Trends in Mathematics

Sandeep Singh
Mehmet Ali Sarigöl
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Editors

Algebra, Analysis, and Associated Topics

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Trends in Mathematics

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Algebra, Analysis, and Associated Topics

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ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-3-031-19081-0 ISBN 978-3-031-19082-7 (eBook)
https://doi.org/10.1007/978-3-031-19082-7

Mathematics Subject Classification: 08-xx, 11-xx, 05Cxx

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Dedicated to God

Preface

Analysis and algebra are two key branches of Mathematics. Analysis is a branch of Mathematics which studies continuous changes and includes the theories of integration, differentiation, measure, limits, analytic functions, and infinite series while algebra is the study of algebraic structures, groups, rings, modules, fields, vector spaces, and lattices. A basic premise of this book is that quality assurance is effectively achieved through the selection of quality research articles. This book comprises the contribution of various researchers in 15 chapters. Each chapter identifies the existing challenges in the areas of Algebra, Analysis, and related topics. These chapters are representing the importance of existing results and helpful for generating new ideas for various research problems of pure Mathematics.

This book provides the technique suitable for solving the problem with sufficient mathematical background, and discussions on the obtained results with physical interruptions to understand the domain of applicability of Analysis and Algebra. This book discusses new results in cutting-edge areas of several branches of mathematics and applications, including analysis, algebra number theory, etc. Also, algebra and combinatorics are core areas of mathematics which find broad applications in the sciences and in other mathematical fields. Literature survey is also provided in each of chapter which reveals the challenges, outcomes, and developments of higher- level mathematics in this decade.

The book comprised of the following interesting topics of Pure Mathematics:

- Maximal Rotational Hypersurfaces
- k-Horadam Sequences
- Lauricella Function
- Absolute Nörlund Summable Factor
- Derivations and Special Functions over Fields
- Central Automorphism of a Group
- Brandt Semigroup B_n
- Δ Convergence in $CAT(0)$ Spaces
- Quantum Dynamical Semi-Group
- Cardinality of Sum-Sets

- Cantor Dyadic Groups
- $\mathcal{I}_2$-Statistical Limit Points and Cluster Points
- Bessel and Whittaker Functions
- Neutrosophic *e* Space

This book promotes a vision of pure mathematics as integral to modern science and engineering. Theoretically oriented readers will find an overview of Mathematics and its applications. Readers will find a variety of current research topics with sufficient discussion in terms of physical point of view to adapt for solving the particular application. The book stimulates the advancement of mathematics and its applications.

As editors, we would like to express our sincere thanks to the Akal University for providing us excellent facilities and support for further research. We are also grateful to all referees for spending their valuable time to reviews the chapters. The editors are thankful to Chris Eder Associate Editor at Springer, for his continuous support toward the publication of this book.

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June 2022

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Chapter 1
Maximal Rotational Hypersurfaces Having Spacelike Axis, Spacelike Profile Curve in Minkowski Geometry

Erhan Güler and Ömer Kişi

1.1 Introduction

Mathematicians and also geometers have researched on the differential geometry of the surfaces (faces) and hypersurfaces (hypfaces) in space forms for almost 300 years. Some of these studies are given in alphabetical order:

Arslan et al. [1] worked on the generalized rotation (rot) faces in Euclidean four space $\mathbb{E}^4$; Arslan et al. [2] studied the Weyl pseudosymmetric hypfaces; Arslan and Milousheva [3] introduced the meridian faces in Minkowski 4-space $\mathbb{E}_1^4$; Arvanitoyeorgos et al. [4] focused on the Lorentz hypfaces satisfying $\Delta H = \alpha H$ in $\mathbb{E}_1^4$; Beneki et al. [5] worked the helicoidal faces in Minkowski 3-space; Chen [6] served the total mean curvature H and the finite-type submanifolds; Cheng and Wan [7] introduced the complete hypfaces in $\mathbb{R}^4$ with CMC; Cheng and Yau [8] stated the hypfaces with constant scalar curvature; Dillen et al. [9] studied the rot-hypfaces in product space forms; Do Carmo and Dajczer [10] considered the rot-hypfaces in spaces of constant curvature.

Ferrandez et al. [11] worked some class of conformally Euclidean hypfaces; Ganchev and Milousheva [12] introduced the general rot-faces in $\mathbb{E}_1^4$; Güler [13] considered the helical hypfaces in $\mathbb{E}_1^4$; Güler [14] worked on the fundamental form IV with the curvatures of the hypersphere; Güler [15] obtained the rot-hypfaces holding $\Delta^I R = AR$ in $\mathbb{E}^4$; Güler et al. [16] presented the Gauss map, the operator of the third Laplace-Beltrami (LBo) of the rot-hypfaces in $\mathbb{E}^4$; Güler et al. [17] examined LBo of a helicoidal hypface in $\mathbb{E}^4$.

Hasanis and Vlachos [18] worked hypfaces in $\mathbb{E}^4$ with harmonic H vector field; Lawson [19] served the minimal submanifolds; Magid et al. [20] revealed the affine

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_1

umbilical faces in $\mathbb{R}^4$; Moore [21] focused the rot-faces in $\mathbb{E}^4$; Moore [22] also gave the rot-faces of constant curvature in $\mathbb{E}^4$; O'Neill [23] served the semi-Riemannian geometry; Takahashi [24] stated the only faces in $\mathbb{E}^3$ holding $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$ are minimal faces and spheres.

We introduce the maximal rotational hypersurfaces having spacelike axis, spacelike profile curve in Minkowski 4-space $\mathbb{E}_1^4$ in this chapter. We give the notions of $\mathbb{E}_1^4$ such as Lorentz metric, triple vector product, spacelike vector, Lorentzian inner product, spacelike hypersurface, fundamental forms I and II, Gauss map e, shape operator matrix $\mathbf{S}$, curvatures $\mathfrak{C}_{i=1,2,3}$, rotational hypersurface, maximal hypersurface, profile curve γ, axis ℓ, and Laplace-Beltrami operator Δ. We consider a spacelike vector with semi-orthogonal matrix. Giving a spacelike profile curve, we construct the rotational hypersurface having spacelike axis, spacelike profile curve. Henceforth, we serve the definition of a rotational hypersurface having spacelike axis, spacelike profile curve. Then, we compute the first and second fundamental forms, obtain the determinants of fundamental forms I and II, and reveal the shape operator matrix $\mathbf{S}$ and the curvature $\mathfrak{C}_i$ of the rotational hypersurface $\mathbf{x}$ in Minkowski 4-space $\mathbb{E}_1^4$. Moreover, we calculate the curvatures $\mathfrak{C}_{i=1,2,3}$ of the rotational hypersurface. Next, we present the results for the maximality conditions of the rotational hypersurface $\mathbf{x}$. In these findings, we solve the differential equations and reveal the general solution functions on the profile curve. Additionally, we define the Laplace-Beltrami operator of any hypersurface in $\mathbb{E}_1^4$. Hence, we consider the rotational hypersurface x having spacelike axis, spacelike profile curve holding $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A}$ is a 4×4 matrix in $\mathbb{E}_1^4$. Then, we indicate the operator of the Laplace$-$Beltrami of rotational hypersurface $\mathbf{x}$ having spacelike axis, spacelike profile curve is equal to zero. This means hypersurface is maximal in $\mathbb{E}_1^4$. Finally, we give a theorem indicating the maximality condition of a rotational hypersurface having spacelike axis, spacelike profile curve.

Briefly, we describe the notions of $\mathbb{E}_1^4$ in Sect. 1.2. We introduce the definition of a rot-hypface having spacelike axis, spacelike profile curve and compute the fundamental forms I and II, the shape operator matrix $\mathbf{S}$, and the curvature $\mathfrak{C}_i$ of the hypface $\mathbf{x} = \mathbf{x}(u, v, w)$ in Minkowski 4-space $\mathbb{E}_1^4$ in Sect. 1.3. In addition, we consider the rot-hypface $\mathbf{x}$ having spacelike axis supplying $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A}$ is a 4×4 matrix in $\mathbb{E}_1^4$ in Sect. 1.4. Finally, we give a conclusion in the last section.

1.2 Preliminaries

We assign a vector (k, l, m, n) with transpose of it. We determine the first and second fundamental forms, the shape operator matrix $\mathbf{S}$, and the curvature $\mathfrak{C}_i$ of the hypface $\mathbf{x} = \mathbf{x}(u, v, w)$ in Minkowski 4-space $\mathbb{E}_1^4$.

Let us see some notions of $\mathbb{E}_1^4$.

Definition 1.1 Let $\mathbf{x}$ be an immersion of a hypface from M_1^3 to $\mathbb{E}_1^4 = (\mathbb{R}^4, ds^2)$. A Lorentz metric is given by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where x_i are the pseudo-Euclidean coordinates.

Definition 1.2 Let $\overrightarrow{a} = (a_1, a_2, a_3, a_4)$, $\overrightarrow{b} = (b_1, b_2, b_3, b_4)$, and $\overrightarrow{c} = (c_1, c_2, c_3, c_4)$ be the vectors of $\mathbb{E}_1^4$. The vectorial product of is represented by

$$\overrightarrow{a} \times \overrightarrow{b} \times \overrightarrow{c} = \det \begin{pmatrix} e_1 & e_2 & e_3 & -e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix},$$

where e_i are the standard base of $\mathbb{E}_1^4$.

Definition 1.3 A vector $\overrightarrow{a}$ is named spacelike if $\overrightarrow{a} \cdot \overrightarrow{a} > 0$. Here, "$\cdot$" is the Lorentz dot product, i.e., $\overrightarrow{a} \cdot \overrightarrow{a} = a_1^2 + a_2^2 + a_3^2 - a_4^2$.

Definition 1.4 In $\mathbb{E}_1^4$, the hypface $\mathbf{x}$ has the following first and second fundamental form matrices:

$$I = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \quad II = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}, \tag{1.1}$$

and

$$\det I = (EG - F^2)C - B^2 E + 2AFB - A^2 G,$$
$$\det II = \left(NL - M^2\right) V - LT^2 + 2PTM - NP^2.$$

Here,

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u, \; F = \mathbf{x}_u \cdot \mathbf{x}_v, \; G = \mathbf{x}_v \cdot \mathbf{x}_v, \\ A &= \mathbf{x}_u \cdot \mathbf{x}_w, \; B = \mathbf{x}_v \cdot \mathbf{x}_w, \; C = \mathbf{x}_w \cdot \mathbf{x}_w, \\ L &= \mathbf{x}_{uu} \cdot e, \; M = \mathbf{x}_{uv} \cdot e, \; N = \mathbf{x}_{vv} \cdot e, \\ P &= \mathbf{x}_{uw} \cdot e, \; T = \mathbf{x}_{vw} \cdot e, \; V = \mathbf{x}_{ww} \cdot e, \end{aligned}$$

e is the Gauss map given by

$$e = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|}. \tag{1.2}$$

Definition 1.5 A hypface $\mathbf{x}$ is called the spacelike hypface if $\det I > 0$.

Definition 1.6 The shape operator matrix $\mathbf{S} = I^{-1} \cdot II$ is given by

$$\mathbf{S} = \frac{1}{\det I} \left(s_{ij} \right),$$

where

$$\begin{aligned}
s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2 L, \\
s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2 M, \\
s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2 P, \\
s_{21} &= ABL - CFL + AFP - BPE + CME - A^2 M, \\
s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2 N, \\
s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2 T, \\
s_{31} &= -AGL + BFL + AFM - BME + PEG - PF^2, \\
s_{32} &= -MAG + MBF + FAN - BNE + TEG - TF^2, \\
s_{33} &= -PAG + PBF + FTA - BTE + VEG - VF^2.
\end{aligned}$$

On the other side, we give the following:

Theorem 1.1 *In $\mathbb{E}^4_1$, the hypface* $\mathbf{x}$ *has the following formulas to its curvatures,* $\mathfrak{C}_0 = 1$ *(by definition):*

$$\mathfrak{C}_1 = \frac{\left\{ \begin{array}{c} (EN + LG - 2MF)C + (GE - F^2)V - LB^2 - NA^2 \\ -2(PGA - PFB - TFA + BTE - ABM) \end{array} \right\}}{3\left((GE - F^2)C - EB^2 + 2FAB - GA^2\right)}, \tag{1.3}$$

$$\mathfrak{C}_2 = \frac{\left\{ \begin{array}{c} (EN + GL - 2FM)\, V + \left(NL - M^2\right) C - ET^2 - P^2 G \\ -2\,(PNA - PMB - ATM + BTL - PTF) \end{array} \right\}}{3\left((EG - F^2)C - EB^2 + 2FAB - GA^2\right)}, \tag{1.4}$$

$$\mathfrak{C}_3 = \frac{\left(LN - M^2\right) V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}. \tag{1.5}$$

Here, $\mathfrak{C}_1 = H$ *and* $\mathfrak{C}_3 = K$, *i.e., mean curvature and Gaussian curvature, respectively.*

Definition 1.7 A hypface $\mathbf{x}$ is called i-maximal, when $\mathfrak{C}_i = 0$, where $i = 1, 2, 3$.

See [14] for Euclidean details. Next, we define the rot-hypface in $\mathbb{E}^4_1$.

Definition 1.8 Let γ be a curve from $\mathrm{I} \subset \mathbb{R}$ to plane Π. A rot-hypface is defined as a hypface with generating, i.e., profile curve γ rotating about a line, i.e., axis ℓ.

Therefore, we introduce the rot-hypfaces having spacelike axis, spacelike profile curve in $\mathbb{E}_1^4$ in the following section.

1.3 Spacelike Rotational Hypersurfaces Having Spacelike Axis, Spacelike Profile Curve

We may suppose ℓ generated by spacelike axis $(1, 0, 0, 0)$. Hence, the semi-orthogonal matrix is described by

$$\mathbf{O}(v, w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sinh w & 0 & \cosh w \\ 0 & \sinh v \cosh w & \cosh v & \sinh v \sinh w \\ 0 & \cosh v \cosh w & \sinh v & \cosh v \sinh w \end{pmatrix}. \tag{1.6}$$

Here, $v, w \in \mathbb{R}$,, and the following hold

$$\det \mathbf{O} = 1, \quad A.\ell = \ell, \quad \mathbf{O}^t.\varepsilon.\mathbf{O} = \varepsilon,$$

with $\varepsilon = diag(1, 1, 1, -1)$. While the rotation axis be ℓ, there exists a Lorentz transformation, and then ℓ formed to the x_1-axis of $\mathbb{E}_1^4$. Therefore, the profile curve is defined by

$$\gamma(u) = (\varphi(u), f(u), 0, 0).$$

Here, $\gamma' \cdot \gamma' = \varphi'^2 + f'^2 > 0$; the functions $f, \varphi : \mathrm{I} \subset \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable for all u. Since $\gamma' \cdot \gamma' > 0$, the curve γ is spacelike. Hence, the rot-hypface spanned by the vector $(1, 0, 0, 0)$ is obtained by

$$\mathbf{x} = \mathbf{O}.\gamma^t.$$

Then, we present the following clearer form of the rot-hypface having spacelike axis, spacelike profile curve:

$$\mathbf{x}(u, v, w) = \begin{pmatrix} \varphi(u) \\ f(u) \sinh w \\ f(u) \sinh v \cosh w \\ f(u) \cosh v \cosh w \end{pmatrix}. \tag{1.7}$$

Taking the first derivatives of (1.7), we obtain the first objects of $\mathbf{x}$ as follows:

$$I = diag\left(\varphi'^2 - f'^2,\ f^2\cosh^2 w,\ f^2\right), \tag{1.8}$$

where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$. We obtain

$$\det I = (\varphi'^2 - f'^2) f^4 \cosh^2 w.$$

Since $\varphi'^2 + f'^2 > 0$, then $\det I > 0$. So, the hypface $\mathbf{x}$ is a spacelike rot-hypface having spacelike axis, spacelike profile curve. By taking the second differentials of $\mathbf{x}$ which depends on $u, v, w,$, we get the second objects of $\mathbf{x}$

$$II = diag\left(\frac{f''\varphi' - f'\varphi''}{\left|f'^2 - \varphi'^2\right|^{1/2}}, \frac{f\varphi'\cosh^2 w}{\left|f'^2 - \varphi'^2\right|^{1/2}}, \frac{f\varphi'}{\left|f'^2 - \varphi'^2\right|^{1/2}}\right),$$

and

$$\det II = \frac{\varphi'^2\left(f''\varphi' - f'\varphi''\right) f^2\cosh^2 w}{\left|f'^2 - \varphi'^2\right|^{3/2}}.$$

Therefore, the shape operator matrix of the hypface $\mathbf{x}$ is given by

$$\mathbf{S} = diag\left(\frac{f''\varphi' - f'\varphi''}{\left|f'^2 - \varphi'^2\right|^{3/2}}, \frac{\varphi'}{f\left|f'^2 - \varphi'^2\right|^{1/2}}, \frac{\varphi'}{f\left|f'^2 - \varphi'^2\right|^{1/2}}\right).$$

We compute the curvatures of the rot-hypface $\mathbf{x}$ having spacelike axis, spacelike profile curve, and state the following:

Proposition 1.1 *The rot-hypfaces having spacelike axis, spacelike profile curve (1.7) has the following curvatures:*

$$\mathfrak{C}_1 = \frac{ff'\varphi'' - 2\varphi'^3 + \left(2f'^2 - ff''\right)\varphi'}{3f\left|f'^2 - \varphi'^2\right|^{3/2}},$$

$$\mathfrak{C}_2 = \frac{2ff'\varphi'\varphi'' + \left(-\varphi'^2 - 2ff'' + f'^2\right)\varphi'^2}{3f^2\left|f'^2 - \varphi'^2\right|^2},$$

$$\mathfrak{C}_3 = \frac{\left(f\varphi'' - f''\varphi'\right)\varphi'^2}{f^2\left|f'^2 - \varphi'^2\right|^{5/2}}.$$

Proof By using Eqs. (1.3)–(1.5) for the rot-hypface having spacelike axis (1.7), we get the curvatures. □

In the end, we give the following maximality conditions of $\mathbf{x}$:

Corollary 1.1 *The rot-hypface* (1.7) *is* 1*-maximal iff it has the following general* φ *solutions:*

$$\varphi = \mp \frac{i.EllipticF\left[i \arg\sinh\left(\left(i.c^{1/2}f\right)^{1/2}\right), -1\right]}{\left(i.c^{1/2}\right)^{1/2}} + d.$$

Here, $EllipticF\,[\Psi, t]$ *denotes the elliptic integral of the first type for* $-\pi/2 < \Psi < \pi/2$, $F\,[\Psi, t] = \int_0^{\Psi} \left(1 - t\sin^2\kappa\right)^{-1/2} d\kappa$, *and* $i = (-1)^{1/2}$, $c, d \in \mathbb{R}$.

Corollary 1.2 *The rot-hypface* (1.7) *is* 2*-maximal iff it has the following general* φ *solutions, respectively:*

$$\varphi = \mp 2c^{-1}\,(cf + 1)^{1/2} + d \text{ or } \varphi = c,$$

where $c, d \in \mathbb{R}$.

Corollary 1.3 *The rot-hypface* (1.7) *is* 3*-maximal iff it has the following general* φ *solutions, respectively:*

$$\varphi = cf + d \text{ or } \varphi = c,$$

where $c, d \in \mathbb{R}$.

1.4 Maximal Rotational Hypersurface Having Spacelike Axis, Spacelike Profile Curve Satisfying $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$

Leu us see the LBo of any hypface and some relations of it.

Definition 1.9 *The* LBo *of the hypface* $\mathbf{x} = \mathbf{x}\,(u, v, w)\,|_{D\subset\mathbb{R}^4}$ *of class* C^3 *is given by*

$$\Delta\mathbf{x} = -\frac{1}{(\det I)^{1/2}} \left\{ \begin{array}{l} \frac{\partial}{\partial u}\left(\frac{\left(CG-F^2\right)\mathbf{x}_u - (AB-CF)\mathbf{x}_v + (BF-AG)\mathbf{x}_w}{(\det I)^{1/2}}\right) \\ -\frac{\partial}{\partial v}\left(\frac{(AB-CF)\mathbf{x}_u - \left(CE-A^2\right)\mathbf{x}_v + (AF-BG)\mathbf{x}_w}{(\det I)^{1/2}}\right) \\ +\frac{\partial}{\partial w}\left(\frac{(BF-AG)\mathbf{x}_u - (AF-BG)\mathbf{x}_v + \left(EG-F^2\right)\mathbf{x}_w}{(\det I)^{1/2}}\right) \end{array} \right\}. \tag{1.9}$$

By using above definition on (1.7), we have the following LBo:

$$\Delta \mathbf{x} = -\frac{ff'\varphi'' - 2\varphi'^3 + \left(2f'^2 - ff''\right)\varphi'}{f\,(W)^2} \begin{pmatrix} f' \\ \varphi' \sinh w \\ \varphi' \sinh v \cosh w \\ \varphi' \cosh v \cosh w \end{pmatrix}.$$

Here, $W = \left|f'^2 - \varphi'^2\right|$. The Gauss map of the rot-hypface having spacelike axis, spacelike profile curve (1.7) is given by

$$e = -\frac{1}{W^{1/2}} \begin{pmatrix} f' \\ \varphi' \sinh w \\ \varphi' \sinh v \cosh w \\ \varphi' \cosh v \cosh w \end{pmatrix}.$$

By using $-3\mathfrak{C}_1 e = \mathcal{A}\mathbf{x}$, we obtain

$$\begin{pmatrix} \Phi \\ -\left(\Phi\varphi' + fa_{22}\right)\sinh w - fa_{23}\sinh v \cosh w - fa_{24}\cosh v \cosh w \\ -fa_{32}\sinh w - \left(\Phi\varphi' + fa_{33}\right)\sinh v \cosh w - fa_{34}\cosh v \cosh w \\ -fa_{42}\sinh w - fa_{43}\sinh v \cosh w - \left(\Phi\varphi' + fa_{44}\right)\cosh v \cosh w \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\varphi + a_{12}f\sinh w + a_{13}f\sinh v \cosh w + a_{14}f\cosh v \cosh w \\ a_{21}\varphi \\ a_{31}\varphi \\ a_{41}\varphi \end{pmatrix}.$$

Here, $\mathcal{A} \in mat(4,4)$, and also $\Phi(u) = \frac{3\mathfrak{C}_1}{W^{1/2}}$. Equations $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$ with respect to the first quantity I, and $\Delta\mathbf{x} = -3\mathfrak{C}_1 e$ give rise to the following system:

$$\begin{aligned} a_{11}\varphi + a_{12}f\sinh w + a_{13}f\sinh v\cosh w + a_{14}f\cosh v\cosh w &= \Phi, \\ -\left(\Phi\varphi' + a_{22}f\right)\sinh w - a_{23}f\sinh v\cosh w - a_{24}f\cosh v\cosh w &= a_{21}\varphi, \\ -a_{32}f\sinh w - \left(\Phi\varphi' + a_{33}f\right)\sinh v\cosh w - a_{34}f\cosh v\cosh w &= a_{31}\varphi, \\ -a_{42}f\sinh w - a_{43}f\sinh v\cosh w - \left(\Phi\varphi' + a_{44}f\right)\cosh v\cosh w &= a_{41}\varphi. \end{aligned}$$

Differentiating above ODE's two times depending on v, we have

$$a_{41} = a_{31} = a_{21} = a_{11} = 0, \quad \Phi = 0. \tag{1.10}$$

Then, from (1.10), we obtain the following:

$$\begin{aligned} a_{13} f \sinh v + a_{14} f \cosh v &= 0, \\ -a_{23} f \sinh v - a_{24} f \cosh v &= 0, \\ -a_{33} f \sinh v - a_{34} f \cosh v &= 0, \\ -a_{43} f \sinh v - a_{44} f \cosh v &= 0. \end{aligned}$$

Here, the functions $f \neq 0$, cosh, and sinh are the linear independent of v. Then, we obtain $a_{ij} = 0$. While $\Phi = \frac{3\mathfrak{C}_1}{W^{1/2}}$, we find $\mathfrak{C}_1 = 0$. Hence, $\mathbf{x}$ is a maximal hypface having spacelike axis.

In the end, we reveal the following theorem.

Theorem 1.2 *Let spacelike* $\mathbf{x}$*:* $M_1^3 \longrightarrow \mathbb{E}_1^4$ *be an immersion given by* (1.7). $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$, *where* $\mathcal{A} \in mat(4,4)$ *matrix iff* $\mathbf{x}$ *is a* 1*-maximal hypface having spacelike axis, spacelike profile curve, i.e.,* $\mathfrak{C}_1 = 0$.

1.5 Conclusion

In this research, we examine the rotational hypersurface $\mathbf{x} = \mathbf{x}(u, v, w)$ with spacelike axis, spacelike profile curve and calculate its fundamental forms I and II, the shape operator matrix $\mathbf{S}$, and the curvature $\mathfrak{C}_i$ in Minkowski 4-space $\mathbb{E}_1^4$. Additionally, we reveal that the rotational hypersurface $\mathbf{x}$ has $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A}$ is a 4×4 matrix in $\mathbb{E}_1^4$. The topic can be extended for the other space forms.

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Chapter 2
On the Generalized k-Horadam-Like Sequences

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MSC: 11B37, 11B39, 11B83.

2.1 Introduction

Horadam sequence is a special second-order sequence like Fibonacci, Pell, Lucas, etc. It is usually denoted with four tuples $H_n(a, b; p, q)$ and defined recursively as

$$H_n = pH_{n-1} + qH_{n-2} \quad \text{with} \quad H_0 = a \quad \text{and} \quad H_1 = b. \tag{2.1}$$

In particular for $p = q = 1$ and $a = 0, b = 1$, Eq. (2.1) gives the standard Fibonacci sequence where as for $p = q = 1$ and $a = 2, b = 1$ gives standard Lucas sequence. Thus, $H_n(0, 1; 1, 1)$ represents Fibonacci sequence, while $H_n(2, 1; 1, 1)$ is Lucas sequence. Recursive integer sequences like Fibonacci, Lucas, and Pell and their generalizations have many applications to various branches of science [5]. Frontczak [3] gave Horadam identities with binomial coefficients and many other known identities.

In the direction of the construction of polynomials from integer sequences, Horzum et al. [4] introduced a new generalization of the second-order Horadam polynomial sequences and investigated some identities and its properties. Recently, TD Şentürk et al. in his study of Horadam hybrid numbers [15] established many of identities and properties like Binet formula, generating function, Catalan identity, etc., whereas in [14], G.Y Şentürk introduced the extended Horadam numbers by using dual-generalized complex, hyperbolic-generalized complex, and complex-generalized complex numbers, proved some well-known identities and formulas for it, and also investigated some properties for its special matrix representations. Munesh et al. [6] have proposed some new families of identities of k-Mersenne and generalized k-Gaussian Mersenne numbers and their polynomials, which consist of

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_2

many well-known identities. Further, the construction of recursive matrices from recursive sequences like Fibonacci and Lucas and their application in cryptography has been demonstrated by Kalika et al. in [10, 11], where it has been observed that recursive matrices play an important role in the formation of keyspace for cryptosystem.

2.1.1 *The Catalan Transform*

The Catalan numbers (OEIS A000108) are given by $c_n = \frac{1}{1+n}\binom{2n}{n}$, equivalently $\frac{2n!}{(1+n)!(n)!}$. For $n = 1, 2, 3, ...,$ the first few Catalan numbers are $\{1, 1, 2, 5, 14, 42, 132, 429, ...\}$. The Catalan transform $\{C_n\}$ of sequence $\{a_n\}_{n\geq 0}$ is a sequence transform given by Barry [1] as follows:

$$C_n = \sum_{r=0}^{n} \frac{r}{2n-r}\binom{2n-r}{n-r} a_r, \ n \geq 1 \ \text{ with } \ C_0 = 0. \tag{2.2}$$

Some recent work on Catalan transform and associated Hankel transform of k-Fibonacci, k-Jacobsthal, k-Lucas, k-Pell, etc. sequences can be seen in [2, 8, 9, 16, 17].

2.2 Generalized k-Horadam Sequences

In this section, we define a new generalized third-order recursive sequence $\{H_{k,n}\}_{k\in\mathbb{N}}$ as follows. Also, we establish some well-known identities for this generalization.

Definition 2.1 For $k \in \mathbb{R}^+$, let $f(k), g(k)$, and $h(k)$ be scalar-valued polynomials. Then for $n \geq 0$, the generalized k-Horadam sequence $\{H_{k,n}\}$ is given by

$$H_{k,n+3} = f(k)H_{k,n+2} + g(k)H_{k,n+1} + h(k)H_{k,n} \tag{2.3}$$

with initial values $H_{k,0} = a$, $H_{k,1} = b$, and $H_{k,2} = c$.

Remark 2.1 For $h(k) = 0$, Eq. (2.3) becomes linear difference equation of second order which is given by

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n} \text{ with initial values } H_{k,0} = a,\ H_{k,1} = b. \quad (2.4)$$

And for this, we use the notation $H_{k,n}(a, b;\ f, g)$.

The first few terms of the sequence are

$$H_{k,0} = a,\ H_{k,1} = b,\ H_{k,2} = ag(k)+bf(k),\ H_{k,3} = af(k)g(k)+b[f^2(k)+g(k)],$$

$$H_{k,4} = a[f^2(k)g(k) + g^2(k)] + b[f^3(k) + 2f(k)g(k)], ...$$

As a special case of Eq. (2.3), when $h(k)$ =0, we have listed definitions of some well-known integer sequences in the next subsection.

2.2.1 Generalized k-Horadam Sequence of Second Order (When $h(k) = 0$)

When $h(k) = 0$, the characteristic equation becomes $x^2 - f(k)x - g(k) = 0$, and its characteristic roots, r_1 and r_2, clearly hold the following relations:

$$r_1 + r_2 = f(k),\ r_1 - r_2 = \sqrt{f^2(k) + 4g(k)}, \quad r_1r_2 = -g(k). \quad (2.5)$$

Definition 2.2 ($H_{k,n}(a, b;\ f, g)$) For particular values of $f, g, a, and\ b$, the following definitions can be achieved:

1. The generalized k-Fibonacci and k-Lucas sequence is achieved if $f(k) = k$ and $g(k) = 1$, i.e., $H_{k,n+2} = kH_{k,n+1} + H_{k,n}$, with $H_{k,0} = a\ and\ H_{k,1} = b$.
2. The k-Fibonacci sequence is obtained if $f(k) = k$, $g(k) = 1$, and $a = 0, b = 1$, i.e., $H_{k,n+2} = kH_{k,n+1} + H_{k,n}$, with $H_{k,0} = 0\ and\ H_{k,1} = 1$.
3. The k-Lucas sequence is obtained if $f(k) = k$, $g(k) = 1$, and $a = 2, b = 1$, i.e., $H_{k,n+2} = kH_{k,n+1} + H_{k,n}$, with $H_{k,0} = 2\ and\ H_{k,1} = 1$.
4. The Horadam sequence is obtained if $f(k) = p$ and $g(k) = q$, i.e., $H_{k,n+2} = pH_{k,n+1} + qH_{k,n}$, with $H_{k,0} = a\ and\ H_{k,1} = b$.
5. The Fibonacci and the Lucas sequence is obtained if $f(k) = 1$ and $g(k) = 1$, i.e., $H_{k,n+2} = H_{k,n+1} + H_{k,n}$, with $H_{k,0} = 0,\ and\ H_{k,1} = 1$ and $H_{k,0} = 2,\ and\ H_{k,1} = 1$, respectively.
6. The sequence Pell and Pell-Lucas is obtained if $f(k) = 2$ and $g(k) = 1$, i.e., $H_{k,n+2} = 2H_{k,n+1} + H_{k,n}$, with $H_{k,0} = 0\ and\ H_{k,1} = 1$ and $H_{k,0} = 2 = H_{k,1}$, respectively.

7. The sequence Jacobsthal and Jacobsthal-Lucas is obtained if $f(k) = 1$ and $g(k) = 2$, i.e., $H_{k,n+2} = H_{k,n+1} + 2H_{k,n}$, $H_{k,0} = 0$ *and* $H_{k,1} = 1$, and $H_{k,0} = 2$, *and* $H_{k,1} = 1$, respectively.
8. The sequence balancing and balancing Lucas [13] is obtained if $f(k) = 6k$ and $g(k) = -1$, i.e., $H_{k,n+2} = 6kH_{k,n+1} - H_{k,n}$, $H_{k,0} = 0$ *and* $H_{k,1} = 1$, and $H_{k,0} = 1$, *and* $H_{k,1} = 3$, respectively.

On the basis of k-Fibonacci, Yasin Yazlik and Necati Taskara [18] defined the generalized k-Horadam sequences $\{H_{k,n}\}_{n\in\mathbb{N}}$, studied their properties, and also obtained the Binet formula and a generating function in terms of $f(k)$ and $g(k)$. Here, Horadam sequence [19] is a special second-order sequence like Fibonacci, Pell, Lucas, etc.

It is assumed that $k \neq 0$ and $f^2(k) + 4g(k) > 0$ throughout the section.

Binet's formulas for integer sequences are useful tools to derive many well-known identities. They are given in the following results:

Theorem 2.1 (Binet Formula [18]) *For every positive integer n, we have* $H_{k,n} = \dfrac{Pr_1^n - Qr_2^n}{r_1 - r_2}$ *where* $P = b - ar_2$ *and* $Q = b - ar_1$.

The next proposition gives the generalized k-Horadam numbers for the negative subscript.

Proposition 2.2 *For* $n \geq 1$, *we have* $H_{k,-n} = (-1)^n \dfrac{Pr_2^n - Qr_1^n}{(g(k))^n(r_1 - r_2)}$ *where* $P = b - ar_2$ *and* $Q = b - ar_1$. □

Proof Replacing n by $-n$ in the Binet's formula (2.1), we get

$$H_{k,-n} = \frac{\frac{P}{r_1^n} - \frac{Q}{r_2^n}}{r_1 - r_2} = \frac{Pr_2^n - Qr_1^n}{(r_1r_2)^n(r_1 - r_2)} = \frac{Pr_2^n - Qr_1^n}{(-g(k))^n(r_1 - r_2)} = (-1)^n \frac{Pr_2^n - Qr_1^n}{(g(k))^n(r_1 - r_2)}.$$

Thus, this completes the proof. □

In order to establish some well-known identities, we first need to prove the following theorem:

Theorem 2.3 *For positive integer n, let* $X^{(n)}$ *and* $Y^{(n)}$ *be a sequence of matrices given by* $X^{(n)} = \begin{bmatrix} H_{k,n+1} & H_{k,n} \\ H_{k,n} & H_{k,n-1} \end{bmatrix}$ *and* $Y^{(n)} = \begin{bmatrix} H_{k,-n+1} & H_{k,-n} \\ H_{k,-n} & H_{k,-n-1} \end{bmatrix}$ *whose each entries are the generalized k-Horadam numbers. Then, we have*

$$|X^{(n)}| = (-g(k))^{n-1}(a^2g(k) + abf(k) - b^2) \tag{2.6}$$

$$\text{and} \quad |Y^{(n)}| = \frac{1}{(-g(k))^{n+1}}(a^2g(k) + abf(k) - b^2). \tag{2.7}$$

Proof For $n \geq 1$, from Theorem (2.1), we have

$$
\begin{aligned}
|X^{(n)}| &= H_{k,n+1}H_{k,n-1} - H_{k,n}^2 \\
&= \frac{Pr_1^{n+1} - Qr_2^{n+1}}{r_1 - r_2}\frac{Pr_1^{n-1} - Qr_2^{n-1}}{r_1 - r_2} - \left(\frac{Pr_1^n - Qr_2^n}{r_1 - r_2}\right)^2 \\
&= \frac{1}{(r_1 - r_2)^2}\left[PQr_1^{n-1}r_2^{n-1}(2r_1r_2 - r_1^2 - r_2^2)\right] \\
&= \frac{1}{(r_1 - r_2)^2}\left[PQ(-g(k))^{n-1}(-(r_1 - r_2)^2)\right] \\
&= -PQ(-g(k))^{n-1} \\
&= -(-g(k))^{n-1}(b^2 - ab(r_1 + r_2) + a^2r_1r_2) \\
&= (-g(k))^{n-1}(a^2g(k) + abf(k) - b^2)
\end{aligned}
$$

and

$$
\begin{aligned}
|Y^{(n)}| &= H_{k,-n+1}H_{k,-n-1} - H_{k,-n}^2 \\
&= \frac{Pr_2^{n-1} - Qr_1^{n-1}}{(r_1r_2)^{n-1}(r_1 - r_2)}\frac{Pr_2^{n+1} - Qr_1^{n+1}}{(r_1r_2)^{n+1}(r_1 - r_2)} - \left(\frac{Pr_2^n - Qr_1^n}{(r_1r_2)^n(r_1 - r_2)}\right)^2 \\
&= \frac{1}{(r_1r_2)^{2n}(r_1 - r_2)^2}\left[PQr_1^{n-1}r_2^{n-1}(2r_1r_2 - r_1^2 - r_2^2)\right] \\
&= \frac{1}{(r_1r_2)^{2n}(r_1 - r_2)^2}\left[PQ((r_1r_2))^{n-1}(-(r_1 - r_2)^2)\right] \\
&= \frac{-PQ}{(r_1r_2)^{n+1}} \\
&= \frac{-(b^2 - ab(r_1 + r_2) + a^2r_1r_2)}{(-g(k))^{n+1}} \\
&= \frac{1}{(-g(k))^{n+1}}(a^2g(k) + abf(k) - b^2).
\end{aligned}
$$

Thus, this completes the proof. □

Hence, from the above theorem, we can conclude that the Cassini's identity for the generalized k-Horadam sequence is given by

$$
H_{k,n+1}H_{k,n-1} - H_{k,n}^2 = (-g(k))^{n-1}(a^2g(k) + abf(k) - b^2). \tag{2.8}
$$

Here, by using particular values of $f, g, a, and\ b$, we can obtain the known Cassini's identity for integer sequences which are given in Definition (2.2).

Further, to establish the Catalan's identity, we need to prove the following lemma and theorem:

Lemma 2.1 *For $n \in \mathbb{N}$, we have*

$$r_1^m - r_2^m = \frac{(r_1 - r_2)}{PQ}(bH_{k,m} - aH_{k,m+1}).$$

Proof Using values of P and Q from Theorem (2.1), we write

$$\begin{aligned}
PQ(r_1^m - r_2^m) &= PQr_1^m - PQr_2^m \\
&= (b - ar_1)Pr_1^m - (b - ar_2)Qr_2^m \\
&= b(Pr_1^m - Qr_2^m) - a(Pr_1^m - Qr_2^m) \\
&= b[(r_1 - r_2)H_{k,m}] - a[(r_1 - r_2)H_{k,m+1}] \quad \text{using Theorem (2.1)} \\
&= (r_1 - r_2)[bH_{k,m} - aH_{k,m+1}]
\end{aligned}$$

which gives

$$r_1^m - r_2^m = \frac{(r_1 - r_2)}{PQ}(bH_{k,m} - aH_{k,m+1}).$$

□

Theorem 2.4 *Let $n \in \mathbb{N}$ and each entry of matrix $Z^{(n)} = \begin{bmatrix} H_{k,n+m} & H_{k,n} \\ H_{k,n} & H_{k,n-m} \end{bmatrix}$ be the terms of generalized k-Horadam sequence. Then, we have*

$$|Z^{(n)}| = \frac{(-g(k))^{n-m}}{a^2 g(k) + abf(k) - b^2}\left[bH_{k,m} - aH_{k,m+1}\right]^2.$$

Proof For natural number n, using Binet's formula (2.1), we write

$$\begin{aligned}
|Z^{(n)}| &= H_{k,n+m}H_{k,n-m} - H_{k,n}^2 \\
&= \frac{Pr_1^{n+m} - Qr_2^{n+m}}{r_1 - r_2}\frac{Pr_1^{n-m} - Qr_2^{n-m}}{r_1 - r_2} - \left(\frac{Pr_1^n - Qr_2^n}{r_1 - r_2}\right)^2 \\
&= \frac{1}{(r_1 - r_2)^2}[(Pr_1^{n+m} - Qr_2^{n+m})(Pr_1^{n-m} - Qr_2^{n-m}) - (Pr_1^n - Qr_2^n)^2] \\
&= \frac{1}{(r_1 - r_2)^2}\left[PQ(2r_1^n r_2^n - r_1^{n-m}r_2^{n+m} - r_1^{n+m}r_2^{n-m})\right] \\
&= \frac{PQ}{(r_1 - r_2)^2}\left[r_1^n r_2^n(2 - r_1^{-m}r_2^m - r_1^m r_2^{-m})\right]
\end{aligned}$$

$$= \frac{r_1^n r_2^n PQ}{(r_1 - r_2)^2}\left[2 - \left(\frac{r_1}{r_2}\right)^m - \left(\frac{r_2}{r_1}\right)^m\right]$$
$$= \frac{(r_1 r_2)^n PQ}{(r_1 - r_2)^2}\left[\frac{2(r_1 r_2)^m - (r_1^{2m} + r_2^{2m})}{(r_1 r_2)^m}\right]$$
$$= \frac{(-1)(r_1 r_2)^{n-m}}{(r_1 - r_2)^2} PQ\left[(r_1^m - r_2^m)^2\right].$$

Thus, from Lemma (2.1), we have

$$= \frac{(-1)(r_1 r_2)^{n-m}}{(r_1 - r_2)^2} PQ\left[\frac{(r_1 - r_2)}{PQ}(bH_{k,m} - aH_{k,m+1})\right]^2$$
$$= \frac{(-1)(r_1 r_2)^{n-m}}{PQ}\left[bH_{k,m} - aH_{k,m+1}\right]^2$$
$$= \frac{(-1)(-g(k))^{n-m}}{PQ}\left[bH_{k,m} - aH_{k,m+1}\right]^2$$
$$= \frac{(-g(k))^{n-m}}{a^2 g(k) + abf(k) - b^2}\left[bH_{k,m} - aH_{k,m+1}\right]^2.$$

This completes the proof. □

Thus, for generalized k-Horadam sequences, the Catalan's identity for the first assumption is given by

$$H_{k,n+m}H_{k,n-m} - H_{k,n}^2 = \frac{(-g(k))^{n-m}}{a^2 g(k) + abf(k) - b^2}\left[bH_{k,m} - aH_{k,m+1}\right]^2.$$

2.2.2 Catalan Transformation and Generating Function

Now, we aim to obtain the Catalan transform [17] of the generalized k-Horadam sequence. Furthermore, we deduce the generating function of the Catalan transform of the generalized k-Horadam sequence and denote it by $ch_k(x)$. Following Eq. (2.2), the Catalan transform of the generalized k-Horadam sequence $\{H_{k,n}\}_{n\geq 0}$ is given by

$$CH_{k,1} = \sum_{r=0}^{1} \frac{r}{2-r}\binom{2-r}{1-r} H_{k,r} = 0.H_{k,0} + 1.H_{k,1} = H_{k,1} = b,$$
$$CH_{k,2} = \sum_{r=0}^{2} \frac{r}{4-r}\binom{4-r}{2-r} H_{k,r} = \frac{1}{3}\binom{3}{1} H_{k,1} + \frac{2}{2}\binom{4}{0} H_{k,2} = H_{k,1} + H_{k,2}$$

$$= ag(k) + b(1 + f(k)),$$

$$CH_{k,3} = \frac{1}{5}\binom{5}{2}H_{k,1} + \frac{2}{4}\binom{4}{1}H_{k,2} + \frac{3}{3}\binom{3}{0}H_{k,3} = 2H_{k,1} + 2H_{k,2} + H_{k,3},$$

$$= a[2g(k) + f(k)g(k)] + b[2 + f(k) + f^2(k) + g(k)],$$

Similarly,

$$\begin{aligned} CH_{k,4} &= 5H_{k,1} + 5H_{k,2} + 3H_{k,3} + H_{k,4}, \\ &= a[5g(k) + 3f(k)g(k) + f^2(k)g(k) + g^2(k)] + b[5 + 5f(k) \\ &\quad +3f^2(k) + 3g(k) + f^3(k) + 2f(k)g(k)] \\ CH_{k,5} &= 14H_{k,1} + 14H_{k,2} + 9H_{k,3} + 4H_{k,4} + H_{k,5}, ...etc. \end{aligned}$$

In matrix form, the above system of equations can be represented as $C^T = AG^T$, where $C = \{CH_{k,1}, CH_{k,2}, CH_{k,3}, ...\}$ and $G = \{H_{k,1}, H_{k,2}, H_{k,3}, ...\}$ are row vectors and A is lower triangular matrix, as follows:

$$\begin{bmatrix} CH_{k,1} \\ CH_{k,2} \\ CH_{k,3} \\ CH_{k,4} \\ CH_{k,5} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 5 & 5 & 3 & 1 & 0 & 0 & \ldots \\ 14 & 14 & 9 & 4 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix} \begin{bmatrix} H_{k,1} \\ H_{k,2} \\ H_{k,3} \\ H_{k,4} \\ H_{k,5} \\ \vdots \end{bmatrix}. \tag{2.9}$$

Here, the entries of matrix A satisfy the relation $a_{i,j} = \sum_{t=j-1}^{i-1} a_{i-1,t}$ and for $i > 1$, the first column equals to the second which are the Catalan numbers.

By virtue of [1], the ordinary generating function $c(x)$ of the Catalan numbers is $c(x) = \frac{1-\sqrt{1-4x}}{2x}$, and it is observed that $A(c(x))$ is the generating function of the Catalan transform of any integer sequence A_n. Since, by virtue of [18], it is clearly observed, the generating function of $H_{k,n}$ is

$$g(x) = \frac{H_{k,0} + x(H_{k,1} - f(k)H_{k,0})}{1 - f(k)x - g(k)x^2}.$$

Thus, the generating function $ch_k(x)$ of the Catalan transform is given by

$$\begin{aligned} ch_k(x) &= g(x.c(x)) \\ &= \frac{H_{k,0} + (x.c(x))(H_{k,1} - f(k)H_{k,0})}{1 - f(k)(x.c(x)) - g(k)(x.c(x))^2} \end{aligned}$$

$$= \frac{H_{k,0} + (x.\frac{1-\sqrt{1-4x}}{2x})(H_{k,1} - f(k)H_{k,0})}{1 - f(k)(x.\frac{1-\sqrt{1-4x}}{2x}) - g(k)(x.\frac{1-\sqrt{1-4x}}{2x})^2}$$

$$= \frac{2(2a + (1 - \sqrt{1-4x}))(b - f(k)a)}{4 - 2f(k)(1 - \sqrt{1-4x}) - g(k)(1 - \sqrt{1-4x})^2}.$$

2.2.3 *The Hankel Transform*

The Hankel transform of a sequence $A = \{h_0, h_1, h_2, h_3, ...\}$ of real numbers is the sequence of Hankel determinants $\{|G_1|, |G_2|, |G_3|, ...\}$ (see [7, 12]) where the Hankel matrix G is defined by an infinite matrix as follows:

$$G = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & \dots \\ h_1 & h_2 & h_3 & h_4 & h_5 & \dots \\ h_2 & h_3 & h_4 & h_5 & h_6 & \dots \\ h_3 & h_4 & h_5 & h_6 & h_7 & \dots \\ h_4 & h_5 & h_6 & h_7 & h_8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \text{with entries } a_{ij} = h_{i+j-2};\ i, j \geq 1.$$

The Hankel matrix G_n of order n is the left-upper submatrix of size $n \times n$ of G, and the corresponding sequence of determinants $(det(G_n) = |G_n|_{n\geq 1})$ produces Hankel transform. For the Catalan sequence, the Hankel transform is the sequence $\{1, 1, 1, ...\}$ [7], and for the sum of consecutive generalized Catalan numbers, the Hankel transform is the bisection of standard Fibonacci sequence [12].

Considering the Catalan transform of the generalized k-Horadam sequences of order 2 of the preceding subsection, we get:

$$GCH_{k,1} = |CH_{k,1}| = b,$$

$$GCH_{k,2} = \begin{vmatrix} CH_{k,1} & CH_{k,2} \\ CH_{k,2} & CH_{k,3} \end{vmatrix} = b^2(1 - f(k) + g(k)) - 2abf(k)g(k) - a^2g^2(k),$$

$$GCH_{k,3} = \begin{vmatrix} CH_{k,1} & CH_{k,2} & CH_{k,3} \\ CH_{k,2} & CH_{k,3} & CH_{k,4} \\ CH_{k,3} & CH_{k,4} & CH_{k,5} \end{vmatrix}.$$

Thus, continuing in this manner, we get the Hankel transform of the Catalan transform of the generalized k-Horadam sequences.

2.3 Generalized k-Horadam Sequences of Order 3 (When $h(k) \neq 0$)

In this section, we present a new family of extended generalized k-Horadam sequences which is achieved by considering $h(k) \neq 0$ in Eq. (2.3), i.e., for nonnegative integer n, we have

$$H_{k,n+3} = f(k)H_{k,n+2} + g(k)H_{k,n+1} + h(k)H_{k,n} \tag{2.10}$$

with initial values $H_{k,0} = a$, $H_{k,1} = b$, and $H_{k,2} = c$.

It is denoted by $H_{k,n}(a, b, c;\ f, g, h)$, and we usually omit the $(a, b, c;\ f, g, h)$ if it does not cause ambiguity. Equation (2.10) is the generalized third-order linear difference equation, and its characteristic equation is given by

$$x^3 = f(k)x^2 + g(k)x + h(k) \text{ equivalently, } x^3 - f(k)x^2 - g(k)x - h(k) = 0. \tag{2.11}$$

Characteristic equation (2.11) has three roots, say $r_1,\ r_2$, and r_3, which satisfy the following relations:

$$r_1 + r_2 + r_3 = f(k),\ r_1r_2 + r_2r_3 + r_3r_1 = -g(k),\ r_1r_2r_3 = h(k). \tag{2.12}$$

In the following definitions, some of the special well-known third-order integer sequences are shown.

Definition 2.3 **($H_{k,n}(a, b, c;\ f, g, h)$)** For particular values of $f(k)$, $g(k)$, $h(k)$, a, b, and c, we achieve the following definitions:

1. The Perrin number sequence is obtained if $f(k) = 0, g(k) = h(k) = 1, a = 3, b = 0$, and $c = 2$, i.e., $H_{k,n+3} = H_{k,n+1} + H_{k,n};\quad H_{k,0} = 3, H_{k,1} = 0$ *and* $H_{k,2} = 2$.
2. The Padovan number sequence is obtained if $f(k) = 0, g(k) = h(k) = 1$, and $a = b = c = 1$, i.e., $H_{k,n+3} = H_{k,n+1} + H_{k,n};\ H_{k,0} = H_{k,1} = H_{k,2} = 1$.
3. The tribonacci number sequence is obtained if $f(k) = g(k) = h(k) = 1, a = b = 0$, and $c = 1$, i.e., $H_{k,n+3} = H_{k,n+2} + H_{k,n+1} + H_n;\ H_{k,0} = H_{k,1} = 0$ *and* $H_{k,2} = 1$.
4. The Trucas number sequence is obtained if $f(k) = g(k) = h(k) = 1, a = 3, b = 1$, and $c = 3$, i.e., $H_{k,n+3} = H_{k,n+2} + H_{k,n+1} + H_{k,n};\ H_{k,0} = 3, H_{k,1} = 1$ *and* $H_{k,2} = 3$.

2.3.1 Generating Function of Extended Generalized k-Horadam Sequences $\{H_{k,n}\}_{n\in\mathbb{N}}$

In this section, we establish the generating function for generalized k-Horadam sequences of third order in terms of initial terms and give in the following theorem:

Theorem 2.5 *The generating function for extended generalized k-Horadam sequences* $\{H_{k,n}\}_{n\in\mathbb{N}}$ *is given by*

$$\sum_{n=0}^{\infty} H_{k,n}x^n = \frac{H_{k,0} + x[H_{k,1} - H_{k,0}f(k)] + x^2[H_{k,2} - H_{k,1}f(k) - H_{k,0}g(k)]}{[1 - xf(k) - x^2g(k) - x^3h(k)]}.$$

Proof Let $g(x)$ be the generating function for extended generalized k-Horadam sequences $\{H_{k,n}\}_{n\in\mathbb{N}}$. Then, we have

$$g(x) = H_{k,0} + xH_{k,1} + x^2H_{k,2} + ... + x^nH_{k,n} + ... \tag{2.13}$$

Now, multiply both sides of recurrence relation (2.10) by x^{n+3} and then take summation. Thus, we get

$$\sum_{n=0}^{\infty} x^{n+3}H_{k,n+3} - f(k)\sum_{n=0}^{\infty} x^{n+3}H_{k,n+2} - g(k)\sum_{n=0}^{\infty} x^{n+3}H_{k,n+1} - h(k)\sum_{n=0}^{\infty} x^{n+3}H_{k,n} = 0. \tag{2.14}$$

From Eq. (2.13), we write

$$\sum_{n=0}^{\infty} x^{n+3}H_{k,n+3} = x^3H_{k,3} + ... + x^nH_{k,n} + ... = g(x) - (H_{k,0} + xH_{k,1} + x^2H_{k,2}),$$

$$\sum_{n=0}^{\infty} x^{n+3}H_{k,n+2} = x(x^2H_{k,2} + x^3H_{k,3} + ... + x^nH_{k,n} + ...) = x[g(x) - H_{k,0} - xH_{k,1}],$$

$$\sum_{n=0}^{\infty} x^{n+3}H_{k,n+1} = x^2(xH_{k,1} + x^2H_{k,2} + x^3H_{k,3} + ... + x^nH_{k,n} + ...) = x^2[g(x) - H_{k,0}].$$

Thus, on putting these values in Eq. (2.14), we get

$$[g(x) - H_{k,0} - xH_{k,1} - x^2H_{k,2}] - f(k)x[g(x) - H_{k,0} - xH_{k,1}] - g(k)x^2[g(x) - H_{k,0}] - h(k)x^3g(x) = 0$$

$$\implies g(x)[1 - xf(k) - x^2g(k) - x^3h(k)] - H_{k,0}[1 - xf(k) - x^2g(k)] - H_{k,1}[x - x^2f(k)] - H_{k,2}x^2 = 0$$

$$\implies g(x) = \frac{H_{k,0}[1 - xf(k) - x^2g(k)] + H_{k,1}[x - x^2f(k)] + H_{k,2}x^2}{[1 - xf(k) - x^2g(k) - x^3h(k)]}$$

$$\implies g(x) = \frac{H_{k,0} + x[H_{k,1} - H_{k,0}f(k)] + x^2[H_{k,2} - H_{k,1}f(k) - H_{k,0}g(k)]}{[1 - xf(k) - x^2g(k) - x^3h(k)]}$$

as required. □

For instance, on taking $f(k) = g(k) = h(k) = 1, a = b = 0$ and $c = 1$, we get the generating function for tribonacci sequences as $\sum_{n=0}^{\infty} H_{k,n}x^n = \dfrac{x^2}{1 - x - x^2 - x^3}$.

2.3.2 Binet Formula

Now, we establish the Binet formula for the extended generalized k-Horadam sequences of third order. Binet formula in the theory of difference equation plays an important role, which helps us to obtain the nth term of the corresponding sequence and to establish many well-known identities like d'Ocagne identity, Catalan's identity, Cassini's identity, and various transformations. Binet's formula for the extended generalized k-Horadam sequences of third order is given in the following theorem:

Theorem 2.6 *For every positive integer n, we write the Binet formula as*

$$H_{k,n} = Ar_1{}^n + Br_2{}^n + Kr_3{}^n, \tag{2.15}$$

where $K = \left[\dfrac{c - (r_1 + r_2)b + r_1r_2a}{r_3^2 - (r_1 + r_2)r_3 + r_1r_2}\right]$, $A = \dfrac{(r_2 - r_3)K - ar_2 + b}{r_1 - r_2}$, *and* $B = \dfrac{(r_3 - r_1)K + ar_1 - b}{r_1 - r_2}$.

Proof For the characteristic equation corresponding to the difference equation (2.10), we have three roots r_1, r_2, and r_3, and these roots satisfy Eq. (2.12). Now, using relation (2.12) in Eq. (2.10), we have

$$H_{k,n+3} = (r_1 + r_2 + r_3)H_{k,n+2} - (r_1r_2 + r_2r_3 + r_3r_1)H_{k,n+1} + r_1r_2r_3H_{k,n}.$$

We can also write it as

$$\begin{aligned} H_{k,n+3} - (r_1 + r_2)H_{k,n+2} + (r_1r_2)H_{k,n+1} &= r_3H_{k,n+2} - r_3(r_1 + r_2)H_{k,n+1} + r_1r_2r_3H_{k,n} \\ &= r_3[H_{k,n+2} - (r_1 + r_2)H_{k,n+1} + r_1r_2H_{k,n}]. \end{aligned} \tag{2.16}$$

Similarly, we have

$$H_{k,n+2}-(r_1+r_2)H_{k,n+1}+r_1r_2H_{k,n} = r_3[H_{k,n+1}-(r_1+r_2)H_{k,n}+r_1r_2H_{k,n-1}]. \tag{2.17}$$

Now, substituting Eq. (2.17) in Eq. (2.16), we get

$$H_{k,n+3}-(r_1+r_2)H_{k,n+2}+(r_1r_2)H_{k,n+1} = r_3^2[H_{k,n+1}-(r_1+r_2)H_{k,n}+r_1r_2H_{k,n-1}].$$

On continuing this substitution process, ultimately we obtain

$$H_{k,n+3}-(r_1+r_2)H_{k,n+2}+(r_1r_2)H_{k,n+1} = r_3^{n+1}[H_{k,2}-(r_1+r_2)H_{k,1}+r_1r_2H_{k,0}].$$

Now, dividing both sides of the above equation by r_3^{n+3}, we get

$$\frac{H_{k,n+3}}{r_3^{n+3}}-\frac{(r_1+r_2)}{r_3^{n+3}}H_{k,n+2}+\frac{(r_1r_2)}{r_3^{n+3}}H_{k,n+1}=\frac{1}{r_3^2}[H_{k,2}-(r_1+r_2)H_{k,1}+r_1r_2H_{k,0}]. \tag{2.18}$$

For simplicity, let $H_{k,2}-(r_1+r_2)H_{k,1}+r_1r_2H_{k,0} = R$ and $\frac{H_{k,n+3}}{r_3^{n+3}} = T_{k,n+3}$; then from Eq. (2.18), we write

$$T_{k,n+3} - \frac{(r_1+r_2)}{r_3}T_{k,n+2} + \frac{(r_1r_2)}{r_3^2}T_{k,n+1} = \frac{1}{r_3^2}R \tag{2.19}$$

which is the second-order nonhomogeneous linear difference equation, and its solution is given as $T_{k,n} = T(C) + T(P)$, where $T(C)$ represents the solution corresponding to homogeneous part and $T(P)$ is particular solution.

Here, we first obtain the solution of the corresponding homogeneous difference equation, and then by adding a particular solution, we get the required result. The roots of characteristic equation for a homogeneous part of Eq. (2.19) are $\lambda_1 = \frac{r_1}{r_3}$ and $\lambda_2 = \frac{r_2}{r_3}$. So, the solution for homogeneous part is given as

$$T(C) = A\left(\frac{r_1}{r_3}\right)^n + B\left(\frac{r_2}{r_3}\right)^n.$$

Furthermore, the nonhomogeneous part is a constant (see RHS of Eq. (2.19)), so particular solution is also a constant, and it is given by

$$T(P) = \frac{R}{r_3^2-(r_1+r_2)r_3+r_1r_2}.$$

Thus, the general solution of Eq. (2.19) is

$$T_{k,n} = T(C) + T(P) = A\left(\frac{r_1}{r_3}\right)^n + B\left(\frac{r_2}{r_3}\right)^n + \frac{R}{r_3^2 - (r_1 + r_2)r_3 + r_1 r_2}.$$

Replacing $T_{k,n}$ by $\dfrac{H_{k,n}}{r_3^n}$ and R by $H_{k,2} - (r_1 + r_2)H_{k,1} + r_1 r_2 H_{k,0}$ in above equation, we get

$$H_{k,n} = A{r_1}^n + B{r_2}^n + {r_3}^n K,\ \text{ where } K = \left[\frac{H_{k,2} - (r_1 + r_2)H_{k,1} + r_1 r_2 H_{k,0}}{r_3^2 - (r_1 + r_2)r_3 + r_1 r_2}\right]. \tag{2.20}$$

And on using initial conditions from Eq. (2.10) in Eq. (2.20), we have

$$A = \frac{(r_2 - r_3)K - ar_2 + b}{r_1 - r_2} \text{ and } B = \frac{(r_3 - r_1)K + ar_1 - b}{r_1 - r_2}.$$

This completes the proof. □

Remark 2.2 By setting different values of $f(k)$, $g(k)$, $h(k)$, a, b, and c in Theorem (2.6) as shown in the Definitions (2.2) and (2.3), we obtain the Binet formula for the corresponding sequences.

Binet Formula for Tribonacci Sequence
For $f(k) = g(k) = h(k) = 1$, $a = b = 0$, and $c = 1$ in Theorem (2.6), we get the Binet formula for tribonacci sequence.

However, in the next subsection, we present the Binet formula for tribonacci sequence $H_{k,n}(0, 0, 1; 1, 1, 1)$ by direct method.

2.3.3 Binet Formula for Tribonacci Sequence by Direct Method

The characteristic equation for tribonacci sequences $H_{k,n}(0, 0, 1; 1, 1, 1)$ is given by

$$x^3 - x^2 - x - 1 = 0. \tag{2.21}$$

Characteristic Eq. (2.21) has three roots, say α_1, α_2 , and α_3, which are given by

$$\alpha_1 = \frac{(19+3\sqrt{33})^{\frac{2}{3}} + (19+3\sqrt{33})^{\frac{1}{3}} + 4}{(19+3\sqrt{33})^{\frac{1}{3}}},$$

$$\alpha_2 = \frac{(19+3\sqrt{33})^{\frac{2}{3}}(-1+i\sqrt{3}) + 2(19+3\sqrt{33})^{\frac{1}{3}} - 4(1+i\sqrt{3})}{6(19+3\sqrt{33})^{\frac{1}{3}}}, \text{ and}$$

$$\alpha_3 = \frac{(19+3\sqrt{33})^{\frac{2}{3}}(1-i\sqrt{3}) - 2(19+3\sqrt{33})^{\frac{1}{3}} + 4(1-i\sqrt{3})}{6(19+3\sqrt{33})^{\frac{1}{3}}}.$$

Thus, for the natural number n, a formula for nth term of tribonacci is given by

$$F_n = C_1(\alpha_1)^n + C_2(\alpha_2)^n + C_3(\alpha_3)^n, \quad n \in \mathbb{N} \tag{2.22}$$

where

$$C_1 = \frac{\frac{1}{2}\left(\sqrt[3]{19+3\sqrt{3}\sqrt{11}}+2\right)\left(19+3\sqrt{3}\sqrt{11}\right)}{\sqrt{33}\left(19+3\sqrt{33}\right)^{\frac{2}{3}} + 2\sqrt{11}\sqrt[3]{19+3\sqrt{33}}\sqrt{3} + 23\sqrt{33} + 9\left(19+3\sqrt{33}\right)^{\frac{2}{3}} + 18\sqrt[3]{19+3\sqrt{33}} + 135},$$

$$C_2 = \frac{-i/12\left(-9i\sqrt{11} + 4i\sqrt[3]{19+3\sqrt{33}}\sqrt{3} - 9\sqrt{33} - 19i\sqrt{3} - 12\sqrt[3]{19+3\sqrt{33}} - 57\right)\sqrt{3}\sqrt[3]{19+3\sqrt{33}}}{99+19\sqrt{3}\sqrt{11}},$$

$$C_3 = \frac{-i/12\sqrt{3}\sqrt[3]{19+3\sqrt{33}}\left(-9i\sqrt{11} + 4i\sqrt[3]{19+3\sqrt{33}}\sqrt{3} + 9\sqrt{33} - 19i\sqrt{3} + 12\sqrt[3]{19+3\sqrt{33}} + 57\right)}{99+19\sqrt{33}}.$$

Acknowledgments The authors are very grateful to anonymous reviewers for their comments and valuable suggestions. The first and third authors acknowledge the University Grants Commission (UGC), India, for providing fellowship for this research work.

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Chapter 3
New Results on $(p_1, p_2, \cdots, p_n, k)$ Analogue of Lauricella Function with Transforms and Fractional Calculus Operator

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MSC: 26A33; 33C05; 33B15; 33C65.

3.1 Introduction

Special functions emerge as an essential tool in problem-solving that comes from many different research fields such as astronomy, engineering, mathematical analysis, aerodynamics, astrophysics, and other technological field. Hypergeometric functions are the most prominent part of the special functions because these functions have extensive scope in a scientific research.

In the modern age, the hypergeometric functions represented by two parameters in the numerator and one parameter in the denominator have received attention of many researchers due to its immense applications in many fields. For the generalization and improvement of the hypergeometric functions by using Pochhammer symbol, numerous efforts have been made by the researchers. For comprehensive review of the exclusive results on hypergeometric functions, we refer [3, 8–10, 12, 19].

Diaz and Pariguan [7] introduced the k-analogue of gamma, beta and hypergeometric functions as well as a variety of other features. Since then, many researchers have identified numerous distinct results concerning the k-hypergeometric and associated functions. For literature related to k-hypergeometric function, we may refer [5, 6, 16, 17].

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_3

Fractional calculus is the discipline which provides the extension of derivatives and integrals to non-integer order. In the evolution of series and integrals, fractional derivatives and fractional integrals of special functions of one and multiple variables are crucial. Many researchers have used the concept of fractional calculus on the special function and obtained different theorems and results of the function. For literature review, the reader may consult [2, 4].

The (p, k) extended Gauss hypergeometric function [1] and (p, k) analogue of the Gauss hypergeometric function linked with fractional calculus [11] were introduced by Abdalla and Hidan. Inspired by few of these preceding analysis of the k-hypergeometric function and (p, k) extended Gauss hypergeometric function, we introduce $(p_1, p_2, \cdots, p_n, k)$ extension of the Lauricella function and derived its some new theorems, integral transforms and fractional calculus operator. The following is the structure of the research paper: we provide several fundamental definitions and terminologies in Sect. 3.1 that we will utilize later. We introduced $(p_1, p_2, \cdots, p_n, k)$ extension of Lauricella function which will be involved in Sect. 3.2. In Sect. 3.3, we established some new theorems. We derived some integral transform like—beta transform and Laplace transform of this function which is given in Sect. 3.4. In Sect. 3.5, we obtained a new theorem of this function with fractional calculus operator. Finally in Sect. 3.6, the conclusion is given.

Preliminaries

The Gauss hypergeometric function has the following definition [18, Eq(6), p.46]:

$$ {}_2F_1(\Theta_1, \Theta_2;\ \Theta_3; t) = \sum_{s=0}^{\infty} \frac{(\Theta_1)_s (\Theta_2)_s}{(\Theta_3)_s} \frac{t^s}{s!} \tag{3.1}$$

where $|t| < 1$, $\Theta_1, \Theta_2, \Theta_3 \in \mathbb{C}$, and $\Theta_3 \neq 0, -1, -2, -3, \cdots$

which is absolutely and uniformly convergent whenever $|t| < 1$ and divergent whenever $|t| > 1$. Also, if $Re(\Theta_3 - \Theta_2 - \Theta_1) > 0$,then it is absolutely convergent whenever $|t| = 1$

The Pochhammer symbol $(\Theta_1)_s$ is defined as [18, Eq(6), p.22]

$$ (\Theta_1)_s = \begin{cases} (\Theta_1)(\Theta_1 + 1) \cdots (\Theta_1 + s - 1), & s \in \mathbb{N}, \Theta_1 \in \mathbb{C} \\ 1, & s = 0, \Theta_1 \in \mathbb{C} \setminus \{0\} \end{cases} \tag{3.2}$$

and

$$ (\Theta_1)_s = \frac{\Gamma(\Theta_1 + s)}{\Gamma(\Theta_1)} $$

The confluent hypergeometric function is defined by Rainville [18, Eq.(1), p.123]

$$ {}_1F_1(\Theta_1;\ \Theta_2; t) = \sum_{s=0}^{\infty} \frac{(\Theta_1)_s}{(\Theta_2)_s} \frac{t^s}{s!} \tag{3.3}$$

where $|t| < 1$, $\Theta_1, \Theta_2 \in \mathbb{C}$ and $\Theta_2 \neq 0, -1, -2, -3, \cdots$

The classical beta function is represented as [18, Eq.(16), p.18]

$$B(\Theta_1, \Theta_2) = \int_0^1 t^{\Theta_1 - 1}(1-t)^{\Theta_2 - 1} dt \tag{3.4}$$

where $Re(\Theta_1)$, $Re(\Theta_2) > 0$ and 'Re' is the real part of the function.

The classical Euler's gamma function is described as follows [18, Eq.(1), p.15]:

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \tag{3.5}$$

where $Re(z) > -1$.

Diaz and Pariguan [7] provided the following interesting generalizations of k-analogue of the beta, gamma, hypergeometric functions and Pochhammer symbol:

Definition 3.1 The k-gamma function $\Gamma^K(v)$, for $k \in \mathbb{R}^+$ is described as follows [11, Eq.(1), p.1]:

$$\Gamma^k(v) = \int_0^\infty x^{(v-1)} e^{-(v^k/k)} dx, \tag{3.6}$$

where $v \in \mathbb{C} \setminus k\mathbb{Z}^-$.

In particular if we take $k \to 1$, then $\Gamma^k(v) = \Gamma(v)$, where $\Gamma(v)$ denotes the classical Euler's gamma function. Also, the connection between $\Gamma^k(v)$ and the Euler's gamma function $\Gamma(v)$ is given by

$$\Gamma^k(v) = k^{(\frac{v}{k})-1} \Gamma(v/k) \tag{3.7}$$

Definition 3.2 The k- Pochhammer symbol for $k \in \mathbb{R}^+$ is denoted by Hidan et al. [11, Eq.(2), p.1]:

$$(v)_{s,k} = \frac{\Gamma^k(v+sk)}{\Gamma^k(v)} = \begin{cases} v(v+k)\cdots(v+(s-1)k), & s \in \mathbb{N}, v \in \mathbb{C} \\ 1, & s = 0, v \in \mathbb{C} \setminus \{0\} \end{cases} \tag{3.8}$$

Definition 3.3 The k-beta function $B_k(\Theta_1,\Theta_2)$ for $k \in \mathbb{R}^+$ and $\Theta_1, \Theta_2 \in \mathbb{C}$ is represented as follows [11, Eq.(4), p.2]:

$$B_k(\Theta_1, \Theta_2) = \begin{cases} \frac{1}{k}\int_0^1 t^{(\Theta_1/k)-1}(1-t)^{(\Theta_2/k)-1} dt, & min\{Re(\Theta_1), Re(\Theta_2)\} > 0 \\ \frac{\Gamma^k(\Theta_1)\Gamma^k(\Theta_2)}{\Gamma^k(\Theta_1+\Theta_2)}, & (\Theta_1, \Theta_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases} \tag{3.9}$$

where $Re(\Theta_1) > 0$ and $Re(\Theta_2) > 0$.

In particular, if we take $k = 1$, then k-beta function reduces in the original beta function. Also, the relation between $B_k(\Theta_1, \Theta_2)$ and $B(\Theta_1, \Theta_2)$ is given by

$$B_k(\Theta_1, \Theta_2) = \frac{1}{k} B(\frac{\Theta_1}{k}, \frac{\Theta_2}{k}) \tag{3.10}$$

Definition 3.4 The k-hypergeometric function for $k \in \mathbb{R}^+$ is written in the form of [11, Eq.(6), p.2]:

$${}_2F_1^k(\Theta_1, \Theta_2;\ \Theta_3; t) = \sum_{s=0}^{\infty} \frac{(\Theta_1)_{s,k}\,(\Theta_2)_{s,k}}{(\Theta_3)_{s,k}} \frac{t^s}{s!}, \tag{3.11}$$

where $|t| < \frac{1}{k}$ and $\Theta_1, \Theta_2, \Theta_3 \in \mathbb{C}$ and also $\Theta_3 \neq 0, -1, -2, -3, \cdots$

In particular, if we take $k = 1$, then the k-hypergeometric function reduces to the Gaussian hypergeometric function ${}_2F_1$.

Definition 3.5 The (p, k) extended hypergeometric function, for $k \in \mathbb{R}^+$ is represented as [11, Eq.(9), p.2]:

$${}_2F_1^{(p,k)}(\Theta_1, \Theta_2;\ \Theta_3; t) = \sum_{s=0}^{\infty} \frac{(\Theta_1)_{s,k}\,(\Theta_2)_{s,k}}{(\Theta_3)_{s,k}} \frac{t^s}{ps!}, \tag{3.12}$$

where $k \in \mathbb{R}^+$ and $\Theta_1, \Theta_2, \Theta_3 \in \mathbb{C}$ also $\Theta_3 \neq 0, -1, -2, -3, \cdots$

Also, this function is an entire function whenever $p > 1$, where $(\Theta_1)_{s,k}$ denotes the k-Pochhammer symbol.

Definition 3.6 The Lauricella function defined by Lauricella is given as [4, Eq(14–17), p.3]

$$F_A^{(n)}\left[\begin{matrix} a, b_1, b_2, \cdots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; u_1, u_2, \cdots, u_n\right] = \sum_{s_1, s_2, \ldots, s_n = 0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n} (b_1)_{s_1} (b_2)_{s_2} \ldots (b_n)_{s_n}}{(c_1)_{s_1} (c_2)_{s_2} \ldots (c_n)_{s_n}} \frac{(u_1)^{s_1} (u_2)^{s_2} \ldots (u_n)^{s_n}}{(s_1)!(s_2)!\ldots(s_n)!} \tag{3.13}$$

$$F_B^{(n)}\left[\begin{matrix} a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \\ c \end{matrix}; u_1, u_2, \cdots, u_n\right] = \sum_{s_1, s_2, \ldots, s_n = 0}^{\infty} \frac{(a_1)_{s_1} (a_2)_{s_2} \ldots (a_n)_{s_n} (b_1)_{s_1} (b_2)_{s_2} \ldots (b_n)_{s_n}}{(c)_{s_1+s_2+\ldots+s_n}} \frac{(u_1)^{s_1} (u_2)^{s_2} \ldots (u_n)^{s_n}}{(s_1)!(s_2)!\ldots(s_n)!} \tag{3.14}$$

$$F_C^{(n)}\left[\begin{matrix} a, b \\ c_1, c_2, \cdots, c_n \end{matrix}; u_1, u_2, \cdots, u_n\right] = \sum_{s_1, s_2, \ldots, s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n} (b)_{s_1+s_2+\ldots+s_n}}{(c_1)_{s_1} (c_2)_{s_2} \ldots (c_n)_{s_n}} \frac{(u_1)^{s_1} (u_2)^{s_2} \ldots (u_n)^{s_n}}{(s_1)!(s_2)! \ldots (s_n)!} \tag{3.15}$$

$$F_D^{(n)}\left[\begin{matrix} a, b_1, b_2, \cdots, b_n \\ c \end{matrix}; u_1, u_2, \cdots, u_n\right] = \sum_{s_1, s_2, \ldots, s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n} (b_1)_{s_1} (b_2)_{s_2} \ldots (b_n)_{s_n}}{(c)_{s_1+s_2+\ldots+s_n}} \frac{(u_1)^{s_1} (u_2)^{s_2} \ldots (u_n)^{s_n}}{(s_1)!(s_2)! \ldots (s_n)!} \tag{3.16}$$

The conditions of convergence for these functions are given by

- When $|u_1| + |u_2| + \ldots + |u_n| < 1$, then $F_A^{(n)}$ converges.
- When $|u_1| < 1, |u_2| < 1, \ldots, |u_n| < 1$, then $F_B^{(n)}$ converges.
- When $|\sqrt{u_1}| + |\sqrt{u_2}| + \ldots + |\sqrt{u_n}| < 1$, then $F_C^{(n)}$ converges.
- When $|u_1| < 1, |u_2| < 1, \ldots, |u_n| < 1$, then $F_D^{(n)}$ converges.

In particular if we take $n = 2$, the Lauricella functions reduce in “Appell series” $F_2, F_3, F_4, and F_1$ respectively. On taking $n = 1$, the functions reduce to Gauss hypergeometric function ${}_2F_1$.

3.2 $(p_1, p_2, \cdots, p_n, k)$ Analogue of Lauricella Function

Now on behalf of (p, k) analogue of the Gauss hypergeometric function which is introduced by M. Abdalla and M. Hidan, we define a new function $(p_1, p_2, \cdots, p_n, k)$ analogue of the Lauricella function, which has the following representation:

$$F_A^{(p_1, p_2, \ldots, p_n, k)}\left[\begin{matrix} a, b_1, b_2, \cdots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; x_1, x_2, \cdots, x_n\right] = \sum_{s_1, s_2, \ldots, s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n} (b_1)_{s_1} (b_2)_{s_2} \ldots (b_n)_{s_n}}{(c_1)_{s_1} (c_2)_{s_2} \ldots (c_n)_{s_n}} \frac{(x_1)^{s_1}}{p_1 s_1!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!} \tag{3.17}$$

The conditions of convergence for this function are described as

$$|x_1| + |x_2| + ... + |x_n| < 1.$$

Now in the next section, we will derive some new results and theorems of this function.

3.3 Integral Representations

Theorem 3.1 *The integral representation of the function* $F_A^{(p_1,p_2,\ldots,p_n,k)}$ *is represented as follows which holds true in (3.17):*

$$\frac{\Gamma^k(c_1+\delta_1+\cdots+c_n+\delta_n)}{\Gamma^k(c_1)\cdots\Gamma^k(c_n)\Gamma^k(\delta_1+\cdots+\delta_n)}\int_0^n u^{\left(\frac{c_1+c_2+\cdots+c_n}{k}\right)-1}(n-u)^{\left(\frac{\delta_1+\delta_2+\cdots+\delta_n}{k}\right)-1}$$
$$\times F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}a,b_1,b_2,\cdots,b_n\\c_1,c_2,\cdots,c_n\end{matrix};\frac{x_1}{u_1},\frac{x_2}{u_2},\cdots,\frac{x_n}{u_n}\right]du$$
$$=kn^{\left(\frac{c_1+\delta_1-km_1+\cdots+c_n+\delta_n-km_n}{k}\right)-1}F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}a,b_1,b_2,\cdots,b_n\\c_1,c_2,\cdots,c_n\end{matrix};x_1,x_2,\cdots,x_n\right] \tag{3.18}$$

$(a\in\mathbb{C}; b_i,c_i,\delta i\in\mathbb{C}; Re(a)>0, Re(b_i)>0, Re(c_i)>0, Re(\delta i)>0;$
$|x_i|<1, p_i\in\mathbb{N})$
where $i=1,2,\cdots,n$ and $k\in\mathbb{R}^+$

Proof Consider L.H.S of the equation which is denoted by S. Now

$$S=\frac{\Gamma^k(c_1+\delta_1+\cdots+c_n+\delta_n)}{\Gamma^k(c_1)\cdots\Gamma^k(c_n)\Gamma^k(\delta_1+\cdots+\delta_n)}\int_0^n u^{\left(\frac{c_1+c_2+\cdots+c_n}{k}\right)-1}(n-u)^{\left(\frac{\delta_1+\delta_2+\cdots+\delta_n}{k}\right)-1}$$
$$\times F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}a,b_1,b_2,\ldots,b_n\\c_1,c_2,\cdots,c_n\end{matrix};\frac{x_1}{u_1},\frac{x_2}{u_2},\ldots,\frac{x_n}{u_n}\right]du$$
$$=\frac{\Gamma^k(c_1+\delta_1+\cdots+c_n+\delta_n)}{\Gamma^k(c_1)\cdots\Gamma^k(c_n)\Gamma^k(\delta_1+\cdots+\delta_n)}\sum_{s_1,\ldots,s_n=0}^{\infty}\frac{(a)_{s_1+\ldots+s_n,k}(b_1)_{s_1,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}}$$
$$\times\frac{(x_1)^{s_1}}{p_1s_1!}\cdots\frac{(x_n)^{s_n}}{p_ns_n!}\times\int_0^n u^{\left(\frac{c_1+c_2+\cdots+c_n}{k}\right)-1}(n-u)^{\left(\frac{\delta_1+\delta_2+\cdots+\delta_n}{k}\right)-1}\frac{du}{u^{(s_1+s_2+\cdots+s_n)}}$$

on taking $u=nv$, and after arranging the terms, the equation is

$$=n^{\left(\frac{c_1+\delta_1-km_1+\cdots+c_2+\delta_2-km_n}{k}\right)-1}\times\frac{\Gamma^k(c_1+\delta_1+\cdots+c_n+\delta_n)}{\Gamma^k(c_1)\cdots\Gamma^k(c_n)\Gamma^k(\delta_1+\cdots+\delta_n)}$$

$$\times \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}} \frac{(x_1)^{s_1}}{p_1 s_1!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!}$$

$$\times \int_0^1 v^{\left(\frac{c_1-ks_1+c_2-ks_2+\cdots+c_n-ks_n}{k}\right)-1}(1-v)^{\left(\frac{\delta_1+\delta_2+\cdots+\delta_n}{k}\right)-1}dv.$$

$$= kn^{\left(\frac{c_1+\delta_1-ks_1+\cdots+c_2+\delta_2-ks_n}{k}\right)-1} \times \frac{\Gamma^k(c_1+\delta_1+\cdots+c_n+\delta_n)}{\Gamma^k(c_1)\cdots\Gamma^k(c_n)\Gamma^k(\delta_1+\cdots+\delta_n)}$$

$$\times \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}} \frac{(x_1)^{s_1}}{p_1 s_1!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!}$$

$$\times B^k(c_1-ks_1+c_2-ks_2+\cdots+c_n-ks_n, \delta_1+\delta_2+\cdots+\delta_n)$$

After calculating this equation, we achieve the intended outcome. □

Theorem 3.2 *The integral representation of the function* $F_A^{(p_1,p_2,\ldots,p_n,k)}$ *represented as follows in (3.17) holds true:*

$$\frac{\Gamma^k(c_1+\cdots+c_n)}{\Gamma^k(b_1+b_2+\cdots+b_n)\Gamma^k(c_1-b_1+\cdots+c_n-b_n)} \int_0^\infty u^{\left(\frac{b_1+b_2+\cdots+b_n}{k}\right)-1} \times$$

$$(1+u)^{-\left(\frac{c_1+c_2+\cdots+c_n}{k}\right)} F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, c_1, \cdots, c_n \\ c_1, c_2, \cdots, c_n \end{matrix}; \frac{x_1 u}{u+1}, \cdots, \frac{x_n u}{u+1}\right] du$$

$$= F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, b_1+b_2+\cdots+b_n \\ c_1+c_2+\cdots+c_n \end{matrix}; x_1, x_2, \cdots, x_n\right] \tag{3.19}$$

$$\Big(a \in \mathbb{C}; b_i, c_i, \delta_i \in \mathbb{C}; Re(a) > 0, Re(b_i) > 0, Re(c_i) > 0, Re(\delta_i) > 0;$$
$$|x_i| < 1, p_i \in \mathbb{N}, k \in \mathbb{R}^+\Big),$$

where $i = 1, 2, \cdots, n$.

Proof Consider the L.H.S of the equation which is denoted by S

$$i.e., S = \frac{\Gamma^k(c_1+c_2+\cdots+c_n)}{\Gamma^k(b_1+b_2+\cdots+b_n)\Gamma^k(c_1-b_1+\cdots+c_n-b_n)}$$
$$\times \int_0^\infty u^{\left(\frac{b_1+b_2+\cdots+b_n}{k}\right)-1}(1+u)^{-\left(\frac{c_1+c_2+\cdots+c_n}{k}\right)}$$
$$\times F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, c_1, c_2, \cdots, c_n \\ c_1, c_2, \cdots, c_n \end{matrix}; \frac{x_1 u}{u+1}, \frac{x_2 u}{u+2}, \cdots, \frac{x_n u}{u+1}\right] du$$

on taking, $v = \frac{u}{u+1}$, and after arranging the terms, the equation is

$$= \frac{\Gamma^k(c_1 + c_2 + \cdots + c_n)}{\Gamma^k(b_1 + b_2 + \cdots + b_n)\Gamma^k(c_1 - b_1 + \cdots + c_n - b_n)}$$

$$\times \int_0^1 v^{\left(\frac{b_1+b_2+\cdots+b_n}{k}\right)-1}(1-v)^{\left(\frac{c_1-b_1+c_2-b_2\cdots+c_n-b_n}{k}\right)-1}$$

$$\times F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix} a, c_1, c_2, \cdots, c_n \\ c_1, c_2, \cdots, c_n \end{matrix}; x_1 v, x_2 v, \cdots, x_n v\right] du$$

$$= \frac{\Gamma^k(c_1 + c_2 + \cdots + c_n)}{\Gamma^k(b_1 + b_2 + \cdots + b_n)\Gamma^k(c_1 - b_1 + \cdots + c_n - b_n)}$$

$$\times \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} (a)_{s_1+\ldots+s_n,k} \frac{(x_1)^{s_1}}{p_1 s_1!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!}$$

$$\times \int_0^1 v^{\left(\frac{b_1+b_2+\cdots+b_n+k(s_1+s_2+\cdots+s_n)}{k}\right)-1}(1-v)^{\left(\frac{c_1-b_1+c_2-b_2+\cdots+c_n-b_n}{k}\right)-1} dv$$

$$= \frac{\Gamma^k(c_1 + c_2 + \cdots + c_n)}{\Gamma^k(b_1 + b_2 + \cdots + b_n)\Gamma^k(c_1 - b_1 + \cdots + c_n - b_n)}$$

$$\times \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} (a)_{s_1+\ldots+s_n,k} \frac{(x_1)^{s_1}}{p_1 s_1!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!}$$

$$\times B^k\{(b_1 + b_2 + \cdots + b_n) + k(s_1 \cdots + s_n), (c_1 - b_1 + \cdots + c_n - b_n)\}$$

$$= \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} (a)_{s_1+\ldots+s_n,k} \times \frac{\Gamma^k(c_1 + c_2 + \cdots + c_n)}{\Gamma^k\{(c_1 + c_2 + \cdots + c_n) + k(s_1 + \cdots + s_n)\}}$$

$$\times \frac{\Gamma^k\{(b_1 + b_2 + \cdots + b_n) + k(s_1 + \cdots + s_n)\}}{\Gamma^k(c_1 + c_2 + \cdots + c_n)} \times \frac{(x_1)^{s_1}}{p_1 s_1!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!}$$

$$= F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix} a, b_1 + b_2 + \cdots + b_n \\ c_1 + c_2 + \cdots + c_n \end{matrix}; x_1, x_2, \cdots, x_n\right] \qquad \square$$

3.4 Integral Transforms

Theorem 3.3 *The k-beta transform for the function* $F_A^{(p_1,p_2,\ldots,p_n,k)}$ *in (3.17) is represented in the following form:*

$$B_k\left\{F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}(a+b), b_1, b_2, \cdots, b_n\\ c_1, c_2, \cdots, c_n\end{matrix}; x_1u, x_2u, \cdots, x_nu\right]\right\}$$
$$= B^k(a,b)F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}a, b_1, b_2, \cdots, b_n\\ c_1, c_2, \cdots, c_n\end{matrix}; x_1, x_2, \cdots, x_n\right] \tag{3.20}$$

($a \in \mathbb{C}$; $b_i, c_i, \in \mathbb{C}$; $Re(a) > 0$, $Re(b_i) > 0$, $Re(c_i) > 0$, $|x_i| < 1$, $p_i \in \mathbb{N}$), where $i = 1, 2, \cdots, n$ and $k \in \mathbb{R}^+$.

Proof Consider the L.H.S of the equation and using the property of k-beta function, i.e.,

$$B_k\left\{F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}(a+b), b_1, b_2, \cdots, b_n\\ c_1, c_2, \cdots, c_n\end{matrix}; x_1u, x_2u, \cdots, x_nu\right]: a, b\right\}$$

$$= \frac{1}{k}\int_0^1 u^{\left(\frac{a}{k}\right)-1}(1-u)^{\left(\frac{b}{k}\right)-1}$$
$$\times F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}(a+b), b_1, b_2, \cdots, b_n\\ c_1, c_2, \cdots, c_n\end{matrix}; x_1u, x_2u, \cdots, x_nu\right]du$$
$$= \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} \frac{(a+b)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}}\frac{(x_1)^{s_1}}{p_1s_1!}$$
$$\frac{(x_2)^{s_2}}{p_2s_2!}\cdots\frac{(x_n)^{s_n}}{p_ns_n!}\times\frac{1}{k}\int_0^1 u^{\left(\frac{a}{k}\right)+s_1+s_2+\cdots+s_n-1}(1-u)^{\left(\frac{b}{k}\right)-1}du$$
$$= \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} \frac{(a+b)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}}\frac{(x_1)^{s_1}}{p_1s_1!}$$
$$\frac{(x_2)^{s_2}}{p_2s_2!}\cdots\frac{(x_n)^{s_n}}{p_ns_n!}\times\frac{\Gamma^k\{a+k(s_1+s_2+\cdots+s_n)\}\Gamma^k(b)}{\Gamma^k\{a+k(s_1+s_2+\cdots+s_n)+b\}}$$
$$= \frac{\Gamma^k(b)}{\Gamma^k(a+b)}\sum_{s_1,s_2,\ldots,s_n=0}^{\infty}\frac{\Gamma^k\{a+k(s_1+\cdots+s_n)\}(b_1)_{s_1,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}}$$
$$\times\frac{(x_1)^{s_1}}{p_1s_1!}\frac{(x_2)^{s_2}}{p_2s_2!}\cdots\frac{(x_n)^{s_n}}{p_ns_n!}$$

$$= \frac{\Gamma^k(b)\Gamma^k(a)}{\Gamma^k(a+b)} \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\cdots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\cdots(c_n)_{s_n,k}}$$

$$\times \frac{(x_1)^{s_1}}{p_1 s_1!} \frac{(x_2)^{s_2}}{p_n s_n!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!}$$

$$= B^k(a,b) F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, b_1, b_2, \cdots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; x_1, x_2, \cdots, x_n \right]$$ □

Theorem 3.4 *The Laplace transform of the function* $F_A^{(p_1,p_2,\ldots,p_n,k)}$ *in (3.17) is described in the following form:*

$$L\left\{ F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, b_1, b_2, \cdots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; x_1 u, x_2 u, \cdots, x_n u \right] \right\}$$

$$= \frac{\Gamma^k(k)}{w} F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, k, b_1, b_2, \cdots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; \frac{x_1}{kw}, \frac{x_2}{kw}, \cdots, \frac{x_n}{kw} \right] \quad (3.21)$$

$(a, w \in \mathbb{C}; b_i, c_i, \in \mathbb{C}; Re(a) > 0, Re(b_i) > 0, Re(c_i) > 0; |\frac{x_i}{kw}| < 1, p_i \in \mathbb{N})$ where $i = 1, 2, \cdots, n$ and $k \in \mathbb{R}^+$.

Proof Consider the L.H.S of the equation and using the property of Laplace integral transform,

i.e.,

$$L\left\{ F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, b_1, b_2, \ldots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; x_1 u, x_2 u, \cdots, x_n u \right] \right\}$$

$$= \int_0^\infty e^{-wu} F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, b_1, b_2, \cdots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; x_1 u, x_2 u, \cdots, x_n u \right] du$$

$$= \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\cdots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\cdots(c_n)_{s_n,k}} \frac{(x_1)^{s_1}}{p_1 s_1!}$$

$$\times \frac{(x_2)^{s_2}}{p_2 s_2!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!} \int_0^\infty e^{-wu} u^{(s_1+s_2+\cdots+s_n)} du$$

Substitute $wu = \frac{\theta^k}{k}$, and after solving, we get

$$= \sum_{s_1,s_2,\ldots,s_n=0}^{\infty} \frac{(a)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\cdots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\cdots(c_n)_{s_n,k}} \frac{(x_1)^{s_1}}{p_1 s_1!}$$

$$\frac{(x_2)^{m_2}}{p_2 s_2!} \cdots \frac{(x_n)^{s_n}}{p_n s_n!} \times \frac{1}{w} \frac{1}{(kw)^{s_1+s_2+\cdots+s_n}} \int_0^\infty e^{-\left(\frac{\theta^k}{k}\right)} \theta^{k(s_1+\cdots+s_n)+k-1} d\theta$$

$$= \sum_{s_1, s_2, \ldots, s_n=0}^{\infty} \frac{(a)_{s_1+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}} \frac{(x_1)^{s_1}}{p_1 s_1!}$$

$$\frac{(x_2)^{s_2}}{p_2 s_2!} \cdots \frac{(x_n)^{m_n}}{p_n m_n!} \times \frac{1}{w} \frac{\Gamma^k(k)}{(kw)^{s_1+\cdots+s_n}} \cdot \frac{\Gamma^k\{k(s_1+s_2+\cdots+s_n)+k\}}{\Gamma^k(k)}$$

$$= \frac{\Gamma^k(k)}{w} F_A^{(p_1,p_2,\ldots,p_n,k)} \left[\begin{matrix} a, k, b_1, b_2, \ldots, b_n \\ c_1, c_2, \cdots, c_n \end{matrix}; \frac{x_1}{kw}, \frac{x_2}{kw}, \cdots, \frac{x_n}{kw} \right] \qquad \square$$

3.5 Fractional Calculus

Presently, various analyses and extensions of fractional calculus operators were investigated by different researchers. With the help of the k-gamma function, the k-Riemann-Liouville fractional integral is described as (see [15])

$$(I_k^v \phi(\tau))(\xi) = \frac{1}{k\Gamma^k(v)} \int_0^\xi \phi(\tau)(\xi-\tau)^{\left(\frac{v}{k}\right)-1} d\tau \tag{3.22}$$

Also, the k-Riemann-Liouville fractional derivative is defined as (see [13])

$$D_k^v \phi(\xi) = D(I_k^{(1-v)} \phi(\xi)) \tag{3.23}$$

Now, we formulate the k-fractional derivative of $(p_1, p_2, \cdots, p_n, k)$ analogue of Lauricella function by considering extended Riemann-Liouville k-fractional derivative with the parameters $\delta_1, \delta_2, \delta_3 \in \mathbb{C}$, which is defined as

$$D_k^{\delta_1,\delta_2,\delta_3}\{f(\xi)\} = I_k^{\delta_1(n-\delta_3)} \frac{d^n}{d\xi^n} (I_k^{(1-\delta_1)(n-\delta_2)}) \tag{3.24}$$

And the lemma which is defined as (see [14])

$$D_k^{\delta_1,\delta_2,\delta_3}\{\xi^{\left(\frac{\lambda}{k}\right)-1}\} = \frac{\Gamma^k(\lambda)}{k^n \Gamma^k(\lambda - nk + (1-\delta_1)(n-\delta_2) + \delta_1(n-\delta_3))} \times \xi^{\frac{(1-\delta_1)(n-\delta_2)+\delta_1(n-\delta_3)+\delta_1}{k} - (n+1)}$$

$\left((n-1) < Re(\delta_2), Re(\delta_3) < n, Re(\delta_1) < 1, Re(\xi) > 0, n \in \mathbb{N} \text{ and } k \in \mathbb{R}^+\right)$

Theorem 3.5 *The following relation holds true:*

$$D_k^{\delta_1,\delta_2,\delta_3}\{\xi^{(\frac{\lambda}{k})-1}F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}a,b_1,b_2,\cdots,b_n\\ c_1,c_2,\cdots,c_n\end{matrix};\frac{x_1}{kw},\frac{x_2}{kw},\cdots,\frac{x_n}{kw}\right]\}$$
$$=\frac{\Gamma^k(\lambda)}{k^n\Gamma^k(\lambda-nk+(1-\delta_1)(n-\delta_2)+\delta(n-\delta_3))}\times\xi^{\frac{(1-\delta_1)(n-\delta_2)+\delta(n-\delta_3)}{k}-(n+1)} \tag{3.25}$$

$(a\in\mathbb{C}; b_i,c_i,\tau_i,\in\mathbb{C}; Re(a)>0, Re(b_i)>0, Re(c_i)>0, Re(\tau_i)>0,$
$(n-1)<Re(\delta_2), Re(\delta_3)<n, Re(\delta_1)<1,\ p_i\in\mathbb{N}\ )$
where $i=1,2,\cdots,n$ and $k\in\mathbb{R}^+$

Proof Consider the left hand side of Equation (3.25), which is denoted by T. Now

$$T=D_k^{\delta_1,\delta_2,\delta_3}\left\{\xi^{\left(\frac{\lambda}{k}\right)-1}F_A^{(p_1,p_2,\ldots,p_n,k)}\left[\begin{matrix}a,b_1,b_2,\cdots,b_n\\ c_1,c_2,\cdots,c_n\end{matrix};\tau_1\xi,\tau_2\xi,\cdots,\tau_n\xi\right]\right\}$$
$$=\sum_{s_1,s_2,\ldots,s_n=0}^{\infty}\frac{(a)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}}\frac{(\tau_1)^{s_1}}{p_1s_1!}$$
$$\frac{(\tau_2)^{s_2}}{p_2s_2!}\cdots\frac{(\tau_n)^{s_n}}{p_ns_n!}\times D_k^{\delta_1,\delta_2,\delta_3}\xi^{\left(\frac{\lambda+k(s_1+s_2+\cdots+s_n)}{k}\right)-1}$$
$$=\sum_{s_1,s_2,\ldots,s_n=0}^{\infty}\frac{(a)_{s_1+s_2+\ldots+s_n,k}(b_1)_{s_1,k}(b_2)_{s_2,k}\ldots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\ldots(c_n)_{s_n,k}}\frac{(\tau_1)^{s_1}}{p_1s_1!}\frac{(\tau_2)^{s_2}}{p_2m_2!}\cdots\frac{(\tau_n)^{s_n}}{p_ns_n!}$$
$$\times\frac{1}{k^n}\frac{\Gamma^k(\lambda+k(s_1+s_2+\cdots+s_n))\times\xi^{\frac{(1-\delta_1)(n-\delta_2)+\lambda+k(s_1+s_2+\cdots+s_n)+\delta(n-\delta_3)}{k}-(n+1)}}{\Gamma^k(\lambda+k(s_1+s_2+\cdots+s_n)-nk+(1-\delta_1)(n-\delta_2)+\delta_1(n-\delta_3)}$$
$$=\frac{\Gamma^k(\lambda)}{k^n\Gamma^k(\lambda-nk+(1-\delta_1)(n-\delta_2)+\delta(n-\delta_3))}\cdot\xi^{\frac{(1-\delta_1)(n-\delta_2)+\lambda+\delta(n-\delta_3)}{k}-(n+1)}$$
$$\times\sum_{s_1,s_2,\ldots,s_n=0}^{\infty}\frac{(a)_{s_1+s_2\cdots+s_n,k}\cdot(\lambda)_{s_1+s_2\cdots+s_n,k}}{(\lambda-nk+(1-\delta_1)(n-\delta_2)+\delta_1(n-\delta_3))_{s_1+\cdots+s_n,k}}$$
$$\frac{(b_1)_{s_1,k}(b_2)_{s_2,k}\cdots(b_n)_{s_n,k}}{(c_1)_{s_1,k}(c_2)_{s_2,k}\cdots(c_n)_{s_n,k}}\times\frac{(\tau_1\xi)^{s_1}}{p_1s_1!}\frac{(\tau_2\xi)^{s_2}}{p_2s_2!}\cdots\frac{(\tau_n\xi)^{s_n}}{p_ns_n!}$$
$$=\frac{\Gamma^k(\lambda)}{k^n\Gamma^k(\lambda-nk+(1-\delta_1)(n-\delta_2)+\delta(n-\delta_3))}\cdot\xi^{\frac{(1-\delta_1)(n-\delta_2)+\lambda+\delta(n-\delta_3)}{k}-(n+1)}$$
$$F_A^{(p_1,p_2,\cdots,p_n,k)}\left[\begin{matrix}a,\lambda,b_1,\cdots,b_n\\ \lambda-nk+(1-\delta_1)(n-\delta_2)+\delta_1(n-\delta_3),c_1,\cdots,c_n\end{matrix};\tau_1\xi,\cdots,\tau_n\xi\right]\square$$

3.6 Concluding Remarks

In this article, we have defined $(p_1, p_2, \cdots, p_n, k)$ extension of the Lauricella function. Some new results and integral representation of this extended function have been obtained. The beta transform and Laplace transform have been also derived from this function. By using the k-Riemann-Liouville integral and derivative on this function, we have obtained a new theorem. The results and theorems that we have derived in this paper are obtained on the basis of (p, k) extended hypergeometric function. The future scope of this extension is that one can define some interesting results and integrals by fitting some suitable parameter.

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Chapter 4
Absolute Linear Method of Summation for Orthogonal Series

Alka Munjal

4.1 Introduction

The absolute convergence of orthogonal series is closely linked to the quantitative measurement of the uniform continuity and the bounded variation of the functions. For the study of absolute convergence of orthogonal series, several classical criteria have been established. A methodical proof for these criteria can be achieved by various summability methods. Out of these criteria, the best way to achieve a methodical proof is to determine a least set of sufficient conditions for absolute summable factor of orthogonal series.

Non-absolute convergent factor of orthogonal series can be studied using the concept of absolute summability. So, absolute summability is extremely contributive in understanding the concept of the absolute convergence of orthogonal series. Consequently, several criteria can be used systematically to get a non-absolute convergent factor for the orthogonal series.

Okuyama [3, 4] and Leindler [5–7] studied the orthogonal series with the help of various absolute summability factors. Bor [8–11, 14, 16] and Rhoades [2, 12] have also derived a number of theorems on absolute Nörlund summability. In the present study, an almost everywhere absolute convoluted Nörlund summability factor for orthogonal series has been worked out, and a set of new and well-known results has been deduced from the presented theorems. Furthermore, it has been shown that an orthogonal series can be made absolute Nörlund summable with some sufficient conditions.

Let $f(x)$ be a periodic function integrable over (a, b) and $\Phi_n(x)$ be an orthonormal system defined in the interval (a, b). The orthogonal series of $f(x)$ belongs to $L^2(a, b)$, which is given by

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_4

$$f(x) := \sum_{n=0}^{\infty} a_n \Phi_n(x), \tag{4.1}$$

where

$$a_n := \int_a^b f(x)\Phi_n(x)dx. \tag{4.2}$$

Nörlund summability method: Let p be a sequence as $p = (p_v) \in w$ and w is defined as

$$w := \mathbb{K}^{\mathbb{N}^0} := \{x = (x_k)_{k\in\mathbb{N}^0} | x : \mathbb{N}^0 \to \mathbb{K},\ k \to x_k := x(k)\} \tag{4.3}$$

and

$$P_n := \sum_{v=0}^{n} p_v \neq 0 \ \ (n \in \mathbb{N}).$$

Then the matrix $N_p := (N, p) := (N, p_n) := (p_{nk})$ is defined by

$$p_{nk} := \begin{cases} \frac{p_{n-k}}{P_n}, & k \leq n, \\ 0, & otherwise, \end{cases} \tag{4.4}$$

where $k,\ n \in \mathbb{N}^0$.

The associated summability method to a Nörlund matrix is called Nörlund summability method (with respect to the sequence p).

The Nörlund mean t_n of series $\sum a_n$ is defined by

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \ \ (P_n \neq 0). \tag{4.5}$$

If the series

$$\sum_{n=0}^{\infty} |t_n - t_{n-1}| \tag{4.6}$$

converges, then the sequence of partial sums $\{s_n\}$ of the series or the series $\sum a_n$ is said to be $|N, p_n|$ summable.

Convoluted Nörlund (p∗q) summability method: Let p and q be the sequences as

$$p = (p_v) \in w \text{ and } P_n := \sum_{v=0}^{n} p_v \neq 0 \ \ (n \in \mathbb{N})$$

$$q = (q_v) \in w \text{ and } Q_n := \sum_{v=0}^{n} q_v \neq 0 \ \ (n \in \mathbb{N})$$

The convolution $(p*q)_n$ is defined by

$$(p*q)_n := \sum_{v=0}^{n} p_{n-v} q_v = \sum_{v=0}^{n} p_v q_{n-v}. \tag{4.7}$$

Apparently,

$$P_n := (p * 1)_n = \sum_{v=0}^{n} p_v \text{ and } Q_n := (1 * q)_n = \sum_{v=0}^{n} q_v.$$

If $(p*q)_n \neq 0$ at all values of n, the convoluted Nörlund transform sequence of $\{s_n\}$ is $\{t_n^{p,q}\}$, which is given by

$$t_n^{p,q} := \frac{1}{(p*q)_n} \sum_{v=0}^{n} p_{n-v} q_v s_v. \tag{4.8}$$

Definition 4.1 (Almost Convoluted $|\mathbf{N}, \mathbf{p}, \mathbf{q}; \mathbf{m}|_{\mathbf{k}}$ Summability Method:) The series $\sum a_n$ is $|N, p, q; m|_k$ summable of order k for $k \geq 1$ if the series

$$\sum_{n=1}^{\infty} n^{k-1} |t_{n,m}^{p,q} - t_{n-1,m}^{p,q}|^k \tag{4.9}$$

uniformly converges with reference to m and the relationship can be represented as

$$\sum_{n=0}^{\infty} a_n \in |N, p, q; m|_k. \tag{4.10}$$

Definition 4.2 (Almost Convoluted $|\mathbf{N}, \mathbf{p}, \mathbf{q}; \delta; \mathbf{m}|_{\mathbf{k}}$ Summability Method:) The series $\sum a_n$ is $|N, p, q; \delta; m|_k$ summable of order k for $k \geq 1$ and $\delta \geq 0$ if the series

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |t_{n,m}^{p,q} - t_{n-1,m}^{p,q}|^k \tag{4.11}$$

uniformly converges with reference to m and the relationship can be represented as

$$\sum_{n=0}^{\infty} a_n \in |N, p, q; \delta; m|_k. \tag{4.12}$$

Definition 4.3 (Almost Convoluted $|\mathbf{N}, \mathbf{p}, \mathbf{q}; \gamma; \delta; \mathbf{m}|_{\mathbf{k}}$ Summability Method:) The series $\sum a_n$ is $|N, p, q; \gamma; \delta; m|_k$ summable for $k \geq 1$, $\delta \geq 0$ and γ is real number if the series

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)} |t_{n,m}^{p,q} - t_{n-1,m}^{p,q}|^k \tag{4.13}$$

uniformly converges with reference to m and the relationship can be represented as

$$\sum_{n=0}^{\infty} a_n \in |N, p, q; \gamma; \delta; m|_k. \tag{4.14}$$

Definition 4.4 (Almost Indexed Convoluted $\varphi - |\mathbf{N}, \mathbf{p}, \mathbf{q}; \mathbf{m}|_{\mathbf{k}}$ Summability Method:) Let $\varphi := \varphi(n)$ be a sequence of positive real numbers; then series $\sum a_n$ is indexed absolute almost convoluted Nörlund summable of order k for $k \geq 1$ if the series

$$\sum_{n=1}^{\infty} |\varphi(n)(t_{n,m}^{p,q} - t_{n-1,m}^{p,q})|^k \tag{4.15}$$

or

$$\sum_{n=1}^{\infty} |\varphi(n)\Delta t_{n,m}^{p,q}|^k \tag{4.16}$$

uniformly converges with reference to m and the relationship can be represented as

$$\sum_{n=0}^{\infty} a_n \in \varphi - |N, p, q; m|_k. \tag{4.17}$$

Notations Let us denote

$$R_n := (p * q)_n, \quad R_n^j := \sum_{v=j}^{n} p_{n-v} q_v,$$

$$R_{n-1}^{n} = 0, \quad R_n^0 = R_n$$

and

$$\widehat{R}_n^j := \sum_{v=j}^{n} \frac{p_{n-v} q_v}{v+1}, \quad \widehat{R}_{n-1}^{n} = 0.$$

4.2 Notable Results

Okuyama's [13] previous findings are as follows:.

Theorem 4.1 *In order to make an orthogonal series* $\sum_{n=0}^{\infty} c_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q|$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{1/2} \textit{must converge.} \tag{4.18}$$

Theorem 4.2 *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of nonnegative numbers. In order to make an orthogonal series* $\sum_{n=0}^{\infty} c_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q|$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

and

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w^{(1)}(n)$$

must converge, where

(i) $\{\Omega(n)\}$ *be a sequence of positive numbers.*
(ii) $\{\Omega(n)/n\}$ *be a sequence of nonincreasing numbers.*
(iii) $w^{(1)}(n)$ *is defined by.*

$$w^{(1)}(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2. \tag{4.19}$$

4.3 Main Results

The established theorems are as follows:

Theorem 4.3 *In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $\varphi - |N, p, q; m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ \varphi^2(n) \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{4.20}$$

must uniformly converge with reference to m for $1 \le k \le 2$.

Theorem 4.4 *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of nonnegative numbers. In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $\varphi - |N, p, q; m|_k$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{\varphi^{\frac{k}{1-k}}(n)}{\Omega(n)}$$

must converge and

$$\sum_{n=1}^{\infty} |a_{m+n}|^2 \Omega^{2/k-1}(n) \mathfrak{R}^{(k)}(n)$$

must uniformly converge with reference to m for $1 < k \le 2$, *where*

(i) $\{\Omega(n)\}$ *be a sequence of positive numbers.*
(ii) $\{\Omega(n)/\varphi^{\frac{k}{k-1}}(n)\}$ *be a sequence of nonincreasing numbers.*
(iii) $\mathfrak{R}^{(k)}(n)$ *is defined by*

$$\mathfrak{R}^{(k)}(j) := \frac{1}{\varphi^{\frac{1}{k-1}-1}(j)} \sum_{n=j}^{\infty} \varphi^{\frac{2}{k-1}}(n) \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{4.21}$$

Beppo Levi [1] established the Lemma 4.1 and used it to prove results based on the functions of series and integrals. Lemma 4.1 has also been used to prove the presented theorems.

Lemma 4.1 *Let a nonnegative function* $V_n(t) \in L(U)$, *if*

$$\sum_{n=1}^{\infty} \int_U V_n(t) < \infty, \tag{4.22}$$

then the series

$$\sum_{n=1}^{\infty} V_n(t) \tag{4.23}$$

almost (absolutely) converges everywhere to a function $V(t) \in L(U)$ over U.

4.4 Proof of the Theorems

Let the vth partial sum of the series is given by $s_v(x) = \sum_{j=0}^{v} a_j \Phi_j(x)$ for $1 < k < 2$:

$$\begin{aligned} s_{v,m}(x) &= \frac{1}{v+1} \sum_{k=0}^{v} s_{k+m}(x) \\ &= \frac{1}{v+1} \sum_{k=0}^{v} \sum_{j=0}^{k+m} a_j \Phi_j(x) \\ &= \sum_{j=0}^{v} \left(1 - \frac{j}{v+1}\right) a_{m+j} \Phi_{m+j}(x) + s_{m-1}(x). \end{aligned} \tag{4.24}$$

Almost Nörlund transform $t_{n,m}^{p,q}(x)$ is given by

$$\begin{aligned} t_{n,m}^{p,q}(x) &= \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_{v,m}(x) \\ &= s_{m-1}(x) + \frac{1}{R_n} \sum_{j=0}^{n} a_{m+j} \Phi_{m+j}(x) \sum_{v=j}^{n} p_{n-v} q_v \\ &\quad - \frac{1}{R_n} \sum_{j=0}^{n} j a_{m+j} \Phi_{m+j}(x) \sum_{v=j}^{n} \frac{p_{n-v} q_v}{v+1} \\ &= s_{m-1}(x) + \frac{1}{R_n} \sum_{j=0}^{n} \left(R_n^j - j \widehat{R}_n^j\right) a_{m+j} \Phi_{m+j}(x). \end{aligned} \tag{4.25}$$

$$\Delta t_{n,m}^{p,q}(x) = t_{n,m}^{p,q}(x) - t_{n-1,m}^{p,q}(x)$$
$$= \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right] a_{m+j} \Phi_{m+j}(x). \quad (4.26)$$

Proof of the Theorem 4.3 The series

$$\sum_{n=1}^{\infty} \int_a^b |\varphi(n) \Delta t_{n,m}^{p,q}(x)|^k dx$$
$$\leq \sum_{n=1}^{\infty} |\varphi(n)|^k \int_a^b |\Delta t_{n,m}^{p,q}(x)|^k dx$$
$$= O(1) \sum_{n=1}^{\infty} |\varphi(n)|^k (b-a)^{1-\frac{k}{2}} \left(\int_a^b |\Delta t_{n,m}^{p,q}(x)|^2 dx \right)^{\frac{k}{2}}$$
$$= O(1) \sum_{n=1}^{\infty} \left\{ \varphi^2(n) \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}}$$
$$= O(1). \quad (4.27)$$

It has been observed that $|\Delta t_{n,m}^{p,q}(x)|$ is a nonnegative function due to which (4.27) converges because

$$\sum_{n=1}^{\infty} \left\{ \varphi^2(n) \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}}$$

uniformly converges with reference to m by the assumption.

Hence, by Lemma 4.1,

$$\sum_{n=1}^{\infty} |\varphi(n) \Delta t_{n,m}^{p,q}(x)|^k \quad (4.28)$$

almost converges everywhere. Schwartz's inequality is applicable for $k = 1$ and used up to $k = 2$. Hence, proof of the theorem is complete. □

Proof of the Theorem 4.4 The series

$$\sum_{n=1}^{\infty} \int_a^b |\varphi(n) \Delta t_{n,m}^{p,q}(x)|^k dx$$

$$\leq \sum_{n=1}^{\infty} |\varphi(n)|^k \left\{ \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}}$$

$$= O(1) \sum_{n=1}^{\infty} \left(\frac{\varphi^{\frac{k}{1-k}}(n)}{\Omega(n)} \right)^{1-\frac{k}{2}} \times$$

$$\times \left\{ \frac{\Omega^{\frac{2}{k}-1}(n)}{\varphi^{\frac{k}{1-k}}(n)} \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}}$$

$$= O(1) \left(\sum_{n=1}^{\infty} \frac{\varphi^{\frac{k}{1-k}}(n)}{\Omega(n)} \right)^{1-\frac{k}{2}} \times$$

$$\times \left\{ \sum_{n=1}^{\infty} \frac{\Omega^{\frac{2}{k}-1}(n)}{\varphi^{\frac{k}{1-k}}(n)} \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}}$$

$$= O(1) \left\{ \sum_{j=1}^{\infty} |a_{m+j}|^2 \sum_{n=j}^{\infty} \frac{\Omega^{\frac{2}{k}-1}(n)}{\varphi^{\frac{k}{1-k}}(n)} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 \right\}^{\frac{k}{2}}$$

$$= O(1) \left\{ \sum_{j=1}^{\infty} |a_{m+j}|^2 \left(\frac{\Omega(j)}{\varphi^{\frac{k}{k-1}}(j)} \right)^{\frac{2}{k}-1} \times \right.$$

$$\left. \times \sum_{n=j}^{\infty} \varphi^{\frac{2}{k-1}}(n) \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 \right\}^{\frac{k}{2}}$$

$$= O(1) \left\{ \sum_{j=1}^{\infty} |a_{m+j}|^2 \Omega^{\frac{2}{k}-1}(j) \mathfrak{R}^{(k)}(j) \right\}^{\frac{k}{2}} \tag{4.29}$$

where

$$\mathfrak{R}^{(k)}(j) := \frac{1}{\varphi^{\frac{k-2}{1-k}}(j)} \sum_{n=j}^{\infty} \varphi^{\frac{2}{k-1}}(n) \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{4.30}$$

Based on the assumption, Eq. (4.29) is uniformly finite with reference to m. This can be proved using the same concept as used for proving Theorem 4.3. The detail of the proof is out of the scope and hence omitted. □

4.5 Special Cases of the Main Theorems

4.5.1 Case I: $\varphi(n) = n^{\gamma(\delta+1-\frac{1}{k})}$

Theorems 4.3 and 4.4 will give the results for $|N, p, q; \delta; \gamma; m|_k$ summability (Definition 4.3).

Corollary 4.1 *In order to make an orthogonal series* $\sum\limits_{n=0}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q; \delta; \gamma; m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ n^{2\gamma(\delta+1-\frac{1}{k})} \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{4.31}$$

must uniformly converge with reference to m for $1 \leq k \leq 2$.

Corollary 4.2 *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of nonnegative numbers. In order to make an orthogonal series* $\sum\limits_{n}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q; \delta; \gamma; m|_k$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\gamma(\delta k+k-1)}{k-1}} \Omega(n)}$$

must converge and

$$\sum_{n=1}^{\infty} |a_{m+n}|^2 \Omega^{2/k-1}(n) \mathfrak{R}^{(k)}(n)$$

must uniformly converge with reference to m for $1 < k \leq 2$, *where*

(i) $\{\Omega(n)\}$ *be a sequence of positive numbers.*
(ii) $\left\{ \frac{\Omega(n)}{n^{\frac{\gamma(\delta k+k-1)}{k-1}}} \right\}$ *be a sequence of nonincreasing numbers.*
(iii) $\mathfrak{R}^{(k)}(n)$ *is defined by*

$$\mathfrak{R}^{(k)}(j) := \frac{1}{j^{\frac{\gamma(\delta k+k-1)(k-2)}{k(1-k)}}} \sum_{n=j}^{\infty} n^{\frac{2\gamma(\delta k+k-1)}{k(k-1)}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{4.32}$$

4.5.2 Case II: $\varphi(n) = n^{(\delta+1-\frac{1}{k})}$

The Theorems 4.3 and 4.4 will give the results for $|N, p, q; \delta; m|_k$ summability (Definition 4.2).

Corollary 4.3 *In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q; \delta; m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ n^{2(\delta+1-\frac{1}{k})} \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{4.33}$$

must uniformly converge with reference to m for $1 \le k \le 2$.

Corollary 4.4 *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of nonnegative numbers. In order to make an orthogonal series* $\sum_{n}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q; \delta; m|_k$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{(\delta k+k-1)}{k-1}} \Omega(n)}$$

must converge and

$$\sum_{n=1}^{\infty} |a_{m+n}|^2 \Omega^{2/k-1}(n) \mathfrak{R}^{(k)}(n)$$

must uniformly converge with reference to m for $1 < k \le 2$, *where*

(i) $\{\Omega(n)\}$ *be a sequence of positive numbers.*
(ii) $\{\Omega(n)/n^{\frac{(\delta k+k-1)}{k-1}}\}$ *be a sequence of nonincreasing numbers.*
(iii) $\mathfrak{R}^{(k)}(n)$ *is defined by*

$$\mathfrak{R}^{(k)}(j) := \frac{1}{j^{\frac{(\delta k+k-1)(k-2)}{k(1-k)}}} \sum_{n=j}^{\infty} n^{\frac{2(\delta k+k-1)}{k(k-1)}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{4.34}$$

4.5.3 Case III [15] $\varphi(n) = n^{1-\frac{1}{k}}$

Theorems 4.3 and 4.4 will give the results for $|N, p, q; m|_k$ summability (Definition 4.1).

Corollary 4.5 *In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q; m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ n^{2(1-\frac{1}{k})} \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{4.35}$$

must uniformly converge with reference to m for $1 \le k \le 2$.

Corollary 4.6 *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of nonnegative numbers. In order to make an orthogonal series* $\sum_{n}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, p, q; m|_k$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

must converge and

$$\sum_{n=1}^{\infty} |a_{m+n}|^2 \Omega^{2/k-1}(n) \mathfrak{R}^{(k)}(n)$$

must uniformly converge with reference to m for $1 \le k \le 2$, *where*

(i) $\{\Omega(n)\}$ *be a sequence of positive numbers.*
(ii) $\{\Omega(n)/n\}$ *be a sequence of nonincreasing numbers.*
(iii) $\mathfrak{R}^{(k)}(n)$ *is defined by*

$$\mathfrak{R}^{(k)}(j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{4.36}$$

4.5.4 Case IV: $\varphi(n) = n^{1-\frac{1}{k}}$ and $p_v = 1$ or $q_v = 1$

Theorem 4.3 will give the following results for $p_v = 1$ for all v :

Corollary 4.7 *In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, q; m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n^{(1-\frac{1}{k})}q_n}{Q_n Q_{n-1}}\right)^2 \sum_{j=1}^{n}\left[Q_{j-1}+j\left(\frac{Q_n}{n+1}-\sum_{v=j}^{n}\frac{q_v}{v+1}\right)\right]^2 |a_{m+j}|^2\right\}^{k/2} \tag{4.37}$$

must uniformly converge with reference to m for $1 \leq k \leq 2$.

Theorem 4.3 will give the following results for $q_v = 1$ for all v :

Corollary 4.8 *In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *to be almost everywhere* $|N, p; m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n^{(1-\frac{1}{k})}p_n}{P_n P_{n-1}}\right)^2 \sum_{j=1}^{n} p_{n-j}^2 [\mathfrak{L}_n^j]^2 |a_{m+j}|^2\right\}^{k/2} \tag{4.38}$$

must uniformly converge with reference to m for $1 \leq k \leq 2$, *where*

$$\mathfrak{L}_n^j = 1 - \frac{P_{n-j-1}}{p_{n-j}} + j\sum_{v=0}^{n-j}\frac{P_n - (n-v+1)p_n p_v}{(n-v)(n+1-v)p_n p_{n-j}}. \tag{4.39}$$

4.6 Conclusion

The paper focuses on the study of absolute convoluted Nörlund $\varphi - |\mathbf{N}, \mathbf{p}, \mathbf{q}; \mathbf{m}|_{\mathbf{k}}$ summability factor which is useful for achieving the stability of the system. The BIBO stability of the system can be achieved by the condition of absolute summability, as absolute summable is a necessary and sufficient condition, i.e.,

$$\textbf{BIBO stability} \quad \Leftrightarrow \quad \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{I}(\mathbf{M})| < \infty.$$

where $I(M)$ is input impulse response of the system.

The absolute summability plays an important role in signal processing as a double digital filter in finite and infinite impulse response (FIR and IIR, respectively). Employing convoluted Nörlund $\varphi - |\mathbf{N}, \mathbf{p}, \mathbf{q}; \mathbf{m}|_{\mathbf{k}}$ summability (a generalized summability), the functions of the filters (like removal of unwanted frequency components, enhancement of the required frequency components, permanent unit power factor, overcoming unbalancing situation, etc.) have been improved. The results are also useful in engineering, for example, the load signal can be represented as a summation of orthonormal functions (orthogonal series).

Based on this investigation, it can be concluded that our theorem is a generalized version which can be reduced to well-known summabilities as shown in the corollaries. Under certain suitable conditions $\varphi(n) = n^{1-\frac{1}{k}}$, the main theorems render the result of Krasniqi [15] on $|\mathbf{N}, \mathbf{p}, \mathbf{q}; \mathbf{m}|_{\mathbf{k}}$ summability, which explains the importance and validation of the presented work.

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Chapter 5
Derivations and Special Functions Over Fields

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2010 Mathematics Subject Classification 12H05, 13N15, 33B20, 33B30.

The objective of this chapter is to study derivatives, integrals, and special functions (especially error functions, logarithmic integrals, and dilogarithmic integrals) from an algebraic point of view. Special functions exemplified above are non-elementary functions, which means the functions whose integrals cannot be determined in terms of logarithm or exponential. We also present an interesting method to determine whether an arbitrary function is elementary or non-elementary.

The notion of expressing derivations and integrals in algebraic manner is novel and intriguing. The area of mathematics that consists of such studies is called *Differential Algebra*. For a better understanding of history of the subject, we refer to the works by I. Kaplansky [2], E .R. Kolchin [5], A. Magid [6], J. F. Ritt [7], M. Rosenlicht [9], and M. Singer [10].

5.1 Basic Conventions

Definition 5.1 A field F equipped with an additive map $' : F \to F$ satisfying the Leibnitz rule, that is, $(\alpha\beta)' = \alpha\beta' + \alpha'\beta$ for $\alpha, \beta \in F$, is called *differential field*, and the map $'$ is called a *derivation*.

For an element α in F and a nonzero element β in F, the derivation on the fraction $\frac{\alpha}{\beta}$ is given by

$$\left(\frac{\alpha}{\beta}\right)' = \frac{\alpha'\beta - \alpha\beta'}{\beta^2}\,.$$

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_5

For a natural number n, $(\alpha^n)' = n\alpha^{n-1}\alpha'$. It is easy to check that the derivation on identity $1' = 0$.

Definition 5.2 Elements $c \in F$ such that $c' = 0$ are called *constants.* The kernel of the derivation $\{c \in F : c' = 0\}$ forms a field. We call the kernel as *constant field* of F and denote it by C_F.

Definition 5.3 A field $E \supseteq F$ is called a *differential field extension* of F if there is a differential field structure on E which is compatible with the differential field structure on F.

Indeed, there always exist a differential structure on the extension field that makes it a differential field extension.

Theorem 5.1 ([9]) *Let F be a differential field with derivation $'$. Let $E \supset F$ be a field extension of F. Then the derivation on F extends to a derivation on E. If E is algebraic over F, then the extended derivation on E is unique.*

Proof Assume $E = F(t)$ be rational function field over F. Consider the ring of dual numbers over $F(t)$, i.e., the ring $F(t)[\epsilon] = F(t) + F(t)\epsilon$ where $\epsilon^2 = 0$. Since ϵ is nilpotent, an element $\alpha = a + b\epsilon$ of $F(t)[\epsilon]$ is a unit if and only if a is a unit of $F(t)$.

Define a map $a_D = (id, D) : F[t] \to F(t)[\epsilon]$ as $a_D(\alpha) = \alpha + \alpha'\epsilon$ for every $\alpha \in F$ and $a_D(t) = t + f(t)\epsilon$ for some $f(t) \in F(t)$. One can easily check that the map $D : F[t] \to F(t)$, defined as $D(\alpha) = \alpha'$ for $\alpha \in F$ and $D(t) = f(t)$, satisfies a differential structure on $F[t]$ if and only if a_D is a ring homomorphism. Since $t + f(t)\epsilon$ is a unit in $F(t)[\epsilon]$, we can extend a_D to the homomorphism $a_\Delta = (id, \Delta) : F(t) \to F(t)[\epsilon]$, where Δ is a derivation on $F(t)$ extending D as well as $'$.

Now assume t is algebraic over F and the minimal polynomial of t is $P(X) = \sum_{0\le i\le n} a_i X^i$, where X is an indeterminate. As observed above, $D : F(X) \to F(X)$ mapping X to any rational function $Q(X) \in F(X)$ gives a derivation on $F(X)$. Observe that $D(P(X)) = \sum_{0\le i\le n} a_i' X^i + \sum_{1\le i\le n} i a_i X^{i-1} Q(X)$ and thus $\sum_{0\le i\le n} a_i' t^i + \sum_{1\le i\le n} i a_i t^{i-1} Q(t) = 0$. Clearly, $\sum_{1\le i\le n} i a_i t^{i-1} \neq 0$, and therefore, we get a unique derivation on $F(t)$ given by

$$D(t) = Q(t) = -\frac{\sum_{0\le i\le n} a_i' t^i}{\sum_{1\le i\le n} i a_i t^{i-1}} .$$

One can observe that D maps the polynomial $P(X)$ into a polynomial which is a multiple of $P(X)$. Thus, the ideal $P(X)F[X]$ of the polynomial ring $F[X]$ is mapped into itself. Therefore, D induces a derivation on the factor ring $\frac{F[X]}{P(X)F[X]} \simeq F(t)$.

Thus, a simple extension of F is a differential extension with derivation D. Using Zorn's lemma, we extend the derivation D to an arbitrary field extension of F. □

One can look into [9] and [1] for different proofs of above theorem. In literature, differential field refers to a field with a family of derivations, but throughout this chapter, we fix a single derivation map $'$ on the differential field F.

Definition 5.4 Let F be a differential field. Let $\alpha \in F$ and $\beta \in F \setminus \{0\}$ be such that $\alpha' = \frac{\beta'}{\beta}$. In correspondence to the classical analytic theory, α will be called *logarithm* of β, and β will be called *exponential* of α. We denote logarithm and exponential as $\alpha = \log \beta$ and $\beta = e^{\alpha}$, respectively.

If β has a logarithm in the field F, then it is unique up to an additive constant, and if α has an exponential in F, then it is unique up to a multiplicative constant. Therefore, for some constants c *and* d and elements β_1 *and* β_2 in F, $\log(\beta_1\beta_2) = \log \beta_1 + \log \beta_2 + c$ and $\log(-\beta_1) = \log(\beta_1) + d$.

Notations We fix a field F with characteristic 0 and a derivation map $'$ on F. We denote the constant subfield of F by C_F. For $\alpha, \beta \in F$, we denote the exponential of α by e^{α} and logarithm of β by $\log \beta$. We fix an algebraic closure $\overline{F}$ of F.

5.2 Liouvillian Extensions

In this section, we study a special type of field extensions of a field F, namely, liouvillian extensions.

Definition 5.5 Let $E \supseteq F$ be differential fields and $t \in E$. If $t' \in F$, then we say that t is *primitive* over F.

Note that for $\alpha \in F$, $\log(\alpha)$ is primitive over F.

Definition 5.6 A field extension $E = F(t_1, \ldots, t_n)$, $F_0 := F$, $F_i = F_{i-1}(t_i)$ is called *liouvillian extension* over F if for each $1 \leq i \leq n$, one of the following holds:

(a) t_i is algebraic over F_{i-1}.
(b) $t_i'/t_i \in F_{i-1}$.
(c) $t_i' \in F_{i-1}$.

Now, we shall describe some properties of liouvillian field extensions.

Theorem 5.2 ([3]) *Let $F(t) \supsetneq F$ be an algebraic differential field extension.*

(a) If $t' \in F$, then there is an element $\alpha \in F$ such that $\alpha' = t'$ and $C_{F(t)} \supsetneq C_F$.
(b) If $\frac{t'}{t} \in F$, then there exists $\alpha \in F \setminus \{0\}$ and an integer n such that $\frac{\alpha'}{\alpha} = n\frac{t'}{t}$. Moreover, if $C_{F(t)} = C_F$, then the monic minimal polynomial of t is $x^n + c\alpha$ for some $c \in C_F$.
(c) $C_{F(t)}$ is an algebraic extension of C_F.

Proof For $n \geq 2$, let $a_0, \ldots, a_{n-1}, a_n = 1 \in F$ be such that $p(x) = \sum_{0 \leq i \leq n} a_i x^i$ denotes the monic minimal polynomial of t over F. We apply the derivation on equation$\sum_{0 \leq i \leq n} a_i t^i = 0$ and get that t satisfies the polynomial

$$p'(x) = (nt' + a'_{n-1})x^{n-1} + \cdots + \left(ia_i t' + a'_{i-1}\right) x^{i-1} + \cdots + a_1 t' + a'_0.$$

If $t \in F$, then $\deg(p'(x)) < \deg(p(x))$. Since $p(x)$ is the minimal polynomial, we get that $p'(x) = 0$. In particular, $\left(\frac{-a_{n-1}}{n}\right)' = t'$ and $t + \frac{a_{n-1}}{n}$ is a constant outside C_F. Through similar calculations for the monic minimal polynomial of $c \in C_{F(t)}$ over F, we obtain that $a'_i = 0$ for all $0 \leq i \leq n$. Therefore, c is algebraic over C_F.

If $\frac{t'}{t} = a \in F$, then

$$p'(x) = nax^n + (a'_{n-1} + (n-1)a_{n-1}a)x^{n-1} + \cdots + a'_0.$$

Observe that $p'(x) = nap(x)$. Thus, for $0 \leq i \leq n-1$, we obtain

$$a'_i = (n-i)aa_i.$$

Since $a_0 \neq 0$, we get $\frac{a'_0}{a_0} = na = n\frac{t'}{t}$. Thus, $t^n + ca_0 = 0$ gives the minimal monic polynomial for some $c \in C_F$. □

Remark 5.1 From the part (c) of Theorem 5.2, it is clear that for an algebraic closure $\overline{F}$ of F, we have $C_{\overline{F}} = C_F$ if and only if C_F is algebraically closed.

Theorem 5.3 ([3]) *Let $F(t) \supset F$ be a rational function field and $t' \in F$. Consider a polynomial $v = \sum_{0 \leq i \leq m} b_i t^i \in F[t]$, and suppose that there is an element $w \in F(t) \setminus F$ such that $v = w'$.*

- *(a) If $v = 0$, that is, $C_{F(t)} \supsetneq C_F$, then there exist $c \in C_F \setminus \{0\}$ and $a_0 \in F$ such that $(ct + a_0)' = 0$:*
- *(b) If $C_{F(t)} = C_F$, then w is a polynomial in $F[t]$, say, $w = \sum_{0 \leq i \leq n} a_i t^i$, where $n \geq 1$, $a_i \in F$, and $a_n \neq 0$.*
- *(c) If $C_{F(t)} = C_F$ and $m =\deg(v)$, then either $n = m$ or $m + 1$. If $n = m$, then $a'_n = b_m$, and if $n = m + 1$, then $a_n \in C_F$ and $(na_n t + a_{n-1})' = b_m$.*
- *(d) Let $\alpha \in F \setminus \{0\}$ be such that $t' = \alpha$ and $x' \neq \alpha$ for any $x \in F$ and then $C_{F(t)} = C_F$. In general, for $\alpha_1, \ldots, \alpha_n \in F \setminus \{0\}$, there exists a differential field extension E of F such that $E = F(t_1, \ldots, t_n)$, $t'_i = \alpha_i$ and $C_E = C_F$.*

Proof Let $p, q \in F[t]$, where q is a monic polynomial and $\gcd(p, q) = 1$, such that $w = \frac{p}{q}$. Since $w' = v$, we take derivative of w and obtain

$$q^2 v = p'q - q'p. \tag{5.1}$$

(a) Since $v = 0$, if $q = 1$, then $p \in F[t] \setminus F$ and clearly $p' = 0$. Now, if $q \neq 1$, then $q \in F[t] \setminus F$. Since q is a monic polynomial and $t' \in F$, we get $\deg(q') < \deg(q)$. Thus, Eq. (5.1) holds only if $q' = 0$ for q divides q'. Therefore, in any case, we obtain a polynomial whose derivative vanishes. Let $\left(\sum_{0 \le i \le n} a_i t^i\right)' = 0$, $n \geq 1$, and $a_n \neq 0$. Now, we equate the coefficients to zero and obtain that $a_n' = 0$ and $(na_n t + a_{n-1})' = na_n t' + a_{n-1}' = 0$. This proves (a).

(b) Since $C_{F(t)} = C_F$, $w' = v \neq 0$. Then, it is immediate from Eq. (5.1) that q divides q'. As observed earlier, we see $\deg(q') < \deg(q)$ and thus $q' = 0$. Then, it is obvious that $q = 1$ and $w = p \in F[t] \setminus F$.

(c) Substituting $v = \sum_{0 \le i \le m} b_i t^i$ and $w = \sum_{0 \le i \le n} a_i t^i$ in $v' = w$, we obtain

$$a_n' t^n + (na_n t' + a_{n-1}')t^{n-1} + \cdots + a_1 t' + a_0' = \sum_{0 \le i \le m} b_i t^i = v. \tag{5.2}$$

It is obvious that n equals m or $m+1$. If $n = m$, then $a_n' = b_m$, and if $n = m+1$, then $a_n \in C_F$ and $(na_n t + a_{n-1})' = na_n t' + a_{n-1}' = b_m$.

(d) If $C_{F(t)} \supsetneqq C_F$, then from (a), there exists $c \in C_F \setminus \{0\}$ and $a_0 \in F$ such that $0 = (ct + a_0)' = c\alpha + a_0'$. Thus $(-a_0/c)' = \alpha$, which is a contradiction. Now, we prove the result in general for n elements $\alpha_1, \ldots, \alpha_n$ in F using the induction on n. Let $F_{n-1} = F(t_1, \ldots, t_{n-1})$, $t_i' = \alpha_i$ for all $1 \le i \le n-1$, and $C_{F_{n-1}} = C_F$. If there does not exist an element $x \in F_{n-1}$ whose derivative $x' = \alpha_n$, then we consider a transcendental element t_n over F_{n-1}, and assume $t_n' = \alpha_n$. This defines a derivation on $E := F_{n-1}(t_n)$. It is clear that $C_E = C_{F_{n-1}} = C_F$. If there exists an element $x \in F_{n-1}$ such that $x' = \alpha_n$, then we consider $t_n = x$.

□

We repeatedly use partial fraction expansions in our results. Thus, in this spirit, it is useful to note the following theorem.

Theorem 5.4 ([3]) *Let $F(t)$ be a Liouvillian extension of F and $C_{F(t)} = C_F$. Assume that t is transcendental over F. Consider an element v in $F(t)$ and its partial fraction expansion $v = \eta \prod_{1 \le j \le m} (t - a_j)^{m_j}$, where $\eta \in F$, $0 = a_1, \ldots, a_m$ are distinct elements in an algebraic closure $\overline{F}$ of F, and m_j are integers.*

(a) Let $t' \in F$. Then $t' - a_j' \neq 0$ for $1 \le j \le m$ and

$$\frac{v'}{v} = \frac{\eta'}{\eta} + \sum_{1 \le j \le m} m_j \frac{t' - a_j'}{t - a_j}. \tag{5.3}$$

(b) Let $\frac{t'}{t} \in F$.

b(i) Then

$$\frac{v'}{v} = \mu + \sum_{2 \le j \le m} m_j \frac{\mu_j}{t - a_j}, \tag{5.4}$$

where $\mu = \frac{\eta'}{\eta} + \sum_{1 \le j \le m} m_j \frac{t'}{t} \in F$ *and* $\mu_j = a_j \frac{t'}{t} - a'_j \in \overline{F} \setminus \{0\}$.

b(ii) *If* v_0 *is the constant term in a partial fraction expansion of* v*, then* v'_0 *is the constant term in partial fraction expansion of* v'.

(c) *Let* $a \in \overline{F}$*. If* $t' \in F$ *and* v *has a pole of order* n *at* a*, then* v' *has a pole at* a *of order* $n + 1$*. Similarly, if* $a \neq 0$*,* $\frac{t'}{t} \in F$ *and* v *has a pole of order* n *at* a*, then* v' *has a pole at* a *of order* $n + 1$.

Proof Let $t' \in F$ and $t' - a'_j = 0$ for some $1 \le j \le m$; then by Theorem 5.2, there exists $x \in F$ such that $x' = a'_j = t'$. Then $t - x \notin F$ is a constant in $F(t)$, which contradicts the fact that $C_{F(t)} = C_F$. Similarly, if $\frac{t'}{t} = x \in F$ and $a'_j = xa_j$, then Theorem 5.2 implies the existence of an integer r and an element $y \in F$ such that $rx = r\frac{t'}{t} = \frac{y'}{y}$. Thus, $\frac{t^r}{y} \in F(t) \setminus F$ is a constant, which is a contradiction. By a straightforward calculation, one can see that Eqs. (5.3) and (5.4) give partial fraction expansion of $\frac{v'}{v}$. Let

$$v = \sum_{1 \le i \le m} \sum_{1 \le j \le m_i} \frac{v_{ij}}{(t - a_i)^j} + v_0 + v_1 t + \cdots + v_n t^n,$$

where elements a_i, v_i, and v_{ij} belong to $\overline{F}$. Note that

$$\left(\frac{v_{ij}}{(t - a_i)^j}\right)' = \begin{cases} \frac{v'_{ij}}{(t-a_i)^j} + \frac{-jv_{ij}(x-a'_i)}{(t-a_i)^{j+1}} & \text{if } t' = x \in F \\ \frac{v'_{ij}}{(t-a_i)^j} + \frac{-jv_{ij}(xa_i-a'_i)}{(t-a_i)^{j+1}} + \frac{-jv_{ij}x}{(t-a_i)^j} & \text{if } \frac{t'}{t} = x \in F \end{cases} \tag{5.5}$$

and

$$(v_i t^i)' = \begin{cases} iv_i x t^{i-1} + v'_i t^i & \text{if } t' = x \in F \\ (v'_i + iv_i x)t^i & \text{if } \frac{t'}{t} = x \in F. \end{cases} \tag{5.6}$$

It is clear that if $\frac{t'}{t} \in F$, then the constant term of v' is v'_0. Suppose that v has a pole of order m_i at a_i. Then, $-m_i v_{im_i}(x - a'_i) \neq 0$ when $t' \in F$ and $-m_i v_{im_i}(xa_i - a'_i) \neq 0$ when $a_i \neq 0$ and $\frac{t'}{t} \in F$. Thus, Eq. (5.5) shows that v' has a pole at a_i of order $m_i + 1$. □

The following result is due to M. Rosenlicht [9]. Note that in the proof, only partial fraction expansions are required.

Theorem 5.5 ([9]) *Let* $F(t)$ *be a liouvillian extension of* F*. Assume that* t *is transcendental over* F *and* $C_{F(t)} = C_F$*. Let* $c_1, \ldots, c_n$ *be* $\mathbb{Q}$ *linearly independent constants,* $v, u_1, \ldots, u_n \in F(t)$*, and* $w \in F$ *such that*

$$v' + \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} = w.$$

(a) If $t' \in F$, then for each $1 \le i \le n$, $u_i \in F$ and $v = ct + \beta$ for some $c \in C_F$ and $\beta \in F$.
(b) If $\frac{t'}{t} \in F$, then $v \in F$, and for each $1 \le i \le n$, $u_i = \eta_i t^{m_i}$ for some integer m_i and $\eta_i \in F$.

Proof Let $u_i = \eta_i \prod_{1 \le j \le m} (t - a_j)^{m_{ij}}$ for $1 \le i \le n$, where $\eta_i \in F$, $0 = a_1, \dots, a_m$ are distinct elements in an algebraic closure $\overline{F}$ of F and m_{ij} are integers.

(a) From Theorem 5.4 part a,c, $\frac{u_i'}{u_i}$ has poles of order 1 only, and if v has poles, then v' has poles of order greater than 1. For cancellation to take place, we must have $\sum_{1 \le i \le n} c_i m_{ij} = 0$ for each j. Since $c_i's$ are $\mathbb{Q}$ linearly independent constants, every $m_{ij} = 0$. Thus, for each i, $u_i \in F$ and $v' \in F$. Using Theorem 5.3, we have $v = ct + \beta$ for some constant c and $\beta \in F$.
(b) Again from Theorem 5.4 part b,c, $\frac{u_i'}{u_i}$ has nonzero poles of order 1 only, and if v has nonzero poles, then v' has poles of order greater than 1. Thus, it follows that for all i and $j = 2, \dots, m$, $m_{ij} = 0$, $u_i = \eta_i t^{m_{i1}}$, and $v \in F$.

□

Theorem 5.6 ([10]) *Let $E \supset F$ be an algebraic extension of F and $C_E = C_F$. Assume that F is a liouvillian extension of C_F and there are $\mathbb{Q}$ linearly independent constants $c_1, \dots, c_n$, elements $u_1, \dots, u_n \in E^*$, and $v \in E$ such that*

$$v' + \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} \in F.$$

Then $v \in F$, and there is a nonzero integer m such that $u_i^m \in F$ for all $1 \le i \le n$.

Proof We use induction on tr degF/C_F. If tr deg$F/C_F = 0$, then $C_F = F = E$ and the result is trivial. Assume tr deg$F/C_F > 0$ and suppose that the result holds for smaller degrees.

Case I Let $v' + \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} = 0$. Choose a subfield F_0 of F that is a liouvillian extension of C_F and an element t such that t is transcendental over F_0 and F is algebraic over $F_0(t)$.

If $t' \in F_0$, then from Theorem 2 of [8], we conclude that $u_1, \dots, u_n$ are algebraic over F_0 and there exists a constant $c \in C_F$ such that $v + ct$ is algebraic over F_0. Thus, $(v+ct)' + \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} \in F_0$. Using induction hypothesis, we get $v+ct \in F_0$, that is, $v \in F$, and there exists a nonzero integer m such that $u_i^m \in F_0 \subset F$.

If $\frac{t'}{t} \in F_0$ then again from Theorem 2 of [8], observe that v is algebraic over F_0 and there exist integers $m_0 \neq 0, m_1, \dots, m_n$ such that for each i, $u_i^{m_0} t^{m_i}$ is algebraic over F_0. Thus

$$m_0 v' + \sum_{1 \le i \le n} c_i \frac{(u_i^{m_0} t^{m_i})'}{u_i^{m_0} t^{m_i}} - \sum_{1 \le i \le n} c_i m_i \frac{t'}{t} \in \overline{F_0}.$$

We again apply induction hypothesis to conclude that $v \in F_0 \subset F$ and that there exists a nonzero integer r such that $(u_i^{m_0} t^{m_i})^r \in F_0$. This implies $(u_i^{m_0})^r \in F_0(t) \subset F$. This proves the result in this particular case.

Case II Let $w \in F$ be such that $v' + \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} = w$. Let L be the smallest normal algebraic extension of F containing u_i, v. Let $N = [L : F]$ and consider the trace with respect to L and then

$$\sum_{1 \le i \le n} c_i \frac{\mathrm{Nr}(u_i)'}{\mathrm{Nr}(u_i)} + \mathrm{Tr}(v)' = Nw = N \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} + Nv'$$

and

$$\sum_{1 \le i \le n} c_i \frac{(\mathrm{Nr}(u_i) u_i^{-N})'}{\mathrm{Nr}(u_i) u_i^{-N}} + (\mathrm{Tr}(v) - Nv)' = 0,$$

where $\mathrm{Nr}(u_i)$ and $\mathrm{Tr}(v)$ denote the norm of u_i and v, respectively, in L over F. Now this reduces to case I. Therefore, $\mathrm{Tr}(v) - Nv \in F$ and $(\mathrm{Nr}(u_i) u_i^{-N})^r \in F$ for some nonzero integer r. Hence, $v \in F$ and $u_i^{rN} \in F$ for $1 \le i \le n$.

□

Remark 5.2 The condition that F must be liouvillian over C_F is essential to the proof of Theorem 5.6 (see [10], p.339). Consider the field of formal power series $F = \mathbb{C}((x))$ and the derivation $x' = 1$ on $\mathbb{C}((x))$. Let $E = F((x^{1/2}))$ and then $\frac{u'}{u} = v'$ where $u = exp(x^{1/2})$ and $v = x^{1/2}$. Here, $v, u^n \notin F$ for any number n.

5.3 Liouville's Theorem

We are now in a position to discuss a problem of integration in finite terms. One of the main results in this area is Liouville's theorem on integration in finite terms. In this section, we will discuss a proof of Liouville's theorem studied in purely algebraic terms by M. Rosenlicht [9].

Definition 5.7 A liouvillian extension E over F is called *elementary extension* if there is a tower of differential fields:

$$F = F_0 \subset F_1 \subset \cdots \subset F_n = E,$$

where for each $1 \le i \le n$, $F_i = F_{i-1}(t_i)$ and t_i satisfies one of the following:

(a) t_i is algebraic over F_{i-1}.
(b) $t_i' = \alpha' t_i$ for some $\alpha \in F_{i-1}$ (i.e. $t_i = e^{\alpha}$).
(c) $t_i' = \frac{\alpha'}{\alpha}$ for some $\alpha \in F_{i-1}$ (i.e. $t_i = \log(\alpha)$).

Elements of an elementary extension are called *elementary functions*.

Remark 5.3 Note that if $u_1, u_2 \in F$ and $c \in C_F$, then for any $\frac{r}{s} \in \mathbb{Q}$, we have

$$c\frac{u_1'}{u_1} + c\frac{r}{s}\frac{u_2'}{u_2} = \frac{c}{s}\frac{(u_1^s u_2^r)'}{u_1^s u_2^r}.$$

In general, if $u_1, \ldots, u_m \in F$ and $a_1, \ldots, a_m \in C_F$, then

$$\sum_{1 \le i \le m} a_i \frac{u_i'}{u_i} = \sum_{1 \le i \le n} b_i \frac{v_i'}{v_i},$$

where $b_1, \ldots, b_n$ is a $\mathbb{Q}-$basis for the vector space spanned by $a_1, \ldots, a_m$ over $\mathbb{Q}$. Then $v_i = \prod_{1 \le j \le m} u_j^{r_j}$ for some $r_j \in \mathbb{Z}$.

Consider an algebraic extension K of F. Let $\alpha \in K$ and $\sum_{0 \le i \le n} a_i x^i$ be the monic minimal polynomial of α over F. Then $\mathrm{tr}(\alpha) := -a_{n-1}$ and $\mathrm{nr}(\alpha) = (-1)^n a_0$. Let L be a finite Galois extension of F containing K. Let G be the Galois group of L over F and $n := [L : F]$. It is well known that trace and norm of α are given by $\mathrm{Tr}(\alpha) := \sum_{\sigma \in G} \sigma(\alpha)$ and $\mathrm{Nr}(\alpha) := \prod_{\sigma \in G} \sigma(\alpha)$. One can easily see that $\mathrm{Tr}(\alpha)$ and $\mathrm{Nr}(\alpha)$ lie in F and

$$\mathrm{Tr}(\alpha) = \frac{n}{m}\mathrm{tr}(\alpha) \quad \text{and} \quad \mathrm{Nr}(\alpha) = \mathrm{nr}(\alpha)^{\frac{n}{m}}.$$

Theorem 1 (Liouville) *Let E be an elementary field extension of F and $C_E = C_F$. Let $u \in E$ be such that $u' \in F$. Then there exists $\mathbb{Q}$ linearly independent constants $c_1, \ldots, c_n$, $g_1, \ldots, g_n \in F \setminus \{0\}$ and $w \in F$ such that*

$$u' = \sum_{1 \le i \le n} c_i \frac{g_i'}{g_i} + w'.$$

Proof We use induction on the length m of the tower

$$F = F_0 \subset F_1 \subset \cdots \subset F_m = E.$$

If $m = 0$, the result is trivial. Let $m > 0$; then by induction, the result holds for the tower $F_1 \subset \cdots \subset F_m = E$, that is,

$$u' = \sum_{1 \le i \le n} c_i \frac{g_i'}{g_i} + w',$$

where $c_i \in C_F$ and $g_i, w \in F_1$ for all i. Let $F_1 = F(t)$.

Case I Assume that t is transcendental over F. This case further reduces to two subcases.

Subcase I. If t is logarithm over F, that is, $t_1' = \frac{\alpha'}{\alpha}$ for some $\alpha \in F$, then we apply Theorem 5.5(a) and obtain $g_i \in F$ for all i and $w = ct + w_0$ for some $c \in C_F$ and $w_0 \in F$. Therefore,

$$u' = \sum_{1 \le i \le n} c_i \frac{g_i'}{g_i} + c\frac{\alpha'}{\alpha} + w_0'.$$

If $c, c_1, \ldots, c_n$ are $\mathbb{Q}$ linearly dependent, then as noted in Remark 5.3, the sum can be reduced further so that the constants are $\mathbb{Q}$ linearly independent.

Subcase II. If t is exponential over F, that is, $t' = \alpha' t$ for some $\alpha \in F$, then we apply Theorem 5.5(b) and obtain $w \in F$ and $g_i = \eta_i t^{m_i}$ where $\eta_i \in F$ and $m_i \in \mathbb{Z}$ for all i. Therefore,

$$u' = \sum_{1 \le i \le n} c_i \frac{\eta_i'}{\eta_i} + \sum_{1 \le i \le n} c_i m_i x' + w'.$$

Case II Assume that t is algebraic over F. Let L be a finite Galois extension of F that contains F_1. Let G be the Galois group of L over F. For any $\sigma \in G$, we have

$$u' = \sum_{1 \le i \le n} c_i \frac{(\sigma g_i)'}{\sigma g_i} + (\sigma w)'$$

and

$$[L : F]u' = \sum_{1 \le i \le n} c_i \sum_{\sigma \in G} \frac{(\sigma g_i)'}{\sigma g_i} + \sum_{\sigma \in G} (\sigma w)' = \sum_{1 \le i \le n} c_i \frac{\mathrm{Nr}(g_i)'}{\mathrm{Nr}(g_i)} + \mathrm{Tr}(w)'.$$

Since $\mathrm{Nr}(g_i), \mathrm{Tr}(w) \in F$, and constants $(1/[L : F])c_i$ are $\mathbb{Q}$ linearly independent, we obtained the desired result.

□

Using this theorem, M. Rosenlicht (see [9], p.160) proved that error functions and logarithmic integrals are nonelementary functions over $F = C(u)$, where $C_F = C$ and u is an indeterminate with derivative $u' = 1$.

Several mathematicians worked out the problem of integration in finite terms considering special functions, like error functions, dilogarithmic integrals, and logarithmic integrals, along with elementary functions. One can look into [1], [3], [4], and [11] for such topics.

5.4 Error Functions

For complex variable z, the error function is defined by the function

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, du\,.$$

This usual definition differs from our algebraic definition of error function by the multiplicative constant $\frac{2}{\sqrt{\pi}}$. The analogous definition in algebraic terminology is given below. This definition is due to Singer, Saunders, and Caviness [11].

Definition 5.8 Let F be a differential field and $\alpha \in F$. The *error function* of α is defined as

$$\mathrm{erf}(\alpha) = \int \alpha' e^{-\alpha^2}$$

and is denoted by $\mathrm{erf}(\alpha)$.

Theorem 5.7 *Error functions* $\mathrm{erf}(\alpha)$ *are non-elementary functions over* $\mathbb{C}(\alpha)$ *where* $\alpha' = 1$.

Proof Suppose $F = \mathbb{C}(\alpha, e^{-\alpha^2})$ and $C_F = \mathbb{C}$. Assume that an antiderivative of $e^{-\alpha^2}$ lies in some elementary extension of F. From Liouville's Theorem, we get that there are elements $u_1, \ldots, u_n \in F \setminus \{0\}$, $w \in F$, and $\mathbb{Q}$ linearly independent constants $c_1, \ldots, c_n$ such that

$$e^{-\alpha^2} = \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} + w'. \tag{5.7}$$

Note that if $e^{-\alpha^2}$ lies in some algebraic normal extension L of $\mathbb{C}(\alpha)$, then for any σ in a Galois group G of L over $\mathbb{C}(\alpha)$, we have

$$-2\,[L : \mathbb{C}(\alpha)]\,\alpha = \sum_{\sigma \in G} \frac{(\sigma e^{-\alpha^2})'}{\sigma e^{-\alpha^2}} = \frac{(\mathrm{Nr}(e^{-\alpha^2}))'}{\mathrm{Nr}(e^{-\alpha^2})}\,.$$

Let $2\,[L : \mathbb{C}(\alpha)] = n$; then we obtain that $v' = 2n\alpha v$ for some element $v \in \mathbb{C}(\alpha)$, which is absurd. Therefore, $e^{-\alpha^2}$ is transcendental over $\mathbb{C}(\alpha)$. Since only $\frac{u_i'}{u_i}$ contains poles of order 1 and c_is are $\mathbb{Q}$ linearly independent, we obtain each u_i is a multiple of some power of $e^{-\alpha^2}$ and $w = w_1 e^{-\alpha^2} + w_0$ for some $w_1, w_0 \in \mathbb{C}(\alpha)$. Comparing the coefficient of $e^{-\alpha^2}$ in Eq. (5.7), we have $1 = w_1' - 2\alpha w_1$, but there is no such element w_1 in $\mathbb{C}(\alpha)$. Hence, $\mathrm{erf}(\alpha)$ is non-elementary function over $F = \mathbb{C}(\alpha, e^{-\alpha^2})$ and hence over $\mathbb{C}(\alpha)$. □

5.5 Logarithmic Integrals

The integral representation for logarithmic integral for a positive real number $x \neq 1$ is given by

$$\mathrm{li}(x) = \int_0^x \frac{dv}{\log(v)}\,.$$

The analogous definition of logarithmic integrals in differential algebra is given as follows. This definition was given by Singer, Saunders, and Caviness [11].

Definition 5.9 A *logarithmic integral* of an element $\alpha \in F$ is defined as

$$\mathrm{li}(\alpha) = \int \frac{\alpha'}{\log \alpha}$$

and is denoted by $\mathrm{li}(\alpha)$.

Theorem 5.8 *Logarithmic integrals* $\mathrm{li}(\alpha)$ *are non-elementary functions over* $\mathbb{C}(\alpha)$ *where* $\alpha' = 1$.

Proof Suppose $F = \mathbb{C}(\alpha, \log(\alpha))$, where $C_F = \mathbb{C}$. Assume that an antiderivative of $\log(\alpha)$ is in an elementary extension of F. Using Liouville's Theorem, we obtain that there are elements $u_1, \ldots, u_n \in F \setminus \{0\}$, $w \in F$ and $\mathbb{Q}$ linearly independent constants $c_1, \ldots, c_n$ such that

$$\frac{1}{\log(\alpha)} = \sum_{1 \le i \le n} c_i \frac{v_i'}{v_i} + w'. \tag{5.8}$$

Since $\log(\alpha)$ is transcendental over $\mathbb{C}(\alpha)$, only $\frac{u_i'}{u_i}$ contains poles of order 1 and c_i's are $\mathbb{Q}$ linearly independent, we obtain $w \in \mathbb{C}(\alpha)$ and each $v_i = a_i \log(\alpha)$ for some constants a_i. Comparing the coefficient of $\frac{1}{\log(\alpha)}$, we have $1 = \sum_{1 \le i \le n} \frac{c_i a_i}{\alpha}$. That implies α is a constant and $\alpha' = 0$ which is a contradiction. Hence, $\mathrm{li}(\alpha)$ is nonelementary function over $F = \mathbb{C}(\alpha, \log(\alpha))$ and hence over $\mathbb{C}(\alpha)$. □

5.6 Dilogarithmic Integrals

The dilogarithmic function has been known for more than two centuries, but it remained a matter of interest to only few individuals for a long time. The functions have become much known in recent years due to their usage in mathematical physics, particularly conformal field theory, hyperbolic geometry, and algebraic K-theory. D. Zageir explained several properties and applications of dilogarithms in [12].

A *dilogarithm or Spence's function*, named after William Spence, a Scottish mathematician in early nineteenth century, is a function defined as

$$Li_2(x) = \sum_{1 \le i < \infty} \frac{x^n}{n^2}, \quad |x| < 1.$$

We recall that the Taylor series of logarithm around 1 is given by

$$-\log(1-x) = \sum_{1 \le i < \infty} \frac{x^n}{n}.$$

Thus, the name and definition of dilogarithm are analogous to the Taylor series of logarithm.

In general, such power series gives the functions named polylogarithms. We define a *polylogarithm* as

$$Li_m(x) = \sum_{1 \le i < \infty} \frac{x^n}{n^m}, \quad |x| < 1, \quad m \in \mathbb{N},$$

where $Li_1(x) = -\log(1-x)$. For $m > 1$, we have

$$\frac{d}{dx} Li_m(x) = Li_{m-1}(x)\frac{1}{x}.$$

Thus, its analytic continuation can be considered as

$$Li_2(x) = -\int_0^x \log(1-u)\frac{u'}{u} \qquad \text{for} \quad x \in \mathbb{C} \setminus [1, \infty).$$

We keep this analytic theory in mind and study dilogarithmic integrals from a purely algebraic standpoint. The following algebraic definition of dilogarithmic integrals is due to Singer, Saunders, and Caviness [11] and can also be found in [1].

Definition 5.10 Let $E \supseteq F$ be differential fields and $\alpha \in F \setminus \{0, 1\}$. The integral

$$\ell_2(\alpha) = -\int \frac{\alpha'}{\alpha} \log(1-\alpha)$$

in E is called *dilogarithmic integral* and is denoted by $\ell_2(\alpha)$.

From the definition, it is clear that if $\ell_2(\alpha) \in E$, then $\log(1-\alpha) \in E$ and $\ell_2(\alpha)$ is primitive over $F(\log(1-\alpha))$. We shall now explain some basic identities satisfied by dilogarithmic integrals.

Theorem 5.9 *Let E be a differential field extension of F and $\ell_2(\alpha) \in E$ for some $\alpha \in F \setminus \{0, 1\}$. Then $\ell_2(1/\alpha)$, $\ell_2(1-\alpha)$ lies in $E(\log \alpha)$ and for a constant c,*

(i) $\ell_2\left(\frac{1}{\alpha}\right) = -\ell_2(\alpha) - \frac{1}{2}\log^2 \alpha + c \log \alpha.$
(ii) $\ell_2(1-\alpha) = -\ell_2(\alpha) - \log \alpha \log(1-\alpha).$

Proof From the definition of dilogarithmic integral,

$$\ell_2\left(\frac{1}{\alpha}\right)' = -\frac{(1/\alpha)'}{(1/\alpha)} \log\left(1 - \frac{1}{\alpha}\right) = \frac{\alpha'}{\alpha} \log\left(\frac{\alpha-1}{\alpha}\right)$$

and

$$\ell_2(1-\alpha)' = -\frac{(1-\alpha)'}{1-\alpha} \log \alpha = -(\log \alpha \log(1-\alpha))' + \frac{\alpha'}{\alpha} \log(1-\alpha).$$

since $\log\left(\frac{\alpha-1}{\alpha}\right) = \log(1-\alpha) - \log \alpha + c$ for some constant c. Thus integrating the above equations, we shall obtain the desired expressions for $\ell_2(1/\alpha)$ and $\ell_2(1-\alpha)$, and clearly $\ell_2(1/\alpha)$, $\ell_2(1-\alpha)$ lies in $E(\log \alpha)$. □

Theorem 5.10 *Let C be a differential field with zero derivation. Dilogarithmic integrals $\ell_2(\alpha)$ are non-elementary functions over $C(\alpha)$, where $\alpha \notin \{0, 1\}$ and $\alpha' = 1$.*

Proof Note that the constant field of $C(\alpha)$ is C. Let $F = C(\alpha, \log(1-\alpha))$ and E be an elementary extension of F containing $\ell_2(\alpha)$. Then Liouville's Theorem's theorem implies that there exists $\mathbb{Q}$ linearly independent constants $c_1, \ldots, c_n$, elements $u_1, \ldots, u_n \in F \setminus \{0\}$, and $w \in F$ such that

$$\ell_2(\alpha)' = -\frac{\alpha'}{\alpha} \log(1-\alpha) = \sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} + w'. \tag{5.9}$$

Clearly, $\log(1-\alpha)$ is transcendental over $C(\alpha)$. Consider the partial fraction expansion of w and u_i for each i as done in Theorem 5.4, and note that w is a polynomial in $C(\alpha)[\log(1-\alpha)]$ with $\deg(w) \le 2$. Since only $\frac{u_i'}{u_i}$ contains poles of order 1, c_is are $\mathbb{Q}$ linearly independent, and $\sum_{1 \le i \le n} c_i \frac{u_i'}{u_i} \in C(\alpha)$, using

Theorem 5.5, we obtain $u_i \in C(\alpha)$. Let $w = c\log^2(1-\alpha) + w_1\log(1-\alpha) + w_0$, where $c \in C$, $w_1, w_0 \in C(\alpha)$. We compare the coefficients of $\log(1-\alpha)$ and obtain that $w_1' = -\frac{\alpha'}{\alpha} - 2c\frac{(1-\alpha)'}{1-\alpha}$. But there is no such element in $C(\alpha)$; therefore, we arrive at a contradiction. Thus, E must be a nonelementary extension of F,, and hence, $\ell_2(\alpha)$ is non-elementary function over $C(\alpha)$. □

Acknowledgments We thank the referee for thoroughly reading this chapter and for his valuable suggestions.

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Chapter 6
On Equalities of Central Automorphism Group with Various Automorphism Groups

Harpal Singh and Sandeep Singh

6.1 Introduction

Throughout the chapter, p denotes a prime number. For group G, we denote by $G^{'}$, $Z(G)$, $cl(G)$, $d(G)$, $\Phi(G)$, and $\mathrm{Aut}(G)$, respectively, the commutator subgroup, the center, the nilpotency class, the rank, the Frattini subgroup, and the automorphism group of G. An automorphism σ of group G is called central if σ commutes with every automorphism in $\mathrm{Inn}(G)$, the group of inner automorphisms of G, (equivalently, if $g^{-1}\sigma(g)$ lies in the center $Z(G)$ of G, for all g in G.)

The central automorphisms of G fix the commutator subgroup of G elementwise and form a normal subgroup of the full automorphism group $\mathrm{Aut}(G)$; we denote this subgroup by $\mathrm{Aut}_{\mathrm{z}}(G)$ in this paper. For groups G having $\mathrm{Aut}(G)$ abelian, it is necessarily the case that $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}(G)$. The non-abelian groups G with $\mathrm{Aut}(G)$ abelian are called as Miller groups (see [19]). However, several people constructed various groups G for which $\mathrm{Aut}(G)$ is non-abelian and $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}(G)$ (see [7, 11, 15, 18]). In 2001, Curran and McCaughan [6] considered the case where the central automorphisms are just the inner automorphisms of G, that is, $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Inn}(G)$; one can also see [4, 23]. Continuing in this direction, in 2004, Curran [8], for group G, derived the equality $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Z(Inn(G))}$,; the same is derived in [1, 12, 22]. Let $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)$ be the set of all central automorphisms of a group G which fixes the center $Z(G)$ of G elementwise. In 2007, Attar [2] characterized finite p-groups for which $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \mathrm{Inn}(G)$ holds. In 2009, Yadav

The second author is thankful to DST for the funding support under the file no MTR/2022/000331

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_6

[25] characterized p-groups of nilpotency class 2 for which $\mathrm{Aut}_z(G) = \mathrm{Aut}_z^z(G)$ (for the same equality, also see [14]).

An automorphism ϕ of a group G is called class preserving if $\phi(x)$ is conjugate to x for all $x \in G$. The set $\mathrm{Aut_c(G)}$ of all class-preserving automorphisms of G forms a normal subgroup of $\mathrm{Aut}(G)$ and contains $\mathrm{Inn}(G)$. In 2013, Yadav [26] characterized finite p-groups, and Kalra and Gumber [16] characterized all finite p-groups of order $\leq p^6$ (for any prime p) and $\leq p^5$ (for odd prime p) for which the set of all central automorphisms is equal to the set of all class-preserving automorphisms, that is, $\mathrm{Aut}_z(G) = \mathrm{Aut_c(G)}$; the same equality is derived in [10].

An automorphism σ of a group G is called IA-automorphism if it induces the identity automorphism on the abelian quotient $G/G^{'}$. Let $\mathrm{IA_z(G)}$ be the group of those IA automorphisms which fix the center of G elementwise. In 2014, Rai [21] characterized finite p-groups in which $\mathrm{Aut}_z(G) = \mathrm{IA_z(G)}$ if and only if $\gamma_2(G) = Z(G)$. In 2016, Kalra and Gumber [17], characterized finite non-abelian p-groups G for which $\mathrm{Aut}_z(G) = \mathrm{IA_z(G)}$ if and only if $G^{'} = Z(G)$.

Hegarty [13] defined the notions of absolute center and autocommutator of a group G (analogous to $Z(G)$ and G^* as follows:

$$L(G) = \{g \in G | \, \alpha(g) = g \, \forall \alpha \in \mathrm{Aut}(G)\}$$

$$G^* = \langle g^{-1} \alpha(g) | g \in G, \, \alpha \in \mathrm{Aut}(G) \rangle$$

These are clearly characteristic subgroups of G. Also, $Z(G) \supset L(G)$ and $G^{'} \subset G^*$. Hegarty [13] also defined absolute central automorphism of G as follows: an automorphism γ of a group G is called an absolute central automorphism if it induces identity automorphism on $G/L(G)$. The set $\mathrm{Aut_l}(G)$ of all absolute central automorphisms of G forms a normal subgroup of $\mathrm{Aut}(G)$; it is also a subgroup of $\mathrm{Aut}_z(G)$. Let $\mathrm{Aut}_l^z(G)$ denote the group of absolute central automorphisms of G which fix $Z(G)$ elementwise.

In 2020, Singh and Gumber [24] gave a necessary and sufficient condition on finite p-group G for which $\mathrm{Aut}_z(G) = \mathrm{Aut_l}(G)$ and also for which $\mathrm{Aut}_z(G) = \mathrm{Aut}_l^z(G)$.

6.2 Equalities of Central Automorphisms

6.2.1 Equalities with Group of All Automorphisms

Definition 6.1 Following Earnley, a non-abelian group with abelian automorphism group is called Miller group.

If $\mathrm{Aut}(G)$ is abelian, then it is clear that all the automorphisms are central, i.e., $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$. The obvious examples of groups with abelian automorphism group are the cyclic groups. There are non-abelian groups with abelian automorphism groups; these are called Miller groups ($see Earnley$ [9]). Several researchers constructed various examples of groups for which $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$, even if $\mathrm{Aut}(G)$ is non-abelian. Curran, in 1982, found first such example. He constructed a group of order 2^7 for which $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian.

Theorem 6.1 ([7], Proposition, p. 394) *There exists a non-abelian group G of order 2^7 which has a non-abelian automorphism group of order 2^{12} in which every automorphism is central, that is,* $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$.

Example of such group is given below:

Let M be the Miller group of order 2^6, and let

$$G = M \times Z_2 = \langle a, b, c, d \mid a^8 = b^4 = c^2 = d^2 = 1,\ a^b = a^5,\ b^c = b^{-1},$$

$$[a, c] = [a, d] = [b, d] = [c, d] = 1\rangle$$

This result of Curran leads the motivation to p-groups for p an odd prime in which $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian. In 1984, Malone proved the following result:

Theorem 6.2 ([18], Proposition, p. 36) *For each odd prime p, there exists a non-abelian p-group with a non-abelian automorphism group in which each automorphism is central, that is,* $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$.

For each odd prime p, we consider the group

$$F = \langle a_1,\ a_2,\ a_3,\ a_4,\ \mid (a_i,\ a_j,\ a_k) = 1\, and\, a_i^{p^2} = 1\ \ for$$

$$1 \leq i, j, k \leq 4;\ (a_1, a_2) = a_1^p;\ (a_1, a_3) = a_3^p;\ (a_1, a_4) = a_4^p$$

$$(a_2, a_3) = a_2^p;\ (a_2, a_4) = 1\,;\, (a_3, a_4) = a_3^p\rangle$$

$\mathrm{Aut}(F)$ is abelian group. We set $B = \langle\, b \mid b^p = 1\,\rangle$. Group $G = F \times B$ is non-abelian group which has $\mathrm{Aut}(G)$ non-abelian in which each automorphism is central.

Curran in [7] and Malone in [18] derived the examples of groups with direct factors for which $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian. The question was left if there is a group G with no direct factors for which $\mathrm{Aut}_z(G) = \mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is non-abelian. Continuing in this direction, in 1986, Glasby produced an infinite family of 2-groups having no direct factors and which have a non-abelian automorphism group in which all automorphisms are central.

Definition 6.2 Define G_n to be the group generated by $x_1, \ldots, x_n$ $x_i^{2^i} = 1$ $(1 \leq i \leq n)$ $[x_i, x_{i+1}] = x_{i+1}^{2^i}$, $(1 \leq i < n)$. $[x_i, x_j] = 1$, $(1 < i + 1 < j \leq n)$

Theorem 6.3 ([11], Theorem, p. 234) *For $n \geq 3$, G_n has no direct factors, and* $\mathrm{Aut}(G_n)$ *is non-abelian of order* $2^{p(n)}$*, where* $p(n) = (n-1)(2n^2 - n = 6/6)(n \geq 4)$*, in which every* automorphism *is central.*

In 2012, Jain and Yadav [15] constructed the following family of groups G_n with no direct factor, for which $\mathrm{Aut_z}(G_n) = \mathrm{Aut}(G_n)$.

Definition 6.3 Let n be a natural number greater than 2 and p an odd prime. Define G_n to be the group generated by x_1, x_2, x_3, x_4

$$x_1^{p^n} = x_2^{p^3} = x_3^{P^2} = x_4^{p^2} = 1,$$

$$[x_1, x_2] = x_2^{p^2}, \quad [x_1, x_3] = x_3^{p}$$

$$[x_1, x_4] = x_4^{p}, \quad [x_2, x_3] = x_1^{p^{n-1}}$$

$$[x_2, x_4] = x_2^{p^2}, \quad [x_3, x_4] = x_4^{p}.$$

This group G is a regular p-group of nilpotency class 2 having order p^{n+7} and exponent p^n. Further, $Z(G) = \Phi(G)$ and therefore G is purely non-abelian.

Theorem 6.4 ([15], Theorem A, p. 228) *Let $m = n + 7$ and p be an odd prime, where n is a positive integer greater than or equal to* 3*. Then there exists a group G of order p^m, exponent p^n, and with no nontrivial abelian direct factor such that* $\mathrm{Aut_z}(G) = \mathrm{Aut}(G)$ *is non-abelian.*

6.2.2 Equalities with Group of Inn(G) *and* Z(Inn(G))

In 2001, Curran and McCaughan [6] characterized finite p-groups in which central automorphisms are precisely the inner automorphisms.

Theorem 6.5 ([6], Theorem, p. 2081) *If G is a finite p-group, then* $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ *if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.*

Definition 6.4 A group G, whose only element of finite order is the identity, is called torsion-free group.

Definition 6.5 A non-abelian group G is purely non-abelian if G has no nontrivial abelian direct factor.

In 2016, Azhdari characterized all finitely generated groups G for which the equality $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ holds. He proved the following:

Theorem 6.6 ([4], Theorem 2, p. 4134) *Let G be a finitely generated group. Then* $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ *if and only if one of the following assertion holds:*

- *G is purely non-abelian and $Z(G) = G^{'}$ is cyclic.*
- *$G \cong C_2 \times N$ where N is purely non-abelian with $|Z(N)|$ odd and $Z(N) = N^{'}$ is cyclic (or $Z(G) = C_2 \times G^{'}$ is cyclic).*
- *G is torsion-free with $Z(G) = G^{'}$ is cyclic and $det(M_G) = 1$ where M_G is skew-symmetric matrix corresponding to G.*

In 2018, Sharma et al. [23] verified the equality $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ for the finite p-groups of order up to p^7 as follows:

Theorem 6.7 ([23], Theorem 2.1, p. 3) *There is no* p-group *G of order up to p^6 satisfying* $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$.

Theorem 6.8 ([23], Theorem 2.2, p. 3) *A* p-group *G of order p^7 satisfies* $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ *if and only if $Z(G) \cong C_p^2$, $|G^{'}| = p^4$ and $cl(G) = 4$.*

In 2004, Curran [8] considered the case where the central automorphism group is as small as possible. Clearly, $\mathrm{Z(Inn(G))} \leq \mathrm{Aut_z}(G)$, for any group G. When G is arbitrary, $\mathrm{Aut_z}(G)$ and $\mathrm{Z(Inn(G))}$ may coincide because both these subgroups of $\mathrm{Aut}(G)$ can be trivial. However, the situation becomes interesting if G is a p-group, since both subgroups are nontrivial.

Theorem 6.9 ([8], Theorem 1.1, p. 223) *Let G be a finite non-abelian p-group. If* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$*, then $Z(G) \leq G^{'}$, and furthermore,* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $\mathrm{Hom}(G/G, Z(G)) \approx Z(G/Z(G))$.

In 2013, Sharma and Gumber [22] characterized p-groups of order $\leq p^5$(for any prime p) and of order p^6(for p odd), for which $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$.

Theorem 6.10 ([22], Theorem 3.2, p. 3) *Let G be p-group of order p^5 and $cl(G) = 3$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if $d(G) = 2$ and $Z(G) \cong C_p$.*

Theorem 6.11 ([22], Theorem 3.3, p. 3) *Let G be a p-group of order p^6, for an odd prime p, and $cl(G) = 3\,0r4$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if $d(G) = 2$ and $Z(G) \cong C_p$.*

Continuing the study of Curran [8] of minimum order of $\mathrm{Aut_z}(G)$, Gumber and Kalra [12] obtained the following:

Let $G/G^{'} \cong C_{p^{r_1}} \times \ldots C_{p^{r_n}}$ $(r_1 \geq \cdots \geq r_n \geq 1)$ and $Z_2(G)/Z(G) \cong C_{p^{s_1}} \times \ldots C_{p^{s_m}}$ $(s_1 \geq \cdots \geq s_m \geq 1)$.

Theorem 6.12 ([12], Theorem 2.1, p. 1803) *Let G be a finite p-group with $Z(G) \cong C_{p^{b_1}}$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if either $G/G^{'} \cong Z_2(G)/Z(G)$ or $d(G) = d(Z_2(G)/Z(G))$, $s_i = b_1$ for $1 \leq i \leq c$, and $s_i = r_i$ for $c + 1 \leq i \leq n$, where c, $1 \leq c \leq n$ is the largest such that $r_c \geq b_1$.*

Definition 6.6 The coclass of a finite p-group G of order p^n is $n-c$, where c is the class of the group.

Corollary 6.1 ([12], Corollary 2.2, p. 1804) *Let G be a finite p-group of coclass* 2. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2$.

Corollary 6.2 ([12], Corollary 2.3, p. 1804) *Let G be a finite p-group of coclass* 3. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2,\ 3$ *or* $Z(G) \cong C_{p^2}$ *and* $Z_2(G)/Z(G) \cong G/G^{'}$.

Corollary 6.3 ([12], Corollary 2.4, p. 1804) *Let G be a finite p-group of coclass* 4. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if one of the following conditions holds:*

(a) $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2, 3, 4$.
(b) $Z(G) \cong C_{p^2}$ *and either* $Z_2(G)/Z(G) \cong G/G^{'}$ *or* $Z_2(G)/Z(G) \cong C_{p^2} \times C_p$ *and* $G/G^{'} \cong C_{p^3} \times C_p$ *or* $Z_2(G)/Z(G) \cong C_{p^2} \times C_p$ *and* $G/G^{'} \cong C_{p^4} \times C_p$.
(c) $Z(G) \cong C_{p^3}$ *and* $Z_2(G)/Z(G) \cong G/G^{'}$.

Gumber and Kalra also generalized the results of Sharma and Gumber [22] as follows:

Theorem 6.13 ([12], Theorem 3.1, p. 1804) *Let G be p-group of order* $= p^5$ *and* $cl(G) = 3$. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2$.

Theorem 6.14 ([12], Theorem 3.2, p. 1805) *Let G be a finite* p-group *such that* $cl(G) = 3$ *or* 4. *Then,* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $d(G) = d(Z_2(G)/Z(G)) = 2$.

Also, Gumber and Kalra obtained the result for $|G| = p^7$ *as in [22]; it was up to* p^6.

Theorem 6.15 ([12], Theorem 3.3, p. 1805) *Let G be a p-group of order* p^7. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if one of the following holds:*

$cl(G) = 3$, $Z(G) \simeq C_p$ *and* $rank(G) = rank(Z_2(G)/Z(G)) = 2,\ 3,\ 4$.
$cl(G) = 4$ *and either* $Z(G) \simeq C_p$ *and* $rank(G) = rank(Z_2(G)/Z(G)) = 2,\ 3$ *or* $Z(G)$ *is cyclic group of order* p^2 *and* $Z_2(G)/Z(G) \simeq G/G^{'}$.
$cl(G) = 5$, $Z(G) \simeq C_p$ *and* $rank(G) = rank(Z_2(G)/Z(G)) = 2$.

Let G be a non-abelian p-group G. Let $G/G^{'} \cong C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_r}}$ $(c_1 \geq \cdots \geq c_r \geq 1)$ and $Z_2G/Z(G) \cong C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_s}}$ $(d_1 \geq d_2 \geq \ldots d_s \geq 1)$, where $C_{p^{a_i}}$ is a cyclic group of order p^{a_i}.

In 2020, Attar [1] characterized the finite p-groups in some special cases, including p-groups G with $C_G(Z(\Phi(G)) \neq \Phi(G)$, p-groups with an abelian maximal subgroup, metacyclic p-groups with $p \geq 2$, p-groups of order p^n and exponent p^{n-2}, and Camina p-groups, for which $\mathrm{Aut_z}(G)$ is of minimal order, as follows:

Theorem 6.16 ([1], Theorem 3.1, p. 4) *Let G be a finite* p-group *such that $C_G(Z(\Phi(G)) \neq \Phi(G)$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if $Z(G)$ is cyclic and one of the following is true:*

- $G/G' \cong Z_2(G)/Z(G)$.
- *$r = s$, $d_i = h$ for $1 \leq i \leq t$, $d_i = c_i$ for $t + 1 \leq i \leq r$, where $p^h = exp(Z(G))$ and t is the largest integer between* 1 *and s such that $c_t > h$.*

Corollary 6.4 ([1], Corollary 3.2, p. 5) *Let G be a non-abelian finite p-group with an abelian maximal subgroup. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.*

Theorem 6.17 ([1], Theorem 3.3, p. 6) *Let G be a non-abelian metacyclic finite p-group with $p > 2$. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if $Z(G) \leq G'$.*

Corollary 6.5 ([1], Corollary 3.4, p. 6) *The finite non-abelian* p-groups *G of order p^n and exponent p^{n-1} for which* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *are of the following isomorphism types:*

(1) $M(p^3) = \langle \alpha, \beta \mid \alpha^{p^2} = \beta^p = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+p}\rangle (p > 2)$.
(2) $D_8 = \langle \alpha, \beta \mid \alpha^4 = \beta^2 = 1,\ \beta^{-1}\alpha\beta = \alpha^{-1}\rangle$.
(3) $Q_8 = \langle \alpha, \beta \mid \alpha^4 = 1,\ \beta^2 = \alpha^2,\ \beta^{-1}\alpha\beta = \alpha^{-1}\rangle$.

Corollary 6.6 ([1], Corollary 3.5, p. 7) *Let p be an odd prime. Then finite non-abelian* p-groups *of order p^n and exponent p^{n-2} for which* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *are one of the following isomorphism types:*

(1) $G = \langle \alpha,\ \beta,\ \gamma \mid \alpha^p = \beta^p = \gamma^p = 1,\ \alpha\beta = \beta\alpha,\ \gamma^{-1}\alpha\gamma = \alpha\beta,\ \beta\gamma = \gamma\beta\rangle$.
(2) $G = \langle \alpha,\ \beta \mid \alpha^{p^3} = \beta^{p^2} = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+p}\rangle$.
(3) $G = \langle \alpha,\ \beta \mid \alpha^{p^4} = \beta^{p^2} = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+p^2}\rangle$.

Corollary 6.7 ([1], Corollary 3.6, p. 8) *The finite non-abelian* 2*-groups G of order 2^n and exponent 2^{n-2} for which* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *are one of the following:*

(1) $G = \langle \alpha,\ \beta,\ \gamma \mid \alpha^8 = \beta^2 = \gamma^2 = 1,\ \beta^{-1}\alpha\beta = \alpha^5,\ \gamma^{-1}\alpha\gamma = \alpha\beta,\ \beta\gamma = \gamma\beta\rangle$.
(2) $G = \langle \alpha,\ \beta,\ \gamma \mid \alpha^{2^{n-2}} = 1,\ \beta^2 = 1,\ \gamma^2 = \beta,\ \beta^{-1}\alpha\beta = \alpha^{1+2^{n-3}},\ \gamma^{-1}\alpha\gamma = \alpha^{-1}\beta,\rangle$.
(3) $G = \langle \alpha,\ \beta \mid \alpha^{16} = \beta^4 = 1,\ \beta^{-1}\alpha\beta = \alpha^5\rangle$.
(4) $G = \langle \alpha,\ \beta \mid \alpha^{2^{n-2}} = 1,\ \beta^4 = 1,\ \beta^{-1}\alpha\beta = \alpha^{-1+2^{n-4}}\rangle$*, where $n \geq 6$.*
(5) $G = \langle \alpha,\ \beta,\ \gamma \mid \alpha^{2^{n-2}} = 1,\ \beta^2 = 1,\ \gamma^2 = 1,\ \beta^{-1}\alpha\beta = \alpha^{1+2^{n-3}},\ \gamma^{-1}\alpha\gamma = \alpha^{-1+2^{n-4}}\beta,\ \beta\gamma = \gamma\beta\rangle$*, where $n \geq 6$.*

(6) $G = \langle \alpha, \beta, \gamma \mid \alpha^{2^{n-2}} = 1, \beta^2 = 1, \gamma^2 = \alpha^{2^{n-3}}, \beta^{-1}\alpha\beta = \alpha^{1+2^{n-3}}, \gamma^{-1}\alpha\gamma = \alpha^{-1+2^{n-4}}\beta, \beta\gamma = \gamma\beta \rangle$, *where* $n \geq 6$.

(7) $G = \langle \alpha, \beta, \gamma \mid \alpha^8 = 1, \beta^2 = 1, \gamma^2 = \alpha^4, \beta^{-1}\alpha\beta = \alpha^5, \gamma^{-1}\alpha\gamma = \alpha\beta, \beta\gamma = \gamma\beta \rangle$.

A pair (G, N) is called Camina pair if $1 < N < G$ is normal subgroup of G and for every element $g \in G/N$, the element g is conjugate to all gN.

Theorem 6.18 ([1], Theorem 3.7, p. 12) *Let G be a non-abelian finite* p-group *such that* $(G, Z(G))$ *is a Camina pair. Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $Z(G) \cong C_p$ *and* $G/G' \cong Z_2(G)/Z(G)$.

Theorem 6.19 ([1], Corollary 3.8, p. 12) *Let G be a finite non-abelian Camina* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Z(Inn(G))}$ *if and only if* $G' = Z(G)$ *and* $Z(G)$ *is cyclic.*

6.2.3 Equalities with Class-Preserving Automorphisms

For a finite p-group G, the subgroup $\Omega_m(G)$ is defined as $\langle x \in G \mid x^{p^m} = 1 \rangle$, and $\mho_m(G)$ is defined as $\langle x^{p^m} \mid x \in G \rangle$. For a finite p-group G with $cl(G) = 2$, $G/Z(G)$ is abelian. Consider the following cyclic decomposition of $G/Z(G)$:

$$G/Z(G) \cong C_{p^{e_1}} \times \ldots \times C_{p^{e_k}} \quad (e_1 \geq e_2 \geq \cdots \geq e_k \geq 1).$$

In 2013, Yadav (see [26]) and Kalra and Gumber (see [16]) characterized p-groups of class 2 with $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ as follows:

Theorem 6.20 ([26], Theorem A, p. 2) *Let G be a finite* p-group *of class* 2. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $G' = Z(G)$ *and* $|\mathrm{Aut_c(G)}| = \Pi_{i=1}^{d} |\Omega_{m_i}(G')|$

Theorem 6.21 ([26], Theorem B, p. 2) *Let G be a finite* p-group *and* $cl(G) = 2$ *with* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *and then rank of G is even.*

Theorem 6.22 ([16], Theorem 3.1, p. 3) *Let G be a finite* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $\mathrm{Aut_c(G)} \cong Hom(G/Z(G), G')$ *and* $G' = Z(G)$.

Theorem 6.23 ([16], Theorem 3.3, p. 4) *Let G be a finite non-abelian* p-group *such that the center of the group is elementary abelian. Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if G is a Camina* p-group *and* $cl(G) = 2$.

Theorem 6.24 ([16], Theorem 3.4, p. 4) *Let G be a finite non-abelian* p-group *such that* $Z(G)$ *is cyclic. Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $Z(G) = G'$.

Definition 6.7 A finite p-group G of class 2 is said to have property $(*)$ if for some $\pi\ \mho_{m_i^\pi}(\Omega_{n_i}(Z(G)) \leq [x, G]$ for all $x \in G/Z(G)$ and $i \in \{1, \ldots, k\}$.

In 2015, Ghoraishi found a necessary and sufficient condition for a finite p-group G to satisfy $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$, as follows:

Theorem 6.25 *Let G be a finite* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_c(G)}$ *if and only if* $Z(G) = G^{'}$ *and G has property* $(*)$.

6.2.4 Equalities with Absolute Central and IA Automorphisms

Definition 6.8 A finite non-Abelian group G is said to be purely non-Abelian if it has no nontrivial Abelian direct factor.

Let $C_{\mathrm{Aut}(G)}(\mathrm{Aut_l}(G)) = \{\alpha \in \mathrm{Aut}(G) \,|\, \alpha\,\beta = \beta\,\alpha,\ \forall\ \beta \in \mathrm{Aut_l}(G)\}$ denote the centralizer of $\mathrm{Aut_l}(G)$ in $\mathrm{Aut}(G)$. In [20], Moghaddam and Safa defined $E(G) = [G, C_{\mathrm{Aut}(G)}(\mathrm{Aut_l}(G))] = \langle g^{-1}\alpha(g) \,|\, g \in G,\ \alpha \in C_{Aut(G)}(\mathrm{Aut_l}(G))\rangle$. One can easily see that $E(G)$ is a characteristic subgroup of G containing the derived group $G^{'} = [G, \mathrm{Inn}(G)]$, and each absolute central automorphism of G fixes $E(G)$ elementwise [20, Theorem C].

Let

$$G/E(G) \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_k}}, \quad (e_1 \geq \ldots e_k \geq 1)$$

$$G/G^{'} \cong C_{p^{f_1}} \times C_{p^{f_2}} \times \cdots \times C_{p^{f_l}}, \quad (f_1 \geq \ldots f_l \geq 1)$$

$$L(G) \cong C_{p^{g_1}} \times C_{p^{g_2}} \times \cdots \times C_{p^{g_m}}, \quad (g_1 \geq \ldots g_m \geq 1)$$

$$Z(G) \cong C_{p^{h_1}} \times C_{p^{h_2}} \times \cdots \times C_{p^{h_n}} \quad (h_1 \geq \ldots h_n \geq 1).$$

Since $G/E(G)$ is a quotient group of $G/G^{'}$, it follows that $k \leq l$ and $e_i \leq f_i$ for all $1 \leq i \leq k$.

In the same year, M. Singh and D. Gumber [24] obtained the equalities of $\mathrm{Aut_z}(G)$ with $\mathrm{Aut_l}(G)$, the group of absolute central automorphisms, and $\mathrm{Aut_l^z}(G)$, the group of absolute central automorphisms that fix the center elementwise, as follows:

Theorem 6.26 ([24], Theorem 1, p. 864) *Let G be a finite non-Abelian* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{Aut_l^z}(G)$ *if and only if either* $L(G) = Z(G)$ *or* $Z(G) \leq \Phi(G)$, $G^{'} = E(G)$, $m = n$, *and* $e_1 \leq g_t$, *where t is the largest integer between* 1 *and m such that* $g_t < h_t$.

Theorem 6.27 ([24], Theorem 2, p. 865) *Let G be a finite non-abelian* p-group *such that $L(G) < Z(G)$. Then* $\mathrm{Aut_z}(G) = \mathrm{Aut}_1^{\mathrm{z}}(G)$ *if and only if $Z(G) \leq \Phi(G)$, $G^{'} = E(G)Z(G)$, $m = n$, $e_1 \leq g_t$, where t is the largest integer between* 1 *and m such that $g_t < h_t$.*

In 2014, Rai [21] characterized finite p-groups for which $\mathrm{Aut_z}(G) = \mathrm{IA_z(G)}$, where $\mathrm{IA_z(G)}$ denote the group of those IA automorphisms which fix the center elementwise, as follows:

Theorem 6.28 ([21], Theorem B(1), p. 170) *Let G be a finite p-group. Then* $\mathrm{Aut_z}(G) = \mathrm{IA_z(G)}$ *if and only if $G^{'} = Z(G)$.*

Let X and Y be the two finite abelian p-groups, and let $X \cong C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_i}}$ and $Y \cong C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_j}}$ be the cyclic decomposition of X and Y, where $a_t \geq a_{t+1}$ and $b_s \geq b_{s+1}$ are positive integers. If either X is proper subgroup or proper quotient group of Y and $d(X) = d(Y)$, then there certainly exists r, $1 \leq r \leq i$ such that $a_r < b_r$, $a_k = b_k$ for $r + 1 < k < i$. For this unique fixed r, let $var(X, Y) = p^r$. In other words, $var(X, Y)$ denotes the order of the last cyclic factor of X whose order is less than that of corresponding cyclic factor of Y.

In 2016, Kalra and Gumber obtained $\mathrm{Aut_z}(G) = \mathrm{IA_z(G)}$ for finite non-abelian p-groups as follows:

Theorem 6.29 ([17], Theorem 2.12, p. 5) *Let G be a finite non-abelian* p-group. *Then* $\mathrm{Aut_z}(G) = \mathrm{IA_z(G)}$ *if and only if either $G^{'} = Z(G)$ or $G^{'} < Z(G)$, $d(G^{'}) = d(Z(G))$ and $exp(G/G^{'}) \leq var(G^{'}, Z(G))$.*

6.2.5 Equalities with Central Automorphisms Fixing the Center Elementwise

In 2007, Attar [2] characterized groups in which the central automorphisms fixing the center elementwise are precisely inner automorphisms, as follows:

Theorem 6.30 ([2], Theorem, p. 297) *If G is a* p-group *of finite order, then* $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{Z}}(G) = \mathrm{Inn}(G)$ *if and only if G is abelian or nilpotency class of G is* 2 *and $Z(G)$ is cyclic.*

Let G be a finite p-group of class 2. Then $G/Z(G)$ and $G^{'}$ have equal exponent $p^C(say)$. Let

$$G/Z(G) \cong C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_m}} \quad (c_1 \geq \cdots \geq c_m \geq 1)$$

where $C_{p^{c_i}}$ is a cyclic group of order p^{c_i}, $1 \leq i \leq r$. Let k be the largest integer between 1 and r such that $c_1 = c_2 = c_k = e$. Note that $k \geq 2$. “Let M be the subgroup of G containing $Z(G)$ such that

$$\bar{M} = M/Z(G) = C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_k}}.\text{”}$$

Let

$$G/G^{'} \cong C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_n}} \;\; d_1 \geq d_2 \geq \ldots d_s \geq 1$$

be a cyclic decomposition of $G/G^{'}$ such that $\bar{M}$ is isomorphic to a subgroup of

$$\bar{N} = N/G^{'} := C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_k}}.$$

In 2009, using the above terminology, Yadav proved the following:

Theorem 6.31 ([25], Theorem, p. 4326) *Let G be a finite p-group of class 2. Then* $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)$ *if and only if $m = n$, $G/Z(G)/\bar{M} \cong (G/G^{'})/\bar{N}$, and $exp(Z(G)) = exp(G^{'})$.*

In 2011, Azhdari and Akhavan-Malayeri [5] generalized the result of Attar in [2] for the finitely generated groups of nilpotency class 2. They got the following:

Theorem 6.32 ([5], Theorm 0.1, p. 1284) *Let G be a finitely generated of $cl(G) = 2$. Then* $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \mathrm{Inn}(G)$ *if and only if $Z(G) \cong C_p$ or $Z(G) \cong C_n \times \mathbb{Z}^s$ where $exp(G/Z(G))/n$ and s is torsion-free rank of $Z(G)$.*

Theorem 6.33 ([5], Corollary 0.2) *Let G be a finitely generated group of class 2, which is not torsion-free. Then* $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \mathrm{Inn}(G)$ *if and only if $cl(G) = 2$ and $Z(G)$ is cyclic or $Z(G) \cong C_n \times \mathbb{Z}^s$ with $exp(G/Z(G))$ divides n and s is torsion-free rank of $Z(G)$.*

Theorem 6.34 ([5], Corollary 0.3) *Let G be a finitely generated of $cl(G) = 2$. $G^{'}$ is torsion-free, and* $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \mathrm{Inn}(G)$ *if and only if $Z(G)$ is infinite cyclic.*

In the same year, Jafari also found a necessary and sufficient condition on a finite p-group G such that $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)$, as follows:

Theorem 6.35 *Let G be a finite* p-group. *Then* $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)$ *if and only if $Z(G)G' \subseteq G^{p^n}G'$, where* $\exp(Z(G)) = p^n$.

Let G be a non-abelian finite p-group. Let

$$G/G^{'} = C_{p^{c_1}} \times C_{p^{c_2}} \times \cdots \times C_{p^{c_r}} \;\; (c_1 \geq \ldots c_r \geq 1).$$

$$G/G^{'}Z(G) \cong C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_s}} \;\; (d_1 \geq \ldots d_s \geq 1).$$

and $\quad Z(G) \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_t}} \;\; (e_1 \geq \ldots e_t \geq 1).$

since $G/G^{'}Z(G)$ is a quotient of $G/G^{'}$.

In 2012, Attar [3] gave a necessary and sufficient condition on finite p-group G such that $\mathrm{Aut}_{\mathrm{z}}(G)$ to be $\mathrm{Aut}_{\mathrm{Z}}^{\mathrm{z}}(G)$, as follows:

Theorem 6.36 ([3], Theorem A, p. 1097) *Let G be a non-abelian finite* p-group. *Then* $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}_{\mathrm{Z}}^{\mathrm{z}}(G)$ *if and only if* $Z(G) \leq G^{'}$ *or* $Z(G) \leq \Phi(G)$*,* $r = s$*, and* $c_1 \leq b_m$ *where m is the largest integer between* 1 *and r such that* $a_m > b_m$*.*

Theorem 6.37 ([3], Corollary 2.1, p. 1098) *Let G be a non-abelian finite* p-group *such that exponent of* $Z(G)$ *is p. Then* $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}_{\mathrm{Z}}^{\mathrm{z}}(G)$ *if and only if* $Z(G) \leq \Phi(G)$.

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Chapter 7
Automorphism Group and Laplacian Spectrum of a Graph Over Brandt Semigroups

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7.1 Introduction

Various researchers studied the graphs constructed over algebraic structure. In 1964, certain graphs are considered over semigroups by Bosak [7]. The most important and oldest graph in such class is the Cayley graphs associated with semigroups (cf. [8, 31, 33]), because they have many applications (cf. [21, 22]). The two foremost classes of directed graphs, viz., divisibility and power graph over semigroups, were introduced by Kelarev and Quinn in [19, 20]. Later, undirected power graph $\mathcal{P}(S)$ of a semigroup S was defined by Chakraborty et al. is whose vertex set is S, and there is an edge between vertices a and b if one of them is a power of the other. For more information on power graph, we recommend the reader to survey article [2]. The commuting graph of a semigroup S is a simple and undirected graph with the vertex set $\Theta \subseteq S$, and two distinct elements are joined by an edge whenever they commute, and it is denoted by $\Delta(S, \Theta)$. Various parameters of commuting graphs have been investigated such as distant properties (cf. [3, 4, 23]). The commuting graph $\Delta(S, \Theta)$ is a super graph of power graph $\mathcal{P}(S)$ whenever $\Theta = S$. In [1], Aalipour et al. provide ncessary condition on finite group G when the power $\mathcal{P}(G)$ and commuting graph $\Delta(G)$ of a group G are equal. Whenever power graph and commuting graph of G are not equal, then they defined a new graph which lies in between power graph and commuting graph over a group G called enhanced power graph, denoted by $\mathcal{P}_e(G)$, which is the graph with vertex set G, and two distinct elements a and b are joined by an edge if both are contained in some cyclic subgroup of a group G. For each pair of these three graphs $(\mathcal{P}(G), \mathcal{P}_e(G), \Delta(G))$, Aalipour et al. classified the finite group G such that two graphs are equal in that

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_7

pair (see [1]). Further, various researchers studied algebraic and graph theoretic properties of enhanced power graphs associated with groups or semigroups. In [5], Bera et al. studied the dominatable property of enhanced power graph of the abelian groups and the non-abelian p-groups. For finite group G, the rainbow connection number was calculated by Dupont et al. (see [13]). Later, in [12], the concept of enhanced quotient graph of a finite group G was defined by Dupont et al. and discussed the graph theoretic properties. Ma et al. [25] provided an explicit formula for the metric dimension of an enhanced power graph $\mathcal{P}(G)$, where G is a finite group. Zahirović et al. [34] classified the finite group. Also, they gave a sufficient condition for a finite group G whose enhanced power graphs are perfect. Further certain graph theoretical properties such as minimum degree along with matching number, perfectness, independence number, and strong metric dimension of enhanced power graph associated with finite abelian groups and some non-abelian groups such as dihedral groups, dicyclic groups, and the group U_{6n} were studied by Panda et al. [28]. Dalal et al. [11] investigated the unperturbed parameters of enhanced power graphs over the group V_{8n} and semidihedral group SD_{8n}. Bera et al. [6] characterized the abelian group G for which the vertex connectivity of enhanced power graph $\mathcal{P}_e(G)$ is 1. Furthermore, they discussed some special class of non-abelian group G such that its proper enhanced power graph is connected and found their vertex connectivity.

For $n \in \mathbb{N}$, we denote $[n] = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Let G be a finite group. The *Brandt semigroup* $B_n(G)$ has underlying set $([n] \times G \times [n]) \cup \{0\}$ with the binary operation '$\cdot$' on $B_n(G)$ which is defined as

$$(p, g, q) \cdot (r, h, s) = \begin{cases} (p, h, s) & \text{if } q = r; \\ 0 & \text{if } q \neq r \end{cases}$$

and, for all $\beta \in B_n(G)$, $\beta \cdot 0 = 0 \cdot \beta = 0$. Further, we observe that 0 is the (two-sided) zero element in $B_n(G)$.

Theorem 7.1 ([14, Theorem 5.1.8]) *A finite semigroup S is both completely 0-simple and an inverse semigroup if and only if S is isomorphic to the semigroup $B_n(G)$ for some group G.*

Since all completely 0-simple inverse semigroups are exhausted by Brandt semigroups, their consideration seems interesting and useful in various aspects. Brandt semigroups have been examined extensively by many researchers (see [17, 29, 30] and the references therein). The Brandt semigroup $B_n(\{e\})$ is denoted by B_n when G is the trivial group. Thus, the semigroup B_n can be narrated by the set $([n] \times [n]) \cup \{0\}$, where 0 is the zero element and the product $(i, j) \cdot (k, l) = (i, l)$, if $j = k$ and 0, otherwise. As we know that Green's $\mathcal{H}$-class of B_n is trivial, it is also recognized as aperiodic Brandt semigroup. In an inverse semigroup theory, Brandt semigroup B_n plays an important role and arises in a number of different ways (see [9, 18] and the references therein). Gilbert and Samman [13] characterized the endomorphism seminear-rings on B_n. Further, researchers studied the combinatorial properties of affine near-semirings which are generated by affine maps on B_n in

[24]. Further, various properties of B_n have been interrelated with the concept of simplicial complexes and matroids in [26]. Various ranks of B_n have been obtained by using the graph theoretic properties of graphs constructed over Brandt semigroup B_n in [15, 16, 27]. Cayley graphs on Brandt semigroup B_n have been extensively studied by various authors (see [15, 16, 27]).

In this chapter, various parameters of enhanced power graph of B_n have been investigated. The chapter is picked up as follows: In Sect. 7.2, we provide required framework definition and notations used throughout the chapter. In Sect. 7.3, we described the structure of $\mathcal{P}_e(S)$ of a Brandt semigroup B_n. Consequently, we discussed the connectivity of $\mathcal{P}_e(B_n)$. Further, we obtain the automorphism group of the enhanced power graph of Brandt semigroup B_n.

7.2 Premliminary

In this section, we recall necessary definitions, results, and notations of graph theory from [32]. A graph Γ consists of a finite nonempty set $V = V(\Gamma)$ called vertex set and a set $E = E(\Gamma) \subset V \times V$ called edge set. Two different vertices $aandb$ are said to be *adjacent*, denoted by $a \sim b$, if there is an edge between a and b. In this chapter, we are assuming simple graphs, i.e., undirected graphs with no loops or repeated edges. If a and b are not adjacent, then we denote $a \nsim b$. The *neighbourhood* of a vertex a is the set of all vertices adjacent to a in Γ denoted by $N(a)$. Furthermore, we indicate $N[a] = N(a) \cup \{a\}$. A graph Γ' is said to be a subgraph of a graph Γ if $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. A sequence of vertices $u = v_1, v_2, \cdots, v_m = w$ $(m > 1)$ is called a *walk* γ in Γ from the vertex u to the vertex w if $v_i \sim v_{i+1}$ for every $i \in \{1, 2, \ldots, m-1\}$. A walk is called a *trail* in Γ if no edge is repeated in γ. A trail is said to be *closed trail* when initial and end vertices are identical. If no vertex is repeated in a walk, then it is called a *path*, and the length of a path is the number of edges it contains. If $U \subseteq V(\Gamma)$, then the subgraph Γ_U of Γ is induced by U whose vertex set is U, and two vertices are adjacent in Γ_U if and only if they are adjacent in Γ. A graph Γ is called *connected* if there is a path between every pair of vertex. A graph Γ is called *complete* if any two distinct vertices are adjacent. A path is called *cycle* when it begins and ends on the same vertex. A cycle in a graph Γ that contains each vertex of Γ is called a *Hamiltonian cycle* of Γ. A graph Γ is said to be a *Hamiltonian graph* if Γ contains a Hamiltonian cycle. Also, we recall that the *degree* of a vertex x is the number of vertices adjacent to x and it is denoted by $\deg(x)$.

7.3 Main Results

The enhanced power graph $\mathcal{P}_e(B_n)$ of a Brandt semigroup B_n is a simple graph whose vertex set is B_n, and two vertices $x, y \in S$ are adjacent if and only if both are contained in some monogenic subsemigroup of B_n. The following results will be useful in sequel:

Lemma 7.1 *In the graph $\mathcal{P}_e(B_n)$, we have the following:*

1. $N[(i,i)] = \{(i,i)\}$.
2. $N[(i,j)] = \{0\} \cup \{(i,j)\}$, *where* $i \neq j$.
3. $N[0] = B_n \setminus \{(i,i) : i \in [n]\}$.

Proof

(i) Let $(i,i) \in B_n$. If $(i,i) \sim x$, for some, $x \neq (i,i) \in B_n$. Then $x, (i,i) \in \langle y \rangle$, for some, $y \in B_n$. In view of the structure of B_n, y must be of the form (j,k), where $j \neq k$ because 0 and (l,l) are idempotent elements in B_n for each $l \in [n]$. Consequently, we get a contradiction as $(i,i) \notin \langle y \rangle = \langle (j,k) \rangle = \{(j,k), 0\}$. Thus, $N[(i,i) = \{(i,i)\}] \forall i \in [n]$.

(ii) For $i \neq j$, let $(i,j) \in B_n$. Since $0, (i,j) \in \langle (i,j) \rangle$, so $0 \sim (i,j)$. Therefore, we have $0 \in N[(i,j)]$. Further, we observe that $(l,l) \nsim (i,j)$ for all $\in [n]$. Thus, N $[(i,j)] = \{0\} \cup \{(i,j)\}$, where $i \neq j$.

(iii) By part (i), (ii), and structure of B_n, the results hold.

□

The following corollary is a consequence Lemma 7.1.

Corollary 7.1 *In the enhanced power graph $\mathcal{P}_e(B_n)$, we have*

1. *deg*$((i,i)) = 0$ *for all* $i \in [n]$.
2. *deg*$((i,j)) = 1$ *for all* $i \neq j \in [n]$.
3. *deg*$(0) = n^2 - n$.
4. $P_e(B_n)$ *is not complete.*
5. *The number of connected components of* $P_e(B_n)$ *is* $n + 1$.
6. $P_e(B_n)$ *is planar.*
7. *The independence number of* $P_e(B_n)$ *is* n^2.
8. $P_e(B_n)$ *is not Hamiltonian.*
9. *The chromatic number of* $P_e(B_n)$ *is* 2.

Now, we examined the algebraic properties of the enhanced power graph of Brandt semigroup B_n. First, we hark back some basic notion and definition of automorphism of graphs. A mapping f on a graph Γ is said to be an *automorphism* if f is a permutation on $V(\Gamma)$ such that $xf \sim yf$ if and only if $x \sim y$ for all distinct vertices x and y. The set Aut(Γ) is the collection of all graph automorphisms of a graph Γ, and it forms a group with respect to the composition of mappings. We denote S_n as the symmetric group of degree n, and the set $T_{m,n}$ consists of all mapping from the $[n]$ to $[m]$. Now, we begin with the following lemma.

Lemma 7.2 *For each* $\phi_1 \in S_n$ *and* $\phi_2 \in S_A$, *there exists* $\alpha \in \mathrm{Aut}(\mathcal{P}_e(B_n))$.

Proof Let $\phi_1 \in S_n$ and $\phi_2 \in S_A$. We define a map

$$\alpha : B_n \to B_n \quad \text{such that}$$

$$\alpha(i, j) = \begin{cases} (\phi_1(i), \phi_1(i)) \text{ if} i = j; \\ \\ \phi_2(i, j) \qquad \text{if} i \neq j \end{cases}$$

and $\alpha(0) = 0$. We show that α is an automorphism. First, we claim that $\alpha(x) \sim \alpha(y)$ if and only if $x \sim y$. Let $x, y \in B_n$ and suppose $x \sim y$. Then one of them must be zero, so without loss of generality, we may assume that $x = 0$. By Lemma 7.1(iii), we must have $y = (i, j)$ for some $i \neq j \in [n]$. Consequently, we get $alpha(y) = \alpha(i, j) = \phi_2(i, j) = (k, l)$, where $k \neq l$. Since $\alpha(x) = \alpha(0) = 0$, so $\alpha(x) \sim \alpha(y) = (k, l)$ (see Lemma 7.1(iii)). Now, we assume that $x \nsim y$. If $x = 0$, then $y = (i, i)$ for some $i \in [n]$. Clearly, note that $\alpha(x) \nsim \alpha(y)$. For $x, y \neq 0$, one can show that $\alpha(x) \nsim \alpha(y)$. Thus, we have $\alpha(x) \sim \alpha(y)$ if and only if $x \sim y$. Since $\phi_1 \in S_n$ and $\phi_2 \in S_A$, so f is a bijective map on B_n. Hence, $\alpha \in \mathrm{Aut}(\mathcal{P}_e(B_n))$. □

Theorem 7.2 *For* $n \geq 1$, $\mathrm{Aut}(\mathcal{P}_e(B_n)) \cong S_n \times S_{n^2-n}$.

Proof Consider the set $A = \{(i, j) : i \neq j \in [n]\}$. Note that $|A| = n^2 - n$ so it is sufficient to prove that $\mathrm{Aut}(\mathcal{P}_e(B_n)) \cong S_n \times S_{n^2-n}$. Let $f \in \mathrm{Aut}(\mathcal{P}_e(B_n))$. Further, we observe that $\deg(f(v)) = \deg(v)$. In view of Corollary 7.1, we must have $f(0) = 0$, $f((i, i)) = (j, j)$ and $f(p, q) = (a, b)$ for all $i, p \neq q, a \neq b \in [n]$. Now, we define maps:

$$\phi_1^f : [n] \to [n] \qquad \text{and} \qquad \phi_2^f : A \to A$$

by $\phi_1^f(i) = j$ if $f(i, i) = (j, j)$ and $\phi_2^f(j, k) = (l, m)$ whenever $f(j, k) = (l, m)$, where $j \neq k$ and $l \neq m$. Now, we define a map:

$$\psi : \mathrm{Aut}(\mathcal{P}_e(B_n)) \cong S_n \times S_A$$

by $\psi(f) = (\phi_1^f, \phi_2^f)$. By the construction of ϕ_1^f and ϕ_2^f, ψ is a well-defined map. We show that ψ is an isomorphism. By Lemma 7.2, it is enough to prove that ψ is a homomorphism map. Let $f_1, f_2 \in \mathrm{Aut}(\mathcal{P}_e(B_n))$. Then, there exist $\phi_1^{f_1}, \phi_1^{f_2} \in S_n$ and $\phi_2^{f_1}, \phi_2^{f_2} \in S_A$. One can verify that $\phi_1^{f_1}\phi_1^{f_2} = \phi_1^{f_1 f_2}$ and $\phi_2^{f_1}\phi_2^{f_2} = \phi_2^{f_1 f_2}$. Now, $\psi(f_1 f_2) = (\phi_1^{f_1 f_2}, \phi_2^{f_1 f_2}) = (\phi_1^{f_1}\phi_1^{f_2}, \phi_2^{f_1}\phi_2^{f_2}) = (\phi_1^{f_1}, \phi_2^{f_1})(\phi_1^{f_2}, \phi_2^{f_2})$. Thus, ψ is an isomorphism map on $\mathcal{P}_e(B_n)$ and $\mathrm{Aut}(\mathcal{P}_e(B_n)) \cong S_n \times S_{n^2-n}$. □

7.3.1 Laplacian Spectrum of $\mathcal{P}_e(B_n)$

Now, we compute the Laplacian spectrum of $\mathcal{P}_e(B_n)$ and begin with the definition of Laplacian spectrum of any finite simple and undirected graph Γ. For vertex set $V(\Gamma) = \{x_1, x_2, \ldots, x_n\}$, the *adjacency matrix* $A(\Gamma)$ is a square matrix of order $n \times n$ whose (i, j)th entry is 1 whenever vertices x_i and x_j are adjacent and 0 otherwise. The *Laplacian matrix* $L(\Gamma)$ of Γ is the matrix $D(\Gamma) - A(\Gamma)$, where $D(\Gamma) = \text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix with d_i as the degree of the vertex x_i of Γ. Note that the Laplacian matrix of a graph Γ is positive semidefinite and symmetric; therefore, its eigenvalues are real and nonnegative. Moreover, the sum of each row (column) of $L(\Gamma)$ is zero. We denote $\Psi(L(\Gamma), x)$ as the *characteristic polynomial* of a Laplacian matrix of a graph Γ, and the eigenvalues of $L(\Gamma)$ are called the *Laplacian eigenvalues* of Γ represented by $\gamma_1(\Gamma) \geq \gamma_2(\Gamma) \geq \cdots \geq \gamma_n(\Gamma) = 0$.

Further, suppose $\gamma_{n_1}(\Gamma) \geq \gamma_{n_2}(\Gamma) \geq \cdots \geq \gamma_{n_r}(\Gamma) = 0$ be the distinct eigenvalues of Γ and their multiplicities $t_1, t_2, \ldots, t_r$, respectively. The *Laplacian spectrum* of Γ, that is, the spectrum of $L(\Gamma)$, is denoted by $\begin{pmatrix} \gamma_{n_1}(\Gamma) & \gamma_{n_2}(\Gamma) & \cdots & \gamma_{n_r}(\Gamma) \\ t_1 & t_2 & \cdots & t_r \end{pmatrix}$. We examine the characteristic polynomial of $L(\mathcal{P}_e(B_n))$ in the following theorem.

Theorem 7.3 *The characteristic polynomial of the Laplacian matrix of $\mathcal{P}_e(B_n)$ is given by*

$$\Psi(L(\mathcal{P}_e(B_n)), x) = x^{n+1}(x-1)^{n^2-n-1}[x - (n^2 - n + 1)].$$

Proof The Laplacian matrix $L(\mathcal{P}_e(B_n))$ is the $n^2 + 1 \times n^2 + 1$ matrix given below, where the rows and columns are indexed in order by the vertices $(1, 1), (2, 2), \ldots, (n, n), (1, 2), (1, 3), \ldots, (1, n), (2, 1), (2, 3), \ldots, (2, n), \ldots,$ $(n, 1), (n, 2), \ldots, (n, n-1)$ and then 0.

$$L(\mathcal{P}_e(B_n)) = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & & & & & & & \\ \vdots & \vdots & & & & & & & \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 1 & 0 & \cdots\cdots & 0 & -1 \\ 0 & 0 & \cdots\cdots & 0 & 0 & 1 & \cdots\cdots & 0 & -1 \\ \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \\ \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 1 & -1 \\ 0 & 0 & \cdots\cdots & 0 & -1 & -1 & \cdots\cdots & -1 & n^2 - n \end{pmatrix}.$$

Then the characteristic polynomial of $L(\mathcal{P}_e(B_n))$ is

$$\Psi(L(\mathcal{P}_e(B_n)), x) = \begin{vmatrix} x & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & x & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & & & & & & & \\ \vdots & \vdots & & & & & & & \\ 0 & 0 & \cdots\cdots & x & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & x-1 & 0 & \cdots\cdots & 0 & 1 \\ 0 & 0 & \cdots\cdots & 0 & 0 & x-1 & \cdots\cdots & 0 & 1 \\ \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \\ \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \\ 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & x-1 & 1 \\ 0 & 0 & \cdots\cdots & 0 & 1 & 1 & \cdots\cdots & 1 & x-(n^2-n) \end{vmatrix}.$$

Solving the determinant by using the first row, we get

$$\Psi(L(\mathcal{P}_e(B_n)), x) = x \begin{vmatrix} x & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ 0 & \cdots\cdots & x & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & \cdots\cdots & 0 & x-1 & 0 & \cdots\cdots & 0 & 1 \\ 0 & \cdots\cdots & 0 & 0 & x-1 & \cdots\cdots & 0 & 1 \\ \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \\ \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \\ 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & x-1 & 1 \\ 0 & \cdots\cdots & 0 & 1 & 1 & \cdots\cdots & 1 & x-(n^2-n) \end{vmatrix}.$$

Continuing this process up to $n-1$ time, we obtain

$$\Psi(L(\mathcal{P}_e(B_n)), x) = x^n \begin{vmatrix} x-1 & 0 & \cdots\cdots & 0 & 1 \\ 0 & x-1 & \cdots\cdots & 0 & 1 \\ \vdots & \vdots & \cdots\cdots & \vdots & \\ \vdots & \vdots & \cdots\cdots & \vdots & \\ 0 & 0 & \cdots\cdots & x-1 & 1 \\ 1 & 1 & \cdots\cdots & 1 & x-(n^2-n) \end{vmatrix}$$

$$= x^n \begin{vmatrix} (x-1)I_{n^2-n} & A \\ A^{'} & x-(n^2-n) \end{vmatrix},$$

where A is a column of $n^2 - n$ length and $A^{'}$ its transpose. In view of Schur's decomposition theorem [10], we get

$$\Psi(L(\mathcal{P}_e(B_n)), x) = x^n \mid (x-1)I_{n^2-n} \mid \cdot \mid x - (n^2-n) - \frac{(n^2-n)}{x-1} \mid .$$

Therefore, we have $\Psi(L(\mathcal{P}_e(B_n)), x) = x^n(x-1)^{n^2-n-1}(x^2 - x(n^2-n) - x)$

$$= x^{n+1}(x-1)^{n^2-n-1}[x - (n^2-n+1)].$$

□

Corollary 7.2 *The Laplacian spectrum of $\mathcal{P}_e(B_n)$ is given by*

$$\begin{pmatrix} 0 & 1 & n^2-n+1 \\ n+1 & n^2-n-1 & 1 \end{pmatrix}.$$

7.3.2 Distance Spectrum of $\mathcal{P}_e(B_n)$

Now, we compute the distance spectrum of $\mathcal{P}_e(B_n)$ and recall the definition of distance Laplacian spectrum of any finite simple, undirected, and connected graph Γ. For vertex set $V(\Gamma) = \{x_1, x_2, \ldots, x_n\}$, the *distance matrix* $\mathcal{D}(\Gamma)$ is a square matrix of order $n \times n$ whose (i, j)th entry is d_{x_i,x_j} whenever the vertices x_i and x_j are distinct and 0 otherwise. Further, we defined *transmission* of a vertex x by the sum of the distances from x to y for all y in a graph Γ, and it is denoted by $Tr(x)$, i.e.,

$$Tr(x) = \sum_{y \in V(\Gamma)} d_{x,y}.$$

The *distance Laplacian matrix* $\mathcal{D}^L(\Gamma)$ of a graph Γ is the matrix $Tr(\Gamma) - \mathcal{D}(\Gamma)$, where $Tr(\Gamma)$ is the diagonal matrix with $d_i = Tr(x_i)$. We denote $\Psi(\mathcal{D}^L(\Gamma), x)$ as the *characteristic polynomial* of a distance Laplacian matrix of a graph Γ. Since $\mathcal{P}_e(B_n)$ has exactly one nontrivial connected component, so we obtain the characteristic polynomial of the distance Laplacian matrix $\mathcal{D}^L(\mathcal{P}_e(B_n))$ of connected component $\mathcal{P}_e(B_n)$ which contains 0 element in the following theorem.

Theorem 7.4 *The characteristic polynomial of the Laplacian matrix of $\mathcal{P}_e(B_n)$ is given by*

$$\Psi(\mathcal{D}^L(\Gamma), x) = x^{n+1}(x-1)^{n^2-n-1}[x-(n^2-n+1)].$$

Proof The distance Laplacian matrix $L(\mathcal{P}_e(B_n))$ is the $n^2 - n + 1 \times n^2 - n + 1$ matrix given below, where the rows and columns are indexed in order by the vertices $0, (1,2), (1,3), \ldots, (1,n), (2,1), (2,3), \ldots, (2,n), \ldots, (n,1), (n,2), \ldots, (n, n-1)$:

$$\mathcal{D}^L(\mathcal{P}_e(B_n)) = \begin{pmatrix} n^2-n & -1 & -1 & \cdots\cdots\cdots & -1 & -1 \\ -1 & 2n^2-2n-1 & -2 & \cdots\cdots\cdots & -2 & -2 \\ -1 & -2 & 2n^2-2n-1 & \cdots\cdots\cdots & -2 & -2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -2 & -2 & \cdots\cdots\cdots & 2n-2n-1 & -2 \\ -1 & -2 & -2 & \cdots\cdots\cdots & -2 & 2n-2n-1 \end{pmatrix}.$$

Then the characteristic polynomial of $L(\mathcal{P}_e(B_n))$ is

$$\Psi(\mathcal{D}^L(\Gamma), x) = \begin{vmatrix} x-(n^2-n) & 1 & \cdots & 1 & 1 \\ 1 & x-(2n^2-2n-1) & \cdots & 2 & 2 \\ 1 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & \cdots & x-(2n^2-2n-1) & 2 \\ 1 & 2 & \cdots & 2 & x-(2n^2-2n-1) \end{vmatrix}.$$

Applying row operation $R_1 \to (x-1)R_1 - R_2 - \cdots - R_{n^2-n+1}$ and then expanding by using the first row, we get

$$\Psi(\mathcal{D}^L(\Gamma), x) = \frac{x[x-(n^2-n+1)]}{x-1} \begin{vmatrix} x-(2n^2-2n-1) & \cdots & 2 & 2 \\ 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots \\ 2 & \cdots & x-(2n^2-2n-1) & 2 \\ 2 & \cdots & 2 & x-(2n^2-2n-1) \end{vmatrix}.$$

Solving the determinant, we get

$$\Psi(\mathcal{D}^L(\mathcal{P}_e(B_n)), x)$$
$$= \frac{x[x-(n^2-n+1)][x-2n^2+2n+1+(n^2-n-1)2](x-2n^2+2n-1)^{n^2-n-1}}{x-1}.$$
$$\Psi(\mathcal{D}^L(\mathcal{P}_e(B_n)), x) = x[x-(n^2-n+1)](x-2n^2+2n-1)^{n^2-n-1}. \qquad \square$$

Corollary 7.3 *The distance Laplacian spectrum of the nontrivial connected component of $\mathcal{P}_e(B_n)$ is given by*

$$\begin{pmatrix} 0 & n^2-n+1 & 2n^2-2n+1 \\ 1 & 1 & n^2-n-1 \end{pmatrix}.$$

Acknowledgments The author was financially supported by the Postdoctoral Fellowship (NISER/ FA/ PDF/ 2022-23) provided by the school of mathematical sciences, National Institute of Science Education and Research (NISER), India.

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Chapter 8
Unified Iteration Scheme in $CAT(0)$ Spaces and Fixed Point Approximation of Mean Nonexpansive Mappings

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AMS Subject Classifications: 47H09; 47H10; 47J25

8.1 Introduction

A significant proportion of this chapter's discussion is composed of two research [1, 2]. These articles sparked a significant resurgence in the analysis of metric fixed point theory in non-positive curvature spaces. The analysis of non-positive curvature spaces actually started with J. Hadamard's emergence of hyperbolic spaces at the beginning of the last century and E. Cartan's work in the 1920s. The framework of a geodesic metric space possessing non-positive curvature (or, more extensively, curvature bounded above by a real number) dates all the way back to the 1950s research of H. Busemann and A.D. Alexandrov.

The renowned Banach contraction principle, which implies that every contraction on a complete metric space has a unique fixed point, and that the fixed point can be approximated by Picard's iterates, influenced the evolution of fixed point theory of metric spaces. Browder [3] and Gohde [4] independently proved that every nonexpansive self-mapping of a closed, convex, and bounded subset of a uniformly convex Banach space has a fixed point, which is perhaps the most persuasive fixed point theorem in topological fixed point theory. Kirk [5, 6] pioneered the study of

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_8

fixed point theory in Cartan–Alexandrov–Toponogov spaces, or more colloquially in CAT(0) spaces.

Zhang [7] developed the notion of a mean nonexpansive mapping in Banach spaces as a generalization of nonexpansive mappings and established the existence and uniqueness of fixed points with normal structure (see Wu and Zhang [8] and Yang and Cui [9]). Nakprasit [10] proved that mean nonexpansiveness implies Suzuki-generalized nonexpansiveness by using an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive. In the framework of CAT(0) spaces, Rastgoo and Abkar [11] looked at mean nonexpansive mappings.

Several of the standard conceptions and methodologies of nonlinear analysis and Banach space theory is applicable to the class of spaces Gromov calls $CAT(0)$ spaces, as described in [1]. (Cartan, Alexandrov, and Toponogov are represented by the letters C, A, and T.) If a metric space $\mathcal{M}$ is geodesically connected as well as every geodesic triangle in $\mathcal{M}$ is at least as thin as its Euclidean plane comparison triangle, it is indeed a $CAT(0)$ space. The $CAT(0)$ inequality, on the other extreme, encapsulates the concept of non-positive curvature in Riemannian geometry and enables one to interpret the same notion in a much broader framework. Hadamard spaces are $CAT(0)$ spaces that are completely empty. The geometrical formation of $CAT(0)$ spaces seems to be quite enticing. Angles exist in a keen sense in these kinds of spaces, the distance function is convex, uniform convexity and orthogonal projection onto convex subsets are both possible, and so on. $CAT(0)$ spaces also arise in a broad array of applications due to their applicability. Nonexpansive mappings arise naturally in $CAT(0)$ spaces when studying isometries or, more extensively, local isometries. Reich and Shafrir discussed a class of "hyperbolic" metric spaces as "an advantageous foundation specifically for the analysis of nonlinear operator theory, and iterative processes for nonexpansive mappings" in [1]. The CAT(0) spaces are analogous to the Hilbert spaces in conventional nonlinear analysis within the hyperbolic framework. An analogy like this, though, could be deceptive. All R-trees are included in $CAT(0)$ spaces, which are unlike Hilbert spaces in many respects. For more details on $CAT(0)$ spaces, one can read [12–23]. The following definitions are required to make this chapter self-contained.

A self-mapping $\overset{\circ}{\mathcal{J}} : \Theta_1 \to \Theta_1$ is called nonexpansive if

$$\partial_{\mathcal{U}}(\overset{\circ}{\mathcal{J}}\theta, \overset{\circ}{\mathcal{J}}\rho) \leq \partial_{\mathcal{U}}(\theta, \rho)\$$$

$\forall\, \theta,\ \rho \in \Theta_1$. The mapping $\overset{\circ}{\mathcal{J}}$ is called quasi-nonexpansive if

$$F_{\overset{\circ}{\mathcal{J}}} = \{\theta : \overset{\circ}{\mathcal{J}}\theta = \theta\} \neq \emptyset$$

and

$$\partial_{\mathcal{U}}(\overset{\circ}{\mathcal{J}}\theta, \sigma) \leq \partial_{\mathcal{U}}(\theta, \sigma)$$

$\forall\, \theta \in \Theta_1$ and $\sigma \in F_{\overset{\circ}{\mathcal{J}}}$.

8.2 Analysis of Existing Iteration Schemes and Unified Iteration Scheme

Apart from Mann and Ishikawa, there is existence of many iteration schemes with better convergence rate (refer to [24]). Also many iteration schemes are defined in the setting of more generalized mappings (refer to [25, 26]). The following iteration methods are referred to as Noor [26], SP [27], Picard-S [28], Garodia's [29], K [30], Abbas and Nazir [31], and CR [32] iteration methods, respectively:

$$\begin{cases} w_0 \in \Theta, \\ w_{n+1} = (1-\sigma_n^0)w_n + \sigma_n^0 \mathring{\mathcal{J}} w_n, \\ v_n = (1-\sigma_n^1)w_n + \sigma_n^1 \mathring{\mathcal{J}} u_n, \\ u_n = (1-\sigma_n^2)w_n + \sigma_n^2 \mathring{\mathcal{J}} w_n, \qquad n \in \mathbb{N}, \end{cases} \tag{$\mathcal{I}_1$}$$

$$\begin{cases} q_0 \in \Theta, \\ q_{n+1} = (1-\sigma_n^0)r_n + \sigma_n^0 \mathring{\mathcal{J}} r_n, \\ r_n = (1-\sigma_n^1)s_n + \sigma_n^1 \mathring{\mathcal{J}} s_n, \\ s_n = (1-\sigma_n^2)q_n + \sigma_n^2 \mathring{\mathcal{J}} q_n, \qquad n \in \mathbb{N}, \end{cases} \tag{$\mathcal{I}_2$}$$

$$\begin{cases} \rho_0 \in \Theta, \\ y_{n+1} = \mathring{\mathcal{J}} k_n, \\ k_n = (1-\sigma_n^0)\mathring{\mathcal{J}} \rho_\eta + \sigma_n^0 \mathring{\mathcal{J}} \sigma_n, \\ \sigma_n = (1-\sigma_n^1)\rho_\eta + \sigma_n^1 \mathring{\mathcal{J}} \rho_\eta, \qquad n \in \mathbb{N}, \end{cases} \tag{$\mathcal{I}_3$}$$

$$\begin{cases} x_0'' \in \Theta, \\ x_{n+1}'' = \mathring{\mathcal{J}} y_n'', \\ y_n'' = (1-\sigma_n^0)z_n'' + \sigma_n^0 \mathring{\mathcal{J}} \sigma_n'', \\ z_\eta'' = \mathring{\mathcal{J}} \theta_n'', \qquad n \in \mathbb{N}, \end{cases} \tag{$\mathcal{I}_4$}$$

$$\begin{cases} x_{n+1}' = (1-\alpha_n)\mathring{\mathcal{J}} y_n' + \alpha_n \mathring{\mathcal{J}} \sigma_n', \\ y_n' = (1-\beta_n)\mathring{\mathcal{J}} \theta_n' + \beta_n \mathring{\mathcal{J}} \sigma_n' \\ z_\eta' = (1-\gamma_n)\theta_n' + \gamma_n \mathring{\mathcal{J}} \theta_n', \qquad n \in \mathbb{N} \end{cases} \tag{$\mathcal{I}_5$}$$

$$\begin{cases} x'_{n+1} = (1-\alpha_n)y'_n + \alpha_n \overset{\circ}{\mathcal{J}} y'_n, \\ y'_n = (1-\beta_n)\overset{\circ}{\mathcal{J}}\theta'_n + \beta_n \overset{\circ}{\mathcal{J}}\sigma'_n \\ z'_\eta = (1-\gamma_n)\theta'_n + \gamma_n \overset{\circ}{\mathcal{J}}\theta'_n, \qquad n \in \mathbb{N}. \end{cases} \tag{$\mathcal{I}_6$}$$

It is also important to note that various existing iteration schemes which are claimed to be novel and independent are special cases of some pre-existing schemes. Inspired and motivated by the results of existing three-step iteration schemes, we used a new iteration scheme, namely, unified standard three-step iteration scheme proposed by Almusawa et al. [33], which is a general defining of all the existing iteration schemes. The unified three-step iteration scheme is defined as follows:

For any $\theta_0 \in \Theta$,

$$\begin{cases} \sigma_\eta &= a^0_\eta\theta_\eta + a^1_\eta\overset{\circ}{\mathcal{J}}\theta_\eta + a^2_\eta\rho_\eta + a^3_\eta\overset{\circ}{\mathcal{J}}\rho_\eta; \\ \rho_\eta &= b^0_\eta\theta_\eta + b^1_\eta\overset{\circ}{\mathcal{J}}\theta_\eta + b^2_\eta\sigma_\eta + b^3_\eta\overset{\circ}{\mathcal{J}}\sigma_\eta; \\ \theta_{n+1} &= c^0_\eta\theta_\eta + c^1_\eta\overset{\circ}{\mathcal{J}}\theta_\eta + c^2_\eta\rho_\eta + c^3_\eta\overset{\circ}{\mathcal{J}}\rho_\eta + c^4_\eta\sigma_\eta + c^5_\eta\overset{\circ}{\mathcal{J}}\sigma_\eta, \end{cases} \tag{$\mathcal{I}_u$}$$

where sequence $\{a^i_\eta\}$, $\{b^i_\eta\}$ for $i = 0, 1, 2, 3$ and $\{c^i_\eta\}$ for $i = 0, 1, 2, 3, 4, 5$ are sequences in (0,1) $\ni$ one of the following condition holds $\ni$

$$\sum_{i=0}^{3} a^i_\eta = 1, \sum_{i=0}^{3} b^i_\eta = 1, \sum_{i=0}^{5} c^i_\eta = 1,$$

and

($\mathcal{N}_1$) $a^i_\eta \in [\zeta_1, \zeta_2]$ and $b^i_\eta \in [0, \zeta_2]$ for some $\zeta_1,\ \zeta_2$ with $0 < \zeta_1 \le \zeta_2 < 1$,
($\mathcal{N}_2$) $a^i_\eta \in [\zeta_1, 1]$ and $b^i_\eta \in [0, \zeta_2]$ for some $\zeta_1,\ \zeta_2$ with $0 < \zeta_1 \le \zeta_2 < 1$,
($\mathcal{N}_3$) $a^i_\eta \in [\zeta_1, \zeta_2]$ and $c^i_\eta \in [0, \zeta_2]$ for some $\zeta_1,\ \zeta_2$ with $0 < \zeta_1 \le \zeta_2 < 1$,
($\mathcal{N}_4$) $a^i_\eta \in [\zeta_1, 1]$ and $c^i_\eta \in [0, \zeta_2]$ for some $\zeta_1,\ \zeta_2$ with $0 < \zeta_1 \le \zeta_2 < 1$,
($\mathcal{N}_5$) $b^i_\eta \in [\zeta_1, \zeta_2]$ and $c^i_\eta \in [0, \zeta_2]$ for some $\zeta_1,\ \zeta_2$ with $0 < \zeta_1 \le \zeta_2 < 1$,
($\mathcal{N}_6$) $b^i_\eta \in [\zeta_1, 1]$ and $c^i_\eta \in [0, \zeta_2]$ for some $\zeta_1,\ \zeta_2$ with $0 < \zeta_1 \le \zeta_2 < 1$.

Remark 8.1 For distinct values of c^4_η, c^5_η a^i_η, b^i_η, c^i_η for $i = 0, 1, 2$, we have well-known distinct iteration schemes. On substituting

($\mathcal{B}_1$) For $a^2_\eta = a^3_\eta = b^2_\eta = b^1_\eta = c^2_\eta = c^1_\eta = c^4_\eta = c^5_\eta = 0$, $a^0_\eta = (1 - a^1_\eta)$, $b^0_\eta = (1 - b^3_\eta)$, $b^0_\eta = (1 - c^3_\eta)$, $\mathcal{I}_u$ is known as *Noor* [26].
($\mathcal{B}_2$) For $a^2_\eta = a^3_\eta = b^2_\eta = b^0_\eta = b^1_\eta = c^2_\eta = c^3_\eta = c^4_\eta = c^5_\eta$, $a^0_\eta = (1 - a^1_\eta)$, $b^2_\eta = (1 - b^3_\eta)$ and $c^2_\eta = (1 - c^3_\eta)$, $\mathcal{I}_u$ is known as *SP* [27].

($\mathcal{B}_3$) For $a_\eta^2 = a_\eta^3 = b_\eta^2 = b_\eta^0 = b_\eta^1 = c_\eta^0 = c_\eta^2 = c_\eta^3 = c_\eta^4 = c_\eta^5$, $a_\eta^2 = (1 - a_\eta^1)$, $b_\eta^1 = (1 - b_\eta^3)$ and $c_\eta^1 = 1$, $\mathcal{I}_u$ is known as $Picard - S$ [28].

($\mathcal{B}_4$) For $a_\eta^0 = a_\eta^1 = a_\eta^2 = a_\eta^3 = b_\eta^2 = b_\eta^3 = c_\eta^0 = c_\eta^1 = c_\eta^4 = c_\eta^5 = 0$ and $a_\eta^0 = (1 - a_\eta^1)$, $b_\eta^1 = (1 - b_\eta^3)$ and $c_\eta^2 = (1 - c_\eta^3)$, $\mathcal{I}_u$ is known as CR iterative algorithm [32].

($\mathcal{B}_5$) $a_\eta^0 = a_\eta^1 = b_\eta^2 = b_\eta^0 = c_\eta^0 = c_\eta^2 = c_\eta^4 = c_\eta^5 = 0$, $b_\eta^0 = (1 - b_\eta^1)$, $b_\eta^1 = (1 - b_\eta^3)$, $\mathcal{I}_u$ is known as $Abbas\ and\ Nazir$ iterative algorithm [31].

($\mathcal{B}_6$) For $a_\eta^2 = a_\eta^3 = b_\eta^0 = b_\eta^1 = c_\eta^1 = c_\eta^2 = c_\eta^3 = c_\eta^4 = 0$, $a_\eta^0 = (1 - a_\eta^1)$, $b_\eta^2 = (1 - b_\eta^3)$ and $\sigma_\eta^2 = (1 - c_\eta^3)$, $\mathcal{I}_u$ is known as P iterative algorithm [34].

($\mathcal{B}_7$) For $a_\eta^2 = a_\eta^3 = b_\eta^0 = b_\eta^2 = c_\eta^1 = c_\eta^2 = c_\eta^0 = c_\eta^4 = 0$, $a_\eta^0 = (1 - a_\eta^1)$, $b_\eta^1 = (1 - b_\eta^3)$ and $\sigma_\eta^2 = (1 - c_\eta^3)$ $\mathcal{I}_u$ is known as D iterative algorithm [35].

($\mathcal{B}_8$) $a_\eta^2 = a_\eta^3 = a_\eta^0 = a_\eta^1 = b_\eta^2 = b_\eta^3 = b_\eta^0 = b_\eta^1 = c_\eta^3 = c_\eta^2 = c_\eta^4 = 0$, $c_\eta^0 = (1 - c_\eta^1)$ in the n_v iteration scheme, we obtain the $Mann$ iterative algorithm [24].

($\mathcal{B}_9$) For $a_\eta^2 = a_\eta^3 = a_\eta^0 = a_\eta^1 = b_\eta^2 = b_\eta^3 = c_\eta^1 = c_\eta^2 = c_\eta^4 = c_\eta^5 = 0$, $b_\eta^0 = (1 - b_\eta^1)$ and $c_\eta^0 = (1 - c_\eta^3)$, $\mathcal{I}_u$ is known as $Ishikawa$ iterative algorithm [36].

The purpose of the chapter is to study the convergence of the earlier described iterative algorithm in $CAT(0)$ spaces for generalized nonexpansive mappings enabling us to enlarge the classes of mappings studied by Takahashi and Kim [25].

8.3 Tools and Notations

In this section, we use the notations which we are going to use in the entire manuscript. The framework in which we will prove our results from now on is $CAT(0)$ space $(\Theta_{\mathcal{C}}, \partial_{\mathcal{U}})$ and $\Delta_{\mathcal{B}}$ is a Banach space. Also, $\Delta_1 \subset \Delta_{\mathcal{B}}$ endowed with a mapping $\overset{\circ}{\mathcal{J}}$ defined from Δ_1 to a Banach space $\Delta_{\mathcal{B}}$. Let Θ_1 be a nonempty bounded, closed, and convex subset of a complete $CAT(0)$ space $(\Theta_{\mathcal{C}}, \partial_{\mathcal{U}})$. The notation $\mathbb{R}$ will now be used to represent the set of real numbers, whereas $\mathbb{N}$ will represent the set of all natural numbers.

Throughout the entire chapter, $(\Theta_{\mathcal{C}}, \partial_{\mathcal{U}})$ is used to represent the $CAT(0)$ space. For a mapping $F_{\overset{\circ}{\mathcal{J}}}$, an element σ is known as a fixed point if

$$\overset{\circ}{\mathcal{J}}(\sigma) = \sigma.$$

$F_{\overset{\circ}{\mathcal{J}}}$ denotes the collection of all fixed points for the mapping $\overset{\circ}{\mathcal{J}}$. The symbol $\ni$ refers such that whereas $\forall$ represents for all.

8.4 Definitions and Preliminaries

Definition 8.1 Let $\emptyset \neq \Theta_1 \subseteq \Theta_{\mathcal{C}}$ and a mapping $\overset{\circ}{\mathcal{J}} : \Theta_1 \to \Theta_1$. Then the mapping $\overset{\circ}{\mathcal{J}}$ is said to be nonexpansive if

$$\partial_{\mathcal{U}}(\overset{\circ}{\mathcal{J}}\theta, \overset{\circ}{\mathcal{J}}\rho) \leq \partial_{\mathcal{U}}(\theta, \rho),$$

$\forall\, \theta,\ \rho \in \Theta_1$.

Definition 8.2 Let $\emptyset \neq \Theta_1 \subseteq \Theta_{\mathcal{C}}$ and a mapping $\overset{\circ}{\mathcal{J}} : \Theta_1 \to \Theta_1$. Then the mapping $\overset{\circ}{\mathcal{J}}$ is said to be mean nonexpansive if

$$\partial_{\mathcal{U}}(\overset{\circ}{\mathcal{J}}\theta, \overset{\circ}{\mathcal{J}}\rho) \leq \tilde{\mathfrak{a}}\partial_{\mathcal{U}}(\theta, \rho) + \tilde{\mathfrak{b}}\partial_{\mathcal{U}}(\theta, \rho)$$

$\forall\, \theta,\ \rho \in \Theta_1$. Also, $\tilde{\mathfrak{a}},\ \tilde{\mathfrak{b}} \in \mathbb{R}^+ \ni \tilde{\mathfrak{a}} + \tilde{\mathfrak{b}} \leq 1$.

Every nonexpansive mapping (with $\tilde{\mathfrak{a}} = 1$ and $\tilde{\mathfrak{b}} = 0$) is obviously a mean nonexpansive mapping. It is important to note that a mean nonexpansive mapping is not always continuous, hence meaning nonexpansive mappings are not always nonexpansive [11]. Nakprasit [10] observed that increasing mean nonexpansive mappings are Suzuki-generalized nonexpansive by providing an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive.

The following lemmas are important to make this chapter self-contained and are as follows:

Lemma 8.1 *For* $\theta, \rho, \sigma \in \Theta_{\mathcal{C}}$ *and* $\overset{\circ}{\mathcal{J}} \in (0, 1)$*, we have*

$$\partial_{\mathcal{U}}((1-\tau)\theta \oplus \tau\rho, \rho_1)) \leq (1-\tau)\partial_{\mathcal{U}}(\theta, \rho_1) + \tau\partial_{\mathcal{U}}(\rho, \rho_1)$$

$\forall\, \tau \in [0, 1)$ *and* $\theta,\ \rho,\ \rho_1 \in \Theta_{\mathcal{C}}$.

To carve out the detailed analysis about $CAT(0)$ space, authors are suggested to refer to [37]. As per [37], a geodesic triangle $\triangle(\theta_1,\ \theta_2,\ \theta_3)$ in a geodesic metric space $(\theta,\ d)$ is consist of three points of $\Theta_{\mathcal{C}}$ (as the vertices of $\triangle$) and a geodesic segment between each pair of points (as the edges of $\triangle$). A comparison triangle for $\triangle(\theta_1,\ \theta_2,\ \theta_3)$ in $(\Theta,\ \partial_{\mathcal{U}})$ is a triangle

$$\overline{\triangle}(\theta_1,\ \theta_2,\ \theta_3) := \triangle(\overline{\theta}_1,\ \overline{\theta}_2, \overline{\theta}_3)$$

in the Euclidean plane $\mathbb{R}^2 \ni d_{\mathbb{R}^2}(\overline{\theta}_i,\ \overline{\theta}_j) = \partial_{\mathcal{U}}(\theta_i,\ \theta_j)$ for $i,\ j \in \{1, 2, 3\}$. A point $\overline{\theta} \in [\overline{\theta}_1,\ \overline{\theta}_2]$ is said to be comparison point for $\theta \in [\theta_1, \theta_2]$ if

$$\partial_{\mathcal{U}}(\theta_1, \theta) = \partial_{\mathcal{U}}(\overline{\theta}_1, \overline{\theta}).$$

Comparison points on $[\overline{\theta}_2, \overline{\theta}_3]$ and $[\overline{\theta}_3, \overline{\theta}_1]$ are defined in the same way. A geodesic metric space $\Theta_{\mathcal{C}}$ is called a $CAT(0)$ space if all geodesic triangles satisfy the following comparison axiom:

Let $\Theta_{\mathcal{C}}$ be complete $CAT(0)$ space and $\{\theta_\eta\}$ be a bounded sequence in $\Theta_{\mathcal{C}}$. For $\theta \in \Theta_{\mathcal{C}}$ set:

$$r(\theta, \{\theta_\eta\} = \limsup_{n\to\infty} \partial_{\mathcal{U}}(\theta, \theta_\eta),$$

whereas the asymptotic radius $r(\{\theta_\eta\})$ is defined as follows:

$$r(\{\theta_\eta\}) = \inf\{r(\theta, \theta_\eta) : \theta \ \in \Theta_{\mathcal{C}}\},$$

and the asymptotic center $A(\{\theta_\eta\})$ of $\{\theta_\eta\}$ is as follows:

$$A(\{\theta_\eta\}) = \{\theta \in \Theta_{\mathcal{C}} : r(\theta, \theta_\eta) = r(\{\theta_\eta\})\}.$$

Note that for $CAT(0)$ space, $A(\{\theta_\eta\})$ is a singleton set [38].

Lemma 8.2 ([39]) *Every bounded sequence in a complete $CAT(0)$ space admits a Δ-convergent subsequence.*

Lemma 8.3 ([40]) *If $\{\theta_\eta\}$ is a bounded sequence in Θ_1, then the asymptotic center of $\{\theta_\eta\}$ is in $\Theta_{\mathcal{C}}$.*

Lemma 8.4 ([41]) *For $\theta, \ \rho \in \Theta_{\mathcal{C}}$ and $\mathring{\mathcal{J}} \ \in \ (0,1)$, $\exists$ a unique $\sigma \in [\theta, \ \ \rho]$ such that*

$$\partial_{\mathcal{U}}(\theta, \sigma) = t\partial_{\mathcal{U}}(\theta, \rho) \quad \textit{and} \quad \partial_{\mathcal{U}}(\rho, \sigma) = (1-t)\partial_{\mathcal{U}}(\theta, \rho).$$

Now, we present the unified iterative algorithm in the framework of $CAT(0)$ spaces. For $\theta_1 \in \Theta_{\mathcal{C}}$, the unified iteration is defined as

$$\begin{cases} \sigma_\eta & = a_\eta^0\theta_\eta \oplus a_\eta^1\mathring{\mathcal{J}}\theta_\eta \oplus a_\eta^2\rho_\eta \oplus a_\eta^3\mathring{\mathcal{J}}\rho_\eta; \\ \rho_\eta & = b_\eta^0\theta_\eta \oplus b_\eta^1\mathring{\mathcal{J}}\theta_\eta \oplus b_\eta^2\sigma_\eta \oplus b_\eta^3\mathring{\mathcal{J}}\sigma_\eta; \\ \theta_{n+1} & = c_\eta^0\theta_\eta \oplus c_\eta^1\mathring{\mathcal{J}}\theta_\eta \oplus c_\eta^2\rho_\eta \oplus c_\eta^3\mathring{\mathcal{J}}\rho_\eta \oplus c_\eta^4\sigma_\eta \oplus c_\eta^5\mathring{\mathcal{J}}\sigma_\eta, \end{cases} \qquad (\mathcal{I}_{\mathcal{C}})$$

where sequence $\{a_\eta^i\}$, $\{b_\eta^i\}$ for $i = 0, 1, 2, 3$ and $\{c_\eta^i\}$ for $i = 0, 1, 2, 3, 4, 5$ are sequences in (0,1) $\ni$ one of the following condition holds $\ni$ $\sum_{i=0}^3 a_\eta^i = 1$, $\sum_{i=0}^3 b_\eta^i = 1$, $\sum_{i=0}^5 c_\eta^i = 1$.

Lemma 8.5 *Let $\Theta_{\mathcal{C}}$ be a complete $CAT(0)$ space and let $\theta \in \Theta_{\mathcal{C}}$. Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and$\{\theta_\eta\}$, $\{\rho_\eta\}$ are sequences in $\Theta_{\mathcal{C}}$ $\ni$*

$$\limsup_{n\to\infty} \eth_{\mathcal{U}}(\theta_\eta, \eth) \leq r,$$

$$\limsup_{n\to\infty} \eth_{\mathcal{U}}(\rho_\eta, \eth) \leq r,$$

and

$$\lim_{n\to\infty} \eth_{\mathcal{U}}((1-t_n)\theta_\eta \oplus t_n\rho_\eta, \eth) = r$$

for some $r \geq 0$. *Then*

$$\lim_{n\to\infty} \eth_{\mathcal{U}}(\theta_\eta, \rho_\eta) = 0.$$

Lemma 8.6 ([16]) *For* Θ_1, *let* $\mathring{\mathcal{J}} : \Theta_1 \to \Theta_1$ *be a mean nonexpansive mapping. If* $\{\theta_n\}$ *is a sequence in* Θ_1 *such that*

$$lim_{\eta\to\infty}\eth_{\mathcal{U}}(\theta_\eta, \mathring{\mathcal{J}}\theta_\eta) = 0$$

and

$$\Delta - lim_{\eta\to\infty}\theta_\eta = \sigma,$$

then $\mathring{\mathcal{J}}\sigma = \sigma$.

8.5 Main Results

Theorem 8.1 *Let a mapping* $\mathring{\mathcal{J}} : \Theta_1 \to \Theta_1$ *be a mean nonexpansive mapping with* $0 \leq \tilde{b} < 1$. *Consider a sequence* $\{\theta_\eta\}_{\eta=1}^{\infty}$ *and a sequence defined by* $\mathcal{I}_{\mathcal{C}}$. *Then* $\{\theta_\eta\}_{\eta=1}^{\infty}$ *is* $\Delta-$ *convergent to* σ *where* $\sigma \in F_{\mathring{\mathcal{J}}}$, *if the following conditions also hold:*

1. $((b_\eta^0 + b_\eta^1) + (b_\eta^2 + b_\eta^3)(a_\eta^0 + a_\eta^1))(1 - (b_\eta^2 + b_\eta^2)(a_\eta^2 + a_\eta^3))^{-1} \leq 1$,
2. $(c_\eta^0 + c_\eta^1 + c_\eta^2 + c_\eta^3 + (c_\eta^4 + c_\eta^5)(a_\eta^0 + a_\eta^1 + a_\eta^2 + a_\eta^3)) \leq 1$.

Proof We claim that $\lim_{\eta\to\infty} \eth_{\mathcal{U}}(\theta_\eta, \eth)$ exists for the sequence $\{\theta_\eta\}$ defined by $\mathcal{I}_{\mathcal{C}}$, let $F_{\mathring{\mathcal{J}}} \neq \emptyset$ and $\eth \in F_{\mathring{\mathcal{J}}}$. Considering the fact that $0 < \{\theta_\eta\}_{\eta=1}^{\infty} < 1$

$$\eth_{\mathcal{U}}(\sigma_\eta, \eth) = \eth_{\mathcal{U}}(a_\eta^0\theta_\eta \oplus a_\eta^1\mathring{\mathcal{J}}\theta_\eta \oplus a_\eta^2\rho_\eta \oplus a_\eta^3\mathring{\mathcal{J}}\rho_\eta, \eth)$$

$$\leq a_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + a_\eta^1 \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \eth) + a_\eta^2 \partial_{\mathcal{U}}(\rho_\eta, \eth)$$
$$+ a_\eta^3 \Big[\tilde{a} \partial_{\mathcal{U}}(\rho_\eta, \eth) + \tilde{b} \partial_{\mathcal{U}}(\rho_\eta, \eth) \Big]$$

$$\begin{aligned}
\partial_{\mathcal{U}}(\sigma_\eta, \eth) &\leq a_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + a_\eta^1 \Big[\tilde{\mathfrak{a}} \partial_{\mathcal{U}}(\theta_\eta, \eth) + \tilde{\mathfrak{b}} \partial_{\mathcal{U}}(\theta_\eta, \eth) \Big] + a_\eta^2 \partial_{\mathcal{U}}(\rho_\eta, \eth) \\
&+ a_\eta^3 \Big[\tilde{a} \partial_{\mathcal{U}}(\rho_\eta, \eth) + \tilde{b} \partial_{\mathcal{U}}(\rho_\eta, \eth) \Big] \\
&\leq a_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + a_\eta^1 \partial_{\mathcal{U}}(\theta_\eta, \eth)(\tilde{\mathfrak{a}} + \tilde{\mathfrak{b}}) + a_\eta^2 \partial_{\mathcal{U}}(\rho_\eta, \eth) \\
&+ a_\eta^3 \partial_{\mathcal{U}}(\rho_\eta, \eth)(\tilde{a} + \tilde{b}) \\
&\leq a_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + a_\eta^1 \partial_{\mathcal{U}}(\theta_\eta, \eth) + a_\eta^2 \partial_{\mathcal{U}}(\rho_\eta, \eth) + a_\eta^3 \partial_{\mathcal{U}}(\rho_\eta, \eth) \\
&= (a_\eta^0 + a_\eta^1) \partial_{\mathcal{U}}(\theta_\eta, \eth) + (a_\eta^2 + a_\eta^3) \partial_{\mathcal{U}}(\rho_\eta, \eth). \qquad (8.1)
\end{aligned}$$

Also,

$$\begin{aligned}
\partial_{\mathcal{U}}(\rho_\eta, \eth) &= \partial_{\mathcal{U}}(b_\eta^0 \theta_\eta \oplus b_\eta^1 \mathring{\mathcal{J}}\theta_\eta \oplus b_\eta^2 \sigma_\eta \oplus b_\eta^3 \mathring{\mathcal{J}}\sigma_\eta, \eth) \\
&\leq b_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1 \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \eth) + b_\eta^2 \partial_{\mathcal{U}}(\sigma_\eta, \eth) + b_\eta^3 \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\sigma_\eta, \eth) \\
&\leq b_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1 \Big[\tilde{a} \partial_{\mathcal{U}}(\theta_\eta, \eth) + \tilde{b} \partial_{\mathcal{U}}(\theta_\eta, \eth) \Big] + b_\eta^2 \partial_{\mathcal{U}}(\sigma_\eta, \eth) \\
&+ b_\eta^3 \Big[\tilde{a} \partial_{\mathcal{U}}(\sigma_\eta, \eth) + \tilde{b} \partial_{\mathcal{U}}(\sigma_\eta, \eth) \Big] \\
&\leq b_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1 \partial_{\mathcal{U}}(\theta_\eta, \eth)(\tilde{a} + \tilde{b}) + b_\eta^2 \partial_{\mathcal{U}}(\sigma_\eta, \eth) \\
&+ b_\eta^3 \partial_{\mathcal{U}}(\sigma_\eta, \eth)(\tilde{a} + \tilde{b})
\end{aligned}$$

$$\begin{aligned}
&\leq b_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^2 \partial_{\mathcal{U}}(\sigma_\eta, \eth) + b_\eta^3 \partial_{\mathcal{U}}(\sigma_\eta, \eth) \\
&= (b_\eta^0 + b_\eta^1) \partial_{\mathcal{U}}(\theta_\eta, \eth) + (b_\eta^2 + b_\eta^3) \partial_{\mathcal{U}}(\sigma_\eta, \eth). \qquad (8.2)
\end{aligned}$$

By using the value of $\partial_{\mathcal{U}}(\sigma_\eta, \eth)$ from (13.115), we have

$$\begin{aligned}
\partial_{\mathcal{U}}(\rho_\eta, \eth) &\leq (b_\eta^0 + b_\eta^1) \partial_{\mathcal{U}}(\theta_\eta, \eth) + (b_\eta^2 + b_\eta^3)((a_\eta^0 + a_\eta^1) \partial_{\mathcal{U}}(\theta_\eta, \eth) \\
&+ (a_\eta^2 + a_\eta^3) \partial_{\mathcal{U}}(\rho_\eta, \eth))
\end{aligned}$$

$$\leq ((b_\eta^0 + b_\eta^1 + (b_\eta^2 + b_\eta^3)(a_\eta^0 + a_\eta^1))\eth_{\mathcal{U}}(\theta_\eta, \eth))$$
$$+ (b_\eta^2 + b_\eta^3)(a_\eta^2 + a_\eta^3)\eth_{\mathcal{U}}(\rho_\eta, \eth)$$
$$\eth_{\mathcal{U}}(\rho_\eta, \eth) \leq (\frac{(b_\eta^0 + b_\eta^1) + (b_\eta^2 + b_\eta^3)(a_\eta^0 + a_\eta^1)}{1 - (b_\eta^2 + b_\eta^3)(a_\eta^2 + a_\eta^3)})\eth_{\mathcal{U}}(\theta_\eta, \eth). \tag{8.3}$$

It is important to note that

$$((b_\eta^0 + b_\eta^1) + (b_\eta^2 + b_\eta^3)(a_\eta^0 + a_\eta^1))(1 - (b_\eta^2 + b_\eta^2)(a_\eta^2 + a_\eta^3))^{-1} \leq 1,$$

which results in

$$\eth_{\mathcal{U}}(\rho_\eta, \eth) \leq \eth_{\mathcal{U}}(\theta_\eta, \eth). \tag{8.4}$$

Now, it follows that

$$\eth_{\mathcal{U}}(\theta_{n+1}, \eth) \leq (c_\eta^0\theta_\eta \oplus c_\eta^1\mathring{\mathcal{J}}\theta_\eta \oplus c_\eta^2\rho_\eta \oplus c_\eta^3\mathring{\mathcal{J}}\rho_\eta \oplus c_\eta^4\sigma_\eta \oplus c_\eta^5\mathring{\mathcal{J}}\sigma_\eta, \eth)$$
$$\leq c_\eta^0\eth_{\mathcal{U}}(\theta_\eta, \eth) + c_\eta^1\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \eth) + c_\eta^2\eth_{\mathcal{U}}(\rho_\eta, \eth) + c_\eta^3\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \eth)$$
$$+ c_\eta^4\eth_{\mathcal{U}}(\sigma_\eta, \eth) + c_\eta^5\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\sigma_\eta, \eth)$$
$$\leq c_\eta^0\eth_{\mathcal{U}}(\theta_\eta, \eth) + c_\eta^1\Big[\tilde{a}\eth_{\mathcal{U}}(\theta_\eta, \eth) + \tilde{b}\eth_{\mathcal{U}}(\theta_\eta, \eth)\Big] + c_\eta^2\eth_{\mathcal{U}}(\rho_\eta, \eth)$$
$$+ c_\eta^3\Big[\tilde{a}\eth_{\mathcal{U}}(\rho_\eta, \eth) + \tilde{b}\eth_{\mathcal{U}}(\rho_\eta, \eth)\Big] + c_\eta^4\eth_{\mathcal{U}}(\sigma_\eta, \eth)$$
$$+ c_\eta^5\Big[\tilde{a}\eth_{\mathcal{U}}(\sigma_\eta, \eth) + \tilde{b}\eth_{\mathcal{U}}(\sigma_\eta, \eth)\Big]$$
$$\leq c_\eta^0\eth_{\mathcal{U}}(\theta_\eta, \eth) + c_\eta^1(\tilde{a} + \tilde{b}) + c_\eta^2\eth_{\mathcal{U}}(\rho_\eta, \eth) + c_\eta^3\eth_{\mathcal{U}}(\rho_\eta, \eth)(\tilde{a} + \tilde{b})$$
$$+ c_\eta^4\eth_{\mathcal{U}}(\sigma_\eta, \eth) + c_\eta^5\eth_{\mathcal{U}}(\sigma_\eta, \eth)(\tilde{a} + \tilde{b})$$
$$\leq (c_\eta^0 + c_\eta^1)\eth_{\mathcal{U}}(\theta_\eta, \eth) + (c_\eta^2 + c_\eta^3)\eth_{\mathcal{U}}(\rho_\eta, \eth) + (c_\eta^4 + c_\eta^5)\eth_{\mathcal{U}}(\sigma_\eta, \eth).$$

As it is given that

$$\eth_{\mathcal{U}}(\rho_\eta, \eth) \leq \eth_{\mathcal{U}}(\theta_\eta, \eth),$$

we have

$$\eth_{\mathcal{U}}(\theta_{n+1}, \eth) \leq (c_\eta^0 + c_\eta^1 + c_\eta^2 + c_\eta^3)\eth_{\mathcal{U}}(\theta_\eta, \eth) + (c_\eta^4 + c_\eta^5)\eth_{\mathcal{U}}(\sigma_\eta, \eth).$$

By using the above inequalities, we have

$$\partial_{\mathcal{U}}(\sigma_\eta, \eth) = (a_\eta^0 + a_\eta^1)\partial_{\mathcal{U}}(\theta_\eta, \eth) + (a_\eta^2 + a_\eta^3)\partial_{\mathcal{U}}(\rho_\eta, \eth),$$

which further results in

$$\begin{aligned}\partial_{\mathcal{U}}(\theta_{n+1}, \eth) &\leq (c_\eta^0 + c_\eta^1 + c_\eta^2 + c_\eta^3)\partial_{\mathcal{U}}(\theta_\eta, \eth) + (c_\eta^4 + c_\eta^5)((a_\eta^0 + a_\eta^1)\partial_{\mathcal{U}}(\theta_\eta, \eth) \\ &\quad + (a_\eta^2 + a_\eta^3)\partial_{\mathcal{U}}(\rho_\eta, \eth)) \\ &\leq (c_\eta^0 + c_\eta^1 + c_\eta^2 + c_\eta^3)\partial_{\mathcal{U}}(\theta_\eta, \eth) + (c_\eta^4 + c_\eta^5)((a_\eta^0 + a_\eta^1) \\ &\quad + (a_\eta^2 + a_\eta^3))\partial_{\mathcal{U}}(\theta_\eta, \eth) \\ &= (c_\eta^0 + c_\eta^1 + c_\eta^2 + c_\eta^3 + (c_\eta^4 + c_\eta^5)(a_\eta^0 + a_\eta^1 + a_\eta^2 + a_\eta^3))\partial_{\mathcal{U}}(\theta_\eta, \eth).\end{aligned} \tag{8.5}$$

Furthermore, it is given that

$$(c_\eta^0 + c_\eta^1 + c_\eta^2 + c_\eta^3 + (c_\eta^4 + c_\eta^5)(a_\eta^0 + a_\eta^1 + a_\eta^2 + a_\eta^3)) \leq 1;$$

we have

$$\partial_{\mathcal{U}}(\theta_{n+1}, \eth) \leq \partial_{\mathcal{U}}(\theta_\eta, \eth).$$

This implies that $\{\partial_{\mathcal{U}}(\theta_\eta, \eth)\}$ is bounded and non-increasing $\forall\, \theta \in F_{\overset{\circ}{\mathcal{J}}}$. Hence, $\lim_{n\to\infty} \partial_{\mathcal{U}}(\theta_\eta, \eth)$ exists. □

Theorem 8.2 *Let a mapping $\overset{\circ}{\mathcal{J}} : \Theta_1 \to \Theta_1$ be a mean nonexpansive mapping with $0 \leq \tilde{b} < 1$. Consider a sequence $\{\theta_\eta\}_{\eta=1}^{\infty}$ and a sequence defined by $\mathcal{I}_{\mathcal{C}}$. Then*

$$\lim_{\eta\to\infty} \partial_{\mathcal{U}}(\theta_\eta, \overset{\circ}{\mathcal{J}}\theta_\eta) = 0,$$

if the following conditions are satisfied:

1. $((b_\eta^0 + b_\eta^1) + (b_\eta^2 + b_\eta^3)(a_\eta^0 + a_\eta^1))(1 - (b_\eta^2 + b_\eta^2)(a_\eta^2 + a_\eta^3))^{-1} \leq 1,$
2. $(c_\eta^0 + c_\eta^1 + c_\eta^2 + c_\eta^3 + (c_\eta^4 + c_\eta^5)(a_\eta^0 + a_\eta^1 + a_\eta^2 + a_\eta^3)) \leq 1.$

Proof Without affecting the generality of the statement, let

$$\overset{\circ}{\eth} := \lim_{\eta\to\infty} \partial_{\mathcal{U}}(\theta_\eta, \eth);$$

therefore, it follows that

$$\begin{aligned}
\lim_{\eta\to\infty} sup\ \eth_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \mathring{\mathcal{J}}\eth) &= \lim_{\eta\to\infty} sup\ \eth_{\mathcal{U}}(\mathring{\eth}\theta_\eta, \eth) \\
&\le \lim_{\eta\to\infty} sup \left[\tilde{a}\eth_{\mathcal{U}}(\theta_\eta, \eth) + \tilde{b}\eth_{\mathcal{U}}(\theta_\eta, \eth)\right] \\
&\le \lim_{\eta\to\infty} sup\ \eth_{\mathcal{U}}(\theta_\eta, \eth)(\tilde{a} + \tilde{b}) \\
&\le \lim_{\eta\to\infty} sup\ \eth_{\mathcal{U}}(\theta_\eta, \eth) \\
&\le \mathring{\eth}.
\end{aligned}$$

By using Eq. (13.115), we have

$$\lim_{\eta\to\infty} sup\ \eth_{\mathcal{U}}(\sigma_\eta, \eth) \le (a_\eta^0 + a_\eta^1)\eth_{\mathcal{U}}(\theta_\eta, \eth) + (a_\eta^2 + a_\eta^3)\eth_{\mathcal{U}}(\rho_\eta, \eth).$$

By using Eq. (8.5), we have

$$\begin{aligned}
\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\eth, \mathring{\mathcal{J}}\rho_\eta) &= \eth_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \mathring{\mathcal{J}}\eth) \\
&= \tilde{a}\eth_{\mathcal{U}}(\rho_\eta, \eth) + \tilde{b}\eth_{\mathcal{U}}(\rho_\eta, \mathring{\mathcal{J}}\eth) \\
&= \eth_{\mathcal{U}}(\rho_\eta, \eth).
\end{aligned}$$

Similarly, we have

$$\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\eth, \mathring{\mathcal{J}}\theta_\eta) \le \eth_{\mathcal{U}}(\eth, \theta_\eta)$$

and

$$\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\eth, \mathring{\mathcal{J}}\sigma_\eta) \le \eth_{\mathcal{U}}(\eth, \sigma_\eta),$$

which means that

$$\begin{aligned}
\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \eth) &\le \eth_{\mathcal{U}}(\rho_\eta, \eth) \\
&\le \eth_{\mathcal{U}}((b_\eta^0\theta_\eta \oplus b_\eta^1\mathring{\mathcal{J}}\theta_\eta \oplus b_\eta^2\sigma_\eta \oplus b_\eta^3\mathring{\mathcal{J}}\sigma_\eta, \eth) \\
&\le b_\eta^0\eth_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \eth) + b_\eta^2\eth_{\mathcal{U}}(\sigma_\eta, \eth) + b_\eta^3\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\sigma_\eta, \eth) \\
&\le b_\eta^0\eth_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1\left[\tilde{a}\eth_{\mathcal{U}}(\theta_\eta, \eth) + \tilde{b}\eth_{\mathcal{U}}(\theta_\eta, \eth)\right] + b_\eta^2\eth_{\mathcal{U}}(\sigma_\eta, \eth) \\
&+ b_\eta^3\left[\tilde{a}\eth_{\mathcal{U}}(\sigma_\eta, \eth) + \tilde{b}\tilde{a}\eth_{\mathcal{U}}(\sigma_\eta, \eth)\right]
\end{aligned}$$

$$\leq b_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1 \partial_{\mathcal{U}}(\theta_\eta, \eth)(\tilde{a} + \tilde{b}) + b_\eta^2 \partial_{\mathcal{U}}(\sigma_\eta, \eth)$$
$$+ b_\eta^3 \partial_{\mathcal{U}}(\sigma_\eta, \eth)(\tilde{a} + \tilde{b})$$
$$\leq b_\eta^0 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^1 \partial_{\mathcal{U}}(\theta_\eta, \eth) + b_\eta^2 \partial_{\mathcal{U}}(\sigma_\eta, \eth) + b_\eta^3 \partial_{\mathcal{U}}(\sigma_\eta, \eth)$$
$$\leq (b_\eta^0 + b_\eta^1)\partial_{\mathcal{U}}(\theta_\eta, \eth) + (b_\eta^2 + b_\eta^3)\partial_{\mathcal{U}}(\sigma_\eta, \eth).$$

Therefore, we have

$$\limsup_{n\to\infty} \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \eth) \leq \limsup_{n\to\infty} \partial_{\mathcal{U}}(\rho_\eta, \eth) \leq \mathring{\eth}.$$

It is important to note that

$$\lim_{n\to\infty} \partial_{\mathcal{U}}(c_\eta^0\theta_\eta + c_\eta^1\mathring{\mathcal{J}}\theta_\eta + c_\eta^2\rho_\eta + c_\eta^3\mathring{\mathcal{J}}\rho_\eta + c_\eta^4\sigma_\eta + c_\eta^5\mathring{\mathcal{J}}\sigma_\eta, \eth) = \lim_{n\to\infty} \partial_{\mathcal{U}}(\theta_{n+1}, \eth) = c.$$

Case 1 If $0 < a \leq \beta_n \leq b < 1$ and $0 \leq \alpha_n \leq 1$, then by foregoing discussion and by Lemma 8.5, we have

$$\lim_{n\to\infty} \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \theta_\eta) = 0.$$

$\forall\, \alpha_n \in [0, \zeta_2]$, we have

$$\begin{aligned}\partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \theta_\eta) &\leq \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \rho_\eta) + \partial_{\mathcal{U}}(\rho_\eta, \theta_\eta)\\ &= \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \alpha_n\mathring{\mathcal{J}}\theta_\eta \oplus (1-\alpha_n)\theta_\eta) + \partial_{\mathcal{U}}(\rho_\eta, \theta_\eta)\\ &\leq (1-\alpha_n)\partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \theta_\eta) + \partial_{\mathcal{U}}(\rho_\eta, \theta_\eta),\end{aligned}$$

so that

$$\alpha_\eta \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \theta_\eta) \leq \partial_{\mathcal{U}}(\rho_\eta, \theta_\eta).$$

As $\alpha_n \in [0, \zeta_2]$, in view of condition Θ_λ, we have

$$\partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \mathring{\mathcal{J}}\rho_\eta) \leq \partial_{\mathcal{U}}(\theta_\eta, \rho_\eta).$$

Now, consider

$$\begin{aligned}\partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \theta_\eta) &\leq \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \mathring{\mathcal{J}}\rho_\eta) + \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \theta_\eta)\\ &\leq \partial_{\mathcal{U}}(\theta_\eta, \rho_\eta) + \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \theta_\eta)\\ &\leq \alpha_n \partial_{\mathcal{U}}(\theta_\eta, \mathring{\mathcal{J}}\theta_\eta) + \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \theta_\eta),\end{aligned}$$

Now, by using the inequality

$$(1-b)\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \theta_\eta) \leq (1-\alpha_n)\eth_{\mathcal{U}}(\theta_\eta, \mathring{\mathcal{J}}\theta_\eta) \leq \eth_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \theta_\eta),$$

we have

$$\lim_{n\to\infty} \eth_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \theta_\eta) \leq \frac{1}{1-b} \lim_{n\to\infty} \eth_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \theta_\eta) = 0.$$

Case 2 $0 < a \leq \beta_n \leq 1$ and $0 < a \leq \alpha_n \leq b < 1$. Since

$$\eth_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \eth) \leq \eth_{\mathcal{U}}(\theta_\eta, \eth),$$

$\forall\ \eta \geq 1$, we have

$$\limsup_{n\to\infty} \eth_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \eth) \leq c.$$

Consider

$$\begin{aligned}\eth_{\mathcal{U}}(\theta_{n+1}, \eth) &\leq \beta_n \eth_{\mathcal{U}}(\mathring{\mathcal{J}}\rho_\eta, \eth) + (1-\beta_n)\eth_{\mathcal{U}}(\theta_\eta, \eth)\\ &\leq \beta_n \eth_{\mathcal{U}}(\rho_\eta, \eth) + (1-\beta_n)\eth_{\mathcal{U}}(\theta_\eta, \eth)\end{aligned}$$

which means that

$$\frac{\eth_{\mathcal{U}}(\theta_{n+1}, \eth) - \eth_{\mathcal{U}}(\theta_\eta, \eth)}{\alpha_n} \leq \eth_{\mathcal{U}}(\rho_\eta, \eth) - \eth_{\mathcal{U}}(\theta_\eta, \eth).$$

Taking $\liminf\limits_{n\to\infty}$ of both sides of the above inequality, we have

$$\liminf_{n\to\infty} \frac{\eth_{\mathcal{U}}(\theta_{n+1}, \eth) - \eth_{\mathcal{U}}(\theta_\eta, \eth)}{\alpha_n} \leq \liminf_{n\to\infty}(\eth_{\mathcal{U}}(\rho_\eta, \eth) - \eth_{\mathcal{U}}(\theta_\eta, \eth)).$$

As

$$\lim_{n\to\infty} \eth_{\mathcal{U}}(\theta_{n+1}, \eth) = \lim_{n\to\infty} \eth_{\mathcal{U}}(\theta_\eta, \eth) = c,$$

it means that

$$0 \leq \liminf_{n\to\infty}(\eth_{\mathcal{U}}(\rho_\eta, \eth) - \eth_{\mathcal{U}}(\theta_\eta, \eth)).$$

While owing to

$$\partial_{\mathcal{U}}(\rho_\eta, \eth) - \partial_{\mathcal{U}}(\theta_\eta, \eth) \leq 0,$$

we have

$$\liminf_{n\to\infty}(\partial_{\mathcal{U}}(\rho_\eta, \eth) - \partial_{\mathcal{U}}(\theta_\eta, \eth)) \leq 0.$$

Therefore,

$$\liminf_{n\to\infty}(\partial_{\mathcal{U}}(\rho_\eta, \eth) - \partial_{\mathcal{U}}(\theta_\eta, \eth)) = 0.$$

By this, we get

$$\liminf_{n\to\infty} \partial_{\mathcal{U}}(\theta_\eta, \eth) \leq \liminf_{n\to\infty} \partial_{\mathcal{U}}(\rho_\eta, \eth).$$

That is,

$$c \leq \liminf_{n\to\infty} \partial_{\mathcal{U}}(\rho_\eta, \eth).$$

Now, by combining foregoing observations, we have

$$c \leq \liminf_{n\to\infty} \partial_{\mathcal{U}}(\rho_\eta, \eth) \leq \limsup_{n\to\infty} \partial_{\mathcal{U}}(\rho_\eta, \eth) \leq c,$$

so that

$$c = \lim_{n\to\infty} \partial_{\mathcal{U}}(\rho_\eta, \eth) = \lim_{n\to\infty} \partial_{\mathcal{U}}(\alpha_n \mathring{\mathcal{J}}\theta_\eta \oplus (1-\alpha_n)\theta_\eta, \eth).$$

Hence,

$$\lim_{n\to\infty} \partial_{\mathcal{U}}(\mathring{\mathcal{J}}\theta_\eta, \theta_\eta) = 0.$$

□

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Chapter 9
Semigroups of Completely Positive Maps

Preetinder Singh

In the beginning, we discuss results from the Hilbert space theory, C^*-algebras, von Neumann algebras, and general semigroup theory on the Banach spaces to completely understand the theory of quantum dynamical semigroups. Due to the enormity of the topic, the proofs of the theorems are generally not given, and statements of the results are recalled to make this chapter self-contained. For proofs of the theorems, the reader may look at the references like [3, 6, 9, 10, 13, 14, 16, 18].

9.1 Hilbert Space Theory

Let $\mathcal{X}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, that is, conjugate linear in the first coordinate and linear in the second coordinate. Let $\| \cdot \|$ be the norm on $\mathcal{X}$. Considering two such Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote the Banach space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and the Banach space of bounded linear operators on $\mathcal{X}$ by $\mathcal{B}(\mathcal{X})$. For a linear subspace $\mathcal{L}$ of $\mathcal{X}$, we define an orthogonal complement as $\mathcal{L}^\perp = \{x \in \mathcal{X};\ \ \langle y, x \rangle = 0, \forall y \in \mathcal{L}\}$.

Exercise 9.1 $\mathcal{L}$ is dense in $\mathcal{X}$ if and only if $\mathcal{L}^\perp = 0$.

Definition 9.1 An operator $\eta \in \mathcal{B}(\mathcal{X})$ is called positive if $\langle \eta x, x \rangle \geq 0$ for all $x \in \mathcal{X}$. For a positive operator η, we write $\eta \geq 0$.

Let η and ξ be two operators in $\mathcal{B}(\mathcal{X})$; then $\eta \geq \xi$ if $\eta - \xi \geq 0$.

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_9

Definition 9.2 An operator $\eta \in \mathcal{B}(\mathcal{X})$ is called compact if the closure of the image of a unit ball under η is compact.

In other words, η is compact if and only if for every bounded sequence $\{x_n\}$ in $\mathcal{X}$, $\{\eta x_n\}$ has a convergent subsequence.

Exercise 9.2 The set of compact operators $\eta \in \mathcal{B}(\mathcal{X})$ forms a closed maximal ideal of the ring $\mathcal{B}(\mathcal{X})$.

Suppose η is a finite rank operator, that is, the range space of η is finite dimensional. Since every closed and bounded set is compact in a finite-dimensional Hilbert space, and the image of a unit ball is bounded, we observe that η is compact. In particular, for $\alpha, \beta, \gamma \in \mathcal{X}$, the rank 1 operators on $\mathcal{X}$, defined by $|\alpha\rangle\langle\beta|(\gamma) := \langle\beta, \gamma\rangle\alpha$, are compact. In fact, each compact operator in $\mathcal{B}(\mathcal{X})$ is the uniform limit of the finite rank operators.

Proposition 9.1 *Let $\eta \in \mathcal{B}(\mathcal{X})$ be a compact normal operator; then the set of eigenvalues of η is countable.*

Proof See [6]. □

Proposition 9.2 *Let $\eta \in \mathcal{B}(\mathcal{X})$ be a compact normal operator and $(\lambda_n)_{n\geq 1}$ be a sequence of eigenvalues of η. Then the eigenspace M_n associated with λ_n is a finite-dimensional Hilbert space. The sequence $\lambda_n \to 0$ if there are infinitely many eigenvalues.*

Proof See [6]. □

We shall now state the **spectral theorem** for compact normal operators:

Theorem 9.1 *Let $\eta \in \mathcal{B}(\mathcal{X})$ be a compact normal operator and $(\lambda_n)_{n\geq 1}$ be a sequence of eigenvalues of η. Let P_n be the orthonormal projection of $\mathcal{X}$ onto the eigenspace $M_n = Ker((\eta - \lambda_n)I)$. Assume that $m = n$ and*

$$\eta = \sum_{n\geq 1} \lambda_n P_n,$$

where in the norm topology on $\mathcal{B}(\mathcal{X})$, the series is convergent. Then $P_n\ P_m = 0 = P_m\ P_n$. In addition, if η is self-adjoint, then λ_ns can be ordered in a decreasing sequence, $|\lambda_1| \geq |\lambda_2| \geq \ldots$, that converges to 0.

Proof See [6]. □

Let η be a compact operator. For a self-adjoint compact operator $\eta^*\eta$, let $(\lambda_n)_{n\geq 1}$ be the decreasing sequence of eigenvalues in the above sense. We define the nth **singular value** of η to be the positive square root of the nth eigenvalue of $\eta^*\eta$. The nth singular value of η is denoted by $s_n(\eta)$.

Definition 9.3 A compact operator η is said to be a **trace-class operator** if the series $\sum_{n\geq 1} s_n(\eta)$ is convergent. We denote the set of all trace-class operator by $\mathcal{B}_1(\mathcal{X})$.

For $\eta \in \mathcal{B}_1(\mathcal{X})$, we define the **trace** of η as $\mathrm{tr}(\eta) = \sum_{n\geq 1} \langle e_n, \eta e_n\rangle$, where $(e_n)_{n\geq 1}$ is an orthonormal basis for $\mathcal{X}$. The trace norm $\|\cdot\|_1$ on $\mathcal{B}_1(\mathcal{X})$ is defined as $\|\eta\|_1 = \sum_{n\geq 1} s_n(\eta)$.

The space $\mathcal{B}_1(\mathcal{X})$ is a Banach space w.r.t. the trace norm. One can easily see that the series $\sum_{n\geq 1} \langle e_n, \eta e_n\rangle$ converges and the sum is independent of the choice of basis. There is an interesting relation between the classes $\mathcal{B}_0(\mathcal{X})$, $\mathcal{B}_1(\mathcal{X})$ and $\mathcal{B}(\mathcal{X})$ which is shown in the following theorem.

Theorem 9.2 *For the spaces $\mathcal{B}_0(\mathcal{X})$, $\mathcal{B}_1(\mathcal{X})$ and $\mathcal{B}(\mathcal{X})$, the following is true:*

(i) $\mathcal{B}_1(\mathcal{X}) \cong \mathcal{B}_0(\mathcal{X})^*$. *That is, the map $\xi \mapsto tr(\xi\ \cdot)$ is an isometric isomorphism of $\mathcal{B}_1(\mathcal{X})$ on $\mathcal{B}_0(\mathcal{X})^*$.*
(ii) $\mathcal{B}(\mathcal{X}) \cong \mathcal{B}_1(\mathcal{X})^*$. *That is, the map $\zeta \mapsto tr(\zeta\ \cdot)$ is an isometric isomorphism of $\mathcal{B}(\mathcal{X})$ on $\mathcal{B}_1(\mathcal{X})^*$.*

Proof See [14]. □

Note that the operators which are of significance and which arise from the study of physical systems are generally not bounded. In this spirit, we mention some results concerning the unbounded operators, necessary to understand the semigroup theory. The closed graph theorem states that an operator which is everywhere defined and whose graph is closed must be bounded. This further suggests that a nice unbounded operator will be defined only on a dense linear subset of $\mathcal{X}$. Thus, we have seen that an operator(unbounded) η is a linear map whose domain is a linear subspace, which is usually dense into $\mathcal{X}$. The domain of the operator η is denoted by $\mathrm{Dom}(\eta)$.

Definition 9.4 The **graph** of a linear operator η is the set

$$\Gamma(\eta) := \{(x, \eta x) : x \in \mathrm{Dom}(\eta)\}$$

and is denoted by $\Gamma(\eta)$.

The dual of the graph Γ is given by $\Gamma^*(\eta) := \{(-\eta x, x); x \in \mathrm{Dom}(\eta)\}$.

A linear operator η is said to be **closed** if $\Gamma(\eta)$ is a closed subset of the space $\mathcal{X} \times \mathcal{X}$. Let η and ξ be two linear operators on $\mathcal{X}$. Then we say ξ is an extension of η if $\Gamma(\eta) \subseteq \Gamma(\xi)$, and we write $\eta \subseteq \xi$.

Definition 9.5 An operator η is said to be **closable** if there exists a closed extension of η. The **closure** of η, denoted by $\overline{\eta}$, is the smallest closed extension, if it exists.

Definition 9.6 Let η be a densely defined linear operator on $\mathcal{X}$. For a fixed $\alpha \in \mathrm{Dom}(\eta)$, consider the linear map $\Phi_\alpha(x) = \langle \alpha, \eta x\rangle$ such that the domain $\mathrm{Dom}(\eta)$ can be extended to a bounded linear functional on $\mathcal{X}$ given by $\langle y, x\rangle$. Then we say $y \in Dom(\eta^*)$ and $\eta^*(\alpha) = y$. The operator η^* is called the **adjoint** of η.

Observe that

$$\Gamma(\eta^*) = [\Gamma^*(\eta)]^{\perp},$$

where the space $S^{\perp} := \{x \in \mathcal{X} : \langle x, s\rangle = 0 \ \forall\ s \in S\}$. If η^* has a dense domain, then we define $\eta^{**} = (\eta^*)^*$. For a linear operator η, the adjoint and the closure are related by the following result.

Theorem 9.3 *Let η be a densely defined operator on $\mathcal{X}$. Then*

(i) η^* *is closed.*
(ii) η *is closable if and only if* $Dom(\eta^*)$ *is dense. Further,* $\overline{\eta} = \eta^{**}$.
(iii) $(\overline{\eta})^* = \eta^*$ *if* η *is closable.*

Proof See [17]. □

Proposition 9.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be the Hilbert spaces and $\eta : \mathcal{X} \to \mathcal{Y}$ be densely defined. Then*

$$(Range\ \eta)^{\perp} = Ker\ \eta^*.$$

If η is closed, then

$$(Range\ \eta^*)^{\perp} = Ker\ \eta.$$

Proof See [17]. □

Now we shall define the resolvent of an operator. The knowledge of a resolvent helps us to understand the nature of semigroups.

Definition 9.7 Let η be a closed operator on $\mathcal{X}$. A **resolvent set**, denoted by $\rho(\eta)$, is the set of complex numbers λ such that $\lambda I - \eta$ is a bijection from $\mathrm{Dom}(\eta)$ onto the dense range of $(\lambda I - \eta)$ with a bounded inverse. The **resolvent** of η at λ is given by $\mathcal{R}(\lambda, \eta) = \mathcal{R}_{\lambda}(\eta) := (\lambda I - \eta)^{-1}$.

Definition 9.8 A densely defined operator η on $\mathcal{X}$ is said to be **symmetric** if $\eta \subseteq \eta^*$.

Equivalently, η is symmetric if and only if $\langle \eta x, y\rangle = \langle x, \eta y\rangle$ for $x, y \in \mathrm{Dom}(\eta)$.

Definition 9.9 An operator η is said to be **self-adjoint** if η is symmetric and $\mathrm{Dom}(\eta^*) = \mathrm{Dom}(\eta)$.

For a symmetric densely defined η, the adjoint η^* is an extension of η, but η^* is not symmetric always. The symmetry of η^* requires the condition that $\eta^* = \eta^{**}$. We recall that η^{**} is the closure of η, and in general, we have $\eta \subset \eta^{**} \subset \eta^*$. Since η^{**} is the closure of η, η^{**} is symmetric.

The distinction between the self-adjoint operators and the closed symmetric operators is significant. For a self-adjoint operator, the spectral theorem holds, and such operators generate the one-parameter unitary groups.

Definition 9.10 Let η be a symmetric operator. If its closure $\overline{\eta}$ is self-adjoint, then η is said to be **essentially self-adjoint**. For a closed operator η, the subset d $\subseteq$Dom(η) is called **core** for η if closure of the restriction $\overline{\eta \restriction \mathrm{d}} = \eta$.

In general, a symmetric densely defined operator does not have a unique self-adjoint extension. On contrary, an essentially self-adjoint operator always possesses a unique self-adjoint extension. So, one need not to give the exact domain of η but just some core if η is a self-adjoint operator. The following results show equivalence conditions that may help in determining whether an operator is self-adjoint or essentially self-adjoint.

Theorem 9.4 *For a symmetric operator η on $\mathcal{X}$, the following are equivalent:*

(i) *η is self-adjoint.*
(ii) *η is closed and $Ker(\eta^* \pm iI) = \{0\}$.*
(iii) *$Range(\eta^* \pm iI) = \mathcal{X}$.*

Proof See [17]. □

Theorem 9.5 *For a symmetric operator η on $\mathcal{X}$, the following are equivalent:*

(i) *η is essentially self-adjoint.*
(ii) *$Ker(\eta^* \pm iI) = \{0\}$.*
(iii) *$Range(\eta^* \pm iI)$ is dense in $\mathcal{X}$.*

Proof See [17]. □

Theorem 9.6 (Spectral Theorem) *Let $\eta \in \mathcal{B}(\mathcal{X})$ be a self-adjoint operator. Then there is a right continuous projection valued function $\mathbb{E} : \mathbb{R} \to \mathcal{P}(\mathcal{X})$, where $\mathcal{P}(\mathcal{X})$ is the space of orthogonal projections on $\mathcal{X}$, such that*

$$\eta = \int_{\mathbb{R}} \lambda \, \mathbb{E}(d\lambda). \tag{9.1}$$

The above function $\mathbb{E} : \mathbb{R} \to \mathcal{P}(\mathcal{X})$ satisfies the following:

(i) $\lim_{t\to\infty} \mathbb{E}(t) = I$ strongly.
(ii) $\lim_{t\to-\infty} \mathbb{E}(t) = 0$ strongly.
(iii) $\mathbb{E}(s)\mathbb{E}(t) = \mathbb{E}(s \wedge t), \; s \wedge t = \min\{s, t\}$.

and is called **spectral measure** for η. The spectral integration in (9.1) is in the sense that

$$\langle u, \eta v\rangle = \int t \; \mu_{u,v}(dt),$$

where $\mu_{u,v}$ is the complex measure given by $\mu_{u,v}((-\infty, t]) = \langle u, E(t)v\rangle$.

Proof See [17]. □

Analogous to the decomposition $z = |z|e^{i \arg z}$ for complex numbers, there is a special decomposition called polar decomposition for operators on a Hilbert space. Any bounded operator η can be written as $\eta = U|\eta|$ *uniquely*, where $|\eta|$ is a positive self-adjoint operator and U is a partial isometry. Now we shall discuss the polar decomposition in the case of unbounded operators. For the bounded case, polar decomposition is easy to construct since we shall set $|\eta| = \sqrt{\eta^*\eta}$ in a view of the existence of positive square root. In the case of unbounded operators, the following result helps us to generalize the existence of a polar decomposition for the unbounded operators.

Theorem 9.7 *Let η be a closed, densely defined operator on $\mathcal{X}$. Then $\eta^*\eta$ is a self-adjoint operator and $Dom(\eta^*\eta)$ is a core for η.*

Proof See [17]. □

Moreover, $\eta^*\eta$ is a positive self-adjoint operator on $\mathcal{X}$. By spectral theorem, we shall define $|\eta| = \sqrt{\eta^*\eta}$. The strategy to construct polar decomposition for bounded operators also works in the case of unbounded operators.

Theorem 9.8 *Let η be a closed, densely defined operator on $\mathcal{X}$. Then there are a positive self-adjoint operator $|\eta| = \sqrt{\eta^*\eta}$, with $Dom(|\eta|) = Dom(\eta)$, and a partial isometry U with domain $(Ker\ \eta)^\perp$ and co-domain $\overline{Range\ \eta}$, such that $\eta = U|\eta|$. The operators $|\eta|$ and U are uniquely determined by these properties together with the property that $Ker(|\eta|) = Ker\ (\eta)$.*

Proof See [17]. □

9.2 C^*-Algebras

Here, we give a brief introduction to C^*-algebras and von Neumann algebra on which quantum dynamical semigroups shall be discussed.

Definition 9.11 A complete normed algebra $\mathcal{A}$ with norm $\|\cdot\|$ is said to be **Banach algebra** if $\|\alpha\beta\| \le \|\alpha\|\|\beta\|$ for $\alpha, \beta \in \mathcal{A}$. It is called C^***-algebra** if it has a $*$-structure and $\|\alpha^*\alpha\| = \|\alpha\|^2$ for all $\alpha \in \mathcal{A}$.

Commutative C^*-Algebra

Let $\mathcal{H}$ be a locally compact Hausdorff space. Consider the complex valued continuous functions on $\mathcal{H}$ that vanishes at infinity and has the supremum norm and the complex conjugation as $*$-operation. Then the space $C_0(\mathcal{H})$ of all such functions forms a commutative C^*-algebra under the point-wise multiplication and addition.

If the algebra contains identity, then it is called **unital**; otherwise, **non-unital**. However, each C^*-algebra can be made unital by adjoining the identity to it. The above example is important as each commutative C^*-algebra is essentially of such form. Explicitly, the following result gives the characterization of commutative C^*-algebras.

Theorem 9.9 (Gelfand-Naimark) *A commutative C^*-algebra $\mathcal{A}$ is isometrically isomorphic to $C_0(\mathcal{H})$, for some locally compact Hausdorff space $\mathcal{H}$. If $\mathcal{A}$ is unital, then the Hausdorff space $\mathcal{H}$ is compact.*

Proof See [9, 14]. □

For a C^*-algebra $\mathcal{A}$, a linear functional $\psi : \mathcal{A} \to \mathbb{C}$ is said to be positive if $\psi(\alpha^*\alpha) \geq 0$, for $\alpha \in \mathcal{A}$. It can be seen that $\alpha \in \mathcal{A}$ is positive if and only if the image $\psi(\alpha)$ is positive for all positive functionals ψ on $\mathcal{A}$. If $\psi(1) = 1$, then the positive linear functional ψ is called a **state** on $\mathcal{A}$. It can be shown that positivity implies boundedness. A state ψ is called **tracial** if $\psi(\alpha\beta) = \psi(\beta\alpha)$ for all $\alpha, \beta \in \mathcal{A}$. It is called **faithful** if $\psi(\alpha^*\alpha) = 0$ implies $\alpha = 0$.

Definition 9.12 Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{X}$ be a Hilbert space such that $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ is a $*$-homomorphism. The pair $(\pi, \mathcal{X})$ is called a **representation** of $\mathcal{A}$. If $\mathcal{A}$ is unital, it is assumed that $\pi(1) = 1$.

Theorem 9.10 (Gelfand-Naimark-Segal Construction) *Let $\mathcal{A}$ be a C^*-algebra and ψ be a state on $\mathcal{A}$. Then there exist a Hilbert space $\mathcal{X}_\psi$, a representation $\pi_\psi : \mathcal{A} \to \mathcal{B}(\mathcal{X}_\psi)$, and a vector $\xi_\psi \in \mathcal{X}_\psi$ that is cyclic, meaning the set $\{\pi_\psi(\alpha)\xi_\psi\ ;\ \alpha \in \mathcal{A}\}$ is total in $\mathcal{X}_\psi$, such that*

$$\psi(\alpha) = \left\langle \xi_\psi, \pi_\psi(\alpha)\xi_\psi \right\rangle.$$

Proof See [9, 14]. □

This triplet $(\mathcal{X}_\psi, \pi_\psi, \xi_\psi)$ is named the GNS triple for $(\mathcal{A}, \psi)$. The space $\mathcal{X}_\psi$ is said to be **GNS Hilbert space** for $(\mathcal{A}, \psi)$ and is denoted by $L^2(\mathcal{A}, \psi)$.

9.3 von Neumann Algebras

The Banach space of bounded linear operators $\mathcal{B}(\mathcal{X})$ on a Hilbert space $\mathcal{X}$ is usually studied with the operator-norm topology, whereas there are several other important topologies on $\mathcal{B}(\mathcal{X})$ such as weak, strong, ultra-weak, and ultra-strong topologies. The algebra of operators $\mathcal{B}(\mathcal{X})$ is locally convex complete topological vector space for these topologies.

Definition 9.13 The norm of a bounded operator defines a topology on the Banach space $\mathcal{B}(\mathcal{X})$ called the **norm topology**.

For a bounded operator η, the function $\eta \rightarrow \|\eta\|$ is a semi-norm on $\mathcal{B}(\mathcal{X})$ and gives rise to the topology of uniform convergence over the bounded subsets of $\mathcal{X}$.

Definition 9.14 For every $x \in \mathcal{X}$, the function $\eta \rightarrow \|\eta x\|$ is a semi-norm on $\mathcal{B}(\mathcal{X})$. The collection of these semi-norms determines the Hausdorff locally convex topology. This is the topology of strong point-wise convergence called the **strong (operator) topology**.

For strong topology, a base of neighborhoods around origin is obtained by taking subsets

$$\{\eta \in \mathcal{B}(\mathcal{X})\ ;\ \|\eta x_i\| < \epsilon\ ,\ 1 \leq i \leq n\},$$

for each finite sequence $(x_i)_{i=1}^n$ in $\mathcal{X}$ and $\epsilon > 0$. We can also define the strong topology on $\mathcal{B}(\mathcal{X})$ as the coarsest topology for which the maps $\eta \rightarrow \eta h$ from $\mathcal{B}(\mathcal{X})$ into $\mathcal{X}$ are continuous.

Definition 9.15 For $x, y \in \mathcal{X}$, the collection of the semi-norms $\eta \rightarrow |\langle \eta x, y\rangle|$ determines the Hausdorff locally convex topology know as **weak (operator) topology** or the topology of weak convergence.

In the view of polarization identity, we see that the semi-norms $\eta \rightarrow |\langle \eta x, x\rangle|$ are enough to define the weak topology. For this topology, a base of neighborhoods around origin is obtained by taking subsets

$$\{\eta \in \mathcal{B}(\mathcal{X})\ ;\ |\langle \eta x_i, y_i\rangle| < \epsilon\ ,\ 1 \leq i \leq n\},$$

for each pair of finite sequences $(x_i)_{i=1}^n$; $(y_i)_{i=1}^n$ in $\mathcal{X}$, $\epsilon > 0$. We shall also define the weak topology on $\mathcal{B}(\mathcal{X})$ as the coarsest topology for which the maps $\eta \rightarrow \langle \eta h, g\rangle$ from $\mathcal{B}(\mathcal{X})$ into $\mathbb{C}$ are continuous.

Definition 9.16 Let $(x_i)_{i=1}^{\infty}$ be a sequence in $\mathcal{X}$ such that $\sum\limits_{i=1}^{\infty} \|x_i\|^2 < \infty$. Since the series $\sum\limits_{i=1}^{\infty} \|\eta x_i\|^2$ is convergent, the map $\eta \rightarrow \left(\sum\limits_{i=1}^{\infty} \|\eta x_i\|^2\right)^{\frac{1}{2}}$ defines a semi-norm on $\mathcal{B}(\mathcal{X})$. The collection of all these semi-norms determines the Hausdorff locally convex topology called **ultra-strong topology**.

A base of neighborhoods around origin is obtained by taking subsets

$$\left\{\eta \in \mathcal{B}(\mathcal{X})\ ;\ \sum_{i=1}^{\infty} \|\eta x_i^k\|^2 < \epsilon\ ,\ 1 \leq k \leq n\right\},$$

for each $\epsilon > 0$ and for every finite family of sequences $\{(x_i^1), (x_i^2), \cdots, (x_i^n)\}_{1 \leq i < \infty}$ in $\mathcal{X}$ such that for $1 \leq k \leq n$,

$$\sum_{i=1}^{\infty} \|x_i^k\|^2 < \infty.$$

For this topology, the maps, $\eta \to (\eta x_1, \eta x_2, \cdots)$ from $\mathcal{B}(\mathcal{X})$ into direct sum $\bigoplus \mathcal{X}_i$: $\mathcal{X}_i = \mathcal{X}$ for all i, are continuous.

Definition 9.17 In the view of Cauchy-Schwarz inequality and Hölder's inequality, we see that, for each pair of sequences $(x_i)_{i=1}^{\infty}$; $(y_i)_{i=1}^{\infty}$ in $\mathcal{X}$ such that

$$\sum_{i=1}^{\infty} \|x_i\|^2 < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \|y_i\|^2 < \infty,$$

the map $\eta \to |\sum_{i=1}^{\infty} \langle \eta x_i, y_i \rangle|$ defines a semi-norm on $\mathcal{B}(\mathcal{X})$. The collection of these semi-norms determines the Hausdorff locally convex topology called the **ultra-weak topology**.

For this topology, a base around origin is given by taking subsets

$$\left\{ \eta \in \mathcal{B}(\mathcal{X}) \ ; \ |\sum_{i=1}^{\infty} \left\langle \eta x_i^k, y_i^k \right\rangle| < \epsilon \ , \ 1 \leq k \leq n \right\},$$

for $\epsilon > 0$ and family of pair of sequences $\{\left((x_i^1); (y_i^1)\right), \left((x_i^2); (y_i^2)\right), \ldots, \left((x_i^n); (y_i^n)\right)\}_{i=1}^{\infty}$ in $\mathcal{X}$, such that for $1 \leq k \leq n$,

$$\sum_{i=1}^{\infty} \|x_i^k\|^2 < \infty, \quad \text{and} \quad \sum_{i=1}^{\infty} \|y_i^k\|^2 < \infty.$$

This topology is also coarsest for which the maps $\eta \to \sum_{i=1}^{\infty} \langle \eta x_i, y_i \rangle$ from $\mathcal{B}(\mathcal{X})$ into $\mathbb{C}$ are continuous.

The above topologies are compared to give the following diagram, where the symbol $<$ means "finer than":

$$\begin{array}{ccccc} \text{Norm topology} & < & \text{Ultra-strong topology} & < & \text{Strong topology} \\ & & \wedge & & \wedge \\ & & \text{Ultra-weak topology} & < & \text{Weak topology} \end{array}$$

For an infinite-dimensional Hilbert space, the symbol $<$ shall be taken to mean "strictly finer than." On bounded subsets of $\mathcal{B}(\mathcal{X})$, the strong(or weak) and ultra-strong(or ultra-weak) topologies coincide. The space $\mathcal{B}(\mathcal{X})$ is complete with respect to each of these topologies, but in general, a C^*- subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{X})$ need not be complete. It is well known that for all locally convex topologies except norm topology, $\mathcal{A}$ is complete if and only if it is complete in any one of them. In such a case, $\mathcal{A}$ is said to be a **von Neumann algebra**. Furthermore, on norm bounded convex subsets of $\mathcal{A}$, the strong(or weak) and ultra-strong(or ultra-weak) topologies coincide.

Let $\mathcal{A}$ be a von Neumann algebra. The commutant of $\mathcal{A}$ is the set $\{\alpha \in \mathcal{B}(\mathcal{X}) \mid \alpha x = x\alpha,\ \forall\ x \in \mathcal{A}\}$. We denote it as $\mathcal{A}'$ and we have $\mathcal{A}'' = (\mathcal{A}')'$. The following result holds a fundamental importance in the study of von Neumann algebras. The result is due to von Neumann.

Theorem 9.11 (Double Commutant Theorem) *Let $\mathcal{A} \in B$ be a non-degenerate C^*-algebra. Then $\mathcal{A}'' = \overline{\mathcal{A}}^w = \overline{\mathcal{A}}^s$, where $\overline{\mathcal{A}}^w$ and $\overline{\mathcal{A}}^s$ are the closures of $\mathcal{A}$ in weak and strong topologies, respectively.*

Proof See [14, 18]. □

Note that an unital C^*-algebra $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A}'' = \mathcal{A}$.

Let ψ be a state on a von Neumann algebra $\mathcal{A}$. Then ψ is said to be **normal** if for a given net $\{a_\alpha\}$ of positive elements in $\mathcal{A}$, $\psi(a_\alpha)$ increases to $\psi(a)$ whenever a_α increases to a.

Let $\mathcal{A}, \mathcal{B}$ be von Neumann algebras. A linear map $\Psi : \mathcal{A} \to \mathcal{B}$ is said to be **normal** if for a net $\{a_\alpha\}$ of positive elements in $\mathcal{A}$, whenever a_α increases to a, we have $\Psi(a_\alpha)$ increases to $\Psi(a)$ in $\mathcal{B}$. Observe that a positive linear map $\Psi : \mathcal{A} \to \mathcal{B}$ is normal if and only if it is continuous w.r.t. ultra-weak topologies. Thus, we shall conclude the same for a bounded linear map between two von Neumann algebras.

Normal states or, in general, the normal positive linear maps, like normal $*$-homomorphism, play an important role in the study of von Neumann algebras. The structure of a normal state is described in the following result.

Theorem 9.12 *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{X})$ be a von Neumann algebra. A state ψ on $\mathcal{A}$ is normal if and only if there is a positive trace-class operator ξ on $\mathcal{X}$ such that $\psi(\alpha) = tr(\xi\alpha)$ for $\alpha \in \mathcal{A}$.*

Proof See [14, 18]. □

Consider a von Neumann algebra $\mathcal{A}$. A Banach space $\mathcal{A}_*$ is called the **predual** of $\mathcal{A}$ if the Banach dual $(\mathcal{A}_*)^*$, w.r.t. the norm topology, coincides with $\mathcal{A}$ and, w.r.t. the weak-$*$ topology, it coincides with the ultra-weak topology on $\mathcal{A}$. Moreover, one can characterize a von Neumann algebra in the class of C^*-algebras using a predual as a Banach space. Subsequently, we give the explicit description of the predual of $\mathcal{A} \subseteq \mathcal{B}(\mathcal{X})$. Since from Theorem 9.2, we see that $\mathcal{A}_*$ is the some quotient space of $\mathcal{B}_1(\mathcal{X})$. Let $\mathcal{B}^{s.a.}(\mathcal{X})$ denotes the real linear space of all bounded self-adjoint operators on $\mathcal{X}$ and $\mathcal{B}_1^{s.a.}(\mathcal{X})$ denotes all trace-class self-adjoint operators on $\mathcal{X}$. We shall denote the subset of all self-adjoint elements in $\mathcal{A}$ by $\mathcal{A}^{s.a.}$. Let $\mathcal{A}_*^{s.a.}$ be the predual of $\mathcal{A}^{s.a.}$. We define an equivalence relation $\sim$ on $\mathcal{B}_1(\mathcal{X})$ as $\xi_1 \sim \xi_2$ if and only if $\mathrm{tr}(\xi_1\alpha) = \mathrm{tr}(\xi_2\alpha)$ for all $\alpha \in \mathcal{A}$. We denote the closed subspace $\{\xi \in \mathcal{B}_1(\mathcal{X})\ ;\ \xi \sim 0\}$ by $\mathcal{A}^\perp$. For $\xi \in \mathcal{B}_1(\mathcal{X})$, we denote its equivalence class w.r.t. $\sim$ as $\tilde{\xi}$ and $\|\tilde{\xi}\| = \inf_{\eta\sim\xi} \|\eta\|_1$. The set $(\mathcal{A}^\perp)^{s.a.}$ denotes the set of all self-adjoint elements in $\mathcal{A}^{\perp}$. Clearly, $(\mathcal{A}^{\perp})^{s.a.}$ is a closed subspace of $\mathcal{B}_1^{s.a.}(\mathcal{X})$ and so one can make sense of the quotient space $\mathcal{B}_1^{s.a.}(\mathcal{X})/(\mathcal{A}^{\perp})^{s.a.}$.

The following theorem determines the predual explicitly.

Theorem 9.13 *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{X})$ be a von Neumann algebra.*

(i) *There exists an isometric isomorphism*

$$\mathcal{A}_* \;\cong\; \frac{\mathcal{B}_1(\mathcal{X})}{\mathcal{A}^{\perp}} \;\cong\; \Omega_{\mathcal{A}},$$

where $\Omega_{\mathcal{A}}$ is the space of all bounded normal complex linear functionals on $\mathcal{A}$.

(ii) *There exists an isometric isomorphism*

$$\mathcal{A}_*^{s.a.} \;\cong\; \frac{\mathcal{B}_1^{s.a.}(\mathcal{X})}{(\mathcal{A}^{\perp})^{s.a.}} \;\cong\; \Omega_{\mathcal{A}^{s.a.}},$$

where $\Omega_{\mathcal{A}^{s.a.}}$ is the space of all bounded normal complex linear functionals on $\mathcal{A}^{s.a.}$.

Proof See [14, 18]. □

The map $\alpha \mapsto \psi_\alpha$ where $\psi_\alpha(\tilde{\xi}) = \mathrm{tr}(\xi\alpha)$ provides the canonical identification between $\mathcal{A}$ and $(\mathcal{B}_1(\mathcal{X})/\mathcal{A}^{\perp})^*$. Moreover, $\tilde{\xi} \in \mathcal{B}_1(\mathcal{X})/\mathcal{A}^{\perp}$ is canonically associated with $\psi_{\tilde{\xi}} \in \Omega_{\mathcal{A}}$, where $\psi_{\tilde{\xi}}(\alpha) = \mathrm{tr}(\xi\alpha)$ for $\alpha \in \mathcal{A}$.

For quantum dynamical semigroups, the condition of complete positivity is fundamental, and it has very important mathematical and physical consequences.

9.4 Completely Positive Maps

Let $\mathcal{A}$ and $\mathcal{B}$ be unital $*$-algebras. We shall first recall that a linear map $\eta : \mathcal{A} \to \mathcal{B}$ is said to be **positive** if $\eta(\alpha^*\alpha) \geq 0$ in $\mathcal{B}$ for all $\alpha \in \mathcal{A}$. A general element $\alpha \in \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ can be written as

$$\sum_{1 \leq i,j \leq n} \alpha_{ij} \otimes E_{ij},$$

where E_{ij} is the $n \times n$ matrix with all entries 0 except 1 at the (ij)th place. For $1 \leq i, j \leq n$, we define the linear operator

$$\eta^{(n)} : \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \to \mathcal{B} \otimes \mathcal{M}_n(\mathbb{C}) \quad \text{as}$$
$$(\alpha \otimes E_{ij}) \mapsto \eta(\alpha) \otimes E_{ij}. \tag{9.2}$$

The map $\eta^{(n)}$ need not be positive.

Definition 9.18 Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras. A linear map $\eta : \mathcal{A} \to \mathcal{B}$ is called **n-positive** if $\eta^{(n)}$ as defined above is positive. If for all $n \geq 1$, $\eta^{(n)}$ is positive, then η is called **completely positive**.

Proposition 9.4 *Let $\eta : \mathcal{A} \to \mathcal{B}$ be a completely positive linear map. Then for all $n \geq 1,\ (\alpha_i)_{i=1}^n \subset \mathcal{A},\ (\beta_i)_{i=1}^n \subset \mathcal{B}$, we have*

$$\sum_{1 \leq i, j \leq n} \beta_i^* \eta(\alpha_i^* \alpha_j) \beta_j \geq 0.$$

Proof See [16]. □

Proposition 9.5 *Consider a sequence $(\eta_n)_{n \geq 1}$ of completely positive maps $\eta_n : \mathcal{A} \to \mathcal{B}(\mathcal{X})$. Assume that for every $\alpha \in \mathcal{A}$, the sequence $(\eta_n(\alpha))_{n \geq 1}$ converges weakly. Then the map $\eta_n : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ given by*

$$\eta(a) = \lim_{n \to \infty} \eta_n(a)$$

is completely positive.

Proof See [16]. □

A $*$-homomorphism is a completely positive map, but converse need not be true. The following theorem by Stinespring establishes that completely positive maps essentially come from $*$-homomorphisms.

Theorem 9.14 (Stinespring) *Let $\mathcal{A}$ be a C^*-algebra and $\eta : \mathcal{A} \to \mathcal{B}(\mathcal{X})$ be a completely positive map. Then there is another Hilbert space $\mathcal{Y}$, representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{Y})$ and $V \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that the set $\{\pi(\alpha)Vx\ :\ \alpha \in \mathcal{A},\ x \in \mathcal{X}\}$ is total in $\mathcal{Y}$ and the map η has the form*

$$\eta(\alpha) = V^* \pi(\alpha) V, \quad \textit{for all} \quad \alpha \in \mathcal{A}.$$

Proof See [16]. □

Such a triple $(\mathcal{Y}, \pi, V)$ is called **Stinespring's triple** associated with η. This triple is unique in the sense that if there is another such triple $(\mathcal{Y}', \pi', V')$, then there exists a unitary operator $\Gamma : \mathcal{Y} \to \mathcal{Y}'$ satisfying $\pi'(\alpha) = \Gamma \pi(\alpha) \Gamma^*$ and $V' = \Gamma V$. Further, if $\mathcal{A}$ is a von Neumann algebra and η is normal, then we can choose π to be normal. A positive map $\eta : \mathcal{A} \to \mathcal{B}$ is completely positive if either $\mathcal{A}$ or $\mathcal{B}$ is abelian.

Theorem 9.15 (Kraus) *A linear map $\eta : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{Y})$ is normal, completely positive if and only if it can be expressed as*

$$\eta(\alpha) = \sum_{1 \leq n < \infty} V_n^* \alpha V_n,$$

where $\{V_n\}_{1 \leq n < \infty}$ *is a sequence in* $\mathcal{B}(\mathcal{K}, \mathcal{H})$ *for which the series* $\sum\limits_{1 \leq n < \infty} V_n^* \alpha V_n$ *converges strongly.*

Proof See [16]. □

In the semigroup theory, a class of operators called conditionally completely positive maps play an important role. We shall now introduce this notion of complete positivity.

Definition 9.19 Let $\mathcal{A}$ be a $*$-algebra. A linear map η on $\mathcal{A}$ is called **conditionally completely positive** (CCP) map if the map $\eta^{(n)}$ defined as in (9.2) satisfies the following inequality:

$$\eta^{(n)}(\alpha^* \alpha) - \alpha^* \eta^{(n)}(\alpha) - \eta^{(n)}(\alpha^*)\alpha + \alpha^* \eta^{(n)}(1)\alpha \geq 0, \tag{9.3}$$

for $n \geq 1$ and $\alpha \in \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$.

Proposition 9.6 *A map* $\eta : \mathcal{A} \to \mathcal{A}$ *is* CCP *if and only if for each pair of finite sequences* $(\alpha_i)_{i=1}^n$, $(\beta_i)_{i=1}^n$ *in* $\mathcal{A}$*, we have*

$$\sum_{i,j=1}^{n} \beta_i^* \eta(\alpha_i^* \alpha_j)\beta_j \geq 0, \text{ whenever } \sum_{i,j=1}^{n} \alpha_i \beta_i = 0.$$

Proof See [14, 16]. □

Proposition 9.7 *Let* $\eta : \mathcal{A} \to \mathcal{A}$ *be a bounded map such that for each* $t \geq 0$, $e^{t\eta}$ *is a contraction map. If* η *is* CCP*, then* $e^{t\eta}$ *is completely positive.*

Proof See [14, 16]. □

The following theorem characterizes all the $*$-preserving CCP maps on von Neumann algebra $\mathcal{B}(\mathcal{X})$.

Theorem 9.16 *A bounded linear map* η *on* $\mathcal{B}(\mathcal{X})$*, satisfying the condition* $\eta(\alpha^*) = (\eta(\alpha))^*$ *for every* $\alpha \in \mathcal{B}(\mathcal{X})$*, is* CCP *if and only if there exists a completely positive map* ξ *on* $\mathcal{B}(\mathcal{X})$ *and* $\zeta \in \mathcal{B}(\mathcal{X})$ *such that*

$$\eta(\alpha) = \xi(\alpha) + \zeta^* \alpha + \alpha \zeta \tag{9.4}$$

for all $\alpha \in \mathcal{B}(\mathcal{X})$*. Moreover,* $\zeta + \zeta^* \leq \eta(1)$.

Proof See [14, 16]. □

9.5 One-Parameter Semigroups

For this section, we denote a complex Banach space as $\mathcal{X}$. The notion of semigroup of bounded linear operators has its roots in the basic observation that the Cauchy functional equation $\rho(a+b) = \rho(a)\rho(b) : \rho(0) = 1$ has only continuous solutions of the form $e^{t\alpha}$, $\alpha \in \mathbb{R}$. In general, the theory was developed by taking into account the Cauchy problem in infinite-dimensional framework, i.e., to find all the maps $\tau : \mathbb{R}_+ \to \mathcal{B}(\mathcal{X})$ that satisfies the functional equation

$$\begin{cases} \tau_{a+b}(\alpha) = \tau_a(\tau_b(\alpha)) \text{ for all } \alpha \in \mathcal{X},\ \forall\, a, b \geq 0, \\ \tau_0(\alpha) = \alpha. \end{cases} \tag{9.5}$$

Definition 9.20 A family $\tau = (\tau_a)_{a\geq 0}$ of bounded linear operators on a Banach space $\mathcal{X}$ that satisfies the functional equation (9.5) is called **one-parameter semigroup** or simply **semigroup** on $\mathcal{X}$.

Definition 9.21 The **infinitesimal generator** or simply **generator** of a semigroup $(\tau_a)_{a\geq 0}$ is the linear operator $L : \mathcal{X} \to \mathcal{X}$ given by

$$Dom(L) = \left\{ \alpha \in \mathcal{X};\ \text{such that } \lim_{a\downarrow 0} \frac{1}{a}(\tau_a - I)\alpha \text{ exists} \right\}$$

and $L\alpha = \lim_{a\downarrow 0} \frac{1}{a}(\tau_a - I)\alpha,\ \alpha \in Dom(L)$. We write $\tau_a = e^{aL}$ and $L = \left.\frac{d}{da}\right|_{a=0} \tau_a$, whenever L generates τ.

Definition 9.22 If the map

$$\mathbb{R}_+ \ni a \to \tau_a \in \mathcal{B}(\mathcal{X})$$

is continuous w.r.t. the norm topology on $\mathcal{B}(\mathcal{X})$, then the semigroup $(\tau_a)_{a\geq 0}$ on $\mathcal{X}$ is called **uniformly continuous** (norm continuous) semigroup.

Theorem 9.17 *A semigroup $(\tau_a)_{a\geq 0}$ on $\mathcal{X}$ is uniformly continuous if and only if the generator L is bounded.*

Proof See [10]. □

9.5.1 Strongly Continuous (C_0)-Semigroups

To describe many important physical processes, we come across unbounded operators, and thus to describe the dynamics of these physical systems, we strongly require uniform continuity. So we study the semigroups with some weak continuity conditions.

Definition 9.23 Let $\mathcal{X}$ be a Banach space. A semigroup $(\tau_a)_{a\geq 0}$ on $\mathcal{X}$ is called **strongly continuous** semigroup if the map

$$\mathbb{R}_+ \ni a \to \tau_a(\alpha) \in \mathcal{X}$$

is continuous for every $\alpha \in \mathcal{X}$. Equivalently, we can say the map $a \to \tau_a$ is continuous with strong operator topology on $\mathcal{B}(\mathcal{X})$.

Every strongly continuous semigroups $(\tau_a)_{a\geq 0}$ is **quasi-bounded**, that is, there exist constants $c \in \mathbb{R}$ and $C \geq 1$ such that for all $a \geq 0$

$$\|(\tau_a)\| \leq Ce^{ca}.$$

A semigroup $(\tau_a)_{a\geq 0}$ is called **isometric** or **contractive** if every τ_a is so. The resolvent of the generator L for a strongly continuous contraction semigroup $(\tau_a)_{a\geq 0}$ is given by the Laplace transform of the semigroup, that is, for $\mathrm{Re}\lambda > 0$,

$$\mathcal{R}(\lambda, L) = \int_0^\infty e^{-\lambda a}\tau_a(\alpha)\, da.$$

Lemma 9.1 *Let L be the generator of a strongly continuous semigroup $(\tau_a)_{a\geq 0}$. Then*

(i) *For every $a \geq 0$ and $\alpha \in Dom(L)$, we have $\tau_a(\alpha) \in Dom(L)$ and*

$$\frac{d}{da}\tau_a(\alpha) = \tau_a(L\alpha) = L(\tau_a(\alpha)). \tag{9.6}$$

(ii) *For each $a \geq 0$ and $\alpha \in \mathcal{X}$, we have $\int_0^a \tau_b(\alpha)\, db \in Dom(L)$ and*

$$\tau_a(\alpha) - \alpha = L\Big(\int_0^a \tau_b(\alpha)\, db\Big) = \int_0^a \tau_b(L\alpha)\, db, \ \text{if}\ \alpha \in Dom(L). \tag{9.7}$$

Proof See [10]. □

Theorem 9.18 *The generator L of a strongly continuous semigroup $(\tau_a)_{a\geq 0}$ is a closed and densely defined linear operator, and it determines the semigroup uniquely.*

Proof See [10]. □

It is often seen that in the results which are true for the generator, it is sufficient if we could prove the same for some core for the generator. So life becomes easy if we are able to identify a nice core for the generator. Proving a set d to be a core for L is equivalent to showing d is dense in $\mathrm{Dom}(L)$ with the graph norm

$$\|\alpha\|_\Gamma := \|\alpha\| + \|L\alpha\|.$$

Proposition 9.8 (Nelson) *Let $(\tau_a)_{a\geq 0}$ be a strongly continuous semigroup on $\mathcal{X}$ and L be its generator. A subspace* d *of Dom(L) which is dense in $\mathcal{X}$ and is invariant under $(\tau_a)_{a\geq 0}$ is a core for L.*

Proof See [10]. □

We shall now state a major theorem in the theory of strongly continuous semigroups, which characterize the strongly continuous semigroups in terms of the generator. Hille-Yosida in [12] proved it for the contraction semigroups, which then extended for the general case by Feller-Miyadera-Phillips.

Theorem 9.19 (Hille-Yosida) *Let $(\tau_a)_{a\geq 0}$ be a strongly continuous contraction semigroup on a Banach space $\mathcal{X}$. Then a linear operator L is a generator of $(\tau_a)_{a\geq 0}$ if and only if*

(i) L is closed and densely defined
(ii) for every $\gamma > 0$, $\gamma \in \tau(L)$
(iii) the resolvent of L at γ satisfies $\|\mathcal{R}(\gamma, L)\| \leq \dfrac{1}{\gamma}$.

Proof See [10]. □

The Lumer-Phillips characterization of strongly continuous semigroups in terms of dissipative operator is important because it does not need the knowledge of resolvent explicitly. For completion of hierarchy of the generation theorems similar to Theorem 9.19, we are incorporating the following results.

Let $\mathcal{X}^*$ be the Banach dual of $\mathcal{X}$. For $\alpha \in \mathcal{X}$, we denote the value $\alpha^*(\alpha)$ by $\langle\alpha, \alpha^*\rangle$ or $\langle\alpha^*, \alpha\rangle$. We define the dual set $F(\alpha) \subseteq \mathcal{X}^*$ of $\alpha \in \mathcal{X}$ as

$$F(\alpha) := \{\alpha^* \in \mathcal{X}^* \,;\; \big\langle\alpha, \alpha^*\big\rangle = \|\alpha^*\|^2 = \|\alpha\|^2\}.$$

Definition 9.24 A linear operator L is called **dissipative** if for each $\alpha \in \mathcal{X}$, there exists $\alpha^* \in F(\alpha)$ such that $Re\langle L\alpha, \alpha^*\rangle \leq 0$.

Remark 9.1 In particular, when $\mathcal{X}$ is a Hilbert space, then $F(\alpha) = \alpha$. Thus, in such case, a linear operator L is dissipative if for each $\alpha \in \mathrm{Dom}(L)$, we have $\langle\alpha, L\alpha\rangle \leq 0$, that is, if $-L$ is a positive operator.

Theorem 9.20 *A linear operator L is dissipative if and only if for each* $\alpha \in Dom(L)$ *and* $\gamma > 0$*, we have*

$$\|(\gamma I - L)x\| \geq \gamma \|x\|.$$

Proof See [10]. □

Theorem 9.21 (Lumer-Phillips) *Let* $(\tau_a)_{a\geq 0}$ *be a strongly continuous contraction semigroup on a Banach space* $\mathcal{X}$*. Then a linear operator L is a generator of* $(\tau_a)_{a\geq 0}$ *if and only if L is closed, densely defined, and dissipative and for* $\gamma > 0$*,* $Range(\gamma I - L)$ *is dense in* $\mathcal{X}$*.*

Proof See [10]. □

The following theorems describe an important fact, that is, how the convergence of strongly continuous semigroups is related to the convergence of generators as well as with the convergence of their resolvents.

Theorem 9.22 *Let* $(\tau_a^{(n)})_{a\geq 0}$ *and* $(\tau_a)_{a\geq 0}$ *be strongly continuous contraction semigroups on* $\mathcal{X}$ *with generators* $L^{(n)}$ *and L, respectively. Then* $(\tau_a^{(n)})_{a\geq 0}$ *is strongly convergent to* $(\tau_a)_{a\geq 0}$ *if and only if* $(\gamma I - L^{(n)})^{-1}$ *is strongly convergent to* $(\gamma I - L)^{-1}$*.*

Proof See [10]. □

Theorem 9.23 (Chernoff) *Let* $(\tau_a^{(n)})_{a\geq 0}$ *and* $(\tau_a)_{a\geq 0}$ *be strongly continuous contraction semigroups on* $\mathcal{X}$ *with generators* $L^{(n)}$ *and L, respectively, with a common core* d *such that* $L^{(n)}\alpha \to L\alpha$ *for* $\alpha \in$ d*. Then* $(\tau_a^{(n)})_{a\geq 0}$ *converges strongly to* $(\tau_a)_{a\geq 0}$*.*

Proof See [10]. □

9.6 Quantum Dynamical Semigroups

Now, we shall introduce the semigroups of completely positive maps. Consider a separable Hilbert pace $\mathcal{X}$ and the von Neumann algebra $\mathcal{B}(\mathcal{X})$ of bounded linear operators on $\mathcal{X}$.

Definition 9.25 Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{X})$ be a C^*-algebra. A **quantum dynamical semigroup** on $\mathcal{A}$ is a semigroup $\tau = (\tau_a)_{a\geq 0}$ of completely positive maps on $\mathcal{A}$ with properties:

(a) $\tau_0(\alpha) = \alpha$, for $\alpha \in \mathcal{A}$.
(b) $\tau_a(I) \leq I$, for all $a \geq 0$.
(c) τ_a is strongly continuous for all $a \geq 0$.
(d) the map $a \to \tau_a(\alpha)$ is continuous w.r.t. strong topology on $\mathcal{A}$, for each $\alpha \in \mathcal{A}$.

For von Neumann algebras, the continuity conditions (c) and (d) change to ultra-weak continuity, that is, τ_a are normal maps, and for each $\alpha \in \mathcal{A}$, the maps $a \to \tau_a(\alpha)$ must be continuous w.r.t. ultra-weak topology on $\mathcal{A}$. A quantum dynamical semigroup is called **Markov** or **Conservative** if $\tau_a(I) = I$ for each $a \geq 0$. The generator of a quantum dynamical semigroup is defined similarly as infinitesimal generator in 9.21, with existence of limit in respective topologies.

Quantum Dynamic Semigroup
Let $(\mathcal{S}_a)_{a\geq 0}$ be a strongly continuous contraction semigroup on $\mathcal{X}$. Then the family $\{\tau_a\}$ defined as

$$\tau_a(\alpha) = \mathcal{S}_a^* \alpha \mathcal{S}_a$$

forms a quantum dynamical semigroup. The strong continuity of $\mathcal{S}_a$ and the result 9.15 provides the continuity properties of τ.

A quantum dynamical semigroup is **uniformly continuous (norm-continuous)** if in (c), the maps are continuous w.r.t. the norm topology along with conditions (a) and (b). The general semigroup theory gives that the generator of a uniform continuous semigroup is a bounded operator. Moreover, the generator for a uniformly continuous quantum dynamical semigroup is bounded conditionally completely positive map and has a nice structure. In [15], Lindblad characterized the structure of uniformly continuous quantum dynamical semigroup on hyper-finite von Neumann algebras in terms of the generator. The generator is called "Lindbladian" by many authors.

Theorem 9.24 (Lindblad) *Let $\mathcal{B}(\mathcal{X})$ be a von Neumann algebra and $\mathcal{L}$ be a bounded operator on $\mathcal{B}(\mathcal{X})$. Consider a uniformly continuous quantum dynamical semigroup $(\tau_a)_{a\geq 0}$. Then $\mathcal{L}$ is the infinitesimal generator of $(\tau_a)_{a\geq 0}$ if and only if*

$$\mathcal{L}(\alpha) = \sum_{1\leq n<\infty} L_n^* \alpha L_n + L^*\alpha + \alpha L \; for \; \alpha \in \mathcal{B}(\mathcal{X}),$$

where L_ns and L are the elements of $\mathcal{B}(\mathcal{X})$. The series on the right is strongly convergent, and $-Re(L)$ generates a contraction semigroup. If quantum dynamical semigroup is unital, then

$$Re(L) = -\frac{1}{2} \sum 1 \leq n < \infty L_n^* L_n.$$

Proof See [15]. □

We shall state the structure theorem for uniform continuous quantum dynamical semigroup on the C^*-algebras which was proved by Christensen-Evans in [5].

Theorem 9.25 (Christensen-Evans) *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ be a C^*-algebra and $(\tau_a)_{a\geq 0}$ be a uniformly continuous quantum dynamical semigroup on $\mathcal{A}$ with generator $\mathcal{L}$.*

There exist a completely positive map Φ of $\mathcal{A}$ into an ultra-weak closure $\mathcal{A}''$ and an operator ξ in $\mathcal{A}''$ such that the generator is given by

$$\mathcal{L}(\alpha) = \Phi(\alpha) + \xi^*\alpha + \alpha\xi.$$

Proof See [5, 14]. □

Since Φ is completely positive, using Stinespring Theorem 9.14, we get that there are a Hilbert space $\mathcal{Y}$, a unital $*$-representation $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{Y})$, $L \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and a self-adjoint element A of $\mathcal{A}''$ such that the generator $\mathcal{L}$ is

$$\mathcal{L}(\alpha) = L^*\Pi(\alpha)L - \frac{1}{2}(L^*L - \mathcal{L}(1))\alpha - \frac{1}{2}\alpha(L^*L - \mathcal{L}(1)) + i[A, \alpha], \text{ for } \alpha \in \mathcal{A}. \tag{9.8}$$

This representation is minimal in the way that $\{(L\alpha - \Pi(\alpha)L)\beta : \beta \in \mathcal{X}, \alpha \in \mathcal{A}\}$ is total in $\mathcal{Y}$.

Remark 9.2 Any characterization of the generator of an arbitrary strongly continuous quantum dynamical semigroup is not complete. At least, when the domain of the generator is an algebra, the problem of constructing strongly continuous quantum dynamical semigroup with unbounded generator can be handled with the Theorem 9.19 so that the conditional complete positivity could make sense. Nonetheless, in general, the infinitesimal generator $\mathcal{L}$ may not make sense but can be understood as an unbounded quadratic form on the Hilbert space $\mathcal{X}$.

Davies in [7] constructed the minimal predual semigroup on the space of positive trace-class operators (density matrices) on some Hilbert space $\mathcal{X}$, a method similar to that of Kato. Chebotarev in [4] constructed the minimal quantum dynamical semigroup on the Banach space $\mathcal{B}(\mathcal{X})$ using the iteration method. In general, such an unbounded operator is usually mentioned as Lindbladian.

In [8], under some assumptions, Davies proved that the unbounded generator has the similar form as in the bounded case. Thus, he extended the Lindblad's result to strongly continuous QDS. An expository article giving the development of QDS theory is written by Fagnola [11]. In [2], Bahn, Ko, and Park discuss conservative QDS generated by noncommutative unbounded elliptic operators. Recently, in [1], the authors give a structure theorem for ultra-weakly continuous QDS on $\mathcal{B}(\mathcal{X})$ under the assumption of existence of rank 1 projection in the domain of generator.

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Chapter 10
On Sumset Problems and Their Various Types

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10.1 Introduction

We begin with some motivation toward the sumset problems.

What is combinatorial number theory?

Mainly, it is combinatorics filled up with some of the computation properties of integers. In contrast with algebra, analytic and other areas of number theory that are directly related to algebra, it is the study of integral sets. As compared to most of the areas of mathematics, it is an extremely operative research field.

A major area of combinatorial number theory is to study the partition of sets of integers or any finite set that is divided into smaller sumsets. Nowadays, it has become interest to identify the largest property for the sets having positive integers, in other words, to check that some subset is large, when the positive integers are divided into a bunch of smaller sets. An example of this is van der Waerden's theorem that declares that on dividing a set of all positive integers into finitely many subsets, we get some subset containing arbitrarily long arithmetic progressions. Another largest area of combinatorial number theory is *additive number theory* that can also be considered as the one of the vast areas. It is not only connected to combinatorics but also to analysis and algebra that is the examination of what happens when the sets of integers are added together. One of the best known unsolved problems in additive number theory is *Goldbach Conjecture* that states that every even number greater than 4 has a representation as the sum of two odd primes. Many combinatorial and algebraic approaches gave positive response for this conjecture. These approaches include: *Pigeonhole principle*, analyzing sets of numbers to be added, the use of minimal and maximal elements, counting sets in

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_10

different ways, and the representations of a sum of numbers as a difference of two other sums. Hence, these combinatorial problems are directly related to sumsets, and theory of sumsets has been become of major concern at the present stage.

In this chapter, we present literature survey on sumsets of different types including sum of dilated set of integers, h-fold sumset, restricted h-fold sumset, h-fold signed sumset, and restricted h-fold signed sumset. We also present an extensive survey on sumsets in different groups and pose some research directions.

Throughout this chapter, the sets considered are finite sets. We fix some notations and present some already defined terms that will be directly used in the whole chapter. All the other unexplained notations (if any) are standard.

- $A_1 + A_2 = \{a_1 + a_2;\ a_1 \in A_1,\ a_2 \in A_2\}$, where $A_1,\ A_2 \subset \mathbb{Z}$.
- $k \cdot A = \{ka;\ a \in A\}$; k dilate of set $A \subset \mathbb{Z}$ and $k \in \mathbb{Z}^+$.
- $\mu_G(r, s) = \min\{|A * B|;\ A,\ B \subset G,\ |A| = r \geq 1,\ |B| = s \geq 1\}$ for $(G, *)$ to be group.
- *Direct problems* are the problems in which one starts with the structure of sets and tries to describe the size of sumsets (of any type).
- *Inverse problems* are the problems in which one starts with the cardinality of sumsets (of any type) and tries to analyze the structure of sets.
- For $A = \{a_0, a_1, \ldots, a_{k-1}\}$, $h^{(r)}A := \left\{\sum_{i=0}^{k-1} r_i a_i\ :\ 0 \leq r_i \leq r\ for\ i = 0, 1, \ldots, k-1\ with\ \sum_{i=0}^{k-1} r_i = h\right\}$; *generalized h-fold sumset*.
- For $A = \{a_0, a_1, \ldots, a_{k-1}\}$, $h_{\pm}A := \{\sum_{i=0}^{k-1} \lambda_i a_i\ :\ (\lambda_0, \ldots, \lambda_{k-1}) \in \mathbb{Z}^k,\ \sum_{i=0}^{k-1} |\lambda_i| = h\}$; *h-fold signed sumset of* A.
- For $A = \{a_0, a_1, \ldots, a_{k-1}\}$, $h\hat{\pm}A := \{\sum_{i=0}^{k-1} \lambda_i a_i\ :\ \lambda_i \in \{-1, 0, 1\}^k\ for\ i = 0, 1, \ldots, k-1,\ \sum_{i=0}^{k-1} |\lambda_i| = h\}$; *h-fold restricted signed sumset of* A.
- $\lceil x \rceil$ denotes the least integer greater than or equal to x.
- If $-s \in S\ for all\ s \in S$, then S is the symmetric set.

In 2007, the general problem of giving a lower bound on the sum of dilates when A, B are subsets of $\mathbb{Z}$ was examined by Nathanson [27]. In the same year, Bukh [7] initiated to deal with the problem of sum of dilates. He gave a sharp lower bound for the cardinality of sumsets of the form $\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_k \cdot A$ for large integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ and integer set A. Instead of sharp error term of the result given by Nathanson, Bukh got the lower bound for $\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_k \cdot A$ with some weaker error term.

In 2010, Cilleruelo et al. [10] initiated to deal with the problem of finding the cardinality of $A + 3 \cdot A$ and also tackled with inverse problems. In the same paper, they proposed the conjecture that $|A + k \cdot A| \geq (k+1)|A| - \lceil \frac{k^2+2k}{4} \rceil$, where A is any set of sufficiently large cardinality. This conjecture has been well studied in the past and is being studied presently. An affirmative answer to this conjecture was given in 2009, by Cilleruelo et al. [9] for k to be prime provided $|A| \geq 3(k-1)^2(k-1)!$. In 2014, Du et al. [11] verified the conjecture for k to be prime power and a product of two distinct primes provided $|A| \geq (k-1)^2 k!$.

In 2017, Freiman et al. [18] handled the direct sumset problem for the cardinality of $A + r \cdot A$, where $r \geq 3$, and they also obtained an extended inverse result. In the same year, Mistri [21] generalized the result given by Freiman et al. [18] for the sum of dilates $A+r\cdot B$ for A and B finite subsets of integers and r to be any integer with $|r|\geq 3$. In 2019, Bhanja et al. [4] gave a new proof of the result given by Freiman et al. [18]. They also generalized the extended inverse result to the sum of dilates $A+2\cdot B$ of two sets A and B. Motivated by the work done on the cardinality of $A+k\cdot A$, various results on the cardinality of $m\cdot A+k\cdot A$ for different values of m and k were constructed by several authors. In 2011, Hamidoune and Rue [19] got the lower bound for $|2\cdot A+k\cdot A|$ for k to be an odd prime and $|A|\geq 8k^k$. Extending this result, in 2013, Ljujic [20] obtained the same bound for the case k a power of an odd prime and the case k a product of two distinct odd primes. In 2013, Balog et al. [1] constructed lower bound for $|p\cdot A+q\cdot A|$ for any relatively prime integers $1<p<q$ and for a set A of integers. In 2020, Chahal and Pandey [8] handled the case $3\cdot A+k\cdot A$ in a special situation, under some suppositions on A, and generalized for $q\cdot A+k\cdot A$, where q is an odd prime $<k$ under the same suppositions.

For a given finite set A of integers and for a positive integer h, the sumset hA and the restricted sumset $h^\wedge A$ are also the major concerns in the field of additive number theory. Two of the major problems in additive number theory are to find the best possible lower bounds for the size of the sumset hA and $h^\wedge A$ and to find the structure of the set A, when the cardinality of hA and $h^\wedge A$ is known. These two problems have been well studied in the group of integers. Nathanson [26] got the lower bound for the cardinality of hA and $h^\wedge A$ for A to be a finite set of k integers and h to be positive integer. Moreover, he dealt with inverse problems also.

In 2014, Mistri et al. [22] defined generalized sumset that includes the above-mentioned two types of sumsets as particular cases. Further, they obtained some results for the direct and inverse problems related to this generalized sumsets. In 2015, Monopoli [24] generalized the result by giving a lower bound for $|h^{(r)}A|$, when $G=\mathbb{Z}/p\mathbb{Z}$ for prime p. In 2018, Mistri and Pandey [23] gave a new proof of the theorem of Monopoli. In the same paper, they developed new proofs of direct and inverse theorems for the case $G=\mathbb{Z}$.

In 2021, Bhanja [6] found the lower bound for the size of sumsets HA and $H^\wedge A$ and the structure of the sets H, A for which the sumsets HA and $H^\wedge A$ contain the minimum number of elements, where $HA:=\bigcup_{h\in H} hA$, and $H^\wedge A:=\bigcup_{h\in H} h^\wedge A$, where H is the finite set of nonnegative integers.

There are also authors who dealt with direct and inverse problems for signed sumsets ($h_{\pm}A$) and restricted signed sumsets ($h\hat{_{\pm}}A$) that are two more types of sumsets in combinatorial number theory. In 2018, Bhanja et al. [3] obtained the results for direct and inverse problems for $|h_{\pm}A|$ for the set A containing positive integers and nonnegative integers separately. In 2019, Bhanja et al. [5] solved some cases of both direct and inverse problems for $|h\hat{_{\pm}}A|$(restricted signed sumset of A) for A to be a subset of group of integers. Also they proposed conjectures for the remaining cases.

Great work has been done on the theory of sumsets in different groups, and finding the cardinality of $\mu_G(r,s)$ has been a major concern for the authors. This includes the works by Plagne [28], Eliahou [17], Eliahou [13], Eliahou [14], Eliahou et al. [16], and Eliahou et al. [15]. These authors examined $\mu_G(r, s)$ for abelian groups, non-abelian groups, solvable groups, non-solvable groups, and dihedral groups. And conjecture has been proposed in this problem also.

10.2 Sum of Dilated Set of Integers

In this section, we summarize all the known results of direct and inverse problems for dilated sets of integers (to the best of our knowledge). One of the classical problems in additive combinatorics is to find the best lower bound for the size of sumsets of the form $\lambda_1 \cdot A + \cdots + \lambda_k \cdot A$. The case $A_1 = A_2 = \cdots = A_n$ is special and has been studied by many authors (see [1, 4, 7–11, 18–21, 25–27] and the references therein). One of the major contribution in this study is due to Bukh [7], in which he proved the following theorem:

Theorem 10.1 ([7], p.9) *For every vector $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$ of k coprime integers, we have $|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \geq (|\lambda_1| + \cdots + |\lambda_k|)|A| - o(|A|)$ for every finite set A subset of integers with the error term $o(|A|)$ depending on $\bar{\lambda}$ only.*
The case when $\lambda_1, \ldots, \lambda_k$ are not coprime can be reduced to the case by making them coprime by the relation $\lambda_1 \cdot A + \cdots + \lambda_k \cdot A = \gcd(\lambda_1, \ldots, \lambda_k).\big(\frac{\lambda_1}{\gcd(\lambda_1,\ldots,\lambda_k)} \cdot A + \cdots + \frac{\lambda_k}{\gcd(\lambda_1,\ldots,\lambda_k)} \cdot A\big)$.

Instead of weaker error term $o(|A|)$, in 2007, Nathanson [27] got sharp $o(1)$ error term for $|A + 2 \cdot A|$. He obtained:

Theorem 10.2 ([27], p.6) *For every finite set $A \subset \mathbb{Z}$, $|A + 2 \cdot A| \geq 3|A| - 2$ and $|A + \lambda \cdot A| \geq 7|A|/2 - o(1)$ for $\lambda \neq 1, 2$.*

There are various authors who improved this error term for different values of λ. In 2010, Cilleruelo et al. [10] dealt with direct and inverse problems for the size of $A + 3 \cdot A$. They constructed:

Theorem 10.3 ([10], p.3) *For any set A, we have $|A + 3 \cdot A| \geq 4|A| - 4$ and that the equality holds only for $A = \{0, 1, 3\}$, $A = \{0, 1, 4\}$, $A = 3\{0, 1, \ldots n\} \cup (3\{0, 1, \ldots, n\} + 1)$ and all the affine transforms of these sets.*

In the same paper, they formulated the conjecture and gave us a research direction.

Conjecture 10.4 (Cilleruelo, Silva and Vinuesa [10], p.7) If k is a positive integer and A is a finite set of integers with sufficiently large cardinality, then

$$|A + k \cdot A| \geq (k + 1)|A| - \lceil k(k + 2)/4 \rceil.$$

This conjecture has been well studied in the past and is being studied presently. Cilleruelo et al. [9] and Du et al. [11] gave affirmative answers for the validity of this conjecture. Cilleruelo et al. [9] got the following result as a corollary for the validity of Conjecture 10.4.

Corollary 10.1 ([9], p.873) *For k to be an odd prime and A a finite set of integers satisfying $|A| \geq 3(k-1)^2(k-1)!$, $|A + k \cdot A| \geq (k+1)|A| - \lceil k(k+2)/4 \rceil$. Moreover, equality holds only if $A = k \cdot \{0, 1, \ldots, n\} + \{0, 1, \ldots, (k-1)/2\}$ for some n and affine transforms of this set.*

Du et al. [11] confirmed the conjecture for k to be an odd prime power or a product of two distinct odd primes.

Theorem 10.5 ([11], p.4, p.10) *Let A be a finite set of integers with $|A| \geq (k-1)^2 k!$, and if k is a prime power or a product of two distinct primes, then $|A + k \cdot A| \geq (k+1)|A| - \lceil k(k+2)/4 \rceil$. They also established the inequality $|A + 4 \cdot A| \geq 5|A| - 6$ for $|A| \geq 5$.*

For dealing with direct and inverse problems for sum of dilates, Freiman et al. [18, p.43] also contributed at a large scale. In 2014, they defined that when the exact bound of cardinality of sumset is known and one wants to know the properties of set, those types of problems are called ordinary inverse problems, but the problems in which some deviation from exact bound is assumed and even then it helps us to determine the structure of set are known as extended inverse problems. They raised a question regarding extended inverse problems:

What is the structure of set A if $A \geq 3$ and $|A + 2 * A| < 4|A| - 4$?

Also answered this question by the next theorem:

Theorem 10.6 ([18], p.50) *If $|A| \geq 3$ and $|A + 2 * A| < 4|A| - 4$, then A is a subset of an arithmetic progression of length at most $|A + 2 * A| - 2|A| + 2 \leq 2|A| - 3$.*

They also proved one more result related to the cardinality of sum of dilates:

Theorem 10.7 ([18], p.47) *Let A be a nonempty finite set of integers, and let $r \geq 3$ be a positive integer. Then $|A + r * A| \geq 4|A| - 4$.*

In the same year, Mistri [21] generalized this result for the sum of dilates $A + r \cdot B$ for A and B distinct sets, where r is any integer with $|r| \geq 3$. In this paper, he defined some notations: If $A = \{a_0, a_1, \ldots, a_{k-1}\} \subseteq \mathbb{Z}$ having all the elements in increasing order, then $\ell(A) := \max(A) - \min(A)$ and for $k \geq 2$, $d(A) := \gcd(a_1 - a_0, a_2 - a_0, \ldots, a_{k-1} - a_0)$.

Theorem 10.8 ([21], p.16) *Let r be an integer with $|r| \geq 3$. Let $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Z}$ be nonempty finite sets satisfying the following conditions:*

- *$|A| \leq |B|$ and $\ell(A) \leq \ell(B)$.*
- *$d(A) = d(B) = 1$ if $|A| \geq 2$ and $|B| \geq 2$.*

Then $|A + r \cdot B| \geq 4|A| - 4$.

Bhanja et al. [4] presented a new proof of the result given by Freiman et al. [18]. They also generalized the extended inverse result to the sum of dilates $A + 2 \cdot B$ for two sets A and B.

Theorem 10.9 ([4], p.2) *Let A be a nonempty finite set of integers, and let $r \geq 3$ be a positive integer. Then $|A + r \cdot A| \geq 4|A| - 4$.*

Theorem 10.10 ([4], p.5) *Let A and B be two nonempty finite sets of integers with* $|A| \geq 3$ *such that:*

- $d(A) = d(B) = 1$.
- $l(A) \leq l(B)$.
- $h_A \leq h_B$.

If $|A + 2 \cdot B| = |A| + 2(|B| - 1) + h < 2(|A| + |B| - 2)$, *where* $h_A := l(A) + 1 - |A|$, *then both A and B are subsets of arithmetic progressions of length at most* $|B| + h = |A + 2 \cdot B| - |A| - |B| + 2 \leq |A| + |B| - 3$.

They also verified that these conditions are sufficient but not necessary by presenting two examples. There are also various authors who worked on the cardinality of $m \cdot A + k \cdot A$ for different values of m and k. In 2011, Hamidoune and Rue [19] handled the case for $m = 2$ and k to be an odd prime. They obtained the following result as a corollary.

Corollary 10.2 ([19], p.8) *Let k be an odd prime. If A is a finite set of integers with* $|A| > 8k^k$, *then* $|2 \cdot A + k \cdot A| \geq (k + 2)|A| - k^2 - k + 2$.

Extending this result, in 2013, Ljujic [20] obtained the same bound for the case k a power of an odd prime and the case k a product of two distinct odd primes, i.e., he proved following results:

Theorem 10.11 ([20], p.6, p.13) *Let A be a finite set of integers such that* $|A| > 8k^k$. *If for* $\alpha \in \mathbb{Z}_{\geq 1}$, $k = p_1^\alpha$ *or pq for* p_1 *to be an odd prime and* p, q *distinct odd primes, then* $|2 \cdot A + k \cdot A| \geq (k + 2)|A| - k^2 - k + 2$.

In 2013, Balog et al. [1] handled the case for $m = p$ and $k = q$ where $1 < p < q$ are relatively prime integers. They constructed:

Theorem 10.12 ([1], p.2) *For any relatively prime integers* $1 < p < q$ *and for any finite* $A \subset \mathbb{Z}$, *one has* $|p \cdot A + q \cdot A| \geq (p + q)|A| - (pq)^{(p+q-3)(p+q)+1}$.

In 2020, Chahal and Pandey [8] handled the case $3 \cdot A + k \cdot A$ in a particular situation, under some suppositions on A, and generalized for $q \cdot A + k \cdot A$, where q is an odd prime $< k$ under the same suppositions. They proved:

Theorem 10.13 ([8], p.4) *Let* $k \geq 5$ *be a prime number and A be a nonempty finite set of integers with* $C_3(X_i) = 3$ *for each* $1 \leq i \leq |\hat{A}|$, *where* $\hat{A}$ *is the projection of A on* $\mathbb{Z}/k\mathbb{Z}$. *Then for* $|A| \geq 9|\hat{A}|^2(k - 1)!$, $|3 \cdot A + k \cdot A| \geq (k + 3)|A| - 3k|\hat{A}|$.

They also generalized this result for $q \cdot A + k \cdot A$, where q is an odd prime $< k$. More precisely, they developed a remark:

Remark 10.1 ([8], p.5) If q and k are odd primes satisfying the inequality $q < k$ and A is a nonempty finite set of integers with $C_q(X_i) = q$ for each $1 \leq i \leq |\hat{A}|$, where $\hat{A}$ is the projection of A on $\mathbb{Z}/k\mathbb{Z}$. Then for $|A| \geq 3q|\hat{A}|^2(k - 1)!$,

$$|q \cdot A + k \cdot A| \geq (k + q)|A| - qk|\hat{A}|.$$

10.3 An Approach to Cardinality of $h^{(r)}A$

We record here the known information about the direct and inverse problems on the cardinality of $h^{(r)}A$ for some particular values of r and on the cardinality of generalized h-fold sumsets with some restrictions on h and r.

Generalized h-fold sumset has already been defined, and from that definition, two particular cases can be visualized by the following remark:

Remark 10.2 If $r = h$, then $h^{(r)}A = h^{(h)}A = hA$, the usual $h - fold$ sumset of A. If $r = 1$, then $h^{(r)}A = h^{(1)}A = h^{\wedge}A$, the restricted $h - fold$ sumset of A.

There are various authors who studied these two particular types of sumsets, but there are also some authors who investigated some properties of generalized h-fold sumset $h^{(r)}A$. First, Nathanson initiated to deal with this type of sumset problems. In [26], he proved following results for the cardinality of h-fold and restricted h-fold sumsets:

Theorem 10.14 ([26], p.6) *If A is a set of k integers, then $|2A| \geq 2k - 1$. And if for A set of k integers $|2A| = 2k - 1$, then A is an arithmetic progression.*

Theorem 10.15 ([26], p.8) *For $h \geq 2$ and A be a finite set of k integers, $|hA| \geq h(k-1) + 1$.*

He also tackled the inverse problem, more precisely he proved:

Theorem 10.16 ([26], p.12) *Let $h \geq 2$. Let A be a finite set of integers with $|A| = k$. Then equality holds in Theorem 10.15 if and only if A is a k-term arithmetic progression.*

Theorem 10.17 ([26], p.14) *Let A be a set of k integers, and let $1 \leq h \leq k$. Then $|h^{\wedge}A| \geq h(k-h) + 1$.*
This lower bound is best possible.

Theorem 10.18 ([26], p.16) *Let $k \geq 5$, and let $2 \leq h \leq k - 2$. If A is a set of k integers such that $|h^{\wedge}A| = h(k-h) + 1$, then A is an arithmetic progression.*

In 2014, Mistri and Pandey [22] defined generalized h-fold sumset that contains the above discussed two types of sumsets as particular cases. Also they derived the following results for the direct and inverse problems related to this generalized sumset.

Theorem 10.19 ([22], p.342) *Let A be nonempty set of k integers. Let r and h be integers such that $1 \leq r \leq h \leq rk$. Let $m = \lfloor h/r \rfloor$. Then*

$$|h^{(r)}A| \geq mr(k-m) + (h-mr)(k-2m-1) + 1.$$

This lower bound is best possible.

Theorem 10.20 ([22], p.349) *Let $k \geq 3$. Let r and $h \geq 2$ be integers satisfying the following conditions:*

1. $h \equiv 0 (mod r)$.
2. $1 \leq r \leq h \leq rk - 2$.
3. $(k, h, r) \neq (4, 2, 1)$.

Set $m = h/r$. If A is a set of k integers such that

$$|h^{(r)}A| = mr(k - m) + 1,$$

then A is a k-term arithmetic progression.

Theorem 10.21 ([22], p.355) *Let $k \geq 3$. Let r and h be integers satisfying the following conditions:*

1. $h \not\equiv 0 (mod r)$.
2. $2 \leq r \leq h \leq rk - 2$.

Set $m = \lfloor h/r \rfloor$. If A is a set of k integers such that

$$|h^{(r)}A| = mr(k - m) + (h - mr)(k - 2m - 1) + 1,$$

then A is a k-term arithmetic progression.

In 2015, Monopoli [24] generalized the result by giving a lower bound for $|h^{(r)}A|$, where A is a subset of $G = \mathbb{Z}/p\mathbb{Z}$ for prime p.

Theorem 10.22 ([24], p.5) *Let h, r be nonnegative integers, $h = mr + \epsilon$, $0 \leq \epsilon \leq r - 1$. Let A be a nonempty finite set of k integers such that $1 \leq h \leq rk$. Then $|h^{(r)}A| \geq hk - m^2r + 1 - 2m\epsilon - \epsilon$.*

Theorem 10.23 ([24], p.8) *Let $h = mr + \epsilon$, $0 \leq \epsilon \leq r - 1$. Let $A \subseteq \mathbb{Z}/p\mathbb{Z}$ be a nonempty set with $|A| = k$ such that $1 \leq r \leq h \leq rk$. Then $|h^{(r)}A| \geq min(p, hk - m^2r + 1 - 2m\epsilon - \epsilon)$.*

Theorem 10.24 ([24], p.10) *Let $k \geq 5$. Let r and $h = mr + \epsilon$, $0 \leq \epsilon \leq r - 1$, be integers with $2 \leq r \leq h \leq rk - 2$. Then any set of k integers A such that $|h^{(r)}A| = hk - m^2r + 1 - 2m\epsilon - \epsilon$ is a k-term arithmetic progression.*

In 2018, Mistri et al. [23] gave a new proof of the Theorem 10.23 given by Monopoli. Also they presented new methods to prove the Theorems 10.19, 10.20, and 10.21 given by Mistri and Pandey in [22]. Mainly, in this chapter, they demonstrated that generalized sumset $h^{(r)}A$ may be expressed in terms of the regular (hA) and restricted $(h^\wedge A)$ sumsets. More precisely, they proved the following theorem:

Theorem 10.25 ([23], p.3) *Let r, h, and k be integers such that $1 \leq r \leq h \leq rk$ and $h = mr + \epsilon$, where $0 \leq \epsilon \leq r - 1$. Let A be a subset of an abelian group G with k elements. Then $h^{(r)}A = \epsilon((m+1)^\wedge A) + (r - \epsilon)(m^\wedge A)$.*

In 2021, Bhanja [6] found the lower bound for the size of sumsets HA and $H^\wedge A$ and the structure of the sets H, A for which the sumsets HA and $H^\wedge A$ contain the minimum number of elements, where $HA := \bigcup_{h \in H} hA$ and $H^\wedge A := \bigcup_{h \in H} h^\wedge A$, where H is the finite set of nonnegative integers. He obtained the results:

Theorem 10.26 ([6], p.2) *Let A be a set of k positive integers. Let H be a set of r positive integers with $\max(H) = h_r$. Then*

$$|HA| \geq h_r(k-1) + r.$$

This lower bound is optimal.

Theorem 10.27 ([6], p.4) *Let A be a set of k positive integers and $H = \{h_1, h_2, \ldots, h_r\}$ be a set of positive integers with $h_1 < h_2 < \cdots < h_r \leq k$. Set $h_0 = 0$. Then*

$$|H^\wedge A| \geq \sum_{i=1}^{r} (h_i - h_{i-1})(k - h_i) + r.$$

This lower bound is optimal. He tackled the inverse problem also.

Theorem 10.28 ([6], p.3) *Let A be a set of $k \geq 2$ positive integers and H be a set of $r \geq 2$ positive integers with $\max(H) = h_r$. If $|HA| = h_r(k-1) + r$, then H is an arithmetic progression of difference d and A is an arithmetic progression of difference $d \cdot \min(A)$.*

Theorem 10.29 ([6], p.6) *Let A be a set of $k \geq 6$ positive integers. Let $H = \{h_1, h_2, \ldots, h_r\}$ be a set of $r \geq 2$ positive integers with $h_1 < h_2 < \cdots < h_r \leq k - 1$. Set $h_0 = 0$. If*

$$|H^\wedge A| = \sum_{i=1}^{r} (h_i - h_{i-1})(k - h_i) + r,$$

then $H = h_1 + [0, r-1]$ and $A = \min(A) \cdot [1, k]$.

10.4 Signed and Restricted Signed Sumset Problems

In 2018, Bhanja et al. [3] obtained the results for direct and inverse problems for $|h_{\pm}A|$ for the set A containing positive integers and nonnegative integers separately.

They also got a bound for $|h_{\pm}A|$ for A containing both positive and negative integers. The obtained results have been discussed below:

Theorem 10.30 ([3], p.3) *Let h be a positive integer, and let A be a finite set of k positive integers. We have*

$$|h_{\pm}A| \geq 2(h(k-1)+1).$$

This is the best possible lower bound for $h \leq 2$.

Theorem 10.31 ([3], p.4) *Let $h \geq 2$, and let A be a finite set of k positive integers. If the equality holds in the result proved in 10.30, then $h = 2$ and $A = d * \{1, 3, \ldots, 2k-1\}$, for some positive integer d.*

Theorem 10.32 ([3], p.5) *Let $h \geq 3$ be a positive integer. Let A be a finite set of positive integers such that $|A| = k \geq 3$. Then*

$$|h_{\pm}A| \geq h(2k-1)+1.$$

This lower bound is best possible.

Theorem 10.33 ([3], p.10) *Let $h \geq 3$, and let A be a finite set of $k \geq 3$ positive integers. If equality holds in the result proved in 10.32, then $A = d * \{1, 3, \ldots, 2k-1\}$, for some positive integer d.*

Following are the results for the finite set A containing nonnegative integers with $0 \in A$:

Theorem 10.34 ([3]. p.11) *Let $h \geq 1$. Let A be a finite set of k nonnegative integers with $0 \in A$. Then*

$$|h_{\pm}A| \geq 2h(k-1)+1.$$

This lower bound is best possible.

Theorem 10.35 ([3], p.12) *Let $h \geq 2$, and let A be a finite set of k nonnegative integers with $0 \in A$. Then $|h_{\pm}A| = 2h(k-1)+1$ if and only if $A = d * [0, k-1]$, for some positive integer d.*

Theorem 10.36 ([3], p.12) *Let $h \geq 1$, and let A be a finite set of k integers containing both positive and negative integers. Then $|h_{\pm}A| \geq h(k-1)+1$.*

Theorem 10.37 ([3], p.12) *Let A be a finite set of $k \geq 2$ integers. Let $|h_{\pm}A| = h(k-1)+1$. Then A is an arithmetic progression, and it is a symmetric set, where the symmetric set $S = \{-s;\ \forall s \in S\}$.*

In 2019, Bhanja et al. [5] solved some cases of both direct and inverse problems for size of $h\hat{\pm}A$ for set A (a subset of additive group of integers). Also they proposed conjectures for the remaining cases. Following are the results for set A containing positive integers only.

Theorem 10.38 ([5], p.3) *Let A be a set of k positive integers, and let* $1 \leq h \leq k$ *be an integer. Then*

$$|h\hat{\pm}A| \geq 2h(k-h) + \binom{h+1}{2} + 1.$$

This lower bound is best possible for $h = 1, 2$ *and* k.

In the next two theorems [10.39, 10.40], they checked the structure of A if the cardinality of $h\hat{\pm}A$ is known for the cases $h = 2$ and $h = k$, respectively,

Theorem 10.39 ([5], p.5) *Let A be a set of* $k(\geq 3)$ *positive integers such that* $|2\hat{\pm}A| = 4k - 4$. *Then,* $A = d * \{1, 3, \ldots, 2k-1\}$ *for some positive integer* d.

Theorem 10.40 ([5], p.6) *Let A be a set of* $k(\geq 3)$ *positive integers such that* $|k\hat{\pm}A| = \binom{k+1}{2} + 1$. *Then*

$$A = \begin{cases} \{a_0, a_1, a_0 + a_1\} \, with \, 0 < a_0 < a_1 & if k = 3; \\ d * [1, k] \, for \, some \, positive \, integer \, d, & if k \geq 4. \end{cases}$$

Also they proposed the conjecture for the cardinality of $h\hat{\pm}A$ if A is known and also for the structure of set A if the cardinality of $h\hat{\pm}A$ is known for $h \geq 3$ and proved that conjectures hold for $h = 3$.

Conjecture 10.41 ([5], p.8) Let A be a set of $k(\geq 4)$ positive integers and let $3 \leq h \leq k - 1$. Then

$$|h\hat{\pm}A| \geq h(2k - h) + 1.$$

This lower bound is best possible. □

Conjecture 10.42 ([5], p.11) Let A be a set of $k(\geq 4)$ positive integers, and let $3 \leq h \leq k - 1$. If $|h\hat{\pm}A| = h(2k - h) + 1$, then $A = d * \{1, 3, \ldots, 2k - 1\}$ for some positive integer d. □

Now we discuss the results proved by Bhanja in [5] for set A containing nonnegative integers with $0 \in A$.

Theorem 10.43 ([5], p.11) *Let A be a set of k nonnegative integers with* $0 \in A$ *and let* $1 \leq h \leq k$ *be a positive integer. Then*

$$|h\hat{\pm}A| \geq 2h(k-h) + \binom{h}{2} + 1.$$

This lower bound is best possible for $h = 1, 2,$ *and* k.

Similarly in the Theorems [10.44, 10.45], they dealt with inverse problems for $h = 2$ and $h = k$.

Theorem 10.44 ([5], p.12) *Let* A *be a set of* $k(\geq 3)$ *nonnegative integers with* $0 \in A$ *such that* $|2\hat{\pm}A| = 4k - 6$. *Then,* $A = d * [0, k-1]$ *for some positive integer* d.

Theorem 10.45 ([5], p.13) *Let* $A = \{0, a_0, a_1, \ldots, a_{k-2}\}$ *be a set of* $k(\geq 4)$ *nonnegative integers such that* $|k\hat{\pm}A| = \binom{k}{2} + 1$. *Then*

$$A = \begin{cases} \{0, a_0, a_1, a_0 + a_1\} \, with \, 0 < a_0 < a_1 & if k = 4; \\ d * [0, k-1] \, for \, some \, positive \, integer \, d, & if k \geq 5. \end{cases}$$

They also gave conjectures for the direct and inverse problems for the cardinality of restricted signed sumsets for the set A containing nonnegative integers with $0 \in A$. Also they checked the validity of conjectures for $h = 3$.

Conjecture 10.46 ([5], p.14) Let A be a set of $k(\geq 5)$ nonnegative integers with $0 \in A$. Let $3 \leq h \leq k - 1$ be a positive integer. Then

$$|h\hat{\pm}A| \geq h(2k - h - 1) + 1.$$

The lower bound is best possible.

Conjecture 10.47 ([5], p.17) Let A be a set of $k(\geq 5)$ nonnegative integers with $0 \in A$. Let $3 \leq h \leq k - 1$ be an integer. If $|h\hat{\pm}A| = h(2k - h - 1) + 1$, then $A = d * [0, k - 1]$ for some positive integer d.

10.5 Sumsets in Groups

There are also various authors who studied the theory of sumsets in various groups (abelian or non-abelian). First in 1998, Eliahou et al. [12] defined $\mu_G(r, s)$, and then finding the bounds of cardinality of $\mu_G(r, s)$ for various groups has become a major concern. Plagne discussed the cardinality of $\mu_G(r, s)$, where $G = \mathbb{Z}/g\mathbb{Z}$ is a cyclic group. He obtained:

Theorem 10.48 ([28], p.111) *For* n *to be an integer and* $1 \leq r, s \leq n$, *we have* $\mu_{\mathbb{Z}/g\mathbb{Z}} = \min_{d|g} \left\{ \left(\lceil \frac{r}{d} \rceil + \lceil \frac{s}{d} \rceil - 1 \right) d \right\}$.

He also obtained both upper and lower bounds on $\mu_G(r, s)$, for G to be an abelian group of order g and exponent e.

Theorem 10.49 ([28], p.111) *For a finite abelian group G and for* $1 \leq r, s \leq g$, $\min_{d|g} \left\{ \left(\lceil \frac{r}{d} \rceil + \lceil \frac{s}{d} \rceil - 1 \right) d \right\} \leq \mu_G(r, s) \leq \min_{\frac{g}{e}|d|g} \left\{ \left(\lceil \frac{r}{d} \rceil + \lceil \frac{s}{d} \rceil - 1 \right) d \right\}$ *holds:*

In 2003, Eliahou et al. [17] discussed the cardinality of $\mu_G(r, s)$, for finite abelian group G. They got the result:

Theorem 10.50 ([17], p.111) *Let G be any finite abelian group of order g. For all r, s satisfying* $1 \leq r, s \leq g$, *then* $\mu_G(r, s) = \min_{d|g} \left\{ \left(\lceil \frac{r}{d} \rceil + \lceil \frac{s}{d} \rceil - 1 \right) d \right\}$.

In 2005, Eliahou et al. [13] handled arbitrary abelian group (finite or infinite), and they proved:

Theorem 10.51 ([13], p.453) *Let G be an arbitrary abelian group; then* $\mu_G(r, s) = \min_{h \in H(G)} \left\{ \left(\lceil \frac{r}{h} \rceil + \lceil \frac{s}{h} \rceil - 1 \right) h \right\}$ *for all positive integers r, s such that* $1 \leq r,\ s \leq |G|$.

In 2006, Eliahou [14] worked on solvable and non-solvable group. In this chapter, they proved that solvable groups satisfy the nested small sumsets property, and they also proved that non-solvable groups satisfy nested small sumsets property under some conditions and restrictions. More precisely, they obtained the following results:

Theorem 10.52 ([14], p.1105) *If G is a solvable finite group, then G has the nested small sumsets property. In other words, given two integers r, s satisfying* $1 \leq r \leq s \leq |G|$, *there exists a pair A, B* $\subset G$ *such that* $A \subset B$, *i.e., nested subsets of G of order r and s, respectively, and* $|A \cdot B| \leq r + s - 1$.

Theorem 10.53 ([14], p.1107) *Let G be a finite group. If* $1 \leq r, s \leq |G|$ *are integers such that* $r \leq 12$, *then* $\mu_G(r, s) \leq r + s - 1$.

Theorem 10.54 ([14], p.1108) *Let G be a finite group. If* $1 \leq r, s \leq |G|$ *are integers such that* $r + s \geq |G| - 11$, *then* $\mu_G(r, s) \leq r + s - 1$.

In [16], Eliahou et al. tried to check in what way the formula proved in 10.50 remains true in case of non-abelian groups also. They constructed a function $k_G(r, s) = \min_{d \in H(G)} \left\{ \left(\lceil \frac{r}{h} \rceil + \lceil \frac{s}{h} \rceil - 1 \right) h \right\}$. So that formula can be examined as $\mu_G(r, s) = k_G(r, s)$. In 2007, Eliahou et al. [16] confirmed the equality of $\mu_G(r, s)$ and $k_G(r, s)$ for non-abelian group under some conditions. Consequently, they proved that $\mu_G(r, s) = k_G(r, s)$ if:

- $|G| \leq r + s$.
- $k_G(r, s) < r/2 + s$.
- s arbitrary and $1 \leq r \leq 3$.

They also obtained that if $r + s = |G| - 1$, then only $\mu_G(r, s) \leq k_G(r, s)$ holds. Then they concluded that instead of these affirmative answers, $\mu_G(r, s) = k_G(r, s)$

does not hold true generally by constructing an example of non-abelian group of order 21 (namely, the semi-direct product $G = C_7 \rtimes C_3$.) They derived the pairs (r, s) with $r \le s$ satisfying the property $\mu_G(r, s) \ne k_G(r, s)$. The pairs are: (6, 8), (5, 9), (6, 9), (8, 9), (9, 9). Also they proposed a conjecture:

Conjecture 10.55 ([16], p.236) For every finite group G of order g and every pair of integers $r,\ s$ with $1 \le r, s \le g$,

$$k_G(r, s) \le \mu_G(r, s)$$

can be expected to have.

In 2006, Eliahou et al. [15] studied the sumsets in dihedral group D_n of order $2n$. They got the following two results:

Theorem 10.56 ([15], p.619) *For all positive integers $r, s \le 2n$, one has the inequality $\mu_{D_n}(r, s) \le k_{D_n}(r, s)$ for every positive integer.*

Also they gave a proof of reverse inequality for n to be a prime power.

Theorem 10.57 ([15], p.622) *Let D_q be a dihedral group where $q = p^m$ a prime power. Let r, s be integers satisfying $1 \le r, s \le 2q$. Then $\mu_{D_n}(r, s) \ge k_{D_n}(r, s)$.*

As a consequence, there is a wonderful corollary.

Corollary 10.3 ([15]) *For q to be a prime power, $\mu_{D_q}(r, s) = k_{D_q}(r, s)$.*

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Chapter 11
Vector-Valued Affine Bi-Frames on Local Fields

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11.1 Introduction

Balan in [2] introduced the novel concept of superframes, whereas Han and Larson introduced it in the context of "multiplexing" in [22]. In recent times, the research of superframes lays focus on wavelet and Gabor frames in $L^2(K, \mathbb{C}^L)$. We refer to [2, 17, 20, 21, 25, 28], for better understanding of super Gabor frames in $L^2(\mathbb{R}, \mathbb{C}^L)$. It was Han and Larson in [22], who showed that there is no MRA in $L^2(\mathbb{R}, \mathbb{C}^L)$ with $L > 1$ according to the usual definition of dilation and translation. Along with this, they were successful in obtaining a Fourier domain characterization of super-wavelets. Bildea et al. in [15] were successful in obtaining super-wavelets using modified MRA, whereas Dai et al. in [16] obtained the properties of super-wavelets other than in [16].

Recently, a considerable attention is given for the construction of wavelet bases on various groups. It was R.L. Benedetto and J.J. Benedetto [14] who initiated a wavelet theory for local fields and their allied groups. We know that local fields can be categorized into two types: zero characteristic and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Under the headings of zero characteristic, we can include the p-adic field $\mathbb{Q}_p$, whereas under those of positive characteristic, one can include the Vilenkin p-groups and Cantor dyadic group.

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_11

Even though we have similarities in the structures and metrics of local fields of both characteristic groups, their wavelet and multiresolution stories are too different.

The concept of vector-valued subspace on local field of positive characteristic was defined by Shah and Bhat [18]. Besides constructing wavelet packets, they developed the concept of reducing subspaces on these non-Archimedean fields. These two authors have also constructed vector-valued nonuniform wavelets on local fields. Abdullah [1] has constructed vector-valued multiresolution analysis on local fields. Shah and Bhat [19] have constructed semi-orthogonal wavelet frames on these fields. We continued the studies and define vector-valued affine bi-frames on such subspaces. We also provide the characterization of such affine bi-frames on local fields of positive characteristic. For more about local fields, we refer to [3–13].

This chapter is organized as follows. Section 11.2 is devoted to some lemmas for later use. In Sect. 11.3, we focus on proving the main theorem which characterizes the vector-valued affine bi-frames on local fields of positive characteristic.

11.2 The Auxiliary Results

We say that a countable sequence $\{e_i\}_{i\in I}$ is called a Bessel sequence in a separable Hilbert space$H$ if we have a $\mathfrak{C} > 0$ such that

$$\sum_{i\in I} |\langle f, e_i\rangle|^2 \leq \mathfrak{C}||f||^2 \quad for \quad f \in H, \tag{11.1}$$

where $\mathfrak{C}$ is noted as Bessel bound; it will be called a frame for separable Hilbert space$H$ if there exist $0 < \mathfrak{C}_1 \leq \mathfrak{C}_2 < \infty$ such that

$$\mathfrak{C}_1||f||^2 \leq \sum_{i\in I} |\langle f, e_i\rangle|^2 \leq \mathfrak{C}_2||f||^2 \quad for \quad f \in H, \tag{11.2}$$

where $\mathfrak{C}_1, \quad \mathfrak{C}_2$ are treated as frame bounds. $\{e_i\}_{i\in I}$ will be a tight frame (Parseval frame) if $\mathfrak{C}_1 = \mathfrak{C}_2(\mathfrak{C}_1 = \mathfrak{C}_2 = 1) \quad in \quad (11.2)$. Having a frame $\{e_i\}_{i\in I}$ for H in hand, we say a sequence $\{\tilde{ei}\}_{i\in I}$ to be a dual of $\{e_i\}_{i\in I}$ if it is a frame with the condition that

$$f = \sum_{i\in I} \langle f, \tilde{e_i}\rangle e_i \quad for \quad f \in H. \tag{11.3}$$

For a positive integer L, the direct sum on separable Hilbert spaces $H_1, H_2, ..., H_L$ is denoted by $\bigoplus_{l=l}^{L} H_l$ and is endowed with inner product

$$\langle f, \tilde{f}\rangle = \sum_{l=1}^{L} \langle f_l, \tilde{f_l} \tag{11.4}$$

for $f = (f_1, f_2, ..., f_L)$, $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_L) \in \bigoplus_{l=l}^{L} H_l$. Furthermore, $\bigoplus_{l=l}^{L} H_l$ will be considered as superspace. In particular, $\bigoplus_{l=l}^{L} L^2(K)$ is exactly the vector-valued space $L^2(K, \mathbb{C}^L)$. Superframes are also designated as vector-valued frames in literature. Given $c > 0$ *and* $x_0 \in K$, the dilation operator $D_{\mathfrak{p}}$ and translation operator $T_{u(k)}$ *on* $L^2(K, \mathbb{C}^L)$ are defined by

$$D_{\mathfrak{p}} f(x) = (\sqrt{q} f_1(\mathfrak{p}^{-1}x), \sqrt{q} f_2(\mathfrak{p}^{-1}x), ..., \sqrt{q} f_L(\mathfrak{p}^{-1}x)), \tag{11.5}$$

$$T_{u(k)} f(x) = (f_1(x - u(k)), f_2(x - u(k)), ..., f_L(x - u(k))) \tag{11.6}$$

for $f \in L^2(K, \mathbb{C}^L)$, respectively. The Fourier transform of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\cdot) = \int_K f(x)\chi(x)dx$$

and can be extended to $L^2(K)$ using the Plancherel theorem. The vector-valued Fourier transform is defined by $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_L)$ *for* $f \in L^2(K, \mathbb{C}^L)$. With a nonzero measurable subset E of K, one can define the closed subspaces $L^2(E, \mathbb{C}^L)$ and $FL^2(E, \mathbb{C}^L)$ of $L^2(K, \mathbb{C}^L)$ by

$$L^2(E, \mathbb{C}^L) = \{f \in L^2(K, \mathbb{C}^L) : supp(f) \subset E\} \tag{11.7}$$

and

$$FL^2(E, \mathbb{C}^L) = \{f \in L^2(K, \mathbb{C}^L) : supp(\tilde{f}) \subset E\} \tag{11.8}$$

where $supp(h) = \{\xi \in K : h(\xi) \neq 0\}$ for h which is a vector-valued measurable function. For the sake of brevity, we designate $L^2(E)$ and $FL^2(E)$ for $L^2(E, C)$ and $FL^2(E, C)$, respectively.

Let $L \in \mathbb{N}$ and $\mathfrak{p} \in K$. For $f \in L^2(K, \mathbb{C}^L)$, the affine system generated by f is defined by

$$X(f) = \{D_{\mathfrak{p}^j} T_{u(k)} f : j \in \mathbb{Z}, k \in \mathbb{N}_0\}. \tag{11.9}$$

For a finite subset F of $L^2(K, \mathbb{C}^L)$, one can define the affine system $X(F)$ generated by F as

$$X(F) = \{D_{\mathfrak{p}^j} T_{u(k)} f : f \in F, j \in \mathbb{Z}, k \in \mathbb{N}_0\}.$$

Any nonzero and closed subspace X of $L^2(K, \mathbb{C}^L)$ will be treated as a reducing subspace if $D_{\mathfrak{p}} X = X$ *and* $T_{u(k)} X = X$ for each $k \in \mathbb{N}_0$.

Proposition 11.1 *Let $a > 1$. A nonzero closed subspace X of $L^2(K, \mathbb{C}^L)$ is a reducing subspace of $L^2(K, \mathbb{C}^L)$ if and only if $X = FL^2(\Gamma, \mathbb{C}^L)$ for some $\Gamma \subset R$ with the condition $\Gamma = a\Gamma$.*

By Proposition 11.1, to be concise, we designate a reducing subspace by $FL^2(\Gamma, \mathbb{C}^L)$ in place of X. In particular, $FL^2(K, \mathbb{C}^L) = L^2(K, \mathbb{C}^L)$, and it will be a reducing subspace of $L^2(K, \mathbb{C}^L)$, and $FL^2((0, \infty), \mathbb{C}^L)$ is also a reducing subspace of $L^2(K, \mathbb{C}^L)$. Let M represent a closed subspace of $L^2(K, \mathbb{C}^L)$. For $\mathfrak{f}, \tilde{\mathfrak{f}} \in L^2(K, \mathbb{C}^L)$, $\Big(\mathcal{A}(\mathfrak{f}), X\tilde{\mathfrak{f}}\Big)$ is called an affine bi-frame (ABF) for M if it is a bi-frame for to M, i.e., $\mathcal{A}(\mathfrak{f})$ *and* $X(\tilde{\mathfrak{f}})$ are two frames with the condition

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}} \rangle D_{\mathfrak{p}^j} T_{u(k)} \mathfrak{f} \quad for \quad f \in M; \tag{11.10}$$

$\Big(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\tilde{\mathfrak{f}})\Big)$ is treated as a weak affine bi-frame (WABF) for M if we have a dense subset V of M with the condition

$$\langle f, g\rangle = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}} \rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \mathfrak{f}, g\rangle \quad for \quad f, g \in \nu, \tag{11.11}$$

where there is an unconditional convergence of the series in (11.10) in $L^2(K)$-norm, while, in (11.11),

$$\sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}} \rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \mathfrak{f}, g\rangle$$

converges unconditionally, and

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}} \rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \mathfrak{f}, g\rangle = \lim_{(J',J)\to(\infty,\infty)} \sum_{j=-J'}^{J} \sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}} \rangle \times \langle D_{\mathfrak{p}^j} T_{u(k)} \mathfrak{f}, g\rangle.$$

For two finite subsets $\mathfrak{f} = \{\mathfrak{f}^{(n)} : 1 \le n \le N\}$, $\tilde{\mathfrak{f}} = \{\tilde{\mathfrak{f}}^{(n)} : 1 \le n \le N\}$ of $L^2(K, \mathbb{C}^L)$, we say that $\Big(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\tilde{\mathfrak{f}})\Big)$ is a weak affine bi-frame (WABF)for M if an equation similar to (11.11) holds, i.e., there exists a dense subset ν of M such that

$$\langle f, g\rangle = \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}^{(n)} \rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \mathfrak{f}^{(n)}, g\rangle \quad for \quad f, g \in \nu$$

with the convergence similar to the above.

Lemma 11.1 *Suppose $F, G \in L^2(K)$, and supp(F), supp(G) are bounded. Then*

$$\sum_{k\in\mathbb{N}_0} \tilde{F}(u(k))\overline{\tilde{G}(u(k))} = \int_K \left\{\sum_{m\in\mathbb{N}_0} F(\xi + u(m))\right\} \overline{G(\xi)} d\xi. \tag{11.12}$$

Proof Since F and G are periodic, therefore, we have

$$\widetilde{F}(\xi) = \sum_{m\in\mathbb{N}_0} F(\xi + u(m)),\ \widetilde{G}(\xi) = \sum_{m\in\mathbb{N}_0} G(\xi + u(m)).$$

Clearly, $\widetilde{F}, \widetilde{G} \in L^2(\mathbb{D})$, because only a finite number of terms contribute to the above sum. Since

$$\widehat{\widetilde{F}}(k) = \int_{\mathfrak{D}} \widetilde{F}(\xi)\overline{\chi_{u(k)}(\xi)}\, d\xi = \widehat{F}(k),\ k \in \mathbb{N}_0,$$

hence by the Plancherel formula,

$$\int_K \widetilde{F}(\xi)\overline{G(\xi)}\, d\xi = \int_{\mathfrak{D}} \widetilde{F}(\xi)\, \overline{\widetilde{G}(\xi)}\, d\xi = \sum_{k\in\mathbb{N}_0} \widehat{F}(k)\overline{\widehat{G}(k)}.$$

□

Lemma 11.2 *For* $\mathfrak{f} \in L^2(K)$ *and* $J \in \mathbb{N}$, *we have*

$$\int_K \sum_{j=J}^{\infty} |\tilde{\mathfrak{f}}(\mathfrak{p}^j\xi)|^2 d\xi < \infty \tag{11.13}$$

and thus

$$\sum_{j=J}^{\infty} |\tilde{\mathfrak{f}}(\mathfrak{p}^j\xi)|^2 < \infty \ a.e.\ on K. \tag{11.14}$$

Proof Inequality (11.14) immediately follows from (11.13). So we need to prove only (11.13).

$$\int_K \sum_{j=J}^{\infty} |\tilde{\mathfrak{f}}(\mathfrak{p}^j\xi)|^2 d\xi = \sum_{j=J}^{\infty} \int_K |\tilde{\mathfrak{f}}(\mathfrak{p}^j\xi)|^2 d\xi = \|\hat{\mathfrak{f}}\|^2 \sum_{j=J}^{\infty} q^{-j}$$
$$= \frac{q^{-J+1}}{q-1} \|\hat{\mathfrak{f}}\|^2 < \infty.$$

□

Lemma 11.3 *Let* $\mathfrak{f} \in L^2(K)$ *and* $f \in \tilde{D}$. *Then*

$$\lim_{j\to\infty} \sum_{j<-J} \sum_{k\in\mathbb{N}_0} |\langle f, \mathfrak{f}_{j,k}\rangle|^2 = 0.$$

Proof We have

$$\sum_{k\in\mathbb{N}_0} \left|\langle f, \mathfrak{f}_{j,k}\rangle\right|^2 = q^j \int_K \overline{\hat{f}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}(\xi) \sum_{m\in\mathbb{N}_0} \hat{f}\left(\mathfrak{p}^j(\xi + u(m)\right) \overline{\hat{\mathfrak{f}}(\xi + u(m))},$$

and thus

$$\begin{aligned}\sum_{j<-J} \sum_{k\in\mathbb{N}_0} &= \sum_{j<-J} q^j \int_K \left|\hat{f}(\mathfrak{p}^j\xi)\hat{\mathfrak{f}}(\xi)\right|^2 d\xi + P_J \\ &\leq \|\hat{f}\|_\infty^2 \|\hat{\mathfrak{f}}\|^2 \sum_{j<-J} q^J + P_J \\ &= \frac{q^{-J}}{q-1} \|\hat{f}\|_\infty^2 \|\hat{\mathfrak{f}}\|^2 \end{aligned} \tag{11.15}$$

where

$$P_J = \sum_{j<-J} q^j \int_K \overline{\hat{f}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}(\xi) \sum_{m\in\mathbb{N}} \hat{f}\left(\mathfrak{p}^j(\xi + u(m)\right) \overline{\hat{\mathfrak{f}}(\xi + u(m))}.$$

Now we proceed to estimate P_J.

$$\begin{aligned}|P_J| &\leq \frac{1}{2} \sum_{j<-J} q^j \sum_{m\in\mathbb{N}} \int_K \left|\hat{f}\left(\mathfrak{p}^j(\xi + u(m)\right) \hat{f}(\mathfrak{p}^j\xi)\right| \left(|\hat{\mathfrak{f}}(\xi)|^2 + |\hat{\mathfrak{f}}(\xi + u(m))|^2\right) d\xi \\ &= \int_K \sum_{j<-J} q^j \sum_{m\in\mathbb{N}} \left|\hat{f}\left(\mathfrak{p}^j(\xi + u(m)\right) \hat{f}(\mathfrak{p}^j\xi)\right| |\hat{\mathfrak{f}}(\xi)|^2 \, d\xi.\end{aligned}$$

By Lemma 11.1, $\lim_{J\to\infty} \sum_{j<-J} q^j \sum_{m\in\mathbb{N}} \left|\hat{f}\left(\mathfrak{p}^j(\xi + u(m)\right) \hat{f}(\mathfrak{p}^j\xi)\right| = 0$, and the integral is dominated by $M\|\hat{f}\|_\infty^2 |\hat{\mathfrak{f}}(\xi)|^2$ which belongs to $L^1(K)$. Therefore, by Lebesgue-dominated convergence theorem, we obtain $\lim_{J\to\infty} P_J = 0$, and by (11.15), we get

$$\lim_{j\to\infty} \sum_{j<-J} \sum_{k\in\mathbb{N}_0} |\langle f, \mathfrak{f}_{j,k}\rangle|^2 = 0.$$

□

Lemma 11.4 *Let $f, g \in \tilde{D}$. Then*

$$\int_K \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\hat{f}(\xi + \mathfrak{p}^j u(k)) h_1(\mathfrak{p}^{-j}\xi + u(k))\hat{g}(\xi) h_2(\mathfrak{p}^{-j}\xi)| d\xi = \mathfrak{C}_{h_1,h_2} < \infty$$

for $h_1, h_2 \in L^2(K)$.

Lemma 11.5 *Assume that $\mathfrak{f}, \tilde{\mathfrak{f}} \in L^2(K, \mathbb{C}^L)$. Then*

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} & \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}\rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}, g\rangle \\ &= \int_K \left\{ \sum_{k\in\mathbb{N}_0} \sum_{\ell=1}^{L} \hat{f}_\ell(\xi + \mathfrak{p}^j u(k)) \overline{\hat{\tilde{\mathfrak{f}}}_\ell(\mathfrak{p}^{-j}\xi + u(k))} \right\} \\ &\quad \times \left\{ \sum_{\ell=1}^{L} \hat{\mathfrak{f}}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{g}_\ell(\xi)} \right\} d\xi \end{aligned} \tag{11.16}$$

for $f, g \in D$ and a fixed $j \in \mathbb{Z}$.

Proof Observe that

$$\sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}\rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}, g\rangle = \sum_{k\in Z} \sum_{1\le \ell_1,\ell_2 \le L} \langle f_{\ell_1}, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}_{\ell_1}\rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}_{\ell_2}, g_{\ell_2}\rangle;$$

therefore, by the Plancherel theorem,

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} & \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}\rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}, g\rangle \\ &= \sum_{k\in\mathbb{N}_0} \sum_{1\le l_1,l_2\le L} q^{-j} \int_K \hat{f}_{l_1}(\xi)\overline{\hat{\tilde{\mathfrak{f}}}_{l_1}(\mathfrak{p}^{-j}\xi)\chi_k(\mathfrak{p}^{-j}\xi)} d\xi \int_K \overline{\hat{g}_{l_2}(\xi)} \hat{\mathfrak{f}}_{l_2}(\mathfrak{p}^{-j}\xi)\chi_k(\mathfrak{p}^{-j}\xi) d\xi.\end{aligned}$$

Making the substitution by $\mathfrak{p}^{-j}\xi = \xi'$ and $k' = -k$, we have

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} & \langle f, D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}\rangle \langle D_{\mathfrak{p}^j} T_{u(k)} \tilde{\mathfrak{f}}, g\rangle \\ &= \sum_{1\le \ell_1,\ell_2\le L} \sum_{k\in\mathbb{N}_0} q^{j} \int_K \hat{f}_{\ell_1}(\mathfrak{p}^j\xi)\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\xi)\chi_k(\xi)} d\xi \int_K \overline{\hat{g}_{\ell_2}(\mathfrak{p}^j\xi)} \hat{\mathfrak{f}}_{\ell_2}(\xi)\chi_k(\xi) d\xi.\end{aligned}$$

For fixed $j \in \mathbb{Z}$ and let $F(\xi) = \hat{f}_{\ell_1}(\mathfrak{p}^j\xi)\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\xi)}$, $\;G(\xi) = \hat{g}_{\ell_2}(\mathfrak{p}^j\xi)\overline{\hat{\mathfrak{f}}_{\ell_2}(\xi)}$, then

$$\sum_{k\in\mathbb{N}_0}\langle f, D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}\rangle\langle D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}, g\rangle = \sum_{1\le\ell_1,\ell_2\le L} q^j \sum_{k\in\mathbb{N}_0}\hat{F}(u(k))\overline{\hat{G}(u(k))}.$$

By Lemma 11.1, we have

$$\sum_{k\in\mathbb{N}_0}\langle f, D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}\rangle\langle D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}, g\rangle$$

$$= \sum_{1\le\ell_1,\ell_2\le L} q^j\int_K\left\{\sum_{k\in\mathbb{N}_0}\hat{f}_{\ell_1}(\mathfrak{p}^j(\xi+u(k)))\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\xi+u(k))}\right\}\overline{\hat{g}_{\ell_2}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_{\ell_2}(\xi)d\xi$$

$$= \sum_{1\le\ell_1,\ell_2\le L}\int_K\left\{\sum_{k\in\mathbb{N}_0}\hat{f}_{\ell_1}(\mathfrak{p}^j(\xi+\mathfrak{p}^ju(k)))\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi+u(k))}\right\}\overline{\hat{g}_{\ell_2}(\xi)}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^{-j}\xi)d\xi,$$

by a change of variable. Then

$$\sum_{k\in\mathbb{N}_0}\langle f, D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}\rangle\langle D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}, g\rangle \tag{11.17}$$

$$= \int_K\left\{\sum_{k\in\mathbb{N}_0}\sum_{\ell=1}^{L}\hat{f}_{\ell_1}(\mathfrak{p}^j(\xi+\mathfrak{p}^ju(k)))\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi+u(k))}\right\}\left\{\sum_{\ell=1}^{L}\hat{\mathfrak{f}}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{g}_\ell(\xi)}\right\}d\xi$$

for $f, g \in D$. This finishes the proof. □

Lemma 11.6 *Given $J\in\mathbb{N}$, assume that $\mathfrak{f}, \tilde{\mathfrak{f}} \in L^2(K, \mathbb{C}^L)$. Then*

$$\sum_{j<J}\sum_{k\in\mathbb{N}_0}\langle f, D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}\rangle\langle D_{\mathfrak{p}^j}T_{u(k)}\mathfrak{f}, g\rangle$$

$$= \int_K\sum_{1\le\ell_1,\ell_2\le L}\hat{f}_{\ell_1}(\xi)\overline{\hat{g}_{\ell_2}(\xi)}\sum_{j>-J}\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi)d\xi$$

$$+\int_K\sum_{\gamma\in\mathbb{N}_0\backslash q\mathbb{N}_0}\sum_{j\in\mathbb{Z}}\sum_{1\le\ell_1,\ell_2\le L}\hat{f}_{\ell_1}(\xi+\mathfrak{p}^ju(\gamma))\overline{\hat{g}_{\ell_2}(\xi)}$$

$$\times\sum_{m=0}^{\infty}\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^m(\mathfrak{p}^{-j}\xi+u(\gamma)))}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^m.\mathfrak{p}^{-j}\xi)d\xi$$

for $f, g \in D$ and J sufficiently large.

Proof Write

$$I_1(J)=\sum_{j<J}\int_K\left\{\sum_{l=1}^{L}\hat{f}_{l_1}(\xi)\overline{\hat{\tilde{\mathfrak{f}}}_l(\mathfrak{p}^{-j}\xi)}\right\}\left\{\sum_{l=1}^{L}\hat{\mathfrak{f}}_l(\mathfrak{p}^{-j}\xi)\overline{\hat{g}_l(\xi)}\right\}d\xi,$$

$$I_2(J)=\sum_{j<J}\int_K\left\{\sum_{k\in\mathbb{N}_0}\sum_{l=1}^{L}\hat{f}_{l_1}(\mathfrak{p}^j(\xi+\mathfrak{p}^j u(k)))\overline{\hat{\tilde{\mathfrak{f}}}_{l_1}(\mathfrak{p}^{-j}\xi+u(k))}\right\}\left\{\sum_{l=1}^{L}\hat{\mathfrak{f}}_l(\mathfrak{p}^{-j}\xi)\overline{\hat{g}_l(\xi)}\right\}d\xi.$$

Then

$$\sum_{j<J}\sum_{k\in\mathbb{N}_0}\langle f, D_{\mathfrak{p}^j}T_{u(k)}\tilde{\mathfrak{f}}\rangle\langle D_{\mathfrak{p}^j}T_{u(k)}\mathfrak{f}, g\rangle=I_1(J)+I_2(J) \tag{11.18}$$

Since $|\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)\hat{\tilde{\mathfrak{f}}}_{\ell_2}(\mathfrak{p}^j\xi)|\le\frac{1}{2}\left(|\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)|^2+|\hat{\tilde{\mathfrak{f}}}_{\ell_2}(\mathfrak{p}^j\xi)|^2\right)$, we have

$$\begin{aligned}\int_K\sum_{j<J}\sum_{1\le\ell_1,\ell_2\le L}&|\hat{f}_{\ell_1}(\xi)\overline{\hat{g_{\ell_2}}(\xi)}\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi)|d\xi\\ &\le\sum_{1\le\ell_1,\ell_2\le L}\|\hat{f}_{\ell_1}\|_\infty\|\hat{g_{\ell_2}}\|_\infty\int_K\sum_{j>-J}|\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi)|d\xi\end{aligned}$$

$$\le\sum_{1\le\ell_1,\ell_2\le L}\|\hat{f}_{\ell_1}\|_\infty\|\hat{g_{\ell_2}}\|_\infty\int_K\sum_{j>-J}\left\{|\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)|^2+|\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi)|^2\right\}d\xi<\infty$$

by Lemma 11.2. Then we can rewrite $I_1(J)$ as

$$I_1(J)=\int_K\sum_{1\le\ell_1,\ell_2\le L}\hat{f}_{\ell_1}(\xi)\overline{\hat{g_{\ell_2}}(\xi)}\sum_{j>-J}\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi)d\xi. \tag{11.19}$$

Now we turn to $I_2(J)$. Observing that, for an arbitrarily fixed $j\in\mathbb{Z}$, there are at most finitely many $k\in\mathbb{N}_0$ such that $\hat{f}_{\ell_1}(\xi+\mathfrak{p}^j uk)\,\overline{\hat{g}_{\ell_2}(\xi)}\neq 0$, we have

$$\begin{aligned}I_2(J)&=\sum_{j<J}\int_K\sum_{k\in\mathbb{N}_0}\sum_{1\le\ell_1,\ell_2\le L}\hat{f}_{\ell_1}(\xi+\mathfrak{p}^j u(k))\overline{\hat{g_{\ell_2}}(\xi)}\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi+u(k))}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^{-j}\xi)d\xi\\ &=\sum_{j<J}\int_K\sum_{1\le\ell_1,\ell_2\le L}\sum_{k\in\mathbb{N}_0}\hat{f}_{\ell_1}(\xi+\mathfrak{p}^j u(k))\overline{\hat{g_{\ell_2}}(\xi)}\overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi+u(k))}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^{-j}\xi)d\xi.\end{aligned}$$

Since $f_\ell, g_\ell \in \tilde{D}$ for $1 \le \ell \le L$, we have

$$\int_K \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\hat{f}_{\ell_1}(\xi + \mathfrak{p}^j u(k))\overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi + u(k))}\hat{g_{\ell_2}}(\xi)\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^{-j}\xi)|d\xi < \infty$$

for $1 \le \ell_1, \ell_2 \le L$ by Lemma 11.3, which implies that

$$\int_K \sum_{j<J} \sum_{k\in\mathbb{N}_0} |\hat{f}_{\ell_1}(\xi + \mathfrak{p}^j u(k))\overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi + u(k))}\hat{g_{\ell_2}}(\xi)\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^{-j}\xi)|d\xi < \infty$$

for $1 \le \ell_1, \ell_2 \le L$. So

$$I_2(J) = \int_K \sum_{1\le\ell_1,\ell_2\le L} \sum_{k\in\mathbb{N}_0} \sum_{j<J} \hat{f_{\ell_1}}(\xi + \mathfrak{p}^j u(k))\overline{\hat{g_{\ell_2}}(\xi)}\overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi + u(k))}\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^{-j}\xi)d\xi$$

$$= \int_K \sum_{1\le\ell_1,\ell_2\le L} \sum_{\gamma\in\mathbb{N}_0\setminus q\mathbb{N}_0} \sum_{m=0}^{\infty} \sum_{j<J} \hat{f_{\ell_1}}(\xi + \mathfrak{p}^{j+m}u(\gamma))\overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{-j}\xi + \mathfrak{p}^m u(\gamma))}$$
$$\times \overline{\hat{g_{\ell_2}}(\xi)}\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^{-j}\xi)d\xi$$
$$= \int_K \sum_{1\le\ell_1,\ell_2\le L} \sum_{\gamma\in\mathbb{N}_0\setminus q\mathbb{N}_0} \sum_{m=0}^{\infty} \sum_{j<J+m} \hat{f_{\ell_1}}(\xi + \mathfrak{p}^j u(\gamma))\overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^{m-j}\xi + \mathfrak{p}^m u(\gamma))}$$
$$\times \overline{\hat{g_{\ell_2}}(\xi)}\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^{m-j}\xi)d\xi \tag{11.20}$$

The equality (11.20) is obtained by the fact that $\mathbb{N}_0 = \bigcup_{m=0}^{\infty} \mathfrak{p}^m(\mathbb{N}_0 \setminus q\mathbb{N}_0)$. Since if J large enough, $\hat{f_{l_1}}(\xi + \mathfrak{p}^j u(\gamma))\overline{\hat{g_{l_1}}(\xi)} = 0$ for all $j \ge J$ and $\gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0$. It leads to

$$I_2(J) = \int_K \sum_{1\le\ell_1,\ell_2\le L} \sum_{\gamma\in\mathbb{N}_0\setminus q\mathbb{N}_0} \sum_{m=0}^{\infty} \sum_{j\in\mathbb{Z}} \hat{f_{\ell_1}}(\xi + \mathfrak{p}^j u(\gamma))\overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^m(\mathfrak{p}^{-j}\xi + u(\gamma)))}$$
$$\times \overline{\hat{g_{\ell_2}}(\xi)}\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^{m-j}\xi)d\xi$$
$$= \int_K \sum_{\gamma\in\mathbb{N}_0\setminus q\mathbb{N}_0} \sum_{j\in\mathbb{Z}} \sum_{1\le\ell_1,\ell_2\le L} \hat{f_{\ell_1}}(\xi + \mathfrak{p}^j u(\gamma))\overline{\hat{g_{l_2}}(\xi)} \sum_{m=0}^{\infty} \overline{\hat{\hat{\mathfrak{f}}}_{l_1}(\mathfrak{p}^m(\mathfrak{p}^{-j}\xi + u(\gamma)))}$$
$$\times \hat{\mathfrak{f}_{l_2}}(\mathfrak{p}^m.\mathfrak{p}^{-j}\xi)d\xi \tag{11.21}$$

for J sufficiently large. The proof is finished by (11.18), (11.19), and (11.21). □

11.3 Main Results

In this section, we characterize the vector-valued affine bi-frames on reducing subspaces $L^2(K, \mathbb{C}^L)$.

Theorem 11.1 *Let $FL^2(\Gamma, \mathbb{C}^L)$ be a reducing subspace of $L^2(K, \mathbb{C}^L)$ and $\mathfrak{f}, \tilde{\mathfrak{f}} \in FL^2(\Gamma, \mathbb{C}^L)$. Then $\left(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\tilde{\mathfrak{f}})\right)$ is a WABF associated with $D \cap FL^2(\Gamma, \mathbb{C}^L)$ if and only if*

$$\lim_{j\to\infty} \sum_{j>-J} \overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi) = \delta_{\ell_1,\ell_2}\chi_\Gamma(\xi)\,\textit{for}\ 1 \le \ell_1, \ell_2 \le L \tag{11.22}$$

weakly in $L^1(K)$, for all compact $K \subset K \setminus \{0\}$, and

$$\sum_{j=0}^{\infty} \overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j(\xi + u(\gamma)))}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi) = 0 \tag{11.23}$$

for $1 \le \ell_1, \ell_2 \le L$, $\gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0$ and a.e. $\xi \in \Gamma$.

Proof It is easy to check that if $\left(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\tilde{\mathfrak{f}})\right)$ is a WABF associated with $D \cap FL^2(\Gamma, \mathbb{C}^L)$, then $\left(\mathcal{A}(\mathfrak{f}_\ell), \mathcal{A}(\tilde{\mathfrak{f}_\ell})\right)$ is a WABF associated with $\tilde{D} \cap FL^2(\Gamma)$ for every $1 \le \ell \le L$. We may as well assume that (11.22) holds for $1 \le \ell_1 = \ell_2 \le L$. Next, we prove the theorem under this assumption. By Lemma 11.2 and the Cauchy-Schwarz inequality, the series in (11.22) and (11.23) are absolutely convergent. And by Lemma 11.3, we have

$$\lim_{j\to\infty} \sum_{j<-J} \sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)}\tilde{\mathfrak{f}}\rangle \langle D_{\mathfrak{p}^j} T_{u(k)}\tilde{\mathfrak{f}}, g\rangle = 0\ \textit{for}\ f, g \in D. \tag{11.24}$$

Then $\left(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\tilde{\mathfrak{f}}\right)$ is a WABF associated with $D \cap FL^2(\Gamma, \mathbb{C}^L)$ if and only if

$$\lim_{j\to\infty} \sum_{j<J} \sum_{k\in\mathbb{N}_0} \langle f, D_{\mathfrak{p}^j} T_{u(k)}\tilde{\mathfrak{f}}\rangle \langle D_{\mathfrak{p}^j} T_{u(k)}\tilde{\mathfrak{f}}, g\rangle = \langle f, g\rangle\ \textit{for}\ f, g \in D \cap FL^2(\Gamma, \mathbb{C}^L). \tag{11.25}$$

By Lemma 11.5, we see that (11.25) is equivalent to

$$\lim_{j\to\infty} (I_1(J) + I_2(J)) = \langle f, g\rangle\ \textit{for}\ f, g \in D \cap FL^2(\Gamma, \mathbb{C}^L). \tag{11.26}$$

where

$$I_1(J) = \int_K \sum_{1\le \ell_1,\ell_2\le L} \hat{f}_{\ell_1}(\xi)\overline{\hat{g_{\ell_2}}(\xi)} \sum_{j>-J} \overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi)d\xi \tag{11.27}$$

and

$$I_2(J) = \int_K \sum_{\gamma\in\mathbb{N}_0\setminus q\mathbb{N}_0} \sum_{j\in\mathbb{Z}} \sum_{1\le\ell_1,\ell_2\le L} \hat{f}_{\ell_1}(\xi+\mathfrak{p}^j u(\gamma))\overline{\hat{g}_{\ell_2}(\xi)} \sum_{m=0}^{\infty} \overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^m(\mathfrak{p}^{-j}\xi+u(\gamma)))}$$
$$\times\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^m.\mathfrak{p}^{-j}\xi)d\xi \tag{11.28}$$

for J large enough.

Now we prove (11.26) is equivalent to both (11.22) and (11.23) holding to finish the proof. First suppose (11.22) and (11.23) hold. Then $I_2(J) = 0$ for J large enough, and thus

$$\begin{aligned}\lim_{j\to\infty}(I_1(J)+I_2(J)) &= \lim_{j\to\infty} I_1(J)\\ &= \lim_{j\to\infty}\int_K \sum_{\ell=1}^{L} \hat{f}_\ell(\xi)\overline{\hat{g_\ell}(\xi)} \sum_{j>-J} \overline{\hat{\hat{\mathfrak{f}}}_\ell(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_\ell(\mathfrak{p}^j\xi)d\xi. \end{aligned}\tag{11.29}$$

Also observe that for every $1 \le l \le L$, $\hat{f}_\ell(\xi)\hat{g}_\ell(\xi) \in L^\infty(K)$ with some compact $K \subset K\setminus\{0\}$ if $f, g \in D\cap FL^2(\Gamma,\mathbb{C}^L)$. It follows that

$$\lim_{j\to\infty}\int_R \hat{f}_\ell(\xi)\overline{\hat{g_\ell}(\xi)} \sum_{j>-J} \overline{\hat{\hat{\mathfrak{f}}}_\ell(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_\ell(\mathfrak{p}^j\xi)d\xi = \int_K \hat{f}_\ell(\xi)\overline{\hat{g_\ell}(\xi)}d\xi$$

for $1\le\ell\le L$. So we have

$$\begin{aligned}\lim_{j\to\infty}(I_1(J)+I_2(J)) &= \int_K \hat{f}_\ell(\xi)\overline{\hat{g_\ell}(\xi)}d\xi\\ &= \langle f, g\rangle \end{aligned}\tag{11.30}$$

by (11.30) and the Plancherel theorem. Therefore, (11.26) holds. Next, we prove the converse implication. Suppose (11.26) holds. First we prove (11.23) for $1 \le \ell_1,\ell_2 \le L$ with $\ell_1 \ne \ell_2$. Fix $\gamma_0 \in \mathbb{N}_0\setminus q\mathbb{N}_0$ *and* $1\le\ell_1,\ell_2\le L$ with $\ell_1\ne\ell_2$. Define t_γ by

$$t_\gamma = \sum_{m=0}^{\infty} \overline{\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^m(\xi+u(\gamma)))\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^m\xi)}$$

for $\gamma \in \mathbb{N}_0\setminus q\mathbb{N}_0$. By Lemma 11.2 and the Cauchy-Schwarz inequality, $t_\gamma \in L^1(K)$ for each $\gamma\in\mathbb{N}_0\setminus q\mathbb{N}_0$. So almost every $\xi\in K$ is a Lebesgue point of

$t_\gamma(.)\xi_{\Gamma\cap(\Gamma-u(\gamma))}(.)$ for all $\gamma\in\mathbb{N}_0\backslash q\mathbb{N}_0$. Arbitrarily fix such a point $\xi_0\neq 0$. Without loss of generality, we assume that $\xi_0>0$. Take $f,g\in D\cap FL^2(\Gamma,\mathbb{C}^L)$ in (11.26) such that $f_\ell(.)=0$ for $1\leq\ell\leq L$ with $\ell\neq\ell_1$, $g_\ell(.)=0$ for $1\leq\ell\leq L$ with $\ell\neq\ell_2$ and

$$\hat{f}_{\ell_1}(\xi+u(\gamma_0))=\hat{g}_{\ell_2}(.)=\frac{1}{\sqrt{q\varepsilon}}\chi_{\xi_0+\varepsilon_1\xi_0+\varepsilon}(.)\chi_{\Gamma\cap(\Gamma+u(\gamma_0))}(.)$$

with $0<\varepsilon<\min\{\frac{1}{q},\frac{a-1}{a+1}\xi_0\}$. Then we have

$$\begin{aligned}0&=I_2(J)\\&=\int_K\sum_{\gamma\in\mathbb{N}_0\backslash q\mathbb{N}_0}\sum_{j\in\mathbb{Z}}\hat{f}_{l_1}(\xi+\mathfrak{p}^ju(\gamma))\overline{\hat{g_{l_2}}(\xi)}\sum_{m=0}^{\infty}\overline{\hat{\hat{\mathfrak{f}}}_{l_1}(\mathfrak{p}^m(\mathfrak{p}^{-j}\xi+u(\gamma)))}\hat{\mathfrak{f}_{l_2}}(\mathfrak{p}^m.\mathfrak{p}^{-j}\xi)d\xi\\&\quad I_3(\varepsilon)+I_4(\varepsilon),\end{aligned}\tag{11.31}$$

for J large enough, and we have

$$I_3(\varepsilon)=\frac{1}{q\varepsilon}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon}\chi_{\Gamma\cap(\Gamma-u(\gamma_0))}(\xi)\gamma t_{\gamma_0}(\xi)d\xi,\gamma\gamma\gamma$$

$$I_4(\varepsilon)=\int_K\sum_{(\gamma,j)\in((\mathbb{N}_0\backslash q\mathbb{N}_0)\times\mathbb{N}_0)\backslash\{(\gamma_0,0)\}}\mathfrak{p}^j\hat{f}_{\ell_1}(\mathfrak{p}^j(\xi+u(\gamma)))\overline{\hat{g_{\ell_2}}(\mathfrak{p}^j\xi)}t_\gamma(\xi)d\xi.$$

Observing that $|t_\gamma(\xi)|\leq\frac{1}{q}\left[\sum_{m=0}^{\infty}|\hat{\hat{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^m(\xi+u(\gamma)))|^2+\sum_{m=0}^{\infty}|\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^m\xi)|^2\right]$, we have

$$|I_4(\varepsilon)|\leq I_5(\varepsilon)+I_6(\varepsilon),\tag{11.32}$$

where

$$I_5(\varepsilon)=\int_K\sum_{(\gamma,j)\in((\mathbb{N}_0\backslash q\mathbb{N}_0)\times\mathbb{N}_0)\backslash\{(\gamma_0,0)\}}\mathfrak{p}^j|\hat{f_{\ell_1}}(\mathfrak{p}^j(\xi+u(\gamma)))\hat{g_{\ell_2}}(\mathfrak{p}^j\xi)\sum_{m=0}^{\infty}|\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^m\xi)|^2d\xi,$$

$$I_6(\varepsilon)=\int_K\sum_{(\gamma,j)\in((\mathbb{N}_0\backslash q\mathbb{N}_0)\times\mathbb{N}_0)\backslash\{(\gamma_0,0)\}}\mathfrak{p}^j|\hat{f_{\ell_1}}(\mathfrak{p}^j(\xi+u(\gamma)))\hat{g_{\ell_2}}(\mathfrak{p}^j\xi)\sum_{m=0}^{\infty}|\hat{\mathfrak{f}_{\ell_1}}(\mathfrak{p}^m\xi)|^2d\xi.$$

Let us first estimate $I_5(\varepsilon)$. Our argument is borrowed from the proof of [[29], Theorem 2.4]. But, for reader's convenience, we state it here. If $j>0$, $|\mathfrak{p}^ju(\gamma)-u(\gamma_0)\geq 1>2\varepsilon$. *If* $j<0$, *take* $j_0=max\{j\in Z:a_j\leq$

$2\varepsilon\}$*and then* $j_0 < 0, |\mathfrak{p}^j u(\gamma) - u(\gamma_0)| = \mathfrak{p}^j u(\gamma) - \mathfrak{p}^{-j} u(\gamma_0) \leq \mathfrak{p}^j > 2\varepsilon$. Thus, $\hat{f_{\ell_1}}(\mathfrak{p}^j(\xi + u(\gamma)))\hat{g_{\ell_2}}(\mathfrak{p}^j\xi) = 0$ for $\gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0$ and $j_0 < j \in \mathbb{Z}$ by the definition of $\hat{f_{\ell_1}}$ and $\hat{g_{\ell_2}}$. It follows that

$$I_5(\varepsilon) \leq \frac{1}{q\varepsilon} \sum_{j=n-\infty}^{j_0} \sum_{\gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0} \mathfrak{p}^j \int_{\mathfrak{p}^{-j}(\xi_0-\varepsilon)}^{\mathfrak{p}^{-j}(\xi_0+\varepsilon)} \chi_{(\xi_0-\varepsilon,\xi_0+\varepsilon)}(\mathfrak{p}^j\xi + \mathfrak{p}^j u(\gamma) - u(\gamma_0))$$

$$\times \sum_{m=0}^{\infty} |\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^m\xi)\|^2 d\xi$$

$$= \frac{1}{q\varepsilon} \sum_{j=-\infty}^{j_0} \mathfrak{p}^j \int_{\mathfrak{p}^{-j}(\xi_0-\varepsilon)}^{\mathfrak{p}^{-j}(\xi_0+\varepsilon)} \sum_{\gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0} \chi_{(\xi_0-\varepsilon,\xi_0+\varepsilon)}(\mathfrak{p}^j\xi + \mathfrak{p}^j u(\gamma) - u(\gamma_0)) \sum_{m=0}^{\infty} |\hat{\mathfrak{f}_{\ell_2}}(\mathfrak{p}^m\xi)\|^2 d\xi.$$

It is easy to check that the cardinality of the set $\left\{\gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0 : \chi_{(\xi_0-\varepsilon,\xi_0+\varepsilon)}(\mathfrak{p}^j\xi + \mathfrak{p}^j u(\gamma) - u(\gamma_0)) \neq 0 \text{ for some } \xi \in (\mathfrak{p}^{-j}(\xi_0-\varepsilon), \mathfrak{p}^{-j}(\xi_0+\varepsilon))\right\}$ is less than $8\mathfrak{p}^{-j}\varepsilon$. So we have

$$\begin{aligned} I_5(\varepsilon) &\leq 4 \sum_{j=-\infty}^{j_0} \int_{\mathfrak{p}^{-j}(\xi_0-\varepsilon)}^{\mathfrak{p}^{-j}(\xi_0+\varepsilon)} \sum_{m=0}^{\infty} |\hat{\mathfrak{f}_{l_2}}(\mathfrak{p}^m\xi)\|^2 d\xi \\ &\leq 4 \int_{\mathfrak{p}^{-j_0}(\xi_0-\varepsilon)}^{\infty} \sum_{m=0}^{\infty} |\hat{\mathfrak{f}_{l_2}}(\mathfrak{p}^m\xi)\|^2 d\xi, \end{aligned} \tag{11.33}$$

where in the last inequality we used the fact that $\left(\mathfrak{p}^{-j}(\xi_0-\varepsilon), \mathfrak{p}^{-j}(\xi_0+\varepsilon)\right)$, $j \in Z$ are mutually disjoint when $\varepsilon < \frac{a-1}{a+1}\xi_0$. Also observing that

$$\int_K \sum_{m=0}^{\infty} |\hat{\mathfrak{f}_{l_2}}(\mathfrak{p}^m\xi)\|^2 d\xi = \sum_{m=0}^{\infty} \mathfrak{p}^{-m} ||\hat{\mathfrak{f}_{l_2}}||^2 < \infty \ \ and \ \ that \ \ \mathfrak{p}^{-j_0}(\xi_0-\varepsilon) \geq \frac{\xi_0}{2\varepsilon} - \frac{1}{2},$$

we have $\lim_{\varepsilon\to 0} T_5(\varepsilon) = 0$ by (3.10). For $I_6(\varepsilon)$, we have

$$I_6(\varepsilon) = \int_K \sum_{(\gamma,j)\in((\mathbb{N}_0-q\mathbb{N}_0)\times\mathbb{N}_0)-\{\gamma_0,0\}} \mathfrak{p}^j |\hat{f_{l_1}}(\mathfrak{p}^j\xi)\hat{g_{l_2}}(\mathfrak{p}^j(\xi - u(\gamma)))| \sum_{m=0}^{\infty} |\hat{\hat{\mathfrak{f}_{l_1}}}(\mathfrak{p}^m\xi)|^2 d\xi$$

by a change of variables. Then, by an argument similar to the above, we can prove that $\lim_{\varepsilon\to\infty} I_6(\varepsilon) = 0$. So we have $I_4(\varepsilon) = 0$ by (11.32) and thus $I_3(\varepsilon) = 0$. by (11.31), that is, $\chi_{\Gamma\cap(\Gamma-u(\gamma_0))}(\xi_0)t_{\gamma_0}(\xi_0) = 0$. It follows that $\chi_{\Gamma\cap(\Gamma-u(\gamma))}(.)t_\gamma(.) = 0$ a.e. on R for $\gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0$ by the arbitrariness of ξ_0 and γ_0. This is equivalent to $t_\gamma(\cdot) = 0 \ \ for \ \ \gamma \in \mathbb{N}_0 \setminus q\mathbb{N}_0$ since $supp(t_\gamma) \subset \Gamma \cap (\Gamma - \gamma)$. Now we prove (I) for

$1 \leq \ell_1, \ell_2 \leq L$ with $\ell_1 \neq \ell_2$. For $h \in L^\infty(K)$ with compact $K \subset K \setminus \{0\}$, define $f, g \in D \cap FL^2(\Gamma, \mathbb{C}^L)$ such that $f_\ell(\cdot) = 0$ for $1 \leq \ell \leq L$ with $l \neq \ell_1$, $g_\ell(\cdot) = 0$ for $1 \leq \ell \leq L$ with $\ell \neq \ell_2$, and $\hat{f_{\ell_1}} = \frac{|h|^{\frac{1}{2}}}{argh}\chi_\Gamma$ and $\hat{g_{\ell_2}} = |h|^{\frac{1}{2}}\chi_\Gamma$, where

$$\arg z = \{\frac{|z|}{z}, \quad z \neq 0, 1, \quad z = 0 \tag{11.34}$$

for $z \in C$. Then

$$\langle f, g\rangle = \sum_{\ell=1}^{L} \int_K f_\ell(x)\overline{g_\ell(x)}dx = 0.$$

From (11.23) and (11.26)–(11.28), we deduce that

$$\lim_{j\to\infty} \int_K \left\{ \sum_{j>-J} \overline{\hat{\tilde{\mathfrak{f}}}_{\ell_1}(\mathfrak{p}^j\xi)}\hat{\mathfrak{f}}_{\ell_2}(\mathfrak{p}^j\xi) \right\} h(\xi)d\xi = 0.$$

The proof is completed. □

Acknowledgments This work is supported by the Research Grant (No. JKST&IC/SRE/J/357-60) provided by JKST&IC, UT, of J & K, India.

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Chapter 12
A New Perspective on $\mathcal{I}_2$-Statistical Limit Points and $\mathcal{I}_2$-Statistical Cluster Points in Probabilistic Normed Spaces

Ömer Kişi and Erhan Güler

12.1 Introduction

In the early 1960s, A. N. Šerstnev [1] investigated random normed spaces (RNS), put forward questions regarding the completeness and the completion of RNS, and then examined the problem of best approximation in RNS. The theory of PNS had gone through significant advancements before Alsina et al. [2] investigated a new, wider recognized definition of PN spaces. The theory of PNS supplies a significant method of generalizing the conclusions of normed linear spaces. It has beneficial implementation, in different fields such as topological spaces [3], continuity features [4], work of boundedness [5], convergence of random variables [6], etc. A comprehensive study in this direction can be examined from the book by Guillen and Harikrishnan [7].

Fast [8] and Schoenberg [9] put forward the thought of statistical convergence for a sequence of $\mathbb{R}$, individually. After the studies of Šalát [10] and Fridy [11], meaningful works have been made in this direction over the years. Fridy [12] gave the concepts of statistical limit points (*SLP*) and statistical cluster points (*SCP*) of a sequence of $\mathbb{R}$ and discussed the relationship between these notions. Statistical convergence *SLP* and *SCP* for a sequence in a PNS was defined and worked by Karakuş [13]. One may refer to the studies [14–23] related to this topic.

The opinion of ideal convergence was initially given by Kostyrko et al. [24]. This notion was constructed for double sequences by Das et al. [25] in a metric space. In [26], the researchers acquired the conception of $\mathcal{I}$-statistical convergence and suggested remarkable features of it. $\mathcal{I}$-statistical cluster points and $\mathcal{I}$-statistical limit points for a sequence of $\mathbb{R}$ were revealed by Debnath et al. [27] and by Malik

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_12

et al. [28] individually. Savaş and Gürdal proposed $\mathcal{I}$-statistical convergence for a sequence in PN spaces [29]. For various works on this area, we refer to [30–45].

As a result, it seems acceptable to examine the concept of $\mathcal{I}_2$-*SLP* and $\mathcal{I}_2$-*SCP* in the theory of PNS. And in this chapter, we do that.

Influenced by this, in this study, a further research into the mathematical features of $\mathcal{I}_2$-*SLP* and $\mathcal{I}_2$-*SCP* of a double sequence in PN spaces will be presented. In addition, we plan to establish several theorems in PNS analogue to the theorems of [29, 36, 37] by utilizing condition additive property for $\mathcal{I}_2$-asymptotic density zero sets. As a consequence, our findings generalize the conclusions of [36].

Summability theory and convergence of sequence in various spaces have become significant topics in mathematical analysis. Researchers have presented some elegant works about PNS in literature. It is known that ideal convergence is more general than statistical convergence for sequences. This has conducted us to work on ideal version concepts of sequences in PNS. The whole research work is done in more of a theoretical direction. Theorems are proved in the light of PNS theory approach. Results are obtained via different perspectives, and new examples are produced to justify the counterparts and show existence of introduced notions. The results established in this research work supply an exhaustive foundation in PNS and make a significant contribution in the theoretical development of PNS in literature. The original aspect of this chapter is the first wholly up-to-date and thorough examination of the features and implementation of $\mathcal{I}_2$-*SLP* and $\mathcal{I}_2$-*SCP* of a double sequences in PN spaces, based upon the standard definition. The conclusions of the chapter are expected to be a source for statistics and mathematics researchers in the areas of convergence methods for sequences and implementations in PNS.

12.2 Main Results

At the beginning, we examine the conceptions of $\mathcal{I}_2$-*SLP* and $\mathcal{I}_2$-*SCP* of double sequences in a PNS $(Y, \mathcal{N}, \circ)$ w.r.t. the PN $\mathcal{N}$; also, we obtain an $\mathcal{I}_2$-statistical analogue of several theorems in PNS related to these studies.

Presume that $(Y, \mathcal{N}, \circ)$ is a PNS and $w = \left\{w_{pr}\right\}_{p,r\in\mathbb{N}}$ is a sequence in Y. Then, in that case, $d^{\mathcal{I}_2}(P) = 0$, and $\{w\}_P$ is named a subsequence of $\mathcal{I}_2$-asymptotic density zero, or an $\mathcal{I}_2$-thin subsequence of w, when P fails to have $\mathcal{I}_2$-asymptotic density zero; in another way, either $d^{\mathcal{I}_2}(P) > 0$ or P does not have $\mathcal{I}_2$-asymptotic density.

Definition 12.1 Take $(Y, \mathcal{N}, \circ)$ as a PNS. Then, $\eta \in Y$ is named to be an $\mathcal{I}_2$-SLP of a sequence $w = \left\{w_{pr}\right\}_{p,r\in\mathbb{N}}$ in Y w.r.t. the PN $\mathcal{N}$, when there is an $\mathcal{I}_2$-nonthin subsequence of w that converges to η.

We note $\Lambda_w (\mathcal{I}_2 (S))_{\mathcal{N}}$ to signify the set of all $\mathcal{I}_2$-SLP of w.

Remark 12.1 Identically, the description of $\mathcal{I}_2$-SLP of a sequence in PNS can be considered in the subsequent way.

Definition 12.2 Take $(Y, \mathcal{N}, \circ)$ as a PNS. Then $\eta \in Y$ is known as $\mathcal{I}_2$-SLP of a sequence $w = \{w_{pr}\}_{p,r\in\mathbb{N}}$ in Y w.r.t. the PN $\mathcal{N}$, when there is an $\mathcal{I}_2$-nonthin subsequence of w that statistically converges to η.

Theorem 12.1 *Definitions 12.1 and 12.2 are equivalent.*

Proof At the beginning, we signify that Definition 12.1 gives Definition 12.2. Assume that $\eta \in Y$ be $\mathcal{I}_2$-*SLP* of a sequence $w = \{w_{pr}\}_{p,r\in\mathbb{N}}$ in Y w.r.t. the PN $\mathcal{N}$. Then, there is a set

$$P = \{(p_1, r_1) < (p_2, r_2) < \ldots < (p_j, r_k) < \ldots\} \subset \mathbb{N} \times \mathbb{N}$$

and $d^{\mathcal{I}_2}(P) \neq 0$ so that $\lim_{j,k\to\infty} \mathcal{N}(w_{p_j,r_k}) = \eta$. We know that each convergent sequence is also statistically convergent to the same limit in Y w.r.t. the PN $\mathcal{N}$. As a result, $st - \lim_{j,k\to\infty} \mathcal{N}(w_{p_j,r_k}) = \eta$. So, Definition 12.2 is acquired.

On the other hand, we presume that Definition 12.2 satisfies; we demonstrate that Definition 12.1 supplies. Take $\eta \in Y$ as an $\mathcal{I}_2$-*SLP* of a sequence $w = \{w_{pr}\}_{p,r\in\mathbb{N}}$ in Y w.r.t. the PN $\mathcal{N}$. Then, there is a set

$$P = \{(p_1, r_1) < (p_2, r_2) < \ldots < (p_j, r_k) < \ldots\} \subset \mathbb{N} \times \mathbb{N}$$

and $d^{\mathcal{I}_2}(P) \neq 0$ so that $st - \lim_{j,k\to\infty} \mathcal{N}(w_{p_j,r_k}) = \eta$. Then, there is a set

$$R = \{(p_{n_1}, r_{n_1}) < (p_{n_2}, r_{n_2}) < \ldots < (p_{n_j}, r_{n_k}) < \ldots\} \subseteq P$$

and $d(R) \neq 0$ so that $\lim_{j,k\to\infty} \mathcal{N}\left(w_{p_{n_j},r_{n_k}}\right) = \eta$. As $\mathcal{I}_2$ is an admissible ideal and $d(R) \neq 0$, we acquire $d^{\mathcal{I}_2}(R) \neq 0$. Hence, $\left\{w_{p_{n_j},r_{n_k}}\right\}_{j,k\in\mathbb{N}}$ is an $\mathcal{I}_2$-nonthin subsequence of w. We obtain $\lim_{j,k\to\infty} \mathcal{N}\left(w_{p_{n_j},r_{n_k}}\right) = \eta$. So, Definition 12.1 supplies. □

Now, we put forward an example of an $\mathcal{I}_2$-*SLP* in a PNS.

Example 12.2 Contemplate the PNS $(\mathbb{R}, \mathcal{N}, \circ)$. We determine a sequence $w = \{w_{pr}\}_{p,r\in\mathbb{N}}$:

$$w_{pr} = \begin{cases} 1, & \text{when } p, r \text{ are odd} \\ 0, & \text{when } p, r \text{ are even.} \end{cases}$$

Suppose that P is the set of whole odd numbers. Then, $d(P) = \frac{1}{2}$. As $\mathcal{I}_2$ is a strongly admissible ideal, $d^{\mathcal{I}_2}(P) = \frac{1}{2}$. So, $\{w\}_P$ is an $\mathcal{I}_2$-nonthin subsequence of w. Also, it converges to 1 w.r.t. norm $\mathcal{N}$. As a result, 1 is an $\mathcal{I}_2$-SLP of w. □

Definition 12.3 Take $(Y, \mathcal{N}, \circ)$ as a PNS. Then, $\eta \in Y$ is named to be an $\mathcal{I}_2$-SCP of a sequence $w = \{w_{pr}\}_{p,r \in \mathbb{N}}$ in Y w.r.t. the PN $\mathcal{N}$, provided that for each $\gamma > 0$ and $\sigma \in (0, 1)$ the set $\{(p, r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(\gamma) > 1 - \sigma\}$ fails to have $\mathcal{I}_2$-asymptotic density zero.

We denote $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ to signify the set of all $\mathcal{I}_2$-SCP of w.

Now, we put forward an example of an $\mathcal{I}_2$-*SCP* in a PNS.

Example 12.3 Contemplate the PNS $(\mathbb{R}, \mathcal{N}, \circ)$. We determine $w = \{w_{pr}\}_{p,r \in \mathbb{N}}$:

$$w_{pr} = \begin{cases} 1, & \text{when } p, r \text{ are perfect squares} \\ 0, & \text{otherwise.} \end{cases}$$

Presume that P is the set of whole perfect squares. Then, $d(P) = 0$. As $\mathcal{I}_2$ is a strongly admissible ideal, $d^{\mathcal{I}_2}(P) = 0$. Now

$$\{(p, r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-0}(\gamma) > 1 - \sigma\} = (\mathbb{N} \times \mathbb{N}) \setminus P$$

for all $\gamma > 0$ and $\sigma \in (0, 1)$. As $d^{\mathcal{I}_2}\left((\mathbb{N} \times \mathbb{N}) \setminus P\right) = 1$, so

$$d^{\mathcal{I}_2}\left(\{(p, r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-0}(\gamma) > 1 - \sigma\}\right) \neq 0.$$

As a result, 0 is an $\mathcal{I}_2$-SCP of w. □

Proposition 12.1 *When $\mathcal{I}_2 = \mathcal{I}_2^{fin} = \{A \subset \mathbb{N} \times \mathbb{N} : |A| < \infty\}$, then the conceptions $\mathcal{I}_2$-SLP and $\mathcal{I}_2$-SCP in Y w.r.t. the PN $\mathcal{N}$ coincide with the conceptions of SLP and SCP in Y w.r.t. the PN $\mathcal{N}$, respectively. So, in a PNS, the concepts of $\mathcal{I}_2$-SLP and $\mathcal{I}_2$-SCP generalize the concepts of SLP and SCP, respectively.*

Theorem 12.4 *Take $(Y, \mathcal{N}, \circ)$ as a PNS. Then, for a sequence $w = \{w_{pr}\}_{p,r \in \mathbb{N}}$ in Y, we acquire $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}} \subseteq \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} \subseteq L_w^{\mathcal{N}}$.*

Proof Take $\nu \in \Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$. Then, there is a subsequence $\{w_{p_j, r_k}\}_{j,k \in \mathbb{N}}$ of w so that

$$\lim_{j,k \to \infty} \mathcal{N}\left(w_{p_j, r_k}\right) = \nu \text{ and } d^{\mathcal{I}_2}\left(\left(p_j, r_k\right) : j, k \in \mathbb{N}\right) \neq 0.$$

Take $\gamma > 0$ and $\sigma \in (0, 1)$. As $\lim_{j,k \to \infty} \mathcal{N}\left(w_{p_j, r_k}\right) = \nu$, so

$$F = \left\{\left(p_j, r_k\right) : \mathcal{N}_{w_{p_j, r_k}-\nu}(\gamma) \leq 1 - \sigma\right\}$$

is finite. Also,

$$\left\{(p,r)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\nu}(\gamma)>1-\sigma\right\}\supset\left\{\left(p_j,r_k\right):j,k\in\mathbb{N}\right\}\setminus F$$
$$\Rightarrow T=\left\{\left(p_j,r_k\right):j,k\in\mathbb{N}\right\}\subset\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\nu}(\gamma)>1-\sigma\right\}\cup F.$$

Now, when

$$d^{\mathcal{I}_2}\left(\left\{(p,r)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\nu}(\gamma)>1-\sigma\right\}\right)=0,$$

then we obtain $d^{\mathcal{I}_2}(T)=0$, which is a contradiction. So, ν is an $\mathcal{I}_2$-*SCP* of w. As $\nu\in\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$ is arbitrary, $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}\subseteq\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$.

Now we evidence that $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}\subseteq L_w^{\mathcal{N}}$. Let $\nu\in\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$. Also, take $\gamma>0$ and $\sigma\in(0,1)$. Consider the subsequent set

$$R=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{jk}-\nu}(\gamma)>1-\sigma\right\}.$$

Then, $d^{\mathcal{I}_2}(R)\neq 0$. Thus, $R\subseteq\mathbb{N}\times\mathbb{N}$ is an infinite subset, so we get

$$R=\left\{\left(j_p,k_r\right):(j_1,k_1)<(j_2,k_2)<\ldots\right\}.$$

We acquire a subsequence $\{w\}_R$ of w that converges to ν w.r.t. PN $\mathcal{N}$. So $\nu\in L_w^{\mathcal{N}}$. As a result,

$$\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}\subseteq\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}\subseteq L_w^{\mathcal{N}}.$$

□

Theorem 12.5 *Assume* $w=\left\{w_{pr}\right\}_{p,r\in\mathbb{N}}$ *and* $t=\left\{t_{pr}\right\}_{p,r\in\mathbb{N}}$ *be sequences in* Y *such that*

$$d^{\mathcal{I}_2}\left(\left\{(p,r):w_{pr}\neq t_{pr}\right\}\right)=0.$$

Then, $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}=\Lambda_t(\mathcal{I}_2(S))_{\mathcal{N}}$ *and* $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}=\Gamma_t(\mathcal{I}_2(S))_{\mathcal{N}}$.

Proof Take $\varpi\in\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$, $\gamma>0$ and $\sigma\in(0,1)$. Then,

$$d^{\mathcal{I}_2}\left(\left\{(p,r)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\varpi}(\gamma)>1-\sigma\right\}\right)\neq 0.$$

Take $Q=\left\{(p,r):w_{pr}=t_{pr}\right\}$. Then, $d^{\mathcal{I}_2}(Q)=1$. So

$$d^{\mathcal{I}_2}\left(\left\{(p,r)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\varpi}(\gamma)>1-\sigma\right\}\cap Q\right)\neq 0.$$

As a result, $\varpi\in\Gamma_t(\mathcal{I}_2(S))_{\mathcal{N}}$. As $\varpi\in\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ is arbitrary, so $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}\subset\Gamma_t(\mathcal{I}_2(S))_{\mathcal{N}}$. Also, it can be proved $\Gamma_t(\mathcal{I}_2(S))_{\mathcal{N}}\subset\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ in a similar way. Hence, $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}=\Gamma_t(\mathcal{I}_2(S))_{\mathcal{N}}$.

Now we demonstrate that $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}} = \Lambda_t(\mathcal{I}_2(S))_{\mathcal{N}}$. Take $\varkappa \in \Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$. Then, w has an $\mathcal{I}_2$-nonthin subsequence $\left\{w_{p_j,r_k}\right\}_{j,k\in\mathbb{N}}$ which converges to $\varkappa$ w.r.t. the PN $\mathcal{N}$. Take $R = \left\{\left(p_j, r_k\right) : j, k \in \mathbb{N}\right\}$. As

$$d^{\mathcal{I}_2}\left(\left\{\left(p_j, r_k\right) : w_{p_j r_k} \neq t_{p_j r_k}\right\}\right) = 0,$$

we get

$$d^{\mathcal{I}_2}\left(\left\{\left(p_j, r_k\right) : w_{p_j r_k} = t_{p_j r_k}\right\}\right) \neq 0.$$

Therefore, we obtain an $\mathcal{I}_2$-nonthin subsequence $\{t\}_{R'}$ of $\{t\}_R$ that converges to $\varkappa$ w.r.t. the PN $\mathcal{N}$. So, $\varkappa \in \Lambda_t(\mathcal{I}_2(S))_{\mathcal{N}}$. As $\varkappa \in \Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$ is arbitrary, hence, $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}} \subset \Lambda_t(\mathcal{I}_2(S))_{\mathcal{N}}$. And also we obtain $\Lambda_t(\mathcal{I}_2(S))_{\mathcal{N}} \subset \Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$. As a result, $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}} = \Lambda_t(\mathcal{I}_2(S))_{\mathcal{N}}$. □

Example 12.6 Contemplate the PNS $(\mathbb{R}, \mathcal{N}, \circ)$. We determine $w = \left\{w_{pr}\right\}_{p,r\in\mathbb{N}}$ and $t = \left\{t_{pr}\right\}_{p,r\in\mathbb{N}}$:

$$w_{pr} = \begin{cases} 1, & \text{when } p, r \text{ are perfect squares} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$t_{pr} = \begin{cases} 2, & \text{when } p, r \text{ are perfect squares} \\ 0, & \text{otherwise.} \end{cases}$$

Take R as the set of whole perfect squares. Then, $d(R) = 0$. Contemplate $\mathcal{I}_2$ as an admissible ideal; we acquire $d^{\mathcal{I}_2}(R) = 0$. So

$$d^{\mathcal{I}_2}\left(\left\{(p, r) : w_{pr} \neq t_{pr}\right\}\right) = d^{\mathcal{I}_2}(R) = 0.$$

Obviously, $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} = \Gamma_t(\mathcal{I}_2(S))_{\mathcal{N}} = \{0\}$ and $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}} = \Lambda_t(\mathcal{I}_2(S))_{\mathcal{N}} = \{0\}$. □

Theorem 12.7 *Take $(Y, \mathcal{N}, \circ)$ as a PNS and $w = \left\{w_{pr}\right\}_{p,r\in\mathbb{N}} \in Y$. At that time, $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ is a closed subset of Y.*

Proof When $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} = \emptyset$, then it is obvious. We presume $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} \neq \emptyset$. Obviously, it is adequate to demonstrate that $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ includes whole its limit points. Assume that $\zeta \in \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$. Take $\gamma > 0$ and $\sigma \in (0, 1)$. Select $\beta \in (0, 1)$ so that $(1-\beta) \circ (1-\beta) > 1-\sigma$. Then

$$\mathcal{B}\left(\zeta, \beta, \frac{\gamma}{2}\right) \cap \left(\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} \setminus \{\zeta\}\right) \neq \emptyset.$$

Select $\alpha \in \mathcal{B}\left(\zeta, \beta, \frac{\gamma}{2}\right) \cap \left(\Gamma_w\left(\mathcal{I}_2\left(S\right)\right)_{\mathcal{N}} \diagdown \{\zeta\}\right)$. As $\alpha \in \Gamma_w\left(\mathcal{I}_2\left(S\right)\right)_{\mathcal{N}}$,

$$d^{\mathcal{I}_2}\left(\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\alpha}\left(\frac{\gamma}{2}\right) > 1-\beta\right\}\right) \neq 0.$$

Now, we denote that

$$\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\alpha}\left(\frac{\gamma}{2}\right) > 1-\beta\right\} \subset \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}\left(\gamma\right) > 1-\sigma\right\}.$$

Take

$$(p,r) \in \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\alpha}\left(\frac{\gamma}{2}\right) > 1-\beta\right\}.$$

Then, $\mathcal{N}_{w_{pr}-\alpha}\left(\frac{\gamma}{2}\right) > 1-\beta$. As $\alpha \in \mathcal{B}\left(\zeta, \beta, \frac{\gamma}{2}\right)$, $\mathcal{N}_{\zeta-\alpha}\left(\frac{\gamma}{2}\right) > 1-\beta$. So

$$\mathcal{N}_{w_{pr}-\zeta}\left(\gamma\right) \geq \mathcal{N}_{w_{pr}-\alpha}\left(\frac{\gamma}{2}\right) + \mathcal{N}_{\zeta-\alpha}\left(\frac{\gamma}{2}\right) > (1-\beta) \circ (1-\beta) > 1-\sigma.$$

As

$$(p,r) \in \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\alpha}\left(\frac{\gamma}{2}\right) > 1-\beta\right\}$$

is arbitrary, so

$$\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\alpha}\left(\frac{\gamma}{2}\right) > 1-\beta\right\} \subset \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}\left(\gamma\right) > 1-\sigma\right\}.$$

We obtain

$$d^{\mathcal{I}_2}\left(\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}\left(\gamma\right) > 1-\sigma\right\}\right) \neq 0.$$

As a result, $\zeta \in \Gamma_w\left(\mathcal{I}_2\left(S\right)\right)_{\mathcal{N}}$. $\square$

Now, utilizing (additive property for $\mathcal{I}_2$-asymptotic density zero sets) (*API(2)O*) condition, we establish some theorems.

Definition 12.4 The $\mathcal{I}_2$-asymptotic density $d^{\mathcal{I}_2}$ is named to fulfill *API(2)O* provided that, given any countable collection of sets $\{R_i\}_{i\in\mathbb{N}}$ with $d^{\mathcal{I}_2}\left(R_i\right) = 0$, for all $i \in \mathbb{N}$, there is a collection of sets $\{S_i\}_{i\in\mathbb{N}}$ under the features $|R_i \Delta S_i| < \infty$ for every $i \in \mathbb{N}$ and $d^{\mathcal{I}_2}(S = \bigcup_{i=1}^{\infty} S_i) = 0$.

Theorem 12.8 *Take $(Y, \mathcal{N}, \circ)$ as a PNS and $\mathcal{I}_2$ as an ideal in $\mathbb{N} \times \mathbb{N}$ so that $d^{\mathcal{I}_2}$ has feature API(2)O. Then, $\mathcal{I}_2 - st \lim w_{pr} = \zeta$ iff there is a subset Q of $\mathbb{N} \times \mathbb{N}$ with $d^{\mathcal{I}_2}\left(Q\right) = 1$ and $\lim_{(p,r)\in Q,\, p,r\to\infty} \mathcal{N}\left(w_{pr}\right) = \zeta$.*

Proof Take $w = \left\{w_{pr}\right\}_{p,r\in\mathbb{N}} \in Y$ and assume that $\mathcal{I}_2 - st \lim w_{pr} = \zeta$ w.r.t. PN $\mathcal{N}$. Then, for all $\gamma > 0$ and $\sigma \in (0, 1)$, we have

$$d^{\mathcal{I}_2}\left(\left\{(p,r) \in \mathbb{N}\times\mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}(\gamma) \geq 1-\sigma\right\}\right) = 0.$$

Put

$$W_1 = \left\{(p,r) \in \mathbb{N}\times\mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}(\gamma) \in \left[0, \tfrac{1}{2}\right)\right\},$$
$$W_s = \left\{(p,r) \in \mathbb{N}\times\mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}(\gamma) \in \left[1-\tfrac{1}{s}, 1-\tfrac{1}{s+1}\right)\right\}$$

for $s \geq 2$, $s \in \mathbb{N}$. Then, $\{W_s\}_{s\in\mathbb{N}}$ is a countable sequence of mutually disjoint sets with $d^{\mathcal{I}_2}(W_s) = 0$ for all $s \in \mathbb{N}$. Then, by presumption, there is a countable sequence of sets $\{Q_s\}_{s\in\mathbb{N}}$ with $|W_s \Delta Q_s| < \infty$ and $d^{\mathcal{I}_2}(Q = \bigcup_{s=1}^{\infty} Q_s) = 0$. We argue that $\lim_{(p,r)\in(\mathbb{N}\times\mathbb{N})\setminus Q, p,r\to\infty} \mathcal{N}\left(w_{pr}\right) = \zeta$. To create our argue, take $\gamma > 0$ and $\rho \in (0, 1)$. Select $j \in \mathbb{N}$ so that $\frac{1}{j+1} < \rho$. Then

$$\left\{(p,r) \in \mathbb{N}\times\mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}(\gamma) \leq 1-\rho\right\} \subset \bigcup_{s=1}^{j+1} W_s.$$

As $|W_s \Delta Q_s| < \infty$ for all $s = 1, 2, \ldots, j+1$, there is $l' \in \mathbb{N}$ so that

$$\bigcup_{s=1}^{j+1} W_s \cap \left(l', \infty\right) = \bigcup_{s=1}^{j+1} Q_s \cap \left(l', \infty\right).$$

Now, when $(p,r) \notin Q$, $p, r > l'$, then $(p,r) \notin \bigcup_{s=1}^{j+1} Q_s$. As a consequence, $(p,r) \notin \bigcup_{s=1}^{j+1} W_s$, which gives $\mathcal{N}_{w_{pr}-\zeta}(\gamma) > 1-\rho$. It finalizes the proof of the necessity section.

On the other hand, assume there is a subset $Q \subset \mathbb{N}\times\mathbb{N}$ with $d^{\mathcal{I}_2}(Q) = 1$ and $\lim_{(p,r)\in Q, p,r\to\infty} \mathcal{N}\left(w_{pr}\right) = \zeta$. We have to denote that $\mathcal{I}_2 - st \lim w_{pr} = \zeta$ w.r.t. PN $\mathcal{N}$. As Q is an infinite set, we can create $Q = \left\{\left(p_j, r_k\right) : (p_1, r_1) < (p_2, r_2) < \ldots\right\}$. Take $\gamma > 0$ and $\sigma \in (0, 1)$. As $\lim_{(p,r)\in Q, p,r\to\infty} \mathcal{N}\left(w_{pr}\right) = \zeta$, then there are j_0, $k_0 \in \mathbb{N}$ so that for all $j > j_0$, $k > k_0$, $\mathcal{N}\left(w_{p_j,r_k}\right)(\gamma) > 1-\sigma$. Consider

$$H = \left\{(p,r) \in \mathbb{N}\times\mathbb{N} : \mathcal{N}_{w_{pr}-\zeta}(\gamma) \leq 1-\sigma\right\}.$$

Then,

$$H \subset (\mathbb{N} \times \mathbb{N}) \setminus \{(p_j, r_k) : j > j_0, k > k_0\}.$$

As

$$d^{\mathcal{I}_2}(Q) = 1, d^{\mathcal{I}_2}\left((\mathbb{N} \times \mathbb{N}) \setminus \{(p_j, r_k) : j > j_0, k > k_0\}\right) = 0.$$

Thus, $d^{\mathcal{I}_2}(H) = 0$. Henceforth, we get $\mathcal{I}_2 - st \lim w_{pr} = \zeta$ w.r.t. PN $\mathcal{N}$. □

Corollary 12.1 *Take $(Y, \mathcal{N}, \circ)$ as a PNS and $w = \{w_{pr}\}_{p,r\in\mathbb{N}} \in Y$. Presume that $\mathcal{I}_2$ is an ideal in $\mathbb{N} \times \mathbb{N}$ so that $d^{\mathcal{I}_2}$ has feature API(2)O. When $\mathcal{I}_2 - st \lim_{p,r\to\infty} \mathcal{N}(w_{pr}) = \eta$, then $\eta \in \Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$.*

Theorem 12.9 *Take $(Y, \mathcal{N}, \circ)$ as a PNS and $w = \{w_{pr}\}_{p,r\in\mathbb{N}} \in Y$. Presume that $\mathcal{I}_2$ is an ideal in $\mathbb{N}\times\mathbb{N}$ so that $d^{\mathcal{I}_2}$ has feature API(2)O. When $\mathcal{I}_2 - st \lim_{p,r\to\infty} \mathcal{N}(w_{pr}) = \eta$, then $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}} = \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} = \{\eta\}$.*

Proof According to Corollary 12.1, we get $\eta \in \Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$. Let $\gamma \in \Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}}$ be an $\mathcal{I}_2$-SLP such that $\eta \neq \gamma$. Then, there are two subsets

$$H = \{(p_j, r_k) : (p_1, r_1) < (p_2, r_2) < \ldots\} \subset \mathbb{N} \times \mathbb{N}$$

and

$$G = \{(j_p, k_r) : (j_1, k_1) < (j_2, k_2) < \ldots\} \subset \mathbb{N} \times \mathbb{N}$$

so that

$$d^{\mathcal{I}_2}(H) \neq 0, \lim_{j,k\to\infty} \mathcal{N}(w_{p_j, r_k}) = \eta$$

and

$$d^{\mathcal{I}_2}(G) \neq 0, \lim_{p,r\to\infty} \mathcal{N}(w_{j_p, k_r}) = \gamma.$$

Take $m > 0$ and $\sigma \in (0, 1)$. Select $\beta \in (0, 1)$ so that $(1 - \beta) \circ (1 - \beta) > 1 - \sigma$. As the subsequence $\{w\}_G$ of w converges to γ w.r.t. PN $\mathcal{N}$, there are $p, r > K_0$ so that $\mathcal{N}_{w_{j_p,k_r} - \gamma}\left(\frac{m}{2}\right) > 1 - \beta$. Hence,

$$\begin{aligned} K &= \left\{(j_p, k_r) \in G : \mathcal{N}_{w_{j_p,k_r} - \gamma}\left(\tfrac{m}{2}\right) \leq 1 - \beta\right\} \\ &\subset \{(j_p, k_r) \in G : (j_1, k_1) < (j_2, k_2) < \ldots < (j_{K_0}, k_{K_0})\}. \end{aligned}$$

Let

$$L=\left\{\left(j_p,k_r\right)\in G:\mathcal{N}_{w_{j_p,k_r}-\gamma}\left(\frac{m}{2}\right)>1-\beta\right\}.$$

As $\mathcal{I}_2$ is an admissible ideal and $K\in\mathcal{I}_2$, we acquire

$$d^{\mathcal{I}_2}\left(L\right)\neq 0. \tag{12.1}$$

Again, $\mathcal{I}_2-st\lim_{p,r\to\infty}\mathcal{N}\left(w_{pr}\right)=\eta$, which gives $d^{\mathcal{I}_2}\left(F\right)=0$, where

$$F=\left\{(p,r)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\eta}\left(\frac{m}{2}\right)\leq 1-\beta\right\}.$$

As a result, $d^{\mathcal{I}_2}\left((\mathbb{N}\times\mathbb{N})\diagdown F\right)\neq 0$.

Since $\eta\neq\gamma$, we get $L\cap\left((\mathbb{N}\times\mathbb{N})\diagdown F\right)=\emptyset$; otherwise, for $(p,r)\in L\cap\left((\mathbb{N}\times\mathbb{N})\diagdown F\right)$

$$\mathcal{N}_{\eta-\gamma}\left(m\right)\geq\mathcal{N}_{w_{pr}-\gamma}\left(\frac{m}{2}\right)\circ\mathcal{N}_{w_{pr}-\eta}\left(\frac{m}{2}\right)>(1-\beta)\circ(1-\beta)>1-\sigma.$$

As $\sigma>0$ is arbitrary and $\mathcal{N}_{\eta-\gamma}\left(m\right)>1-\sigma$, we acquire $\mathcal{N}_{\eta-\gamma}\left(m\right)=1$, for each $m>0$, which means $\eta=\gamma$.

So $L\subset F$. As $d^{\mathcal{I}_2}\left(F\right)=0$, we get $d^{\mathcal{I}_2}\left(L\right)=0$, which contradicts (12.1). Therefore, $\Lambda_w\left(\mathcal{I}_2\left(S\right)\right)_{\mathcal{N}}=\{\eta\}$.

Again take $\eta,\gamma\in\Gamma_w\left(\mathcal{I}_2\left(S\right)\right)_{\mathcal{N}}$ and $\eta\neq\gamma$. Take $m>0$ and $\sigma\in(0,1)$. Select $\beta\in(0,1)$ so that $(1-\beta)\circ(1-\beta)>1-\sigma$. Then, we get

$$P=\left\{(p,r)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\eta}\left(\frac{m}{2}\right)>1-\beta\right\},d^{\mathcal{I}_2}\left(P\right)\neq 0 \tag{12.2}$$

and

$$R=\left\{(p,r)\in\mathbb{N}\times\mathbb{N}:\mathcal{N}_{w_{pr}-\gamma}\left(\frac{m}{2}\right)>1-\beta\right\},d^{\mathcal{I}_2}\left(R\right)\neq 0 \tag{12.3}$$

As $\eta\neq\gamma$, we get $P\cap R=\emptyset$; otherwise, for $(p,r)\in P\cap R$

$$\mathcal{N}_{\eta-\gamma}\left(m\right)\geq\mathcal{N}_{w_{pr}-\eta}\left(\frac{m}{2}\right)\circ\mathcal{N}_{w_{pr}-\gamma}\left(\frac{m}{2}\right)>(1-\beta)\circ(1-\beta)>1-\sigma.$$

Considering $\sigma>0$ is arbitrary and $\mathcal{N}_{\eta-\gamma}\left(m\right)>1-\sigma$, we acquire $\mathcal{N}_{\eta-\gamma}\left(m\right)=1$, for each $m>0$, which means $\eta=\gamma$.

So $R\subset(\mathbb{N}\times\mathbb{N})\diagdown P$ and $\mathcal{I}_2-st\lim_{p,r\to\infty}\mathcal{N}\left(w_{pr}\right)=\eta$, which gives $d^{\mathcal{I}_2}\left((\mathbb{N}\times\mathbb{N})\diagdown P\right)=0$. So $d^{\mathcal{I}_2}\left(R\right)=0$, which contradicts (12.3). Therefore, $\Gamma_w\left(\mathcal{I}_2\left(S\right)\right)_{\mathcal{N}}=\{\eta\}$. □

Example 12.10 Contemplate the ideal $\mathcal{I}_2 = \mathcal{I}_2^{fin}$ of each finite subset of $\mathbb{N} \times \mathbb{N}$. Here, $\mathcal{I}_2$ is an admissible ideal and $d^{\mathcal{I}_2}$ has feature API(2)O. Contemplate the PNS $(\mathbb{R}, \mathcal{N}, \circ)$. We determine a sequence $w = \{w_{pr}\}_{p,r\in\mathbb{N}}$:

$$w_{pr} = \begin{cases} 1, & \text{when } p, r \text{ are prime squares} \\ 0, & \text{otherwise.} \end{cases}$$

Take R as the set of whole prime numbers. Then $d(R) = 0$. When we take $\mathcal{I}_2 = \mathcal{I}_2^{fin}$, then $d^{\mathcal{I}_2}(R) = 0$. Obviously, $\mathcal{I}_2 - st \lim_{p,r\to\infty} \mathcal{N}(w_{pr}) = 0$ and $\Lambda_w(\mathcal{I}_2(S))_{\mathcal{N}} = \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} = \{0\}$. □

Take $(Y, \mathcal{N}, \circ)$ as a PNS. Now, we investigate the norm topology on Y as follows: A set $\mathcal{V} \subset Y$ is named to be open provided that for all $w = \{w_{pr}\}_{p,r\in\mathbb{N}} \in \mathcal{V}$, there is a $r \in (0, 1)$ so that for all $m > 0$, $\mathcal{B}(w, r, m) \subset \mathcal{V}$.

Definition 12.5 ([37]) Take $(Y, \mathcal{N}, \circ)$ as a PNS. When Y has countable basis, then Y is named to be second countable PNS (SCPNS) under its norm topology.

Lemma 12.1 ([37]) *Take $(Y, \mathcal{N}, \circ)$ as a SCPNS. We say that each subspace of Y is SCPNS under subspace norm topology.*

Lemma 12.2 ([37]) *Take $(Y, \mathcal{N}, \circ)$ as a SCPNS. Then, all open covering of Y includes a countable subcollection covering Y.*

Theorem 12.11 *Presume that $(Y, \mathcal{N}, \circ)$ is a SCPNS. Consider that $\mathcal{I}_2$ is an admissible ideal and $d^{\mathcal{I}_2}$ has feature API(2)O. Then, for any sequence $w = \{w_{pr}\}_{p,r\in\mathbb{N}} \in Y$, there is a sequence $t = \{t_{pr}\}_{p,r\in\mathbb{N}} \in Y$ so that $L_t^{\mathcal{N}} = \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ and $d^{\mathcal{I}_2}(\{(p,r) \in \mathbb{N} \times \mathbb{N} : w_{pr} \neq t_{pr}\}) = 0$.*

Proof At the beginning, we demonstrate that $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} \subset L_w^{\mathcal{N}}$. Let $\eta \in \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$. Take $m > 0$ and $\sigma \in (0, 1)$. Then, we get

$$d^{\mathcal{I}_2}\left(\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(m) > 1 - \sigma\}\right) \neq 0.$$

We argue that

$$d\left(\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(m) > 1 - \sigma\}\right) \neq 0.$$

If possible, assume that

$$d\left(\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(m) > 1 - \sigma\}\right) = 0.$$

Then, $\lim_{m,n\to\infty} \frac{|S(m,n)|}{mn} = 0$, where

$$S = \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(m) > 1-\sigma\right\}.$$

As $\mathcal{I}_2$ is admissible ideal, so $\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{|S(m,n)|}{mn} = 0$, where

$$S = \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(m) > 1-\sigma\right\}.$$

Thus,

$$d^{\mathcal{I}_2}\left(\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(m) > 1-\sigma\right\}\right) = 0,$$

which is contradiction. Hence,

$$d\left(\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : \mathcal{N}_{w_{pr}-\eta}(m) > 1-\sigma\right\}\right) \neq 0.$$

Hence, η is *SCP* of w and so a limit point of w. As a result, we acquire $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} \subset L_w^{\mathcal{N}}$. When $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}} = L_w^{\mathcal{N}}$, we can take $w = \left\{w_{pr}\right\}_{p,r\in\mathbb{N}} = \left\{t_{pr}\right\}_{p,r\in\mathbb{N}} = t$, and we are proved. Take $\Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ as a proper subset of $L_w^{\mathcal{N}}$. Take $\gamma \in L_w^{\mathcal{N}} \setminus \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$. Select a ball

$$\mathcal{B}\left(\gamma, r_\gamma, m\right) = \left\{\kappa \in Y : \mathcal{N}_{\gamma-\kappa}(m) > 1 - r_\gamma\right\}$$

with radius $r_\gamma \in (0,1)$ and the center at γ so that

$$d^{\mathcal{I}_2}\left(\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : w_{pr} \in \mathcal{B}\left(\gamma, r_\gamma, m\right)\right\}\right) = 0.$$

We say that the collection of all such $\mathcal{B}\left(\gamma, r_\gamma, m\right)$'s is an open cover for $L_w^{\mathcal{N}} \setminus \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$. As $L_w^{\mathcal{N}} \setminus \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$ is a subspace of a *SCS* Y, it is clear that it is *SC*. Then, there is a countable subcover $\{\mathcal{B}(\gamma_i, r_i, m)\}_{i\in\mathbb{N}}$ of

$$\left\{\mathcal{B}\left(\gamma, r_\gamma, m\right) : \gamma \in L_w^{\mathcal{N}} \setminus \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}\right\}$$

for $L_w^{\mathcal{N}} \setminus \Gamma_w(\mathcal{I}_2(S))_{\mathcal{N}}$. As each γ_i is a limit point of w and

$$d^{\mathcal{I}_2}\left(\left\{(p,r) \in \mathbb{N} \times \mathbb{N} : w_{pr} \in \mathcal{B}(\gamma_i, r_i, m)\right\}\right) = 0,$$

as a result, all $\mathcal{B}(\gamma_i, r_i, m)$ includes an $\mathcal{I}_2$-thin subspace of w. Take

$$J_1 = \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : w_{pr} \in \mathcal{B}(\gamma_1, r_1, m)\right\},$$
$$J_k = \left\{(p,r) \in \mathbb{N} \times \mathbb{N} : w_{pr} \in \mathcal{B}(\gamma_k, r_k, m)\right\} \setminus (J_1 \cup J_1 \cup \ldots \cup J_{k-1}), \forall k \geq 2, k \in \mathbb{N}.$$

Then $\{J_k\}_{k\in\mathbb{N}}$ is a countable collection of mutually disjoint sets with $d^{\mathcal{I}_2}(J_k) = 0$, $\forall k \geq 2$. As $d^{\mathcal{I}_2}$ has feature *API(2)O*, there is a countable collection of sets $\{T_k\}_{k\in\mathbb{N}}$

so that $|J_k \Delta T_k| < \infty$ for all $k \in \mathbb{N}$ and $d^{\mathcal{I}_2}\left(T = \bigcup_{k=1}^{\infty} T_k\right) = 0$. Then $J_k \setminus T$ is finite and hence

$$\left\{(p, r) \in \mathbb{N} \times \mathbb{N} : (p, r) \in J_{\gamma_k}\right\} \setminus T$$

is finite for all $k \in \mathbb{N}$. Take

$$(\mathbb{N} \times \mathbb{N}) \setminus T = \left\{(j_1, k_1) < (j_2, k_2) < \ldots < \left(j_p, k_r\right) < \ldots\right\}$$

and we determine a sequence $t = \left\{t_{pr}\right\}_{p,r \in \mathbb{N}}$:

$$t_{pr} = \begin{cases} w_{j_p k_r}, & \text{when } (p, r) \in T \\ w_{pr}, & \text{when } (p, r) \in \mathbb{N} \times \mathbb{N} \setminus T. \end{cases}$$

Clearly, when

$$d^{\mathcal{I}_2}\left(\left\{(p, r) \in \mathbb{N} \times \mathbb{N} : w_{pr} \neq t_{pr}\right\} (\subset T)\right) = 0$$

satisfies, we get $\Gamma_w (\mathcal{I}_2 (S))_{\mathcal{N}} = \Gamma_t (\mathcal{I}_2 (S))_{\mathcal{N}}$. Now, we prove that $L_t^{\mathcal{N}} = \Gamma_t (\mathcal{I}_2 (S))_{\mathcal{N}}$. If at all possible, take $\Gamma_t (\mathcal{I}_2 (S))_{\mathcal{N}} \subsetneq L_t^{\mathcal{N}}$ and $l \in L_t^{\mathcal{N}} \setminus \Gamma_t (\mathcal{I}_2 (S))_{\mathcal{N}}$. Then, there is a subsequence of t converging to l. Clarify that the subsequence has to be $\mathcal{I}_2$-thin; however, $\{t\}_T$ has no limit points. So no such l can exist. Therefore, $L_t^{\mathcal{N}} = \Gamma_t (\mathcal{I}_2 (S))_{\mathcal{N}}$. As a result, $L_t^{\mathcal{N}} = \Gamma_w (\mathcal{I}_2 (S))_{\mathcal{N}}$. □

12.3 Conclusion

In this chapter, we examine the notions of $\mathcal{I}_2$-statistical limit points and $\mathcal{I}_2$-statistical cluster points in probabilistic normed spaces, investigate their relationship, and make some observations about these classes. These results unify and generalize the existing results. It may attract the forthcoming researchers in this direction.

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Chapter 13
Evaluation of Integral Transforms in Terms of Humbert and Lauricella Functions and Their Applications

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13.1 Introduction and Preliminaries

The solution of many problems of scientific areas such as mathematics, physics, optimization, cybernetic technology, biology, etc. is reduced to the evaluation of integral transforms involving special functions such as Bessel, Whittaker, and Kummer functions (see [2–5, 15, 17]). The introduction of these integral transforms has initiated a great interest in mathematical physics and its applications to laser physics and linear or non-linear optics. Earlier, we have published a series of papers in laser physics (see [6–12]), and we have elaborated some applications of these theories in this field. Recently, several authors have studied the generation of new laser beams (see [2, 18, 20–22, 24, 25]) and their propagation through some special optical elements such as axicon, opaque disk, free space, ABCD optical system, photonic crystals, chiral medium, turbulent and oceanic atmospheres, maritime turbulence and biological tissue.

In this present work, motivated by some recent studies in laser physics, we will derive several integral transforms involving special functions such as Exponential, Bessel, modified Bessel, Whittaker, Kummer, parabolic cylindrical functions and Laguerre and Hermite polynomials. We will evaluate the considered integral transforms in terms of Humbert and Lauricella hypergeometric functions. To our knowledge, the findings of the actual study have not been investigated previously.

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_13

In the following, we recall some definitions as the hypergeometric ${}_pF_q$ given by (see [14, 19, 23])

$$ {}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots\left(\alpha_p\right)_n}{(\beta_1)_n\ldots\left(\beta_q\right)_n}\frac{z^n}{n!}. \tag{13.1}$$

Here, we assume that $\beta_j\neq 0,-1,-2,\ldots;\ j=1,\ldots,q$.

In (13.1), $(\lambda)_\nu$ is the Pochhammer symbol defined by (see [23])

$$(\lambda)_\nu=\frac{\Gamma\left(\lambda+\nu\right)}{\Gamma\left(\lambda\right)}, \tag{13.2}$$

where Γ denotes the gamma function.

From (13.1), we have the following expression:

$$ {}_1F_1\left(a;c;z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}, \tag{13.3}$$

which is known as Kummer's series or the confluent hypergeometric series, which obeys to the following relationship so-called Kummer's first formula:

$$ {}_1F_1\left(\alpha;\beta;z\right)=e^{z}\,{}_1F_1\left(\beta-\alpha;\beta;-z\right). \tag{13.4}$$

We require also the following well-known definition of Humbert's double hypergeometric function (see [23]):

$$\Psi_2\left[\alpha;\gamma,\gamma';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{13.5}$$

where $|x|<\infty$ and $|y|<\infty$.

The Lauricella function of n variables $F_A^{(n)}$, defined by (see [23])

$$\begin{aligned}&F_A^{(n)}\left[a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right]\\&=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},\end{aligned} \tag{13.6}$$

with $|x_1|+\ldots+|x_n|<1$. The denominator parameters of the above functions are neither zero or negative integers.

We deduce from this last equation the triple hypergeometric function $F_A^{(3)}$ given by

$$
\begin{aligned}
&F_A^{(3)}\left[a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z\right] \\
&= \sum_{l,m,n=0}^{\infty} \frac{(a)_{l+m+n}(b_1)_l(b_2)_m(b_3)_n}{(c_1)_l(c_2)_m(c_3)_n} \frac{x^l}{l!} \frac{y^m}{m!} \frac{z^n}{n!},
\end{aligned} \tag{13.7}
$$

with $|x| + |y| + |z| < 1$.

The interesting particular case of (13.7) are as follows:

$$
F_A^{(3)}\left[a, b, -, -; c_1, c_2, c_3; x, y, z\right] = \sum_{l,m,n=0}^{\infty} \frac{(a)_{l+m+n}(b)_l}{(c_1)_l(c_2)_m(c_3)_n} \frac{x^l}{l!} \frac{y^m}{m!} \frac{z^n}{n!}. \tag{13.8}
$$

With the help of (13.3), Whittaker function $M_{s,\xi}$ was established in the Whittaker theory of the confluent hypergeometric function as (see [23])

$$
M_{s,\xi}(z) = z^{\xi+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) {}_1F_1\left(\frac{1}{2} + \xi - s; 2\xi + 1; z\right). \tag{13.9}
$$

We recall also some simple special cases of the Whittaker function by following the notation of Buchholz [14], which are deduced from (13.3) and (13.9) as follows:

$$
M_{\mp r, -r-\frac{1}{2}}(z) = z^{-r} \exp\left(\mp\frac{z}{2}\right), \tag{13.10}
$$

$$
M_{0,\frac{1}{2}}(z) = 2\sinh\left(\frac{z}{2}\right), \tag{13.11}
$$

$$
M_{a+\frac{1}{4},-\frac{1}{4}}\left(z^2\right) = (-1)^a \frac{a!}{(2a)!} \exp\left(-\frac{z^2}{2}\right) \sqrt{z} H_{2a}(z), \tag{13.12}
$$

$$
M_{a+\frac{3}{4},\frac{1}{4}}\left(z^2\right) = \frac{(-1)^a}{2} \frac{a!}{(2a+1)!} \exp\left(-\frac{z^2}{2}\right) \sqrt{z} H_{2a+1}(z), \tag{13.13}
$$

$$
M_{\pm\left(b+\frac{1+a}{2}\right),\frac{a}{2}}(z) = \frac{b!}{(a+1)_b} z^{\frac{1+a}{2}} \exp\left(\pm\frac{z}{2}\right) L_b^{(a)}(\mp z), \tag{13.14}
$$

$$
M_{0,r}(z) = 2^{2r}\Gamma(r+1)\sqrt{z} I_r\left(\frac{z}{2}\right), \tag{13.15}
$$

$$
M_{r-\frac{1}{2},r}(z) = 2r \exp\left(\frac{z}{2}\right) z^{\frac{1}{2}-r} \gamma(2r, z), \tag{13.16}
$$

$$
M_{-\frac{1}{4},\frac{1}{4}}\left(z^2\right) = \frac{\exp\left(z^2/2\right)}{2} \sqrt{\pi z}\, erf(z), \tag{13.17}
$$

and

$$M_{\pm\left(\frac{1}{2}-r\right),r}(z) = z^{r+\frac{1}{2}} \exp\left(\pm\frac{z}{2}\right) {}_1F_1(2r; 2r+1; \mp z). \tag{13.18}$$

In the above equations, $L_b^{(a)}$, H_a, and J_μ are the generalized Laguerre polynomial, the Hermite polynomial and the Bessel function of the first kind respectively. The functions erf and γ are known as error and incomplete Gamma functions.

We recall here the classical Bessel function J_ν of the first order expressed as (see [23])

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{\left(-z^2/4\right)^m}{m!} \frac{1}{\Gamma(\nu+1+m)}; \ \forall z \in \mathbb{C}\backslash(-\infty, 0), \tag{13.19}$$

and the relation between the modified Bessel I_ν and J_ν is

$$I_\nu(z) = i^{-\nu} J_\nu(iz). \tag{13.20}$$

13.2 Main Results

In this section, we establish two integral transforms involving Bessel functions and the product of Bessel and Whittaker functions. The used weight in the integrand of our transformation is $e^{-p\rho^2+2q\rho}$ with $\Re(p) > 0$. We establish some closed forms of integral transforms in terms of Humbert and Lauricella functions of two and three variables.

Theorem 13.1 *For $\Re(p) > 0$, and $\Re(\mu) > -1$, the undermentioned integral transform holds true:*

$$\int_0^\infty \rho^\mu J_\nu(\chi\rho) e^{-p\rho^2+2q\rho} d\rho$$
$$= I_0 \left\{\Gamma(\alpha)\Psi_2\left[\alpha; \nu+1, \frac{1}{2}; x, y\right] + \frac{2q}{\sqrt{p}}\Gamma(\beta)\Psi_2\left[\beta; \nu+1, \frac{3}{2}; x, y\right]\right\}, \tag{13.21}$$

where

$$I_0 = \frac{1}{2p^{\frac{\mu+1}{2}}} \frac{\left(\frac{\chi}{2\sqrt{p}}\right)^\nu}{\nu!}, \tag{13.22}$$

$$x = -\frac{\chi^2}{4p}, \quad y = \frac{q^2}{p}\ , \tag{13.23}$$

$$\alpha = \frac{\mu+\nu+1}{2} \ \textit{and} \ \beta = \frac{\mu+\nu}{2} + 1\ . \tag{13.24}$$

Proof Designating the left-hand side of (13.21) by I and by the use of the relation $e^x = chx + shx$ and by substituting the expansion of Bessel function of J_ν given by (13.19), we get after some simplifications the following expression

$$I = (\chi/2)^\nu \sum_{k=0}^{\infty} \frac{\left(-\chi^2/4\right)^k}{k!\Gamma(\nu+k+1)} I_k, \tag{13.25}$$

where

$$I_k = \int_0^\infty \rho^{\mu+\nu+2k} e^{-p\rho^2} ch(2q\rho)d\rho + \int_0^\infty \rho^{\mu+\nu+2k} e^{-p\rho^2} sh(2q\rho)d\rho. \tag{13.26}$$

By using the following well known identities (see [16])

$$\int_0^\infty x^{2\alpha-1} e^{-\beta x^2} sh(\gamma x)dx = \frac{\Gamma(2\alpha)}{2(2\beta)^\alpha} e^{\gamma^2/8\beta} \left[D_{-2\alpha}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) - D_{-2\alpha}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right], \tag{13.27}$$

with $\Re(\alpha) > -\frac{1}{2},\ \Re(\beta) > 0$,

$$\int_0^\infty x^{2\alpha-1} e^{-\beta x^2} ch(\gamma x)dx = \frac{\Gamma(2\alpha)}{2(2\beta)^\alpha} e^{\gamma^2/8\beta} \left[D_{-2\alpha}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) + D_{-2\alpha}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right], \tag{13.28}$$

with $\Re(\alpha) > 0,\ \Re(\beta) > 0$,

Equation (13.25) becomes

$$I = \frac{(\chi/2)^\nu}{(2p)^{\frac{\mu+\nu+1}{2}}} e^{\frac{q^2}{2p}} \sum_{k=0}^{\infty} \frac{\left(-\chi^2/8p\right)^k}{k!} \frac{\Gamma(\mu+\nu+1+2k)}{\Gamma(\nu+1+k)} D_{-(\mu+\nu+1+2k)}\left(-\sqrt{\frac{2}{p}}q\right). \tag{13.29}$$

In these last equations, we have used $D_{-2\delta}$, the parabolic cylinder function, which is expressed in terms of Kummer's function as

$$D_{-2\delta}(z) = \frac{\sqrt{\pi}}{2^\delta} e^{-\frac{z^2}{4}} \left[\frac{1}{\Gamma\left(\frac{1}{2}+\delta\right)}\, {}_1F_1\left(\delta; \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma(\delta)}\, {}_1F_1\left(\frac{1}{2}+\delta; \frac{3}{2}; \frac{z^2}{2}\right)\right]. \tag{13.30}$$

This last equation yields

$$I=\frac{\sqrt{\pi}}{(4p)^{\frac{\mu+1}{2}}}\left(\chi/_{4\sqrt{p}}\right)^{\nu}\left\{\sum_{k=0}^{\infty}\frac{\left(-\chi^{2}/_{16p}\right)^{k}}{k!\Gamma\left(\nu+1+k\right)}\varepsilon_{1}^{k}{}_{1}F_{1}\left(\frac{\mu+\nu+1}{2}+k;\frac{1}{2};\frac{q^{2}}{2}\right)\right.$$
$$\left.+\frac{2q}{\sqrt{p}}\sum_{k=0}^{\infty}\frac{\left(-\chi^{2}/_{16p}\right)^{k}}{k!\Gamma\left(\nu+1+k\right)}\varepsilon_{2}^{k}{}_{1}F_{1}\left(\frac{\mu+\nu}{2}+1+k;\frac{3}{2};\frac{q^{2}}{2}\right)\right\}, \tag{13.31}$$

where

$$\varepsilon_{1}^{k}=\frac{\Gamma\left(\mu+\nu+1+2k\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1+k\right)} \tag{13.32}$$

and

$$\varepsilon_{2}^{k}=\frac{\Gamma\left(\mu+\nu+1+2k\right)}{\Gamma\left(\frac{\mu+\nu+1}{2}+k\right)}. \tag{13.33}$$

To evaluate the expression of ε_1^k and ε_2^k, we use the following identities (see [23]):

$$(\lambda+m)_n=\frac{(\lambda)_{m+n}}{(\lambda)_m}, \tag{13.34}$$

and

$$\Gamma\left(2z\right)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right),\quad z\neq 0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ldots \tag{13.35}$$

By the use of (13.32), (13.33), (13.34), and (13.35), (13.31) is expressed in terms of the Humbert function of two variables ψ_2, and one finds (13.21) easily. This completes the proof of our first result. □

Theorem 13.2 *For* $\Re(p)>0, \Re(\mu)>-1$*, and* $\frac{|\chi|^2}{4}+|q|^2<|p|$*, the undermentioned integral transform holds true:*

$$\int_0^{\infty}\rho^{\mu}J_{\nu}(\chi\rho)M_{s,\xi}(2\gamma\rho^2)e^{-p\rho^2+2q\rho}d\rho$$
$$=K_0\left\{\Gamma\left(\alpha_K\right)F_A^{(3)}\left[\alpha_K,\frac{1}{2}+\xi-s,\_,\_;2\xi+1,\nu+1,\frac{1}{2};x,y,z\right]\right.$$

$$+\frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\beta_K\right)F_A^{(3)}\left[\beta_K,\frac{1}{2}+\xi-s,\_,\_;2\xi+1,\nu+1,\frac{3}{2};x,y,z\right]\Bigg\},\tag{13.36}$$

where

$$K_0=\frac{(2\gamma)^{\xi+\frac{1}{2}}}{2(p+\gamma)^{\alpha_K}}\frac{\left(\frac{\chi}{2}\right)^{\nu}}{\nu!},\tag{13.37}$$

$$x=\frac{2\gamma}{(p+\gamma)},\quad y=-\frac{\chi^2}{4\,(p+\gamma)},\quad z=\frac{q^2}{(p+\gamma)},\tag{13.38}$$

$$\alpha_K=\frac{\mu+\nu}{2}+\xi+1\ \textit{and}\ \beta_K=\frac{\mu+\nu}{2}+\xi+\frac{3}{2}.\tag{13.39}$$

Proof With the help of the expression of the Whittaker function

$$M_{s,\,\xi}(2\gamma\rho^2)=(2\gamma)^{\xi+\frac{1}{2}}\sum_{l=0}^{\infty}\frac{\left(\frac{1}{2}+\xi-s\right)_l}{(2\xi+1)_l}\frac{(2\gamma)^l}{l!}e^{-\gamma\rho^2}\rho^{2l+2\xi+1},\tag{13.40}$$

and by naming the left-hand side of (13.36) by K, we obtain

$$K=(2\gamma)^{\xi+\frac{1}{2}}\sum_{l=0}^{\infty}\frac{\left(\frac{1}{2}+\xi-s\right)_l}{(2\xi+1)_l}\frac{(2\gamma)^l}{l!}K_l,\tag{13.41}$$

where

$$K_l=\int_0^{\infty}\rho^{\mu+2l+2\zeta+1}J_\nu(\chi\rho)e^{-(p+\gamma)\rho^2+2q\rho}d\rho.\tag{13.42}$$

By using Theorem 13.1, (13.42) can be rearranged as

$$K_l=\frac{1}{2(p+\gamma)^{\frac{\mu}{2}+l+\xi+1}}\frac{\left(\frac{\chi}{2\sqrt{p+\gamma}}\right)^{\nu}}{\nu!}\Bigg\{\Gamma\left(\alpha_K+l\right)\Psi_2\left[\alpha_K+l;\nu+1,\frac{1}{2};y,z\right]$$
$$+\frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\beta_K+l\right)\Psi_2\left[\beta_K+l;\nu+1,\frac{3}{2};y,z\right]\Bigg\},\tag{13.43}$$

where the variables x, y, and z are given by (13.38).

Employing this last equation in (13.41), we get

$$K = K_0\left\{ \sum_{l,m,n=0}^{\infty} \frac{\left(\frac{1}{2}+\xi-s\right)_l \Gamma\left(\alpha_K+l\right)\left(\frac{\mu+\nu}{2}+\xi+1+l\right)_{m+n}}{(2\xi+1)_l(\nu+1)_m\left(\frac{1}{2}\right)_n} \frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!} \right.$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \sum_{l,m,n=0}^{\infty} \frac{\left(\frac{1}{2}+\xi-s\right)_l \Gamma\left(\beta_K+l\right)\left(\frac{\mu+\nu+1}{2}+\xi+l+1\right)_{m+n}}{(2\xi+1)_l(\nu+1)_m\left(\frac{3}{2}\right)_n} \frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!} \right\}. \tag{13.44}$$

Now with the help of the following identity

$$(\lambda)_{l+m+n} = (\lambda+l)_{m+n}(\lambda)_l, \tag{13.45}$$

and the definition of Lauricella's hypergeometric function of three variables denoted by $F_A^{(3)}$, (13.36) is proved. This completes the proof of our second result. □

13.3 Special Cases

In this section, we will investigate several special cases of our main results given by (13.21) and (13.36) corresponding to several particular parameters of the classical Bessel function. For the following corollaries, we will use (13.23) to evaluate the two variables of the Humbert function and (13.38) for the three variables of Lauricella function.

13.3.1 Case of $\mu = -\nu = \frac{1}{2}$

In this case, we use the following well-known identity (see [16]):

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos(z). \tag{13.46}$$

Corollary 13.1 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \cos(\chi\rho) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{1}{2\sqrt{p}} \left\{ \sqrt{\pi}\Psi_2\left[\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; x, y\right] + \frac{2q}{\sqrt{p}} \Psi_2\left[1; \frac{1}{2}, \frac{3}{2}; x, y\right] \right\}. \tag{13.47}$$

Under the conditions $\Re(p) > 0$ *and by using Theorem 13.1, we arrive at* (13.47).

Corollary 13.2 *The undermentioned integral transform holds true:*

$$\int_0^\infty cos(\chi\rho) M_{s,\xi}(2\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$
$$= (2\gamma)^{\xi+1} \frac{1}{2(p+\gamma)^{\xi+1}} \left\{ \Gamma(\xi+1) F_A^{(3)} \left[\xi+1, \frac{1}{2}+\xi-s, \_, \_; 2\xi+1, \frac{1}{2}, \frac{1}{2}; x, y, z \right] \right.$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\xi+\frac{3}{2}\right) F_A^{(3)} \left[\xi+\frac{3}{2}, \frac{1}{2}+\xi-s, \_, \_; 2\xi+1, \frac{1}{2}, \frac{3}{2}; x, y, z \right] \right\}. \tag{13.48}$$

Under the conditions $\Re(p) > 0$ *and* $\frac{|\chi|^2}{4} + |q|^2 < |p|$ *and by using Theorem 13.2, we arrive at* (13.48).

13.3.2 *Case of* $\mu = \nu = \frac{1}{2}$

In this case, we use the following well-known identity (see [16]):

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z). \tag{13.49}$$

Corollary 13.3 *The undermentioned integral transform holds true:*

$$\int_0^\infty \sin(\chi\rho) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{\chi}{2p} \left\{ \Psi_2 \left[1; \frac{3}{2}, \frac{1}{2}; x, y \right] + q\sqrt{\frac{\pi}{p}} \Psi_2 \left[\frac{3}{2}; \frac{3}{2}, \frac{3}{2}; x, y \right] \right\}. \tag{13.50}$$

Under the conditions $\Re(p) > 0$ *and by using Theorem 13.1, we arrive at* (13.50).

Corollary 13.4 *The undermentioned integral transform holds true:*

$$\int_0^\infty sin(\chi\rho) M_{s,\zeta}(2\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$
$$= (2\gamma)^{\zeta+\frac{1}{2}} \frac{1}{2(p+\gamma)^{\zeta+\frac{3}{2}}} \left\{ \Gamma\left(\zeta+\frac{3}{2}\right) \right.$$

$$\times F_A^{(3)}\left[\zeta+\frac{3}{2},\frac{1}{2}+\zeta-s,\_,\_;2\zeta+1,\frac{3}{2},\frac{1}{2};x,y,z\right]$$
$$+\frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\zeta+2\right)F_A^{(3)}\left[\zeta+2,\frac{1}{2}+\zeta-s,\_,\_;2\zeta+1,\frac{3}{2},\frac{3}{2};x,y,z\right]\Bigg\}. \tag{13.51}$$

Under the conditions $\Re(p)>0$ *and* $\frac{|\chi|^2}{4}+|q|^2<|p|$ *and by using Theorem 13.2, we arrive at* (13.51).

In the following, we present some new integral transforms involving Exponential function, Sine hyperbolic function, Hermite polynomial, Laguerre polynomial, modified Bessel function, incomplete Gamma function, Error function and confluent Hypergeometric functions. By taking some particular values of the index of the Whittaker function, we will deduce with the help of (13.21) and (13.36) some interesting special cases of our main results.

13.3.3 Let $s=\mp r$ and $\xi=-r-\frac{1}{2}$

The Whittaker function is expressed in this case as

$$M_{\mp r,-r-\frac{1}{2}}\left(z\right)=z^{-r}\exp\left(\mp\frac{z}{2}\right). \tag{13.52}$$

Corollary 13.5 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\mu-2r}J_\nu(\chi\rho)e^{-(p\pm\gamma)\rho^2+2q\rho}d\rho$$
$$=K_5\Bigg\{\Gamma\left(\alpha_K\right)F_A^{(3)}\left[\alpha_K,-r\pm r,\_,\_;-2r,\nu+1,\frac{1}{2};x,y,z\right]$$
$$+\frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\beta_K\right)F_A^{(3)}\left[\beta_K,-r\pm r,\_,\_;-2r,\nu+1,\frac{3}{2};x,y,z\right]\Bigg\}, \tag{13.53}$$

where

$$K_5=\frac{1}{2(p+\gamma)^{\alpha_k}}\frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{13.54}$$

$$\alpha_k=\frac{\mu+\nu}{2}-r+\frac{1}{2}\ \textit{and}\ \ \beta_K=\frac{\mu+\nu}{2}-r+1. \tag{13.55}$$

Under the conditions $\Re(p) > 0$ and $\frac{|\chi|^2}{4}+|q|^2 < |p|$ and by using Theorem 13.2, we arrive at (13.53).

Corollary 13.6 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{-2r} cos(\chi\rho)e^{-(p\pm\gamma)\rho^2+2q\rho} d\rho$$
$$= \frac{1}{2(p+\gamma)^{\frac{1}{2}-r}} \left\{ \Gamma\left(\frac{1}{2}-r\right) F_A^{(3)}\left[\frac{1}{2}-r, -r\pm r, \_, \_; -2r, \frac{1}{2}, \frac{1}{2}; x, y, z\right] \right. \tag{13.56}$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma(1-r) F_A^{(3)}\left[1-r, -r\pm r, \_, \_; -2r, \frac{1}{2}, \frac{3}{2}; x, y, z\right] \right\}.$$

Applying (13.36) with the condition $\mu = -\nu = \frac{1}{2}$, we arrive at (13.56).

Corollary 13.7 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{-2r} sin(\chi\rho)e^{-(p\pm\gamma)\rho^2+2q\rho} d\rho$$
$$= \frac{\chi}{2(p+\gamma)^{1-r}} \left\{ \Gamma(1-r) F_A^{(3)}\left[1-r, -r\pm r, \_, \_; -2r, \frac{3}{2}, \frac{1}{2}; x, y, z\right] \right.$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{3}{2}-r\right) F_A^{(3)}\left[\frac{3}{2}-r, -r\pm r, \_, \_; -2r, \frac{3}{2}, \frac{3}{2}; x, y, z\right] \right\}. \tag{13.57}$$

The proof of (13.57) is the same as (13.56) by letting the condition $\mu = \nu = \frac{1}{2}$.

13.3.4 Case of $s = \pm\left(\frac{1}{2}-r\right)$ and $\xi = r$

The Whittaker function becomes

$$M_{\pm\left(\frac{1}{2}-r\right),r}(z) = z^{r+\frac{1}{2}} \exp\left(\pm\frac{z}{2}\right) {}_1F_1(2r; 2r+1; \pm z). \tag{13.58}$$

Corollary 13.8 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\mu+2r+1} J_\nu(\chi\rho)\, {}_1F_1(2r; 2r+1; \pm 2\gamma\rho^2) e^{-(\pm\gamma+p)\rho^2+2q\rho} d\rho$$

$$= K_8\left\{\Gamma\left(\alpha_K\right) F_A^{(3)}\left[\alpha_K, \frac{1}{2}+r \pm\left(\frac{1}{2}-r\right), \_, \_; 2r+1, \nu+1, \frac{1}{2}; x, y, z\right]\right.$$
$$\left.+\frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\beta_K\right) F_A^{(3)}\left[\beta_K, \frac{1}{2}+r \pm\left(\frac{1}{2}-r\right), \_, \_; 2r+1, \nu+1, \frac{3}{2}; x, y, z\right]\right\}, \tag{13.59}$$

where

$$K_8=\frac{1}{2(p+\gamma)^{\alpha_k}} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{13.60}$$

$$\alpha_k=\frac{\mu+\nu}{2}+r+1 \text{ and } \beta_K=\frac{\mu+\nu}{2}+r+\frac{3}{2}. \tag{13.61}$$

By the use of Theorem 13.2, one finds explicitly (13.59).

Corollary 13.9 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho^{2r+1} cos(\chi\rho)\, {}_1F_1(2r; 2r+1; \pm 2\gamma\rho^2) e^{-(\pm\gamma+p)\rho^2+2q\rho} d\rho \tag{13.62}$$
$$=\frac{1}{2(p+\gamma)^{1+r}}\left\{\Gamma(1+r) F_A^{(3)}\left[1+r, \frac{1}{2}+r \pm\left(\frac{1}{2}-r\right), \_, \_; 2r+1, \frac{1}{2}, \frac{1}{2}; x, y, z\right]\right.$$

$$\left.+\frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{3}{2}+r\right) F_A^{(3)}\left[\frac{3}{2}+r, \frac{1}{2}+r \pm\left(\frac{1}{2}-r\right), \_, \_; 2r+1, \frac{1}{2}, \frac{3}{2}; x, y, z\right]\right\}. \tag{13.63}$$

Taking $\mu=-\nu=\frac{1}{2}$ *and using* (13.36)*, one proves* (13.63).

Corollary 13.10 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho^{2r+1} sin(\chi\rho)\, {}_1F_1(2r; 2r+1; \pm 2\gamma\rho^2) e^{-(\pm\gamma+p)\rho^2+2q\rho} d\rho$$
$$=\frac{\chi}{2(p+\gamma)^{\frac{3}{2}+r}}\left\{\Gamma\left(\frac{3}{2}+r\right)\right.$$
$$\times F_A^{(3)}\left[\frac{3}{2}+r, \frac{1}{2}+r \pm\left(\frac{1}{2}-r\right), \_, \_; 2r+1, \frac{3}{2}, \frac{1}{2}; x, y, z\right]$$
$$\left.+\frac{2q}{\sqrt{p+\gamma}} \Gamma(r+2) F_A^{(3)}\left[r+2, \frac{1}{2}+r \pm\left(\frac{1}{2}-r\right), \_, \_; 2r+1, \frac{3}{2}, \frac{3}{2}; x, y, z\right]\right\}. \tag{13.64}$$

This corollary can be proved by the use of Theorem 13.2 and by taking $\mu = \nu = \frac{1}{2}$.

13.3.5 Case of $s = \pm\left(\frac{a+1}{2} + b\right)$ and $\xi = \frac{a}{2}$

The Whittaker function becomes

$$M_{\pm\left(b+\frac{1+a}{2}\right),\frac{a}{2}}(z) = \frac{L_b^{(a)}(\pm z)}{L} z^{\frac{a+1}{2}} \exp\left(\mp\frac{z}{2}\right), \tag{13.65}$$

where

$$L = \frac{(a+1)_b}{b!}. \tag{13.66}$$

Corollary 13.11 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho^{\mu+a+1} J_\nu(\chi\rho)\, L_b^{(a)}\left(\pm 2\gamma\rho^2\right) e^{-(\pm\gamma+p)\rho^2+2q\rho} d\rho \tag{13.67}$$

$$= \frac{(1+a)_b}{b!} K_{11}\left\{\Gamma(\alpha_K)\, F_A^{(3)}\left[\alpha_K, \frac{1}{2} + \frac{a}{2} \pm \left(\frac{a+1}{2} + b\right), \_, \_; a+1, \nu+1, \frac{1}{2}; x, y, z\right]\right.$$

$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma(\beta_K)\, F_A^{(3)}\left[\beta_K, \frac{1}{2} + \frac{a}{2} \pm \left(\frac{a+1}{2} + b\right), \_, \_; a+1, \nu+1, \frac{3}{2}; x, y, z\right]\right\}, \tag{13.68}$$

where

$$K_{11} = \frac{1}{2(p+\gamma)^{\alpha_k}} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{13.70}$$

$$\alpha_k = \frac{\mu+\nu}{2} + \frac{a}{2} + 1 \ \mathit{and}\ \ \beta_K = \frac{\mu+\nu}{2} + \frac{a}{2} + \frac{3}{2}. \tag{13.71}$$

Applying Theorem 13.2, one finds easily (13.68).

Corollary 13.12 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho^{a+1} cos(\chi\rho)\, L_b^{(a)}\left(\pm 2\gamma\rho^2\right) e^{-(\pm\gamma+p)\rho^2+2q\rho} d\rho$$

$$= \frac{(1+a)_b}{b!} \frac{1}{2(p+\gamma)^{\frac{a}{2}+1}} \left\{ \Gamma\left(\frac{a}{2}+1\right) \right.$$
$$\times F_A^{(3)} \left[\frac{a}{2}+1, \frac{1}{2}+\frac{a}{2} \pm \left(\frac{a+1}{2}+b\right), \_, \_; a+1, \frac{1}{2}, \frac{1}{2}; x, y, z \right]$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{a+3}{2}\right) F_A^{(3)} \left[\frac{a+3}{2}, \frac{1}{2}+\frac{a}{2} \pm \left(\frac{a+1}{2}+b\right), \_, \_; a+1, \frac{1}{2}, \frac{3}{2}; x, y, z \right] \right\}. \tag{13.72}$$

With $\mu = -\nu = \frac{1}{2}$ and under the conditions of (13.36)*, one arrives at* (13.72)*.*

Corollary 13.13 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{a+1} sin(\chi\rho)\, L_b^{(a)}\left(\pm 2\gamma\rho^2\right) e^{-(\pm\gamma+p)\rho^2+2q\rho} d\rho$$
$$= \frac{(1+a)_b}{b!} \frac{\chi}{2(p+\gamma)^{\frac{a+3}{2}}} \left\{ \Gamma\left(\frac{a+3}{2}\right) \right.$$
$$\times F_A^{(3)} \left[\frac{a+3}{2}, \frac{1}{2}+\frac{a}{2} \pm \left(\frac{a+1}{2}+b\right), \_, \_; a+1, \frac{3}{2}, \frac{1}{2}; x, y, z \right]$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{a}{2}+2\right) F_A^{(3)} \left[\frac{a}{2}+2, \frac{1}{2}+\frac{a}{2} \pm \left(\frac{a+1}{2}+b\right), \_, \_; a+1, \frac{3}{2}, \frac{3}{2}; x, y, z \right] \right\}. \tag{13.73}$$

With $\mu = \nu = \frac{1}{2}$ and under the conditions of Theorem 13.2, (13.73) *is proved.*

13.3.6 Case of $s = 0$ and $\xi = r$

The Whittaker function is given as

$$M_{0,r}(z) = \frac{1}{C\sqrt{2\gamma}} \sqrt{z} I_r\left(\frac{z}{2}\right), \tag{13.74}$$

where

$$C = \frac{1}{2^{2r}\sqrt{2\gamma} r!}. \tag{13.75}$$

Corollary 13.14 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho^{\mu+1} J_\nu(\chi\rho)\, I_r\left(\gamma\rho^2\right) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{(2\gamma)^r}{2^{2r} r!} K_{14}\Bigg\{\Gamma(\alpha_K)\, F_A^{(3)}\left[\alpha_K, \frac{1}{2}+r, \_, \_; 2r+1, \nu+1, \frac{1}{2}; x, y, z\right]$$
$$+ \frac{2q}{\sqrt{p+\gamma}} \Gamma(\beta_K)\, F_A^{(3)}\left[\beta_K, \frac{1}{2}+r, \_, \_; 2r+1, \nu+1, \frac{3}{2}; x, y, z\right]\Bigg\}, \tag{13.76}$$

where

$$K_{14} = \frac{1}{2(p+\gamma)^{\alpha_k}} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{13.77}$$

$$\alpha_k = \frac{\mu+\nu}{2} + r + 1 \text{ and } \beta_K = \frac{\mu+\nu}{2} + r + \frac{3}{2}. \tag{13.78}$$

With the use of Theorem 13.2, one finds (13.76).

Corollary 13.15 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho cos(\chi\rho)\, I_r\left(\gamma\rho^2\right) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{(2\gamma)^r}{2^{2r} r!} \frac{1}{2(p+\gamma)^{1+r}}\Bigg\{\Gamma(1+r)\, F_A^{(3)}\left[1+r, \frac{1}{2}+r, \_, \_; 2r+1, \frac{1}{2}, \frac{1}{2}; x, y, z\right]$$
$$+ \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{3}{2}+r\right) F_A^{(3)}\left[\frac{3}{2}+r, \frac{1}{2}+r, \_, \_; 2r+1, \frac{1}{2}, \frac{3}{2}; x, y, z\right]\Bigg\}. \tag{13.79}$$

With $\mu = -\nu = \frac{1}{2}$ and under the conditions of Theorem 13.2, one proves (13.79).

Corollary 13.16 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho \sin(\chi\rho)\, I_r\left(\gamma\rho^2\right) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{(2\gamma)^r}{2^{2r} r!} \frac{\chi}{2(p+\gamma)^{\frac{3}{2}+r}}\Bigg\{\Gamma\left(\frac{3}{2}+r\right) F_A^{(3)}\left[\frac{3}{2}+r, \frac{1}{2}+r, \_, \_; 2r+1, \frac{3}{2}, \frac{1}{2}; x, y, z\right]$$
$$+ \frac{2q}{\sqrt{p+\gamma}} \Gamma(2+r)\, F_A^{(3)}\left[2+r, \frac{1}{2}+r, \_, \_; 2r+1, \frac{3}{2}, \frac{3}{2}; x, y, z\right]\Bigg\}. \tag{13.80}$$

Under the conditions of Theorem 13.2, and by taking $\mu = \nu = \frac{1}{2}$, one obtains (13.80).

13.3.7 Case of $s = 0$ and $\xi = \frac{1}{2}$

In this case, the Whittaker function is expressed as

$$M_{0,\frac{1}{2}}(z) = 2\sinh\left(\frac{z}{2}\right). \tag{13.81}$$

Corollary 13.17 *The undermentioned integral transform holds true:*

$$\begin{aligned}&\int_0^\infty \rho^{\mu+1} J_\nu(\chi\rho)\ \sinh(\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho\\ &= K_{17}\gamma\left\{\Gamma(\alpha_K)\, F_A^{(3)}\left[\alpha_K, 1, \_, \_; 2, \nu+1, \frac{1}{2}; x, y, z\right]\right.\\ &\left.+\frac{2q}{\sqrt{p+\gamma}}\Gamma(\beta_K)\, F_A^{(3)}\left[\beta_K, 1, \_, \_; 2, \nu+1, \frac{3}{2}; x, y, z\right]\right\},\end{aligned} \tag{13.82}$$

where

$$K_{17} = \frac{1}{2(p+\gamma)^{\alpha_k}}\frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{13.83}$$

$$\alpha_k = \frac{\mu+\nu}{2} + \frac{3}{2}\ \text{and}\ \ \beta_K = \frac{\mu+\nu}{2} + 2. \tag{13.84}$$

The use of Theorem 13.2 yields easily (13.82).

Corollary 13.18 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho\cos(\chi\rho)\ \sinh(\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$

$$= \frac{1}{2(p+\gamma)^{\frac{3}{2}}}\gamma\left\{\frac{\sqrt{\pi}}{2}F_A^{(3)}\left[\frac{3}{2}, 1, \_, \_; 2, \frac{1}{2}, \frac{1}{2}; x, y, z\right]\right. \tag{13.85}$$

$$\left.+\frac{2q}{\sqrt{p+\gamma}}F_A^{(3)}\left[2, 1, \_, \_; 2, \frac{1}{2}, \frac{3}{2}; x, y, z\right]\right\}. \tag{13.86}$$

With $\mu = -\nu = \frac{1}{2}$ *and under the conditions of Theorem 13.2, we arrive at* (13.85).

Corollary 13.19 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho \sin(\chi\rho)\ \sinh(\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{\chi}{2(p+\gamma)^2}\gamma \left\{ F_A^{(3)}\left[2, 1, \_, \_; 2, \frac{3}{2}, \frac{1}{2}; x, y, z\right]\right.$$
$$\left. + \frac{3q\sqrt{\pi}}{2\sqrt{p+\gamma}} F_A^{(3)}\left[\frac{5}{2}, 1, \_, \_; 2, \frac{3}{2}, \frac{3}{2}; x, y, z\right]\right\}. \tag{13.87}$$

The use of $\mu = \nu = \frac{1}{2}$ *yields* (13.87).

13.3.8 Case of $s = a + \frac{1}{4}$ *and* $\xi = -\frac{1}{4}$

The Whittaker function is expressed, with these parameters, as

$$M_{a+\frac{1}{4},-\frac{1}{4}}\left(z^2\right) = \frac{S}{(2\gamma)^{1/4}} \exp\left(-\frac{z^2}{2}\right)\sqrt{z}H_{2a}\left(z\right), \tag{13.88}$$

where

$$S = (-1)^a \frac{a!}{(2a)!}(2\gamma)^{1/4}, \tag{13.89}$$

with $\gamma \neq 0$.

Corollary 13.20 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\mu+\frac{1}{2}} J_\nu(\chi\rho)\ H_{2a}(\sqrt{2\gamma}\rho) e^{-(p+\gamma)\rho^2+2q\rho} d\rho$$
$$= K_{20}\frac{(2a)!}{(-1)^a a!}\left\{\Gamma\left(\alpha_K\right) F_A^{(3)}\left[\alpha_K, -a, \_, \_; \frac{1}{2}, \nu+1, \frac{1}{2}; x, y, z\right]\right.$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\beta_K\right) F_A^{(3)}\left[\beta_K, -a, \_, \_; \frac{1}{2}, \nu+1, \frac{3}{2}; x, y, z\right]\right\}, \tag{13.90}$$

where

$$K_{20} = \frac{1}{2(p+\gamma)^{\alpha_k}} \frac{\left(\frac{\chi}{2}\right)^{\nu}}{\nu!}, \tag{13.91}$$

$$\alpha_k = \frac{\mu+\nu}{2} + \frac{3}{4} \; and \; \beta_K = \frac{\mu+\nu}{2} + \frac{5}{4}. \tag{13.92}$$

With Theorem 13.2 and the conditions of this case, one finds (13.90).

Corollary 13.21 *The undermentioned integral transform holds true:*

$$\begin{aligned} &\int_0^{\infty} \rho^{\frac{1}{2}} cos(\chi\rho)\, H_{2a}(\sqrt{2\gamma}\,\rho) e^{-(p+\gamma)\rho^2+2q\rho} d\rho \\ &= \frac{(2a)!}{(-1)^a a!} \frac{1}{2(p+\gamma)^{\frac{3}{4}}} \left\{ \Gamma\left(\frac{3}{4}\right) F_A^{(3)}\left[\frac{3}{4}, -a, \_, \_; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z\right] \right. \\ &\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{5}{4}\right) F_A^{(3)}\left[\frac{5}{4}, -a, \_, \_; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z\right] \right\}. \end{aligned} \tag{13.93}$$

By applying (13.36) *with the condition* $\mu = -\nu = \frac{1}{2}$*, the corollary is proved.*

Corollary 13.22 *The undermentioned integral transform holds true:*

$$\begin{aligned} &\int_0^{\infty} \rho^{\frac{1}{2}} sin(\chi\rho)\, H_{2a}(\sqrt{2\gamma}\,\rho) e^{-(p+\gamma)\rho^2+2q\rho} d\rho \\ &= \frac{(2a)!}{(-1)^a a!} \frac{\chi}{2(p+\gamma)^{\frac{5}{4}}} \left\{ \Gamma\left(\frac{5}{4}\right) F_A^{(3)}\left[\frac{5}{4}, -a, \_, \_; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z\right] \right. \\ &\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{7}{4}\right) F_A^{(3)}\left[\frac{7}{4}, -a, \_, \_; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z\right] \right\}. \end{aligned} \tag{13.94}$$

With $\mu = \nu = \frac{1}{2}$ *and under the conditions of Theorem 13.2, we arrive at* (13.94).

13.3.9 Case of $s = a + \frac{3}{4}$ and $\xi = \frac{1}{4}$

The Whittaker function is expressed as

$$M_{a+\frac{3}{4},\frac{1}{4}}\left(z^2\right) = \frac{S'}{(2\gamma)^{1/4}} \exp\left(-\frac{z^2}{2}\right) \sqrt{z} H_{2a+1}(z), \tag{13.95}$$

where

$$S' = \frac{(-1)^a}{2} \frac{a!(2\gamma)^{1/4}}{(2a+1)!}, \tag{13.96}$$

with $\gamma \neq 0$.

Corollary 13.23 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\mu+\frac{1}{2}} J_\nu(\chi\rho)\, H_{2a+1}(\sqrt{2\gamma}\,\rho) e^{-(p+\gamma)\rho^2+2q\rho} d\rho$$
$$= K_{23} \frac{2\,(a+1)!}{(-1)^a a!} \sqrt{2\gamma} \Bigg\{ \Gamma\,(\alpha_K)\, F_A^{(3)} \left[\alpha_K, -a, \_, \_; \frac{3}{2}, \nu+1, \frac{1}{2}; x, y, z \right]$$
$$+ \frac{2q}{\sqrt{p+\gamma}} \Gamma\,(\beta_K)\, F_A^{(3)} \left[\beta_K, -a, \_, \_; \frac{3}{2}, \nu+1, \frac{3}{2}; x, y, z \right] \Bigg\}, \tag{13.97}$$

where

$$K_{23} = \frac{1}{2(p+\gamma)^{\alpha_k}} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{13.98}$$

$$\alpha_k = \frac{\mu+\nu}{2} + \frac{5}{4} \;\; \textit{and} \;\; \beta_K = \frac{\mu+\nu}{2} + \frac{7}{4}. \tag{13.99}$$

(13.97) *is proved by the help of Theorem 13.2.*

Corollary 13.24 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\frac{1}{2}} \cos(\chi\rho)\, H_{2a+1}(\sqrt{2\gamma}\,\rho) e^{-(p+\gamma)\rho^2+2q\rho} d\rho$$
$$= \frac{2\,(a+1)!}{(-1)^a a!} \frac{1}{2(p+\gamma)^{\frac{5}{4}}} \sqrt{2\gamma} \Bigg\{ \Gamma\left(\frac{5}{4}\right) F_A^{(3)} \left[\frac{5}{4}, -a, \_, \_; \frac{3}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z \right]$$
$$+ \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{7}{4}\right) F_A^{(3)} \left[\frac{7}{4}, -a, \_, \_; \frac{3}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z \right] \Bigg\}. \tag{13.100}$$

With $\mu = -\nu = \frac{1}{2}$ and under the conditions of Theorem 13.2, we arrive at (13.100).

Corollary 13.25 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\frac{1}{2}} \sin(\chi\rho)\, H_{2a+1}(\sqrt{2\gamma}\,\rho) e^{-(p+\gamma)\rho^2+2q\rho} d\rho$$

$$= \frac{2\,(a+1)!}{(-1)^a a!}\frac{\chi}{2(p+\gamma)^{\frac{7}{4}}}\sqrt{2\gamma}\left\{\Gamma\left(\frac{7}{4}\right)F_A^{(3)}\left[\frac{5}{4}, -a, \_, \_; \frac{3}{2}, \frac{3}{2}, \frac{1}{2}; x, y, z\right]\right.$$

$$\left. + \frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\frac{7}{4}\right)F_A^{(3)}\left[\frac{9}{4}, -a, \_, \_; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; x, y, z\right]\right\}. \tag{13.101}$$

With $\mu = \nu = \frac{1}{2}$ *and under the conditions of Theorem 13.2, we arrive at* (13.101).

13.3.10 Case of $s = r - \frac{1}{2}$ and $\xi = r$

The Whittaker function becomes

$$M_{r-\frac{1}{2},r}(z) = 2r \exp\left(\frac{z}{2}\right) z^{\frac{1}{2}-r}\gamma(2r, z), \tag{13.102}$$

where $\gamma(a, x)$ is the incomplete gamma function.

Corollary 13.26 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\mu-2r+1} J_\nu(\chi\rho)\,\gamma(2r, 2\gamma'\rho^2)e^{-(p-\gamma')\rho^2+2q\rho}d\rho$$

$$= K_{26}\frac{(2\gamma')^{2r}}{2r}\left\{\Gamma(\alpha_K)\,F_A^{(3)}[\alpha_K, 1, \_, \_; 2r+1, \nu+1, \frac{1}{2}; x, y, z]\right.$$

$$\left. + \frac{2q}{\sqrt{p+\gamma'}}\Gamma(\beta_K)\,F_A^{(3)}[\beta_K, 1, \_, \_; 2r+1, \nu+1, \frac{3}{2}; x, y, z]\right\}, \tag{13.103}$$

where

$$K_{26} = \frac{1}{2(p+\gamma)^{\alpha_k}}\frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{13.104}$$

$$\alpha_k = \frac{\mu+\nu}{2} + r + 1 \text{ and } \beta_K = \frac{\mu+\nu}{2} + r + \frac{3}{2}. \tag{13.105}$$

By applying Theorem 13.2, one finds (13.103).

Corollary 13.27 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{-2r+1} cos(\chi\rho)\,\gamma(2r, 2\gamma'\rho^2)e^{-(p-\gamma')\rho^2+2q\rho}d\rho$$

$$= \frac{\left(2\gamma'\right)^{2r}}{2r}\frac{1}{2(p+\gamma)^{1+r}}\left\{\Gamma\left(1+r\right)F_A^{(3)}\left[r+1,1,\_,\_;2r+1,\frac{1}{2},\frac{1}{2};x,y,z\right]\right.$$
$$\left.+\frac{2q}{\sqrt{p+\gamma'}}\Gamma\left(\frac{3}{2}+r\right)F_A^{(3)}\left[\frac{3}{2}+r,1,\_,\_;2r+1,\frac{1}{2},\frac{3}{2};x,y,z\right]\right\}. \tag{13.106}$$

By the use of $\mu=-\nu=\frac{1}{2}$ *and under the conditions of Theorem 13.2, it's easy to deduce* (13.106).

Corollary 13.28 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{-2r+1}sin(\chi\rho)\,\gamma(2r,2\gamma'\rho^2)e^{-(p-\gamma')\rho^2+2q\rho}d\rho$$
$$= \frac{\left(2\gamma'\right)^{2r}}{2r}\frac{\chi}{2(p+\gamma)^{\frac{3}{2}+r}}\left\{\Gamma\left(\frac{3}{2}+r\right)F_A^{(3)}\left[r+\frac{3}{2},1,\_,\_;2r+1,\frac{3}{2},\frac{1}{2};x,y,z\right]\right.$$
$$\left.+\frac{2q}{\sqrt{p+\gamma'}}\Gamma\left(2+r\right)F_A^{(3)}\left[2+r,1,\_,\_;2r+1,\frac{3}{2},\frac{3}{2};x,y,z\right]\right\}. \tag{13.107}$$

With the help of $\mu=\nu=\frac{1}{2}$*, and by applying Theorem 13.2, we get* (13.107).

13.3.11 Case of $s=-\frac{1}{4}$ and $\xi=\frac{1}{4}$

In this case, the Whittaker function is given in terms of the error function erf as

$$M_{-\frac{1}{4},\frac{1}{4}}\left(z^2\right)=\frac{\exp\left(z^2/2\right)}{2}\sqrt{\pi z}erf\left(z\right). \tag{13.108}$$

Corollary 13.29 *The undermentioned integral transform holds true:*

$$\int_0^\infty \rho^{\mu+\frac{1}{2}}J_\nu(\chi\rho)\,erf(\sqrt{2\gamma}\rho)e^{-(p-\gamma)\rho^2+2q\rho}d\rho$$
$$=2\sqrt{\frac{2\gamma}{\pi}}K_{29}\left\{\Gamma\left(\alpha_K\right)F_A^{(3)}\left[\alpha_K,\frac{1}{2},\_,\_;\frac{3}{2},\nu+1,\frac{1}{2};x,y,z\right]\right.$$
$$\left.+\frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\beta_K\right)F_A^{(3)}\left[\beta_K,\frac{1}{2},\_,\_;\frac{3}{2},\nu+1,\frac{3}{2};x,y,z\right]\right\}, \tag{13.109}$$

where

$$K_{29} = \frac{1}{2(p+\gamma)^{\alpha_k}} \frac{\left(\frac{\chi}{2}\right)^{\nu}}{\nu!}, \tag{13.110}$$

$$\alpha_k = \frac{\mu+\nu}{2} + \frac{5}{4} \textit{ and } \beta_K = \frac{\mu+\nu}{2} + \frac{7}{4}. \tag{13.111}$$

Use of Theorem 13.2 gives the expression of (13.109).

Corollary 13.30 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho^{\frac{1}{2}} cos(\chi\rho)\, erf(\sqrt{2\gamma}\rho) e^{-(p-\gamma)\rho^2+2q\rho} d\rho$$
$$= 2\sqrt{\frac{2\gamma}{\pi}} \frac{1}{2(p+\gamma)^{\frac{5}{4}}} \left\{ \Gamma\left(\frac{5}{4}\right) F_A^{(3)}\left[\frac{5}{4}, \frac{1}{2}, -, -; \frac{3}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z\right] \right.$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{7}{4}\right) F_A^{(3)}\left[\frac{7}{4}, \frac{1}{2}, -, -; \frac{3}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z\right] \right\}. \tag{13.112}$$

With the help of $\mu = -\nu = \frac{1}{2}$, *and by applying Theorem 13.2, we get* (13.112).

Corollary 13.31 *The undermentioned integral transform holds true:*

$$\int_0^{\infty} \rho^{\frac{1}{2}} sin(\chi\rho)\, erf(\sqrt{2\gamma}\rho) e^{-(p-\gamma)\rho^2+2q\rho} d\rho$$
$$= 2\sqrt{\frac{2\gamma}{\pi}} \frac{\chi}{2(p+\gamma)^{\frac{7}{4}}} \left\{ \Gamma\left(\frac{7}{4}\right) F_A^{(3)}\left[\frac{7}{4}, \frac{1}{2}, -, -; \frac{3}{2}, \nu+1, \frac{1}{2}; x, y, z\right] \right.$$
$$\left. + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{9}{4}\right) F_A^{(3)}\left[\frac{9}{4}, \frac{1}{2}, -, -; \frac{3}{2}, \nu+1, \frac{3}{2}; x, y, z\right] \right\}. \tag{13.113}$$

With the help of $\mu = \nu = \frac{1}{2}$, *and by applying Theorem 13.2, we get* (13.113).

13.4 Numerical Simulations

In this section, based on the Laguerre-Gauss quadrature method, we will compare our theoretical results of (13.21) and (13.36) given by its second member and its first member. These integrals denoted by R, can be expressed as follows:

$$R = \sum_{i=1}^{N} \omega_i e^{x_i} g(x_i), \tag{13.114}$$

where the values of abscissas x_i and weight factors $w_i \exp(x_i)$ are listed below (see Table 13.1) and $g(x_i) = {x_i}^{\mu} J_\nu(\chi x_i) e^{-p{x_i}^2+2qx_i}$ with N=15, which relates to (13.21), and $g(x_i) = {x_i}^{\mu} J_\nu(\chi x_i) \mathrm{M}_{s,\xi}\left(2\gamma {x_i}^2\right) e^{-p{x_i}^2+2qx_i}$, which corresponds to (13.36).

In Fig. 13.1, we illustrate the closed form expressed by (13.21) and the numerical evaluation exposited above. This figure shows that there is an agreement between the two curves which represent the numerical and theoretical predictions.

Furthermore, we present in Fig. 13.2 a comparison between our theoretical and numerical results. Figure 13.2 illustrates an agreement between the result afforded by the Laguerre-Gauss quadrature method and the closed form established by (13.21).

In order to compare our numerical simulations given in (13.114) and the second member of (13.36), we display the two expressions in Fig. 13.3, with the parameters $\mu = 1$, $\xi = 1$, $s = 1$, and $\gamma = 1$, where the condition of (13.36) is verified. So, from this figure, one can see clearly the concordance between the theoretical result and the numerical one.

In addition, we depict a comparison of our numerical and theoretical findings in Fig. 13.4, and the other computational parameters are the same as those used in Fig. 13.3. One can deduce from this figure that there is an excellent agreement between the two considered evaluations.

Table 13.1 Values of x_i and $w_i \exp(x_i)$ used for Laguerre-Gauss integration (see [1])

i	x_i	$w_i e^{x_i}$
1	0.093307812017	0.239578170311
2	0.492691740302	0.560100842793
3	1.215595412071	0.887008262919
4	2.269949526204	1.22366440215
5	3.667622721751	1.57444872163
6	5.425336627414	1.94475197653
7	7.565916226613	2.34150205664
8	10.120228568019	2.77404192683
9	13.130282482176	3.25564334640
10	16.654407708330	3.80631171423
11	20.776478899449	4.45847775384
12	25.623894226729	5.27001778443
13	31.407519169754	6.35956346973
14	38.530683306486	8.03178763212
15	48.026085572686	11.5277721009

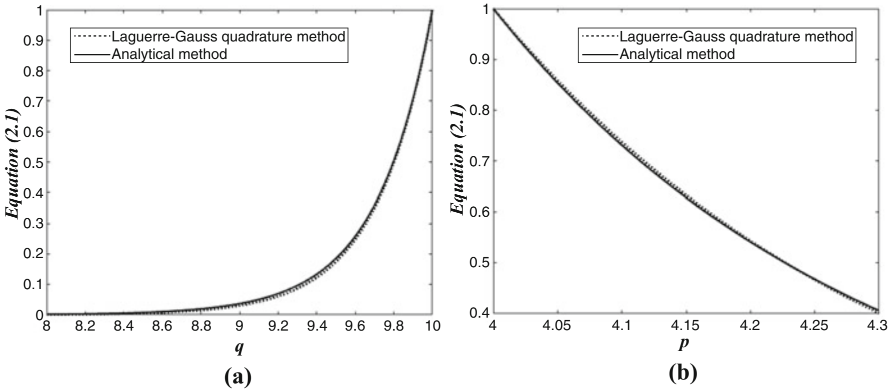

Fig. 13.1 Illustration of (13.21) as a function of (**a**) q with p=6, $\chi = 1$, $\mu = 1$ and $\nu = 1$, and (**b**) p with q=7, $\chi = 1$, $\mu = 0$, and $\nu = 1$

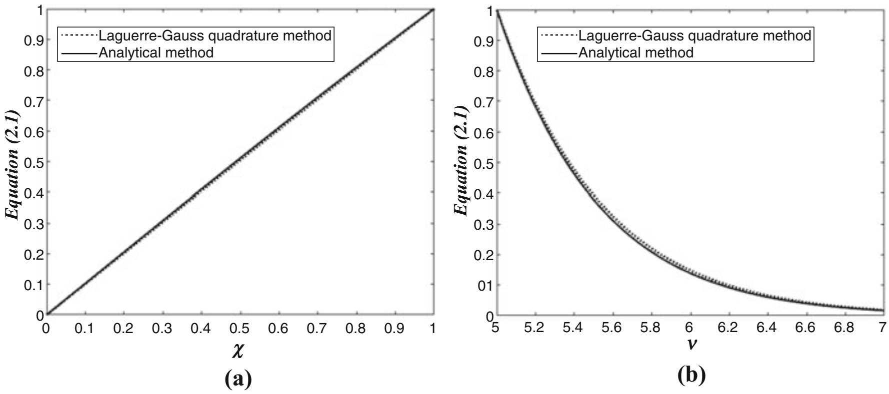

Fig. 13.2 Illustration of (13.21) as a function of (**a**) χ with q=1, $p = 8$, $\mu = 1$, and $\nu = 1$ and (**b**) ν with q=8, $p = 8$, $\mu = 1$, and $\chi = 1$

13.5 Application

In this section, we will evaluate the properties of a diffracted abruptly autofocusing beam, referenced in the following by AAB, by a radial phase shift modulated spiral zone plate (RSSZP).

In the polar coordinate system, the incident beam AAB is given by (see [20, 25])

$$U_0\left(r\right) = \exp\left(-\frac{\left(r-a\right)^2}{\omega_0}\right)\exp\left(iQ\left(r-a\right)\right), \tag{13.115}$$

where a is the initial coordinate, ω_0 is the beam waist of the beam, and Q is the corresponding phase.

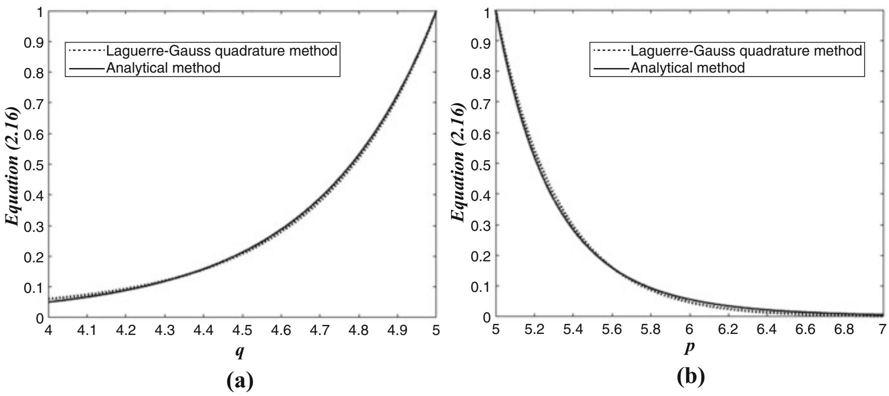

Fig. 13.3 Illustration of (13.36) as a function of (**a**) q with p=4, $\chi = 2$, and $\nu = 1$ and (**b**) p with q=10, $\chi = 2$, and $\nu = 1$

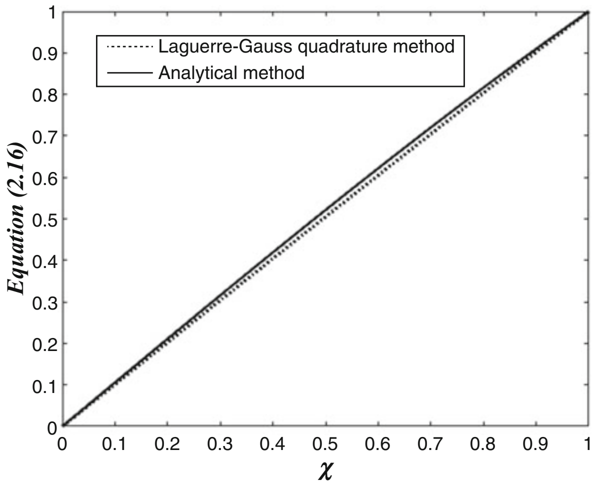

Fig. 13.4 Illustration of (13.36) as a function of χ with q=5, $p = 12$, and $\nu = 1$

The RSSZP's transmittance function can be represented as

$$t(r, \phi) = e^{-i\frac{\pi\left(r-\alpha' R\right)^2}{\lambda f}+ip\phi}. \tag{13.116}$$

In this expression, (r, ϕ) are polar coordinates, p is the topological charge, λ is the wavelength of the incident beam, R and f are the radius and the focal length of the element, and α is the shifting parameter.

In order to study the creation of optical vortices created by illuminating a RSSZP with our considered beam, the field situated at a distance z from the RSSZP plane is determined by the following Fresnel-Kirchhoff integral (see [13])

$$U(\rho, z, \theta) = \frac{ik}{2\pi z} e^{-ik\left(z+\frac{\rho^2}{2z}\right)} \int_0^\infty \int_0^{2\pi} t(r, \phi)\, U_0(r)$$

$$\times \exp\left[-\frac{ik}{2z}\left(r^2 - 2r\rho\cos(\phi - \theta)\right)\right] r dr d\phi. \tag{13.117}$$

Additionally, using (13.115) and (13.116) in (13.117), one finds

$$U(\rho, z, \theta) = \frac{ik}{2\pi z} e^{-ik\left(z+\frac{\rho^2}{2z}\right)} \int_0^\infty \int_0^{2\pi} \exp\left(-\frac{(r-a)^2}{\omega_0}\right) \exp\left(iQ(r-a)\right)$$

$$\times \exp\left(-i\frac{\pi\left(r - \alpha' R\right)^2}{\lambda f} + ip\phi\right) \exp\left[-\frac{ik}{2z}\left(r^2 - 2r\rho\cos(\phi - \theta)\right)\right] r dr d\phi. \tag{13.118}$$

We recall the following integral formulae:

$$\int_0^{2\pi} \exp\left(iL\phi'\right) \exp\left(\frac{ik\rho\rho'\cos\left(\phi - \phi'\right)}{B}\right) d\phi' = 2\pi (i)^L \exp(iL\phi)\, J_L\left(\frac{k\rho\rho'}{B}\right), \tag{13.119}$$

and after tedious algebraic calculations over the azimuthal variable ϕ, (13.118) is reduced to

$$U(\rho, z, \phi) = 2\pi (i)^p \frac{ik}{2\pi z} e^{-ik\left(z+\frac{\rho^2}{2z}\right)} \exp\left(-ik\frac{\alpha'^2 R^2}{2f}\right) \exp\left(-\frac{a^2}{\omega} + iQa - ip\theta\right) I, \tag{13.120}$$

where

$$I = \int_0^\infty \exp\left(-\varepsilon r^2\right) \exp(\gamma r)\, J_p\left(\frac{k\rho}{z} r\right) r dr, \tag{13.121}$$

with

$$\varepsilon = \frac{1}{\omega} + i\frac{k}{2}\left(\frac{1}{z} + \frac{1}{f}\right), \tag{13.122}$$

and

$$\gamma = \frac{2a}{\omega} + iQ + i\frac{\pi\alpha R}{\lambda f}. \tag{13.123}$$

Equation (13.121) can be evaluated with the help of Theorem 13.1. So, (13.120) can be expressed as

$$U(\rho, z, \phi) = 2\pi (i)^p \frac{ik}{2\pi z} e^{-ik\left(z+\frac{\rho^2}{2z}\right)} \exp\left(-ik\frac{\alpha'^2 R^2}{2f}\right) \exp\left(-\frac{a^2}{\omega} + iQa - ip\theta\right)$$
$$\times I_0 \left\{\Gamma(\alpha)\, \Psi_2\left[\alpha; p+1, \frac{1}{2}; x, y\right] + \frac{\gamma}{\sqrt{\varepsilon}}\Gamma(\beta)\, \Psi_2\left[\beta; p+1, \frac{3}{2}; x, y\right]\right\}, \tag{13.124}$$

where

$$I_0 = \frac{1}{2\varepsilon} \frac{\left(\frac{k\rho}{2z\sqrt{\varepsilon}}\right)^p}{p!}, \tag{13.125}$$

$$x = -\frac{\left(\frac{k\rho}{z}\right)^2}{4\varepsilon},\ \ y = \frac{\gamma^2}{4\varepsilon}, \tag{13.126}$$

$$\alpha = \frac{p}{2} + 1 \ and\ \beta = \frac{p+3}{2}. \tag{13.127}$$

Finally, (13.124) is the closed form of the diffracted wave by a RSSZP.

By using the main result established by (13.124), some numerical simulations are elaborated to present the intensity distribution of the diffracted AAB by a RSSZP. In the following, we use the numerical values λ=632.8 nm, $\omega = 0.5$ mm, and $z = 1000$ mm. The parameters corresponding to the RSSZP are chosen as $f = 500$ mm and $R = 6$ mm. Two values of shifting parameter are $\alpha' = 0.05$ and 0.1.

Figure 13.5 illustrates the evolution of the intensity of the diffracted AAB for two values of a (a=0 mm and a=0.75 mm) and two values of α at various values of p. The plots of this figure show that, when p=0, the intensity presents a central bright spot with side lobe located sideways. From Figs. 13.5.B(a) and B(b), we see that the intensity patterns are plotted for a=0.75 mm. From these plots, it becomes very clear that the output wave has in the center a maximum intensity if p is equal to zero. A dark-centered distribution occurs with non-zero topological charge p. Also, we note that the dark region grows larger with the increase of p.

We present in Fig. 13.6 the intensity profiles of the diffracted beam for two values of the shifting parameter α (=0.05 and 0.1) and for different values of the phase Q with fixed topological charge p=1 and a=0.75 mm. The plots of this figure show that if Q increases, the intensity maximum increases and the radius of the dark-centered distribution is unaffected by the value of Q.

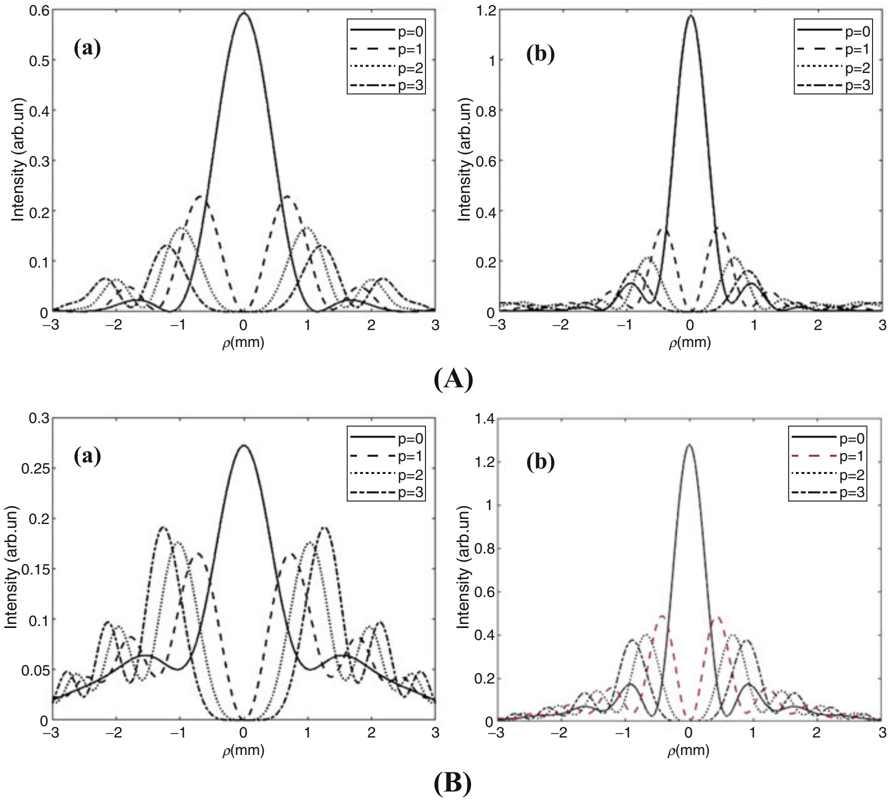

Fig. 13.5 Intensity of the diffracted beam for (**A**) a = 0 and (**B**) a = 0.75 mm with (**a**) $\alpha' = 0.05$ and (**b**) $\alpha' = 0.1$

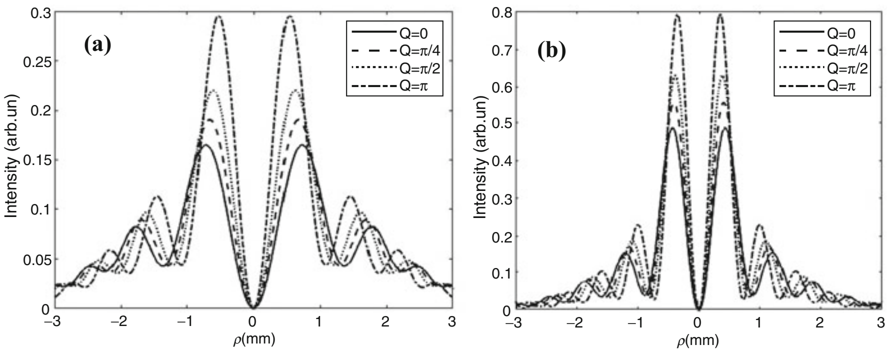

Fig. 13.6 Intensity profiles of the diffracted AAB p = 1 for (**a**) $\alpha' = 0.05$ and (**b**) $\alpha' = 0.1$

13.6 Conclusion

In this chapter, we have elaborated some transformations involving the product of the Bessel functions and some special functions. Some corollaries are derived as particular cases from our main results. To compare our theoretical and numerical results, some numerical evaluations have been done. Our results show that there is a compatibility between the numerical solution obtained using the Laguerre-Gauss quadrature method and our theoretical results.

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Chapter 14
Some Spaces in Neutrosophic *e*-Open Sets

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AMS Subject Classification Number: 03E72, 54A10, 54A40

14.1 Introduction

In the field of logic and set theory, Zadeh [31] was the first to establish the concept of a fuzzy set between intervals. Chang [6] has adopted the general topology framework with fuzzy set called as fuzzy topological space. In 1983, Atanassov [4] began developing an intuitionistic fuzzy set with membership and non-membership values. Coker [7] developed intuitionistic fuzzy topological spaces, which are intuitionistic fuzzy sets in a topology. Smarandache [19, 20] established the concepts of neutrosophy and neutrosophic set in the beginning of the $twentieth$ century. In 2012, Salama and Alblowi [16] created a neutrosophic set in a neutrosophic topological space.

In fuzzy topological spaces, Saha [15] defined δ-open sets. In a neutrosophic topological space, Vadivel et al. [24–26] introduced δ-open sets. In a general topology, Ekici [8–13] developed the concept of e-open sets in 2008. Seenivasan et al. [18] introduced fuzzy e-open sets and fuzzy e-continuity in a topological space in 2014. Vadivel et al. [5] investigated intuitionistic fuzzy e-open sets in an intuitionistic fuzzy topological space. In neutrosophic topological spaces, Vadivel et al. [27, 28] investigated neutrosophic e-open sets and continuous and open mapping

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_14

functions, and Vadivel et al. [21–23, 29, 30] investigated N_{nc} e-open sets and continuous mapping functions.

Recently, Murad Arar [2] discussed about countly compactness, Ahu Acikgoza and Ferhat Esenbel [1] discussed neutrosophic connectedness, and Parimala et al. [14] introduced $\alpha\psi$-connectedness in neutrosophic topological spaces.

In this chapter, we study connectedness, compactness, and separated sets of neutrosophic e-open sets in neutrosophic topological spaces.

14.2 Preliminaries

Definition 1 ([17]) Let U be a non-empty set. A neutrosophic set (briefly, $N_s s$) K is an object having the form $K = \{\langle u, \mu_K(u), \sigma_K(u), \nu_K(u)\rangle : u \in U\}$ where $\mu_K \to [0, 1]$ is a membership function, $\sigma_K \to [0, 1]$ is an indeterminacy function, and $\nu_K \to [0, 1]$ is a non-membership function, respectively, of each element $u \in U$ to the set K and $0 \leq \mu_K(u) + \sigma_K(u) + \nu_K(u) \leq 3 \ \forall \ u \in U$.

Definition 2 ([17]) Let U be a non-empty set & the Nss's K & M in the form $K = \{\langle u, \mu_K(u), \sigma_K(u), \nu_K(u)\rangle : u \in U\}$, $M = \{\langle u and \mu_M(u), \sigma_M(u), \nu_M(u)\rangle : u \in U\}$; then

1. $0_N = \langle u, 0, 0, 1\rangle$ and $1_N = \langle u, 1, 1, 0\rangle$,
2. $K \subseteq M$ iff $\mu_K(u) \leq \mu_M(u), \sigma_K(u) \leq \sigma_M(u),$ & $\nu_K(u) \geq \nu_M(u) : u \in U$,
3. $K = M$ iff $K \subseteq M$ and $M \subseteq K$,
4. $1_N - K = \{\langle u, \nu_K(u), 1 - \sigma_K(u), \mu_K(u)\rangle : u \in U\} = K^c$,
5. $K \cup M = \{\langle u, \max(\mu_K(u), \mu_M(u)), \max(\sigma_K(u), \sigma_M(u)), \min(\nu_K(u), \nu_M(u))\rangle : u \in U\}$,
6. $K \cap M = \{\langle u, \min(\mu_K(u), \mu_M(u)), \min(\sigma_K(u), \sigma_M(u)), \max(\nu_K(u), \nu_M(u))\rangle : u \in U\}$.

Definition 3 ([16]) A neutrosophic topology (briefly, $N_s t$) on a non-empty set U is a family τ_N of neutrosophic subsets of U satisfying

1. $0_N, 1_N \in \tau_N$,
2. $K_1 \cap K_2 \in \tau_N$ for any $K_1, K_2 \in \tau_N$,
3. $\bigcup K_a \in \tau_N, \forall \ K_a : a \in A \subseteq \tau_N$.

Then (U, τ_N) is called a neutrosophic topological space (briefly, $N_s ts$) in U. The τ_N elements are called neutrosophic open sets (briefly, $N_s os$) in U. A $N_s s$ C is called a neutrosophic closed sets (briefly, $N_s cs$) iff its complement C^c is $N_s os$.

Definition 4 ([16]) Let (U, τ_N) be a $N_s ts$ on U and H be a $N_s s$ on U; then the neutrosophic

1. interior of H (in short, $N_s int(H)$) is $N_s int(H) = \bigcup\{K : K \subseteq H \,\&\, K$ is a $N_s os$ in $U\}$.
2. closure of H (in short, $N_s cl(H)$) is $N_s cl(H) = \bigcap\{K : H \subseteq K \,\&\, K$ is a $N_s cs$ in $U\}$.
3. regular open set [3] (briefly, $N_s ros$) if $H = N_s int(N_s cl(H))$.
4. δ interior of H [24] (in short, $N_s \delta int(H)$) is $N_s \delta int(H) = \bigcup\{K : K \subseteq H \,\&\, K$ is a $N_s ros$ in $U\}$.
5. δ closure of H [24] (in short, $N_s \delta cl(H)$) is $N_s \delta cl(H) = \bigcap\{A : H \subseteq A \,\&\, A$ is a $N_s rcs$ in $U\}$.
6. e-open set (briefly, $N_s eos$) [27, 28] if $H \subseteq N_s cl(N_s \delta int(H)) \cup N_s int(N_s \delta cl(H))$.
7. e interior of H [27, 28] (briefly, $N_s eint(H)$) is $N_s eint(H) = \bigcup\{K : K \subseteq H \,\&\, K$ is a $N_s eos$ in $U\}$.
8. e closure of H [27, 28] (briefly, $N_s ecl(H)$) is $N_s ecl(H) = \bigcap\{A : H \subseteq A \,\&\, A$ is a $N_s ecs$ in $U\}$.

The complement of a $N_s ros$ (resp. $N_s eos$) is called a neutrosophic regular (resp. e) closed set (briefly, $N_s rcs$ (resp. $N_s rcs$)) in U.

Definition 5 ([27]) A map $h : (U, \tau_N) \to (V, \tau_N)$ is called neutrosophic

1. e-continuous (briefly, $N_s eCts$) function if $h^{-1}(K)$ is a $N_s eos$ in (U, τ_N) for every $N_s os$ K in (V, τ_N).
2. e-irresolute (briefly, $N_s eIrr$) function if $h^{-1}(K)$ is a $N_s eos$ in (U, τ_N) for every $N_s eos$ K of (V, τ_N).
3. e-open (in short, $N_s eO$) [28] mapping if image of every $N_s o$ set of (U, τ_N) is $N_s eo$ set in (V, τ_N).

14.3 Neutrosophic *e* Connected Spaces

Definition 6 Let (U, τ_N) be a $N_s ts$ is neutrosophic e disconnected (briefly, $N_s eDCon$) spaces if there exists $N_s eo$ sets H, L in U, $H \neq 0_N$, $L \neq 0_N$ such that $H \cup L = 1_N$ and $H \cap L = 0_N$. That is,

1. $\mu_H(u) \vee \mu_L(u) = 1_N, \sigma_H(u) \wedge \sigma_L(u) = 1_N, \nu_H(u) \wedge \nu_L(u) = 0_N$.
2. $\mu_H(u) \vee \mu_L(u) = 1_N, \sigma_H(u) \vee \sigma_L(u) = 1_N, \nu_H(u) \wedge \nu_L(u) = 0_N$.
3. $\mu_H(u) \wedge \mu_L(u) = 0_N, \sigma_H(u) \wedge \sigma_L(u) = 0_N, \nu_H(u) \vee \nu_L(u) = 1_N$.
4. $\mu_H(u) \wedge \mu_L(u) = 0_N, \sigma_H(u) \vee \sigma_L(u) = 0_N, \nu_H(u) \vee \nu_L(u) = 1_N$.

If U is not $N_s eDCon$, then it is said to be neutrosophic e connected (briefly, $N_s eCon$) spaces.

Example 1 Let $U = \{c_1, c_2, c_3\}$, and define $N_s s$ $U_1, U_2, U_3, \,\&\, U_4$ in U as

$$U_1 = \langle U, (\frac{\mu_{c_1}}{0.1}, \frac{\mu_{c_2}}{0.3}, \frac{\mu_{c_3}}{0.4}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.9}, \frac{\nu_{c_2}}{0.7}, \frac{\nu_{c_3}}{0.6})\rangle,$$
$$U_2 = \langle U, (\frac{\mu_{c_1}}{0.2}, \frac{\mu_{c_2}}{0.3}, \frac{\mu_{c_3}}{0.4}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.8}, \frac{\nu_{c_2}}{0.7}, \frac{\nu_{c_3}}{0.6})\rangle,$$
$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.2}, \frac{\mu_{c_2}}{0.3}, \frac{\mu_{c_3}}{0.7}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.8}, \frac{\nu_{c_2}}{0.7}, \frac{\nu_{c_3}}{0.3})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{0.7}, \frac{\mu_{c_2}}{0.6}, \frac{\mu_{c_3}}{0.8}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.3}, \frac{\nu_{c_2}}{0.4}, \frac{\nu_{c_3}}{0.2})\rangle.$$

Then we have $\Psi = \{0, U_1, U_2, 1\}$. The sets U_3 & U_4 are N_seo sets. Then U is N_seCon.

Example 2 In Example 1, let

$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.0}, \frac{\mu_{c_2}}{0.0}, \frac{\mu_{c_3}}{1.0}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{1.0}, \frac{\nu_{c_2}}{1.0}, \frac{\nu_{c_3}}{0.0})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{1.0}, \frac{\mu_{c_2}}{1.0}, \frac{\mu_{c_3}}{0.0}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.0}, \frac{\nu_{c_2}}{0.0}, \frac{\nu_{c_3}}{1.0})\rangle.$$

The sets U_3 & U_4 are N_seo sets. Then U is N_seDCon.

Definition 7 Let (U, τ_N) be a N_sts on U. Let H be a N_ss of U.

1. If there exist N_seo sets K_1 and K_2 in U satisfying the following properties, then H is called neutrosophic e C_i-disconnected $(i = 1, 2, 3, 4)$:
   a. $C_1 : H \subseteq K_1 \cup K_2, K_1 \cap K_2 \subseteq H^c, H \cap K_1 \neq 0_N, H \cap K_2 \neq 0_N$.
   b. $C_2 : H \subseteq K_1 \cup K_2, H \cap K_1 \cap K_2 = 0_N, H \cap K_1 \neq 0_N, H \cap K_2 \neq 0_N$.
   c. $C_3 : H \subseteq K_1 \cup K_2, K_1 \cap K_2 \subseteq H^c, K_1 \nsubseteq H^c, K_2 \nsubseteq H^c$.
   d. $C_4 : H \subseteq K_1 \cup K_2, H \cap K_1 \cap K_2 = 0_N, K_1 \nsubseteq H^c, K_2 \nsubseteq H^c$.
2. H is said to be neutrosophic e C_i-connected $(i = 1, 2, 3, 4)$ if H is not N_seC_iDCon $(i = 1, 2, 3, 4)$. Obviously, the following implications are true.
   a. $N_seC_1Con \Rightarrow N_seC_2Con$.
   b. $N_seC_1Con \Rightarrow N_seC_3Con$.
   c. $N_seC_3Con \Rightarrow N_seC_4Con$.
   d. $N_seC_1Con \Rightarrow N_seC_4Con$.

Example 3 In Example 1, let

$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.8}, \frac{\mu_{c_2}}{0.5}, \frac{\mu_{c_3}}{0.6}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.2}, \frac{\nu_{c_2}}{0.5}, \frac{\nu_{c_3}}{0.4})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{0.8}, \frac{\mu_{c_2}}{0.5}, \frac{\mu_{c_3}}{0.4}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.2}, \frac{\nu_{c_2}}{0.5}, \frac{\nu_{c_3}}{0.6})\rangle,$$
$$U_5 = \langle U, (\frac{\mu_{c_1}}{0.2}, \frac{\mu_{c_2}}{0.4}, \frac{\mu_{c_3}}{0.9}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.8}, \frac{\nu_{c_2}}{0.6}, \frac{\nu_{c_3}}{0.1})\rangle.$$

The sets U_4 & U_5 are N_seo sets. Then U_3 is

1. N_seC_2Con but not N_seC_1Con.
2. N_seC_3Con but not N_seC_1Con.
3. N_seC_4Con but not N_seC_1Con.

Example 4 In Example 1, let

$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.8}, \frac{\mu_{c_2}}{0.5}, \frac{\mu_{c_3}}{0.6}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.2}, \frac{\nu_{c_2}}{0.5}, \frac{\nu_{c_3}}{0.4})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{0.8}, \frac{\mu_{c_2}}{0.5}, \frac{\mu_{c_3}}{0.4}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.2}, \frac{\nu_{c_2}}{0.5}, \frac{\nu_{c_3}}{0.6})\rangle,$$
$$U_5 = \langle U, (\frac{\mu_{c_1}}{0.2}, \frac{\mu_{c_2}}{0.5}, \frac{\mu_{c_3}}{0.9}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.8}, \frac{\nu_{c_2}}{0.5}, \frac{\nu_{c_3}}{0.1})\rangle.$$

The sets U_4 & U_5 are N_seo sets. Then U_3 is N_seC_4Con but not N_seC_3Con.

Definition 8 Let (U, τ_N) be a N_sts which is neutrosophic e C_5-disconnected (briefly, N_seC_5DCon) if there exists neutrosophic subset L in U which is both N_seo and N_sec in U, such that $L \neq 0_N$, $L \neq 1_N$. If U is not N_seC_5DCon, then it is called as neutrosophic e C_5-connected (briefly, N_seC_5Con).

Example 5 In Example 1, let
$U_3 = \langle U, (\frac{\mu_{c_1}}{0.1}, \frac{\mu_{c_2}}{0.2}, \frac{\mu_{c_3}}{0.3}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.9}, \frac{\nu_{c_2}}{0.8}, \frac{\nu_{c_3}}{0.7})\rangle$ is N_seC_5DCon.

Theorem 14.1 *Let (U, τ_N) be a N_sts on U. Then N_seC_5DCon-ness implies N_seCon-ness.*

Proof Suppose that there exist non-empty N_seo sets L & $M \ni L \cup M = 1_N$ & $L \cap M = 0_N$. Then $\mu_L \vee \mu_M = 1_N$, $\sigma_L \wedge \sigma_M = 0_N$, $\nu_L \wedge \nu_M = 0_N$, and $\mu_L \vee \mu_M = 0_N$, $\sigma_L \wedge \sigma_M = 1_N$, and $\nu_L \wedge \nu_M = 1_N$. In other words, $M^c = L$. Hence, L is N_seclo which implies U is N_seC_5Con. □

The following example shows the converse is not true.

Example 6 In Example 1, let

$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.2}, \frac{\mu_{c_2}}{0.3}, \frac{\mu_{c_3}}{0.7}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.8}, \frac{\nu_{c_2}}{0.7}, \frac{\nu_{c_3}}{0.3})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{0.7}, \frac{\mu_{c_2}}{0.6}, \frac{\mu_{c_3}}{0.8}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.3}, \frac{\nu_{c_2}}{0.4}, \frac{\nu_{c_3}}{0.2})\rangle.$$

The sets U_3 & U_4 are N_seo sets. Then U is N_seCon but not N_seC_5DCon.

Theorem 14.2 *Let $h : (U, \tau_N) \to (V, \tau_N)$ be a N_seIrr surjection and U be a N_seCon. Then V is N_seCon.*

Proof Assume that V is not N_seCon, then $\exists$ non-empty N_seo sets K_1 and K_2 in V such that $K_1 \cup K_2 = 1_N$ and $K_1 \cap K_2 = 0_N$. Since h is N_seIrr mapping, $K = h^{-1}(K_1) \neq 0_N$, and $M = h^{-1}(K_2) \neq 0_N$, which are N_seo sets in U, & $h^{-1}(K_1) \cup h^{-1}(K_2) = h^{-1}(1_N) = 1_N$, which implies $K \cup M = 1_N$. Also, $h^{-1}(K_1) \cap h^{-1}(K_2) = h^{-1}(0_N) = 0_N$, which implies $K \cap M = 0_N$. By hypothesis, this is a contradiction to U which is N_seDCon. Hence, V is N_seCon. □

Theorem 14.3 *Let (U, τ_N) be a N_sts which is N_seC_5Con if and only if there exist no non-empty N_seo sets L & M in $U \ni L = M^c$.*

Proof Suppose that L and M are N_seo sets in U such that $L \neq 0_N$, $M \neq 0_N$, and $L = M^c$. Since $L = M^c$, M^c is a N_seo set, and M is a N_sec set, and $L \neq 0_N$ implies $M \neq 1_N$. But it is contradiction to U is N_seC_5Con.

Conversely, let L & M be both N_seos & N_sec set in U such that $L \neq 0_N$, $L \neq 1_N$. Now take $L^c = M$ as a N_seo set and $L \neq 1_N$ which implies $L^C = M \neq 0_N$, a contradiction. Hence, U is N_seC_5Con. □

Theorem 14.4 *Let (U, τ_N) be a N_sts is N_seCon space iff $\exists$ no non-zero N_seo set K & M in U, $\ni K = M^c$.*

Proof Necessity: Let K & M be two N_seo sets in $U \ni K \neq 0_N$, $M \neq 0_N$, and $K = M^c$. Therefore, M^c is a N_sec set. Since $K \neq 0_N$, $M \neq 1_N$. This implies M is a proper neutrosophic subset which is both N_seo set and N_sec set in U. Hence, U is not a N_seCon space. By hypothesis, it is a contradiction. Thus, there exists no non-zero N_seo sets K & M in U, $\ni K = M^c$.

Sufficiency: Let K be both N_seo & N_sec, $U \ni K \neq 0_N$, and $K \neq 1_N$. Now let $M = K^c$. Then M is a N_seo set and $M \neq 1_N$. This implies $K^c = M \neq 0_N$; by hypothesis, it is a contradiction. Therefore, U is N_seCon space. □

Theorem 14.5 *Let (U, τ_N) be a N_sts is N_seCon space iff $\exists$ no non-zero neutrosophic subsets K & M in U, $\ni K = M^c$, $M = (N_secl(K))^c$ and $K = (N_scl(M))^c$.*

Proof Necessity: Let K and M be two neutrosophic subsets in $U \ni K \neq 0_N$, $M \neq 0_N$ and $K = M^c$, $M = (N_secl(K))^c$, & $K = (N_secl(M))^c$. Since $(N_secl(K))^c$ and $(N_secl(M))^c$ are N_seo sets in U, K and M are N_seo set in U. This implies U is not a N_seCon space, a contradiction. Thus, there exists no non-zero N_seo set K & M in U, $\ni K = M^c$, $M = (N_secl(K))^c$ and $K = (N_secl(M))^c$.

Sufficiency: Let K be both N_seo and N_sec set in U such that $K \neq 0_N$ and $K \neq 1_N$. Now let $M = K^c$; by hypothesis, we get a contradiction. Hence, U is N_seCon space. □

Definition 9 Let (U, τ_N) be a N_sts is neutrosophic e strongly connected (briefly, $N_seStCon$), if $\exists$ no nonempty N_sec sets L & M in $U \ni \mu_L + \mu_M \geq 1_N$, $\sigma_L + \sigma_M \geq 1_N$, $\nu_L + \nu_M \leq 1_N$ or $\mu_L + \mu_M \geq 1_N$, $\sigma_L + \sigma_M \leq 1_N$, $\nu_L + \nu_M \leq 1_N$.

In other words, a N_sts U is $N_seStCon$, if there exist no nonempty N_sec sets L & M in $U \ni L \cap M = 0_N$.

Example 7 In Example 1, let

$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.0}, \frac{\mu_{c_2}}{0.0}, \frac{\mu_{c_3}}{1.0}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{1.0}, \frac{\nu_{c_2}}{1.0}, \frac{\nu_{c_3}}{0.0})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{1.0}, \frac{\mu_{c_2}}{1.0}, \frac{\mu_{c_3}}{0.0}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.0}, \frac{\nu_{c_2}}{0.0}, \frac{\nu_{c_3}}{1.0})\rangle.$$

The sets U_3 & U_4 are N_seo sets. Then U is $N_seStCon$.

Theorem 14.6 *Let (U, τ_N) be a N_sts which is $N_seStCon$, if there exist no nonempty N_seo sets L & M in U, $L \neq 1_N \neq M \ni \mu_L + \mu_M \geq 1_N$, $\sigma_L + \sigma_M \geq 1_N$, and $\nu_L + \nu_M \leq 1_N$.*

Proof Let L & M be N_seo sets in U such that $L \neq 1 \neq M$ and $\mu_L + \mu_M \geq 1_N$, $\sigma_L + \sigma_M \geq 1_N$, and $\nu_L + \nu_M \leq 1_N$. If we take $C = L^c$ and $D = M^c$, then C & D have N_sec sets in U and $C \neq 0_N \neq D$, $\nu_C + \nu_D = \mu_L + \mu_M \geq 1_N$, $\mu_C + \mu_D = \sigma_L + \sigma_M \geq 1_N$, and $\sigma_C + \sigma_D = \nu_L + \nu_M \leq 1_N$, a contradiction.

Conversely, use a similar technique as above. □

Theorem 14.7 *Let $h : (U, \tau_N) \to (V, \tau_N)$ be a N_seIrr surjection and U be a $N_seStCon$. Then V is also $N_seStCon$.*

Proof Assume that V is not $N_seStCon$, then $\exists$ nonempty N_sec sets K & M in $V \ni K \neq 0_N$, $M \neq 0_N$ & $K \cap M = 0_N$. Since h is N_seIrr mapping, $A = h^{-1}(K) \neq 0_N$, and $B = h^{-1}(M) \neq 0_N$, which are N_sec sets in U, & $h^{-1}(K) \cap h^{-1}(M) = h^{-1}(0_N) = 0_N$, which implies $A \cap B = 0_N$. By hypothesis, this is a contradiction to U which is not a $N_seStCon$. Hence, V is $N_seStCon$. □

Remark 14.1 $N_seStCon$ and N_seC_5Con are independent.

Example 8 In Example 1, let

$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.0}, \frac{\mu_{c_2}}{0.0}, \frac{\mu_{c_3}}{1.0}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{1.0}, \frac{\nu_{c_2}}{1.0}, \frac{\nu_{c_3}}{0.0})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{1.0}, \frac{\mu_{c_2}}{1.0}, \frac{\mu_{c_3}}{0.0}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.0}, \frac{\nu_{c_2}}{0.0}, \frac{\nu_{c_3}}{1.0})\rangle.$$

The sets U_3 & U_4 are N_seo sets. Then U is $N_seStCon$ but not N_seC_5Con.

Example 9 In Example 1, let

$$U_3 = \langle U, (\frac{\mu_{c_1}}{0.8}, \frac{\mu_{c_2}}{0.6}, \frac{\mu_{c_3}}{0.7}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.2}, \frac{\nu_{c_2}}{0.4}, \frac{\nu_{c_3}}{0.3})\rangle,$$
$$U_4 = \langle U, (\frac{\mu_{c_1}}{0.3}, \frac{\mu_{c_2}}{0.6}, \frac{\mu_{c_3}}{0.8}), (\frac{\sigma_{c_1}}{0.5}, \frac{\sigma_{c_2}}{0.5}, \frac{\sigma_{c_3}}{0.5}), (\frac{\nu_{c_1}}{0.7}, \frac{\nu_{c_2}}{0.4}, \frac{\nu_{c_3}}{0.2})\rangle.$$

The sets U_3 & U_4 are N_seo sets. Then U is N_seC_5Con but not $N_seStCon$.

14.4 Neutrosophic *e* Separated Sets

Definition 10 Let (U, τ_N) be a N_sts. If L and M are non-zero neutrosophic subsets in U, then L and M are said to be

1. neutrosophic e weakly separated (briefly, N_seWSep) if $N_secl(L) \subseteq M^c$ & $N_secl(M) \subseteq L^c$.
2. neutrosophic e separated (briefly, N_seSep) if $N_secl(L) \cap M = L \cap N_secl(M) = 0_N$.

Remark 14.2 Let (U, τ_N) be a N_sts. Any two disjoint non-empty N_sec sets are N_seSep.

Proof Suppose L and M are disjoint non-empty N_sc sets. Then $N_secl(L) \cap M = L \cap N_secl(M) = L \cap M = 0_N$. This shows that L and M are N_seSep. □

Theorem 14.8 *Let (U, τ_N) be a N_sts. If L and M are non-zero neutrosophic subsets in U.*

1. *If L and M are N_seSep and $C \subseteq L$, $D \subseteq M$, then C and D are also N_seSep.*
2. *If L and M are both N_seo sets and if $H = L \cap M^c$ and $G = M \cap L^c$, then H and G are N_seSep.*

Proof

1. Let L and M be N_seSep sets in N_sts U. Then $N_secl(L) \cap M = 0_N = L \cap N_secl(M)$. Since $C \subseteq L$ and $D \subseteq M$, then $N_secl(C) \subseteq N_secl(L)$ and $N_secl(D) \subseteq N_secl(M)$. This implies that $N_secl(C) \cap D \subseteq N_secl(L) \cap M = 0_N$ and hence $N_secl(C) \cap D = 0_N$. Similarly, $N_secl(D) \cap C \subseteq N_secl(M) \cap L = 0_N$ and hence $N_secl(D) \cap C = 0_N$. Therefore, C and D are N_seSep.
2. Let L and M both N_seo subsets in U. Then L^c and M^c are N_sec sets. Since $H \subseteq M^c$, then $N_secl(H) \subseteq N_secl(M^c) = M^c$ and so $N_secl(H) \cap M = 0_N$. Since $G \subseteq M$, then $N_secl(H) \cap G \subseteq N_secl(H) \cap M = 0_N$. Thus, $N_secl(H) \cap G = 0_N$. Similarly, $N_secl(G) \cap H = 0_N$. Hence, H and G are N_seSep. □

Theorem 14.9 *Let (U, τ_N) be a N_sts. If A & B are non-zero neutrosophic subsets in U are N_seSep if and only if there exist L and M in N_seo set in $U \ni A \subseteq L$, $B \subseteq M$ & $A \cap M = 0_N$ and $B \cap L = 0_N$.*

Proof Let A & B be N_seSep. Then $A \cap N_secl(B) = 0_N = N_secl(A) \cap B$. Take $M = (N_secl(A))^c$ & $L = (N_secl(B))^c$. Then L & M are N_so sets $\ni A \subseteq L$, $B \subseteq M$, & $A \cap M = 0_N$ and $B \cap L = 0_N$.

Conversely, let L and M be N_so sets $\ni A \subseteq L$, $B \subseteq M$, & $A \cap M = 0_N$ and $B \cap L = 0_N$. Then $A \subseteq M^c$ and $B \subseteq L^c$ and M^c and L^c are N_sc. This implies $N_secl(A) \subseteq N_secl(M^c) = M^c \subseteq B^c$ and $N_secl(B) \subseteq N_secl(L^c) = L^c \subseteq A^c$. That is, $N_secl(A) \subseteq B^c$ and $N_secl(B) \subseteq A^c$. Therefore, $A \cap N_secl(B) = 0_N = N_secl(A) \cap B$. Hence, A and B are N_seSep. □

Proposition 14.1 *Let (U, τ_N) be a N_sts. Each two N_seSep sets are always disjoint.*

Proof Let K & M be N_seSep. Then $K \cap N_secl(M) = 0_N = N_secl(K) \cap M$. Now, $K \cap M \subseteq K \cap N_secl(M) = 0_N$. Therefore, $K \cap M = 0_N$, and hence, K & M are disjoint. □

Theorem 14.10 *Let (U, τ_N) be a N_sts. Then U is N_seCon iff $1_N \neq K \cup M$, where K & M are N_seSep sets.*

Proof Assume that U is a N_seCon space. Suppose $1_N = K \cup M$, where K and M are N_seSep sets. Then $N_secl(K) \cap M = K \cap N_secl(M) = 0_N$. Since $K \subseteq N_secl(K)$, we have $K \cap M \subseteq N_secl(K) \cap M = 0_N$. Therefore, $N_secl(K) \subseteq M^c = K$ & $N_secl(M) \subseteq K^c = M$. Hence, $K = N_secl(K)$ and $M = N_secl(M)$. Therefore, K and M are N_sec sets, and hence, $K = M^c$ and $M = K^c$ are disjoint N_seo sets. Thus $K \neq 0_N$, $M \neq 0_N \ni K \cup M = 1_N$ & $K \cap M = 0_N$, K and M are N_seo sets. That is, U is not N_seCon, which is a contradiction to U which is a N_seCon space. Hence, 1_N is not the union of any two N_seSep sets.

Conversely, assume that 1_N is not the union of any two N_seSep sets. Suppose U is not N_seCon. Then $1_N = K \cup M$, where $K \neq 0_N$, $M \neq 0_N$ such that and $K \cap M = 0_N$, K and M are N_seo sets in U. Since $K \subseteq M^c$ & $M \subseteq K^c$, $N_secl(K) \cap M \subseteq M^c \cap M = 0_N$ and $K \cap N_secl(M) \subseteq K \cap K^c = 0_N$. That is, K and M are N_seSep sets. This is a contradiction. Therefore, U is N_seCon. □

Definition 11 Let (U, τ_N) be a N_sts. If L is a non-zero neutrosophic subset in U, then

1. neutrosophic e regular open (briefly, N_sero) set if $L = N_seint(N_secl(L))$.
2. neutrosophic e regular closed (briefly, N_serc) set if $L = N_secl(N_seint(L))$.
3. The complement of N_sero set is N_serc set.

Proposition 14.2 *Let (U, τ_N) be a N_sts.*

1. *Every N_sero set is N_seo.*
2. *Every N_serc set is N_sec.*

Proof

1. Let L be a N_sero in U. Then $L = N_seint(N_secl(L))$. Since the union of N_seo sets in U is also N_seo set in U, $N_seint(N_secl(L))$ is N_seo. Therefore, L is N_seo.
2. Similar proof of 1. □

Definition 12 Let (U, τ_N) be a N_sts. Then U is neutrosophic e super disconnected (briefly, $N_sesuperDCon$) if $\exists$ a N_sero set L in $U \ni L \neq 0_N$ & $L \neq 1_N$. A N_sts U is called neutrosophic e super connected (briefly, $N_sesuperCon$) if U is not $N_sesuperDCon$.

Example 10 In Example 1, let

$U_3 = \langle U, (\frac{\mu_{c1}}{0.1}, \frac{\mu_{c2}}{0.2}, \frac{\mu_{c3}}{0.3}), (\frac{\sigma_{c1}}{0.5}, \frac{\sigma_{c2}}{0.5}, \frac{\sigma_{c3}}{0.5}), (\frac{\nu_{c1}}{0.9}, \frac{\nu_{c2}}{0.8}, \frac{\nu_{c3}}{0.7})\rangle$ is $N_s esuperDCon$.

Theorem 14.11 *Let (U, τ_N) be a $N_s ts$; the following*

1. *U is $N_s esuperCon$.*
2. *For each $N_s eo$ set $L \neq 0_N$ in U, we have $N_s ecl(L) = 1_N$.*
3. *For each $N_s ec$ set $L \neq 1_N$ in U, we have $N_s eint(L) = 0_N$.*
4. *There exist no $N_s eo$ subsets L & M in U, $\ni L \neq 0_N$, $M \neq 0_N$, & $L \subseteq M^c$.*
5. *There exist no $N_s eo$ subsets L & M in U, $\ni L \neq 0_N$, $M \neq 0_N$, $M = (N_s ecl(L))^c$, and $L = (N_s ecl(M))^c$.*
6. *There exist no $N_s ec$ subsets L & M in $U \ni L \neq 1_N$, $M \neq 1_N$, $M = (N_s ecl(L))^c$, and $L = (N_s ecl(M))^c$*

are equivalent.

Proof $1 \Rightarrow 2$: Assume that, there exists a $N_s o$ set $L \neq 0_N$ such that $N_s ecl(L) \neq 1_N$. Now take $M = N_s eint(N_s ecl(L))$. Then M is proper $N_s ro$ set in U which contradicts that U is $N_s esuperCon$-ness.
$2 \Rightarrow 3$: Let $L \neq 1_N$ be a $N_s c$ set in U. If $M = L^c$, then M is $N_s o$ set in U and $M \neq 0_N$. Hence, $N_s ecl(L) = 1_N$, and $(N_s ecl(M))^c = 0_N \Rightarrow N_s eint(M^c) = 0 \Rightarrow N_s eint(L) = 0_N$.
$3 \Rightarrow 4$: Let L and M be $N_s o$ sets in U such that $L \neq 0_N \neq M$ and $L \subseteq M^c$. Since M^c is an $N_s c$ set in U and $M \neq 0_N$ implies $M^c \neq 1_N$, we obtain $N_s eint(M^c) = 0_N$. But, from $L \subseteq M^c$, $0_N \neq L = N_s eint(L) \subseteq N_s eint(M^c) = 0_N$, which is a contradiction.
$4 \Rightarrow 1$: Let $0_N \neq L \neq 1_N$ be a $N_s ro$ set in U. If we take $M = (N_s ecl(L))^c$, we get $M \neq 0_N$. Otherwise, we have $M \neq 0_N$ which implies $(N_s ecl(L))^c = 0_N$. That implies $N_s ecl(L) = 1_N$. That shows $L = N_s eint(N_s ecl(L)) = N_s eint(1_N) = 1_N$. But this is a contradiction to $L \neq 1_N$.
Further, $L \subseteq M^c$, this is also a contradiction.
$1 \Rightarrow 5$: Let L and M be $N_s o$ sets in U such that $L \neq 0_N \neq M$ and $M = (N_s ecl(L))^c$ and $L = (N_s eint(M))^c$. Now $N_s eint(N_s ecl(L)) = N_s eint(M^c) = (N_s ecl(M))^c = L$ and $L \neq 0_N$ and $L \neq 1_N$. Suppose not; if $L = 1_N$, then $1_N = (N_s ecl(M))^c$ implies $0 = N_s ecl(M) \Rightarrow M = 0$. This is a contradiction.
$5 \Rightarrow 1$: Let L be a $N_s o$ set in U such that $L = N_s eint(N_s ecl(L))$, $0_N \neq L \neq 1_N$. Now $M = (N_s ecl(L))^c$ and $(N_s ecl(M))^c = (N_s ecl(N_s ecl(L))^c)^c = N_s eint(N_s ecl(L)) = L$. This is a contradiction.
$5 \Rightarrow 6$: Let L & M be a $N_s c$ set in U such that $L \neq 1_N \neq M$. $M = (N_s eint(L))^c$, and $L = (N_s eint(M))^c$. Taking $C = L^c$ & $D = M^c$, C & D become $N_s o$ set in U and $C \neq 0_N \neq D$, $(N_s ecl(C))^c = (N_s ecl(L^c))^c = ((N_s eint(L))^c)^c = N_s eint(L) = M^c = D$, and similarly $(N_s ecl(D))^c = C$. But this is a contradiction.
$6 \Rightarrow 5$: Similar as in above. □

14.5 Neutrosophic *e* Compact Spaces

Definition 13 Let (U, τ_N) be a $N_s ts$. A collection M of $N_s o$ (resp. $N_s eo$) sets in U is called a neutrosophic (resp. e) open cover (briefly, $N_s ocov$ (resp. $N_s eocov$)) of a subset M of U if $M \subseteq \bigvee\{L_\alpha : L_\alpha \in M\}$.

Definition 14 Let (U, τ_N) be a $N_s ts$; then U is said to be neutrosophic (resp. e) compact (briefly, $N_s Comp$ (resp. $N_s eComp$)) if every $N_s ocov$ (resp. $N_s eocov$) of U has a finite subcover.

Definition 15 Let (U, τ_N) be a $N_s ts$. A subset L of U is said to be $N_s Comp$ (resp. $N_s eComp$) relative to U if every $N_s ocov$ (resp. $N_s eocov$) of U has a finite subcover.

Theorem 14.12 *Let (U, τ_N) be a $N_s ts$. Every $N_s eComp$ space is $N_s Comp$.*

Proof Let U be $N_s eComp$. Suppose U is not $N_s Comp$. Thus, there exists no non-zero $N_s ocov$ M of U has no finite subcover. Since every $N_s o$ set is a $N_s eo$ set, then we have $N_s eocov$ M of U, which has no finite subcover. This is a contradiction to U which is $N_s Comp$. Hence, U is $N_s Comp$. □

Theorem 14.13 *Let (U, τ_N) be a $N_s ts$. A $N_s ec$ subset of a $N_s eComp$ space U is $N_s eComp$ relative to U.*

Proof Let L be a $N_s c$ subset of a $N_s Comp$ space U. Then L^c is $N_s o$ in U. Let $M = \{L_i : i \in I\}$ be a $N_s eocov$ of L. Then $M \bigvee \{L^c\}$ is a $N_s eocov$ of U. Since U is $N_s Comp$, it has a finite subcover say $\{P_1, P_2, \cdots P_n, L^c\}$. Then $\{P_1, P_2, \ldots, P_n\}$ is a finite $N_s eocov$. Thus, L is $N_s eComp$ relative to U. □

Theorem 14.14 *Let $h : (U, \tau_N) \to (V, \tau_N)$ be a $N_s eCts$ surjection and U be $N_s eComp$. Then V is $N_s eComp$.*

Proof Let h be a $N_s eCts$ surjection & U be $N_s eComp$. Let $\{M_\alpha\}$ be a $N_s eocov$ for V. Since h is $N_s eCts$, $\{h^{-1}(M_\alpha)\}$ is a $N_s eocov$ of U. Since U is $N_s eComp$, $\{h^{-1}(M_\alpha)\}$ contains a finite subcover, namely, $\{h^{-1}(M_{\alpha_1}), h^{-1}(M_{\alpha_2}), \ldots, h^{-1}(M_{\alpha_n}$ $)\}$. Since h is a surjection, $\{M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}\}$ is a finite subcover for V. Thus, V is $N_s eComp$. □

Theorem 14.15 *Let $h : (U, \tau_N) \to (V, \tau_N)$ be a $N_s eO$ function and V be $N_s eComp$. Then U is $N_s eComp$.*

Proof Let h be a $N_s eO$ function & V be $N_s eComp$. Let $\{M_\alpha\}$ be a $N_s eocov$ for U. Since h is $N_s eO$, $\{h(M_\alpha)\}$ is a $N_s eocov$ of V. Since V is $N_s eComp$, $\{h(M_\alpha)\}$ contains a finite sub-$N_s eocov$, namely. $\{h(M_{\alpha_1}), h(M_{\alpha_2}), \ldots, h(M_{\alpha_n})\}$. Then $\{M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}\}$ is a finite subcover for U. Thus, U is $N_s eComp$. □

Theorem 14.16 *Let $h : (U, \tau_N) \to (V, \tau_N)$; then the image of a N_seComp space under a N_seCts map is N_seComp.*

Proof Let $h : (U, \tau_N) \to (V, \tau_N)$ be a N_seCts map from a N_seComp space (U, τ_N) onto (V, τ_N). Let $\{L_i : i \in K\}$ be a N_seocov of (V, τ_N). Since h is N_seCts, $\{h^{-1}(L_i) : i \in K\}$ is a N_seocov of (U, τ_N). As (U, τ_N) is N_seComp, the N_seocov $\{h^{-1}(L_i) : i \in K\}$ of (U, τ_N) has a finite subcover $\{h^{-1}(L_i) : i = 1, 2, 3, \dots, n\}$. Therefore, $L = \bigcup_{i \in K} h^{-1}(L_i)$. Then $h(L) = \bigcup_{i \in K} L_i$, that is, $M = \bigcup_{i \in K} L_i$. Thus, $\{L_1, L_2, \dots, L_n\}$ is a finite subcover of $\{L_i : i \in K\}$ for (V, τ_N). Hence, (V, τ_N) is N_seComp. □

14.6 Conclusion

In the neutrosophic topological space, neutrosophic e connectedness and neutrosophic e disconnectedness have been addressed. In addition, the neutrosophic e compactness in neutrosophic topological space has been addressed. Normal and regular spaces on the neutrosophic e open set can be built based on this. We believe that the discoveries in this chapter will aid scholars in furthering and promoting research on neutrosophic topology in order to develop a general framework for their practical applications.

Acknowledgments The authors would like to thank the editors and the anonymous reviewers for their valuable comments and suggestions which have helped immensely in improving the quality of the paper.

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Chapter 15
Generalized Finite Continuous Ridgelet Transform

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AMS Subject Classification: **46F12, 46F10, 44A20, 44A45**

15.1 Introduction

The definition of continuous Ridgelet transform for two variable function $f(x, y)$ is given in [14]. In [15], the author defined and studied Ridgelet transform of the distributions. In [16], the Ridgelet transform of Schwartz distributions was studied.

The aim of this chapter is to extend classical finite continuous Ridgelet transform [4] analogous to generalized functions on certain spaces, which is defined on the interval $[-\rho, \rho] \times [-\sigma, \sigma]$:

$$\Re h\left(I\left(\omega_{s,t}, \mu, \nu\right)\right) = \frac{\mu^{-1/2}}{4\rho\sigma} \int_{-\sigma}^{\sigma} \int_{-\rho}^{\rho} h(x, y)\,\kappa(\omega)\,dxdy \tag{15.1}$$

for every $s, t = 1, 2, 3, \cdots$, where $\kappa(\omega) = \psi\left(\left(\omega_{s,t} \cdot (x, y) - \nu\right)/\mu\right)$ and $\omega_{s,t} = \left(\frac{\pi\mu p}{\rho} + \frac{\pi\mu q}{\sigma}\right) \in \mathbb{R}^2$ and $\mu, \nu \in \mathbb{R}$. The bounded variation in $[-\rho_1, \rho_1] \times [-\sigma_1, \sigma_1]$, $[-\rho < -\rho_1 < \rho_1 < \rho], [-\sigma < -\sigma_1 < \sigma_1 < \sigma]$ and $(t_1, t_2) \in [-\rho_1, \rho_1] \times [-\sigma_1, \sigma_1]$, then the series $c_{s,t}\mu^{1/2}\left(1 + \sum_{s,t=1}^{\infty} e^{((p\mu\pi x/\rho)+(q\mu\pi y/\sigma)-\nu)/\mu}\right)^{-1}$ converges to $\frac{1}{2}[f(-\rho, -\sigma) + f(\rho, \sigma)]$. Orthogonal series representation for generalized function was studied by Pathak [12]. In [8, 9], the authors developed expansions of distributions. The author in [17] studied series of orthogonal functions on distributional sense. The author in [6] studied finite generalized Hankel transformation on different spaces extended to class of generalized functions.

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S. Singh et al. (eds.), *Algebra, Analysis, and Associated Topics*, Trends in Mathematics, https://doi.org/10.1007/978-3-031-19082-7_15

Gelfand and Shilov [10] applied generalized functions in harmonic analysis. The generalization of absolute value equations was studied in [18].

In [19], using the finite Fourier sine transform method, the authors solved the boundary value problem for Kirchhoff plates that was supported by simply supported rectangular beams. The numerical results in [11] confirmed that the Radon transform formulation is valid when applied to the vibrations of rectangular thin plates. Bending of fully clamped an orthotropic rectangular thin plates solution is presented in [7] using finite continuous Ridgelet transforms subjected to loadings. In this text, notation and terminology are from [1], and interval is considered as $I = [-\rho_1, \rho_1] \times [-\sigma_1, \sigma_1]$. From [4], we get

$$\Omega_{x,y,\theta} = \left(\sin^2\theta\right)\Omega_x - \left(\cos^2\theta\right)\Omega_y \tag{15.2}$$

where $\Omega_x = D_x^2$ and $\Omega_y = D_y^2$; ρ, σ are real constant and $D_x = \frac{d}{dx}$ and $D_y = \frac{d}{dy}$.

The following operational formula is computable as follows:

$$\Omega^k_{x,y,\theta}\kappa\left(\omega\right) = \left[\left(-\eta_s^2\right)^k \sin^2\theta\cos^{2k}\theta - \left(-\eta_t^2\right)^k \cos^2\theta\sin^{2k}\theta\right]\kappa\left(\omega\right) \tag{15.3}$$

for every $k = 0, 1, 2, \cdots$.

15.2 $W_{\mu,\nu}(I)$ as Testing Function Space with Its Dual $W'_{\mu,\nu}(I)$

Theorem 15.1 *$W_{\mu,\nu}(I)$ is a countably multinormed space for every real number μ, ν s.t. $1 < \mu, \nu < \infty$.*

Proof Assuming the topology of linear space $W_{\mu,\nu}(I)$ is generated by the collection of seminorms, $\left(\chi_k^{\mu,\nu}\right)$ is given by

$$\chi_k^{\mu,\nu}\left(\psi\right) = \sup_I \left|\Omega_{x,y,\theta}\psi\left(x,y\right)\right| < \infty \tag{15.4}$$

where $\psi\left(x,y\right)$ is an infinitely differentiable complex-valued function on I. Hence, analogous to [5], $W_{\mu,\nu}(I)$ is a countably multinormed space. □

Some results analogous to [1] are:

Property 1. $|\langle f, \psi\rangle| = \int_{-\sigma}^{\sigma}\int_{-\rho}^{\rho} h\left(x,y\right)\psi\left(x,y\right)dxdy$, where $h\left(x,y\right)$ generates a regular generalized function in $W'_{\mu,\nu}(I)$.

Property 2. $|\langle f, \psi\rangle| \leq K \max_{0\leq k\leq s} \chi_k^s\left(\psi\right)$ for every $\psi \in W_{\mu,\nu}(I)$ for a positive constant K and a non-negative integer s.

Property 3. For each $k = 0, 1, 2, \cdots$ and from (15.3):

$$\chi_k^{\mu,\nu}\kappa(\omega) = \sup_I \left| \left[\left(-\eta_s^2\right)^k \sin^2\theta \cos^{2k}\theta - \left(-\eta_t^2\right)^k \cos^2\theta \sin^{2k}\theta \right] \kappa(\omega) \right| < \infty \tag{15.5}$$

where $\kappa(\omega)$ is a member of $W_{\mu,\nu}(I)$.

Property 4. $\Omega_{x,y,\theta}\psi(x,y) = \sum_{j=0}^{2k} \left[\left(\sin^2\theta\right) \Omega_x^{2k-j} - \left(\cos^2\theta\right) \Omega_y^{2k-j} \right] \psi(x,y).$

Theorem 15.2 *If $W_{\mu,\nu}(I)$ is a testing function space, then it is a Fréchet space.*

Proof Let $(x_1, y_1) \in I$ and $z = (x, y)$ be variable in I. The result can be proved using [1, p. 253] and [3]. Considering L as an arbitrary compact subset of I and $D_x^{-1} = \int_{x_1}^{x} \cdots dx$ and $D_y^{-1} = \int_{y_1}^{y} \cdots dy$ are integral operators.

Assuming $\{\psi_{s,t}\}$ be Cauchy sequence in $W_{\mu,\nu}(I)$ and from (15.4), $\Omega_{x,y,\theta}\psi_{s,t}(x,y)$ converges uniformly on L as $s, t \to \infty$.

Hence, if $k = 0$, then $\{\psi_{s,t}\}$ converges uniformly on L as $s, t \to \infty$.

If $k = 1$, then

$$\Omega_{x,y,\theta}\psi_{s,t}(x,y) = \left[\left(\sin^2\theta\right) \Omega_x - \left(\cos^2\theta\right) \Omega_y \right] \psi_{s,t}(x,y). \tag{15.6}$$

From (15.6)

$$\begin{aligned} D_x^{-1}\left[\left(\sin^2\theta\right)\Omega_x\psi_{s,t}(x,y)\right] &= \left(\sin^2\theta\right) D_x^{-1} D_x^2 \left(\psi_{s,t}(x,y)\right) \\ &= \psi_{s,t}(x,y) - \psi_{s,t}(x_1,y) - (x - x_1) D_{x_1}\left(\psi_{s,t}(x_1,y)\right). \end{aligned} \tag{15.7}$$

$$\begin{aligned} D_y^{-1}\left[\left(\cos^2\theta\right)\Omega_y\psi_{s,t}(x,y)\right] &= \left(\cos^2\theta\right) D_y^{-1} D_y^2 \left(\psi_{s,t}(x,y)\right) \\ &= \psi_{s,t}(x,y) - \psi_{s,t}(x,y_1) - (y - y_1) D_{y_1}\left(\psi_{s,t}(x,y_1)\right). \end{aligned} \tag{15.8}$$

Equations (15.7) and (15.8) show that $D_x^k\psi_{s,t}(x,y)$ and $D_y^k\psi_{s,t}(x,y)$ converge uniformly on L as $s, t \to \infty$, for each $k \geq 0$ such that $D_x^k\psi_{s,t}(x,y) \to D_x^k\psi(x,y)$ and $D_y^k\psi_{s,t}(x,y) \to D_y^k\psi(x,y)$ as $s, t \to \infty$.

Hence, $\psi \in W_{\mu,\nu}(I)$ and $\psi(x,y)$ is the limit of the sequence $\psi_{s,t}(x,y)$ in this space. Consider dual space of $W_{\mu,\nu}(I)$ as $W'_{\mu,\nu}(I)$, where $W'_{\mu,\nu}(I)$ is weak convergence and thus the proof of the theorem. □

15.3 Inversion Theorem

The inner product from (15.1) can be written as

$$\Re h\left(I\left(\omega_{s,t},\mu,\nu\right)\right)=\left\langle f\left(X\right),\frac{\mu^{-1/2}}{4\rho\sigma}\kappa\left(\omega\right)\right\rangle \tag{15.9}$$

where $h\left(x,y\right)=h\left(X\right)$, where $X=\left(x,y\right)$.

Using [2], $\upsilon_{s,t}\left(R,X\right)$ is defined as

$$T_{s,t}\left(R,X\right)=\sum_{t=1}^{T}\sum_{s=1}^{S}\frac{\mu^{-1/2}}{4\rho\sigma}\psi\left(\left(\omega_{s,t}\cdot\left(R\right)-\nu\right)/\mu\right)\psi\left(\left(\omega_{s,t}\cdot\left(X\right)-\nu\right)/\mu\right)$$

for $S,T\in I^{+}$, $\forall\ S\neq T$.

For $X\in I$, the properties are stated below.

Theorem 15.3 *From (15.1), if* $\Re f\left(I\left(\omega_{s,t},\mu,\nu\right)\right)$ *is GFCRT of* f *, then*

$$h\left(x,y\right)=\lim_{T,S\to\infty}\sum_{t=1}^{T}\sum_{s=1}^{S}\mu^{-1/2}\,\Re h\left(I\left(\omega_{s,t},\mu,\nu\right)\right)\psi\left(u\cdot X\right) \tag{15.10}$$

converges in $D'\left(I\right)$.

Proof For $\psi\left(x,y\right)$ in $-\rho<-k<k<\rho,-\sigma<-l<l<\sigma$.

To prove (15.10) converges in $D'\left(I\right)$, this is the equivalent to proving as

$$\left\langle T_{s,t}\left(R,X\right),\psi\left(X\right)\right\rangle\to\left\langle h\left(R\right),\psi\left(R\right)\right\rangle \tag{15.11}$$

as $S,T\to\infty$.

Therefore, the theorem can be demonstrated as

$$\left\langle\sum_{t=1}^{T}\sum_{s=1}^{S}\Re h\left(I\left(\omega_{s,t},\mu,\nu\right)\right)\psi\left(\left(\omega_{s,t}\cdot\left(X\right)-\nu\right)/\mu\right),\psi\left(X\right)\right\rangle \tag{15.12}$$

$$=\int_{-\rho}^{\rho}\int_{-\sigma}^{\sigma}\sum_{t=1}^{T}\sum_{s=1}^{S}\Re h\left(I\left(\omega_{s,t},\mu,\nu\right)\right)\psi\left(\left(\omega_{s,t}\cdot\left(X\right)-\nu\right)/\mu\right)\psi\left(X\right)dxdy \tag{15.13}$$

$$=\left\langle h\left(R\right),\int_{-\rho}^{\rho}\int_{-\sigma}^{\sigma}T_{s,t}\left(R,X\right)\psi\left(X\right)dxdy\right\rangle \tag{15.14}$$

$$\to\left\langle h\left(R\right),\psi\left(R\right)\right\rangle. \tag{15.15}$$

Using property 15.2 and $\forall\ \Omega_{x,y,\theta}\psi\left(X\right)\in D'\left(I\right)$:

$$\Omega_{t,\theta}^{k}\left[\int_{-\rho}^{\rho}\int_{-\sigma}^{\sigma} T_{s,t}(R,X)\,\psi(X)dxdy-\psi(R)\right]$$

$$=\int_{-\rho}^{\rho}\int_{-\sigma}^{\sigma} T_{s,t}(R,X)\left\{\begin{array}{l}\left[\left(\sin^2\theta\right)\Omega_x^k-\left(\cos^2\theta\right)\Omega_y^k\right]\psi(X)\\ -\left[\left(\sin^2\theta\right)\Omega_{t_1}^k-\left(\cos^2\theta\right)\Omega_{t_2}^k\right]\psi(R)\end{array}\right\}dxdy$$

$$=\int_{-\rho}^{\rho}\int_{-\sigma}^{\sigma} T_{s,t}(R,X)\left[\psi(X)-\psi(R)\right]dxdy$$

which completes the proof. □

Theorem 15.4 *Let* $f\in W'_{\mu,\nu}(I)$ *and* $\forall\ R=(t_1,t_2)\in I$*:*

$$\int_{-\sigma}^{\sigma}\int_{-\rho}^{\rho}\left\langle h(R),T_{s,t}(R,X)\right\rangle\psi(x,y)dxdy=\left\langle h(R),\int_{-\sigma}^{\sigma}\int_{-\rho}^{\rho}T_{s,t}(R,X)\,\psi(x,y)dxdy\right\rangle \tag{15.16}$$

and if $k,l\in\mathbb{R}$*;* $k\in(-\rho,\rho)$ *and* $l\in(-\sigma,\sigma)$*, then*

$$\lim_{S,T\to\infty}\int_{-\rho}^{\rho}\int_{-\sigma}^{\sigma}T_{s,t}(R,X)\,\psi(x,y)dxdy=1,\quad (t_1,t_2)\in(-k,k)\times(-l,l).$$

Theorem 15.5 *Let* $f,g\in W'_{\mu,\nu}(I)$ *and* $\Re f\left(I\left(\omega_{s,t},\mu,\nu\right)\right)=\Re g\left(I\left(\omega_{s,t},\mu,\nu\right)\right)$*; then*

$$\left\langle f(X),\frac{\mu^{-1/2}}{4\rho\sigma}\kappa(\omega)\right\rangle=\left\langle g(X),\frac{\mu^{-1/2}}{4\rho\sigma}\kappa(\omega)\right\rangle \tag{15.17}$$

for $f=g$ *in the sense of equality in* $D'(I)$ *for all* $s,t=1,2,3,\cdots$*.*

The proof is obvious and is called as uniqueness theorem.

Example 1 Let $\delta\left(\omega_{s,t}\cdot(R)-k_0\right))\in E'(I)$, $\forall\ k_0\in I$ and $E'(I)$ is a subspace of $W'_{\mu,\nu}(I)$, which gives $\delta\left(\omega_{s,t}\cdot(R)-k_0\right))\in W'_{\mu,\nu}(I)$.

Applying (15.1) to $\delta\left(\omega_{s,t}\cdot(R)-k_0\right))$ and using [13, p. 25],

$$\Re\delta\left(I\left(\omega_{s,t},\mu,\nu\right)\right)=\left\langle\delta\left(\omega_{s,t}\cdot(R)-k_0\right)),\psi\left(\left(\omega_{s,t}\cdot(R)-\nu\right)/\mu\right)\right\rangle$$

$$=\frac{\mu^{-1/2}}{4\rho\sigma}\psi\left(\left(\omega_{s,t}\cdot(k_0)-\nu\right)/\mu\right).$$

Considering $\psi(X)\in D'(I)$,

$$\left\langle \sum_{t=1}^{T}\sum_{s=1}^{S}\frac{\mu^{-1/2}}{4\rho\sigma}\psi\left(\left(\omega_{s,t}\cdot(k_0)-\nu\right)/\mu\right)\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right),\psi(X)\right\rangle$$
$$=\int_{-\rho}^{\rho}\int_{-\sigma}^{\sigma}T_{s,t}(k_0,X)\,\psi(X)dxdy$$
$$\to\psi(k_0)$$

where $\psi(k_0)=\langle T_{s,t}(k_0,X),\psi(R)\rangle$.

15.4 Operational Calculus

The generalized finite continuous Ridgelet transform (GFCRT) is applicable to solve partial differential equations with boundary conditions.

Now let us define an operator for GFCRT $\Omega^*_{x,y,\theta}:W_{\mu,\nu}\to W'_{\mu,\nu}$ with the relation given below

$$\left\langle\Omega^*_{x,y,\theta}h(X),\psi(X)\right\rangle=\left\langle h(X),\Omega_{x,y,\theta}\psi(X)\right\rangle. \tag{15.18}$$

Here,

$$\Omega^*_{x,y,\theta}h=\Omega_{x,y,\theta}h. \tag{15.19}$$

As $\Omega^*_{x,y,\theta}$ is linear and continuous on $W_{\mu,\nu}(I)$, since the operator $\Omega^*_{x,y,\theta}$ on $W'_{\mu,\nu}(I)$ is adjoint of the operator $\Omega_{x,y,\theta}$ on $W_{\mu,\nu}(I)$ and by [4] is a self-adjoint operator.

It is a simple exercise to show for any integer $k=1,2,3\cdots$:

$$\left\langle\Omega^{*k}_{x,y,\theta}h(X),\psi(X)\right\rangle=\left\langle h(X),\Omega^{k}_{x,y,\theta}\psi(X)\right\rangle. \tag{15.20}$$

From (15.3) and (15.20) follows

$$\left\langle\Omega^{*k}_{x,y,\theta}hf(X),\frac{\mu^{-1/2}}{4\rho\sigma}\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right)\right\rangle$$
$$=\frac{\mu^{-1/2}}{4\rho\sigma}\left[\left(-\eta_s^2\right)^k\sin^2\theta\cos^{2k}\theta-\left(-\eta_t^2\right)^k\cos^2\theta\sin^{2k}\theta\right]\left\langle h(X),\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right)\right\rangle$$

$\forall\ s,t=1,2,3,\cdots$ which gives an operational formula.

$\mathfrak{R}\left\{\Omega^{*k}_{x,y,\theta}h\right\}\left(I\left(\omega_{s,t},\mu,\nu\right)\right)$

$$= \left[\left(-\eta_s^2\right)^k \sin^2\theta \cos^{2k}\theta - \left(-\eta_t^2\right)^k \cos^2\theta \sin^{2k}\theta \right] \Re h \left(I \left(\omega_{s,t}, \mu, \nu \right) \right). \tag{15.21}$$

We can use (15.21) to solve the operator equation given by

$$P \left(\Omega_{x,y,\theta}^* \right) u = g \tag{15.22}$$

where P is a polynomial and g and u are given members of $W'_{\mu,\nu}$.

Note that $P \left[\left(-\eta_s^2\right)^k \sin^2\theta \cos^{2k}\theta - \left(-\eta_t^2\right)^k \cos^2\theta \sin^{2k}\theta \right] \neq 0$.

Using (15.1), (15.21), and (15.22) follows
$P \left[\left(-\eta_s^2\right) \sin^2\theta \cos^2\theta - \left(-\eta_t^2\right) \cos^2\theta \sin^2\theta \right] U = G$, where $U = \Re u \left(I \left(\omega_{s,t}, \mu, \nu \right) \right)$ and $G = \Re g \left(I \left(\omega_{s,t}, \mu, \nu \right) \right)$ are GFCRT of u and g, respectively.

Therefore,

$$U = \frac{G}{P \left[\left(-\eta_s^2\right) \sin^2\theta \cos^2\theta - \left(-\eta_t^2\right) \cos^2\theta \sin^2\theta \right]}. \tag{15.23}$$

Applying Inversion Theorem 3.1 to (15.23), we get

$$u(x, y) = \lim_{S,T \to \infty} \sum_{t=1}^{T} \sum_{s=1}^{S} \frac{G}{P \left[\left(-\eta_s^2\right) \sin^2\theta \cos^2\theta - \left(-\eta_t^2\right) \cos^2\theta \sin^2\theta \right]} \Omega (u \cdot X) \tag{15.24}$$

for every $P \left[\left(-\eta_s^2\right) \sin^2\theta \cos^2\theta - \left(-\eta_t^2\right) \cos^2\theta \sin^2\theta \right] \neq 0$; then solution exists in $D'(I)$.

15.5 Applications

Example 2 (Two-Dimensional Heat Equation) Find the conventional function $u(x, y, z)$ on domain $\{(x, y, z) : -\pi < x < \pi, -\pi < y < \pi, -\pi < z < \pi\}$, satisfying the following differential equation:

$$\left(\sin^2\theta\right) \frac{\partial^2 u}{\partial x^2} + \left(\cos^2\theta\right) \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial t} = 0 \tag{15.25}$$

with the boundary conditions:

(i) if $0 < T \le t < \infty$ and in $D'(I)$, $u(x, y, z) \to \infty$
(ii) for $t \to 0^+$, $u(x, y, z)$ converges in the sense of $D'(I)$ to $f(x, y) \in W'_{\mu,\nu}(I)$.

Now (15.25) can be written as

$$\left[\left(\sin^2\theta\right)\Omega_x - \left(\cos^2\theta\right)\Omega_y\right]u + \frac{\partial u}{\partial t} = 0. \tag{15.26}$$

Applying GFCRT to (15.26), it is obtained as

$$\left(\eta_t^2 - \eta_s^2\right)\sin^2(2\theta)\,\Re u\left(I\left(\omega_{s,t},\mu,\nu\right)\right) + \frac{\partial\Re u\left(I\left(\omega_{s,t},\mu,\nu\right)\right)}{\partial t} = 0 \tag{15.27}$$

for every $s, t = 1, 2, 3, \cdots$ where

$$\Re u\left(I\left(\omega_{s,t},\mu,\nu\right)\right) = \left\langle u(X,z), \frac{\mu^{-1/2}}{4\rho\sigma}\psi\left(\left(\omega_{s,t}\cdot(R)-\nu\right)/\mu\right)\right\rangle. \tag{15.28}$$

Hence,

$$\Re v\left(I\left(\omega_{s,t},\mu,\nu\right)\right) = C\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\left(\eta_t^2-\eta_s^2\right)\sin^2 2\theta\right)t} \tag{15.29}$$

where $C\left(\omega_{s,t},\mu,\nu\right)$ is a constant.

From boundary conditions (ii) and $\lim_{z\to 0^+}\Re u\left(I\left(\omega_{s,t},\mu,\nu\right)\right) = F\left(\omega_{s,t},\mu,\nu\right)$ gives $C\left(\omega_{s,t},\mu,\nu\right) = F\left(\omega_{s,t},\mu,\nu\right)$ in (15.29), gives

$$\Re u\left(I\left(\omega_{s,t},\mu,\nu\right)\right) = F\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\left(\eta_t^2-\eta_s^2\right)\sin^2 2\theta\right)t}. \tag{15.30}$$

Using (15.10) and (15.30), we get

$$u(x,y,z) = \lim_{S,T\to\infty}\sum_{t=1}^{T}\sum_{s=1}^{S}F\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right).$$

Also,
$\langle u(X,z), \psi(X)\rangle$

$$= \int_I \lim_{S,T\to\infty}\sum_{t=1}^{T}\sum_{s=1}^{S}F\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right)dX.$$

Therefore, $u(X,z)$ can be represented as a classical function:

$$u(x,y,z) = \sum_{t=1}^{\infty}\sum_{s=1}^{\infty}F\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right). \tag{15.31}$$

Further, we get
$\lim_{z\to 0^+}\langle u(x,y,z), \psi(X)\rangle$

$$= \lim_{z\to 0^+} \sum_{t=1}^{\infty}\sum_{s=1}^{\infty} F\left(\omega_{s,t}, \mu, \nu\right) e^{-(\sqrt{\eta_s - \eta_t}\sin 2\theta)z} \left\langle \psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right), \psi(X)\right\rangle. \tag{15.32}$$

$$\lim_{z\to 0^+} \langle u(x,y,z), \psi(X)\rangle = \int_I \sum_{t=1}^{\infty}\sum_{s=1}^{\infty} F\left(\omega_{s,t}, \mu, \nu\right) \psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right) \psi(X)\, dX. \tag{15.33}$$

From (15.31) follows

$$\lim_{z\to 0^+} \langle u(x,y,z), \psi(X)\rangle = \langle f, \psi\rangle . \tag{15.34}$$

And thus

$$|u(x,y,z)| \le \sum_{t=1}^{\infty}\sum_{s=1}^{\infty} \left|F\left(\omega_{s,t}, \mu, \nu\right)\right| e^{-(\sqrt{\eta_s - \eta_t}\sin 2\theta)z} \left|\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right)\right| \to 0$$

as $z \to \infty$, thus verifying the boundary condition (ii).

Example 3 (Dirichlet Problem) Find a function $v(x,y,z)$ satisfying

$$\left(\sin^2\theta\right)\frac{\partial^2 v}{\partial x^2} + \left(\cos^2\theta\right)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0,\ \ \pi < x < \pi,\ -\pi < y < \pi,\ -\pi < z < \pi, \tag{15.35}$$

such that:

(i) $v(x,y,z) \to \infty$ in the sense of $D'(I)$ for $0 < Z \le z < \infty$.
(ii) $t \to 0^+$, $v(x,y,z)$ converges to $f(x,y) \in W'_{\mu,\nu}(I)$ in the sense of $D'(I)$.

On solving (15.35), we obtain

$$\left[\left(\sin^2\theta\right)\Omega_x - \left(\cos^2\theta\right)\Omega_y\right] v + \frac{\partial^2 v}{\partial z^2} = 0. \tag{15.36}$$

Using (15.1) and (15.36), we get

$$\left(\eta_t^2 - \eta_s^2\right)\sin^2(2\theta)\, \Re v\left(I\left(\omega_{s,t}, \mu, \nu\right)\right) + \frac{\partial^2 \Re v\left(I\left(\omega_{s,t}, \mu, \nu\right)\right)}{\partial z^2} = 0 \tag{15.37}$$

where

$$\Re v\left(I\left(\omega_{s,t}, \mu, \nu\right)\right) = \left\langle v(X,z), \frac{\mu^{-1/2}}{4\rho\sigma}\psi\left(\left(\omega_{s,t}\cdot(R)-\nu\right)/\mu\right)\right\rangle. \tag{15.38}$$

$$\Re v\left(I\left(\omega_{s,t},\mu,\nu\right)\right)=A\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}+B\left(\omega_{s,t},\mu,\nu\right)e^{\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z} \tag{15.39}$$

where $A\left(\omega_{s,t},\mu,\nu\right)$ and $B\left(\omega_{s,t},\mu,\nu\right)$ are constants.

Using boundary conditions (ii) and (iii) and (15.39), $\lim_{z\to\infty}\Re v\left(I\left(\omega_{s,t},\mu,\nu\right)\right)=0$, and $\lim_{z\to 0^+}\Re v\left(I\left(\omega_{s,t},\mu,\nu\right)\right)=F(u)$, respectively.

Hence, $B\left(\omega_{s,t},\mu,\nu\right)=0$ and $A\left(\omega_{s,t},\mu,\nu\right)=F\left(\omega_{s,t},\mu,\nu\right)$.

Thus, (15.39) gives

$$\Re v\left(I\left(\omega_{s,t},\mu,\nu\right)\right)=F\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}. \tag{15.40}$$

From (15.10) and (15.40), we get

$$v(x,y,z)=\lim_{S,T\to\infty}\sum_{t=1}^{T}\sum_{s=1}^{S}F\left(\omega_{s,t},\mu,\nu\right)e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right). \tag{15.41}$$

Using [2], (15.41) is obtained as

$$|v(x,y,z)|\le\sum_{t=1}^{\infty}\sum_{s=1}^{\infty}\left|F\left(\omega_{s,t},\mu,\nu\right)\right|e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}\left|\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right)\right| \tag{15.42}$$

which verifies the boundary condition (i).

Similarly,

$$|v(x,y,z)|\le\sum_{t=1}^{\infty}\sum_{s=1}^{\infty}\left|F\left(\omega_{s,t},\mu,\nu\right)\right|e^{-\left(\sqrt{\eta_s-\eta_t}\sin 2\theta\right)z}\left|\psi\left(\left(\omega_{s,t}\cdot(X)-\nu\right)/\mu\right)\right|\to 0$$

as $z\to\infty$; this verifies the boundary condition (ii).

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