Chapter 15
Visualization: A Pathway to Mathematical Challenging Tasks

Isabel Vale and Ana Barbosa

15.1 Introduction

Mathematics learning is strongly dependent on the teacher and the tasks proposed to students (e.g. Doyle, 1988; Stein & Smith, 1998; Sullivan et al., 2013). Thus, the teacher must develop students' mathematical understanding, creating situations to ensure that they have the opportunity to engage and be challenged in high-level thinking, through the tasks proposed. The teacher's choices will determine the quality of students' learning (e.g. Chapman, 2015; Stein & Smith, 1998). This implies the use of tasks that meet different ways of thinking displayed by the students, confronting them with multiple-solution tasks, that challenge them to see outside of the box, motivating them to learn and to work with each other. We are interested in visualization because it plays an important cognitive role in the teaching and learning of mathematics, as an aid to thinking, as a means of communicating mathematical ideas and as a useful tool in problem-solving (Arcavi, 2003). So, pre-service and in-service teacher training should promote an insight into the nature of mathematics and its teaching, meaning that teachers need to have different teaching and learning experiences similar to the ones they are expected to use with their own students (Ponte & Chapman, 2008; Vale & Barbosa, 2020).

Thus, after a theoretical discussion about challenging tasks with multiple solutions (e.g. Leikin, 2016; Stein & Smith, 1998) and visualization (e.g. Arcavi, 2003; Duval, 1999; Presmeg, 2014, 2020), we emphasize the use of visual processes in teachers' practices and their potential in the teaching and learning of mathematics. This discussion will be illustrated with examples based on studies carried out with pre-service teachers of elementary education (6–12 years). This perspective emerges from the work we have been developing in teacher training through which we have

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come to conclude that visualization is not valued in school practices, neither as a problem-solving strategy nor as a way to support the understanding of mathematical concepts.

15.2 The Mathematics Classroom

15.2.1 The Teacher and the Tasks

All students should have the opportunity to engage in meaningful mathematical activity, and it's the teacher's role to unlock their potential through the choice of adequate tasks and teaching strategies. Although tasks have the power to trigger mathematical activity, they may not be sufficient to implicate mathematical challenges. Teachers must establish a classroom environment that guarantees students' engagement in embracing mathematical challenges. This implies the use of strategies that contemplate the heterogeneity of a classroom, and stimulate students to think and interact with each other, leading to rich discussions.

An effective teaching approach implies the orchestration of productive discussions, giving students opportunities to communicate, reason, be creative, think critically, solve problems, make decisions and understand mathematical ideas (e.g. NCTM, 2014; Vale & Barbosa, 2020). This context requires an exploratory approach, anchored on inquiry-based learning (Engel et al., 2013), allowing students to learn mathematics by understanding, criticizing, comparing and being encouraged to use different approaches to solve non-routine tasks, discussing the multiple solutions and processes used. This approach is demanding for teachers and is often the reason for continuing to perpetuate the common classroom practice that some refer to as the Triple X teaching (exposition, examples, exercises) (Evans & Swan, 2014). The option for more traditional approaches is related to teachers' beliefs regarding mathematics teaching and learning, which influence the type of tasks and strategies used (Sullivan et al., 2013). Many of the fragilities that students have in learning mathematics are due to those options, but also to the gaps in the teachers' mathematical knowledge and in the use of innovative teaching strategies. Hence, mathematics learning is strongly dependent upon the teacher and the tasks that are proposed (e.g. Doyle, 1988; Smith & Stein, 2013; Stein & Smith, 1998). To improve the conceptual understanding of mathematical ideas, teachers must select challenging tasks that promote flexible thinking and problem-solving abilities (Smith & Stein, 2013; Stein & Smith, 1998).

Therefore, teachers should be able to take advantage of all the potential embedded in a task and, in order to do this, they need opportunities to explore and solve tasks in the same way that they are expected to explore with their own students (Stein & Smith, 1998; Sullivan et al., 2013). Teacher education programs should include experiences that stimulate teachers' knowledge, using the same principles as teaching mathematics to school students, in particular focusing on problem-solving situations that combine mathematical and pedagogical issues (e.g. Cooney & Krainer, 1996). The use of challenging tasks in these programs may lead to pre-service teachers not only being able to reproduce tasks/solutions/strategies presented, but also produce new and original proposals (Guberman & Leikin, 2013).

15.2.2 Challenging Tasks

Mathematical tasks may have different levels of demand, inducing different learning modes, so teachers should pay special attention to their choice of task. Therefore, task design triggers the activity developed by students, allowing teachers to introduce new ideas and procedures and students to have the opportunity to think differently (e.g. Chapman, 2015; Smith & Stein, 2013). Among the different tasks that we use in mathematics classes, we privilege the tasks that can be solved in different ways, which is very similar to the idea of multiple solution tasks (MSTs) proposed by Leikin (2016), because they develop mathematical knowledge and encourage flexibility and creativity in the individual's mathematical thinking (Leikin, 2016; Polya, 1973). Tasks should have an impact on mathematical activity, allowing students to assess their mathematical understanding, establishing relationships between concepts, and have enough flexibility to use divergent thinking.

Challenge is an important variable in the mathematics classroom because students can become demotivated and bored very easily in a "routine" class. Some may even have difficulties in learning unless they are challenged (Barbeau & Taylor, 2009; Holton et al., 2009). A mathematical challenge occurs when the individual is not aware of procedural or algorithmic tools that are critical to solve a problem and seems to have no standard method of solution. It includes a strong affective call involving curiosity, imagination, inventiveness, and creativity and it is placed intentionally to attract students to their solution (Barbeau, 2009). In this sense, for some problem-solving authors (e.g. Kadijevich, 2007; Polya, 1973; Schoenfeld, 1985) a problem is a mathematical task that challenges learners to solve it.

The expression *challenging* task is normally used to describe a task that is interesting and perhaps enjoyable, but not always easy to deal with or attain, and should actively engage students, developing a diversity of thinking and learning styles. Thus, even when it is not easy to deal with or to solve, it is perceived by the solver as an interesting and enjoyable problem. The engagement in productive struggle allows learners to widen their understanding (NCTM, 2014). Challenging tasks may particularly be those that require the learner to relate mathematical concepts or procedures, by considering, for example, different representations, views or applications (Kadijevich, 2007). According to Leikin (2014) a mathematical challenge is a mathematical difficulty that a person is able and willing to overcome. However mathematical challenges are not just difficult problems. The same problem may be a challenge for one student and a routine problem for another (Holton et al., 2009).

As the one who introduces challenges in the classroom, the teacher must be aware of some particular circumstances. For instance, appropriate challenges can be given to mathematically able students as well as to less qualified ones. The solution for the same task may also be scaffolded differently to different students, providing challenges at several levels. Our difficult role and goal are to engage students with different mathematical backgrounds in different settings so that they can further develop their mathematical ideas, reasoning and problem-solving strategies, as well as their enjoyment in solving mathematical tasks. According to Leikin (2014), we need to develop students' mathematical potential through an adequate level of mathematical challenge. Tasks are considered rich or good because they give students the opportunity to learn, choosing from several different areas of mathematics and different mathematical and non-mathematical abilities, and using these in an integrated, creative and meaningful way. This is in accordance with the use of MSTs, suggested by Leikin (2016) as "a didactical and research tool in the majority of the studies that focus on the identification, development, and role of creativity in the teaching and learning of mathematics to students and teachers" (p.7). The author considers the link of MSTs to creativity, expressed in the differences in learners with varying levels of excellence in school mathematics (or in teachers with varying levels of expertise), whether they use insight-based solutions (related to an *aha!* moment) or, in contrast, learning-based solutions (the standard ones) of the problems. In this sense, for Leikin (2016), challenging mathematical tasks can "require solving insight-based problems, proving, posing new questions and problems, and investigating mathematical objects and situations" (p. 1–28). Insight-based problems are the ones that have a relatively simple solution which is difficult to discover until solution-relevant features are recognized (Weisberg, 2015, cited by Leikin, 2016). These kinds of tasks are "challenging either for novices or expert students requiring flexibility when finding additional solutions and raising different conjectures as well as originality when finding new mathematical facts and new mathematical solutions" (Leikin, 2016, p. 10).

We define MSTs as tasks that invite different ways of solving a problem, which constitutes a challenge for the solver. This only makes sense in an exploratory teaching where the teacher is the orchestrator (Smith & Stein, 2013), according to effective teaching of mathematics, that engages students in solving and discussing challenging tasks. This environment promotes mathematical reasoning and problem-solving with multiple and varied solution strategies, including visual ones, promoting creativity (Leikin, 2016; NCTM, 2014; Presmeg, 2014).

15.3 The Potential of Visualization

Throughout the history of mathematics, it is possible to identify moments when visualization and arguments of visual nature played a major role in mathematical activity, but also periods of time when this way of thinking was avoided. However, for the last two decades, we have seen a growing interest in the use of images as a general cultural change. Considering that mathematics requires the frequent use of diagrams, figures, tables, spatial arrangements of symbols and/or other types of representations, the recognition of the importance of visual processing and external representations associated with visualization has been progressively evident. Visualization has acquired an important status in mathematics, not only due to its illustrative functions but also to its recognized relevance as an important component of mathematical reasoning and proof (Arcavi, 2003).

The role of visualization in mathematics learning has been subject of much research and discussion as has been the delimitation of its meaning (e.g. Arcavi, 2003; Dreyfus, 1995; Presmeg, 2006, 2020; Stylianou & Silver, 2004). Many authors embrace the definition of visualization proposed by Arcavi (2003), which is broad enough to include product and process, visualization as an artifact, as well as the meanings constructed by individual learners (Presmeg, 2014):

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

This definition contemplates different facets of visualization, considering it as a bidirectional process between mathematical understanding and the external environment.

The discussion about the nature and role of visualization in the teaching and learning of mathematics is not simple. Much has been written about the potential of this ability in the development of an intuitive perspective and in the understanding of concepts associated with different areas of mathematics (Zimmermann & Cunningham, 1991). One thing is settled, visualization must not be reduced to the mere production or appreciation of figures or drawings, or even to the development of knowledge within the scope of geometry, on the contrary, it fosters an intuition that contributes to the clarification of mathematical ideas of different nature (Dreyfus, 1995).

Adding to the previous ideas, we can also highlight the relevance of visualization in problem-solving. This relation is unavoidable because visualization provides the use of intuitive and effective strategies that inspire creative findings (Nelson, 1993; Presmeg, 2006; Vale et al., 2018; Zimmermann & Cunningham, 1991). Actually, several studies have analyzed the advantages of using visualization in problem-solving (e.g. Presmeg, 2014, 2020; Stylianou & Silver, 2004; Vale et al., 2018) and it is a common idea that visual thinking contributes to the use of powerful strategies, different from those applied in more traditional approaches, where formalism and symbolism prevail. The use of visual forms of representation, like a drawing or a model, are frequently important aids to solve a diversity of problems, geometric or not, and can act either as unique strategies that lead to a solution or as a crucial starting point to solve a problem (e.g. Polya, 1973; Schoenfeld, 1985; Stylianou & Silver, 2004). In the scope of problem-solving we come across problems of a visual nature or which are presented in a visual context and, for that reason, may more easily be solved using a visual approach. Vale et al. (2018) propose the use of an additional and specific strategy called *seeing*. According to these authors:

> seeing involves an activity that may be associated with a more traditional range of strategies (e.g. draw a picture or diagram, solve a simpler problem, look for a pattern), but it is specifically considered as a strategy of thought that involves visual perception of mathematical objects and is blended with knowledge and past experiences. It includes imagining, which is related with having creative insights or *Aha!* experiences and intuitions; it can also be expressed in terms of drawing, which means translating one's ideas in some visual form. (p. 253)

The *seeing* strategy does not replace any other traditional problem-solving strategy, it is rather a way to tackle a problem. In spite of being a very useful strategy, as will be seen in the examples further ahead in this chapter, unfortunately, this approach is not always encouraged or used by teachers.

Visualization can also have a fundamental role as a complement to analytical thinking. For example, Fischbein (1987) comments that a visual image "is an essential factor for creating the feeling of self-evidence and immediacy" (p. 101) and "not only organizes data at hand into meaningful structures, but it is also an important factor guiding the analytical development of a solution" (p. 104). Visualization can act as a catalyst in understanding the meaning of concepts and in producing inductive reasoning, but it can also be an informal way of understanding deductive reasoning, with the algebraic treatment being done later.

Although visual approaches are considered to be a basis for learning in mathematics and also for problem-solving, the literature often mentions that many students show reluctance to explore visual support systems (Dreyfus, 1995; Presmeg, 2006, 2020). This phenomenon can be enhanced by several factors. On one hand, it is possible that mathematics, by its nature, favors the non-visual thinker, taking into account that the logical-verbal component is considered the sine qua non of mathematical abilities, while the visual-spatial component is not considered mandatory (Krutetskii, 1976). Another aspect is the relevance attributed to non-visual methods in the instruction process, under the conception that visual approaches are harder to teach and difficult for students to understand. Based on the ideas of Presmeg (2014) and Vale et al. (2018), we consider that visual solutions are understood as the way in which mathematical information is presented and/or processed in the initial approach or during problem-solving. They include the use of different representations of visual nature as an essential part of the process of reaching the solution (e.g. graphics, charts, figures, drawings). On the contrary, non-visual solutions or analytical solutions do not depend on visual representations as an essential part of achieving the solution, using other representations/procedures, such as numerical, algebraic and verbal ones (e.g. Presmeg, 2014; Vale et al., 2018).

Despite the fact that mathematical educators apparently recognize the potential of visual thinking, frequently this idea is not reflected in their practices, continuing to attribute a secondary role to this type of method. This should be faced as a concern since visual abilities are not self-evident or innate, but created, developed and learned, through teaching (e.g. Hoffmann, 1998; Whitley, 2004). So, regardless of the reasons, if teachers do not include visual approaches in their practices, it is unlikely that students are able to develop the visual-spatial component of their reasoning. Consequently, teacher training courses should include this discussion and awareness in the respective programs.

Other than this, not all students have the same preferences when it comes to learning mathematics. This is noticeable, for example, in the preference for the different themes in the curriculum, in the way they understand these themes and solve the respective tasks, privileging words, formulas or figures. Thus, teachers have to consider that students may have different learning styles and that they may also have different preferences in relation to mathematical communication, which has direct influence on the representations used. Emphasizing this perspective, psychologists and mathematical educators (e.g. Borromeo Ferri, 2012; Clements, 1982; Krutetskii, 1976; Presmeg, 2014) propose a typology of problem-solving strategies used by students, according to their learning styles: (1) *Visualizers* or geometric - prefer to use visual solution methods (figures, diagrams, pictures) or pictorial-visual schemes, even when problems could more easily be solved with analytical tools; (2) *Verbalizers*, non-visual or analytical - prefer to use verbal-logic approaches or non-visual solution methods (algebraic, numeric, verbal representations), even with problems where it could be simpler to use a visual approach; and (3) *Harmonic*, mixed or integrated - have no specific preference for either logical-verbal or visual-pictorial thinking, and tend to combine analytical and visual methods, showing an integrated thinking style. These styles of thinking are of great importance and influence the way that each student processes information. However, it is extremely complex to apply this categorization and difficult to distinguish, in absolute terms, an individual tendency of a student for a certain type of thinking (Clements, 1982). Nonetheless, these issues have strong implications in the classroom practices and, in particular, in the teachers' choices. Whether with the intention of meeting the diversity of learning styles, with the purpose of broadening the students' repertoire of strategies, or showing the potential of certain mathematical tools, teachers should promote the use of analytical and visual approaches and, if possible, integrate them in order to construct rich understandings of mathematical concepts (Zazkis et al., 1996). Despite the different learning styles, a teacher may find in a classroom, students should experience the use of different approaches to the same problem, either of visual or non-visual nature. This is fundamental to the development of a more flexible reasoning and to make more conscious decisions about the choice of strategies.

Taking into account the ideas discussed in this section, it is important to establish a connection between visualization and mathematical challenge, clarifying our perspective. The use of MSTs that allow the application of either analytical or visual methods can be effective instructional resources to promote the abovementioned flexibility, but also enhance the level of mathematical challenge.

This connection can be translated into different situations. Challenging tasks are thought-provoking mathematical problems that aim to include all students in the mathematical activity (Sullivan & Mornane, 2014) and, in this sense, thinking of the students' learning styles and preferences, these tasks generate an opportunity to extend their knowledge. A task can become more challenging, for example, when students are required to use specific mathematical knowledge in the solution process (e.g. concepts, procedures, representations, rules, or reasoning). For example, some may find the generation and use of a diagram in the solution challenging (e.g. Diezmann & English, 2001).

Visualization can elicit the development of intuition and the ability to see new relationships, producing a cut with mental fixations that enable creative thinking (Haylock, 1997), especially with students who are used to apply analytical methods. Krutetskii (1976) and Polya (1973) also point out that one of the characteristics of mathematically competent students is being able to look for a clear, simple, short, and therefore elegant, visual solution to a problem. This endeavour can be seen as a challenge. Reinforcing this discussion, we would like to draw attention to the importance of developing the students' *mathematical eye* or *geometrical eye* (Fujita & Jones, 2002), referring to the use of mathematics as a lens to see and interpret things/elements that surround us. It means to see the unseen, interpret things in the world as a boundless opportunity, and discover the mathematics involved by seeing the world around us with new eyes. For most people, the mathematics that surrounds them often remains "invisible" to their untrained or inattentive eye. That is why it's necessary to educate the mathematical eye so that they can identify contexts and elements that can become more competent in tackling rich and challenging mathematical tasks (Vale & Barbosa, 2020). Also, certain tasks can be difficult to solve with analytical tools either because of their strong visual structure or because of the students' lack of specific knowledge to solve them. In these cases, visualization may help face the mathematical challenge, focusing on the visual cues of the task or using a dynamic solution to understand key mathematical relations, acting as a support for understanding (Duval, 1999; Presmeg, 2020).

Our perspective on the connection between visualization and challenging tasks is using MSTs that allow different methods (analytical or non-visual, visual or mixed) and invite students to go beyond the conventional knowledge or their personal style of thinking and push them towards visual approaches, which are normally out their comfort zone.

15.4 Visual Contexts and Challenge in Mathematics

The potential and limitations of visual reasoning are recognized as part of the mathematical culture of the classroom (e.g. Arcavi, 2003; Presmeg, 2014), as well as being particularly beneficial for all students, especially those with more difficulties (e.g. Gates, 2015; Vale et al., 2018). However, visual strategies that use different representations are not always fully used to solve a problem. They are usually overlooked by the routine use of rules and procedures learned without meaning, which reduces teaching to a mechanized and monotonous process of numerical, symbolic and/or algorithmic manipulation, diminishing the challenge that is intended in a task. We agree with Roche and Clarke (2014) when they state that all students should experience challenging tasks, but sometimes teachers are reluctant to pose these types of tasks. Often due to lack of knowledge, insecurity, the type of teaching and learning they practice, or because they do not have an available repertoire of tasks.

In the next sections, we start by presenting two examples of tasks that illustrate the importance of visualization and of visual skills. In the first example, we intend to highlight the potential of visual solutions in the context of problem-solving and in the second one the power of a visual approach in situations involving proof or mathematical validation of a statement. The following examples refer to tasks proposed to our students, future elementary education teachers (6–12 years old), during their teacher training course. We focus on MSTs with different themes, cognitive demands and contexts, privileging visual features, as an alternative to more traditional approaches. These students were subjected to instruction that highlighted the potential of visual solutions, and contacting with different strategies of this nature.

We advocate, in teacher training, the use of MSTs, recurring to visual contexts, to show that the same problem can be solved in many different ways and that visualization can be helpful, not only as a strategy per se but also as a means to help give meaning to analytical approaches, promoting the establishment of connections between different representations. These are thought-provoking mathematical problems that generate the opportunity for solvers to extend their knowledge, specifically related to visual representations and this is the challenge.

15.4.1 Example 1: Visual Solutions in Problem-Solving

Some problems may be complicated for individuals who are analytical in the solutions they adopt, or, at least can be more laborious due to the number of calculations required. However, after the discovery of the visual relations involved, with some intuition or *aha!* experience to begin the solving process, these problems become much simpler, hence accessible to more students (Vale et al., 2018). They can include conventional (i.e. learning-based) and unconventional (not learning-based that usually require insight) solutions (Leikin, 2016).

This first example, presented in Fig. 15.1 (Vale et al., 2020), illustrates the idea that looking for a visual solution can either be helpful to solve a problem for

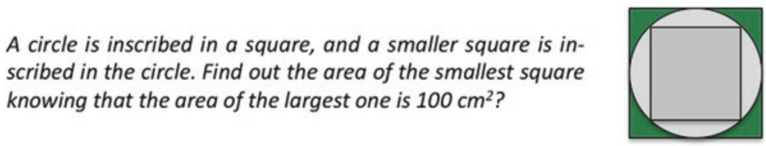


Fig. 15.1 Problem-solving task

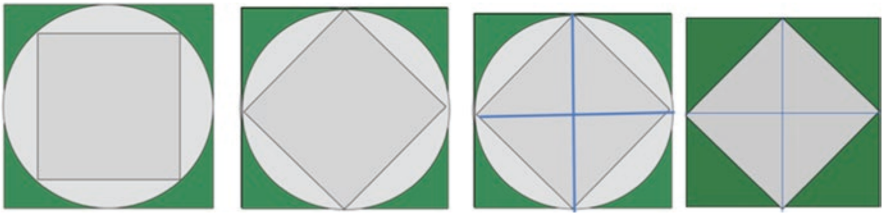


Fig. 15.2 Visual sequence of the steps to achieve the solution

students who don't have certain concepts internalized or to propose some complex problems to more elementary levels.

The traditional way to solve this task is to use the Pythagoras Theorem and the formulas of the area of the circle and of the square. However, if we see the smaller square in another position, the solution is immediate and free of errors or unknowledgeable formulas. According to Duval (1999), this transformation gives an insight into the solution of the problem since it is quickly understood and retained longer, than a sequence of words. Fig. 15.2 shows a visual sequence of steps to achieve the solution $\left(\frac{1}{2}\times 100 = 50\right)$.

This is a simple, clear and elegant solution, or a dynamic solution (Krutetskii, 1976; Presmeg, 2014). As Duval (1999) claims we reconfigured the figure, changing its position, and this kind of transformation does not require a mathematical justification because it is immediate and obvious. The insight of the solution manifests itself in breaking with the conventional or established set of knowledge (e.g. Haylock, 1997; Presmeg, 2014), such as the use of formulas that can be induced by the word area. This is an insight-based problem that has a relatively simple solution which is difficult to discover until solution-relevant features are recognized (Weisberg, 2015, cited by Leikin, 2016).

We may say that the *seeing* strategy is not essential because more traditional analytical strategies could have been used, like the use of calculation methods/procedures or formulas (non-visual). However, this strategy simplifies the process of solving the problem and, at the same time, allows us to relate other knowledge and develop the flexibility that underlies divergent thinking, which is one of the characteristics of creativity. Furthermore, *seeing* can serve to "unpack" the structure of a problem and direct the foundation for its solution (Diezmann & English, 2001).

15.4.2 Example 2: Visual Solutions in Proof

Nelson (1993), in his book *Proofs without words*, was able to draw attention to the importance of the visual approach in mathematical proofs, where he argues that "a picture or a diagram helps the solver see why a particular statement may be true, and also see how one might begin to go about proving it true" (p. vi). However, some mathematicians consider such visual arguments to be of poor value. On the contrary, there are mathematicians who defend the potential of visualization, among them Polya, when he says that drawing a figure is a powerful strategy to solve a problem, adding to the perspectives of Einstein and Poincaré on the importance of using visual intuitions in their work. In the same way, Gardner (1973, cited by Nelson, 1993) refers to the power of visualization when faced with a boring test, it can often be overcome by a simpler and more pleasant analogue geometric proof that allows the truth of the statement to be understood at first glance. Arcavi (2003) goes further when he says that visual representations are legitimate elements of mathematical proofs.

According to Nelson (1993), figures or diagrams can help to see why a certain statement can be true and, at the same time, see how to start proving its veracity. Often the use of algebra can help guide this process, but the emphasis is clearly on providing visual cues to the observer to stimulate mathematical thinking.

The arguments must be rigorous as they can easily lead to misinterpretations and, therefore, lead to wrong inferences. In any case, the importance of visual representations is recognized as a support in the discovery of new results and in the production of more formal tests, and above all for its role in the teaching and learning of mathematics (e.g. Presmeg, 2014).

The second example, shown in Fig. 15.3 (Vale, 2017), illustrates that many mathematical statements can be simply proved by translating the numeric information of the statement into a visual interpretation and discover the relationships and properties that can be established in that figure.

This is an usual example when working with numerical sequences, in particular the sum of geometric progressions. The traditional way to solve this task is to use a formula to calculate the sum of n terms of that geometric progression. It is a task that involves numerical manipulation and the use of formulas without much meaning for most students. But translating the task into a geometric model, the problem acquires another meaning, more challenging, allowing to visualize the infinite sum of the sequence. It starts from a square of unitary area (Fig. 15.4) from which we successively obtain figures with half the area of the previous figure and so on, until physically possible to divide (which allows to have the notion of pattern, generalization, infinity, infinite sum, limit, convergence). That is, the total area of the different squares and rectangles is the same as the sum of all terms of the progression. Since this area is equal to the area of the starting square, the sum of the progression terms is 1.

Prove that $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1.$$

Fig. 15.3 Proof task

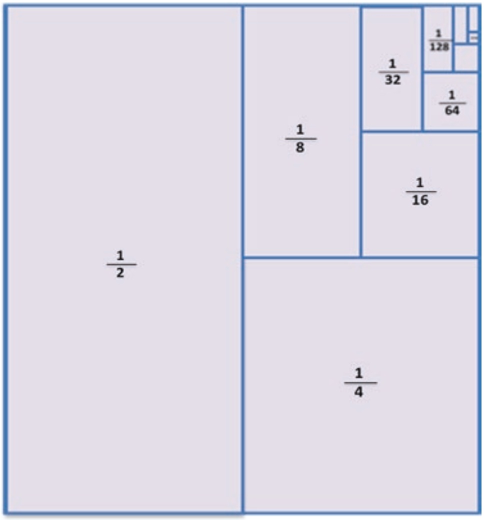


Fig. 15.4 Visual proof

This is a problem that is not normally presented to elementary school students (6–12 years old) as they do not yet have the necessary knowledge to prove the statement analytically. But the solution presented in Fig. 15.4 allows a visual understanding of the underlying meaning in the statement, as they build the figures, based on the concept of fraction in its part-whole interpretation and the intuitive notion of limit. Furthermore, students with more advanced mathematics knowledge quickly and simply understand the meaning of the statement.

15.4.3 Examples in Pre-service Teacher Training

The following examples involve diversified mathematical contexts and, in addition, illustrate the use of the strategy *seeing*. The tasks elicit multiple solutions, promoting some of the dimensions of creativity, apart from mathematical knowledge, and have the potential to promote visual strategies and eventually provide an *aha!* experience. The pre-service teachers were challenged to solve the tasks in as many different ways as they could, being encouraged to present several solutions.

The visual approach to these tasks was unfamiliar to this public, due to the lack of previous experience and visual literacy. In this sense, in spite of frequently generating simpler and elegant solutions, visual methods are challenging for these students, which implies that their training programs contemplate this perspective.

15.4.4 Symmetries

Geometric transformations, and in particular symmetries, are one of the most important applications of mathematics in daily life and nature, allowing the establishment of rich connections. It enables students to explore/create patterns, solve problems and think spatially. However, students generally show a low level of learning when geometric transformations are concerned. This is a theme where the spatial and visual abilities of the solver are essential to attack specific problems and to recognize the different transformations in everyday situations.

Our students, future teachers, were exposed to the teaching of geometric transformations (translations, rotations, reflections and glide reflections), analyzing examples of applications in mathematics and other areas.

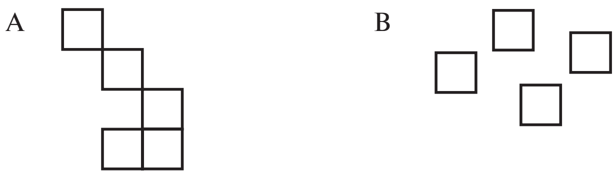
This is a purely visual task, where the solution results from the drawing of different shapes of A, which we propose in Fig. 15.5 (Barbosa & Vale, 2019). It is challenging for the solver: it allows to identify the students' knowledge of symmetries but also, being a multiple solution tasks, with more than one correct answer, enables students' creativity.

Figure 15.6 illustrates some of the productions presented by the students. The solutions to this task are, by nature, visual ones, because the task was proposed in a purely visual context. In the first question, the students had no difficulty to reach the solution, however, some presented only one possibility.

However, in the second question, when asked to build a figure with rotation symmetry, less than a quarter of the students succeeded. Figure 15.7 shows three correct solutions (the first three images) and two incorrect solutions (the last two images).

It is important that teachers promote these kind of abilities in students for the duration of their compulsory education, mainly for the non-visualizers, because a mathematics course for higher levels may not be enough to accomplish these goals.

Image A has five squares.



Add to image A the four isolated squares, observed in image B, so that the resulting figure has:

1. *reflectional symmetry.*
2. *rotational symmetry.*

Present two different solutions for each case.

Fig. 15.5 Task involving symmetries

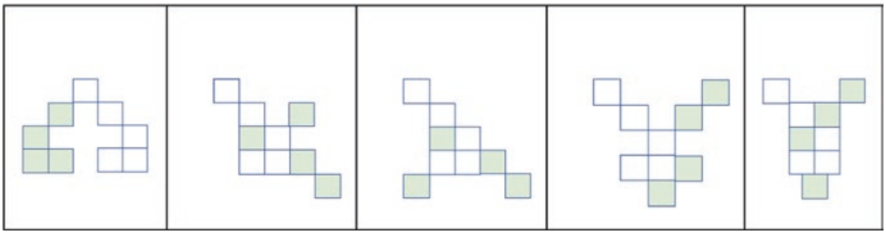


Fig. 15.6 Some solutions with reflectional symmetry

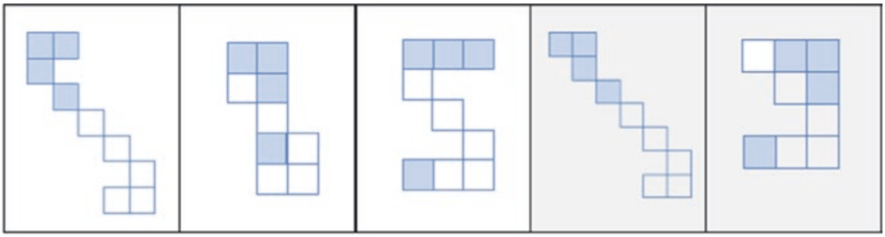


Fig. 15.7 Some answers for the rotational symmetry question

15.4.5 The Vasarely Rhombus

The Vasarely Rhombus task (Fig. 15.8) has a geometrical nature and the aim is to find the area of the shape.

This geometric problem (Vale et al., 2016) confirmed our expectations, that the students would present analytical strategies at first. But, as they were asked to solve the task using more than one process, other categories of solutions appeared. This allowed to identify fluency and flexibility. Figure 15.9 shows examples illustrating those categories.

The first solution is the most usual in this type of problem. As you ask for the area, students start immediately by writing the formula for the area of the rhombus or the triangle. Thus, this solution is purely analytical, that is, a learning-based solution (Leikin, 2016). The other two following solutions reveal the use of similar strategies and are considered mixed solutions. Despite using calculations, they are based on reasoning resulting from the properties of the figure and the concept of area measurement. These were the most common solutions used by the students. The last solution illustrates the application of a simpler and intuitive strategy, resulting from *seeing* the relationships between the different shapes identified in the rhombus and the square, getting a dynamic solution (Presmeg, 2014) or a reconfiguration of parts of the figure (Duval, 1999). This was the most original solution. In our opinion, this unique solution was original because it differs from the expected outcome for this type of task. Anyway, this solution was simpler since, merely by observation, we can conclude that the area of the rhombus is 1/3 of the area of the square.

What is the area of the rhombus, if M_1, M_2, M_3, M_4 are middle points Of each side of the square and the square has 1 unit of area? Find out more than one process to get to the solution.

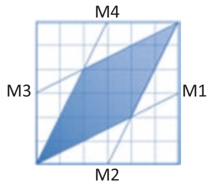


Fig. 15.8 The Vasarely Rhombus task

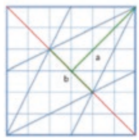
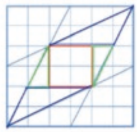
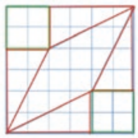
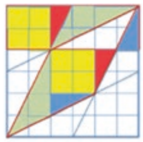
			
Area formula – Pythagoras Theorem $A_\Delta = \frac{\frac{\sqrt{2}}{3} \times \frac{\sqrt{2}}{2}}{2} = \frac{1}{6}$ $A = \frac{1}{3}$	Decomposition $A =$ $= 4 + 3 + 1 + 3 + 1 =$ $= 12$ $A = \frac{12}{36} = \frac{1}{3}$ $A = \frac{1}{3}$	Framing $A = 36 - 2x(4 + 8)$ $= 12$ $A = \frac{12}{36} = \frac{1}{3}$ $A = \frac{1}{3}$	Dynamic solution – decomposition/composition $A = \frac{1}{3}$

Fig. 15.9 Some solutions to this task

This problem, in addition to enhancing the use of different strategies, allows one to approach various contents (e.g. areas, relationships between figures, rational numbers, Pythagoras Theorem), promoting the establishment of connections between mathematical concepts, which can be further explored by the teacher. The cognitive level of the task can increase if we have the same situation without the square grid.

15.4.6 Rational Numbers

The following example is a word problem involving rational numbers. With this example, we see the adequacy and potential of visual solutions even in non-visual contexts. Usually, it is proposed when students are learning concepts and procedures related to this topic, like the different interpretations of fractions or the operations with fractions (Fig. 15.10).

Traditionally word problems with fractions are solved using analytical approaches, but in this example, the whole is an unknown quantity, which normally makes this problem more complex. This can sustain the poor results obtained by some pre-service teachers who chose to solve the task using numerical tools and computation (CHP, 2011; Vale et al., 2018). However, as these students had previous instruction about the use of visual strategies, such as the bar model, the majority used this approach.

The most common analytical solution was to start by determining the part of the whole that remained after taking away the part of the students that use the bus, $1-\frac{1}{3}=\frac{2}{3}$. Calculating the part of the students that go to school by car, we have $\frac{1}{4}\times\frac{2}{3}=\frac{2}{12}=\frac{1}{6}$. As the part of the students that go by car is $\frac{1}{4}$, or 90 students, the whole will be 6 × 90 = 540. We conclude that the school has 540 students. For many students, this numerical manipulation is not always understood, even for pre-service teachers, as it involves conceptual and procedural knowledge to solve the task.

The use of a visual model can be helpful in a first stage of learning and dealing with fractions or to make sense of analytical procedures. One approach that fits this criterium is the bar/rectangular model. Solving the same task with this strategy, some students started by using the bar to represent the unknown quantity, and then the needed data can be obtained using successive bars (Fig. 15.11).

In alternative to the previous solution, some students chose to use only one bar, concentrating all the needed information in one representation (Fig. 15.12):

Students go to school using different means of transportation. One third of the students go by bus. One quarter of the remaining students goes by car. The others take a bike or walk to school. Knowing that 90 students go to school by car, how many students attend this school?

Fig. 15.10 Task involving rational numbers

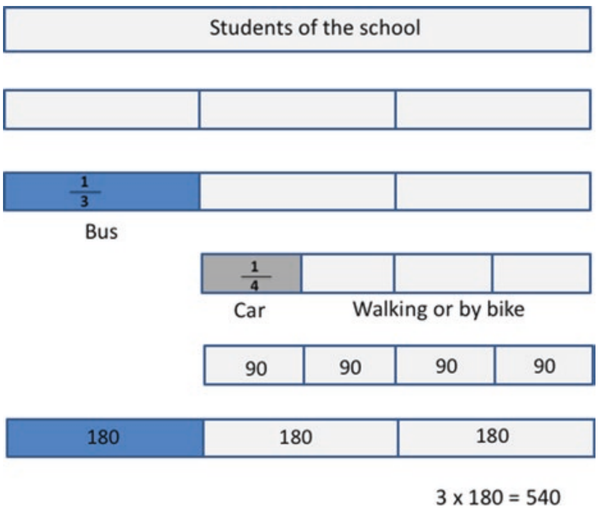


Fig. 15.11 Visual solution of the task

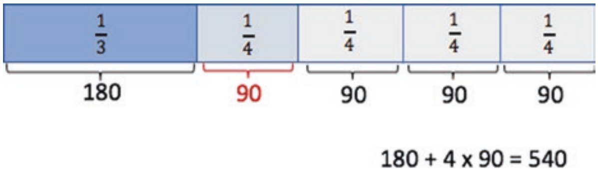


Fig. 15.12 Visual solution of the task

As can be seen by the examples, the bar model may be sufficient to solve the problem or to help clarify some misunderstandings in the interpretation of the task. The students may use different and complementary representations to make sense of the calculations or vice versa. This model is especially valued by underachieving or intermediate-level students, but we also must say that it can be helpful in solving more complex problems and in the understanding of fractions related concepts.

15.4.7 Paper Folding: The Cube

Spatial visualization is viewed as an essential part of geometric thought described as building and manipulating mental representations of two and three-dimensional objects and perceiving an object from different perspectives (NCTM, 2014) and includes the ability to manipulate the information presented in a visual, diagrammatic or symbolic form (Diezmann & Watters, 2000). Paper folding is a useful teaching tool to enable those skills in students and a way to promote their spatial thinking as an impact on the understanding of geometry (Boakes, 2009). It associates itself very naturally with visualization and geometric reasoning, making it possible to approach different mathematical themes, as well as a diversity of transversal skills (e.g. communication, problem-solving, proof).

The actions of folding applied to the paper allow it to be transformed into different shapes, either two or three-dimensional, opening the opportunity to investigate and discover relationships of different nature. In this way, paper folding can be a dynamic, creative and challenging strategy to approach several concepts in the mathematics classroom, facilitating visualization and problem-solving (Vale et al., 2020). Paper folding involves students cognitively in the challenges it provides and physically, because it requires auditory abilities and visual stimuli, and it is through these actions that it also involves spatial skills, which promotes the construction and discussion of meanings and mathematical ideas.

The example presented in Fig. 15.13 (Vale et al., 2020), refers to a task proposed to our students. This task had two main goals, to find the optimal solution and use spatial abilities to transform a 2D figure into a 3D figure.

This problem involves geometric and spatial reasoning, since the students have to construct a net (a two-dimensional figure that can be folded into a three-dimensional object) of a cube. Many nets can be built on a square sheet, but only one fits the conditions. It is a problem with some complexity for the elementary level. The students mostly began by exploring the most obvious possibilities, in

Use a square sheet of paper to draw the net of a cube with the maximum volume. Then construct the cube by folding that net.

Fig. 15.13 Paper folding cube

Fig. 15.14 Some incorrect solutions

which the segments representing the edges in the planning were parallel to the sides of the square or took advantage of the diagonal of the square. Despite having made different trials to reach a solution, none of them led to the expected outcome, because they did not achieve the highest volume (Fig. 15.14).

After many net trials, calculations and group discussions, the students discovered the correct answer. Figure 15.15 shows one of the analytical productions where they compared the volume of two nets.

The students made the design of the possible traditional nets. In fact, it is necessary to have mathematical knowledge to apply to this situation, and also intuition linked to the visualization of the different nets of a cube. In addition, exploration required divergent thinking to imagine and admit a completely different net from the classical approaches. Another way to approach it was the use of trial and error, doing the folds on the square paper and coming up with more positive results. After discovering the right net, the bigger challenge was to fold the paper to get the cube, without cutting. They did many attempts, but not all of them got the solution by themselves (Fig. 15.16).

This is a task with some complexity to use at an elementary level, but students were challenged and engaged, and the discussions that emerged at the end allowed a better understanding of the importance of the use of different approaches to solve a mathematical situation.

15.4.8 The Cup

Consider the following task (Fig. 15.17):

This task (Vale et al., 2016) can motivate several solutions involving the properties of the observed figures. Students who attempt a solution using formulas applied

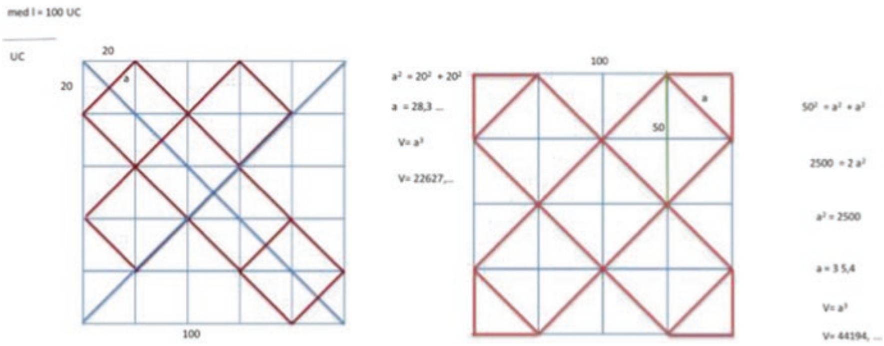


Fig. 15.15 Analytical solutions to determine the volume

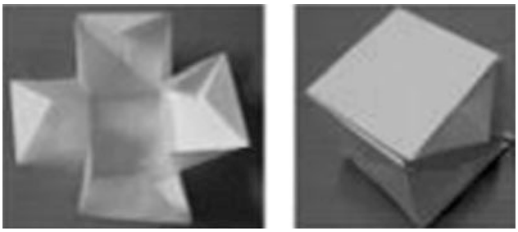


Fig. 15.16 Optimal solution and paper folding cube

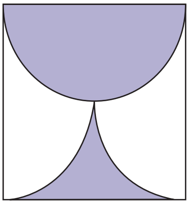
to different parts of the figure may find some difficulty to solve it. The students who attempted a method using formulas applied to different parts of the figure considered that this is a difficult problem, especially if the square is not shown. The most common numerical solution was in Fig. 15.18.

There are only calculations, looking for the best way to use the formula of the area of the circles. We can say that this solution is blind, that is, there is no attempt to see any relationship between the upper parts and the lower parts of the figure. Yet, there are some solutions that we called mixed (Fig. 15.19) because, although they are numerical, students can see beyond the given figure, even adding some geometric construction.

However, the challenge for the solvers was to get a visual solution, but it did not appear. If the students have more visual abilities, they can discover a dynamic visual process, *seeing* transformations of the initial figure (see the arrows), and doing a reconfiguration of parts of the figure (Fig. 15.19). We can mentally slide the two parts that make up the “foot” of the “cup” to the top, forming a rectangle. It follows the trivial conclusion that the “cup” has an area equal to half of the square, i.e., 1/2 unit area (the first of Fig. 15.19). Another dynamic solution could be the last one shown in Fig. 15.19, in which, after drawing the diagonals of the square, we easily

The figure shows a unitary side square. The curved lines are circumference arcs. What is the area of the shaded region? Find out more than one process of getting the solution.

Fig. 15.17 The cup task



$$\text{Up: } \frac{\pi\times\left(\frac{1}{2}\right)^2}{2}=\frac{\pi}{8}$$

$$\text{Down: } 2\times\left[\left(\frac{1}{2}\right)^2-\frac{\pi\times\left(\frac{1}{2}\right)^2}{4}\right]=2\times\left(\frac{1}{4}-\frac{\pi}{16}\right)=\frac{4-\pi}{8}$$

$$\text{Total area: } \frac{\pi}{8}+\frac{4-\pi}{8}=\frac{1}{2}$$

Fig. 15.18 The common numerical solution

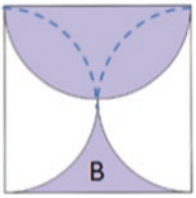
see that the area of the "cup" corresponds to 2/4 of the square area, i.e., half of the square. In each case, the visual elements convey the thinking process (Fig. 15.20).

What makes such a solution creative and also simpler, being a challenge for these students, as some authors (Haylock, 1997; Leikin, 2016; Presmeg, 2014) suggest, is the fact of being necessary to break the mental set that suggests the use of formulas or conventional/learning-based solutions. It is necessary to use divergent thinking in looking for other ways to solve the task. This can happen if instruction push learners to find new ways to solve some tasks, where thinking by analogy can be a helpful strategy to attack new problems (Polya, 1973).

15.5 Concluding Remarks

In this chapter, we intended to discuss and illustrate some ideas concerning the use of visualization as a meaningful pathway to pose and solve challenging tasks in mathematics education. The choice and use of tasks are nuclear to effective teaching and learning of mathematics since they are the driving force that triggers mathematical activity. The importance of the teachers' role in this matter is undeniable, but it goes far beyond task selection. Students must be motivated and engaged as solvers to be successful, being incited to think, discuss, reflect and overall be challenged.

We believe that multiple solution tasks give students the opportunity to apply their thinking styles, whatever their nature, and also to come into contact with a variety of strategies that will contribute to the expansion of their repertoire. In this framework, and according to our own experience with pre-service teachers, we consider that visualization can have great potential, either as the context in which the



$$\text{area of the circle} = A\circ = \pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$$

$$B = \frac{1 - A\circ}{2} = \frac{4-\pi}{8}$$

$$\frac{1}{2}A\circ + B = \frac{\pi}{8} + \frac{4-\pi}{8} = \frac{1}{2}$$

Fig. 15.19 A mixed solution

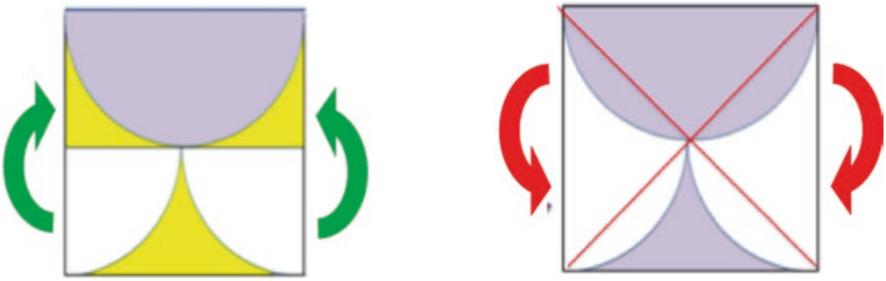


Fig. 15.20 Two visual solutions

task is presented or as a vehicle to reach a solution. Visual strategies are not new in the literature but usually, they are deprecated over analytical ones. This situation does not benefit the students since visual approaches can be an excellent complement to analytical thinking or can even help generate simpler and more meaningful solutions. The visual approach, being transversal, mediated by tasks with multiple solutions, contributes not only to a better understanding of mathematics and to the development of students' creativity, but above all to show a new perspective of mathematics. This means that students may overcome the idea of seeing mathematics as an isolated collection of themes, made up of a set of formulas and techniques that they have to memorize and master, when, in fact, they can have the opportunity to see it as a set of meaningful great ideas and connections.

Some studies and projects that we have been carrying out with pre-service teachers of elementary education have generated results that we consider relevant in the context of multiple solutions tasks, with a close connection to visualization. There are several aspects that stand out, namely the fact that, through these tasks, we can invoke a diversity of mathematical themes; establish connections between visual and analytical processes, as well as between representations; allow a more fluent, flexible and even original reasoning, which fosters creativity. Our intention in developing this type of work with pre-service teachers is that they understand the potential of *seeing* in order to develop this ability with their future students. We have to consider the background of the candidates to this teacher training course: not everyone has the same solid mathematical knowledge; the majority of them are non-visualizers due to the predominance of analytical methods in their previous experiences, which makes them unaware of the utility and power of visual approaches, undermining their *visual literacy*. Many of the tasks presented in this chapter are not necessarily high-level and do not involve complex mathematical concepts, but they are challenging from our perspective. The challenge rises from the fact that we use multiple solution tasks, chosen with the intent of allowing the application of analytical and visual methods, that must be solved in different ways. So, our interest was that solvers were challenged by discovering a visual solution.

For the non-visualizers this request may trigger the need to discover a strategy of a different nature, making them face the challenge of *seeing* the visual cues; and the same happens with the visualizers that may need to use non visual methods, which brings the opportunity to connect mathematical ideas in a rich and meaningful way. Subsequently, challenge can be faced as a situation that enhances the learning process, experiencing something new and unforeseen and trying to come to grips with it. With regard to visualization and visual thinking in particular, being less valued in mathematics classes and even in textbooks, it's pertinent to provoke the use of this way of reasoning, enhancing the level of mathematical challenge and flexibility in problem-solving.

We consider this didactical approach an asset, given the positive results of previous research projects and, for this reason, we include it in these pre-service teachers training programs, believing that it can contribute to both visualizers and non-visualizers to better understand certain mathematical themes and be challenged to find solutions out of their comfort zone. To conclude, in this chapter, we chose to present examples in different contexts, dealing with a wide range of themes and abilities, in detriment of an in-depth discussion of the results obtained, in order to focus on the potentialities of the tasks and contribute to broaden the repertoire of tasks that may help teachers challenge students through this perspective.

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