Chapter 3
Development and Stimulation of Early Core Mathematical Competencies in Young Children: Results from the Leuven Wis & C Project

L. Verschaffel, B. De Smedt, K. Luwel, P. Onghena, J. Torbeyns, and W. Van Dooren

3.1 Introduction

Mathematics has always been a central curricular domain in elementary and secondary education worldwide (De Corte et al., 1996; Kilpatrick, 1992). For a long time it was common to pay only little attention to mathematics education in the preschool years, both by teachers in preschool and by parents and other caretakers at home. The general idea was that preschool children should essentially spend their time at developing their psychomotor and social-emotional skills, together with their language and emergent literacy skills. Little or no attention was paid to interventions aimed at children's early mathematical growth. And, if some attention was paid to it, there was a remarkably narrow focus on the acquisition of some basic numerical abilities, such as reciting the counting words, identifying the numerosity of a small set of objects, indicating which set has the largest numerosity, and solving simple additive problem situations involving small whole numbers (Dede, 2010). However, the past two decades have witnessed a great research interest in early mathematical cognition, early mathematical development, and early mathematics education, both in home and preschool settings.

A starting point of this line of research – with its main origins in cognitive (neuro) science – is the idea that young children are equipped with some foundational core systems to process quantities (Butterworth, 2015; Dehaene, 2011). This allows them to exactly identify small (i.e., below 4) non-symbolic quantities, to compare non-symbolic quantities that are too numerous to enumerate exactly, or to perform some very basic approximate arithmetic on these quantities (Andrews & Sayers, 2015; Butterworth, 2015; Torbeyns et al., 2015; Verschaffel et al., 2017). Within

L. Verschaffel (✉) · B. De Smedt · K. Luwel · P. Onghena · J. Torbeyns · W. Van Dooren
KU Leuven, Leuven, Belgium
e-mail: lieven.verschaffel@kuleuven.be

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these foundational core number sense systems, magnitudes are represented non-verbally and non-symbolically, but, over development and through early (mathematics) education, verbal and symbolic representations are gradually mapped on these foundational representations, to evolve into a more elaborate system for number sense and more complex mathematical concepts and skills (Torbeyns et al., 2015). The dynamics of this development remain one of the most debated areas in research on these foundational representations (e.g., Leibovich & Ansari, 2016). This research has shown large individual differences in these early numerical abilities, which predict later general math achievement (De Smedt et al., 2013; Schneider et al., 2017; Siegler & Lortie-Forgues, 2014). Furthermore, researchers working within this research tradition have also tried to stimulate children's foundational numerical abilities with (game-based) intervention programs before or at the beginning of formal instruction in number and arithmetic in elementary school (e.g., Maertens et al., 2016; Wilson et al., 2009).

As shown in the above description, this prominent line of research has strongly focused on young children's basic numerical abilities (Cohen Kadosh & Dowker, 2015). More recently, this narrow focus on only early numerical and arithmetic abilities with non-symbolic and symbolic entities has been increasingly questioned on various grounds (e.g., Bailey et al., 2014; Dede, 2010; English & Mulligan, 2013). From a disciplinary perspective, it is evident that mathematics is much more than understanding whole numbers, counting, and basic arithmetic. Therefore, even in the early years of education, mathematics education should already represent a broader coverage of the richness of the discipline, including early reasoning about mathematical relations, shapes, and patterns and structures (Clements & Sarama, 2013; Mulligan & Mitchelmore, 2009). From an empirical perspective, recent meta-analytic work has shown that children's early numerical and counting skills explain only a small percentage of the individual differences in general mathematics achievement (Schneider et al., 2017). Accordingly, some other scholars have suggested that early quantitative reasoning about additive and multiplicative relations may be more predictive for later achievement in school mathematics (Nunes et al., 2012). Finally, and in line with the results of the above developmental studies, the abovementioned intervention studies on the early enhancement of children's foundational numerical abilities yielded mixed findings, with mainly marginal effects in terms of retention and transfer (Torbeyns et al., 2015).

As a result of the abovementioned critiques on early numerical cognition, researchers have started to go beyond analyzing young children's basic numerical abilities and to look at the early development of other, more complex, mathematical competencies in younger ages than is currently the case, i.e., already before the start of elementary school and/or while children are making the transition from preschool to elementary school (Bryant & Nunes, 2012; Dowker, 2003; Mulligan & Mitchelmore, 2009). This complementary research line has started to provide evidence of the possibility and value of broadening and deepening the scope of mathematics for young children beyond initial experiences with small whole numbers and simple arithmetic with them.

As part of that complementary approach, we embarked some years ago on a research project involving a large-scale longitudinal study about the early development of four such additional core mathematical competences: mathematical patterning, computational estimation, proportional reasoning, and probabilistic reasoning, followed by four intervention studies on the same four competencies.

This kind of research project may first of all help us rethink the traditional early mathematics curriculum with a view to make it more challenging both in terms of breadth and depth. Second, the analysis of the developmental steps children take as revealed by the longitudinal study may lead to well-articulated research-based learning trajectories. Armed with these learning trajectories, one can assess any child's thinking, locate the child on a trajectory, and determine the next step in the child's learning related to these additional mathematical topics, analogous to the trajectories developed for number, counting, and early addition and subtraction by Clements and Sarama (2013). Third, the diagnostic tools designed for the developmental studies and the instructional materials and techniques developed for the intervention studies may yield valuable building blocks for implementing these challenging curricula and designs. Finally, paying special attention in these studies to the children at the lower and the higher ends of the continuum of mathematical ability may help make these early mathematics curricula and designs more challenging and inclusive for all children.

In the present chapter, we provide a selective overview of some provisional results of this ongoing longitudinal research project. After a brief presentation of the overall aims and scope of the study and its overall methodology, available data from the various parts of the study are used to provide illustrative evidence for the basic claim that early mathematical development involves much more than children's early numerical abilities, that also with respect to these other core mathematical competencies important initial steps are being made much earlier in children's development than traditionally thought, and that these core mathematical competencies develop in close relation to each other and to the development of children's early numerical abilities. At the end we formulate some general conclusions of the research being reviewed in this chapter and we summarize its contribution to understanding how a focus in the curriculum and instructional design on challenging domains such as patterns, computational estimation, proportional reasoning, and probabilistic reasoning may enhance the mathematical competence of all young children.

3.2 A Research Project Consisting of Four Parts

In 2016, we started a 6-year-long research project on the development of 4- to 9-year-olds' competencies in four early core mathematics-related domains[1]: mathematical patterns, computational estimation, proportional reasoning, and probabilistic reasoning. While the rationale for this selection was partly pragmatically

[1] C16/16/001 project "Early development and stimulation of core mathematical competencies" of the Research Council of the KU Leuven, with Nore Wijns, Elke Sekeris, Elien Vanluydt, Anne-Sophie Supply, and Merel Bakker as PhD researchers and, consecutively, Joke Torbeyns, Greet Peters, and Laure De Keyser as project coordinator.

grounded in the fields of expertise of our research team, a common characteristic of these four domains is that they all receive little or no instructional attention in current early mathematics education curricula, while they do represent important domains of mathematics and while there is increasing empirical evidence that they start to emerge much earlier than traditionally thought, and therefore, children may be challenged in these domains at an earlier age than is currently the case.

For each of these four domains, we tried to document the emergence and early development of intuitive concepts and basic skills related to the domain, to look for interrelations between these emerging concepts and skills, with a view to explore ways to organize early and elementary school mathematics such that this organization does not undermine these intuitive concepts and emerging skills but rather creates an environment wherein they can be acknowledged and stimulated.

In order to longitudinally map the emergence and development of these core mathematical competencies, as well as children's early numerical abilities, a cohort of over 400 children from 17 schools is followed from the second year of preschool (±4 years of age) to the third year of elementary school (±9 years of age). Using a stratified cluster sampling strategy to ensure an SES distribution that is representative for the Flemish context, schools were selected based on the relative number of pupils who receive study allowance and/or whose mother did not obtain a secondary school certificate. In Flanders, children go to preschool from the age of 2.5 years onwards. Preschool consists of three years (P1, P2, and P3). It is fully government subsidized and non-mandatory, yet it is attended by nearly all children. In September of the year children turn 6, they start in elementary school, which consists of six grades and which is mandatory.

As shown in Fig. 3.1, a rich battery of measures was administered during the 5 years of data collection. This battery comprised tasks and instruments assessing children's mathematical patterning, computational estimation, proportional reasoning, and probabilistic reasoning (parts 1 to 4, see further), as well as children's domain-specific early numerical abilities, domain-general cognitive abilities, and general mathematics achievement. The domain-specific early numerical abilities test comprised a wide variety of tasks measuring verbal counting, object counting, Arabic number recognition, number comparison, number order, and non-verbal

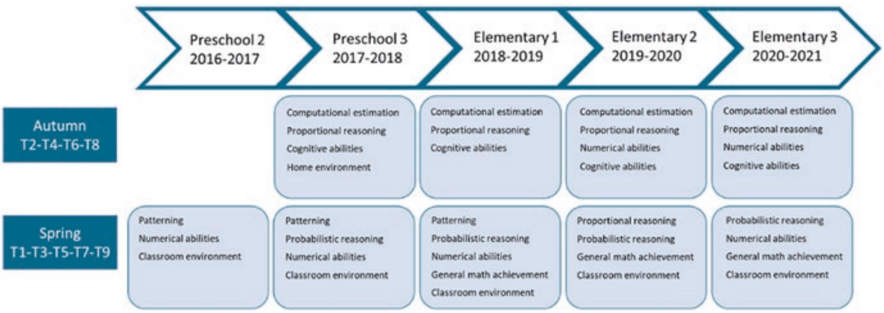

Fig. 3.1 Timeline for school years, time points of assessments, and administered measures

calculation; the domain-general cognitive abilities measures were two working memory and one spatial ability test; and general mathematics achievement was measured by means of a Flemish standardized mathematics achievement test. Finally, data on children's home and class environment were, respectively, collected via parent and teacher questionnaires. For more details about these instruments see Bakker et al. (2019).

3.3 Early Mathematical Patterning

Patterning is an aspect of early mathematical ability that is defined as children's performance on a wide set of tasks, such as extending, translating, or identifying a pattern's structure, that can be done with regular configurations of elements in the environment (Wijns et al., 2019c). These regular configurations are called patterns. There are different types of patterns, and a distinction is often made between repeating (e.g., ABABAB), growing (e.g., 2 4 6), and spatial structure patterns (e.g.,::::; Mulligan & Mitchelmore, 2009). Repeating and growing patterns are both sequences that can be continued indefinitely. Their underlying structure or rule describes how the sequence continues. For repeating patterns (e.g., ABABAB, Δ□□Δ□□Δ□□), the structure is defined on the basis of a unit (e.g., AB, Δ□□) that is reiterated. The structure of growing patterns (e.g., 2 4 6, Δ□Δ□□Δ□□□) involves a systematic increase or decrease between the units in the sequence (e.g., +2, + □). Spatial structure patterns, by contrast, represent two-dimensional configurations of elements. In part 1 of the project, we investigate the development of 4- to 6-year-olds' repeating and growing patterning competencies, and their associations with these children's numerical abilities.

Children are confronted with repeating patterns from a very young age in their daily life activities (e.g., day-night-day-night and yellow-red-yellow-red lines on their T-shirt). Repeating patterns are also the most common type of patterns in early childhood education and research (for a review, see Wijns et al., 2019c). At the start of the project, a number of empirical studies on young children's repeating patterning competencies had provided evidence for preschoolers' ability to solve tasks involving repeating patterns, and for the association between children's repeating patterning abilities and both concurrent and later numerical and mathematical abilities (e.g., Collins & Laski, 2015; Lüken, 2012; Rittle-Johnson et al., 2015, 2017; Zippert et al., 2019). Repeating patterning competencies were also shown to uniquely contribute to later mathematical performance, in addition to children's early numerical abilities (Lüken, 2012; Nguyen et al., 2016; Rittle-Johnson et al., 2017). However, systematic analyses of the mechanisms that might explain the association between young children's patterning and early numerical ability as well as the developmental associations between these two early mathematical competencies were non-existent. Moreover, researchers were criticized for their exclusive focus on repeating patterning abilities, arguing that young children are already capable of handling more complex patterns, such as growing patterns (Pasnak et al., 2019). As far as growing patterns were included in empirical studies, they were analyzed in view of their contribution to the development of elementary school children's algebraic skills (e.g., Warren & Cooper, 2008). To the best of our knowledge, no research had investigated preschool children's ability to successfully deal with activities focusing on growing patterns. Finally, although several researchers had already hinted toward the idea that children who by themselves look for patterns in their environment are good mathematicians, young children's spontaneous attention for patterns was not yet systematically investigated. This contrasted with the domain of number, where researchers had already documented the pivotal role of young children's spontaneous attention for quantities (SFON; Hannula & Lehtinen, 2005) and number symbols (SFONS; Rathé et al., 2019) for their concurrent and later mathematical development. Part 1 of the present project aimed to increase current insight into the role of patterning within early mathematical development by addressing the abovementioned weaknesses and systematically analyzing (a) young children's spontaneous attention for patterns in their environment, (b) their ability in handling repeating as well as growing patterns, and (c) the association between their repeating and growing patterning ability and their numerical ability.

For this part of our longitudinal research project, we followed the development of children's early patterning and number abilities between 4 and 6 years. This age range covers a critical developmental period in which several aspects of patterning and numerical ability are known to be acquired rapidly. In the spring of preschool year 2, preschool year 3, and elementary school Grade 1, children were offered two patterning ability measures, one focusing on repeating patterns and one focusing on growing patterns. Both patterning ability measures consisted of three types of patterning activities: extending the pattern (i.e., what comes next in the pattern?), translating the pattern (i.e., make the same pattern using different materials), and identifying the structure of the pattern (i.e., identifying the unit of repeat that defines the repeating pattern, identifying the systematic increase or decrease that defines the growing pattern). Figure 3.2 provides an example item for the three patterning activities in the repeating patterns and growing patterns ability measure. The patterning ability measures consisted of 18 items (6 items per activity) that were scored as either correct or incorrect, resulting in a maximum score of 18 per measure. Before they solved the two patterning ability measures, children engaged in an activity that addressed their spontaneous attention for patterns, the so-called tower task, in which children were asked, in a free-play context, to make a tower construction with 15 building blocks in three colors (five per color). Children's tower constructions were scored as (a) pattern, when the tower included at least two full units and the start of the third unit of a pattern, (b) sorting, when all the blocks were sorted per color, or (c) random, indicating no pattern or sorting construction.

In a first study (Wijns et al., 2019a) we focused on 4-year-olds' spontaneous attention for patterns when solving the tower task (i.e., SFOP). We looked for individual differences in 4-year-olds' SFOP as well as their associations with children's repeating patterning and numerical ability. We found individual differences in 4-year-olds' SFOP and showed that children who spontaneously created a pattern had higher repeating patterning ability and numerical ability than children who

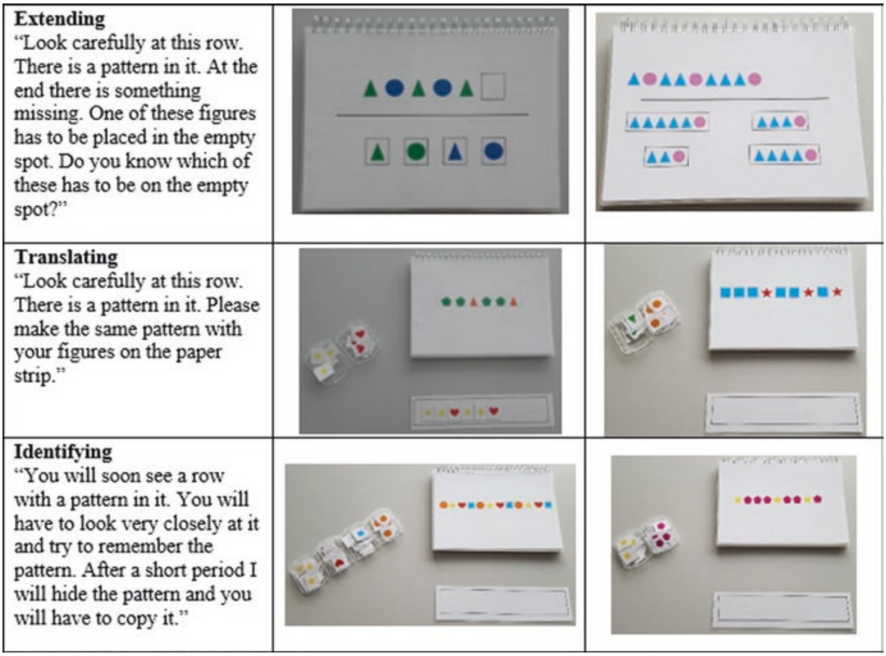

Fig. 3.2 Example items for the two pattern types (repeating and growing) and the three activities (extending, translating, and identifying)

made a random arrangement. The positive associations between 4-year-olds' SFOP and their repeating patterning and numerical ability can be hypothetically explained via the mechanism of self-initiated practice (cf. Hannula & Lehtinen, 2005, and Rathé et al., 2019, for a similar explanation related to, respectively, SFON and SFONS). This mechanism suggests that children with a spontaneous tendency to focus on mathematical elements in their environment will have more opportunities to practice their mathematical abilities and therefore improve them. Related to SFOP, children who spontaneously look for and create patterns during daily life activities are assumed to have more opportunities to practice their patterning abilities and, by extension, numerical skills. Although viable, this hypothetical explanation requires further research attention.

In a second study (Wijns et al., 2019b), we analyzed 4-year-olds' ability in both repeating and growing patterning tasks, and their association with children's numerical ability. A confirmatory factor analysis showed that the 2 × 3 structure of our patterning ability measure (two types of patterns, three patterning activities) could also be found in our data, confirming the validity of our measure. Additionally, both the pattern type and the patterning activity had an impact on children's patterning performance. Concerning the pattern type, we found that growing patterns were more difficult than repeating patterns. This difference in difficulty level might be due to differences in the complexity of the structure of the different pattern types (i.e., a clearly visible unit that repeats versus a systematic increase or decrease that needs to be deduced from the visible pattern) as well as the emphasis on mainly repeating patterns in current educational practice, which might lead children to think that all patterns are repeating patterns. Importantly, the study also showed that, despite the high difficulty level of activities with growing patterns, growing patterns are already feasible for a significant number of children of this young age. Turning to the impact of patterning activity, our study indicated that translating patterns was easier than both extending patterns and identifying the structure of patterns. The difference in difficulty level of extending and translating patterns is not in line with earlier studies with only repeating patterns (Lüken, 2012; Rittle-Johnson et al., 2015) but might be due to children's use of a one-one matching strategy to solve the latter type of tasks. Although our observational data do not support this explanation, future studies on the strategies that children use when solving different types of patterning tasks are required (cf. the recent study of Lüken & Sauzet, 2020). Finally, we also found that most patterning tasks uniquely predicted children's numerical ability.

Our third study involved a longitudinal analysis of the direction of the associations between repeating patterning, growing patterning, and numerical ability from age 4 to 6 (Wijns et al., 2021a). Although several studies had already provided evidence for an association between patterning and numerical ability, little was known about the direction of this association. Moreover, at the start of the project, it was unclear whether the association with numerical ability was different for distinct pattern types. Our cross-lagged panel analysis revealed bidirectional associations between all three abilities from age 4 to 5, suggesting that performance on one ability supports performance on another ability 1 year later. From age 5 to 6, patterning abilities predicted numerical ability, but the reverse was no longer true. Also, from age 5 to 6, repeating patterning abilities predicted growing patterning abilities, but not vice versa. These findings suggest that children's repeating and growing patterning ability supports the acquisition of later numerical ability, and that, within children's patterning ability, repeating patterning ability supports the acquisition of growing patterning ability. Although several researchers already hinted at the possibility to explore regularities in both patterning and number tasks (e.g., the base-10 structure of our number system with repeating units across decades, or the systematic increase with one of our counting row) as a mechanism that explains their associations, theoretical models are missing. It is a challenge for future work in the domain to first develop these theoretical models and next conduct focused intervention studies that help reveal the mechanisms underlying the frequently observed associations between patterning ability and numerical ability.

In our fourth study, we evaluated the effectiveness of an intervention aiming to enhance 5-year-olds' repeating and growing patterning ability for their development of early patterning competency (Wijns et al., 2021b). A 20-week intervention program (with 30 minutes patterning activities per week, focusing on the patterns' structure) resulted in significant improvements in the patterning competency of the children following the intervention compared to the control group, but there was no transfer effect to their numerical ability.

The findings of the longitudinal and intervention study provide important building blocks for optimizing current early mathematics education. First, the longitudinal analyses add to current insights into children's learning trajectories in the domains of number and patterning, pointing to the pivotal role of preschoolers' patterning competencies. Second, our focused intervention on the structure of repeating and growing patterns greatly enhanced preschoolers' patterning competency. These findings call for more attention than is currently the case for stimulating patterning competency in preschool curricula, integrating also more complex, growing patterns and more challenging patterning activities, such as translating the pattern or identifying its structure.

3.4 Early Computational Estimation

Computational estimation can be described as providing an approximate answer to an arithmetic problem without calculating it precisely. This mathematical skill shows a commonality with the approximate arithmetic competence that is assumed to be part of young children's foundational approximate number system (ANS), in the sense that in both cases one has to mentally perform an arithmetic operation on two operands in an approximate way. However, the two skills are also fundamentally different: while in approximate arithmetic children have to process the operands approximately, the numerical value of the operands is known in computational estimation (Sekeris et al., 2019).

Computational estimation is viewed as an important mathematical competence in our daily life since many situations only require calculations with a reasonable degree of accuracy, such as splitting the bill among a group of friends in restaurant. In addition, it is widely agreed that computational estimation should play an important role in the elementary mathematics curriculum (Siegler & Booth, 2005; Sowder, 1992; van den Heuvel-Panhuizen, 2000) as it involves a complex interplay of various types of mathematical knowledge and skills, including conceptual knowledge (e.g., accepting more than one value as an outcome of an estimation), procedural knowledge (e.g., being able to modify the problem to arrive at a mentally more manageable problem), and arithmetic knowledge and skills (e.g., mental computation skills). Given that computational estimation problems can be solved in many different ways, it allows children to develop number sense (LeFevre et al., 1993) and strategy flexibility (Siegler & Booth, 2005). Although computational estimation is nowadays widely recognized as an important part of the elementary mathematics curriculum (e.g., NCTM, 2000), it has, compared to its counterpart exact arithmetic, received far less attention from curriculum developers and researchers (Dowker, 2003; Siegler & Booth, 2005).

A recent literature review by our team revealed that the vast majority of studies on computational estimation investigated this skill from the age of eight and onward (Sekeris et al., 2019). This could be related to the fact that computational estimation is typically only instructed from the middle grades of elementary school onwards, when children have already acquired ample experience with whole-number exact arithmetic (Common Core State Standards Initiative, 2010). However, a few studies suggested that children are already able to engage in computational estimation at a younger age than is traditionally expected (Dowker, 1997, 2003; Jordan et al., 2009). This might not be so surprising, given that recent studies provided empirical evidence that preschool children can use their basic numerical abilities to solve approximate arithmetic problems with both non-symbolically and symbolically presented comparisons (e.g., “15 + 13 vs. 49, which is more?”) before they have been taught exact arithmetic in school.

Part 2 of our research project therefore aimed at charting the emergence and early development of computational estimation from the age of 5 (third grade of preschool) until the age of 9 (third grade of elementary school). Children’s computational estimation skills were tested on an individual basis once each year. To that aim, we developed a task in which children had to estimate the outcome of addition problems and that consisted of a non-verbal and a verbal variant. In the non-verbal variant, which was used in third grade of preschool and first grade of elementary school, the estimation problems were presented by means of manipulatives. Both addends were represented by a number of cows that were consecutively positioned in a horizontal row in front of the child, verbally labeled by the experimenter (“Here are N cows”), and hidden in a stable afterward. Next, children were asked to indicate about how many cows there were altogether in the stable by putting a number of cows from their own pile on the table (see Fig. 3.3).

In the verbal variant, which was used in the first three grades of elementary school, the estimation problems were presented with Arabic numerals on a computer screen for 20 seconds and simultaneously read out loud by the experimenter after which children had to respond verbally. To ensure that children would engage in computational estimation, we presented them with addition problems that were numerically just too difficult to be solved by means of exact arithmetic (Dowker, 1997, 2003). The level of exact arithmetic in each grade was based on children’s curriculum and extensive pilot testing. Over the entire duration of the longitudinal study, children had to estimate the outcome of 24 addition problems of different difficulty levels, which were defined by the size of the exact outcome of the estimation problems. More specifically, these outcomes ranged between 11 and 30 in third

Fig. 3.3 Example of a computational estimation problem from the non-verbal task variant with (**a**) the first addend, (**b**) the second addend, and (**c**) the child’s answer

year of preschool, 11 and 100 in first and second years of elementary school, and 51–10.000 in third year of elementary school.

We focused both on computational estimation performance and strategy use (see e.g., Sekeris et al., in press). At present, data have been collected from third year of preschool until second year of elementary school. Estimation performance in both tasks was measured in terms of children's accuracy and was operationalized in terms of percentage absolute error (PAE) of children's estimates relative to the exact answer. We observed that children's PAE evolved from 34% in third year of preschool to 19% in second year of elementary school, indicating that children became more accurate in their estimates when growing older. Interestingly, in the first year of elementary school – where both task variants were administered with exactly the same problems – we found that children were, as expected, more accurate in the non-verbal (34%) than in the verbal task variant (27%). Presumably, this lower performance on the verbal task variant could be attributed to children being insufficiently familiar with two-digit numbers being represented with Arabic numerals. Similar findings have been reported by Dowker (2003) for computational estimation and Levine et al. (1992) for exact arithmetic. In both task variants we also observed an effect of problem size. Children's estimates became less accurate with increasing problem size, suggesting that children were not merely guessing the outcome of the estimation problems.

Children's strategy use was examined for both task variants separately. For the non-verbal variant we looked at two aspects of children's externally observable behavior when lying down their answer by means of the manipulatives: (a) the way in which they constructed the answer set and (b) their counting behavior while constructing the answer set. With respect to the construction of the answer set, we distinguished, based on previous studies in arithmetic (Carpenter & Moser, 1982; De Corte & Verschaffel, 1987), among three different strategies that might reflect different representations of numbers and arithmetic operations: (a) creating two sets of manipulatives representing both addends which were either kept separate (addends only) or (b) put together afterward (combining), and (c) immediately putting all manipulatives in one group (result-only). For their counting behavior we looked at whether children counted or not when constructing the answer set. Results showed that both in third year of preschool and first year of elementary school about 95% of the problems were solved by means of the result-only strategy. This frequency did not change with age or problem size. The frequency of children's counting behavior showed an age-related increase and decreased with increasing problem size. A structural equation model showed that in preschool none of the two aspects of children's material solution strategies were predictive of their estimation performance, whereas in first grade the result-only strategy was a negative predictor and counting frequency a positive predictor of their estimation performance. These findings might indicate that children in third year of preschool lack the insight that the way in which they use the manipulatives or their counting skills could help them make better estimates. By the first year of elementary school, they might have come to the understanding that a purposeful use of the manipulatives (i.e., by representing both addends first separately) and counting might lead to improved estimations.

In the verbal variant of the computational estimation task, children's strategy use was identified on the basis of immediate trial-by-trial verbal strategy reports. Strategies were classified according to an a priori classification scheme which distinguished among four broad strategy categories: (a) exact arithmetic in which children calculated the answer exactly instead of estimating it, (b) exact-calculation-and-adjusting in which the answer was calculated exactly and then adjusted to make it look like an estimate, and (c) rudimentary computational estimation strategies that showed some basic and rough conceptual understanding of the principles of computational estimation, and genuine computational estimation strategies in which the estimation problem is first simplified (e.g., by rounding the operands) before calculating the approximate outcome. We observed that children hardly used any genuine computational estimation strategies, presumably because children at this age did not yet possess the necessary mathematical knowledge and skills for applying such advanced estimation strategies. However, children already had a basic understanding of some of the underlying principles of computational estimation, as was evidenced by the fact that they referred to the proximity principle (i.e., the idea that an estimate should be close to the exact outcome) when applying a rudimentary computational estimation strategy or that they took into account the approximation principle (i.e., the estimate should be an approximation of the exact outcome) when using the exact-calculation-and-adjusting strategy. Interestingly, the use of the exact arithmetic and exact-calculation-and-adjusting strategies increased from first to second grade of elementary school. Probably, the strong focus on exact arithmetic in mathematics education at the beginning of elementary school makes children increasingly convinced that each arithmetic problem has only one correct answer.

To conclude, the present findings indicate that young children are already able to engage in computational estimation at a much younger age than is generally assumed. Their estimation performance increases with age, even in the absence of instruction in computational estimation. Young children already use a variety of strategies to solve computational estimation problems. This strategy use reveals traces of a beginning conceptual understanding of the principles underlying computational estimation. Taking into account the aforementioned multi-componential nature of computational estimation, its potential for developing number sense, and the recurrent finding that people are generally bad at it (Siegler & Booth, 2005), our findings suggest that computational estimation could be incorporated much earlier in the mathematics curriculum. Such an early learning trajectory for computational estimation could start by familiarizing young children with the concept of estimation, its underlying principles (e.g., the proximity and approximation principle), and the specific language of estimation (e.g., "about," "near," and "close to"). This earlier incorporation in the curriculum might prevent that the early development of children's estimation skills becomes too much hampered by their strong focus on being exact as a result of their confrontation with formal school mathematics.

3.5 Early Proportional Reasoning

Proportional reasoning plays a critical role in people's mathematical development. It is essential in the learning of numerous advanced mathematical topics, such as algebra, geometry, statistics, or probability, but people also encounter it in numerous daily life situations (e.g., recipes, sales). Unfortunately, it is also considered to be hard to apprehend for children, and achieving a full understanding of proportionality is considered a major challenge (Kaput & West, 1994). In the research literature, there is no unanimity about the age range in which proportional reasoning abilities develop.

The traditional Piagetian stance on the development of proportional reasoning is that it is a rather late achievement (Inhelder & Piaget, 1958). They see it as an indicator of formal operational thought, typically only starting to develop from the age of 12. Typical evidence comes from tasks like the Paper Clip Task (Karplus & Peterson, 1970): learners get the height of Mr. Tall and Mr. Short expressed in a number of buttons, and the height of Mr. Short in expressed in a number of paper clips. They need to find the height of a Mr. Tall expressed in paper clips. Academically upper-track or upper middle-class students used proportional reasoning increasingly at the age of 12 years, but only a small fraction of urban low-income and academically lower-track students used proportions at the age of 14 or even 17 years. Similar findings come from Noelting (1980), who used Orange Juice Problems: comparing mixtures of varying numbers of glasses of orange juice and water. He reported that proportional reasoning is a concept that finds its achievement only in late adolescence and that children did not reach the formal operational level before the age of 12.

However, more recent studies suggested that proportional reasoning may start to develop much earlier than suggested by Piaget and colleagues. We mention a few examples. Resnick and Singer (1993) presented 5- to 7-year-old children with a proportional missing-value task. Children had to feed fish of different lengths. All children tended to give proportionally larger amounts of food to larger fish. Boyer and Levine (2012) used an orange juice task to assess proportional reasoning in 6-to 9-year-old children. Results showed that these young children could already match equal proportional mixtures, but performance depended on the scaling magnitudes in the problems. Finally, in preparation for part 3 of the current longitudinal study we also found early traces of proportional reasoning in 4- to 5-year-old children (Vanluydt et al., 2018). Many of them were able to make the ratio between puppets and grapes in a set B equal to the ratio between puppets and grapes in a set A, and strategies pointed to the emergence of a notion of one-to-many correspondence, which is an important first step in the development of proportional reasoning. While the full understanding of proportionality might only be achieved at the age of 12, the development of proportional reasoning seems to begin much earlier, allowing young children to reason proportionally in certain tasks (involving specific contexts and ratios) and under certain conditions (i.e., individual interview settings with hands-on activities).

In part 3 of the longitudinal study, we are mapping the development of children's proportional reasoning ability from the age of 5 (last year of kindergarten) until 9 (third year of elementary school). For this purpose, we developed and validated a task about a fair sharing context, involving manipulatives, and avoiding the need to use number symbols (Vanluydt et al., 2019). Children are given missing-value problems involving discrete and/or continuous quantities. In tasks with discrete quantities, they have to construct a set B equivalent to a comparison set A by putting the elements in set B in the same ratio as the elements in set A. The two discrete quantities are puppets and grapes that need to be shared among them. In tasks with continuous quantities, the context is similar, but the grapes are replaced by a continuous quantity, chocolate bars of varying lengths. Example items are shown in Fig. 3.4.

We are currently awaiting the results of the longitudinal study to map the development in detail, and to link it to various learner characteristics. A cross-sectional exploration with a comparable sample (Vanluydt et al., 2019) already revealed several qualitatively different early stages of proportional reasoning, in which the nature of the quantities involved in the problem (discrete vs. continuous) as well as the unknown quantity (the grapes/chocolate or the puppets) played a role. For instance, while performing equally well in general, some children showed a greater ability to reason proportionally when the problem involved only discrete quantities, whereas others performed better when continuous quantities were involved. Some children already showed full mastery on the proportional reasoning tasks at the age of 9, but most children were still developing this ability. Our longitudinal data will allow to reveal which children progress fastest and furthest by the age of 9: those who can reason about discrete quantities at an early age or those who can reason about continuous quantities.

Instruction: "All puppets are equally hungry. If I give four grapes to these puppets, how many grapes do you have to give to these puppets for it to be fair?"

Instruction: "All puppets are equally hungry. If I give this chocolate bar to these puppets, which chocolate bar do you have to give these puppets for it to be fair? You can give a chocolate bar to the puppets so that it's fair."

Fig. 3.4 Example items of the proportional reasoning tasks involving discrete and continuous quantities

Along with the study of the development of early proportional reasoning, we were also able to investigate how other mathematical competencies, such as mathematical patterning, are associated with children's proportional reasoning ability. We have shown the predictive association between patterning in the second year of kindergarten and proportional reasoning ability in the first year of elementary school. Two measures of patterning ability (repeating and growing patterns, see paragraph 3) were used as a predictor for two measures of proportional reasoning ability (involving a discrete or a continuous quantity). Patterning ability turned out to be a unique predictor of proportional reasoning ability over and above sex and general cognitive and numerical abilities. More specifically and quite remarkably, performance on repeating patterns was uniquely related to performance on proportional reasoning with a discrete quantity, whereas performance on growing patterns was uniquely related to performance on proportional reasoning with a continuous quantity.

Another aspect that we investigated is the role of language abilities in proportional reasoning. It is generally known that language – be it language in general or language related to mathematics – plays a crucial role in mathematical thinking and learning (Peng et al., 2020). However, so far no studies studied the role of language in proportional reasoning at an early age. We longitudinally investigated if specific mathematical vocabulary related to proportional reasoning (e.g., understanding expressions like "half" or "three times more") in the first year of elementary school predicts proportional reasoning abilities in the second year of elementary school. A hierarchical linear regression analysis showed that specific mathematical vocabulary related to early proportional reasoning in the first year of elementary school is a unique predictor for proportional reasoning abilities in the second year of elementary school over and above age, socio-economic status (SES), and general vocabulary (Vanluydt et al., 2021). Although more evidence based on intervention studies is needed to reveal the causal nature and the direction of this relation, these results suggest more attention to specific mathematical vocabulary related to proportional reasoning in young children might stimulate early proportional reasoning.

Several other studies are planned using the available longitudinal data, in order to obtain a deeper understanding of the development of proportional reasoning abilities at a young age. From second grade on, we started to offer arithmetic word problems, in addition to the proportional reasoning fair sharing tasks with manipulatives that were described above. Some of these word problems are proportional, but also additive word problems are included, such as the following:

> Roos and Loes are running around a track. They run equally fast, but Loes started later. When Loes has run 2 rounds, Roos has run 8 rounds. When Loes has run 4 rounds, how many has Roos run?

The literature (e.g., Van Dooren et al., 2010) reports that young children often erroneously solve proportional problems additively while older children solve additive problems proportionally (in the problem above, they would answer that Roos has run 16 rounds). Our longitudinal data will reveal whether early individual differences in proportional reasoning abilities predict these two kinds of errors.

So far, our findings indicate that children in the third grade of kindergarten and the first years of elementary school can make sense of the one-to-many correspondences in proportional situations and suggest that these may already be stimulated and developed into an understanding of many-to-many situations. This seems possible even before the arithmetic skills for addition and multiplication are extensively practiced. Attention to the specific mathematical vocabulary involved in proportional situations seems important in doing so. We are currently developing instructional materials for this purpose, which will be tested in an intervention study.

3.6 Early Probabilistic Reasoning

Parallel to the research and discussion about early proportional reasoning, there is a growing body of developmental research showing that very young children have basic intuitions about chance events and that these intuitions develop into a more formal probability concept during elementary school (Bryant & Nunes, 2012; Piaget & Inhelder, 1951/1975). The successive developmental stages of probabilistic reasoning have been given several labels, but boil down to three main stages: non-probabilistic reasoning (preoperational; until the average age of 6 years), emergent probabilistic reasoning (concrete operational; from 6 to 11 years old), and finally quantification of probability (formal operational; from about the average age of 11 years) (Green, 1991; Jones et al., 1999; Way, 2003).

Preliminary results on these basic intuitions in young children have already been obtained with respect to several components of probabilistic reasoning: understanding randomness, working out the sample space, comparing and quantifying probabilities, and understanding relations between events (Bryant & Nunes, 2012). However, the developmental pathways of these components and their relation to the development of other competencies remain largely unexplored.

Based on these descriptive developmental studies, many countries around the world have introduced probability calculus as part of the curriculum in elementary school in the 1990s (Way, 2003). More recently, in two southern German states, Baden-Württemberg and Bayern, the basics of probabilistic thinking are included in the elementary school curriculum partly as a result of the rising awareness of the importance of "risk competency" (Granzer et al., 2009; Martignon & Erickson, 2014; Till, 2014). However, little is known about the effects of teaching probability and statistics in elementary school or about the processes involved.

With respect to probabilistic reasoning, our project had three main objectives. First, we aimed to construct a more comprehensive view on the development of different components of probabilistic reasoning in children from the age of five to nine. Second, we wanted to explore the relationship between the development of numerical abilities, mathematical patterning, computational estimation, and proportional reasoning on the one hand, and the development of probabilistic reasoning among elementary school children on the other hand. Our expectation was that these other abilities are important building blocks for emergent probabilistic reasoning. A third objective was to investigate whether it is possible to stimulate probabilistic reasoning at a younger age than is currently the case in Flemish schools.

This part of the project is still ongoing, but we already have some first results from pilot studies and analyses from the first wave of the longitudinal study. Because we needed an instrument for the early assessment of probabilistic reasoning, we constructed several tasks that tapped into children's ability to recognize (un)certainty and children's ability to compare probabilities. The basic setup is an individually administered binary choice task in which children have to select one out of two boxes that has the best chance to blindly pick a winning element. The concrete setup is an adapted version of the setup proposed by Falk et al. (2012) and goes as follows (see Fig. 3.5):

> Children sit in front of a laptop screen. They are introduced to a blindfolded bird and are told that the bird loves black berries but hates white berries. In each trial, the bird blindly picks a berry from one of two boxes that are filled with different number of berries of the desired and undesired color (see Fig. 3.5). Unlike the bird, children can see the content of each box and they are asked to help the bird by deciding which of two boxes is best for the bird to blindly pick a berry from.

An interesting property of this setup is that the difficulty of the items can be varied meaningfully by manipulating their features. For example, it is possible to vary the total number of berries, the proportion of black berries, and even more than two colors of berries can be used (after slightly adapting the instruction). Based on the study by Falk et al. (2012), we expected that items would become particularly challenging to the children if the optimal box would contain a smaller absolute number of black berries (see Fig. 3.6); and even if there are no white berries left in that box (see Fig. 3.7).

After pilot testing, the final instrument consisted of 29 items. For an independent validation and feasibility study, we presented the instrument to a cross-sectional sample of 177 5- to 9-year-olds in a school who did not participate in our larger longitudinal data collection. We found that our instrument was fit to use in kindergarten and elementary school. The children understood our instruction and it took no longer than 10 minutes to administer the task. Furthermore, the results were encouraging from the perspective of assessing probabilistic reasoning at these

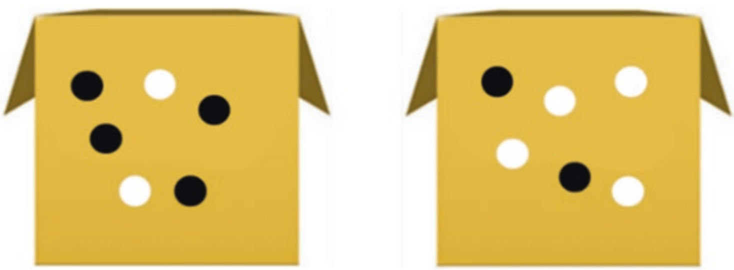

Fig. 3.5 Example item for the probabilistic reasoning task: Select the box that gives you the best chance to randomly draw a black berry from

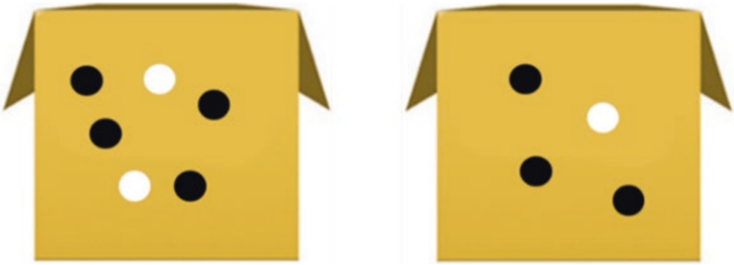

Fig. 3.6 Example of a difficult item for the probabilistic reasoning task: The box with the smaller number of black berries has a larger probability to randomly draw a black berry from

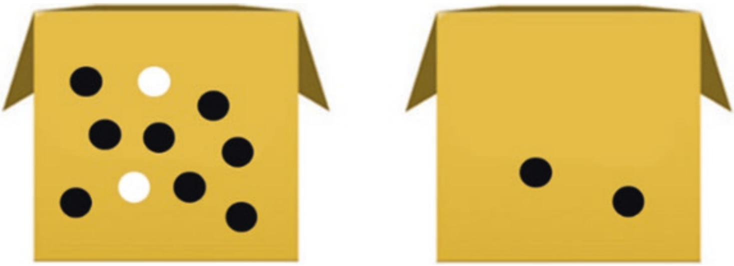

Fig. 3.7 Example of a difficult item for the probabilistic reasoning task: The box with the smaller number of black berries has a 100% probability to randomly draw a black berry from

young ages: item difficulty varied as expected, older children obviously had better performance than younger ones, and we found no indications for floor or ceiling effects in any age group. The extensions that we added to the setup by Falk et al. (2012) also seemed to improve the reliability and validity of the instrument (Supply et al., 2018, 2020).

When we applied this instrument to the 5- and 6-year-olds in our longitudinal study, we found that children within the same year of kindergarten strongly differed in their performance on the items that had one box with 100% probability to randomly draw a black berry from. Furthermore, children's performance on these items was predictive for their performance on the items that required a comparison of probabilities of only uncertain outcomes. These results demonstrate that, although conventional developmental theory assumes that there is no probabilistic reasoning in the preoperational stage, kindergarten children already have good performance in certain tasks that require probabilistic judgments. In addition, the recognition of uncertainty may act as a precursor for emergent probabilistic reasoning (Supply et al., 2019a). In these 5- and 6-year-old children, we also explored the relation between the performance on the numerical tasks that were administered as part of the longitudinal study (see Sect. 3.1) and our binary choice instrument, extended with a construction task. For the construction task, children were introduced to two representations of identical birds, two rectangular boxes containing white and black marbles, and one larger square box containing 10 black marbles (see Fig. 3.8).

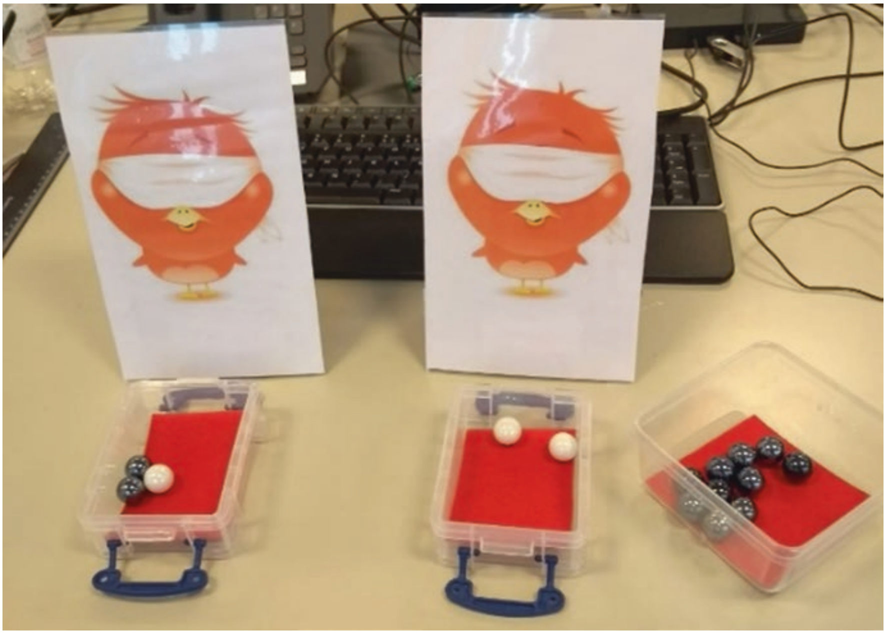

Fig. 3.8 Example of an item for the construction task: Add black berries to the right-side rectangular box to have an equal probability to randomly draw a black berry from each of the boxes (the square box on the right hand contains the black berries that can be used to make the adjustment)

As with the binary choice instrument, the construction task was administered individually. The instruction was as follows:

> These are Flip and Flap. Flip and Flap are twins. Flip and Flap are both blindfolded because we are going to play a game with them. Flip and Flap both like black berries (experimenter shows child black marbles), but get sick of these white berries (experimenter shows child white marbles). I will always give a box to each Flip and Flap and they can each blindly pick one berry from their own box. Of course, Flip and Flap cannot see what is in the box, because they are wearing that blindfold. Flip's box contains white and black berries, but Flap's box always contains only sickening white berries. That is not fair of course. You can add black berries to the box of Flap so that it becomes a fair game. You can add as many berries, until you think that Flip and Flap are just as likely to blindly pick a black berry when they are blindfolded and allowed to pick only once in their own box.

We found a strong general association between the performance on the numerical tasks and the items that required a comparison of probabilities of only uncertain outcomes. There was no association between numerical skills and the ability to distinguish uncertain from certain events, and we also found no association with the performance in the construction task. In the latter task, children with better numerical skills tended to add as many winning elements to the new box as there were in the box that was given, thereby ignoring the number of losing elements in the given box. This suggests that at this young age, good early numerical skills might promote the use of erroneous strategies in probabilistic situations. Future research could investigate whether these erroneous strategies can be seen as the first step in reasoning about probabilities or whether they impede proper probabilistic reasoning (Supply et al., 2019b).

In sum, our preliminary findings suggest that probabilistic situations are already intelligible for 5- to 6-year-olds. At this age, children have not been formally introduced to addition, multiplication, and proportionality, but nevertheless are able to give meaningful answers in binary choice tasks that involve probabilistic optimization. These findings challenge the common notion that probability as a mathematical topic is too difficult for elementary school children and should only be included in the curriculum of secondary school or university. As such, these findings open up a perspective for a learning trajectory on probability and statistics from kindergarten (e.g., by playing games of chance) to elementary (e.g., calculating probabilities) and secondary school (e.g., deriving Bayes' theorem).

In our opinion, this perspective is of paramount importance because the inclusion of probability as a topic in the elementary school curriculum can act as a counterweight to current mathematics and science curricula that – from the first years on – put a strong emphasis on exact arithmetic with small cardinal numbers, deterministic causal explanations, and certitude and that instill a view of science and a view of the world that leaves no room for doubt, uncertainty, intrinsic stochastic processes, or measurement error. However, we must also acknowledge that our finding of developing probabilistic reasoning in 5- to 6-year-olds does not imply that education can improve or accelerate this development. Therefore, an additional intervention study is planned to investigate whether it is possible to stimulate probabilistic reasoning at a younger age than is currently the case in Flemish schools.

3.7 Conclusion

In this chapter we gave a snapshot of the main results of a 6-year-long research project that started in 2016 and in which we longitudinally follow the integrated development of 4- to 9-year-olds' competencies in four challenging mathematical domains – mathematical patterns, computational estimation, proportional reasoning, and probabilistic reasoning – using a rich battery of measures.

The preliminary findings of the longitudinal study confirm our basic claim that, with respect to these four core mathematical competencies, important initial steps are being made in children's development (much) earlier than traditionally thought. Many preschoolers were able to handle repeating patterns and some even showed beginning mastery of growing patterns; a significant number of them solved computational estimation problems in ways that suggest a nascent conceptual understanding of the principles underlying computational estimation; many of them were already able to reason proportionally and to make probabilistic judgments in certain tasks and under certain conditions.

We found that these four early mathematical competencies showed unique associations with children's numerical abilities. These associations were observed both cross-sectionally and longitudinally. For example, we observed for the first time that the association between patterning and numerical skills changed from bidirectional to unidirectional (i.e., from patterning to numerical ability) in 4- to 6-year old children (Wijns et al., 2021a, b), but further work is needed to further pinpoint the direction of associations between these two abilities.

Furthermore, we also observed that these four early mathematical competencies were interrelated. For example, patterning in 4- to 5-year-olds turned out to be a unique predictor of proportional reasoning one and a half year later over and above various general cognitive and numerical abilities (Vanluydt et al., in press).

As was exemplarily shown for patterning, it is important to look not only at the ability side of young children's early core mathematical competencies, but to look at the dispositional side of these competencies too. For this competence, we found individual differences in 4-year-olds' spontaneous focusing on mathematical patterns (SFOP), as well as significant associations between their SFOP scores and their scores on the patterning and numerical ability measures, which might be explained via the mechanism of self-initiated practice (cf. Hannula & Lehtinen, 2005), in line with what has already been reported for other spontaneous mathematical focusing tendencies, such as spontaneous focusing on numerosity (SFON), spontaneous focusing on number symbols (SFONS), and spontaneous focusing on mathematical relations (SFOR) (Verschaffel et al., 2020).

An outstanding strand is the understanding of the cognitive origins of individual differences in the abovementioned four mathematical competencies. There is a large body of research that has examined individual differences in children's mathematical development (e.g., Dowker, 2005), but again, this work is largely restricted to the study of numerical abilities and arithmetic, both in children with high and low achievement in mathematics. On the one hand, this strand will be informative for the study of children who excel in their mathematical achievement. Research on excellence in mathematics almost exclusively focused on adolescents and adults (e.g., Lubinski & Benbow, 2006; Preckel et al., 2020) and hardly anything is known about the early seeds of this excellence in elementary school and earlier. It has been posited that numerical and arithmetic abilities, although useful, do not necessarily represent the quintessence of excellence in mathematical achievement (Krutetskii, 1976). As the abovementioned challenging domains are mathematically more complex than number and arithmetic, they might allow high achievers to show their mathematical potential. Our longitudinal data will allow us to investigate whether children who excel in mathematics in Grades 2 and 3 of elementary school also excel in the abovementioned mathematical competencies in earlier grades of elementary school and even preschool, and verify to which extent this excellence can be explained by domain-general cognitive capacities, such as spatial skills or working memory. On the other hand, this strand also has implications for the study of children with low mathematics achievement, a research area that has traditionally been focused on the study of numbers and arithmetic. Our longitudinal data will also allow us to investigate whether children with low achievement in mathematics are also at risk for developing difficulties in patterning, computational estimation, proportional reasoning, and probabilistic reasoning. Again, we will be able to identify to which extent these difficulties can be explained by domain-general cognitive capacities.

Finally, throughout the chapter we have pointed at several places how the findings of our longitudinal study may contribute to the development of educational standards, learning trajectories, and instructional tasks and techniques that give mathematical patterns, computational estimation, proportional reasoning, and probabilistic reasoning a more prominent place in early mathematics education. In doing so, these changes in the early mathematics education curriculum and practice will make early mathematics education more challenging and inclusive for all young children, and provide them a better preparation for the challenges of the mathematics curriculum of the upper elementary school. However, we are well aware that concrete educational recommendations should be based on findings coming from intervention studies that test the feasibility and effectiveness of these more challenging early mathematical curricula and designs in real educational settings.

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