Chapter 12
Probing Beneath the Surface of Resisting and Accepting Challenges in the Mathematics Classroom

John Mason

12.1 Introduction

What interests me most is the lived experience of thinking, doing, learning, and teaching mathematics. In taking up the challenge to write about mathematical challenge I have interrogated my own experience and used this to probe beneath the surface of common reactions to being challenged mathematically.

Ten years after Bill Brookes (1976) suggested that something is a problem only when a person experiences it *as* a problem, Christiansen and Walther (1986), following Vygotsky (1978), distinguished between a *task* as what students are offered or inveigled to undertake, and *activity* as what happens as they attempt to carry out their interpretation of the task. Combining these, some thing or some situation can usefully be described as a 'problem' only when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in sense-making activity.

The notion of mathematical challenge has an inbuilt ambiguity. On the one hand, someone can challenge me to resolve a problem. On the other hand the challenge may be taken up and experienced *as* a challenge, or it may be resisted in some way. In this paper, the focus is on the latter so that a mathematical task is considered to be a challenge only when someone experiences a state of 'feeling challenged' and takes action to try to meet that perceived challenge. I shall use the word *challenge* in this sense, not as a description of qualities of any particular stimulus or prompt but as an indicator of someone's state within a situation with affective, cognitive, enactive, and other consequences. In other words, *challenge* depends on the current state of the psyche of individuals within the current social setting.

J. Mason (✉)
Open University and University of Oxford, Oxford, UK
e-mail: John.Mason@open.ac.uk

R. Leikin (ed.), *Mathematical Challenges For All*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-031-18868-8_12

The notion of *challenge* in relation to learning, doing, and teaching mathematics is of vital importance: if a mathematical task is thought to be too challenging, learners are likely to resist or even reject it, while if it seems insufficiently challenging, learners can become de-motivated and-or complacent, which is itself a form of resisting personal investment. Something perceived as a routine exercise rather than as a challenge to their powers is likely to reinforce a separation between schooling and living. Furthermore, what is a reasonable challenge to some may be routine to others, yet out of reach for still others, and this can change with different circumstances and at different times.

I am interested in the constellation of conditions in which learners might accept and take on a challenge, in which they might (learn to) park a challenge, and in which they might resist, defer, or reject a challenge. An important aspect of the psyche in this regard is trust, whether in the source of the challenge or in themselves, in their self-confidence (literally trust-in-self), which of course may sometimes be inflated or deflated inappropriately.

12.2 Human Psyche

In order to probe beneath the surface of mathematical challenge, I make use of discourses which have to do with both distinctions between and coordination among aspects of the human psyche. It is important to note, however, that these discourses are partial. They make no claim to completeness, no claim that they are either necessary or universal. They are based on observations which, being made by human beings with history and predilections, are necessarily biased. Their potential validity resides not in statistical studies but in whether their use is found to inform personal action in the future.

12.2.1 Six Aspects of the Human Psyche

Traditional (Western) psychology has focused largely on three aspects of the human psyche: *enaction*, *affect* and *cognition,* with only scant recognition of *attention*, of the role of *will*, and of the presence or absence of an inner *witness* or monitor. Each of these aspects has both an experienced and a description version.

Thus *enaction* is what is experienced when an action is initiated and continued, whereas *behaviour* is what others describe when reporting observed activity. For example, I find myself constructing and then working through an example (action) that the observer describes as specialising, even though I am not aware of specialising as such.

Similarly, *affect* points to the experience of changes in physiological conditions of the individual such as perspiration, change in pulse or breathing, etc. and may be distinguished from *emotions* which are descriptions of imagined states such as

anger, fear, excitement, etc. (Mandler, 1989; Barrett, 2017). Notice how affect and emotion have significant physiological and cognitive components. For example, I find myself disinclined to work at a particular task, which an observer might describe as resisting or rejecting the challenge, and might confuse with my observed behaviour of not-engaging or working slowly.

Again, *cognition* can be used to refer to an act of mentation (having ideas popping into consciousness, such as words to say; changes in the oxygen levels in relevant parts of the brain, …), while *intellect* can be used to describe the effects of such mentation such as thinking, pondering, considering, problem-solving, etc.

Of course, descriptions cross over between these components: someone observed staring out a window may be interpreted as thinking, or as resisting activity due to a surge in affect (which they interpret as an emotion), perhaps blocking the possibility of enacting some action, or obscuring a lack of any available action.

The descriptive terms have the effect of generalising, labelling habitual patterns of action, emotion and intellect, attributed to the individual (psychologically) and to a group of people (sociologically). Such descriptions become fossilised, obscuring subtle differences which could otherwise have had different pedagogical implications (Mason, 1989).

In addition to the usual trio of aspects, it seems clear that *attention*, both what is attended to and how it is attended to, plays a vital role in mathematical thinking, and again there is a difference between what is experienced and how that experience is described (Mason, 1989). In chapter XI of his *Principles of Psychology,* William James (1890) proposed that "My experience is what I agree to attend to" (Green website). This begs the question of what constitutes the "I" that does the agreeing. My own observations suggest that attention is the centre of experience; its scope and range, breadth, focus and locus *are* the "I" that claims to be the subject of the predicates that describe my various states and actions. It is not surprising therefore that it is very difficult to observe my own attention. This is the role of the inner *witness*.

The inner *witness* is the voice that suddenly asks "Why are we doing this?", or "Isn't there a better way?". It observes but does not act. It alone is able to observe the locus and focus, range, scope, and intensity of attention, and how attention is functioning mathematically (Mason, 2001). It has been referred to as an *inner monitor* or *inner executive* (Schoenfeld, 1985), and its recognition as part of the psyche has historical roots in a stanza in the Rg Veda (Bennett, 1943):

> Two birds, close yoked companions, both clasp the self-same tree;
> One eats of the sweet fruit, the other looks on without eating.

The witness is the bird that observes without eating, without being caught up in the action.

Human *will* is another elusive notion. Usually expressed in terms of *will-power*, it is used, for example, to describe someone sustaining activity in the face of opposition or difficulty, but this is always in relation to an observer's expectation and has an affective component as well. Here it is used to refer to summoning or organising of energies, perhaps to persevere, perhaps to change direction, to pause or park activity. It is associated with the taking of initiative, of initiating some action. It feels like a releasing and channelling of the energy of intention and desire, of

precipitated action, of insight, so that attention is sustained and focussed, despite distractions.

It is often assumed that actions are initiated by intellect-cognition, although sometimes it is acknowledged that they may be triggered through affect, or even enaction. Norretranders (1998) summarised neurological studies which suggest that the common assumption that conscious cognition is in control of actions is a *User Illusion*. Rather, many if not most actions are actually re-actions based on previously developed habits, previously coordinated adherences amongst the aspects of the psyche making predictions based on past experience (Nave et al., 2020). These habits are not simply actions, but repeated patterns of action, affect, thought, propensity to attend to certain things in certain ways, exercise of will and characteristic observations made by the inner witness. Over time these patterns of interactions become adherences which manifest themselves as micro-selves, personalities or distinctive selves.

12.2.2 Initiating Action

One of the ways in which action is initiated is illustrated by the cliché that, "to a child with a hammer, the world looks like nails". In other words, when a new tool becomes available, there is a tendency to use it everywhere. For example, when a new word is encountered, it is often used initially rather more broadly than most people are accustomed to. Over time, its use settles and, judging from the use, the meaning contracts. So too with other tools. Mathematicians do the same: upon encountering a fresh way of proving something, they are likely to try using that same or similar approach in the near future. The extent to which this happens will be influenced by the self-confidence and the vibrancy of whatever psycho-social habits are dominant at the time.

Using a recently acquired tool almost indiscriminately is but one instance of action which is enacted without reference to cognition. Terms such as 'habits below the level of consciousness', 'unformulated action', 'theorems-in-action', 'tacit knowing', and even 'poetic knowing' have been used similarly: see Mason and Johnston-Wilder (2004, pp. 298–291) for a partial trail. Kahneman and Frederick (2002) and Kahneman (2012) drew on the ideas of Wason and Evans (1974) concerning Dual Process Theory. Kahneman's elaboration was based on experiments which suggested that the human psyche has two different 'systems': System 1 (S1) is immediate, reactive, enacted by the musculature, and often associated with intuition (or is it habit?), while System 2 (S2) involves consideration (literally *sitting with*) by the conscious cognitive apparatus. Norretranders (1998), and others such as Mandler (1989) point out, again in alignment with more ancient knowledge of the human psyche such as presented in the Upanishads or the Bhagavad Gita, that under stress the body reacts first, the emotions second, and cognition a slow third.

Liam Hudson (1968) used the notion of *frames of mind* to express something akin to coordinated adherences, in much the same way as Marvin Minsky (1975)

who, being a computer scientist, thought in terms of actions being enacted by computer-like programmes which have default parameters to be used in place of absent information. As soon as all the parameters have 'values', the action is initiated, without requiring contribution, consideration, or permission from cognition, which aligns with the notion of the *predictive brain* (Nave et al., 2020). This provides one explanation for why human beings so often act without being consciously aware of acting and is captured nicely in a traditional teaching story:

> A horse suddenly came galloping quickly down the road. It seemed as though the rider had somewhere important to go. A bystander shouted out, "Where are you going?" and the rider on the horse replied, "I don't know! Ask the horse!" (Hanh, 1986)

The horses of the human psyche, the emotions, all too readily carry us away! Being carried away is sometimes seen as a positive state, associated with letting go of what has previously been shackling so as to enable a state of *flow* (Csikszentmihalyi, 1997). Certainly, it is relatively easy to be carried away sufficiently so as to lose the sense of time and place and to be so caught up that it is difficult to pay attention to the focus and nature of one's own attention, to the powers one is using, or to mathematical themes which are being played out. However, sometimes it *is* possible to be aware of, to 'be present to' these aspects of mathematical activity, however subconsciously or even consciously. So flow has both a 'carried away' version and a 'being present to' version.

Although it is clear from self-observation that action is often initiated spontaneously, as it were, its origins sometimes lie in emotions which may be triggered metonymically through idiosyncratic association, making certain actions available; sometimes they lie in the arising of a thought and sometimes they lie in a stimulated shift of attention. It is useful therefore to extend the processes idea in line with the psychology articulated by, among others, Ouspensky (1950), to include an intermediate System 1.5 (affective, later narrated as emotion) and a further System 3 which concerns access to deeper or higher energy (Mason & Metz, 2017). System 3 is the source of sudden insight which is often described as coming from 'the muse', and attributed to 'creativity'. It is closely aligned with the Gestalt notion of *form* (Zwicky, 2019), and is experienced in brief moments, which are almost always immediately overlaid by thoughts, emotions and activity. It is accessed through periods of relaxation of tensions in the body, emotions and thoughts, in what is sometimes referred to as fallow periods, or centredness. Waiting, or gazing, seen as one way of attending to something, may be most effectively thought about as coordination of enaction, affect and cognition, with corresponding things to attend to and ways of attending, which leave the person open to unexpected possibilities, sometimes leading to a momentary experience of S3.

Neville (1989) describes a variety of educational initiatives with psychological backing based on the notion that learning is most efficiently undertaken not by consciously focused attention but by peripheral attention. Gattegno (1970, 1987) used the same principle to suggest that in order to internalise an action so that it is available to be enacted, it is best to provoke the action peripherally, as a side-line of some other action. He called the process of sensitising oneself to the possibility of an

action, that is, recognising contexts in which it might be appropriate, and internalising that action, as *educating awareness.* Hence the role of mathematical exploration is in order to create conditions in which learners spontaneously rehearse some procedure so as to make sense of some apparently unrelated phenomenon. This aligns with the Eastern teaching method illustrated several times in the movie *Karate Kid* (1984), in which the student is inveigled into rehearsing an action while attending to something else entirely.

The four 'systems' S1, S1.5, S2 and S3 describe four different ways in which action can be initiated and actively pursued, making use of combinations of or coordinated aspects of the psyche. Emotions provide the energy (cf. the etymological roots of *emotion*), but attention is the core of presence.

12.2.3 Psycho-Social-Coordinations

The various aspects or components of the psyche do not operate in isolation. It seems that particular emotions become associated with and amplify particular thoughts and together these energise certain actions over others. Particular thoughts and emotions stress certain behaviours, and the will to continue on any path of action, emotion, thought and attention is influenced by perceptions about actual and likely success (or failure) which in turn are influenced by current emotions (Skemp, 1979). Things deemed worthy of attention, and particular ways of attending to them, become salient, even to the extent of blocking out other possibilities. The inner witness observes the sorts of things it has become accustomed to observing and issues alerts which have become part of the adherence.

Co-ordinations can be self-amplifying and self-sustaining, preserving the psychological state, and in turn may be amplified, sustained or ameliorated by the social milieu. These stimulate characteristic actions, emotions, dispositions, patterns of thought, foci and functioning of attention, activated willpower and even types of observations made by the witness. Over time a repeated coordination 'takes hold'. It becomes an adherence manifested as a habit, hence the notion of *psycho-social coordinated adherences*. They are like *micro-identities* (Varela, 1999) or *multiple selves* (Bennett, 1964; Hudson, 1968; Minsky, 1986; Hanson, 1986; Kahn, 1983; Davies & Harré, 1990; Eakin, 1999; Lester, 2012). Since many people see themselves as trying to locate their 'true self' and reject out of hand the notion of multiple selves, the language of *coordinated adherences* seems to be more generally acceptable.

Over time, coordinations can become stable, so that characteristic flows of energy adhere to each other to form habits not just of behaviour, but of psycho-social states. The adjective *psycho-social* emphasises that coordinations are influenced by perceived social conditions as well as by psychological states, and so although adherences are in the psyche, which adherence becomes dominant at any particular time, and how it became coordinated in the first place, is influenced by the social situation and relationships as perceived and experienced by the psyche.

An example of this is a collection of socio-mathematical norms which are enacted by teachers and then may be picked up by learners (Yakel & Cobb, 1996). Some learners may reject them out of hand, while others may take them up with alacrity, and still others may gradually become inured to them. Another example can be found in the report of Brown and Coles (2000) in which students picked up the practice of considering what is the same and what is different about two or more mathematical 'objects', and then began to initiate this action for themselves.

I noticed an example recently when a friend showed me an intriguing book consisting of drawings of geometrical configurations, using both solid and dashed lines in various places (Akopyan, 2011). Each diagram can be taken as a challenge to discern and articulate a property which relates the dashed (construction) lines to the solid lines. When I subsequently received my own copy I was initially entranced, but then overcome with lethargy and a sense of burden. My witness recognised this state as one which I have experienced with other problem collections. The immense potential, the scale of commitment implied, and the fear of not being able to work them all out combine to stifle action and lead me to reject the challenge, at least for a time. A psycho-social adherence is brought to the surface which finds it all too much and saps away any initial energy and disposition to engage.

Carol Dweck (2000) is well known for her investigations of how background assumptions can establish adherences which have their own narrative (eg. "I resist the unfamiliar because I associate it with failure") but how inner incantations can be replaced and the unfamiliar embraced. She reports a lifetime of work developing ways to assist people to release themselves from habitual patterns based on perceiving failure as inbuilt rather than as happenstance (See also Neville, 1989).

12.3 Resisting, Accepting and Parking Challenges

Responses to challenge are many and various. They cover a spectrum from outright rejection to enthusiastic take-up. Observing learners respond to challenges set by their teachers, it may be useful to think broadly in terms of *resistance,* which extends from outright rejection through to grudging compliance, transmuting into *acceptance* which extends from grudging compliance through to enthusiastic take-up. At almost any stage there is the possibility of *deflecting* or *parking* the challenge, with intentions varying from long-term parking, amounting to rejection, to waiting for fresh ideas, further resources, or sufficient time to direct attention to it.

Resisting and accepting, deflecting and parking are only superficial descriptions by observers of learner behaviour. However, bearing in mind the proposal by Maturana (1988) that "everything said, is said by an observer", even self-report involves observation whose quality depends markedly on the presence and inner separation or objectivity of the witness. The claim here is that what is being observed is likely to be a coordination of various aspects of the learner psyche, and likely to contribute to the creation of adherence as the basis of a habit. These in turn activate one or other *systems*, whether S1, S1.5, S2 or S3.

For example, learners who, on being given a task, wait until the teacher comes round so they can ask for specific guidance on 'what are we supposed to do?' are in danger of developing a habit which will diminish, even stifle opportunities in the future. It is ever so easy for an initial resistance to develop into a reluctance, and then into a rejection. Their S1 or S1.5 triggers inaction, and rather than shift into S2, they remain inactive until someone tells them what to do. Unfortunately, teaching assistants are often all too ready to meet this demand. The tension between telling and prompting is captured by the notion of the *didactic tension* (Brousseau, 1984 p. 110; Mason & Johnston-Wilder, 2004 p. 82) which can be described as

> the more clearly and specifically the teacher indicates the behaviour expected from the learner, the easier it is for the learner to enact that behaviour without actually generating it for and from themselves, and so without the likelihood of internalising that action.

Bob Davis (1984) presented this to students as an ethical dilemma: would they rather be told, or be allowed to search for something for themselves?

As another example, there are many learners who, on being set a task, immediately enact the first action that becomes available. They may in retrospect account for it on the grounds of 'getting it over with as soon as possible' or as an outcome of their eagerness to learn or engage, but in either case, reacting immediately can become a habit, coordination of adherences which waist time and sometimes obscure access to a more fruitful approach.

12.3.1 Recognising Challenge

The first question is how the psycho-social system recognises challenge (as distinct from simply a task). Usually, there are somatic changes in pulse, breathing and perspiration, often arising from an increase in adrenalin, triggering emotions such as fight-flight or fear-fancy, with concomitant coordinated adherences in the rest of the psyche. Unfortunately, these coordinated adherences are often inappropriate and over-rated and may block other adherences from coming into play.

Somatic changes need not be interpreted in such drastic ways. Adrenalin flow can be perceived as stimulation and excitement, leading to a sharpening of the senses. I find that one situation in which I become aware of the mathematical challenge is when something disturbs my current adherence of enaction, affect and cognition, when something shifts or alters what I am attending to, or how I am attending to it, when my will power feels tested, when my inner witness signals that something is awry. Often it can be something quite simple but which becomes fodder to my propensity to try to generalise, to place a result in a wider context. Only then am I aware of feeling challenged. However, that 'feeling' is not simply cognitive or affective in nature. It comes from coordination of cross-linked habits between the various 'components' of my psyche which adhere to, and consequently both feed and limit each other.

For example, encountering the idea of constructing a decimal number by writing down the digits sequentially (so 0.12345678910111213141516 17…) immediately raises the question for me as to how to tell whether it is rational or irrational. What constitutes a convincing justification? What about other sequences? And that is only a starting point. Suppose only one new decimal place is allocated to each numeral so that there are carries to the left which may affect earlier decimal places (e.g. 0.1234567901234…), or perhaps two or three new decimal places are allocated; what if the number of allocated decimal places changes in some systematic fashion? What if some other sequence is used, such as triangular numbers or Fibonacci numbers?

Having an action become available is essential. For example, the action of presenting such strings in terms of powers of 10, followed by, in some cases, recognising a geometric series, provides a method of dealing with many of the questions posed above, and for many different sequences. I immediately want to start exploring, which is an imprecise way of saying that my attention shifted, actions became available and I recognised a desire to find out what is going on. Without any sense of exercising will, but rather of the will being dragged along, one of my 'explorer' adherences took over. This happened not once or twice but several times with the same idea on different occasions.

Furthermore, I notice (my inner witness notices and brings to cognition) a resonance with two questions posed by David Fowler (1985a, b):

> Guess the length of the period of the square of 0.001 001 001 … . Then and only then, work out the answer.
> Use a procedure for multiplying decimal numbers to calculate the first significant digit of 1.2222… × 0.818181…

The associated lesson is that arithmetic with infinite decimals can be tricky! Expressing repeating decimals as fractions may be necessary in order to be certain.

Mathematically, I also perceive myself to be challenged when there is some situation or assertion that I cannot readily explain or justify, yet which appeals to my affect by striking me as surprising or unexpected. This often happens with geometrical configurations. A good example for me arose by taking a convex quadrilateral and joining each vertex to the midpoint of the next-but-one edge taken clockwise. An inner quadrilateral is formed and in dynamic geometry software, it often appears to have an area of one-fifth of the area of the original quadrilateral (Mason & Zazkis, 2019). It turns out that this is due to rounding errors … but what in fact is the case? And what happens when midpoints are replaced by some other construction?

A sense of being challenged can also take the form of something which alters the way I perceive or attend to something, which again needs explaining or justifying. I particularly enjoy situations in which there are dual perceptions to be reconciled, for example thinking of chords of functions as made up of families in each of which the chords all have a fixed endpoint, or as families each of which consists of chords whose midpoints are all vertically aligned; finding tangents to a curve passing through a given point P in terms of a tangent at a particular point Q on the curve as Q runs along the curve, and as a line through P rotating to positions of tangency;

thinking of $\sqrt{17}$ as known by its properties (positive, square is 17) and as a real number with an essentially unknown infinite sequence of decimal digits; thinking of a straight line as an instance of a circle of infinite radius with centre at infinity, and so on.

12.3.2 Responding to Challenge

Although people are accustomed to believe that they 'choose' actions to enact, that choice is cognitive, this is a *User Illusion* (Norretranders, 1998). Close observations suggest that more often than not, some habit, some psycho-social adherence of coordination between action, affect, cognition, attention, will and even witness is what drives behaviour. Brief moments of true choice are glimpses of freedom.

While responses can have positive, negative and neutral influences, let us concentrate on positive responses. What lies behind different responses? In my experience, there is an immediate evaluation of the scope and potential of the task, not as a question to be carefully considered, but arising immediately. Does the challenge seem recognisable, and is some immediate action available? Does it appear to be attainable, or do I have confidence in the person posing it that it will be attainable, even if I do not immediately have a suitable mathematical action available? Does it appear to align with my current or past interests and successes? This is modified by the energy released, ranging from surprise or intrigue, through the desire to make sense which in turn is supported by my predisposition to tackle such challenges, to attempts to minimise the impact of the situation on my current well-being. So the pressure to perform, or in an examination situation, to perform quickly and efficiently, is a different kind of challenge to desire to resolve or comprehend some situation.

My immediate reaction then is either to take up the perceived challenge, to resist it by investing as little energy as possible in it, or even to reject it altogether. I also recognise that sometimes it is necessary to defer or park a challenge and that what was once a rejection can turn into parking because later it is actually taken up. This confirms Bill Brookes' observation (earlier) that challenge (having a problem) is about psycho-social states experienced by human beings in a particular situation, rather than any objective and universal quality. I have upon occasion rejected a problem posed by someone in one situation, and then later accepted it when posed by someone different in a different situation. A lot depends on my perception of, and social relation to, the situation.

Perception of the degree of challenge is necessarily idiosyncratic, depending as it does on a person's history, including the development of particular ways of responding to challenge and current state, and on the current situation as perceived, including who or what is posing the challenge. It is hardly a matter of cause-and-effect, more a matter of a soup of multiple forces, impulses and tendencies which play out differently despite only minor changes to the apparent situation (Mason, 2016).

Feedback of pleasure/endorphins arising from success, particularly unanticipated or striven-for success, can reinforce the disposition to engage in the future. Undertaking a challenge after a period of perceived failure is quite different to undertaking it during a period of perceived success: emotions are likely to be different, which may channel different flows of energy, thereby directing thoughts arising from attending to particular things in different ways. All of these interconnections tend, over time, to become habitual. A future stimulus may awaken or evoke thoughts, emotions or actions which bring a particular adherence into dominance in the way the person functions. An adherence may come to the fore for unexpected reasons, and afford access to associated actions with thought and emotion patterns, to ways of attending and to what, and to strength of will as to whether to persist, because of coordination of these aspects of the psyche.

12.3.2.1 Accepting

To accept a challenge there must be some sense of hope or possibility, whether based on a false sense of personal competence or on intensity of commitment. One important feature is trust in the source of the challenge, that the challenge is doable but not trivial, and worthwhile (Jackson, 2011; Mason, 2020). Something about the task has to appeal to the psyche, whether through emotions (surprise, intrigue), intellect (resonance with past experience), enaction (putative actions become available) or attention (perhaps a sense of generalisability). The appeal has to bring a coordinated set of adherence to the surface.

With some possible exceptions, the most alluring and persistent challenges are ones which I have set for myself. This even applies to challenges arriving from other sources, for it is only when my state is "in challenge" that I can truly be said to have taken it up. There is some sort of transformation, not always recognisable as such, which takes place so that an externally sourced challenge becomes a 'challenge for me'. The intensity with which it is taken up often waxes and wanes over time according not only to current feelings of (partial) success or progress but also according to exterior conditions of a psycho-social nature.

For example, in the 1970s the following problem circulated widely (Gardner, 1979; Klarner, 1981, pp. 285–307):

> There are four symmetrically placed (and so indistinguishable) doors in a circular table. Behind each door is a tumbler which is either up or down. If all the tumblers are in the same state, a bell rings. You may open any two doors and adjust the positions of those two tumblers, but the doors then close, and an unknown rotation takes place so you do not know which tumblers are beneath the doors you last opened. Can you make the bell ring?

Having eventually resolved it with ad hoc reasoning, I wanted to know what was going on structurally, so I posed myself the challenge of d doors, a symmetry group G acting on the doors so that which tumblers are behind which doors is not known, and allowing myself to use h hands (ie. to open h doors and make adjustments to any or all of these in a single move). My explorations revealed the structure of chains of

subgroups of G with indices bounded by h in order to be sure to be able to ring the bell.

There is a weaker form of accepting challenge which is more apparent than real and applies particularly to classrooms. Care must be taken about interpreting activity as acceptance of challenge: I may simply display the appearance of accepting a challenge, when in fact I am resigned or compliant to it out of perceived lack of choice. It is a task, not a challenge. Throughout history we are presented with examples in which forced acquiescence is mistakenly taken as agreement, only to feed resentment and negative disposition generally. Browbeating learners into acquiescing rather than engaging wholeheartedly may be one of the reasons why so many learners suddenly leave mathematics, even those who undertake undergraduate studies.

12.3.2.2 Rejecting

Putting a challenge aside immediately may at first be seen as a rejection, an act to conserve energy and not be diverted from more pressing tasks. This may arise from the inner witness asking questions and alerting both cognition and affect to a need to focus attention elsewhere. It may also arise from a habit of rejecting or blocking the unfamiliar, established coordination between affect, cognition, and enaction that may have become habitual.

There are far too many mathematical challenges to undertake them all. For example, although questions about phenomena in the material world often occur to me, I also know that my modelling skills are limited, so I usually simply note the situation but reject the challenge as such (Fig. 12.1).

Fig. 12.1 Swing scooter (copyright free image)

An instance of this is the swing scooter which my granddaughter uses effortlessly, the design of which intrigues me: how did people decide the optimal angle of separation, the optimal size of wheels, and the optimal length of the wheel extensions? Similar questions apply to designs of overhead cranes, skip-transporters and many other things which I encounter. I wonder how design choices are made so as to optimise the functioning of the apparatus.

Often I reject a challenge because I do not see immediately how to get started, what action to enact in order to begin. More specifically, it is usually necessary to have an action become available within my current threshold of resilience: the period of time within which I am likely to persist, which will vary between individuals, and for individuals at different times and in different conditions. Even the Pólya-based advice to specialise in order to comprehend underlying structural relationships may seem to require too much effort, if it even seems possible.

A notable counter-example for me is the problem I posed myself many years ago, arising from the following mathematical challenge:

> In how many different ways can a circle be cut into four congruent pieces?

The problem is attractive pedagogically because it offers an opportunity to work with learners on how the meaning of *different* changes as examples accumulate, and how care is needed to justify conjectures, especially when they become rather too optimistically general. I noticed that in the only examples I could construct, at least some of the pieces always have the centre of the circle on their boundary, even when for 12 and 24 not all the pieces have to have the centre on their boundary.

Out of this came the problem:

> Is it possible to divide a disk into congruent pieces so that the centre is not on the boundary of any piece?

Here ‘piece’ is taken to be simply connected, acting like a jigsaw piece rather than exercising topological concern about boundaries and interiors. I cannot even really see how to specialise, as changing the circle to polygons opens up different possibilities altogether with no sense of how these might inform the circle case. I made myself a jigsaw featuring twelve congruent pieces as depicted in the central figure of Fig. 12.2, which can be assembled so that only some of them have the centre on their boundary, but it has not helped me see a way forward. I have returned to this

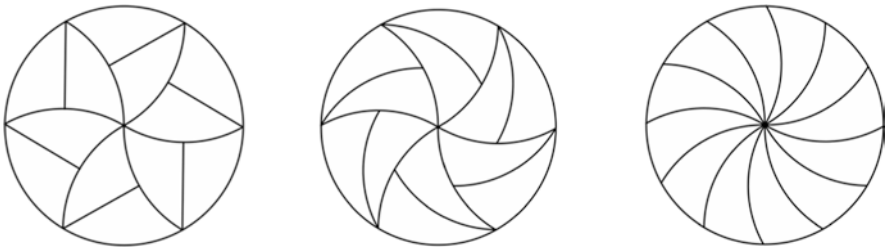

Fig. 12.2 Two variations for 12 congruent pieces; the second admits other variations such as the third

challenge several times, but, having tried combinatorial, geometric and function-analytic thinking at different times without success, I have been unable to find a way to make progress, and have again put it aside. I have not rejected it so much as deferred it, resisting for the time being and parking it for another time. So even though I have no actions to enact, I have not totally rejected it. This shows up something about the nature of resilience and persistence, but borders on obsession.

Another example for me is the following problem:

> Is it possible to glue congruent regular tetrahedra together so that they form a ring or torus (even if they cut through each other in 3-space)?

I posed this when I first struggled with simplicial complexes and chains in topology as an undergraduate. I returned to it several times when a fresh idea came to me, and eventually, some 7 years later, I managed to prove not only that it is impossible, but to extend my method to deal with other similar problems. There was something about the challenge that appealed to me, meaning that I persisted (Mason, 1972; see Elgersma and Wagon (2016) and Stewart (2019) for further developments). Here it was the recognition of a fresh action to try out, or of an action to retry with more persistence that kept me from rejecting it altogether.

Sometimes I am already working intensively on another challenge and can muster neither the energy nor the will to put the current one to one side, to shift my attention to the new challenge, especially if I cannot immediately see a way to get started. Again the word *obsession* comes to mind as a possibility, and at the time of writing, I am obsessed with a family of problems whose challenge I seem unable to put aside.

12.3.2.3 Resisting

In school and as an undergraduate and graduate student, I had to accept challenges presented to me in courses and examinations. I trusted the lecturer, aware that I had some very bright and accomplished colleagues, so I did not hope to succeed at everything. I have a vivid memory of a night spent trying to complete a take-home exam in Hilbert spaces in graduate school. As I became stuck on one of the problems, I would shift to another, returning to each again and again. I remember spending a good deal of time staring at the line between the wall and the ceiling of my study, waiting for inspiration. I trusted that the problems were within my capability, and I was desperate to do well in the course. So I persisted. In the end I completed them, only to be told by the lecturer when I handed them in that I need not have done more than one or two!

As I have become older, and slower, I find myself more able to resist challenges. For example, during corona-time conversations on-line we were posed a problem that I recognised, but had never really appreciated:

> Place four integers at the corners of a square. On the edges, record the differences in the adjacent numbers (the absolute values). Treat these as the corners of a square and continue the process. What happens?

Despite a slight resonance with arithmogons which I have exploited pedagogically (Mason & Houssart, 2000), I had (and still have) a strong sense that manipulating compositions of absolute values is not going to be attractive, and I have a vague memory of not enjoying what I found out when I last looked at it. So I resisted. I recognised a familiar coordination between affect (heaviness, concern), cognition (sense of other commitments, particularly to my current problem), enaction (initiating parking-type behaviour), attention (sustained to current commitments) and will (directing attention to current commitments). It turned out not to be a full rejection, because as it came close to time to report what we had noticed in the way of shifts of attention, I felt it necessary to have something to report. I looked for some actions that could be used to reduce the number of cases needing to be considered: take all the numbers to be non-negative; take at least one of the numbers to be 0. The point is that I tried to invest as little energy as possible, cutting down the number of examples I was willing to try in order to detect what was possible in the long run.

Often it is the case that response to a challenge is half-hearted, or an instance of being resigned to a challenge rather than actually taking it up wholeheartedly. This can even turn into a habit, summoning up familiar coordination based on a desire to invest only as little energy time and effort as possible, in the hope that that will be sufficient to get through the lesson. Closely involved of course is the implicit contract (*contrat didactique* see Brousseau, 1984) in which learners act as though their job is simply to attempt the tasks set by the teacher, and that somehow this will be sufficient to produce the learning that is expected of them. The teacher's side of the contract is to choose, set, and support work on the tasks so as to achieve this learning.

This form of the contract is of course rather inadequate and essentially vacuous, even for very cleverly chosen tasks. "One thing that we do not seem to learn from experience is that we do not often learn from experience alone. Something more is required" (Mason & Johnston-Wilder, 2004, p. 263). It is vital that learners do actually learn from their experience (Pólya, 1954 called it 'looking back'), which means, among other things, articulating a personal narrative about the topic. This involves recalling, and then imagining re-using actions that were effective, mathematical themes which emerged, and personal powers which were exercised. It would also include reviewing any relevant personal example space and its associated construction methods (Watson & Mason, 2005).

Learning mathematics is as much about developing a disposition to try some initial actions, if only to specialise in order to uncover underlying structural relationships, or to clarify what is being asked for and what other ideas or actions might possibly be relevant, as it is about mastering specific procedures in order to resolve routine questions. Personal narratives or *self-explanations* (Chi & Bassok, 1989) play a key role in learning from experience.

12.3.2.4 Deferring or Parking: Letting-Go, Hanging-on, and Pausing

While recalling my various experiences of mathematical challenge, I realised that one important, natural, and often necessary action worthy of being internalised is to park work. For example, the first action that becomes available is not always the most helpful, so parking that action before it is automatically enacted can avoid wasting time and energy. This applies whether it is a task, a reaction to a task or an action within a task. Not that this is easy to do. Often it is only after an initial action has been enacted but fails to result in progress, that real thinking takes place. There is a parallel with teaching, where it may only be on hearing a students' reply to my question that I realise I have asked a question with a prepared answer I wished to hear, placing me in danger of playing 'guess what is in my mind'.

Considering deeply, and allowing thinking to go on in the background provides access to an important part of the human psyche (S3) that may go undetected and unused in the constant push to 'cover topics' and 'reach solutions'. The following teaching story illustrates the point.

> A person was looking closely at the ground under a street lamp. Asked what they were doing, the reply was, "looking for my keys". When asked "Where did you lose them?", the reply was "over there, but it is brighter here".

Despite the absurdity of the story, most people have experienced persisting at something using the same available actions over and over (the light from street lamp) despite lack of progress. In the absence of any other action it is difficult not to keep trying an available action every so often in order to see if perchance it will now work, even though the difficulty lies elsewhere.

Any behaviour can become obsessive, which means persisting at carrying out available actions, coupled with an emotional state of desire uninformed or uninfluenced by cognition. Something in the will becomes stuck. Distinguishing between persistence and obsession is never easy, especially in oneself, as there can be a lingering hope that 'this time things will work out'. It is a state I recognise all too well in myself, and as such it is difficult to trap the coordinations between action, affect, cognition, attention and will. My witness observes, but is powerless to act! Andrew Wiles (2017) in interview observed that:

> You need a particular kind of personality that will struggle with things, will focus, won't give up. … we learn how to adapt to that struggle. Mathematicians struggle with mathematics even more than the general public does … We really struggle. It's hard. I am always quite encouraged when people say something like: 'You can't do it that way'.

More importantly, perhaps the real issue of challenge is recognising when progress is not being made, and learning to put a problem aside, at least for a time.

When emotional energy drains out and the will to continue begins to ebb, when the inner witness keeps asking "why are we doing this; isn't there something else we could be doing?" but without any suitable reply, when attention wanders and fails to concentrate, it may be time to let-go, either temporarily by parking, or by abandoning for the foreseeable future. These witness-questions are likely to emerge when no fresh actions are available. In the absence of suitable tools, it is wise to defer for a

while. Thus an important aspect of *challenge* is to recognise and acknowledge when more tools or more ideas are required, leading to parking the challenge at least until conditions change. Wiles (2017) refers to the 'three B's', namely "bus, bath and bed", pointing to the need to let the unconscious (S3) create new associations, access forms or 'senses-of', and open up new vistas.

Periods of letting-go or parking can afford access to S3, yielding insight and new possibilities (Hadamard, 1945). As a friend and colleague reported recently:

> … I'm still working on the question in odd moments. It's interesting how questions like this can be like a staircase, with stair-like times where you're following a direction, discerning details, recognising relationships, and then you reach a 'landing' where you get a sense of a whole, but the next flight of stairs feels too much for the moment. Simon Gregg. (personal communication June 2020)

Mathematicians know from experience that even if there is little prospect of picking up the challenge later, it is always wise to make a summary of what has been achieved, listing conjectures and notes about what evidence there might be for them. This is part of a personal narrative, and it makes it so much easier to pick up the challenge at a later date than is the case if the only record is a sheaf of scribbles. What seems curious is that this is such a good habit to form, such a powerful coordinated adherence to develop for learning from the experience of a challenge in order to facilitate actions in the future, that one might expect it to be a core focus in mathematics classrooms in every phase, yet this does not seem to be the case.

Another example of the appeal of action before considering it properly (parking S1 and activating S2) is the desire to turn to electronic support, whether to perform algebra correctly, to generate examples, or to look for invariance in the midst of change. The form and nature of thinking on a machine using computer algebra or dynamic geometry are quite different from sitting quietly and contemplating, or from thinking in the background while doing other things. The greater the intrigue, desire, sense of possibility and trust in self and source, the harder it is to resist the impulse to enact some mathematical action without further consideration.

12.3.2.5 Giving-Up

It may be a moot point whether a pause, perhaps intended to be brief, turns into abandoning altogether, or retains the challenge on a 'back burner' for subsequent consideration. I myself have a long list of problems that I have worked on for a time but have had to put aside for various reasons. Sometimes I am expecting it to be temporary, sometimes permanent. I had hoped to return to many during retirement, but there always seem to be fresh things to think about!

Giving-up is not always intentional. I have several times wanted to use Isaac Newton's algebra problem of the grazing cows as an example in some writing:

> Problem 11. If cattle a should eat up a meadow b in time c, and cattle d an equally fine meadow e in time f, and if the grass grows at a uniform rate, how many cattle will eat up a similar meadow g in time h? (Newton, 1707, in Whiteside, 1972, p. 147)

Each time I have puzzled over the modelling assumptions, and finally resorted to assuming that the grass grows uniformly over a week, that the cows graze uniformly over a week, and that what matters is that at the end of the week the cows have not grazed more than the grass has grown. Each time I have then pondered the question of how to parametrise the problem so as to guarantee integer solutions. Initial forays have not been successful, and in each case my attention has been drawn away to some other problem, leaving this one behind.

12.4 An Indication of Pedagogical Issues

How can the notion of four Systems and the notion of psycho-social coordinations of adherences contribute to setting and sustaining mathematical challenge in the classroom and beyond?

While it is beyond the scope of this paper to develop these ideas, it is worth noting that setting tasks for others which might be taken up as challenges is the first and relatively easy step, in which the aspect of trust plays a dominant role. But the real pedagogical challenge is how to respond to the ways in which the students respond to the tasks set. Learners display various psycho-social coordinations of adherences, and the real challenge is pedagogical: how to respond to learners responses; how to enable them to resist immediate strong but debilitating emotional reactions so as to use that energy positively and productively. The discourse of psycho-social coordinations of adherences applies to the teacher as well, bringing to the surface various pedagogically oriented habits. I conjecture that a significant factor in the activating of pedagogical actions concerns learner and teacher search for affirmation, from colleagues and from respected-others. How these interact with those of learners will influence the outcome.

12.5 Final Reflections

I can be challenged by something or some person, and I can feel challenged by something or some person, but I can also choose whether or not to accept that challenge. Such a choice might even take the form of appearing to accept a challenge but in fact resisting or rejecting it by ‘going through the motions’, displaying the behaviour I anticipate is being looked for.

Since it seems clear that challenge is not a quality of a mathematical task itself, but of the relationship between a person’s current state, the cultural and social milieu, and how the task is perceived at the moment, it is important to work at increasing sensitivity to the psycho-social coordinations experienced by learners, and to help them work against unhelpful adherences. This requires personal work on one’s own adherences, particularly in relation to propensities, dispositions, and pedagogical habits, in short, to one’s own mathematical being, so that one can be mathematical with and in front of learners (Mason, 2008).

References

Akopyan, A. (2011). *Geometry in figures*. Privately published.

Barrett, L. (2017). *How emotions are made: The secret life of the brain*. Houghton Mifflin Harcourt.

Bennett, J. (1943). *Values*. Coombe Springs Press.

Bennett, J. (1964). *Energies: material, vital, cosmic*. Coombe Springs Press.

Brookes, B. (1976). Philosophy and action in education: When is a problem? *ATM Supplement, 19*, 11–13.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. Steiner (Ed.), *Theory of mathematics education; paper 54* (pp. 110–119). Institut fur Didaktik der Mathematik der Universitat.

Brown, L., & Coles, A. (2000). Same/different: A 'natural' way of learning mathematics. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (2-153-2-160).

Chi, M., & Bassok, M. (1989). Learning from examples via self-explanation. In L. Resnick (Ed.), *Knowing, learning and instruction: Essays in honour of Robert Glaser*. Erlbaum.

Christiansen, B. & Walther, G. (1986). Task and Activity. In B. Christiansen, G. Howson & M. Otte. (Eds.) *Perspectives in Mathematics Education* (p.243–307). Reidel.

Csikszentmihalyi, M. (1997). *Finding flow: The psychology of engagement with everyday life*. Basic Books.

Davies, P., & Harré, R. (1990). Positioning: The discursive production of selves. *Journal for the Theory of Social Behaviour*, 43–63. https://doi.org/10.1111/j.1468-5914.1990.tb00174.x

Davis, R. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Ablex.

Dweck, C. (2000). *Self-theories: Their role in motivation, personality and development*. Psychology Press.

Eakin, P. (1999). *How our lives become stories: Making selves*. Cornell University Press.

Elgersma, M., & Wagon, S. (2016). The quadrahelix: A nearly perfect loop of tetrahedra. arXiv:1610.00280.

Fowler, D. (1985a). 400 years of decimal fractions. *Mathematics Teaching, 110*, 20–21.

Fowler, D. (1985b). 400.25 years of decimal fractions. *Mathematics Teaching, 111*, 30–31.

Gardner, M. (1979, February 16). *Mathematical games*. Scientific American.

Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning*. Routledge & Kegan Paul.

Gattegno, C. (1987). *The science of education part I: Theoretical considerations*. Educational Solutions.

Green, C. (n.d.). (accessed August 2020): psychclassics.yorku.ca/James/Principles/

Hadamard, J. (1945). *An essay on the psychology of invention in the mathematical field*. Princeton University Press.

Hanh, T. (1986). Being aware. *Fellowship, 53*(12), 21.

Hanson, K. (1986). *The self imagined: Philosophical reflections on the social character of psyche*. Routledge & Kegan Paul.

Hudson, L. (1968). *Frames of mind*. Methuen.

Jackson, Y. (2011). *The pedagogy of confidence: Inspiring high intellectual performance in urban schools*. Teachers College press.

James, W. (1890 reprinted 1950). *Principles of psychology* (Vol. 1). Dover.

Kahn, M. (1983 reprinted 1989). *Hidden selves: Between theory and practice in psychoanalysis*. Maresfield Library.

Kahneman, D. (2012). *Thinking fast, thinking slow*. Penguin.

Kahneman, D., & Frederick, S. (2002). Representativeness revisited: Attribute substitution in intuitive judgement. In T. Gilovich, D. Griffin, & D. Kahneman (Eds.), *Heuristics and biases: The psychology of intuitive judgment* (pp. 49–81). Cambridge Univ. Press.

Klarner, D. (Ed.). (1981). *The mathematical Gardner*. Prindle, Weber & Schmidt.

Lester, D. (2012). A multiple self theory of the mind. *Comprehensive Psychology, 1*(5), Doi 10.2466/02.09.28.Cp.1.5.
Mandler, G. (1989). Affect and learning: Causes and consequences of emotional interactions. In D. McLeod & V. Adams (Eds.), *Affect and mathematical problem solving: New perspectives* (pp. 3–19). Springer-Verlag.
Mason, J. (1972). Can regular Tetrahedra be glued together face to face to form a ring? *Mathematical Gazette, 56*(397), 194–197.
Mason, J. (1989). Does description = experience? A fundamental epistemological error with far-reaching consequences. *Cambridge Journal of Education, 19*(3), 311–321.
Mason, J. (2001). Teaching for flexibility in mathematics: Being aware of the structures of attention and intention. *Questiones Mathematicae, 24*(Suppl 1), 1–15.
Mason, J. (2008). Being mathematical with & in front of learners: Attention, awareness, and attitude as sources of differences between teacher educators, teachers & learners. In T. Wood (Series Ed.) & B. Jaworski (Vol. Ed.), *International handbook of mathematics teacher education: Vol. 4. The mathematics teacher educator as a developing professional* (pp. 31–56). Sense Publishers.
Mason, J. (2016). Rising above a cause-and-effect stance in mathematics education research. *JMTE, 19*(4), 297–300.
Mason, J. (2020). Generating worthwhile mathematical tasks in order to sustain and develop mathematical thinking. *Sustainability, 2*, 5727. https://doi.org/10.3390/su12145727
Mason, J., & Houssart, J. (2000). Arithmogons: A case study in locating the mathematics in tasks. *Primary Teaching Studies., 11*(2), 34–42.
Mason, J., & Johnston-Wilder, S. (2004). *Fundamental constructs in mathematics education*. RoutledgeFalmer.
Mason, J., & Metz, M. (2017). Digging beneath dual systems theory and the bicameral brain: Abductions about the human psyche from experience in mathematical problem solving. In U. Eligio (Ed.), *Understanding emotions in mathematical thinking and learning*. Elsevier/Academic Press.
Mason, J., & Zazkis, R. (2019). Fooled by rounding. *Digital Experiences in Mathematics Education, 5*, 252. https://doi.org/10.1007/s40751-019-00055-2
Maturana, H. (1988). Reality: The search for objectivity or the quest for a compelling argument. *The Irish Journal of Psychology, 9*(1), 25–82.
Minsky, M. (1975). A framework for representing knowledge. In P. Winston (Ed.), *The psychology of computer vision* (pp. 211–280). McGraw Hill.
Minsky, M. (1986). *The Society of Mind*. Simon and Schuster.
Nave, K., Deane, G., Miller, M., & Clark, A. (2020). Wilding the predictive brain. *WIREs Cognitive Science, 11*, e1542. https://doi.org/10.1002/wcs.1542
Neville, B. (1989). *Educating psyche: Emotion, imagination, and the unconscious in learning*. Collins Dove.
Newton, I. (1707). *Arithmetica Universalis* (Ed. William Whiston). Cambridge.
Norretranders, T. (1998). (J. Sydenham Trans.). *The user illusion: Cutting consciousness down to size*. Allen Lane.
Ouspensky, P. (1950). *In search of the miraculous: Fragments of an unknown teaching*. Routledge & Kegan Paul.
Pólya, G. (1954). *Induction and analogy in mathematics. Mathematics and plausible reasoning. Vol 1*. Princeton University Press.
Schoenfeld, A. (1985). *Mathematical problem solving*. Academic Press.
Skemp, R. (1979). *Intelligence, learning and action*. Wiley.
Stewart, I. (2019). Tetrahedral chains and a curious semigroup. *Extracta Mathematicae, 34*(1), 99–102. https://doi.org/10.17398/2605-5686.34.1.99
Varela, F. (1999). *Ethical know-how: Action, wisdom, and cognition*. Stanford University Press.

Vygotsky, L. (1978). *Mind in Society: The development of the higher psychological processes*. Harvard University Press.
Wason, P., & Evans, J. (1974). Dual processes in reasoning? *Cognition, 3*(2), 141–154.
Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Erlbaum.
Whiteside, D. (Ed.). (1972). *The mathematical papers of Isaac Newton Vol V 1683–1684*. Cambridge University Press.
Wiles, A. (2017). blog.sciencemuseum.org.uk/sir-andrew-wiles-on-the-struggle-beauty-rapture-of-mathematics
Yakel, U., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.
Zwicky, J. (2019). *The experience of meaning*. McGill-Queens University Press.