# Chapter 23
# Taiwanese Teachers' Collection of Geometry Tasks for Classroom Teaching: A Cognitive Complexity Perspective

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## 23.1 Introduction

Many researchers have pointed out the crucial role of mathematical instructional tasks in student learning outcomes (Boston & Smith, 2009; Henningsen & Stein, 1997; Silver & Stein, 1996; Stein et al., 1996). Mathematical tasks can direct students' attention to particular aspects of mathematics and structure their ways of thinking about mathematics (Doyle, 1983, 1988). The work students do determines how they think about a curricular domain and understand the meaning of mathematics. The types of tasks may also influence instruction, subsequently leading to different opportunities for students to learn mathematics (Doyle, 1988; Stein et al., 2000).

Of particular research interest is the relationship between mathematical tasks and the levels of cognitive demand, as this dramatically influences student learning outcomes (Boston & Smith, 2009; Henningsen & Stein, 1997; Silver & Stein, 1996; Stein et al., 1996). Leikin (2014) further proposed a more comprehensive conception of mathematical tasks, namely *mathematical challenge*, which highlights the importance of students thinking of tasks as interesting, thus motivating them to engage with mathematically difficult tasks. One key to determining mathematical challenge is the cognitive complexity[1] that a task entails. During instruction, teachers

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[1] Cognitive complexity and cognitive demand share a similar construct that denotes task features entailed, which influence the kinds of cognitive processes students may need to perform to solve the task (Stein et al., 1996). Cognitive complexity particularly refers to cognitively demanding or cognitively complex tasks. It possesses features that appear to require students to engage in high-level cognitive processes such as making connections or mathematical reasoning (Magone et al., 1994).

 
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have to maintain or increase the cognitive complexity of tasks to challenge students to move to a higher level of thinking (Leikin, 2009; Stein & Lane, 1996).

This study focuses on Taiwan mathematics instruction, as Taiwanese students are consistently in the top group in cross-national assessments (e.g., Mullis et al., 2012; OECD, 2014). One of the main reasons for these students' out-performance could be the mathematical tasks that Taiwanese mathematics teachers collect for classroom teaching. Hsu and Silver (2014) examined the type of geometry tasks used by Taiwanese mathematics teachers. They reported several significant findings concerning the collection of tasks and the cognitive complexity those tasks entail. The type of geometry task examined by Hsu and Silver was geometric calculation with numbers (GCN), which refers to tasks that involve numerical calculations done based on geometric properties or formulas in a geometric diagram environment. GCN tasks often require cognitive complexity as problem-solving requires high-level thought and reasoning processes (Magone et al., 1994). Hsu and Silver (2014) reported that Taiwanese teachers used tasks not just from textbooks but from other sources as well, and GCN tasks from non-textbook sources tended to be more cognitively challenging than those found in textbooks. This finding implies that the opportunity to practice tasks from non-textbook sources may be one of the critical factors in the superb mathematics achievements of East Asian students. Hsu and Silver's study also anchored a study by Silver et al. (2009) that showed that tasks used by teachers for the assessment of mathematical understanding tended to have higher cognitive demand characteristics than tasks used to develop mathematical understanding.

The study reported here is a follow-up to Hsu and Silver (2014), with an attempt to further examining different Taiwanese teachers' collections of sources of instructional/curricular materials from a cognitive complexity perspective. In particular, we intended to learn if students' mathematics performance influences teachers' collection of tasks. The research question for the study was as follows:

> What is the cognitive complexity of geometry tasks collected by Taiwanese mathematics teachers, and does the cognitive complexity of geometry tasks differ between schools with different mathematics performance levels?

## 23.2 Analytical Framework

Hsu and Silver (2014) extended the construct of cognitive complexity and proposed an analytical framework that can examine the cognitive complexity of geometry tasks. As shown in Fig. 23.1, the analytical framework includes two dimensions—diagram complexity and problem-solving complexity—each of which describes the kind of cognitive activity involved in geometry problem-solving. Diagram complexity refers to the segments and lines comprising a geometric diagram, which can influence cognitive complexity in solving geometry problems. Problem-solving complexity identifies four kinds of cognitive activity involved in geometry problem-solving processes. The details of each category of the dimensions are as follows.

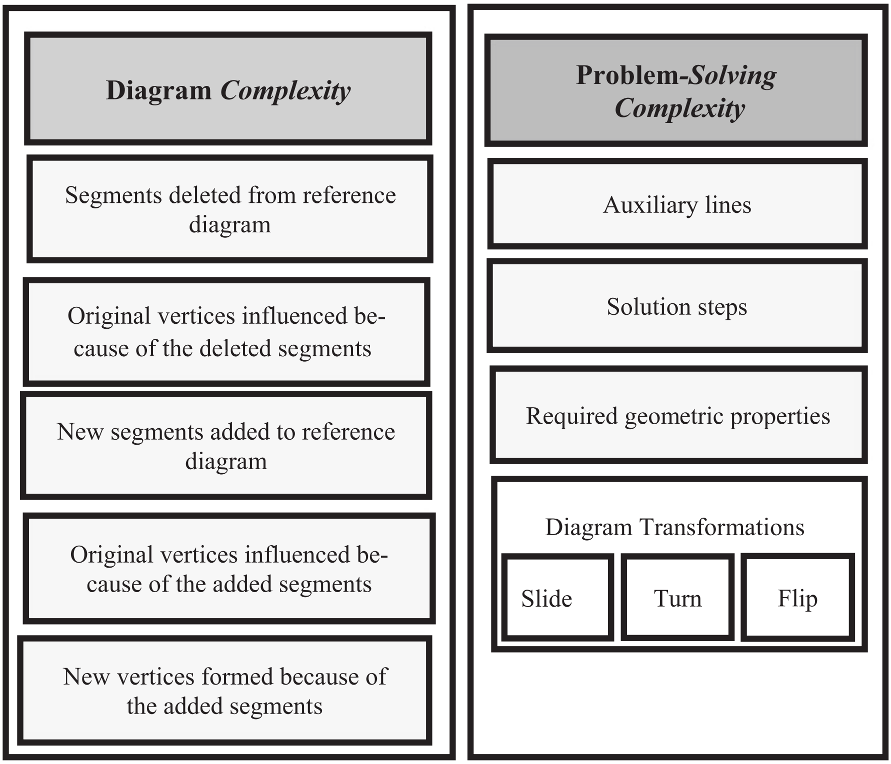



**Fig. 23.1** The cognitive-complexity framework (Hsu & Silver, 2014)

### *23.2.1 Diagram Complexity Dimension*

Hsu and Silver (2014) recognized the cognitive complexity that a geometric diagram might cause. The psychology literature confirms that the schemas used in problem-solving processes are strongly tied to diagrams, especially when dealing with high-level cognitive activities (Carlson et al., 2003; Greeno, 1978; Koedinger & Anderson, 1990; Larkin & Simon, 1987; Lovett & Anderson, 1994; Mousavi et al., 1995). Here, schemas refer to a "cluster of knowledge that contains information about the core concepts, the relations between concepts and knowledge about how and when to use these concepts" (Chinnappan, 1998, p. 202). To this end, geometry problem solving requires diagram parsing to identify familiar configurations with corresponding schemas in the diagram, which can be used to formulate a solution plan by reasoning forward and backward between the givens and the goals (Koedinger & Anderson, 1990).

Hsu and Silver proposed the construct of reference diagrams to analyze cognitive complexity concerning schema searching in geometry diagrams. They used it as the basis to analyze diagram complexity in a geometry task. They defined a *reference* *diagram* as a geometric diagram shown with a geometric property that is formally introduced in a textbook. A reference diagram provides a common point of contact through which many geometric concepts and properties are linked.

Hsu and Silver (2014) further explained the reasons for using a reference diagram as the basis for examining the cognitive complexity embedded in geometry diagrams. First, analyzing a geometry task diagram by comparing it to a reference diagram provides information regarding possible visual obstacles that students may encounter when identifying the reference diagram and its corresponding geometric properties in the given geometry task. Second, a reference diagram is an *external representation* (Laborde, 2005) presented in textbooks, thereby preventing coding inconsistencies that can arise when making inferences about the mental images of a diagram as processed internally by individuals.

Figure 23.2 shows a reference diagram of an isosceles triangle that is usually shown in textbooks. The reference diagram conveys not only the definition (e.g., that two of the three sides in the triangle are congruent) but also other related geometric properties (e.g., the sum of the interior angles of the triangle is 180°). A reference diagram also possesses visual features that can help draw attention to the salient geometric properties. For instance, one can easily recognize the congruence of the segments in an isosceles triangle. Figure 23.2 presents its reference diagram that has the lengths of the two congruent legs standing symmetrically on the two sides with the base side on the bottom parallel to the horizontal axis.

The categories of diagram complexity describe how a diagram given in a geometry task is altered compared to a reference diagram. Diagram complexity includes five categories used to describe the changes in terms of segments and vertices in a geometric diagram. Those five categories are the number of segments deleted from the reference diagram (category 1), the number of original vertices influenced by the deleted segments (category 2), the number of segments added to the reference diagram (category 3), the number of original vertices influenced by the added segments (category 4), and the number of new vertices created because of the added segments (category 5). One can see the details of analyzing diagram complexity for a geometry task along with the five categories in 2.3 in the session.

### *23.2.2 Problem-Solving Complexity Dimension*

The problem-solving complexity in the analytical framework includes four categories, each of which refers to the cognitive processing that appears to be essential in geometry problem-solving. The categories are auxiliary lines, solution steps, required geometric properties, and diagram transformations.

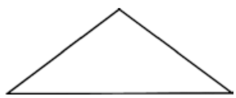


**Fig. 23.2** Reference diagram for an isosceles triangle

The auxiliary lines category concerns cognitive complexity in analyzing geometry tasks to determine if drawing auxiliary lines is needed and, if so, where to draw the lines such that new subconfigurations and new geometric properties can be created and used to generate a solution. Drawing the lines requires recalling prior knowledge and previous problem-solving experiences (Pólya, 1945; 2nd edition 1957). Drawing auxiliary lines on a diagram is often cognitively demanding. It forces one to anticipate creating subconfigurations associated with corresponding geometric properties that can be used to generate a solution plan.

The solution steps category involves the analysis of the reasoning steps required to obtain a solution. A reasoning step is defined as a problem-solving action taken based on a geometric property. Hsu and Silver (2014) counted the number of reasoning steps required to solve a geometry task, as the number can significantly influence cognitive demand. Researchers have indicated that generating a multi-step solution is cognitively demanding as it requires students to identify geometric properties for each reasoning step and chain the reasoning steps into a logic sequence (Ayres & Sweller, 1990; Heinze et al., 2005). Thus, the number can be an indicator used to describe cognitive complexity in reasoning a geometry task. It is also recognized that a geometry task can be solved in multiple ways, which might lead to different numbers of reasoning steps. Hsu and Silver (2014) stipulated that the solution used to classify the number of steps for a task should require the minimum number of reasoning steps to obtain the correct answer. They noted that each reasoning step in the solution should be supported by a geometric property that students have learned or will learn in the current instructional unit. Thus, classifying geometry tasks based on the minimum number of reasoning steps provides information regarding what prior geometric knowledge students have to access to successfully solve the tasks.

In addition to using the solution steps to describe the cognitive complexity of a geometry task, Hsu and Silver (2014) also considered the analysis of the number of geometric properties needed for a solution. They included this category because the number of geometric properties needed to solve a geometry task may not be the same as the number of solution steps. The reason is that different reasoning steps in a solution may require using the same geometric property. Analyzing the number of geometric properties required in a solution could provide richer information for describing the cognitive complexity of a geometry task. Geometric properties are those geometric statements or definitions that have been formally introduced in textbooks.

The diagram transformation category focuses on analyzing the cognitive complexity involved in diagram transformations (e.g., rotating). When solving a geometry task, one may need to perform a diagram transformation to map reference diagrams onto the task diagram. The performance enables recognizing and retrieving the geometric properties embedded in subconfigurations in the diagram. The mapping process requires mentally or physically transforming the reference diagrams to check if they resemble a diagram configuration in the geometry task. Operations on diagrams cause cognitive challenges for students as the orientation and position of a geometry task diagram may influence the identification of the corresponding reference diagrams (Fischbein & Nachlieli, 1998). Hsu and Silver (2014) included three types of diagram transformation in this category: slide (translation), turn (rotation), and flip (reflection).

The analytical framework proposed by Hsu and Silver allows one to systematically and scientifically analyze geometry tasks without constraints caused by the diversity of students' prior knowledge and learning experiences. This is because the basis for analysis is the geometric properties and diagrams presented in textbooks. As the properties and reference diagrams offered in textbooks are the materials used by students to learn, analysis based on the proposed framework is still closely tied to student cognition.

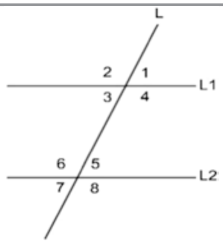
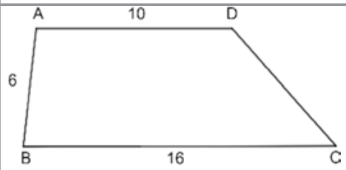
### 23.2.3 Analysis Examples of GCN Tasks

To unpack the cognitive complexity embedded in a GCN task, we provide two analysis examples based on the analytical framework (see Table 23.1).

Task A is considered as a low cognitive-complexity task, whereas Task B is a high cognitive-complexity task. Details of the analysis of Task B can be seen in the appendix as an external link to Hsu and Silver (2014). The elaboration of the cognitive complexity of the two tasks begins from the problem-solving dimension. The problem-solving processes influence the cognitive complexity related to decomposing and recomposing diagram configurations into subconfigurations in order to retrieve the geometric properties for a solution (Gal & Linchevski, 2010; Hsu & Silver, 2014).

**Analysis Based on the Categories in Problem-Solving Complexity Dimension**
Solving Task A does not require the cognitive work of drawing the auxiliary line as the given information is enough to generate a solution. However, Task B cannot be solved unless an auxiliary line is drawn. Figure 23.3 shows a strategy to draw an

**Table 23.1** Descriptions of two GCN tasks

| | Task A | Task B |
|---|---|---|
| The given diagram |  |  |
| The written task | Given that L1‖L2 and L intersects L1 and L2, and $m\angle 1 = 66°$. Find $m\angle 5$. | In a quadrilateral ABCD, given that AD‖BC and AD = 10, BC = 16, AB = 6, $m\angle DCB = 48°$. Find $m\angle BAD$. |

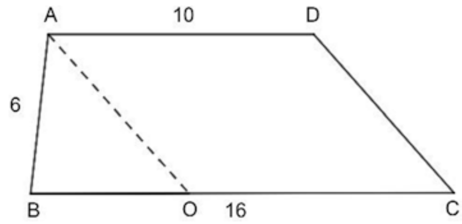



**Fig. 23.3** The drawing of auxiliary line AO

auxiliary line AO such that AO is parallel to DC. The drawing of the auxiliary line allows one to reason that ABCD is a parallelogram as well as OC = 10 and BO = 6.

Table 23.2 shows the minimum solution steps and the geometric properties required as supportive reasons for Task A and Task B. As can be seen, Task A can be solved in one reasoning step and with one geometry property. In contrast, Task B involves higher cognitive complexity as it requires five reasoning steps and five geometric properties to find the answer.

Concerning the analysis of diagram transformation, it has to identify reference diagrams corresponding to each geometric property required in the solution (see Table 23.3). Identifying individual reference diagrams forms the basis for analyzing what transformation actions are needed to map the reference diagrams onto the GCN task diagram. After checking the reference diagrams for the geometric properties required in the solution, diagram transformations are examined. For Task A, as its task diagram structure is the same as that of the reference diagram (e.g., a pair of parallel lines and a transversal), no diagram transformation action is needed. However, Task B necessitates diagram transformations to map the reference diagrams onto the GCN task diagram (see Table 23.4). As a result, five diagram transformation actions are required for Task B.

**Analysis Based on the Categories in Diagram Complexity Dimension**

For Task A, as its task diagram shares the same structure as the reference diagram, the analysis of diagram complexity is denoted as 0 because no changes in terms of the segments and vertices can be identified. For Task B, one of the reference diagrams shown in Table 23.3 is used as the basis for the analysis of its diagram complexity. The reference diagram for the corresponding angles property is determined as the basis for the analysis of diagram complexity because the geometric property is one of the main contents to be learned in the lessons. Figure 23.4 shows how the reference diagram for the corresponding angles property resembles part of the Task B diagram. The diagram shown on the left side in Fig. 23.4 is the reference diagram for the corresponding angles property. The diagram shown on the right side is how the reference diagram resembles the GCN task diagram.

The analysis of diagram complexity along with the five categories is used to describe the changes to the reference diagram so that it becomes the Task B diagram. As shown in Table 23.5, the analysis of diagram complexity for Task B based on the analytical framework shows ten changes.

Table 23.6 summarizes the analysis results for Task A and Task B based on the categories of problem-solving complexity and diagram complexity in the analytical

**Table 23.2** Solution steps and required geometric properties for Task A and Task B

| Steps | Calculating sentences | Supportive reasons | Calculating sentences | Supportive reasons |
|---|---|---|---|---|
| Step one | As L1 and L2 are parallel,<br>$m\angle 5 = m\angle 1 = 66°$ | The corresponding angles property | As AD∥OC,<br>AD = OC = 10,<br>AOCD is a parallelogram | If one pair of the opposite sides is parallel and congruent, the quadrilateral is a parallelogram. |
| Step two | | | $m\angle DCO = m\angle DAO = 48°$ | Opposite angles of a parallelogram are congruent |
| Step three | | | AO∥CD | The opposite sides of a parallelogram are parallel |
| Step four | | | $m\angle DCO = m\angle AOB = 48°$ | The corresponding angles property |
| Step five | | | BA = BO = 6<br>$m\angle AOB = m\angle BAO = 48°$<br>So that<br>$m\angle BAD = 48° + 48° = 96°$ | Properties of an isosceles triangle |

**Table 23.3** Reference diagrams corresponding to the geometric properties required

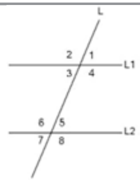
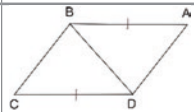
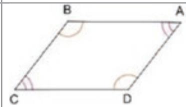
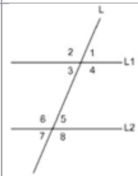
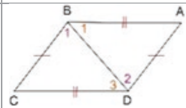
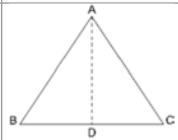
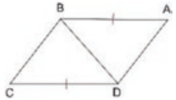
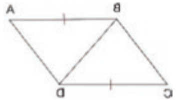
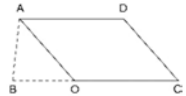
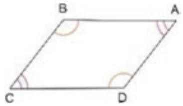
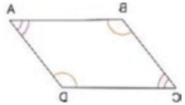
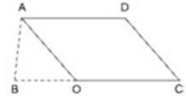
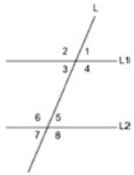
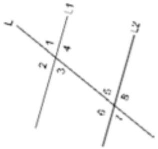
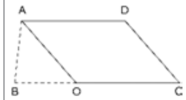
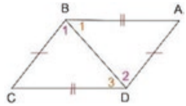
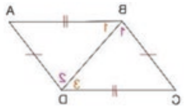
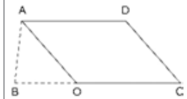
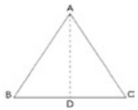
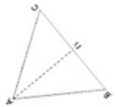
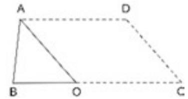
| | Task A | | Task B | |
|---|---|---|---|---|
| Steps | Geometric properties | Corresponding reference diagrams | Geometric properties | Corresponding reference diagrams |
| Step one | The corresponding angles property |  | If one pair of the opposite sides is parallel and congruent, the quadrilateral is a parallelogram |  |
| Step two | | | Opposite angles of a parallelogram are congruent |  |
| Step three | | | The corresponding angles property |  |
| Step four | | | Opposite sides of a parallelogram are congruent |  |
| Step five | | | If two sides of a triangle are congruent, then their corresponding angles are congruent |  |

**Table 23.4** Diagram transformations for Task B

| Solution steps | Transformation actions | Reference diagrams appearing in the textbooks | Reference diagrams after transformations | Subconfigurations in GCN task diagram |
|---|---|---|---|---|
| Step one | Flip |   |  |   |
| Step two | Flip |   |  |   |
| Step three | Turn |   |  |   |
| Step four | Flip |   |  |   |
| Step Five | Turn |   |  |   |

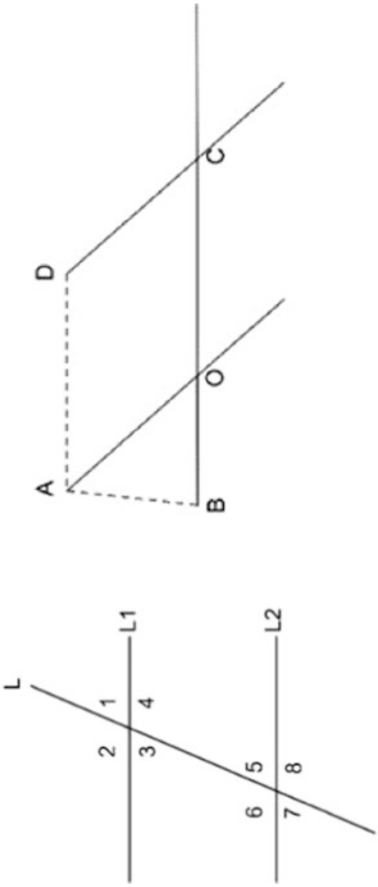



**Fig. 23.4** Reference diagram (left) and how it resembles the Task B diagram (right)

**Table 23.5** Analysis of diagram complexity for Task B

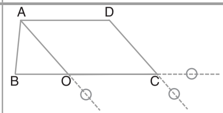
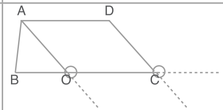
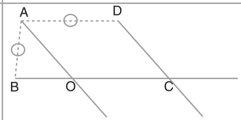
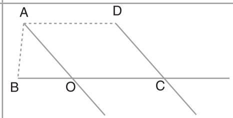
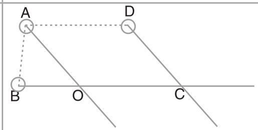
| Categories of diagram complexity | Changes in diagram | Coding results |
|---|---|---|
| Segments deleted from reference diagram |   | Three segments are deleted |
| Original vertices influenced because of the deleted segments |   | Two original vertices are influenced because of the deleted segments |
| New segments added to reference diagram |   | Two new segments are added to the reference diagram |
| Original vertices influenced because of the added segments |   | No original vertex is influenced because of the added segments |
| New vertices formed because of the added segments |   | Three new vertices are formed because of the added segments |
| Sum of the changes | | 10 |

**Table 23.6** Summary of the analysis of cognitive complexity for Task A and Task B

| Coding | | Task A | Task B |
|---|---|---|---|
| Problem-solving complexity | Auxiliary line | 0 | 1 |
| | Solution steps | 1 | 5 |
| | Required geometric properties | 1 | 5 |
| | Diagram transformation | 0 | 5 |
| Diagram complexity | | 0 | 10 |
| Cognitive complexity | | 2 | 26 |

framework. As can be seen, the coding for Task A is denoted as 2, whereas that for Task B is 26. The numbers allow one to understand how geometry tasks are made cognitively complex based on the requirements with respect to diagram complexity and problem-solving complexity. The bigger the coding number, the more cognitive complexity the GCN task entails.

## 23.3 Methodology

### *23.3.1 Selection of Teachers*

To select subjects for this study, we first searched for experienced Taiwanese mathematics teachers with more than 5 years of teaching experience who were willing to participate in the study. We then checked if those teachers taught eighth-grade students because a geometry topic designed for those students was the focus of the analysis. We then identified students' overall mathematics performance in the schools those teachers taught. In Taiwan, student mathematics performance varies between schools, which can be due to factors such as school reputation, the socio-economic status of the students' parents, and residential areas (e.g., remote areas) (Huang, 2017). In general, overall student mathematics performance in a school does not change much over the years. We decided to use overall school mathematics performance as an indicator for the students taught by the teachers who participated in the study. The first reason was that students in the same school have to be randomly grouped into classes. Thus, overall school mathematics performance can represent the students in classes due to the random assignment process.

The second reason was that overall school mathematics performance on high school entrance examinations could be obtained, which allowed a fair comparison among the classes the participating teachers taught. Taiwan's high school entrance examination is a nationwide examination that has to be taken by Taiwanese middle school students as they need an examination score to apply to high schools. The examination ranks students as level A, B, or C, where A represents a high-performance level, and C indicates a low-performance level. According to a report on the mathematics subject in the high school entrance examination (Comprehensive Assessment Program for Junior High School Students, 2020), 22% of students who took the examination were identified as level A, 50% were identified as level B, and about 28% were identified as Level C. We considered a school as having high mathematics performance if more than 50% of its students achieved level A, middle performance if more than 50% of its students achieved level B. Low performance if more than 50% of its students identified as level C. In the end, a total of six Taiwanese mathematics teachers participated in the study. Two of them represented high mathematics performance schools, two middle mathematics performance schools, and two low mathematics performance schools.

As shown in Table 23.7, the highest number of years of teaching was 27, and the lowest was 5. The mathematics teachers' majors were either in mathematics or mathematics-related areas (e.g., mathematics education). Teacher Jyu and Teacher Ing taught at high math performance schools; Teacher Sheng and Teacher Yao taught at middle math performance schools; Teacher May and Teacher Wen taught at the same school identified as a low math performance school.

### *23.3.2 Data Collection*

The teaching of a geometric topic—properties related to parallel lines—in eighth-grade textbooks in Taiwan was the data collection context. The instructional goals set up with the geometric topic included helping students become familiar with geometric properties related to parallel lines and the concept of geometric proofs. In this regard, a high percentage of GCN tasks were included in the textbooks. All mathematical tasks situated in sources of instructional/curricular materials collected by the six teachers when they taught the geometric topic were the data for the analysis. The six teachers' teaching of the geometric topic was videotaped and analyzed. As a result, four sources of instructional materials were identified, including textbooks, supplementary materials, tests, and tasks created by the teacher during classroom teaching. The textbooks included both student textbooks and student workbooks. Different textbooks published in Taiwan all have to be evaluated based on the national mathematics curriculum, but the mathematical tasks included in the textbooks can be slightly different. The student textbook contained both instructional blocks comprised of diverse mathematical activities (e.g., diagram construction and proving) and exercise blocks. The student workbook included additional exercises for students to practice.

Teachers may feel that textbooks and workbooks are not enough for their students and decide to include supplementary materials for classroom teaching. Supplementary materials are either designed by mathematics teachers themselves or are published by textbook companies. No matter whether designed by mathematics teachers or by textbook publishers, supplementary materials usually have several characteristics. First, they are often arranged in the same sequence as textbooks. Second, they often include a large number of tasks. Third, they usually summarize the main mathematical content (e.g., definitions and geometry properties). Four out of the six teachers in our sample used supplementary materials for their teaching. However, the underlying reasons for the use of supplementary materials were different. Teacher Jyu indicated that she thinks textbooks are too easy to prepare students to obtain high scores on examinations. Thus, she decided to include supplementary materials in her teaching. Teacher Wen pointed out that she uses supplementary materials as they can provide students with extra opportunities to practice tasks and learn mathematical concepts. In particular, she indicated that she often chooses supplementary materials that include tasks with similar cognitive complexity to those included in textbooks.

**Table 23.7** Teachers' background and information on instructional/curricular materials

| Teacher Pseudonym | Years of teaching | School math performance | Sources of instructional/curricular materials used | Number of pages and tasks | Number of lessons |
|---|---|---|---|---|---|
| Jyu | 27 | High | Textbooks | 21 pages with 77 tasks | 9 |
| | | | Supplementary materials | 7 pages with 111 tasks | |
| | | | Tests | 7 tests with 173 tasks | |
| | | | Problems created by the teacher | 10 tasks | |
| Ing | 6 | High | Textbooks | 23 pages with 59 tasks | 11 |
| | | | Supplementary materials | 7 pages with 91 tasks | |
| | | | Tests | 3 tests with 62 tasks | |
| | | | Problems created by the teacher | 2 tasks | |
| Sheng | 9 | Middle | Textbooks | 22 pages with 63 tasks | 7 |
| | | | Supplementary materials | Not used | |
| | | | Tests | 2 tests with 38 tasks | |
| | | | Problems created by the teacher | 9 tasks | |
| Yao | 5 | Middle | Textbooks | 21 pages with 77 tasks | 8 |
| | | | Supplementary materials | Not used | |
| | | | Tests | 2 tests with 31 tasks | |
| | | | Problems created by the teacher | 2 tasks | |
| Teacher May | 17 | Low | Textbooks | 22 pages with 63 tasks | 7 |
| | | | Supplementary materials | 9 pages with 52 tasks | |
| | | | Tests | 1 test with 18 tasks | |
| | | | Problems created by the teacher | 2 tasks | |
| Teacher Wen | 14 | Low | Textbooks | 22 pages with 63 tasks | 7 |
| | | | Supplementary materials | 9 pages with 52 tasks | |
| | | | Tests | 1 test with 18 tasks | |
| | | | Problems created by the teacher | Not used | |

Another primary source of curricular/instructional material collected during classroom teaching was the tests often used for formative or summative purposes. Teachers may use tests to evaluate students' learning outcomes or assign test sheets as homework. All six teachers used tests in their teaching. In addition, they all created tasks as they thought those tasks would benefit student learning during classroom teaching. Table 23.7 shows the number of pages and tasks for each source of curricular/instructional materials collected by the six teachers. Table 23.7 also presents the number of lessons each teacher spent on teaching the geometric topic. In Taiwan, a lesson at the middle school level lasts for 45 minutes.

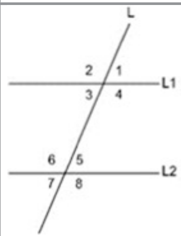
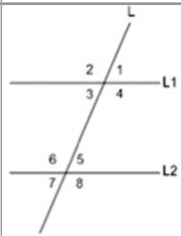
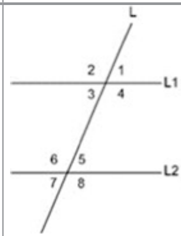
Interviews with the teachers were also implemented to better understand the reasons underlying their collection of tasks situated in sources of instructional/curricular materials.

### *23.3.3 Data Analysis*

The data analysis started by identifying the types of tasks from the different kinds of instructional/curricular material collected from the six Taiwanese mathematics teachers. Different task types were identified, including exploration activities, geometric proof (GP) tasks, GCN tasks, geometric algebra (GA) tasks, and diagram construction tasks. Exploration refers to those activities that aim to help students understand geometric concepts through manipulation work. Construction refers to the work of drawing a geometric diagram using a compass and straightedge. Regarding the similarities and differences among GCN, GA, and GP, Table 23.8 shows examples of the three kinds of tasks. As can be seen, the three tasks use the same diagrams and given information to describe the diagram. The only difference among the three tasks is that GCN includes numerical information that can be used to reason unknown measures. GA requires applying algebraic skills in order to obtain a solution. GP involves finding reasons based on geometric properties that can be used to prove that a statement is always true. We counted the number of each type of task situated in the sources of instructional/curricular materials, where a task was defined as a problem asking for an answer (Charalambous et al., 2010).

After identifying a GCN task from sources of curricular/instructional materials, the problem-solving complexity dimension with its four analysis categories (auxiliary lines, solution steps, required geometric properties, and rigid transformation) was performed. The number of minimum solution steps was determined, which consequently became the basis for checking if drawing auxiliary lines was necessary. We also counted the number of geometric properties used to support the reasoning steps in the identified solution. In particular, we checked if those geometric properties had been formally introduced in the textbooks or had been learned previously by the students. Any geometric properties that 8th grade students have not learned were excluded, even if they could be used to generate a solution to a GCN task. The task analysis then focused on identifying diagram transformations with sliding, turning, and flipping actions. The diagram transformation analysis required

**Table 23.8** Examples of GCN, GP, and GA tasks

| | GCN | GP | GA |
|---|---|---|---|
| The given diagram |  |  |  |
| The written task | *Given*: that L1‖L2 and L intersects L1 and L2, | | |
| | $m\angle 1 = 56°$ | $m\angle 1 = m\angle 5$ | $m\angle 1 = (3X + 5)°$<br>$m\angle 5 = (23 - 6X)°$ |
| | *Find*:<br>$m\angle 5$. | *Prove*:<br>$m\angle 4 + m\angle 5 = 180°$. | *Find*:<br>value of $X$. |

Note: *GA* geometric algebra, *GP* geometric proof the presence of given diagrams in the GCN tasks. Tasks in which a diagram was not provided were excluded. The minimum number of transformation actions necessary to map the reference diagrams representing the identified solution's geometric properties onto the given GCN task diagram was determined.

The next step was to analyze the diagram complexity of the GCN task. We determined the reference diagram as the basis for the examination of diagram complexity in a GCN task. As a solution often is generated by more than one geometric property, the reference diagram was decided based on two criteria. The first was that the reference diagram identified had to correspond to the geometric properties needed to obtain the minimum number of solution steps in a GCN task. The second was that the identified reference diagram represented one of the to-be-learned geometric properties in the current teaching topic. Once a reference diagram was determined for a GCN task, diagram complexity and its five coding categories were analyzed. Finally, we counted the number of changes needed to transform a reference diagram into a GCN task diagram and used the number to describe the GCN task's diagram complexity.

The author and a coder were responsible for the data analysis. To ensure the consistency of the coding results, tasks that were difficult to classify were selected for checking their reliability. Two coders analyzed those complex tasks individually and then discussed the coding results together. If an inconsistency occurred, both coders discussed the inconsistency until an agreement was reached. Regarding the interviews, we used a back-and-forth analysis process. Once we found something interesting from the data analysis, we showed those findings to the teachers to learn the reasons. It was also possible for the teachers' responses from the interviews to inform how we analyzed the collected data.

## 23.4 Results

### *23.4.1 Collections of Types of Mathematical Instructional Tasks*

Among the multiple sources of instructional/curricular materials collected from the six teachers, we first identified the types of tasks and activities. Table 23.9 shows the types of tasks and the number that the six teachers collected. As can be seen, the types of tasks collected by the teachers included exploration activities, diagram construction activities, GCN tasks, GP tasks, and GA tasks. It is worth noting that teachers who taught in high mathematics performance schools collected more tasks than those teaching in middle and low mathematics performance schools (Teacher Jyu: 371 tasks; Teacher Ing: 214 tasks; Teacher Seng: 110 tasks; Teacher Yao: 110 tasks; Teacher May: 135 tasks; Teacher Wen: 140 tasks). It is recognized that tasks may entail different cognitive complexity and may be used in different ways (e.g., worked example vs. exercise) and for different instructional purposes (e.g.,

**Table 23.9** Types of mathematical tasks collected by the teachers

| | | | Mathematical tasks | | | | | |
|---|---|---|---|---|---|---|---|---|
| Teacher | Math performance | Lessons | Exploration | Construction | GCN | GP | GA | Total |
| Jyu | High | 9 | 0 (0%)[a] | 10 (3%) | 229 (62%) | 72 (19%) | 60 (16%) | 371 (100%) |
| Ing | High | 11 | 0 (0%) | 9 (4%) | 132 (62%) | 44 (21%) | 29 (14%) | 214 (100%) |
| Sheng | Middle | 7 | 0 (0%) | 7 (6%) | 84 (76%) | 10 (9%) | 9 (8%) | 110 (100%) |
| Yao | Middle | 8 | 0 (0%) | 2 (2%) | 76 (69%) | 22 (20%) | 10 (9%) | 110 (100%) |
| May | Low | 7 | 0 (0%) | 7 (5%) | 94 (70%) | 15 (11%) | 19 (14%) | 135 (100%) |
| Wen | Low | 7 | 7 (5%) | 7 (5%) | 92 (66%) | 15 (11%) | 19 (14%) | 140 (100%) |
| Total | | | 7 | 42 | 707 | 178 | 146 | 1080 |

[a](number of tasks)/(total number of tasks collected by the teacher) × 100%

understanding the mathematical concept vs. applying the concept to a more complex task context). We found that high mathematics performance school teachers were inclined to collect more tasks for their students. Teacher Jyu said the following:

> I intended to include a high amount of tasks for my students as they can learn mathematics from a variety of tasks….By practicing the tasks, they can correct their misconceptions and understand what they did not understand previously….This is a very useful strategy to prepare students for the high school entrance examination." (Transcript of interview data, 20200302)

Among the types of tasks, GCN tasks occupied the highest percentage of tasks collected by the six teachers (Teacher Jyu: 62%; Teacher Ing: 62%; Teacher Sheng: 76%; Teacher Yao: 69%; Teacher May: 70%; Teacher Wen: 66%). The result made it reasonable to compare the cognitive complexity of the tasks collected for classroom teaching among the teachers. In addition, for the high and middle mathematics performance schools, the second-highest percentage of tasks collected by the teachers was GP tasks (Teacher Jyu: 19%; Teacher Ing: 21%; Teacher Sheng: 9%; Teacher: Yao 20%). Interviews with those teachers showed that they think their students can learn proofs even though textbooks do not include tasks that require students to construct geometric proofs themselves. For the low mathematics performance schools, the second-highest percentage of tasks collected for classroom teaching was GA (14% for both Teacher May and Teacher Wen). High-performance schools also used many GA tasks for teaching (Teacher Jyu: 16%; Teacher Ing: 14%). Only Teacher Wen included exploration tasks in her classroom teaching (5%).

### *23.4.2 Cognitive Complexity of the Tasks Collected by Taiwanese Teachers*

We further examined the cognitive complexity of the GCN tasks collected by the six teachers. Table 23.10 shows the number of GCN tasks analyzed and the average diagram complexity, problem-solving complexity, and cognitive complexity for each teacher. Only GCN tasks accompanied by diagrams collected by the teachers were surveyed. As shown in Table 23.10, the tasks collected by the six teachers tended to entail both diagram complexity and problem-solving complexity, no matter if they taught at high mathematics performance or low mathematics performance schools. The average diagram complexity for the GCN tasks collected by all six teachers was 6.52, indicating that a GCN task was made about seven changes on average. The average problem-solving complexity was 5.39, implying that the GCN tasks were inclined to require multiple reasoning steps, multiple geometric properties for a solution, and the performance of diagram transformation. It is also likely that those GCN tasks asked students to draw auxiliary lines to obtain enough geometric properties to generate a solution. The average cognitive complexity for the GCN tasks collected by all six teachers was 11.90.

**Table 23.10** Cognitive complexity of the GCN tasks collected by the teachers

| Teacher | Math performance | Number of tasks | Diagram complexity | Problem-solving complexity | Cognitive complexity | Average |
|---|---|---|---|---|---|---|
| Jyu | High | 216[a] | 6.95 | 5.44 | 12.40 | 13.23 |
| Ing | High | 129 | 8.87 | 5.76 | 14.63 | |
| Sheng | Middle | 80 | 5.65 | 5.61 | 11.26 | 11.08 |
| Yao | Middle | 74 | 6.54 | 4.35 | 10.89 | |
| May | Low | 94 | 5.63 | 4.88 | 10.50 | 10.44 |
| Wen | Low | 92 | 5.53 | 4.85 | 10.37 | |
| Average | | 114.17 | 6.52 | 5.39 | 11.90 | 11.90 |

[a]The number of GCN tasks refers to those accompanying a diagram, which may be inconsistent with the number of GCN tasks reported in Table 23.9

Of interest is the relationship between school mathematics performance and the cognitive complexity of GCN tasks. Table 23.10 shows that the better the mathematics performance of a school, the higher the cognitive complexity of the tasks that the teachers tended to collect for their students. The average cognitive complexity for Teacher Jyu and Teacher Ing, who taught high mathematics performance students, was the highest (13.23). The average cognitive complexity for Teacher May and Teacher Wen, who taught lower mathematics performance students, was the lowest (10.44). The average cognitive complexity for Teacher Sheng and Teacher Yao was in between, at 11.08. This finding shows that Taiwanese mathematics teachers consider cognitive complexity when collecting tasks from sources of instructional/curricular materials for classroom teaching.

### *23.4.3 Cognitive Complexity of Tasks Situated in Sources of Curricular/Instructional Materials*

Four sources of instructional/curricular materials were identified from the six Taiwanese teachers, including textbooks, supplementary materials, tests, and tasks created by the teachers during classroom teaching. Table 23.11 shows the analysis of the cognitive complexity specific to each source of instructional/curricular materials collected by the six teachers. As can be seen, textbooks collected by the teachers possessed lower cognitive complexity than non-textbook sources. The average cognitive complexity of tasks situated in textbooks was 9.15. For the six teachers, the cognitive complexity of the tasks situated in the textbooks they used was similar. Cognitive complexity was 9.21 for Teacher Jyu, 9.82 for Teacher Ing, 9.12 for Teacher Sheng, 9.74 for Teacher Yao, and 9.12 for Teacher May and Teacher Wen.

The average cognitive complexity of the tasks situated in supplementary materials was slightly higher than that in textbooks, which was 10.7. Four out of the six teachers used supplementary materials in their teaching. An analysis of the supplementary materials showed that the cognitive complexity of the tasks collected by Teacher Ing (14.68) was much higher than those managed by Teacher Jyu (9.68), Teacher May (11.65), and Teacher Wen (11.65). Of interest is that the cognitive complexity of the tasks situated in supplementary materials used by Teacher Jyu, who taught high mathematics performance students, was lower than those used by Teacher May and Teacher Wen, who taught low mathematics performance students.

The interviews with Teacher Jyu and Teacher Wen revealed the underlying reason. Teacher Jyu said

> ...The supplementary materials used in my classes were designed by my colleagues and me….I use it [supplementary materials] for my teaching…but not the textbooks…because we have our own ideas on selecting tasks and sequencing them for our students….We use the materials to develop students mathematics concepts. (Transcript of interview data, 20190810)

**Table 23.11** Cognitive complexity of tasks from sources of instructional/curricular materials collected by the teachers

| Teacher Name | Math performance | Cognitive complexity | Sources of instructional/curricular materials | | | | Average |
|---|---|---|---|---|---|---|---|
| | | | Textbooks | Supplementary materials | Tests | Tasks created by teacher | |
| Jyu | High | Number of tasks | 57 | 62 | 94 | 3 | 216 |
| | | Diagram complexity | 5.51 | 5.23 | 8.85 | 10.67 | 6.95 |
| | | Problem-solving complexity | 3.7 | 4.45 | 7.04 | 9.00 | 5.44 |
| | | Cognitive complexity | 9.21 | 9.68 | 15.89 | 19.67 | 12.40 |
| Ing | High | Number of tasks | 35 | 44 | 49 | 1 | 129 |
| | | Diagram complexity | 6.13 | 9.32 | 10.04 | 27.00 | 8.87 |
| | | Problem-solving complexity | 3.69 | 5.36 | 7.41 | 15.00 | 5.76 |
| | | Cognitive complexity | 9.82 | 14.68 | 17.45 | 42.00 | 14.63 |
| Sheng | Middle | Number of tasks | 49 | 0 | 28 | 3 | 80 |
| | | Diagram complexity | 4.31 | 0 | 7.43 | 11.00 | 5.65 |
| | | Problem-solving complexity | 4.82 | 0 | 7.00 | 5.50 | 5.61 |
| | | Cognitive complexity | 9.12 | 0 | 14.43 | 16.50 | 11.26 |
| Yao | Middle | Number of tasks | 57 | 0 | 17 | 0 | 74 |
| | | Diagram complexity | 5.82 | 0 | 8.94 | 0 | 6.54 |
| | | Problem-solving complexity | 3.91 | 0 | 5.82 | 0 | 4.35 |
| | | Cognitive complexity | 9.74 | 0 | 14.76 | 0 | 10.89 |
| May | Low | Number of tasks | 49 | 33 | 10 | 2 | 94 |
| | | Diagram complexity | 4.31 | 6.89 | 7.03 | 10.00 | 5.63 |
| | | Problem-solving complexity | 4.82 | 4.76 | 5.26 | 6.50 | 4.88 |
| | | Cognitive complexity | 9.12 | 11.65 | 12.29 | 16.50 | 10.50 |
| Wen | Low | Number of tasks | 49 | 33 | 10 | 0 | 92 |
| | | Diagram complexity | 4.31 | 6.89 | 7.03 | 0 | 5.53 |
| | | Problem-solving complexity | 4.82 | 4.76 | 5.26 | 0 | 4.85 |
| | | Cognitive complexity | 9.12 | 11.65 | 12.29 | 0 | 10.37 |
| Average | | Number of tasks | 296 | 172 | 208 | 9 | 685 |
| | | Diagram complexity | 4.65 | 6.09 | 8.31 | 10.63 | 6.52 |
| | | Problem-solving complexity | 4.51 | 4.61 | 6.78 | 7.06 | 5.39 |
| | | Cognitive complexity | 9.15 | 10.70 | 15.10 | 17.69 | 11.90 |

Teacher Jyu indicated that she and her school colleagues write supplementary materials themselves and use them for classroom teaching. They use supplementary materials to scaffold students in building up new mathematical concepts. For the use of supplementary materials, Teacher Wen said

> ….We often use textbooks to teach our students as our students' mathematics is not very good….However, sometimes we select more challenging tasks from supplementary materials and discuss the tasks with our students….Even our students do not perform mathematics very well, they can learn from practicing those tasks from the supplementary materials (Transcript of interview data, 20190810)

Teacher Wen expressed a different way of using supplementary materials. She thinks textbooks are the appropriate instructional materials that fit her students' mathematical competence. Thus, Teacher Wen often teaches students mainly based on textbooks. Concerning the supplementary materials, she thinks they can provide her students with more opportunities to practice mathematics. In this regard, she collects tasks from supplementary materials to challenge her students. Different ways of using supplementary materials also influence the design of tasks concerning cognitive complexity. If the materials are used to help students build up mathematical concepts, the tasks included in the materials cannot be too cognitively demanding. If the materials are used to create more opportunities to practice mathematics, the tasks' cognitive complexity will increase.

The cognitive complexity of tests and tasks created by the teachers was much higher than those in textbooks and supplementary materials (cognitive complexity in tests: 15.10; cognitive complexity in tasks created by teachers: 17.69). The data shows that teachers intended to collect more cognitively complex tasks for formative and summative assessment purposes. They were also inclined to use very cognitively demanding tasks created by themselves during classroom teaching. The cognitive complexity of the tasks situated in the tests for Teacher Jyu was 15.89, for Teacher Ing was 17.45, for Teacher Sheng was 14.43, for Teacher Yao was 14.76, and for both Teacher May and Teacher Wen was 12.29. This finding reveals that the better the student's mathematics performance, the higher the cognitive complexity of the tasks in tests the teachers collected. Regarding the cognitive complexity of tasks created by the teachers, it was also higher than that of tasks in textbooks and supplementary materials (19.67 for Teacher Jyu, 42 for Teacher Ing, 16.5 for Teacher Sheng, and 16.5 for Teacher May). This finding suggests that the teachers tended to create more cognitively complex tasks during their classroom teaching.

## 23.5 Discussion

As Taiwanese students consistently perform at the top in cross-national mathematics assessments, this study investigated how Taiwanese mathematics teachers collect mathematical instructional tasks for their students. In particular, we examined if the mathematics performance of schools influences teachers in collecting tasks for their students. Based on the cognitive-complexity framework developed by Hsu and

Silver (2014), we analyzed six Taiwanese mathematics teachers who represented schools with different levels of mathematics performance.

The empirical analysis revealed that Taiwanese mathematics teachers tended to collect geometry tasks that entailed diagram complexity and problem-solving complexity, no matter the level of mathematics performance at the school where they taught. The diagrams accompanying geometry tasks are made complex, so they may not look like the reference diagrams accompanying geometric properties. The complex diagrams may consequently cause visual obstacles in identifying geometric properties that can be used to generate a solution to a geometry task. The geometry tasks collected by Taiwanese mathematics teachers also tended to require multiple reasoning steps and multiple geometric properties for a solution. Such tasks may also require the cognitive work of performing diagram transformations to identify the geometric properties embedded in the task diagrams successfully. The cognitive work required of solving various cognitive-complexity and non-routine tasks may subsequently equip Taiwanese students with abilities to attack high-level problems found in cross-national mathematics assessments (e.g., PISA and TIMSS).

The analysis also showed that the mathematics performance in the schools where the teachers taught did influence their collection of geometry tasks. The better the mathematics performance of the school, the higher the cognitive complexity of the geometry tasks the teacher collected. In addition, the cognitive complexity of tasks collected from non-textbook sources was higher than those from textbooks. Tasks situated in tests and those created by the teachers possessed the most increased cognitive complexity compared to textbooks and supplementary materials. Hsu and Silver (2014) reported a case study of a Taiwanese mathematics teacher. They indicated a tendency to include multiple sources of instructional/curricular materials with high cognitive-complexity tasks for classroom teaching. This study further confirmed this tendency by examining six Taiwanese mathematics teachers who taught students with different levels of mathematics performance.

Although the tasks situated in multiple instructional/curricular materials entail cognitive complexity, Taiwanese teachers consider students' mathematics performance when collecting tasks for classroom teaching. This finding implies a cultural script (Stigler & Hiebert, 1998) for teaching in East Asian countries, as teachers tend to increase the cognitive complexity as much as they can through the collection of tasks. Meanwhile, they also have to consider students' mathematics competence when collecting the tasks. The finding also brings several follow-up research questions. For example, researchers have indicated that challenging students by maintaining or increasing the cognitive complexity of tasks is vital for high-quality instruction (Leikin, 2009; Stein & Lane, 1996). In this regard, it is important to know how Taiwanese teachers manage to teach with those cognitive complexity tasks, especially when teaching in Taiwan is often described as teacher-centered (Lin & Tsao, 1999). Researchers from other countries may also expect to know the keys to determining the high quality of student learning outcomes. Can it be the case that collecting the cognitive complexity of tasks for classroom teaching ensures the high quality of student learning outcomes? Those questions require further investigations.

## References


Ayres, P., & Sweller, J. (1990). Locus of difficulty in multistage mathematics problems. *The American Journal of Psychology, 103*(2), 167–193.

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education, 40*(2), 119–156.

Carlson, R., Chandler, P., & Sweller, J. (2003). Learning and understanding science instructional material. *Journal for Educational Psychology, 95*(3), 629–640.

Charalambous, C. Y., Delaney, S., Hsu, H.-Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *International Journal for Mathematical Thinking and Learning, 12*(2), 117–151.

Chinnappan, M. (1998). Schemas and mental models in geometry problem solving. *Educational Studies in Mathematics, 36*(3), 201–217.

Comprehensive Assessment Program for Junior High School Students (2020). 2020 statistical results for the high school entrance examination. Retrieved from https://cap.rcpet.edu.tw/examination.html.

Doyle, W. (1983). Academic work. *Review of Educational Research, 53*(2), 159–199.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167–180.

Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning. *International Journal of Science Education, 20*(10), 1193–1211.

Gal, H., & Linchevski, L. (2010). To see or not to see: Analyzing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics, 74*, 163–183. https://doi.org/10.1007/s10649-010-9232-y

Greeno, J. G. (1978). A study of problem solving. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 1). Erlbaum.

Heinze, A., Reiss, K., & Rudolph, F. (2005). Mathematics achievement and interest in mathematics from a differential perspective. *ZDM, 37*(3), 212–220.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524–549.

Hsu, H.-Y., & Silver, E. A. (2014). Cognitive complexity of mathematics instructional tasks in a Taiwanese classroom: An examination of task sources. *Journal for Research in Mathematics Education, 45*(4), 460–496.

Huang, M.-H. (2017). Excellence without equity in student mathematics performance: The case of Taiwan from an international perspective. *Comparative Education Review, 61*(2), 391–412. https://doi.org/10.1086/691091

Koedinger, K. R., & Anderson, J. R. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science, 14*, 511–550.

Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose, & P. Valero (Eds.), *Meaning in Mathematics Education* (Vol. 37, pp. 157–179). New York: Springer.

Larkin, J. H., & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11*, 65–99.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Sense Publishers.

Leikin, R. (2014). Challenging mathematics with multiple solution tasks and mathematical investigations in geometry. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 59–80). Springer International Publishing.

Lin, F.-L., & Tsao, L.-C. (1999). Exam math re-examined. In C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Rethinking mathematics curriculum* (pp. 228–239). Falmer Press.

Lovett, M. C., & Anderson, J. R. (1994). Effects of solving related proofs on memory and transfer in geometry problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 20*(2), 366–378.

Magone, M. E., Cai, J., Silver, E. A., & Wang, N. (1994). Validating the cognitive complexity and content quality of a mathematics performance assessment. *International Journal of Educational Research, 21*(3), 317–340. https://doi.org/10.1016/S0883-0355(06)80022-4

Mousavi, S. Y., Low, R., & Sweller, J. (1995). Reducing cognitive load by mixing auditory and visual presentation modes. *Journal for Educational Psychology, 87*(2), 319–334.

Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 international results in mathematics*. TIMSS & PIRLS International Study Center.

OECD. (2014). *PISA 2012 results: What students know and can do–Student performance in mathematics, reading and science* (Vol. 1). OECD Publishing.

Pólya, G. (1945; 2nd edition 1957). *How to solve it*. Princeton University Press.

Silver, E. A., & Stein, M. K. (1996). The quasar project: The “revolution of the possible” in mathematics instructional reform in urban middle schools. *Urban Education, 30*(4), 476–521. https://doi.org/10.1177/0042085996030004006

Silver, E. A., Mesa, V., Morris, K. A., Star, J. R., & Benken, B. M. (2009). Teaching mathematics for understanding: An analysis of lessons submitted by teachers seeking NBPTs certification. *American Educational Research Journal, 46*(2), 501–531.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50–80.

Stein, M. K., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455–488.

Stein, M. K., Smith, M. S., Henningsen, M., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. Teachers College Press.

Stigler, J. W., & Hiebert, J. (1998). Teaching is a cultural activity. *American Educator, 22*(4), 4–11.