 6.5 ([1], Corollary 3.4, p. 6) *The finite non-abelian* p-groups G *of order* p^n *and exponent* p^{n-1} *for which* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *are of the following isomorphism types:*

(1) $M(p^3) = \langle \alpha, \beta \,|\, \alpha^{p^2} = \beta^p = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+p} \rangle (p > 2)$.
(2) $D_8 = \langle \alpha, \beta \,|\, \alpha^4 = \beta^2 = 1,\ \beta^{-1}\alpha\beta = \alpha^{-1} \rangle$.
(3) $Q_8 = \langle \alpha, \beta \,|\, \alpha^4 = 1,\ \beta^2 = \alpha^2,\ \beta^{-1}\alpha\beta = \alpha^{-1} \rangle$.

Corollary 6.6 ([1], Corollary 3.5, p. 7) *Let* p *be an odd prime. Then finite non-abelian* p-groups *of order* p^n *and exponent* p^{n-2} *for which* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *are one of the following isomorphism types:*

(1) $G = \langle \alpha,\ \beta,\ \gamma \,|\, \alpha^p = \beta^p = \gamma^p = 1,\ \alpha\beta = \beta\alpha,\ \gamma^{-1}\alpha\gamma = \alpha\beta,\ \beta\gamma = \gamma\beta \rangle$.
(2) $G = \langle \alpha,\ \beta \,|\, \alpha^{p^3} = \beta^{p^2} = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+p} \rangle$.
(3) $G = \langle \alpha,\ \beta \,|\, \alpha^{p^4} = \beta^{p^2} = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+p^2} \rangle$.

Corollary 6.7 ([1], Corollary 3.6, p. 8) *The finite non-abelian* 2*-groups* G *of order* 2^n *and exponent* 2^{n-2} *for which* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *are one of the following:*

(1) $G = \langle \alpha,\ \beta,\ \gamma \,|\, \alpha^8 = \beta^2 = \gamma^2 = 1,\ \beta^{-1}\alpha\beta = \alpha^5,\ \gamma^{-1}\alpha\gamma = \alpha\beta,\ \beta\gamma = \gamma\beta \rangle$.
(2) $G = \langle \alpha,\ \beta,\ \gamma \,|\, \alpha^{2^{n-2}} = 1,\ \beta^2 = 1,\ \gamma^2 = \beta,\ \beta^{-1}\alpha\beta = \alpha^{1+2^{n-3}},\ \gamma^{-1}\alpha\gamma = \alpha^{-1}\beta, \rangle$.
(3) $G = \langle \alpha,\ \beta \,|\, \alpha^{16} = \beta^4 = 1,\ \beta^{-1}\alpha\beta = \alpha^5 \rangle$.
(4) $G = \langle \alpha,\ \beta \,|\, \alpha^{2^{n-2}} = 1,\ \beta^4 = 1,\ \beta^{-1}\alpha\beta = \alpha^{-1+2^{n-4}} \rangle$, *where* $n \geq 6$.
(5) $G = \langle \alpha,\ \beta,\ \gamma \,|\, \alpha^{2^{n-2}} = 1,\ \beta^2 = 1,\ \gamma^2 = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+2^{n-3}},\ \gamma^{-1}\alpha\gamma = \alpha^{-1+2^{n-4}}\beta,\ \beta\gamma = \gamma\beta \rangle$, *where* $n \geq 6$.

(6) $G = \langle \alpha, \beta, \gamma \mid \alpha^{2^{n-2}} = 1, \beta^2 = 1, \gamma^2 = \alpha^{2^{n-3}}, \beta^{-1}\alpha\beta = \alpha^{1+2^{n-3}}, \gamma^{-1}\alpha\gamma = \alpha^{-1+2^{n-4}}\beta, \beta\gamma = \gamma\beta \rangle$, *where* $n \geq 6$.

(7) $G = \langle \alpha, \beta, \gamma \mid \alpha^8 = 1, \beta^2 = 1, \gamma^2 = \alpha^4, \beta^{-1}\alpha\beta = \alpha^5, \gamma^{-1}\alpha\gamma = \alpha\beta, \beta\gamma = \gamma\beta \rangle$.

A pair (G, N) is called Camina pair if $1 < N < G$ is normal subgroup of G and for every element $g \in G/N$, the element g is conjugate to all gN.

Theorem 6.18 ([1], Theorem 3.7, p. 12) *Let G be a non-abelian finite* p-group *such that* $(G, Z(G))$ *is a Camina pair. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $G/G' \cong Z_2(G)/Z(G)$.

Theorem 6.19 ([1], Corollary 3.8, p. 12) *Let G be a finite non-abelian Camina* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $G' = Z(G)$ *and* $Z(G)$ *is cyclic.*

6.2.3 *Equalities with Class-Preserving Automorphisms*

For a finite p-group G, the subgroup $\Omega_m(G)$ is defined as $\langle x \in G \mid x^{p^m} = 1 \rangle$, and $\mho_m(G)$ is defined as $\langle x^{p^m} \mid x \in G \rangle$. For a finite p-group G with $cl(G) = 2$, $G/Z(G)$ is abelian. Consider the following cyclic decomposition of $G/Z(G)$:

$$G/Z(G) \cong C_{p^{e_1}} \times \ldots \times C_{p^{e_k}} \quad (e_1 \geq e_2 \geq \cdots \geq e_k \geq 1).$$

In 2013, Yadav (see [26]) and Kalra and Gumber (see [16]) characterized p-groups of class 2 with $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ as follows:

Theorem 6.20 ([26], Theorem A, p. 2) *Let G be a finite* p-group *of class* 2. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $G' = Z(G)$ *and* $|\mathrm{Aut_c(G)}| = \Pi_{i=1}^{d} |\Omega_{m_i}(G')|$

Theorem 6.21 ([26], Theorem B, p. 2) *Let G be a finite* p-group *and* $cl(G) = 2$ *with* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *and then rank of G is even.*

Theorem 6.22 ([16], Theorem 3.1, p. 3) *Let G be a finite* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $\mathrm{Aut_c(G)} \cong Hom(G/Z(G), G')$ *and* $G' = Z(G)$.

Theorem 6.23 ([16], Theorem 3.3, p. 4) *Let G be a finite non-abelian* p-group *such that the center of the group is elementary abelian. Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if G is a Camina* p-group *and* $cl(G) = 2$.

Theorem 6.24 ([16], Theorem 3.4, p. 4) *Let G be a finite non-abelian* p-group *such that* $Z(G)$ *is cyclic. Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $Z(G) = G'$.

Definition 6.7 A finite p-group G of class 2 is said to have property $(*)$ if for some $\pi\ \mho_{m_i^\pi}(\Omega_{n_i}(Z(G)) \leq [x, G]$ for all $x \in G/Z(G)$ and $i \in \{1, \ldots, k\}$.

In 2015, Ghoraishi found a necessary and sufficient condition for a finite p-group G to satisfy $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$, as follows:

Theorem 6.25 *Let G be a finite* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $Z(G) = G^{'}$ *and G has property* $(*)$.

6.2.4 Equalities with Absolute Central and IA Automorphisms

Definition 6.8 A finite non-Abelian group G is said to be purely non-Abelian if it has no nontrivial Abelian direct factor.

Let $C_{\mathrm{Aut}(G)}(\mathrm{Aut_l}(G)) = \{\alpha \in \mathrm{Aut}(G) \,|\, \alpha\,\beta = \beta\,\alpha,\ \forall\ \beta \in \mathrm{Aut_l}(G)\}$ denote the centralizer of $\mathrm{Aut_l}(G)$ in $\mathrm{Aut}(G)$. In [20], Moghaddam and Safa defined $E(G) = [G, C_{\mathrm{Aut}(G)}(\mathrm{Aut_l}(G))] = \langle g^{-1}\alpha(g)\,|g \in G,\ \alpha \in C_{Aut(G)}(\mathrm{Aut_l}(G))\rangle$. One can easily see that $E(G)$ is a characteristic subgroup of G containing the derived group $G^{'} = [G, \mathrm{Inn}(G)]$, and each absolute central automorphism of G fixes $E(G)$ elementwise [20, Theorem C].

Let

$$G/E(G) \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_k}},\ \ (e_1 \geq \ldots e_k \geq 1)$$

$$G/G^{'} \cong C_{p^{f_1}} \times C_{p^{f_2}} \times \cdots \times C_{p^{f_l}},\ \ (f_1 \geq \ldots f_l \geq 1)$$

$$L(G) \cong C_{p^{g_1}} \times C_{p^{g_2}} \times \cdots \times C_{p^{g_m}},\ \ (g_1 \geq \ldots g_m \geq 1)$$

$$Z(G) \cong C_{p^{h_1}} \times C_{p^{h_2}} \times \cdots \times C_{p^{h_n}}\ \ (h_1 \geq \ldots h_n \geq 1).$$

Since $G/E(G)$ is a quotient group of $G/G^{'}$, it follows that $k \leq l$ and $e_i \leq f_i$ for all $1 \leq i \leq k$.

In the same year, M. Singh and D. Gumber [24] obtained the equalities of $\mathrm{Aut_z}(G)$ with $\mathrm{Aut_l}(G)$, the group of absolute central automorphisms, and $\mathrm{Aut_l^z}(G)$, the group of absolute central automorphisms that fix the center elementwise, as follows:

Theorem 6.26 ([24], Theorem 1, p. 864) *Let G be a finite non-Abelian* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_l^z}(G)$ *if and only if either* $L(G) = Z(G)$ *or* $Z(G) \leq \Phi(G)$, $G^{'} = E(G)$, $m = n$, *and* $e_1 \leq g_t$, *where t is the largest integer between* 1 *and m such that* $g_t < h_t$.

Theorem 6.27 ([24], Theorem 2, p. 865) *Let G be a finite non-abelian* p-group *such that* $L(G) < Z(G)$. *Then* $\mathrm{Aut}_z(G) = \mathrm{Aut}_1^z(G)$ *if and only if* $Z(G) \leq \Phi(G)$, $G^{'} = E(G)Z(G)$, $m = n$, $e_1 \leq g_t$, *where t is the largest integer between* 1 *and m such that* $g_t < h_t$.

In 2014, Rai [21] characterized finite p-groups for which $\mathrm{Aut}_z(G) = \mathrm{IA}_z(\mathrm{G})$, where $\mathrm{IA}_z(\mathrm{G})$ denote the group of those IA automorphisms which fix the center elementwise, as follows:

Theorem 6.28 ([21], Theorem B(1), p. 170) *Let G be a finite p-group. Then* $\mathrm{Aut}_z(G) = \mathrm{IA}_z(\mathrm{G})$ *if and only if* $G^{'} = Z(G)$.

Let X and Y be the two finite abelian p-groups, and let $X \cong C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_i}}$ and $Y \cong C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_j}}$ be the cyclic decomposition of X and Y, where $a_t \geq a_{t+1}$ and $b_s \geq b_{s+1}$ are positive integers. If either X is proper subgroup or proper quotient group of Y and $d(X) = d(Y)$, then there certainly exists r, $1 \leq r \leq i$ such that $a_r < b_r$, $a_k = b_k$ for $r + 1 < k < i$. For this unique fixed r, let $var(X, Y) = p^r$. In other words, $var(X, Y)$ denotes the order of the last cyclic factor of X whose order is less than that of corresponding cyclic factor of Y.

In 2016, Kalra and Gumber obtained $\mathrm{Aut}_z(G) = \mathrm{IA}_z(\mathrm{G})$ for finite non-abelian p-groups as follows:

Theorem 6.29 ([17], Theorem 2.12, p. 5) *Let G be a finite non-abelian* p-group. *Then* $\mathrm{Aut}_z(G) = \mathrm{IA}_z(\mathrm{G})$ *if and only if either* $G^{'} = Z(G)$ *or* $G^{'} < Z(G)$, $d(G^{'}) = d(Z(G))$ *and* $exp(G/G^{'}) \leq var(G^{'}, Z(G))$.

6.2.5 *Equalities with Central Automorphisms Fixing the Center Elementwise*

In 2007, Attar [2] characterized groups in which the central automorphisms fixing the center elementwise are precisely inner automorphisms, as follows:

Theorem 6.30 ([2], Theorem, p. 297) *If G is a* p-group *of finite order, then* $\mathrm{Aut}_z^Z(G) = \mathrm{Inn}(G)$ *if and only if G is abelian or nilpotency class of G is* 2 *and* $Z(G)$ *is cyclic.*

Let G be a finite p-group of class 2. Then $G/Z(G)$ and $G^{'}$ have equal exponent $p^C(say)$. Let

$$G/Z(G) \cong C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_m}} \quad (c_1 \geq \cdots \geq c_m \geq 1)$$

where $C_{p^{c_i}}$ is a cyclic group of order p^{c_i}, $1 \leq i \leq r$. Let k be the largest integer between 1 and r such that $c_1 = c_2 = c_k = e$. Note that $k \geq 2$. "Let M be the subgroup of G containing $Z(G)$ such that

$$\bar{M} = M/Z(G) = C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_k}}.\text{''}$$

Let

$$G/G^{'} \cong C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_n}} \quad d_1 \geq d_2 \geq \ldots d_s \geq 1$$

be a cyclic decomposition of $G/G^{'}$ such that $\bar{M}$ is isomorphic to a subgroup of

$$\bar{N} = N/G^{'} := C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_k}}.$$

In 2009, using the above terminology, Yadav proved the following:

Theorem 6.31 ([25], Theorem, p. 4326) *Let G be a finite p-group of class* 2. *Then* $\mathrm{Aut}_z(G) = \mathrm{Aut}_z^z(G)$ *if and only if* $m = n$, $G/Z(G)/\bar{M} \cong (G/G^{'})/\bar{N}$, *and* $exp(Z(G)) = exp(G^{'})$.

In 2011, Azhdari and Akhavan-Malayeri [5] generalized the result of Attar in [2] for the finitely generated groups of nilpotency class 2. They got the following:

Theorem 6.32 ([5], Theorm 0.1, p. 1284) *Let G be a finitely generated of* $cl(G) = 2$. *Then* $\mathrm{Aut}_z^z(G) = \mathrm{Inn}(G)$ *if and only if* $Z(G) \cong C_p$ *or* $Z(G) \cong C_n \times \mathbb{Z}^s$ *where* $exp(G/Z(G))/n$ *and s is torsion-free rank of* $Z(G)$.

Theorem 6.33 ([5], Corollary 0.2) *Let G be a finitely generated group of class* 2, *which is not torsion-free. Then* $\mathrm{Aut}_z^z(G) = \mathrm{Inn}(G)$ *if and only if* $cl(G) = 2$ *and* $Z(G)$ *is cyclic or* $Z(G) \cong C_n \times \mathbb{Z}^s$ *with* $exp(G/Z(G))$ *divides n and s is torsion-free rank of* $Z(G)$.

Theorem 6.34 ([5], Corollary 0.3) *Let G be a finitely generated of* $cl(G) = 2$. $G^{'}$ *is torsion-free, and* $\mathrm{Aut}_z^z(G) = \mathrm{Inn}(G)$ *if and only if* $Z(G)$ *is infinite cyclic.*

In the same year, Jafari also found a necessary and sufficient condition on a finite p-group G such that $\mathrm{Aut}_z(G) = \mathrm{Aut}_z^z(G)$, as follows:

Theorem 6.35 *Let G be a finite* p-group. *Then* $\mathrm{Aut}_z(G) = \mathrm{Aut}_z^z(G)$ *if and only if* $Z(G)G' \subseteq G^{p^n}G'$, *where* $\exp(Z(G)) = p^n$.

Let G be a non-abelian finite p-group. Let

$$G/G^{'} = C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_r}} \quad (c_1 \geq \ldots c_r \geq 1).$$

$$G/G^{'}Z(G) \cong C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_s}} \quad (d_1 \geq \ldots d_s \geq 1).$$

and $\quad Z(G) \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_t}} \quad (e_1 \geq \ldots e_t \geq 1).$
since $G/G^{'}Z(G)$ is a quotient of $G/G^{'}$.

In 2012, Attar [3] gave a necessary and sufficient condition on finite p-group G such that $\mathrm{Aut_z}(G)$ to be $\mathrm{Aut_z^z}(G)$, as follows:

Theorem 6.36 ([3], Theorem A, p. 1097) *Let G be a non-abelian finite* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_z^z}(G)$ *if and only if* $Z(G) \leq G^{'}$ *or* $Z(G) \leq \Phi(G)$, $r = s$, *and* $c_1 \leq b_m$ *where* m *is the largest integer between* 1 *and* r *such that* $a_m > b_m$.

Theorem 6.37 ([3], Corollary 2.1, p. 1098) *Let G be a non-abelian finite* p-group *such that exponent of* $Z(G)$ *is* p. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_z^z}(G)$ *if and only if* $Z(G) \leq \Phi(G)$.

References

1. M.S. Attar, *Some finite p-groups with central automorphism group of minimal order*, J. Algebra and Appl.,2050167 (2020), (13pages).
2. M. S. Attar, *On the central automorphisms that fix the center element-wise*, Arch. Math., 89 (2007), 296–297.
3. M.S. Attar, *Finite* p-groups *in which each central* automorphism *fixe the center element-wise*, Comm. Algebra, 40 (2012), 1096–1102.
4. Z. Azhdari, *Central automorphisms and inner automorphisms in finitely generated groups*, Comm. Algebra, 44(2016), 4133–4139.
5. Z. Azhdari and M. Akhavan-Malayeri, *On inner automorphisms and central automorphisms of nilpotent groups of class* 2, J. Algebra Appl., 10 (2011), pp. 1283–1290.
6. M. J. Curran and D.J. McCaughan, *central automorphisms that are almost inner*, Comm. Algebra, 29 (2001), no. 5, pp. 2081–2087.
7. M. J. Curran, *A non-abelian automorphism group with all automorphisms central*, Bull. Austral. Math. Soc., Vl. 26 (1982), pp. 393–397.
8. M. J. Curran, *Finite groups with central automorphism group of minimal order*, Math. Proc. Royal Irish Acad., 104 A(2) (2004), pp. 223–229.
9. B. E. Earnley, *On finite groups whose group of automorphisms is abelian*, Ph.D. Thesis, Wayne State University(1975).
10. S. M. Ghoraishi *Finite groups in which class preserving and central automorphisms coincide*, Algebra Colloq., 22(Spec 1) (2015), pp. 969–974.
11. S. P. Glasby, 2-*groups with every automorphism central*, J. Austral. Math. Soc., 41 (1986), pp. 233–236.
12. D. Gumber and H. Kalra, *Finite p-groups with central automorphism group of minimal order*, Comm. Algebra, 43 (2015), pp. 1802–1806.
13. P. Hegarty *The Absolute Centre of a group*, J. Algebra, 169 (1994), pp. 929–935.
14. S. H. Jafari, *Central automorphism groups fixing the center element-wise*. Int. Electron. J. Algebra, 9 (2011), pp. 167–170.
15. V. K. Jain and M.K. Yadav, *On finite p-groups whose automorphisms are all central*, Israel J. Math., 189(2012), pp. 225–236.
16. H. Kalra and Deepak Gumber, *On equality of central and class preserving* automorphisms *of finite* p-groups, Indian J. Pure Appl. Math., 44(2013), pp. 711–725.
17. H. Kalra and D. Gumber, *Equality of certain automorphism groups of finite* p-groups, J. Algeb. Appl., 15(2016), pp. 1650056.
18. J. J. Malone, *p-groups with non-abelian automorphism groups and all automorphisms central*, Bull. Austral. Math. Soc., 29 (1984), pp. 35–37.

19. G. A. Miller, *A non-abelian group whose group of isomorphisms is abelian*, The Messenger of Mathematics, 18 (1913), no. 1, pp. 124–125.
20. M.R.R. Moghaddam and H. Safa, *Some properties of autocentral automorphisms of a group* Ricerche Math., 59 (2010), pp. 255–264.
21. P.K. Rai, *On IA-automorphisms that fix the center elementwise*, Proc. Indian Acad. Sci. (Math. Sci.), vol. 124 (2014), no. 2, pp. 169–173.
22. M. Sharma and D. Gumber, *On central automorphisms of finite p-groups*, Comm. Algebra, 41 (2013), pp. 1117–1122.
23. M. Sharma, H. Kalra and D. Gumber, *Some finite p-groups with central automorphism group of non-minimal order*, J. Algebra Appl., 17 (2018), no. 2, 1850026 (5 pages).
24. M. Singh and D. Gumber, *On the coincidence of the central and absolute central* automorphism *groups of finite p-groups*, Mathematical notes, vol. 107 (2020), no. 5, pp. 863–866.
25. M.K. Yadav, *On central automorphisms fixing the center elementwise*, Comm. Algebra, Vol. 37 (2009), pp. 4325–4331.
26. M. K. Yadav, *On finite* p-groups *whose central* automorphisms *are all class preserving*, Comm. Algebra, 41(2013), pp. 4576–4592.