Chapter 1
Introduction to Mathematical Challenges for All Unraveling the Intricacy of Mathematical Challenge

Roza Leikin

1.1 Introduction

In mathematics education literature the concept of "challenging mathematics" or "mathematical challenge" does not frequently appear. In contrast to problem solving, problem posing, and proving, which all can be challenging for learners, mathematical challenge is not considered to be a core element of mathematical instruction. For example, Stacy and Turner (2015) mention only once "challenging mathematics situation that call for the activation of a particular competency" (Niss, 2015) and mention "mathematical challenge" three times when considering real-world context categories as a source of challenge (Stacy & Turner, 2015). In Li, Silver and Li (2014) and in Felmer et al. (2019) all instances of "mathematical challenge" are concentrated in chapters by Leikin (2014, 2019). Huang and Li (2017) and Hanna and De Villiers (2012) do not include this terminology. In Amado et al. (2018), Amado and Carreira address "challenging mathematics" in their chapter, in connection to affect and aesthetics in mathematics mainly related to extracurricular activities. Amado and Carreira (2018) discuss inclusive competitions aimed at all students, regardless of their school achievements, through which students deal with (mathematical) challenges.

The authors in this volume consider mathematical challenge essential for mathematical development and attempt to put it at the forefront of mathematics education discourse. The essence of mathematical challenge is its call for mental attempt appropriate to an individual or group of individuals in association with positive affect evoked in the process of tackling a problem or as a result of succeeding in solving it.

R. Leikin (✉)
Department of Mathematics Education, Faculty of Education, RANGE Center, University of Haifa, Haifa, Israel
e-mail: rozal@edu.haifa.ac.il

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The 16th ICMI Study focused explicitly on the concept of “challenging mathematics” (Barbeau & Taylor, 2009). Barbeau regards challenge as “a question posed deliberately to entice its recipient to attempt a resolution, while at the same time stretching their understanding and knowledge of the same topic” (Barbeau, 2009, p. 5). In the volume edited by Barbeau and Taylor, ICMI Study participants in eight groups discussed mathematical challenges in connection to various practices, problems, and tasks accompanied by examples of challenging mathematical problems. Freiman et al. (2009) addressed “challenging mathematics” as an expression that describes mathematical tasks that are “enjoyable but not easy to deal with” (p.103) and discussed the role of educational technologies for creating challenging mathematics beyond the classroom at the primary level. Holton et al. (2009) connected challenge with “what mathematics is” (p. 2005) and argued that this covers content that was historically developed for, and traditionally taught in, school, as well as the creative side that is connected, for example, to solving an open problem. Holton et al. (2009) stressed that a problem is a challenge with respect to the mathematical proficiency of solvers and emphasized the centrality of teachers’ proficiency in managing mathematical challenges. Stillman et al. (2009) discussed classroom practices and heuristic behaviors associated with challenging mathematics. Falk de Losada et al. (2009) described curriculums and assessments that provide challenge in mathematics, using examples from school exams in Singapore and in Norway, Brazilian Olympiads, and Iranian university entrance exams.

The goal of the current volume is to advance the centrality of mathematical challenge for the mathematical development of all students, and to provide research-based characterization and exemplification of different types of mathematical challenge. The book is composed of three interrelated sections: Part I: Mathematical challenges in curriculum and instructional design (edited by Demetra Pitta-Pantazi and Costantinos Christou); Part II: Kinds and variation of mathematically challenging tasks (edited by Rina Zazkis); and Part III: Collections of mathematical problems (edited by Alexander Karp). Twenty-nine chapters by researchers from universities from different continents present various views on mathematical challenges for all. All of the authors explore theoretically grounded ideas related to the effectiveness of mathematical instruction. Some chapters develop new theoretical perspectives on mathematical challenges, supported by empirical evidence, while in other chapters theoretical lenses from the theories of mathematics education are used for the analysis of mathematically challenging experiences. In this chapter, I suggest a theoretical framing of mathematical challenge and use this framework to connect between challenging mathematical tasks, challenging collections of mathematical tasks and curricular approaches to challenge-rich mathematical instruction.

1.2 Challenge as a Springboard to Human Development

All human development is related to overcoming difficulties and striving to progress on an individual and societal level. It can be motivated either externally by social norms, expectations, and environmental requirements or intrinsically by individual goals, curiosity, and desire to succeed. The same applies to mathematical development linked to different branches of mathematics as well as to different contexts, situations, and settings. Since any human activity is goal oriented (Leontiev, 1983), a disposition oriented to success in mathematics differs among different individuals and groups of individuals, depending on their goals.

The word "challenge" has multiple meanings, which are not necessarily associated with optimal experiences (https://www.merriam-webster.com/thesaurus). We choose to refer to challenge as an integral part of experiences that

- Require thought and skill for resolution (entry 1.2), or
- Demand proof of truth or rightness (entry 2.1), or
- Invite (someone) to take part in a contest or to perform a feat (entry 2.2).

According to Csikszentmihalyi and Csikzentmihaly (1990) (addressed in detail in Liljedahl, Chap. 28 in this volume), the development of a person is associated with *optimal experiences* of "stretching the limits" by "accomplishing something difficult and worthwhile" (p. 3). In this sense, overcoming a challenge is an optimal experience directed at learning. The process of overcoming a challenge is not necessarily pleasant, but once attained is associated with enjoyment and satisfaction. Mason (Chap. 12, in this volume) examines a combination of cognitive and affective conditions associated with mathematical challenge and the recognition of something as a challenge. According to Mason, positive affect can develop in tackling the challenge.

We apply these meanings to mathematical challenges that contribute to mathematics learning and development and ask the following questions:

- Do "optimal experiences" exist in mathematics learning?
- What makes a mathematical activity "optimal" for a student?
- Can mathematical activity be "optimal for all"?

As one of the possible answers this book suggests that mathematically challenging curricula, sets of tasks, and tasks make learning experiences optimal.

1.3 Mathematical Challenge

Mathematical education is aimed at the maximal development of the mathematical potential of each and every student. *Mathematical potential* is a function of the following:

- Cognitive (domain-specific (mathematical) and domain-general) abilities
- Affective characteristics associated with learning mathematics, including (but not limited to) motivation to learn mathematics and enjoyment from learning mathematics, which are mutually related
- Personality, which includes persistence, risk taking, teachability, and adaptability
- Learning opportunities from the past, present, and future.

Engagement with mathematical challenges is a core element of the learning opportunities that can lead to mathematical development. Mathematical challenge is a mathematical difficulty that an individual is able and willing to overcome (Leikin, 2009, 2014). The concept of mathematical challenge is rooted in Vygotsky's (1978) notion of zone of proximal development – what a student can do today with the help of an adult or a more proficient peer, tomorrow the student will be able to alone. In addition, Davydov's (1996) principles of developing education propose that learning tasks used to develop students' mathematical reasoning should not be too easy or too difficult. Per cognitive load theory (Sweller et al., 1998), intrinsic cognitive load is linked to the cognitive resources a person must activate in order to satisfy task demands, and germane cognitive load is linked to the cognitive resources needed for the learning of new schema. As such, mathematically challenging tasks are cognitively demanding (in the terms used by Silver & Mesa, 2011). At the same time, the concept of mathematical challenge goes beyond the cognitive demand of a task and acknowledges such affective aspects as willingness, curiosity, and motivation associated with being engaged with a task or with a set of tasks.

The connection between the cognitive and affective components of mathematical challenge is reflected in the concept of "flow," defined by Csikszentmihalyi and Csikzentmihaly (1990) as a function of the balance between a person's proficiency and the level of complexity of the task. Accordingly, flow stands in contrast to boredom or frustration, which occur when the level of challenge and that of problem-solving proficiency are unbalanced. According to Liljedahl (2018), flow is a necessary condition for the development of mathematical skills by means of raising the level of mathematical challenge. In contrast, I consider mathematical challenge to be a function of the suitability of the task's complexity to students' mathematical potential (including its cognitive and affective components). Correspondingly, a task is challenging only when it embeds a difficulty that is appropriate for an individual, and that individual has the motivation to take on the challenge.

The concept of mathematical challenge is an intricate concept within the educational terrain. Its intricacy is linked to multiple components that include:

- The notion of challenge and its relative nature (Csikszentmihalyi & Csikzentmihaly, 1990; Jaworski, 1992),
- The complexity of mathematics as a scientific field; hierarchy of mathematical concepts and principles (Barbeau & Taylor, 2009; https://undergroundmathematics.org/),

- Goals of mathematics education in general and of specific mathematical activities in particular (cf. Leont'ev, 1978),
- The varied characteristics of mathematical tasks (Goldin & McClintock, 1979; Kilpatrick, 1985; Silver & Zawodjewsky, 1997),
- Educational policy and subjective decisions about curricula and task design,
- The complexity of learners' mathematical potential (Leikin, 2009, 2019), and
- Teachers' professional potential in terms of monitoring mathematically challenging instruction (Jaworski, 1992; Leikin, 2019).

The intricate nature of mathematical challenge is obvious and is addressed in different chapters of this book. Figure 1.1 depicts components that influence the mathematical challenge embedded in a task. Note that there are multiple interpretations of the terms "mathematical tasks" and "problems." Some researchers consider a mathematical problem to be a task that requires the individual, or group of individuals, to invest effort while solving it. On the other hand, a problem can also be defined as a question that requires an attempt to find an answer. In this case, a task is a problem accompanied by a requirement to do something about that problem or situation. Most of the chapters in this volume use the former interpretation.

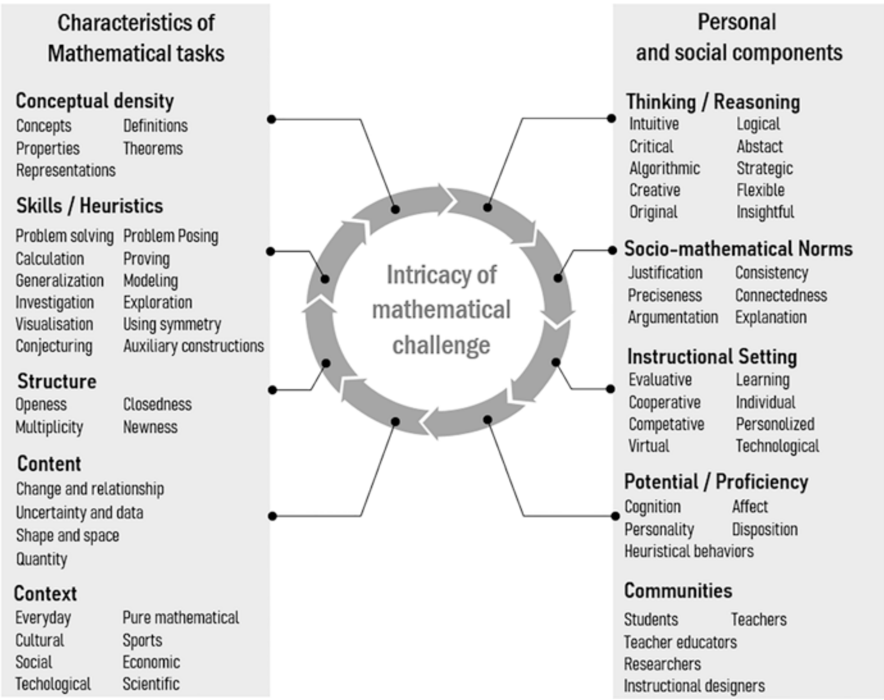

Fig. 1.1 Model of factors influencing mathematical challenge

1.4 Model of Factors Influencing Mathematical Challenge

Analysis of mathematical challenge is usually associated with task complexity. This, in turn, can be examined theoretically according to predetermined task variables, or empirically according to the success of groups of participants in engaging with the task. Lester (1994) performed a meta-analysis of mathematical problem-solving research published from 1970 to 1994. He found that during the period of 1970–1982, problem-solving research was directed at “isolation of key determinants of problem difficulty” along with “identification of successful problem solvers and heuristic training” (p. 664). Between 1982 and1990, there was a shift in researchers’ attention to metacognition and training for metacognition, and affect related to problem solving. From 1990 to 1994, social influence and problem solving in context appeared to be the focus of problem-solving research:

> In 1979 a landmark book was published that synthesized the research on what were then referred to as "task variables" in mathematical problem solving (Goldin & McClintock, 1979). To briefly summarize this and other closely related research, four classes of variables were identified that contribute to problem difficulty: content and context variables, structure variables, syntax variables, and heuristic behavior variables. Initially, these classes were studied via linear regression models, later via information-processing techniques. This line of inquiry was replaced eventually by investigations of the interaction between task variables and the characteristics of the problem solver (Kilpatrick, 1985). (Lester, 1994, p. 664).

All of the topics studied from the 1970s through the 1990s are relevant to contemporary problem-solving research, with clear emphasis on the interaction between different variables and attempts to understand how problem solving can be monitored, and how it can provide mathematical challenge for all. The current volume nicely reflects recent advances in research on problem solving and problem solving in mathematical instruction and assessment.

Over the past three decades, mathematics education has undergone significant changes, with an emphasis on mathematical competencies (e.g., NAEP, TIMMS studies – Carpenter et al., 1983; Hiebert & Stigler, 2000), mathematical understanding (e.g., BUSUN and Balanced Assessment projects – Silver & Zawodjewsky, 1997, Burke, 2010), contextualization and mathematical literacy (PISA studies – Cobb & Couch, 2022, Stacy & Turner, 2015), and mathematics creativity (OECD, 2021). These changes are largely determined by technological and scientific progress, which has led to an emphasis on twenty-first century skills and deep learning (Pellegrino & Hilton, 2012), and by societal changes as expressed, for example, in the Sustainable Development Goals (https://sdgs.un.org/goals), such as quality education (SDG-4) and reduced inequalities (SDG-10). The latter leads to the requirement of sustainable learning and to the understanding of the importance of students’ literacy in and beyond mathematics. Additionally, it directs us to see tools for attaining SDGs globally in education in general and mathematics education in particular. This volume, which is centered on mathematical challenge for all, clearly demonstrates that challenge-rich mathematical instruction strives for quality education, educational equity, and a “no child left behind” ideology.

As mentioned above, mathematical challenge is relative to a person's mathematical potential. As agreed in multiple chapters in this book, the effectiveness of mathematically challenging instruction is a function of teachers' proficiency in conveying mathematical activity suited to each student's potential. Taking into account the heterogeneity of a mathematical classroom, this seems to be almost a fantasy. However, the authors in this volume provide creative ideas which can make mathematically challenging tasks and collections of tasks accessible to all students.

Figure 1.1 demonstrates the major factors influencing mathematical challenge. It is an elaborated version of the model of mathematical challenge suggested in Leikin (2018) that included three characteristics: Conceptual characteristics of mathematical tasks, of which conceptual density and openness were at the center of the discussion, the setting, including use of digital technologies and learning/teaching methods, and socio-mathematical norms. This elaborated model reflects the intricacy of mathematical challenge as connected to multiple faces of mathematical instruction and is inspired by my long-term discussion with Avi Berman and Dina Tirosh (Sept 2021–May 2022) of the structure and nature of "advanced mathematical thinking in school." Below I describe and exemplify the main component of this model in connection to different chapters in this book.

In what follows, I explain the model of factors influencing mathematical challenge using examples from different chapters in this book. Additional theoretical framing of mathematical challenge in curricula design, tasks, and collections of tasks can be found in the introductory chapters by the section editors (Demetra Pitta-Pantazi and Constantinos Christou (Chap. 2), Rina Zazkis (Chap. 11) and Alexander Karp (Chap. 20) and in commentary chapters by Jeremy Hodgen (Chap. 10), David Pimm (Chap. 19). Alan Schoenfeld (Chap. 29) contributed retrospective analysis of problems, problem solving, and thinking mathematically).

1.5 Explaining the Model Using Works Presented in This Volume

It seems almost trivial that mathematical challenge is determined by the *mathematical content* taught. A fine illustration of different levels of complexity of mathematical content can be seen in underground mathematics concept maps that choose (https://undergroundmathematics.org/) numbers, algebra, geometry, functions, and calculus as major lines along which the concepts and mathematical principles are developed and become increasingly more complex at the unions of different lines. PISA (OECD, 2021) suggest slightly different lines of mathematical content: Quantity, shape and space, change and relationship, uncertainty, and data (which are depicted in Fig. 1.1). Independent of terminology used and independent of design principles, development of mathematical content in any mathematical curriculum is hierarchical with an increasing level of complexity of mathematical concepts, their properties, and mathematical skills and presumed heuristic behaviors. Verschaffel et al. (Chap.

3) explore teaching and learning of quantity domain with a special focus on mathematical patterns and structures, computational estimations, proportional reasoning, and probabilistic reasoning. They justify the importance of these topics using previous cognitive and neurocognitive research as well as the longitudinal study they conducted for their project. The chapter stresses the importance of early acquired numerical abilities – much earlier than have been traditionally studied. The study of Wasserman (Chap. 13) focuses on challenges associated with binary operations and the links between university and school *mathematical content*.

The underground mathematics maps mentioned above demonstrate raising *conceptual density* across the different topics and branches of mathematics. At the same time, mathematical tasks within the topics may differ in their conceptual density, determined by both complexity of the concept included in the task as well as the need for using different concepts and different rules and theorems (cf. Silver & Zawodjewsky, 1997) concurrently. Leikin and Elgrably (Chap. 27) demonstrate systematic bottom-up variation in conceptual density of geometry problems through construction of chains of problems posed by participants. Top-down structuring of mathematical challenge is illustrated in this chapter using stepped tasks. Verschaffel et al. (Chap. 3) also present examples of tasks of different levels of conceptual density and explain variations in the challenge embedded in related tasks.

Complexity of mathematical tasks is a function of the *skills* required from solvers. The relationships between skills, such as proving, problem solving, problem-posing, modeling, and generalization, are not necessarily hierarchical and depend on other variables included in the model of mathematical challenge (Fig. 1.1). However, while problem posing precedes solving the posed problems, the level of complexity of mathematical activity increases. Cai and Hwang (Chap. 7) present a rich collection of problem-posing tasks framed by a theoretical analysis of the types of problem-posing tasks and accompanied by analysis of the challenges embedded in problem-posing activities. Similarly, modeling tasks, in which participants must develop a mathematical model of a *contextual situation*, are more complex than problems in which an identical mathematical model is introduced to students along with the problem conditions. Applebaum and Zazkis (Chap. 14) describe how simple computational tasks can be transformed into challenging tasks by requiring a generalization process. Solutions by different groups of participants – mathematicians, teachers, and students – are discussed to illustrate how mathematical proficiency affects mathematical performance.

Goos et al. (Chap. 4) apply "the curriculum policy, design and enactment system" (with reference to Remillard & Heck, 2014) in their discussion of mathematical curriculum "enhanced" by modeling activities. They introduce contextual opportunities and constraints embedded in the implementation of modeling activities on a systematic basis. Borromeo Ferri et al. (Chap. 22) present a collection of modeling tasks to demonstrate the developmental power of these tasks. The complexity of the modeling activity is described in terms of a mathematical modeling cycle (borrowed from Kaiser & Stender, 2013). They demonstrate how modeling activities are inherently integrated in differentiated *instructional settings* based on self-differentiation of the learning process. Note here that Borromeo Ferri et al. clearly demonstrate the *openness* of mathematical modeling tasks expressed in a variety of student-generated models.

Vale and Barbosa (Chap. 15) provide examples of different types of visual problems and draw connections between mathematical challenges and *visualization skills*. They show that visual thinking is a tool that helps indicate the level of mathematical challenge embedded in the task, and can be applied to multiple solution tasks (considered also in Leikin & Guberman Chap. 17) to develop creative thinking. Interestingly, the principles described by Vale and Barbosa can be applied to multiple problems included in different chapters in the book. Using symmetry in solving problems, which is analyzed by Vale and Barbosa, is also one of the heuristics described by Polya (1973/45). Symmetry of geometric diagrams is implemented in the design of mathematical problems of varying levels of mathematical challenge in Waisman et al. (Chap. 26). Empirically, they demonstrate that when solving equivalent problems, a symmetrical diagram decreases the complexity of a problem. In addition Waisman et al. address field dependency of geometry diagrams and Hsu (Chap. 23) analyzes complexity of geometry diagrams as a meaningful variable of solving geometry problems. I invite readers to solve Problem 18 presented in Marushina (Chap. 25) both visually and using symmetry. The solution is elegant and enjoyable.

Personalized mathematics and mathematical inquiry is introduced in Chap. 5 by Christo et al. through a precise analysis of tasks that require and further develop exploration and investigation skills, and an accurate distinction between these *skills* is integrated in a technology supported *instructional setting* with applets designed to support students' explorations and investigations. The ideas presented in this chapter are illustrated using specific tasks that are mathematically challenging for all students. *Contextualization* of the mathematical content, inquiry that leads to curious experiences, and the connection of mathematical fluency with mathematical understanding create an activity challenging for all because of its cognitive complexity, social involvement, and positive affect.

The readers of this volume can learn about different levels of openness of exploration and investigation tasks in Christo et al., in the modeling tasks in Goos et al. and in Borromeo Ferri et al. The openness is related both to the multiplicity of ways in which students can approach the tasks and in the solution outcomes attained by different students. Such openness emphasizes the discursive nature of a mathematics lesson, with *socio-mathematical norms* of justification, consistency, and explanation. The requirement of explaining or justifying each idea presented in the classroom contributes to the creation of challenging-for-all mathematical activities. Leikin et al. (Chap. 6) stress the importance of the integration of open tasks in mathematical instruction. They present examples of tasks that are recommended in a computer-based *instructional setting* since exploratory applets are designed to support students' understanding of the *task structure*. Positive affect is related to multiple tasks outcomes, which are usually surprising and develop curiosity linked to the completeness of the solution spaces. Attaining a complete set of solution outcomes or production of multiple solution strategies (which become a norm in Math-Key classrooms) allows variations in the level of mathematical challenge as connected to the mathematical potential of different students.

Each of the chapters in the book describes different types of *instructional settings* associated with mathematically challenging activities. Common to all the settings are *norms* of preciseness and justification as well as mathematical discussion, whether in small groups or as a whole class. Sinclair and Ferrara (Chap. 16) suggest a *socio-material framing* of mathematically challenging tasks based on Leikin's (2014) concept of mathematical challenge. While solving challenging tasks with technological tools the first grade students make progress in solving problems through interaction with the environment. They are motivated as a result of finding a solution and the socio-material system is reactive to students' progress. This variation in mathematical challenge that ensures challenges-for-all is rooted in students' interactions in working groups and with technological tools. In this learning environment the students experience moments of insight related to knowledge advancement. All the chapters in this book either implicitly or explicitly address the development of students' creativity through engagement with mathematical challenges. The development of creativity is linked to the openness of the tasks, multiplicity of solution strategies and solution outcomes, and students' mathematical learning through engagement with new tasks. Leikin and Guberman (Chap. 17) analyze the relationship between mathematical challenge and insight-based tasks and make a distinction between insight-requiring and insight-allowing mathematical tasks to discuss different levels of mathematical challenge embedded in insight-based tasks.

The level of challenge is related to *mathematical reasoning*, which includes conjecturing, generalization, and justification. Da Ponte et al. (Chap. 8) focus their study on an exploratory approach to developing students' mathematical reasoning. They discuss tasks and a learning environment that allows students to develop new knowledge through conjecturing, generalization, and justification of findings. The tasks in this study allow a variety of solution strategies. Special skills and beliefs are required from the teachers when monitoring exploratory mathematical instruction. Lloyd and Murphy present conceptualization of argumentative practices and connected features of mathematical reasoning. In their study they discuss requirements for teachers' knowledge and skills essential for conducting scaffolding moves, while developing critical-analytic thinking in their students. Liljedahl (Chap. 28) discusses the thinking classroom and classifies instructional settings based on collections of problems with varying levels of conceptual density. In Chap. 8, Lloyd and Murphy, Wasserman (Chap. 13), and Applebaum and Zazkis (Chap. 14) analyze the development of teachers' proficiency in monitoring challenging mathematical tasks. In addition to the discussion of mathematically challenging activities in mathematics teacher education, Biza and Nardi (Chap. 18) describe how they use mathematics education research in the education of undergraduate students. They introduce design principles using Math Tasks within learning situations emerging in the mathematics classroom. They stress the importance of teachers' awareness of the interaction between mathematical challenge and pedagogical challenge.

The characteristics of challenging mathematical tasks and curricular design observed above can be applied to *collections of problem*s as well (Fig. 1.1). Bass (Chap. 21) introduces five principles of deliberate production of collections of problems to readers. The principles are related to curricular principles, task structure, mathematical content, task models, and task outcomes. In addition, Karp (Chap. 20) considers "the *morphology* of problem sets – the role of each problem within a set, its position in it, and the mental processes that take place during the transition from one problem to another" (Karp, 2002). For example, in Chap. 7, Cai and Hwang present characterization of problem-posing activities and include collections of problem-posing tasks that exemplify these characterizations. The collections of problems are differentiated depending on whether they are designed by mathematicians, mathematics educators, teachers, or researchers and can be differentiated depending on their goals and morphological structures. For example, Karp (Chap. 24) analyzes collections of problems in mathematical textbooks and Marushina (Chap. 25) analyzes sets of exam problems – all created by instructional designers who are professional mathematicians. While in Waisman et al. the collections of problems are designed by researchers, Hsu examines complexity of problems designed by mathematics teachers. In addition, there are culturally dependent and policy-related characteristics of the sets of problems. Analyses of school textbooks in the United States (Karp, Chap. 24), of problem sets in exams in Russia (Marushina, Chap. 25), and of collections of problems generated by Taiwanese teachers are examples of culturally dependent collections that also reflect decisions related to educational policies in different countries at different periods of time. Both in Marushina and Karp and in Liljedahl (Chap. 28), the collections of problems include variations based on the conceptual density of the tasks included in the collections of problems borrowed from the education documents (in Karp and Marushina) or created by the author.

Collections of mathematical problems of varying levels of mathematical challenge, theoretically justified and connected to different mathematical and cognitive skills, can be found in Krutetskii (1976) in his seminal research on characterization of higher mathematical abilities. Collections of mathematical problems can be created for mathematical textbooks, evaluation tools, Olympiad problem collections, and sets of problems for particular instructional activities. The construction of the set of problems in Wiseman et al. (Chap. 26) is based on integration of psychological domain-specific characteristics (symmetry and field dependency) with domain-specific geometry properties.

1.6 Concluding Notes and Questions for Future Research

This volume "Mathematical Challenges for All" considers mathematical challenge to be an "optimal experience" in mathematics education (cf. Csikszentmihalyi & Csikzentmihaly, 1990). Optimal experiences in mathematics education are directed to the realization of the mathematical potential of each and every student.

As presented in the book chapters, mathematical challenge integrates the following:

- Cognitive demand determined by the characteristics of mathematical activity, including:
  - Characteristics of tasks that include conceptual density, contextual framing, and task structure with an emphasis on openness
  - Required domain of general and mathematical skills and associated mathematical thinking, reasoning, and argumentative practices
  - Setting in which the participants are involved in the mathematical activity including competitive and cooperative elements (coopetition), personalization and technological affordance of the activity
  - Socio-mathematical norms of preciseness, justification, and explanation.
- Affective components evoked by the mathematical activity including motivation to overcome the difficulty, frustration or curiosity caused by the task's complexity, and enjoyment from the process or outcome of engagement with the task.

Figure 1.1 suggests an elaborated view of the factors that influence the intricacy of a mathematical challenge.

The volume proposes multiple *ways in which mathematically challenging activities become "optimal for all."* The characteristics of mathematical activities emphasized by the majority of the authors in this volume include but are not limited to the following:

- The novelty of mathematical content or context or types of tasks for learners. Novelty develops motivation to learn mathematics and evokes mathematical curiosity.
- Variations in the level of mathematical challenge by means of collections of tasks of different levels of challenge. This includes problem sets, problem chains, and stepped tasks designed by instructional designers or teachers.
- Openness of tasks and tasks of an explorative nature that allows self-regulation of the level of mathematical challenge by students. This includes engagement with mathematical investigations, problem posing, mathematical modeling, multiple solution-strategy tasks or multiple solution-outcome tasks.
- Use of digital technologies that support self-regulation of mathematical challenge by students, development of new knowledge through mathematical experiences, and encouragement of interpersonal interactions. Explorative dynamic applets are among the recommended technological tools.
- Discursive and argumentative practices that support variations in levels of mathematical challenge to fit the mathematical potential of each and every student, including norms of asking hypothetical and elaborative rather than verification questions, and norms of preciseness and of justification of all mathematical conjectures raised.
- Scaffolding practices with no funneling that "reduces the challenge" or "closes the openness." Such practices require a high level of proficiency of teachers,

deep mathematical knowledge, and belief in the centrality of mathematical challenge for high quality mathematical instruction for all students.

- Teacher training directed at advancing teachers' professional potential that combines their mathematical and pedagogical knowledge and skills with the ability to identify students' mathematical potential. Developing teachers' proficiency in navigating mathematically challenging activities includes giving them experience in varying the level of mathematical challenge in accordance with students' mathematical potential and with enhancing students' enjoyment from doing mathematics and from successful task completion, and will allow teachers to enjoy their own pedagogical challenges.

This book has the potential to be useful for a broad range of mathematics educators, educational researchers, and mathematicians who work with mathematics teachers and instructional designers. I am certain that readers will be able to learn from the variety of theoretical, practical, and methodological ideas that the authors present. More research about mathematical challenge for all is required, and this book opens new venues of research in mathematics education. Finally, as noted above, the volume includes a wonderful collection of mathematical challenges at different levels that readers are invited to explore and enjoy.

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