Chapter 5
Personalized Mathematics and Mathematics Inquiry: A Design Framework for Mathematics Textbooks

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5.1 Introduction

Almost 30 years ago, Wigley (1992) discussed two contrasting models for the teaching of mathematics, the path-smoothing model and the challenging model. The path-smoothing model involves more expository methods of instruction which provide students with structured and secured work pathways. The challenging model involves more exploratory, inquiry-based methods of instruction which allow students to interact with challenging tasks. More recent articles and research studies, mainly in science education (Alfieri et al., 2011; Blair & Hindle, 2019; Lazonder & Harmsen, 2016), seem to suggest that inquiry-based methods can be more effective than expository methods of instruction. Still, the integration of mathematical challenge in the instructional process and more specifically in mathematics inquiry approaches is neither clear nor explicit in the way in which mathematics curricula and textbooks may promote this.

The effective integration of mathematical challenge in the instructional process was one of the main principles in the design of the Cypriot Mathematics Curriculum (Cyprus Ministry of Education and Culture, 2016a). The series of mathematics textbooks, which were designed to translate this policy into pedagogy tried to fulfill this principle. Research studies support that textbooks play a vital role in translating the educational policy into pedagogy (Valverde et al., 2002). In this chapter, we aim to present the design framework for the Cypriot Mathematics Textbooks and indicative examples, to illustrate the way in which these textbooks may evoke mathematical challenge in heterogeneous classes. It is beyond the scope of the current chapter to

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present any empirical findings related to teachers' training or findings regarding the impact of the textbooks on students' learning.

In this chapter, we start by first looking at the role of mathematics textbooks in teaching and give some information about the Cypriot Mathematics Textbooks. Then we discuss the most frequently used design models for mathematics teaching. We then proceed to the presentation of the theoretical framework "Personalized Mathematics and Mathematics Inquiry" (PMMI) which we used for the development of the Cypriot Mathematics Textbooks. We exemplify this theoretical model with some indicative examples from the mathematics textbooks of primary and secondary education.

5.2 Role of Mathematics Textbooks

The structure and content of mathematics textbooks is likely to have an impact on actual classroom instruction (Pepin et al., 2013; Rezat, 2006). Valverde et al. (2002) argued that the form of textbooks promotes a distinct pedagogical model and thus embodies a plan for the particular succession of educational opportunities. The development and design of mathematics textbooks are assessed through the opportunity they offer to students to learn and thus they are considered an important contributing factor in learning outcomes (Törnroos, 2005). Empirical studies have shown that the quality of mathematics textbooks has a significant effect on learning outcomes (Sievert et al., 2019).

The development of textbooks should be based on the idea that improvement of mathematics learning in classrooms is fundamentally related to the development of teaching, and that teaching develops through a learning process in which teachers and students grow into the practices in which they engage. Since textbooks strongly influence what students learn and what teachers teach, teachers and students should have suitable and appropriate textbooks (Reys et al., 2004). Textbooks facilitate teachers to modify their methods for teaching mathematics, in such a way as to align with the principles of the textbooks. The philosophy and teaching procedures are often guided by the mathematics textbooks.

5.2.1 The Cypriot Mathematic Textbooks

Textbooks are probably one of the most important curriculum resources which help teachers transform the mathematics curriculum into practice. According to Koutselini (2012), Cypriot teachers adhere to textbooks' teaching. The TIMSS study (2003) showed that 71% of the fourth-grade students were taught by teachers who used the mathematics textbooks as their primary source, while the remaining 29% of teachers used textbooks as a supplementary resource.

The Cypriot Mathematics Textbooks, published by the Ministry of Education, are used in all state schools in Cyprus. They are the only resources provided to schools and are based on the recently developed Cypriot Mathematics Curriculum. There are different types of schools in Cyprus and the textbooks intend to cater a diverse group of students. Thus, one of the main roles of textbooks is to help teachers teach in the spirit of the curriculum and the only way for the curriculum to be implemented properly and consistently was to develop a good set of textbooks. These textbooks come with teacher guides which serve as manual for mastering teaching and learning. During the first years of the implementation of the new textbooks, in-service training was organized by the Ministry of Education to familiarize teachers with the new textbooks and their main principles.

The Cypriot Mathematics Curriculum was launched in schools in September 2012. In the same year, the mathematics textbooks which aligned with the new curriculum were introduced in Grade 1 (the first grade of primary education) and Grade 7 (the first grade of secondary education). In September 2013, Grade 2 and Grade 8 textbooks were introduced and the same pattern of introduction of new textbooks continued for 4 years, until 2017, when the whole series of textbooks from Grade 1 to Grade 12 were completed. Two types of textbooks were introduced for Grade 10 to Grade 12, one for students who take mathematics as a specialization subject and one for students who take mathematics as a common core subject. The textbooks are reviewed almost every year, based on the comments and suggestions received from teachers implementing them.

5.3 Design Models

Thirty years ago, Wigley (1992) argued that in order to develop mathematics textbooks, it is common to follow one of the two designs, the path-smoothing model or the challenging model. The essential methodology of the first model is to smoothen the path for the learner. The textbook, in this case, states the kind of problem which the class will be working on. The problem attempts to classify the subject matter into a limited number of categories and to present them one at a time. The key principle is to establish secure pathways for the pupils. Thus, it is important to present ways of solving problems in a series of steps and exercises to practice the methods. The path-smoothing model is the instructional approach in which teachers, following the textbooks, prescribe the content, present the content, and measure student acquisition of that content.

The challenging model promotes what its name denotes, the use of tasks that are challenging to students. The teacher provides sufficient time for students to work on a task, suggest their own approaches, and try different solving pathways. The teacher may have considered beforehand a syllabus, but this is not presented to students from the beginning. The teacher has a critical role in helping students share their ideas with the whole class and discuss different strategies. Students are encouraged to reflect on their work, recognize what they have learned, and how new knowledge links to previous knowledge. Wigley (1992) and more recently Blair and Hindle (2019) argued that the challenging model can create better learning and more positive attitudes towards mathematics.

Since Wigley (1992) discussed the path-smoothing and the challenging model, there have been several meta-analysis studies, which did look not only at the two extremes but also at intermediate points of this spectrum. These more recent studies (see, for example, Alfieri et al., 2011; Lazonder & Harmsen, 2016) did not use the words smoothing and challenging model but referred to "explicit instruction and unassisted discovery" (Alfieri et al., 2011) or "guided and unguided inquiry learning" (Lazonder & Harmsen, 2016), which we believe resemble the path-smoothing and challenging models. They also referred to "enhanced discovery or minimally guided approach" which lies somewhere between these two extremes of the challenging spectrum.

Although the extent and the type of the guidance that students should receive is not yet completely clear, research studies have consistently shown that enhanced discovery is more effective than explicit instruction or unassisted discovery, as long as students are adequately supported (Lazonder & Harmsen, 2016). This is the reason that we decided to follow an enhanced discovery instructional approach for the Cypriot Mathematics Textbooks.

We adopted an inquiry-based approach to mathematics with focus on problem solving, understanding, problems within a context, learning processes, and strategies. We considered that the implementation of these central concepts improves students' attitudes towards mathematics and their ability to use mathematics both in the "real world" and inner mathematical contexts. However, an inquiry-based approach is fundamentally based on the humanized aspects of mathematics which are inherent in the nature of mathematics. Thus, the essential characteristics of the challenging model, as it was implemented for the design of the Cypriot Mathematics Textbooks, has two interrelated elements that defined the design of the mathematics textbooks: *Personalized Mathematics* and *Mathematics Inquiry*. In the next section we present the structure and underlying principles of this framework.

5.4 The Structure of the PMMI Framework

We propose the "Personalized Mathematics and Mathematics Inquiry (PMMI)," as the overarching and fundamental theoretical framework for the design of mathematics textbooks and pedagogical instruction in Cyprus for K-12 grades in order to achieve desirable teaching–learning practices. The PMMI framework (Fig. 5.1) involves two major elements: (a) Personalized Mathematics and (b) Mathematics Inquiry. We set off from "Personalized Mathematics" where we present the fundamental practices of mathematics teaching: mathematics goals, reasoning, problem solving, mathematization, connections of mathematical representations, development of

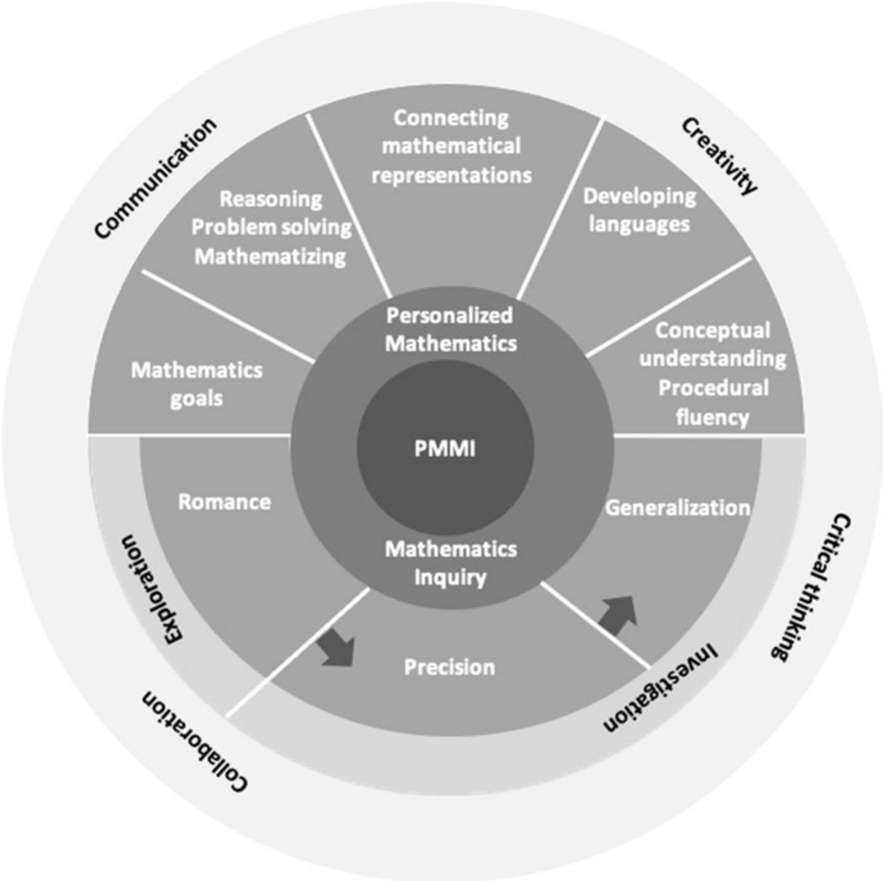
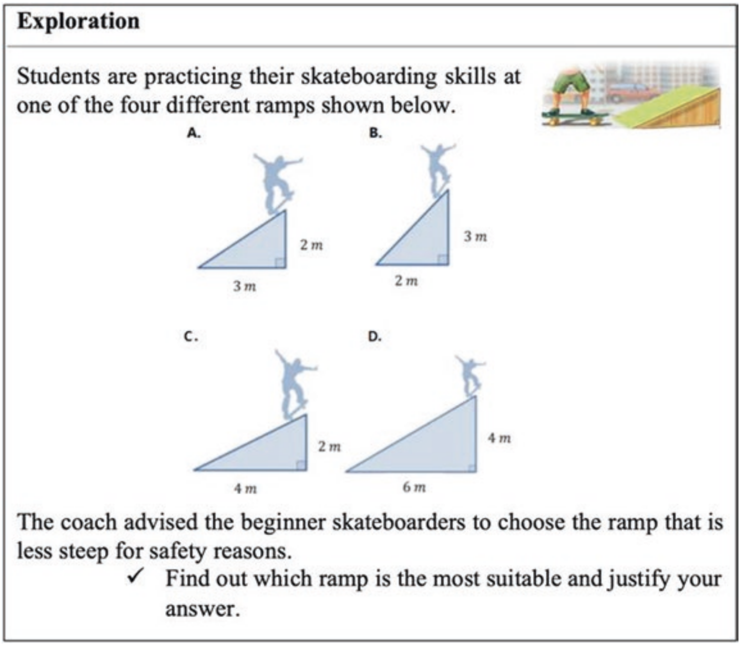


Fig. 5.1 The PMMI framework

language, conceptual understanding, and procedural fluency. Then, we proceed to the second element of the PMMI framework "Mathematics Inquiry," which is based on Whitehead's (1929) theory of learning (Romance, Precision, Generalization), and present the rationale for the design of the phases in which students are intended to go through, while being taught a mathematics chapter.

The overarching aim of the PMMI framework is that the interweaving of "Personalized Mathematics" and "Mathematics Inquiry" will lead to the enhancement of students' positive attitude towards mathematics, development of mathematical concepts and procedures, as well as the development of more general skills, such as critical thinking, creativity, collaboration, and communication.

To make the PMMI framework explicit, Fig. 5.2a, b illustrates the way in which the teaching of slope of a straight line is introduced in the Cypriot Mathematics Textbooks in Grade 8. In the rest of the chapter, we will discuss the main principles



Exploration

Students are practicing their skateboarding skills at one of the four different ramps shown below.

The coach advised the beginner skateboarders to choose the ramp that is less steep for safety reasons.

✓ Find out which ramp is the most suitable and justify your answer.

Fig. 5.2 (**a**) Exploration of slope in Grade 8 mathematics textbook (**b**) Investigation of slope in Grade 8 mathematics textbook (Cyprus Ministry of Education and Culture, 2016b)

of the PMMI framework alongside indicative examples from the mathematics textbooks in order to make the link between the theoretical principles and their implementation more transparent to the reader.

As illustrated in Fig. 5.2a, b, each chapter begins with an exploration and an investigation, which constitute important ingredients of the PMMI framework. These explorations and investigations are followed by further tasks to give students the opportunity to develop both conceptual understanding, procedural fluency, as well as use of clear and precise mathematical language. Real-life applications are often utilized throughout the textbooks. These applications are opportunities for students to connect classroom lessons to realistic scenarios and assist teachers transforming mathematical learning into an engaging and meaningful way to explore the real world. Attention was also paid to organize content in a way in which learning mathematics would be an active, constructive, cumulative, and goal-oriented process.

5.4.1 Personalized Mathematics

As shown in Fig. 5.1, the two elements of the PMMI framework, "Personalized Mathematics" and "Mathematics Inquiry," are intertwined in such a way as to provide a framework for teaching and learning that fully aligns with the learning of

Investigation

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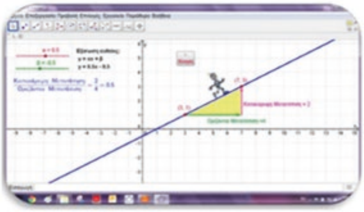
A robot is at point A and moving to point B on the line represented on the axes.

- Move the cursors α and β, to construct the equation, of the line $y = 2x$.



- ✓ Select point A and then change the coordinates of point B, so that the robot moves one unit to the right (horizontal change). Calculate the vertical shift of the above movement. Consider whether this applies to any starting point A.
- ✓ If the robot moves from point A to two units to the right, how large will the vertical change be? Does this apply to any starting point A?

- ✓ What do you think will be the vertical change if it moves from point A, 5 units to the right? Check your answer with the use of the app.
- ✓ Consider the ratio $\frac{\textit{vertical change}}{\textit{horizontal change}}$ for any two points A and B of the line.
- Change the cursors α and β and observe how the ratio $\frac{\textit{vertical change}}{\textit{horizontal change}}$ relates with the equation of the line.

Fig. 5.2 (continued)

mathematics. Personalized mathematics learning is an instructional approach which encompasses a number of practices to support mathematics teaching and learning. However, the lack of a consensus on the definition of personalized learning allows for a range of ideas on what it might entail.

In the context of the PMMI framework, personalized means that learning focuses primarily on improving students' achievement without ignoring the humanizing and social aspects of mathematics teaching and learning. For example, our purpose was to focus on tailoring mathematics tasks and problems to learners. This means, for students, to find solutions based on their own mathematical understanding and what makes sense to them. We consider personalized mathematics learning as the space in which learners give voice to their own ways of mathematical thinking, represent and discuss their mathematical ideas, and use mathematics to make sense of their worlds. "Personalized Mathematics" can help learners see themselves as doers of mathematics by providing support for developing perseverance and understanding. It is also personalized because the design of lessons provides students with multiple entry points and encourages different ways of students' active engagement with various mathematical ideas through discussions, presentations, and use of various representations.

To encourage Personalized Mathematics instruction, the textbooks were based on five teaching practices (see Fig. 5.1), as these were defined by NCTM policy document, "NCTM's Principles to Actions: Ensuring Mathematical Success for All" (NCTM, 2014). In particular, these practices were implemented by engaging students with tasks, which are expected to facilitate mathematics learning through the following:

- Establishing mathematics goals to focus learning (personalized goals that build up students' mathematical understanding, increase student confidence, goals that ensure that each and every student has the opportunity to learn rigorous mathematics content and develop mathematical processes and practices).
- Implementing tasks that promote reasoning and problem solving (Personalized Mathematics supports tasks that require reasoning, problem solving, and mathematizing our world through mathematical modeling and culturally relevant mathematics tasks).
- Using and connecting mathematical representations (Personalized Mathematics allows students to use representations familiar to them and age appropriate, use multiple representations so that students can draw on multiple resources, and develop connections among multiple representations to deepen their understanding of mathematical concepts and procedures).
- Facilitating meaningful mathematical discussions (Personalized Mathematics allows students to develop language to express mathematical ideas).
- Building procedural fluency from conceptual understanding (Personalized Mathematics routinely connects conceptual understanding and procedural fluency to deepen learning and reduce mathematical anxiety. Procedural and conceptual knowledge is more solid when it is built on students' prior personalized knowledge and experiences).

For example, in the exploration in Fig. 5.2a, students are invited to decide which one of the four skateboard ramps is the least dangerous and justify their solution. It is likely that students will find this exploration interesting since it arises from real life and some of them may even have tried this activity themselves. Thus, students are expected to interpret the problem based on their prior knowledge and personal experience. Students may also become curious as to when the ramp is more dangerous and find challenging how to respond or explain their intuitive feelings. Students may try to communicate their own experiences or ideas, look critically at the problem, and try to be creative as to the way in which to investigate it or justify their answers. In this sense, students might formulate a personal goal for understanding the problem and offer a reasonable justification, while the problem reflects a situation of the real world which students should mathematize. Furthermore, through students' collaboration and communication it is likely that students may realize the need of a new mathematical concept which is neither the height nor the length of the ramp, but actually the combination of the two measures. Hence, the representations of the skateboard ramps which provide information about the height and the length are expected to trigger students' curiosity. This is the sort of curiosity we expect to develop in the mathematics classroom, the romance of learning, which will eventually bring the evolution of a new mathematical concept for the students, that of slope. It is likely that if the mathematical concept of slope emerges through students' curiosity and need to respond to a problem will remain in students' memory longer (Gruber et al., 2014; Kang et al., 2009; Knuth, 2002; Peterson & Cohen, 2019). Students will develop conceptual understanding of the concept of slope and not depend on the rote memorization of the formula *slope = rise/run* (Bos et al., 2020).

In the investigation that follows immediately after the exploration, students are invited to use a mathematical applet. The technological tool is recommended so as to offer students the opportunity to experiment with the slope of a straight line. In this activity, students are again offered with relevant representations which they may link to the skateboard ramp. Students are prompted to observe the changes that occur as a robot moves along the tilted line. Specifically, students are asked what the robot's vertical displacement is, when it makes one-unit move to the right along the slope, then when it makes two-units move to the right and so on. Students are offered this applet to experiment, construct hypotheses, and investigate whether these hypotheses are confirmed or rejected. It is anticipated that this investigation will support the development of procedural fluency from conceptual understanding. Students' active engagement with the technological tool, precision in calculations, and generalization is expected to lead to the conceptual development of the concept of slope and not the rote memorization of the formula *slope = rise/run*.

5.4.2 Mathematics Inquiry

The second element of the PMMI framework refers to "Mathematics Inquiry." Mathematics inquiry-based learning is seen as the approach and pathway for implementing the practices of Personalized Mathematics. On a European level, most educational policy documents and curriculum guidelines suggest inquiry-based instructional methods to school subjects (Dorier & Garcia, 2013; Supovitz & Turner, 2000). Inquiry-based instruction in mathematics can loosely be defined as "a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work" (Artigue & Blomhøj, 2013, p. 797). This approach promotes problem solving and involves addressing questions that are epistemologically relevant from a mathematical perspective and triggering students to work autonomously in order to provide valid answers (Artigue & Blomhøj, 2013). It entails observing, asking questions, creating representations, making conjectures, looking for relationships, generalizing, modeling, and communicating ideas (Dorier & Maass, 2020).

Although the level of guidance that the inquiry-based learning should involve may vary, a meta-analysis of 72 studies carried out by Lazonder and Harmsen (2016) suggests that guidance has a positive effect on inquiry learning activities, performance success and learning outcomes. Therefore, we chose to apply an approach which may be characterized as minimal guided or enhanced discovery approach.

Mathematics inquiry could be seen as a process that starts from a wonder, a question or a problem, for which students seek answers through exploration and/or investigation following an enhanced discovery approach. The context for mathematics inquiry often relies on problems arising from the world around us, and problems that emerge from history, art, or the science of mathematics. Designing lessons based on the inquiry approach entails consideration of the mathematical concepts involved, incorporation of artifacts that support exploration and experimentation, and the use of language and symbolic tools accessible to students for expressing and discussing their ideas.

In inquiry-based learning, the role of the teacher involves challenging students, probing questions, utilizing their prior knowledge, encouraging discussion, and structuring students' opportunities for developing understanding (Dorier & Maass, 2020). This presupposes that task sequences are developed to scaffold students' work in reinventing and creating mathematics new to them (Laursen & Rasmussen, 2019). Hence, the design of textbooks intends to provide these learning sequences in order to promote inquiry-based mathematics learning.

As described earlier, for the teaching of straight-line slope, students are presented with an exploration, where they need to decide which ramp is the steepest. At this point students are invited to explore and hypothesize, and are not offered any specific guidance. However, if students are unable to reach an answer or make conjectures, they are invited to work on an applet and are given more guided questions. Students are asked to explore the concept of slope as the constant ratio of $\frac{\text{vertical change}}{\text{horizontal change}}$.

The development of mathematics inquiry is affected by the nature of the mathematical concepts involved, students' conceptualization, language, symbolic tools accessible to students for expressing and discussing their ideas, and the artifacts accessible for supporting exploration and experimentation. In this sense, a variety of artifacts could support the experimental dimension of mathematics, like digital technologies. The history of mathematics shows that such an experimental dimension is not new, but over the last decades technological developments have put a large number of new resources at the disposal of teachers and students. Researchers generally agree that the strategic use of ICT could support students to develop understanding and advanced mathematical proficiencies, like problem solving, reasoning, and justifying (NCTM, 2015). Digital tools offer dynamic representations and classroom connectivity, which could optimize students' access to basic mathematical concepts and procedures (Hegedus & Moreno-Armella, 2014).

The PMMI framework adopts the idea that digital tools should be used in a way that corresponds to the conceptual schemas that students are expected to develop. This process is widely known as "instrumental genesis" (Artigue, 2002; Drijvers, 2020). Therefore, the thoughtful use of digital tools in carefully designed ways at appropriate times, mostly through exploration and investigation settings could support schema construction, experimenting, sense making, communicating, and doing mathematics. For example, the use of the applet in the investigation about the concept of straight-line slope (see Fig. 5.2b) is expected to facilitate the construction of a relevant schema that relates the concept of straight-line slope with the idea of coordinating the vertical and horizontal covariation.

Concluding, "Mathematics Inquiry" in the PMMI framework refers to an instructional approach that is expected to serve enhanced learning and promote the practices of "Personalized Mathematics." In the following section, we present in detail the three phases through which "Mathematics Inquiry" is expected to evolve: romance, precision, and generalization. The notions of exploration and investigation are also revisited to further illustrate their special characteristics, significance, and contribution to "Personalized Mathematics" and "Mathematics Inquiry."

5.4.2.1 Mathematics Inquiry Phases

Mathematics inquiry, in the PMMI framework, evolves in three phases (see Fig. 5.1) which are based on Whitehead's theory of education "Rhythm of learning" (Whitehead, 1929). According to Whitehead's theory, the natural way that individuals learn is through the pattern of Romance–Precision–Generalization. Learners should be introduced with something interesting, something they care about, and be offered the opportunity to explore. Then, the precision phase follows where learners develop knowledge and skills that are needed for the development of a new concept. Of course, at this phase, romance should not be lost, since this will sustain the interest and therefore the development of skills, knowledge, and applications. The last phase is generalization. At this phase learners link what they have learned with prior knowledge, apply this knowledge in a new context, make generalizations, return to romance with new competences, and become ready to explore new concepts. Students are anticipated to pass through these three phases through two activities: the exploration and the investigation.

The romance of learning is introduced in the textbooks with the idea of explorations while the phases of precision and generalization are substantiated by investigations. Afterward, examples and activities allow students to practice and sharpen their skills as they work towards mathematical understanding. At the end of each chapter there are enrichment tasks which serve four purposes: (1) tasks for students who struggle with mathematics, (2) more challenging tasks for students who excel in mathematics, (3) tasks for all students who need further practice, and (4) tasks which offer a different approach to teaching to the one already presented in the preceding chapter.

5.4.2.2 Exploration: Romance

The inquiry process starts with an exploration which, as reported earlier, has a unique goal, to create wonder, curiosity, engagement in mathematics, and the romance of learning, as mentioned by Whitehead (1929). Thus, exploration provides students with lived, curious experiences, in which they are expected to shape their own learning, as they work on mathematical problems. Students grow and change with opportunities to identify problems, generate personal wonderings, and engage in dialogue around these problems. They reflect as they apply their new knowledge, by discussing possible solutions in ways that transform thinking. Offering learners space to generate their own wonderings about problems helps them connect their own interests to real-life issues in ways that can lead to real change (Alberta Learning, 2004). One of the most valuable things that an exploration can serve is to make students become more aware of and deliberate about their curiosity. This is why, in the Cypriot Mathematics Textbooks, exploration is the starting activity of each unit in the mathematics textbooks. Usually explorations are real-life or life-like learning experiences that are open and provide opportunities for students to wonder and develop their imagination. It is the essence of a mathematics inquiry. However, to "enquire" does not mean that we always have to reach an answer to a problem or to complete a task.

Exploration is the means to personalize mathematics since it is a purposeful, self-directed inquiry fulfilling learning experience (Pink, 2009). This is why it is important that individuals set their own goals, in this case mathematical goals, and seek to satisfy them. In order for individuals to set a mathematical goal, the topic must be of interest to them and trigger their curiosity. In addition, explorations request students to reflect, by discussing what they bring to the content and what ideas they actively construct as they interact. For example, during explorations, students can use their wonderings and meaning to reflect on their process and seek feedback from others.

The exploratory approach provides an opportunity for mathematics to occur in a context, providing a balance between problem solving and skills-based activities and engage students in deep mathematical learning (Boaler, 2008). Explorations allow students to "do" mathematics, to "make sense" of their world, and "be mathematicians" (Marshman et al., 2011). The exploratory tasks aim to get students involved in "problem formulation, problem solving, and mathematical reasoning" (Battista, 1994, p. 463).

In explorations, a substantial task can thus be presented, in which students help define the problem; develop ways of tackling it; generate examples; and predict and generalize. Explorations direct students to the realization that there is a need to learn or discover a new mathematical concept, or strategy which is useful for mathematics and life in general. Explorations have multiple entry points allowing students to think creatively in order to respond to complex challenging tasks "allowing students to think in a creative manner in the framework of challenging complex tasks" (Swan, 2009, p. 1). These tasks facilitate the process of discussion and contextual use of mathematical vocabulary.

In the context of Cypriot Mathematics Textbooks in Cyprus, the PMMI framework situates classroom exploration experiences at the beginning of a chapter on a new concept or procedure. Approximately, every week students encounter new explorations and/or investigations. The time anticipated for the completion of an exploration and investigation varies, from 10 min to 40 min sessions. The exploration is quite open, and teachers may ask further questions to orient the students towards the topic under investigation.

Figures 5.3 and 5.4 present examples of explorations. Figure 5.3 presents an exploration on the concept of exponents which is taught in Grade 7. Figure 5.4 presents an exploration on the concept of equivalent fractions which is taught in Grade 4.

In the exploration presented in Fig. 5.3, students are presented with the legend of Sissa which aims to engage students to the concept of exponent. Students are asked to explore why the emperor could not fulfill his promise and deliver to Sissa the grains of wheat that he had promised. This question anticipates to trigger students' curiosity, make them wonder why this happened, and want to explore the problem. Some of the students may make certain hypothesis, others may bring to the fore their own experiences about the number of squares that a chess has and suggest possible strategies to address the problem. Thus, it is expected that this problem will direct students to set mathematical goals and they may try to respond to the problem

Exploration

According to a legend, long ago in one of the kingdoms of ancient India there was a powerful and rich emperor named Velchib. A Brahmin priest, named Sissa, invented and offered as a present to the emperor, a chess. The emperor was so impressed and excited with the present to the emperor that he decided to offer him a gift. Velchib asked Sissa what present he wanted.

Sissa thought for a moment and replied "I want two grains of wheat in the first square, four in the second, eight in the third and so on…"

The emperor was puzzled and angry about the cheap gift that Sissa had asked for and ordered his storekeepers to give him the wheat he wanted. However, as things turned out he could not deliver his promise.

✓ Why couldn't the emperor deliver his promise?

Fig. 5.3 Exploration of exponents in Grade 7 mathematics textbook (Cyprus Ministry of Education and Culture, 2016c)

Exploration

In a school party there were three pizzas of the same size. Each pizza was cut into equal pieces.

- Sophie ate 2 pieces from Pizza A.
- Andrea ate 3 pieces from Pizza B.
- Michael ate 4 pieces from Pizza C.
- All children ate the same quantity of pizza.

How is this possible? Explain

Fig. 5.4 Exploration of fraction equivalence in Grade 4 mathematics textbook (Cyprus Ministry of Education and Culture, 2019)

by sharing their personal experiences and activating various procedures, such as multiplication. The exchange of ideas amongst students may also require the development of specific mathematical language related to multiplication, the use of tools (such as a calculator) and creation of various representations (such as repeated multiplication). After their initial calculations, students may realize that numbers become very large. This is when some of them may intuitively feel the need of a new mathematical concept which will make this mathematical process simpler. Thus, this exploration triggers students' curiosity, encourages them to collaborate and communicate by exchanging ideas, building appropriate language, and using various representations, and offers them the opportunity to be critical and creative.

In the exploration presented in Fig. 5.4 students are asked to explore a real-life situation involving the concept of equivalent fractions. Based on the scenario, three pizzas of the same size were offered at a school party. One child ate two pieces from the first pizza, one child ate three pieces from the second one and one child ate four pieces from the third pizza; yet, all of them ate exactly the same quantity of pizza. Students are asked to explain how this could be possible. This question aims to trigger students' curiosity and wonder how this is feasible and probably start thinking that the three pizzas were cut in different ways. It is expected that students will make conjectures based on their own experience of cutting pizzas into slices. This discussion will probably lead to a realization that the three pizzas were cut in different ways and students will need to find the way in which the pizzas were cut and what fractions are involved. Students may hypothesize about the number of pizza slices that each student had. Based on these hypotheses, students could be prompted to collaborate to construct representations or use tools (e.g., fraction circles) to show the way in which each pizza was cut. Students' work and ideas will contribute to orchestrating a productive mathematical discussion about the fact that the same quantity of pizza could be expressed using different fractions. This discussion is anticipated to facilitate the introduction of a new mathematical concept, that of equivalent fractions. The concept could be explained through appropriate language and use of various representations arising from the scenario that students were invited to explore. Based on students' answers, they could be prompted to find further alternative solutions (be creative), communicate their mathematical ideas through words, symbols, drawings, and representations, and explain how the different cuts result to the same quantity of pizza.

5.4.2.3 Investigations: Precision and Generalization

Investigation is an activity originating in mathematics or the real world which lends itself to inquiry. A mathematics investigation allows students to satisfy their curiosity created in the exploration using various techniques. In the process of the investigation, students develop skills that can be applied to other problems (da Ponte, 2007). Students develop creative and critical thinking abilities and apply them to the expansion of their knowledge and skills. The intellectual satisfaction that one gets when discovering concepts and procedures as well as the generalizations of rules in different contexts are the major components of personalized mathematics.

The investigative approach is illustrated by posing questions, collecting data, hypothesizing, reflecting on, and drawing conclusions. These processes need to take place individually, in small groups and in the classroom as a whole. Explorations and investigations appear from Grade 1 to Grade 12. The level of difficulty and guidance offered varies, based on students' age and experiences. In the textbooks three kinds of investigations were designed, following Harris and Hofer's (2009) categories of activity types that provide students opportunities for *knowledge building* (i.e., students are expected to build the same content and process knowledge), *convergent knowledge expression* (i.e., students are expected to develop and express understanding of content which is similar to what they were introduced), and *divergent knowledge expression* (i.e., students are encouraged to express their own understanding of a given topic). In an analogous way, we developed investigations for knowledge building, convergent knowledge expressions, and divergent knowledge expressions with purpose to deepen and extend learning.

In the mathematics textbooks, the investigations follow the explorations. After students' curiosity and wonder, students need an explanation and the information which demystifies the mathematical content. The knowledge of mathematical concepts, skills, and procedures are the tools to justify mathematical phenomena through investigations. Mathematical investigation allows students to learn about mathematics, especially the nature of mathematical activity and thinking. It also makes them realize that learning mathematics involves intuition, conjecturing, and reasoning, and is not about memorizing and following existing procedures. The main component of an investigation is conjecturing which is followed by refinement of conjectures, refutation or proof of conjectures, and monitoring of proofs (Leikin, 2014). Investigations stimulate a way of thinking that goes beyond the application of knowledge or isolated procedures and implies the mobilization of ideas from different areas of mathematics. They deal with complex thinking processes, but reinforce the learning of facts, concepts, and procedures, making an important contribution to their consolidation (Abrantes et al., 1999). The ultimate aim of mathematical investigation is to develop students' mathematical habits of mind.

Explorations encourage students to pursue their curiosity, helping students figure out just what they want to know, while investigations are showing them how to systematically go about getting the answers to the investigations and explorations. In a mathematical exploration, one begins with a very general question or from a set of little structured information from which one seeks to formulate a more precise question and then produce a number of conjectures. Afterward, one tests those conjectures and proceeds to investigations in a systematic manner. If someone finds counterexamples, those conjectures may be improved or put completely aside (rejected or discarded). In this process, sometimes new questions are formulated and the initial questions are abandoned, completely or partially. The conjectures that resist to several tests gain credibility, stimulating a proof that, if achieved, will confer mathematical validity.

Both explorations and investigations call for creativity and critical thinking. They require abilities that are much beyond simple computation and memorization of definitions and procedures (da Ponte, 2007). These abilities, sometimes called "higher order abilities," are important not only for the mathematical development of the individual but for one's overall development as an individual and as an active member of society (da Ponte, 2007).

In the following section (Figs. 5.5 and 5.6), we present the investigations on the concept of exponents and on the concept of equivalent fractions that follow the explorations presented earlier.

Investigation

Fill in the table:

Square	Number of wheat grains	Result
1	2	2
2	$2 \cdot 2$	4
3	$2 \cdot 2 \cdot 2$	
4		
⋮		
8		
10		
⋮		
32		
⋮		
64		

✓ Explain your strategy

To produce this huge quantity of grains, which is actually a 20 - digit number, one has to plant the whole Earth 76 times!

It is said that the emperor, in order to avoid the insult for not keeping his promise, he was consulted by his advisors to ask Sissa to count all the grains. Something that would take forever!

Fig. 5.5 Investigation of exponents in Grade 7 mathematics textbook (Cyprus Ministry of Education and Culture, 2016c)

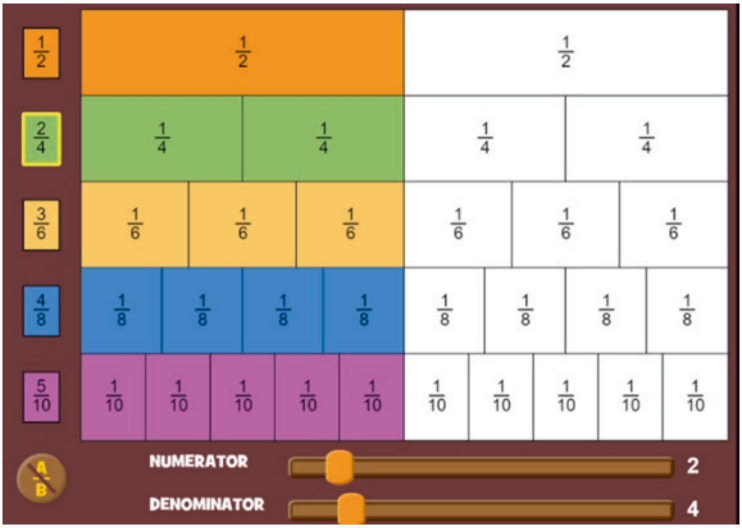


Fig. 5.6 Investigation of fraction equivalence in Grade 4 mathematics textbook (mathplayground.com) (Cyprus Ministry of Education and Culture, 2019)

For the concept of exponents, the investigation aims to offer students a scaffold toward the solution of the problem. A table is presented where students are asked to find the number of grains, in the 1st, 2nd, 3rd, 4th, 8th, 10th, 32nd, and 64th square. The investigation is presented in this form in order to provide students the opportunity to calculate with precision the answer in the first four cases (1st, 2nd, 3rd, 4th) and then try to find a pattern and a general rule that would apply for the number of grains in the subsequent squares and eventually in any number of squares. The table facilitates students' observation and deduction of a general rule that when a number is multiplied by itself it may be represented in the form a^b, where b indicates the number of times the number a will be multiplied by itself. Therefore, precision in the calculation of specific cases is expected to lead to generalization about a rule for calculating the number of wheat grains for any square.

In the investigation for equivalent fractions, students are invited to find at least 4 equivalent fractions to $\frac{1}{2}, \frac{1}{3}$, and $\frac{3}{4}$ by utilizing the affordances of an interactive applet (see Fig. 5.5). The applet involves the representation of fraction bars (mathplayground.com). Students can experiment by dragging sliders that define the numerator and denominator of each fraction bar. The goal of the activity is to align the colored fractions. The visualization and experimentation affordances of the applet might prompt students to make conjectures regarding the relation between the given fraction and the equivalent ones. Students will be asked to test the validity of their conjectures and find the relation between the denominators of the equivalent fractions and then the relation of the corresponding numerators. It is expected that students will study a number of examples and observe that the denominator of the equivalent fractions is a multiple of the denominator in the given fraction and this can be done by aligning the vertical lines (precision). Students will be given the opportunity to gain a deep conceptual understanding of the fact that finding an equivalent fraction equals dividing *n* times each piece of the given fraction; this process results to a new fraction with a denominator that is *n* times bigger (generalization). The experimentation with the applet aims to both facilitate the generalization of a procedure for finding any equivalent fraction and justification as to why it is possible to find an infinite number of equivalent fractions. We anticipate that students will be able to argue that the general rule for determining an equivalent fraction is to multiply the numerator or denominator of a fraction with the same number, or divide the numerator or denominator of a fraction with the same number.

5.5 The PMMI and the 4C's

The twenty-first-century skills of communication, collaboration, creativity, and critical thinking, often referred to as the "four C's" may be developed through the PMMI framework. The PMMI design prompts students to engage with the four C's (see Fig. 5.1).

Designing purposeful tasks that incorporate students' wonder is the foundation upon which the PMMI framework is built. Explorations lead to student action or investigations, through both participation and creation. Participation is an essential step in the inquiry process. In fact, Casey (2013) argued that it is the ultimate goal of learning. Through participation, individuals assert their autonomy and ownership of learning; in turn, their inquiry becomes more personal and engaging (Pink, 2009; Zhao, 2012). Creation is viewed mainly in two ways: first, through tasks where students are invited to create a new concept that was not previously known to them and second, through tasks where students are invited to offer multiple or unique solutions (Kaufman & Beghetto, 2009). For example, in investigations students are required to put forward a proposition about objects and operation which may involve unexpected relationships; thus, both creativity and critical thinking are essential (Leikin, 2014). Ultimately, creation and participation are essential elements for knowledge construction. Creation is one common form of participation (Reilly et al., 2012).

5.6 The PMMI and the Role of the Teacher

In the PMMI framework the role of the teacher also changes. The teacher would need to become a co-learner in the classroom context in order to implement the lessons as intended according to the PMMI design framework. The teacher is not the one asking and answering questions, but the teacher facilitates students' engagement with the explorations and investigations. The teacher encourages students to be in control of their leaning. It is the context of the explorations and investigations that supports this role of the teacher. Depending on whether the exploration and investigations are structured, guided or open, the control that teacher has over students' learning varies. However, in all types of explorations and investigations the students' initiative is always high.

5.7 Conclusion

This chapter presents the PMMI framework which guided the design of the mathematics textbooks in both elementary and secondary school grades. Despite the fact that PMMI was developed in the context of a particular country with certain traditions, goals, and aspirations, we suggest that it could provide a reference point that elaborates how teaching and learning mathematics might look like out of regional circumstances. In this final section, we highlight the affordances and strengths of PMMI as a potential framework for designing mathematics textbooks. We also suggest some fruitful forthcoming pathways and challenges in implementation and teacher professionalization.

Looking across the elements of the PMMI framework, two cross-cutting themes arise. First, the importance of blending practices of teaching and learning enables students to experience "Personalized Mathematics." These practices include establishment of mathematical goals, emergence and growth of reasoning, perseverance to problem solving, connections among mathematical representations, formation of concepts, and fluency with procedures (NCTM, 2014). In this sense, the PMMI framework deals with an issue that has been largely left subtle, i.e., how mathematical inquiry interweaves with and supports the needs of individual students for personalized learning. Second, it entails the importance of identifying a possible route through which students could engage with "Mathematical Inquiry." This route needs to integrate opportunities for students to experience the romance, exhibit precision, and achieve generalization of mathematical ideas (Whitehead, 1929). In addition, this route adopts an enhanced discovery approach, by varying the guidance and feedback offered to students, based on their needs (Lazonder & Harmsen, 2016).

The PMMI framework suggests that explorations and investigations constitute concrete examples of mathematical tasks that promote the realization of such opportunities in the classroom environment (Pink, 2009). Promoting inquiry-based learning is only part of the solution for achieving quality learning outcomes; students should also be given the opportunity to express and discuss their own ways of thinking mathematically. Hence, the PMMI framework contributes into comprehending the process through which students could make sense of the mathematics and at the same time develop skills such as critical thinking, creativity, collaboration, and communication.

Furthermore, the PMMI framework elaborates the position that textbook' tasks can be viewed as "shapers of the curriculum rather than merely presenting a given curriculum" (Thompson & Watson, 2013, p. 279). In this perspective, mathematics textbooks should include lessons that "shape" the underlying principles of the PMMI framework about the way in which mathematics are expected to be taught and learnt. Specifically, each lesson in the textbooks provides insight into how "Personalized Mathematics" and "Mathematics Inquiry" could effortlessly be implemented through the enactment of explorations and investigations. In this way, teachers are given a concrete learning context that outlines the conceptual objectives of their instruction through structured, innovative, or even unusual tasks. Of course, we acknowledge that a coherent and well-structured textbook does not always ensure that different teachers in different school classrooms will implement the tasks in the same way and trigger similar learning outcomes. Efforts should be placed in training and supporting teachers in students' engagement with the textbook tasks as designed based on the PMMI framework. Future empirical research may also examine the way teachers perceive the PMMI framework, how they implement it in their classroom or what are the more challenging aspects of this implementation.

The PMMI framework was designed to be applicable for teaching and learning mathematics across K-12. The underline assumption is that students can perform inquiry-based learning and mathematical practices from their early years. This does not mean that advanced mathematical thinking or complex concepts will be pushed down in elementary school. Rather, the PMMI framework suggests that how the content of elementary school is approached and taught should be reformed. Still, several questions remain to be addressed. We agree with other researchers (Alfieri et al., 2011), that future studies should investigate the type and the extent of guidance appropriate for various age groups. Additionally, further studies may investigate what type of support students of various ages need in order to become more efficient in posing questions, collecting data, hypothesizing, reflecting, or drawing conclusions. In addition, further research is needed to define explicit design principles for explorations and investigations and evaluation criteria based on the targeted age. Furthermore, teaching interventions and design research studies may yield empirical data regarding the effectiveness of the framework, elaborate the design principles of explorations and investigations, and offer insightful details regarding the effect of the guidance given during the learning route. Finally, an empirical study may reveal the exploration and investigation characteristics that contribute to further enhancing the personalized dimension of the model, by providing opportunities to find solutions based on students' own mathematical understanding.

Concluding, the PMMI framework yields insights into how the ongoing goal for fostering Personalized Mathematics learning and mathematics inquiry-based learning could be served. Using the notions of exploration and investigation, the PMMI framework defines how the content of mathematics textbooks might look like, in order to boost students' engagement in developing and reinventing mathematical concepts and ideas by linking relevant contexts with individual, sustainable conceptions. It offers a suggestion as to how mathematical challenge may look like in the mathematics classroom. Needless to say, that enacting "Personalized Mathematics" and "Mathematical Inquiry" requires investment in curriculum and textbook development, as well as long-term teacher professional development.

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