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Mathematical Challenges For All

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Research in Mathematics Education

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Roza Leikin
Editor

Mathematical Challenges For All

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Chapter 1
Introduction to Mathematical Challenges for All Unraveling the Intricacy of Mathematical Challenge

Roza Leikin

1.1 Introduction

In mathematics education literature the concept of "challenging mathematics" or "mathematical challenge" does not frequently appear. In contrast to problem solving, problem posing, and proving, which all can be challenging for learners, mathematical challenge is not considered to be a core element of mathematical instruction. For example, Stacy and Turner (2015) mention only once "challenging mathematics situation that call for the activation of a particular competency" (Niss, 2015) and mention "mathematical challenge" three times when considering real-world context categories as a source of challenge (Stacy & Turner, 2015). In Li, Silver and Li (2014) and in Felmer et al. (2019) all instances of "mathematical challenge" are concentrated in chapters by Leikin (2014, 2019). Huang and Li (2017) and Hanna and De Villiers (2012) do not include this terminology. In Amado et al. (2018), Amado and Carreira address "challenging mathematics" in their chapter, in connection to affect and aesthetics in mathematics mainly related to extracurricular activities. Amado and Carreira (2018) discuss inclusive competitions aimed at all students, regardless of their school achievements, through which students deal with (mathematical) challenges.

The authors in this volume consider mathematical challenge essential for mathematical development and attempt to put it at the forefront of mathematics education discourse. The essence of mathematical challenge is its call for mental attempt appropriate to an individual or group of individuals in association with positive affect evoked in the process of tackling a problem or as a result of succeeding in solving it.

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The 16th ICMI Study focused explicitly on the concept of "challenging mathematics" (Barbeau & Taylor, 2009). Barbeau regards challenge as "a question posed deliberately to entice its recipient to attempt a resolution, while at the same time stretching their understanding and knowledge of the same topic" (Barbeau, 2009, p. 5). In the volume edited by Barbeau and Taylor, ICMI Study participants in eight groups discussed mathematical challenges in connection to various practices, problems, and tasks accompanied by examples of challenging mathematical problems. Freiman et al. (2009) addressed "challenging mathematics" as an expression that describes mathematical tasks that are "enjoyable but not easy to deal with" (p.103) and discussed the role of educational technologies for creating challenging mathematics beyond the classroom at the primary level. Holton et al. (2009) connected challenge with "what mathematics is" (p. 2005) and argued that this covers content that was historically developed for, and traditionally taught in, school, as well as the creative side that is connected, for example, to solving an open problem. Holton et al. (2009) stressed that a problem is a challenge with respect to the mathematical proficiency of solvers and emphasized the centrality of teachers' proficiency in managing mathematical challenges. Stillman et al. (2009) discussed classroom practices and heuristic behaviors associated with challenging mathematics. Falk de Losada et al. (2009) described curriculums and assessments that provide challenge in mathematics, using examples from school exams in Singapore and in Norway, Brazilian Olympiads, and Iranian university entrance exams.

The goal of the current volume is to advance the centrality of mathematical challenge for the mathematical development of all students, and to provide research-based characterization and exemplification of different types of mathematical challenge. The book is composed of three interrelated sections: Part I: Mathematical challenges in curriculum and instructional design (edited by Demetra Pitta-Pantazi and Costantinos Christou); Part II: Kinds and variation of mathematically challenging tasks (edited by Rina Zazkis); and Part III: Collections of mathematical problems (edited by Alexander Karp). Twenty-nine chapters by researchers from universities from different continents present various views on mathematical challenges for all. All of the authors explore theoretically grounded ideas related to the effectiveness of mathematical instruction. Some chapters develop new theoretical perspectives on mathematical challenges, supported by empirical evidence, while in other chapters theoretical lenses from the theories of mathematics education are used for the analysis of mathematically challenging experiences. In this chapter, I suggest a theoretical framing of mathematical challenge and use this framework to connect between challenging mathematical tasks, challenging collections of mathematical tasks and curricular approaches to challenge-rich mathematical instruction.

1.2 Challenge as a Springboard to Human Development

All human development is related to overcoming difficulties and striving to progress on an individual and societal level. It can be motivated either externally by social norms, expectations, and environmental requirements or intrinsically by individual goals, curiosity, and desire to succeed. The same applies to mathematical development linked to different branches of mathematics as well as to different contexts, situations, and settings. Since any human activity is goal oriented (Leontiev, 1983), a disposition oriented to success in mathematics differs among different individuals and groups of individuals, depending on their goals.

The word "challenge" has multiple meanings, which are not necessarily associated with optimal experiences (https://www.merriam-webster.com/thesaurus). We choose to refer to challenge as an integral part of experiences that

- Require thought and skill for resolution (entry 1.2), or
- Demand proof of truth or rightness (entry 2.1), or
- Invite (someone) to take part in a contest or to perform a feat (entry 2.2).

According to Csikszentmihalyi and Csikzentmihaly (1990) (addressed in detail in Liljedahl, Chap. 28 in this volume), the development of a person is associated with *optimal experiences* of "stretching the limits" by "accomplishing something difficult and worthwhile" (p. 3). In this sense, overcoming a challenge is an optimal experience directed at learning. The process of overcoming a challenge is not necessarily pleasant, but once attained is associated with enjoyment and satisfaction. Mason (Chap. 12, in this volume) examines a combination of cognitive and affective conditions associated with mathematical challenge and the recognition of something as a challenge. According to Mason, positive affect can develop in tackling the challenge.

We apply these meanings to mathematical challenges that contribute to mathematics learning and development and ask the following questions:

- Do "optimal experiences" exist in mathematics learning?
- What makes a mathematical activity "optimal" for a student?
- Can mathematical activity be "optimal for all"?

As one of the possible answers this book suggests that mathematically challenging curricula, sets of tasks, and tasks make learning experiences optimal.

1.3 Mathematical Challenge

Mathematical education is aimed at the maximal development of the mathematical potential of each and every student. *Mathematical potential* is a function of the following:

- Cognitive (domain-specific (mathematical) and domain-general) abilities
- Affective characteristics associated with learning mathematics, including (but not limited to) motivation to learn mathematics and enjoyment from learning mathematics, which are mutually related
- Personality, which includes persistence, risk taking, teachability, and adaptability
- Learning opportunities from the past, present, and future.

Engagement with mathematical challenges is a core element of the learning opportunities that can lead to mathematical development. Mathematical challenge is a mathematical difficulty that an individual is able and willing to overcome (Leikin, 2009, 2014). The concept of mathematical challenge is rooted in Vygotsky's (1978) notion of zone of proximal development – what a student can do today with the help of an adult or a more proficient peer, tomorrow the student will be able to alone. In addition, Davydov's (1996) principles of developing education propose that learning tasks used to develop students' mathematical reasoning should not be too easy or too difficult. Per cognitive load theory (Sweller et al., 1998), intrinsic cognitive load is linked to the cognitive resources a person must activate in order to satisfy task demands, and germane cognitive load is linked to the cognitive resources needed for the learning of new schema. As such, mathematically challenging tasks are cognitively demanding (in the terms used by Silver & Mesa, 2011). At the same time, the concept of mathematical challenge goes beyond the cognitive demand of a task and acknowledges such affective aspects as willingness, curiosity, and motivation associated with being engaged with a task or with a set of tasks.

The connection between the cognitive and affective components of mathematical challenge is reflected in the concept of "flow," defined by Csikszentmihalyi and Csikzentmihaly (1990) as a function of the balance between a person's proficiency and the level of complexity of the task. Accordingly, flow stands in contrast to boredom or frustration, which occur when the level of challenge and that of problem-solving proficiency are unbalanced. According to Liljedahl (2018), flow is a necessary condition for the development of mathematical skills by means of raising the level of mathematical challenge. In contrast, I consider mathematical challenge to be a function of the suitability of the task's complexity to students' mathematical potential (including its cognitive and affective components). Correspondingly, a task is challenging only when it embeds a difficulty that is appropriate for an individual, and that individual has the motivation to take on the challenge.

The concept of mathematical challenge is an intricate concept within the educational terrain. Its intricacy is linked to multiple components that include:

- The notion of challenge and its relative nature (Csikszentmihalyi & Csikzentmihaly, 1990; Jaworski, 1992),
- The complexity of mathematics as a scientific field; hierarchy of mathematical concepts and principles (Barbeau & Taylor, 2009; https://undergroundmathematics.org/),

- Goals of mathematics education in general and of specific mathematical activities in particular (cf. Leont'ev, 1978),
- The varied characteristics of mathematical tasks (Goldin & McClintock, 1979; Kilpatrick, 1985; Silver & Zawodjewsky, 1997),
- Educational policy and subjective decisions about curricula and task design,
- The complexity of learners' mathematical potential (Leikin, 2009, 2019), and
- Teachers' professional potential in terms of monitoring mathematically challenging instruction (Jaworski, 1992; Leikin, 2019).

The intricate nature of mathematical challenge is obvious and is addressed in different chapters of this book. Figure 1.1 depicts components that influence the mathematical challenge embedded in a task. Note that there are multiple interpretations of the terms "mathematical tasks" and "problems." Some researchers consider a mathematical problem to be a task that requires the individual, or group of individuals, to invest effort while solving it. On the other hand, a problem can also be defined as a question that requires an attempt to find an answer. In this case, a task is a problem accompanied by a requirement to do something about that problem or situation. Most of the chapters in this volume use the former interpretation.

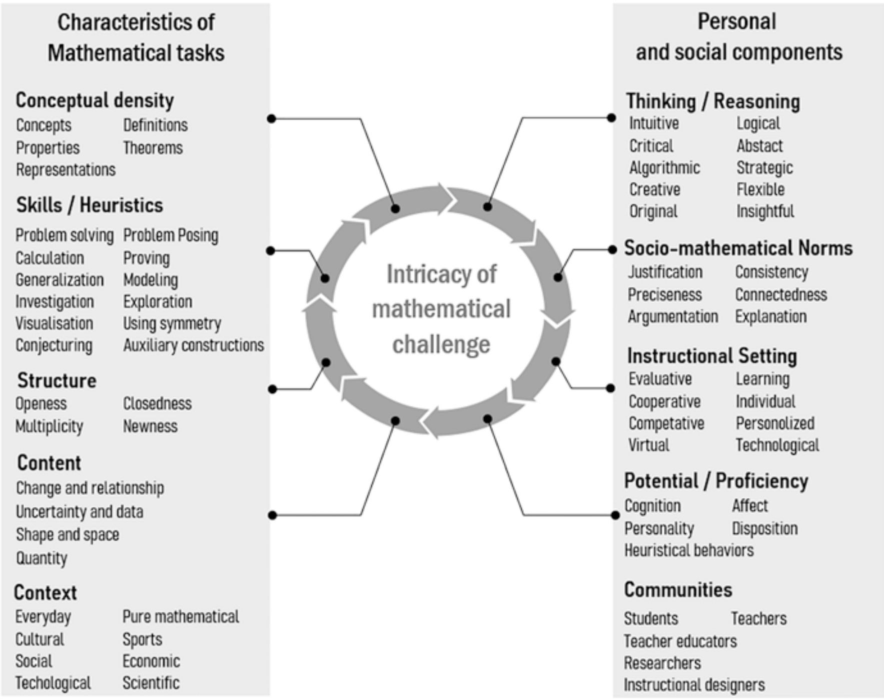

Fig. 1.1 Model of factors influencing mathematical challenge

1.4 Model of Factors Influencing Mathematical Challenge

Analysis of mathematical challenge is usually associated with task complexity. This, in turn, can be examined theoretically according to predetermined task variables, or empirically according to the success of groups of participants in engaging with the task. Lester (1994) performed a meta-analysis of mathematical problem-solving research published from 1970 to 1994. He found that during the period of 1970–1982, problem-solving research was directed at "isolation of key determinants of problem difficulty" along with "identification of successful problem solvers and heuristic training" (p. 664). Between 1982 and1990, there was a shift in researchers' attention to metacognition and training for metacognition, and affect related to problem solving. From 1990 to 1994, social influence and problem solving in context appeared to be the focus of problem-solving research:

> In 1979 a landmark book was published that synthesized the research on what were then referred to as "task variables" in mathematical problem solving (Goldin & McClintock, 1979). To briefly summarize this and other closely related research, four classes of variables were identified that contribute to problem difficulty: content and context variables, structure variables, syntax variables, and heuristic behavior variables. Initially, these classes were studied via linear regression models, later via information-processing techniques. This line of inquiry was replaced eventually by investigations of the interaction between task variables and the characteristics of the problem solver (Kilpatrick, 1985). (Lester, 1994, p. 664).

All of the topics studied from the 1970s through the 1990s are relevant to contemporary problem-solving research, with clear emphasis on the interaction between different variables and attempts to understand how problem solving can be monitored, and how it can provide mathematical challenge for all. The current volume nicely reflects recent advances in research on problem solving and problem solving in mathematical instruction and assessment.

Over the past three decades, mathematics education has undergone significant changes, with an emphasis on mathematical competencies (e.g., NAEP, TIMMS studies – Carpenter et al., 1983; Hiebert & Stigler, 2000), mathematical understanding (e.g., BUSUN and Balanced Assessment projects – Silver & Zawodjewsky, 1997, Burke, 2010), contextualization and mathematical literacy (PISA studies – Cobb & Couch, 2022, Stacy & Turner, 2015), and mathematics creativity (OECD, 2021). These changes are largely determined by technological and scientific progress, which has led to an emphasis on twenty-first century skills and deep learning (Pellegrino & Hilton, 2012), and by societal changes as expressed, for example, in the Sustainable Development Goals (https://sdgs.un.org/goals), such as quality education (SDG-4) and reduced inequalities (SDG-10). The latter leads to the requirement of sustainable learning and to the understanding of the importance of students' literacy in and beyond mathematics. Additionally, it directs us to see tools for attaining SDGs globally in education in general and mathematics education in particular. This volume, which is centered on mathematical challenge for all, clearly demonstrates that challenge-rich mathematical instruction strives for quality education, educational equity, and a "no child left behind" ideology.

As mentioned above, mathematical challenge is relative to a person's mathematical potential. As agreed in multiple chapters in this book, the effectiveness of mathematically challenging instruction is a function of teachers' proficiency in conveying mathematical activity suited to each student's potential. Taking into account the heterogeneity of a mathematical classroom, this seems to be almost a fantasy. However, the authors in this volume provide creative ideas which can make mathematically challenging tasks and collections of tasks accessible to all students.

Figure 1.1 demonstrates the major factors influencing mathematical challenge. It is an elaborated version of the model of mathematical challenge suggested in Leikin (2018) that included three characteristics: Conceptual characteristics of mathematical tasks, of which conceptual density and openness were at the center of the discussion, the setting, including use of digital technologies and learning/teaching methods, and socio-mathematical norms. This elaborated model reflects the intricacy of mathematical challenge as connected to multiple faces of mathematical instruction and is inspired by my long-term discussion with Avi Berman and Dina Tirosh (Sept 2021–May 2022) of the structure and nature of "advanced mathematical thinking in school." Below I describe and exemplify the main component of this model in connection to different chapters in this book.

In what follows, I explain the model of factors influencing mathematical challenge using examples from different chapters in this book. Additional theoretical framing of mathematical challenge in curricula design, tasks, and collections of tasks can be found in the introductory chapters by the section editors (Demetra Pitta-Pantazi and Constantinos Christou (Chap. 2), Rina Zazkis (Chap. 11) and Alexander Karp (Chap. 20) and in commentary chapters by Jeremy Hodgen (Chap. 10), David Pimm (Chap. 19). Alan Schoenfeld (Chap. 29) contributed retrospective analysis of problems, problem solving, and thinking mathematically).

1.5 Explaining the Model Using Works Presented in This Volume

It seems almost trivial that mathematical challenge is determined by the *mathematical content* taught. A fine illustration of different levels of complexity of mathematical content can be seen in underground mathematics concept maps that choose (https://undergroundmathematics.org/) numbers, algebra, geometry, functions, and calculus as major lines along which the concepts and mathematical principles are developed and become increasingly more complex at the unions of different lines. PISA (OECD, 2021) suggest slightly different lines of mathematical content: Quantity, shape and space, change and relationship, uncertainty, and data (which are depicted in Fig. 1.1). Independent of terminology used and independent of design principles, development of mathematical content in any mathematical curriculum is hierarchical with an increasing level of complexity of mathematical concepts, their properties, and mathematical skills and presumed heuristic behaviors. Verschaffel et al. (Chap.

3) explore teaching and learning of quantity domain with a special focus on mathematical patterns and structures, computational estimations, proportional reasoning, and probabilistic reasoning. They justify the importance of these topics using previous cognitive and neurocognitive research as well as the longitudinal study they conducted for their project. The chapter stresses the importance of early acquired numerical abilities – much earlier than have been traditionally studied. The study of Wasserman (Chap. 13) focuses on challenges associated with binary operations and the links between university and school *mathematical content*.

The underground mathematics maps mentioned above demonstrate raising *conceptual density* across the different topics and branches of mathematics. At the same time, mathematical tasks within the topics may differ in their conceptual density, determined by both complexity of the concept included in the task as well as the need for using different concepts and different rules and theorems (cf. Silver & Zawodjewsky, 1997) concurrently. Leikin and Elgrably (Chap. 27) demonstrate systematic bottom-up variation in conceptual density of geometry problems through construction of chains of problems posed by participants. Top-down structuring of mathematical challenge is illustrated in this chapter using stepped tasks. Verschaffel et al. (Chap. 3) also present examples of tasks of different levels of conceptual density and explain variations in the challenge embedded in related tasks.

Complexity of mathematical tasks is a function of the *skills* required from solvers. The relationships between skills, such as proving, problem solving, problem-posing, modeling, and generalization, are not necessarily hierarchical and depend on other variables included in the model of mathematical challenge (Fig. 1.1). However, while problem posing precedes solving the posed problems, the level of complexity of mathematical activity increases. Cai and Hwang (Chap. 7) present a rich collection of problem-posing tasks framed by a theoretical analysis of the types of problem-posing tasks and accompanied by analysis of the challenges embedded in problem-posing activities. Similarly, modeling tasks, in which participants must develop a mathematical model of a *contextual situation*, are more complex than problems in which an identical mathematical model is introduced to students along with the problem conditions. Applebaum and Zazkis (Chap. 14) describe how simple computational tasks can be transformed into challenging tasks by requiring a generalization process. Solutions by different groups of participants – mathematicians, teachers, and students – are discussed to illustrate how mathematical proficiency affects mathematical performance.

Goos et al. (Chap. 4) apply "the curriculum policy, design and enactment system" (with reference to Remillard & Heck, 2014) in their discussion of mathematical curriculum "enhanced" by modeling activities. They introduce contextual opportunities and constraints embedded in the implementation of modeling activities on a systematic basis. Borromeo Ferri et al. (Chap. 22) present a collection of modeling tasks to demonstrate the developmental power of these tasks. The complexity of the modeling activity is described in terms of a mathematical modeling cycle (borrowed from Kaiser & Stender, 2013). They demonstrate how modeling activities are inherently integrated in differentiated *instructional settings* based on self-differentiation of the learning process. Note here that Borromeo Ferri et al. clearly demonstrate the *openness* of mathematical modeling tasks expressed in a variety of student-generated models.

Vale and Barbosa (Chap. 15) provide examples of different types of visual problems and draw connections between mathematical challenges and *visualization skills*. They show that visual thinking is a tool that helps indicate the level of mathematical challenge embedded in the task, and can be applied to multiple solution tasks (considered also in Leikin & Guberman Chap. 17) to develop creative thinking. Interestingly, the principles described by Vale and Barbosa can be applied to multiple problems included in different chapters in the book. Using symmetry in solving problems, which is analyzed by Vale and Barbosa, is also one of the heuristics described by Polya (1973/45). Symmetry of geometric diagrams is implemented in the design of mathematical problems of varying levels of mathematical challenge in Waisman et al. (Chap. 26). Empirically, they demonstrate that when solving equivalent problems, a symmetrical diagram decreases the complexity of a problem. In addition Waisman et al. address field dependency of geometry diagrams and Hsu (Chap. 23) analyzes complexity of geometry diagrams as a meaningful variable of solving geometry problems. I invite readers to solve Problem 18 presented in Marushina (Chap. 25) both visually and using symmetry. The solution is elegant and enjoyable.

Personalized mathematics and mathematical inquiry is introduced in Chap. 5 by Christo et al. through a precise analysis of tasks that require and further develop exploration and investigation skills, and an accurate distinction between these *skills* is integrated in a technology supported *instructional setting* with applets designed to support students' explorations and investigations. The ideas presented in this chapter are illustrated using specific tasks that are mathematically challenging for all students. *Contextualization* of the mathematical content, inquiry that leads to curious experiences, and the connection of mathematical fluency with mathematical understanding create an activity challenging for all because of its cognitive complexity, social involvement, and positive affect.

The readers of this volume can learn about different levels of openness of exploration and investigation tasks in Christo et al., in the modeling tasks in Goos et al. and in Borromeo Ferri et al. The openness is related both to the multiplicity of ways in which students can approach the tasks and in the solution outcomes attained by different students. Such openness emphasizes the discursive nature of a mathematics lesson, with *socio-mathematical norms* of justification, consistency, and explanation. The requirement of explaining or justifying each idea presented in the classroom contributes to the creation of challenging-for-all mathematical activities. Leikin et al. (Chap. 6) stress the importance of the integration of open tasks in mathematical instruction. They present examples of tasks that are recommended in a computer-based *instructional setting* since exploratory applets are designed to support students' understanding of the *task structure*. Positive affect is related to multiple tasks outcomes, which are usually surprising and develop curiosity linked to the completeness of the solution spaces. Attaining a complete set of solution outcomes or production of multiple solution strategies (which become a norm in Math-Key classrooms) allows variations in the level of mathematical challenge as connected to the mathematical potential of different students.

Each of the chapters in the book describes different types of *instructional settings* associated with mathematically challenging activities. Common to all the settings are *norms* of preciseness and justification as well as mathematical discussion, whether in small groups or as a whole class. Sinclair and Ferrara (Chap. 16) suggest a *socio-material framing* of mathematically challenging tasks based on Leikin's (2014) concept of mathematical challenge. While solving challenging tasks with technological tools the first grade students make progress in solving problems through interaction with the environment. They are motivated as a result of finding a solution and the socio-material system is reactive to students' progress. This variation in mathematical challenge that ensures challenges-for-all is rooted in students' interactions in working groups and with technological tools. In this learning environment the students experience moments of insight related to knowledge advancement. All the chapters in this book either implicitly or explicitly address the development of students' creativity through engagement with mathematical challenges. The development of creativity is linked to the openness of the tasks, multiplicity of solution strategies and solution outcomes, and students' mathematical learning through engagement with new tasks. Leikin and Guberman (Chap. 17) analyze the relationship between mathematical challenge and insight-based tasks and make a distinction between insight-requiring and insight-allowing mathematical tasks to discuss different levels of mathematical challenge embedded in insight-based tasks.

The level of challenge is related to *mathematical reasoning*, which includes conjecturing, generalization, and justification. Da Ponte et al. (Chap. 8) focus their study on an exploratory approach to developing students' mathematical reasoning. They discuss tasks and a learning environment that allows students to develop new knowledge through conjecturing, generalization, and justification of findings. The tasks in this study allow a variety of solution strategies. Special skills and beliefs are required from the teachers when monitoring exploratory mathematical instruction. Lloyd and Murphy present conceptualization of argumentative practices and connected features of mathematical reasoning. In their study they discuss requirements for teachers' knowledge and skills essential for conducting scaffolding moves, while developing critical-analytic thinking in their students. Liljedahl (Chap. 28) discusses the thinking classroom and classifies instructional settings based on collections of problems with varying levels of conceptual density. In Chap. 8, Lloyd and Murphy, Wasserman (Chap. 13), and Applebaum and Zazkis (Chap. 14) analyze the development of teachers' proficiency in monitoring challenging mathematical tasks. In addition to the discussion of mathematically challenging activities in mathematics teacher education, Biza and Nardi (Chap. 18) describe how they use mathematics education research in the education of undergraduate students. They introduce design principles using Math Tasks within learning situations emerging in the mathematics classroom. They stress the importance of teachers' awareness of the interaction between mathematical challenge and pedagogical challenge.

The characteristics of challenging mathematical tasks and curricular design observed above can be applied to *collections of problem*s as well (Fig. 1.1). Bass (Chap. 21) introduces five principles of deliberate production of collections of problems to readers. The principles are related to curricular principles, task structure, mathematical content, task models, and task outcomes. In addition, Karp (Chap. 20) considers "the *morphology* of problem sets – the role of each problem within a set, its position in it, and the mental processes that take place during the transition from one problem to another" (Karp, 2002). For example, in Chap. 7, Cai and Hwang present characterization of problem-posing activities and include collections of problem-posing tasks that exemplify these characterizations. The collections of problems are differentiated depending on whether they are designed by mathematicians, mathematics educators, teachers, or researchers and can be differentiated depending on their goals and morphological structures. For example, Karp (Chap. 24) analyzes collections of problems in mathematical textbooks and Marushina (Chap. 25) analyzes sets of exam problems – all created by instructional designers who are professional mathematicians. While in Waisman et al. the collections of problems are designed by researchers, Hsu examines complexity of problems designed by mathematics teachers. In addition, there are culturally dependent and policy-related characteristics of the sets of problems. Analyses of school textbooks in the United States (Karp, Chap. 24), of problem sets in exams in Russia (Marushina, Chap. 25), and of collections of problems generated by Taiwanese teachers are examples of culturally dependent collections that also reflect decisions related to educational policies in different countries at different periods of time. Both in Marushina and Karp and in Liljedahl (Chap. 28), the collections of problems include variations based on the conceptual density of the tasks included in the collections of problems borrowed from the education documents (in Karp and Marushina) or created by the author.

Collections of mathematical problems of varying levels of mathematical challenge, theoretically justified and connected to different mathematical and cognitive skills, can be found in Krutetskii (1976) in his seminal research on characterization of higher mathematical abilities. Collections of mathematical problems can be created for mathematical textbooks, evaluation tools, Olympiad problem collections, and sets of problems for particular instructional activities. The construction of the set of problems in Wiseman et al. (Chap. 26) is based on integration of psychological domain-specific characteristics (symmetry and field dependency) with domain-specific geometry properties.

1.6 Concluding Notes and Questions for Future Research

This volume "Mathematical Challenges for All" considers mathematical challenge to be an "optimal experience" in mathematics education (cf. Csikszentmihalyi & Csikzentmihaly, 1990). Optimal experiences in mathematics education are directed to the realization of the mathematical potential of each and every student.

As presented in the book chapters, mathematical challenge integrates the following:

- Cognitive demand determined by the characteristics of mathematical activity, including:
  - Characteristics of tasks that include conceptual density, contextual framing, and task structure with an emphasis on openness
  - Required domain of general and mathematical skills and associated mathematical thinking, reasoning, and argumentative practices
  - Setting in which the participants are involved in the mathematical activity including competitive and cooperative elements (coopetition), personalization and technological affordance of the activity
  - Socio-mathematical norms of preciseness, justification, and explanation.
- Affective components evoked by the mathematical activity including motivation to overcome the difficulty, frustration or curiosity caused by the task's complexity, and enjoyment from the process or outcome of engagement with the task.

Figure 1.1 suggests an elaborated view of the factors that influence the intricacy of a mathematical challenge.

The volume proposes multiple *ways in which mathematically challenging activities become "optimal for all."* The characteristics of mathematical activities emphasized by the majority of the authors in this volume include but are not limited to the following:

- The novelty of mathematical content or context or types of tasks for learners. Novelty develops motivation to learn mathematics and evokes mathematical curiosity.
- Variations in the level of mathematical challenge by means of collections of tasks of different levels of challenge. This includes problem sets, problem chains, and stepped tasks designed by instructional designers or teachers.
- Openness of tasks and tasks of an explorative nature that allows self-regulation of the level of mathematical challenge by students. This includes engagement with mathematical investigations, problem posing, mathematical modeling, multiple solution-strategy tasks or multiple solution-outcome tasks.
- Use of digital technologies that support self-regulation of mathematical challenge by students, development of new knowledge through mathematical experiences, and encouragement of interpersonal interactions. Explorative dynamic applets are among the recommended technological tools.
- Discursive and argumentative practices that support variations in levels of mathematical challenge to fit the mathematical potential of each and every student, including norms of asking hypothetical and elaborative rather than verification questions, and norms of preciseness and of justification of all mathematical conjectures raised.
- Scaffolding practices with no funneling that "reduces the challenge" or "closes the openness." Such practices require a high level of proficiency of teachers,

deep mathematical knowledge, and belief in the centrality of mathematical challenge for high quality mathematical instruction for all students.

- Teacher training directed at advancing teachers' professional potential that combines their mathematical and pedagogical knowledge and skills with the ability to identify students' mathematical potential. Developing teachers' proficiency in navigating mathematically challenging activities includes giving them experience in varying the level of mathematical challenge in accordance with students' mathematical potential and with enhancing students' enjoyment from doing mathematics and from successful task completion, and will allow teachers to enjoy their own pedagogical challenges.

This book has the potential to be useful for a broad range of mathematics educators, educational researchers, and mathematicians who work with mathematics teachers and instructional designers. I am certain that readers will be able to learn from the variety of theoretical, practical, and methodological ideas that the authors present. More research about mathematical challenge for all is required, and this book opens new venues of research in mathematics education. Finally, as noted above, the volume includes a wonderful collection of mathematical challenges at different levels that readers are invited to explore and enjoy.

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Part I
Mathematical Challenges in Curriculum and Instructional Design

Editors
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Chapter 2
Introduction to Part I of Mathematical Challenges For All: Mathematical Challenges in Curriculum and Instructional Design

Demetra Pitta-Pantazi and Constantinos Christou

This section of the volume "mathematical challenges for all" is about mathematical challenges in curriculum and instructional design. Taylor (2006) stated that "challenge is not only an important component of the learning process but also a vital skill for life. People are confronted with challenging situations each day and need to deal with them" (p. 349). Mathematics is among the subjects that offer the most opportunities for students to confront challenges. Mathematics is about productive struggle, solving everyday problems, and seeing patterns in the world around us. Challenges are a natural part of mathematics. The scope of this section is to address mathematical challenge from two aspects, that of mathematics curriculum, and that of instructional design. Thus, this section is divided into two parts, where the first part discusses the challenges of mathematics curriculum and the second one refers to the designing of learning experiences taking into account the components of effective learning of mathematics, such as the learning objectives and learning activities.

The first part of this section includes two chapters discussing curriculum development and how mathematical ideas translate into classroom practicalities. In the first chapter Verschaffel, De Smedt, Luwel, Onghena, Torbeyns, and Van Dooren (Chap. 3, this volume) provide a suggestion for how we may rethink the traditional early mathematics curriculum with a view to make it more challenging both in terms of breadth and depth. The chapter focuses on a longitudinal study which emphasizes four key strands of the curriculum: (1) mathematical patterns and structures, (2) computational estimation, (3) proportional reasoning, and (4) probabilistic reasoning. The authors claim that the competencies young students develop in early years involve much more than numerical abilities and that the four strands

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mentioned above develop in close relation to each other and to the development of numerical abilities. They suggest that the developmental steps that students take during preschool years may lead learning trajectories useful for teachers in preparing their students to understand mathematics and dealing with more mathematical challenging topics in the future. The diagnostic tools designed for the longitudinal study presented by the authors and the instructional materials and techniques developed for the intervention studies may yield valuable building blocks for implementing these challenging curricula. However, as the authors clearly state, the effectiveness of the results in this longitudinal study remain open depending on the future implementation of their recommendations in real classroom situations.

In Chap. 4, Goos, O'Meara, Johnson, Fitzmaurice, and Guerin (this volume) consider mathematical modeling as a kind of mathematical challenge which is rarely found in elementary or secondary school curricula and classrooms. The authors first address the similarities and differences between applications and modeling. Then, setting off from Remillard and Heck's (2014) model of the curriculum enactment processes, they articulate the main purpose of their research which aims at finding out the factors supporting or hindering the implementation of modeling as a mathematical challenge in the school curriculum. To answer this main question, the authors analyze two different strategies which may promote modeling as a stimulus for curriculum and instructional reform. The first strategy Goose et al. (Chap. 4, this volume) suggest is approaching curriculum developers in forming challenging curricular aims and objectives, while the second strategy supports mathematics teachers in translating curricular goals into classroom interactions with students. The authors also present two case studies in which they illustrate how modeling was introduced into the secondary school mathematics curriculum in Ireland. In both case studies, they presented in an interesting way the factors that influenced the introduction of modeling in mathematics curriculum. Finally, in their conclusions, the authors referred to meta-challenges, meaning the idea that it is challenging to introduce teachers and students to challenging tasks and the ways of working with modeling.

The second part of this section includes five chapters discussing instructional design and how textbooks, tasks, and teaching may offer mathematical challenge to students. In Chap. 5, Christou, Pitta-Pantazi, Demosthenous, Pittalis, and Chimoni (this volume) present an instructional design whose corner stone is mathematical challenge. The authors describe the theoretical framework "Personalized Mathematics and Mathematics Inquiry" (PMMI) which supported the instructional design and development of the Cyprus Mathematics Textbooks and exemplify it through examples taken from the mathematics textbooks. Mathematics Inquiry is at the core of the PMMI theoretical framework and consists of challenging problems, namely, explorations and investigations, which capture students' curiosity and invite them to make hypotheses and pursue their hunches. During these challenging explorations and investigations, the teachers' role is to provide students with key elements which will facilitate them to take control and ownership of their learning. The chapter describes this framework and also provides specific examples from the mathematics textbooks in order to bring closer together theory and practice.

In Chap. 6, Leikin, Klein, and Ovodenko (this volume) propose the integration of the Math-Key program within the regular mathematics curricula of junior schools which offers the opportunity to teachers and students to engage with mathematically challenging tasks. The aim of the Math-Key program is to open mathematical minds through the use of Multiple Solution-Strategies Tasks (MSTs) and Multiple Outcome Tasks (MOTs). These tasks support the development of mathematical creativity and mental flexibility along with the advancement of mathematical knowledge and skills. At the same time, the Math-Key program provides opportunities to teachers to enrich their instructional strategies of problem solving. Leikin et al. (Chap. 6, this volume) provide numerous and rich examples of Multiple Solution-Strategies Tasks (MSTs) and Multiple Outcome Tasks (MOTs) explaining in detail the types of openness, the number of possible solution strategies, and different types of solution outcomes. What is also of interest to the readers of this article is that most of the problems are accompanied by dynamic applets. These applets allow students to explore and investigate the situation of the problems in a way to support their understanding of the mathematical concepts which are needed in order to reach a meaningful solution. The inclusion of applets also provides an opportunity to develop positive "growth mindset." Teaching with Math-Key tasks requires changing the classroom culture, an important factor for changing students' mindset (Boaler, et al., 2021).

Cai's and Hwang's (Chap. 7, this volume) chapter focuses on another particular kind of instructional task which offers mathematical challenge to students, that of mathematical problem posing. They consider problem posing from the perspective of the student, who may be asked to pose problems, and from the perspective of the teacher, who can either pose problems for students to solve or use problem-posing tasks as instructional tasks with students (Cai & Hwang, 2020). They investigate how teachers could present problems to engage students and provide a more productive learning environment for them. They discuss various types of problem posing situations and prompts that can be used in problem-posing tasks. They also highlight that one of the affordances of problem posing is that it provides levels of mathematical challenge that correspond to students' level of understanding. Furthermore, Cai and Hwang (Chap. 7, this volume) offer suggestions and examples of problem-posing tasks and outline ideas for professional development that can help teachers use problem posing in teaching mathematics. The authors indicate that unfortunately in many curricula there is lack of problem-posing tasks. Thus, to assist teachers in promoting problem posing, one needs to develop their knowledge and beliefs about teaching through problem posing. The authors suggest that a useful approach to help teachers understand how problem posing can be enacted in classrooms is through the use of teaching cases. They close their chapter with some interesting results from empirical studies on problem posing and suggestions for future directions for research.

In the last two chapters of this section, Ponte, Mata-Pereira, and Quaresma (Chap. 8, this volume) and Lloyd and Murphy (Chap. 9, this volume), the attention shifts to the work done in the classroom and the role of mathematical challenge. Ponte et al. (Chap. 8, this volume) investigate the way in which teachers' actions supported by the work on mathematically rich tasks enhance students' development of mathematical reasoning. They present two situations, one in a grade 8 and one in a grade 9 class, and discuss the features of the tasks and the learning environment of the students. They explore the guiding actions and suggestions or challenges teachers may provide in their mathematics classroom that will offer a fruitful environment for students to extend their thinking, consider other possibilities, and communicate their justifications and arguments. The authors present and discuss teachers' actions aimed to develop students' mathematical reasoning, generalizations, and justifications in various instances during mathematics lessons. Both teachers organized their lessons according to the three phases of the exploratory approach. The first phase is the task launching, the second one is the students' autonomous work, and the third phase of the lesson is the collective discussion, where all students' solutions are presented and discussed. What is of interest to the readers is the new knowledge about the general and specific actions that teachers may use in their lessons in order to help students develop mathematics reasoning and find out students' strategies and difficulties.

Lloyd and Murphy (Chap. 9, this volume) also address mathematical challenge from the perspective of the work done during mathematics lessons. They are interested in both classroom discourse elements and the way in which prospective teachers and elementary students co-construct mathematical arguments related to complex mathematical tasks. Specifically, they describe an on-going project in which prospective teachers gain experience by participating in and facilitating small-group discussions which emphasize the construction of mathematical arguments during the solution of complex mathematical tasks. To do so, Lloyd and Murphy (Chap. 9, this volume) introduce the discussion approach, the Quality Talk which can support prospective teachers' learning to facilitate small-group discussions that advance learners' mathematical argumentation. Through two episodes, with elementary prospective teachers and with second-grade students, the authors explore how prospective teachers can use the experiences gained in their university methods course about Quality Talk to plan and facilitate productive discussions during their field experience classrooms. This investigation leads them to the identification of key discourse elements in enhancing the co-construction of mathematical arguments as prospective teachers and elementary school students work on complex solving tasks. In addition, Lloyd and Murphy (Chap. 9, this volume) discuss the importance of carefully selected questions and scaffolding moves to support students' engagement and discussion-based mathematical argumentation while solving complex mathematical tasks.

The section closes with a commentary chapter by Hodgen. Hodgen (Chap. 10, this volume) addresses the "challenge" in mathematics to deal with the terminology of "challenge" itself, since it can carry different meanings and may refer to different aspects of mathematics education. He looks at challenge from various angles, in terms of curriculum materials and tasks, the role of teachers, and that of the curriculum.

As a conclusion, the chapters in this section highlight the authors' more recent findings and perspectives on mathematical challenge. In particular, the authors provided their suggestions as to the way in which mathematical challenge may be addressed, for instance, by rethinking the context of mathematics curriculum in early years and by considering the inclusion of modeling tasks. The authors also discussed role of textbooks and types of tasks which may promote mathematical challenge and highlighted the benefits of explorations, investigations, Multiple Solution-Strategies Tasks (MSTs), Multiple Outcomes Tasks (MOTs), modellng problems and problem-posing tasks. Moreover, the authors explored the work done in mathematics classrooms. Specifically, they investigated and discussed the types of tasks used, the discourse, and key actions that teachers take in order to support students' development of mathematical reasoning and arguments. These chapters can be challenging to read. They tackle different issues in the broad field of mathematical challenge and provide insights into this area of research from various angles. We believe such insights have the potential to support the work of teachers, teacher educators, curriculum developers, and researchers in the field of mathematics education.

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Chapter 3
Development and Stimulation of Early Core Mathematical Competencies in Young Children: Results from the Leuven Wis & C Project

L. Verschaffel, B. De Smedt, K. Luwel, P. Onghena, J. Torbeyns, and W. Van Dooren

3.1 Introduction

Mathematics has always been a central curricular domain in elementary and secondary education worldwide (De Corte et al., 1996; Kilpatrick, 1992). For a long time it was common to pay only little attention to mathematics education in the preschool years, both by teachers in preschool and by parents and other caretakers at home. The general idea was that preschool children should essentially spend their time at developing their psychomotor and social-emotional skills, together with their language and emergent literacy skills. Little or no attention was paid to interventions aimed at children's early mathematical growth. And, if some attention was paid to it, there was a remarkably narrow focus on the acquisition of some basic numerical abilities, such as reciting the counting words, identifying the numerosity of a small set of objects, indicating which set has the largest numerosity, and solving simple additive problem situations involving small whole numbers (Dede, 2010). However, the past two decades have witnessed a great research interest in early mathematical cognition, early mathematical development, and early mathematics education, both in home and preschool settings.

A starting point of this line of research – with its main origins in cognitive (neuro) science – is the idea that young children are equipped with some foundational core systems to process quantities (Butterworth, 2015; Dehaene, 2011). This allows them to exactly identify small (i.e., below 4) non-symbolic quantities, to compare non-symbolic quantities that are too numerous to enumerate exactly, or to perform some very basic approximate arithmetic on these quantities (Andrews & Sayers, 2015; Butterworth, 2015; Torbeyns et al., 2015; Verschaffel et al., 2017). Within

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these foundational core number sense systems, magnitudes are represented non-verbally and non-symbolically, but, over development and through early (mathematics) education, verbal and symbolic representations are gradually mapped on these foundational representations, to evolve into a more elaborate system for number sense and more complex mathematical concepts and skills (Torbeyns et al., 2015). The dynamics of this development remain one of the most debated areas in research on these foundational representations (e.g., Leibovich & Ansari, 2016). This research has shown large individual differences in these early numerical abilities, which predict later general math achievement (De Smedt et al., 2013; Schneider et al., 2017; Siegler & Lortie-Forgues, 2014). Furthermore, researchers working within this research tradition have also tried to stimulate children's foundational numerical abilities with (game-based) intervention programs before or at the beginning of formal instruction in number and arithmetic in elementary school (e.g., Maertens et al., 2016; Wilson et al., 2009).

As shown in the above description, this prominent line of research has strongly focused on young children's basic numerical abilities (Cohen Kadosh & Dowker, 2015). More recently, this narrow focus on only early numerical and arithmetic abilities with non-symbolic and symbolic entities has been increasingly questioned on various grounds (e.g., Bailey et al., 2014; Dede, 2010; English & Mulligan, 2013). From a disciplinary perspective, it is evident that mathematics is much more than understanding whole numbers, counting, and basic arithmetic. Therefore, even in the early years of education, mathematics education should already represent a broader coverage of the richness of the discipline, including early reasoning about mathematical relations, shapes, and patterns and structures (Clements & Sarama, 2013; Mulligan & Mitchelmore, 2009). From an empirical perspective, recent meta-analytic work has shown that children's early numerical and counting skills explain only a small percentage of the individual differences in general mathematics achievement (Schneider et al., 2017). Accordingly, some other scholars have suggested that early quantitative reasoning about additive and multiplicative relations may be more predictive for later achievement in school mathematics (Nunes et al., 2012). Finally, and in line with the results of the above developmental studies, the abovementioned intervention studies on the early enhancement of children's foundational numerical abilities yielded mixed findings, with mainly marginal effects in terms of retention and transfer (Torbeyns et al., 2015).

As a result of the abovementioned critiques on early numerical cognition, researchers have started to go beyond analyzing young children's basic numerical abilities and to look at the early development of other, more complex, mathematical competencies in younger ages than is currently the case, i.e., already before the start of elementary school and/or while children are making the transition from preschool to elementary school (Bryant & Nunes, 2012; Dowker, 2003; Mulligan & Mitchelmore, 2009). This complementary research line has started to provide evidence of the possibility and value of broadening and deepening the scope of mathematics for young children beyond initial experiences with small whole numbers and simple arithmetic with them.

As part of that complementary approach, we embarked some years ago on a research project involving a large-scale longitudinal study about the early development of four such additional core mathematical competences: mathematical patterning, computational estimation, proportional reasoning, and probabilistic reasoning, followed by four intervention studies on the same four competencies.

This kind of research project may first of all help us rethink the traditional early mathematics curriculum with a view to make it more challenging both in terms of breadth and depth. Second, the analysis of the developmental steps children take as revealed by the longitudinal study may lead to well-articulated research-based learning trajectories. Armed with these learning trajectories, one can assess any child's thinking, locate the child on a trajectory, and determine the next step in the child's learning related to these additional mathematical topics, analogous to the trajectories developed for number, counting, and early addition and subtraction by Clements and Sarama (2013). Third, the diagnostic tools designed for the developmental studies and the instructional materials and techniques developed for the intervention studies may yield valuable building blocks for implementing these challenging curricula and designs. Finally, paying special attention in these studies to the children at the lower and the higher ends of the continuum of mathematical ability may help make these early mathematics curricula and designs more challenging and inclusive for all children.

In the present chapter, we provide a selective overview of some provisional results of this ongoing longitudinal research project. After a brief presentation of the overall aims and scope of the study and its overall methodology, available data from the various parts of the study are used to provide illustrative evidence for the basic claim that early mathematical development involves much more than children's early numerical abilities, that also with respect to these other core mathematical competencies important initial steps are being made much earlier in children's development than traditionally thought, and that these core mathematical competencies develop in close relation to each other and to the development of children's early numerical abilities. At the end we formulate some general conclusions of the research being reviewed in this chapter and we summarize its contribution to understanding how a focus in the curriculum and instructional design on challenging domains such as patterns, computational estimation, proportional reasoning, and probabilistic reasoning may enhance the mathematical competence of all young children.

3.2 A Research Project Consisting of Four Parts

In 2016, we started a 6-year-long research project on the development of 4- to 9-year-olds' competencies in four early core mathematics-related domains[1]: mathematical patterns, computational estimation, proportional reasoning, and probabilistic reasoning. While the rationale for this selection was partly pragmatically

[1] C16/16/001 project "Early development and stimulation of core mathematical competencies" of the Research Council of the KU Leuven, with Nore Wijns, Elke Sekeris, Elien Vanluydt, Anne-Sophie Supply, and Merel Bakker as PhD researchers and, consecutively, Joke Torbeyns, Greet Peters, and Laure De Keyser as project coordinator.

grounded in the fields of expertise of our research team, a common characteristic of these four domains is that they all receive little or no instructional attention in current early mathematics education curricula, while they do represent important domains of mathematics and while there is increasing empirical evidence that they start to emerge much earlier than traditionally thought, and therefore, children may be challenged in these domains at an earlier age than is currently the case.

For each of these four domains, we tried to document the emergence and early development of intuitive concepts and basic skills related to the domain, to look for interrelations between these emerging concepts and skills, with a view to explore ways to organize early and elementary school mathematics such that this organization does not undermine these intuitive concepts and emerging skills but rather creates an environment wherein they can be acknowledged and stimulated.

In order to longitudinally map the emergence and development of these core mathematical competencies, as well as children's early numerical abilities, a cohort of over 400 children from 17 schools is followed from the second year of preschool (±4 years of age) to the third year of elementary school (±9 years of age). Using a stratified cluster sampling strategy to ensure an SES distribution that is representative for the Flemish context, schools were selected based on the relative number of pupils who receive study allowance and/or whose mother did not obtain a secondary school certificate. In Flanders, children go to preschool from the age of 2.5 years onwards. Preschool consists of three years (P1, P2, and P3). It is fully government subsidized and non-mandatory, yet it is attended by nearly all children. In September of the year children turn 6, they start in elementary school, which consists of six grades and which is mandatory.

As shown in Fig. 3.1, a rich battery of measures was administered during the 5 years of data collection. This battery comprised tasks and instruments assessing children's mathematical patterning, computational estimation, proportional reasoning, and probabilistic reasoning (parts 1 to 4, see further), as well as children's domain-specific early numerical abilities, domain-general cognitive abilities, and general mathematics achievement. The domain-specific early numerical abilities test comprised a wide variety of tasks measuring verbal counting, object counting, Arabic number recognition, number comparison, number order, and non-verbal

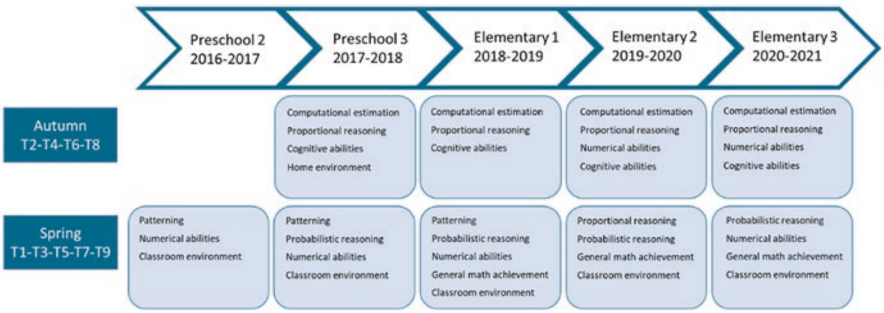


Fig. 3.1 Timeline for school years, time points of assessments, and administered measures

calculation; the domain-general cognitive abilities measures were two working memory and one spatial ability test; and general mathematics achievement was measured by means of a Flemish standardized mathematics achievement test. Finally, data on children's home and class environment were, respectively, collected via parent and teacher questionnaires. For more details about these instruments see Bakker et al. (2019).

3.3 Early Mathematical Patterning

Patterning is an aspect of early mathematical ability that is defined as children's performance on a wide set of tasks, such as extending, translating, or identifying a pattern's structure, that can be done with regular configurations of elements in the environment (Wijns et al., 2019c). These regular configurations are called patterns. There are different types of patterns, and a distinction is often made between repeating (e.g., ABABAB), growing (e.g., 2 4 6), and spatial structure patterns (e.g.,::::; Mulligan & Mitchelmore, 2009). Repeating and growing patterns are both sequences that can be continued indefinitely. Their underlying structure or rule describes how the sequence continues. For repeating patterns (e.g., ABABAB, Δ□□Δ□□Δ□□), the structure is defined on the basis of a unit (e.g., AB, Δ□□) that is reiterated. The structure of growing patterns (e.g., 2 4 6, Δ□Δ□□Δ□□□) involves a systematic increase or decrease between the units in the sequence (e.g., +2, + □). Spatial structure patterns, by contrast, represent two-dimensional configurations of elements. In part 1 of the project, we investigate the development of 4- to 6-year-olds' repeating and growing patterning competencies, and their associations with these children's numerical abilities.

Children are confronted with repeating patterns from a very young age in their daily life activities (e.g., day-night-day-night and yellow-red-yellow-red lines on their T-shirt). Repeating patterns are also the most common type of patterns in early childhood education and research (for a review, see Wijns et al., 2019c). At the start of the project, a number of empirical studies on young children's repeating patterning competencies had provided evidence for preschoolers' ability to solve tasks involving repeating patterns, and for the association between children's repeating patterning abilities and both concurrent and later numerical and mathematical abilities (e.g., Collins & Laski, 2015; Lüken, 2012; Rittle-Johnson et al., 2015, 2017; Zippert et al., 2019). Repeating patterning competencies were also shown to uniquely contribute to later mathematical performance, in addition to children's early numerical abilities (Lüken, 2012; Nguyen et al., 2016; Rittle-Johnson et al., 2017). However, systematic analyses of the mechanisms that might explain the association between young children's patterning and early numerical ability as well as the developmental associations between these two early mathematical competencies were non-existent. Moreover, researchers were criticized for their exclusive focus on repeating patterning abilities, arguing that young children are already capable of handling more complex patterns, such as growing patterns (Pasnak et al., 2019). As far as growing patterns were included in empirical studies, they were analyzed in view of their contribution to the development of elementary school children's algebraic skills (e.g., Warren & Cooper, 2008). To the best of our knowledge, no research had investigated preschool children's ability to successfully deal with activities focusing on growing patterns. Finally, although several researchers had already hinted toward the idea that children who by themselves look for patterns in their environment are good mathematicians, young children's spontaneous attention for patterns was not yet systematically investigated. This contrasted with the domain of number, where researchers had already documented the pivotal role of young children's spontaneous attention for quantities (SFON; Hannula & Lehtinen, 2005) and number symbols (SFONS; Rathé et al., 2019) for their concurrent and later mathematical development. Part 1 of the present project aimed to increase current insight into the role of patterning within early mathematical development by addressing the abovementioned weaknesses and systematically analyzing (a) young children's spontaneous attention for patterns in their environment, (b) their ability in handling repeating as well as growing patterns, and (c) the association between their repeating and growing patterning ability and their numerical ability.

For this part of our longitudinal research project, we followed the development of children's early patterning and number abilities between 4 and 6 years. This age range covers a critical developmental period in which several aspects of patterning and numerical ability are known to be acquired rapidly. In the spring of preschool year 2, preschool year 3, and elementary school Grade 1, children were offered two patterning ability measures, one focusing on repeating patterns and one focusing on growing patterns. Both patterning ability measures consisted of three types of patterning activities: extending the pattern (i.e., what comes next in the pattern?), translating the pattern (i.e., make the same pattern using different materials), and identifying the structure of the pattern (i.e., identifying the unit of repeat that defines the repeating pattern, identifying the systematic increase or decrease that defines the growing pattern). Figure 3.2 provides an example item for the three patterning activities in the repeating patterns and growing patterns ability measure. The patterning ability measures consisted of 18 items (6 items per activity) that were scored as either correct or incorrect, resulting in a maximum score of 18 per measure. Before they solved the two patterning ability measures, children engaged in an activity that addressed their spontaneous attention for patterns, the so-called tower task, in which children were asked, in a free-play context, to make a tower construction with 15 building blocks in three colors (five per color). Children's tower constructions were scored as (a) pattern, when the tower included at least two full units and the start of the third unit of a pattern, (b) sorting, when all the blocks were sorted per color, or (c) random, indicating no pattern or sorting construction.

In a first study (Wijns et al., 2019a) we focused on 4-year-olds' spontaneous attention for patterns when solving the tower task (i.e., SFOP). We looked for individual differences in 4-year-olds' SFOP as well as their associations with children's repeating patterning and numerical ability. We found individual differences in 4-year-olds' SFOP and showed that children who spontaneously created a pattern had higher repeating patterning ability and numerical ability than children who

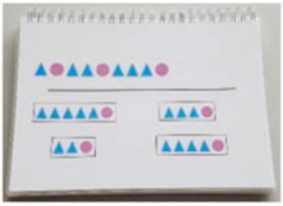
Extending "Look carefully at this row. There is a pattern in it. At the end there is something missing. One of these figures has to be placed in the empty spot. Do you know which of these has to be on the empty spot?"		
Translating "Look carefully at this row. There is a pattern in it. Please make the same pattern with your figures on the paper strip."		
Identifying "You will soon see a row with a pattern in it. You will have to look very closely at it and try to remember the pattern. After a short period I will hide the pattern and you will have to copy it."		

Fig. 3.2 Example items for the two pattern types (repeating and growing) and the three activities (extending, translating, and identifying)

made a random arrangement. The positive associations between 4-year-olds' SFOP and their repeating patterning and numerical ability can be hypothetically explained via the mechanism of self-initiated practice (cf. Hannula & Lehtinen, 2005, and Rathé et al., 2019, for a similar explanation related to, respectively, SFON and SFONS). This mechanism suggests that children with a spontaneous tendency to focus on mathematical elements in their environment will have more opportunities to practice their mathematical abilities and therefore improve them. Related to SFOP, children who spontaneously look for and create patterns during daily life activities are assumed to have more opportunities to practice their patterning abilities and, by extension, numerical skills. Although viable, this hypothetical explanation requires further research attention.

In a second study (Wijns et al., 2019b), we analyzed 4-year-olds' ability in both repeating and growing patterning tasks, and their association with children's numerical ability. A confirmatory factor analysis showed that the 2 × 3 structure of our patterning ability measure (two types of patterns, three patterning activities) could also be found in our data, confirming the validity of our measure. Additionally, both the pattern type and the patterning activity had an impact on children's patterning performance. Concerning the pattern type, we found that growing patterns were more difficult than repeating patterns. This difference in difficulty level might be due to differences in the complexity of the structure of the different pattern types (i.e., a clearly visible unit that repeats versus a systematic increase or decrease that needs to be deduced from the visible pattern) as well as the emphasis on mainly repeating patterns in current educational practice, which might lead children to think that all patterns are repeating patterns. Importantly, the study also showed that, despite the high difficulty level of activities with growing patterns, growing patterns are already feasible for a significant number of children of this young age. Turning to the impact of patterning activity, our study indicated that translating patterns was easier than both extending patterns and identifying the structure of patterns. The difference in difficulty level of extending and translating patterns is not in line with earlier studies with only repeating patterns (Lüken, 2012; Rittle-Johnson et al., 2015) but might be due to children's use of a one-one matching strategy to solve the latter type of tasks. Although our observational data do not support this explanation, future studies on the strategies that children use when solving different types of patterning tasks are required (cf. the recent study of Lüken & Sauzet, 2020). Finally, we also found that most patterning tasks uniquely predicted children's numerical ability.

Our third study involved a longitudinal analysis of the direction of the associations between repeating patterning, growing patterning, and numerical ability from age 4 to 6 (Wijns et al., 2021a). Although several studies had already provided evidence for an association between patterning and numerical ability, little was known about the direction of this association. Moreover, at the start of the project, it was unclear whether the association with numerical ability was different for distinct pattern types. Our cross-lagged panel analysis revealed bidirectional associations between all three abilities from age 4 to 5, suggesting that performance on one ability supports performance on another ability 1 year later. From age 5 to 6, patterning abilities predicted numerical ability, but the reverse was no longer true. Also, from age 5 to 6, repeating patterning abilities predicted growing patterning abilities, but not vice versa. These findings suggest that children's repeating and growing patterning ability supports the acquisition of later numerical ability, and that, within children's patterning ability, repeating patterning ability supports the acquisition of growing patterning ability. Although several researchers already hinted at the possibility to explore regularities in both patterning and number tasks (e.g., the base-10 structure of our number system with repeating units across decades, or the systematic increase with one of our counting row) as a mechanism that explains their associations, theoretical models are missing. It is a challenge for future work in the domain to first develop these theoretical models and next conduct focused intervention studies that help reveal the mechanisms underlying the frequently observed associations between patterning ability and numerical ability.

In our fourth study, we evaluated the effectiveness of an intervention aiming to enhance 5-year-olds' repeating and growing patterning ability for their development of early patterning competency (Wijns et al., 2021b). A 20-week intervention program (with 30 minutes patterning activities per week, focusing on the patterns' structure) resulted in significant improvements in the patterning competency of the children following the intervention compared to the control group, but there was no transfer effect to their numerical ability.

The findings of the longitudinal and intervention study provide important building blocks for optimizing current early mathematics education. First, the longitudinal analyses add to current insights into children's learning trajectories in the domains of number and patterning, pointing to the pivotal role of preschoolers' patterning competencies. Second, our focused intervention on the structure of repeating and growing patterns greatly enhanced preschoolers' patterning competency. These findings call for more attention than is currently the case for stimulating patterning competency in preschool curricula, integrating also more complex, growing patterns and more challenging patterning activities, such as translating the pattern or identifying its structure.

3.4 Early Computational Estimation

Computational estimation can be described as providing an approximate answer to an arithmetic problem without calculating it precisely. This mathematical skill shows a commonality with the approximate arithmetic competence that is assumed to be part of young children's foundational approximate number system (ANS), in the sense that in both cases one has to mentally perform an arithmetic operation on two operands in an approximate way. However, the two skills are also fundamentally different: while in approximate arithmetic children have to process the operands approximately, the numerical value of the operands is known in computational estimation (Sekeris et al., 2019).

Computational estimation is viewed as an important mathematical competence in our daily life since many situations only require calculations with a reasonable degree of accuracy, such as splitting the bill among a group of friends in restaurant. In addition, it is widely agreed that computational estimation should play an important role in the elementary mathematics curriculum (Siegler & Booth, 2005; Sowder, 1992; van den Heuvel-Panhuizen, 2000) as it involves a complex interplay of various types of mathematical knowledge and skills, including conceptual knowledge (e.g., accepting more than one value as an outcome of an estimation), procedural knowledge (e.g., being able to modify the problem to arrive at a mentally more manageable problem), and arithmetic knowledge and skills (e.g., mental computation skills). Given that computational estimation problems can be solved in many different ways, it allows children to develop number sense (LeFevre et al., 1993) and strategy flexibility (Siegler & Booth, 2005). Although computational estimation is nowadays widely recognized as an important part of the elementary mathematics curriculum (e.g., NCTM, 2000), it has, compared to its counterpart exact arithmetic, received far less attention from curriculum developers and researchers (Dowker, 2003; Siegler & Booth, 2005).

A recent literature review by our team revealed that the vast majority of studies on computational estimation investigated this skill from the age of eight and onward (Sekeris et al., 2019). This could be related to the fact that computational estimation is typically only instructed from the middle grades of elementary school onwards, when children have already acquired ample experience with whole-number exact arithmetic (Common Core State Standards Initiative, 2010). However, a few studies suggested that children are already able to engage in computational estimation at a younger age than is traditionally expected (Dowker, 1997, 2003; Jordan et al., 2009). This might not be so surprising, given that recent studies provided empirical evidence that preschool children can use their basic numerical abilities to solve approximate arithmetic problems with both non-symbolically and symbolically presented comparisons (e.g., "15 + 13 vs. 49, which is more?") before they have been taught exact arithmetic in school.

Part 2 of our research project therefore aimed at charting the emergence and early development of computational estimation from the age of 5 (third grade of preschool) until the age of 9 (third grade of elementary school). Children's computational estimation skills were tested on an individual basis once each year. To that aim, we developed a task in which children had to estimate the outcome of addition problems and that consisted of a non-verbal and a verbal variant. In the non-verbal variant, which was used in third grade of preschool and first grade of elementary school, the estimation problems were presented by means of manipulatives. Both addends were represented by a number of cows that were consecutively positioned in a horizontal row in front of the child, verbally labeled by the experimenter ("Here are N cows"), and hidden in a stable afterward. Next, children were asked to indicate about how many cows there were altogether in the stable by putting a number of cows from their own pile on the table (see Fig. 3.3).

In the verbal variant, which was used in the first three grades of elementary school, the estimation problems were presented with Arabic numerals on a computer screen for 20 seconds and simultaneously read out loud by the experimenter after which children had to respond verbally. To ensure that children would engage in computational estimation, we presented them with addition problems that were numerically just too difficult to be solved by means of exact arithmetic (Dowker, 1997, 2003). The level of exact arithmetic in each grade was based on children's curriculum and extensive pilot testing. Over the entire duration of the longitudinal study, children had to estimate the outcome of 24 addition problems of different difficulty levels, which were defined by the size of the exact outcome of the estimation problems. More specifically, these outcomes ranged between 11 and 30 in third

(a) (b) (c)


Fig. 3.3 Example of a computational estimation problem from the non-verbal task variant with (**a**) the first addend, (**b**) the second addend, and (**c**) the child's answer

year of preschool, 11 and 100 in first and second years of elementary school, and 51–10.000 in third year of elementary school.

We focused both on computational estimation performance and strategy use (see e.g., Sekeris et al., in press). At present, data have been collected from third year of preschool until second year of elementary school. Estimation performance in both tasks was measured in terms of children's accuracy and was operationalized in terms of percentage absolute error (PAE) of children's estimates relative to the exact answer. We observed that children's PAE evolved from 34% in third year of preschool to 19% in second year of elementary school, indicating that children became more accurate in their estimates when growing older. Interestingly, in the first year of elementary school – where both task variants were administered with exactly the same problems – we found that children were, as expected, more accurate in the non-verbal (34%) than in the verbal task variant (27%). Presumably, this lower performance on the verbal task variant could be attributed to children being insufficiently familiar with two-digit numbers being represented with Arabic numerals. Similar findings have been reported by Dowker (2003) for computational estimation and Levine et al. (1992) for exact arithmetic. In both task variants we also observed an effect of problem size. Children's estimates became less accurate with increasing problem size, suggesting that children were not merely guessing the outcome of the estimation problems.

Children's strategy use was examined for both task variants separately. For the non-verbal variant we looked at two aspects of children's externally observable behavior when lying down their answer by means of the manipulatives: (a) the way in which they constructed the answer set and (b) their counting behavior while constructing the answer set. With respect to the construction of the answer set, we distinguished, based on previous studies in arithmetic (Carpenter & Moser, 1982; De Corte & Verschaffel, 1987), among three different strategies that might reflect different representations of numbers and arithmetic operations: (a) creating two sets of manipulatives representing both addends which were either kept separate (addends only) or (b) put together afterward (combining), and (c) immediately putting all manipulatives in one group (result-only). For their counting behavior we looked at whether children counted or not when constructing the answer set. Results showed that both in third year of preschool and first year of elementary school about 95% of the problems were solved by means of the result-only strategy. This frequency did not change with age or problem size. The frequency of children's counting behavior showed an age-related increase and decreased with increasing problem size. A structural equation model showed that in preschool none of the two aspects of children's material solution strategies were predictive of their estimation performance, whereas in first grade the result-only strategy was a negative predictor and counting frequency a positive predictor of their estimation performance. These findings might indicate that children in third year of preschool lack the insight that the way in which they use the manipulatives or their counting skills could help them make better estimates. By the first year of elementary school, they might have come to the understanding that a purposeful use of the manipulatives (i.e., by representing both addends first separately) and counting might lead to improved estimations.

In the verbal variant of the computational estimation task, children's strategy use was identified on the basis of immediate trial-by-trial verbal strategy reports. Strategies were classified according to an a priori classification scheme which distinguished among four broad strategy categories: (a) exact arithmetic in which children calculated the answer exactly instead of estimating it, (b) exact-calculation-and-adjusting in which the answer was calculated exactly and then adjusted to make it look like an estimate, and (c) rudimentary computational estimation strategies that showed some basic and rough conceptual understanding of the principles of computational estimation, and genuine computational estimation strategies in which the estimation problem is first simplified (e.g., by rounding the operands) before calculating the approximate outcome. We observed that children hardly used any genuine computational estimation strategies, presumably because children at this age did not yet possess the necessary mathematical knowledge and skills for applying such advanced estimation strategies. However, children already had a basic understanding of some of the underlying principles of computational estimation, as was evidenced by the fact that they referred to the proximity principle (i.e., the idea that an estimate should be close to the exact outcome) when applying a rudimentary computational estimation strategy or that they took into account the approximation principle (i.e., the estimate should be an approximation of the exact outcome) when using the exact-calculation-and-adjusting strategy. Interestingly, the use of the exact arithmetic and exact-calculation-and-adjusting strategies increased from first to second grade of elementary school. Probably, the strong focus on exact arithmetic in mathematics education at the beginning of elementary school makes children increasingly convinced that each arithmetic problem has only one correct answer.

To conclude, the present findings indicate that young children are already able to engage in computational estimation at a much younger age than is generally assumed. Their estimation performance increases with age, even in the absence of instruction in computational estimation. Young children already use a variety of strategies to solve computational estimation problems. This strategy use reveals traces of a beginning conceptual understanding of the principles underlying computational estimation. Taking into account the aforementioned multi-componential nature of computational estimation, its potential for developing number sense, and the recurrent finding that people are generally bad at it (Siegler & Booth, 2005), our findings suggest that computational estimation could be incorporated much earlier in the mathematics curriculum. Such an early learning trajectory for computational estimation could start by familiarizing young children with the concept of estimation, its underlying principles (e.g., the proximity and approximation principle), and the specific language of estimation (e.g., "about," "near," and "close to"). This earlier incorporation in the curriculum might prevent that the early development of children's estimation skills becomes too much hampered by their strong focus on being exact as a result of their confrontation with formal school mathematics.

3.5 Early Proportional Reasoning

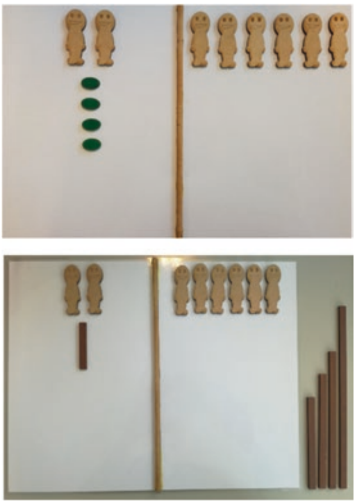
Proportional reasoning plays a critical role in people's mathematical development. It is essential in the learning of numerous advanced mathematical topics, such as algebra, geometry, statistics, or probability, but people also encounter it in numerous daily life situations (e.g., recipes, sales). Unfortunately, it is also considered to be hard to apprehend for children, and achieving a full understanding of proportionality is considered a major challenge (Kaput & West, 1994). In the research literature, there is no unanimity about the age range in which proportional reasoning abilities develop.

The traditional Piagetian stance on the development of proportional reasoning is that it is a rather late achievement (Inhelder & Piaget, 1958). They see it as an indicator of formal operational thought, typically only starting to develop from the age of 12. Typical evidence comes from tasks like the Paper Clip Task (Karplus & Peterson, 1970): learners get the height of Mr. Tall and Mr. Short expressed in a number of buttons, and the height of Mr. Short in expressed in a number of paper clips. They need to find the height of a Mr. Tall expressed in paper clips. Academically upper-track or upper middle-class students used proportional reasoning increasingly at the age of 12 years, but only a small fraction of urban low-income and academically lower-track students used proportions at the age of 14 or even 17 years. Similar findings come from Noelting (1980), who used Orange Juice Problems: comparing mixtures of varying numbers of glasses of orange juice and water. He reported that proportional reasoning is a concept that finds its achievement only in late adolescence and that children did not reach the formal operational level before the age of 12.

However, more recent studies suggested that proportional reasoning may start to develop much earlier than suggested by Piaget and colleagues. We mention a few examples. Resnick and Singer (1993) presented 5- to 7-year-old children with a proportional missing-value task. Children had to feed fish of different lengths. All children tended to give proportionally larger amounts of food to larger fish. Boyer and Levine (2012) used an orange juice task to assess proportional reasoning in 6-to 9-year-old children. Results showed that these young children could already match equal proportional mixtures, but performance depended on the scaling magnitudes in the problems. Finally, in preparation for part 3 of the current longitudinal study we also found early traces of proportional reasoning in 4- to 5-year-old children (Vanluydt et al., 2018). Many of them were able to make the ratio between puppets and grapes in a set B equal to the ratio between puppets and grapes in a set A, and strategies pointed to the emergence of a notion of one-to-many correspondence, which is an important first step in the development of proportional reasoning. While the full understanding of proportionality might only be achieved at the age of 12, the development of proportional reasoning seems to begin much earlier, allowing young children to reason proportionally in certain tasks (involving specific contexts and ratios) and under certain conditions (i.e., individual interview settings with hands-on activities).

In part 3 of the longitudinal study, we are mapping the development of children's proportional reasoning ability from the age of 5 (last year of kindergarten) until 9 (third year of elementary school). For this purpose, we developed and validated a task about a fair sharing context, involving manipulatives, and avoiding the need to use number symbols (Vanluydt et al., 2019). Children are given missing-value problems involving discrete and/or continuous quantities. In tasks with discrete quantities, they have to construct a set B equivalent to a comparison set A by putting the elements in set B in the same ratio as the elements in set A. The two discrete quantities are puppets and grapes that need to be shared among them. In tasks with continuous quantities, the context is similar, but the grapes are replaced by a continuous quantity, chocolate bars of varying lengths. Example items are shown in Fig. 3.4.

We are currently awaiting the results of the longitudinal study to map the development in detail, and to link it to various learner characteristics. A cross-sectional exploration with a comparable sample (Vanluydt et al., 2019) already revealed several qualitatively different early stages of proportional reasoning, in which the nature of the quantities involved in the problem (discrete vs. continuous) as well as the unknown quantity (the grapes/chocolate or the puppets) played a role. For instance, while performing equally well in general, some children showed a greater ability to reason proportionally when the problem involved only discrete quantities, whereas others performed better when continuous quantities were involved. Some children already showed full mastery on the proportional reasoning tasks at the age of 9, but most children were still developing this ability. Our longitudinal data will allow to reveal which children progress fastest and furthest by the age of 9: those who can reason about discrete quantities at an early age or those who can reason about continuous quantities.



Instruction: "All puppets are equally hungry. If I give four grapes to these puppets, how many grapes do you have to give to these puppets for it to be fair?"

Instruction: "All puppets are equally hungry. If I give this chocolate bar to these puppets, which chocolate bar do you have to give these puppets for it to be fair? You can give a chocolate bar to the puppets so that it's fair."

Fig. 3.4 Example items of the proportional reasoning tasks involving discrete and continuous quantities

Along with the study of the development of early proportional reasoning, we were also able to investigate how other mathematical competencies, such as mathematical patterning, are associated with children's proportional reasoning ability. We have shown the predictive association between patterning in the second year of kindergarten and proportional reasoning ability in the first year of elementary school. Two measures of patterning ability (repeating and growing patterns, see paragraph 3) were used as a predictor for two measures of proportional reasoning ability (involving a discrete or a continuous quantity). Patterning ability turned out to be a unique predictor of proportional reasoning ability over and above sex and general cognitive and numerical abilities. More specifically and quite remarkably, performance on repeating patterns was uniquely related to performance on proportional reasoning with a discrete quantity, whereas performance on growing patterns was uniquely related to performance on proportional reasoning with a continuous quantity.

Another aspect that we investigated is the role of language abilities in proportional reasoning. It is generally known that language – be it language in general or language related to mathematics – plays a crucial role in mathematical thinking and learning (Peng et al., 2020). However, so far no studies studied the role of language in proportional reasoning at an early age. We longitudinally investigated if specific mathematical vocabulary related to proportional reasoning (e.g., understanding expressions like "half" or "three times more") in the first year of elementary school predicts proportional reasoning abilities in the second year of elementary school. A hierarchical linear regression analysis showed that specific mathematical vocabulary related to early proportional reasoning in the first year of elementary school is a unique predictor for proportional reasoning abilities in the second year of elementary school over and above age, socio-economic status (SES), and general vocabulary (Vanluydt et al., 2021). Although more evidence based on intervention studies is needed to reveal the causal nature and the direction of this relation, these results suggest more attention to specific mathematical vocabulary related to proportional reasoning in young children might stimulate early proportional reasoning.

Several other studies are planned using the available longitudinal data, in order to obtain a deeper understanding of the development of proportional reasoning abilities at a young age. From second grade on, we started to offer arithmetic word problems, in addition to the proportional reasoning fair sharing tasks with manipulatives that were described above. Some of these word problems are proportional, but also additive word problems are included, such as the following:

> Roos and Loes are running around a track. They run equally fast, but Loes started later. When Loes has run 2 rounds, Roos has run 8 rounds. When Loes has run 4 rounds, how many has Roos run?

The literature (e.g., Van Dooren et al., 2010) reports that young children often erroneously solve proportional problems additively while older children solve additive problems proportionally (in the problem above, they would answer that Roos has run 16 rounds). Our longitudinal data will reveal whether early individual differences in proportional reasoning abilities predict these two kinds of errors.

So far, our findings indicate that children in the third grade of kindergarten and the first years of elementary school can make sense of the one-to-many correspondences in proportional situations and suggest that these may already be stimulated and developed into an understanding of many-to-many situations. This seems possible even before the arithmetic skills for addition and multiplication are extensively practiced. Attention to the specific mathematical vocabulary involved in proportional situations seems important in doing so. We are currently developing instructional materials for this purpose, which will be tested in an intervention study.

3.6 Early Probabilistic Reasoning

Parallel to the research and discussion about early proportional reasoning, there is a growing body of developmental research showing that very young children have basic intuitions about chance events and that these intuitions develop into a more formal probability concept during elementary school (Bryant & Nunes, 2012; Piaget & Inhelder, 1951/1975). The successive developmental stages of probabilistic reasoning have been given several labels, but boil down to three main stages: non-probabilistic reasoning (preoperational; until the average age of 6 years), emergent probabilistic reasoning (concrete operational; from 6 to 11 years old), and finally quantification of probability (formal operational; from about the average age of 11 years) (Green, 1991; Jones et al., 1999; Way, 2003).

Preliminary results on these basic intuitions in young children have already been obtained with respect to several components of probabilistic reasoning: understanding randomness, working out the sample space, comparing and quantifying probabilities, and understanding relations between events (Bryant & Nunes, 2012). However, the developmental pathways of these components and their relation to the development of other competencies remain largely unexplored.

Based on these descriptive developmental studies, many countries around the world have introduced probability calculus as part of the curriculum in elementary school in the 1990s (Way, 2003). More recently, in two southern German states, Baden-Württemberg and Bayern, the basics of probabilistic thinking are included in the elementary school curriculum partly as a result of the rising awareness of the importance of "risk competency" (Granzer et al., 2009; Martignon & Erickson, 2014; Till, 2014). However, little is known about the effects of teaching probability and statistics in elementary school or about the processes involved.

With respect to probabilistic reasoning, our project had three main objectives. First, we aimed to construct a more comprehensive view on the development of different components of probabilistic reasoning in children from the age of five to nine. Second, we wanted to explore the relationship between the development of numerical abilities, mathematical patterning, computational estimation, and proportional reasoning on the one hand, and the development of probabilistic reasoning among elementary school children on the other hand. Our expectation was that these other abilities are important building blocks for emergent probabilistic reasoning. A third objective was to investigate whether it is possible to stimulate probabilistic reasoning at a younger age than is currently the case in Flemish schools.

This part of the project is still ongoing, but we already have some first results from pilot studies and analyses from the first wave of the longitudinal study. Because we needed an instrument for the early assessment of probabilistic reasoning, we constructed several tasks that tapped into children's ability to recognize (un)certainty and children's ability to compare probabilities. The basic setup is an individually administered binary choice task in which children have to select one out of two boxes that has the best chance to blindly pick a winning element. The concrete setup is an adapted version of the setup proposed by Falk et al. (2012) and goes as follows (see Fig. 3.5):

> Children sit in front of a laptop screen. They are introduced to a blindfolded bird and are told that the bird loves black berries but hates white berries. In each trial, the bird blindly picks a berry from one of two boxes that are filled with different number of berries of the desired and undesired color (see Fig. 3.5). Unlike the bird, children can see the content of each box and they are asked to help the bird by deciding which of two boxes is best for the bird to blindly pick a berry from.

An interesting property of this setup is that the difficulty of the items can be varied meaningfully by manipulating their features. For example, it is possible to vary the total number of berries, the proportion of black berries, and even more than two colors of berries can be used (after slightly adapting the instruction). Based on the study by Falk et al. (2012), we expected that items would become particularly challenging to the children if the optimal box would contain a smaller absolute number of black berries (see Fig. 3.6); and even if there are no white berries left in that box (see Fig. 3.7).

After pilot testing, the final instrument consisted of 29 items. For an independent validation and feasibility study, we presented the instrument to a cross-sectional sample of 177 5- to 9-year-olds in a school who did not participate in our larger longitudinal data collection. We found that our instrument was fit to use in kindergarten and elementary school. The children understood our instruction and it took no longer than 10 minutes to administer the task. Furthermore, the results were encouraging from the perspective of assessing probabilistic reasoning at these

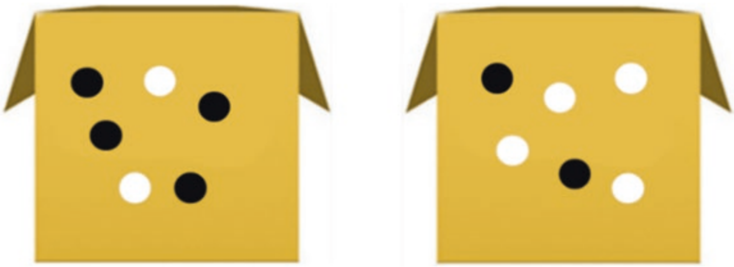


Fig. 3.5 Example item for the probabilistic reasoning task: Select the box that gives you the best chance to randomly draw a black berry from

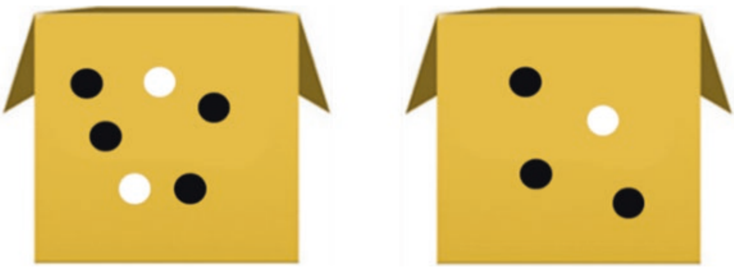


Fig. 3.6 Example of a difficult item for the probabilistic reasoning task: The box with the smaller number of black berries has a larger probability to randomly draw a black berry from

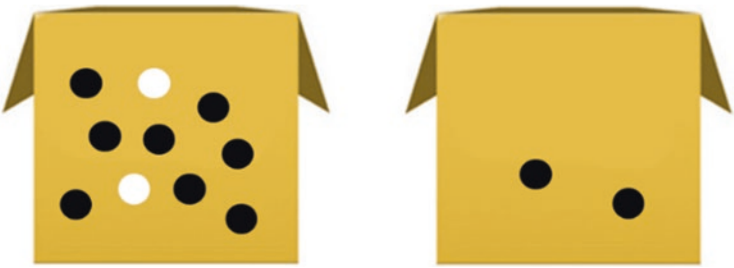


Fig. 3.7 Example of a difficult item for the probabilistic reasoning task: The box with the smaller number of black berries has a 100% probability to randomly draw a black berry from

young ages: item difficulty varied as expected, older children obviously had better performance than younger ones, and we found no indications for floor or ceiling effects in any age group. The extensions that we added to the setup by Falk et al. (2012) also seemed to improve the reliability and validity of the instrument (Supply et al., 2018, 2020).

When we applied this instrument to the 5- and 6-year-olds in our longitudinal study, we found that children within the same year of kindergarten strongly differed in their performance on the items that had one box with 100% probability to randomly draw a black berry from. Furthermore, children's performance on these items was predictive for their performance on the items that required a comparison of probabilities of only uncertain outcomes. These results demonstrate that, although conventional developmental theory assumes that there is no probabilistic reasoning in the preoperational stage, kindergarten children already have good performance in certain tasks that require probabilistic judgments. In addition, the recognition of uncertainty may act as a precursor for emergent probabilistic reasoning (Supply et al., 2019a). In these 5- and 6-year-old children, we also explored the relation between the performance on the numerical tasks that were administered as part of the longitudinal study (see Sect. 3.1) and our binary choice instrument, extended with a construction task. For the construction task, children were introduced to two representations of identical birds, two rectangular boxes containing white and black marbles, and one larger square box containing 10 black marbles (see Fig. 3.8).

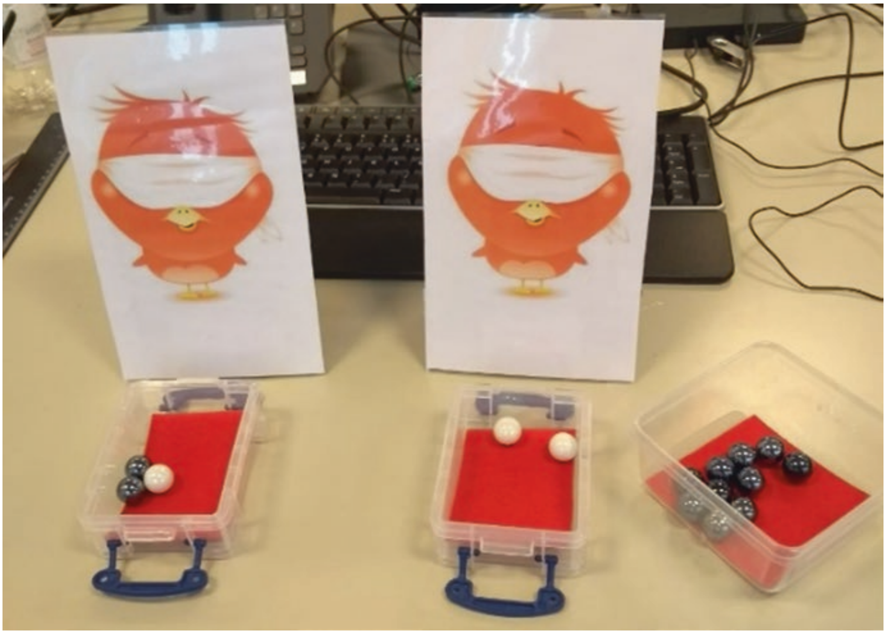


Fig. 3.8 Example of an item for the construction task: Add black berries to the right-side rectangular box to have an equal probability to randomly draw a black berry from each of the boxes (the square box on the right hand contains the black berries that can be used to make the adjustment)

As with the binary choice instrument, the construction task was administered individually. The instruction was as follows:

> These are Flip and Flap. Flip and Flap are twins. Flip and Flap are both blindfolded because we are going to play a game with them. Flip and Flap both like black berries (experimenter shows child black marbles), but get sick of these white berries (experimenter shows child white marbles). I will always give a box to each Flip and Flap and they can each blindly pick one berry from their own box. Of course, Flip and Flap cannot see what is in the box, because they are wearing that blindfold. Flip's box contains white and black berries, but Flap's box always contains only sickening white berries. That is not fair of course. You can add black berries to the box of Flap so that it becomes a fair game. You can add as many berries, until you think that Flip and Flap are just as likely to blindly pick a black berry when they are blindfolded and allowed to pick only once in their own box.

We found a strong general association between the performance on the numerical tasks and the items that required a comparison of probabilities of only uncertain outcomes. There was no association between numerical skills and the ability to distinguish uncertain from certain events, and we also found no association with the performance in the construction task. In the latter task, children with better numerical skills tended to add as many winning elements to the new box as there were in the box that was given, thereby ignoring the number of losing elements in the given box. This suggests that at this young age, good early numerical skills might promote the use of erroneous strategies in probabilistic situations. Future research could investigate whether these erroneous strategies can be seen as the first step in reasoning about probabilities or whether they impede proper probabilistic reasoning (Supply et al., 2019b).

In sum, our preliminary findings suggest that probabilistic situations are already intelligible for 5- to 6-year-olds. At this age, children have not been formally introduced to addition, multiplication, and proportionality, but nevertheless are able to give meaningful answers in binary choice tasks that involve probabilistic optimization. These findings challenge the common notion that probability as a mathematical topic is too difficult for elementary school children and should only be included in the curriculum of secondary school or university. As such, these findings open up a perspective for a learning trajectory on probability and statistics from kindergarten (e.g., by playing games of chance) to elementary (e.g., calculating probabilities) and secondary school (e.g., deriving Bayes' theorem).

In our opinion, this perspective is of paramount importance because the inclusion of probability as a topic in the elementary school curriculum can act as a counterweight to current mathematics and science curricula that – from the first years on – put a strong emphasis on exact arithmetic with small cardinal numbers, deterministic causal explanations, and certitude and that instill a view of science and a view of the world that leaves no room for doubt, uncertainty, intrinsic stochastic processes, or measurement error. However, we must also acknowledge that our finding of developing probabilistic reasoning in 5- to 6-year-olds does not imply that education can improve or accelerate this development. Therefore, an additional intervention study is planned to investigate whether it is possible to stimulate probabilistic reasoning at a younger age than is currently the case in Flemish schools.

3.7 Conclusion

In this chapter we gave a snapshot of the main results of a 6-year-long research project that started in 2016 and in which we longitudinally follow the integrated development of 4- to 9-year-olds' competencies in four challenging mathematical domains – mathematical patterns, computational estimation, proportional reasoning, and probabilistic reasoning – using a rich battery of measures.

The preliminary findings of the longitudinal study confirm our basic claim that, with respect to these four core mathematical competencies, important initial steps are being made in children's development (much) earlier than traditionally thought. Many preschoolers were able to handle repeating patterns and some even showed beginning mastery of growing patterns; a significant number of them solved computational estimation problems in ways that suggest a nascent conceptual understanding of the principles underlying computational estimation; many of them were already able to reason proportionally and to make probabilistic judgments in certain tasks and under certain conditions.

We found that these four early mathematical competencies showed unique associations with children's numerical abilities. These associations were observed both cross-sectionally and longitudinally. For example, we observed for the first time that the association between patterning and numerical skills changed from bidirectional to unidirectional (i.e., from patterning to numerical ability) in 4- to 6-year old children (Wijns et al., 2021a, b), but further work is needed to further pinpoint the direction of associations between these two abilities.

Furthermore, we also observed that these four early mathematical competencies were interrelated. For example, patterning in 4- to 5-year-olds turned out to be a unique predictor of proportional reasoning one and a half year later over and above various general cognitive and numerical abilities (Vanluydt et al., in press).

As was exemplarily shown for patterning, it is important to look not only at the ability side of young children's early core mathematical competencies, but to look at the dispositional side of these competencies too. For this competence, we found individual differences in 4-year-olds' spontaneous focusing on mathematical patterns (SFOP), as well as significant associations between their SFOP scores and their scores on the patterning and numerical ability measures, which might be explained via the mechanism of self-initiated practice (cf. Hannula & Lehtinen, 2005), in line with what has already been reported for other spontaneous mathematical focusing tendencies, such as spontaneous focusing on numerosity (SFON), spontaneous focusing on number symbols (SFONS), and spontaneous focusing on mathematical relations (SFOR) (Verschaffel et al., 2020).

An outstanding strand is the understanding of the cognitive origins of individual differences in the abovementioned four mathematical competencies. There is a large body of research that has examined individual differences in children's mathematical development (e.g., Dowker, 2005), but again, this work is largely restricted to the study of numerical abilities and arithmetic, both in children with high and low achievement in mathematics. On the one hand, this strand will be informative for the study of children who excel in their mathematical achievement. Research on excellence in mathematics almost exclusively focused on adolescents and adults (e.g., Lubinski & Benbow, 2006; Preckel et al., 2020) and hardly anything is known about the early seeds of this excellence in elementary school and earlier. It has been posited that numerical and arithmetic abilities, although useful, do not necessarily represent the quintessence of excellence in mathematical achievement (Krutetskii, 1976). As the abovementioned challenging domains are mathematically more complex than number and arithmetic, they might allow high achievers to show their mathematical potential. Our longitudinal data will allow us to investigate whether children who excel in mathematics in Grades 2 and 3 of elementary school also excel in the abovementioned mathematical competencies in earlier grades of elementary school and even preschool, and verify to which extent this excellence can be explained by domain-general cognitive capacities, such as spatial skills or working memory. On the other hand, this strand also has implications for the study of children with low mathematics achievement, a research area that has traditionally been focused on the study of numbers and arithmetic. Our longitudinal data will also allow us to investigate whether children with low achievement in mathematics are also at risk for developing difficulties in patterning, computational estimation, proportional reasoning, and probabilistic reasoning. Again, we will be able to identify to which extent these difficulties can be explained by domain-general cognitive capacities.

Finally, throughout the chapter we have pointed at several places how the findings of our longitudinal study may contribute to the development of educational standards, learning trajectories, and instructional tasks and techniques that give mathematical patterns, computational estimation, proportional reasoning, and probabilistic reasoning a more prominent place in early mathematics education. In doing so, these changes in the early mathematics education curriculum and practice will make early mathematics education more challenging and inclusive for all young children, and provide them a better preparation for the challenges of the mathematics curriculum of the upper elementary school. However, we are well aware that concrete educational recommendations should be based on findings coming from intervention studies that test the feasibility and effectiveness of these more challenging early mathematical curricula and designs in real educational settings.

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Chapter 4
Mathematical Modelling as a Stimulus for Curriculum and Instructional Reform in Secondary School Mathematics

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4.1 Mathematical Modelling in the School Curriculum

Curriculum documents often advocate making connections to the real world through the use of mathematical applications and modelling as a means of motivating and engaging students, as well as illustrating the usefulness of mathematics to describe and analyse real-world situations. Although applications and modelling are often coupled together, they have received differing attention in school curricula, with modelling viewed as being more challenging and complex than applications. Applications tasks are typically well structured and demonstrate the relevance of particular mathematical content for solving a problem set in a real-world context. The task situation is fully described; all assumptions about the situation are made explicit; and students know they will normally use all the data provided in order to find the solution. Modelling tasks, on the other hand, are usually more open and require mathematisation of a real-world situation. It is up to the modeller to define the real-world problem, specify assumptions and choose variables, identify relevant mathematical knowledge and tools, formulate and solve the mathematical problem, interpret and validate the solution, and modify the model if necessary. Teaching the modelling process is often viewed as a worthwhile educational goal in itself.

The inclusion of mathematical modelling and applications within school curricula has a history dating back to the early twentieth century. In 1904 the German mathematician Felix Klein developed a new curriculum that placed a larger emphasis on the inclusion of applications in the instruction of secondary mathematical education (Krüger, 2019). Much later, in the 1970s and 1980s, the focus on mathematical modelling and applications came to the fore in many English-speaking countries as large-scale curriculum projects encompassing modelling and

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applications were developed (see Schukajlow et al., 2018, for more on this). This resurgence in the focus on modelling and applications led to the first biennial conference series on the teaching and learning of mathematical modelling and applications in the University of Exeter in 1983, which in 1987 was rebranded as the International Conference on the Teaching of Mathematical Modelling and Applications (ICTMA). A regular working/topic group on mathematical modelling and applications was also included at the quadrennial International Congresses on Mathematical Education (ICMEs). Although they are linked, the distinction between modelling and applications is evident when we realise that when considering mathematical applications, we are looking for ways to use mathematics that has already been chosen, that is, moving from the mathematical world to the real world. In these situations, the necessary mathematical tools and models are already learnt and exist. On the other hand, with modelling we are focusing on the process of finding some mathematics that will help us understand and potentially solve the real-world problem. In this case the model must be constructed through understanding, simplifying, and mathematising the real-world scenario.

While modelling and applications play more significant roles in many countries' curricula and classrooms than in the past, the difficulties of implementing widespread curriculum change represent core barriers to bringing about changes in mathematics teaching and learning (Burkhardt, 2018). Consequently, the inclusion of authentic modelling activities in mathematics classrooms is still rather scarce and sporadic. Recent mathematics curriculum reform in Ireland has promoted a move away from calculation using learned procedures towards engaging students in authentic, challenging tasks. Since 2008 the newly introduced secondary mathematics curriculum has advocated for the use of contexts and applications to develop students' problem-solving abilities and to assist them in seeing the value and relevance of the mathematics being taught (National Council for Curriculum and Assessment [NCCA], n.d.). Additionally, the recent inclusion of a "classroom-based assessment" component at lower secondary school level, which requires students to apply their mathematical knowledge to address a problem of their own choosing, again highlights the emphasis on mathematical modelling and applications in the curriculum.

Although the Irish secondary mathematics curriculum has undergone several major changes in recent years, targeted at making it more applicable and relevant in nature, there is a lack of evidence that this is happening effectively in practice. One possible reason for this is because mathematical modelling is seen to be challenging for both students and teachers. The facilitation of a modelling activity is challenging for teachers as it requires them to mediate a lesson in a manner that they may not have previously received training in nor been sufficiently exposed to as a viable approach to the teaching and learning of mathematics. Moreover, mathematical modelling may not explicitly be listed in the curriculum documentation, and therefore, the incentive to regularly engage in modelling activities in the mathematics classroom may be lacking as they are viewed by many teachers as time consuming and challenging to assess (Blum, 2015). Additionally, in Ireland many teachers still rely heavily on textbooks and use them primarily as their main source of information when it comes to the planning and conducting of lessons (O'Keeffe, 2011; O'Sullivan, 2017). While textbooks may be able to assist teachers in delivering a more standardised curriculum, they offer insufficient advice and ideas regarding the planning and execution of modelling activities and so many teachers may find themselves lacking in confidence and knowledge to properly carry out appropriate mathematical modelling activities (Ang, 2010). Finally, teachers may be reluctant to utilise modelling as a teaching strategy because of the open nature of the tasks and the fact that it is not always clear in advance what mathematical tools and models are available, what assumptions need to be made, and what outcomes can be expected. This lack of certainty can leave teachers feeling underprepared and requires a significant paradigm shift in how teachers view their role in the classroom; moving away from the position of being the authority on the subject knowledge towards acting as a facilitator whose role is to question and query students' approaches and strategies rather than provide answers and guide students towards a single correct solution.

From the student's perspective, mathematical modelling is a demanding activity that requires them to possess a rich and connected mathematical knowledge in parallel with possessing other traits such as perseverance, curiosity, and creativity. The ability to "understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" is defined by Niss (2003, p. 7) as mathematical competence. A key element within the development of mathematical competence is the ability to model mathematically, that is, to be able to analyse and build models. For this reason, and others, many countries around the world now accept that the ability of students to model mathematically should be a key component within their school curricula. Additionally, modelling develops within students the ability to solve real-world problems that they may encounter outside of school, in society, or even in their future careers and so is deemed to be a valuable skill to foster and develop within a school curriculum (Mousoulides, 2009).

This brief analysis suggests that there are many elements of curriculum development and implementation that come into play when considering how to introduce mathematical modelling into the school curriculum. These considerations are captured in the analytical framework presented next, which guides our case study investigations.

4.2 Analytical Framework: Curriculum Policy, Design, and Enactment

Remillard and Heck (2014) defined curriculum as "a *plan for the experiences* that learners will encounter, as well as the *actual experiences* they do encounter, that are designed to help them reach specified mathematics objectives" (p. 707, original emphasis). This definition indicates that a curriculum is more than a list of topics or learning objectives, and it points to the distinction between what the curriculum intends and what actually happens in classrooms. Remillard and Heck synthesised and extended existing conceptual frameworks for curriculum that examine relationships between curricular intent and educational outcomes as well as how different actors reformulate curriculum at different levels within an educational system. Their resulting model (shown in Fig. 4.1) delineates the features of a broader curriculum policy, design, and enactment system.

The *official curriculum* is specified by governing authorities and sets out expectations for students' learning. Remillard and Heck (2014) identify three components of the official curriculum: (a) curricular aims and objectives, (b) the content of consequential assessments, and (c) the designated curriculum. In Ireland, the official school curriculum is prepared by the National Council for Curriculum and Assessment (NCCA), and there are separate curriculum specifications for subjects in the junior and senior secondary school. At both these levels, expectations for student learning are expressed as learning outcomes that describe what students should know, understand, and be able to do as a result of having studied mathematics. For example, in every content strand of the senior secondary mathematics curriculum, students are expected to "devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions" (NCCA, 2015b, p. 15). Inclusion of the content of consequential assessments in the official curriculum acknowledges the influence of high-stakes assessment on

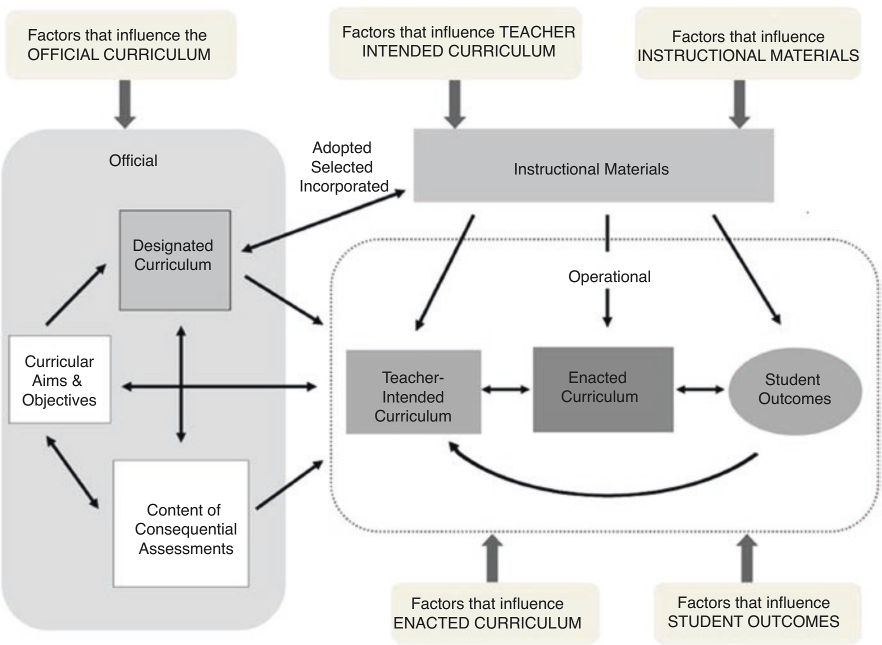


Fig. 4.1 The curriculum policy, design, and enactment system (Remillard & Heck, 2014)

curricular goals. Yet it is common for state mandated assessments to sample only a subset of curriculum goals, especially those goals related to knowledge and skills that are amenable to measurement via a timed written test. In Ireland, the consequential assessment of students' performance in each senior secondary school subject takes place in the final Leaving Certificate examination. The third component of the official curriculum, the designated curriculum, refers to the instructional plans, textbooks, and other materials that might be specified by a ministry of education to offer guidance towards addressing the curriculum's goals. Remillard and Heck note that across educational systems there is variation in the form and specificity of the designated curriculum. In Ireland, the education system does not specify a designated curriculum and schools are free to select from instructional resources produced by commercial publishers, professional associations, or support services within the education system.

In Remillard and Heck's (2014) model, the *operational curriculum* includes (a) the teacher-intended curriculum, (b) the enacted curriculum, and (c) student outcomes. Thus, the operational curriculum represents the transformation of the official curriculum into teachers' personal plans, whether these are in writing or in the teachers' minds, and how these plans play out in the interactions between teachers and students in the classroom.

Around the perimeter of Fig. 4.1, Remillard and Heck (2014) pointed to factors that influence elements of the official curriculum and the operational curriculum. They noted that these factors "may be social, political, cultural, structural, or cognitive" (p. 714). Drawing on existing research, they identified influencing factors such as societal needs, values, expectations, and beliefs; views of individuals and groups wielding power; research on learning, teaching, and assessment; teacher knowledge, beliefs and practices; teachers' access to resources and support; contextual opportunities and constraints; and a range of student characteristics and cultural resources. These factors interact with each other in complex ways, and their degree of influence on the curriculum system may be either direct or subtle. Remillard and Heck also commented that further research is needed in order to explore and elaborate on the ways in which these factors exercise influence.

In this chapter, we draw on Remillard and Heck's (2014) curriculum system model to address the following research question:

What factors support or hinder the implementation of modelling as an exemplar of mathematical challenge in the school curriculum?

To answer this question we present two case studies illustrating how modelling is being introduced into the secondary school mathematics curriculum in Ireland. The first case study explores contested attempts to infuse a modelling focus into the specialist Applied Mathematics curriculum at senior secondary level. The second discusses a university-led professional development project that exploited the Transition Year – a non-academic year between the junior and senior secondary school examination cycles – as an opportunity to introduce teachers to modelling tasks and pedagogical strategies. Each case study begins with an account of the curriculum and educational context. This is followed by an analysis of curriculum change in terms of factors that are influencing either the official or operational curriculum.

4.3 Case Study 1: Modelling as a Stimulus for Mathematics Curriculum Reform

4.3.1 Background to Applied Mathematics Curriculum Reform

In Ireland, senior secondary students typically study between six and eight subjects for the final Leaving Certificate examination (O'Meara & Prendergast, 2017). Mathematics is considered a core subject that can be taken at Ordinary or Higher level, but without being compulsory. However, it is treated as such by schools due to the fact that mathematics is a gatekeeper for the vast majority of tertiary courses (Prendergast et al., 2020). On the other hand, Applied Mathematics is an additional, optional subject which is only available to students in a small number of secondary schools. Even in these schools, Applied Mathematics is not usually offered as part of the daily timetable. Instead, students take the subject either "off timetable", that is, with lessons before or after school, or with a private teacher outside school hours.

Applied Mathematics is viewed as a subject which "mirror(s) a section of the Leaving Certificate Physics syllabus" (NCCA, 2014, p. 1). Its subject matter differs from that of the mainstream Mathematics subject at senior secondary level, which focuses on statistics, probability, geometry, trigonometry, number, algebra, functions, and calculus. Applied Mathematics instead deals with topics from the domain of physics known as mechanics, including laws of motion, projectiles, and statics (State Examinations Commission, 2018). First introduced in Ireland over 40 years ago, the Applied Mathematics syllabus has undergone very few changes in the intervening years. The syllabus lacks an explicit aim or rationale and consists only of the list of topics to be examined. However, in late 2014 the National Council for Curriculum and Assessment undertook a review process with the ultimate aim of revising this very dated curriculum. There were a multitude of concerns which led to the review of the Applied Mathematics curriculum and shaped the revised curriculum.

Firstly, there were concerns about the low numbers of students choosing Applied Mathematics. Figure 4.2 summarises data collected by the State Examinations Commission between 2014 and 2019, which shows the number of students who sat the Leaving Certificate examination across a range of different science subjects.

Figure 4.2 clearly highlights how, in the period from 2014 to 2019, Applied Mathematics was the least popular of all the science subjects among Leaving Certificate students. Data collected by the State Examinations Commission (2018) show that, from 2014 to 2018, Applied Mathematics candidates comprised around 3.2% to 3.8% of the full Leaving Certificate cohort.

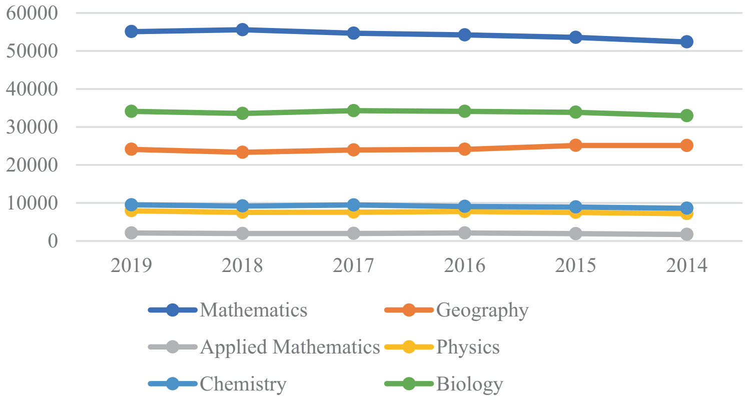


Fig. 4.2 Number of candidates in Leaving Certificate science subjects from 2014 to 2019

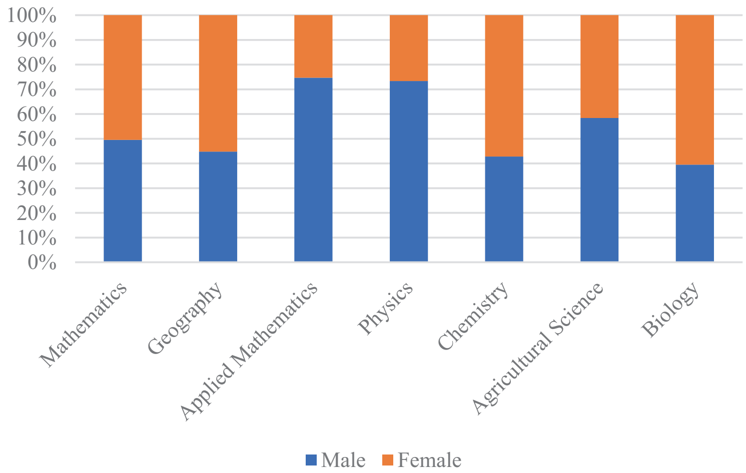


Fig. 4.3 Gender participation in Leaving Certificate science subjects 2019

In addition to poor uptake, concerns were also expressed regarding the gender imbalance in participation in Applied Mathematics. As demonstrated in Fig. 4.3, significantly more males than females opted for Applied Mathematics in 2019. The statistics revealed that 25.3% of those who sat the Applied Mathematics examination were female. Corresponding figures for physics, chemistry, and biology were 26.6%, 57.2%, and 60.5%, respectively. Similar statistics were reported by the State Examinations Commission in previous years also.

A third concern in relation to the dated Applied Mathematics curriculum was connected to the misalignment between the Leaving Certificate Mathematics and Applied Mathematics curricula. In 2006 efforts began to completely overhaul the senior secondary Mathematics curriculum in Ireland with a new curriculum, known locally as Project Maths, being introduced nationally in 2010. As discussed previously, the new curriculum promoted a fundamental shift in the teaching and learning of mathematics in Ireland. The emphasis changed from examination-driven teaching that promoted memorisation and practice of learned procedures to student-centred teaching that promoted conceptual understanding (Department of Education and Skills, 2010). However, this change to the Mathematics curriculum did not occur in tandem with a change to the Applied Mathematics curriculum and resulted in the differences between the two curricula becoming even more stark. For example, while problem solving and modelling were at the heart of the new Mathematics curriculum, Applied Mathematics continued to focus on particular types of problems in mechanics that were "quite disconnected from 'applications'" (O'Reilly, 2002, p. 1). Furthermore, the change to the Mathematics curriculum resulted in examination questions becoming much less predictable (Shiel & Kelleher, 2017). However, this was not the case with Applied Mathematics. In fact, many teachers involved in the consultation process for Applied Mathematics were of the opinion that "...you can train people to do well in Applied Mathematics because the exam is so predictable" (NCCA, 2015a, p. 7).

Due to these concerns a consultation process began in 2014 with the publication of the *Background Paper for the Review of Applied Mathematics* (NCCA, 2014). The consultation process subsequently led to the publication of a *Consultation Report* in December 2015 (NCCA, 2015a) and following this the Mathematics Development Group agreed that the revised Leaving Certificate Applied Mathematics subject would aim to develop the learner's capacity to use mathematics to solve real-world, twenty-first century problems. The new Applied Mathematics syllabus is still under development, but from draft versions it is clear that it will focus on all aspects of the problem-solving cycle. In doing so it is envisaged that learners will see beyond calculating procedures and gain experience in asking appropriate questions, formulating mathematical representation of problems and interpreting and verifying results. Based on the consultation process it was also decided that the new Applied Mathematics specification would place a strong emphasis on mathematical modelling, as this was seen to be "...at the heart of modern applications of mathematics" (NCCA, 2015a, p. 18).

The draft Applied Mathematics syllabus resulting from the consultation and review process describes the subject as involving the use of the language of mathematics to study and solve real-world problems. It introduces mathematical modelling as the process through which real-world phenomena are represented, analysed, and understood. The syllabus aims and objectives are centred on the modelling process: formulating a real-world problem, translating the problem into a mathematical representation, solving the mathematical problem, and interpreting the solution in the original real-world context. The syllabus is organised around four strands:

- Mathematical modelling
- Mathematical modelling with networks and graphs
- Mathematically modelling the physical world
- Mathematically modelling a changing world

This organisation foregrounds the modelling process as a means by which to explore mathematical content.

4.3.2 Factors That Support or Hinder the Implementation of Modelling in the Applied Mathematics Curriculum

We analysed the two key documents that initiated the Applied Mathematics curriculum reform (the *Background Paper*) and collected stakeholder responses (the *Consultation Report*), looking for evidence of the influencing factors on the official curriculum that were identified by Remillard and Heck (2014).

The first factor acknowledges the *perceived and expressed needs of society*, and this was a source of tension in the consultation process. In the *Background Paper*, the context for curriculum reform was framed in terms of economic needs and the STEM (science, technology, engineering, and mathematics) agenda, with STEM education underpinned by mathematical knowledge and skills seen as "help[ing] Ireland to generate the capable and flexible workforce needed to compete in a global marketplace" (NCCA, 2014, p. 5). However, participants in the consultation process "who were critical of the background paper claimed it had been overly influenced by the needs of third level and industry and that it was prepared in response to PISA" (NCCA, 2015a, p. 7).

A second influencing factor is related to *advancements in the fields of mathematics, learning, educational practice, and technology*. As we have previously indicated, there had been many advancements to mathematics education in Ireland since the introduction of the original Applied Mathematics curriculum. In particular, the revised senior secondary Mathematics curriculum, launched in 2010, seemed to act as a catalyst for the reform of the Applied Mathematics curriculum. The *Consultation Report* indicated that many stakeholders believed there was a need for better alignment between these two subjects. Similar to curriculum reforms internationally, the revised Mathematics curriculum in Ireland placed increased emphasis on problem solving, and so there was now scope for the Applied Mathematics curriculum to continue to further develop students' problem-solving skills through the lens of mathematical modelling. Giving more weight to problem solving led to removal of some content from the Mathematics syllabus, and this too led to calls for a reimagining of the content of Applied Mathematics. For example, the *Background Paper* raised the possibility of shifting some of this excluded content, such as vectors and matrices, into the Applied Mathematics curriculum. However, this proposal was not welcomed by some participants in the consultation process who claimed that the content "is there for political reasons because it was left out of Project Maths and this was seen as a deficit and will be lumped into the new spec [i.e. into the Applied Mathematics subject] to say it is there" (NCCA, 2015a, p. 7).

The *Background Paper* and consultation process gave explicit attention to the potential impact of *advancements in technology* on the new Applied Mathematics curriculum. Technology in mathematics education was now seen as “a tool with the potential to change how concepts are demonstrated, projects assigned and progress assessed” (NCCA, 2014, p. 22). The *Consultation Report* indicated all stakeholders agreed that technology had the potential to facilitate a shift towards mathematical modelling, but emphasised that technology should support, and not compromise, the development of skills. One challenge that arises here involves providing teachers with professional development opportunities. A substantial amount of research on technology in mathematics education has found that developing teachers’ technology skills in isolation is insufficient; instead, professional development must seek to simultaneously develop teachers’ technological and pedagogical knowledge (e.g. Li et al., 2019). A second challenge in the Irish context is the need for investment in technology resources to complement the revised Applied Mathematics curriculum. Research has shown that the textbook is the primary resource used in mathematics classrooms in Ireland and internationally (Nathan et al., 2002; O’Meara et al., 2020). However, textbooks alone will not facilitate the use of technology in the Applied Mathematics classroom and will not promote student-centred learning experiences in the area of mathematical modelling.

A third factor influencing the official curriculum is the *values and beliefs about mathematics and the goals of education held publicly and by individuals and groups wielding power*. The values and beliefs of different stakeholders played an important role in the instigation of reform to Advanced Mathematics, but led to some challenges in the framing/design of the revised curriculum. In particular, values and beliefs relating to what constituted problem solving and how problem solving and content knowledge could co-exist within a curriculum document with a strong emphasis on mathematical modelling came to the fore in both documents, with different stakeholders, at times, holding contrasting views. For example, the *Consultation Report* indicated that proponents of the old Applied Mathematics syllabus believed that problem solving was already central to that curriculum. They feared that any additional content included in a revised curriculum would lead to a dilution of the problem-solving aspects of the course. Hence, calls to include content omitted from the revised mathematics curriculum were not welcomed by all, with some expressing the view that “...those involved in teaching Applied Mathematics do not believe that they should have to teach mathematics. Instead they see their focus as being on teaching problem solving skills” (NCCA, 2015a, p. 12). On the other hand, the *Background Document* suggested that “with its [the old curriculum] emphasis on content as opposed to the development of skills and mathematical reasoning students are not problem solving per se” (NCCA, 2014, p. 3).

These contrasting views and beliefs present their own set of challenges for this curriculum reform effort as they will have a significant impact on the content that is included in the syllabus, as well as the skills and dispositions that the curriculum promotes. Such differences could potentially lead to an overcrowded curriculum that lacks depth and causes practical difficulties with timetabling and the allocation of class time. Similar issues have plagued previous mathematics curriculum reform in Ireland (see O’Meara & Prendergast, 2017). However, another view expressed in the *Consultation Document* in relation to the content versus problem-solving debate could offer a potential solution. One respondent suggested that Applied Mathematics should be viewed as a "subject [that] draws upon concepts and methods of mathematics from the fields of application and in turn, brings ideas, techniques and scientific knowledge back to influence the development of mathematics" (NCCA, 2015a, p. 12). Adopting this outlook would help overcome the "either/or" debate and instead mathematical content, problem solving, and mathematical understanding could be seen as key components which complement each other when engaging in the mathematical modelling process.

4.4 Case Study 2: Modelling as a Stimulus for Mathematics Instructional Reform

4.4.1 Background to the Young Modellers Transition Year Project

Transition Year (TY), an optional non-academic school year between the junior and senior secondary examination cycles, is unique to Ireland (Clerkin, 2012; Jeffers, 2007; Smyth & Calvert, 2011). It was introduced as a pilot scheme in three schools in 1974 and was mainstreamed in secondary schools by 1994. Currently, it is offered in 75% of secondary schools in Ireland (Jeffers, 2011) with just over half of the potential student cohort participating (Clerkin, 2012). The guidelines distributed to schools emphasise that TY is to be neither viewed nor utilised as an extra year to prepare students for the Leaving Certificate examination. Indeed,

> Where Leaving Certificate material is chosen for study it should be done so on the clear understanding that it is to be explored in an original and stimulating way that is significantly different from the way in which it would have been treated in the two years to Leaving Certificate. (Department of Education, 1993, p. 4).

The purpose of TY, therefore, is to replace formal study with a broad range of non-academic educational and vocational experiences in the absence of examination pressure (Department of Education, 1993; Smyth & Calvert, 2011). Teachers have great flexibility, and indeed are strongly encouraged, to create a TY programme to suit their students' needs (Clerkin, 2012; Smyth & Calvert, 2011). As such, it offers an ideal period to implement innovative educational interventions. The Young Modellers project was one such intervention that aimed to introduce mathematically challenging tasks to teachers and students.

The significance of the Young Modellers project needs to be understood in the context of the high-stakes summative assessment environment in Irish secondary schools, and teachers' perception of their role, "at least in part, as that of exam coach" committed to "covering" in class all question types that might be asked in the Leaving Certificate examination (NCCA, 2014, p. 16). Although the Leaving Certificate Mathematics curriculum purports to develop students' ability to solve mathematical problems in familiar and unfamiliar contexts, there is little evidence that this actually happens. For example, the most recent Chief Examiner's Report on Leaving Certificate Mathematics stated that, in the Ordinary level examination, "the majority of candidates seemed unable to deal with problems presented in an unfamiliar context" (State Examinations Commission, 2015, p. 23). It was observed that students were more inclined to abandon their work than persevere when difficulties arose. Students taking the Higher level examination also struggled with problem solving, and applying knowledge in unfamiliar contexts. The report concluded by recommending that teachers should

> provide students with opportunities to practise solving problems involving real-life applications of mathematics, and to get used to dealing with "messy data" in such problems. Students should also be encouraged to construct algebraic expressions or equations to model these situations, and / or to draw diagrams to represent them. (p. 30)

Young Modellers was a 10-week teaching and learning module implemented in 15 secondary schools in Ireland that served to address many of the issues referred to in the Chief Examiner's report mentioned above. The programme was designed and delivered by university-based mathematicians and mathematics educators, who have significant experience in both mathematical modelling and teacher professional development. The purpose of the initiative was to challenge students and teachers to exploit problem-solving skills to solve real-world problems that appear in science, engineering, technology, and industry, using mathematical techniques. Young Modellers aimed to engage students in how to use mathematics in realistic problems providing them with an insight into mathematics in action. It was hoped that participation in the Young Modellers programme would help develop perseverance skills and encourage different ways of approaching a problem. The Young Modellers development team wanted to encourage teachers and students to appreciate the links between mathematical concepts and skills acquired at school with the utility of mathematics in the real world by applying the mathematics that they learn at school to solve real-world problems, giving them a first-hand experience of mathematical and statistical modelling.

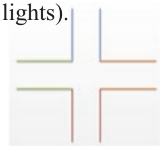
TY teachers were provided with a 2-day professional development programme on how to move from a problem formulated in non-mathematical terms to developing a mathematical solution. Throughout the programme the use of collaborative and communication skills was emphasised and encouraged. Emphasis was placed on how to formulate a problem, represent it in mathematical terms, investigate various different methods for solving the problem, and interpret that solution in terms of the real-world problem. Figure 4.4 shows an example of a modelling task from the professional development programme that illustrated important modelling strategies, such as using an appropriate representation, making simplifying assumptions, developing a simulation, specialising and generalising, and considering extreme cases. A package of teaching and learning materials was also developed for use in the classroom. This package consisted of two parts. Part 1 comprised problem-solving tasks, some of which introduced students, in a structured way, to strategies which are also useful in modelling (using appropriate representations and identifying assumptions). Part 2 provided real problems which required students to engage

Example of a modelling problem:

At which kind of intersection (a roundabout or a crossroads with traffic lights) can more cars pass a crossing?

Strategies :

- Use an appropriate representation.
- Make simplifying assumptions (e.g. number of cars/hour; direction of travel at intersection; car length, acceleration, separation; timing of traffic lights).
- Make use of symmetry.



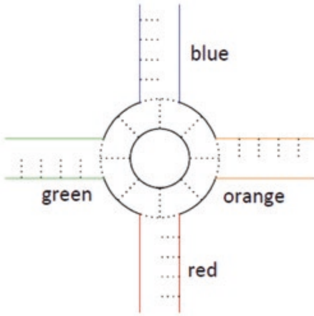


Fig. 4.4 Task from the Young Modellers professional development programme (Stender, 2019, p. 202)

in mathematical modelling. Figures 4.5 and 4.6 show two tasks from Part 1 of the package of teaching and learning materials.

Teachers then implemented Young Modellers in the classroom over a 10-week period. They were supported by members of the Young Modellers development team and received three school visits from a team member along with two PhD students during the project. Participating teachers first introduced mathematical modelling to their students and explained how modelling fits into everyday life using simple examples. Students then were presented with a selection of real problems (e.g. modelling disease spread in a population) from Part 2 of the package of teaching and learning materials. Students selected one of these real problems to work on with a small group of their peers for the period. Throughout the school visits, the research team supported students in selecting parameters, identifying assumptions, producing a simple model, and eventually finalising and testing their models. Students were required to create a presentation of their work and present it as a team in front of other school participants, and answer questions from a panel of judges at a showcase event in the host university.

4.4.2 Factors Supporting or Hindering the Implementation of Modelling in the Young Modellers Transition Year Project

The Young Modellers project tried to address several factors among those identified by Remillard and Heck (2014) as influencing both the teacher-intended and the enacted components of the operational curriculum. We discuss the nature and role

On December 28th 2015, the average price for unleaded petrol was €1.50 per litre.

A driver in Cork plans to drive to Donegal and back, a distance of approximately 850km return in one day.

The driver owns a BMW with fuel consumption of 10km per litre.

Her tz Rent a Car, Ltd., offers a Golf car rental in Cork at €45 per day.

It is estimated that the car has fuel consumption of 25km per litre.

(i) Should the driver rent a Golf to drive to Donegal or should he drive the BMW to Donegal?

(ii) Solve this problem in a different way from how you did for (i) and compare both solutions. Is your solution correct?

(iii) From your solutions:

- For what range of distances is it more expensive to rent the Golf?
- For what range of distances is it more expensive to drive the BMW?
- Illustrate your answers graphically if you have not already done so.

Fig. 4.5 Car hire task from the Young Modellers teaching and learning package (Guerin, 2017)

The back of a truck passes the exit sign for Shannon airport, travelling at 42km/hr. One and a half hours later, the back of a car passes the same exit sign for Shannon airport traveling at a speed of 63km/hr. The length of the truck is three and a half times the length of the car.

It is 3am and there is no traffic.

(i) At what distance from that exit sign, will the back of the car be in line with the back of the truck?

(ii) State all assumptions you are making in solving this problem.

(iii) Under what conditions will it be possible that the car will catch up with the truck?

(iv) Explain a few scenarios where it might be not possible for the car to catch up with the truck.

Fig. 4.6 Motorway task from the Young Modellers teaching and learning package (Guerin, 2017)

of each of these factors as they played out in the Young Modellers project. To do so we draw on relevant literature as well as three sources of data from the project:

teacher surveys completed at the project's beginning and end, teacher reflective diary entries collected from week 4 and week 8 of the classroom implementation period, and an interview conducted via email with the project leader.

The first influencing factor comprised *contextual opportunities and constraints* that create expectations about the value and feasibility of implementing a modelling focus. The decision to situate the project in the Transition Year took advantage of the contextual opportunities afforded by this unique curriculum context and avoided the constraints experienced by teachers and students in the senior secondary years leading to the Leaving Certificate examination. Firstly, teachers report experiencing tremendous pressure to complete the senior secondary mathematics syllabus, particularly at Higher level, within the 2 year time frame (O'Meara & Prendergast, 2017). That, added to a heavy emphasis on performance in the Leaving Certificate examination, lends little opportunity to spend time on content that will not appear in examination papers (Gill, 2006). In contrast, in Transition Year there is a lack of examination pressure and teachers have much more flexibility in terms of what will be taught. Furthermore, activities which promote the application of mathematical skills and concepts to real-life problems and "problem-solving using interpretation, approximation, model making" are strongly endorsed in the TY Guidelines (Department of Education, 1993, p.13).

Teacher knowledge, beliefs, and practices in relation to mathematics and mathematical modelling also influence how teachers interpret the generalised aims and objectives of the official curriculum, and then plan, adapt, and enact instruction with the students in their classes. All teachers who participated in the Young Modellers project were new to modelling and they faced challenges to their conceptions of mathematics and mathematics teaching. In mathematical modelling students must figure out and formulate their own responses to problems and so a more student-centred approach is warranted in these situations. Dealing with these non-routine problems, from real-world situations, in a more student-centred classroom environment places the majority of mathematics teachers in a situation which is not routine for them and outside of their past experiences of teaching mathematics (Burkhardt, 2013). Facilitating mathematical modelling requires a broader range of teaching approaches than most teachers currently use (Burkhardt, 2006), such as knowing when and how to help, orchestrating student discussion, and providing a wide range of authentic, non-routine tasks. Thus, teacher knowledge, beliefs, and practices were recognised and explicitly addressed as potential implementation barriers in the Young Modellers project.

Some teachers in the Young Modellers project reported feeling challenged because they were no longer in the position of "expert" knower in the classroom:

> There were times I felt out of my depth as a teacher because one particular group found this interesting formula they wanted to use but there were elements of that formula that were very complex for them and I was unable to explain it to them. (Teacher KML3AHTU diary, week 4)

> I felt out of my depth at times when students asking about coastal heights of areas across Ireland – where could they get answers to certain questions. (Teacher BRL6NOHG diary, week 8)

There were also mixed responses from teacher surveys regarding perceived changes in their self-reported levels of anxiety when engaging their students with modelling tasks, and levels of confidence in their mathematical content knowledge and mathematical pedagogical knowledge. While it might be unrealistic to expect significant change in teaching practice in the relatively short time frame of the project, some teachers did report changes in their questioning practice by "asking more higher order questions, getting students to lead their own learning" (Teacher TBR2RIYU, post-survey). Others described change in terms of what they learned about students, especially in relation to their desire to arrive at a correct answer:

> [I learned] that students are too used to getting a definite answer in maths and can't deal with a general answer problem. (Teacher IOG0IAMU, post-survey)

> [I learned} that there is always more than one correct answer. Allow students to explore all answers. (Teacher TBR2RIYU, post-survey)

These comments from teachers show that *student knowledge, beliefs, and practices* in relation to mathematics can also influence the implementation of a modelling focus. The Young Modellers project leader, a research mathematician and modeller with many years of experience in working with school teachers, commented:

> The ability and willingness to try and fail is crucial to good modelling. We find that most students' concept of mathematics is heavily influenced by the notion of "there is one correct answer", but good mathematical models cannot be created without trial and error. We emphasised to the teachers, and repeated on visits to schools, that the modelling process is an iterative one: we start with a very crude and probably inaccurate model, and only by solving this simple model do we gain insight into how it can be improved and made more accurate. Students who are not willing to have "wrong" models find this process very difficult to accept; indeed, it is the source of the "this isn't mathematics!" comments that we sometimes get in feedback! Indeed, it is not mathematics as these students have learned it in the traditional classroom, but of course it very much *is* mathematics as it is applied in real-world industry and research applications!

Perseverance in the face of challenge and failure is an essential trait that students need to develop if they are to be successful at modelling. Many teachers were pleasantly surprised by the sustained engagement of their students in modelling tasks.

> There is a lot more resilience. They don't throw in the towel so quickly. (Teacher EBU0IANU diary, week 4)

> Students seem more invested in the amount of time they are spending to solve a problem. They are committed to solving it and they are not giving up on the problem straight away. (Teacher KML3AHTU diary, week 4)

> I noticed a huge difference in the time students were willing to spend to solve a problem. (Teacher HJR8GAAG diary, week 8)

Others commented on the increased levels of independence demonstrated by students:

> It was good to see them working with problems on their own without the teacher dictating the approach or pace. (Teacher EBU0IANU diary, week 4)

> One group decided to carry out experimental work and did so independently. They didn't look for prompts as how to design and implement the experiment and they worked in a very efficient and conscientious manner. I was hugely impressed with this as I felt it was a huge departure from the typical scenarios where students usually would need to be heavily coached when it comes to experimental work. (Teacher SMP diary, week 8)

A further factor that influences implementation of modelling in the operational curriculum is teachers' *access to resources and support*. We have already pointed to Irish mathematics teachers' strong reliance on textbooks (O'Keeffe, 2011; O'Sullivan, 2017) for planning and instruction; however, the Young Modellers project viewed textbooks as being inadequate for teaching modelling, and especially for developing the ability and willingness to try and fail that is crucial to a modelling disposition. Instead, a range of modelling tasks and resources was created or accessed from other sources, such as COMAP's Mathematical Contest in Modelling (Consortium for Mathematics and Its Applications, n.d.). Teachers who were unfamiliar with modelling and accustomed to having textbooks were keen to have access to these exemplar tasks and strategies.

4.5 Discussion

The aim of this chapter was to explore strategies for promoting mathematical modelling as a stimulus for curriculum and instructional reform, where modelling is considered to exemplify a kind of mathematical challenge that is still rarely found in secondary school curricula and classrooms. Remillard and Heck's (2014) model of the curriculum policy, design, and enactment system provided the framework for analysing factors that influence implementation of modelling in the official curriculum and the operational curriculum in Irish secondary schools. While much research attention has been directed at the distinction between what the official curriculum endorses and how teachers translate this curriculum into classroom practice, there are other factors that mediate curriculum enactment. In this chapter we showed how a range of institutional constraints; needs, values, and beliefs expressed by various stakeholders; advances in mathematics, technology, and educational research and practice; and access to resources and support interacted to influence curriculum development and enactment.

There is no doubt that it is challenging to introduce teachers and students to the mathematically challenging tasks and ways of working that characterise modelling. We think of this as the *meta-challenge* of institutionalising mathematical challenge in the school curriculum. Nevertheless, the two case studies presented in this chapter illustrate some ways in which the meta-challenge can be addressed. Common to both is a strategy for taking advantage of contextual opportunities which, while unique to Ireland, might find application in other curriculum contexts. In the first case, a dated curriculum for a mathematical physics subject, currently taken by very small numbers of students, is being overhauled by infusing a modelling focus. While this subject is offered in a high-stakes assessment environment that might otherwise act as a constraint to curriculum and instructional reform, its "niche" character provides a small-scale and potentially less risky context for innovation than the mainstream Mathematics subject that is taken by almost all senior secondary students. In the second case, a low-stakes curriculum and assessment environment in the form of Transition Year presents ideal opportunities for small-scale innovation and experimentation with modelling, away from the pressures of external examinations. In both cases, there is potential for teachers to become more comfortable with modelling, gradually building confidence and expertise without the expectation of implementing a full-scale modelling focus in an examinable mathematics subject taken by nearly all senior secondary students across the country. While the numbers of teachers involved in these initiatives is relatively small, their participation creates an existence proof for implementing modelling that, over time, might encourage others to try this approach.

Good modelling requires the ability and willingness to try and fail, and these requirements apply just as much to teachers as to students. We would argue that, in addition, curriculum authorities and education systems need to embrace "trying and failing", by taking a long-term view of the time and support that teachers need in order to meet the meta-challenge of embedding modelling into the school mathematics curriculum.

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Chapter 5
Personalized Mathematics and Mathematics Inquiry: A Design Framework for Mathematics Textbooks

Constantinos Christou, Demetra Pitta-Pantazi, Marios Pittalis, Eleni Demosthenous, and Maria Chimoni

5.1 Introduction

Almost 30 years ago, Wigley (1992) discussed two contrasting models for the teaching of mathematics, the path-smoothing model and the challenging model. The path-smoothing model involves more expository methods of instruction which provide students with structured and secured work pathways. The challenging model involves more exploratory, inquiry-based methods of instruction which allow students to interact with challenging tasks. More recent articles and research studies, mainly in science education (Alfieri et al., 2011; Blair & Hindle, 2019; Lazonder & Harmsen, 2016), seem to suggest that inquiry-based methods can be more effective than expository methods of instruction. Still, the integration of mathematical challenge in the instructional process and more specifically in mathematics inquiry approaches is neither clear nor explicit in the way in which mathematics curricula and textbooks may promote this.

The effective integration of mathematical challenge in the instructional process was one of the main principles in the design of the Cypriot Mathematics Curriculum (Cyprus Ministry of Education and Culture, 2016a). The series of mathematics textbooks, which were designed to translate this policy into pedagogy tried to fulfill this principle. Research studies support that textbooks play a vital role in translating the educational policy into pedagogy (Valverde et al., 2002). In this chapter, we aim to present the design framework for the Cypriot Mathematics Textbooks and indicative examples, to illustrate the way in which these textbooks may evoke mathematical challenge in heterogeneous classes. It is beyond the scope of the current chapter to

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present any empirical findings related to teachers' training or findings regarding the impact of the textbooks on students' learning.

In this chapter, we start by first looking at the role of mathematics textbooks in teaching and give some information about the Cypriot Mathematics Textbooks. Then we discuss the most frequently used design models for mathematics teaching. We then proceed to the presentation of the theoretical framework "Personalized Mathematics and Mathematics Inquiry" (PMMI) which we used for the development of the Cypriot Mathematics Textbooks. We exemplify this theoretical model with some indicative examples from the mathematics textbooks of primary and secondary education.

5.2 Role of Mathematics Textbooks

The structure and content of mathematics textbooks is likely to have an impact on actual classroom instruction (Pepin et al., 2013; Rezat, 2006). Valverde et al. (2002) argued that the form of textbooks promotes a distinct pedagogical model and thus embodies a plan for the particular succession of educational opportunities. The development and design of mathematics textbooks are assessed through the opportunity they offer to students to learn and thus they are considered an important contributing factor in learning outcomes (Törnroos, 2005). Empirical studies have shown that the quality of mathematics textbooks has a significant effect on learning outcomes (Sievert et al., 2019).

The development of textbooks should be based on the idea that improvement of mathematics learning in classrooms is fundamentally related to the development of teaching, and that teaching develops through a learning process in which teachers and students grow into the practices in which they engage. Since textbooks strongly influence what students learn and what teachers teach, teachers and students should have suitable and appropriate textbooks (Reys et al., 2004). Textbooks facilitate teachers to modify their methods for teaching mathematics, in such a way as to align with the principles of the textbooks. The philosophy and teaching procedures are often guided by the mathematics textbooks.

5.2.1 The Cypriot Mathematic Textbooks

Textbooks are probably one of the most important curriculum resources which help teachers transform the mathematics curriculum into practice. According to Koutselini (2012), Cypriot teachers adhere to textbooks' teaching. The TIMSS study (2003) showed that 71% of the fourth-grade students were taught by teachers who used the mathematics textbooks as their primary source, while the remaining 29% of teachers used textbooks as a supplementary resource.

The Cypriot Mathematics Textbooks, published by the Ministry of Education, are used in all state schools in Cyprus. They are the only resources provided to schools and are based on the recently developed Cypriot Mathematics Curriculum. There are different types of schools in Cyprus and the textbooks intend to cater a diverse group of students. Thus, one of the main roles of textbooks is to help teachers teach in the spirit of the curriculum and the only way for the curriculum to be implemented properly and consistently was to develop a good set of textbooks. These textbooks come with teacher guides which serve as manual for mastering teaching and learning. During the first years of the implementation of the new textbooks, in-service training was organized by the Ministry of Education to familiarize teachers with the new textbooks and their main principles.

The Cypriot Mathematics Curriculum was launched in schools in September 2012. In the same year, the mathematics textbooks which aligned with the new curriculum were introduced in Grade 1 (the first grade of primary education) and Grade 7 (the first grade of secondary education). In September 2013, Grade 2 and Grade 8 textbooks were introduced and the same pattern of introduction of new textbooks continued for 4 years, until 2017, when the whole series of textbooks from Grade 1 to Grade 12 were completed. Two types of textbooks were introduced for Grade 10 to Grade 12, one for students who take mathematics as a specialization subject and one for students who take mathematics as a common core subject. The textbooks are reviewed almost every year, based on the comments and suggestions received from teachers implementing them.

5.3 Design Models

Thirty years ago, Wigley (1992) argued that in order to develop mathematics textbooks, it is common to follow one of the two designs, the path-smoothing model or the challenging model. The essential methodology of the first model is to smoothen the path for the learner. The textbook, in this case, states the kind of problem which the class will be working on. The problem attempts to classify the subject matter into a limited number of categories and to present them one at a time. The key principle is to establish secure pathways for the pupils. Thus, it is important to present ways of solving problems in a series of steps and exercises to practice the methods. The path-smoothing model is the instructional approach in which teachers, following the textbooks, prescribe the content, present the content, and measure student acquisition of that content.

The challenging model promotes what its name denotes, the use of tasks that are challenging to students. The teacher provides sufficient time for students to work on a task, suggest their own approaches, and try different solving pathways. The teacher may have considered beforehand a syllabus, but this is not presented to students from the beginning. The teacher has a critical role in helping students share their ideas with the whole class and discuss different strategies. Students are encouraged to reflect on their work, recognize what they have learned, and how new knowledge links to previous knowledge. Wigley (1992) and more recently Blair and Hindle (2019) argued that the challenging model can create better learning and more positive attitudes towards mathematics.

Since Wigley (1992) discussed the path-smoothing and the challenging model, there have been several meta-analysis studies, which did look not only at the two extremes but also at intermediate points of this spectrum. These more recent studies (see, for example, Alfieri et al., 2011; Lazonder & Harmsen, 2016) did not use the words smoothing and challenging model but referred to “explicit instruction and unassisted discovery” (Alfieri et al., 2011) or “guided and unguided inquiry learning” (Lazonder & Harmsen, 2016), which we believe resemble the path-smoothing and challenging models. They also referred to “enhanced discovery or minimally guided approach” which lies somewhere between these two extremes of the challenging spectrum.

Although the extent and the type of the guidance that students should receive is not yet completely clear, research studies have consistently shown that enhanced discovery is more effective than explicit instruction or unassisted discovery, as long as students are adequately supported (Lazonder & Harmsen, 2016). This is the reason that we decided to follow an enhanced discovery instructional approach for the Cypriot Mathematics Textbooks.

We adopted an inquiry-based approach to mathematics with focus on problem solving, understanding, problems within a context, learning processes, and strategies. We considered that the implementation of these central concepts improves students’ attitudes towards mathematics and their ability to use mathematics both in the “real world” and inner mathematical contexts. However, an inquiry-based approach is fundamentally based on the humanized aspects of mathematics which are inherent in the nature of mathematics. Thus, the essential characteristics of the challenging model, as it was implemented for the design of the Cypriot Mathematics Textbooks, has two interrelated elements that defined the design of the mathematics textbooks: *Personalized Mathematics* and *Mathematics Inquiry*. In the next section we present the structure and underlying principles of this framework.

5.4 The Structure of the PMMI Framework

We propose the “Personalized Mathematics and Mathematics Inquiry (PMMI),” as the overarching and fundamental theoretical framework for the design of mathematics textbooks and pedagogical instruction in Cyprus for K-12 grades in order to achieve desirable teaching–learning practices. The PMMI framework (Fig. 5.1) involves two major elements: (a) Personalized Mathematics and (b) Mathematics Inquiry. We set off from “Personalized Mathematics” where we present the fundamental practices of mathematics teaching: mathematics goals, reasoning, problem solving, mathematization, connections of mathematical representations, development of

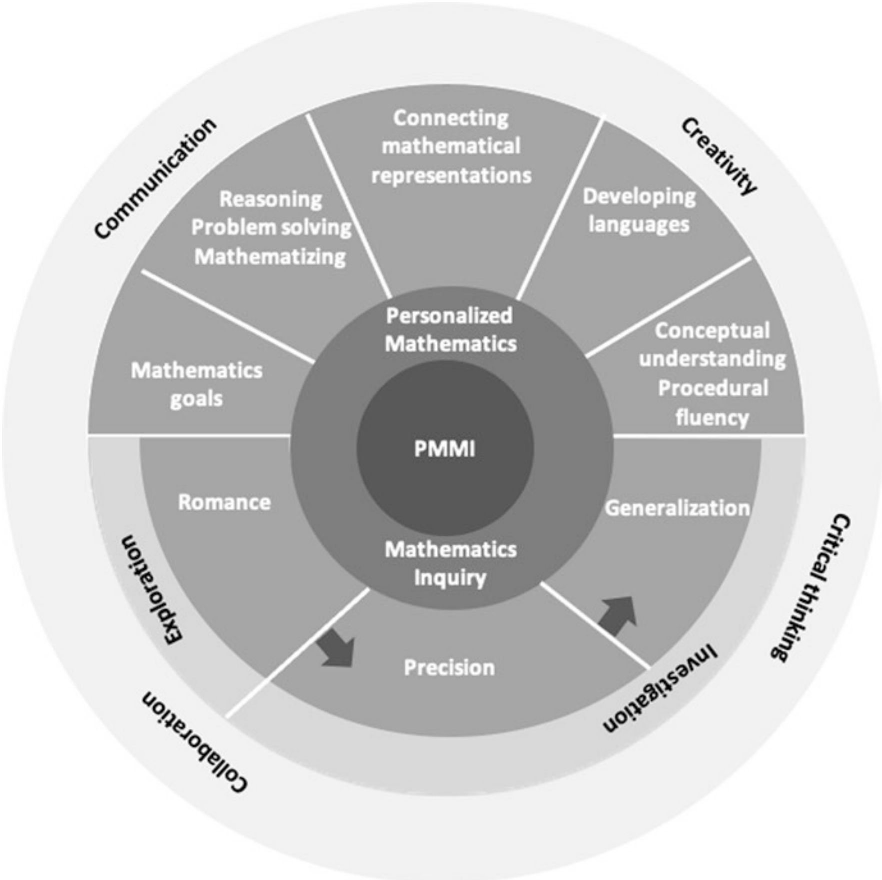
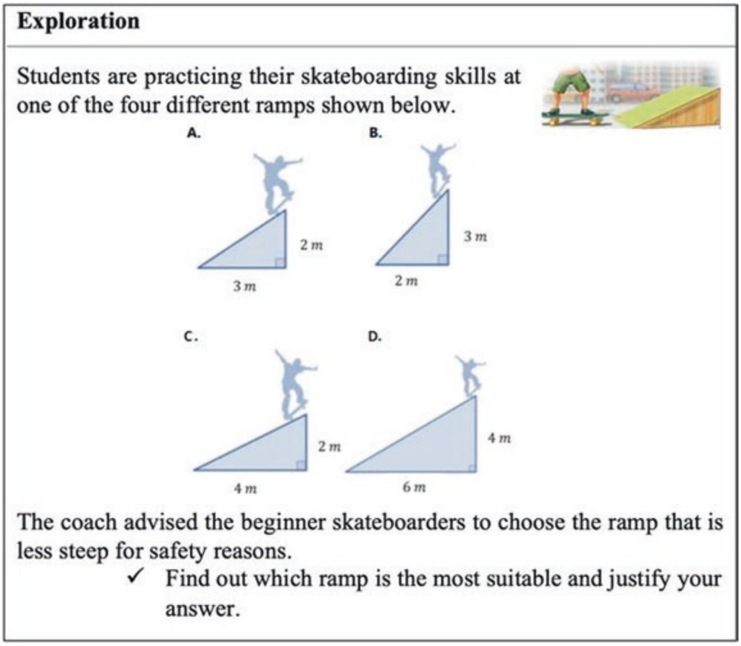


Fig. 5.1 The PMMI framework

language, conceptual understanding, and procedural fluency. Then, we proceed to the second element of the PMMI framework "Mathematics Inquiry," which is based on Whitehead's (1929) theory of learning (Romance, Precision, Generalization), and present the rationale for the design of the phases in which students are intended to go through, while being taught a mathematics chapter.

The overarching aim of the PMMI framework is that the interweaving of "Personalized Mathematics" and "Mathematics Inquiry" will lead to the enhancement of students' positive attitude towards mathematics, development of mathematical concepts and procedures, as well as the development of more general skills, such as critical thinking, creativity, collaboration, and communication.

To make the PMMI framework explicit, Fig. 5.2a, b illustrates the way in which the teaching of slope of a straight line is introduced in the Cypriot Mathematics Textbooks in Grade 8. In the rest of the chapter, we will discuss the main principles


Exploration

Students are practicing their skateboarding skills at one of the four different ramps shown below.

The coach advised the beginner skateboarders to choose the ramp that is less steep for safety reasons.

✓ Find out which ramp is the most suitable and justify your answer.

Fig. 5.2 (**a**) Exploration of slope in Grade 8 mathematics textbook (**b**) Investigation of slope in Grade 8 mathematics textbook (Cyprus Ministry of Education and Culture, 2016b)

of the PMMI framework alongside indicative examples from the mathematics textbooks in order to make the link between the theoretical principles and their implementation more transparent to the reader.

As illustrated in Fig. 5.2a, b, each chapter begins with an exploration and an investigation, which constitute important ingredients of the PMMI framework. These explorations and investigations are followed by further tasks to give students the opportunity to develop both conceptual understanding, procedural fluency, as well as use of clear and precise mathematical language. Real-life applications are often utilized throughout the textbooks. These applications are opportunities for students to connect classroom lessons to realistic scenarios and assist teachers transforming mathematical learning into an engaging and meaningful way to explore the real world. Attention was also paid to organize content in a way in which learning mathematics would be an active, constructive, cumulative, and goal-oriented process.

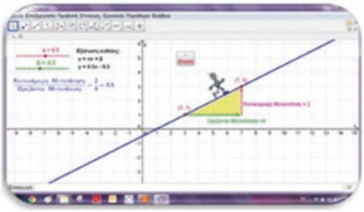
5.4.1 Personalized Mathematics

As shown in Fig. 5.1, the two elements of the PMMI framework, "Personalized Mathematics" and "Mathematics Inquiry," are intertwined in such a way as to provide a framework for teaching and learning that fully aligns with the learning of

Investigation

Open the file «B_En5_Klisi.ggb»

A robot is at point A and moving to point B on the line represented on the axes.



- Move the cursors α and β, to construct the equation, of the line $y = 2x$.

✓ Select point A and then change the coordinates of point B, so that the robot moves one unit to the right (horizontal change). Calculate the vertical shift of the above movement. Consider whether this applies to any starting point A.

✓ If the robot moves from point A to two units to the right, how large will the vertical change be? Does this apply to any starting point A?

✓ What do you think will be the vertical change if it moves from point A, 5 units to the right? Check your answer with the use of the app.

✓ Consider the ratio $\frac{vertical\ change}{horizontal\ change}$ for any two points A and B of the line.

- Change the cursors α and β and observe how the ratio $\frac{vertical\ change}{horizontal\ change}$ relates with the equation of the line.

Fig. 5.2 (continued)

mathematics. Personalized mathematics learning is an instructional approach which encompasses a number of practices to support mathematics teaching and learning. However, the lack of a consensus on the definition of personalized learning allows for a range of ideas on what it might entail.

In the context of the PMMI framework, personalized means that learning focuses primarily on improving students' achievement without ignoring the humanizing and social aspects of mathematics teaching and learning. For example, our purpose was to focus on tailoring mathematics tasks and problems to learners. This means, for students, to find solutions based on their own mathematical understanding and what makes sense to them. We consider personalized mathematics learning as the space in which learners give voice to their own ways of mathematical thinking, represent and discuss their mathematical ideas, and use mathematics to make sense of their worlds. "Personalized Mathematics" can help learners see themselves as doers of mathematics by providing support for developing perseverance and understanding. It is also personalized because the design of lessons provides students with multiple entry points and encourages different ways of students' active engagement with various mathematical ideas through discussions, presentations, and use of various representations.

To encourage Personalized Mathematics instruction, the textbooks were based on five teaching practices (see Fig. 5.1), as these were defined by NCTM policy document, "NCTM's Principles to Actions: Ensuring Mathematical Success for All" (NCTM, 2014). In particular, these practices were implemented by engaging students with tasks, which are expected to facilitate mathematics learning through the following:

- Establishing mathematics goals to focus learning (personalized goals that build up students' mathematical understanding, increase student confidence, goals that ensure that each and every student has the opportunity to learn rigorous mathematics content and develop mathematical processes and practices).
- Implementing tasks that promote reasoning and problem solving (Personalized Mathematics supports tasks that require reasoning, problem solving, and mathematizing our world through mathematical modeling and culturally relevant mathematics tasks).
- Using and connecting mathematical representations (Personalized Mathematics allows students to use representations familiar to them and age appropriate, use multiple representations so that students can draw on multiple resources, and develop connections among multiple representations to deepen their understanding of mathematical concepts and procedures).
- Facilitating meaningful mathematical discussions (Personalized Mathematics allows students to develop language to express mathematical ideas).
- Building procedural fluency from conceptual understanding (Personalized Mathematics routinely connects conceptual understanding and procedural fluency to deepen learning and reduce mathematical anxiety. Procedural and conceptual knowledge is more solid when it is built on students' prior personalized knowledge and experiences).

For example, in the exploration in Fig. 5.2a, students are invited to decide which one of the four skateboard ramps is the least dangerous and justify their solution. It is likely that students will find this exploration interesting since it arises from real life and some of them may even have tried this activity themselves. Thus, students are expected to interpret the problem based on their prior knowledge and personal experience. Students may also become curious as to when the ramp is more dangerous and find challenging how to respond or explain their intuitive feelings. Students may try to communicate their own experiences or ideas, look critically at the problem, and try to be creative as to the way in which to investigate it or justify their answers. In this sense, students might formulate a personal goal for understanding the problem and offer a reasonable justification, while the problem reflects a situation of the real world which students should mathematize. Furthermore, through students' collaboration and communication it is likely that students may realize the need of a new mathematical concept which is neither the height nor the length of the ramp, but actually the combination of the two measures. Hence, the representations of the skateboard ramps which provide information about the height and the length are expected to trigger students' curiosity. This is the sort of curiosity we expect to develop in the mathematics classroom, the romance of learning, which will eventually bring the evolution of a new mathematical concept for the students, that of slope. It is likely that if the mathematical concept of slope emerges through students' curiosity and need to respond to a problem will remain in students' memory longer (Gruber et al., 2014; Kang et al., 2009; Knuth, 2002; Peterson & Cohen, 2019). Students will develop conceptual understanding of the concept of slope and not depend on the rote memorization of the formula *slope = rise/run* (Bos et al., 2020).

In the investigation that follows immediately after the exploration, students are invited to use a mathematical applet. The technological tool is recommended so as to offer students the opportunity to experiment with the slope of a straight line. In this activity, students are again offered with relevant representations which they may link to the skateboard ramp. Students are prompted to observe the changes that occur as a robot moves along the tilted line. Specifically, students are asked what the robot's vertical displacement is, when it makes one-unit move to the right along the slope, then when it makes two-units move to the right and so on. Students are offered this applet to experiment, construct hypotheses, and investigate whether these hypotheses are confirmed or rejected. It is anticipated that this investigation will support the development of procedural fluency from conceptual understanding. Students' active engagement with the technological tool, precision in calculations, and generalization is expected to lead to the conceptual development of the concept of slope and not the rote memorization of the formula *slope = rise/run*.

5.4.2 Mathematics Inquiry

The second element of the PMMI framework refers to "Mathematics Inquiry." Mathematics inquiry-based learning is seen as the approach and pathway for implementing the practices of Personalized Mathematics. On a European level, most educational policy documents and curriculum guidelines suggest inquiry-based instructional methods to school subjects (Dorier & Garcia, 2013; Supovitz & Turner, 2000). Inquiry-based instruction in mathematics can loosely be defined as "a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work" (Artigue & Blomhøj, 2013, p. 797). This approach promotes problem solving and involves addressing questions that are epistemologically relevant from a mathematical perspective and triggering students to work autonomously in order to provide valid answers (Artigue & Blomhøj, 2013). It entails observing, asking questions, creating representations, making conjectures, looking for relationships, generalizing, modeling, and communicating ideas (Dorier & Maass, 2020).

Although the level of guidance that the inquiry-based learning should involve may vary, a meta-analysis of 72 studies carried out by Lazonder and Harmsen (2016) suggests that guidance has a positive effect on inquiry learning activities, performance success and learning outcomes. Therefore, we chose to apply an approach which may be characterized as minimal guided or enhanced discovery approach.

Mathematics inquiry could be seen as a process that starts from a wonder, a question or a problem, for which students seek answers through exploration and/or investigation following an enhanced discovery approach. The context for mathematics inquiry often relies on problems arising from the world around us, and problems that emerge from history, art, or the science of mathematics. Designing lessons based on the inquiry approach entails consideration of the mathematical concepts involved, incorporation of artifacts that support exploration and experimentation, and the use of language and symbolic tools accessible to students for expressing and discussing their ideas.

In inquiry-based learning, the role of the teacher involves challenging students, probing questions, utilizing their prior knowledge, encouraging discussion, and structuring students' opportunities for developing understanding (Dorier & Maass, 2020). This presupposes that task sequences are developed to scaffold students' work in reinventing and creating mathematics new to them (Laursen & Rasmussen, 2019). Hence, the design of textbooks intends to provide these learning sequences in order to promote inquiry-based mathematics learning.

As described earlier, for the teaching of straight-line slope, students are presented with an exploration, where they need to decide which ramp is the steepest. At this point students are invited to explore and hypothesize, and are not offered any specific guidance. However, if students are unable to reach an answer or make conjectures, they are invited to work on an applet and are given more guided questions. Students are asked to explore the concept of slope as the constant ratio of $\frac{\text{vertical change}}{\text{horizontal change}}$.

The development of mathematics inquiry is affected by the nature of the mathematical concepts involved, students' conceptualization, language, symbolic tools accessible to students for expressing and discussing their ideas, and the artifacts accessible for supporting exploration and experimentation. In this sense, a variety of artifacts could support the experimental dimension of mathematics, like digital technologies. The history of mathematics shows that such an experimental dimension is not new, but over the last decades technological developments have put a large number of new resources at the disposal of teachers and students. Researchers generally agree that the strategic use of ICT could support students to develop understanding and advanced mathematical proficiencies, like problem solving, reasoning, and justifying (NCTM, 2015). Digital tools offer dynamic representations and classroom connectivity, which could optimize students' access to basic mathematical concepts and procedures (Hegedus & Moreno-Armella, 2014).

The PMMI framework adopts the idea that digital tools should be used in a way that corresponds to the conceptual schemas that students are expected to develop. This process is widely known as "instrumental genesis" (Artigue, 2002; Drijvers, 2020). Therefore, the thoughtful use of digital tools in carefully designed ways at appropriate times, mostly through exploration and investigation settings could support schema construction, experimenting, sense making, communicating, and doing mathematics. For example, the use of the applet in the investigation about the concept of straight-line slope (see Fig. 5.2b) is expected to facilitate the construction of a relevant schema that relates the concept of straight-line slope with the idea of coordinating the vertical and horizontal covariation.

Concluding, "Mathematics Inquiry" in the PMMI framework refers to an instructional approach that is expected to serve enhanced learning and promote the practices of "Personalized Mathematics." In the following section, we present in detail the three phases through which "Mathematics Inquiry" is expected to evolve: romance, precision, and generalization. The notions of exploration and investigation are also revisited to further illustrate their special characteristics, significance, and contribution to "Personalized Mathematics" and "Mathematics Inquiry."

5.4.2.1 Mathematics Inquiry Phases

Mathematics inquiry, in the PMMI framework, evolves in three phases (see Fig. 5.1) which are based on Whitehead's theory of education "Rhythm of learning" (Whitehead, 1929). According to Whitehead's theory, the natural way that individuals learn is through the pattern of Romance–Precision–Generalization. Learners should be introduced with something interesting, something they care about, and be offered the opportunity to explore. Then, the precision phase follows where learners develop knowledge and skills that are needed for the development of a new concept. Of course, at this phase, romance should not be lost, since this will sustain the interest and therefore the development of skills, knowledge, and applications. The last phase is generalization. At this phase learners link what they have learned with prior knowledge, apply this knowledge in a new context, make generalizations, return to romance with new competences, and become ready to explore new concepts. Students are anticipated to pass through these three phases through two activities: the exploration and the investigation.

The romance of learning is introduced in the textbooks with the idea of explorations while the phases of precision and generalization are substantiated by investigations. Afterward, examples and activities allow students to practice and sharpen their skills as they work towards mathematical understanding. At the end of each chapter there are enrichment tasks which serve four purposes: (1) tasks for students who struggle with mathematics, (2) more challenging tasks for students who excel in mathematics, (3) tasks for all students who need further practice, and (4) tasks which offer a different approach to teaching to the one already presented in the preceding chapter.

5.4.2.2 Exploration: Romance

The inquiry process starts with an exploration which, as reported earlier, has a unique goal, to create wonder, curiosity, engagement in mathematics, and the romance of learning, as mentioned by Whitehead (1929). Thus, exploration provides students with lived, curious experiences, in which they are expected to shape their own learning, as they work on mathematical problems. Students grow and change with opportunities to identify problems, generate personal wonderings, and engage in dialogue around these problems. They reflect as they apply their new knowledge, by discussing possible solutions in ways that transform thinking. Offering learners space to generate their own wonderings about problems helps them connect their own interests to real-life issues in ways that can lead to real change (Alberta Learning, 2004). One of the most valuable things that an exploration can serve is to make students become more aware of and deliberate about their curiosity. This is why, in the Cypriot Mathematics Textbooks, exploration is the starting activity of each unit in the mathematics textbooks. Usually explorations are real-life or life-like learning experiences that are open and provide opportunities for students to wonder and develop their imagination. It is the essence of a mathematics inquiry. However, to "enquire" does not mean that we always have to reach an answer to a problem or to complete a task.

Exploration is the means to personalize mathematics since it is a purposeful, self-directed inquiry fulfilling learning experience (Pink, 2009). This is why it is important that individuals set their own goals, in this case mathematical goals, and seek to satisfy them. In order for individuals to set a mathematical goal, the topic must be of interest to them and trigger their curiosity. In addition, explorations request students to reflect, by discussing what they bring to the content and what ideas they actively construct as they interact. For example, during explorations, students can use their wonderings and meaning to reflect on their process and seek feedback from others.

The exploratory approach provides an opportunity for mathematics to occur in a context, providing a balance between problem solving and skills-based activities and engage students in deep mathematical learning (Boaler, 2008). Explorations allow students to "do" mathematics, to "make sense" of their world, and "be mathematicians" (Marshman et al., 2011). The exploratory tasks aim to get students involved in "problem formulation, problem solving, and mathematical reasoning" (Battista, 1994, p. 463).

In explorations, a substantial task can thus be presented, in which students help define the problem; develop ways of tackling it; generate examples; and predict and generalize. Explorations direct students to the realization that there is a need to learn or discover a new mathematical concept, or strategy which is useful for mathematics and life in general. Explorations have multiple entry points allowing students to think creatively in order to respond to complex challenging tasks "allowing students to think in a creative manner in the framework of challenging complex tasks" (Swan, 2009, p. 1). These tasks facilitate the process of discussion and contextual use of mathematical vocabulary.

In the context of Cypriot Mathematics Textbooks in Cyprus, the PMMI framework situates classroom exploration experiences at the beginning of a chapter on a new concept or procedure. Approximately, every week students encounter new explorations and/or investigations. The time anticipated for the completion of an exploration and investigation varies, from 10 min to 40 min sessions. The exploration is quite open, and teachers may ask further questions to orient the students towards the topic under investigation.

Figures 5.3 and 5.4 present examples of explorations. Figure 5.3 presents an exploration on the concept of exponents which is taught in Grade 7. Figure 5.4 presents an exploration on the concept of equivalent fractions which is taught in Grade 4.

In the exploration presented in Fig. 5.3, students are presented with the legend of Sissa which aims to engage students to the concept of exponent. Students are asked to explore why the emperor could not fulfill his promise and deliver to Sissa the grains of wheat that he had promised. This question anticipates to trigger students' curiosity, make them wonder why this happened, and want to explore the problem. Some of the students may make certain hypothesis, others may bring to the fore their own experiences about the number of squares that a chess has and suggest possible strategies to address the problem. Thus, it is expected that this problem will direct students to set mathematical goals and they may try to respond to the problem

Exploration

According to a legend, long ago in one of the kingdoms of ancient India there was a powerful and rich emperor named Velchib. A Brahmin priest, named Sissa, invented and offered as a present to the emperor, a chess. The emperor was so impressed and excited with the present to the emperor that he decided to offer him a gift. Velchib asked Sissa what present he wanted.

Sissa thought for a moment and replied "I want two grains of wheat in the first square, four in the second, eight in the third and so on…"

The emperor was puzzled and angry about the cheap gift that Sissa had asked for and ordered his storekeepers to give him the wheat he wanted. However, as things turned out he could not deliver his promise.

✓ Why couldn't the emperor deliver his promise?

Fig. 5.3 Exploration of exponents in Grade 7 mathematics textbook (Cyprus Ministry of Education and Culture, 2016c)

Exploration

In a school party there were three pizzas of the same size. Each pizza was cut into equal pieces.

- Sophie ate 2 pieces from Pizza A.
- Andrea ate 3 pieces from Pizza B.
- Michael ate 4 pieces from Pizza C.
- All children ate the same quantity of pizza.

How is this possible? Explain

Fig. 5.4 Exploration of fraction equivalence in Grade 4 mathematics textbook (Cyprus Ministry of Education and Culture, 2019)

by sharing their personal experiences and activating various procedures, such as multiplication. The exchange of ideas amongst students may also require the development of specific mathematical language related to multiplication, the use of tools (such as a calculator) and creation of various representations (such as repeated multiplication). After their initial calculations, students may realize that numbers become very large. This is when some of them may intuitively feel the need of a new mathematical concept which will make this mathematical process simpler. Thus, this exploration triggers students' curiosity, encourages them to collaborate and communicate by exchanging ideas, building appropriate language, and using various representations, and offers them the opportunity to be critical and creative.

In the exploration presented in Fig. 5.4 students are asked to explore a real-life situation involving the concept of equivalent fractions. Based on the scenario, three pizzas of the same size were offered at a school party. One child ate two pieces from the first pizza, one child ate three pieces from the second one and one child ate four pieces from the third pizza; yet, all of them ate exactly the same quantity of pizza. Students are asked to explain how this could be possible. This question aims to trigger students' curiosity and wonder how this is feasible and probably start thinking that the three pizzas were cut in different ways. It is expected that students will make conjectures based on their own experience of cutting pizzas into slices. This discussion will probably lead to a realization that the three pizzas were cut in different ways and students will need to find the way in which the pizzas were cut and what fractions are involved. Students may hypothesize about the number of pizza slices that each student had. Based on these hypotheses, students could be prompted to collaborate to construct representations or use tools (e.g., fraction circles) to show the way in which each pizza was cut. Students' work and ideas will contribute to orchestrating a productive mathematical discussion about the fact that the same quantity of pizza could be expressed using different fractions. This discussion is anticipated to facilitate the introduction of a new mathematical concept, that of equivalent fractions. The concept could be explained through appropriate language and use of various representations arising from the scenario that students were invited to explore. Based on students' answers, they could be prompted to find further alternative solutions (be creative), communicate their mathematical ideas through words, symbols, drawings, and representations, and explain how the different cuts result to the same quantity of pizza.

5.4.2.3 Investigations: Precision and Generalization

Investigation is an activity originating in mathematics or the real world which lends itself to inquiry. A mathematics investigation allows students to satisfy their curiosity created in the exploration using various techniques. In the process of the investigation, students develop skills that can be applied to other problems (da Ponte, 2007). Students develop creative and critical thinking abilities and apply them to the expansion of their knowledge and skills. The intellectual satisfaction that one gets when discovering concepts and procedures as well as the generalizations of rules in different contexts are the major components of personalized mathematics.

The investigative approach is illustrated by posing questions, collecting data, hypothesizing, reflecting on, and drawing conclusions. These processes need to take place individually, in small groups and in the classroom as a whole. Explorations and investigations appear from Grade 1 to Grade 12. The level of difficulty and guidance offered varies, based on students' age and experiences. In the textbooks three kinds of investigations were designed, following Harris and Hofer's (2009) categories of activity types that provide students opportunities for *knowledge building* (i.e., students are expected to build the same content and process knowledge), *convergent knowledge expression* (i.e., students are expected to develop and express understanding of content which is similar to what they were introduced), and *divergent knowledge expression* (i.e., students are encouraged to express their own understanding of a given topic). In an analogous way, we developed investigations for knowledge building, convergent knowledge expressions, and divergent knowledge expressions with purpose to deepen and extend learning.

In the mathematics textbooks, the investigations follow the explorations. After students' curiosity and wonder, students need an explanation and the information which demystifies the mathematical content. The knowledge of mathematical concepts, skills, and procedures are the tools to justify mathematical phenomena through investigations. Mathematical investigation allows students to learn about mathematics, especially the nature of mathematical activity and thinking. It also makes them realize that learning mathematics involves intuition, conjecturing, and reasoning, and is not about memorizing and following existing procedures. The main component of an investigation is conjecturing which is followed by refinement of conjectures, refutation or proof of conjectures, and monitoring of proofs (Leikin, 2014). Investigations stimulate a way of thinking that goes beyond the application of knowledge or isolated procedures and implies the mobilization of ideas from different areas of mathematics. They deal with complex thinking processes, but reinforce the learning of facts, concepts, and procedures, making an important contribution to their consolidation (Abrantes et al., 1999). The ultimate aim of mathematical investigation is to develop students' mathematical habits of mind.

Explorations encourage students to pursue their curiosity, helping students figure out just what they want to know, while investigations are showing them how to systematically go about getting the answers to the investigations and explorations. In a mathematical exploration, one begins with a very general question or from a set of little structured information from which one seeks to formulate a more precise question and then produce a number of conjectures. Afterward, one tests those conjectures and proceeds to investigations in a systematic manner. If someone finds counterexamples, those conjectures may be improved or put completely aside (rejected or discarded). In this process, sometimes new questions are formulated and the initial questions are abandoned, completely or partially. The conjectures that resist to several tests gain credibility, stimulating a proof that, if achieved, will confer mathematical validity.

Both explorations and investigations call for creativity and critical thinking. They require abilities that are much beyond simple computation and memorization of definitions and procedures (da Ponte, 2007). These abilities, sometimes called "higher order abilities," are important not only for the mathematical development of the individual but for one's overall development as an individual and as an active member of society (da Ponte, 2007).

In the following section (Figs. 5.5 and 5.6), we present the investigations on the concept of exponents and on the concept of equivalent fractions that follow the explorations presented earlier.

Investigation

Fill in the table:

Square	Number of wheat grains	Result
1	2	2
2	2 · 2	4
3	2 · 2 · 2	
4		
⋮		
8		
10		
⋮		
32		
⋮		
64		

✓ Explain your strategy

To produce this huge quantity of grains, which is actually a 20 - digit number, one has to plant the whole Earth 76 times!

It is said that the emperor, in order to avoid the insult for not keeping his promise, he was consulted by his advisors to ask Sissa to count all the grains. Something that would take forever!

Fig. 5.5 Investigation of exponents in Grade 7 mathematics textbook (Cyprus Ministry of Education and Culture, 2016c)

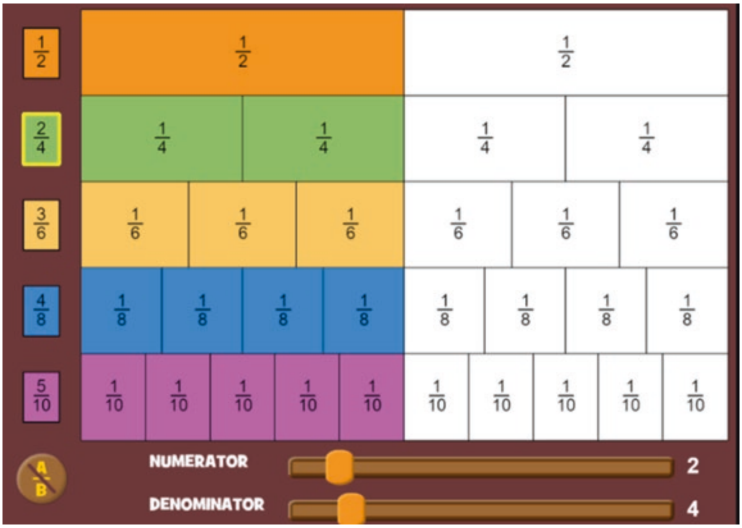


Fig. 5.6 Investigation of fraction equivalence in Grade 4 mathematics textbook (mathplayground.com) (Cyprus Ministry of Education and Culture, 2019)

For the concept of exponents, the investigation aims to offer students a scaffold toward the solution of the problem. A table is presented where students are asked to find the number of grains, in the 1st, 2nd, 3rd, 4th, 8th, 10th, 32nd, and 64th square. The investigation is presented in this form in order to provide students the opportunity to calculate with precision the answer in the first four cases (1st, 2nd, 3rd, 4th) and then try to find a pattern and a general rule that would apply for the number of grains in the subsequent squares and eventually in any number of squares. The table facilitates students' observation and deduction of a general rule that when a number is multiplied by itself it may be represented in the form a^b, where b indicates the number of times the number a will be multiplied by itself. Therefore, precision in the calculation of specific cases is expected to lead to generalization about a rule for calculating the number of wheat grains for any square.

In the investigation for equivalent fractions, students are invited to find at least 4 equivalent fractions to $\frac{1}{2}, \frac{1}{3}$, and $\frac{3}{4}$ by utilizing the affordances of an interactive applet (see Fig. 5.5). The applet involves the representation of fraction bars (mathplayground.com). Students can experiment by dragging sliders that define the numerator and denominator of each fraction bar. The goal of the activity is to align the colored fractions. The visualization and experimentation affordances of the applet might prompt students to make conjectures regarding the relation between the given fraction and the equivalent ones. Students will be asked to test the validity of their conjectures and find the relation between the denominators of the equivalent fractions and then the relation of the corresponding numerators. It is expected that students will study a number of examples and observe that the denominator of the equivalent fractions is a multiple of the denominator in the given fraction and this can be done by aligning the vertical lines (precision). Students will be given the opportunity to gain a deep conceptual understanding of the fact that finding an equivalent fraction equals dividing n times each piece of the given fraction; this process results to a new fraction with a denominator that is n times bigger (generalization). The experimentation with the applet aims to both facilitate the generalization of a procedure for finding any equivalent fraction and justification as to why it is possible to find an infinite number of equivalent fractions. We anticipate that students will be able to argue that the general rule for determining an equivalent fraction is to multiply the numerator or denominator of a fraction with the same number, or divide the numerator or denominator of a fraction with the same number.

5.5 The PMMI and the 4C's

The twenty-first-century skills of communication, collaboration, creativity, and critical thinking, often referred to as the "four C's" may be developed through the PMMI framework. The PMMI design prompts students to engage with the four C's (see Fig. 5.1).

Designing purposeful tasks that incorporate students' wonder is the foundation upon which the PMMI framework is built. Explorations lead to student action or investigations, through both participation and creation. Participation is an essential step in the inquiry process. In fact, Casey (2013) argued that it is the ultimate goal of learning. Through participation, individuals assert their autonomy and ownership of learning; in turn, their inquiry becomes more personal and engaging (Pink, 2009; Zhao, 2012). Creation is viewed mainly in two ways: first, through tasks where students are invited to create a new concept that was not previously known to them and second, through tasks where students are invited to offer multiple or unique solutions (Kaufman & Beghetto, 2009). For example, in investigations students are required to put forward a proposition about objects and operation which may involve unexpected relationships; thus, both creativity and critical thinking are essential (Leikin, 2014). Ultimately, creation and participation are essential elements for knowledge construction. Creation is one common form of participation (Reilly et al., 2012).

5.6 The PMMI and the Role of the Teacher

In the PMMI framework the role of the teacher also changes. The teacher would need to become a co-learner in the classroom context in order to implement the lessons as intended according to the PMMI design framework. The teacher is not the one asking and answering questions, but the teacher facilitates students' engagement with the explorations and investigations. The teacher encourages students to be in control of their leaning. It is the context of the explorations and investigations that supports this role of the teacher. Depending on whether the exploration and investigations are structured, guided or open, the control that teacher has over students' learning varies. However, in all types of explorations and investigations the students' initiative is always high.

5.7 Conclusion

This chapter presents the PMMI framework which guided the design of the mathematics textbooks in both elementary and secondary school grades. Despite the fact that PMMI was developed in the context of a particular country with certain traditions, goals, and aspirations, we suggest that it could provide a reference point that elaborates how teaching and learning mathematics might look like out of regional circumstances. In this final section, we highlight the affordances and strengths of PMMI as a potential framework for designing mathematics textbooks. We also suggest some fruitful forthcoming pathways and challenges in implementation and teacher professionalization.

Looking across the elements of the PMMI framework, two cross-cutting themes arise. First, the importance of blending practices of teaching and learning enables students to experience "Personalized Mathematics." These practices include establishment of mathematical goals, emergence and growth of reasoning, perseverance to problem solving, connections among mathematical representations, formation of concepts, and fluency with procedures (NCTM, 2014). In this sense, the PMMI framework deals with an issue that has been largely left subtle, i.e., how mathematical inquiry interweaves with and supports the needs of individual students for personalized learning. Second, it entails the importance of identifying a possible route through which students could engage with "Mathematical Inquiry." This route needs to integrate opportunities for students to experience the romance, exhibit precision, and achieve generalization of mathematical ideas (Whitehead, 1929). In addition, this route adopts an enhanced discovery approach, by varying the guidance and feedback offered to students, based on their needs (Lazonder & Harmsen, 2016).

The PMMI framework suggests that explorations and investigations constitute concrete examples of mathematical tasks that promote the realization of such opportunities in the classroom environment (Pink, 2009). Promoting inquiry-based learning is only part of the solution for achieving quality learning outcomes; students should also be given the opportunity to express and discuss their own ways of thinking mathematically. Hence, the PMMI framework contributes into comprehending the process through which students could make sense of the mathematics and at the same time develop skills such as critical thinking, creativity, collaboration, and communication.

Furthermore, the PMMI framework elaborates the position that textbook' tasks can be viewed as "shapers of the curriculum rather than merely presenting a given curriculum" (Thompson & Watson, 2013, p. 279). In this perspective, mathematics textbooks should include lessons that "shape" the underlying principles of the PMMI framework about the way in which mathematics are expected to be taught and learnt. Specifically, each lesson in the textbooks provides insight into how "Personalized Mathematics" and "Mathematics Inquiry" could effortlessly be implemented through the enactment of explorations and investigations. In this way, teachers are given a concrete learning context that outlines the conceptual objectives of their instruction through structured, innovative, or even unusual tasks. Of course, we acknowledge that a coherent and well-structured textbook does not always ensure that different teachers in different school classrooms will implement the tasks in the same way and trigger similar learning outcomes. Efforts should be placed in training and supporting teachers in students' engagement with the textbook tasks as designed based on the PMMI framework. Future empirical research may also examine the way teachers perceive the PMMI framework, how they implement it in their classroom or what are the more challenging aspects of this implementation.

The PMMI framework was designed to be applicable for teaching and learning mathematics across K-12. The underline assumption is that students can perform inquiry-based learning and mathematical practices from their early years. This does not mean that advanced mathematical thinking or complex concepts will be pushed down in elementary school. Rather, the PMMI framework suggests that how the content of elementary school is approached and taught should be reformed. Still, several questions remain to be addressed. We agree with other researchers (Alfieri et al., 2011), that future studies should investigate the type and the extent of guidance appropriate for various age groups. Additionally, further studies may investigate what type of support students of various ages need in order to become more efficient in posing questions, collecting data, hypothesizing, reflecting, or drawing conclusions. In addition, further research is needed to define explicit design principles for explorations and investigations and evaluation criteria based on the targeted age. Furthermore, teaching interventions and design research studies may yield empirical data regarding the effectiveness of the framework, elaborate the design principles of explorations and investigations, and offer insightful details regarding the effect of the guidance given during the learning route. Finally, an empirical study may reveal the exploration and investigation characteristics that contribute to further enhancing the personalized dimension of the model, by providing opportunities to find solutions based on students' own mathematical understanding.

Concluding, the PMMI framework yields insights into how the ongoing goal for fostering Personalized Mathematics learning and mathematics inquiry-based learning could be served. Using the notions of exploration and investigation, the PMMI framework defines how the content of mathematics textbooks might look like, in order to boost students' engagement in developing and reinventing mathematical concepts and ideas by linking relevant contexts with individual, sustainable conceptions. It offers a suggestion as to how mathematical challenge may look like in the mathematics classroom. Needless to say, that enacting "Personalized Mathematics" and "Mathematical Inquiry" requires investment in curriculum and textbook development, as well as long-term teacher professional development.

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Chapter 6
Math-Key Program: Opening Mathematical Minds by Means of Open Tasks Supported by Dynamic Applets

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6.1 Introduction

The aim of the Math-Key program described in this chapter is to open mathematical minds through the use of open tasks. The program rationale is based on the position that intellectual openness determines the future of individuals and of society, determines learning motivation and curiosity, and advances technological and scientific progress as well as equity in a multicultural and highly heterogeneous society. We believe that mathematics in general and open mathematical tasks in particular are tools for the advancement of intellectual openness, flexibility, and creativity and that they promote collaborative skills. We suggest systematic integration of the Math-Key instructional approach in teaching and learning mathematics. The Math-Key program is designed for junior-high school mathematics as a problem-solving path designed to complement the regular curricular instructional activities. It can be used though integration in the regular lessons or as an enrichment program.

The *goal* of the Math-Key program is the development of mathematical creativity and mental flexibility along with the advancement of mathematical knowledge and skills. Of equal importance, it is aimed at making mathematics lessons enjoyable and attainable for all students. This is done by piquing their curiosity by exposing them to a variety of problem-solving strategies applied to a particular mathematical problem or a variety of solution outcomes attained.

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The openness of Math-Key tasks determines their mathematical challenge. To allow regulation of mathematical challenges with respect to students' mathematical potential, the majority of Math-Key tasks are accompanied by dynamic applets (DA) that allow regulation of the level of mathematical challenge embedded in the tasks. The applets are open to allow students to investigate the situation given in the task, to support their understanding of the mathematical structure of the task and, if needed, to simplify the situation through experimentation. Teaching with Math-Key tasks requires changing the classroom culture and requires flexibility and openness to ideas of students on the part of the teachers. In what follows, we characterize the mathematical challenge embedded in Math-Key tasks, with a focus on its association with task openness. Then we turn to the characterization of the dynamic applets integrated into Math-Key tasks as a major tool for various mathematical challenges.

6.2 Openness of Math-Key Tasks

One of the well-advertised instructional approaches to teaching and learning mathematics is solving open problems. Open mathematical tasks are broadly discussed in mathematics education and are accompanied by strong arguments about their effectiveness for the development of knowledge and creativity (Pehkonen, 1995; Silver, 1995; Leikin, 2018). However, open tasks are seldom used in mathematics classrooms and are rarely included in mathematics textbooks. The Math-Key program makes open tasks available for systematic use in mathematics classes.

Math-Key tasks are of several types:

- Multiple Solution-Strategies Tasks (*MSTs*) that explicitly require solving a mathematical problem in multiple ways (using multiple strategies). MSTs are *open-start tasks,* the openness of which is linked to the possibility of producing different individual solution-strategies spaces.
- Multiple Outcomes Tasks (*MOTs*) are associated with solving problems that have multiple solution outcomes independently of the solution-strategy used. MOTs are usually ill defined (cf., Krutetskii, 1976). In order to solve the problems, participants are required to change/ choose/add problem givens. MOTs can be either be *complete-ended tasks* or *open-ended tasks*.
  - *Complete-ended tasks* require finding a complete set of solution outcomes. The challenge is in examining the completeness of the set of solution outcomes of an ill-defined problem.
  - *Open-ended tasks* allow finding multiple outcomes, without attaining a complete set of solutions.
- *Combined open-start and open-ended tasks are both MSTs and MOTs* that can be solved using different solution strategies and have multiple solution outcomes that do not have a complete solution set. Examples of such tasks are problem-posing tasks and investigation tasks.

6.2.1 *Examples of Math-Key Tasks*

This section presents examples of Math-Key tasks. The tasks are analyzed according to the type of openness, spaces of possible solution strategies, and types of solution outcomes. The solution outcome spaces can be composed of finite or infinite, discreet or continuous sets of outcomes. The solution strategies can differ in representations, frequency, conventionality, and insight imbedded in the solution.

6.2.1.1 Task 1: Birthday Party

Tal and Limor both have a birthday. Tal plans to invite five friends to her birthday party. Limor invites seven friends. They decide to celebrate their birthdays together. How many children will be at the party?

Solution strategies:

1. Venn Diagram;
2. Trying different cases;
3. Insight about min-max-all between;
4. Algebraic: sum of inequalities.

Solution outcomes: {8, 9, 10, 11, 12, 13, 14}

The task is based on a task presented in Verschaffel et al. (1994). Its solution presumes students' understanding that the two groups of friends can have common participants. In the term used by Krutetskii (1976) this problem is an ill-defined task since there is missing information in the givens, i.e., the number of common friends in the two groups is unknown. The complexity of the solution is related to the requirement of completing the missing information. Additional complexity is related to the unconventionality of the solution outcome, which is a set of numbers and not one particular number as is usually attained when solving textbook word problems.

6.2.1.2 Task 2: Distance from School

This task is based on a problem presented in Verschaffel et al. (1997). There is a missing given about the exact placement of the houses. An infinite number of correct solution outcomes is possible as related to the two circles around the school which depict loci of the houses' positions. The solution outcome is an inequality which is rarely the format for an answer to a word problem.

Distance from school

Eran's house is 100 meters from his school, and Alex's house is 300 meters from the school. What is the distance (S) between Eran's house and Alex's house?

Solution strategies:

1. Numerical.
2. Insight about min-max-all between.
3. Diagram.

Solution outcomes: Real numbers S: $100 \leq S \leq 500$

6.2.1.3 Task 3: Car Speed

A motorcycle leaves Haifa at 10:00, traveling to Ashdod at 50 km/h. The length of the road between Haifa and Ashdod is 150 km. A car leaves Haifa at 10:30, following the same route as the motorcycle. How fast does the car need to be travelling in order to catch up with the motorcycle before reaching Ashdod? Solve the problem in multiple ways.

Solution strategies:

1. Numerical.
2. Algebraic.
3. Graphical.

Solution outcomes: $60 < v \leq 110$

In the car speed task, the speed of the car that travels from Haifa to Ashdod is missing and, thus, the problem is ill defined and requires students to consider different conditions as in Tasks 1 and 2. Speed limitation by law, not given in the problem, is a constraint that the solvers have to take into account. The task enables solving the problem in multiple ways: using numerical or algebraic expressions, and using graphs of functions. The solution outcome is an interval of real numbers, the outcome of which is very rare for school algebra.

6.2.1.4 Task 4: Polygon from Two Squares

This task asks us to find possible polygons constructed of two squares, while the sizes and orientation of the two given squares are unknown. A finite number of n-corner polygons (with various infinite rotated figures) can solve the task. The task can be solved in multiple ways. The complete set of natural numbers {4, 5, 6, 7, 8, 9, 10, 11, 13, 16} presents the number of corners in the resulting polygons. However, this task is open-ended because for the same n in this set there is an infinite number of polygons that can be attained.

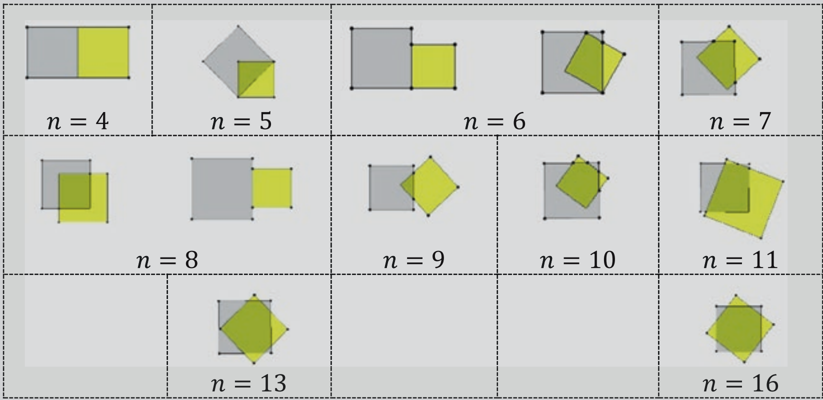
Polygon from two squares

For which values of n can n-corner polygons be constructed of two squares?

Solution strategies:

1. Using manipulatives
2. Paper and pencil
3. Using dynamic geometry

Solution outcomes: {4, 5, 6, 7, 8, 9, 10, 11, 13, 16} examples of polygons:



6.2.1.5 Task 5: Average Test Grade

A class has 24 students. All of the students took a math test. The teacher graded the tests and found that the average was 74.5. Below are the grades: 79, 80, 82, 63, 70, 70, 80, 80, 82, 63, 56, 76, 82, 90, 56, 44, 90, 90, 82, 82, 72, and 70. The next day, the teacher discovered that she had forgotten to grade two tests. She added the two missing tests to the list and found that the average did not change. What could be the grades of the two tests she added? Explain.

(continued)

(continued)
Solution strategies:

1. Numerical – Trial and error
2. Algebraic
3. Using Excel

Solution outcomes: 36 (non-ordered) pairs of scores: (100, 49), (99, 50), (98, 51), …, (75, 74).

This task requires solvers to complete the missing test grades to attain the given average score. Multiple combinations of pairs of grades constitute solutions of the problem. Multiple solution strategies include numerical, algebraic, and graphical solutions. The completeness of the solution is determined by the constraints of the highest (100) and the lowest (0) school grades.

6.2.1.6 Task 6: Tiles on a Square

Find different ways to calculate the number of (colored) tiles on the perimeter of the square (a) Square 6×6 (b) Square $n \times n$

Solution strategies:

1. Numerical (counting)
2. Graphical (coloring)
3. Generalization of numerical expressions for $n \times n$ (from (a) to (b))
4. Substitution of numbers in the algebraic expressions (from (b) to (a))

Solution outcomes:

(a) Set of numerical expressions
(for specific n, e.g. $n = 6$)
$4(6 - 1)$;
$2 \bullet 6 + 2(6 - 2)$;
$4 \bullet 6 - 4$;
$6^2 - (6 - 2)^2$;
$4 + 4(6 - 2)$.
(b) Set of algebraic expressions:
$4(n - 1)$; $2n + 2(n - 2)$;
$4n - 4$; $n^2 - (n - 2)^2$; $4 + 4(n - 2)$.

The openness of the "tiles on a square" task is linked to the requirement to solve the tiles problem in multiple ways. The task is both open-start and open-ended. The task's outcomes are numerical (or algebraic) expressions that reflect the way the problem is solved. Moreover, while numerical solutions can be used as generic examples to attain generalized solutions, when starting from an algebraic solution, numerical solutions can be attained by substitution of concrete numbers in algebraic expressions.

6.2.1.7 Task 7: Expressions of Parabolas

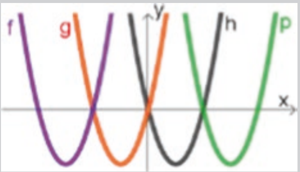
The expressions of parabolas tasks requires the translation of a set of mutually related graphs of quadratic functions into a set of algebraic expressions. The openness from the start is related both to the possibility of solving the problem in multiple ways and with the need to decide the values of the intercepts of the graphs with the x-axis and the min values of the given functions, which are not given in the task. Among multiple solution strategies, representation of functions as a product of two linear expressions is the most elegant way. Thus, an infinite number of solutions is possible and can be generalized using parametric representation of the functions.

Expressions of parabolas

Write expressions for the parabolas pictured. Find at least two more possible expressions for the parabolas.

Solution strategies:

1. Paper-pencil solution
2. Solution using dynamic software



Solution outcomes:

(a) Mutually related function equations,
e.g.,
$f(x) = 2(x + 6)(x + 3)$;
$g(x) = 2x(x + 3)$;
$h(x) = 2x(x - 3)$;
$p(x) = 2(x - 6)(x - 3)$.

(b) Generalized solution: Infinite number of mutually related algebraic expressions for functions $f(x)$, $g(x)$, $h(x)$, $p(x)$
For: $a > 0$, $d > 0$:
$f(x) = a(x + 2d)(x + d)$;
$g(x) = ax(x + d)$;
$h(x) = ax(x - d)$;
$p(x) = a(x - 2d)(x - d)$.

Table 6.1 summarizes the task analysis presented above. As described, the openness of Math-Key tasks increases the complexity of the tasks. These tasks require a high level of cognitive demand, which is linked to the mental flexibility needed to relate

Table 6.1 Characterization of the openness of Math-Key tasks

		Solution outcomes			Solution strategies			
		Set of solution outcomes	No of solutions	Continuity	Insight	Visual	Algebraic	Trial and error
1	*Birthday Party*	Natural numbers: $8 \leq n \leq 14$ Completeness is required	Finite	Discrete	Min-Max	Venn Diagram	Sum of inequalities	Different cases
2	*Distance from school*	Real numbers: $100 \leq S \leq 500$ Completeness is required	Infinite	Continuous	Min-Max	Geometric diagram	Triangle inequality	Different cases
3	*Car speed*	Real numbers: $60 < v \leq 110$ Completeness is required	Infinite	Continuous	Min-Max	Graphs of functions	Algebraic inequalities	Concrete examples
4	*Polygons*	$\begin{Bmatrix} 4,5,6,7,8, \\ 9,10,11,13,16 \end{Bmatrix}$ Completeness is required Openness with regard to different figures for the same n	Finite	Discrete		Dynamic diagram		Manipulatives
5	*Average grades*	36 pairs of scores: (100, 49), (99, 50), (98, 51), ….,(75, 74). Completeness is required	Finite	Pairs of numbers	Min-Max			Excel worksheet

6	*Tiles on a square*	$4(n-1)$ $2n+2(n-2)$ $4n-4$ $n^2-(n-2)^2$ $4+4(n-2)$ Open problem since completeness is required	Finite	Algebraic expressions	Equivalent expressions	Coloring	Generalization	Numerical expressions
7	*Expressions of parabolas*	$f(x)$, $g(x)$, $h(x)$, $p(x)$ Generalized solution: For: $a>0$, $d>0$ $f(x)=a(x+2d)(x+d)$ $g(x)=ax(x+d)$ $h(x)=ax(x-d)$ $(x)=a(x-2d)(x-d)$	Infinite	Parametric expressions	Generalized - parametric	Dynamic diagram	Concrete sets functions	Dynamic diagram

to the multiplicity of solutions and the novelty of mathematical reasoning necessary to complete the missing givens of the problems. To make these tasks accessible to students at different levels, we design dynamic diagrams (applets).

6.2.2 Solution Spaces and Mathematical Challenges Embedded in Math-Key Program Tasks

Solving Math-Key tasks is a fundamentally creativity-directed activity: Using different problem-solving strategies requires and develops mental flexibility and opens opportunities for using original strategies. Through task exploration, solvers search for data or knowledge they have acquired and build new structures that are matched with the task‘s information, learn new mathematical concepts, and develop new skills (Cai, 2010; Leikin, 2014; Nohda, 1995; Silver, 1995, 1997; Vale et al., 2018). Solving open tasks involves divergent and convergent thinking, decision-making, mathematical reasoning and critical reasoning.

Solving Math-Key tasks provides multiple opportunities for the development of social skills through cooperative learning and group competition directed at finding original solutions. Solving these tasks usually evokes surprises since different students can find different solutions. Positive affect is associated with surprise and "Aha!" moments when solving Math-Key tasks lead to the development of students' mathematical curiosity and motivation to learn mathematics (Boaler, 2015; NCTM, 2014). Due to the openness of the tasks, solving Math-Key tasks also leads to the development of students' self-regulated learning skills, self-esteem, and other twenty-first century skills (Kim et al., 2019; Leikin, 2018; Pellegrino & Hilton, 2012). Moreover, solving open tasks transforms mathematical instruction, and leads to enhanced classroom discussions in which students share different approaches and ideas as well as difficulties and successes that they have experienced (Peled & Leikin, 2017).

Figure 6.1 depicts the main components of the Math-Key program with the emphasis on typical goals, activities, conditions, and tools (cf. Leontiev's (1978) Activity Theory). Additionally it draws attention to the construct of solution spaces and their transformation that promote students' intellectual development. There are two kinds of solution spaces for Math-Key tasks: Spaces of Solution Strategies and Spaces of Solution Outcomes. We distinguish between individual and collective solution spaces. Due to the multiplicity of solution strategies and solution outcomes, individual solution spaces differ from one another and can be broadened by exposure to collective solution spaces (Leikin, 2007). Through broadening spaces of solution strategies, students develop problem-solving skills. Classroom discussion focuses on the elegance of the solutions, the level of complexity of the solutions, and their originality. Broadening the spaces of solution outcomes leads to either complete solution spaces or to the consideration of the quality of solution outcomes and their originality.

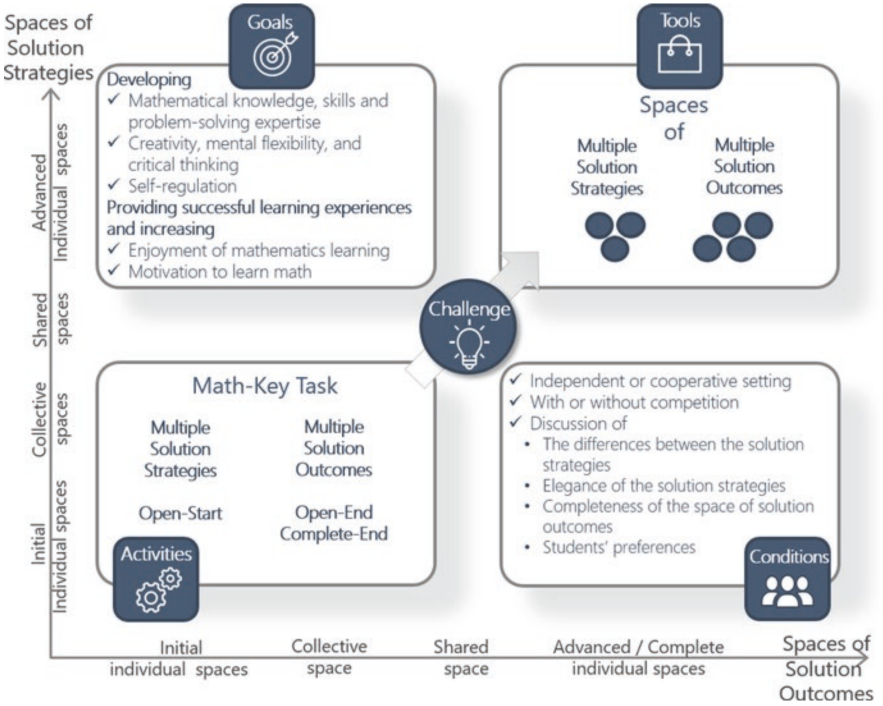


Fig. 6.1 Main components of the Math-Key activities

When solving Math-Key tasks, upon finding a suitable strategy or outcome, the solvers have to search for a different strategy or outcome that fits the situation of the displayed problem (Dorfman et al., 2008; Lin & Lien, 2013). The inhibition process requires mental effort on the part of the solvers. Thus, high-level students usually display higher creativity capabilities than lower-level students (Kattou et al., 2013; Levav-Waynberg & Leikin, 2012). Usually, teachers state that MOTs and ITs are more difficult to solve than MSTs since higher cognitive skills are required. In these problems, various strategies have to be considered, as well as different outcomes (Klein & Leikin, 2020). Moreover, sometimes, during the problem-solving process, solvers experience an "Aha!" moment like a missing piece of a puzzle that falls into place, and are suddenly inspired to solve the problem (Liljedahl, 2013; Presmeg, 2018; Vale et al., 2018).

To summarize, Math-Key tasks are inherently challenging (Fig. 6.1). The challenge is associated with five main characteristics of the problem-solving process. The required multiplicity of the solutions induces activation of

(a) Math-Key tasks requiring broader mathematical knowledge and advanced mathematical skills
(b) Inhibition, which is essential for inhibiting a particular way of thinking which led to an already performed solution

(c) Mathematical flexibility, which is necessary for finding a different solution strategy
(d) Critical reasoning needed for the evaluation of the significance of differences between the strategies used or when examining the completeness of the space of solution outcomes
(e) Novel thinking, which is entailed in the solver's search for

  – New (for the solver) solution strategies
  – New (still unknown) solution outcomes (when solving open multiple-outcomes tasks)
  – Additional solution outcomes when completeness of the solution set is required (due to the unconventionality of the completeness criterion).

As mentioned in the introduction, dynamic applets are specially designed to allow the regulation of mathematical challenge with respect to students' mathematical potential. The applets allow investigation of the situation given in the task, support students' understanding of the mathematical structure of the tasks, and, if needed, simplify the situation through experimentation. In the next section, we characterize the dynamic applets that complement the Math-Key tasks and are a major tool for varying the mathematical challenge therein.

6.3 Varying Mathematical Challenge with Dynamic Applets

6.3.1 Math-Key Applets

In recent years, different types of technological tools have been developed to support teaching and learning practices. These tools have a wide variety of applications with respect to mathematical focus and didactical functionality. The notion of didactical functionality is a compound of a set of characteristics of the tool, a specific educational goal exploiting these characteristics, and a set of modalities of employing the tool in a teaching/learning process referred to the chosen educational goal (Cerulli et al., 2005). Dynamic diagrams are among the most popular technological tools used in teaching and learning mathematics.

Yerushalmi (2005) defined "a dynamic diagram" as a pre-constructed software application (often called an applet) built around an example or a problem. She distinguished between different types of diagrams: *Illustrating diagrams* usually offer a single graphic representation with relatively simple actions, such as viewing an animated example. *Elaborating diagrams* present occurrences relevant to the problem, to be explored while working on the task. *Narrating diagrams* are the principal delivery channel of the activity's message.

Barzel et al. (2005) defined the four dimensions for structuring and categorizing dynamic diagrams linked to the availability of technological tools in learning and problem solving in mathematics. We apply these dimensions to Math-Key applets as follows: Math-key applets are *explorative*. They allow students to express and develop their ideas while exploring the mathematical relations and structures embedded in the applets. They are also Math-Key *user driven*. A user, not the applet, makes decisions about what to do with the applet and how. Math-Key applets are *open* to work with applets in different ways and find multiple solution outcomes. These Math-Key applets are *specific* since they are designed for use with concrete Math-Key tasks.

Implementation of Math-Key tasks accompanied by the Math-Key applets is performed according to the following principles suggested by Drijvers et al. (2010): The implementation of the Math-Key tasks requires didactic configuration based on the choice teachers make to use specific tools in order to create the desired learning outcome. Math-Key tasks support an exploitation mode of teaching by choosing how the task is presented to the students and the expected solution process needed for developing the students' knowledge. Finally, Math-Key tasks support teachers' didactical performance while expected results are attained by combinations of the technological tool, teaching method, and the potential of students.

Math-key applets are designed to help students to discover the structure of problems. The use of GeoGebra software simulates the situation of a given task and helps students explore it. Employing technology during math lessons is considered by some of the students as a game and not as study (Kebritchi et al., 2010). This may reduce mathematical anxiety, contribute to a positive atmosphere, and increase motivation to discover strategies and solutions. Furthermore, the applets enable the refinement of concepts and the relationships between them. When looking for solutions, sometimes the applets reveal options that students did not think of. Additionally, low-level students can reach solutions more easily using the applets, while high-level students can verify their answers. The applets can also be used as a summary tool to review the subjects studied. In conclusion, Math-Key applets make the problems displayed more approachable for students. Additionally, they make teaching more effective because mathematical concepts and structures are discovered by students through exploration of tasks with Math-Key applets.

6.3.2 Examples of Math-Key Applets

In this section we describe applets developed for the Math-Key tasks described above. The applets are analyzed from the point of view of applets' functions: (a) *technological features used*: dragging, measuring, coloring, animation, slider, translation between representations, counting, writing expressions, value substitution; (b) focus of *investigation:* comparing, analyzing specific cases, observing regularity, searching for generalization. As mentioned above, all the applets are user driven, explorative, task directed, and ready to use.

6.3.2.1 Applet for Task 1: Birthday Party Applet

The birthday party applet (Fig. 6.2a) displays two disjoint groups of icons, representing the girls at the party, marked in two different colors. Each group is framed by a rectangle of a matching color. One of the groups is static (the number of girls icon for Tal's group is constant). The second (Limor's) group is dynamic, i.e., the rectangle can be dragged (Fig. 6.2b), and the number of icons can be changed (Fig. 6.2c). By dragging Limor's rectangle, a solver can change the overall number of icons while the icons of different colors overlap. By dragging on the slider ($1 \le n \le 10$), the number of icons that represent Limor's friends changes. The applet provides a dynamic illustration of the task's structure. The dragging enables the users to focus on specific cases by creating additional solutions (including a situation where the number of participants is minimal), and as a result, enables them to analyze the outcomes. The applet enables the users to explore the situation and investigate the number of the participants at the party. The applet employs a Venn diagram with dynamic features that displays the minimal and the maximal numbers of party participants as well as all the numbers between. Changes in givens allow generalization using parameters for the invited people (Fig. 6.2).

6.3.2.2 Applet for Task 2: Distance from School

This applet enables the students to visualize the task structure by visualizing the mutual position of two houses relative to the school. By continuous dragging, the students can observe the whole range of solution outcomes. The users can drag two points on the circles and understand that there is an infinite number of solutions to the problem. The applet has two versions. In one version, it does not display distances between the school and the houses; in the other version, the numerical values are displayed (Fig. 6.3).

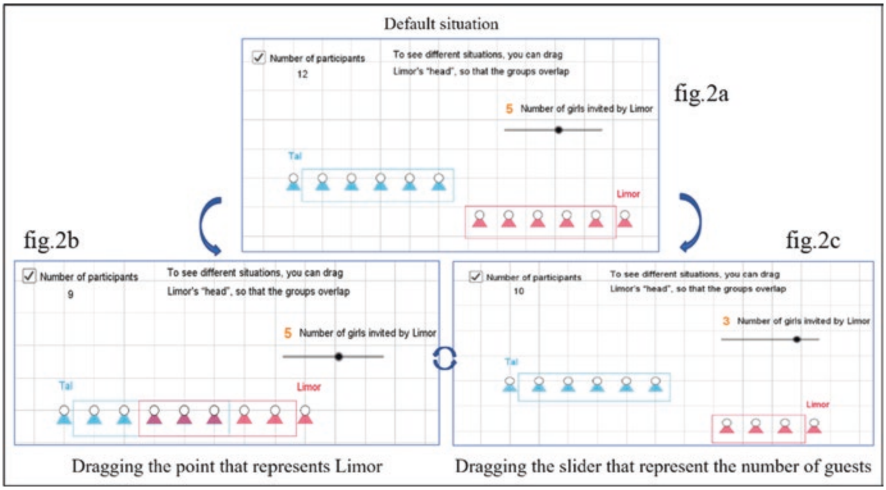


Fig. 6.2 Birthday party applet

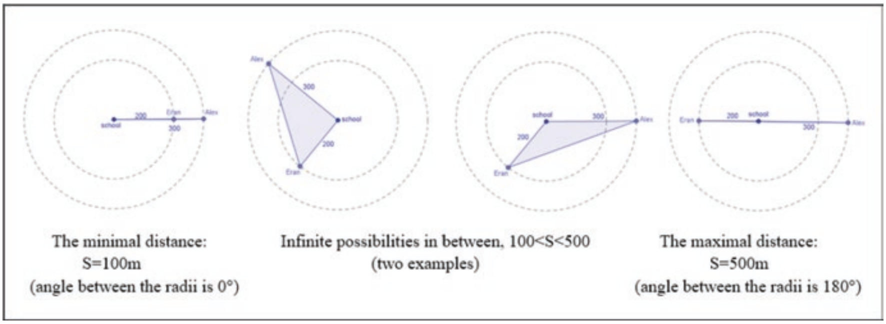


Fig. 6.3 Distance from school applet – solution space (including the extremes)

6.3.2.3 Applet for Task 3: Car Speed

The car speed applet includes animations of the motion of the car. Students can observe specific cases using the applet. The clock tracks the time that elapses from the beginning of the movement and can track the motion of the vehicles. They can stop the motion by clicking on the “stop” button in order to estimate the moment the vehicles meet. Additionally, two graphs of distance correspond to the speed of the vehicles and are displayed simultaneously with the animations. The applet enables students to understand the connections between numerical (car’s speed, clock’s time), algebraic (graph), and visual (animation) representations (Fig. 6.4).

6.3.2.4 Applet for Task 4: Polygons from Two Squares

The polygon from two squares applet displays as a default two congruent squares. The squares can be rotated and dragged. The sliders enable changing the side length of each square to create incongruent squares. The dragging in the applet allows changing the positions of the squares. The users can focus on specific cases and analyze the outcomes. They can understand that there is an infinite number of polygons that can be created, while the number of vertices varies from four to sixteen (Fig. 6.5).

6.3.2.5 Applet for Task 5: Average Grades

The applet of this task makes use of an excel table with students’ grades that enables the calculation of the mean score. Students can add and then change grades, to see how the average changes. The applet displays the constraints of the task helping students to understand the structure of the givens of the task, to focus on specific cases, and to analyze the outcomes. Conclusions that are related to the sum of the missing grades can be reached as well conclusions regarding whether or not the order of the paired available set of scores is important (Fig. 6.6).

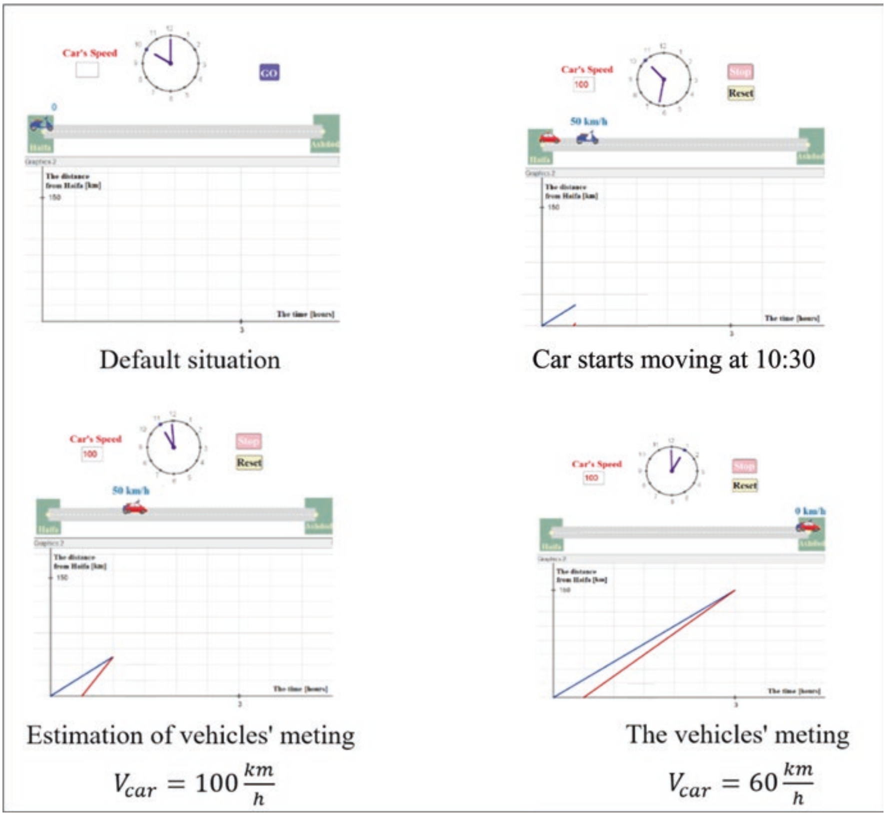


Fig. 6.4 Car speed applet

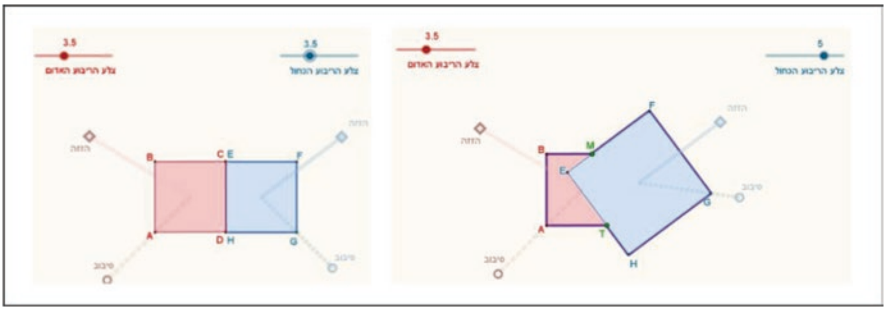


Fig. 6.5 Polygon from two squares applet: congruent and incongruent squares

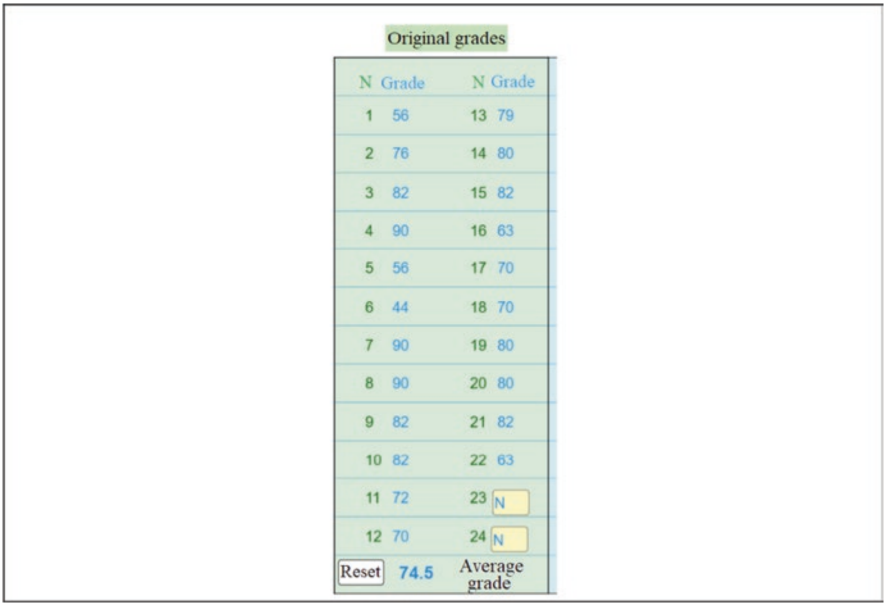


Fig. 6.6 Average grades applet

6.3.2.6 Applet for Task 6: Tiles on a Square

The tiles on a square applet depicts a square with the possibility of painting the tiles using different colors. The number of colored tiles can be displayed according to the user's requirements. The default situation displayed is a 6x6 square. Users can change the length of the square's sides ($1 \le n \le 11$). Coloring the tiles enables users to focus on specific cases and different patterns that reflect different numerical patterns. Changes of the square's size, for which similar numerical patterns that express the number of tiles on the perimeter of the square allow geberalisation (Fig. 6.7).

6.3.2.7 Applet 7: Expressions of Parabolas

The applet for the expressions of parabolas task depicts a set of parabolas in a coordinate system. The scale on the axes is missing. The values and gridlines allow students to use specific cases. The applet allows users to write expressions for the functions and depicts corresponding graphs of parabolas. The applet allows students to check if the functions are correct by visual examination and also by using the checkbox. The correct connections between the functions are marked in blue. If there is a mistake, the incorrect connection between functions will appear in red (Fig. 6.8).

The users can focus on specific cases, find connections between numerical, algebraic, and visual representations, analyze the outcomes, and discover the regularity of the expressions. Using the applet, students can think of different solution strategies. For instance, they can start the solution process by focusing on the function

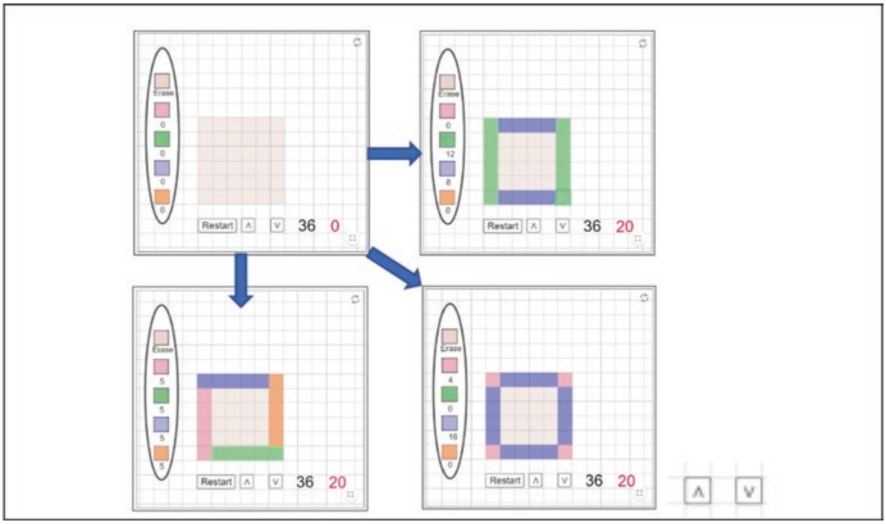


Fig. 6.7 Tiles on a square applet

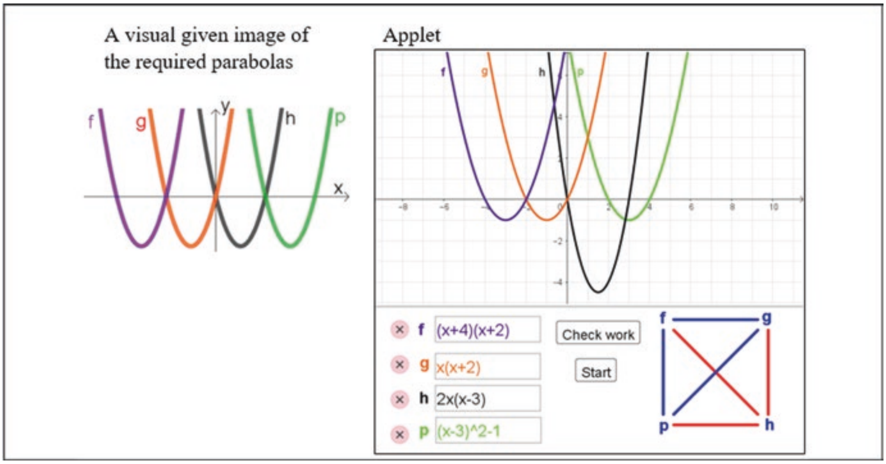


Fig. 6.8 Expressions of parabolas applet (right side)

that looks most familiar, and use it as a building block for the other functions. A student who chooses the function $h(x)$ may notice that this is a function of the form $h(x) = ax(x - d)$ for each value of $d > 0$, $a > 0$, and decide on values for a and d. The students may determine whether the algebraic expression they wrote matches the graph of the given function. This is done by conducting a visual comparison between the parabola appearing in the upper window and the original image of parabolas, and by doing calculations. How do they proceed? The students may note that every function in the given image meets the following conditions: (i) The graph of each function intersects the x-axis at two points; (ii) All of the functions are horizontal translations of the function $h(x)$, respectively: $(x) = a(x - 2d)(x - d)$, $g(x) = ax(x + d)$, $f(x) = a(x + 2d)(x + d)$. Students are familiar with the three forms of a parabola (standard, vertex, and factored form) with transitions between them, and with the significance of the parameters. They may decide which form to use in order to find an algebraic representation of the functions: First, they can focus on a specific instance with chosen parameters, and then they can try to generalize and to discuss the family of functions that fill the conditions given in the task. Some students may notice the symmetry between the functions with respect to the y-axis: $g(x) = h(-x)$ and $f(x) = p(-x)$.

6.3.3 *Math-Key Applets Characteristics*

Math-key applets have different goals related to each task, such as focusing on specific cases, connections between numerical, algebraic, and visual representations, discovery of regularity (expression / function), and visualization of the structure of task givens, analyzing the outcomes, and sorting (Fig. 6.9).

Additionally, the goals of the applets can be reached by diverse functionalities of different types of dragging (continuous or discrete values; measurement; with or without change in objects' size), coloring, animation, writing expressions or numerical values that match constraints. Moreover, each Math-Key applet is user driven – in each case, the student decides what strategy to choose.

Table 6.2 summarizes several aspects of the applets of each of the Math-Key examples discussed in this paper. These include (a) functionality of changing/transforming (through dragging, sliding, animation, and coloring) and investigations by means of measurement, writing, and substitution of values; (b) strategies used in the

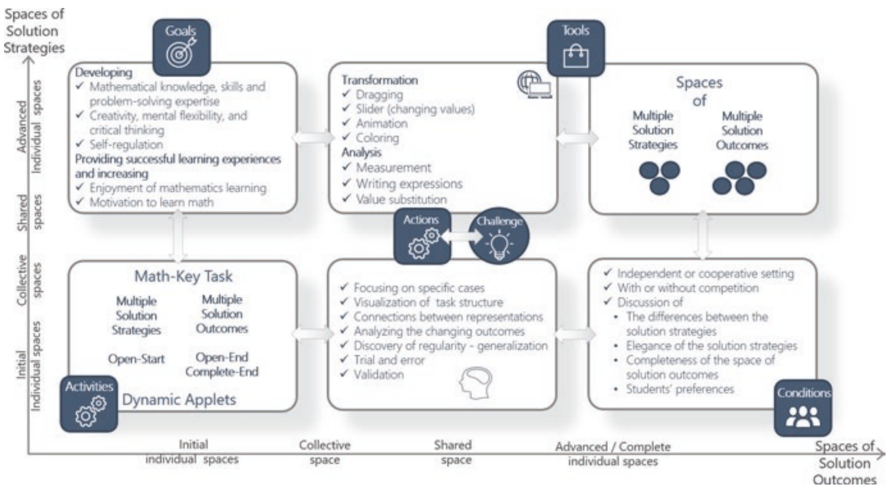


Fig. 6.9 Main components of the Math-Key program

Table 6.2 Type of applets' goals, functioning, structure, and mathematical domains according to Math-Key open tasks

Task Applet functioning		Birthday Party	Distance from school	Car speed	Polygons	Average grades	Tiles on a square	Expressions of parabolas
Technological functions	Dragging	Discrete	Continuous		Continuous			
	Measuring or counting	Number	Length	Time			Tiles	
	Coloring						Tiles	
	Animation			Car motion				
	Substituting/ writing					Numbers		Alg. expressions
	Slider	Given numbers		Speed	Square size			
Mathematical / cognitive functions	Specific cases	Numbers	Location	Speed	Square size Orientation	Scores	Size	Alg. expressions
	Changing the givens							
	Examining task structure	Venn diagram	Triangle inequality	Graphs of functions			Different coloring	Family of functions
	Connections / translations between representations	numbers and images		Graphs and numbers	Figures and numbers		Tiles' coloring and alg. expressions	Graphs and formulas
	Generalization (corresponding to completeness)	Set of natural numbers (min-max)	Interval of rational numbers			Set of pairs of natural numbers		Parametric representation
	Outcome validation							Validation diagram
Domain	Algebra	✓		✓		✓	✓	✓
	Geometry		✓		✓		✓	
	Statistics					✓		
Structure	User driven	✓	✓	✓	✓	✓	✓	✓
	Explorative	✓	✓	✓	✓	✓	✓	✓
	Ready to use	✓	✓	✓	✓	✓	✓	✓
	Open	✓	✓	✓	✓	✓	✓	✓
	Goal directed	✓	✓	✓	✓	✓	✓	✓

applets, as part of the students' mental processing to solve the tasks (via focusing on specific cases, visualization, connections between representations, generalization, trial and error, and getting feedback from the applet); (c) the structure of the applets (user driven, explorative, open, and specific to the displayed problem); (d) the mathematical domain that the applet was built for (algebra, geometry, and statistics).

6.4 Summary

In conclusion, technology in mathematics classrooms has great potential to develop students' knowledge. Coping with the challenge of Math-Key tasks using their dynamic applets is a key element in providing positive experiences, developing student knowledge, and increasing their motivation to learn. The applets provide a comfortable environment to discuss different solution strategies, produce different outcomes, and connect mathematical concepts and structures through visualization. This type of technology provides opportunities both for teachers and students through an exciting and enjoyable learning process, and should be further developed.

The variability of solution strategies and outcomes results in significant changes in classroom culture. In contrast to common instructional practices, students arriving at different solutions is acceptable and even desirable. The constructs of collective solution spaces – spaces of solution strategies and spaces of solution outcomes – are central for monitoring mathematics lessons.

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Chapter 7
Making Mathematics Challenging Through Problem Posing in the Classroom

Jinfa Cai and Stephen Hwang

7.1 Introduction

A well-established observation in education is that challenge can provide fertile ground for students to learn. In order to understand mathematical concepts deeply, students need opportunities to struggle productively with challenging mathematical problems (Hiebert & Grouws, 2007). This kind of productive struggle, in which students actively grapple with concepts that are within their grasp but that they do not yet clearly understand, forms the basis for constructing conceptual understanding. A primary question, then, is how to provide appropriate challenges for students so they may engage in productive struggle that leads to deeper understanding of mathematics. Instructional tasks that engage students in effortful reasoning and problem solving constitute a critical part of the answer to this question (Arbaugh et al., 2010).

In this chapter, we begin by discussing instructional tasks and their role in fostering productive struggle through appropriate challenge. We then focus on a particular kind of instructional task—mathematical problem posing—that can promote productive struggle. In particular, we consider problem posing from both the perspective of the student, who may be asked to pose problems based on given situations or by reformulating existing problems, and the perspective of the teacher, who can either pose problems for students to solve or use problem-posing tasks as instructional tasks with students (Cai & Hwang, 2020). In particular, we argue that problem-posing tasks have an inherent benefit in that they are able to provide levels of mathematical challenge that scale to the level of understanding of the student. We

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then consider how problem posing could be implemented effectively in the mathematics classroom, both by providing examples of problem-posing tasks and by outlining ideas for professional development that can help teachers use problem posing to teaching mathematics. Finally, we discuss some findings from empirical research (and future directions for such research) on teaching mathematics through problem posing.

7.2 Challenging and Worthwhile Instructional Tasks

Instructional tasks, referred to by some researchers as "academic tasks" or "mathematical tasks" (e.g., Doyle, 1983; Hiebert & Wearne, 1993), can be defined broadly as activities related to projects, questions, problems, constructions, applications, and exercises with which students engage. Instructional tasks provide intellectual environments for students' learning and the development of mathematical thinking. Tasks govern not only students' attention to particular aspects of content but also their ways of processing information. Doyle (1988) argued that tasks with different cognitive demands are likely to induce different kinds of learning. In particular, instructional tasks that are truly challenging have the potential to provide the intellectual contexts for students' mathematical development. Such tasks can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture students' interests and curiosity (National Council of Teachers of Mathematics [NCTM], 1991). Indeed, standards documents have recommended that, in classrooms, students should be exposed to challenging tasks specifically to promote mathematical sense making (NCTM, 1991, 2000).

A number of studies support this connection between the nature of instructional tasks and student learning (Cai, 2014; Hiebert & Wearne, 1993; Stein & Lane, 1996). In the QUASAR project, classrooms using instructional tasks that are cognitively demanding produced the highest gains in students' conceptual understanding (Stein & Lane, 1996). Similarly, in a longitudinal analysis of the effects of the Connected Mathematics Project (CMP) middle school mathematics curriculum versus traditional curricula, CMP students exhibited greater gains in conceptual understanding (and comparable gains in procedural skill) than their non-CMP counterparts (Cai, 2014). Notably, in classrooms using the CMP curriculum, a significantly larger proportion of implemented instructional tasks were at a higher level of cognitive demand than the instructional tasks implemented in the non-CMP classrooms.

Given that high-cognitive-demand instructional tasks are particularly "worthwhile" for creating opportunities for students to solidify and extend what they know and stimulate mathematics learning (NCTM, 1991), it would be useful to know what makes a task worthwhile. Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites speculation and hard work. Most importantly, worthwhile mathematical tasks should direct students to investigate important mathematical ideas and ways of thinking toward the learning goals. Lappan and Phillips (1998) developed a set of useful criteria to choose worthwhile problems for mathematics instruction:

- The problem has important, useful mathematics embedded in it.
- Students can approach the problem in multiple ways using different solution strategies.
- The problem has various solutions or allows different decisions or positions to be taken and defended.
- The problem encourages student engagement and discourse.
- The problem requires higher-level thinking and problem solving.
- The problem contributes to the conceptual development of students.
- The problem connects to other important mathematical ideas.
- The problem promotes the skillful use of mathematics.
- The problem provides an opportunity to practice important skills.
- The problem creates an opportunity for the teacher to assess what his or her students are learning and where they are experiencing difficulty.

The two tasks below exhibit some of these criteria. The first task comes from Heid (1995):

Given the two job offers below, determine the better-paying summer job. Justify your answer.

Offer 1:	*At Timmy's Tacos you will earn $4.50 an hour. However, you will be required to purchase a uniform for $45.00. You will be expected to work 20 hours each week.*
Offer 2:	*At Kelly's Car Wash you will earn $3.50 an hour. No special attire is required. You must agree to work 20 hours each week.*

In response to this task, students could generate a wide variety of solutions, such as the following:

SOLUTION 1:

In a 20 hr. week,

Offer 1 will pay $4.50 × 20 = $90.00.

Offer 2 will pay $3.50 × 20 = $70.00.

Since the difference is $20 per week and the uniform for Offer 1 costs $45.00, it will take ($45.00/$20/week =) 2.25 weeks to pay for the uniform and break even. If you keep the job for 3 weeks or more, you should take Offer 1.

SOLUTION 2:
At Timmy's you make $1.00 more for each hour of work. After 45 hours of work, you'd make $45 more at Timmy's than Kelly's. This extra money would pay for the uniform. From that point on, you'd make $1 more an hour at Timmy's than Kelly's.

SOLUTION 3:
Let x be the number of weeks you intend to work. The total amount for Offer 1 $=90x - 45$ and the total amount for Offer 2 $=70x$.

If $90x - 45 = 70x$, then $x = 2.25$. So if you work less than 3 weeks, you should take Offer 2, otherwise take Offer 1.

SOLUTION 4:
Let x be the number of weeks you intend to work, y_1 be the total amount for Offer 1 after working x weeks, and y_2 be the total amount for Offer 2 after working x weeks. Therefore, $y_1 = 90x - 45$ and $y_2 = 70x$. Using a graphing calculator to graph them, you will see they intersect at (2.25, 157.5). From the graph, you will see that if you have the job for 3 weeks or more, you take Offer 1.

SOLUTION 5:
Construct a table to show the amount of income for Offers 1 and 2 for 1 week, 2 weeks, and 3 weeks…, and then compare the information from the table to determine which offer you will take.

SOLUTION 6:

Let x be the number of weeks you intend to work. The total amount for

Offer 1 $=4.5 \times 20\,x - 45$ and the total amount for Offer 2 $=3.5 \times 20\,x$.

If $(90x - 45) < 70x$, then $x < 2.25$. So if you work less than 3 weeks, you should take Offer 2, otherwise take Offer 1.

After students solve a given problem like this, they often think they have accomplished their mission and stop further exploration. However, generating alternative solutions and then analyzing and discussing them in class can create additional learning opportunities for students. Each of the solutions above highlights how the total amount of earnings for each offer is related to the payment for each hour and the expense required for taking the offer. However, the total amount of earnings for each offer is represented differently in these solutions. This first task is not only embedded in important and useful mathematics but can also be approached in multiple ways using different solution strategies. In addition, the problem allows different decisions or positions to be taken and defended and contributes to students' conceptual development.

The second task focuses on detecting and correcting errors in the use of the long-division algorithm:

John was asked to divide 1308 by 12. His work is shown below.

```
     19
12 ) 1308
     12
      108
      108
        0
```

Is John's work correct? Why or why not?

Although long division problems are a staple of elementary school mathematics, students frequently make computation errors when using this algorithm. For example, as in this task, when computing 1308 ÷ 12, many students overlook the zero that should be in the tens place of the quotient. This may reflect a careless error or a lack of understanding of the reasoning behind the algorithm. By asking students to analyze work, as in this task, instead of simply applying the algorithm to find the quotient, this task has the potential to provide both the student and the teacher with a new perspective. The student, in order to explain why the work is incorrect, may draw on (or construct) understanding of the conceptual underpinning of the algorithm. The teacher, then, gains an opportunity to perceive the nature of the student's understanding of long division. Thus, this task not only encourages student engagement and discourse but also contributes to students' conceptual development.

Keeping these criteria in mind, we turn our attention to one particular type of worthwhile instructional task: problem-posing tasks. Problem posing has been discussed and studied in different ways. Here, we will focus on three aspects. First, we examine how teachers can pose problems properly to engage students and provide learning opportunities for students. That is, we consider how teachers should present mathematical problems so as to create more learning opportunities for students. Second, we discuss how students can be provided with opportunities to pose their own mathematical problems and how they may better understand mathematics through posing and solving their own problems. Third, we consider how teachers themselves learn to use problem posing to teach mathematics.

7.3 Posing Problems Properly: From Routine Problem Solving to Non-routine Problem Solving

Teachers may engage in problem posing in several ways. We consider teachers' problem posing as consisting of the following specific intellectual activities: (a) Teachers themselves pose mathematical problems based on given problem situations which may include mathematical expressions or diagrams, (b) teachers predict the kinds of problems that students can pose based on given problem situations, (c) teachers pose problems by changing existing problems, (d) teachers generate mathematical problem-posing situations for students to pose problems, and (e) teachers pose mathematical problems for students to solve (Cai & Hwang, 2020).

Perhaps the most common way that teachers engage with problem posing is the last of these, that is, when teachers pose problems for their students to solve. Although this is still in the domain of presenting problems for students to solve, this aspect of teachers' problem posing highlights the importance of the ways teachers can present problems so as to increase learning opportunities for students.

In fact, as early as 1980, Butts discussed the value of posing problems properly, noting that the way in which a problem is posed has a significant impact on the problem solver's motivation to solve it as well as their understanding of key underlying concepts of the problem. He proposed five types of problems—recognition, algorithmic, application, open-search, and problem situations—and provided suggestions for improving the presentation of each type to maximize the learners' motivation and understanding when solving them.

Recognition problems require the solver to recognize or recall something such as a theorem or definition. They often rely on true–false, multiple-choice, and fill-in-the-blank formats. An example of this type is: "The line segment joining a vertex of a triangle to the midpoint of the opposite side is called a …?" (Butts, 1980, p. 24). To encourage the solver to understand the underlying concepts rather than merely memorizing, an effective way to pose these problems is the "give an example of" format, for example: Give "an example of…a proper fraction greater than 3/4" (Butts, 1980, p. 26).

Algorithmic problems require the solver to perform a particular procedure or algorithm, for example: "Solve $2x^2 - 3x - 5 = 0$" (Butts, 1980, p. 24). He notes that the challenge with these problems is to make them interesting to the problem solver rather than routine. One of the ways to make it more interesting is to present application problems. Application problems require the solver to apply an algorithm to a problem that is not formulated symbolically, for example: "If the length and width of a rectangle are each increased by 20%, by what percent is the area increased?" (Butts, 1980, p. 24). He again notes the need to keep these problems interesting and, particularly in the case of word problems, realistic.

The less common problem types are open-search problems and problem situations. According to Butts (1980), "an open-search problem is one that does not contain a strategy for solving the problem in its statement" (p. 25), for example: "How many different triangles with integer sides can be drawn having a longest side (or sides) of length 5 cm? 6 cm? *n* cm?" (Butts, 1980, p. 25). The key function of these problems is to encourage guessing and exploration, which he claims is the preliminary step on the path to proof writing. Thus, the best way to pose these problems is in a way that encourages the solver to make guesses at the solution. One example of this is what he refers to as "whimsical problems" which pose superrealistic situations that keep the solvers' interest by using realistic elements but with outlandish characteristics. Finally, problem situations are, as their name suggests, not explicit problems but rather situations in which the solver has to identify the problem in the situation before identifying the solution that will address that problem. An example of this problem type is:

Design a parking lot. Possible problems to consider could include the following. There are many, many others.

(a) *How large should each space be?*
(b) *At what angle should each space be placed?* (Butts, 1980, p. 25)

It goes without saying that the most important criterion of a worthwhile mathematical problem is that the problem should serve as a means for students to learn important mathematics. Such a problem does not have to be complicated with a fancy format. As long as a problem can reach the goal of fostering students' learning of important mathematics, it is a worthwhile problem. As Hiebert et al. (1996) have noted, a problem as simple as finding the difference in heights between two children, one 62 and the other 37 inches tall, can be a worthwhile problem if teachers use it appropriately for students' learning of multidigit addition. Teachers must decide what mathematical tasks to select or develop according to specific learning goals of a lesson. Textbooks can be a useful resource for selecting worthwhile mathematical tasks. In fact, teachers can develop worthwhile and interesting mathematical tasks by simply modifying problems from the textbooks.

7.4 Students and Problems: From Solving to Posing

Typically, when talking about instructional tasks, educators focus on problem-*solving* tasks, just as in Butts' (1980) analysis above. In that context, the role of teachers is to select and develop problem-solving tasks that are likely to foster students' development of understanding and mastery of procedures in a way that also promotes their development of abilities to solve problems and reason and communicate mathematically (NCTM, 1991). Brown and Walter (1983) took this a step further, examining instructional tasks in which students pose their own problems and then solve them. They described processes for posing new problems from existing situations or problems, including asking "what-if-not" questions that encourage the variation of the conditions and constraints of a mathematical problem or phenomenon (this process echoes the "looking back" phase of Polya's [1945] approach to problem solving). Moreover, Brown and Walter noted that the activity of asking questions in mathematics (that is, mathematical problem posing) may be helpful in addressing students' mathematics anxiety because, although some questions may be more productive than others, questions are not inherently "right" or "wrong." Indeed, the value of a posed problem is frequently not obvious without delving into it and thus thinking more deeply about the underlying mathematics (which is ultimately a positive outcome for an instructional task).

For the purposes of this chapter, when we consider the perspective of students as problem posers, we use the term "problem-posing tasks" to mean instructional tasks that engage students in generating new problems and questions based on given situations (including mathematical expressions or diagrams) or changing (i.e., reformulating) existing problems (Cai & Hwang, 2020; Kilpatrick, 1987; Silver, 1994). Problem-posing tasks put the students into the role of problem generator instead of teacher. Educationally, this switching of roles is theoretically sound based on both constructivist and sociocultural aspects of learning, and it can actually increase students' access to mathematical sense making and learning. When students have the opportunity to pose their own mathematical problems based on a situation, they must make sense of the constraints and parameters that can be mathematized (which also happens to be a mathematical modeling competency). They then extend from that sense-making activity to build connections between their existing understanding and the new context and its related mathematical ideas.

Although problem-posing activities are cognitively demanding tasks, they are adaptable to students' abilities and thus can increase students' access such that students with different levels of understanding can still participate and pose potentially productive problems based on their own sense making. Indeed, what is a challenging problem to one student may be easier for another student and impossible for yet another. In our view of problem posing, we consider challenge (conceived as mathematical difficulty) as somewhat orthogonal to the capacity to successfully pose problems. In fact, prior research has shown that students and teachers without problem-posing experience are quite capable of posing mathematically complex problems. In this context, challenge is not something that needs to be overcome but, instead, is a source of productive struggle. The more students can productively struggle while posing problems, the more they can learn. Because problem posing is an activity with a high ceiling and low floor (Cai et al., 2015), it offers access to all students to opportunities for productive struggle and mathematical sense making. The problems different students pose may reflect different levels of complexity and challenge, but each student still benefits from making sense of the problem situation and the mathematical concepts embedded therein.

More generally, we consider that in problem posing, mathematical challenge does not simply refer to mathematical difficulty. We see mathematical challenge in problem-posing tasks as referring to their capacity to challenge students to be more engaged with the mathematics by making it more accessible to them. In this way, problem-posing tasks increase the potential for students' learning. Certainly, problem-posing tasks can be cognitively demanding, but they also challenge students in this other way.

In addition, problem-posing tasks can foster students' positive mathematics identities by stimulating their creativity (Silver, 1997); sparking their interest and curiosity (NCTM, 1991); and positioning them as agents within the problem, that is, empowering them with agency as explorers of mathematics (NCTM, 2020). It provides a way for them to connect mathematics to their interests, something that is often not the case with routine problem solving, and allows them to personalize their responses. Students can connect to their different experiences and backgrounds and pose very different problems, all of which are related to the mathematical context (Cai & Leikin, 2020). They can make sense of and take ownership of the concepts from which they build their problems. Allowing not just teachers and textbooks but also students to pose the problems considered by the class creates shared mathematical authority and positions students as people who are capable of making sense of mathematics. When students then share their posed problems with their peers in the classroom and solve each other's problems, they expand their horizons and build a shared understanding (Silver, 1994).

7.5 Problem-Posing Tasks

Just as there are many types of problems and problem-solving tasks, there are several types of problem-posing tasks. In this section, we present a number of examples of such tasks, discussing the learning opportunities they offer. We begin by describing a categorization scheme for these tasks that is based on the nature of the problem situations and on the prompts used to initiate students' posing activity. With respect to prompts, there are many choices, some of which result in more open tasks and others of which result in more closed tasks.

We note that other researchers have proposed categorization schemes for problem posing (e.g., Baumanns & Rott, 2020; Stoyanova & Ellerton, 1996). For example, Stoyanova and Ellerton (1996) considered the degree to which a problem-posing task constrained the students' freedom to pose problems, establishing three categories: free problem posing, semi-structured problem posing, and structured problem posing. Free problem posing imposed almost no constraints, even including very little in the way of context on which to build a problem. Semi-structured problem posing provided more context but allowed students to pose relatively freely based on that context. Structured problem posing imposed the most conditions, such as providing a problem or an equation on which to base the posed problem. Baumanns and Rott (2020) built on this categorization by including consideration of whether a situation leads to problem posing or not and whether the initial given problem in a structured posing task can be considered routine or non-routine.

We do not intend to present a competing categorization of problem-posing tasks, but rather we have chosen to frame our categorization with respect to the considerations that have played a part in our own research on problem posing (e.g., Cai & Hwang, 2021). We first discuss problem-posing situations and then discuss the prompts that could be used in problem-posing tasks.

7.5.1 Problem Situations in Problem-Posing Tasks

In a problem-posing task, the problem situation is what provides the context and data that the students may draw on (in addition to their own life experiences and knowledge) to craft problems. Problem-posing tasks can begin from many different kinds of problem situations. The context for posing can involve words, pictures, graphs, patterns, tables, and mathematical expressions. We consider problem situations that are based on real-world referents and problem situations that are purely mathematical or abstract. Within each of these two categories, there are several subcategories of problem situations that differ in how the contextual information or data are presented. Figure 7.1 shows the various types of problem situations in our categorization. Note that, although we highlight real-life contexts and purely mathematical contexts as a way to characterize problem situations, we are not specifically focusing on the interplay between real life and pure mathematics. Rather, we aim to illustrate (and provide some systematization for) the diversity of contexts that can be used for problem-posing tasks. Fundamentally, problem posing in both purely mathematical contexts and real-life situations can provide mathematical challenges which non-problem-posing tasks would ordinarily not provide.

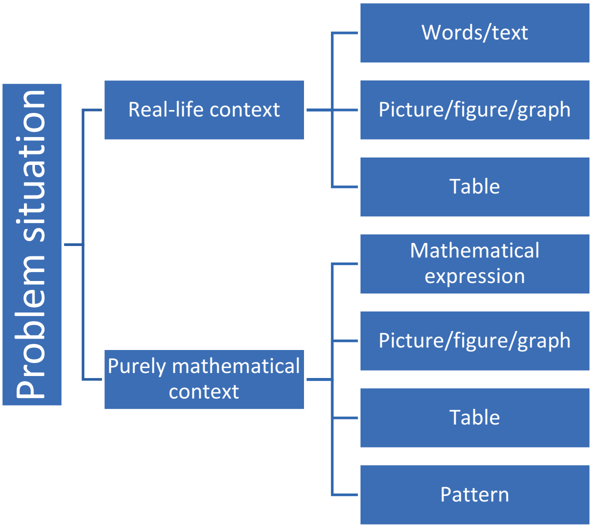


Fig. 7.1 Types of problem situations in problem-posing tasks

7.5.1.1 Real-Life Context Examples

Example 7.1 Text Based (Percent)

> *Some students submitted paintings for a competition. The total number of paintings submitted was 125. Six paintings received first prize, 6% of the total number of paintings received second prize, and the number of paintings receiving third prize was 40% more than the number of paintings receiving second prize. Pose percent-related problems and then solve them.*

This task uses mathematics with a real-life context as the problem situation (Cai & Xu, 2019). The context and data are communicated through text—a story about a painting competition. The teacher tells the students the total number of entries, the quantitative relationship of each award, and other information, and asks the students to ask and answer questions that can be solved by using percents. The student's activity is not to directly answer an existing mathematical problem but to ask questions based on this given information and then solve them. Each student may ask one or more math problems, and different students may ask different questions, leading to potentially useful comparisons across posed problems to highlight specific concepts related to percents.

Example 7.2 Text Based (Waiter)

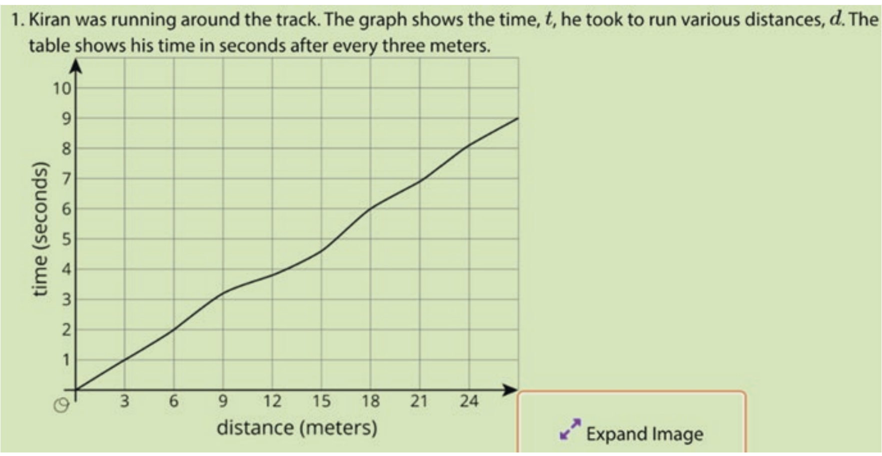
Some curriculum materials include the beginnings of support for student problem posing. For example, the problem-posing task below comes from Illustrative Mathematics (2019a), which includes in its design an instructional routine called Co-Craft Questions. When teachers use the Co-Craft Questions routine, they modify a problem-solving task in the lesson they are teaching into a problem-posing task by only showing the students the problem situation (withholding the rest of the problem) and then asking the students to write possible mathematical questions. (We will return to this strategy for turning problem-solving tasks into problem-posing tasks below.) Drawing on Zwiers et al. (2017), *Illustrative Mathematics* states that the purpose of this routine is “to allow students to get inside of a context before feeling pressure to produce answers” and “to create space for students to produce the language of mathematical questions themselves” (Illustrative Mathematics, n.d.).

During one waiter’s shift, he delivered 13 appetizers, 17 entrées, and 10 desserts. Before students begin work, display the waiter’s situation without revealing the questions. Ask students to write down possible mathematical questions that might be asked about the situation. Invite pairs to compare their questions, and then ask for a few to be shared in a whole-class discussion. Reveal the actual questions about the waiter’s situation that students will answer. This will help students make sense of the problem before attempting to solve it.

The problem situation in this task is again expressed through words. Note that the data from the waiter’s shift is simply expressed through a single sentence. Problem situations need not be overly complex. There is plenty of context and data in this sentence to allow students to generate many kinds of problems.

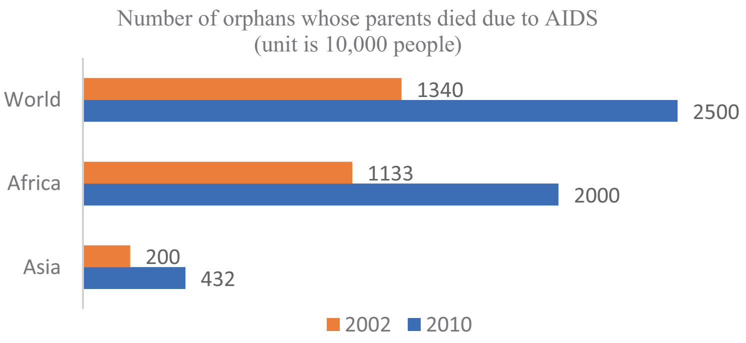
Example 7.3 Graph Based (Running Graph)
This example of a problem-posing task from the *Illustrative Mathematics* Grade 8 curriculum (Illustrative Mathematics, 2019b) uses a graph to present students with the context for posing problems about speed. The instructions for the teacher suggest selecting pairs of students to present their problems to the class in order to bring out the idea of relationships between distance and time.

*Display only the graph and context (*i.e.*, “Kiran was running around the track. The graph shows the time, t, he took to run various distances, d.”). Ask pairs of students to write possible questions that could be answered by the graph. Invite pairs to share their questions with the class. Look for questions that ask students to interpret quantities represented in the graph.*



Example 7.4 Graph Based (Bar Graph)

The graph below shows the number of orphans whose parents died due to AIDS in 2002 and 2010 in the entire world, in Africa, and in Asia. Analyze the data in the graph and pose mathematical problems based on the data that can be solved using linear equations.



In this task, the problem situation includes data presented in the form of a bar graph (Cai & Xu, 2019). In addition to providing students with the opportunity to makes sense of this method of displaying data, the instructions (that is, the prompt) for this task focus the students on posing problems that can be solved by using linear equations. There are many types of problems that could be posed based on these data; this is a key characteristic of many problem-posing tasks. However, in order to use the task effectively to achieve particular learning goals, it is sometimes useful to add constraints to the task. We discuss the role of prompts in greater detail below.

Example 7.5 Table Based (Animal Speed)

Animal	Crawling speed (km/h)
Snail	0.045
Tortoise	0.32
Spider	1.9

Based on the information in the table, about how many times as fast is a spider compared to a tortoise? Can you pose other mathematical problems and solve them?

As with the previous two examples, this task presents students with data related to a real-world phenomenon (Cai & Xu, 2019). This time, the data are presented in a table, which supports a natural tendency to make comparisons across entries in the table. The task includes a given problem that makes a multiplicative comparison. This may help students think of making multiplicative comparisons in their own posed problems rather than relying only on additive comparisons.

7.5.1.2 Purely Mathematical Context Examples

Example 7.6 Mathematical Expression Based (Writing Stories)
A common instructional task involves asking students to interpret a story and solve the so-called "story problem." Story problems are sometimes application problems like those Butts (1980) discussed, although they can also be much more routine and algorithmic. Notably, when teachers (or curriculum writers) generate story problems, they are already setting up the parallel between the mathematical situation and common the story context. A much less common task in mathematics classes is to ask students to make up their own stories. When students are asked to write stories to go with mathematical situations, they must take up the work of setting up the connection between the mathematics and their chosen context. To do so, they need to understand the meaning of the mathematical concepts or procedures as well as the features of the proposed context. For example, writing stories to go with number sentences can provide students with the opportunity to focus on the meaning of the operations and procedures involved. Consider, for example, these problem-posing tasks that ask students to write stories (Ma, 1999):

(1) *Write story problems to show the application of the following computation: 1 3/4 ÷ 1/2=?*
(2) *Make up a word problem that can be solved by using the following expressions:* $295 - 43 \times 4$ *and* $(74 - 52) + (67 - 23)$.
(3) *Write a story problem that can be answered by finding the value of n in the equation* $-4n = -24$.
(4) *Write a story problem that can be answered by finding the value of n in the equation* $x^2 + 2x = 20$.

Example 7.7 Mathematical Expression Based (Distributive Property over Addition)
Problem-posing tasks based on given mathematical expressions can also be used to help students understand other kinds of mathematical concepts. The following whole-class task uses problem posing to develop students' understanding of the distributive property of multiplication over addition (Chen & Cai, 2020):

Divide the class into 4 groups. Provide the expression "(5+7)×4" to the students in groups 1 and 2, and provide the expression "5×4+7×4" to the students in groups 3 and 4. (The teacher deliberately hides the two formulas to prevent the students in each group from knowing the other group's expression.) Students in groups 1, 2, 3, and 4 know each other's calculations, respectively. Ask students to pose mathematical problems based on daily life according to their assigned expression. In the follow-up discussion, let the students guess what expression the other group of students based their problems on.

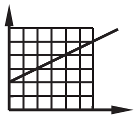
Example 7.8 Mathematical Expression Based (Pythagorean Theorem)

Please help John think of a mathematical problem that can be solved by the Pythagorean Theorem.

In this task, the students' posed problems must be solvable using the Pythagorean Theorem. Thus, the information in the problem should, in some way, satisfy that theorem. Similar to writing stories based on given expressions, this task begins with a given expression. However, in this case, the expression does not use particular numbers. It is a general relationship expressed symbolically. Still, in order to pose problems that satisfy the expression, students must make use of their understanding of the Pythagorean Theorem and potentially deepen that understanding as they work to pose an appropriate problem.

Example 7.9 Graph Based (Linear Function)

Use the graph below to answer the following questions.

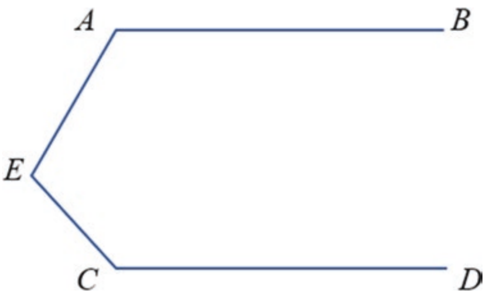


a. *Write an equation that will produce the above graph when x is greater than or equal to zero.*
b. *Write a real-life situation that could be represented by this graph. Be specific.*

This graph task (Cai et al., 2013) provides a purely mathematical context—a linear graph—but invites the student to provide a real-life situation that could be represented by the graph. In one sense, the task is asking the students to transform a purely mathematical context into a real-world context. As with Example 7.6, a goal of this type of task is to assess students' understanding of a mathematical concept by examining how the student creates a connection between the concept and their chosen real-life context. For example, one might check that the student has used their knowledge that a line that points up and to the right represents an increasing relationship by describing an increasing relationship in their word problem. In addition, students' understanding of the meaning of a positive y-intercept could be assessed by how they set up their story to reflect this feature of the graph.

Example 7.10 Figure Based (Parallel Line Geometry)
Consider the following geometry task based on a given figure:

In the figure below, AB//CD.



Show that the sum of the measures of <A, <E, and <C is 360°.

This is a very common, if closed, way of presenting this kind of geometry problem in regular textbooks (Cai et al., 2015). There are several ways to make the problem more open. Instead of prompting the student to show that the sum of the measures of the specified angles is 360°, we may instead ask, "*What is the sum of the measures of <A, <E, and <C?*" Although the second version of the prompt leaves the student in a more open position (in terms of given information), neither of these prompts results in a problem-posing task. In addition, we may ask students to make a generalization of the problem by asking "*What is the sum of the three angle measures with different locations of point E?*" This again opens the problem to a wider set of possibilities, but it remains ultimately a problem-solving task. If, instead, we change the prompt to, "*Please pose as many mathematical problems as you can with respect to the relationships in the figure,*" the task is now a problem-posing task, and it has become even more open, hence increasing the opportunities for students' learning. Some students may indeed generate problems similar to the ones presented above. However, others may explore the figure and pose completely different problems.

Example 7.11 Table Based (Pythagorean Theorem)
Consider the following task, adapted from Brown and Walter (1983), that, like Example 7.8, focuses on the Pythagorean theorem. This time, however, the problem situation proceeds from a table of data:

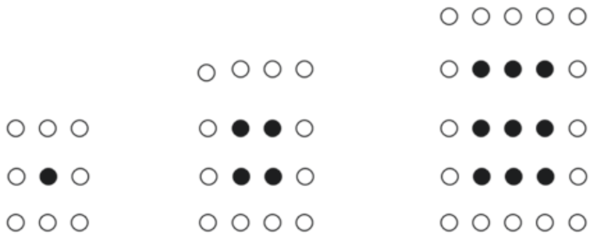
x	y	z
3	4	5
5	12	13
7	24	25
8	15	17
9	40	41
12	35	37

The table above shows several triples that satisfy the Pythagorean Theorem $x^2 + y^2 = z^2$. *Using this table, what questions can you ask?*

In this task, the focus is less on finding a context or problem that satisfies the Pythagorean Theorem and more on exploring patterns in the given data. Students may make any number of conjectures and pose several kinds of questions based on the numbers. For example, students might ask whether z is always odd. Or, they might ask whether y is always divisible by 4 or 5.

Example 7.12 Pattern Based (Black and White Dots)
The pattern-based problem-solving task below (Cai & Hwang, 2002) is interesting because there is no immediate pathway suggested by the task. It requires students to discover underlying mathematical structures.

Look at the figures below.

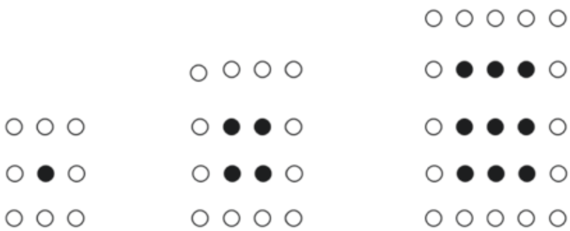


(Figure 1) (Figure 2) (Figure 3)

1. *Draw the 4th figure.*
2. *How many black dots are there in the 6th figure? Explain how you found your answer.*
3. *How many white dots are there in the 6th figure? Explain how you found your answer.*
4. *Figure 1 has 8 white dots. Fig. 3 has 16 white dots. If a figure has 44 white dots, which figure is this? Explain how you found your answer.*

However, we can examine students' thinking from a different perspective if we ask them to generate their own mathematical problems. Research shows that students are capable of generating interesting mathematical problems and that there is a direct link between students' problem-solving and problem-posing skills (Cai & Hwang, 2002; Silver & Cai, 1996). A more open, problem-posing version of this task might look like the following:

Mr. Miller drew the following figures in a pattern, as shown below.



(Figure 1) (Figure 2) (Figure 3)

For his student's homework, he wanted to make up three problems BASED ON THE ABOVE SITUATION: an easy problem, a moderate problem, and a difficult problem. These problems can be solved using the information in the situation. Help Mr. Miller make up problems and write these problems in the space below.

7.5.2 *Prompts in Problem-Posing Tasks*

In addition to a problem situation that provides context and data for students to use in their posed problems, a problem-posing task must include a prompt that lets students know what they are expected to do. Depending on the goal of the task, for the same problem-posing situation, there can be many different kinds of prompts. Some possible prompts include the following:

- Pose as many mathematical problems as possible
- Pose problems of different levels of difficulty (e.g., "Pose one easy problem, one moderately difficult problem, and one difficult problem.")
- Given a sample problem, pose similar problems (or problems that are structurally different)

The choice of prompt can influence both the mathematical focus for the students and the level of challenge that the posing task presents. Consider the problem-posing task in Example 7.4 above (the bar graph task). Certainly, this context could be used to address a variety of content, including interpreting graphs and data or understanding linear equations. However, problem-posing tasks are often constructed specifically to address the learning goals of a particular lesson. Example 7.4 came from a lesson related to linear equations. Thus, a constraint (solvable using linear equations) was included in the prompt in order to increase the likelihood of the task supporting the students' understanding of linear equations. In this case, the prompt constraint does increase the challenge of the task (e.g., by preventing simple arithmetic problems) relative to the lesson and its learning goals, but, generally speaking, the effect of the prompt is still an open area of research. Does the prompt always affect challenge this way? The answer probably depends on the lesson goals. For example, if the learning goals had to do with exponential functions, this prompt might actually decrease the challenge (in addition to making the task disconnected from the goal of the lesson).

Indeed, from a research perspective, it is not yet well understood what prompts are best to pair with a given problem situation or what prompts are most suited to achieving a desired degree of challenge or to address particular learning goals. That is, research has not yet illuminated the connections between different kinds of problem-posing prompts and different cognitive processes in problem posers. Research on problem-solving tasks has established that different prompts can elicit different cognitive processes and impact students' problem-solving performance (Goldin & McClintock, 1984). Thus, it is reasonable to expect that the prompt in a problem-posing task also shapes students' engagement with the task. A few studies have investigated how different prompts in problem-posing tasks impact students' or teachers' problem-posing performance and problem-posing processes (e.g., Silber & Cai, 2017). Silber and Cai (2017) compared preservice teachers' problem posing using structured prompts and free prompts, finding that the preservice teachers in the structured-posing condition more closely attended to the mathematical concepts in each task. Moreover, the effect of the prompt depends, in part, on the setup of the task. For example, in their review of problem posing in textbooks, Cai and Jiang (2017) identified four common types of problem-posing tasks: posing a problem that matches the given/specific kinds of arithmetic operations, posing variations on a question with the same mathematical relationship or structure, posing additional questions based on the given information and a sample question, and posing questions based on given information. A similar prompt (e.g., "Pose a mathematical problem.") could be used with many of these types of tasks, but its meaning to the student could be different for each type.

Based on practice, it does seem that encouraging students to pose different difficulty levels of problems has some advantages for eliciting deeper student thinking about some kinds of problems (Cai & Hwang, 2002) and for adjusting the level of challenge of the task relative to each student. For example, the prompt, "Create a problem that would be difficult for you to solve," can challenge each student to stretch toward the edge of their own ability. Although each student may still engage the problem-posing task at a level that is appropriate for their existing mathematical understanding, such a prompt could result in the overall level of challenge increasing. Ultimately, we believe that the choice of problem-posing prompt has the potential to make a difference in how students engage with problem-posing tasks. More research in this area is needed to explore the effects of problem-posing prompts, including identifying what kinds of prompts are most inviting or engaging to students, determining how providing example problems may shape students' posing activity (for better or worse with respect to the learning goals), and what the effects of other features of prompts (e.g., including conceptual cues, as in Yao et al., 2021) may be.

The following tasks provide examples of various ways that prompts can be used to engage students in problem posing.

Example 7.13 Number Pattern Task (Odd Numbers)

Look at the pattern of numbers in the arrangement below.

1
3 5
7 9 11
13 15 17 19
21 23 25 27 29
……

The pattern continues. I wanted to make up some problems that used this pattern for a group of high school students/college freshmen. Help me by writing as many problems as you can in the space below.

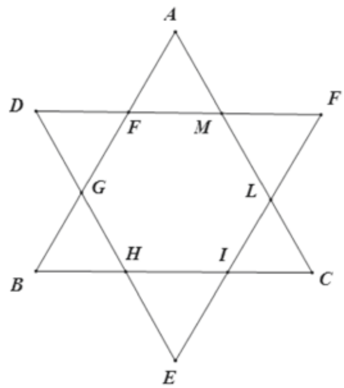
In this task (Cai, 2012), there are many possible observations that students might make about the pattern. By using a prompt that asks for "as many problems" as the student can generate, this task encourages students to explore the pattern in greater depth (i.e., beyond the first thing they notice). For example, students might ask mathematical questions like, “What is the sum of the numbers in the *n*th row?” or “What is the sum of the numbers in the first *n* rows?” or “What is the relationship between the first number in each row?”

Other prompts could also be used with this problem situation. For example, by using a prompt like, “Help me by writing one problem that would be easy for the students to solve and one problem that would be difficult for them to solve,” could help illuminate what features of the pattern the students consider to be more or less accessible. Moreover, a discussion of posed “easy” and “difficult” problems across students could be used to identify characteristics that make problems more or less challenging (e.g., number of steps required to solve, degree to which the problem generalizes beyond the given information).

One could also provide a sample problem (e.g., “What is the sum of the numbers in the 10th row?”) before asking students to pose additional problems. In that case, the choice of sample problem could encourage students to look in a particular direction when posing their own problems, perhaps guiding them toward a particular target concept or generalization.

Example 7.14 Plane Geometry Task (Conjectures)

Problem-posing tasks based on pictures and diagrams can enrich students’ experiences with plane geometry. Often, problem-solving tasks in geometry require students to prove statements that are either obvious from the diagram (although the proof may not be obvious) or statements that are obscure and for which little motivation is provided. By providing a geometric diagram without an obvious statement to prove, instead prompting students to make conjectures and explore them, problem-posing tasks of the type below can stimulate students’ interest, creativity, and initiative in learning.



In the diagram above, △ABC and △DEF are congruent triangles. Please use this diagram to make three conjectures and then prove or disprove them.

In response to this task, adapted from Brown and Walter (1983), a student could, for example, explore the positioning of the two overlapping congruent triangles, realizing that the intersection is only sometimes a regular hexagon. This might lead the student to wonder what conditions the positions of the triangles satisfy in order to make the hexagon GHILMF a regular hexagon (as it appears to be in the picture).

Example 7.15 Mirroring a Given Problem (Clothing Combination Task)

Anna has three shirts: one white, one red, and one yellow, and three skirts: one black, one green, and one blue. (1) How many different ways are there to pair one shirt with one skirt? (2) Pose a new mathematical problem that has the same mathematical structure as the given problem (a combination problem), and solve it.

As with Example 7.1 (the painting contest situation), this task provides a real-life context in the problem situation, and students are invited to ask mathematical questions. The difference here is that an example problem is provided for students to imitate. This type of prompt is what Cai and Jiang (2017) categorized as posing variations on a question with the same mathematical relationship or structure. The intention is for students to vary the context, data, or unknown quantities in the problem situation while keeping the most basic mathematical relationships and structures (in this case, a combinatorics problem) consistent with the example problem.

Example 7.16 Mirroring a Given Problem (Ages)

A similar type of problem-posing task asks students to take a given problem and explicitly change the numbers (without changing the rest of the problem situation) to produce analogous problems. For example,

The sum of the ages of Sana and her father is 45. Sana's age is 2/7 of her father's. How old is Sana? Please pose a similar question by varying the two numbers in the situation.

This task, adapted from Arikan and Unal (2013), relies on students recognizing certain practical, real-life constraints on how they may modify the problem. For example, Sana's age is assumed to be less than her father's age, and one might expect her father's age to be at least 18 years old. A student might also decide that, by convention, the age should be a whole number. Moreover, this task could be broadened to allow students to add conditions or steps. For example, a student might augment the given problem as follows: "The sum of the ages of Sana and her father is 54. Sana's age is one-fifth of her father's age. In 4 years, how old will Sana's father be?"

7.6 Teachers Learning to Teach Mathematics through Problem Posing

If problem-posing tasks have the potential to be a powerful class of instructional tasks that are both challenging to a wide range of students and effective at helping students learn mathematics deeply, a pressing question is how to integrate problem posing into day-to-day school mathematics instruction. Based on standards documents, problem posing is already recognized as an important activity for students. For example, in NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics*, problem posing was advocated for in the hopes that students would "have some experience recognizing and formulating their own problems, an activity that is at the heart of doing mathematics" (p. 138). Paralleling this recommendation for mathematics curriculum was an explicit call for teachers to create problem-posing learning opportunities: "Students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem" (NCTM, 1991, p. 95). Subsequent updates to NCTM's recommendations placed an even stronger emphasis on student thinking and problem posing (e.g., NCTM, 2000; 2020). Moreover, the *Common Core State Standards for Mathematics* promote mathematical modeling, a process in which formulating the problem is a critical step (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Given the ongoing call for problem posing in standards documents, it would seem logical that problem-posing tasks would be built into the curriculum materials that teachers use regularly. Unfortunately, this is not the case. For example, widely used curriculum materials in the United States fail to incorporate problem posing in a substantial and consistent way. Only a very small proportion of problem-posing activities are included in popular elementary and middle school mathematics textbooks (Cai & Jiang, 2017; Silver, 2013). In addition to being sparse, the problem-posing activities in textbooks can be very unevenly distributed across grade levels and content areas. Of the limited number of problem-posing activities, the vast majority are related to number and operations. Very few involve algebra, geometry, measurement, or data analysis (Cai & Jiang, 2017). The unrepresentative distribution of problem-posing tasks reflects a haphazard approach to incorporating problem posing in the intended curriculum.

7.6.1 Changing Beliefs and Increasing Knowledge About Teaching Through Problem Posing

A consequence of the lack of problem-posing tasks in curriculum materials is that teachers, who are at the heart of implementing changes in instruction, do not have consistent support to implement problem posing in their classrooms. Thus, there is a critical need to support teachers to integrate problem posing into their instruction despite the lack of curricular support. Fundamentally, this means supporting teachers to develop their knowledge and beliefs about teaching through problem posing.

Figure 7.2 shows a teacher professional learning model guiding a large research project on teachers' learning to teach through problem posing (Cai et al., 2020). Through teacher learning, teachers increase their knowledge and change their beliefs and then change their classroom instruction, aiming to improve students' learning. Indeed, both teachers' knowledge and teachers' beliefs are consistently

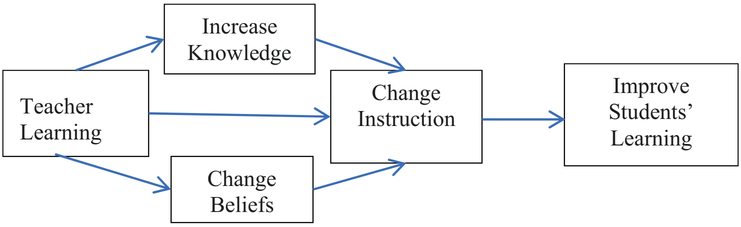


Fig. 7.2 A teacher professional learning model

important factors in teachers' professional learning and classroom instruction (Richardson, 1996; Thompson, 1992), influencing how they interact with curriculum (Handal & Herrington, 2003; Hill & Charalambous, 2012; Lloyd et al., 2017), their instructional practices (Stipek et al., 2001; Wilkins, 2008), and how their students learn (Hill & Chin, 2018; Peterson et al., 1989). Thus, attempts to change instruction by incorporating problem posing as a teaching method will necessarily need to attend to teachers' knowledge and beliefs about teaching through problem posing.

Through professional learning opportunities that offer extended experiences with problem posing, teachers are able to increase their knowledge about teaching with problem posing and develop positive beliefs about it. Indeed, Cai and Hwang (2021) have shown that participation in professional development workshops focused on problem posing results in teachers gaining expertise at posing problems themselves as well as knowledge about the kinds of problems their students might pose. Moreover, they exhibit increased confidence about teaching using problem posing, and they develop more specific and comprehensive beliefs about the advantages and challenges of teaching with problem posing.

Although there is the potential for teachers to change their instructional practice based on developing their knowledge and beliefs about teaching through problem posing, it remains important to consider the degree to which teachers (and the systems within which they operate) buy into the proposed change (Kramer et al., 2015). Without buy-in, instructional reforms cannot be sustained. Thus, to gain buy-in from teachers, it is critical to consider the impact of changing instruction on teachers' resources (e.g., time, energy, attention).

7.6.2 Equipping Teachers to Develop Problem-Posing Tasks

When considering how to integrate problem posing more effectively in mathematics classrooms, it is important to avoid the change being perceived as a burden or a radical change in practice that would require too much time to adapt to. Rather, problem posing may be more readily accepted if it builds on existing, common practices. Ideally, problem posing would be introduced through small, incremental changes that would be accessible to teachers and students but that offer the promise of rich returns in student learning (Cai & Hwang, 2021). One natural way to support teachers to work around the limited support for problem posing in published curricula would therefore be to support them in reshaping existing problem-solving tasks in simple ways that transform lessons to create learning opportunities with problem posing. Teachers are already active participants in translating the intended, written curriculum into the enacted curriculum, engaging with their curriculum materials in a design process (e.g., Cai & Hwang, 2021; Lloyd et al., 2017; Remillard, 2005; Stein et al., 2007). Thus, it makes sense to tap into this existing process to empower teachers to adapt their curriculum materials to introduce more substantial problem-posing tasks and opportunities. This allows teachers to work with familiar tasks that they modify in simple ways to serve a new purpose.

In that vein, we describe two strategies for integrating problem posing into the school mathematics curriculum. The first strategy is empowering teachers as curriculum redesigners to change problem-solving elements of their curriculum materials to create learning opportunities for mathematical problem posing. For example, teachers could add a follow-up problem-posing prompt such as "Can you pose a similar problem that could be solved?" or "Can you pose another problem using this information?" at the end of a problem-solving task. As with Polya's "looking back" step, this would encourage students to make use of the mathematical thinking they have already been engaging in to generalize a mathematical relationship or to find additional connections. Another simple modification would be to change the problem-solving prompt into a problem-posing prompt. For example, in a word problem, such as "Jenna, Eli, and Angela are driving home from a trip. Angela drove 150 miles less than Eli. Eli drove twelve times as far as Jenna drove. Jenna drove 50 miles. How many miles did they drive altogether?" teachers could replace the question "How many miles did they drive altogether?" with a request that students pose problems based on this situation. Research has shown that students are capable of posing linguistically and semantically complex problems based on such situations (Silver & Cai, 1996).

The second strategy is to encourage students to pose problems at different levels of complexity. As we noted above, prompts that ask students to pose different difficulty levels of problems may help elicit deeper student thinking. For example, Cai and Hwang (2002) used parallel pattern-based tasks to examine the problem solving and problem posing of U.S. and Chinese sixth graders, finding that the Chinese students' posed problems reflected their use of abstract problem-solving strategies and the kinds of critical thinking involved in solving pattern-based problems. To gain insight into students' mathematical thinking with problem posing requires more than a single posed problem. Once teachers and students have had some experience with posing mathematical problems in the classroom, asking students to pose more than one problem for a given problem-posing situation is again an incremental change in practice. In addition to providing useful data for the teacher to get a sense of the students' level of understanding of a mathematical situation (Cai & Hwang, 2002), when students generate a sequence of posed problems of varying difficulty, they are again prompted to engage with the mathematics more deeply. Consider the black and white dots task in Example 7.12 above. By asking students to provide three problems of varying difficulty for a problem situation, the teacher is potentially increasing the level of cognitive demand of the task, creating an opportunity for students to think beyond simple pattern recognition. Silver and Cai (1996) have found that when students are asked to pose a sequence of problems, their later problems tend to be more complex and mathematically sophisticated than their earlier ones. Thus, by making this incremental change to a one-shot problem-posing task, a teacher can obtain both a different level of access to student thinking and elicit different levels of cognitive demand from the students.

7.6.3 *Supporting Teachers to Develop Teaching Cases*

With respect to understanding how problem posing can be enacted in classrooms, there is a need for careful analyses of practice that can be shared with teachers, researchers, and professional developers to build a common basis and image of effective problem-posing instruction. One mechanism for documenting practice and disseminating it is the development of teaching cases (Zhang & Cai, 2021). Teaching cases serve as representations of detailed and careful analyses of teaching practice (Merseth, 2003; Stein et al., 2009). A teaching case includes major elements of a lesson and related analysis, but it is not a transcribed lesson. Teaching cases include narratives describing instructional tasks and related instructional moves for the tasks. Cases also include information about the underlying thinking of major instructional decisions as well as reflections on and discussions of those decisions. The development of teaching cases is based on real lessons and typical instructional events from the lessons. Research has documented the effectiveness of using case-based instruction for professional learning. This approach is effective because it situates instruction in meaningful contexts in order to learn to teach (e.g., Hillen & Hughes, 2008; Smith et al., 2014).

Often, teaching cases are created by researchers to support teacher learning (e.g., Merseth, 2003, 2016; Smith et al., 2014; Stein et al., 2009). However, although the effectiveness of using the case-based approach for teacher learning has been demonstrated (e.g., Smith et al., 2014), we anticipate that engaging teachers in creating and writing teacher cases would also be effective (if not more so) for teacher learning. Moreover, we have argued elsewhere that teaching cases could potentially serve as physical artifacts for storing and improving professional knowledge for teaching (Cai et al., in press). More work is needed to accumulate teaching cases in problem posing. With more successfully implemented teaching cases using problem posing as a resource, teachers could learn from the cases to teach using problem posing even though they lack problem-posing tasks in their curriculum materials. In addition, after accumulating more teaching cases, it will be possible to explore multiple discourse patterns for handling students' posed problems as well as to identify the most effective discourse patterns in teaching mathematics through problem posing.

7.7 Summary and Looking to the Future

Teaching through problem posing, like teaching through problem solving, holds great potential for students' learning not only because of the mathematically challenging nature of problem-posing tasks but also because problem posing is a low-floor-and-high-ceiling activity that allows for students at all levels to be challenged through engaging in problem-posing tasks. In this chapter, we have defined problem-posing tasks and examined a variety of examples that vary in the type of problem situation and the type of prompt. Note, however, that we have not attempted to establish an overall ranking of the level of challenge of problem-posing tasks. This is primarily because the appropriateness of a problem-posing task needs to be determined with respect to the learning goals of a particular lesson. If the challenge of a problem-posing task is adaptable to the individual learner, the level of understanding that the learner brings to the task is key to the task's challenge for that learner. Moreover, as we indicated above, it is an open question how to choose and use different prompts with the same situations and how such choices can create different learning opportunities for students. Thus, the ranking of challenge is a focus for problem posing only insofar as it is oriented toward understanding how problem posing can create effective learning opportunities.

Even though there is empirical evidence of using problem posing to effectively assess students' mathematical thinking and learning as well as evidence of the positive effect of teaching mathematics through problem posing on students' learning (Cai & Hwang, 2021; Cai et al., 2015), there are a number of areas which need further research. To conclude this chapter, we point out three areas of future research with respect to problem-posing tasks, teaching through problem posing, and teachers learning to teach through problem posing. We do not claim that these are the only areas of research with respect to the mathematically challenging nature of problem-posing tasks, but we do believe that these three areas of research would help the field capitalize on the mathematically challenging nature of problem-posing tasks to maximize students' learning.

7.7.1 Problem-Posing Tasks

With respect to problem-posing tasks, we have focused on problem situations and prompts. Every problem-posing task includes these two features, but it is an open question how the different situations and prompts influence students' problem posing and their overall mathematical thinking. Given the diversity of problem-posing tasks, extensive and detailed study is needed to increase the field's understanding of how best to design a problem-posing task to help students meet a particular learning goal. With respect to mathematical challenge, we can, for example, ask what kinds of prompts increase (or decrease) the level of challenge for students. What kinds of prompts tend to guide students toward mathematically sound problems that are challenging to themselves?

Fundamentally, the cognitive and affective processes involved in problem posing are still being mapped out. The implications for how problem-posing tasks may be designed to present particular levels of challenge are therefore not yet known. However, preliminary research findings have pointed to directions for further work. For example, sample problems in a problem-posing task can facilitate students' problem posing and help them overcome some challenges in the problem-posing process. Also, when students better understand the problem situation, they are likely to be more successful in posing problems. The link between problem-solving thinking and problem-posing thinking may suggest that one may help the student with challenges they experience in the other (bidirectionally) because these can develop in tandem.

Currently, there is preliminary empirical evidence suggesting that asking students to pose problems at various difficulty levels may deepen their engagement with and exploration of the mathematical concepts in a problem-posing task. Much more detailed research is needed to secure the bases of our understanding of the impact of problem-posing task characteristics. This is true for both cognitive impacts and impacts on non-cognitive aspects of students' learning such as affect, engagement, and creativity (Cai & Leikin, 2020).

7.7.2 Teaching Through Problem Posing

With respect to teaching through problem posing, it is important to recognize that problem-posing tasks do not exist in isolation. Even if they are included in curriculum materials, the implementation of problem-posing tasks in the classroom involves many other considerations. Teachers are responsible for choosing worthwhile and challenging instructional tasks to foster their students' learning. In the realm of problem-posing tasks, the field is still just beginning to conceptualize how to choose the most appropriate problem-posing tasks for a given set of learning goals in a lesson.

Moreover, because problem posing is not a frequent practice in mathematics classrooms at present, we do not yet have a robust understanding of effective classroom routines for using problem-posing tasks to challenge students and teach mathematics. Critical issues to be addressed in this area include how teachers can most effectively handle and make use of students' posed problems to further the learning goals. What makes one student's posed problem the right choice to discuss with the class? How would a teacher make this choice? Also, what patterns of classroom discourse and activity (e.g., social and sociomathematical norms) might need to be established in order for teachers to be able to use problem-posing tasks effectively to engage all students in the class?

7.7.3 Supporting Teachers to Learn to Teach Through Problem Posing

Finally, there is much work yet to do with respect to supporting teachers to learn to teach through problem posing. Although teachers often have experience posing problems for their students, many teachers have little experience with students posing their own problems as a mechanism for teaching mathematics. Teaching through problem posing is therefore a challenging task for teachers. As we have discussed above, there are many elements of teachers' knowledge and beliefs that may be relevant to helping them overcome the challenge of teaching with problem posing: their own conceptual understanding, their own experience with problem posing as posers themselves, their knowledge of their students' problem posing, their beliefs about problem posing and teaching with problem posing (e.g., beliefs about the advantages and challenges), and their buy-in to teaching with problem posing (and the persistence to work on their teaching with problem posing that the buy-in supports). Because current curriculum materials do not substantively include problem-posing opportunities, other kinds of efforts are needed to support teachers to learn to teach through problem posing.

For example, teachers need support to develop productive beliefs about teaching through problem posing, including beliefs about the advantages and challenges to expect. There is early-stage evidence to show that teachers, through professional development, are able to both increase their own problem-posing performance and develop more positive and detailed beliefs about problem posing and teaching mathematics through problem posing (Cai et al., 2020; Cai & Hwang, 2021). This helps support them in their efforts to incorporate problem posing into their practice.

Clearly, more systematic research is needed to explore multiple prongs for supporting teachers to learn to teach mathematics through problem posing, whether that is during preservice teacher preparation (e.g., Crespo, 2003, 2015), through focused in-service professional development experiences (Cai et al., 2020), or through ongoing collaborative work between teachers and researchers. We believe one promising avenue that merits further exploration is engaging teachers with researchers to develop teaching cases for problem posing. Such teaching cases could both serve as a form of professional development for the teachers engaged in creating the cases and act as a type of artifact, sharable with other teachers, for accumulating and storing professional knowledge about specific implementations of problem posing to challenge students and achieve desired mathematical learning goals. As the field moves forward with teaching mathematics through problem posing, we will need longitudinal studies that track the entire process: how teachers learn to teach through problem posing, how problem posing is enacted in their classrooms, and how well problem posing helps students achieve challenging learning goals. Initial studies of this type are currently underway in China (Cai & Hwang, 2021) and as part of a newly launched design and development project in the United States.

7.7.4 Problem Posing and Mathematical Challenge

Problem posing as an instructional activity holds great promise for improving students' learning opportunities by creating situations in which students may productively struggle with challenging mathematics. However, the connection between problem posing and challenge is somewhat complicated. On the one hand, as we noted above, one can consider problem posing (as a cognitive and an instructional activity) as somewhat orthogonal to challenge. This is partly because of the inherent adaptability of problem-posing tasks to students' various ability levels and existing understanding. That is, students who have a greater understanding of the problem situation and associated mathematics may pose both simple problems (from the perspective of a knowledgeable observer) and quite challenging problems (in that the problem is itself challenging for the solver/student and that the act of posing the problem is challenging for the poser). Yet, students who have less robust understanding may also be challenged by the same problem-posing task. Although the problems they pose may be less challenging (again, from the perspective of a knowledgeable observer), they may yet experience a high degree of mathematical challenge relative to their level of understanding.

On the other hand, there are aspects of problem posing that are more intrinsically related to mathematical challenge. For teachers, as we discussed above, there are quite a few potential challenges related to teaching mathematics through problem posing. Some of these challenges are external, such as the lack of curricular support. Others are internal, including understanding how to use problem posing to create learning opportunities and how to make use of students' posed problems to help achieve the learning goals for a lesson. Overcoming these kinds of challenges requires the development of teachers' knowledge and beliefs. For students, the degree of challenge they encounter when posing problems may be related to aspects of our framework such as the nature of the prompts used in problem-posing tasks. For example, as we noted above in the case of constraints, it is possible that some kinds of constraints in the problem-posing prompt may increase (or decrease) the challenge of the activity for students. However, this is always relative to the students or to the learning goals of the lesson. Ultimately, there is much need for research that illuminates how the design of problem-posing tasks may best engage students in productive struggle with challenging mathematics so as to maximize their learning opportunities.

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Chapter 8
Challenging Students to Develop Mathematical Reasoning

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8.1 Introduction

Mathematics learning includes learning basic facts, terminology, concepts, representations, and procedures for solving routine tasks. However, it also includes learning core reasoning processes, such as formulating mathematical conjectures and generalizations and providing justifications. In an exploratory approach, students' development of mathematical reasoning is supported by the work on tasks in which, besides using prior knowledge, they develop new ideas, concepts, and representations and are prompted to establish new conjectures and generalizations and to justify them (Ponte, 2005). Tasks need to be mathematically fruitful and allow for students' involvement. The students need to have the opportunity to work in interaction with their colleagues and with the teacher's support. In addition, different students' solutions may be presented and discussed in a whole-class setting, so that all students in the class may appropriate the main ideas. In this approach, a task represents a challenge – a new idea that the student did not master yet, but that may emerge from the work to be carried out. Such challenge needs to be well established, considering the students' prior knowledge, dispositions, and diversity (NCTM, 2014). In this chapter, we present situations of such work in classes at grades 8 and 9 and discuss the features of the tasks and of the learning environment that may support students' development of mathematics reasoning. In particular, we aim to know what actions the teachers may use during the moments of launching of the work, students' autonomous work in pairs or small groups, and whole-class discussion. We pay special attention to how guiding actions combine with informing/suggestion actions as well as to the role of challenging actions in which the

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teacher prompts the students to generalize, extend their thinking, consider new possibilities, and justify their statements.

8.2 Exploratory Approach and Teacher Actions

The exploratory approach aims for students to play a decisive role in their learning. Thus, unlike the usual lesson that has as its starting point the teacher's explanations, in this lesson the starting point is a task in which students are called to work. This task should present some challenge for students, requiring them to formulate a solution strategy, which may involve the application of already known mathematical ideas and also the formulation of new ideas that may include representations, concepts, procedures, or mathematical properties. As indicated by Ponte and Quaresma (2016),

> Challenges may involve the establishment of connections between mathematical concepts and properties or between the context and the conditions of the problem or may be related to the construction, selection or coordination of representations. Challenges may also be related to promoting the reasoning processes of generalizing and justifying (p. 65).

Thus, a challenging task requires the formulation of a new idea, which is always the result of connections among previous ideas. If students do not hold these previous ideas in an actionable way, the task is much more difficult, if not impossible, for them. Therefore, the identification of previous knowledge to solve a task is an essential element for the work of the teacher.

This approach arises in several national contexts with different names, often with some nuances of meaning, but which basically represent close approaches, such as inquiry-based mathematics education (Artigue & Blomhøj, 2013), guided rediscovery (Gravemeijer, 2005), and reform mathematics education (Confrey, 2017).

The exploratory lesson usually unfolds in three phases (Ponte, 2005; Stein et al., 2008): task launching, students' autonomous work, and collective discussion. The first phase is the task launching, in which the teacher presents the task to the students and seeks to create the necessary conditions for students to actively engage in working on it (Jackson et al., 2013; Jackson et al., 2012). For this, the meanings of the relevant mathematical terms and others that students may be unaware of should be discussed, as well as any aspects of the context of the task that may constitute an obstacle to student involvement. This discussion of terms that may create problems for the students takes place through a process of negotiation of meanings (Bishop & Goffree, 1986), where the teacher seeks that students, from their previous ideas, develop a meaning for mathematically relevant terms coincident with the conventionally accepted meaning in mathematics.

The second phase of the lesson is the students' autonomous work. In this phase, the students work on the proposed task in pairs, in small groups, or even individually (Ponte, 2005; Stein et al., 2008). The aim is to provide students an opportunity for as much independent work as possible. The teacher moves around the classroom, seeking to observe the students' work and support them to progress if necessary. To do this, the teacher pays special attention to the strategies that students are using and to their possible difficulties. However, the teacher should take particular care not to give the students suggestions that represent a decrease in the cognitive level of the task (Stein & Smith, 1998), because the educational value of the work depends on the students' active involvement in their realization. For students who can finish early the work on the task, the teacher can propose extensions, involving analyzing other cases or exploring possible generalizations and other issues related to the initial task (Sullivan et al., 2006).

The third phase of the lesson is the collective discussion, in which the teacher puts to the discussion the solutions of some students, carefully selected (Stein et al., 2008). This discussion should allow students not only to understand the correct solution or solutions of the task but also to develop new mathematical ideas and become aware of possible mistakes to avoid. In the absence of the possibility of presenting the work done by all students, the teacher must select the solutions to consider and sequence them appropriately, usually starting with solutions that have errors and limitations, then moving to mathematically more correct solutions. Students often express correct mathematical ideas in an imperfect language, which can lead the teacher to revoicing actions, expressing the same idea in a mathematically more correct way. This phase creates many dilemmas for the teacher, in striving to honor the work of the students and to provide the opportunity to develop their own mathematical ideas (Sherin, 2002). At this stage, it is particularly important that opportunities for argument are created among students, leading them to disagree with each other and to justify why a given statement is mathematically valid or invalid (Wood, 1999). This phase of the lesson usually ends with a synthesis, highlighting the main ideas underlying the concepts involved in the task and the strategies used in solving it.

The rationale for these three phases of the lesson stands on the idea that, instead of beginning with an explanation from the teacher, illustrated by a few examples, followed by students' practice on similar examples, the lesson should begin with the proposal of a task requiring students' thinking, followed by students' autonomous work on the task, and ended with a discussion of the possible strategies to solve the task and with a summary of the main ideas provided by all this work. This approach promotes a much stronger student involvement in the mathematical work, with important benefits for student learning of specific concepts and representations as well as of transversal processes such as mathematics reasoning.

In all phases of the lesson, the teacher's actions regarding the students can be classified into three major groups (Ponte et al., 2013): (i) guiding; (ii) informing/suggesting; and (iii) challenging. In guiding actions, the teacher seeks to get the students to explain their reasoning or give them a prompt to continue their work. These actions are mostly carried out through questions or suggestions to pursue what the students are doing. In informing/suggesting actions, the teacher validates a statement of a student, introduces new information, or gives him/her suggestions on what he/she can do. Both guiding and informing/suggesting actions may raise problems regarding the maintenance of the cognitive level of the task, but are often necessary, at the risk that students are unable to continue working because they do not know indispensable information or run the serious risk of being discouraged, quitting the work. Finally, challenging actions, the teacher poses new challenges to the students, encouraging them to go further in their reasoning or to foresee new possibilities of work. These actions can be identified in any lesson where there is interaction between teacher and students. This classification of teacher general actions has some affinity with other frameworks, such as those proposed by Fraivillig et al. (1999), Cengiz et al. (2011), and Brodie (2010), but it is intended for a very wide spectrum of different lessons.

8.3 Mathematics Reasoning

Students' learning in mathematics, in addition to concepts and procedures, also includes the development of transversal capabilities, which NCTM designates as process standards. Among these capacities stands out mathematical reasoning, which includes generalizing and justifying as central processes (Jeannotte & Kieran, 2017; Lannin et al., 2011; Mata-Pereira & Ponte, 2017). Formulating a generalization is formulating a mathematical statement applicable to an entire class of objects, a statement that can be true or false. A conjecture is a statement that is supposed to be true, being a generalization in the case when it applies to a class of objects. Formulating a justification is providing a reason for the validity of a statement. A justification based on just a few examples provides some credibility to a statement but is not enough to be mathematically valid. For this to happen, the justification must include all objects to which the claim applies, which can be done through the logical structure, a generic example, or the use of already known properties and claims (Mata-Pereira & Ponte, 2017).

To promote students' reasoning, it is important that the proposed tasks assume certain characteristics, namely, allowing several solution strategies and requesting conjectures, generalizations, and justifications (Table 8.1). In the case of exploratory lessons, where reasoning processes are intended to occupy a central place, specific actions by the teacher can be identified for the various phases of the lesson.

8.4 Methodology

This chapter stands on the work of Project REASON – Mathematical Reasoning and Teacher Education. The aim of this project is to study the mathematical and didactical knowledge teachers need to carry out a practice that promotes students' mathematical reasoning and study ways to foster its development in prospective and practicing teachers. As part of the work of the project, a teacher education course was offered with 8 sessions (2.5 hours each). The main focus of attention in the course was how to create opportunities for students to get involved in reasoning

Table 8.1 Characteristics of tasks and teacher actions to promote students' mathematical reasoning

Characteristics of tasks to select	Teacher actions during the work on the task		
	During the launching	During the autonomous work	During the whole-class discussion
To be diversified and have different degrees of challenge, including exploration questions and problems. To enable a variety of solution strategies. (R) To encourage the formulation of conjectures and generalizations; (R) To request justification for solutions or strategies	To ensure that all students understand the mathematical terms of the statement. To ensure that all students understand the context. To develop a common language to describe the essential aspects of the task. (R) To highlight reasoning processes that may be involved in the task, such as conjecturing, generalizing, and justifying To promote the involvement of students in working on the task without lowering its degree of challenge.	To monitor the solving of the task by giving only small indications, without significantly lowering its degree of challenge. To try to understand students' strategies and check their progress. (R) for students with difficulties in formulating or implementing a solving strategy, generalization or justification, to ask facilitating questions or give suggestions that help them reach a strategy, generalization, or justification on their own For students who quickly solve the task, propose extensions, involving the exploration of new questions, possible conjectures and generalizations, or the formulation of alternative justifications.	To encourage sharing ideas. To explore disagreements between students, leading them to argue their positions. To accept and value incorrect or partial contributions, promoting a discussion that deconstructs, complements, or clarifies them. (R) To request the explanation of the "why," the presentation of justifications for responses or solution strategies and the formulation of alternative justifications (R) To ask students to identify valid and invalid justifications, highlighting what validates them (R) To emphasize mathematical reasoning processes (R) To promote reflection on the reasoning processes used (R) To propose proofs whenever relevant and appropriate to the students' knowledge (R) To challenge students to formulate new questions and establish new conjectures and generalizations.

Adapted from Ponte et al. (2020)
Entries signed with (R) are actions particularly oriented to mathematics reasoning

processes, especially generalizing and justifying. Besides opportunities for discussion and working on mathematical tasks and studying samples of student work, this course included two moments of “Taking into practice,” in which teachers were encouraged to consider with their students the ideas that were discussed in the sessions. To prepare these lessons the teachers worked in groups, but the lessons were conducted individually with their own classes. These experiences in the classroom were afterward presented and discussed in the course sessions.

In this chapter, we present what we consider interesting situations from two lessons, one carried out at grade 8 and another at grade 9 that refer to the first “Taking into practice” moment. These lessons were chosen from the small number of lessons concerning this moment that were videotaped. The tasks were selected by the teachers themselves, as examples of tasks that could create situations in which students could depict and develop their mathematical reasoning. The teachers both have less than 5 years of experience and are graduates of the Universidade de Lisboa. The grade 8 lesson was carried out in a public school in the Lisbon suburban area and the grade 9 lesson in a religious private school in Lisbon. The lessons were transcribed. Data analysis was carried out following the categories of mathematical reasoning processes (generalizing, justifying) and teacher general actions in the classroom (informing/suggesting, guiding, challenging) or specific actions (indicated in Table 8.1). In both lessons, we choose the episodes that best describe situations in which students formulate reasoning processes and then classify teachers’ actions according to our framework. In this chapter, we present some situations from these two lessons that illustrate important ideas at different phases of the development of the lessons.

8.4.1 Grade 8 Lesson: Edges of Pyramids and Prisms

8.4.1.1 The Task

1. Do all pyramids have an even number of edges? Justify your answer.
2. And regarding prisms, will they have an odd or even number of edges? Justify your answer.
3. Can you find another property regarding the total number of edges of a prism? What is that property?

This task, proposed to a grade 8 class, is clearly oriented toward the realization of generalizations and justifications. Question 1 requires a justification on the number of edges of pyramids. Question 2 begins by asking for a generalization about the number of edges of a prism and then asks the student to justify the answer. Question 3 calls for an additional generalization of the number of edges of a prism. The first two questions are oriented, specifying the generalizations or justifications to be made, while the third question is open, asking for the formulation of a property, without specifying which one. The students are seated at double tables and, as it is customary in mathematics classes, they work in pairs.

8.4.1.2 Launching and Autonomous Work

In launching this task, the teacher considers it important to *ensure that the students know the mathematical terms of the statement*. Thus, he begins by asking a student to read the statement of question 1, which leads to a small discussion about the meaning of the term "edge":

Teacher:	OK, this is the first challenge you're going to have to think about. Is everyone comfortable with what edges are?
Jaime:	No...
Teacher:	Ana, what is an edge?
Ana:	It is this from the pyramid.
Teacher:	"It is this." Can anyone define what an edge is? Other than that, "it is this." Diogo.
Diogo:	That part of the sides.
Several students:	That's the sides.
Teacher:	Is the sides of the pyramid? What is it? The faces?
Irina:	The lines that determine the sides.
Teacher:	The lines that determine the sides…
Bernardo:	The segments.
Teacher:	The segments, we already approaching a more correct mathematical language. They are the segments that join any vertices of the pyramid. So, when I join one vertex to the other, that line is called edge and it is a straight segment... So, think if the number of edges of a pyramid is always even.

There are students who say they do not know what edges are. Others show that they can identify edges, but have difficulty verbally defining the concept. The definition is constructed in stages, first from the idea of "it is this" and "that part" to "line" and then to "segment," ideas indicated by the students that the teacher formalizes by saying that edges are the segments that unite the vertices of a pyramid.

Then, the teacher asks Manuela to read question 2, which raises the need to know how to distinguish between pyramids and prisms:

Teacher:	Do you notice the difference between a prism and a pyramid?
Several students:	Yes.
Teacher:	What's the difference?
	(Several students start speaking)
Teacher:	Let me hear Berta.
Berta:	The pyramid has a vertex, and the prism has two faces.
Teacher:	Two faces that are...
Several students:	Bases.

Several students: Equal.
Teacher: Equal, isn't it? Once you have analyzed what's going on with the pyramids, you're going to have to look at what's going on with the prisms.

It is then established that a pyramid has a base and a vertex opposite that base and that a prism has two equal bases, the case that should be analyzed in question 2. In the last intervention, the teacher restates the challenge provided by this question.

The teacher moves to question 3, seeking to lead the students to understand the meaning of the term "property," essential for solving the task:

Teacher: OK? So, first we ask if [the number of edges] is even, at the edges of the pyramid, then whether it is even or odd in the prism. Therefore, that is a property, if the edges are even or are odd [the number of edges is even or odd], this is a property of the number of edges. And, in question 3, you are invited to find out if there is another property of the number of edges that you can identify. So, you can start with question 1.

Thus, during the launching of the task, the teacher seeks to promote the negotiation of meanings of the concepts necessary for the understanding of the task, both those that students should know from previous years (edge, pyramid, prism) and the more abstract concept of property. In addition, at various times, the teacher highlights what is asked in the statement of each question of the task. Simultaneously with this negotiation of meanings and with attention to the statement of the questions, the teacher seeks *to promote the involvement of students in the task, taking care not to suggest solution strategies that lead to lower the degree of challenge*. In the discussion of questions 1 and 2, the teacher does guiding actions, asking questions that progressively lead students to approach a mathematically acceptable formulation of mathematical concepts. In question 3, with the meaning of the term "property," possibly because it is a more abstract concept and not to prolong for too long the launching of the task, the teacher opts instead for directly informing students.

During the students' autonomous work, the teacher circulates around the room, observing the students' work and interacting with them. His interventions have different objectives, depending on what he observes. For example, for students who have trouble in formulating a solution strategy, the *teacher gives suggestions that help them reach a generalization for themselves*:

Teacher: Give examples, give examples to see what happens.

At another moment, in a dialogue with a student regarding question 2, the teacher helps this student *formulate his generalization more clearly, while recalling the need to justify it*:

Duarte: It is the triple.
Teacher: What is the triple?
Duarte: Of the edges…
Teacher: Of the number…
Duarte: Of the number of edges of the base.

Teacher: Think about it and try to put that phrase there. The idea is already there.
Duarte: With the formula?
Teacher: You can write the formula too, but first you must justify it.

In question 3, the teacher seeks to help the students formulate another generalization, in this case that can be understood as a property, once again seeking not to lower the degree of challenge of the task:

Teacher: So, what's going on from each other? [The student says something]. So, you got a property.

For the students who quickly solve the task, the teacher proposes extensions. Thus, speaking with a pair of students who had already reached a generalization, he *formulated a new challenge suggesting the students to formulate this generalization in a more formal language*:

Teacher: Do we manage to get here an expression . . . An algebraic expression?

Thus, during the students´ autonomous work, the teacher's actions alternate between guiding, when he asks questions that lead students to clarify their statements ("What is triple?",…) and informing/suggesting, when he points out paths that students can follow ("You can write the formula too, but first [you] must justify",…). In the case of students who are able to answer the questions proposed in the task, the teacher formulates new challenges. As in the launching of the task, also in this phase of the work, the teacher seeks not to suggest solution strategies that could lead to decrease the degree of challenge of the task.

8.4.1.3 Whole-Class Discussion

The teacher begins the whole-class discussion of question 1 by *encouraging the students to share their ideas*. He requests the participation of a student, Marta, whose answer is represented on the board (Fig. 8.1). The student's justification is based on the analysis of two particular cases:

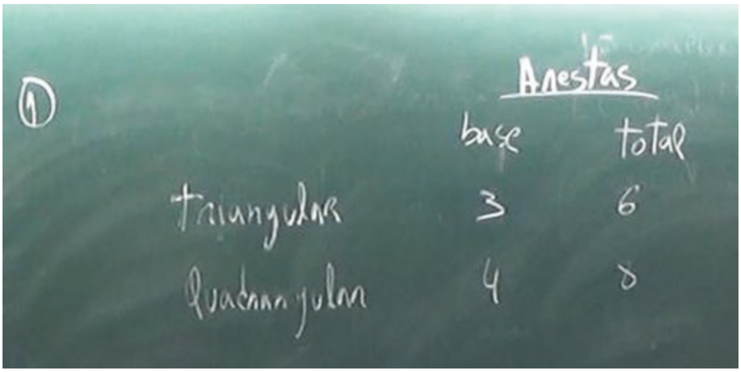

Fig. 8.1 Solution of Marta represented at the board ("arestas" = edges)

Teacher: [Let us] start with Marta. So, first, read the question, so we're all talking about the same thing.
Marta: [Reads the question] Yes, it's correct, because all the edges added up give an even number. Even so, the triangular pyramid has a base with three [edges], odd, the total number of edges is always even.
Teacher: You wrote something else.
Marta: So, in the triangular pyramid the number of edges is six. At the base, the number of edges is three, so, [the total number of edges] is always double.
Teacher: And you concluded that through an example?
Marta: No, two.
Teacher: What was the other example?
Marta: From the quadrangular pyramid.
Teacher: How many edges on the base?
Marta: Four.
Teacher: And how many in total?
Marta: Eight.

In this dialogue, the teacher seeks to get Marta to explain her reasoning. The generalization presented by the student is correct. She uses two examples to justify this generalization, which is mathematically invalid, but the teacher, at this moment, decides to accept and value her contribution.

Next, the teacher *promotes a reflection on the validity of this justification*. He asks students to identify valid and invalid mathematical justifications, highlighting what validates them:

Teacher: So, we're in mathematics, aren't we? And Marta is saying, I have an example here that works, I have another example here that works, so, yes, it's true. In mathematics two examples are enough to prove that something is true?
Several students: No.
Teacher: It could be two things that work, three, four, five, a thousand . . . But [that is not enough for us].

The argument that justifies this answer for all pyramids is the possibility of associating, in a biunivocal way, to each edge of the base, a side edge. The teacher does not introduce this discussion, possibly taking into account the age level of the students. Instead, he promotes the intervention of another student, Berta, who, for the general case of a pyramid with *a* edges at the base, indicates that the total number of edges is "*a* times two."

For question 2, which refers to prisms, the teacher *keeps encouraging the sharing of ideas.* He begins by asking a student to read the question and then her answer:

Rita: It can be even or odd. If the [number of] base edges is even, it's even. If the number of base [edges] is odd, it's odd. Thus, it depends on the number of edges of the base.

The student presents a correct generalization. In order to obtain a justification, the teacher *asks for the explanation of the "why":*

Teacher: And how did you get to that conclusion?
Rita: Doing edges times three.
Teacher: And why times three?
Rita: Because we have to know, we have to add [the edges of] the base, plus the side edges, plus [the edges of] the other base.

Next, the teacher challenges the students to formulate the generalization in a more formal language:

Teacher: We can say here, base 1 and base 2. At base 1 we have… Let us also generalize, a letter.
Bernardo: *a*.
Teacher: *a*. And at the base 2?
Students: *a*.
Teacher: *a*.
Teacher: And at the sides?
Students: *a*.
Teacher: So, how many [edges] do I have in total?
Students: 3*a*.

In question 3, the teacher begins the discussion by *encouraging students' sharing of ideas*. Two students present their answers, indicating generalizations as possible properties:

Teacher: And, in the meantime, here's already the third property on the board. What's the property?... What did you find out, Eduardo?
Eduardo: The triple.
Teacher: The triple, what is the triple?
Eduardo: The [total] number of edges is equal to three times the number of edges of the base.
Teacher: And that is a property.
Berta: The total number of edges of the prism s will always be a multiple of three. Whether it's even or odd.
Faced with Berta's response, the teacher asks the student *to justify her answer:*
Teacher: Why?
Berta: Because we're always going to have to multiply by three. Because it is the base plus the base plus the other [side edges].

During the whole-class discussion, the teacher asks students to present their solutions, starting with partial or incomplete solutions, which he seeks to value. However, he also promotes moments of reflection in order to draw attention to the limitations of these responses. His questions highlight generalizations and justifications. For the most part, they are guiding questions ("And have you concluded this through an example?"…), although there are also informing questions ("And that's a property"...). There are also some challenges, particularly when the teacher seeks to lead students to formulate their answer in a more formal language or when he asks the students to justify their answer ("Why?"). In each question, the teacher seeks that the students' contributions lead to the formulation of a correct answer, which he finally synthesizes in a small informing action.

8.4.2 *Grade 9 Lesson: Comparing Areas of Rectangles*

8.4.2.1 The Task

> The length of a rectangle was enlarged 10% and its width was shortened 10%. What can you say about the area of the rectangle when compared with the area of the initial rectangle? Show your thinking.

This task, proposed to a grade 9 class, calls for a response to a situation (10% increase in length and 10% decrease in the width of a rectangle), which may yield a generalization if applied to any rectangle. In addition, the task allows generalizing for other percentages and justifying the results. The task is interesting because it leads to an unintuitive result, since, at first glance, it could be thought that changes in dimensions would compensate for each other. The task is open as it asks, "what can you say?" In fact, several things can be said, from simply whether the area is maintained, increased or decreased, for all or only for a few rectangles and, in the case of change, how it changes. In this lesson, students work in pairs or groups of three.

8.4.2.2 Launching and Autonomous Work

The launching of the task is based on reading the statement of the task after which the teacher asks, "Does anyone have questions?" The teacher gives indications for the work to be developed, says, "I am now asking you to begin to work, OK?" and wishes students success, thus *seeking to involve the students in solving the task*.

In the phase of autonomous work, the teacher follows the solution of the task by the students. To try to understand what a pair of students had already thought, she asks them *guiding questions, without giving indications that could lower the degree of challenge of the task:*

Abel: These 10% first increase and then decrease, the area will continue to be larger, even if it is close.
Teacher: OK. Can you repeat, please, Abel? What do you think is going to happen to the area?
Abel: It will increase, no matter how little it may be. Because 10% here [sheet length] is greater than 10% here [sheet width]. So if we increase it 10% here...

Abel presents a wrong conjecture made apparently from a particular case (a paper sheet).

As a way to get students to think more about their answer, the teacher asks them *to indicate a justification strategy*:

Teacher: And do you have a way to show that?
Abel: To show? We can try to get there.

Tiago: By drawings.
Teacher: Any justification? By drawings, OK. And more, what else can you try to do?
Abel: Give numbers. We can give numbers.
Teacher: You can give numbers. So try giving numbers there. What else, you were going to suggest something else, Tiago?
Tiago: *x* and *y*.
Teacher: So... [The teacher withdraws and the students continue their dialogue]
Abel: Do we experiment with numbers?

The first question ("And do you have a way of show that?") is a challenge, which is successful because then the students indicate several possible solution strategies. The teacher welcomes the students' first suggestion ("By drawings") but presses them to present other possible strategies. The students then advance other suggestions, such as "giving numbers" (trying a specific case) and using variables, finally deciding to try a specific case. It should be noted that most of the teacher's subsequent questions are guiding question ("More, you can try to do what?",...) although there are also informing/suggesting actions ("Drawings, OK",...). Thus, the initial challenge of this dialogue was important to direct the students' work, but it was supported in guiding and informing/suggesting actions.

Later, the teacher interacts with another student pair, in which one of the students questions what the other student had done. Her first goal is *to understand the strategy followed by the students*. She finds that the students had explored an example, coming to a correct answer, but were having difficulty in interpreting the result. In view of this, she seeks that the students resume the statement of the task in order to identify what is the aim:

Teacher: So, read the question again. What can you...
Fernando: Say about the area of the new rectangle, as it compares with the initial one.
Teacher: So, the area of the new rectangle compared to the area of the initial rectangle, is what?
Fernando: It's smaller.
Teacher: It's smaller. And can you tell how much smaller it is?
Fernando: Yes. No.
Guilherme: It's 10%.

With the help of the teacher's questions, the students recognize that the transformed rectangle will have an area smaller than the original rectangle but are confused about the change that occurs. In view of this, the teacher pressures the students to give a more precise response, quantifying the change that occurs:

Teacher: Is it 10%?
Fernando: No, it is not 10%.
Teacher: It is not 10%, Fernando? How can you show me that it is not 10%?
Fernando: I have to do a rule of three.
[Fernando and Guilherme discuss what values can be or not]

Teacher: So, experiment, Fernando.
Guilherme: I know.
Teacher: So, do it, Guilherme, do the computation.

The students continue undecided about the transformation that takes place, with contradictory ideas, and continue to show difficulty in interpreting the result that they themselves obtained. In view of this, the teacher encourages them to continue to explore the situation, quantifying the change that occurs. Although the task involves simple percentages, the students consider that they need to use the rule of three. The teacher performs mainly guiding actions, although there are also informing/suggesting actions (“Read the question again”; “do the computations there”).

The teacher resumes the interaction with one of the groups, *in order to understand the progress made by the students*:

Teacher: Have you come to a different conclusion or have you been able to show?
Fernando: The area [of the rectangle] is going to be smaller.

The students had explored two specific cases, verifying that the area was reduced. The teacher seeks *to get students to make a generalization*:

Teacher: So, but is this true for these values 10 and 6 or for all?
Fernando: For all, for all.
Teacher: Yes? Show me.
Fernando: We made two, it’s always!

Faced with the answer from Fernando, the teacher questions the students if the justification given is valid, that is, if two examples are sufficient to justify an answer. In view of the students’ indecision, the teacher *suggests that they use variables to verify what happens to any rectangle*:

Teacher: And is [checking] two examples enough to show that it’s always true?
Fernando: I don’t know.
Teacher: And what if the dimensions of the rectangle were x and y? You would do the same computations that you did for these specific values? Would you arrive at the same conclusion? Experiment.
Fernando: But with x and y? But x and y are any numbers.

Considering the difficulty of students in moving from the use of specific measures to the use of variables, the teacher *asks targeted questions to help the students arrive for themselves at a strategy* that allows them to calculate the area of the transformed rectangle:

Teacher: If I have 10, if I increase 10%, what value do I get?
Fernando: 11.
Teacher: So, if I have y? If I increase 10% of y, I get…
Fernando: More 10% of y.
Teacher: How do I write?
Guilherme: You do not write, it is only y.

Teacher: So, is my original measure. If I'm going to increase 10%. Can I say that the *y* corresponds to what percentage if it is my original number?
Fernando: 100%.
Teacher: 100%, so the corresponding amount I'll get, it is worth another 10%, is it worth what percentage?
Fernando: 110%.
Teacher: 110%. So can I work with that? Look at that. For *x* and *y*.
Fernando: All right, we'll think about it.

To lead students to consider the general case of a rectangle, which allows for a mathematically valid justification, the teacher chooses to guide the students by showing them the parallel between the particular cases that they had considered and the general case of the rectangle of dimensions *x* and *y*. A strong support from the teacher was necessary to help the students write the dimensions of the transformed rectangle from the initial rectangle of dimensions *x* and *y*.

In the interaction with another group, the teacher begins by observing the students' solution. They used variables *x* and *y* for the dimensions of the initial rectangle and show, through algebraic calculations, that the area of the transformed rectangle is 99/100 of the initial rectangle (Fig. 8.2). Considering that the students' answer fully responds to the situation, the teacher *proposes an extension of the task, involving the exploration of new questions, leading to possible conjectures and generalizations*:

Teacher: So, and you weren't curious to try to figure it out... For example, in this case, I changed the length, increased it by 10% and the width decreased by 10%. What if instead of 10%, I used 20%? Or 30%? Was anything interesting going to happen? Did you experiment?
Telmo: No.

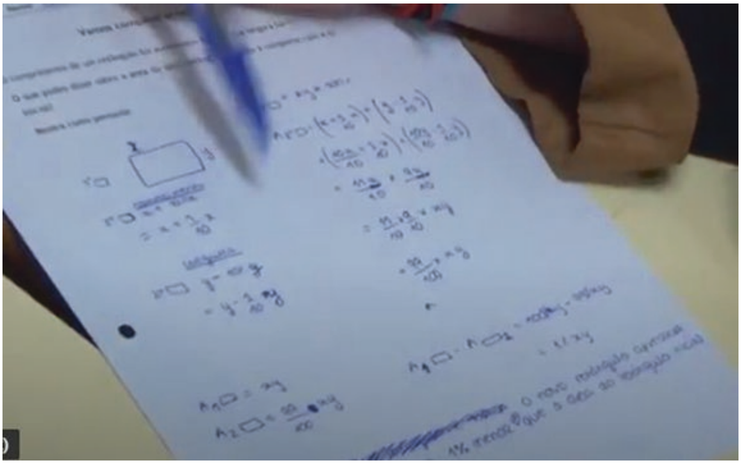

Fig. 8.2 Students' algebraic computations to find the area of the transformed rectangle

Teacher: Try it. If I took a rectangle and increased, it 20% in length, and in that case, how much is it going to decrease in width?

Telmo: The more it increased and decreased, then, we are going to multiply anyway.

Teacher: And what do you think it was going to happen in relation to the area?

Telmo: The difference was going to be larger.

Teacher: But was it still smaller? Or is there a possibility of increasing?

Bianca: It was still smaller.

Teacher: So show me that. Can you show me?

Telmo: But how much? Increases 20% and decreases 20%?

Teacher: And so on, what if it was 30 and 40, what would happen?

In this case, the teacher uses guiding actions to get students to explain their reasoning. Once the students have correctly solved the proposed task, the teacher challenges them to try other cases, involving other variations. The students immediately start making conjectures, but the teacher tells them that they should systematically explore different cases.

The launching of this task was rather straightforward; as the teacher believes that the students will have no trouble understanding the statement, so she simply encourages them to start for students who working. During the autonomous work, using guiding questions, the teacher seeks to understand what the students are thinking. When they have a wrong strategy, she challenges them to justify it. With a group that is having trouble in interpreting the results of their computations, she provides suggestions so that they can see in detail what they got. For students who could solve a specific case but were having great trouble in considering the general case, the teacher provided detailed guidance. Finally, with students who solved correctly the task, she challenges them with an extension involving different variations.

8.4.2.3 Whole-Class Discussion

The moment of collective discussion begins with the presentation of a group that bases the generalization on only two examples:

Teacher: I asked a group to come forward to explain the first strategy they used when they solved the problem . . . Madalena, I want to hear it.

Madalena [speaking with reference to the solution she put on the board (Fig. 8.3)]: So we, for the first rectangle, decided that the width would be 20 and the length would be 40. Then we went to calculate the 10% of each number so that we could reduce or increase.

Teacher: So, Guilherme, help Madalena there, what did you do here?

Guilherme: We calculated 10%.

Teacher: So, the 10% length is how much? The 40?

Guilherme: 4.

Teacher: And then the new rectangle has what dimensions?

Guilherme: 18 and 44.

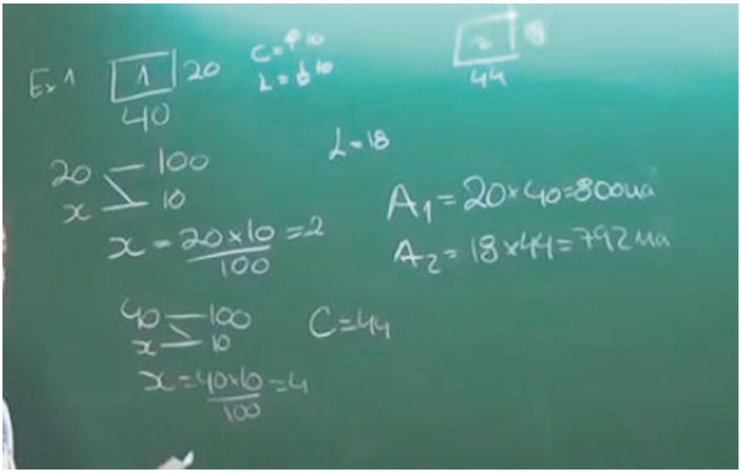

Fig. 8.3 Solution of Madalena at the board

Teacher: We get 18 and 44 units of measurement, right? Then, the colleagues started by using an example, assigning values to the measure of the length of the rectangle and the measure of the width of the rectangle.

Madalena: Now let's calculate the area of the first and second rectangle and then we will compare the two areas. The initial rectangle gives us 800 units of area. And the area of the second gives us 792 units of area. .. So, it's smaller and we wanted to calculate the percentage of how much smaller the final rectangle was to the initial rectangle. That's why we made another rule of three. We made, 800 stands for 100%, as 792 stands for x.

At first the teacher, using mainly guiding actions but also some informing/suggesting actions, seeks to have the group explain to the whole class how the two examples were explored, valuing the students' work. It should be noted that students prefer to use the rule of three rather than do the direct calculation of the percentage.

Later, the teacher raises the question of the *possibility of justification for any rectangle of the result obtained from specific examples.* This leads into a presentation from a group that had explored this possibility:

Teacher: But the question I asked most of you was: You experimented very well with specific values for measurement of length and measurement of width. Would it be possible for us to justify that for any measure of length and any measure of width? If we increased the length by 10% and decreased the width by 10%? Could we generalize this conclusion to any length value and any width measure? Is that possible?

Tiago: Yes.

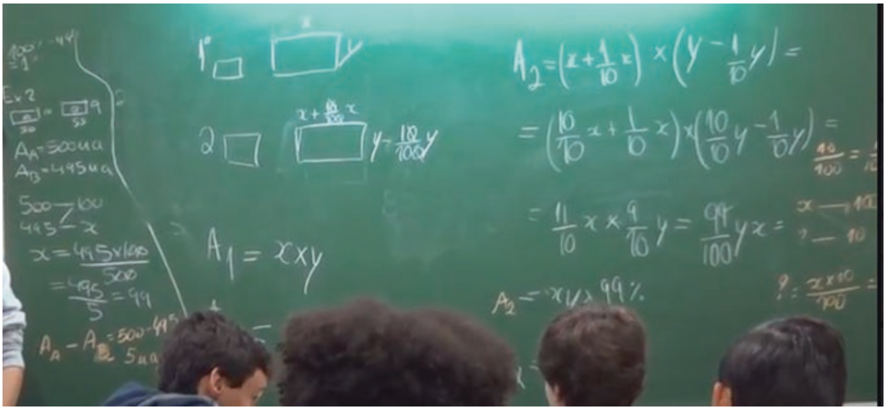

Fig. 8.4 Computations made by the group of Tiago

Teacher: So, here to the group of Tiago, Tomé and Manuela, can I ask you to come and show how you thought? Start there explaining how you did it, what was your first approach?

Tiago: At the beginning, we decided to find how much was the first rectangle, to help us. So, we found it easier to say that the length of the first rectangle was *x* and the width of the first [rectangle] was *y*. After, the second, as we knew the length was 10% longer, we knew that the length was *x* plus 10% of *x* [writes 10/100]. And the width was *y* minus 10% of *y* [writes 10/100]. So, we wanted to know the area, because that's what it's asked for. The area of the first and the area of the second.

Teacher: Just one thing, does everyone understands why the colleagues used 10/100?

Students: It's the same thing. [they refer to various representations of 10%]

Finally, the computations done by this group (Fig. 8.4) show that the area of the transformed rectangle A2 represents 99% of the *xy* area of the initial rectangle A1 of dimensions *x* and *y*.

In this whole-class discussion, a group of students presents two examples and the teacher guides them to show that the transformed rectangle has an area that is 99% of the area of the original rectangle. The teacher asks the question whether, in addition to the use of specific values, it would not be possible to draw the conclusion for the general case of any rectangle. The teacher then invites a group of students to present a generalization based on algebraic manipulation by transforming the percentages into fractions. Thus, the teacher begins by promoting the presentation of incomplete or partial solutions, which she thus values, in order to promote the presentation of the general case.

8.5 Conclusion

In both lessons, the teachers chose tasks that required mathematical reasoning. The grade 8 task required generalizations and justification in an explicit way, whereas the grade 9 task required them indirectly, at the same time allowing for different levels of generalization. Both tasks concerned the application of knowledge to situations that were new for the students rather than the development of new concepts or procedures. In fact, creating tasks that sustain the development of new concepts is much more complex for teachers, even when they search for ideas on the internet or elsewhere.

Both teachers organized their lessons according to the phases of the exploratory approach. The grade 8 teacher made an interesting launching, providing a negotiation of meaning of concepts central to the task. The grade 9 teacher did not foresee the need for such negotiation of meanings and just provided for the organization of the class in pairs or small groups and encouraged the students to begin working on the task. In both cases, the teachers took care not to lower the cognitive level of the tasks. During students' autonomous work, the teachers moved around the class to observe students' progress and difficulties and to provide hints concerning representations to use, trying out examples in order to formulate conjectures and generalizations, or providing justifications. Teachers' actions at this phase considered each group of students and their standpoint, striving to provide opportunities for all students to engage with mathematical challenges. Again, also at this phase of the lesson, the teachers strived to not lower the cognitive demand of the tasks. For the whole-class discussions, the teachers sequenced the presentation of students' solutions, from more incomplete or invalid to the most complete ones, striving that the main ideas would be appropriated by all students in the class. In both cases, they sought to highlight students' generalizations and justifications, moving from the analysis of specific cases to the consideration of the general case. The students had more facility in making generalizations, usually from considering a few examples, than in providing complete justifications. In the grade 9 class, some students were able to use algebraic representations to consider the general case but were far from using it in an efficient way.

This chapter provides new knowledge about the general and specific actions that teachers may use in lessons aimed at the development of students' mathematics reasoning, extending the work of Fraivillig et al. (1999), Cengiz et al. (2011), and Brodie (2010). In these lessons, the teachers used general actions, with much emphasis on guiding actions. When the teachers felt necessary, they used informing/suggesting actions in order to provide students specific information that they need or to strengthen their confidence in their approach or line of reasoning. In both lessons, the main challenge was provided in the statement of the task, but the teachers also used challenging actions during their interaction with students, striving to lead them to develop their reasoning or to improve the mathematical formulation of their answers. As illustrated in the discussion of the episodes, the teachers also used many of the specific actions indicated in Table 8.1, which provides a general orientation toward what teachers may do in seeking to promote students' generalizations and justifications.

In summary, these lessons illustrate that developing mathematical reasoning, with an emphasis on generalizing and justifying, is achievable in mathematics lessons at these grade levels. Such emphasis requires the careful selection of tasks and may be achieved by working in an exploratory approach, through three-phase lessons. It is important that teachers carefully plan these lessons in order to foresee possible students' strategies and difficulties and plan ways to deal with them as they arise in the classroom. Providing these experiences to students and supporting their mathematical growth is part of a mathematical education that honors important mathematical ideas and the same time the students' capabilities for doing mathematics.

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Chapter 9
Mathematical Argumentation in Small-Group Discussions of Complex Mathematical Tasks in Elementary Teacher Education Settings

Gwendolyn M. Lloyd and P. Karen Murphy

9.1 Introduction

Mathematical argumentation – a process of developing, presenting, and evaluating evidence and reasoning in support of mathematical claims about a question or situation – is a central disciplinary practice that is globally considered an essential component of school mathematics programs for students at all grade levels. In the United States, the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010) include eight Standards for Mathematical Practice for students from kindergarten through high school, one of which focuses explicitly on mathematical argumentation, "Construct viable arguments and critique the reasoning of others" (p. 7), consistent with the earlier "reasoning and proof" process standard of the National Council of Teachers of Mathematics (2000). Although standards and curricular recommendations promote visions of classrooms rich with mathematical argumentation, there is a limited body of research about how teachers can promote and support students' engagement in mathematical argumentation in the elementary grades (Krummheuer, 2007, 2013; Stylianides, 2007; Yackel, 2002).

Existing research from classrooms across grade levels suggests that, as students learn to make and evaluate mathematical arguments (Conner et al., 2014a; Ellis, 2011; Stephan & Rasmussen, 2002; Weber et al., 2008), teachers play important roles in supporting students as they gain proficiency with mathematical argumentation (Bieda, 2010; Conner et al., 2014b; Conner & Singletary, 2021). Facilitating mathematical argumentation in the classroom demands changes in existing discourse patterns and other mathematics classroom norms and expectations (Cobb, 1999; Forman et al., 1998; Walshaw & Anthony, 2008; Yackel, 2002). Because many teachers, including those who teach (or will teach) in elementary schools,

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have not benefitted from personal experiences with and knowledge about mathematical argumentation (e.g., Herbert et al., 2015; Melhuish et al., 2020), this area has gained increasing attention in mathematics teacher education in recent years (Conner & Singletary, 2021; Rogers & Steele, 2016; Stylianides et al., 2013; Yopp, 2015; Zambak & Magiera, 2020).

In this chapter, we describe an ongoing project in which we provide prospective teachers [PTs] with first-hand experiences with mathematical argumentation through a discussion approach known as Quality Talk (Murphy et al., 2018a, b, c). In our project, elementary PTs learn about classroom discussions that prioritize mathematical argumentation through both explicit instruction and participation in a series of small-group discussions about complex mathematical tasks during their university methods courses. PTs also plan and facilitate small-group mathematics discussions with children in kindergarten through grade 4 in their supervised field experiences. Our aim in this chapter is to illustrate how elements of the Quality Talk discussion model enhanced our ability to support PTs' developing understandings and use of mathematical argumentation in elementary classrooms.

In the following sections, we describe the foundations of our project by providing an overview of the core components of mathematical argumentation and presenting key elements of the Quality Talk model. Subsequent sections offer details about and examples of our use of Quality Talk to support mathematical argumentation in different teacher education settings involving elementary PTs: a university methods course and field experience classrooms. Guided by the notion that talk is an external representation of thought, our central aim is for PTs to learn to facilitate mathematics discussions that promote critical-analytic thinking, understood as "effortful, cognitive processing through which an individual or group of individuals comes to an examined understanding" (p. 563, Murphy et al., 2014). In this way, we view discussion-based argumentation as a fundamental tool by which students can reach examined understandings of mathematics.

9.2 Mathematical Argumentation

9.2.1 Components of Mathematical Arguments

Our conceptualization of mathematical argumentation is rooted in Toulmin's (1969) model which describes an argument in terms of three core components. A *conclusion* or *claim* is a statement being asserted or argued for, the *data* provide evidence in support of the claim, and the *warrant* connects the data or evidence with the claim. The warrant demonstrates that the claim is valid and provides reasoning for why the evidence supports the claim. In Toulmin's model, an argument may include additional components, including a modal qualifier (an expression of the degree of confidence about the claim), backing (additional support for the warrant), and a rebuttal (valid rejection of a warrant). While some researchers have focused on the three core components *claim–data* (*evidence*)–*warrant* (*reasoning*) to examine students' and teachers' argumentation activity in mathematics classrooms (e.g., Krummheuer, 2007; Yopp, 2015), many have applied Toulmin's full scheme, or an adaptation of it, to characterize mathematical argumentation (e.g., Conner et al., 2014a; Forman et al., 1998; Inglis et al., 2007; Weber et al., 2008; Zambak & Magiera, 2020).

In our work with students, teachers, and teacher educators, we characterize *mathematical argumentation* as a process of developing, presenting, and evaluating evidence and reasoning in support of mathematical claims about a question or situation. Drawing on Toulmin's (1969) model, the three primary elements of this conceptualization of mathematical argumentation, as shown in Fig. 9.1, are *claims*, *evidence*, and *reasoning*. In response to an authentic question[1] about a mathematical task or situation, students make claims in order to take a position on some aspect of the mathematical situation under consideration. A claim is insufficient on its own as a response to an authentic question; evidence and reasoning together provide needed support for a claim. A student might use a mathematical representation, such as a graph or a data table, as evidence in support of a claim. The student's reasoning, then, demonstrates *why* the evidence supports the claim through the logical use of known mathematical definitions, properties, relationships, and concepts.

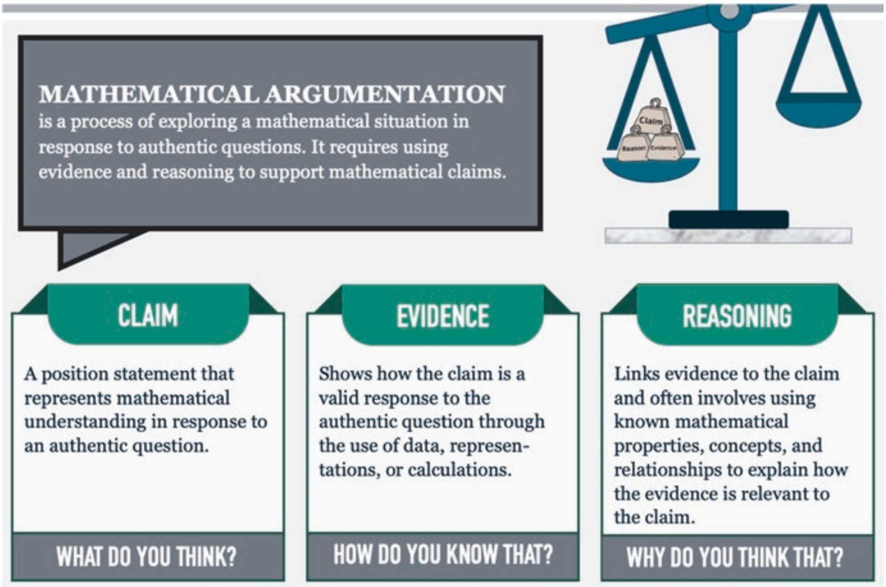

Fig. 9.1 Components of mathematical argumentation

[1] An authentic question is an open-ended question in which the person asking the question does not know the answer or is genuinely interested in knowing how others will answer; in other words, the answer is not pre-specified (Nystrand & Gamoran, 1991; Murphy et al., 2018a). See also Table 9.1.

The strength of a mathematical argument depends upon the quality of the evidence and reasoning and the relationships among the claims, evidence, and reasoning, as portrayed in Fig. 9.2. To provide strong support for a claim, evidence needs to be both accurate and relevant to the claim being made, and the reasoning must clearly and logically link the evidence to the claim. We consider the reasoning exemplified in Fig. 9.3, made in relation to a complex mathematical task, to be strong because it connects the evidence (namely, the labeled diagram that a student sketched, showing a composite shape that is consistent with the constraints of the situation) to the claim. The reasoning draws on familiar properties of rectangles to demonstrate logically how and why the sketch supports the claim that the composite shape is not a square.

When a stated claim is not valid or when the evidence or reasoning is weak, inaccurate, or unclear, the opportunity for refining or rebutting an argument emerges. In accord with Toulmin's (1969) ancillary argument component of rebuttal, our framing of mathematical argumentation also includes *challenge arguments* which counter another argument or explain why the evidence or reasoning in another argument is not valid.

Mathematics education researchers who explicitly frame their analyses of mathematical argumentation in terms of Toulmin's (1969) core components of claims, evidence, and reasoning offer similar conceptualizations to our description above. Consider, for example, two recent studies about elementary and middle grades PTs' engagement in mathematical argumentation during mathematics courses required in their university teacher preparation programs. In a research report about the quality and types of claims that elementary PTs make when presented with a false generalization and a counterexample, Yopp (2015) uses Toulmin's core components to define a *viable argument*:

Viable arguments have a claim, data, and a warrant and meet the following criteria:

1. Express a clear, explicit, unambiguous, prudent, and appropriately worded claim;
2. Express support for that claim that involves acceptable data (or foundations);
3. Express acceptable warrants (or narrative links) that link the data to the claim; and
4. Identify the mathematics (definitions and prior results) on which the argument relies. (p. 82)

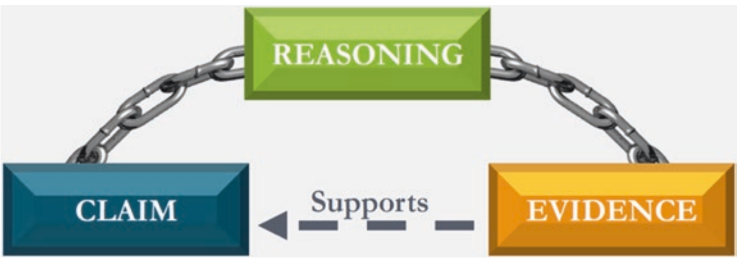

Fig. 9.2 Relationships among claims, evidence, and reasoning

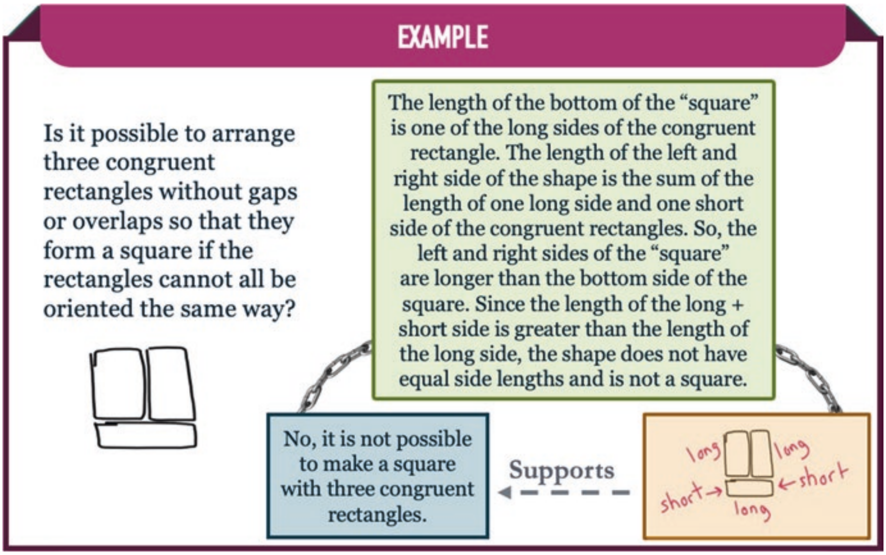

Fig. 9.3 Example of strong reasoning and evidence in support of a claim

Acceptable forms of data or evidence in a viable argument may include "examples, diagrams, prior results, definitions, narrative descriptions, stories, etc., provided that the representation of the data/foundations can be appealed to appropriately in the warrant" (p. 82). This characterization by Yopp largely resonates with the qualities of a strong argument we described in reference to Figs. 9.2 and 9.3, with the exception that we see the possibility for mathematical definitions to provide evidence or to serve as part of reasoning.

Reporting about their study of elementary and middle grades PTs' argumentation related to solving crypto-arithmetic mathematical tasks, Zambak and Magiera (2020) defined *claims* as "final or intermediate statements a [PT] articulates for the solution of a problem," *evidence* as "information a [PT] gathered and used to provide support for the validity of a claim," and *reasons* as "rationales a [PT] provided to eliminate uncertainties about letter-value assignments and to document that the solution is comprehensive and that it addresses all possible cases" (p. 5). In general, these definitions coincide with those presented in Fig. 9.1.[2] Whereas Zambak and Magiera used claims, evidence, and reasoning to examine PTs' individual written arguments, they additionally used *refutations* and *certainty*, similar to rebuttal and qualifier in Toulmin's (1969) model, to analyze PTs' collective classroom

[2] Zambak and Magiera's study primarily deals with claims that are assertions of solutions to mathematical tasks. In contrast, in our conceptualization, a claim is a statement in response to an authentic question about the underlying mathematical ideas of a task or its solution; it is possible that a claim will not speak directly to the solution to a mathematical task (see Fig. 9.3). We explain more about the roles of mathematical tasks and authentic questions in Quality Talk discussions below.

arguments. They describe how, when PTs in their study generated arguments together in the classroom, "the social nature of argumentation provided [them] with the opportunity to engage in refutations or to express the degree of certainty, as they listened, evaluated, challenged, and contributed ideas to each other's arguments" (pp. 4–5). This *collective mathematical argumentation* is the focus of the next section.

9.2.2 Collective Mathematical Argumentation in the Classroom

Although a mathematical argument can be developed individually, our primary focus in this chapter and in our project is on the collective, dialogic argumentation that develops in the context of classroom discussions in which students are encouraged to work together to make claims in response to questions about a mathematical situation (often presented in the form of a mathematical task); support claims with convincing evidence and reasoning; and evaluate the quality and validity of shared claims, evidence, and reasoning in conjunction with peers. Collective mathematical argumentation has multiple purposes in classrooms, including in elementary school classrooms (Lannin et al., 2011). When students co-construct mathematical arguments with peers, they gain opportunities to develop mathematical insights and understandings that they would be unlikely to reach by working on their own (Conner et al., 2014a; Ellis, 2011; Weber et al., 2008; Yackel, 2002) and to begin to build early but foundational understandings related to mathematical proof (Pedemonte, 2007; Stylianides, 2007). Furthermore, collective mathematical argumentation provides students with first-hand experiences with core disciplinary practices that stand in contrast to those that continue to dominate many mathematics classrooms in the United States (Banilower et al., 2013; Hiebert et al., 2003).

For students to engage in collective mathematical argumentation, changes must occur in the kinds of interactions that take place in most classrooms. A large corpus of mathematics education research provides insights into the important roles that teachers play in shaping learning opportunities for students in classroom discussions through the establishment of sociomathematical norms (e.g., Kazemi & Stipek, 2001; Yackel, 2002; Yackel & Cobb, 1996); teachers' selection of complex, cognitively demanding tasks from which to launch discussions (e.g., Gresalfi et al., 2009; Henningsen & Stein, 1997); and teachers' use of particular actions and moves intended to extend students' thinking and understanding through discussions (e.g., Cengiz et al., 2011; Fraivillig et al., 1999; Hintz & Tyson, 2015; Stein et al., 2008). Studies describe the tendency of teachers to ask leading questions (Franke et al., 2009) or to hesitate at times to provide explicit direction for fear of imposing on student thinking (Lobato et al., 2005), highlighting pedagogical tensions faced by teachers as they facilitate mathematics discussions with students. Adding to this complexity, research studies also show that students' different academic and social identities and relations can influence the nature of their participation in mathematics discussions (Esmonde & Langer-Osuna, 2013; Gresalfi et al., 2009).

In light of the important roles of teachers in students' opportunities to participate in mathematical argumentation through classroom discussions, PTs' development with respect to mathematical argumentation has become the focus of numerous research studies in mathematics teacher education (e.g., Conner & Singletary, 2021; Rogers & Steele, 2016; Stylianides et al., 2013; Yopp, 2015; Zambak & Magiera, 2020). For example, Zambak and Magiera's study, introduced previously, offers illustrations of elementary and middle grades PTs' emergent mathematical arguments during a university mathematics course. These researchers analyzed transcripts of whole-class discussions in which PTs presented and discussed problem solutions that they had written prior to the discussions and during which "the instructor skillfully [*sic*] directed PTs' attention" (p. 22) to important argument elements, such as "well-developed support for their assertions" (p. 22). In addition to scoring the argumentation skills exhibited in the written solutions, the researchers traced components of PTs' arguments during the discussions and marked moments when PTs asked questions and "pedagogical moves of the instructor (e.g., prompting, re-voicing)" (p. 13). Zambak and Magiera found that although PTs initially used predominantly inefficient guessing strategies and exhibited inadequate argumentation skills overall, the PTs became increasingly systematic in providing support for their claims and showed gains in argumentation skills over the course of a semester.

Although existing studies indicate that students can effectively co-construct mathematical arguments during classroom discussions, our understandings about how argumentation develops in the context of discussions remain limited. Few studies in mathematics education simultaneously examine core components of mathematical argumentation (i.e., claims, evidence, and reasoning) *and* qualities of productive classroom talk as it relates to students' cognitive processing (e.g., critical-analytic thinking). When studies of collective mathematical argumentation include analysis of classroom discussions, they tend to do so without the guidance of an explicit discourse approach (e.g., Forman et al., 1998; Stylianides, 2007; Weber et al., 2008; Zambak & Magiera, 2020). We propose that intentional instruction about and analysis of argumentation and discourse together may be particularly fruitful in elementary classrooms and elementary teacher education contexts, where deeper understandings of discussion-based mathematical argumentation practices are needed.

9.3 Classroom Discussions

Dewey (1916) described discussion as "…bringing various beliefs together; shaking one against the other and tearing down their rigidity…it is conversation of thoughts; it is dialogue—the mother of dialectic…" (pp. 194–195). It is through a *conversation of thought* that individuals by themselves or as part of some social exchange begin to examine their understanding through logical evaluation and embrace the power of thinking and interthinking. The centrality of discussion as a pedagogical tool is not new. Both Eastern and Western scholarly traditions date the use of discourse-intensive pedagogy to the earliest written records (Palmer, 2001). From the preSocratic period into the Modern era, high-quality discussions have been praised for their utility in enriching students' thinking and argumentation.

Paralleling the support for critical, reflective discussions found in philosophical writings, an array of psychological theories empirically ground the mechanisms by which individuals' thinking and argumentation are enhanced through discussion. Cognitively, discussions promote active engagement in meaning-making from text and content (McKeown et al., 2009), elaboration and explanation of understanding (Fonseca & Chi, 2011; Inagaki & Hatano, 2013), and evaluation of claims and evidence (Greene et al., 2016). Socioculturally, immense value is placed on language as a tool for thinking individually or through co-construction of understanding (Vygotsky, 1978). Essentially, Vygotsky held that children develop language to express their ideas or thoughts using the tools and signs of their culture. With repeated exposure to critical, reflective discussions, children eventually internalize the discourse community as the voice of "social others" guiding their thoughts. Like Dewey, Vygotsky valued discussion for its ability to foster students' co-construction of knowledge and understandings about content, internalize ways of thinking that promote knowledge acquisition and refinement, and to forge habits of mind needed for meaningful learning (Cobb, 1999; Wells, 2007).

We also see empirical support for talk as a valued pedagogical tool in the critical-thinking literature. Results from the meta-analytic studies of Abrami et al. (2008, 2015) provide evidence that discussion is one of the most effective pedagogical tools for increasing critical-analytic thinking and argumentation. In particular, pedagogical approaches that combine a stand-alone discourse model, through which students receive explicit instruction and modeling in productive discourse participation, with content-rich activities that engage students in complex problem solving produce stronger effects than either a generic or content-embedded approach. One such stand-alone discussion model is Quality Talk (Murphy & Firetto, 2018).

9.3.1 The Quality Talk Model

Quality Talk (QT) is an educator-facilitated approach to small-group discussions aimed at increasing learners' critical-analytic thinking *about*, *around*, and *with*[3] text and content. As shown in Fig. 9.4, the QT approach encompasses three interrelated

[3] We use *about*, *around*, and *with* to describe the type of cognitive processing students engage in during discussion. Learners bring basic understandings *about* the text or content to the discussion. Then, during discussion, they broaden their understandings *around* and *with* the text or content by asking questions that elicit generalizations, analyses, and connections to other texts, as well as personal and shared experiences. In the end, learners reach an examined understanding by weighing the arguments presented by peers in response to their questions and scrutinizing their understandings *about*, *around*, and *with* the text or content during discussion.

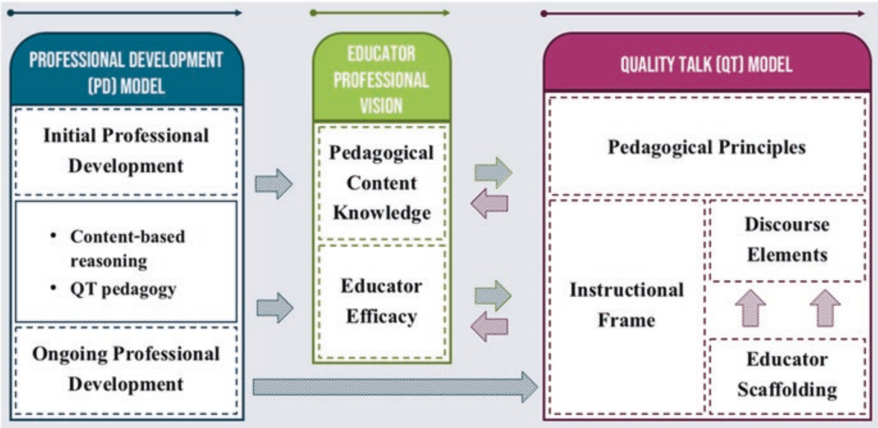

Fig. 9.4 Conceptual model of the Quality Talk intervention

dimensions: (a) a professional development model, (b) educator professional vision, and (c) a discussion model. These dimensions interact reciprocally to promote individual and collective critical-analytic thinking during small-group discussions.

In describing Quality Talk, we use the term "educator" broadly to include practicing teachers, PTs, and teacher educators who are learning to facilitate productive discussions. Similarly, we use the general term "learners" to include those who participate in educator-facilitated discussions. A learner might be an elementary school student or a PT, depending on the context. In our ongoing project, PTs sometimes hold the role of learner (i.e., when they participate in discussions facilitated by teacher educators) and other times serve as educators (i.e., when they facilitate discussions with children in their field experience classrooms).

9.3.2 QT Professional Development Model

For many educators, facilitating QT discussions requires a significant change in how they conceptualize the role of talk in teaching and learning (Wilkinson et al., 2007). In making such shifts, educators often must reconceptualize their role and learners' roles in discussions and make corresponding changes in instructional practices when implementing QT. Research has shown that educators need support in enhancing their professional vision, and associated pedagogical content knowledge, to effectively implement such learner-centered pedagogy (Murphy, 2018; NRC, 2012). The QT professional development model was designed to support this pedagogical change and is composed of both initial and ongoing professional developments (detailed in Murphy & Firetto, 2018; Murphy et al., 2017; 2018b).

As displayed in Fig. 9.4, the *initial professional development* provides educators with an overview of QT pedagogy as well as content-based argumentation.

Specifically, this initial professional development introduces educators to the principles that underlie effective change-making in instructional practice. Educators are taught about the four components of the QT model and how to enact these components with learners in the classroom in ways that will lead more productive discourse. They are also encouraged to construct their own understandings of effective discussion facilitation through participation in sample QT discussions and by learning how to analyze their own and their learners' discourse. *Ongoing professional development* (Garet et al., 2001; Sztajn et al., 2017) reinforces educators' understandings of the QT model by providing opportunities to reflect on discussions, first individually and then collaboratively with a discourse coach, in order to examine whether learners are displaying indicators of critical-analytic thinking during discussion and how their own facilitation practices can promote indicators of such cognitive processing in future discussions.

9.3.3 Educator Professional Vision

Professional vision refers to "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (Goodwin, 1994, p. 606). Within the QT approach, fostering educators' professional vision is seen as essential in supporting them in learning to facilitate productive discourse. Professional vision enables educators to perceive and codify meaningful patterns within classroom discussion, guide and promote richer learner exchanges through those identified patterns, and communicate and share the principles of argumentation with others. Importantly, educators' ability to acquire and enact professional vision is mediated by their pedagogical content knowledge (Shulman, 1986), particularly their discussion-specific pedagogical knowledge (Magnusson et al., 1999). Moreover, as educators develop their professional vision, we expect to see positive changes in their beliefs about their ability to facilitate meaningful discussions with learners (i.e., teacher efficacy; Bandura, 1977; Lakshmanan et al., 2011; Lotter et al., 2018). The notion of educator *efficacy*, rooted in Bandura's (1977) self-efficacy theory, refers to an educator's "judgment of his or her capabilities to bring about desired outcomes of [learner] engagement and learning, even among those [learners] who may be difficult or unmotivated" (Tschannen-Moran & Hoy, 2001, p. 783). Enhanced pedagogical content knowledge and efficacy strengthen educators' ability to promote all learners' critical-analytic thinking through content-rich discourse.

9.3.4 Discussion Model Components

As shown in Fig. 9.5, four components comprise the QT discussion model: an ideal instructional frame, discourse elements, a set of educator scaffolding moves, and six pedagogical principles (Murphy et al., 2022). The ideal *instructional frame* places

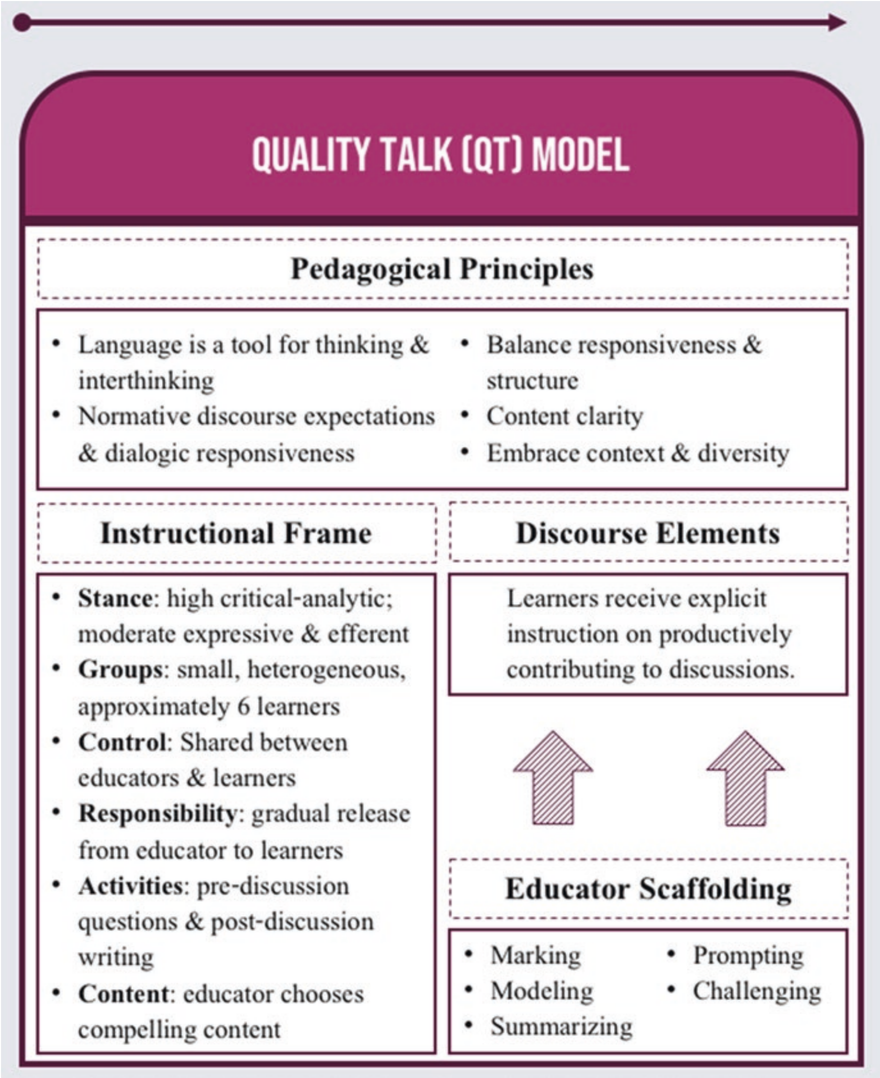

Fig. 9.5 Discussion model elements of the Quality Talk intervention

importance on embracing a critical-analytic stance (i.e., argument-rich querying; Wade et al., 1994) toward the text or content supported by moderate levels of the expressive (i.e., emotional connection; Jakobson, 1987) and efferent stances (i.e., information or knowledge seeking; Rosenblatt, 2004). During the discussion, educators and learners share control of the discussion with educators choosing compelling content while gradually releasing control of the discussion to learners. Such shared control affords learners increased opportunities to govern turn taking and enact interpretive authority over problem solving and learning. Learners also participate in several pre-discussion activities designed to prime their relevant understandings such as establishing authentic questions that will guide the discussion. Discussions take place in small, heterogeneous groups of six to eight learners.

The second component, *discourse elements*, refers to the indicators of productive talk within learners' verbal interactions (Murphy & Firetto, 2018; Murphy et al., 2022). These discourse elements, described in Table 9.1, include types of questions and responses that are simultaneously *indicators* of learners' critical-analytic thinking (Soter et al., 2008) and *tools* that learners can use to engage in critical-analytic thinking during discussions. For example, during productive discussions, learners pose different types of authentic questions about, around, and with content that they find meaningful. In other words, learners ask questions and respond to each other's questions about the specific text or content being discussed, but they are also encouraged to explore related ideas and principles in the domain and to make connections between the text or content and their own learning experiences (see, for example, transcripts later in this chapter for further clarification).

A productive discussion includes *authentic questions* that elicit high-level thinking (e.g., generalization, analysis, or speculation) or forge affective and intertextual connections.[4] In QT discussions, learners are also encouraged to ask questions that build on what has already been said (i.e., *uptake questions*) to open and extend the discursive floor to new or unexamined ideas. The QT model also emphasizes learners' use of argumentation to explain and justify their thinking through *elaborated explanations* while also encouraging them to co-construct knowledge with other learners. They can challenge the claims, evidence, and reasoning of their peers (i.e., *exploratory talk*), or they can support and build on other learners' claims by providing additional evidence and reasoning (i.e., *cumulative talk*).

As part of the QT model, educators implement a set of explicit discourse-specific lessons designed to augment learners' discussion skills (e.g., how to ask and respond to questions). As suggested by Dewey (1916) and Vygotsky (1978), it is expected that over time learners will internalize productive discursive practices about content including how to activate relevant content knowledge, justify claims, scrutinize sources of evidence, or modify understandings to accommodate examined understandings.

Educator scaffolding, the third model component, emphasizes a set of talk moves (i.e., modeling, marking, prompting, challenging, and summarizing), described in Table 9.2, that educators can use to facilitate productive talk (i.e., discourse elements) during QT discussions and that have been shown to promote learners' use of discourse elements (Wei et al., 2018). Importantly, these talk moves enable educators to facilitate productive discussions while allowing learners to maintain interpretive authority.

The final component of the QT model, *pedagogical principles*, refers to a set of guiding principles that provide a foundation for fostering a discursive environment

[4] Note that whereas questions can be initiated by educators or learners, response types (i.e., EE, ET, or CT) can only occur within learner-generated turns.

Table 9.1 Quality Talk discourse elements

Discourse Element	Description	Example
Authentic question (AQ)	Has multiple acceptable answers; speaker is genuinely interested in knowing how others will respond; answer is not pre-specified	Q: *What are some different ways to represent the "part" and the "whole" in this situation*?
Test question (TQ)	Presupposes a particular answer. Answer is explicitly stated in the text or is generally known	Q: *What is one-half of 8*?
Uptake question (UT)	Asks about something that another speaker has uttered previously	R: *We could draw eight ice cream cones and circle every other one. The circles would be the people who want vanilla* Q: *Why would we circle every other ice cream cone*?
High-level thinking question (HLT)	Elicits analysis or generalization; learners engage in inductive or deductive reasoning	Q: *How can the properties of rectangles help us reason about this task*?
Speculation question (SQ)	Elicits consideration of alternative possibilities	Q: *What are some ways that the green rectangle helps us find the missing length of the blue rectangle*?
Personal experience question (PE)	Elicits connections between the text and feelings or about life experiences	Q: *What problems have you solved in the past that remind you of this task*?
Connection question (CQ)	Elicits a connection to another text (books, movies, TV shows, artwork, website) or shared knowledge	Q: *How can we apply what we have been talking about regarding "reasoning with shapes" to this task*?
Elaborated explanation (EE)	A statement with a claim (position, opinion, or belief) that is based on at least two independent, conjunctive, or causally connected forms of support (evidence or reasoning)	R: W*e could use a multiple of 8 to figure out the number of bunnies [claim] because if we knew the number of hutches, then we could multiply by 8 and add 1 [reasoning]. The second sentence of the task says that. Ten hutches would make 81 bunnies because 8 bunnies times 10 hutches plus 1 bunny equals 81 bunnies [Evidence]*
Exploratory talk (ET)	Learners build, evaluate, and share knowledge over several turns; there must be an element of challenge	R1: *You would have to use area, not perimeter* R2: *You could use both!* R3: *But area's faster because you just multiply the two and with perimeter, you have to add up all the sides*
Cumulative talk (CT)	Learners build positively, but uncritically, on what others have said over several turns; does not include an element of challenge	R1: *I think I would start out putting 8 bunnies in each hutch* R2: *I agree, but I also drew 11 bunnies in a hutch each time I put 8 in a hutch. They're separate equations* R3: *Yeah, my initial way of solving it was to make a system of equations with one for eight bunnies per hutch and one for 11 bunnies per hutch*

Table 9.2 Quality Talk educator scaffolding talk moves

Educator talk move	Description	Example
Prompting	The educator prompts a learner to construct a longer response or a response that includes evidence or reasoning, thereby supporting more sophisticated talk	*How do you know that it works*?
Summarizing	The educator tries to slow down the group and overview part of the discussion to help build coherence	*Let's just pause and have someone summarize what we heard*
Marking	The educator reinforces specific aspects of a learner's discourse by explicitly pointing it out	*Really nice explanation. You used evidence and excellent reasoning*
Modeling	The educator explicitly models use of a discourse element for learners. The educator must make the intention of modeling clear	*I'm going to ask an uptake question …*
Challenging	The educator asks a learner to consider another point of view	*I'm not sure I agree with that. Have you considered …*

that empowers learners' perspectives and ideas (Murphy & Firetto, 2018). Among these pedagogical principles are (a) embracing talk as a tool for thinking and interthinking, (b) establishing normative discourse expectations which encourage a learner-centered talk pattern and further promote productive talk, (c) setting the boundaries for learners' interactions by balancing responsiveness and structure so that discussions do not veer too far *around* and *with* the text or content in unproductive ways, (d) ensuring that educators come to the discussion with content clarity, and (e) embracing the context and diversity of the setting and those involved in the discussion. Together, these four components (i.e., instructional frame, discourse elements, educator scaffolding, and pedagogical principles) lay the foundation for a classroom discussion community in which educators and learners engage in productive discussions that lead to deep, meaningful cognitive processing, including critical-analytic thinking and argumentation.

9.4 Adapting Quality Talk to Develop Mathematical Argumentation in Elementary Mathematics Teacher Education Settings

Over the past several years, we have been adapting QT for use in mathematics teacher education. PTs enrolled in an elementary mathematics methods course are also supervised in intensive field experiences in grades K-4 classrooms. PTs participating in our project receive explicit instruction about the four components of the QT discussion model (i.e., instructional frame, discourse elements, educator scaffolding, and pedagogical principles) and are given the opportunity to use their developing knowledge of these components as they participate in QT discussions in which they attempt to solve a complex mathematical task with peers. In addition to participating in QT as learners, PTs learn how QT can be applied in elementary (grades K-4) classrooms to promote generative mathematical argumentation among children. Stratified enactment of the QT model helps PTs understand how productive discussions can undergird critical-analytic thinking and meaningful learning. PTs experience first-hand how the QT discourse elements can be used as *tools* that support learners' engagement in mathematical argumentation and also provide *indicators* of learners' critical-analytic thinking during discussions (Soter et al., 2008).

In this project, we have adapted selected instructional materials and elements in the mathematics methods course to support PTs' conceptualizations of productive discussion centered on mathematical argumentation in elementary classrooms. The mathematical tasks selected for PTs' small-group methods course discussions, which are facilitated by teacher educators who engage in initial and ongoing QT professional development, have several important characteristics: they are non-routine, place a high level of cognitive demand on PTs (Henningsen & Stein, 1997), and would be difficult for one PT to solve individually without insights from peers. An example of such a task appears in Fig. 9.6. Before engaging in discussion, PTs participate in several activities to prime their understandings of the mathematics of the selected task, including writing authentic questions for use in the discussion. After discussions, PTs also reflect upon their experiences and learning in writing.

As an assignment that builds upon their experiences participating in methods course discussions with peers, PTs plan and facilitate small-group discussions with students in their grade K-4 field-based classrooms. Paralleling the task-based discussions facilitated by mathematics teacher educators with the PTs, the discussions

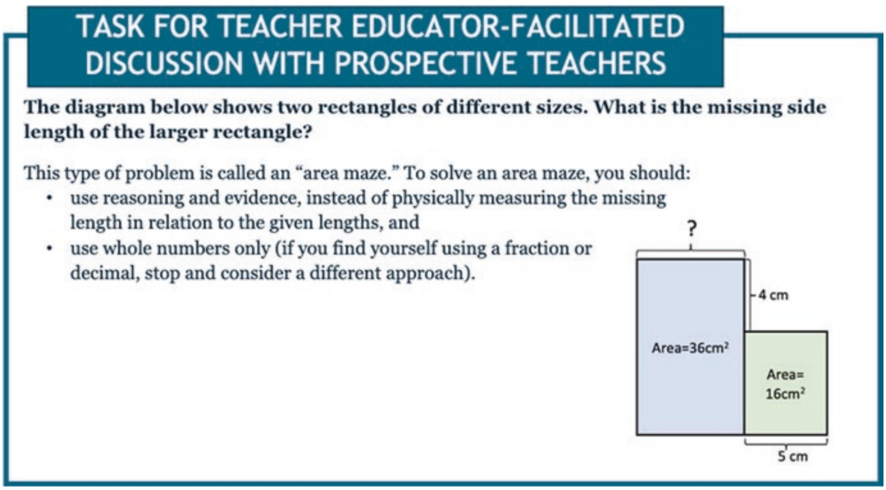

Fig. 9.6 Task for a Quality Talk discussion with prospective teachers

with grades K-4 students also center on complex tasks selected from the mathematics curricular program used in their elementary schools (e.g., see Fig. 9.8). PTs' experiences as both learners and educators in QT discussions allow them to begin to develop their own professional vision for facilitating productive collective mathematical argumentation through discussions in elementary classrooms.

9.5 Collective Mathematical Argumentation: Two Discussion Excerpts

In this section, we share two exemplar transcripts that illustrate the effectiveness of utilizing the QT model as a means to (a) promote PTs' mathematical argumentation and support their facilitation of productive discussions among grades K-4 students and (b) characterize the discourse underlying the mathematical arguments being developed in different teacher education settings (university elementary mathematics methods course and elementary classrooms where PTs complete their fieldwork).

9.5.1 A Discussion in a Methods Course

In the first exemplar discussion, PTs are discussing a mathematical task involving area, congruence, and properties of rectangles (shown in Fig. 9.6; task adapted from Bellos, 2016) that was introduced in their elementary mathematics methods course. PTs were presented with this task and given time to write questions about the task, and they then participated in a QT discussion facilitated by university-based teacher educators who had completed initial professional development and were engaged in ongoing professional development through our project.

In alignment with the pedagogical principles introduced previously, learners pose the majority of questions and produce the majority of the talk during QT discussions. That is, we expect to see fewer educator-generated questions and scaffolding moves as responsibility for interpretive authority gradually shifts from educators to learners. For example, in the excerpt in Fig. 9.7, PTs are actively engaged in asking questions and responding to what others have said, rather than waiting for the teacher educator to choose topics for discussion. As a case in point, in Turn 3, PT 3 demonstrates interpretive authority by telling a peer that she believes she is on the right track. It is important to recognize that, although the teacher educator is not overtly vocal in this excerpt, she is actively analyzing the PTs' talk and listening for opportunities to support the PTs' needs. For example, later in this discussion, the teacher educator prompts a PT to elaborate on her claim when the PT does not support her thinking with sufficient evidence and reasoning. The teacher educator also asks an authentic question when the PTs seem to have stalled in their progress toward constructing a solution to the task. Thus, increases in PT talk and

Turn	Speaker	Notes	Codes/Types
1.	PT 1	What made you think of that?	AQ/UT/ HLT
2.	PT 5	So, I'm guessing the claim would be that we're trying to find the side lengths for each rectangle. And evidence maybe would be that, um, we know what the area is, and maybe by cutting it into triangles it might be a little bit easier to figure out what the side lengths are. Did that make sense?	
3.	PT 3	Yeah. I think actually you're going somewhere with that.	
4.	PT 4	Yeah, so, I think that you can cut it into triangles, but I think you would have to start with the smaller green rectangle first [**Claim**]. My evidence for that is it gives you a full side length already and the area [**Evidence**], so you would have to start with that side length in order to find the adjacent side, is that … the one that's connected to the blue rectangle in order to complete this full side length [**Reasoning**].	EE
5.	PT 3	So, what I'm thinking is … so if we divide it into a triangle, the area should be half of it because we're dividing it into half at the diagonal, so it would be an area of 8 [**Reasoning**], and we know … I guess this is also kind of evidence, the area of a triangle is ½ base times height [**Evidence**]. So, we have one of those, and we have area, so if we put that into the equation, we can figure out what this side is, the one we're trying to figure out [**Claim**].	EE
6.	PT 4	That makes sense.	
7.	PT 5	Ok, so we know the one side of the rectangle is 5 cm [**Evidence**], and since we know the definition of a rectangle is the two parallel sides have to be the same size [**Reasoning**], so we know the other side is also 5 cm [**Claim**].	EE
8.	TE	Ok, this was reasoning because you used a property of the rectangle to explain something.	EM
9.	PT 2	Why are you doing triangles?	AQ/UT/ HLT
10.	PT 5	Well, my question before was asking, could you break the rectangles up into triangles to figure out what the side lengths are.	
11.	PT 2	But why?	
12.	PT 5	Well, I don't know what the formulas are, but there are formulas that you can use for a triangle to figure out what the missing side lengths are. Like $a^2 + b^2 = c^2$. But I'm not sure … that's for a right triangle, but not for a right triangle there are others you can use.	
13.	PT 3	So, I'm going to make a claim. We can use a triangle if that makes it easier for us to work with the numbers, but we don't have to [**Claim**] because even if we divide this little rectangle up into two squares, this side is still going to be the same [**Reasoning**]. So, the area of a rectangle is base times height or length times width. The area of a triangle is one-half base times height [**Evidence**]. So, it's going to be the same answer to the equation either way.	EE
14.	TE	In what ways does it matter that the shapes in this problem are rectangles?	AQ
15.	PT 2	It matters because if it was a square, all of these would be the same height, and then … I don't know.	
16.	PT 1	Because you can use … if it was a triangle, you'd be using a = ½bh, but with the rectangle you can just do the length times width to take the area. Right?	
17.	TE	[PT 6], were you agreeing with that? Can you elaborate or explain?	EM
18.	PT 6	The claim would be that I agree with [PT 1], that it matters because we know what formulas we'd have to use which is different from other shapes. It also matters with what [PT 5] was saying, if you fill in that space, it would just make a bigger rectangle, so we're still using the same formula.	

Fig. 9.7 Transcript from a QT discussion with prospective teachers

commensurate decreases in teacher educator talk illustrate how this teacher educator is *facilitating* this discussion rather than *leading* it; this is indicative of a productive QT discussion.

Prospective Teacher-initiated discourse elements further illustrate PTs' engagement in critical-analytic thinking during the discussion (Soter et al., 2008). More specifically, PTs ask authentic questions and respond to those questions with reasoned, evidence-based arguments. For example, PTs ask two authentic uptake questions in which they query peers' ideas and ask group members to explain their ideas more fully. Following these uptake questions, PTs provide responses that are rich in indicators of high-level thinking (i.e., analysis and generalization). In response to PT 5 suggesting that triangles could be used to reach a solution, PT 1 asks PT 5 to explain what made her think of using triangles in the first place (see Turn 1). This question also elicits analysis (i.e., high-level thinking) as demonstrated when the PTs discuss decomposing the figure into triangles as a potential solution path. Similarly, in Turn 9, PT 2 requires PT 5 to explain why triangles might be used, to which PT 3 replies with a statement about a general relationship between triangles and rectangles (i.e., generalization; see Turn 13). These are not the only turns indicative of high-level thinking, but they also serve as clear examples of how talk can be considered the external representation of thought and thus demonstrate PTs' critical-analytic thinking during discussion.

PTs also generate a number of mathematical arguments in response to their peers' authentic questions about, around, and with the area maze task. When responding to questions posed by peers, PTs regularly provide support for their claims (see Turns 4, 5, 7, and 13). In particular, PTs often contribute full arguments in a single turn (i.e., elaborated explanations), supporting their claims with both evidence and reasoning. For example, in Turn 5, PT 3 presents an argument supporting the use of triangles to help solve the task:

> So, I'm going to make a claim. We can use a triangle if that makes it easier for us to work with the numbers, but we don't have to [*Claim*] because even if we divide this little rectangle up into two squares, this side is still going to be the same [*Reasoning*]. So, the area of a rectangle is base times height or length times width. The area of a triangle is one-half base times height [*Evidence*]. So, it's going to be the same answer to the equation either way.

Importantly, PT 3 is engaged in mathematical argumentation, determining appropriate support for the claim. Although it is not present in this excerpt, we also hope to witness explicit episodes of *interthinking* during productive QT discussions; that is, PTs engaged in co-constructed responses in which they challenge parts of each other's elaborated explanations or build toward some level of consensus in response to a claim (i.e., exploratory or cumulative talk).

In summary, the excerpt depicted in Fig. 9.7 offers insights into some of the kinds of talk that are indicative of productive discussions of a complex mathematical task, including two authentic question events and several instances of mathematical argumentation. Notably, the QT discourse elements (authentic questions, elaborated explanations, etc.) worked in tandem with PTs' emergent use of claims, evidence, and reasoning in the discussion to support their engagement in high-level thinking about, around, and with the mathematical content of the task. For example, in the elaborated explanations responding to the first authentic question in the transcript (Turns 4, 5, and 7), PTs demonstrated that they were thinking around and with the task by describing how they might use their knowledge of triangles, as opposed to the rectangles presented in the figure, to solve the task.

Participating in productive mathematics discussions, such as this one, prepares PTs to enact discourse-intensive pedagogy in their own classrooms by deepening their mathematical thinking and providing them with a model of productive pedagogical practices. In the following section, we share an excerpt from a discussion that a PT enacted in her field experience classroom.

9.5.2 A Discussion in an Elementary Classroom

Figure 9.9 presents a transcript of an exemplar discussion that was facilitated by a PT with a group of second-grade students in her field experience classroom. The discussion centered on the mathematics of a task involving money and cost per unit (Fig. 9.8; task adapted from The Math Learning Center, 2017). Consistent with the QT model, the PT produced grade-level appropriate instructional materials and identified the complex mathematical task in Fig. 9.8 prior to the discussion. She then used those instructional materials to introduce her students to authentic questions and test questions, explored examples of each question type with them, and provided support for students as they formulated authentic questions about the task.

TASK FOR PROSPECTIVE TEACHER-FACILITATED DISCUSSION WITH SECOND GRADE STUDENTS

A customer bought 3 beans and the total price was 55¢. What beans did she buy? Explain how you are sure of your response.

Jack charges for his beans by length, 1 cent per cube. Here are the lengths of his beans.

Bean Color	Length
Green	10 Cubes
Red	12 Cubes
White	30 Cubes
Black	45 Cubes
Purple	15 Cubes
Rainbow	60 Cubes

Fig. 9.8 Task for a Quality Talk discussion with second-grade students

Similar to the previous transcript, the transcript with second-grade students reflects a learner-centered turn pattern; the PT limits herself to posing one question and enacting one discourse control move (i.e., educator moves such as procedural moves that likely have an indirect, rather than direct, effect on learners' critical-analytic thinking compared; Wei et al., 2018). Notably, in Turn 12, the PT communicates that the students have interpretive authority of the discussion when she tells them, "I don't know. Discuss it." In essence, the PT is letting the students know that they need to use talk as a tool for thinking and interthinking and determine what they believe to be correct together. This encouragement of students' autonomy of thought suggests that the PT has begun to internalize the pedagogical principles that undergird the QT approach.

In addition to the PT posing an authentic question in Turn 1, the students themselves pose two authentic questions related to the mathematical task they are working together to solve. Unlike the previous transcript, neither of these student-initiated questions are uptake questions, given that they reflect a shift in the dialogic floor toward a topic rather than asking about what someone said previously. The second authentic question (see Turn 18) does, however, elicit a co-constructed analysis (i.e., HLT) as the students discuss whether or not it is possible to subtract or decompose monetary quantities into smaller units. Given that the students participating in this discussion are in second grade, it is particularly encouraging to see them demonstrate their ability to productively engage in interthinking after one lesson about questioning.

The discourse depicted in Fig. 9.9 also demonstrates the elementary students' use of mathematical argumentation throughout the discussion. Notably, however, the mathematical argumentation here differs from that exemplified by the PTs in some regards. For instance, while PTs offered full arguments, including a claim, evidence, and reasoning, within a single turn (see Fig. 9.7), the second-grade students tend to develop parts of arguments together over a series of turns. Further, in the elaborated explanation in Turn 4, the student provides multiple pieces of evidence in the form of mathematical calculations but does not offer reasoning to link the evidence to the claim. It is worth noting that whereas the PTs received explicit instruction on mathematical argumentation, the second-grade students did not. Thus, it stands to reason that given additional support, the second-grade students would likely be able to construct and pose more sophisticated and connected arguments consisting of a claim, evidence, and reasoning.

Moreover, later in the transcript, Students 1 and 4 question whether money can be subtracted, leading to an episode of exploratory talk (see Turn 20) in which one student challenges another's idea and supports her challenge using a real-world situation in which money is subtracted:

Student 1: Can you subtract money?
Student 3: No, you can't. I don't …
Student 2: You can subtract money. It's a thing. [*Claim*]
Student 4: Well yeah, but …
Student 2: Like, let's say I need to pay 25 dollars and I have a 50 dollar bill … [*Reasoning*]
Student 3: You can take 25 away. (Fig. 9.9)

Turn	Speaker	Notes	Codes/Types
1.	PT	How can the one cent per cube help you figure out which beans the customer bought?	AQ/HLT
2.	Student 2	Well, I would probably start adding up like … so, if it was 55, I would probably put 40 plus … actually, no, you can't do that.	
3.	Student 3	I think it was saying it because it's not like it's 55 dollars.	
4.	Student 2	Yeah, it's not 55 dollars, it's 55 cents. If I had to solve this, I'd probably use 30 … um, then one of these, which would be 40, then 15 which would be 50, and then plus 5 equals 55 **[Evidence]**. So, 30, 10, and 15 **[Claim]**.	EE
5.	Student 3	Yeah, 30, 10, and 15.	
6.	Student 2	That equals 55.	
7.	Student 4	Is there a bean that's 35?	
8.	Student 2	No, but there's a 45.	
9.	Student 4	I tried to use 45 one time, but it didn't work.	
10.	Student 2	Wait, can you only use addition, or can you also use subtraction?	AQ
11.	Student 4	That's a good question.	
12.	PT	I don't know. Discuss it.	
13.	Student 3	Ok.	
14.	Student 1	Well, I would use six …	
15.	Student 2	Another way a kindergartener would probably use it, they would probably do 60 minus 10 equals ... no, 60 minus 15.	
16.	Student 3	No, 60 minus 5. 60 minus 15 is 50.	
17.	Student 2	No, it's not. That's 45.	
18.	Student 1	I just thought of another question. Can you subtract money?	AQ/HLT
19.	Student 4	Yeah, can you subtract?	
20.	Student 3	No, you can't. I don't …	ET
21.	Student 2	You can subtract money. It's a thing. **[Claim]**	
22.	Student 4	Well, yeah, but …	
23.	Student 2	Like, let's say I need to pay 25 dollars and I have a 50-dollar bill … **[Reasoning]**	
24.	Student 3	You can take 25 away.	
25.	Student 2	Yeah, I can take 25 away.	
26.	Student 3	But if it's a dollar bill, you can't take it away.	
27.	Student 2	You can't take it away without ripping it.	
28.	Student 3	Yeah, you just get half.	
29.	Student 2	You have to get change.	
30.	Student 3	Yeah, you have to get change if you want to do that.	

Fig. 9.9 Transcript from a Quality Talk discussion with second-grade students

In this excerpt, students engage in argumentation around and with mathematical concepts of the bean task, leading to deeper understandings of how addition and subtraction can meaningfully occur within a monetary system. Ultimately, the students reached agreement that money can indeed be subtracted and subsequently recognized that subtracting cubes was not a meaningful approach to take, given the context of this task. This exchange also provides a powerful example of how misconceptions can be shared and examined during discussions with a learner-centered talk pattern aimed at critical-analytic thinking and argumentation. QT's ideal instructional frame creates space for misconceptions to surface, and the normative discourse expectations foster a productive environment for them to be addressed as learners work toward an examined understanding.

9.6 Discussion and Conclusions

In this chapter, we have described our use of the Quality Talk (QT) approach to support prospective teachers (PTs) in learning to facilitate meaningful small-group discussions that advance learners' critical-analytical thinking about, around, and with complex mathematical tasks. Through two exemplar discussions, one involving elementary PTs (Fig. 9.7) and the other involving second-grade students (Fig. 9.9), we have illustrated how mathematical argumentation can emerge as learners jointly explore complex mathematical tasks with the support of QT discourse tools and educators who are engaged in learning to facilitate productive discussions. These exemplar discussions also illustrate how PTs can draw upon understandings and experiences developed in their university methods courses (namely, participating with peers in QT discussions about mathematical tasks) to plan and facilitate productive mathematics discussions with elementary students in their field experience classrooms. PTs' ability to put their methods course learning into classroom practice is particularly encouraging and important in light of persistent reports in the teacher education literature about disconnects between methods coursework and field experiences as well as the recent call from Cochran-Smith et al. (2015) for research studies that investigate how teacher education experiences influence teachers' classroom practice and use of specific pedagogical skills and approaches.

Through analyses of coded transcripts, we highlighted connections between PTs' and elementary students' use of components of mathematical argumentation (claims, evidence, and reasoning; Fig. 9.1) and QT discourse elements (e.g., authentic questions, elaborated explanations; Table 9.1) as they made progress toward solving complex tasks (e.g., Figs. 9.6 and 9.8). In addition, we discussed, in relation to the transcripts, the important role of educators' judicious use of authentic questions and scaffolding moves (Table 9.2) as they enact QT pedagogical principles and support learners' mathematical engagement. With this dual attention to mathematical argumentation and discourse elements, our analyses extend current understandings of how sound arguments can be co-constructed during classroom discussions. While identification of mathematical argumentation components can assist researchers (and educators) in observing mathematical arguments as they develop during classroom discussions, identifying discourse elements can provide additional essential information about *how* mathematical arguments develop in the course of discussions. Such insights are needed for noticing and fostering the key discourse actions of educators and learners that will support advances in mathematical argumentation and critical-analytical thinking through classroom activities.

Little prior research in mathematics education, particularly in studies involving elementary students and teachers, has focused explicitly on both argumentation and discourse elements. For example, although Zambak and Magiera (2020) traced elementary and middle grades PTs' emergent mathematical argumentation in discussions, the instructor's role facilitating PTs' argumentation was not revealed through their analysis.[5] And while Conner et al. (2014b) developed an empirically based framework that specifies three categories of teacher support for collective mathematical argumentation (making direct contributions to arguments, asking questions to elicit student contributions, and other supportive actions), these types of educator moves were identified by analyzing discussions of two secondary mathematics PTs – teachers whose mathematical knowledge and classroom experiences are likely quite different from those of most elementary teachers.

Our chapter offers specific examples of ways that elementary learners' and educators' use of particular discourse elements during discussions contribute to advancing the mathematical depth of argumentation and increasing opportunities for learning, even among young learners. As we have described, elementary educators' capacity to foster mathematical argumentation can be enhanced by learning about and participating in such generative discussions. Our hope is that the exemplar discussions and conceptual contributions of this chapter may be informative to others interested in promoting productive discussion-based argumentation in elementary school and teacher education contexts.

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[5] Nonetheless, we recognize some of the instructor's questions in Zambak and Magiera's (2020) transcripts as examples of well-known teacher talk moves and question types that other researchers suggest may allow educators to attend to aspects of learners' arguments or to advance or change the course of a collective argument (Chapin et al., 2009; Forman et al., 1998; Franke et al., 2009; O'Connor & Michaels, 1993).

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Chapter 10
Commentary to Part I of Mathematical Challenges For All: Commentary on 'Challenge' in Terms of Curriculum Materials and Tasks, the Teacher's Role and the Curriculum

Jeremy Hodgen

10.1 Innovations in Textbooks, Curriculum Materials and Tasks to Promote the Mathematical Challenge

Four of the chapters in this section discuss curriculum materials and tasks designed to increase the challenge in mathematics with respect to textbooks (Christou et al., Chap. 5), mathematical modelling (Goos et al., Chap. 4), problem posing (Cai & Hwang, Chap. 7) and representing open tasks using dynamic applets (Leikin et al., Chap. 6).

Christou et al. (Chap. 5) discuss the theoretical framework underlying the design and development of Cyprus Mathematics Textbooks, the Personalised Mathematics and Mathematics Inquiry (PMMI) framework. This is an ambitious textbook project that aims to promote a challenging approach to the school mathematics curriculum across Cyprus. Over the past decade, there has been a growing interest in the design of textbooks and how textbooks can be used as instruments to promote change in the teaching of mathematics (e.g., Rezat et al., 2021). However, Christou et al.'s (Chap. 5) is unusual in textbook research in articulating the theory, and design principles, underlying a system-wide project at scale from the perspective of the developers, thus providing insight into the textbook design at scale.

The PMMI framework draws on two major elements: personalised mathematics, and mathematics inquiry. The focus of personalised mathematics is on facilitating space for learners to 'give voice to their own ways of mathematical thinking' (p. 77) in order to build mathematical understanding. This has echoes of Schoenfeld's (2018) Teaching for Robust Understanding (TRU), particularly the dimension of

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agency, ownership and identity, which focuses on the extent to which learners are encouraged to 'walk the walk and talk the talk' (p. 493) of mathematics.

The mathematics inquiry element of PMMI draws on Whitehead's (1929) ideas of the 'rhythm of learning' to develop three phases of inquiry: romance, precision and generalisation. Romance is not a familiar idea in mathematics and is perhaps more commonly associated with language and arts subjects in the school curriculum. Hence, the idea is somewhat at odds with much of the practices of traditional mathematics classrooms and Rezat et al. (2021) note a paucity of research on the impact of textbooks on learners' attitudes, beliefs and perceptions. I suspect that Christou et al. (Chap. 5) use the term, romance, specifically to draw teachers', and learners', attention to the importance of these affective aspects of mathematics. Romance is conceived of as developing learners' wonder, curiosity and engagement with mathematics. But romance is nevertheless rooted in the 'real world' and Christou et al. illustrate how this is developed through an 'open-ended' exploration of slope in skateboarding.

Precision and generalisation are more familiar notions in mathematics teaching and learning. These are developed through a threefold process of knowledge building, convergent knowledge expression and divergent knowledge expression (Harris & Hofer, 2009), which are used as guides for the design of more focused investigations. Here, one is reminded of *realistic mathematics education* and Streefland's (1991) focus on the need for learners to engage in the *insightful construction* of mathematics.

Christou et al. (Chap. 5) illustrate the framework and the design process using several exemplars from the textbooks, providing considerable insights into how the framework was operationalised in this substantial project. The chapter due to constraints of scope and space necessarily concentrates on just the design and development of the textbooks. Indeed, the design focus itself is described at a relatively high level. However, it is clear from the tasks that research into how students learn mathematics informed the design of the tasks. I would like to know more about the principles underlying this process.

I now turn to the issue of implementation, which I hope the authors will address in further writing and research. As Rezat et al. (2021) observe, whilst textbooks are influential, even the best-designed curriculum resources cannot on their own produce a substantial change in the mathematics curriculum. Christou et al. (Chap. 5) devote a (necessarily) brief section outlining the role of the teacher as a co-learner and facilitator in the classroom. This changed role is far from straightforward even for very committed teachers (see, e.g., Calleja et al., 2021) and, hence, achieving such change at scale is ambitious. It would be helpful for the authors to outline how this change will be supported through, for example, 'educative' features of the textbooks (Davis & Krajcik, 2005) and/or through coaching and other forms of professional development. Teachers are an important part of the process of change, but, without substantial systemic support from advisors and school leaders, the process of change, particularly such an ambitious change as conceived in the Cyprus textbooks, is likely to fail (e.g., Cobb & Smith, 2008). On the other hand, Christou et al.'s textbooks do have considerable support from the educational system in

Cyprus. It would be valuable for the authors to explicitly consider this systemic support, ideally including a Theory of Change (Jankvist et al., 2021b) that could inform research evaluating the implementation of the Cyprus Mathematics Textbooks.

Goos et al. (Chap. 4) examine the introduction of mathematical modelling in Ireland as a case of curriculum reform in secondary school mathematics in order to examine the factors that support or hinder reform. They perceive mathematical modelling to be challenging because it involves the mathematisation of the real world and argue that this creates challenges for both teachers and learners. In contrast to Christou et al. (Chap. 5) they consider that textbooks, at least those commonly used by Irish mathematics teachers, to be a hindrance to the use of modelling, although I suspect that these textbooks do not present explorations that are deliberately informed by a framework like PMMI that is focused on promoting challenge.

Goos et al. (Chap. 4) draw on Remillard and Heck's (2014) model of curriculum policy, design and enactment to analyse two case studies of reforms that introduce mathematical modelling. The first is, like Christou et al. (Chap. 5) a system-wide initiative, the reform of the Applied Mathematics syllabus, a pre-university optional course in upper secondary that is taken by a relatively small proportion of learners (3–4% of the cohort). They examine two documents related to the stakeholder consultation process to highlight tensions around the aims of, and values of, mathematics and mathematics education (including tension around how teachers 'traditionally' use textbooks). Of particular note, they highlight very substantive differences amongst different stakeholders in how they view modelling and problem-solving, with some believing that the introduction of mathematical modelling would reduce, rather than increase, the amount and challenge of problem-solving within the syllabus. The issue of competing values and belief systems is a perennial issue around reform to school mathematics in the UK as well as Ireland and I have argued elsewhere that effective reform needs to be sufficiently pluralist to accommodate these competing values (Hodgen et al., 2022). But, to actually achieve this, we may be well-advised to look beyond our shores to countries like Cyprus and Belgium for examples of more consensual reform.

Goos et al.'s (Chap. 4) second case study is of a smaller-scale initiative, the Young Modellers Transition Year project, in which teachers taught a 10-week mathematical modelling course supported by professional development and coaching. They highlight the well-documented issue of teacher knowledge, beliefs and practices and argue that providing curriculum resources was an important factor in supporting reform. Crucial, however, appears to have been the decision to locate the reform in the Transition Year, an optional school year in Ireland during which teachers are actively encouraged to offer a more rounded and interesting curriculum and are not subject to examination pressures. One challenge for this project, which involved just 15 schools, is how to scale it up to a wider group of schools. There is increasing interest in the issue of implementation and scale-up in mathematics education (e.g., see Jankvist et al., 2021a; Maass et al., 2019). Prediger's (2022/Forthcoming) programme of research in Germany around language and mathematics suggests that developing a range of different types of evidence may be important in scale up.

Cai and Hwang (Chap. 7) discuss the issue of problem posing by learners from the perspective of mathematically challenging tasks. This topic strongly resonates with Christou et al.'s (Chap. 5) focus on explorations and Goos et al.'s focus on mathematical modelling. Problem posing, like Christou et al's (Chap. 5) explorations, can support learners' engagement with mathematics. In addition, as Black et al. (2003) found, posing problems can enable learners to understand the 'game' of mathematics and, thus, can be an important first step in enacting formative assessment and enabling learners to self-regulate (and thus challenge) their mathematics learning (Wiliam & Thompson, 2007).

Cai and Hwang (Chap. 7) devote a great deal of attention to exemplifying a typology of mathematical contexts and real-life situations to guide the design of problem-posing tasks. In doing so, they consider how problem-posing tasks can be constructed to increase the challenge and to achieve particular learning goals. There are many insights into the design and use of such tasks. Over recent years, there has been a great deal of work using variation theory to analyse and support the design of problems and tasks (e.g., Watson & Mason, 2006). I wonder whether variation theory and Marton's (1997) *dimensions of possible variation* and *range of permissible change* might be productive in terms of further insights into the pedagogical role of problem-posing tasks.

Cai and Hwang (Chap. 7) then consider how to support teachers to use problem posing tasks in their teaching and discuss the professional learning model, or theory of change, used in their current research. They briefly discuss the role of professional development in supporting teachers, but they note also the need for teacher buy-in to the value of problem-posing. This raises two issues. First, there is likely to be a role for 'educative' curriculum materials (Davis & Krajcik, 2005) to support teachers' enactment of problem-posing tasks. An important role of such materials would be to enable teachers to 'notice' ways in which problem posing can result in productive learning (Choy & Dindyal, 2021). Second, as Goos et al. discuss, there are many constraints on a teacher's capacity to enact changed practices that go beyond a teacher's knowledge, beliefs or pedagogical expertise. 'Traditional' mathematics classrooms are structured in particular ways (see, e.g., Ruthven, 2009). Enacting substantial change, such as integrating problem posing tasks, into everyday teaching involves a significant change to these structural features and, hence, is likely to require the support, and buy-in, of advisors and school principals (Burkhardt & Schoenfeld, 2003).

Leikin et al. (Chap. 6) focus on and how technology can support students to engage with challenging open tasks as part of the Math-Key program. They begin by considering different ways in which open tasks can be structured (or designed) to place the emphasis on multiple solution strategies and/or on multiple outcomes, then exemplify this using a number of tasks. In doing so, they provide an explicit account of the dimensions of variation that informed the design of the tasks and how the approach is intended to alter the emphasis of mathematics classrooms through changed goals, activities, conditions and tools (Leontiev, 1978). Finally, they discuss the design of applets intended to enable learners to engage with different solution strategies or outcomes.

Leikin et al.'s (Chap. 6) applets certainly provide potential opportunities for learners to explore different approaches and, thus, to consider mathematics as a more open discipline. However, since they are designed in advance, they necessarily constrain this exploration to particular approaches or strategies. It may not be immediately obvious to a learner how these particular approaches relate to the learner's own approaches, even where they are similar. Indeed, as Hodgen et al. (2018) found in a recent review of mathematics teaching and learning, the role of the teacher is crucial in making such connections. I look forward to future work by these authors on how they support teachers to do this and on whether the Math-Key approach is effective in enabling learners to develop more open mathematical minds.

These four chapters provide significant insights into how textbooks, curriculum materials, tasks and technology can be designed to increase mathematical challenges. In doing so, they raise questions about how to support teachers, and others within the system, to implement curricular change that promotes challenges for learners.

10.2 The Teacher's Role in Promoting Mathematical Challenge

Two of the chapters in this section address the role of the teacher in facilitating mathematical challenges in terms of mathematical reasoning (Ponte et al., Chap. 8) and mathematical talk and argumentation (Lloyd & Murphy, Chap. 9).

Ponte et al. (Chap. 8) focus on the teacher's role in enacting an exploratory approach to teaching mathematics as part of Project REASON: Mathematical Reasoning and Teacher Education. The research involves using tasks similar to the explorations and investigations contained in the Cyprus Mathematics Textbooks and has a focus on challenging learners to engage in mathematical reasoning. To do this, they characterise the pedagogic actions that teachers can use to guide, structure and challenge learners' mathematical reasoning during what they see as the three phases of an exploratory approach in mathematics lessons: launching the task, learners' autonomous work and whole class discussion. These are illustrated using the examples of two contrasting lessons.

Ponte et al.'s (Chap. 8) analysis highlights the complexity of the teacher's role in exploratory work, particularly in balancing the need to provide sufficient guidance for learners without reducing the challenge of mathematics. The whole class discussion is perhaps the most challenging aspect for teachers because it requires teachers to 'respond in the moment' (Mason, 2015) and at times improvise (McIvor, 2022/Forthcoming). Ponte et al.'s (Chap. 8) analysis shows how teachers can prepare for this in the learners' autonomous work phase of the lesson, but they emphasise that this is dependent on 'careful planning … in order to foresee possible students' strategies and difficulties and plan ways to deal with them as they arise in the classroom' (p. 166). In order to do this, teachers need support to develop an understanding of challenging, but achievable, learning trajectories (Sztajn et al., 2012).

Ponte et al.'s (Chap. 8) research study involves a professional development programme. Unfortunately, due to obvious space constraints, they do not describe the design of this programme, aside from noting that the planning for teachers' exploratory lessons was collaborative. As I have argued throughout this commentary, a key issue is how to enable and support teachers in enacting challenging mathematics such as Ponte et al.'s exploratory approach. It would seem from the empirical examples presented that the project has had some success in doing this. Hence, I look forward to future work from this team examining how they enabled (or facilitated) this changed practice.

Lloyd and Murphy (Chap. 9) examine *Quality Talk*, an intervention designed to enable teachers to provide opportunities for, and facilitate, mathematical argumentation in elementary classrooms. Quality Talk is a relatively well-established intervention in science (Murphy et al., 2018a) and literacy classrooms (Murphy et al., 2018b) that has shown some promise, albeit on a relatively small scale. Over the past decade, the theoretical and empirical research on language and mathematics has made substantial progress in moving from simply describing classrooms to examining how to intervene pedagogically. See, for example, Erath et al.'s (2021) paper outlining theoretically and empirically grounded design principles for language-enhanced mathematics instruction. Lloyd and Murphy's (Chap. 9) focus on mathematical argumentation in the context of productive mathematical discourse is located in a field of research with much potential.

Quality Talk builds on Toulmin's (1969) model of argumentation. Lloyd and Murphy devote a great deal of space to describing the aims and approach of the intervention, including eight key discourse elements that teachers can use productively to scaffold mathematical argument. This is an impressive programme of work. The (necessarily brief) empirical examples in Lloyd and Murphy's chapter certainly provide evidence of prospective elementary teachers using the structure of claim, reasoning and evidence much more explicitly and of the relationship to the scaffolding using the discourse elements when presented with a challenging task. However, the eight discourse elements for Quality Talk in mathematics appear to be very similar to those described in the Quality Talk intervention targeted at increasing critical engagement with texts in literacy (Murphy et al., 2018b). This is somewhat surprising given their stated aim to develop mathematical argumentation as a 'central *disciplinary* practice' (p. 169). There is a large body of work examining how the argument in mathematics is a distinctive disciplinary practice and that this practice is inextricably linked to the content and norms of the discipline. See, for example, Lampert's (1990) use of Lakatos's (1976) proofs and refutations and Polya's (1954) moral qualities of doing mathematics in facilitating challenging discussions amongst learners.

Like Ponte et al., Lloyd and Murphy's (Chaps. 8 and 9) study highlights the complexities faced by teachers in facilitating productive mathematical talk. The study provides evidence that Quality Talk has some promise in contexts where the teachers have direct contact with the academic developers. It will be interesting to see how the intervention develops over time and how the approach can be implemented more widely.

10.3 Understanding What Constitutes an Appropriately Challenging Curriculum for Learners

The sixth chapter in this section addresses the curriculum. Verschaffel et al. (Chap. 3) report on a large-scale longitudinal study aimed at demonstrating the potential for a broad and challenging early mathematics curriculum. They argue that the mathematics curriculum offered to the majority of young learners is narrow and unchallenging, focused almost entirely on whole numbers, counting and basic arithmetic. Building on a learning trajectories approach (e.g., Clements & Samara, 2014), they investigate young children's development through carefully designed interventions in four domains: mathematical patterns and structure; computational estimation; proportional reasoning and probabilistic reasoning.

This is an impressive study that provides significant insights into children's development and the potential for, and developmental importance of, children at all attainment levels engaging with topics often considered to be too challenging by policymakers and (some) educators. They show, for example, that many young children can handle problems in all four domains and begin to provide much-needed evidence of the kinds of interventions that can support young children's development, an area where recent reviews have indicated that research is much needed (e.g., Hodgen et al., 2020; Simms et al., 2019). Echoing Lloyd and Murphy's (Chap. 9) focus on the importance of quality talk in mathematics, Verschaffel et al. (Chap. 3) show 'that specific mathematical vocabulary related to early proportional reasoning in the first year of elementary school is a unique predictor for proportional reasoning abilities in the second year of elementary school over and above age, SES and general vocabulary' (p. 39). This further suggests that it would be productive to combine Lloyd and Murphy's (Chap. 9) argumentation with a focus on the specifics of mathematical vocabulary, in particular, vocabulary focused on mathematical relations and structure.

Verschaffel et al. (Chap. 3) conceive of children's mathematical development in terms of competencies, consisting of dispositions alongside cognitive abilities. Extending earlier work on numerosity (e.g., Rathé et al., 2016), dispositions are considered in terms of children's tendencies to attend to and focus on aspects of mathematics, such as the *spontaneous focusing on patterns* (SFOP). This focus on the affective in mathematics is welcome and important, although one wonders whether the concept of spontaneous focusing captures the entirety of what Kilpatrick et al. (2001) term a productive disposition, or the 'habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy' (p. 117). It would be interesting to examine the extent to which various aspects of spontaneous focusing on mathematics in very young children are related to, and predictive of, later dispositions such as valuing mathematics, mathematics self-efficacy or resilience in mathematics.

10.4 Conclusion

The chapters in this section represent a diverse range of research and interests all addressing different aspects of challenges in mathematics. They contain many insights into the nature of the challenge in mathematics and how to promote it. In doing so, they demonstrate the complexity of facilitating challenges in school mathematics. This is, of course, a worthy and important endeavour involving change not simply to the task that learners engage with but also to the teacher's role and crucially to the content and structure of the entire mathematics curriculum.

The diversity of approaches itself raises a challenge in how to align, and learn from, the many insights and results from the different approaches. This diversity has been one of the strengths of mathematics education as it has enabled the research community to generate potential strategies and interventions for promoting mathematical challenges for learners. As an increasingly mature academic discipline, it is becoming increasingly important to develop greater theoretical coherence amongst these diverse approaches (e.g., Bikner-Ahsbahs & Prediger, 2010), particularly in evaluating and implementing these approaches.

Finally, as Verschaffel et al., Christou et al., and Leikin et al. (Chaps. 3, 5, and 6) all argue, the mathematical challenge involves the affective in addition to the cognitive, encouraging learners to engage with the 'romance' of mathematics and to develop habits like spontaneous focusing on (and doing) mathematics. As I have emphasised throughout this commentary, such a change to mathematics teaching and learning will require considerable attention to the problem of implementation. I look forward to reading further work from these authors and their studies that addresses the problem of challenge in mathematics and, in particular, how to support teachers and others in facilitating this.

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Part II
Kinds and Variation of Mathematically Challenging Tasks

Editor
Rina Zazkis

Chapter 11
Introduction to Part II of Mathematical Challenges For All: Many Faces of Mathematical Challenge

Rina Zazkis

11.1 What Is a Mathematical Challenge?

What is a mathematical challenge? The first idea that comes to mind when considering this question is associating a mathematical challenge with problem-solving. It is agreed upon in the mathematics education community that what constitutes a problem, as related to mathematical problem solving, is relative to the solver. That is, what is a problem for one learner can be a standard exercise for a teacher or for a more advanced learner. The same can be said about a challenging mathematical problem. That is, the challenge is in the eye of the beholder. What is challenging for some may not be challenging for others, either because they have no knowledge of how to proceed or have no interest in engaging with the problem.

I spent my grade-5 year looking for a formula that generates prime numbers. This was in the "olden days" – before the internet, before cellphones, before handheld calculators. I challenged myself because my teacher told the class that there was no such formula (Of course, she meant no polynomial prime number generator, but this I understood only years later). I did not believe her. Yet, the problem of finding prime number generators did not present a challenge to my classmates, as they were not motivated to find a solution.

However, mathematical challenge extends beyond mathematical problem-solving. As such, rather than focusing exclusively on mathematical problems, I consider mathematical activity as a wide-reaching umbrella. Mathematical activity can be defined broadly to include any engagement that involves doing, learning or teaching mathematics. For teachers and teacher educators, in addition to

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problem-solving, engagement with mathematics includes aspects of teaching such as preparing lessons, designing tasks, choosing instructional examples, assessing student work or addressing student questions. The chapters in this part of "Mathematical challenge for all" describe multiple faces of challenging mathematical activities.

11.2 In the Chapters

The chapters in Part II present readers with a variety of studies. Each chapter brings to mind a multitude of related experiences and ideas. In what follows I describe the chapters and mention just one of these invoked personal encounters in each case.

Mason considers a mathematical challenge "as an indicator of someone's state within a situation with affective, cognitive, enactive, and other consequences. In other words, *challenge* depends on the current state of the psyche of individuals within the current social setting."

Mason outlines his personal relationship with mathematical challenge, describing responses to challenge as resisting, accepting and parking. He considers the ways in which psycho-social adherence accounts for an individual's responses to challenges. The power of his typology is that anyone engaged with mathematics can relate to the description; any reader will recall examples of accepting or resisting mathematical challenges, as well as the feeling of joy having faced a challenge successfully.

Parking a challenge is what I experienced reading Mason's chapter, as well as many others of his writing. The chapter, having elaborated on aspects of human psyche, proceeds with a variety of mathematical examples, where my immediate inclination is to stop reading and accept the challenge of seeking a solution. However, I find myself parking the challenge and returning to it after the term grades are submitted and dinner is ready.

A problem that I have recently parked is from http://www.gogeometry.com/school-college/5/p1494-parallelogram-midpoints-octagon-area.htm

In particular, Geometry Problem 1494 states: Given a parallelogram ABCD of area S with M, N, P, and Q midpoints of AB, BC, CD, and AD. Lines AN, AP, BP, BQ, CM, CQ, DM, and DN determine the octagon EFGHIJKL of area S_1. Prove that. $S_1 = \frac{7}{25}S$.

Acknowledging that affine transformations preserve the ratios of areas, I can embed the octagon in Cartesian coordinates and use "brute force" to prove the ratio of the areas. Simply recognizing this option may take the initial challenge away from the problem. However, the challenge still remains to find an elegant proof; such proof will likely be based on some insight related to the relationship among the sub-areas.

Leikin and Guberman focus on the notion of "insight", which is a sudden realization that leads to a solution. They introduce the distinction between *insight-allowing problems*, in which an insight may lead to an elegant or a "smart" solution that

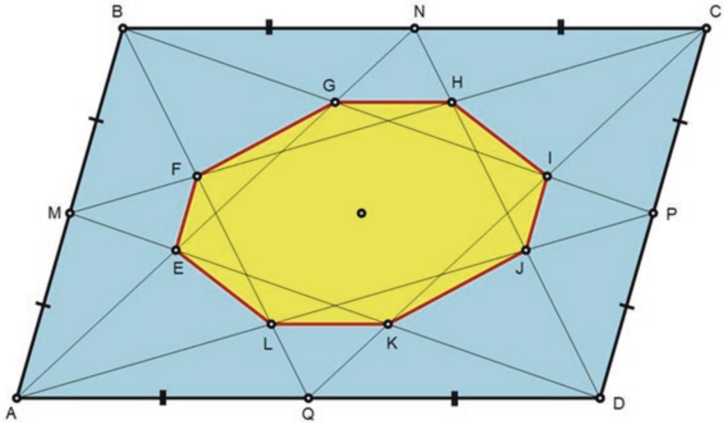

Fig. 11.1 Parallelogram and octagon areas

draws on ideas from a different area, and *insight-requiring problems* in which a solution cannot be found without an insight (According to this distinction, the problem in Fig. 11.1 fits in the former category). The authors present examples of problems of each type and then hypothesize that solving insight-requiring problems is more challenging for students, but allows for demonstrating higher creativity. They share research results that support this hypothesis.

When I scroll through problems found in a variety of problem-solving collections in different outlets, I find a variety of insight-allowing problems, but insight-requiring problems are rare. However, a handful of insight-requiring problems is found in a collection of so-called "Jewish problems",[1] also referred to as "killer problems" or "coffins." These are problems that were used in the 1970s and 1980s in the USSR in oral entry exams at Moscow State University, among others, to fail Jewish applicants and restrict their access to higher education (Sriraman & Dikman, 2017).

Putting the evil history of these collections aside – if it only were possible – I am fascinated with the ingenuity in the design of these problems. The problems do not appear over-complicated, but each solution involves some "trick" (for example, a particular choice of representation or a particular substitution) which is very different from the conventional approaches that come to mind when one solves problems in the same domain. Using the trick – or the required insight – the solution appears very short and straightforward and can be even argued to be "simple." But without the insight, a solution is beyond reach. As an example, determine which is larger, $\log_2 3$ or $\log_3 5$-- without any calculating devices, of course. Or, how many digits are there in the number 125^{100}? The reference in the footnote includes solutions and hints.

Sinclair and Ferrara describe how first graders engage in the activity of distributing 12 or 18 candies among six children using *TouchCounts*, a multi-touch

[1] https://arxiv.org/pdf/1110.1556.pdf

application on iPads. The mathematical challenge that students face in Sinclair and Ferrara's contribution can be seen as the students' engagement with a mathematical problem whose solution is not familiar to them.

Analysing students' work, Sinclair and Ferrara reframe the mathematical activity of students working with *TouchCounts* from the inclusive socio-material perspective. They use this activity to exemplify "the socio-material dimensions of mathematical activity, in line with the inclusive materialist approach of de Freitas and Sinclair (2014)." The chapter proposes a reconceptualization of the conditions for a task to be considered a mathematical challenge, originally described by Leikin (2014), so that they are applicable in cases of digital technology. The reframed conditions are described from the socio-material perspective in which the children-problem-solvers and the digital application form a human/non-human socio-material system. Sinclair and Ferrara note that their "reframing enables researchers to see tasks as being materially and temporally in relation with prior mathematical activity, rather than isolated problem-solving opportunities."

While this reframing claim makes perfect sense, it makes me wonder how my personal engagement with a challenging (for me) problem, showed to me by a colleague during a boring conference lecture, can be described from an inclusive materialism perspective. The problem considered "tridians", lines connecting a vertex of a triangle to a point that marks 1/3 of the opposite side. The task was to prove the relationship between the areas of the "big" triangle and a "small triangle in the middle" (that is, Area (ΔABC): Area (ΔDEF)) (Fig. 11.2).

The relationship can be easily confirmed with Dynamic Geometry software and proved using vector algebra or affine coordinates. But the presented challenge was to prove the relationship using only tools of Euclidean geometry. I pursued this challenge after having it "parked" for a while, using multiple sketches with Dynamic Geometry, filling the trash basket with multiple files, and finding additional ratios on the way. However, considering the reframing of mathematical activity suggested by Sinclair and Ferrara, I keep wondering: was this mathematical challenge an isolated activity or was I an oblivious participant in some socio-material system?

Applebaum and Zazkis discuss the seemingly simple problem of placing digits in a frame □□□×□□ to reach the maximal product. That is, without using a

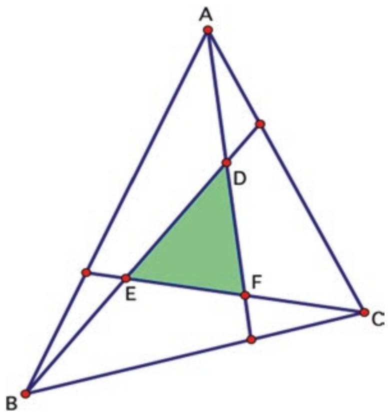

Fig. 11.2 Tridians in a triangle

calculator, the task presented to a group of teachers was to use the digits 1,2,3,4 and 5 (each digit once) in the multiplication of a 3-digit by a 2-digit number to get the largest product.

On the one hand, the solution can be effortlessly confirmed by computation. On the other hand, most people found the solution counterintuitive, and when the task was approached by a "guess and check" method, the solution was not usually found on the first guess. This brought up a challenge – for teachers engaged with the task – of explaining the unexpected result without simply pointing to it. The authors analyzed this challenge in terms of the intellectual need for causality (Harel, 2013), that is, seeking a reason for the phenomenon.

Discussion with the teachers resulted in further challenges for the authors: first to provide a better (or at least an alternative) explanation, and then to generalize the results from considering digits 1,2,3,4,5 to considering any 5 digits $a < b < c < d < e$. The chapter by Applebaum and Zazkis highlights a challenge that teachers face, which is not just providing explanations, but seeking explanations that help face and confront initial, often misleading, intuition. They also provide a visual interpretation of a related task that may help in confronting misleading intuitions.

In my memory, the most powerful visualization that helps with reframing a misleading intuition is provided in Papert's *Mindstorms*. Papert (1980) described the problem, classical by now, as follows:

> Imagine a string around the circumference of the earth, which for this purpose we shall consider to be a perfectly smooth sphere, four thousand miles in radius. Someone makes a proposal to place a string on six-foot-high poles. Obviously, this implies that the string will have to be longer. A discussion arises about how much longer it would have to be. Most people who have been through high school know how to calculate the answer. […]
>
> $$2\pi(R+h)-2\pi R=2\pi h$$
>
> But the challenge here is to "intuit an approximate answer rather than to calculate an exact one. (p. 146)

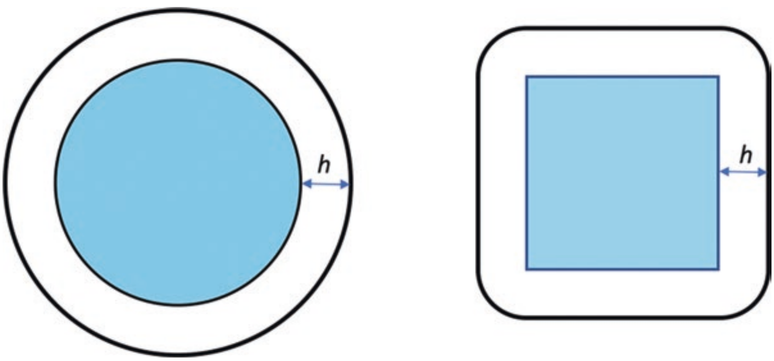

Fig. 11.3 A string around a circle vs. a string around a square

Fig. 11.4 Proof without words

Obviously, the algebraic elaboration is convincing, as is the calculation of the products in the Applebaum and Zazkis chapter. But the convincing power, and thus a more powerful sense of causality, is found in a similar problem: considering a rope around a square, rather than around a circle (Fig. 11.3).

In a square case, it is obvious that the additional amount of string needed is rather small, adjusting only at the corners. This is strongly related to the notion of *seeing* explored by Vale and Barbosa, to which we turn next.

Vale and Barbosa establish a connection between visualization and mathematical challenge when engaging prospective teachers with Multiple Solution Tasks (MST). They discuss multiple uses for visualization – in investigating, in getting a sense of a relationship, in proving – and demonstrate how the challenge of visualization can help in giving meaning to analytic approaches. They advocate for a strategy of "seeing" – as a strategy of thought that involves the visual perception of mathematical objects that is blended with knowledge and past experiences (Vale, Pimentel and Barbosa, 2018).

A beautiful collection of problems is found in this chapter, where several problems are accompanied by multiple solutions presented by prospective teachers. As a reader, I am challenged to find yet another solution. Vale and Barbosa express a hope that presenting teachers with MST tasks and challenging solvers to find visual solutions, among others, will enrich their repertoire and help them use similar strategies in their classrooms.

The notion of *seeing* discussed in this chapter inspired my recent lesson with teachers, in which I present a diagram to students and ask – what do you see?

In fact, the diagram in Fig. 11.4 is a "proof without words" for an infamous inequality, $x + \frac{1}{x} \geq 2$, which students are invited to find.

Biza and Nardi describe a course – *The teaching and learning of mathematics* – in which they introduce undergraduate mathematics students to mathematics education research via MathTASK, a program explicitly developed for this purpose. Tasks in this course present situations that may appear in a mathematics classroom; they include either mathematical or logical errors or focus on disagreement among interlocutors. In their responses, the participants are expected to analyze the situations mathematically and describe their hypothetical pedagogical responses to students. Furthermore, participants are expected to engage with mathematics education literature and use appropriate theory and methodology. Researchers analyzed participants' responses for clarity, coherence, consistency, specificity and the use of mathematics education theoretical constructs and terminology.

In my view, the presented tasks can provide a mathematical and pedagogical challenge not only to undergraduates, but also to practicing teachers, graduate students, and mathematics educators. Biza and Nardi refer to their course as a "bootcamp experience for newcomers into RME discourse". They also claim that their activities "welcome mathematics undergraduates into RME in a manner that balances engagement with mathematics and mathematics education discourses productively."

While not stated explicitly, I infer from the presented excerpts that it is the engagement with the mathematics education literature that appeared to be the most challenging for undergraduates. But maybe in making this suggestion I am simply reflecting on my personal experience. In a related study, inspired by Nardi (2015) but much more limited in scope, we asked practicing teachers of mathematics in the beginning of their Master's program to read and reflect on several research reports in mathematics education (Rouleau et al., 2019). It was notable that teachers experienced difficulty discussing features of the presented research and focused on the relevance of the readings to their pedagogical practice. It was concluded that "making use of their mathematical and pedagogical knowledge could be viewed as an appreciation of the challenge that critical consumption of a research article entails." (p. 57).

As such, an early start with undergraduates, as demonstrated by Biza and Nardi, appears to be a preferable timing for introducing novices to mathematics education research. Biza and Nardi discuss the reification of both mathematical and pedagogical discourse by the participants in their study. While not mentioned explicitly, I detect in the reported data a reification of research discourse, one aspect of the beginning of the nuanced transition from a mathematics learner to a mathematics education researcher.

Wasserman discusses the challenge that teachers face in connecting advanced and secondary mathematics. In a particular example, Wasserman articulates the connection between two ideas: a function, a notion familiar from school, and a binary operation, a notion usually introduced in an Abstract Algebra course. He suggests that "The mathematical challenge was not necessarily "learning" something new; it was in "re-seeing" something familiar from a new perspective."

Wasserman describes two ways of connecting advanced and secondary mathematics: generalization and instantiation. A generalization connection is when concepts of advanced mathematics generalise concepts of school mathematics. This happens, for example, when familiar Cartesian coordinates become an example of a general affine coordinate system. An instantiation connection is when a concept from advanced mathematics serves as an example of a concept familiar from secondary mathematics.

Wasserman frames the discussion of binary operations as an instantiation of the function concept, which appears in secondary mathematics curriculum. He describes the struggle of teachers when exploring this connection and the ways of facing this challenge in pedagogically powerful activity.

However, in order to recognise a binary operation as function, we need first to expand the school idea of a function. As such, I would like to offer a different perspective on such a connection between binary operation and school function, which is neither instantiation nor generalisation. While indeed the concept of a function appears in secondary mathematics, we find there a particular view of functions, often defined as an expression $y = f(x)$ that specifies a relationship between input (or independent variable) and output (dependent variable). In advanced mathematics a function is viewed as a set of ordered pairs, specifically, a function f from A to B uniquely associates $a \in A$ with $f(a) \in B$.

So rather than seeing a binary operation as an instance of a function familiar from secondary mathematics, in my view, both school functions and binary operations become examples of an advanced-function concept. In my experience, it is not obvious, and so can present a mathematical challenge, to place previously learned disjoint concepts under the same umbrella. For example, in a study related to teachers' interpretations of exponent (−1) (Zazkis & Kontorovich, 2016) we wondered how teachers connect the notion of inverse function (denoted f^{-1}) with that of reciprocal (as in 3^{-1}). It was noted that some participants rejected the overarching notion of "inverse with respect to some operation" and insisted on differentiating between the notions of function inverse and multiplicative inverse.

11.3 Conclusion

The chapters in Part 2 spread across mathematical content and ways of engagement. There are descriptions of problem-solving work with young learners using a digital application (Sinclair and Ferrara) and a reflection on a career-long personal engagement with challenging mathematics (Mason). There are accounts of work with undergraduate students (Biza and Nardi, Leikin and Guberman), with prospective teachers (Vale and Barbosa), with practicing teachers (Applebaum and Zazkis) and with both prospective and practicing teachers (Wasserman).

I started with the claim of considering mathematical challenges broadly, extending the focus from mathematical problems to any activity of doing, learning and teaching mathematics. These chapters extend these considerations even further. This includes connecting advanced and secondary mathematics (Wasserman), seeking explanations for counterintuitive results (Applebaum and Zazkis), seeking visual solutions in multiple solutions tasks (Vale and Barbosa) and utilizing research in mathematics education in responding to classroom scenarios (Biza and Nardi).

This is definitely only a partial list of activities in which a mathematical challenge is found, should we decide to embrace it.

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Chapter 12
Probing Beneath the Surface of Resisting and Accepting Challenges in the Mathematics Classroom

John Mason

12.1 Introduction

What interests me most is the lived experience of thinking, doing, learning, and teaching mathematics. In taking up the challenge to write about mathematical challenge I have interrogated my own experience and used this to probe beneath the surface of common reactions to being challenged mathematically.

Ten years after Bill Brookes (1976) suggested that something is a problem only when a person experiences it *as* a problem, Christiansen and Walther (1986), following Vygotsky (1978), distinguished between a *task* as what students are offered or inveigled to undertake, and *activity* as what happens as they attempt to carry out their interpretation of the task. Combining these, some thing or some situation can usefully be described as a 'problem' only when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in sense-making activity.

The notion of mathematical challenge has an inbuilt ambiguity. On the one hand, someone can challenge me to resolve a problem. On the other hand the challenge may be taken up and experienced *as* a challenge, or it may be resisted in some way. In this paper, the focus is on the latter so that a mathematical task is considered to be a challenge only when someone experiences a state of 'feeling challenged' and takes action to try to meet that perceived challenge. I shall use the word *challenge* in this sense, not as a description of qualities of any particular stimulus or prompt but as an indicator of someone's state within a situation with affective, cognitive, enactive, and other consequences. In other words, *challenge* depends on the current state of the psyche of individuals within the current social setting.

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The notion of *challenge* in relation to learning, doing, and teaching mathematics is of vital importance: if a mathematical task is thought to be too challenging, learners are likely to resist or even reject it, while if it seems insufficiently challenging, learners can become de-motivated and-or complacent, which is itself a form of resisting personal investment. Something perceived as a routine exercise rather than as a challenge to their powers is likely to reinforce a separation between schooling and living. Furthermore, what is a reasonable challenge to some may be routine to others, yet out of reach for still others, and this can change with different circumstances and at different times.

I am interested in the constellation of conditions in which learners might accept and take on a challenge, in which they might (learn to) park a challenge, and in which they might resist, defer, or reject a challenge. An important aspect of the psyche in this regard is trust, whether in the source of the challenge or in themselves, in their self-confidence (literally trust-in-self), which of course may sometimes be inflated or deflated inappropriately.

12.2 Human Psyche

In order to probe beneath the surface of mathematical challenge, I make use of discourses which have to do with both distinctions between and coordination among aspects of the human psyche. It is important to note, however, that these discourses are partial. They make no claim to completeness, no claim that they are either necessary or universal. They are based on observations which, being made by human beings with history and predilections, are necessarily biased. Their potential validity resides not in statistical studies but in whether their use is found to inform personal action in the future.

12.2.1 Six Aspects of the Human Psyche

Traditional (Western) psychology has focused largely on three aspects of the human psyche: *enaction*, *affect* and *cognition,* with only scant recognition of *attention*, of the role of *will*, and of the presence or absence of an inner *witness* or monitor. Each of these aspects has both an experienced and a description version.

Thus *enaction* is what is experienced when an action is initiated and continued, whereas *behaviour* is what others describe when reporting observed activity. For example, I find myself constructing and then working through an example (action) that the observer describes as specialising, even though I am not aware of specialising as such.

Similarly, *affect* points to the experience of changes in physiological conditions of the individual such as perspiration, change in pulse or breathing, etc. and may be distinguished from *emotions* which are descriptions of imagined states such as anger, fear, excitement, etc. (Mandler, 1989; Barrett, 2017). Notice how affect and emotion have significant physiological and cognitive components. For example, I find myself disinclined to work at a particular task, which an observer might describe as resisting or rejecting the challenge, and might confuse with my observed behaviour of not-engaging or working slowly.

Again, *cognition* can be used to refer to an act of mentation (having ideas popping into consciousness, such as words to say; changes in the oxygen levels in relevant parts of the brain, …), while *intellect* can be used to describe the effects of such mentation such as thinking, pondering, considering, problem-solving, etc.

Of course, descriptions cross over between these components: someone observed staring out a window may be interpreted as thinking, or as resisting activity due to a surge in affect (which they interpret as an emotion), perhaps blocking the possibility of enacting some action, or obscuring a lack of any available action.

The descriptive terms have the effect of generalising, labelling habitual patterns of action, emotion and intellect, attributed to the individual (psychologically) and to a group of people (sociologically). Such descriptions become fossilised, obscuring subtle differences which could otherwise have had different pedagogical implications (Mason, 1989).

In addition to the usual trio of aspects, it seems clear that *attention*, both what is attended to and how it is attended to, plays a vital role in mathematical thinking, and again there is a difference between what is experienced and how that experience is described (Mason, 1989). In chapter XI of his *Principles of Psychology,* William James (1890) proposed that “My experience is what I agree to attend to” (Green website). This begs the question of what constitutes the “I” that does the agreeing. My own observations suggest that attention is the centre of experience; its scope and range, breadth, focus and locus *are* the “I” that claims to be the subject of the predicates that describe my various states and actions. It is not surprising therefore that it is very difficult to observe my own attention. This is the role of the inner *witness*.

The inner *witness* is the voice that suddenly asks “Why are we doing this?”, or “Isn’t there a better way?”. It observes but does not act. It alone is able to observe the locus and focus, range, scope, and intensity of attention, and how attention is functioning mathematically (Mason, 2001). It has been referred to as an *inner monitor* or *inner executive* (Schoenfeld, 1985), and its recognition as part of the psyche has historical roots in a stanza in the Rg Veda (Bennett, 1943):

> Two birds, close yoked companions, both clasp the self-same tree;
> One eats of the sweet fruit, the other looks on without eating.

The witness is the bird that observes without eating, without being caught up in the action.

Human *will* is another elusive notion. Usually expressed in terms of *will-power*, it is used, for example, to describe someone sustaining activity in the face of opposition or difficulty, but this is always in relation to an observer’s expectation and has an affective component as well. Here it is used to refer to summoning or organising of energies, perhaps to persevere, perhaps to change direction, to pause or park activity. It is associated with the taking of initiative, of initiating some action. It feels like a releasing and channelling of the energy of intention and desire, of precipitated action, of insight, so that attention is sustained and focussed, despite distractions.

It is often assumed that actions are initiated by intellect-cognition, although sometimes it is acknowledged that they may be triggered through affect, or even enaction. Norretranders (1998) summarised neurological studies which suggest that the common assumption that conscious cognition is in control of actions is a *User Illusion*. Rather, many if not most actions are actually re-actions based on previously developed habits, previously coordinated adherences amongst the aspects of the psyche making predictions based on past experience (Nave et al., 2020). These habits are not simply actions, but repeated patterns of action, affect, thought, propensity to attend to certain things in certain ways, exercise of will and characteristic observations made by the inner witness. Over time these patterns of interactions become adherences which manifest themselves as micro-selves, personalities or distinctive selves.

12.2.2 Initiating Action

One of the ways in which action is initiated is illustrated by the cliché that, "to a child with a hammer, the world looks like nails". In other words, when a new tool becomes available, there is a tendency to use it everywhere. For example, when a new word is encountered, it is often used initially rather more broadly than most people are accustomed to. Over time, its use settles and, judging from the use, the meaning contracts. So too with other tools. Mathematicians do the same: upon encountering a fresh way of proving something, they are likely to try using that same or similar approach in the near future. The extent to which this happens will be influenced by the self-confidence and the vibrancy of whatever psycho-social habits are dominant at the time.

Using a recently acquired tool almost indiscriminately is but one instance of action which is enacted without reference to cognition. Terms such as 'habits below the level of consciousness', 'unformulated action', 'theorems-in-action', 'tacit knowing', and even 'poetic knowing' have been used similarly: see Mason and Johnston-Wilder (2004, pp. 298–291) for a partial trail. Kahneman and Frederick (2002) and Kahneman (2012) drew on the ideas of Wason and Evans (1974) concerning Dual Process Theory. Kahneman's elaboration was based on experiments which suggested that the human psyche has two different 'systems': System 1 (S1) is immediate, reactive, enacted by the musculature, and often associated with intuition (or is it habit?), while System 2 (S2) involves consideration (literally *sitting with*) by the conscious cognitive apparatus. Norretranders (1998), and others such as Mandler (1989) point out, again in alignment with more ancient knowledge of the human psyche such as presented in the Upanishads or the Bhagavad Gita, that under stress the body reacts first, the emotions second, and cognition a slow third.

Liam Hudson (1968) used the notion of *frames of mind* to express something akin to coordinated adherences, in much the same way as Marvin Minsky (1975) who, being a computer scientist, thought in terms of actions being enacted by computer-like programmes which have default parameters to be used in place of absent information. As soon as all the parameters have 'values', the action is initiated, without requiring contribution, consideration, or permission from cognition, which aligns with the notion of the *predictive brain* (Nave et al., 2020). This provides one explanation for why human beings so often act without being consciously aware of acting and is captured nicely in a traditional teaching story:

> A horse suddenly came galloping quickly down the road. It seemed as though the rider had somewhere important to go. A bystander shouted out, "Where are you going?" and the rider on the horse replied, "I don't know! Ask the horse!" (Hanh, 1986)

The horses of the human psyche, the emotions, all too readily carry us away! Being carried away is sometimes seen as a positive state, associated with letting go of what has previously been shackling so as to enable a state of *flow* (Csikszentmihalyi, 1997). Certainly, it is relatively easy to be carried away sufficiently so as to lose the sense of time and place and to be so caught up that it is difficult to pay attention to the focus and nature of one's own attention, to the powers one is using, or to mathematical themes which are being played out. However, sometimes it *is* possible to be aware of, to 'be present to' these aspects of mathematical activity, however subconsciously or even consciously. So flow has both a 'carried away' version and a 'being present to' version.

Although it is clear from self-observation that action is often initiated spontaneously, as it were, its origins sometimes lie in emotions which may be triggered metonymically through idiosyncratic association, making certain actions available; sometimes they lie in the arising of a thought and sometimes they lie in a stimulated shift of attention. It is useful therefore to extend the processes idea in line with the psychology articulated by, among others, Ouspensky (1950), to include an intermediate System 1.5 (affective, later narrated as emotion) and a further System 3 which concerns access to deeper or higher energy (Mason & Metz, 2017). System 3 is the source of sudden insight which is often described as coming from 'the muse', and attributed to 'creativity'. It is closely aligned with the Gestalt notion of *form* (Zwicky, 2019), and is experienced in brief moments, which are almost always immediately overlaid by thoughts, emotions and activity. It is accessed through periods of relaxation of tensions in the body, emotions and thoughts, in what is sometimes referred to as fallow periods, or centredness. Waiting, or gazing, seen as one way of attending to something, may be most effectively thought about as coordination of enaction, affect and cognition, with corresponding things to attend to and ways of attending, which leave the person open to unexpected possibilities, sometimes leading to a momentary experience of S3.

Neville (1989) describes a variety of educational initiatives with psychological backing based on the notion that learning is most efficiently undertaken not by consciously focused attention but by peripheral attention. Gattegno (1970, 1987) used the same principle to suggest that in order to internalise an action so that it is available to be enacted, it is best to provoke the action peripherally, as a side-line of some other action. He called the process of sensitising oneself to the possibility of an action, that is, recognising contexts in which it might be appropriate, and internalising that action, as *educating awareness*. Hence the role of mathematical exploration is in order to create conditions in which learners spontaneously rehearse some procedure so as to make sense of some apparently unrelated phenomenon. This aligns with the Eastern teaching method illustrated several times in the movie *Karate Kid* (1984), in which the student is inveigled into rehearsing an action while attending to something else entirely.

The four 'systems' S1, S1.5, S2 and S3 describe four different ways in which action can be initiated and actively pursued, making use of combinations of or coordinated aspects of the psyche. Emotions provide the energy (cf. the etymological roots of *emotion*), but attention is the core of presence.

12.2.3 Psycho-Social-Coordinations

The various aspects or components of the psyche do not operate in isolation. It seems that particular emotions become associated with and amplify particular thoughts and together these energise certain actions over others. Particular thoughts and emotions stress certain behaviours, and the will to continue on any path of action, emotion, thought and attention is influenced by perceptions about actual and likely success (or failure) which in turn are influenced by current emotions (Skemp, 1979). Things deemed worthy of attention, and particular ways of attending to them, become salient, even to the extent of blocking out other possibilities. The inner witness observes the sorts of things it has become accustomed to observing and issues alerts which have become part of the adherence.

Co-ordinations can be self-amplifying and self-sustaining, preserving the psychological state, and in turn may be amplified, sustained or ameliorated by the social milieu. These stimulate characteristic actions, emotions, dispositions, patterns of thought, foci and functioning of attention, activated willpower and even types of observations made by the witness. Over time a repeated coordination 'takes hold'. It becomes an adherence manifested as a habit, hence the notion of *psycho-social coordinated adherences*. They are like *micro-identities* (Varela, 1999) or *multiple selves* (Bennett, 1964; Hudson, 1968; Minsky, 1986; Hanson, 1986; Kahn, 1983; Davies & Harré, 1990; Eakin, 1999; Lester, 2012). Since many people see themselves as trying to locate their 'true self' and reject out of hand the notion of multiple selves, the language of *coordinated adherences* seems to be more generally acceptable.

Over time, coordinations can become stable, so that characteristic flows of energy adhere to each other to form habits not just of behaviour, but of psycho-social states. The adjective *psycho-social* emphasises that coordinations are influenced by perceived social conditions as well as by psychological states, and so although adherences are in the psyche, which adherence becomes dominant at any particular time, and how it became coordinated in the first place, is influenced by the social situation and relationships as perceived and experienced by the psyche.

An example of this is a collection of socio-mathematical norms which are enacted by teachers and then may be picked up by learners (Yakel & Cobb, 1996). Some learners may reject them out of hand, while others may take them up with alacrity, and still others may gradually become inured to them. Another example can be found in the report of Brown and Coles (2000) in which students picked up the practice of considering what is the same and what is different about two or more mathematical 'objects', and then began to initiate this action for themselves.

I noticed an example recently when a friend showed me an intriguing book consisting of drawings of geometrical configurations, using both solid and dashed lines in various places (Akopyan, 2011). Each diagram can be taken as a challenge to discern and articulate a property which relates the dashed (construction) lines to the solid lines. When I subsequently received my own copy I was initially entranced, but then overcome with lethargy and a sense of burden. My witness recognised this state as one which I have experienced with other problem collections. The immense potential, the scale of commitment implied, and the fear of not being able to work them all out combine to stifle action and lead me to reject the challenge, at least for a time. A psycho-social adherence is brought to the surface which finds it all too much and saps away any initial energy and disposition to engage.

Carol Dweck (2000) is well known for her investigations of how background assumptions can establish adherences which have their own narrative (eg. "I resist the unfamiliar because I associate it with failure") but how inner incantations can be replaced and the unfamiliar embraced. She reports a lifetime of work developing ways to assist people to release themselves from habitual patterns based on perceiving failure as inbuilt rather than as happenstance (See also Neville, 1989).

12.3 Resisting, Accepting and Parking Challenges

Responses to challenge are many and various. They cover a spectrum from outright rejection to enthusiastic take-up. Observing learners respond to challenges set by their teachers, it may be useful to think broadly in terms of *resistance,* which extends from outright rejection through to grudging compliance, transmuting into *acceptance* which extends from grudging compliance through to enthusiastic take-up. At almost any stage there is the possibility of *deflecting* or *parking* the challenge, with intentions varying from long-term parking, amounting to rejection, to waiting for fresh ideas, further resources, or sufficient time to direct attention to it.

Resisting and accepting, deflecting and parking are only superficial descriptions by observers of learner behaviour. However, bearing in mind the proposal by Maturana (1988) that "everything said, is said by an observer", even self-report involves observation whose quality depends markedly on the presence and inner separation or objectivity of the witness. The claim here is that what is being observed is likely to be a coordination of various aspects of the learner psyche, and likely to contribute to the creation of adherence as the basis of a habit. These in turn activate one or other *systems*, whether S1, S1.5, S2 or S3.

For example, learners who, on being given a task, wait until the teacher comes round so they can ask for specific guidance on 'what are we supposed to do?' are in danger of developing a habit which will diminish, even stifle opportunities in the future. It is ever so easy for an initial resistance to develop into a reluctance, and then into a rejection. Their S1 or S1.5 triggers inaction, and rather than shift into S2, they remain inactive until someone tells them what to do. Unfortunately, teaching assistants are often all too ready to meet this demand. The tension between telling and prompting is captured by the notion of the *didactic tension* (Brousseau, 1984 p. 110; Mason & Johnston-Wilder, 2004 p. 82) which can be described as

> the more clearly and specifically the teacher indicates the behaviour expected from the learner, the easier it is for the learner to enact that behaviour without actually generating it for and from themselves, and so without the likelihood of internalising that action.

Bob Davis (1984) presented this to students as an ethical dilemma: would they rather be told, or be allowed to search for something for themselves?

As another example, there are many learners who, on being set a task, immediately enact the first action that becomes available. They may in retrospect account for it on the grounds of 'getting it over with as soon as possible' or as an outcome of their eagerness to learn or engage, but in either case, reacting immediately can become a habit, coordination of adherences which waist time and sometimes obscure access to a more fruitful approach.

12.3.1 Recognising Challenge

The first question is how the psycho-social system recognises challenge (as distinct from simply a task). Usually, there are somatic changes in pulse, breathing and perspiration, often arising from an increase in adrenalin, triggering emotions such as fight-flight or fear-fancy, with concomitant coordinated adherences in the rest of the psyche. Unfortunately, these coordinated adherences are often inappropriate and over-rated and may block other adherences from coming into play.

Somatic changes need not be interpreted in such drastic ways. Adrenalin flow can be perceived as stimulation and excitement, leading to a sharpening of the senses. I find that one situation in which I become aware of the mathematical challenge is when something disturbs my current adherence of enaction, affect and cognition, when something shifts or alters what I am attending to, or how I am attending to it, when my will power feels tested, when my inner witness signals that something is awry. Often it can be something quite simple but which becomes fodder to my propensity to try to generalise, to place a result in a wider context. Only then am I aware of feeling challenged. However, that 'feeling' is not simply cognitive or affective in nature. It comes from coordination of cross-linked habits between the various 'components' of my psyche which adhere to, and consequently both feed and limit each other.

For example, encountering the idea of constructing a decimal number by writing down the digits sequentially (so 0.12345678910111213141516 17…) immediately raises the question for me as to how to tell whether it is rational or irrational. What constitutes a convincing justification? What about other sequences? And that is only a starting point. Suppose only one new decimal place is allocated to each numeral so that there are carries to the left which may affect earlier decimal places (e.g. 0.1234567901234…), or perhaps two or three new decimal places are allocated; what if the number of allocated decimal places changes in some systematic fashion? What if some other sequence is used, such as triangular numbers or Fibonacci numbers?

Having an action become available is essential. For example, the action of presenting such strings in terms of powers of 10, followed by, in some cases, recognising a geometric series, provides a method of dealing with many of the questions posed above, and for many different sequences. I immediately want to start exploring, which is an imprecise way of saying that my attention shifted, actions became available and I recognised a desire to find out what is going on. Without any sense of exercising will, but rather of the will being dragged along, one of my 'explorer' adherences took over. This happened not once or twice but several times with the same idea on different occasions.

Furthermore, I notice (my inner witness notices and brings to cognition) a resonance with two questions posed by David Fowler (1985a, b):

> Guess the length of the period of the square of 0.001 001 001 … . Then and only then, work out the answer.
>
> Use a procedure for multiplying decimal numbers to calculate the first significant digit of 1.2222… × 0.818181…

The associated lesson is that arithmetic with infinite decimals can be tricky! Expressing repeating decimals as fractions may be necessary in order to be certain.

Mathematically, I also perceive myself to be challenged when there is some situation or assertion that I cannot readily explain or justify, yet which appeals to my affect by striking me as surprising or unexpected. This often happens with geometrical configurations. A good example for me arose by taking a convex quadrilateral and joining each vertex to the midpoint of the next-but-one edge taken clockwise. An inner quadrilateral is formed and in dynamic geometry software, it often appears to have an area of one-fifth of the area of the original quadrilateral (Mason & Zazkis, 2019). It turns out that this is due to rounding errors … but what in fact is the case? And what happens when midpoints are replaced by some other construction?

A sense of being challenged can also take the form of something which alters the way I perceive or attend to something, which again needs explaining or justifying. I particularly enjoy situations in which there are dual perceptions to be reconciled, for example thinking of chords of functions as made up of families in each of which the chords all have a fixed endpoint, or as families each of which consists of chords whose midpoints are all vertically aligned; finding tangents to a curve passing through a given point P in terms of a tangent at a particular point Q on the curve as Q runs along the curve, and as a line through P rotating to positions of tangency;

thinking of $\sqrt{17}$ as known by its properties (positive, square is 17) and as a real number with an essentially unknown infinite sequence of decimal digits; thinking of a straight line as an instance of a circle of infinite radius with centre at infinity, and so on.

12.3.2 Responding to Challenge

Although people are accustomed to believe that they 'choose' actions to enact, that choice is cognitive, this is a *User Illusion* (Norretranders, 1998). Close observations suggest that more often than not, some habit, some psycho-social adherence of coordination between action, affect, cognition, attention, will and even witness is what drives behaviour. Brief moments of true choice are glimpses of freedom.

While responses can have positive, negative and neutral influences, let us concentrate on positive responses. What lies behind different responses? In my experience, there is an immediate evaluation of the scope and potential of the task, not as a question to be carefully considered, but arising immediately. Does the challenge seem recognisable, and is some immediate action available? Does it appear to be attainable, or do I have confidence in the person posing it that it will be attainable, even if I do not immediately have a suitable mathematical action available? Does it appear to align with my current or past interests and successes? This is modified by the energy released, ranging from surprise or intrigue, through the desire to make sense which in turn is supported by my predisposition to tackle such challenges, to attempts to minimise the impact of the situation on my current well-being. So the pressure to perform, or in an examination situation, to perform quickly and efficiently, is a different kind of challenge to desire to resolve or comprehend some situation.

My immediate reaction then is either to take up the perceived challenge, to resist it by investing as little energy as possible in it, or even to reject it altogether. I also recognise that sometimes it is necessary to defer or park a challenge and that what was once a rejection can turn into parking because later it is actually taken up. This confirms Bill Brookes' observation (earlier) that challenge (having a problem) is about psycho-social states experienced by human beings in a particular situation, rather than any objective and universal quality. I have upon occasion rejected a problem posed by someone in one situation, and then later accepted it when posed by someone different in a different situation. A lot depends on my perception of, and social relation to, the situation.

Perception of the degree of challenge is necessarily idiosyncratic, depending as it does on a person's history, including the development of particular ways of responding to challenge and current state, and on the current situation as perceived, including who or what is posing the challenge. It is hardly a matter of cause-and-effect, more a matter of a soup of multiple forces, impulses and tendencies which play out differently despite only minor changes to the apparent situation (Mason, 2016).

Feedback of pleasure/endorphins arising from success, particularly unanticipated or striven-for success, can reinforce the disposition to engage in the future. Undertaking a challenge after a period of perceived failure is quite different to undertaking it during a period of perceived success: emotions are likely to be different, which may channel different flows of energy, thereby directing thoughts arising from attending to particular things in different ways. All of these interconnections tend, over time, to become habitual. A future stimulus may awaken or evoke thoughts, emotions or actions which bring a particular adherence into dominance in the way the person functions. An adherence may come to the fore for unexpected reasons, and afford access to associated actions with thought and emotion patterns, to ways of attending and to what, and to strength of will as to whether to persist, because of coordination of these aspects of the psyche.

12.3.2.1 Accepting

To accept a challenge there must be some sense of hope or possibility, whether based on a false sense of personal competence or on intensity of commitment. One important feature is trust in the source of the challenge, that the challenge is doable but not trivial, and worthwhile (Jackson, 2011; Mason, 2020). Something about the task has to appeal to the psyche, whether through emotions (surprise, intrigue), intellect (resonance with past experience), enaction (putative actions become available) or attention (perhaps a sense of generalisability). The appeal has to bring a coordinated set of adherence to the surface.

With some possible exceptions, the most alluring and persistent challenges are ones which I have set for myself. This even applies to challenges arriving from other sources, for it is only when my state is "in challenge" that I can truly be said to have taken it up. There is some sort of transformation, not always recognisable as such, which takes place so that an externally sourced challenge becomes a 'challenge for me'. The intensity with which it is taken up often waxes and wanes over time according not only to current feelings of (partial) success or progress but also according to exterior conditions of a psycho-social nature.

For example, in the 1970s the following problem circulated widely (Gardner, 1979; Klarner, 1981, pp. 285–307):

> There are four symmetrically placed (and so indistinguishable) doors in a circular table. Behind each door is a tumbler which is either up or down. If all the tumblers are in the same state, a bell rings. You may open any two doors and adjust the positions of those two tumblers, but the doors then close, and an unknown rotation takes place so you do not know which tumblers are beneath the doors you last opened. Can you make the bell ring?

Having eventually resolved it with ad hoc reasoning, I wanted to know what was going on structurally, so I posed myself the challenge of d doors, a symmetry group G acting on the doors so that which tumblers are behind which doors is not known, and allowing myself to use h hands (ie. to open h doors and make adjustments to any or all of these in a single move). My explorations revealed the structure of chains of subgroups of G with indices bounded by h in order to be sure to be able to ring the bell.

There is a weaker form of accepting challenge which is more apparent than real and applies particularly to classrooms. Care must be taken about interpreting activity as acceptance of challenge: I may simply display the appearance of accepting a challenge, when in fact I am resigned or compliant to it out of perceived lack of choice. It is a task, not a challenge. Throughout history we are presented with examples in which forced acquiescence is mistakenly taken as agreement, only to feed resentment and negative disposition generally. Browbeating learners into acquiescing rather than engaging wholeheartedly may be one of the reasons why so many learners suddenly leave mathematics, even those who undertake undergraduate studies.

12.3.2.2 Rejecting

Putting a challenge aside immediately may at first be seen as a rejection, an act to conserve energy and not be diverted from more pressing tasks. This may arise from the inner witness asking questions and alerting both cognition and affect to a need to focus attention elsewhere. It may also arise from a habit of rejecting or blocking the unfamiliar, established coordination between affect, cognition, and enaction that may have become habitual.

There are far too many mathematical challenges to undertake them all. For example, although questions about phenomena in the material world often occur to me, I also know that my modelling skills are limited, so I usually simply note the situation but reject the challenge as such (Fig. 12.1).

Fig. 12.1 Swing scooter (copyright free image)

An instance of this is the swing scooter which my granddaughter uses effortlessly, the design of which intrigues me: how did people decide the optimal angle of separation, the optimal size of wheels, and the optimal length of the wheel extensions? Similar questions apply to designs of overhead cranes, skip-transporters and many other things which I encounter. I wonder how design choices are made so as to optimise the functioning of the apparatus.

Often I reject a challenge because I do not see immediately how to get started, what action to enact in order to begin. More specifically, it is usually necessary to have an action become available within my current threshold of resilience: the period of time within which I am likely to persist, which will vary between individuals, and for individuals at different times and in different conditions. Even the Pólya-based advice to specialise in order to comprehend underlying structural relationships may seem to require too much effort, if it even seems possible.

A notable counter-example for me is the problem I posed myself many years ago, arising from the following mathematical challenge:

> In how many different ways can a circle be cut into four congruent pieces?

The problem is attractive pedagogically because it offers an opportunity to work with learners on how the meaning of *different* changes as examples accumulate, and how care is needed to justify conjectures, especially when they become rather too optimistically general. I noticed that in the only examples I could construct, at least some of the pieces always have the centre of the circle on their boundary, even when for 12 and 24 not all the pieces have to have the centre on their boundary.

Out of this came the problem:

> Is it possible to divide a disk into congruent pieces so that the centre is not on the boundary of any piece?

Here ‘piece’ is taken to be simply connected, acting like a jigsaw piece rather than exercising topological concern about boundaries and interiors. I cannot even really see how to specialise, as changing the circle to polygons opens up different possibilities altogether with no sense of how these might inform the circle case. I made myself a jigsaw featuring twelve congruent pieces as depicted in the central figure of Fig. 12.2, which can be assembled so that only some of them have the centre on their boundary, but it has not helped me see a way forward. I have returned to this

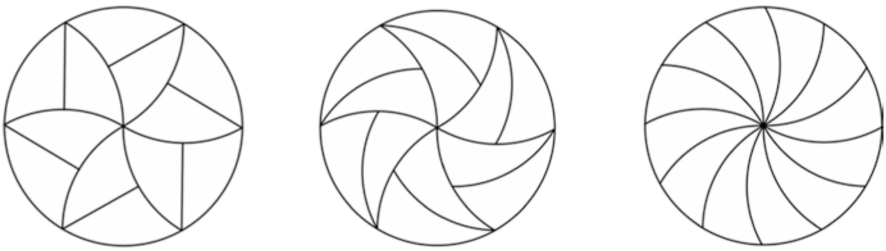

Fig. 12.2 Two variations for 12 congruent pieces; the second admits other variations such as the third

challenge several times, but, having tried combinatorial, geometric and function-analytic thinking at different times without success, I have been unable to find a way to make progress, and have again put it aside. I have not rejected it so much as deferred it, resisting for the time being and parking it for another time. So even though I have no actions to enact, I have not totally rejected it. This shows up something about the nature of resilience and persistence, but borders on obsession.

Another example for me is the following problem:

> Is it possible to glue congruent regular tetrahedra together so that they form a ring or torus (even if they cut through each other in 3-space)?

I posed this when I first struggled with simplicial complexes and chains in topology as an undergraduate. I returned to it several times when a fresh idea came to me, and eventually, some 7 years later, I managed to prove not only that it is impossible, but to extend my method to deal with other similar problems. There was something about the challenge that appealed to me, meaning that I persisted (Mason, 1972; see Elgersma and Wagon (2016) and Stewart (2019) for further developments). Here it was the recognition of a fresh action to try out, or of an action to retry with more persistence that kept me from rejecting it altogether.

Sometimes I am already working intensively on another challenge and can muster neither the energy nor the will to put the current one to one side, to shift my attention to the new challenge, especially if I cannot immediately see a way to get started. Again the word *obsession* comes to mind as a possibility, and at the time of writing, I am obsessed with a family of problems whose challenge I seem unable to put aside.

12.3.2.3 Resisting

In school and as an undergraduate and graduate student, I had to accept challenges presented to me in courses and examinations. I trusted the lecturer, aware that I had some very bright and accomplished colleagues, so I did not hope to succeed at everything. I have a vivid memory of a night spent trying to complete a take-home exam in Hilbert spaces in graduate school. As I became stuck on one of the problems, I would shift to another, returning to each again and again. I remember spending a good deal of time staring at the line between the wall and the ceiling of my study, waiting for inspiration. I trusted that the problems were within my capability, and I was desperate to do well in the course. So I persisted. In the end I completed them, only to be told by the lecturer when I handed them in that I need not have done more than one or two!

As I have become older, and slower, I find myself more able to resist challenges. For example, during corona-time conversations on-line we were posed a problem that I recognised, but had never really appreciated:

> Place four integers at the corners of a square. On the edges, record the differences in the adjacent numbers (the absolute values). Treat these as the corners of a square and continue the process. What happens?

Despite a slight resonance with arithmogons which I have exploited pedagogically (Mason & Houssart, 2000), I had (and still have) a strong sense that manipulating compositions of absolute values is not going to be attractive, and I have a vague memory of not enjoying what I found out when I last looked at it. So I resisted. I recognised a familiar coordination between affect (heaviness, concern), cognition (sense of other commitments, particularly to my current problem), enaction (initiating parking-type behaviour), attention (sustained to current commitments) and will (directing attention to current commitments). It turned out not to be a full rejection, because as it came close to time to report what we had noticed in the way of shifts of attention, I felt it necessary to have something to report. I looked for some actions that could be used to reduce the number of cases needing to be considered: take all the numbers to be non-negative; take at least one of the numbers to be 0. The point is that I tried to invest as little energy as possible, cutting down the number of examples I was willing to try in order to detect what was possible in the long run.

Often it is the case that response to a challenge is half-hearted, or an instance of being resigned to a challenge rather than actually taking it up wholeheartedly. This can even turn into a habit, summoning up familiar coordination based on a desire to invest only as little energy time and effort as possible, in the hope that that will be sufficient to get through the lesson. Closely involved of course is the implicit contract (*contrat didactique* see Brousseau, 1984) in which learners act as though their job is simply to attempt the tasks set by the teacher, and that somehow this will be sufficient to produce the learning that is expected of them. The teacher's side of the contract is to choose, set, and support work on the tasks so as to achieve this learning.

This form of the contract is of course rather inadequate and essentially vacuous, even for very cleverly chosen tasks. "One thing that we do not seem to learn from experience is that we do not often learn from experience alone. Something more is required" (Mason & Johnston-Wilder, 2004, p. 263). It is vital that learners do actually learn from their experience (Pólya, 1954 called it 'looking back'), which means, among other things, articulating a personal narrative about the topic. This involves recalling, and then imagining re-using actions that were effective, mathematical themes which emerged, and personal powers which were exercised. It would also include reviewing any relevant personal example space and its associated construction methods (Watson & Mason, 2005).

Learning mathematics is as much about developing a disposition to try some initial actions, if only to specialise in order to uncover underlying structural relationships, or to clarify what is being asked for and what other ideas or actions might possibly be relevant, as it is about mastering specific procedures in order to resolve routine questions. Personal narratives or *self-explanations* (Chi & Bassok, 1989) play a key role in learning from experience.

12.3.2.4 Deferring or Parking: Letting-Go, Hanging-on, and Pausing

While recalling my various experiences of mathematical challenge, I realised that one important, natural, and often necessary action worthy of being internalised is to park work. For example, the first action that becomes available is not always the most helpful, so parking that action before it is automatically enacted can avoid wasting time and energy. This applies whether it is a task, a reaction to a task or an action within a task. Not that this is easy to do. Often it is only after an initial action has been enacted but fails to result in progress, that real thinking takes place. There is a parallel with teaching, where it may only be on hearing a students' reply to my question that I realise I have asked a question with a prepared answer I wished to hear, placing me in danger of playing 'guess what is in my mind'.

Considering deeply, and allowing thinking to go on in the background provides access to an important part of the human psyche (S3) that may go undetected and unused in the constant push to 'cover topics' and 'reach solutions'. The following teaching story illustrates the point.

> A person was looking closely at the ground under a street lamp. Asked what they were doing, the reply was, "looking for my keys". When asked "Where did you lose them?", the reply was "over there, but it is brighter here".

Despite the absurdity of the story, most people have experienced persisting at something using the same available actions over and over (the light from street lamp) despite lack of progress. In the absence of any other action it is difficult not to keep trying an available action every so often in order to see if perchance it will now work, even though the difficulty lies elsewhere.

Any behaviour can become obsessive, which means persisting at carrying out available actions, coupled with an emotional state of desire uninformed or uninfluenced by cognition. Something in the will becomes stuck. Distinguishing between persistence and obsession is never easy, especially in oneself, as there can be a lingering hope that 'this time things will work out'. It is a state I recognise all too well in myself, and as such it is difficult to trap the coordinations between action, affect, cognition, attention and will. My witness observes, but is powerless to act! Andrew Wiles (2017) in interview observed that:

> You need a particular kind of personality that will struggle with things, will focus, won't give up. … we learn how to adapt to that struggle. Mathematicians struggle with mathematics even more than the general public does … We really struggle. It's hard. I am always quite encouraged when people say something like: 'You can't do it that way'.

More importantly, perhaps the real issue of challenge is recognising when progress is not being made, and learning to put a problem aside, at least for a time.

When emotional energy drains out and the will to continue begins to ebb, when the inner witness keeps asking "why are we doing this; isn't there something else we could be doing?" but without any suitable reply, when attention wanders and fails to concentrate, it may be time to let-go, either temporarily by parking, or by abandoning for the foreseeable future. These witness-questions are likely to emerge when no fresh actions are available. In the absence of suitable tools, it is wise to defer for a while. Thus an important aspect of *challenge* is to recognise and acknowledge when more tools or more ideas are required, leading to parking the challenge at least until conditions change. Wiles (2017) refers to the 'three B's', namely "bus, bath and bed", pointing to the need to let the unconscious (S3) create new associations, access forms or 'senses-of', and open up new vistas.

Periods of letting-go or parking can afford access to S3, yielding insight and new possibilities (Hadamard, 1945). As a friend and colleague reported recently:

> … I'm still working on the question in odd moments. It's interesting how questions like this can be like a staircase, with stair-like times where you're following a direction, discerning details, recognising relationships, and then you reach a 'landing' where you get a sense of a whole, but the next flight of stairs feels too much for the moment. Simon Gregg. (personal communication June 2020)

Mathematicians know from experience that even if there is little prospect of picking up the challenge later, it is always wise to make a summary of what has been achieved, listing conjectures and notes about what evidence there might be for them. This is part of a personal narrative, and it makes it so much easier to pick up the challenge at a later date than is the case if the only record is a sheaf of scribbles. What seems curious is that this is such a good habit to form, such a powerful coordinated adherence to develop for learning from the experience of a challenge in order to facilitate actions in the future, that one might expect it to be a core focus in mathematics classrooms in every phase, yet this does not seem to be the case.

Another example of the appeal of action before considering it properly (parking S1 and activating S2) is the desire to turn to electronic support, whether to perform algebra correctly, to generate examples, or to look for invariance in the midst of change. The form and nature of thinking on a machine using computer algebra or dynamic geometry are quite different from sitting quietly and contemplating, or from thinking in the background while doing other things. The greater the intrigue, desire, sense of possibility and trust in self and source, the harder it is to resist the impulse to enact some mathematical action without further consideration.

12.3.2.5 Giving-Up

It may be a moot point whether a pause, perhaps intended to be brief, turns into abandoning altogether, or retains the challenge on a 'back burner' for subsequent consideration. I myself have a long list of problems that I have worked on for a time but have had to put aside for various reasons. Sometimes I am expecting it to be temporary, sometimes permanent. I had hoped to return to many during retirement, but there always seem to be fresh things to think about!

Giving-up is not always intentional. I have several times wanted to use Isaac Newton's algebra problem of the grazing cows as an example in some writing:

> Problem 11. If cattle a should eat up a meadow b in time c, and cattle d an equally fine meadow e in time f, and if the grass grows at a uniform rate, how many cattle will eat up a similar meadow g in time h? (Newton, 1707, in Whiteside, 1972, p. 147)

Each time I have puzzled over the modelling assumptions, and finally resorted to assuming that the grass grows uniformly over a week, that the cows graze uniformly over a week, and that what matters is that at the end of the week the cows have not grazed more than the grass has grown. Each time I have then pondered the question of how to parametrise the problem so as to guarantee integer solutions. Initial forays have not been successful, and in each case my attention has been drawn away to some other problem, leaving this one behind.

12.4 An Indication of Pedagogical Issues

How can the notion of four Systems and the notion of psycho-social coordinations of adherences contribute to setting and sustaining mathematical challenge in the classroom and beyond?

While it is beyond the scope of this paper to develop these ideas, it is worth noting that setting tasks for others which might be taken up as challenges is the first and relatively easy step, in which the aspect of trust plays a dominant role. But the real pedagogical challenge is how to respond to the ways in which the students respond to the tasks set. Learners display various psycho-social coordinations of adherences, and the real challenge is pedagogical: how to respond to learners responses; how to enable them to resist immediate strong but debilitating emotional reactions so as to use that energy positively and productively. The discourse of psycho-social coordinations of adherences applies to the teacher as well, bringing to the surface various pedagogically oriented habits. I conjecture that a significant factor in the activating of pedagogical actions concerns learner and teacher search for affirmation, from colleagues and from respected-others. How these interact with those of learners will influence the outcome.

12.5 Final Reflections

I can be challenged by something or some person, and I can feel challenged by something or some person, but I can also choose whether or not to accept that challenge. Such a choice might even take the form of appearing to accept a challenge but in fact resisting or rejecting it by ‘going through the motions’, displaying the behaviour I anticipate is being looked for.

Since it seems clear that challenge is not a quality of a mathematical task itself, but of the relationship between a person’s current state, the cultural and social milieu, and how the task is perceived at the moment, it is important to work at increasing sensitivity to the psycho-social coordinations experienced by learners, and to help them work against unhelpful adherences. This requires personal work on one’s own adherences, particularly in relation to propensities, dispositions, and pedagogical habits, in short, to one’s own mathematical being, so that one can be mathematical with and in front of learners (Mason, 2008).

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Chapter 13
Mathematical Challenge in Connecting Advanced and Secondary Mathematics: Recognizing Binary Operations as Functions

Nicholas H. Wasserman

13.1 Introduction

Felix Klein was both an important mathematician and an influential mathematics educator. Perhaps his largest contributions to mathematics education were in describing the 'double discontinuity' that secondary teachers faced in their mathematical education and in his approach of studying 'elementary[1] mathematics from an advanced standpoint' (Klein, 1932; Weigand et al., 2019). His observations about these discontinuities – which pose mathematical "challenges" to those preparing to be secondary mathematics teachers – still ring true today. The first discontinuity is that the elementary and secondary mathematics that students learn bears little resemblance to the advanced (tertiary) mathematics that is taught at universities. The second discontinuity is that the advanced mathematics that prospective secondary mathematics teachers learn in university appears unrelated to their future teaching of school mathematics. In both cases, advanced mathematical study at the university can seem disconnected from the school mathematics they studied and will have to teach. Inherent in Klein's "solution" to this problem is that secondary teachers should understand the fundamentally important elements and connections between advanced mathematics and the mathematics they will teach. Yet, as is turns out, connecting advanced and secondary mathematics is not simple – it is filled with mathematical challenges.

[1] 'Elementary' in this sense can be understood in relation to the fundamental 'elements' of school mathematics, both elementary and secondary school levels, and 'advanced' can be understood in relation to university, or tertiary, level mathematics; his approach was in elaborating the profound connections between seemingly disparate domains.

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In this chapter, I explore the mathematical challenges faced by prospective and practicing teachers (PPTs) in connecting two ideas – the idea of a binary operation from their study in abstract algebra, and the idea of a function in secondary mathematics. The earlier reference to Klein is perhaps doubly important in this context; it was one of Klein's primary aims to put the function concept at the center of mathematical learning (e.g., McCallum, 2019). Based on a study with two PPTs, I elaborate on three conceptual shifts they went through as they came to understand a binary operation as a function itself. I use this example to ground the discussion of the mathematical challenges faced, more broadly, by PPTs as they develop connections from their advanced mathematics coursework; and to explore further how these conceptual shifts might be important for their work teaching secondary students.

13.2 Literature

In this section, I elaborate on extant literature that frames the mathematical challenge of connecting advanced and secondary mathematics. This challenge is mathematical in that it is about connecting two mathematical domains – relating how the mathematics in one domain is connected to the mathematics in the other. Although there will be implications discussed about pedagogy, the primary challenges posed are mathematical ones. In reviewing the literature, first, I consider the domains of advanced and secondary mathematics and, in particular, how and why they might be important for secondary teachers. Then, I consider two different ways there might be connections between these domains. Lastly, I consider specifically the mathematical notion of function, as it serves as the basis for our discussion about mathematical challenges.

13.2.1 Connecting Advanced and Secondary Mathematics in Secondary Teacher Education

Theories about the professional knowledge base for teaching have increasingly drawn on a practice-based lens (e.g., Ball et al., 2008; Shulman, 1986). That is, the mathematics teachers should know is intrinsically linked to the kinds of mathematical knowledge they would use in the act of teaching – including planning for and enacting instruction. Although this might seem to suggest that the mathematics that a teacher needs to know should only be a sufficiently deep understanding of the mathematics they will teach, many, including Klein (1932), have argued for the value of advanced mathematics (e.g., CBMS, 2012). Here, I consider advanced mathematics in relation to developing mathematical knowledge for teaching, arguing that making mathematical connections is an important part of this work.

Developing Mathematical Knowledge for Teaching Silverman and Thompson (2008) laid out a two-step cognitive model for the development of mathematical knowledge for teaching. The model posits that, first, a teacher must develop a mathematically powerful understanding of some mathematical idea, which was connected to what Simon (2006) described as a key developmental understanding. While such mathematical understandings are powerful, they are insufficient for developing knowledge for teaching; second, a teacher must develop a pedagogically powerful understanding. A mathematically powerful understanding must also have pedagogical power for it to become knowledge for teaching.

To summarize this cognitive model, there are two hurdles that PPTs must overcome for their knowledge to be professionally useful; the first is mathematical and the second is pedagogical. Extant literature suggests both of these are challenging for PPTs (e.g., Dubinsky et al., 1994; Even, 1993; Wasserman, 2017; Wasserman et al., 2018; Zazkis & Leikin, 2010). Indeed, in the realm of advanced mathematics, these two hurdles – to some degree – mirror Klein's (1932) two discontinuities. In this chapter, I focus primarily on the mathematical challenges that arise in connecting advanced and secondary mathematics. That is, overcoming these challenges can be a *mathematically powerful activity*. From this study, though, we will also see how it can become a *pedagogically powerful activity* as well.

Nonlocal Mathematical Knowledge for Teaching Wasserman (2018) conceptualized the mathematical landscape relative to the content that a teacher is going to teach. The *local* mathematical neighborhood were aspects relatively close to the content being taught, whereas the *nonlocal* neighborhood were ideas that were much farther away. "Close" in this sense entailed both the degree to which mathematical ideas are closely connected, but also temporally close in relation to when mathematical ideas are typically developed. For a secondary mathematics teacher, at a very broad grain size, one might consider the domain of *secondary mathematics* as being the local neighborhood, and the domain of *advanced mathematics* as being part of the nonlocal neighborhood.

In that paper, I posited three ways that advanced mathematics has been considered in relation to secondary teacher education. I elaborate briefly on the second two here, which are the most pertinent (The first was characterized by a lack of explicit emphasis on connections). The second emphasized the *mathematical* connection between the domains of advanced and secondary mathematics; the third – a relatively novel perspective – emphasized the *pedagogical* connection between advanced mathematics and teaching secondary mathematics. For secondary mathematics teachers, it would be important that their mathematical coursework, including advanced mathematics, helps deepen their understanding of the school mathematics they will teach. That is, making connections between advanced and secondary mathematics is vital for practicing and prospective teachers (PPTs). It would also be important for secondary mathematics teachers that such new understandings become pedagogically powerful for their work in teaching. Both are important. We consider both as they arise through the process of overcoming the challenge of making mathematical connections.

Previous Studies Connecting Advanced and Secondary Mathematics Here, studies about mathematical development are considered as they relate to connecting advanced and secondary mathematics. Much of the literature suggests that it is challenging to make such connections. Many scholars have documented teachers' limited conceptions of the function concept (e.g., Even, 1993; Zazkis & Marmur, 2018); others have pointed to challenges with inverse functions and their notation (e.g., Weber et al., 2020; Zazkis & Kontorovich, 2016); Wasserman et al. (2017a) demonstrated difficulties with grasping variance (and, relatedly, standard deviation) as an unbiased estimator; Mamolo and Zazkis (2012) pointed out challenges in reconciling ideas about derivatives and area formulas. The key point is that it is not trivial work to develop a coordinated and coherent sense of mathematics across both school and advanced conceptions. One study is elaborated on in more detail to give a broader picture of these kinds of difficulties.

Wasserman (2017) explored secondary teachers' understanding of inverse functions in relation to their learning about groups in abstract algebra. From abstract algebra, inverse functions can be understood as inverse elements in the group of invertible functions under composition. The study used semi-structured, task-based interviews with ($n = 7$) secondary teachers. Although there were several tasks, the primary analysis focused on participants' concept maps, in which they were asked to construct a concept map of all ideas related to inverse functions. The key point was whether participants would connect inverse functions (a secondary topic) to their group structure (an advanced mathematics topic) in their concept maps, identifying important pieces such as functional composition as the binary operation, the identity function ($i(x) = x$), etc. In short, very few secondary teachers made such connections. Furthermore, the findings suggest that developing mathematically powerful understandings of inverse functions was dependent not solely on understanding those ideas from advanced mathematics but also on the evocation of those secondary concepts being built upon. Namely,

> …if the first step toward developing mathematical knowledge for teaching rests in acquiring mathematically powerful understandings of that content (e.g., Silverman & Thompson, 2008), then this study suggests that there are many layers where difficulties may be encountered in having ideas from advanced mathematics be mathematically powerful, particularly when the study of the advanced mathematics concepts seem dependent upon inducing certain conceptions of secondary content. (p. 198)

To summarize, making connections between advanced and secondary mathematics is filled with challenges not only in understanding the advanced mathematical ideas but also in understanding the secondary mathematical ideas that the advanced ideas build upon in a sufficiently deep manner. This challenge of making connections between advanced and secondary mathematics is of particular interest to teacher education because the connections they are having to make are between local and nonlocal mathematics, that is, the connections to advanced (nonlocal) mathematics are outside the scope of the mathematics they will teach. It is to this notion of making mathematical connections between these two domains that I turn next.

13.2.2 Connections

Connections between advanced and secondary mathematics may be of different types. Wasserman and Galarza (2018), for example, distinguished two types of connections: a *generalization* connection, and an *instantiation* connection. A generalization connection between advanced and secondary mathematics is when advanced mathematics is serving as a generalization of secondary mathematics, i.e., secondary mathematics is an instance, or example, of a more general idea being studied in advanced mathematics. Connecting the notion of inverse functions in relation to its group structure (as in the previous study) would be one such example. An instantiation connection is when advanced mathematics is serving as an instance of, or being framed in terms of, secondary mathematics. For example, a binary operation is defined as a function and could be considered an instance of a function (a connection I elaborate on in the next section). Now, some may argue that instantiation connections are not possible – that secondary conceptions do not entail such abstract examples. For me, however, the crux of the matter is not that what is learned in secondary mathematics includes these abstract examples, but that what is learned allows for – according to the definition – these abstract examples.

This paper will elaborate on a mathematical connection between secondary and advanced mathematics – one I consider to be an instantiation connection. But the key point is that the mathematical connection discussed in this paper is characterized by being a topic studied in advanced mathematics, and one that would be considered an example of a mathematical idea that is studied and taught in secondary mathematics. That is to say, the advanced topic might add to a teacher's *example space* (Watson & Mason, 2005) of the mathematical idea studied in secondary school. Although the example might not be appropriate for use with secondary students, it nonetheless represents an important connection between these two spaces. And, I contend, this sort of connection has a different "feel" than one intended to demonstrate that ideas in secondary mathematics are examples of more advanced and abstract mathematical structures (e.g., groups).

13.2.3 Binary Operations and Functions: A Mathematical Connection

In this section, I lay out the mathematical ideas that serve as the backbone of the connection discussed in this chapter. Namely, that a binary operation is a function.

Binary operation In abstract algebra, a binary operation is often defined in relation to a set of objects. That is, a binary operation, $*$, defined on a set of objects, A, assigns to each pair of elements in A – the pair being operated upon – a resultant element of A. Addition on the set of integers, for example, is a binary operation. Every pair of integers is assigned a sum, e.g., $2 + 5 = 7$. Inherent in this definition is the notion of *closure*; the result of the binary operation, in this example the sum of 7, also belongs to the original set, in this example an integer. More precisely, the following definition can be used:

A *binary operation*, $*$, *on a set,* A is a function from the Cartesian product $A \times A$ to A. That is, $* : A \times A \to A$.

The Cartesian product of sets A and B is the set of ordered pairs:

$$A \times B = \{(a,b) | a \in A, b \in B\}$$

Hence, in the definition, the Cartesian product $A \times A = \{(a, b)| \ a, b \in A\}$. Thus, a binary operation takes an ordered pair – i.e., two elements from A – and assigns to that ordered pair a single element of A. In the case of our example of addition on the integers, the binary operation of addition assigns the pair (2, 5) to the integer 7. The operation is binary because it is defining how two elements combine; for any two integers (the inputs), there is a single output (the sum).

Function Although there are many ways in which functions are defined in school mathematics, including as a dynamic covariation between quantities, a function is often defined as a relation between sets that maps each element from one set to exactly one element in another set (McCallum, 2019). Although mostly considered in school mathematics as a relation between real numbers, the definition itself allows for abstract objects and elements. There are several component pieces here. The first is a relation. A relation between two sets A and B is any subset, R, of the Cartesian product, $A \times B$. We use $(a, b) \in R$ to mean that the ordered pair is in the relation, i.e., that a is related to b. So, while a relation R is *any* subset of the Cartesian product, a function f is a particular kind of relation – one which meets two criteria: i) *total*: $\forall a \in A, \exists \ b \in B$ such that $(a, b) \in f$ (i.e., each element in A is part of an ordered pair); and ii) *univalent*: if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$ (i.e., each element in A maps to exactly one element in B). I generally refer to A and B as the domain and codomain[2] sets. To be explicit:

A *function* f is a relation between two sets (i.e., a subset of a Cartesian product) that is total and univalent.

Although it is clear that a function is a set of ordered pairs, it is interesting that there is still some disagreement on what makes two functions the same, or different (Mirin et al., 2021). For example, if a function is a Bourbaki triple of (A, B, f), then the functions $(R, R, (x, e^x)| \ x \in R)$ and $(R, R^+, (x, e^x)| \ x \in R)$ are different because the codomain sets are not the same; notably, though, the graphs of the two functions are identical – they contain the same set of ordered pairs. But if a function is just a set of ordered pairs, then the question of totality is relatively unimportant because the

[2] I use codomain to refer to the set of values that might come out of a function, and the image, or range, to refer to the set of values that do come out of a function.

domain and codomain sets are not necessarily pre-stipulated (they can be derived from the set of ordered pairs), and all sets of ordered pairs are total on the set of domain elements present in the collection. (Indeed, some modern definitions of function only stipulate univalence (McCallum, 2019).) Regardless, although there are some interesting challenges to consider in relation to functions, for the purposes of our study, it is sufficient to say that a function is a relation between two sets that is total and univalent.

Mathematical connection The notion of a binary operation studied in abstract algebra can be connected to the notion of function in school mathematics. Specifically, a binary operation is defined as a function – meaning a binary operation is an instance of the broader mathematical notion of a function. I would call this an instantiation connection, because the definition of function introduced in secondary mathematics allows for such abstract examples of function, even if such abstract examples are not discussed there. Specifically, a binary operation, $*$, on a set, A, is a function that maps ordered pairs, (a, b) (with $a, b \in A$), to single elements, c (with $c \in A$). Yet, a function was defined as a relation – a set of ordered pairs? To what ordered pairs is this referring? In the example of a binary operation, it refers to ordered pairs of the form, $((a, b), c)$ (for $a, b, c \in A$). To make this clear, let's return to our example of addition, +, as a binary operation on the set of integers, Z. The goal is to understand addition on Z as a function – specifically, as a set of ordered pairs (a relation) that is total and univalent. In this example, the binary operation of addition on Z would be the following set:

$$+ = \{((0,0),0),((0,1),1),\ldots,((-3,8),5),\ldots\}.$$

From this, for example, $(-3,8)$ is an element of the domain set, and this element gets mapped to its sum, 5, which is unique. The mapping is listed as an ordered pair of these two elements. Notably, each element in the domain set, which is $Z \times Z$ (a set of ordered pairs), is present in the binary operation + (i.e., the relation + is total), and each element in the domain set maps to exactly *one* element in Z, the sum (i.e., the relation + is univalent).

13.3 Methodology

The findings reported in this paper are part of a larger study – one that explored the use of Wasserman et al.'s (2017b, 2019) instructional model of "building up from and stepping down to practice" in designing modules for an abstract algebra course for practicing and prospective teachers (PPTs). Some aspects of this broader study have already been reported in Wasserman and Galarza (2018); this chapter provides a different analysis. For the purposes of this paper, I report only on PPTs' interactions with material from one module, the Functions module – although I also briefly give pertinent information from other modules as needed. Each module always began with a teaching situation; in the Functions module, PPTs were asked to consider a secondary teacher's question (and the ensuing dialogue) about a diagram that mapped elements from one set to another, and whether the depicted relationship was a function.

Using design research (e.g., Cobb et al., 2003) within a teaching experiment, the broader study, including the singular module reported on in this chapter, engaged participants with some specific mathematical ideas and secondary teaching situations. Researcher hypotheses for each module were tested against participants' ways of thinking during the sessions. Two students (PPTs) enrolled in a program in secondary mathematics teacher education agreed to participate. One was a preservice teacher (Pam), the other an in-service teacher (Irene) with 5 years of experience (but not currently teaching). For each module, which was one session, I collected and analyzed two sources of data: (i) a (transcribed) video-recording of PPTs engagement in the materials; and (ii) an (transcribed) audio-recording of a post-teaching experiment, semi-structured interview.

In the analysis, I analyzed what actually transpired during the teaching experiment, comparing it to the hypothesized responses. Here, I report on one activity within the Functions module – the Binary Operations task – in which the participants' responses differed significantly from what was anticipated; namely, the two PPTs faced a mathematical challenge in connecting the advanced and secondary mathematics in that activity. The analysis from the transcribed video recording focuses on instances where PPTs' thinking appeared to shift mathematically in terms of connecting the two domains. The analysis leveraged a grounded theory approach (Strauss & Corbin, 1990) to track PPTs' thinking, as it sought to capture, qualitatively, the theoretically important phases and shifts that were evident in their thinking. The analysis was mathematical in the sense that I paid particular attention to mathematical aspects of their interactions with the task – especially ideas foundational to the concept of function (e.g., domain and codomain sets, totality, univalence). This analysis led to the identification of conceptual shifts in the PPTs' mathematical thinking – changes that were important in their overcoming the challenge of viewing the binary operation table through a functional lens. The analysis of the post-interview focuses on how PPTs reported that the mathematical challenge overcome in this one activity also became pedagogically powerful.

13.3.1 The Binary Operations Task

As one activity within the designed module, PPTs were given the addition modulo 12 operation table (Fig. 13.1) and asked to: "Consider the binary operation table below. Describe the function (i.e., mapping) that this binary operation table represents. What are some elements in the domain of this function? What are some elements in the range of this function? Express the function in the most concise way you can."

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

Fig. 13.1 Addition modulo 12 operation table

As background information, in a previous module, PPTs had been introduced to the definition of a binary operation (and a Cartesian product) – namely, the one(s) given in the previous section: "A binary operation, $*$, on a set, A is a function from the Cartesian product $A \times A$ to A. That is, $* : A \times A \rightarrow A$." During that prior module, PPTs also discussed that binary operations can be expressed using function notation. Participants did so in order to discuss function composition as a binary operation – one that could be used to understand composite transformations (since transformations are functions). Within the Functions module, and immediately prior to the Binary Operations task, PPTs had discussed the definition of a function, first, by defining a relation and, second, by providing the definition of a function: "A function f mapping set A to set B is a relation between A and B (i.e., $f \subseteq A \times B$) such that each $x \in A$ (total) is related to exactly one element in B (univalent)." As part of this conversation, participants were also reminded of functional notation, such as "$f : A \rightarrow B$" and "$f(a) = b$". Furthermore, relations were described as possibly having a one-to-one, many-to-one, or one-to-many correspondence, with the cases of one-to-one and many-to-one being functions.

Within the larger context of the study, researchers hypothesized that PPTs would navigate the Binary Operations task with relative ease – it was deemed a reasonably straightforward application of the definition of a function given that binary operations had been discussed in great depth in a previous module, and they had been explicitly tied to, and talked about as, functions in that module. It was hoped PPTs would recognize and express the function mapping: for $a, b \in A = \{0, 1, 2, \ldots 11\}$, $(a, b) \rightarrow (a + b)$[mod 12]. However, PPTs struggled on this mathematical task. They struggled to make the mathematical connection between advanced and secondary mathematics, specifically, the connection that a binary operation is an example of a function. Due to the difficulty faced by PPTs (and the later discussion of how overcoming this difficulty was powerful for their own thinking about secondary mathematics and teaching), I analyzed PPTs thinking during the task. What was identified were three conceptual shifts (between four conceptual stages) that occurred in their thinking about functions. These stages and shifts were important markers in PPTs' overcoming the mathematical challenge of connecting advanced and secondary mathematics. I elaborate on these results below.

13.4 Findings

The primary result discussed in this chapter is the elaboration of four conceptual stages – and the three shifts between these stages – that PPTs went through in connecting a binary operation from abstract algebra to the notion of function in secondary mathematics. That is, these stages represent how the Binary Operations task – a challenge to connect advanced and secondary mathematics – became a mathematically powerful activity for the PPTs. Afterwards, I briefly describe how PPTs reported this mathematical challenge as also being a pedagogically powerful activity.

13.4.1 Four Conceptual Stages

Equation-view During what is referred to as Stage 1, the PPTs had an *equation-view* of function. When initially given the tasks, PPTs expressed a little confusion, despite having previously talked about binary operations. Their initial reactions to the task were:

Pam:	Can you just explain what's going on here?
Teacher-researcher:	So, this is just a binary operation.
Pam:	So, you're just, like, saying that 0 + 0 = 0?
Teacher-researcher:	Mmhm.
Pam:	Ok. …
Irene:	So, we just say, like, it's taking all the integers 0 to 11, and then…this is what we're mapping to?
Pam:	I don't know, I'm so confused… I'm so confused. I don't if this is the input, or…
Irene:	Are both of these inputs?
Pam:	… wouldn't these be outputs?… Unless it's like x + y = z. I don't know.
Irene:	Is a binary operation…function…hmm…you notice that it starts over…we can only go to 11.
Pam:	Yeah. Right. We can't go…oh, is it like, mod 11.

Irene: Right, so the domain is just 0 to 11…
Pam: And so is the range.
Irene: I think we're confused by the fact that it's both places.

I point out that their initial, admittedly confused, attempts to view this as a function were by defining equations: $0 + 0 = 0$ and $x + y = z$. That is, their attempt to identify the binary operation table as a function was characterized by trying to come up with an equation that captured what was happening in the table. Now, these equations describe individual facts as well as more general truths about the binary operation table at hand. They are, and can be, helpful in describing the functional relationship. However, this equation-view was, ultimately, unproductive. One of the reasons it appears to have been unproductive was that participants were unable to list actual elements in the mapping. The participants struggled to determine the domain and codomain – they cycled back and forth between thinking it was and was not "0 to 11." Difficulty identifying the elements being mapped from and to make understanding a function nearly impossible; an equation-view at this stage did not help make these two sets explicit.

Mapping-view The first shift, to Stage 2, a *mapping-view* of function, was facilitated by prompts to describe the mapping informally and to determine specific elements in the domain and range. This took the form of two prompts. The first one is below:

Teacher-researcher: Okay, so how would you describe a function?
Irene: Oh, so is it…so is it $A \times A$?
Pam: Uh, I don't know.
Irene: …So it's just A, and A is the set 1, 2, 3, 4, all the way up to 11.
Pam: … Is it [the function]…no [the function] contains $A \times A$, so then you're right. The domain is, oh no, 0 to… oh yeah, 0 to 11.
Irene: …I'm still trying to figure out how we do this, cause $A \times A$ would produce like (0, 0) and (0, 1).
Pam: No, I think it's just A plus A.

What is evident from this exchange is the introduction of participants trying to wrestle with a Cartesian product, which likely stemmed from the definition of function (still on the board) which referenced relations as a subset of a Cartesian product. Although they still appear to be struggling with identifying elements in the domain and codomain sets, PPTs introduction of the set

$$A = \{0,1,2,3,4,5,6,7,8,9,10,11\}$$

and the Cartesian product $A \times A$, into the conversation, and the differentiation between these two sets, set the stage for a second mapping-view prompt. Notably, although the researcher induced this prompt into their conversation, it was also already explicitly in the text of the activity.

Teacher-researcher:	Okay, so what's the mapping? So if the domain is A, and the range is A, you should be able to show me the mapping…it doesn't have to be fancy, a mapping can just be arrows…take an element in set A and figure out what it maps to.
Irene:	But it maps to a whole bunch of stuff.
Pam:	Yeah, that's what I'm confused about…Cause 0 can map…
Irene:	It takes two of them… So it takes, it takes…if we do $A \times A$, we get all the ordered pairs, and then the added pairs get added together…like the two pieces of the pairs get added together to get that, but I don't know how we would write that.
Pam:	Ohhh.
Teacher-researcher:	So… You don't have to be technical at this point. Just show me… Not just describe it, but show me things that map to things…
Pam:	0 + 0 maps to 0, 1 + 0 maps to 1. It…
Irene:	Go all the way up to, like, 11 + 1, and 11 + 1 maps to 0.
Teacher-researcher:	So what're you mapping? So what's the domain and what's the range?
Pam:	This [e.g., 0 + 0] is our domain right? Cause this is being mapped to this [e.g., 0].

Although this may seem a trivial difference, I argue that viewing the binary operation table as $(0 + 0) \rightarrow 0$ and *not* $0 + 0 = 0$ was an important conceptual shift. Notably, however, in this current stage, although they are producing a reasonable mapping, their discussion of elements in the domain still makes use of the addition sign – their domain elements are utilizing the binary operation symbol in a way that would not allow them to define the binary operation in terms of its function. Essentially, all that has changed from the equation view is the replacement of the "=" with a mapping symbol "→". Yet, this shift fostered their ability to identify, or at least get closer to identifying, elements in the domain and the range, because the mapping symbol helped solidify what might constitute elements of both sets. Notably, while the codomain set is easily recognizable as A, the domain set is not quite this set – it is *two* elements of A.

Multivariable-view The next shift, to Stage 3, a *multivariable-view* of function, was facilitated by another researcher's prompt. Specifically, in the exchange here:

Teacher-researcher:	Ok. So the '+' [e.g., in Pam's previous comment "0 + 0 maps to 0"] is actually fairly irrelevant. In other words, it could be any operation that I'm talking about.
Irene:	If we have 0 and 0, we get 0. If we have 1 and 0, we get 1.
Pam:	Right…
Irene:	So we can just list the ordered pairs? Ok, so we can list them just like this, all the way down to when we had 11, 0 going to 11, and 11, 1 going to 0.
Pam:	Ok…

Irene: So now our domain is all these ordered pairs.
Pam: Yeah.
Irene: And our range is over here.

What is visible in this exchange is that, by the researcher stating that the operation symbol is actually unimportant, participants connected the two parts from their conversations during the mapping-view: (i) their thinking about the Cartesian product $A \times A$; and (ii) their thinking about mapping. These two pieces coming together transitioned their sense of the elements being mapped. In other words, this shift allowed them to recognize the mapping as $(0, 0) \rightarrow 0$, which is more clearly indicative of the multivariable domain input and which removes the "+" from the domain. Essentially, the multi-variable view evident in this stage is fundamentally about identifying the particular elements of the domain and codomain sets, which are the elements being mapped from and to as part of the function.

Expression-view The last shift, to Stage 4, an *expression-view* of function, was facilitated by a participant-researcher interaction, where participants were specifically asked to consider the last part of the activity prompt – about expressing the function in a concise way. This led participants to think about how to express the mapping, which furthered their conception of the function being expressed in the binary operation table.

Teacher-researcher: Ok. So the general set, the domain is what?…
Irene: So $A \times A$ is the domain, and A is the range.
Teacher-researcher: …You all had written it as $x + y = z$ …you're saying x and y come from here, right, so they both come from that first set…
Irene: Right.
Teacher-researcher: Right, so z could be something… And so probably the easiest way to describe this is as a function is to say our function is taking things of the form here, it's taking two inputs, and it's mapping it to what? So if I have these inputs A and B, it's mapping it to…?
Irene: $A + B$.

This last shift, guided by the researcher, was important. Notably, it allowed writing the mapping not as $(a, b) \rightarrow c$, where c is just *some* element that happened to be mapped to from the pair (a, b), but rather as $(a, b) \rightarrow (a + b)$[mod 12]. In other words, it established the element in the codomain set as being dependent on the input variables. That c could, in fact, be *expressed* in terms of the input; in this case, the input being the two values (a, b) in the ordered pair, and c being simply the sum (mod 12) of those two input values. Interestingly, the binary operation symbol, "+", makes its way as part of the output, not the input (where it had been previously placed); simply put, the binary operation symbol is used as a way to express the sum, which is the *output* of the two inputs. This allowed participants to recognize the equation form of the function as $f(a, b) = (a + b)$[mod 12]. Although writing functions as a dependent equation, like I have done, is not always possible (i.e., one might just list all input-output pairs to define the function), in instances where it is possible to express this dependence, equations provide a concise way to express all pairs of the function.

Secondary Applications These four stages marked significant conceptual shifts in PPTs' thinking on the Binary Operations task, which facilitated their coming to a deeper, mathematically powerful understanding of function. Notably, at later points in the module, PPTs were able to translate these discussions to other areas of secondary mathematics. Later on in the module, participants were asked to consider functional mappings between other abstract sets of objects, and were prompted to identify other "interesting examples of functions" – specifically, "You might consider, for example, the distance formula, $d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$, as a possible function, or other topics studied in secondary mathematics." Conversations during this time demonstrated the depth of PPTs' abstract understanding of function, especially in relation to the Binary Operations task. They were able to identify the distance formula as a function, $d : \left(x_1, y_1, x_2, y_2\right) \to \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$. Moreover, they brought up the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, as another possibly interesting secondary mathematics example to consider. Initially, Irene stated, "So, sometimes we have two answers for whatever we put as the input, the *a*, *b*, *c*…the domain…and it's not a function because it, in some cases, produces more than one [output]." That is, they described the quadratic formula as a one-to-many relation (not a function). Shortly afterwards, though, they realized they could modify the output set to be a pair. Doing so allowed them to discuss the quadratic formula as the function $q : \left(a, b, c\right) \to \left(\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}, \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$. Evident from these examples is the powerful way participants tackled other topics in secondary mathematics, and identified them as instances of functions as well.

13.4.2 Mathematical Challenge as a Pedagogically Powerful Activity

Immediately after having engaged with the module materials, the two PPTs reflected on some of their activity on the module tasks and their responses in the semi-structured interview with the researcher. One of the things that stood out in their reflections was the attribution they made to the mathematical challenge they faced, and overcame, during the Binary Operations task. They described the activity as being very influential for their thinking about functions – a mathematically powerful activity. Moreover, their reflections about teaching exemplified this mathematical challenge as also being a pedagogically powerful activity for their thinking about teaching. During the interview, one of their first reflections was:

Interviewer: …So what were the main ideas that you got going through the abstract algebra content? …

Irene: A deeper understanding of the function being something besides what I traditionally always thought about a mathematical function to be…

Pam: I think that was the one that I had the hardest time—like the binary operation…

Irene: And that one was really hard to think about cause it took us forever…it took us forever for us to figure out what the domain was.

Pam: …it was a good place for us to get stuck.

Irene: That's where I feel we, at least for me, I turned the corner about thinking about a function outside of just some linear situation… The fact that your domain can actually be an ordered pair…

Notably, the two participants described the Binary Operations task as a mathematical challenge – what might be called 'productive struggle' (e.g., Heibert & Grouws, 2007). They also described this as helping them develop depth to their sense of function, outside of some linear situation with real-number inputs. After reflecting back on their initial answers to the teaching situation – in which they suggested they would use pictorial mappings, tables, graphs, etc., of real-valued functions in $R \times R$, to demonstrate the idea of a function, and specifically the univalence property – the two PPTs changed course. Specifically, they both agreed that they would: "…give [students] other examples of things that are functions besides what we traditionally talk about in an algebra classroom." The reason Irene cited was: "I have other examples of things that are functions now that I didn't have before…And maybe some of these are too complicated to show them, but it would cause me to maybe stop and think about…maybe there's another mathematical thing that I could show them outside of the traditional $y = x + 3$… that is a function that's not normally something we would talk about as a function." In other words, it is clear the mathematical challenge they overcame had some pedagogical power as well; namely, their ability to abstract the notion of function (to encompass a binary operation table as an instance of a function) led them to consider how students might experience the notion of function in a more abstract sense as well.

13.5 Discussion

In this discussion section, I elaborate on several primary implications from this study. On the whole, this chapter explores the broad topic of the mathematical challenge of connecting advanced and secondary mathematics. I have done so in particular by looking deeply at one example of a mathematical connection through two PPTs' interactions with a task that asked them to recognize a binary operation table as a function. The study's findings are instructive about mathematical challenges at both specific and general levels.

First, *the mathematical challenges faced by PPTs in connecting advanced and secondary mathematics are a unique form of challenge.* In this study, participants were faced with content in both the domains of advanced mathematics and secondary mathematics. Specifically, a binary operation table from abstract algebra, and functions from secondary mathematics. One of the things that is unique about this sort of mathematical challenge is that students are not necessarily learning something *new*, per se, but rather what they are learning is to make a new *connection*. What I mean is that the participants understood a binary operation table (i.e., they could find results of an operation from the table) and they understood a function (i.e., they easily provided an appropriate definition, examples, and non-examples). What is unique about the mathematical challenge faced in such contexts is that PPTs have already been introduced to these two topics – the challenge they confront is somewhat unique because making a new mathematical connection recalls all of their previous conceptions (and misconceptions) about those topics. As Pam reflected: "[The teaching situation] put my brain in function mindset…Like, I just was thinking about functions, so everything that I know about functions was in my brain." The mathematical challenge was not necessarily "learning" something new; it was in "re-seeing" something familiar from a new perspective. This is a fundamentally different sort of challenge – one that might relate to the literature on cognitive notions of transfer, or backward transfer (e.g., Barnett & Ceci, 2002; Hohensee, 2014). The challenge of connecting advanced and secondary mathematics is also unique in that mathematics is a discipline that builds on previous ideas; this means that sufficiently deep understandings of secondary mathematics are important for making such connections. Indeed, a binary operation (from abstract algebra) was *defined* as a function – advanced mathematics is intentionally trying to build on this idea. As such, PPTs' notion of function is fundamentally important for their ability to make a connection; yet, it is the content of advanced mathematics that often serves as the basis for expanding (and abstracting) notions of function. This is to say, a sufficiently deep understanding of secondary mathematics is unlikely to occur *without* trying to make such connections to advanced mathematics. This demonstrates the cyclical nature of the mathematical challenge of connecting advanced and secondary mathematics. Forming the connection is dependent on a sufficiently deep understanding of secondary mathematics topics, but such sufficiently deep understandings are unlikely to come without more abstract and advanced examples.

Second, *the conceptual shifts evident in PPTs' mathematical challenge on the binary operation task potentially mirror (and inform) the mathematical challenge of secondary students in understanding functions.* One of the interesting findings from the study is that PPTs shifted through four conceptual stages in their coming to understand binary operations as a function. The four-stage process they went through as part of their own mathematical challenge, in fact, potentially mirrors, and elucidates, shifts that secondary students also go through in their own mathematical challenge of understanding the concept of function. That is to say, although nonlocal mathematics would not be taught to students, the process of coming to know some nonlocal mathematical ideas can potentially mirror, and elucidate, how local mathematical ideas might be learned and taught. I elaborate. Functional relationships are regularly introduced through an *equation-view* with two variables, e.g., $y = x + 3$. Students are often asked to explore these relationships. But the transition to function, and functional notation (e.g., $f(x) = x + 3$) is challenging. A first step in helping students is to transition to the second stage, a *mapping-view*; specifically, something like $x + 3 \rightarrow y$. The next transition (the multivariable stage) was essentially about identifying the objects in the domain and range; i.e., in this case, mapping *numbers* to *numbers*, e.g., $x \rightarrow y$. This step makes clearer the nature of the elements of the domain and codomain. The last step is in recognizing the number being mapped to, y, as being *expressed* in terms of the input, x. Specifically, being three more than the value x. This provides a more useful characterization of the mapping, $x \rightarrow x + 3$, and explicitly highlights the dependent nature of the functional relationship. It is from this mapping that the more formal and typical equation notation of a function, $f(x) = x + 3$, can be understood. Without students having an appreciation for mapping (stage 2), input and output elements (stage 3), and expression (stage 4), their ability to interpret functional equations and functional notation such as $f(x)$=, or $f(3)$, will be limited. According to PPTs' own reports, engaging in this process with a more abstract example helped them deepen their sense of function. Now, it is not clear that PPTs' mathematical challenges in connecting advanced and secondary mathematics would always mirror students' mathematical challenges; although I would argue that it was certainly so in this case. I would also suggest that it is very likely that some of the same challenges faced in coming to a deeper understanding of a secondary topic are, in fact, very pertinent for understanding the challenges of coming to understand that secondary topic in the first place. If this were true, it would provide a very natural approach for connecting PPTs' mathematical learning to their teaching (i.e., their pedagogical learning): identify various conceptual challenges faced in advanced mathematical learning and map those back to simpler examples to better understand the challenges students might face.

Third, *connections such as the one discussed in this paper can serve a particular purpose in secondary teacher education.* One reason for the focus of this paper is that, based on my own sense of extant literature, most of the field's approach to making connections between advanced and secondary mathematics highlight how advanced mathematics represents the more abstract and general situation, and topics in secondary mathematics are an instance of them – what I referred to previously as generalization connections. This is sensible; as one progresses in mathematics, the topics often become further abstracted. Groups, fields, and rings are general constructs; addition and multiplication of real numbers exemplify them. Topological spaces are general; the Euclidean plane is one example. The law of cosines is general; the Pythagorean theorem is one instance of this rule. So, it is often the case that, in the study of advanced mathematics, one would like to demonstrate how this new (advanced) construct is more abstract and that previous ideas represent a subset of this new idea. Connections like the one discussed in this paper, what I would call instantiation connections, are different – they are not populating the example space of an advanced mathematics topic, but rather adding to a teacher's example space of a topic discussed in secondary mathematics. Regardless of whether the examples might be used with secondary students or not, these sorts of connections force PPTs to more deeply understand, and wrestle with, secondary mathematical ideas. Perhaps in complement to Klein's idea of 'secondary mathematics from an advanced perspective', through the connection described in this paper, PPTs explored 'advanced mathematics from a secondary perspective'. A binary operation table is an object studied in abstract algebra; through the activity, PPTs were asked to view that binary operation table through a functional lens, i.e., *as* a function. After recognizing this connection, PPTs' example space of functions (which previously included things such as real-valued functions) was expanded. I regard the kinds of connections discussed in this paper as particularly valuable to secondary teacher preparation for at least two reasons. The first is that expanding PPTs' example space of secondary topics adds *diversity* to the kinds of examples they might provide. And the diversity of examples is inherently beneficial in teaching. By expanding the diversity of function examples, teachers can give a broader, more varied, more accurate, and more complete sense of a concept – in the sense of adding to one's concept image (i.e., Tall & Vinner, 1981). Indeed, variation theory (Marton & Tsui, 2004) suggests that it is precisely from examples and the variation of examples, that mathematical learning occurs – which underscores the value of teachers having a broader example space on which to draw. The second is that such connections, because they derive from advanced mathematics topics, frequently add *abstractness* to the kinds of examples they have in their example space. And again, I would argue that such abstractness in one's example space only helps clarify the true nature of the concept. By not just having real-valued functions on which to draw, but other, more abstract functions (e.g., from 2-space), the power, utility, and potential application of the function concept becomes increasingly apparent.

Fourth, *there is a potential conflict in terms of pedagogical power from PPTs engaging in the mathematical challenge of connecting advanced and secondary mathematics.* Lastly, I touch on the notion that such mathematically challenging situations can be a pedagogically powerful activity. In this study, one of these pedagogical aspects was that the two participants reported that they would alter the kinds of functions they might use with secondary students to introduce them to the concept. Namely, they discussed including more abstract examples of functions, beyond just $R \times R$ functions. I found this to be potentially encouraging, and certainly suggestive of pedagogical power in thinking about how to enhance student's notion of function. But I also note here that some of the PPTs' tendencies also included "transporting" (i.e., Wasserman et al., 2018) activities from their own learning to use with their students, which would likely be inappropriate. Moreover, their intended inclusion of abstract examples also had the consequence of a de-emphasis on utilizing multiple representations of functions (i.e., equations, tables, graphs). In contrast to real-valued functions, abstract functions have a tendency to only be represented in a singular manner. And such a lack of different representations might diminish (not enhance) the mathematical quality of students' experiences. So, while the mathematical challenge in connecting advanced and secondary mathematics provides an opportunity for pedagogical power, I still regard there to be important work to be done in this regard. More work would need to occur to help PPTs explicitly address how these ideas might become pedagogically powerful, not just for their thoughts about teaching, but for their students' mathematical learning and development.

13.6 Conclusion

Connecting the mathematical domains of advanced and secondary mathematics presents formidable, and unique, mathematical challenges, which are especially relevant to prospective and practicing secondary teachers. What we find from this study is evidence of this challenge, in the specific context of recognizing a binary operation as a function – a mathematical connection. Notably, participants' work on this task demonstrated the unique nature of this sort of mathematical challenge. The results also documented four specific conceptual stages that were productive in overcoming this challenge – indeed, ones which potentially match conceptual shifts that would be important for secondary students in coming to understand more basic functions – and provided an opportunity to better understand such connections and their potential use in secondary teacher education.

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Chapter 14
Mathematical Challenge of Seeking Causality in Unexpected Results

Mark Applebaum and Rina Zazkis

14.1 Introduction

A mathematical challenge is often associated with a difficult mathematical problem. However, the notion of difficulty depends on the problem solver. A mathematical challenge refers to a difficulty that a problem solver can overcome, that is, there is sufficient motivation, ability and availability of resources (Leikin, 2014). Different types of challenging mathematical tasks were identified by teachers in Applebaum and Leikin's (2014) study. These included non-conventional problems, problems that require integration of knowledge from different areas, problems that require knowledge of extra-curricular topics, and problems that require logical reasoning, among others.

One of the problem types exemplified in Applebaum and Leikin (2014) attracted our attention – it was described as a problem that requires finding a mistake in a solution. While a task of "find a mistake" in a solution can be presented to a student, and there are various examples of erroneous reasoning or erroneous computation that lead to absurd conclusions, finding mistakes in student solutions can present a challenging mathematical task to a teacher. This turned our attention to mathematical challenges in the work of teachers, in particular, to the challenge of explaining.

Explaining mathematical ideas is part of regular teaching work. However, finding an explanation that is convincing and that points to the cause of obtained results may present a challenge. When the results are unexpected, the challenge of

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explaining is stronger. The mathematical challenge of explaining unexpected results is our interest in this chapter.

In what follows, we review the notion of intellectual need formulated by Harel (2013) and focus on one of the intellectual needs that is predominant in teaching, that of causality. We then present the results of a study in which teachers and mathematicians engaged in a simple task with unexpected results. We also address a self-imposed challenge that was provoked by the participants' responses.

14.2 Intellectual Need

The general notion of intellectual need often refers to a person's intrinsic motivation to learn something new or to solve a problem. Harel (2008, 2013) elaborated on this notion with a particular focus on mathematics, considering the historical and epistemological development of the discipline. He considered intellectual need as a necessity for constructing a new piece of knowledge, either by a community or by an individual. He criticized teaching mathematics for failing to provoke students' intellectual needs.

Harel (2013) described five categories of intellectual needs, noting that these categories are not disjoint and may not provide a complete description of intellectual needs.

Briefly,

- The need for certainty is the need to prove, and determine that a claim is true
- The need for causality is the need to explain or determine a cause
- The need for computation is the need to quantify and search for efficient computation
- The need for communication includes the need to persuade others and establish a common frame of reference
- The need for structure includes the need to organize knowledge and connect its various components

We focus in this chapter on the need for causality, in particular, on teachers' need for causality. This is the need to explain the observed phenomenon, to determine its cause, in particular when the phenomenon is counterintuitive or unexpected. The need for causality is not limited to teachers, but we believe that this need is stronger in teachers, in comparison to other individuals working with mathematics. The need for causality "does not refer to physical causality in some real-world situation being mathematically modeled, but to logical explanation within the mathematics itself" (Harel, 2013, pp. 143–144).

We add here that a teacher's need for causality is not necessarily geared towards finding a mathematical explanation that is accessible and convincing for students. It is a need for seeking explanations, at times additional explanations, that result in enriched personal ways of understanding a given phenomenon. In the next section, we exemplify several attempts to satisfy the need for causality which could be applicable in an instructional situation.

14.3 Seeking Causality: Three Examples

14.3.1 Example 1: Rope around Earth

Recall the classical problem of a rope around the Earth[1]:

> If a rope is wrapped around the equator of a spherical earth, then the needed length of the rope is the circumference, the length of the equator. Now imagine that this rope is placed on 1 m high poles. How much more rope is needed?

Common intuition suggests that the answer is in hundreds of kilometers. This is the usual first guess when people encounter the question. The surprising result is that what is needed is only about 6 meters: $2\pi(R + 1) - 2\pi R = 2\pi$. The surprise is amplified by the realization that the result is independent of the actual radius of the earth, the same amount of additional rope is needed for any size of planet.

We used this problem with many groups of students, and the surprise that students face with the counterintuitive conclusion reinforces our desire to find an explanation that helps confront a misleading intuition. The search for such an explanation is our intellectual need for causality, while the need for certainty is satisfied by a simple computation.

One possible explanation is presented in Papert's (1980) book. Rather than thinking of a circle as a simplified model of equator circumference, imagine it as a square. Then, when the rope around this square is raised on a 1 m poles, an additional amount of rope is needed only at the corners. It becomes obvious that the additional amount of rope is rather small, $\leq 8m$, depending on how it is wrapped around the corners.

An additional explanation connects the story of the rope to the rate of change (Yan et al., 2022). In fact, the rope length $2\pi R$ is a function of the radius R. And since the derivative (the rate of change in R) depends on the linear variable R, the derivative is constant. As such, it does not matter if the rope is raised around the earth or around a basketball. The need for causality is satisfied by connecting the story of the rope to introductory calculus.

[1] Several versions are available, see for example, Burger (2007) or https://en.wikipedia.org/wiki/String_girdling_Earth

14.3.2 Example 2: Horizontal Translation of a Parabola

The fact that the graph of $y = (x - 3)^2$ appears to the right of the canonical $y = x^2$ is often initially surprising to learners as the "minus" is associated with the negative direction, and the expectation is that the graph will "move left". When asked to explain the unexpected location of $y = (x - 3)^2$, teachers provided a variety of different explanations, which included consideration of the vertex, plotting different points, finding the zeroes of the function or simply citing the rules (Zazkis et al., 2003). However, none of these explanations appeared to the teachers as satisfactory, which we interpret to mean that none of the explanations satisfied the intellectual need for causality. To satisfy this need, it was suggested (ibid.) to situate the problem in the context of transformation: first, consider the image of a parabola after a horizontal translation of 3 units to the right: $T(x, y) = (x + 3, y)$ and then look for the equation of the resulting image.

Without loss of generality, focus on a point (a, b) of the source set that was translated to the point (c, d) of the image set. According to the specific translation performed, $d = b$ and $c = a + 3$. We wish to connect c and d in an equation. Relating c to d, we obtain the following: $d = b$ and $c = a + 3$, which implies $a = c - 3$. However, $b = a^2$ as (a, b) is a point on the source parabola. Substitution leads to $d = (c - 3)^2$. Since the above is true for every point of the image set, the image of the translation is described by the equation $y = (x - 3)^2$.

The teachers we worked with referred to this explanation as "really convincing". "The participating teachers referred to this view of transformations as "an eye-opening clarification" or a "pedagogical AHA!" (Zazkis et al., 2003, p. 450). We interpret these reactions as an indication that the teachers' need for causality was satisfied.

14.3.3 Example 3: Division by a Fraction

Division by a fraction is often perceived by learners as a strange rule. There are different explanations and justifications, such as considering division as an inverse of multiplication or following some fancy computations, such as

$$\frac{2}{3} \div \frac{4}{5} = \frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times \frac{5}{4}}{\frac{4}{5} \times \frac{5}{4}} = \frac{\frac{2}{3} \times \frac{5}{4}}{1} = \frac{2}{3} \times \frac{5}{4}$$

While we are convinced by the computation, we sought a stronger explanatory power. What can provide such an explanatory power is the following:

Multiplying by 4 and then dividing by 5 can be carried out as multiplying by $\frac{4}{5}$. Since the undoing of multiplication is division, and the undoing of division is multiplication, the undoing of multiplying by 4 and then dividing by 5 is multiplying by 5 and then dividing by 4, that is, multiplying by $\frac{5}{4}$. Notice the switch of order in the operations. But it is also dividing by $\frac{4}{5}$. So multiplying by $\frac{4}{5}$ is undone by dividing by $\frac{4}{5}$ which is the same as multiplying by $\frac{5}{4}$.

While this explanation may appear less straightforward than the familiar "rule" of "invert and multiply", we suggest that it satisfies the need for causality, at least for those who experience this need.

14.3.4 The Need for Causality as a Challenge

What is common in the presented examples is that (a) the need for certainty is satisfied by a presented computation, and (b) the challenge of explanation arises after the solution is found. Considering the three examples, we assert that the challenge of satisfying the need for causality is a mathematical challenge, as it results in mathematical activity. It is the activity of seeking an applicable piece of mathematical knowledge, often not explicitly related to the problem at hand, that can be harnessed for the situation.

In example 1 (the case of a rope) it is knowledge of derivatives and rate of change; in example 2 (the case of a parabola) it is knowledge of transformations; and in example 3 (the case of division of fractions) it is the connection to the general case of finding an inverse of a composition of operations, which is mathematically expressed as $(a * b)^{-1} = b^{-1} * a^{-1}$, where a^{-1} is the inverse of a and b^{-1} is the inverse of b with respect to the operation $*$.

The three examples also demonstrate that the need for causality is satisfied by utilising mathematical knowledge which is not required for the solution of the problem. They also demonstrate how (more) advanced mathematics can support comprehension of school mathematics.

14.4 Creating Challenge in Simple Tasks

Simple actions can be turned to challenging ones by adding a constraint. For a cyclist, a challenge can be added by adding a constraint of cycling without holding the handlebars; for a hiker, a challenge is added by wearing heavy gear. A constraint can also be in limiting available resources for common actions, such as time or light.

Turning to mathematical examples, a simple problem can be turned to a challenging one by limiting the allowable resources. For example, given the length of the legs of a right-angle triangle, determine the length of a hypotenuse without using the Pythagorean theorem. Or, measure exactly 1 litre of water using only 5 and 3 litre jugs, or 6 and 4 litre jugs. Interesting mathematical ideas can emerge by adding and varying constraints that present varying levels of challenge.

We note that some imposed constraints lead not only to challenging tasks but also to exciting mathematical developments. Consider for example geometric constructions with straight edge and compass, in particular, the case of trisecting the angle. Obviously, trisecting is not the goal, the goal is to address the challenge given the constraint of allowable tools. However, attempts to address this challenge stimulated the development of the discipline, where the proof of impossibility came from presumably unrelated mathematical content, abstract algebra of field extensions.

14.5 Teachers Responding to the Five-Digits Task

In this section, we describe how secondary school teachers address a presumably simple task, where a challenge is presented by a constraint of not allowing the use of a calculator. Following the presented solutions, we describe the challenging activity of explaining the unexpected result without explicit computations.

14.5.1 The Five-Digits Task

We presented this task to secondary school mathematics teachers (n = 17) who participated in a professional development program. To focus on how the teachers

> TASK: Without using a calculator, place the digits 1,2,3,4 and 5 (each digit once) in the following multiplication
>
> □□□×□□
>
> to get the largest product.

approach the task, we added the following components:

> Indicate your first guess
>
> □□□×□□

Calculators are NOT allowed.

BUT, if you were allowed to try 3 products using a calculator, what would you try? WHY?

□□□×□□
□□□×□□
□□□×□□

After recording possible candidates for the largest product, the teachers were invited to solve the task, record their solution and indicate to what degree they were convinced by the correctness of their solution, by responding to the following question:

How convinced are you?
(Circle 25% 50% 90% 100% other__%)

Next, the participants were asked to address the following questions:

How would you convince a colleague, in case of disagreement?
How could you convince a colleague without calculating/computing the final result?

Note that the largest product is obtained by 431 × 52, which, as is demonstrated below, was an unexpected result for many of the participants. In fact, while there are 5! options for placing the 5 digits in the 5 designated spots, an a priori reasoning reduces the number of "candidates" for the largest product significantly: the first digit of the two factors should be one of the largest digits, 4 and 5 respectively, and each number must have digits in a decreasing order. This reasoning results in only 6 possible products:

$$531\times 42,\ 532\times 41,\ 521\times 43,\ 431\times 52,\ 432\times 51,\ 421\times 53.$$

We refer to these products as relevant cases.

While calculating the possible products may address the need for certainty, the request to convince a colleague without referring to the result of the calculation addresses the need for causality.

14.5.2 Determining the Largest Product

Table 14.1 summarises the participants' first guesses, the first three products they wished to calculate, the total number of calculated products (as derived from the recorded calculations), their final solution and the percentage that indicates the degree of certainty.

Table 14.1 Summary of participants' responses to the Five-digits Task

		3 products to try after indicating first guess													
Participant	First guess	521 × 43	531 × 42	542 × 31	532 × 41	543 × 21	541 × 32	412 × 53	421 × 53	432 × 51	431 × 52	TC	TCR	Final result	%
P1	421 × 53	✓	✓								✓	4	4	431 × 52	100
P2	531 × 42	✓	✓	✓								5	4	431 × 52	99
P3	531 × 42		✓		✓	✓						4	3	521 × 43	90
P4	532 × 41	✓	✓	✓								6	4	521 × 43	100
P5	521 × 43	✓			✓	✓						7[a]	6	521 × 43	100
P6	531 × 42	✓			✓		✓					5	5	521 × 43	–
P7	421 × 53	✓					✓		✓			5	2	521 × 43	100
P8	531 × 42			✓	✓		✓					4	3	531 × 42	90
P9	531 × 42	✓	✓		✓							5	4	521 × 43	90
P10	531 × 42		✓				✓			✓		5	4	531 × 42	90
P11	531 × 42		✓	✓	✓							6	4	431 × 52	90
P12	531 × 42	✓					✓				✓	7	5	431 × 52	80
P13	531 × 42		✓				✓		✓			6	4	431 × 52	90
P14	531 × 42				✓					✓	✓	5	5	431 × 52	99
P15	531 × 42		✓				✓				✓	5	3	431 × 52	90
P16	531 × 42	✓	✓					✓				4	4	521 × 43	90
P17	432 × 51	✓			✓							4	4	431 × 52	50
	Total	10	10	4	8	2	7	1	2	2	4				

TC Total number of checked products, *TCR* Total number of checked relevant products
[a]Computational mistake

By observing Table 14.1 we note the following:

First Guess. Note that as the first step in addressing the task the teachers were asked to indicate their first guess. None of the "first guesses" included a correct answer.

The most popular answer was 531 × 42. It was found in 12 of the 17 answers.

Most teachers used the form 5 _ _ × 4_, that is, 5 was the first digit of the 3-digit number. Only 3 of the 17 teachers (P1, P8 and P17) suggested for their first guess a product of the form 4 _ _ × 5_, but different from the correct answer; 2 of these 3 teachers (P1 and P17) found the correct answer in their final result.

Three products to try. After indicating the first guess the teachers were asked to suggest 3 candidates for the largest product to be tested. Including the first guess in the 3 suggested products was an implicit option. 10 of the 17 teachers included the first guess as one of their 3 suggestions.

Of these 50 products[2] listed as potential candidates for the largest result, 41 were of the form 5 _ _ × 4_. Of these 41, 531 × 42 and 521 × 43 appeared 10 times each. Only 9 of the indicated 3 candidates were of the form 4 _ _ × 5_. Of these 9 only 4 included the correct solution (P1, P12, P14, and P15).

Number of cases checked. Based on the teachers' worksheets, the number of calculated products varied between 4 and 7. We noted above that there are only 6 relevant cases: 531 × 42, 532 × 41, 521 × 43, 431 × 52, and 432 × 51, and 421 × 53. While three teachers (P5, P12, P14) calculated 5 or 6 products, only one teacher (P5) checked all the 6 relevant cases.

Final Answers. 8 of 17 final answers were correct (431 × 52), 9 were incorrect. 7 of the 9 incorrect final answers indicated 521 × 43, and 2 indicated 531 × 42.

Certainty. There appears no apparent connection between the number of checked cases, correctness of the answer and the indicated percentage for certainty. 100% was indicated for 1 correct and 3 incorrect answer, 99% was indicated for 2 correct answers; the lowest numbers of 50% and 80% were indicated for two correct answers. 90% was the most frequent indicator assigned to 5 incorrect and 3 correct answers.

Convincing without calculating. The notion of place value was featured in most of the suggested explanations. Some explanations were general, such as "consider the place values of the digits". We exemplify below several of the more explicit examples.

Figures 14.1 and 14.2 demonstrate how P1 and P11 would convince a colleague without calculating. They indicated that digits 5 and 4 should appear in the highest place values and digits 2 and 1 in the lowest. We note that these teachers do not

[2] One of the 17 teachers listed only 2 products, so we have 50 rather than the expected 51 products.

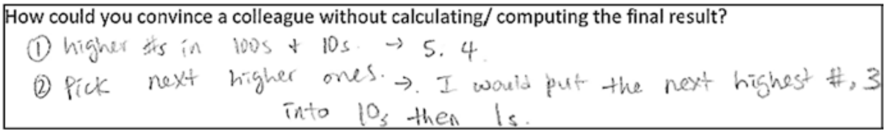

Fig. 14.1 Explanation of P1 for the obtained correct answer 431 × 52

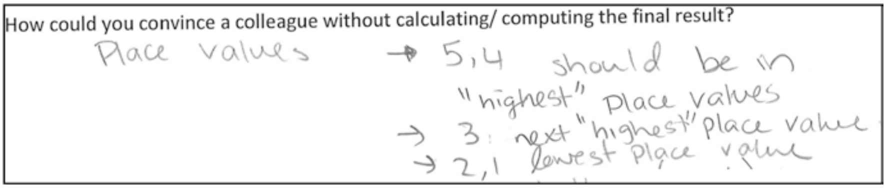

Fig. 14.2 Explanation of P11 for the obtained correct answer 431 × 52

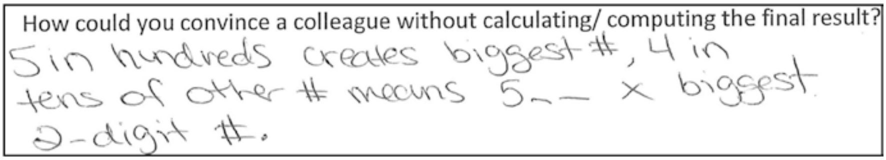

Fig. 14.3 Explanation of P8 for the obtained incorrect result 521 × 43

actually justify what they determined to be the largest product, but provide guidance of what products have to be considered and checked.

Figures 14.3 and 14.4 demonstrate "convincing a colleague" explanations of P8 and P9, respectively, whose final answer was incorrect 521 × 43. P8's explanation places 5 as the first digit of the 3-digit number, which is multiplied by the largest 2-digit number composed of the available digits. While this describes the answer, it does not explain why this answer is considered the largest. P9's explanation implicitly compares 521 × 43 and 531 × 42. She claimed that 5 "should go in the 3-digit number" and then compared (20 + 1) × 3 ("three 20s and 3 ones") and (30 + 1) × 2 ("2 thirties and 2 ones"), which led her to conclude that 521 × 43 > 531 × 42. Focusing only on these two comparisons suggests that other possible products were of no importance for the final decision.

It is evident from the first guesses, from the suggested products to be calculated and from the final incorrect results, that there was a strong prevalence of placing the largest digit in the 3-digit number. For 10 teachers all their first 3 trials were of the form 5 _ _ × 4 _ ; 2 of the 17 teachers did not even consider a product in the form 4 _ _ × 5_ in any of their subsequent calculations. Furthermore, P5 included 521 × 43 and 431 × 52 in her column-multiplication calculations, but despite the correct calculation she indicated 521 × 43 as her final answer. This careless conclusion is in accord with the preference towards a solution of the form 5 _ _ × 4_.

14.5.3 Follow-Up Activity: The Challenge of Causality

After the work on the task was completed, the participants shared their solutions. The result appeared surprising for those who did not indicate the correct solution as well as for those who did. The challenging activity, which invoked the need for causality, was to compare possible answers without referring to the result. Because the results are rather close to each other, the answer cannot be achieved by estimation.

The following 3 products were indicated as "the largest" in the teachers' answers: 521×43, 531×42 and 431×52. As such, the discussion focused on these three products. It appears rather surprising that an incorrect answer 521×43 was given with a rather high certainty. Those who chose 521×43 also computed 531×42 as a possible candidate for the largest product. This could have been the reason for the perceived certainty.

P16 suggested that one possible way to explain why 521×43 was larger than 531×42, without explicit calculation, was to claim that 500×40 appears in both products, and that 21×3 is larger than 31×2. This was consistent with what she indicated in her worksheet, see Fig. 14.5 (Note the similarity with P9 explanation in Fig. 14.4).

This appeared reasonable, as the argument confirms the known result. However, the same argument fails if applied to comparing 431×52 and 421×53, as actually $431 \times 52 > 421 \times 53$.

Another attempt was to decompose the numbers into the sum of their place values and subsequently use the distributive property. For example, in comparing 521×43 and 531×42, P3 wrote the following:

$$521 \times 43 = (500 + 20 + 1) \times (40 + 3) = 500 \times 40 + 20 \times 40 + 40 + 500 \times 3 + 20 \times 3 + 3$$

$$531 \times 42 = (500 + 30 + 1) \times (40 + 2) = 500 \times 40 + 30 \times 40 + 40 + 500 \times 2 + 30 \times 2 + 2$$

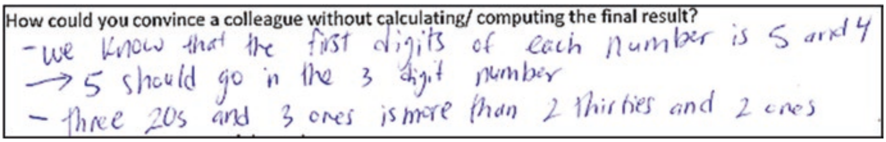

Fig. 14.4 Explanation of P9 for the obtained incorrect result 521×43

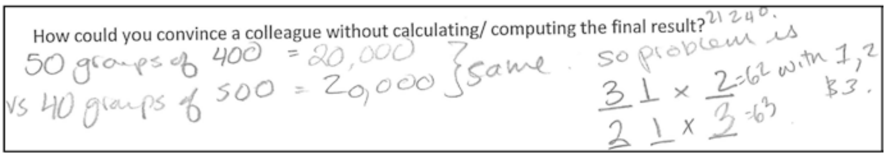

Fig. 14.5 P16's attempt at a convincing argument

Then she suggested to ignore the numbers that were the same in both calculations (500 × 40) and focus only on the different ones. But the fact that the larger components were in different products (that is, 500 × 3 in the first product is larger than its corresponding term in the second product, but 30 × 40 in the second product is larger than its corresponding term in the first product) did not lead to an argument that her classmates found convincing.

Some prompting appeared necessary in order to guide the participants towards a different approach. We discussed how the relative size of two objects can be compared, without determining the measure, especially when estimation is of no use. From this discussion an idea was developed to look at the difference between the two products, without calculating explicitly what the products were. A convincing argument was achieved by direct comparison, invoking distributivity in a different way:

$$521 \times 43 = (531 - 10) \times (42 + 1) = 531 \times 42 - 420 + 531 - 10$$
$$= 531 \times 42 + \text{a positive number}$$

It is easy to see that this positive number is 101, but determining the exact difference is not needed to demonstrate that 521 × 43 is larger than 531 × 42. A slightly more complicated calculation was needed to find the difference between 521 × 43 and 431 × 52.

$$431 \times 52 = (521 - 90) \times (43 + 9) =$$
$$= 521 \times 43 - 90 \times 43 + 521 \times 9 - 90 \times 9 =$$
$$= 521 \times 43 - 90 \times 43 + 431 \times 9 =$$
$$= (521 \times 43) - 430 \times 9 + 431 \times 9 =$$
$$= 521 \times 43 + \text{a positive number}$$

Again, it is clear that this positive number is 9, but the task was to determine the largest product, not its difference from the second in size.

The class also extrapolated the above calculation to the general case: Given 5 consecutive digits (n, $n + 1$, $n + 2$, $n + 3$, $n + 4$) to be used in a product of a 2-digit by a 3-digit number, what combination gives the largest product? The teachers compared the two "winning" combinations for $n = 1$, that is, the choices that result in 521 × 43 and 431 × 52.

Figure 14.6 is a screenshot of the board where the two products are compared. First, both products are presented in the expanded notation. Then, some elements of the product on the right are decomposed to match the elements of the product on the left. At the next step we note that opening the parentheses on the left will result in 4 addends, while opening the parentheses on the right will result in 9 addends, 4 of which match those on the left. As such, only the 5 different elements are considered. With some manipulations, we see that the difference between the two results is $9n$,

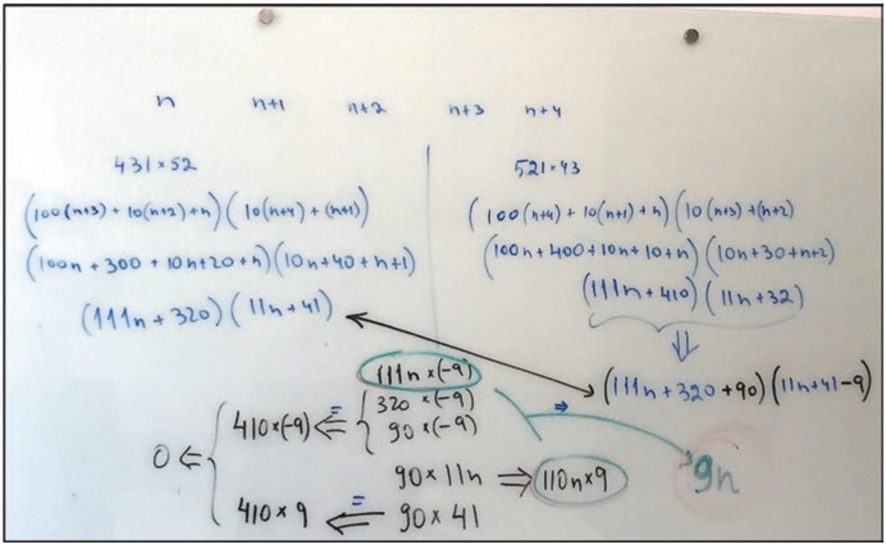

Fig. 14.6 Two compared products in expanded notation

which was previously calculated directly to be 9 in case of $n = 1$. This result was considered by the teachers as satisfying as the observed unexpected phenomenon was extended to the general case.

We note that the "general" proof developed in class only determines which product from the chosen two is larger. It does not attend to the problem in its generality, considering other possible products. While it appealed to the teachers' need for causality, drawing comparison between two cases only was not justified. As a result, we pursued a challenge of proving the general case, which satisfied our personal intellectual needs.

14.6 Mathematicians Responding to the Five-Digits Task

Having considered the teachers' approaches, we wondered how individuals with a stronger mathematics background would respond to the Five-digits Task. Two mathematicians agreed to be interviewed and "think aloud" as they approached the task. We present summaries of the two interviews below.

14.6.1 Interview with Ada

As the first step in her approach, Ada determined how many different cases exist in this task. She determined that there were $5! = 120$ cases, which was too much for a trial and error method. She further noted that not all cases were potential “candidates” for the largest product. As such, she reduced the number of cases that need to be considered.

I:	So what cases will you focus upon?
Ada:	It is clear that in the first positions in 3-digit number and 2-digit number needs to be the largest digit. I mean that the digits 5 and 4 will take the first places in these numbers: 5 _ _ × 4_ or 4 _ _ × 5_. Now I’ll care about place for digit 3. If 3 is placed in a 2-digit number, we’ll have two next products to consider: 521 × 43 and 421 × 53.
I:	Please go on.
Ada:	Without calculating the product we can compare as follows: (a) 521 × 43 = 421 × 43 + 100 × 43 or (b) 421 × 53 = 421 × 43 + 421 × 10 We can see that product in (a) is larger than the product of the task (b).
I:	Ok, what next?
Ada:	To continue, if digit 3 is in the second place in the 3-digit number, then we have to check 4 cases [(c) – (f)]: (c) 532 × 41 = 531 × 41 + 1 × 41 or (d) 531 × 42 = 531 × 41 + 1 × 531 In this competition the winner is product (d). And in the next comparing (e) 432 × 51 = 431 × 51 + 1 × 51 or (f) 431 × 52 = 431 × 51 + 1 × 431 the larger product is in the task (f).
I:	So what are we left with?
Ada:	Now we have only 3 candidates for the largest product: (a) 521 × 43, (d) 531 × 42 and (f) 431 × 52. Now I’ll compare (a) 521 × 43 = 521 × 42 + 521 × 1 and (d) 531 × 42 = 521 × 42 + 10 × 42. Obviously, product of (a) is larger than product of (d). Finally there are only two candidates now: (a) and (f). (a) 521 × 43 = 520 × 43 + 1 × 43 (f) 431 × 52 = 430 × 52 + 1 × 52 It is easy to see that the product of (f) is the largest.
I:	Easy to see?
Ada:	Sure, easy now, after all the previous work.
I:	Interesting…
Ada:	And, it will be interesting to try to generalize this problem: You have 5 different digits: a_1, a_2, a_3, a_4, a_5. Put them (each digit one time only) in the task: _ _ _ × _ _, so that the result will be largest.

14.6.2 Interview with Ben

I:	Ready to start and talk aloud as you are thinking?
Ben:	It is clear that on the first positions in 3-digits number and 2-digits number should be the bigger digits: 5 and 4 will take the first places in these numbers: 5 _ _ × 4_ or 4 _ _ × 5_. In each case, we have only 3! = 6 options for 3 rest digits. Then we have 12 products. But actually, we have to check only 6, because in half of cases the digits will not be in decreasing order. The cases are: 521 × 43 531 × 42 532 × 41 431 × 52 421 × 53 432 × 51
I:	So how will you approach the 6 cases, you do not have a calculator.
Ben:	I believe that the first product is the largest, that is how it appears. But let's see …

Ben chose to calculate the six products.

1. 521 × 43 = 22403
2. 531 × 42 = 22302
3. 532 × 41 = 21812
4. 431 × 52 = 22412
5. 421 × 53 = 22313
6. 432 × 51 = 22032

Having calculated the products, Ben appeared surprised with the result.

Without prompting from the interviewer, he began to analyze what contributed to his erroneous prediction.

Ben:	My argument was to go with the biggest 2-digit number when the 3-digit number will start with digit 5. So I got 521 × 43. In both cases the first one and the fourth we have 500 × 40 = 50 × 400. What made the difference? Then in first case we have in positions of tens: 40 × 20 and in the fourth case: 50 × 30. So we have +700 for the forth case. On the other side we have in the first case 500 × 3 and in the fourth case: 400 × 2. Then we have −700. It means that we have equality. Then we have another equality: 20 × 3 = 30 × 2. And finally 1 × 43 < 1 × 52. That was the reason why we had so close result.

14.6.3 Comments on the Mathematicians' Approaches

Both Ada and Ben used a very systematic approach: they clearly identified the relevant cases to be considered and articulated why there was no need to consider additional cases. In both interviews six products were considered: Ben calculated the six products and examined his incorrect prediction. Ada did not multiply any numbers; she compared the products in pairs, until the largest product was identified, to which she referred as the ‘winner’. Metaphorically, she treated this pairwise comparison as a tournament, where a winner of a game competed in the next round.

This systematic approach can be seen as the evidence of the mathematicians’ intellectual need for certainty, when the certainty is assured by examining all the “candidate” cases. We point out that only one (P5) of the participating teachers attended to the relevant six cases in their calculations. Furthermore, the number of computed products did not correspond to the claimed certainty.

With respect to the need for causality, we believe that causality is embedded in Ada’s comparative method. The need for causality featured differently in Ben’s approach: he was interested in the cause of his mistaken prediction, rather than in explaining why the product 431 × 52 was the largest among the six. We further observe that Ben’s approach in comparing two products was similar to what was suggested in class by P3, the approach that she and her classmates did not pursue to a satisfactory resolution.

14.7 Proceeding with a Self-Challenge

Our analysis of the work of teachers and mathematicians challenged us both mathematically and pedagogically. We describe in this section how we addressed these challenges.

At the end of her interview, Ada wondered about a possible generalization: “You have 5 different digits: a_1, a_2, a_3, a_4, a_5. Put them (each digit one time only) in the task: _ _ _ × _ _, so that the result will be largest.”

We interpreted this generalization as the need for structure and formulated Ada’s proposal as the following conjecture: “for all digits *a, b, c, d, e* such that: $1 \leq a < b < c < d < e \leq 9$, the product $\overline{dca} \times \overline{eb}$ will get the maximum value”. We engaged in this self-challenging task and generated different proofs, starting with the case of five consecutive digits and then pursuing the general case. However, before attempting to prove, we verified our conjecture by using a computer program.

14.7.1 The Case of Five Consecutive Digits

For given 5 consecutive digits (n, $n + 1$, $n + 2$, $n + 3$, $n + 4$) that should be used one time each in the product of a 2-digit number by a 3-digit number: _ _ _ × _ _, what combination gives the largest product?

We should compare the next six products:

$$\overline{n+4, n+2, n+1} \times \overline{n+3, n},$$

$$\overline{n+4, n+2, n} \times \overline{n+3, n+1},$$

$$\overline{n+4, n+1, n} \times \overline{n+3, n+2},$$

$$\overline{n+3, n+2, n+1} \times \overline{n+4, n},$$

$$\overline{n+3, n+2, n} \times \overline{n+4, n+1},$$

$$\overline{n+3, n+1, n} \times \overline{n+4, n+2}, \text{ for each } n = 2, 3, 4, 5.$$

There are only six cases in which all digits in both numbers are in decreasing order.

1. $\left(100(n+4)+10(n+2)+n+1\right)\times\left(10(n+3)+n\right)=1221n^2+7961n+12630$
2. $\left(100(n+4)+10(n+2)+n\right)\times\left(10(n+3)+n+1\right)=1221n^2+8061n+13020$
3. $\left(100(n+4)+10(n+1)+n\right)\times\left(10(n+3)+n+2\right)=1221n^2+8062n+13120$
4. $\left(100(n+3)+10(n+2)+n+1\right)\times\left(10(n+4)+n\right)=1221n^2+7971n+12840$
5. $\left(100(n+3)+10(n+2)+n\right)\times\left(10(n+4)+n+1\right)=1221n^2+8071n+13120$
6. $\left(100(n+3)+10(n+1)+n\right)\times\left(10(n+4)+n+2\right)=1221n^2+8072n+13020.$

We need to compare only between sums of 2 last terms of the 6 algebraic expressions.

It is clear that $7961n + 12630 < 8061n + 13020$, $8062n + 13120 > 7971n + 12840$, and $8071n + 13120 > 8072n + 13020$ for every $n = 2, 3, 4, 5$.

Now we need to check if one of 3 algebraic expressions $8061n + 13020$,

$8062n + 13120$, and $8071n + 13120$ is bigger than others for every $n = 2, 3, 4, 5$.

And it is clear that: $8061n + 13020 < 8062n + 13120 < 8071n + 13120$.

Then in all cases $n = 2, 3, 4, 5$ the product $\overline{n+3, n+2, n} \times \overline{n+4, n+1}$ is the largest.

14.7.2 General Case

Proof A

In the general case, we should show that for all digits *a, b, c, d, e* such that:

$1 \le a < b < c < d < e \le 9$, the product $\overline{dca} \times \overline{eb}$ will give the maximum value of all products of a 2-digit number by a 3-digit number.

We should observe and compare the values of next 6 products:

(1) $\overline{dca} \times \overline{eb}$

(2) $\overline{dba} \times \overline{ec}$

(3) $\overline{dcb} \times \overline{ea}$

(4) $\overline{eba} \times \overline{dc}$

(5) $\overline{eca} \times \overline{db}$

(6) $\overline{ecb} \times \overline{da}$

First we compare the products in pairs: (1) and (3), (2) and (4), (5) and (6).

(1) $\left(10\overline{dc} + a\right) \times \left(10e + b\right) = 100\overline{dc} \times e + 10\left(\overline{dc} \times b + ae\right) + ab$

(3) $\left(10\overline{dc} + b\right) \times \left(10e + a\right) = 100\overline{dc} \times e + 10\left(\overline{dc} \times a + be\right) + ab$

Let us show that (1) > (3). It is clear that we need to show that

$$\overline{dc} \times b + ae > \overline{dc} \times a + be.$$

It follows that $\overline{dc} \times b - \overline{dc} \times a > be - ae$,
and then

$$\overline{dc} \times \left(b - a\right) > e\left(b - a\right),$$

and this last inequality is true ($\overline{dc} > e$; $b > a$).
Next we compare (2) and (4) in a similar way and get that

(4) $\overline{eba} \times \overline{dc} \;>\; \overline{dba} \times \overline{ec}$ (2)

Comparing the products (5) and (6), we get that:

(5) $\overline{eca} \times \overline{db} \;>\; \overline{ecb} \times \overline{da}$ (6)

Now we should find the "winner" between next 3 products:

(1) $\overline{dca} \times \overline{eb}$

(4) $\overline{eba} \times \overline{dc}$

(5) $\overline{eca} \times \overline{db}$

Let's compare the (1) and (5) products

(1) $\left(100d + \overline{ca}\right) \times \left(10e + b\right) = 1000de + 100db + 10\overline{ca} \times e + \overline{ca} \times b$

(4) $\left(100e + \overline{ca}\right) \times \left(10d + b\right) = 1000de + 100eb + 10d \times \overline{ca} + b \times \overline{ca}$

Now it is enough to show that

$$100db + 10\overline{ca} \times e > 100eb + 10d \times \overline{ca}$$

Or

$$10db - 10eb > d \times \overline{ca} - \overline{ca} \times e$$

$$10b(d - e) > \overline{ca} \times (d - e)$$

$$10b < \overline{ca}(d - e < 0, \text{b} < \text{c})$$

In the same way, we can show that (1) > (5).

Then in the general case where $1 \le a < b < c < d < e \le 9$, the largest product is $\overline{dca} \times \overline{eb}$.

Proof B

We are looking for the largest product of $\overline{xyz} \times \overline{uv}$, where $1 \le x, y, z, u, v \le 9$ are distinct natural numbers.

In other words, we are looking for the maximum value of the expression:

$$(100x + 10y + z) \times (10u + v) = 1000xu + 100(xv + uy) + 10(yv + uz) + vz$$

It is clear that if we want to get the maximal value, we should choose x and u the largest digits and z and v the smallest. Then y is the median digit: $1 \le v, z < y < x, u \le 9$.

The terms $1000xu$ and vz do not depend on the relative positions of x and u or v and z. Thus we attend to maximizing the value of $100(xv + uy) + 10(yv + uz)$ or the value of $10(xv + uy) + yv + uz$

What positions should our digits have to make the expression $10(xv + uy) + yv + uz$ yield its maximum value? We have to check the next four cases:

(1) $1 \le v < z < y < x < u \le 9$,
(2) $1 \le z < v < y < x < u \le 9$,
(3) $1 \le v < z < y < u < x \le 9$,
(4) $1 \le z < v < y < u < x \le 9$.

Next we show that case (1) leads to a larger result than case (3):

In case (1) we suppose: $x = u - t$, $t \ge 1$; whereas in case (3) x and u will exchange their places. Then we show that the next inequality holds:

$$10((u - t)v + uy) + yv + uz > 10(uv + (u - t)y) + yv + (u - t)z.$$

Indeed, after simplification we have:

$$10ty + tz - 10tv > 0 \text{ and then } t(10(y - v) + z) > 0$$

Given that $y > v$, the last inequality holds for all digits t, y, v, z.

In the same way, we can show that case (2) leads to a larger result than case (4). And finally comparing cases (1) and (2) we get that in case (2) we have the largest result.

Therefore, the order of the digits $1 \le z < v < y < x < u \le 9$ will give the maximum product, and the largest product is $\overline{xyz} \times \overline{uv}$.

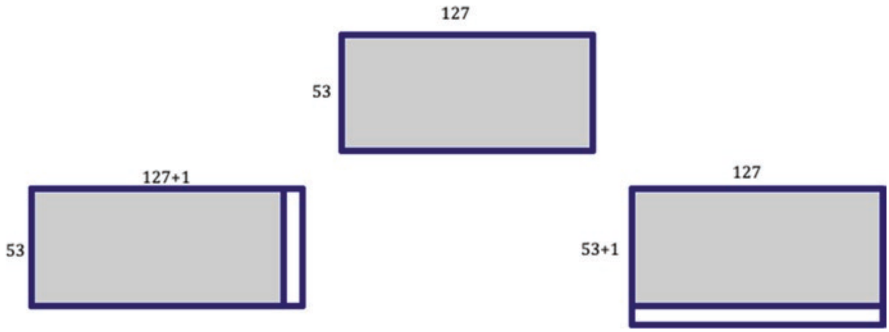

Fig. 14.7 Comparing products by considering areas of rectangles

14.7.3 Detour: On a Simpler but Similar Task

Which product results in a larger value: 128 × 53 or 127 × 54?

This task resembles the Five-digits Task discussed above not only in a search for a larger product but also in the fact that most people guess incorrectly, focusing on 128 vs. 127, that is, on increasing the larger number. In addition to calculating the product (6784 vs. 6858), the larger product can be determined by invoking distributivity, as in the case of the Five-digits Task:

$$128 \times 53 = (127 + 1) \times (54 - 1) = 127 \times 54 + 54 - 127 - 1 < 127 \times 54$$

However, we were challenged to find an alternative explanation, in which the conclusion is more intuitive. Such an explanation is provided in Fig. 14.7. We start with a rectangle with dimensions 127 × 53 and then increase one side or the other by 1.

The larger product (127 × 54) is evident by considering the resulting rectangles. This generic illustration clearly points to a larger product comparing $(a + k) \times b$ and $a \times (b + k)$.

14.8 Discussion

The Five-digits task does not resemble the 'Rope around the Earth' problem at all. Nevertheless, we acknowledge a significant similarity. The similarity – as we see it – is that the results are unexpected and therefore surprising. The unexpected result in the 'Rope around the Earth' task is well acknowledged in the literature (e.g., Arcavi, 2003; Farlow, 2014). While we reported only results from 17 teachers and 2 mathematicians, we presented this task to various groups of students and in most cases, the correct solution did not emerge immediately and was unexpected. As such, both tasks present a challenge in provoking the intellectual need for causality, the desire to find an explanation for the phenomenon, rather than to simply indicate a solution.

"Mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (Steiner, 1978, p. 135). Hanna (1990) articulated this difference further, claiming that proofs that "just" prove establish *that* the statement is true, while the proofs that also explain establish *why* it is true. While our task does not explicitly deal with proofs, we note a similarity. In the case of the Five-digits Task, the result of calculating the relevant products can be seen as proof that does not explain, whereas the comparison of the relevant products without calculating the result provides an explanation to the phenomenon.

Harel (2013) argued for directing students' attention to the cause of phenomena. He suggested that "By repeatedly attending to explanations as well as to proofs, we aimed at enculturating students into the habit of seeking to understand cause, not only attaining certainty" (2013, p. 127). In the case of secondary school teachers and mathematicians, those participating in our study, the desire to find the cause arose naturally, which exemplifies the intellectual need for causality. Furthermore, the generalization proposed for examination by the mathematician Ada, exemplifies her intellectual need for structure, for determining a regularity in which the five digits are not limited to the consecutive and smallest. We pursued the proposed generalization by proving the general case.

As we claimed above, challenge can be found in rather simple tasks, by adding some constraints. The challenge for our participants was to explain the results without any computation. Thus, seeking causality after certainty is achieved is a valuable activity for teachers, especially when the result is rather unexpected.

Continuing with the book's theme of "challenge for all", we conclude with a variation that some readers may find challenging:

What choice of digits results in the smallest product?
Does the knowledge of what product is the largest help in finding the smallest one?
How can the Five-digits task be extended to a Six-digit task?

These questions resemble the "What if not" strategy described by Brown and Walter (1983). We invite the readers to challenge themselves and their students by varying the task and its constraints, while finding different explanations that satisfy both the need for certainty and the need for causality, particularly when obtaining unexpected results.

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Chapter 15
Visualization: A Pathway to Mathematical Challenging Tasks

Isabel Vale and Ana Barbosa

15.1 Introduction

Mathematics learning is strongly dependent on the teacher and the tasks proposed to students (e.g. Doyle, 1988; Stein & Smith, 1998; Sullivan et al., 2013). Thus, the teacher must develop students' mathematical understanding, creating situations to ensure that they have the opportunity to engage and be challenged in high-level thinking, through the tasks proposed. The teacher's choices will determine the quality of students' learning (e.g. Chapman, 2015; Stein & Smith, 1998). This implies the use of tasks that meet different ways of thinking displayed by the students, confronting them with multiple-solution tasks, that challenge them to see outside of the box, motivating them to learn and to work with each other. We are interested in visualization because it plays an important cognitive role in the teaching and learning of mathematics, as an aid to thinking, as a means of communicating mathematical ideas and as a useful tool in problem-solving (Arcavi, 2003). So, pre-service and in-service teacher training should promote an insight into the nature of mathematics and its teaching, meaning that teachers need to have different teaching and learning experiences similar to the ones they are expected to use with their own students (Ponte & Chapman, 2008; Vale & Barbosa, 2020).

Thus, after a theoretical discussion about challenging tasks with multiple solutions (e.g. Leikin, 2016; Stein & Smith, 1998) and visualization (e.g. Arcavi, 2003; Duval, 1999; Presmeg, 2014, 2020), we emphasize the use of visual processes in teachers' practices and their potential in the teaching and learning of mathematics. This discussion will be illustrated with examples based on studies carried out with pre-service teachers of elementary education (6–12 years). This perspective emerges from the work we have been developing in teacher training through which we have

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come to conclude that visualization is not valued in school practices, neither as a problem-solving strategy nor as a way to support the understanding of mathematical concepts.

15.2 The Mathematics Classroom

15.2.1 The Teacher and the Tasks

All students should have the opportunity to engage in meaningful mathematical activity, and it's the teacher's role to unlock their potential through the choice of adequate tasks and teaching strategies. Although tasks have the power to trigger mathematical activity, they may not be sufficient to implicate mathematical challenges. Teachers must establish a classroom environment that guarantees students' engagement in embracing mathematical challenges. This implies the use of strategies that contemplate the heterogeneity of a classroom, and stimulate students to think and interact with each other, leading to rich discussions.

An effective teaching approach implies the orchestration of productive discussions, giving students opportunities to communicate, reason, be creative, think critically, solve problems, make decisions and understand mathematical ideas (e.g. NCTM, 2014; Vale & Barbosa, 2020). This context requires an exploratory approach, anchored on inquiry-based learning (Engel et al., 2013), allowing students to learn mathematics by understanding, criticizing, comparing and being encouraged to use different approaches to solve non-routine tasks, discussing the multiple solutions and processes used. This approach is demanding for teachers and is often the reason for continuing to perpetuate the common classroom practice that some refer to as the Triple X teaching (exposition, examples, exercises) (Evans & Swan, 2014). The option for more traditional approaches is related to teachers' beliefs regarding mathematics teaching and learning, which influence the type of tasks and strategies used (Sullivan et al., 2013). Many of the fragilities that students have in learning mathematics are due to those options, but also to the gaps in the teachers' mathematical knowledge and in the use of innovative teaching strategies. Hence, mathematics learning is strongly dependent upon the teacher and the tasks that are proposed (e.g. Doyle, 1988; Smith & Stein, 2013; Stein & Smith, 1998). To improve the conceptual understanding of mathematical ideas, teachers must select challenging tasks that promote flexible thinking and problem-solving abilities (Smith & Stein, 2013; Stein & Smith, 1998).

Therefore, teachers should be able to take advantage of all the potential embedded in a task and, in order to do this, they need opportunities to explore and solve tasks in the same way that they are expected to explore with their own students (Stein & Smith, 1998; Sullivan et al., 2013). Teacher education programs should include experiences that stimulate teachers' knowledge, using the same principles as teaching mathematics to school students, in particular focusing on problem-solving situations that combine mathematical and pedagogical issues (e.g. Cooney & Krainer, 1996). The use of challenging tasks in these programs may lead to pre-service teachers not only being able to reproduce tasks/solutions/strategies presented, but also produce new and original proposals (Guberman & Leikin, 2013).

15.2.2 Challenging Tasks

Mathematical tasks may have different levels of demand, inducing different learning modes, so teachers should pay special attention to their choice of task. Therefore, task design triggers the activity developed by students, allowing teachers to introduce new ideas and procedures and students to have the opportunity to think differently (e.g. Chapman, 2015; Smith & Stein, 2013). Among the different tasks that we use in mathematics classes, we privilege the tasks that can be solved in different ways, which is very similar to the idea of multiple solution tasks (MSTs) proposed by Leikin (2016), because they develop mathematical knowledge and encourage flexibility and creativity in the individual's mathematical thinking (Leikin, 2016; Polya, 1973). Tasks should have an impact on mathematical activity, allowing students to assess their mathematical understanding, establishing relationships between concepts, and have enough flexibility to use divergent thinking.

Challenge is an important variable in the mathematics classroom because students can become demotivated and bored very easily in a "routine" class. Some may even have difficulties in learning unless they are challenged (Barbeau & Taylor, 2009; Holton et al., 2009). A mathematical challenge occurs when the individual is not aware of procedural or algorithmic tools that are critical to solve a problem and seems to have no standard method of solution. It includes a strong affective call involving curiosity, imagination, inventiveness, and creativity and it is placed intentionally to attract students to their solution (Barbeau, 2009). In this sense, for some problem-solving authors (e.g. Kadijevich, 2007; Polya, 1973; Schoenfeld, 1985) a problem is a mathematical task that challenges learners to solve it.

The expression *challenging* task is normally used to describe a task that is interesting and perhaps enjoyable, but not always easy to deal with or attain, and should actively engage students, developing a diversity of thinking and learning styles. Thus, even when it is not easy to deal with or to solve, it is perceived by the solver as an interesting and enjoyable problem. The engagement in productive struggle allows learners to widen their understanding (NCTM, 2014). Challenging tasks may particularly be those that require the learner to relate mathematical concepts or procedures, by considering, for example, different representations, views or applications (Kadijevich, 2007). According to Leikin (2014) a mathematical challenge is a mathematical difficulty that a person is able and willing to overcome. However mathematical challenges are not just difficult problems. The same problem may be a challenge for one student and a routine problem for another (Holton et al., 2009).

As the one who introduces challenges in the classroom, the teacher must be aware of some particular circumstances. For instance, appropriate challenges can be given to mathematically able students as well as to less qualified ones. The solution for the same task may also be scaffolded differently to different students, providing challenges at several levels. Our difficult role and goal are to engage students with different mathematical backgrounds in different settings so that they can further develop their mathematical ideas, reasoning and problem-solving strategies, as well as their enjoyment in solving mathematical tasks. According to Leikin (2014), we need to develop students' mathematical potential through an adequate level of mathematical challenge. Tasks are considered rich or good because they give students the opportunity to learn, choosing from several different areas of mathematics and different mathematical and non-mathematical abilities, and using these in an integrated, creative and meaningful way. This is in accordance with the use of MSTs, suggested by Leikin (2016) as "a didactical and research tool in the majority of the studies that focus on the identification, development, and role of creativity in the teaching and learning of mathematics to students and teachers" (p.7). The author considers the link of MSTs to creativity, expressed in the differences in learners with varying levels of excellence in school mathematics (or in teachers with varying levels of expertise), whether they use insight-based solutions (related to an *aha!* moment) or, in contrast, learning-based solutions (the standard ones) of the problems. In this sense, for Leikin (2016), challenging mathematical tasks can "require solving insight-based problems, proving, posing new questions and problems, and investigating mathematical objects and situations" (p. 1–28). Insight-based problems are the ones that have a relatively simple solution which is difficult to discover until solution-relevant features are recognized (Weisberg, 2015, cited by Leikin, 2016). These kinds of tasks are "challenging either for novices or expert students requiring flexibility when finding additional solutions and raising different conjectures as well as originality when finding new mathematical facts and new mathematical solutions" (Leikin, 2016, p. 10).

We define MSTs as tasks that invite different ways of solving a problem, which constitutes a challenge for the solver. This only makes sense in an exploratory teaching where the teacher is the orchestrator (Smith & Stein, 2013), according to effective teaching of mathematics, that engages students in solving and discussing challenging tasks. This environment promotes mathematical reasoning and problem-solving with multiple and varied solution strategies, including visual ones, promoting creativity (Leikin, 2016; NCTM, 2014; Presmeg, 2014).

15.3 The Potential of Visualization

Throughout the history of mathematics, it is possible to identify moments when visualization and arguments of visual nature played a major role in mathematical activity, but also periods of time when this way of thinking was avoided. However, for the last two decades, we have seen a growing interest in the use of images as a general cultural change. Considering that mathematics requires the frequent use of diagrams, figures, tables, spatial arrangements of symbols and/or other types of representations, the recognition of the importance of visual processing and external representations associated with visualization has been progressively evident. Visualization has acquired an important status in mathematics, not only due to its illustrative functions but also to its recognized relevance as an important component of mathematical reasoning and proof (Arcavi, 2003).

The role of visualization in mathematics learning has been subject of much research and discussion as has been the delimitation of its meaning (e.g. Arcavi, 2003; Dreyfus, 1995; Presmeg, 2006, 2020; Stylianou & Silver, 2004). Many authors embrace the definition of visualization proposed by Arcavi (2003), which is broad enough to include product and process, visualization as an artifact, as well as the meanings constructed by individual learners (Presmeg, 2014):

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

This definition contemplates different facets of visualization, considering it as a bidirectional process between mathematical understanding and the external environment.

The discussion about the nature and role of visualization in the teaching and learning of mathematics is not simple. Much has been written about the potential of this ability in the development of an intuitive perspective and in the understanding of concepts associated with different areas of mathematics (Zimmermann & Cunningham, 1991). One thing is settled, visualization must not be reduced to the mere production or appreciation of figures or drawings, or even to the development of knowledge within the scope of geometry, on the contrary, it fosters an intuition that contributes to the clarification of mathematical ideas of different nature (Dreyfus, 1995).

Adding to the previous ideas, we can also highlight the relevance of visualization in problem-solving. This relation is unavoidable because visualization provides the use of intuitive and effective strategies that inspire creative findings (Nelson, 1993; Presmeg, 2006; Vale et al., 2018; Zimmermann & Cunningham, 1991). Actually, several studies have analyzed the advantages of using visualization in problem-solving (e.g. Presmeg, 2014, 2020; Stylianou & Silver, 2004; Vale et al., 2018) and it is a common idea that visual thinking contributes to the use of powerful strategies, different from those applied in more traditional approaches, where formalism and symbolism prevail. The use of visual forms of representation, like a drawing or a model, are frequently important aids to solve a diversity of problems, geometric or not, and can act either as unique strategies that lead to a solution or as a crucial starting point to solve a problem (e.g. Polya, 1973; Schoenfeld, 1985; Stylianou & Silver, 2004). In the scope of problem-solving we come across problems of a visual nature or which are presented in a visual context and, for that reason, may more easily be solved using a visual approach. Vale et al. (2018) propose the use of an additional and specific strategy called *seeing*. According to these authors:

> seeing involves an activity that may be associated with a more traditional range of strategies (e.g. draw a picture or diagram, solve a simpler problem, look for a pattern), but it is specifically considered as a strategy of thought that involves visual perception of mathematical objects and is blended with knowledge and past experiences. It includes imagining, which is related with having creative insights or *Aha!* experiences and intuitions; it can also be expressed in terms of drawing, which means translating one's ideas in some visual form. (p. 253)

The *seeing* strategy does not replace any other traditional problem-solving strategy, it is rather a way to tackle a problem. In spite of being a very useful strategy, as will be seen in the examples further ahead in this chapter, unfortunately, this approach is not always encouraged or used by teachers.

Visualization can also have a fundamental role as a complement to analytical thinking. For example, Fischbein (1987) comments that a visual image "is an essential factor for creating the feeling of self-evidence and immediacy" (p. 101) and "not only organizes data at hand into meaningful structures, but it is also an important factor guiding the analytical development of a solution" (p. 104). Visualization can act as a catalyst in understanding the meaning of concepts and in producing inductive reasoning, but it can also be an informal way of understanding deductive reasoning, with the algebraic treatment being done later.

Although visual approaches are considered to be a basis for learning in mathematics and also for problem-solving, the literature often mentions that many students show reluctance to explore visual support systems (Dreyfus, 1995; Presmeg, 2006, 2020). This phenomenon can be enhanced by several factors. On one hand, it is possible that mathematics, by its nature, favors the non-visual thinker, taking into account that the logical-verbal component is considered the sine qua non of mathematical abilities, while the visual-spatial component is not considered mandatory (Krutetskii, 1976). Another aspect is the relevance attributed to non-visual methods in the instruction process, under the conception that visual approaches are harder to teach and difficult for students to understand. Based on the ideas of Presmeg (2014) and Vale et al. (2018), we consider that visual solutions are understood as the way in which mathematical information is presented and/or processed in the initial approach or during problem-solving. They include the use of different representations of visual nature as an essential part of the process of reaching the solution (e.g. graphics, charts, figures, drawings). On the contrary, non-visual solutions or analytical solutions do not depend on visual representations as an essential part of achieving the solution, using other representations/procedures, such as numerical, algebraic and verbal ones (e.g. Presmeg, 2014; Vale et al., 2018).

Despite the fact that mathematical educators apparently recognize the potential of visual thinking, frequently this idea is not reflected in their practices, continuing to attribute a secondary role to this type of method. This should be faced as a concern since visual abilities are not self-evident or innate, but created, developed and learned, through teaching (e.g. Hoffmann, 1998; Whitley, 2004). So, regardless of the reasons, if teachers do not include visual approaches in their practices, it is unlikely that students are able to develop the visual-spatial component of their reasoning. Consequently, teacher training courses should include this discussion and awareness in the respective programs.

Other than this, not all students have the same preferences when it comes to learning mathematics. This is noticeable, for example, in the preference for the different themes in the curriculum, in the way they understand these themes and solve the respective tasks, privileging words, formulas or figures. Thus, teachers have to consider that students may have different learning styles and that they may also have different preferences in relation to mathematical communication, which has direct influence on the representations used. Emphasizing this perspective, psychologists and mathematical educators (e.g. Borromeo Ferri, 2012; Clements, 1982; Krutetskii, 1976; Presmeg, 2014) propose a typology of problem-solving strategies used by students, according to their learning styles: (1) *Visualizers* or geometric - prefer to use visual solution methods (figures, diagrams, pictures) or pictorial-visual schemes, even when problems could more easily be solved with analytical tools; (2) *Verbalizers*, non-visual or analytical - prefer to use verbal-logic approaches or non-visual solution methods (algebraic, numeric, verbal representations), even with problems where it could be simpler to use a visual approach; and (3) *Harmonic*, mixed or integrated - have no specific preference for either logical-verbal or visual-pictorial thinking, and tend to combine analytical and visual methods, showing an integrated thinking style. These styles of thinking are of great importance and influence the way that each student processes information. However, it is extremely complex to apply this categorization and difficult to distinguish, in absolute terms, an individual tendency of a student for a certain type of thinking (Clements, 1982). Nonetheless, these issues have strong implications in the classroom practices and, in particular, in the teachers' choices. Whether with the intention of meeting the diversity of learning styles, with the purpose of broadening the students' repertoire of strategies, or showing the potential of certain mathematical tools, teachers should promote the use of analytical and visual approaches and, if possible, integrate them in order to construct rich understandings of mathematical concepts (Zazkis et al., 1996). Despite the different learning styles, a teacher may find in a classroom, students should experience the use of different approaches to the same problem, either of visual or non-visual nature. This is fundamental to the development of a more flexible reasoning and to make more conscious decisions about the choice of strategies.

Taking into account the ideas discussed in this section, it is important to establish a connection between visualization and mathematical challenge, clarifying our perspective. The use of MSTs that allow the application of either analytical or visual methods can be effective instructional resources to promote the abovementioned flexibility, but also enhance the level of mathematical challenge.

This connection can be translated into different situations. Challenging tasks are thought-provoking mathematical problems that aim to include all students in the mathematical activity (Sullivan & Mornane, 2014) and, in this sense, thinking of the students' learning styles and preferences, these tasks generate an opportunity to extend their knowledge. A task can become more challenging, for example, when students are required to use specific mathematical knowledge in the solution process (e.g. concepts, procedures, representations, rules, or reasoning). For example, some may find the generation and use of a diagram in the solution challenging (e.g. Diezmann & English, 2001).

Visualization can elicit the development of intuition and the ability to see new relationships, producing a cut with mental fixations that enable creative thinking (Haylock, 1997), especially with students who are used to apply analytical methods. Krutetskii (1976) and Polya (1973) also point out that one of the characteristics of mathematically competent students is being able to look for a clear, simple, short, and therefore elegant, visual solution to a problem. This endeavour can be seen as a challenge. Reinforcing this discussion, we would like to draw attention to the importance of developing the students' *mathematical eye* or *geometrical eye* (Fujita & Jones, 2002), referring to the use of mathematics as a lens to see and interpret things/elements that surround us. It means to see the unseen, interpret things in the world as a boundless opportunity, and discover the mathematics involved by seeing the world around us with new eyes. For most people, the mathematics that surrounds them often remains "invisible" to their untrained or inattentive eye. That is why it's necessary to educate the mathematical eye so that they can identify contexts and elements that can become more competent in tackling rich and challenging mathematical tasks (Vale & Barbosa, 2020). Also, certain tasks can be difficult to solve with analytical tools either because of their strong visual structure or because of the students' lack of specific knowledge to solve them. In these cases, visualization may help face the mathematical challenge, focusing on the visual cues of the task or using a dynamic solution to understand key mathematical relations, acting as a support for understanding (Duval, 1999; Presmeg, 2020).

Our perspective on the connection between visualization and challenging tasks is using MSTs that allow different methods (analytical or non-visual, visual or mixed) and invite students to go beyond the conventional knowledge or their personal style of thinking and push them towards visual approaches, which are normally out their comfort zone.

15.4 Visual Contexts and Challenge in Mathematics

The potential and limitations of visual reasoning are recognized as part of the mathematical culture of the classroom (e.g. Arcavi, 2003; Presmeg, 2014), as well as being particularly beneficial for all students, especially those with more difficulties (e.g. Gates, 2015; Vale et al., 2018). However, visual strategies that use different representations are not always fully used to solve a problem. They are usually overlooked by the routine use of rules and procedures learned without meaning, which reduces teaching to a mechanized and monotonous process of numerical, symbolic and/or algorithmic manipulation, diminishing the challenge that is intended in a task. We agree with Roche and Clarke (2014) when they state that all students should experience challenging tasks, but sometimes teachers are reluctant to pose these types of tasks. Often due to lack of knowledge, insecurity, the type of teaching and learning they practice, or because they do not have an available repertoire of tasks.

In the next sections, we start by presenting two examples of tasks that illustrate the importance of visualization and of visual skills. In the first example, we intend to highlight the potential of visual solutions in the context of problem-solving and in the second one the power of a visual approach in situations involving proof or mathematical validation of a statement. The following examples refer to tasks proposed to our students, future elementary education teachers (6–12 years old), during their teacher training course. We focus on MSTs with different themes, cognitive demands and contexts, privileging visual features, as an alternative to more traditional approaches. These students were subjected to instruction that highlighted the potential of visual solutions, and contacting with different strategies of this nature.

We advocate, in teacher training, the use of MSTs, recurring to visual contexts, to show that the same problem can be solved in many different ways and that visualization can be helpful, not only as a strategy per se but also as a means to help give meaning to analytical approaches, promoting the establishment of connections between different representations. These are thought-provoking mathematical problems that generate the opportunity for solvers to extend their knowledge, specifically related to visual representations and this is the challenge.

15.4.1 Example 1: Visual Solutions in Problem-Solving

Some problems may be complicated for individuals who are analytical in the solutions they adopt, or, at least can be more laborious due to the number of calculations required. However, after the discovery of the visual relations involved, with some intuition or *aha!* experience to begin the solving process, these problems become much simpler, hence accessible to more students (Vale et al., 2018). They can include conventional (i.e. learning-based) and unconventional (not learning-based that usually require insight) solutions (Leikin, 2016).

This first example, presented in Fig. 15.1 (Vale et al., 2020), illustrates the idea that looking for a visual solution can either be helpful to solve a problem for

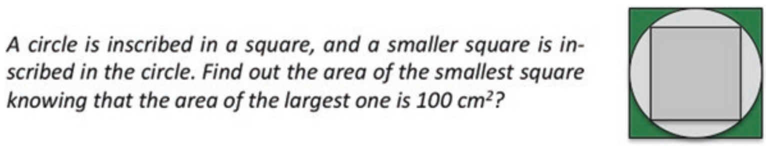


Fig. 15.1 Problem-solving task

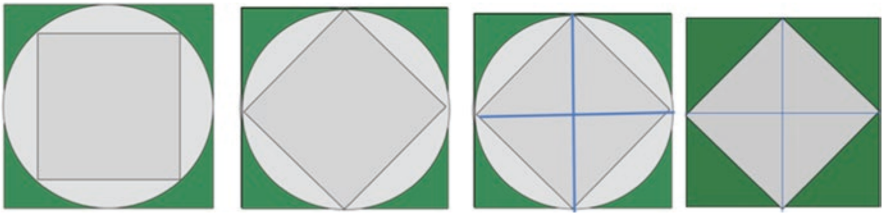


Fig. 15.2 Visual sequence of the steps to achieve the solution

students who don't have certain concepts internalized or to propose some complex problems to more elementary levels.

The traditional way to solve this task is to use the Pythagoras Theorem and the formulas of the area of the circle and of the square. However, if we see the smaller square in another position, the solution is immediate and free of errors or unknowledgeable formulas. According to Duval (1999), this transformation gives an insight into the solution of the problem since it is quickly understood and retained longer, than a sequence of words. Fig. 15.2 shows a visual sequence of steps to achieve the solution $\left(\frac{1}{2}\times 100 = 50\right)$.

This is a simple, clear and elegant solution, or a dynamic solution (Krutetskii, 1976; Presmeg, 2014). As Duval (1999) claims we reconfigured the figure, changing its position, and this kind of transformation does not require a mathematical justification because it is immediate and obvious. The insight of the solution manifests itself in breaking with the conventional or established set of knowledge (e.g. Haylock, 1997; Presmeg, 2014), such as the use of formulas that can be induced by the word area. This is an insight-based problem that has a relatively simple solution which is difficult to discover until solution-relevant features are recognized (Weisberg, 2015, cited by Leikin, 2016).

We may say that the *seeing* strategy is not essential because more traditional analytical strategies could have been used, like the use of calculation methods/procedures or formulas (non-visual). However, this strategy simplifies the process of solving the problem and, at the same time, allows us to relate other knowledge and develop the flexibility that underlies divergent thinking, which is one of the characteristics of creativity. Furthermore, *seeing* can serve to "unpack" the structure of a problem and direct the foundation for its solution (Diezmann & English, 2001).

15.4.2 Example 2: Visual Solutions in Proof

Nelson (1993), in his book *Proofs without words*, was able to draw attention to the importance of the visual approach in mathematical proofs, where he argues that "a picture or a diagram helps the solver see why a particular statement may be true, and also see how one might begin to go about proving it true" (p. vi). However, some mathematicians consider such visual arguments to be of poor value. On the contrary, there are mathematicians who defend the potential of visualization, among them Polya, when he says that drawing a figure is a powerful strategy to solve a problem, adding to the perspectives of Einstein and Poincaré on the importance of using visual intuitions in their work. In the same way, Gardner (1973, cited by Nelson, 1993) refers to the power of visualization when faced with a boring test, it can often be overcome by a simpler and more pleasant analogue geometric proof that allows the truth of the statement to be understood at first glance. Arcavi (2003) goes further when he says that visual representations are legitimate elements of mathematical proofs.

According to Nelson (1993), figures or diagrams can help to see why a certain statement can be true and, at the same time, see how to start proving its veracity. Often the use of algebra can help guide this process, but the emphasis is clearly on providing visual cues to the observer to stimulate mathematical thinking.

The arguments must be rigorous as they can easily lead to misinterpretations and, therefore, lead to wrong inferences. In any case, the importance of visual representations is recognized as a support in the discovery of new results and in the production of more formal tests, and above all for its role in the teaching and learning of mathematics (e.g. Presmeg, 2014).

The second example, shown in Fig. 15.3 (Vale, 2017), illustrates that many mathematical statements can be simply proved by translating the numeric information of the statement into a visual interpretation and discover the relationships and properties that can be established in that figure.

This is an usual example when working with numerical sequences, in particular the sum of geometric progressions. The traditional way to solve this task is to use a formula to calculate the sum of n terms of that geometric progression. It is a task that involves numerical manipulation and the use of formulas without much meaning for most students. But translating the task into a geometric model, the problem acquires another meaning, more challenging, allowing to visualize the infinite sum of the sequence. It starts from a square of unitary area (Fig. 15.4) from which we successively obtain figures with half the area of the previous figure and so on, until physically possible to divide (which allows to have the notion of pattern, generalization, infinity, infinite sum, limit, convergence). That is, the total area of the different squares and rectangles is the same as the sum of all terms of the progression. Since this area is equal to the area of the starting square, the sum of the progression terms is 1.

Prove that $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1.$$

Fig. 15.3 Proof task

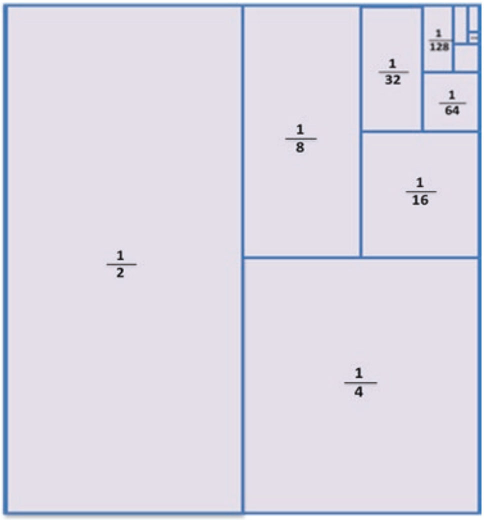


Fig. 15.4 Visual proof

This is a problem that is not normally presented to elementary school students (6–12 years old) as they do not yet have the necessary knowledge to prove the statement analytically. But the solution presented in Fig. 15.4 allows a visual understanding of the underlying meaning in the statement, as they build the figures, based on the concept of fraction in its part-whole interpretation and the intuitive notion of limit. Furthermore, students with more advanced mathematics knowledge quickly and simply understand the meaning of the statement.

15.4.3 Examples in Pre-service Teacher Training

The following examples involve diversified mathematical contexts and, in addition, illustrate the use of the strategy *seeing*. The tasks elicit multiple solutions, promoting some of the dimensions of creativity, apart from mathematical knowledge, and have the potential to promote visual strategies and eventually provide an *aha!* experience. The pre-service teachers were challenged to solve the tasks in as many different ways as they could, being encouraged to present several solutions.

The visual approach to these tasks was unfamiliar to this public, due to the lack of previous experience and visual literacy. In this sense, in spite of frequently generating simpler and elegant solutions, visual methods are challenging for these students, which implies that their training programs contemplate this perspective.

15.4.4 Symmetries

Geometric transformations, and in particular symmetries, are one of the most important applications of mathematics in daily life and nature, allowing the establishment of rich connections. It enables students to explore/create patterns, solve problems and think spatially. However, students generally show a low level of learning when geometric transformations are concerned. This is a theme where the spatial and visual abilities of the solver are essential to attack specific problems and to recognize the different transformations in everyday situations.

Our students, future teachers, were exposed to the teaching of geometric transformations (translations, rotations, reflections and glide reflections), analyzing examples of applications in mathematics and other areas.

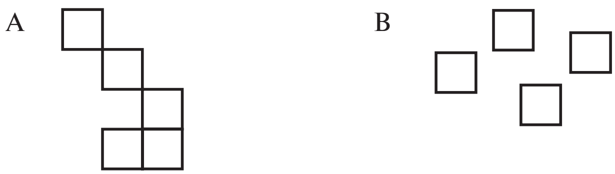
This is a purely visual task, where the solution results from the drawing of different shapes of A, which we propose in Fig. 15.5 (Barbosa & Vale, 2019). It is challenging for the solver: it allows to identify the students' knowledge of symmetries but also, being a multiple solution tasks, with more than one correct answer, enables students' creativity.

Figure 15.6 illustrates some of the productions presented by the students. The solutions to this task are, by nature, visual ones, because the task was proposed in a purely visual context. In the first question, the students had no difficulty to reach the solution, however, some presented only one possibility.

However, in the second question, when asked to build a figure with rotation symmetry, less than a quarter of the students succeeded. Figure 15.7 shows three correct solutions (the first three images) and two incorrect solutions (the last two images).

It is important that teachers promote these kind of abilities in students for the duration of their compulsory education, mainly for the non-visualizers, because a mathematics course for higher levels may not be enough to accomplish these goals.

Image A has five squares.



Add to image A the four isolated squares, observed in image B, so that the resulting figure has:

1. *reflectional symmetry.*
2. *rotational symmetry.*

Present two different solutions for each case.

Fig. 15.5 Task involving symmetries

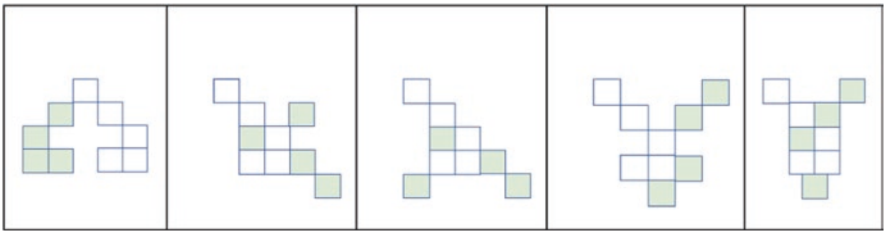


Fig. 15.6 Some solutions with reflectional symmetry

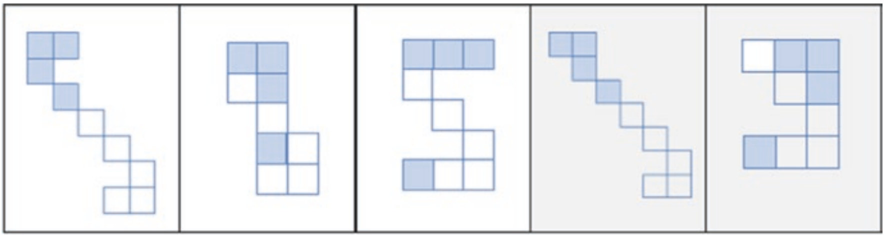


Fig. 15.7 Some answers for the rotational symmetry question

15.4.5 The Vasarely Rhombus

The Vasarely Rhombus task (Fig. 15.8) has a geometrical nature and the aim is to find the area of the shape.

This geometric problem (Vale et al., 2016) confirmed our expectations, that the students would present analytical strategies at first. But, as they were asked to solve the task using more than one process, other categories of solutions appeared. This allowed to identify fluency and flexibility. Figure 15.9 shows examples illustrating those categories.

The first solution is the most usual in this type of problem. As you ask for the area, students start immediately by writing the formula for the area of the rhombus or the triangle. Thus, this solution is purely analytical, that is, a learning-based solution (Leikin, 2016). The other two following solutions reveal the use of similar strategies and are considered mixed solutions. Despite using calculations, they are based on reasoning resulting from the properties of the figure and the concept of area measurement. These were the most common solutions used by the students. The last solution illustrates the application of a simpler and intuitive strategy, resulting from *seeing* the relationships between the different shapes identified in the rhombus and the square, getting a dynamic solution (Presmeg, 2014) or a reconfiguration of parts of the figure (Duval, 1999). This was the most original solution. In our opinion, this unique solution was original because it differs from the expected outcome for this type of task. Anyway, this solution was simpler since, merely by observation, we can conclude that the area of the rhombus is 1/3 of the area of the square.

What is the area of the rhombus, if M_1, M_2, M_3, M_4 are middle points
Of each side of the square and the square has 1 unit of area?
Find out more than one process to get to the solution.

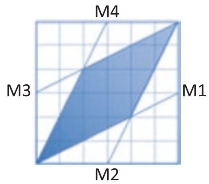


Fig. 15.8 The Vasarely Rhombus task

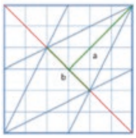
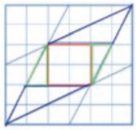
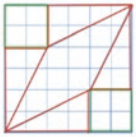
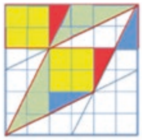
			
Area formula − Pythagoras Theorem $A_\Delta = \frac{\frac{\sqrt{2}}{3} \times \frac{\sqrt{2}}{2}}{2} = \frac{1}{6}$ $A = \frac{1}{3}$	Decomposition $A =$ $= 4 + 3 + 1 + 3 + 1 =$ $= 12$ $A = \frac{12}{36} = \frac{1}{3}$ $A = \frac{1}{3}$	Framing $A = 36 - 2x(4 + 8)$ $= 12$ $A = \frac{12}{36} = \frac{1}{3}$ $A = \frac{1}{3}$	Dynamic solution − decomposition/composition $A = \frac{1}{3}$

Fig. 15.9 Some solutions to this task

This problem, in addition to enhancing the use of different strategies, allows one to approach various contents (e.g. areas, relationships between figures, rational numbers, Pythagoras Theorem), promoting the establishment of connections between mathematical concepts, which can be further explored by the teacher. The cognitive level of the task can increase if we have the same situation without the square grid.

15.4.6 Rational Numbers

The following example is a word problem involving rational numbers. With this example, we see the adequacy and potential of visual solutions even in non-visual contexts. Usually, it is proposed when students are learning concepts and procedures related to this topic, like the different interpretations of fractions or the operations with fractions (Fig. 15.10).

Traditionally word problems with fractions are solved using analytical approaches, but in this example, the whole is an unknown quantity, which normally makes this problem more complex. This can sustain the poor results obtained by some pre-service teachers who chose to solve the task using numerical tools and computation (CHP, 2011; Vale et al., 2018). However, as these students had previous instruction about the use of visual strategies, such as the bar model, the majority used this approach.

The most common analytical solution was to start by determining the part of the whole that remained after taking away the part of the students that use the bus, $1-\frac{1}{3}=\frac{2}{3}$. Calculating the part of the students that go to school by car, we have $\frac{1}{4}\times\frac{2}{3}=\frac{2}{12}=\frac{1}{6}$. As the part of the students that go by car is $\frac{1}{4}$, or 90 students, the whole will be $6 \times 90 = 540$. We conclude that the school has 540 students. For many students, this numerical manipulation is not always understood, even for pre-service teachers, as it involves conceptual and procedural knowledge to solve the task.

The use of a visual model can be helpful in a first stage of learning and dealing with fractions or to make sense of analytical procedures. One approach that fits this criterium is the bar/rectangular model. Solving the same task with this strategy, some students started by using the bar to represent the unknown quantity, and then the needed data can be obtained using successive bars (Fig. 15.11).

In alternative to the previous solution, some students chose to use only one bar, concentrating all the needed information in one representation (Fig. 15.12):

Students go to school using different means of transportation. One third of the students go by bus. One quarter of the remaining students goes by car. The others take a bike or walk to school. Knowing that 90 students go to school by car, how many students attend this school?

Fig. 15.10 Task involving rational numbers

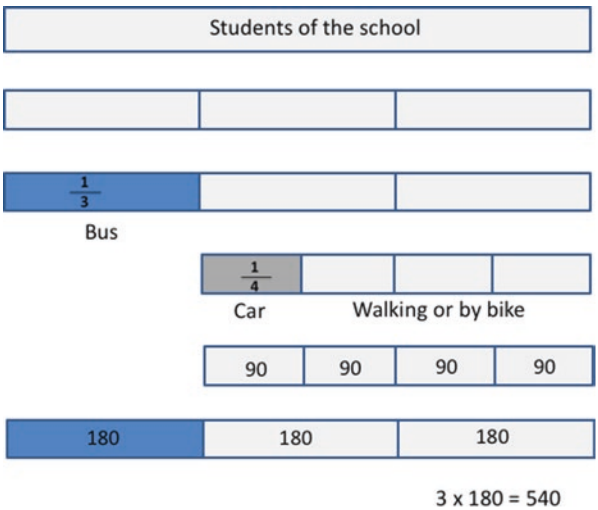


Fig. 15.11 Visual solution of the task

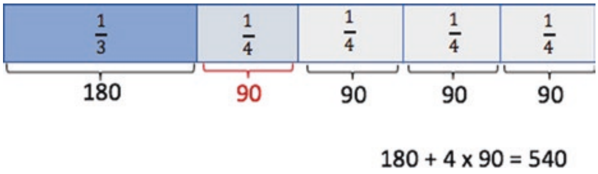


Fig. 15.12 Visual solution of the task

As can be seen by the examples, the bar model may be sufficient to solve the problem or to help clarify some misunderstandings in the interpretation of the task. The students may use different and complementary representations to make sense of the calculations or vice versa. This model is especially valued by underachieving or intermediate-level students, but we also must say that it can be helpful in solving more complex problems and in the understanding of fractions related concepts.

15.4.7 Paper Folding: The Cube

Spatial visualization is viewed as an essential part of geometric thought described as building and manipulating mental representations of two and three-dimensional objects and perceiving an object from different perspectives (NCTM, 2014) and includes the ability to manipulate the information presented in a visual, diagrammatic or symbolic form (Diezmann & Watters, 2000). Paper folding is a useful teaching tool to enable those skills in students and a way to promote their spatial thinking as an impact on the understanding of geometry (Boakes, 2009). It associates itself very naturally with visualization and geometric reasoning, making it possible to approach different mathematical themes, as well as a diversity of transversal skills (e.g. communication, problem-solving, proof).

The actions of folding applied to the paper allow it to be transformed into different shapes, either two or three-dimensional, opening the opportunity to investigate and discover relationships of different nature. In this way, paper folding can be a dynamic, creative and challenging strategy to approach several concepts in the mathematics classroom, facilitating visualization and problem-solving (Vale et al., 2020). Paper folding involves students cognitively in the challenges it provides and physically, because it requires auditory abilities and visual stimuli, and it is through these actions that it also involves spatial skills, which promotes the construction and discussion of meanings and mathematical ideas.

The example presented in Fig. 15.13 (Vale et al., 2020), refers to a task proposed to our students. This task had two main goals, to find the optimal solution and use spatial abilities to transform a 2D figure into a 3D figure.

This problem involves geometric and spatial reasoning, since the students have to construct a net (a two-dimensional figure that can be folded into a three-dimensional object) of a cube. Many nets can be built on a square sheet, but only one fits the conditions. It is a problem with some complexity for the elementary level. The students mostly began by exploring the most obvious possibilities, in

Use a square sheet of paper to draw the net of a cube with the maximum volume. Then construct the cube by folding that net.

Fig. 15.13 Paper folding cube

Fig. 15.14 Some incorrect solutions

which the segments representing the edges in the planning were parallel to the sides of the square or took advantage of the diagonal of the square. Despite having made different trials to reach a solution, none of them led to the expected outcome, because they did not achieve the highest volume (Fig. 15.14).

After many net trials, calculations and group discussions, the students discovered the correct answer. Figure 15.15 shows one of the analytical productions where they compared the volume of two nets.

The students made the design of the possible traditional nets. In fact, it is necessary to have mathematical knowledge to apply to this situation, and also intuition linked to the visualization of the different nets of a cube. In addition, exploration required divergent thinking to imagine and admit a completely different net from the classical approaches. Another way to approach it was the use of trial and error, doing the folds on the square paper and coming up with more positive results. After discovering the right net, the bigger challenge was to fold the paper to get the cube, without cutting. They did many attempts, but not all of them got the solution by themselves (Fig. 15.16).

This is a task with some complexity to use at an elementary level, but students were challenged and engaged, and the discussions that emerged at the end allowed a better understanding of the importance of the use of different approaches to solve a mathematical situation.

15.4.8 The Cup

Consider the following task (Fig. 15.17):

This task (Vale et al., 2016) can motivate several solutions involving the properties of the observed figures. Students who attempt a solution using formulas applied

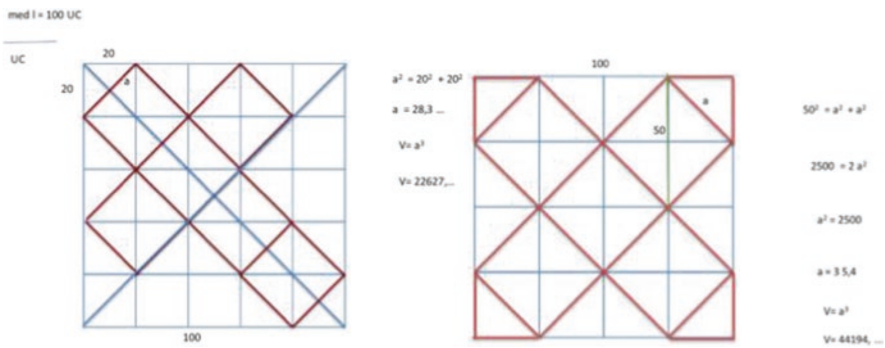


Fig. 15.15 Analytical solutions to determine the volume

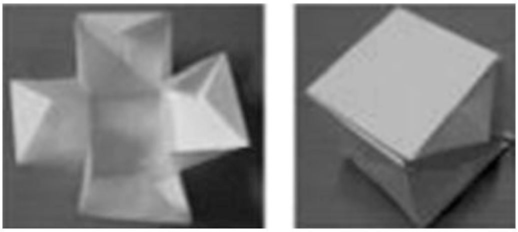


Fig. 15.16 Optimal solution and paper folding cube

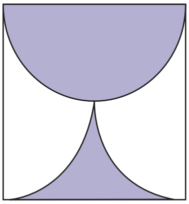
to different parts of the figure may find some difficulty to solve it. The students who attempted a method using formulas applied to different parts of the figure considered that this is a difficult problem, especially if the square is not shown. The most common numerical solution was in Fig. 15.18.

There are only calculations, looking for the best way to use the formula of the area of the circles. We can say that this solution is blind, that is, there is no attempt to see any relationship between the upper parts and the lower parts of the figure. Yet, there are some solutions that we called mixed (Fig. 15.19) because, although they are numerical, students can see beyond the given figure, even adding some geometric construction.

However, the challenge for the solvers was to get a visual solution, but it did not appear. If the students have more visual abilities, they can discover a dynamic visual process, *seeing* transformations of the initial figure (see the arrows), and doing a reconfiguration of parts of the figure (Fig. 15.19). We can mentally slide the two parts that make up the "foot" of the "cup" to the top, forming a rectangle. It follows the trivial conclusion that the "cup" has an area equal to half of the square, i.e., 1/2 unit area (the first of Fig. 15.19). Another dynamic solution could be the last one shown in Fig. 15.19, in which, after drawing the diagonals of the square, we easily

The figure shows a unitary side square. The curved lines are circumference arcs. What is the area of the shaded region? Find out more than one process of getting the solution.

Fig. 15.17 The cup task



Up: $\dfrac{\pi\times\left(\frac{1}{2}\right)^2}{2} = \dfrac{\pi}{8}$

Down: $2 \times \left[\left(\dfrac{1}{2}\right)^2 - \dfrac{\pi\times\left(\frac{1}{2}\right)^2}{4}\right] = 2 \times \left(\dfrac{1}{4} - \dfrac{\pi}{16}\right) = \dfrac{4-\pi}{8}$

Total area: $\dfrac{\pi}{8} + \dfrac{4-\pi}{8} = \dfrac{1}{2}$

Fig. 15.18 The common numerical solution

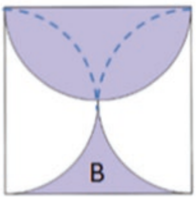
see that the area of the “cup” corresponds to 2/4 of the square area, i.e., half of the square. In each case, the visual elements convey the thinking process (Fig. 15.20).

What makes such a solution creative and also simpler, being a challenge for these students, as some authors (Haylock, 1997; Leikin, 2016; Presmeg, 2014) suggest, is the fact of being necessary to break the mental set that suggests the use of formulas or conventional/learning-based solutions. It is necessary to use divergent thinking in looking for other ways to solve the task. This can happen if instruction push learners to find new ways to solve some tasks, where thinking by analogy can be a helpful strategy to attack new problems (Polya, 1973).

15.5 Concluding Remarks

In this chapter, we intended to discuss and illustrate some ideas concerning the use of visualization as a meaningful pathway to pose and solve challenging tasks in mathematics education. The choice and use of tasks are nuclear to effective teaching and learning of mathematics since they are the driving force that triggers mathematical activity. The importance of the teachers’ role in this matter is undeniable, but it goes far beyond task selection. Students must be motivated and engaged as solvers to be successful, being incited to think, discuss, reflect and overall be challenged.

We believe that multiple solution tasks give students the opportunity to apply their thinking styles, whatever their nature, and also to come into contact with a variety of strategies that will contribute to the expansion of their repertoire. In this framework, and according to our own experience with pre-service teachers, we consider that visualization can have great potential, either as the context in which the



$$\text{area of the circle} = A_\circ = \pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$$

$$B = \frac{1 - A_\circ}{2} = \frac{4-\pi}{8}$$

$$\frac{1}{2}A_\circ + B = \frac{\pi}{8} + \frac{4-\pi}{8} = \frac{1}{2}$$

Fig. 15.19 A mixed solution

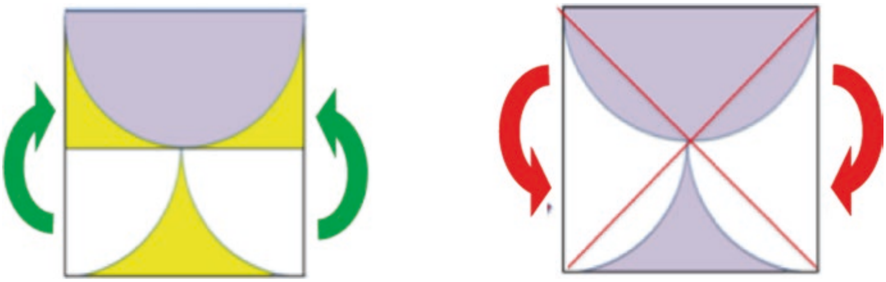


Fig. 15.20 Two visual solutions

task is presented or as a vehicle to reach a solution. Visual strategies are not new in the literature but usually, they are deprecated over analytical ones. This situation does not benefit the students since visual approaches can be an excellent complement to analytical thinking or can even help generate simpler and more meaningful solutions. The visual approach, being transversal, mediated by tasks with multiple solutions, contributes not only to a better understanding of mathematics and to the development of students' creativity, but above all to show a new perspective of mathematics. This means that students may overcome the idea of seeing mathematics as an isolated collection of themes, made up of a set of formulas and techniques that they have to memorize and master, when, in fact, they can have the opportunity to see it as a set of meaningful great ideas and connections.

Some studies and projects that we have been carrying out with pre-service teachers of elementary education have generated results that we consider relevant in the context of multiple solutions tasks, with a close connection to visualization. There are several aspects that stand out, namely the fact that, through these tasks, we can invoke a diversity of mathematical themes; establish connections between visual and analytical processes, as well as between representations; allow a more fluent, flexible and even original reasoning, which fosters creativity. Our intention in developing this type of work with pre-service teachers is that they understand the potential of *seeing* in order to develop this ability with their future students. We have to consider the background of the candidates to this teacher training course: not everyone has the same solid mathematical knowledge; the majority of them are non-visualizers due to the predominance of analytical methods in their previous experiences, which makes them unaware of the utility and power of visual approaches, undermining their *visual literacy*. Many of the tasks presented in this chapter are not necessarily high-level and do not involve complex mathematical concepts, but they are challenging from our perspective. The challenge rises from the fact that we use multiple solution tasks, chosen with the intent of allowing the application of analytical and visual methods, that must be solved in different ways. So, our interest was that solvers were challenged by discovering a visual solution.

For the non-visualizers this request may trigger the need to discover a strategy of a different nature, making them face the challenge of *seeing* the visual cues; and the same happens with the visualizers that may need to use non visual methods, which brings the opportunity to connect mathematical ideas in a rich and meaningful way. Subsequently, challenge can be faced as a situation that enhances the learning process, experiencing something new and unforeseen and trying to come to grips with it. With regard to visualization and visual thinking in particular, being less valued in mathematics classes and even in textbooks, it's pertinent to provoke the use of this way of reasoning, enhancing the level of mathematical challenge and flexibility in problem-solving.

We consider this didactical approach an asset, given the positive results of previous research projects and, for this reason, we include it in these pre-service teachers training programs, believing that it can contribute to both visualizers and non-visualizers to better understand certain mathematical themes and be challenged to find solutions out of their comfort zone. To conclude, in this chapter, we chose to present examples in different contexts, dealing with a wide range of themes and abilities, in detriment of an in-depth discussion of the results obtained, in order to focus on the potentialities of the tasks and contribute to broaden the repertoire of tasks that may help teachers challenge students through this perspective.

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Chapter 16
Towards a Socio-material Reframing of Mathematically Challenging Tasks

Nathalie Sinclair and Francesca Ferrara

16.1 Introduction

Technology-based tasks in open, expressive environments present (at least) two significant challenges. The first concerns the relation between these tasks and their paper-and-pencil counterparts. The second concerns the fact that these tasks are not just mathematical tasks, but digital-mathematical tasks that offer technical as well as conceptual challenges—in other words, task solvers must mobilise tool fluency as well as mathematical fluency. In both cases, the mathematics at stake finds itself being modulated by its imbrication with the digital technology. This will be our starting point in addressing the theme of this book. We will explore challenging tasks, but pay particular attention to the way in which the digital technology context of these tasks opens up new conceptualisations of mathematical challenges. In particular, we propose shifting the constructivist underpinnings that we find in the current conceptualisation of mathematical challenge towards an inclusive materialist (de Freitas & Sinclair, 2014) one. Although we will exemplify this shift through a multi-touch application geared towards numbers and operations, there will be several aspects of the distributed, material context we study that will be relevant to other forms of digital technologies.

Indeed, the inclusive materialist approach shifts attention away from the doer and focuses on the doing, recognising the role of material agency in the mathematics classroom. Tools and other non-human entities are seen as partaking in the mathematical activity—shaping it and affecting it—and not just passive mediators to the intentional, intact subject. Constructivism, in contrast, assumes this intentional,

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intact subject as the locus of knowledge, who comes to know in an autonomous way (Radford, 2008). For inclusive materialism, knowledge is distributed across the system of human and non-human (tools, desks, pencils, etc.) entities. Further, the subject is not autonomous, but emerges (as motivated or knowing) from the system. The decentring of human agency means that we will understand challenge to be neither a subjective determination of individual ability or knowledge nor a property of a given task. Therefore, instead of thinking of challenge in individual terms—differing from one individual to another—and as independent of activity, we will rethink challenge as a social and collective endeavour.

We begin by providing an overview of the recent work on technology-based tasks in mathematics education. We then examine how this literature relates to the concept of mathematical challenge as articulated in Leikin (2014). This will allow us to formulate specific dimensions of challenge that become newly available in particular digital technology environments. We will then illustrate these challenging tasks by drawing on a grade 1 classroom experiment in Italy involving the use of the multi-touch application *TouchCounts* (Jackiw & Sinclair, 2014). The choice of a single digital technology to exemplify our new conceptualisations of challenge allows us to pay adequate attention to the particularities of the mathematical concepts at play, as well as to the impact of the digital technology on the teaching and learning environment. That said, we will make connections between the examples we offer with *TouchCounts* (TC) to other well-known digital technologies.

16.2 Tasks in Digital Technology Environments

In their edited book, *Digital Technologies in Designing Mathematics Education Tasks*, Leung and Baccaglini-Frank (2017) build on the growing research on task design in mathematics education. While none of the chapters refers specifically to supporting mathematical challenge, several chapters discuss considerations that are relevant to mathematical challenge. For example, Leung (2017) discusses the importance of the feedback offered by digital technology, and how it affects the learning potential of a task. Feedback is relevant to mathematical challenges because it can affect student-task interactions, as well as student-teacher ones, potentially changing the difficulty level of the task.

According to Mackrell et al. (2013), there are three distinctive kinds of feedback: *evaluation feedback* is related to the completion of a task or part of a task; *strategy feedback* aims to support or amend student approaches while she is engaged in a task; and, *direct manipulation feedback*, “is the response of the environment to student action” (p. 83). In TC, write Sinclair and Zazkis (2017), feedback mainly occurs through direct manipulation, which enables learners to observe the consequences of their actions and adjust them in order to solve the given task successfully. This means that the teacher is no longer the authority that determines whether or not the task has been correctly solved, thereby altering students’ agency.

Sinclair and Zazkis discuss tasks in TC that involve different kinds of questions and forms of interaction than those typically found in non-digital settings. For instance, since the direct manipulation feedback comes in both visual, aural and tactile forms, students are invited into a multiplicity of forms of reasoning, which is typical in many different digital technology environments (Leung, 2017), and this alters the way mathematical ideas can be encountered. In TC, placing a finger on the screen produces a tactile feedback of the screen on the finger, a visual object, a number name spoken aloud (one) and a symbolic numeral ('1'). This tactile, visual, auditory and symbolic multiplicity can make some tasks easier (a 3-year-old child can press a finger on the screen without knowing how to count) and some harder (connecting the symbol with the visual object). In other words, there are various types of challenge involved, not all of which work in the same direction.

Given that our discussion will draw on an episode in which children are engaged with the Operating World of TC (which is one of the two worlds, the other being the more ordinally focused Enumerating World), we first provide a brief overview of it.[1] In this world, 'herds' can be created by placing one or more fingers on the screen simultaneously. Herds are cardinal quantities that are labelled with their associated numerals (Fig. 16.1a), which are spoken aloud. These herds can be operated on by means of specific screen-contact gestures. Pressing a herd 'highlights' its circumference in fuchsia. Pinching two herds together (Fig. 16.1b) produces a new herd that is the sum of the quantities, whose discs retain the colour of the original herds (Fig. 16.1c). By doing the opposite of the pinching gesture, a herd can be partitioned into two—the size of the part removed from the initial herd depending on how far the fingers separate out from each other on the screen.

We now shift to describe one of the tasks proposed in Sinclair and Zazkis (2017), which refers to the use of the Operating World. "Count by 3s" resembles the common "skip-counting" one found in elementary classrooms where it is usually done orally, with children choral chanting the number names. With TC, the task itself changes, as do the solution strategies and the opportunity for feedback. There are at least three ways of accomplishing this task in the Operating World. A child could take an ordinal approach by placing a finger on the screen and holding it there, then

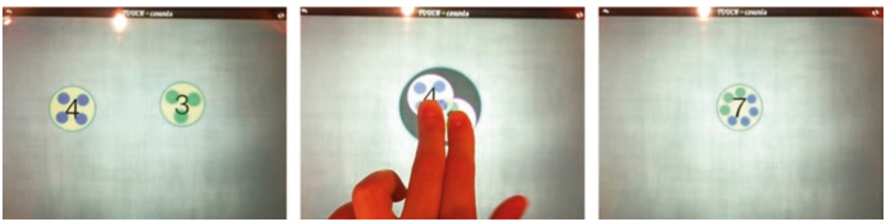

Fig. 16.1 (**a**) Two herds of 4 and 3; (**b**) pinching the two herds together; (**c**) the new herd of 7

[1] Please see the video on this webpage to consider how *TouchCounts* works: http://touchcounts.ca/about.html.

use another finger to tap sequentially until three discs are visible, then let go. This would produce a herd of three, with the accompanying audible "three". In a similar way, the child could produce a 6, 9, 12, etc. A child might also use a cardinal approach by pressing three fingers simultaneously on the screen, which will make a herd of three, along with an audible "three". Then a child could make a herd of six by tapping six fingers simultaneously, along with an audible "six" and so on, producing herds of 9, 12, etc.[2] Another method would be an additive approach, which would begin the same way as the cardinal approach, with the first herd of three, then create a second herd of three and join it to the first herd so as to obtain a herd of six; then, create another herd of three, join it with the herd of six, so as to obtain a herd of nine, and so on. In the cardinal and ordinal approaches, TC would have uttered "three, six, nine, twelve, …" and in the additive approach "three, three, six, three, nine, three, twelve, …".

Visually, in the first case, there would be several herds on the screen, of 3, 6, 9 12, etc., while in the second case, there would be only one herd on the screen. Tactilely, the second case would require repeating gestures of three fingers simultaneously touching the screen and then pinching, which might highlight skip-counting as a constant additive action (Sinclair & Heyd-Metzuyanim, 2014). Of significance is the multiple strategies that this task allows, as well as the multiple forms of non-evaluative feedback.

There are two important aspects of TC that differentiate it from other environments in terms of its epistemic and pragmatic values, as defined in Artigue (2002). First, it takes care of the computation so that children can attend to the result of their successive repeated addition. It is important to note that in doing the calculation, TC offers both symbolic and aural results. This distinguishes the task from what could be done on a calculator, which would also take care of the computation because the calculator does not announce the sums out loud. The second distinguishing feature is the gestural interface for performing the action(s). This gesture, which has both pragmatic and epistemic functions, draws learners' attention to the action as an adding operation (while the 'taking apart' gesture draws learners' attention to the subtraction operation). Finally, as already noted, the feedback that is offered is immediate and is directly related to one's actions, and so can guide further actions, especially if a goal has not been reached.

16.3 Digital Technology-Inflected Mathematical Challenge

Leikin (2014) has provided four conditions for a task to be considered a mathematical challenge: "First, the person who performs the task has to be motivated to find a solution. Second, the person has to have no readily available procedures for finding

[2] Placing more than 10 fingers simultaneously would require the help of another classmate, of course.

a solution. Third, the person has to make an attempt and persist to reach a solution. Fourth, the task or a situation has several solving approaches" (p. 62). In the paragraphs that follow, we will consider each of these conditions, in turn, beginning with the last one.

The *fourth* condition fits well within tasks that make a "strong use of technology", as described above. However, we might ask what accounts for a "different solving approach" for tasks that take their meaning from the technology having several solving approaches. In the TC example given of Counting by 3s, there are certainly different solving approaches, depending on whether one produces the skip-counting numbers directly or additively. Additionally, there are also differences that concern the actions performed, for example, sequential touches, simultaneous touches or the use of the pinching gesture. In other words, the condition "several solving approaches" will be inflected by the broader body-material context of how the multiple potential actions of the students coordinate with the multiple structures of the digital technology—as evidenced in the literature on the embodied and distributed nature of tool use (Nemirovsky et al., 2013; Sinclair & de Freitas, 2019; Ferrara & Ferrari, 2020).

The *third* condition resonates at least in part with the use of digital technology environments that offer feedback—particularly *strategy* or *direct manipulation feedback*, which are forms of feedback that respond to students' approaches and actions and therefore have the potential to lead to changes or adjustments in these actions. Here we are particularly interested in the intensification of student agency in digital technology environments where the feedback loop is not dependent on a teacher or textbook's authority, and where mistakes can be made without incurring teacher judgement (see Herbel-Eisenmann & Wagner, 2010). This third condition also concerns the student's opportunity to "make an attempt", which we also see as being linked to the action-oriented nature of a digital technology, where you can often 'begin the conversation' by doing something and then receiving a response (see Jackiw & Sinclair, 2010). Hence, we will take this third condition to refer specifically to the technology-inflected mathematical challenge's *performative* and *reactive* potential.

The *second* condition is useful in distinguishing a mathematical challenge from an exercise, the latter which usually requires the use of a known procedure. However, given both the epistemic and pragmatic values at play in a digital technology environment, "finding a solution" may involve engaging both the technical and conceptual aspects of the task, which are often closely intertwined.

With respect to the *first* condition, it seems to us closely related to the third one, since it involves affective considerations. If the digital technology in question offers new objects on which to act and with which to interact, we see an important affective dimension that is less about motivating students through new, shiny, animated devices, and more about increasing and diversifying the affective relations at play. These relations may involve feelings of connectedness (from having made the objects with one's own fingers), togetherness (Sinclair & Ferrara, 2021), belonging (Turkle, 2011), power (Jackiw, 2006), and sympathy (de Freitas et al., 2019).

In the next section, we wish to exemplify our inclusive materialist conceptualisation of mathematical challenges. This new conceptualisation will not assume that students are autonomous and intact cognizing subjects, as per the constructivist point of view. It will also not ground mathematical knowledge solely in sociocultural practices. Instead, it seems to reframe Leikin's conditions of mathematical challenge in terms of a distributed, material ontological framing.

16.4 Mathematical Challenging Tasks Using *TouchCounts*

We have chosen an episode from a teaching experiment that was conducted in a Northern Italian grade 1 classroom where the second author (R) has been working for many years in collaboration with the classroom teacher. There was also a Masters student (MDS) in the classroom who video-recorded the whole experiment, which she then analysed as part of her Masters degree thesis. We focus on days 8 and 9 of the teaching experiment, which lasted 10 days in total, and was designed to experiment with the use of TC as an environment to develop learners' ordinal and cardinal conceptions of number. We have chosen this episode because it allows us to speak relatively succinctly about the four conditions offered in Leikin (2014).

During the fifth to seventh days, the students had worked in the Operating World, adding different numbers and reasoning about the result along the way while returning to the original numbers through using addition (pinching herds together) and subtraction (separating a herd into two parts). In the second part of the eighth day, they began to reason about multiplication (in terms of repeated addition) by producing multiple herds of the same size and pinching them together. This continued on the ninth day and extended to division by integers and the concept of the remainder. We now shift attention just to explore and discuss the tasks that emerged during the second part of the eighth day, and those involving division on the ninth day.

16.4.1 The Initial Task: Distributing Candies Amongst Children

The class was divided into three groups, each consisting of eight children. Each group was seated together around a table, with an iPad in the middle. At the end of the day 8 activity, the researcher asked each group to create one herd of 12 and one herd of 18, using whatever combinations of numbers they wanted. We focus on the group composed of Alessandro, Alice, Caterina, Linda, Marco, Matilde, Pietro and Sofia, which was video-recorded. Pietro soon proposed to his classmates that half of the group make the herd of 12 and that the other half make the herd of 18. The first half thus worked together and produced the herd of 12 using herds of 4, 5, 2 and 1. Then, the other half of the group produced a herd of 18 using three 5s, 1 and 2 (Fig. 16.2).

Fig. 16.2 The two herds of 12 and 18 produced by Alessandro, Alice, Caterina, Linda, Marco, Matilde, Pietro and Sofia

Once herds of 12 and 18 were created, the researcher posed the following task:

R: Suppose that these (pointing to the 12) are 12 pretend candies instead, rather than discs and that these (pointing to the 18) are 18 candies. Pretend, we are imagining that, pretend that they are 18 candies. There are six children[3]. Can these six children have the same number of candies?

Most of the children responded by saying "no", but Pietro said "yes". The researcher repeated the question. This time Alice said "no" and Pietro again said "yes". The researcher asked the children to explain why. Pietro proposed that one child could start taking one candy, the next child could take the next candy, etc., until they were finished, and then they could count the number of candies each child had and see whether they were equal. Pietro first stood up to state this, then he got close to the iPad and pointed to the herd of 18, so the researcher asked whether the children could use TC to test Pietro's idea.

We pause here to consider the task in relation to mathematical challenge. In terms of the *first* condition, which states that "the person who performs the task has to be motivated to find a solution", it is clear that the condition would have to apply to a collective rather than to an individual. Of course, there were individual responses given to the researcher's question, notably by Alice and Pietro, but we suggest that the condition under which a solution is sought arises from the *collective uncertainty* expressed. Further, by suggesting that the students test Pietro's hypothesis on the iPad, the condition of finding a solution is distributed across the collective and the tool. In other words, any attribution of motivation must include the tool's agency in determining a solution. As alluded to in the previous section, we would therefore

[3] The children had previously worked on a task that involved 6 children, which the researcher reminded them of.

not rely on the individual emotional state (of motivation), as typical in a constructivist theoretical framing, but turn attention to the motivation arising from the collective students-tool-mathematics *potential for mathematical action*. This point of view aligns more with a socio-cultural perspective, in which autonomy is an outcome rather than a pre-requisite for learning. However, we extend the autonomy to include the non-human material world too because TC does not simply mediate the mathematics—it actually constitutes it. The autonomy does not belong exclusively to the learner, nor to the tool, but to the learner-tool in action.

In terms of Leikin's *second* condition, which is that the children do not have a procedure for solving the task, it is obviously the case. Not only have they not yet encountered the relevant concepts, the children actually disagree about the solution. However, it is important to note that the children also have access to TC, as well as to the researcher, and in this sense, the socio-material does indeed offer readily available actions for finding a solution. As with our discussion about the first condition, we attend to the socio-cultural source of knowledge (in this case, with both the teacher and the tool), but we further stress the material agency of the system. In our socio-material perspective, the ontological assumption shifts from a socio-cultural one, in which knowledge is historically generated, to a processual one, in which knowledge is historically *and* materially generated (Table 16.1).

We continue our analysis of the episode now, which will allow us to address the third and fourth conditions of a mathematical challenge. The children began by re-enacting Pietro's idea, with each imagined child taking turns being given a candy by Pietro, until the bag is empty. They realise that it works.

Pietro: How many do you have?
Cc: Three.
R: So?
Alice & Pietro: It works.
R: Can we see it from there (pointing to the iPad) that six children can have each, how many candies?
Pietro: Yes.
Alice: No.
MDS: How many?
Pietro: Three each.
MDS: Could you see it there, on the iPad?
Pietro: No (the children are all moving closer to the iPad; Fig. 16.3a).
R: Do we find a way to see that six children can have three candies each?
Cc: Yes (getting closer and closer; Fig. 16.3b).

Table 16.1 Reframing Leikin's first two conditions

Leikin's constructivist conditions	Reframed socio-material conditions
1. The person who performs the task has to be motivated to find a solution	Motivation arises from a human/non-human system as a result of finding a solution
2. The person has to have no readily available procedures for finding a solution	Procedures for finding the solution are not dictated by the socio-material system, but potential within it

Fig. 16.3 (**a**) Children gathered around the iPad; (**b**) beginning to act on 18

Fig. 16.4 (**a**) Taking away a herd of 3; (**b**) two herds of 15 and 3

Fig. 16.5 (**a**) Alessandro taking away 3; (**b**) Matilde taking away 3; (**c**) resulting herds of 12, 3 and 3

Pietro explained that they have to start taking three away from the herd of 18 with their fingers. While Pietro pressed on the herd of 18, which became fuchsia-highlighted, Alessandro helped him to take away three discs (Fig. 16.4a), which produced a new herd of three, and left behind a herd of 15 (Fig. 16.4b).

In the meantime, Pietro pressed on the herd of 15, making it fuchsia-highlighted, and Matilde brought her right index finger closer to it to repeat Alessandro's gesture (Fig. 16.5a).

Fig. 16.6 (**a**) Herds of 3 and 9 from 12; (**b**) children watching Marco taking away the 6th herd of 3; (**c**) six herds of 3 left on the screen

R: Now, Matilde is taking her three Fig. 16.5b).
Alice: Release it (Matilde releases her finger and another herd of three is created, leaving behind a herd of 12; Fig. 16.5c).

Caterina then took away another herd of 3 from the herd of 12, leaving a herd of 9 (Fig. 16.6a). Pietro invited Linda to go next. She approached the screen with her left index finger to repeat the previous gesture and took away another herd of three, leaving a herd of 6. Then it was Marco's turn, but he only took away a herd of 1, which left behind a herd of 5. Pietro recreated the herd of 6, by pinching the herds of 5 a 1 together. Marco tried again, while the other children watched closely (Fig. 16.6b), and produced a sixth herd of 3 (Fig. 16.6c).

R: Well, we have been successful.
Pietro: Yes.
R: Each of the six children has three candies. She has light green (referring to Alice and the colour of the discs of 'her' herd), she has light blue (referring to Linda and the colour of her herd of 3), he has dark green (referring to Alessandro), she has brown (referring to Caterina). And Matilde has blue.

We pause here again to consider the task in relation to the third and fourth conditions of a mathematical challenge. Regarding the *third* condition, which affirms that "the person has to make an attempt and persist to reach a solution", we see once again how the children persist to reach the solution as a collective rather than individually. Even though there are individual roles for each child—each of whom is 'receiving' three candies as they take away a herd of three from the total—the solution can only be reached through collective action, which is something that the task itself called for. Success depended on each child taking their part of the candies by performing the take away gesture—and the success was only measured at the end, by the fact that there were 6 herds of three on the screen, one for each child, distinguished from the others by colour. Thus, the condition of persisting to reach a solution is *dispersed* within the socio-material system made of the group and TC, and the actions emerging out of it. In this case, any attribution of persistence towards a solution must be referred to the distributed agency of the children-tool system.

Beyond persistence to reach the solution, we see how the challenge and the activity grow with no external or a-priori determination, out of the 'always something to do' possibilities that the children have when working with TC. For example, the children want to take away herds of any size (not necessarily the 'correct' size), and they can also make new herds or join these herds again, as in the case of Pietro re-joining the herds of 1 and 5 obtained by Marco. These actions are all ones they have made before, which means they are thinking/acting contiguously with their past perceptions, and this can push their actions into an unscripted and contiguous future. We see here how the mathematical challenge is itself *mobile* and *full of potentiality*, open to deformation and indeterminacy (including accidental productions of herds, like the herd of 1 seen in Fig. 16.6c that nobody commented upon). The children's actions constantly reconfigure the challenge, whose alteration taps into the past (first encounters with subtraction, use of more than one child) and expands to the future (a sequence of new, coloured herds). In other words, the mathematical challenge is no longer static and partakes in the activity.

Concerning the *fourth* condition, strictly speaking, there are only two and not "several" solving approaches. The children could have produced a herd of 18 by using repeated addition; each of the six children could have created a herd of 3 and then joined it with all the others. This action-oriented performance would have produced a herd that visibly reflected the creation process, being made of six different colours. In the strategy adopted by Pietro and his classmates, however, the children equi-partitioned 18 into herds of 3 by repeated subtraction, ending up with 6 herds. The task situation itself was open to multiple approaches in that it embedded the possibility of working with the number of children seated around the table as a variable to be used in the task. It is in this sense that the task is implicated in, and inflected by, the socio-material system going beyond the epistemic actions provided by TC. That 18 is a multiple of 6 emerged out of the mathematical challenge as a sequence of subtractions in TC, which became herds first, then candies that the children wanted to have. In touching the screen, memories of previous touches are evoked—colour calls up the past perception of repeated addition. In this way, colour, touch, and memory are different dimensions of the mathematical challenge (Table 16.2).

We use the remaining part of the episode to come back to the four conditions, which we have now characterised according to a socio-material perspective. In particular, the next part of the episode will allow us to show how the mutual evocation of matter and memory mentioned above was an expression of affect, in that the children's actions and gestures are treated as affective states that mobilise new questions (de Freitas & Ferrara, 2015). We will additionally see how the children were

Table 16.2 Reframing Leikin's third and fourth conditions

Leikin's constructivist conditions	Reframed socio-material conditions
3. The person has to make an attempt and persist to reach a solution	The socio-material system becomes performative and reactive as a result of attempting and persisting to reach a solution
4. The task or a situation has several solving approaches	Solving approaches are inflected by the socio-material system, through the mutual evocation of matter and memory

eventually able to compare the herds of 12 and 18 in terms of being multiples of 6. It is in fact not simply that the two numbers emerged from the challenge separately but that they could be related and, therefore, compared. This will also help us think of the mathematical challenge as becoming more and more social instead of more and more difficult.

16.4.2 The Follow-Up Task: Comparing 12 and 18 as Multiples of 6

The researcher asked whether six children could have the same (as each other) number of candies if there were only 12 of them. Pietro immediately answered affirmatively. Alice said nothing, but her right hand was thrust out with two outstretched fingers (Fig. 16.7a). Then suddenly, she says, "Yes, it's possible, it's possible, it's possible" (almost jumping on her chair; Fig. 16.7b).

R: And how can we do it?
Alice: It's possible. It can be done. You have to give two candies to six children (turning towards the iPad).
R: How did you do it?
Alice: I've done, one child two, another child two (adding two fingers on her right hand) and I'm at four (showing four fingers), another child two (Fig. 16.7c).
Pietro: Six.
Alice: Six (turns to Pietro), I'm doing it! Another child, eight (Looking at R), another child ten.
Pietro: Ten.
Alice: Another child twelve.
Pietro: Twelve.

The researcher asked Alice how she had the idea of giving two candies to each child. She shrugged her shoulders and looked at her fingers. Sofia proposed that maybe she took the multiplication table of two. When the researcher asked how she thought of that, Sofia responded:

Fig. 16.7 (**a**) Alice's two fingers jutting out; (**b**) Alice insisting "it's possible"; (**c**) Alice's hands after distributing two candies to 3 children

Fig. 16.8 (**a**) Sofia counting two candies; (**b**) Sofia counting up to four candies; (**c**) Marco putting his fingers out; (**d**) Sofia looking at Alice

Sofia: Since I've heard that she counted two (marking two fingers on the desk) plus two (moving the two fingers on her right along the desk; Fig. 16.8a), which makes four, plus other two which makes six (Fig. 16.8b), and plus others that makes eight (Marco raises his hand to speak; Fig. 16.8c), then plus other two (looking at Alice and getting even closer to Marco; Fig. 16.8d), which makes ten, plus other two it makes ten.
Marco intervened to say that this is not the multiplication table of 2.

Alice: We don't say one child plus one child up to four children, we say two four six, up to… I said in another way. One child, I put two candies, another child, up to six children I've put two candies.
R: Ok.
Pietro: But, how do you do, how do you do to say that those that you have counted are twelve?
Alice: Because I know to count (turns towards Pietro). Two candies are here (showing two fingers to him), another two make four, "doesn't it"? Because I count the number of candies. Two.
MDS: Alice, but why did you think of two?
Alice: I don't know (smiling), I made two (looks at her fingers).
MDS: Why not again 3, for example?
Alice: (smiling and shaking her shoulders; Fig. 16.9), I don't know.

We see Alice and then Sofia entering the challenge of considering the herd of 12 that they previously created as a multiple of 6. The children abandoned the iPad for a while, while Alice took the number of children as the variable in the imagined actions oriented to the goal of distributing candies uniformly and, in the meantime, controlling them by counting ("one child two", "another child two"). Motivation still arose from the students-tool *potential for mathematical action*, to which the number of children as a variable to be used in the activity with TC is related. Indeed, using the number of children (associated with the number of candies) as a variable is potential in the relationship between the tool and the users, which allows for the production of herds-numbers by the multiple finger touches of different children.

The challenge was *altered* at this point from the introduction of the two fingers, in the action of counting, which pushed Sofia to think of an eventual association with the multiplication table of 2. The herd of 12 became a counting process on Alice's fingers, which was shared by Pietro, who counted together with her, but at the same time, this allowed Sofia to bring in the multiplication table, therefore to make the herd of 12 closer to being a multiple of 6 and 2.

Fig. 16.9 Alice smiling and shrugging her shoulders

Thought and matter, and matter and memory are contiguous here in the ways that we can reasonably interpret Alice having recalled the previous reasoning (when quickly gazing to the iPad) while referring to the number of children. Similarly, we can interpret Sofia as having connected the process of counting by 2 on her fingers with the multiplication table of 2.

Both Alice and Sofia moved, in different ways, rendering different but similar thinking. Persistence to reach a solution is again dispersed while the mathematical concept of multiple is emerging out of the *affective* bond of the children with the past activity, expressed by the children's movements, which we can therefore see as affective states in the way they partake of the becoming of the challenge. The equal parts that constitute 12 as a multiple are in this case *anticipated* by Alice and Sofia, and do not need the same performance with TC that was used to show 18 as a multiple of 6 and 3.

The variables of the number of children and number of candies moved the discussion towards a new degree of understanding of the challenge, which introduced the eventual relationship with the herds of 3 previously used with the question posed to Alice about how she was able to think of 2. The performance with TC to create the six herds of 2 only served, this time, as a way to control the solution and to relate it to the previous situation. How this occurred within the group is offered in the next very short extract of the episode, which also invites the children to compare the two herds of 18 and 12 as both being multiples of 6.

The researcher asked the other children what they thought about what Alice was saying (when she tried to explain her answer of two). Matilde and Linda both responded by saying that what Alice had done was right because it worked.

R: Why? How would you explain it? To my brother?
Linda: Because, 2 plus 2 makes 4, plus 2 makes 6, plus 2 makes 8, plus 2 makes 10, plus 2 makes 12.

At this point, the researcher invited the children to compare the two different distributions of candies to six children, going back to the six herds of 3, which were

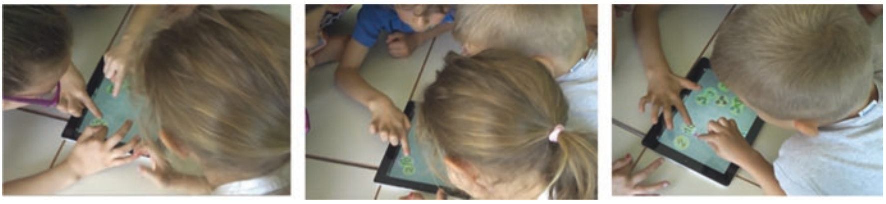

Fig. 16.10 (**a**) Alice producing the first herd of 2; (**b**) and (**c**) Pietro and Marco producing the second herd of 2

still visible on the screen. With the help of Linda's explanation, it became clear that the two situations could be compared, and that each child could have one more candy with a bag of 18 candies, compared to the smaller bag of 12. Then, the researcher asked the children to show on the iPad what they have been discussing and the children worked on the herd of 12 to get the six smaller herds of 2 (Fig. 16.10).

We wanted to provide an account of the extended excerpt in order to draw attention to the choreography of the tasks, which we see as an important feature in making them a mathematical challenge. It is possible to describe this excerpt in terms of three tasks: (a) Can 6 children get the same number of candies each if there are 18 candies? (b) Can 6 children get the same number of candies each if there are 12 candies? (c) Are 18 and 12 both multiples of 6? We have discussed our socio-material framing of mathematical challenge with respect to the first of these three tasks. We have seen how, in a sense, the same considerations are at play for task (b), except for the particular role of memory and matter, since the fact of the candies, or the 6 children and of the repeated taking away replays in the second task, carrying with it, sediments of the first, and therefore necessarily affecting the motivation of the system (our revised condition 1) and the performative nature of the task (related to our revised condition 3).

We can shift attention for a moment to the crucial role of the researcher in relation to task (c). Challenge is not established before or independent of the activity, instead it develops and affectively unfolds through and through, with the researcher posing questions, asking for explanations, introducing the variables of number of children and number of candies as partaking in the material activity of the children with *TouchCounts* and in their imaginative situation in which they all want to have candies.

16.5 Conclusion

In this chapter, we have considered Leikin's concept of a mathematical challenge in the context of a teaching and learning environment involving digital technology. We were interested in pursuing a reframing of mathematical challenge that would adequately address the distributed, material aspects of mathematics teaching and learning, as per the inclusive materialism perspective. We used a specific episode involving a group of eight children using TC in order to illustrate the modifications in Leikin's four conditions. These new conditions are as follows: (1) Motivation arises from a human/non-human system as a result of finding a solution. (2) Procedures for finding the solution are not dictated by the socio-material system, but potential within it. (3) The socio-material system becomes performative and reactive as a result of attempting and persisting to reach a solution; (4) Solving approaches are inflected by the socio-material system, through the mutual evocation of matter and memory.

We propose that our reframing enables researchers to see tasks as being materially and temporally in relation to prior mathematical activity, rather than isolated problem-solving opportunities. It also distributes the agency for task resolution across a wider socio-material system, which reduces the necessity to label certain students as more or less motivated, and more or less able to persist in their mathematical activity. Finally, it enables a more nuanced understanding of the multiplicity of mathematical reasonings, which varies across the epistemic and pragmatic values of a given technological environment, as well as across forms of engagement.

These four conditions resonate strongly with the four characteristics of mathematical inventiveness proposed in Sinclair et al. (2013), which emerged from an analysis of students' engagement with two different digital technology environments (*The Geometer's Sketchpad* and the *Motion Visualizer DV*). For these authors, a creative act: brings forth or makes visible what was not present before; it does not align with current habits and norms of behaviour; it is without prior determination or direct cause; and, its meaning cannot be exhausted by existent meanings. The resonance is not surprising given that in both cases, we are concerned with a novel production that brings forth new mathematical ideas, which are taken to be inseparable from their socio-material contexts. Like these two digital technologies, TC is one in which novel meanings – as well as combinations thereof – are possible, particularly in relation to new bodily interactions (such as gestures), collective forms of expression and mobile mathematical objects. We conjecture that tasks taking advantage of these novelties will be well disposed to offering mathematical challenges.

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Chapter 17
Creativity and Challenge: Task Complexity as a Function of Insight and Multiplicity of Solutions

Roza Leikin and Raisa Guberman

17.1 Introduction: Problem-Solving

It is a commonly shared position that mathematical problem-solving is fundamental to any learning and teaching process in mathematics. English and Sriraman (2010) recommend that problem solving should be integral to the development of an understanding of any given mathematical concept or process, as do Lesh and Zawojewski (2007). Over the past four decades, the importance of problem-solving as one of the central tools in mathematical instruction was analyzed in multiple volumes and papers in mathematics education (Liljedahl & Cai, 2021; Schoenfeld, 1985; Silver, 1985; Verschaffel et al., 2020). In summary, problem-solving is an effective didactical tool that allows pupils to mobilize their existing knowledge, construct new mathematical connections between known concepts and properties, and construct new knowledge in the process of overcoming challenges embedded in the problems (Lampert, 2001; Silver et al., 2005; Thompson, 1985).

Research that examines problem-solving processes and outcomes addresses (among other aspects) the following characteristics of problem-solving:

- Problem-solving strategies and the role of heuristic processes in solving non-routine problems (Polya, 1973, 1981; Silver, 2013; Schoenfeld, 1985, 1992).
- Mathematical modeling (Kaiser et al., 2011; Lesh, 2003; Lu & Kaiser, 2022);

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- The nature and structure of problem-solving expertise (Davidson & Sternberg, 2003; Schoenfeld, 1985, 1992; Silver & Marshall, 1990; Elgrably & Leikin, 2021).
- Effective ways of teaching the solving of mathematical problems (Kilpatrick, 1985; Schoenfeld, 1985; Silver & Marshall, 1990).
- Belief systems regarding problem-solving (Callejo & Vila, 2009; Goldin, 2009; Leder et al., 2006).
- Creativity in problem-solving (Silver, 1997; Pehkonen, 1997; Haylock, 1987; Leikin et al., 2009; Leikin & Sriraman, 2017, 2022).

One of the central components of creative mathematical problem-solving is mathematical insight associated with 'Aha! moments' (Weisberg, 2015) in problem-solving. This chapter analyzes the mathematical challenge embedded in problem-solving tasks from the point of view of the use of multiple solution strategies and mathematical insight required for solving the problem.

17.2 Creativity and Mathematical Ability

Mathematicians and mathematics educators have addressed the connection between mathematical creativity and mathematical ability. Beginning with Poincare (1908/1952), who drew a bridge between mathematical creation in professional mathematicians and their special intuition and understanding of mathematics, several seminal studies examined the relationships between mathematical abilities and mathematical creativity. Hadamard (1945) explored creativity in outstanding mathematicians and scientists and identified four main stages of mathematical creation: *initiation, incubation, illumination, and verification*. Based on publications from the 1950s and 60s, Aiken (1973) stressed that highly gifted scientists and mathematicians are highly creative and suggested that creativity can be seen in novel and useful products, divergent and fruitful processes, and in inspired and immanent experiences. He pointed out three major characteristics of mathematical creativity: First, creative thinking leads to new knowledge and the production of original and unusual solutions to known problems. Second, the creative process involves combining ideas and approaches in new ways, analyzing one particular problem in many ways, and finding a way to tackle an unfamiliar situation. Third, creative activity integrates "the subjective experience known as the ("flash of insight") …which has been vividly described by Poincare (1908/1952), Hadamard (1954)" (Aiken, 1973, p. 409).

The connection between mathematical creativity and high mathematical ability or mathematical talent implies that creative production is challenging. Challenge is one of the core elements that promote any learning process and mathematics learning in particular. Through overcoming difficulties people train their minds, expand their knowledge and develop proficiency in using various skills. "Challenge is not only an important component of the learning process but also a vital skill for life" (Taylor, 2006, p. 2). The "developing education" approach (Davydov, 1996) postulates that in order to develop their mental abilities, learners should approach tasks through a meaningful activity that allows them to tackle difficulties embedded in the tasks. Based on Polya (1973), Schoenfeld (1985), and Charles and Lester (1982), a task is challenging if the solver is motivated to tackle the task, does not have readily available procedures for solving the task; and is ready to invest effort in the solution process. Consequently, Leikin (2009b) suggested that mathematical challenge be defined as an "interesting difficulty" that a person is motivated to overcome (p. 2019) and that a mathematically challenging task is appropriate to students' abilities, neither too easy nor too difficult. No less important, students have to be motivated to complete the task, and thus the task should be directed at the development of students' mathematical curiosity and interest in the subject. One of the effective ways to develop students' motivation and curiosity is by integrating creative activities, which are often surprising.

We believe that mathematics teaching should be aimed not only at the advancement of students' mathematical knowledge and skills and their problem-solving proficiency, but, even more importantly, at the advancement of mathematical creativity. While for many decades creativity in mathematics teaching and learning was largely overlooked (Haylock, 1987; Leikin, 2009a), luckily, in the last decade, we observed an exponential development of research publications related to creativity in mathematics (Leikin & Sriraman, 2022). We see the development of knowledge and skills and the development of creativity in a circular manner: more advanced knowledge and skills allow better creative processing, while creativity can serve as a mechanism for the development of knowledge and skills (Guberman & Leikin, 2013; Levav-Waynberg & Leikin, 2012; Pitta-Pantazi et al., 2022). Thus, we argue that creativity-directed activities are an effective instructional tool.

In this chapter, we examine creativity-directed tasks as inherently challenging activities and specifically focus on mathematical insight as an indicator of the illumination stage of the creative process. Like other creative processes, mathematical creativity is connected to both divergent and convergent thinking; mathematical insight is related to convergent thinking, while a multiplicity of mathematical ideas /solutions is considered to be an instance of divergent mathematical thinking. The paper presents an analysis of mathematical challenge linked to multiplicity of solutions and mathematical insight in the context of mathematical problem-solving.

17.3 Insightful Problem-Solving

Ervynck (1991) suggested distinguishing between algorithmic, strategic, and insight-related problem-solving processing as three levels of mathematical creativity while considering creativity a major characteristic of professional mathematicians. Insight-based problem solutions are relatively simple to perform but are difficult to discover until solution-relevant features are recognized (Metcalfe & Wiebe, 1987; Weisberg, 2015). Insight is "an experience of suddenly realizing how to solve a problem" (Eysenck & Keane, 2010: p. 463) that requires understanding and restructuring and performing a solution that is not rooted in previous experience

(Eysenck & Keane, 2010). This position implies that insight-based problems require unconventional mathematical thinking. Eysenck and Keane (2010) argued that insight-based problem-solving is associated with parallel cognitive processing, in contrast to experience-based problem-solving, which is connected to serial processing related to experience (in our case, learning). Moreover, several studies have demonstrated neuro-cognitive distinctions between insight-based problem-solving and experience-based problem-solving (Leikin et al., 2016).

Researchers draw strong connections between insight, creativity, and exceptional abilities, with exceptional intellectual accomplishments connected to intellectual insights (Sternberg, 1985). Insight is viewed as one of the typical characteristics of general giftedness (as measured with IQ) since gifted children outperform their average-achieving peers in insight-based problem-solving (Davidson & Sternberg, 2003). According to Davidson and Sternberg, there are several different approaches to insight: Gestalt approach, puzzle approach, and great-minds approach, all of which are connected to each other. The Gestalt approach considers previously learned analogues as one of the sources of insight when a structural organization of a familiar situation is used when solving a new problem. The puzzle-problem approach connects insight with retrieval and application of prior knowledge in several ways, with incubation allowing problem solvers to get rid of fixations that are blocking their access to relevant information. The great-minds approach to insight presumes large amounts of knowledge within and across domains.

Sriraman (2005) characterized the links between creativity and giftedness using the Gestalt Principle, which focuses on insight. In addition, he suggested four additional principles that connect creativity with giftedness, all of which are associated with insightful problem-solving. The Aesthetic Principle connects creativity to the beauty of mathematics and elegance of creative processing and products, which are usually insightful. The Uncertainty Principle links creative mathematical processing to solving open-ended and ill-defined problems, the solutions to which often require insight. The Free Market Principle includes risk-taking connected to solving non-conventional and thus insight-based problems. Finally, the Scholarly Principle considers creativity to be the main mechanism of extending mathematics as a scientific field, and to be associated with the four stages of the creative process (Wallas, 1926; Hadamard, 1945) that include incubation which leads to illumination (i.e. insight).

According to all the approaches and principles mentioned above, prior knowledge is crucial for solving problems in new, insightful ways and restructuring mental representations when solving non-routine problems. On the other hand, Davidson and Sternberg (2003) argued that fixation on previously performed solutions can inhibit insightful thinking and the ability to change one's problem-solving strategies. Based on the literature observed above (Davidson & Sternberg, 2003; Ervynck, 1991; Metcalfe & Wiebe, 1987; Weisberg, 2015), the following components of the thinking process seem to be essential for insightful problem-solving:

- Overcoming fixation.
- Discovering the structure of a problem and generating a mental schema for a problem.
- Filling in gaps between the given elements and the goals of a problem, restructuring information related to a problem's goal.
- Generating a set of associations.
- Manipulating with (visual) representation.
- Viewing a problem in a new way.

This list is not hierarchical and not complete; however, it reflects the complexity of the thinking processes associated with different types of insight-based problems.

Davidson and Sternberg (2003) present several examples of puzzle problems that require overcoming fixation when solving the problems, restructuring the givens and filling gaps between the givens and the solutions. The following is an example of a problem that is used in the study described later in this chapter (Fig. 17.1).

Krutetskii (1968/1976) and his research team explored *The psychology of mathematical abilities in schoolchildren* and described what they termed a *mathematical cast of mind* in students with high mathematical abilities. This study also revealed that high mathematical ability is expressed in solving non-routine problems, independent mathematical reasoning, and mathematical flexibility. In his examination of mathematical abilities, Krutetskii's team designed 26 batteries of mathematical problems for 73 tests. Several batteries clearly included problems that are unconventional in regular curriculums, the solutions to which require insight. For example, ill-defined problems included the following: (a) problems with an unstated question, (b) problems with incomplete information, (c) self-restriction problems, and (d) unrealistic problems. The unconventionality and creativity-directedness of problems of types (a) and (b) is determined by the requirement to pose the problems by completing (a) missing questions or (b) missing givens. Self-restriction problems (c) require overcoming fixation with respect to the givens, and the unrealistic problems required overcoming fixation connected to the problem-solving strategies used when solving a problem (examples given in Fig. 17.2).

Solving insight-based problems requires high cognitive effort despite the existing knowledge base required for the solution. Problems presented at mathematical Olympiads and contests often presume insight-based solutions (Carreira & Amaral, 2018; Koichu & Andžāns, 2009; Reznik, 1994). In contrast, due to the exceptional cognitive effort required to solve them insight-based problems are rarely included in school textbooks and everyday instructional practices.

Problem 1[i]**:** *Puzzle problem*
You have red stockings and green stockings mixed in a dresser drawer at a ratio of 4 to 5. How many stockings must you remove in order to guarantee that you have a pair that is the same color?
Davidson & Sternberg (2003)

[i] We invite readers to solve the problems in this section.

Fig. 17.1 Puzzle problem. (We invite readers to solve the problems in this section)

Problem 2.2[ii] *Unrealistic problem*
The perimeter of a right triangle is equal to 3.72 m. Two of its sides are 1.24 m each. Find the third side. (p.132)

Problem 2.3[iii] *Self-restriction problem*
Intersect a quadrilateral with one straight-line segment so as to obtain 4 triangles. (p.142)

[ii, iii] We invite readers to solve the problems in this section. Explanations and solutions appear in Appendix 1.

Fig. 17.2 Unrealistic and self-restriction problems (Krutetskii, 1968/1976). (We invite readers to solve the problems in this section. Explanations and solutions appear in Appendix 1)

17.4 Divergent Problem-Solving

Haylock (1987) stressed the educational value of mathematical creativity in the teaching and learning of mathematics. He described three major ways of developing divergent thinking: *solving open problems, problem-posing,* and *redefinition.* Redefining tasks can be seen as a combination of open problem-solving and problem posing, since (according to Haylock) they require students to change information in the given problems and solve them. He argued that integration of these types of problems in mathematical instruction leads to the development of mathematical creativity in students. Silver (1997) connected "rich in mathematical problem-solving and problem posing" (p. 75) creativity-directed mathematical instruction with the theoretical position that creative thinking includes a combination of divergent and convergent thinking as defined by Guilford (1964). Silver (ibid.) connected the possibility of developing mathematical creativity with Torrance's (1974) models of creativity, composed of fluency, flexibility, originality, and elaboration. Krutetskii (1968/1976) noted that mathematical flexibility is also among the main characteristics of mathematical ability and comprises mathematical creativity, arguing that when students "leave the patterned stereotyped means of solving a problem and find a few different ways of solving it … this is the real appearance of mathematical creativity." (p. 117). Using this perspective Leikin (2009b) introduced a model for the evaluation of mathematical creativity using Multiple Solution Tasks (MSTs), which explicitly require participants to solve a problem using multiple solution strategies. This model also allows drawing a connection between the problem-solving process and the diverse problem-solving strategies used by a participant (Leikin & Elgrably, 2022).

The differences and similarities between the solutions in MSTs can be illustrated by using: (a) different representations of a mathematical concept, (b) different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic, or (c) different mathematical tools and theorems from different branches of mathematics (Leikin, 2009b). By requiring multiple solutions to a particular problem, we raise the complexity of the task and the challenge embedded in it. In accordance with these views on mathematical challenge and its role in mathematics education in general, and teachers‘ education in particular, we focus our attention on MSTs as exemplary tasks that embed mathematical challenge, in that they encourage the performance of insightful solutions alongside conventional ones and can be considered as “new tasks“when “moved” to a new context. Performance of multiple solutions requires and develops mathematical flexibility.

Leikin and Sriraman (2022), in their expansive review of empirical studies on creativity in mathematics (education), demonstrated that researchers in mathematics education have shown significant interest in this topic. Interestingly they demonstrated that only 5 of the 49 (10%) empirical studies that they reviewed addressed mathematical insight when examining mathematical creativity. Use of divergent problem-solving was more frequently addressed (in 36 of 49 (73%) studies). This difference in the frequency (in publications) of examination of uses of insight versus divergent production when examining mathematical creativity can be explained by the complexity of solving insight-based problems and the unconscious components of mathematical insight (Haavold & Sriraman, 2022), which are difficult to analyze.

Haavold and Sriraman (2022) find that there are two types of mathematical insight: insight as the consequence of conscious analytical thinking vs. insight as the result of unconscious processes linked to an impasse. They argue that the differences between these two types of insights cause them to complement each other in problem-solving processes and can explain different aspects of the problem-solving process in experts and non-experts. Through neurocognitive analysis of learning-based and insight-based problem-solving by participants who differed in terms of their level of general giftedness and level of mathematical expertise, Leikin et al. (2016) hypothesized that mathematical expertise includes an insight component at the stage of connecting a problem with an appropriate solution strategy, and general creativity increases participants’ success in solving insight-based problems.

17.5 Insight-Requiring or Insight-Allowing Problems

In this paper, we draw a distinction between insight–based problems (those that require insight to solve a problem) and problems that allow an insight-based solution, which is usually the most elegant. We argue that insight-based problems are the most complex.

We start with two pairs of problems that illustrate the differences between these two types of problems. The first part includes two word problems both of which are missing numerical givens that could simplify solving these problems: The Monk Problem (Duncker, 1945), which we consider to be an insight-requiring problem, and the Half-way – Half-time Problem (Leikin, 2006), which is an example of an insight-allowing problem.

The Monk problem (Fig. 17.3) is an ill-defined problem, in which no information about the trail is provided. Moreover, the solver is not required to determine the exact location or time but should show that this place exists. The problem cannot be solved using ordinary algebraic tools. When performing a graphical solution

Problem 3: Monk Problem

In the morning, a Buddhist monk walks outside from his house at 6 a.m. to climb up the mountain to get to the temple at the peak. He reaches the temple at 6 p.m. A couple days later, he departs from the temple at 6 a.m. to climb back down the mountain on the same road and reaches his house at 6 p.m. Prove that there is a point on the trail where the monk was located at the same time of day when going to the temple and when going down the mountain (based on Duncker, 1945)

Solution 3.1: *Graphical solution*
Diagram 1 demonstrates graphs of functions (distance from house on the trail) corresponding to climbing up ($f(t)$ function) and climbing down ($g(t)$ function) rotes. The graphs of the functions $f(t)$ and $g(t)$ each connect opposite vertices of a rectangle (see the diagram), thus they intersect. This means that there is a point on the trail where the monk was at the same time at the two days.

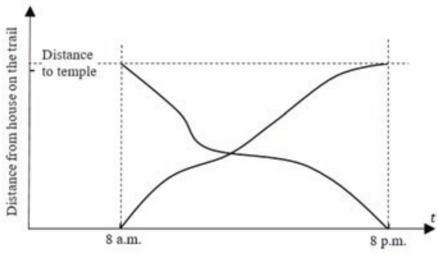


Diagram 1

Solution 3.2: *Verbal imaginary solution*
Imagine two monks one of whom climbs up and the other climbs on the same trail, at the same day from 6 a.m. till 6 p.m. At some point on the trail they will meet, and thus there is a point on the trail where the monk was in the same place at the same time on both days.

Fig. 17.3 Insight-requiring problem – Monk Problem

(Solution 3.1, Fig. 17.3), insight into the arrangement of the graphs in the coordinate system is required. The verbal imaginary solution (Solution 3.2, Fig. 17.3) requires real-life insight – replacing one monk over 2 days with two monks walking on the same day.

In contrast, the Half-time Half-way problem (Fig. 17.4) allows an algebraic solution, which is complex, but relatively standard. The complexity of Solution 4.1 (Fig. 17.4) is related to the choice and the number of variables required for the solution. Solutions 4.3 and 4.4 of Problem 4 we consider insight-based solutions. Both Problem 3 and Problem 4 lack numerical information. Solution 4.3 is a graphical insightful solution, requiring first drawing a graph for Tom's walk that will determine S (the length of the walk). Then half S will be defined as the point at which Dan changes his walking speed. Solution 4.2. can be considered an illustration of Solution 4.4, which is insightful.

In terms of insightful problem-solving activity, the solutions of both problems require discovering the structure of a problem and generating a mental schema for a problem, filling in gaps between the given elements and the goals of a problem, and manipulating with graphical representation (Figs. 17.5 and 17.6).

Problems 5 and 6 are two visual 3D Problems. Problem 5 is an insight-allowing problem. Solution 5.1 is based on considering concrete examples and generalizing observations. Solutions 5.2 and 5.3 are based on discovering that the number of

Problem 4: Half-way Half-time problem

Dan and Tom walk from the train station to the hotel. They start out at the same time. Dan walks half the time at speed v_1 and half the time at speed v_2. Tom walks half way at speed v_1 and half way at speed v_2. Who gets to the hotel first: Dan or Tom?
Leikin (2006)

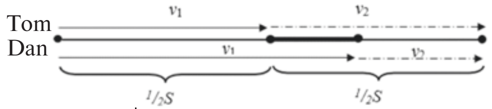
Solution 4.1: *Algebraic solution*

		Speed	*Way*	*Time*
Tom	*Half-way*	v_1	$\frac{S}{2}$	$\frac{S}{2v_1}$
	Half-way	v_2	$\frac{S}{2}$	$\frac{S}{2v_2}$
Dan	*Half time*	v_1	$v_1\frac{T}{2}$	$\frac{T}{2}$
	Half time	v_2	$v_2\frac{T}{2}$	$\frac{T}{2}$

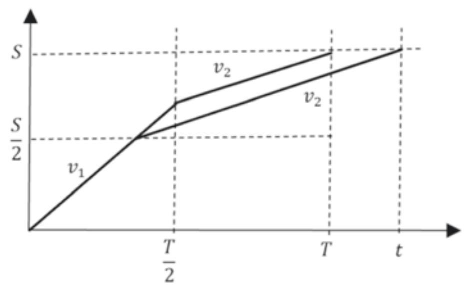
Time spent by Tom	$t = \frac{S}{2v_1} + \frac{S}{2v_2} = \frac{S(v_2 + v_1)}{2v_1v_2}$
Time spent by Dan	T
$S = v_1\frac{T}{2} + v_2\frac{T}{2}$	$T = \frac{2S}{v_2 + v_1}$
Compare t and T	$(v_1 + v_2)^2 \geq 4v_1v_2$ $t \geq T$

In solutions 2, 3, 4 without loss of generality we assume $v_1 > v_2$

Solution 4.2: *Pictorial solution*



Solution 4.3: *Graphical solution*



Solution 4.4: *Logical (verbal) solution*

Dan walks half the time at speed v_1 and half the time at speed v_2.
Assume $v_1 > v_2$, thus during the first half of the time he walks a longer distance than during the second half of the time. Thus he walks at the faster speed v_1 a longer distance than Tom.

Tom gets to the hotel first

Fig. 17.4 Insight-allowing problem – Half-way Half-time

cubes, which varies based on the height of the tower, is an arithmetic series. Solutions 5.3.1, 5.3.2 and 5.3.3 are insight-based solutions that are based on visual restructuring of the tower. Problem 6 calls for the creation of a three-dimensional

Problem 5 - Skeleton Tower

Calculate the number of cubes needed to construct a tower n cubes high (Ridgway, 1998).

Solution 5.1: *Using a table – Indictive solution*

Height	N of cubes

Solution 5.2: *Sum of arithmetic series*

5.2.1 $S_n = 1 + 5 + 9 + \cdots + a_n = 1 + 5 + 9 + \cdots + (4(n-1) + 1)$

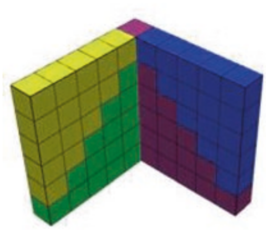
$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n\big(1 + (4(n-1) + 1)\big)}{2} = 2n^2 - n$$

5.2.2 $S_n = n + 4(n-1) + 4(n-2) + \cdots + 4 \times 1 = n + 4 \times \frac{n(n-1)}{2} = 2n^2 - n$

Solution 3.3: *Rearranging cubes*

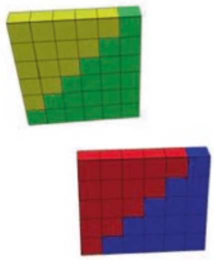
5.3.1

$2n(n-1) + n = 2n^2 - n.$



5.3.2

$n^2 + n(n-1)$



5.3.3

$(n + n - 1)n = 2n^2 - n$

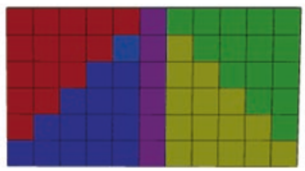


Fig. 17.5 Insight-allowing problem – Skeleton Tower

configuration that satisfies certain conditions. It is described by Polya (1976), and Sharygin and Erganzhiyeva (2001) included it in their *Visual geometry* textbook for 5th and 6th grade. The problem demands verbal, visual, drawing, and applied skills. Yet it requires no knowledge of geometrical theorems or deductive reasoning. Familiarity with basic geometrical objects like rectangular parallelepiped or cube, sphere, cylinder, cone, and triangular right parallelepiped, and the ability to combine them physically or mentally, is adequate for solving the problem. The task, then, requires a high level of visual reasoning, but on the other hand it is regarded as one that allows refinement of mathematical reasoning at an elementary-school level (Leikin & Kawass, 2005). Insightful solutions of both problems require filling in the gaps between the given elements and the goals of a problem and manipulating with (visual) representation.

Following the analysis performed above we raised the hypothesis that insight-requiring problems are more challenging than insight-allowing problems, however, they allow students to exhibit higher creativity. As presented in the next section, we asked students to solve two types of tasks to examine our hypothesis.

Problem 6: Find a plug

Given a wood plate with three hollows of different shapes: a square, a circle, and an isosceles triangle. The side of the square is equal to the diameter of the circle and to the altitude and the base of the triangle. Find a plug that satisfies the two following conditions

(1) It may pass through each one of the three hollows in the wooden plate;

(2) It may close precisely each one of the hollows in the plate above.

Leikin and Kawass (2005)

Solution 6.1:
Combining three plane figures in the space – matching thickness to the holes; adding "a fixed square" to a cone

Solution 6.2:
Real plugs fitting the three holes. Using manipulatives: an apple, cheese, plasticine. Drawing 3D body.

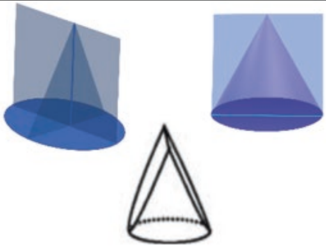


Fig. 17.6 Insight-requiring problem – Find a plug

17.6 Research Experiment

17.6.1 The Tasks

We used 4 problems: Two problems (P1 and P3) were MSTs with at least one insightful solution (see Figs. 17.7 and 17.8). P1 and P3 can be solved with direct calculations or by viewing the structure of the problem as exemplified in Table 17.2. Two other problems (P2 and P4) are insight-based problems that could not be solved algorithmically. For P2 concrete numbers are missing for performing calculations. The solution to P4 (borrowed from Davidson & Sternberg, 2003). We used a scoring scheme to evaluate participants' creativity when solving the tasks (see Appendix 2).

17.6.2 Findings

Sixty-five college engineering students participated in the study. Table 17.1 depicts the number of participants who solved problems in multiple ways (for the evaluation of fluency of the solutions) and in different ways (for the evaluation of their flexibility). Table 17.2 demonstrates that solving insight-based problems (P2 and P4) is more complex and that these problems allow less fluent and flexible solutions. About 50% of participants did not succeed in solving these 2 problems (produced 0 solutions). Only 5 students from Group 1 succeeded in solving P2 in more than 1 way and only 1 student from Group 2 produced 2 solutions to P4. In contrast, 14 students solved P1 in 2 or 3 different ways, and 11 students solved P3 in 2 different ways.

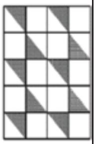
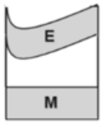
	Insight-allowing problem	Insight-requiting problem
	Try solving the problem in as many different ways as possible.	
Group1	P1: Helen knits a blanket of size $120\ cm \times 150\ cm$. She uses 2 colors to knit the pattern as shown. What is the fraction of the blanket knitted in the darker shade? 	P2: Two painted shapes with areas M and E are located between parallel lines. The distance between the parallel lines in shapes E and M is equal to 1 cm. What is the ratio between the area of shape M and the area of shape E? 
Group 2	P3: In rectangle $ABCD$, AD = 12 cm, $AB = 9\ cm$, $EF = 2\ cm$ and $BE = FC$. Find the black area. 	P4: If you have black socks and brown socks in your drawer, mixed in a ratio of 4 to 5, how many socks will you have to remove to make sure that you have a matching pair? Davidson & Sternberg (2003)

Fig. 17.7 Problems used in the research experiment

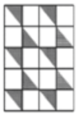
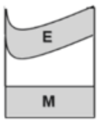
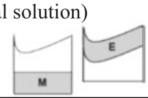
		Examples of solutions	
		Standard Or =0.1 or 1	Insight Or =10
P1		1.1 Calculating area	1.2 $\frac{1}{4}$ in a small unit 
P2			2.1 Proving and using integrals (parametrical solution) 2.2 Translation Ratio = 1 
P3		3.1 Performing calculations	3.2 The black area (composed of triangles) is $\frac{1}{2}$ of the area of rectangle (all the triangles have equal altitudes)
P4		4.1 Drawing different possibilities and generalizing	4.2 Three socks are sufficient (2 will always be the same color) 

Fig. 17.8 Examples of solutions to the 4 problems in the test

Table 17.1 Fluency and flexibility when solving the problems

No.		0	1	2	3	4
No. of participants who produced a particular number of solutions	P1	3	10	13	6	3
	P2	14	16	5		
	P3	0	18	11	0	1
	P4	13	16	1		
No. of participants who produced a particular number of groups of different solutions	P1	3	18	9	5	
	P2	14	16	5		
	P3	0	19	10	1	
	P4	13	16	1		

Table 17.2 Difference in problem complexity

	P1	P2	P3	P4	P1 vs. P2	P3 vs. P4	P1 vs. P3	P2 vs. P4
	Group 1 *N* = 35		Group 2 *N* = 30		Within group diff		Between group diff	
	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	t-test df = 34	t-test df = 29	t-test df = 63	t-test df = 63
Correctness	17.43 (6.57)	8.43 (8.64)	16.34 (5.85)	9.50 (9.50)	5.99***	3.2**	0.70	−0.48
Fluency	1.89 (1.08)	0.74 (0.70)	1.47 (0.68)	0.60 (0.56)	5.92***	5.28***	1.84	0.90
Flexibility	14.97 (8.62)	7.43 (7.01)	13.74 (6.22)	6.33 (5.56)	4.78***	4.66***	0.65	0.69
Originality	1.35 (1.10)	4.86 (5.10)	1.23 (2.61)	1.55 (3.08)	−4.30***	−0.39	0.25	3.13**
Creativity	11.45 (8.79)	48.57 (50.10)	11.57 (26.15)	15.33 (30.26)	−4.54***	−0.48	−0.03	3.13**

*** p<.001, **p<.01

Table 17.2 presents the within-group and between-group differences in correctness, fluency, flexibility, originality, and creativity. The study confirms our hypothesis: it shows that problems P2 and P4 are more difficult than P1 and P3; participants' solutions to P1 and P3 were significantly more likely to be correct and fluent. While P1 and P3 are highly correlated, P2 and P4 are less so, with P2 more difficult to solve. The high levels of originality and creativity in the solutions to problem P2 indicate that the problem's difficulty stems from the fact that it cannot be solved without insight. Problems P1 and P3 yielded the lowest scores for originality and creativity due to the possibility of performing multiple solutions not necessarily based on insight.

17.7 Summary

In this chapter, we introduced a distinction between insight-allowing and insight-requiring problems and demonstrated that tasks of these two types are challenging. As hypothesized, insight-requiring problems appear to be more challenging than insight-allowing MSTs. Moreover, the participants' low scores for originality of solutions for problems of both types demonstrate that only a small number of participants produced insight-based solutions. Based on previous studies (Leikin & Lev, 2013; Leikin et al., 2016; Haavold & Sriraman, 2022) we argue that, as insightful problem-solving is an effective tool for the identification and development of mathematical abilities, both types of insight problems should be integrated in school mathematical instruction. A large-scale longitudinal study could shed further light on the effectiveness of insight-based problem-solving as an instructional tool.

Acknowledgments The research on tool development was supported by Eleusis Benefit Corporation, PBC.

Appendix 1

(i) Puzzle problem (Problem 1 in this chapter).

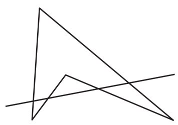
Davidson and Sternberg (2003) explain that "many people incorrectly assume that this is a ratio problem and, therefore, that they must compute the answer using the 4:5 information" (p. 157). We present the solution to Problem 1 in the "Research Experiment" Section.

(ii) Unrealistic problem (Problem 2 in this chapter).

Solution 1 (focusing on the right triangle – stereotypic and incorrect): In any right tringle two equal sides are legs of the triangle, then the hypotenuse is $\sqrt{1.24^2 + 1.24^2}$.

Solution 2 (focusing on the perimeter): the third side is $3.72 - 2 \times 1.24 = 1.24$. The triangle is equilateral and thus cannot be a right triangle.

(iii) The restriction consists of the examinees' considering only convex quadrilaterals and supposing that the triangles ought to lie only "inside" the quadrilateral.



Appendix 2 Model and Scoring scheme for the evaluation of creativity (based on Leikin, 2009b)

	Fluency	Flexibility	Originality	Creativity
Scores per solution	1	$Flx_1 = 10$ for the first solution $Flx_i = 10$ for solutions from a different group of strategies $Flx_i = 1$ for a similar strategy but a different representation $Flx_i = 0.1$ for the same strategy and the same representation	$Or_i = 10$ if $p < 15\%$or for insight/ unconventional solutions $Or_i = 1$ *if* $15\% \leq p \leq 40\%$or for model-based/partly unconventional solutions $Or_i = 0.1$ *if* $p \geq 40\%$ *o*r for algorithm-based/ learning-based conventional solutions	$Cr_i = Flx_i \times Or_i$
Total score	Flu = n	$Flx = \sum_{i=1}^{n} Flx_i$	$Or = \sum_{i=1}^{n} Or_i$	$Cr = \sum_{i=1}^{n} Flx_i \times Or_i$

Note: *Flu* Fluency, *Flex* Flexibility, *Or* Originality, *Cr* Creativity

n is the total number of correct solutions

P = (m_i/n) 9100% where m_i is the number of students who used strategy j

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Chapter 18
Challenging Undergraduate Students' Mathematical and Pedagogical Discourses Through MathTASK Activities

Irene Biza and Elena Nardi

18.1 Welcoming Mathematics Undergraduates to Mathematics Education

Some mathematics undergraduate programs include in their syllabi also courses on mathematics education. The motivation for such courses is to introduce mathematics students to the field of mathematics education research or/and to prepare them for mathematics teaching. Very often, these courses familiarize students not only with the new content of the social science of education but also with the new, to them, practices of educational research. Research in mathematics education, however, is a very different enterprise from research in mathematics (Schoenfeld, 2000). For example, in mathematics education, in comparison to mathematics, the perspective is less absolutist and more contextually bounded. There is less attention to error and more focus on the reasons behind the error. Approaches are more relativist on what constitutes knowledge (Nardi, 2015) and evidence is not in the form of proof, but rather more "cumulative, moving towards conclusions that can be considered to be beyond a reasonable doubt" (Schoenfeld, 2000, p. 649). Thus, findings are rarely definitive; typically, they are more suggestive.

Such epistemological differences affect the experiences of those who, although familiar with mathematics research and practices, are newcomers to mathematics education. Boaler et al. (2003) analyze the challenges of mathematics graduates when they embark on postgraduate studies in mathematics education. They describe the epistemological shift these students experience in their transition from systematic enquiry in mathematics to systematic enquiry in mathematics education. To facilitate such transition, Nardi (2015) addresses challenges with such epistemological shifts in the context of a postgraduate program in mathematics education

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that enrols mathematics graduates by proposing an "activity set designed to facilitate incoming students' engagement with the mathematics education research literature" (ibid, p. 135) through gradual familiarization with the key journals in the field and through co-engineering with the students steps purposefully designed to develop their skills in identifying, reading, summarising and critically reflecting on literature in our field.

In this chapter, we draw on studies that observe and address such shifts at a postgraduate level (Boaler et al., 2003; Kontorovich & Rouleau, 2018; Nardi, 2015; Rouleau et al., 2019) to discuss a course that introduces mathematics education to undergraduate mathematics students. Our work contributes to the broader discussion of the challenges that newcomers face as they enter the field of mathematics education research (Kontorovich & Liljedahl, 2018). Specifically, we propose course activities and an assessment frame for undergraduate engagement with both mathematical and mathematics education discourses. Mathematical discourse is related to the mathematical content seen at school and first-year university level, whereas mathematics education discourse is related to theoretical constructs and findings of mathematics education research. The proposed activities aim to challenge often long-held views about mathematics as well as about its learning and teaching (pedagogy).

These activities aim to pose both mathematical and pedagogical challenge (MC and PC) to undergraduate students. Mathematical challenge in these tasks resonates with (Applebaum & Leikin, 2014; Leikin, 2014) and concerns tackling a piece of mathematics – that is familiar from the school mathematics curriculum – in diverse and alternative ways. Pedagogical challenge is seen in relation to how respondents may tackle the aforementioned mathematical challenge in the context of a specific classroom situation.

In what follows, we describe the theoretical foundations of the course design and its assessment frame and we exemplify the use of this frame in the context of one assessment item. We then describe the course context, objectives, structure, activities and assessment. Subsequently, we outline – and sample from the findings of – a research study of student responses to certain course activities that posed the aforementioned mathematical and pedagogical challenges to the students. We conclude with a broader discussion of the potentialities of such activities in undergraduate programs that introduce mathematics students to mathematics education research.

18.2 A Mathematics Education Course for Mathematics Undergraduates: Theoretical Foundations

The theoretical perspective of this work is discursive and inspired by the commognitive framework proposed by Sfard (2008) which sees mathematics and mathematics education as distinct discourses. Learning of mathematics and learning about mathematics education research (thereafter RME) are then communication acts within these discourses. We are interested in discursive differences – and potential conflicts – between mathematics and RME and we aim toward a balanced engagement with both.

Specifically, we explore how mathematics undergraduates transform what they know about mathematics from their mathematical studies and about mathematics education research (to which they are introduced during the aforementioned courses) into discursive objects that can be used to describe the teaching and learning of mathematics. This transformation is the productive discursive activity of "reification" (Sfard, 2008, p. 118). For example, the reification of the theoretical construct *sociomathematical norms* (Cobb & Yackel, 1996) may be evidenced when the construct is used by an undergraduate to describe a classroom situation in which the teacher and the students customarily negotiate different approaches to solving a mathematical problem. At a "meta-level" (Sfard, 2008, p. 300), we are also interested in how an undergraduate may deploy the theoretical construct *sociomathematical norms* in the analysis of a classroom situation as an opportunity to reflect on whether there is value in seeking diverse and alternative solutions to a mathematical problem or whether a lesson must always privilege a single, optimal solution as sanctioned by the teacher.

Our course design is informed by three principles, set out in Nardi (2015), for supporting post-graduate students' (Master's and doctoral levels) introduction to RME: "engaged pedagogy and participation; cultural sensitivity; and, independence, creativity and critical thinking" (p. 140) and by her proposed set of activities for such introduction. In these activities, students are asked to engage with literature from RME and to produce accounts of their readings. In addition, students are asked to produce accounts of instances in "their personal and professional experiences that can be narrated in the language of the theoretical perspective" (ibid, p. 151) featured in those readings. These accounts of students' experiences are called *Data Samples*. Engagement with literature, together with the production of Data Samples, aims to support students with situating readings in their own experiences and their engagement with the discourse of mathematics education research. Nardi's (2015) analysis of student interviews and written responses identifies four milestones regarding students' transition from studies in mathematics to studies in mathematics education: *learning how to identify appropriate mathematics education literature*; *reading increasingly more complex writings in mathematics education*; *coping with the complexity of literate mathematics education discourse*; and, *working towards a contextualized understanding of literate mathematics education discourse* (ibid). The fourth milestone, contextualization of the mathematics education discourse triggered by the Data Samples (Nardi, 2015), is the inspiration for the course activities that are the focus of this chapter.

Rouleau et al. (2019) also adopt the principles and milestones listed in (Nardi, 2015) to design activities for novice in-service mathematics teachers who study a graduate mathematics education course and "their engagement with scholarly mathematics education literature" (p. 43). In their project, teachers engage with scholarly mathematics education literature in activities such as reading and critiquing pre-set articles; drawing on their own experiences to comment on these articles; using ideas from the articles to design mathematical activities or problems; and, designing a follow-up study to the one reported in the articles they read. The study considers teachers and mathematics education researchers as “members of distinct yet closely related communities” (p. 56) that can mutually benefit from the exchange of experiences and practices. The articles from scholarly mathematics education literature have the potential to “act as boundary objects” (p. 56) between the two communities. Findings highlight the complexity and the challenges of teachers’ engagement with this task that invite them to participate in researchers’ practices which are different to those of the teachers: making “sense of the theories and terminology that the articles used” (p. 57); acquainting themselves with research methodologies that may challenge previously held views and appreciation for certain research designs (e.g. experimental); or, expecting (and experiencing disappointment when not finding) prescriptive suggestions for overcoming students’ mistakes in the research literature. Teachers’ challenges with engagement with research are also rooted in conflicts between the role of the teacher who is tempted to intervene and help the student and the role of the researcher who observes the learning process from a distance (Kontorovich & Rouleau, 2018). Rouleau et al.’s (2019) work exposes the challenges that lie in efforts to engage teachers with research literature “ranging from choosing a research article with which to engage; to turning it into an object that has the potential to transfer praxeologically foreign knowledge; and finally, to the development of reading praxes themselves” (p. 58).

Although the studies we review here (Nardi, 2015; Rouleau et al., 2019) are not about undergraduate students, their relevance to the design of the course and the research study discussed in this chapter is in their focus on engaging newcomers to mathematics education research and the potential challenges such engagement may involve (see also, Nardi & Biza, 2022).

Another inspiration for the study we report in this chapter comes from our work with pre- and in- service mathematics teachers in the MathTASK program in which we engage teachers with fictional but realistic classroom situations and ask them to reflect on these situations. We call these activities *mathtasks* (Biza et al., 2007). Mathtasks are presented to teachers as short narratives that comprise a classroom situation where a teacher and students deal with a mathematical problem and a conundrum that may arise from the different responses to the problem put forward by different students (we discuss a mathtask example in Sect. 18.3).

Teachers engage with these tasks through reflecting, responding in writing and discussing. At the heart of the MathTASK program is the assumption that theoretical discussion related to the teaching and learning of mathematics is not productive unless it becomes focused on particular elements of mathematics and its teaching embedded in classroom situations that are likely to occur in actual practice (Speer, 2005). The MathTASK design underlies the course activities we sample in this chapter and which aims to challenge (mathematically: MC; pedagogically: PC) undergraduate students’ long-held narratives about mathematics and its pedagogy.

The mathematical problem, the students’ responses and the teacher’s reactions to the mathtask are all inspired by the vast array of issues that typically emerge in the complexity of the mathematics classroom and that prior research has highlighted as seminal (Biza et al., 2007). We see the MC in a mathtask as having three components. One component concerns how the mathematical problem in a mathtask is embedded in school mathematics: the task must be appropriate for students at a certain school level, it should motivate students to complete it and it should develop their mathematical curiosity and interest (Leikin, 2014).

A further component concerns the mathematical problem together with the fictional responses to this problem proposed by the students or/and the teacher in the mathtask scenario. These draw on characteristics of a mathematically challenging task identified by the teachers in the study of Applebaum and Leikin (2014): (1) a problem that requires a combination of different mathematical topics; (2) a problem that requires logical reasoning; (3) a problem that has to be solved in different ways; (4) an inquiry-based problem; (5) a nonconventional problem; (6) a problem that requires generalization of problem results; (7) proving a new mathematics statement; (8) a problem that requires auxiliary constructions; (9) finding mistakes in solutions; (10) a paradox; (11) a conventional problem that requires knowledge of extracurricular topics; (12) a problem with parameters (p. 399).

A third component concerns the ways in which the mathtask may invite our undergraduate students to see beyond the school mathematical content of the task when they solve the problem and interpret fictional student/teacher responses in the incident and relate its contents to mathematics they may have learned during their university studies.

We see the PC components in the mathtask as being about bringing to the fore and reflecting upon a classroom situation from the epistemological position of mathematics (which our undergraduates typically hold) and from the epistemological position of mathematics education (which our undergraduates are starting to recognize). Thus, mathtasks aim to challenge narratives about mathematics and its pedagogy that are reported in the literature as dominant.

For example, these narratives include:

PC1. Absolutist and decontextualized views of mathematics (Schoenfeld, 2000);
PC2. Attention to error and less focus on the reasons behind the error(Nardi, 2015);
PC3. Seeking evidence in the form of proof (e.g., experimental studies) in definite findings and less attention to research methods that justify valid evidence (Rouleau et al., 2019; Schoenfeld, 2000);
PC4. Engagement (or lack of) with RME narratives, word use and routines(Nardi, 2015);
PC5. Criticality (or lack of) in the engagement with mathematics education literature (Boaler et al., 2003; Nardi, 2015; Rouleau et al., 2019); and,
PC6. Expectations of pedagogical prescription from mathematics education lit-erature (Rouleau et al., 2019).

The undergraduates' responses to the mathtasks are analyzed (for the purposes of course assessment, as we explain in Sect. 18.4, and for the purposes of research, as we explain in Sect. 18.5) through a typology of four interrelated characteristics (Biza et al., 2018) that emerged from our prior research with mathematics teachers enrolled on a Master's course in Mathematics Education. That research focused on teachers' engagement with mathematics and RME discourses – particularly in relation to mathematics education theories they had been introduced to during the course.

Our typology is as follows:

Consistency: how consistent is a response in the way it conveys the link between the respondent's stated pedagogical priorities and their intended practice? For example, do respondents who prioritize student participation in class propose a response to a classroom situation that involves such participation of students? Or, does their proposed response involve only telling students the expected answer to a mathematical problem?

Specificity: how contextualized and specific is a response to the teaching situation under consideration? For example, do respondents who write generally about their valuing the use of vivid, visual imagery in mathematics teaching, propose a response to a classroom situation that involves specific examples of such imagery? Or, does the response include only a general or generic statement of their preference?

Reification of pedagogical discourse (RPD): how reified is the pedagogical discourse, the theories and findings from research into the teaching and learning of mathematics – that respondents have become familiar with through the course – in their responses? For example, how productively are terms such as "relational understanding" (Skemp, 1976) or "sociomathematical norms" (Cobb & Yackel, 1996) used in the responses?

Reification of mathematical discourse (RMD): how reified is the mathematical discourse – that respondents are familiar with through prior mathematical studies – in their responses? For example, how productively does prior familiarity with natural, integer, rational and real numbers inform a respondent's discussion about fractions in a primary classroom situation? (Biza & Nardi, 2019, p. 46–47).

In the next section, we see the application of the aforementioned theoretical foundations in one mathtask.

18.3 A Mathtask: Students Discuss How to Solve an Algebraic Inequality

In Fig. 18.1, we present an example of a mathtask. The context of this mathtask is a Year 12 lesson in which the teacher asks the students to solve an algebraic inequality that involves fractions. Three fictional students, Mary, Ann and Georgia, discuss solutions to the problem. The classroom incident is inspired by the difficulties students face when dealing with algebraic inequalities (e.g., Tsamir & Almog, 2001) and the benefits of overt use of erroneous responses to tasks about inequalities in classroom discussions (Schreiber & Tsamir, 2012). Mary's response involves reversing the fractions (and the inequality) without distinguishing whether the

In a Year 12 lesson, Mr Smith has asked the students to solve the following inequality:

$$"\frac{1}{x} > \frac{1}{2}"$$

Mary, Georgia and Ann are working on the problem while Mr Smith is observing from a distance without intervening:

Mary: x is less than 2 [she writes in her notebook: $x<2$]
Ann: Well, how do you know this?
Mary: If you inverse the numbers, the big number becomes small …
Georgia: Mmmm?
Mary: Three is bigger than two [she writes: 3>2], so one over three is less than one over two and the other way round [she writes: $\frac{1}{3} < \frac{1}{2}$].
Georgia: So, if one over x is more than one over 2, then x is less than 2. That's it!
Ann: This sounds too simple to me. I do not feel that this explanation is enough to get full marks.

Questions:
P6.1. Solve the inequality: $\frac{1}{x} > \frac{1}{2}$
P6.2. What is the aim of giving this task to the students?
P6.3. What are the issues in Mary's, Georgia's and Ann's responses?
P6.4. How would you respond to Mary, Georgia, Ann and the whole class?

Fig. 18.1 A mathtask from the course's portfolio of learning outcomes

numbers are positive or not. As a result, she misses the point that the inequality does not have a solution for $x \leq 0$. The correct solution to the problem is $0 < x < 2$. Ann and Georgia challenge Mary's choice and trigger an inductive explanation with one example:

$$\text{if } 3 > 2, \text{ then } \frac{1}{3} < \frac{1}{2}.$$

Georgia seems convinced by this explanation. Ann however expresses concerns about the explanation and its capacity to result in receiving full marks.

The undergraduates are invited to solve the problem (Q1); to think about the potential aims of giving such a task to students in the class (Q2); to reflect on potential issues evidenced in Mary's, Ann's and Georgia's responses (Q3); and, to respond to Mary, Ann and Georgia and to the whole class (Q4).

In terms of MC in the mathtask in Fig. 18.1, undergraduates are invited to solve the problem (Q1) and to identify issues in students' responses (Q3). From the Applebaum and Leikin (2014) list of (1)–(12), see Sect. 18.2:

- The problem and the interpretation of fictional student responses require a combination of different mathematical topics (see (1), ibid., p. 399) – e.g., meaning, properties and graphical representations of inequalities and variables.
- The problem and fictional student responses require logical reasoning (see (2), ibid., p. 399) – e.g., why is multiplication with x^2 a correct approach? Or, why

does inversing the numbers not necessarily imply inversing the inequality? Or, does Mary's trial of numbers constitute acceptable justification?

- The problem can be solved in different ways (see (3), ibid., p. 399) – e.g. graphical solution, distinguishing cases, multiplying by x^2.
- Fictional student responses have errors that should be identified, interpreted and acted upon (see (9), ibid., p. 399) – e.g., "[i]f you inverse the numbers, the big number becomes small", Mary says.

In terms of PC in the mathtask in Fig. 18.1, from the list of aforementioned challenges (PC1–PC6), see Sect. 18.2:

- Fictional student responses should be seen in the context of the classroom incident (Year 12 lesson) and the exchanges between Mary, Ann and Georgia (see PC1) – e.g., why does Ann say "[t]his sounds too simple to me. I do not feel that this explanation is enough to get full marks"?
- Errors in fictional student responses should be identified with attention to the reasons behind the error (see PC2) – e.g.: students' intuitive beliefs about the order of inverse numbers are in conflict with formal properties of numbers; or, students draw on inappropriate analogies between processes applied to equations to processes applied in inequalities; or, students tend to multiply both sides of the inequality with x (Schreiber & Tsamir, 2012; Tsamir & Almog, 2001).
- Engagement (or lack of) with RME narratives, word use and routines (see PC4) – e.g., we ask undergraduates to use RME theory and terminology introduced in the sessions in their responses to Q2–Q4.
- Critical engagement with mathematics education literature is necessary (see PC5) – e.g., we ask undergraduates to use the literature in their responses to this item (identification of the issues in question Q3 and response to the students in Q4).
- Moving beyond prescriptive suggestions from mathematics education literature (see PC6) – e.g., we expect undergraduates to provide a response to the students (see Q4) with the expectation to transform the findings from the literature to pedagogical recommendation and not teaching prescriptions.

We now describe the course context, objectives, structure, activities and assessment.

18.4 The Course: Context, Objectives, Structure, Activities and Assessment

The mathematics education course entitled *The Teaching and Learning of Mathematics* is offered as optional to final year (Year 3) mathematics undergraduate students (BSc in Mathematics) in a research-intensive university in the UK. The aim of the course is to introduce undergraduates to the study of the teaching and learning of mathematics typically included in the secondary and post-compulsory curriculum (Biza & Nardi, 2020; Nardi & Biza, 2022). The learning objectives of the course include the following: to become familiar with learning theories in mathematics education; to be able to critically appraise research papers in mathematics education; to be able to compose arguments regarding the learning and teaching of mathematics by appraising and synthesizing recent literature; to become familiar with the requirements of teaching mathematics; to become familiar with key findings in research into the use of technology in the learning and teaching of mathematics; and, to practise reading, writing, problem solving and presentation skills with a particular focus on texts of theoretical content, yet embedded in key issues in RME.

Teaching activities, led by Biza, include 1 hours per week (two for lectures and two for seminars) for a period of 12 weeks. In the lectures, theoretical course content is introduced. In the seminars, undergraduates present and discuss their work that involves preparing presentations of papers they have read, identifying examples from their experience (Data Samples, as per Nardi, 2015), solving problems and reflecting on their solutions; and, responding to mathtasks (Biza et al., 2007). Undergraduates are encouraged to upload their contributions in a shared folder before the session. Discussion during the seminars typically draws on their uploaded contributions. In the middle of the course, for the purpose of formative assessment, they are asked to produce a response of about 800 words to a mathtask. Summative assessment is at the end of the course in the format of a portfolio of learning outcomes that involve questions on mathematics education theory; reflection on undergraduates' own learning experiences in mathematics; solving a mathematical problem and reflecting on the solution; and, responding to mathtasks. Opportunities for verbal and written feedback are interspersed across the seminars. There is also written feedback for formative and summative pieces of writing and a feed-forward session for discussing this feedback once summative assessment is complete.

The mathtask in Fig. 18.1 was in the portfolio of learning outcomes (summative assessment) at the end of the course in a recent academic year. The undergraduates are asked to use mathematics education theory introduced in the course in the preparation of their responses to the task – and their portfolio entries overall:

> In your responses, you are expected to deploy terms that we introduced and used throughout the [course] sessions. You are also expected to refer to a small number (one or two) of research or professional publications in each part […] in addition to the essential publications used in the sessions. (Portfolio guidelines)

Marking criteria are presented in Fig. 18.2 ('arguments and understanding' section adapted from the marking sheet template given to the students). Of those criteria, *consistency*; *specificity*; *use of terms and constructs from mathematics education theory*; and, *use of terms and processes from mathematical theory* are the elaboration of the typology of four characteristics (Biza et al., 2018) – consistency, specificity, reification of pedagogical discourse and reification of mathematical discourse – we introduced in Sect. 18.2.

Once the undergraduates' responses to the mathtasks are marked (for the purposes of course assessment), the work of those students who have consented to

Portfolio marking criteria:

Arguments and understanding

1. Clarity: How clear, justified and transparent the arguments are.
2. Coherence: How logically connected the arguments are.
3. Consistency: How consistent the arguments are across the text.
4. Specificity: How contextualised and specific the arguments are in the used examples and the discussed situations.
5. Use of terms and constructs from mathematics education theory: How precise and accurate the arguments are in relation to the used mathematics education constructs and terms.
6. Use of terms and processes from mathematical theory: How precise and accurate the arguments are in relation to the used mathematical terms and processes, such as definitions and proofs.

Fig. 18.2 Portfolio of learning outcomes marking criteria

the use of their work for research purposes is analyzed through the aforementioned typology of four characteristics (Biza et al., 2018). In what follows, we present findings from this analysis. First, we introduce the participants, the data and the data analysis method.

18.5 A Research Study of Student Responses to a Mathtask: Participants, Data Collection and Data Analysis Method

Of the cohort of 13 mathematics undergraduates enrolled on the course we described above, eight consented to their work being used for research purposes after the completion of the course assessment. These eight undergraduates are the participants of the study and, at the time of data collection, they were in the third year of a 3-year undergraduate course in Mathematics. For the purposes of this study, we analyzed responses to the mathtask in Fig. 18.1.

As this course aimed primarily to introduce undergraduates to the field of RME, a particular focus of our analysis is on manifestations of undergraduates' engagement with reading, writing, reflecting upon and using the constructs of RME (theory and findings in our field) by the end of the course (PC) and in connection to the mathematical accuracy of their responses (MC). To this purpose, our analysis draws on the typology of the four characteristics (Biza et al., 2018) which we introduced in Sect. 18.2 and also underpins the marking criteria of the assessment as we described in Sect. 18.4. Specifically, we aim to identify evidence of reification of the undergraduates' pedagogical discourse (RPD) in tandem with reification of mathematical discourse (RMD). The analysis we sample in what follows naturally weaves in references to the other two characteristics of our typology: specificity and consistency.

18.6 Analysis of Student Responses to a Mathtask

We start the presentation of the analysis of the undergraduates' responses to the mathtask in Fig. 18.1 by discussing first these responses in terms of reification of mathematical discourse (RMD) as evidenced in how respondents engage with the mathematical challenge (MC) of the problem and the responses of the fictional students (Sect. 18.6.1). Then, in the light of the RMD observations, we discuss reification of pedagogical discourse (RPD) as evidenced in how respondents engage with the pedagogical challenge (PC) posed by the situation in the mathtask. Specifically, we explore four themes (Sects. 18.6.2, 18.6.3, 18.6.4, and 18.6.5) that emerged from our commognitive analysis of the undergraduates' responses to this mathtask: *engaging with the RME routine of referencing relevant literature (explicitly or implicitly)*; *endorsing the RME narrative of the importance of considering social interactions during mathematical activity*; *ritualized engagement with RME theory and findings*; and, *RME theory as a descriptor of pedagogical prescription*.

18.6.1 RMD in Responses to the Mathematical Challenge of the Mathtask

Of the eight participants, Isaac, Shaun and Tim, agreed with Mary that $x < 2$ is the right response to the problem. They justified their choice by multiplying both sides of the inequality with 2 and x without noticing that x might be a negative number (see Issac's response in Fig. 18.3).

The remaining five participants spotted the flaw in Mary's response and solved the problem

- By multiplying both sides with $x^2 > 0$ and then solving the inequality $x(x-2) < 0$ (Max, Fig. 18.4);
- By distinguishing cases for $x < 0$, $x = 0$ and $x > 0$ and solving the problem in each case (Nicole and Lawrence);

	$\frac{1}{x} > \frac{1}{2}$	Multiply both sides by 2
=	$\frac{2}{x} > 1$	Multiply both sides by 'x'
=	$2 > x$	Reverse inequality (flip sign and swap sides)
=	$x < 2$	

Fig. 18.3 Isaac's response to the problem

As for all $x, x^2 > 0$, $\frac{1}{x} > \frac{1}{2} \Rightarrow \frac{x^2}{x} > \frac{x^2}{2} \Rightarrow x > \frac{x^2}{2}$ [*provided* $x \neq 0$]

$$\Rightarrow 0 > \frac{x^2}{2} - x \Rightarrow 0 > x^2 - 2x \Rightarrow x^2 - 2x < 0 \Rightarrow x(x-2) < 0$$

Critical Values at $x = 0$ and $x = 2$
Test values of x at the following ranges:

$$x < 0 \Rightarrow x - 2 < -2 < 0.$$

As $x < 0$ and $x - 2 < 0, x(x-2) > 0$, making the original inequality a false statement at $x < 0$

$0 < x < 2 \Rightarrow x > 0$ and $x < 2$. $x < 2$ implies that $x - 2 < 0$
As $x > 0$ and $x - 2 < 0, x(x-2) < 0$, making the original inequality true.

$$x > 2 \Rightarrow x > 0 \textit{ and } x - 2 > 0$$

As $x > 0$ and $x - 2 > 0$, $x(x-2) > 0$, making the original inequality false.

In summary, for the original inequality to be true, $0 < x < 2$

Fig. 18.4 Max's response to the problem

- By saying that the inequality cannot be true if x is not positive and then solving the problem for positive x only (Penny); and,
- By making the graph of the corresponding function and identifying the parts of the graph that satisfy the inequality (Harry).

We focus now on undergraduates' reflections on the incident in the mathtask as evidenced in their responses to Q2–Q4 in Fig. 18.1. We focus particularly on evidence in their responses of engagement with the RME discourse (theories and findings) they had been introduced to during the course.

18.6.2 Engaging with the RME Routine of Referencing Relevant Literature (Explicitly or Implicitly)

Unsurprisingly, given the portfolio guidelines and our emphasis in the course sessions, all undergraduates engage to some extent with RME narratives, word use and routines in their responses to the mathtask in Fig. 18.1. Responses draw on theoretical constructs and findings discussed in the course as well as in additional ones found in publications beyond the course resources (PC4). Such engagement is done however at different levels of criticality (PC5).

In some cases, reflections on the incident are well aligned with RME narratives and word use. For example, Nicole writes in her response to the students and the whole class:

> I would also show a graph of $y = 1/x$ on the whiteboard and the area where $y>1/2$, which would reinforce the learning and illustrate the complications of $x = 0$. Tsamir and Almog (2001) found that inequalities were usually solved correctly when graphs were used, with

> common problems being not rejecting excluded values, and using techniques that apply to equations but not inequalities. (Nicole, Q4)

Nicole proposes the use of a graphical approach as a response to the students and justifies this choice by drawing on relevant literature.

In other cases, the enacted words and narratives are tangentially relevant to the incident under discussion. Harry, for example, in his response to the students, acknowledges the benefits of "constructive conversations" with students and wants to promote more "structure" in student responses through the techniques of problem-solving:

> My response to the students would be to first recognise the constructive convers[at]ions they were having with one another to come up with a solution. However, I would then point out to them that their responses have a lack of direction or structure. To address this issue, I would then recommend the students follow Polya's Problem Solving Process. In his book Polya outlines four stages for solving problems. These stages are (Pólya, 1957):
>
> [… Pólya stages follow]
>
> In studies such as that by (Griffin & Jitendra, 2009) it found when techniques like Polya's was used this led to an increase in student's problem-solving performance. Therefore, by giving learners this instruction, over time it will become a sociomathematical norm to follow this method of problem solving ensuring that proofs are constructed better in the future." (Harry, Q4)

Although problem-solving is at the heart of most mathematical activities, Harry's attempt to connect the situation in the mathtask to Pólya's stages on problem-solving is commendable in principle but arguable in its realization. While we see value in his attempt to establish a new sociomathematical norm related to a structured approach to proving and problem-solving in the classroom he has been invited to imagine teaching in, his recommendation is related to the specific situation in a tangential, generic manner. It seems to us therefore that Harry's response lacks specificity to the situation in the mathtask – this type of response could be given to many, almost any, classroom situation that involves students talking to each other during problem-solving. We may discern here therefore engagement with the "rituals" (Sfard, 2008, p. 241) of RME discourse: Harry knows he is expected to demonstrate awareness of RME works and does so in a generic manner to fulfil his task-completion obligation. We return to this point in the next sections.

While Nicole and Harry engage explicitly with RME literature through referencing specific works – a routine in RME discourse that the undergraduates were explicitly encouraged to engage with – other undergraduates did not do so. However, even in those cases, the RME terms used in the course do appear in their responses to the mathtask. Shaun, for example, writes:

> With regards to Mary, it seems that she does have an understanding of how inequalities work when combined with fractions. Although this shows a low level of <u>relational understanding</u> her explanation of her method lacks formal language showing us that she has not yet fully grasped the <u>sociomathematical norms</u> of the class level. She seems to have applied an <u>inductive reasoning</u> to her approach, and although true cannot be relied upon as a <u>formal proof</u>. (Shaun, Q3, our underlining)

Shaun does not see the flaw in Mary's solution and focuses his critique on her justification and whether such justification (inductive reasoning) is considered acceptable or not in the classroom (sociomathematical norms).

Similarly, Tim, who also did not spot the flaw in Mary's solution, proposes a discussion in the classroom about different "proofs" of the problem, which will be more "deductive" and "convincing":

> I would then ask the other students whether they had any proofs as to why she is right that they prefer to Mary, to see if any of the students would have a proof that is more deductive in style and then ask the class which proof they found more convincing and why. (Tim, Q4, our underlining)

In Shaun's and Tim's responses, we see the enactment of terms (from mathematics and from RME) used in the course as well integrated into their argument, although the relevant literature from which these terms have been drawn is not explicitly referenced. We see such word use as implicit engagement with RME discourse, but we note that engagement with the routine of explicitly referencing the relevant literature is not present. Also, returning to the observation that both Shaun and Tim have not spotted the error in Mary's response, we note that their discussion focuses on the mode of the argument (e.g., inductive vs deductive) and not on the mathematical flaw of the argument. We see this as a missed opportunity to bring in RME literature that proposes potential explanations of the reasons behind such errors and ways to address them.

18.6.3 Endorsing the RME Narrative of the Importance of Considering Social Interactions During Mathematical Activity

In the participants' responses, we observed evidence where the incident in the mathtask was seen beyond its mathematical focus, as an excerpt of student interactions in class. Such responses are attentive to students' learning activity, to the interaction between students or to the norms of the fictional class in the task. We consider such evidence as an indication of the participants' "meta-level learning" (Sfard, 2008, p.300) about a common RME routine: a thoughtful consideration of student contributions in class requires that they are not simply seen as right or wrong (PC1 and PC2).

Nicole, for example, mentions that:

> […] in Vygotsky's (1978) socio-cultural framework the group work enables benefit from the Zone of Proximal Development (ZPD, what they can learn with the support of more knowledgeable others); and from scaffolding (the support they get from others e.g. students, the teacher). (Nicole, Q2)

Later, she returns to this point when she writes "Georgia has understood Mary's response, so has derived some benefit from the ZPD. However she has not noticed Mary's error […]" (Nicole, Q3). Although in Nicole's response the use of ZPD is not precise – it is worded as an approach that can benefit student learning – we can see her attending to the interaction between the students and the potential contribution of this interaction to students' learning. She warrants her support for this type of interaction with her – not-so-precise but appropriately selected – a reference to ZPD.

Similarly, Isaac comments on the interaction (and scaffolding) between Georgia and Mary:

> Georgia is hesitant to give her own answer until she hears Mary's explanation for her answer, where she simply agrees. Georgia's agreement with Mary does imply that her concept is expanding and reveals the working of scaffolding between Georgia and Mary within the class. (Isaac, Q3)

Isaac has not spotted the error in Mary's response and his attention is mostly on the justification of why $x < 2$ is the right response and the communication of this justification. In particular, he discusses how Mary tries to persuade Georgia and Ann:

> With Mary's response, observations show that she seems confident in her answer, and is prepared to give answers for how she solved the question. She has a persuading proof, removing doubts the others have (Harel & Sowder, 2007, p. 6). Mary's justification and mathematical reasoning does not meet the expected standard for a year 12 class. It would be assumed that in her class, there would be socio-mathematical norms set in which mathematically proving and justifying answers. (Isaac, Q3)

Similarly to Isaac, several students aptly – if not always with precise wording – reference the construct of sociomathematical norms (Cobb & Yackel, 1996) to discuss Ann's concerns whether Mary's solution is enough to receive full marks. Nicole writes in Q3: "Ann appears to have understood Mary's response, but thinks the explanation is too simple. She has considered the sociomathematical norm of 'what counts as an acceptable mathematical explanation' (Cobb & Yackel, 1996, p.178)".

Penny also sees the establishment of sociomathematical norms in the aims of using such a problem in a Year 12 class when she writes that

> [the problem] also aims to tackle a sociomathematical norm explored previously, where students may have the idea that longer, more complicated answers are usually worth more marks, so this task can produce evidence of problems that conflict with this notion and challenge it. (Penny, Q2)

Overall, although some of the participant statements are not precise – e.g., "[Georgia] has derived some benefit from the ZPD" (Nicole) – we see participants' attending to issues of student interaction and the social/sociomathematical norms of the mathematics classroom as an endorsement of a common RME narrative: social interactions, not only the mathematical content of such interactions, is a significant and worthy focus of attention when we consider students' contributions in class. The explicit attending to these social interactions in the participants' responses evidences part of what we see as their becoming social scientists, in tandem with being mathematicians: they are endorsing the priorities, foci and methods of the social science of mathematics education while remaining attentive to the mathematical focus of the classroom incident under scrutiny.

18.6.4 Ritualized Engagement with RME Theory and Findings

We have already seen examples in which engagement with RME narratives and words is relatively inaccurate (e.g. Nicole's reference to ZPD) or not so relevant to the situation (e.g. Harry's generic connection to the problem-solving literature). Having in mind that these responses were produced in the context of summative assessment, we acknowledge the undergraduates' understandable effort to appear as knowledgeable and appreciative users of the RME terms introduced in the course (PC4) for the purpose of achieving a higher mark. We detect therefore that they may do so in a ritualized way.

Max uses the constructs of "concept image" and "evoked concept image" (Tall & Vinner, 1981) to describe students' exchanges in the incident. He seems to use "concept image" to describe students' deficiencies:

> Mary's concept image "may cause problems" (Tall & Vinner, 1981) as it does not take into account the cases where x is less than 0 […] Georgia does not have a concept image […] There is "conflict" (Tall & Vinner, 1981) in Ann's concept image, most likely leading to her confusion" (Max, Q3, his quotation marks).

Max's reflection develops around the adequacy or not of Mary, Ann and Georgia's concept image and whether they can see that x might be a negative number. In his response to Q4, he does not attempt a reconstruction of their contributions or he does not address the conflict that may emerge from these contributions. He merely proposes a correct solution to the problem instead. Max's response indicates confidence with the mathematical content (RMD) but also a tendency to focus on what he sees as important: the correctness, or otherwise, of the students' contributions. He takes a largely deficit perspective on these contributions and resorts to the RME literature through a superfluous reference to "concept image" (possibly because he thinks that such a reference may help him gain marks). His alignment with the words, routines and narratives of the RME discourse may therefore be seen as ritualized.

We saw earlier Nicole proposing a visual approach (graphing functions $1/x$ and 1/2 and showing where the former lies above the latter) for her response to the class and grounding this choice in the relevant literature. A similar proposition came from Penny: she "would have them draw the graph of $1/x$ to help them visually understand and identify what values x can take in this situation" (Penny, Q4). She justifies this proposition as follows:

> Using a more visual method could also potentially aid those that would be considered Visual Spatial Learners who may struggle to understand problems without a visual representation, as detailed by Rapp (2009) in her paper on the subject. (Penny, Q4).

The reference here to the "Visual Spatial Learners" is one that does not resonate with the focus and principles of the course that explicitly fostered an avoidance of crude characterization of *learners* (as visual, analytic or kinesthetic, for example) and encouraged characterizations of *learning* (and, even more, of learning *in context*). We are aware though that such characterizations proliferate amongst practitioners who find them readily helpful when they plan differentiated activities in their lessons. It is not unlikely that Penny's response may be influenced by recall of uses of such characterizations by, for example, her teachers when she herself was in school.

Yet, Penny's response continues with at least two references directly from those RME works introduced in the course, Ball et al.'s (2008) Specialized Content Knowledge (SCK) and sociomathematical norms (Cobb & Yackel, 1996).

> This would also fall within Ball, Thames and Phelps' Specialized Content knowledge (Ball et al., 2008), as without my own understanding of how the problem may be related to graphs, this would not be a viable method. It would also be beneficial to explain that when x is taken to be a non-zero positive, Mary's method would work, but not in all cases, with the above example given, so the students could understand from their own work and thinking where the issues arise. Solving the issue alongside the students could also potentially combat the aforementioned sociomathematical norm that answers must be complicated for high marks. (Penny, Q4)

We see in Penny's response an attempt to bring elements from the literature, some of them directly relevant to the incident (sociomathematical norms) and some a little less directly so (specialized content knowledge). We note however the reflective element in her response when she quotes SCK: for a teacher to be able and willing to offer a confident alternative to solutions proposed by their students, her own SCK needs to be confident. We see merit in Penny's efforts to discern teacher-related issues in a mathtask incident that at face value seems to be largely about learners.

18.6.5 RME Theory as a Descriptor of Pedagogical Prescription

We now return to Nicole's response that "group work enables benefit from the Zone of Proximal Development" and that "Georgia has […] derived some benefit from the ZPD". In this statement, ZPD seems enacted not as a tool to explain Georgia's meaning-making in her interaction with Mary, but as a didactic approach with potential benefits for learning. Lawrence's response to Q4 evidences a tendency we discerned in the undergraduates' responses to deploy RME theoretical constructs not as explanatory tools but as recommendations for effective teaching practice (PC6):

> Firstly, I would use the teaching triad which introduce by Barbara Jaworski (1994). According to Management of Learning I have to split the classroom into groups and give them some examples and tasks to check whether the students are familiar with negative numbers and of course negative inequalities. Also, I have to remind them the principle that if we inverse the positive numbers of the inequality then the sign of the inequality changes. For example, if we have two positive numbers $5 > 4$ then $1/5 < 1/4$. Also, if we multiply an inequality with negative number then the sign of the inequality changed. For instance, let $3 < 6$ then if we multiply by -1 both sides then $-3 > -6$. In addition, when I would finish with these examples, I would encourage the students to participate into a dialogue with the aim to realise if they understand these principles (sensitivity to students). Furthermore, I will

> split the problem into three cases. First case, when $x < 0$ second case when $0 < x < 2$ and third when $x > 2$. Also, I can use a program like desmos[1] to sketch graphs and I can sketch the graph $y = 1/x$ and $y = 1/2$ and try to find values where $y = 1/x$ is above $y = 1/2$. Lastly, I would ask the students for any "challenges to engender mathematical thinking and activity" (Potari & Jaworski, 2002, p. 352–353) with this task and any questions that they appear (Mathematical Challenge)." (Lawrence, Q4)

For Lawrence, Jaworski's Teaching Triad (1994) is not, as its author intended, a lens through which to analyze classroom events; it is instead an alert to three areas of concern that a teacher needs to address: how to manage classroom activity; how to address student needs with sensitivity; and, how to provide precise mathematical support. We discern in this, and other responses of this ilk, a tendency of our newcomers to see RME as an applied field that is able and willing to provide a pedagogical prescription. As such, RME theoretical constructs are often construed by our participants not as interpretive instruments but as alerts to what the field prescribes as pedagogically efficient.

18.7 How Facing the MC and PC in Mathtasks Works as a Boot-Camp Experience for Newcomers into RME Discourse

In this chapter, we present a course that aims to introduce third-year undergraduate mathematics students to the field of mathematics education research (RME) by deploying certain course activities and their assessment frame. The course activities are inspired by studies that have identified the epistemological differences between practices in mathematics and mathematics education (Boaler et al., 2003; Kontorovich & Rouleau, 2018; Nardi, 2015; Rouleau et al., 2019; Schoenfeld, 2000) and have addressed these differences in the learning of postgraduate students (Nardi, 2015; Rouleau et al., 2019). Specifically, in this chapter, we focus on one specific type of activity (mathtask) inspired by the principles of the MathTASK program (Biza et al., 2007) that contextualizes the use of RME theory and the mathematical content in specific learning situations. Mathtasks aim to pose both mathematical and pedagogical challenges (MC and PC) to undergraduate students. Undergraduates' responses to such challenges are analyzed for the purposes of course assessment and for the purposes of research through an adaptation of a typology of four interrelated characteristics (Biza et al., 2018): consistency; specificity; reification of pedagogical discourse; and, reification of mathematical discourse. In this chapter, we present findings from the analysis of the evidence of reification of mathematical and pedagogical discourses (RMD and RPD, respectively) in the responses of eight undergraduates.

[1] https://www.desmos.com

With regard to RMD in response to MC (Applebaum & Leikin, 2014; Leikin, 2014), three undergraduates erroneously multiply both sides of the inequality with x (Schreiber & Tsamir, 2012; Tsamir & Almog, 2001) and conclude with the same incorrect solution, $x < 2$ (as student Mary in the mathtask): they cannot see the error in Mary's response but comment on how Mary warrants her response. The remaining five undergraduates present a range of mathematically valid responses.

With regard to RPD in response to PC (PC1–PC6, see Sect. 18.2), our analysis highlights that the undergraduates engage with RME literature either explicitly, with the use of theoretical constructs connected to citations of relevant studies, or implicitly, with the use of theoretical constructs without the appropriate citations (PC4). Sometimes this engagement is at different levels of criticality (PC5). The undergraduates do not always realize that an argument in the social science of RME needs to be supported by evidence, either of a first-order – namely, data they collected themselves – or of second order – namely, findings published in peer-reviewed RME outlets (Nardi, 2015).

The literature that the undergraduates choose to reference varies from specific to the topic under discussion to generic and less relevant. We see this as an attempt to gain marks in the course assessment and earn the lecturer's approval: they need to appear as knowledgeable and appreciative users of the terminology the lecturer introduced in the sessions. We see this as ritualized engagement with RME for the purpose of being accepted as a member of the RME community. One of these rituals is referencing the work of eminent members of the community. We see such engagement as a productive, albeit imperfect, path in the epistemological shift from mathematics to mathematics education. More nuanced enculturation can follow.

Furthermore, we observed how the undergraduates attend to social or institutional aspects of the mathematical activity that is contextualized (PC1), goes beyond considering the mathematical correctness of the students' contributions in class and pays attention to group work, student interaction and sociomathematical norms (PC2). In doing so, the undergraduates sometimes conflate theoretical constructs – intended as interpretive tools in the analysis of learning and teaching situations in mathematics – with pedagogical prescriptions (PC6). We see this as a natural step from the prescriptive and normative position that theory may hold in the natural sciences and mathematics to its more interpretive and reflective role in the social sciences. And, again, we see this as a place from which more nuanced enculturation can follow.

RMD is strongly related to RPD. We observed the interface of attending (or not) to certain mathematical issues in the classroom situation presented in the mathtask with the noticing (or not) of certain details of a pedagogical nature. For example, undergraduates who did not spot the mathematical error in the incident tended to focus their attention on how the solution is communicated. Although they reflected on what an acceptable proof would have been – e.g. differences between deductive and inductive proof, persuading others etc. – the opportunity to discuss the mathematics of the problem and to address associated student needs eluded them. This observation illustrates the potency of activities that pose both MC and PC. Discussion of issues related to mathematics as well as to the learning and teaching of mathematics are better situated when MC and PC are seen in synergy. For those who are engaged, or intend to engage with mathematics teaching, mathematical content can be better seen in the context of classroom situations – and pedagogy can be better supported by relevant mathematical content.

We see the potency of the course activities we present in this chapter to welcome mathematics undergraduates into RME in a manner that balances engagement with mathematics and mathematics education discourses productively (see also Nardi & Biza, 2022). Also, we see how the findings from this study can inform us about how undergraduates' epistemological transition from the sciences to the social sciences can be facilitated and how such findings can provide tools for nuanced and targeted formative feedback. Finally, we see this work as contributing to the ongoing endeavor in our field to support the entry of newcomers with diverse backgrounds to mathematics education research (Kontorovich & Liljedahl, 2018).

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Chapter 19
Commentary on Part II of Mathematical Challenges For All: Making Mathematics Difficult? What Could Make a Mathematical Challenge Challenging?

David Pimm

Back in 1972, after my first mathematics undergraduate year at the University of Warwick in the UK, I came across a very brief book review by Douglas Quadling in *The Mathematics Gazette* (who had become its editor in 1971), a teacher mathematics education journal I had been looking at for the previous couple of years. The book was called *Mathematics Made Difficult* (Linderholm, 1971) and Quadling wrote:

> Professor Linderholm's analogy of Parkinson's law is that "the simpler the things a man gets difficulty out of, the better his mathmanship". […] You will learn far more about number systems and the like from this book than from many far more solemn tomes several times as long. (1972, pp. 255–256)

I got the book from the library and saw that its title was, in part, a reaction to a number of books at the time related to 'mathematics made easy'. I read it as much as I could (and it amused me appreciably, as well as confounding me in places), not yet having engaged with forgetful functors, monoids and categories. As William Thurston (1995) subsequently wrote, "as mathematics advances, we incorporate it into our thinking" (p. 29), something I shall return to in engagement with Nicholas Wasserman's chapter.

There was a second, more detailed, Carl Linderholm book review that I just came across this year, by Geoff Howson (1972). He wrote:

> To take this book as the excuse for a sermon on pedagogy would, however, be wrong. It is a *jeu d'esprit* that should be treated as such. True, it contains much involved and deep mathematics that demonstrate, in particular, how pure mathematicians now concentrate their attention on mappings rather than on the things on which they are defined. Thus a 'modern' view is that "a group is a category with one object in which every morphism is an isomorphism". (p. 83)

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But 50 years later, while reading all the chapters of Section 2 in this book, memories of this old book came back to me, with regard to the notion of 'challenge'. And some questions arose. Is a challenging task one that is not difficult? Is a challenging task one that arises out of a sequence of activities or is it a stand-alone? Is it possible to speak of a challenging task without including the person being tasked and/or the tools at one's disposition? And, based on Roza Leikin and Raisa Guberman's chapter, there are also "insight-allowing" and "insight-requiring" challenges.

One aspect that arose is clearly the potential of a reconceptualisation of the concept involved. I also deliberately challenged myself mathematically by working on all of the tasks on offer in this section. But my response, as a short reflection chapter, primarily consists of detailing some personal associations that arose from my reading (and doing).

Before that takes place, however, I want to add that, of late, I find myself increasingly interested in the etymology of certain core words. In this instance, obviously, it is both the transitive verb 'to challenge' and the noun 'a/the challenge'. In regard to the verb, it is significant both in terms of who or what is the subject and who or what is the direct object. I appreciated John Mason's final reflection comment:

> I can be challenged by something or some person, and I can feel challenged by something or some person, but I can also choose whether or not to accept that challenge.

Historically, 'challenge' is quite a negative term. There is an old French verb *chalenger* (and related noun *chalenge*), derived from Latin, which meant "to accuse falsely" or "to slander", and a related middle-English sense of "accuse/accusation". Even now, it commonly means questioning whether something (or someone) is right. It also is sometimes used comparatively and competitively between individuals engaged in a common activity. According to an etymological online site, the "Meaning of 'difficult task' is [only] from 1954" (https://www.etymonline.com/word/challenge).

But, in this section, the challenge is often seen as potentially positive (though for whom is, on occasion, in question), although in their chapter Nathalie Sinclair and Francesca Ferrara address the notion of mathematical challenge more relationally, based on grade-one students and technology, and broaden and distribute the notion in multiple ways and into plural dimensions.

As with these early instances, from now on in this concluding piece of this section, I shall pick up an item or a mention from some of the earlier chapters and connect it with my (increasingly nostalgic, given post-retirement) experience. For instance, the very first sentence of Irene Biza and Elena Nardi's chapter is:

> Some institutions offer courses on mathematics education in mathematics undergraduate programs with the aim to introduce students to the field of mathematics education research or/and to prepare them for the profession of mathematics teaching.

One of the very first instances of this aspect and location of teaching in the UK was by David Tall in the mathematics department at the University of Warwick (another significant, similar course was taught at the same time by Brian Griffith and Geoff

Howson at the University of Southampton) in the early 1970s. I took it in 1972 and, among other things, we worked intensively on Zoltán Dienes' (1960) book *Building up Mathematics*. But there was no overlap with tertiary mathematics: rather, there was an expansion of secondary mathematics content in terms of pedagogic possibilities.

I felt this was also the case of the inequality task described in Irene Biza and Elena Nardi's chapter, though I misread it as asking to solve $-1/x > 1/2$, which has the solution range $-2 < x < 0$, rather than to solve $1/x > 1/2$, whose solution ($0 < x < -2$) they discussed at length. It raised questions for me about symmetry/rotation/reflection/… in terms of the answer in relation to the functional diagram.

More broadly, I was unsure how having undertaken a degree in mathematics would assist specifically in working on this task. Nevertheless, a mathematician colleague, who had been relearning some norm aspects of linear algebra for a new course he is teaching, said, "Nothing specific from what I have relearnt will be taught, but nonetheless I feel a better teacher about it and some elements may implicitly turn up".

One event that came up when Richard Skemp was applying to be the professor of education at the University of Warwick in 1973 was in a lecture he gave to the mathematics department. It involved an issue of whether examples should be presented before a formal definition or the other way around. Skemp declared the former, while Chris Zeeman preferred the reverse, as long as examples were made evident immediately afterwards. Interestingly, Trevor Hawkes gave a first-year group theory course in 1970 and presented over 20 such varied instances prior to the formal definition being provided, which then created the notion of a group, after which these all became examples of a group. I think an example is not an example until the object of which it is to be an example of surfaces/appears/shows up.

However, the most significant thing for me from David Tall's course was wondering whether the course was at all intended to improve my understanding of mathematics. Recalling it also got me re-interested in the need for distinguishing 'solution' and 'proof', as well as 'task' and 'activity', to which John Mason, in his chapter, also referred.[1] The task frequently is not mine, but the activity always is.

Isabel Vale and Ana Barbosa, in their chapter, indicate both their separation and temporal order by writing, "Although tasks have the power to trigger mathematical activity, they may not be sufficient to implicate mathematical challenge". The temporal order also came up in Mark Applebaum and Rina Zazkis' chapter, both in their abstract, "we turn the conversation from a challenging task or problem to a challenging activity" and in the main text, "the challenge of explanation arises after the solution is found". Nevertheless, according to Jane Austin, "Time alone does not determine intimacy".

I had not come across the Applebaum and Leikin (2014) article detailing aspects of a mathematically challenging task before. However, the first characteristic – "a

[1] Christiansen and Walther (1986) provide a significant discussion and distinction between 'task' and 'activity', something that the negation regularly causes issues in mathematics education by its blurring (not least in regard to activity theory).

problem requires combination of different mathematical topics" – brought to mind the striking mathematician Thurston's (1995) article, where he described his professional work invoking and bringing together a variety of significantly different areas in certain proofs. The third characteristic – "a problem that has to be solved in different ways" – evoked for me Euler's optional multiple different proofs of the same theorem. My intention here is to suggest that professional mathematicians engage in these characteristics too.

The Nicholas Wasserman chapter triggered significant echoes for me of the 'New Math' circumstances from the 1960s and 1970s.[2] This was, in part, because Nathalie Sinclair and I have just been writing a chapter about Canada in that context (Pimm & Sinclair, in press). And for a decade and more in many parts of the world in that period, there were versions and diverse attempts precisely to interact 'secondary' and 'post-secondary mathematics'.[3]

But the term 'advanced' mathematics rather than 'tertiary' also brought up for me the fact that 'elementary' (as opposed to 'primary') also has a connotation of simplicity ("Elementary, my dear Watson"), as well as early. By evoking Felix Klein's translated title from the 1920s as invoking an 'advanced standpoint', it misses the common mistranslation of *höheren* in his title (*Elementarmathematik vom höheren Standpunkte aus*), which actually means "higher" more than "advanced".

Nicholas Wasserman's footnote 1 also engages with these complex connotations:

> 'Elementary' in this sense can be understood in relation to the fundamental 'elements' of school mathematics, both elementary and secondary school levels, and 'advanced' can be understood in relation to university, or tertiary, level mathematics

At ICME 11, Jeremy Kilpatrick presented a significant engagement with Felix Klein's connotations in this regard (and much, much more):

> Throughout his career, Klein saw school mathematics as demanding more dynamic teaching and consequently university mathematics as needing to help prospective teachers "stand above" their subject. […] I suggest why *higher* is a better translation than *advanced* is and end by noting some problems posed when considering *mathematics education* from a higher standpoint (2008, pp. 26–27).

[2] And, relatively recently, there is an engaging article entitled 'The new new math' (Brown, 2015), that, in part, reviews Cheng's (2015) book *How to bake π: An edible exploration of the mathematics of mathematics*. In particular, Aaron Brown discusses Eugenia Cheng involving category theory rather than set theory (hence his article title). And her book is full of metaphors about mathematics. Lastly, Ian Stewart (another of my lecturers at the University of Warwick, coincidentally) commented on her book, "From clotted cream to category theory, neither cookery nor math are what you thought they were. But deep down they're remarkably similar. A brilliant gourmet feast of what math is *really* about."

[3] I started secondary school aged 11 in September 1964 and, as we went up through the years, our year turned out to be the last of 'old math' in that school. In the following year's class, the new students started using Richard Skemp's series of books entitled *Understanding Mathematics: 11–16*. Across desks in school, I subsequently saw scrawled 'Skemp is hard', 'Skemp is mad' and 'Skemp is impossible' (see Pimm, 2002). Nevertheless, I more than happily worked with Skemp for a couple of years 1979–1981.

My interest was also caught up by Rina Zazkis' introduction to this section, where she provided a challenge of sorts: "However, to recognise a binary operation as a function we need first to expand the school idea of a function". I recalled having worked on determinants of matrices (2×2 and 3×3) at school but was taken aback once more in my first-year linear algebra course by David Fowler, when he defined the determinant as a function det: $M^n(R) \rightarrow R$ with the property that $\det(AB) = \det(A) \det(B)$, where A and B are $n \times n$ real matrices in $M^n(R)$. In passing, there is also a notational issue in that the determinant notation is identical to the absolute value notation, but the former can be positive, negative or zero, while the latter has to be non-negative.

In the William Thurston article I mentioned above, he provided seven different notions of derivatives and then provided an eighth one he called "number 37"! But he made sure we were aware that, "This is a list of different ways of thinking about or conceiving of the derivative, rather than a list of different logical definitions" (p. 30). It is clear to me that this is about a mathematical concept rather than a mathematical definition.

There is an engaging book called *What is a mathematical concept?* (de Freitas et al., 2017) and, when writing its final chapter, I listed 16 brief elements, including, "Mathematical concepts are thought-fossils (mostly from extinct species?)", "What is the virtue of the virtual in mathematics?", "A mathematical concept is not a definition, but even definitions mutate (*mutatis mutandis*)" and 'Mathematical concepts are rays" (Pimm, 2017, pp. 276–277).

The issue of addition or multiplication as operations as opposed to functions, in part, has to do with the notation and where it is placed. Yes, it could be '×(3, 5)' – in English, "Multiplying three by five" or, possibly better, "multiplying three and five" – but there could also be '(3, 5)×' (an order known as reverse Polish notation in some contexts, especially technology) and I do recall from it occurring in my first group-theory course, the notation of functions can be placed *after* the elements rather than *before* them. So, will Nicholas Wassermann's proposal that, in school notation, operations can/should be framed as functions mean that the entire conventional notation about operations needs to be adjusted?

My final recollection was of Michael Spivak (1967, pp. 257–258) in his calculus text, where he makes it clear that there are two distinct (albeit related) sine functions – which he symbolises as $\sin^{\circ}(x)$ and $\sin^{r}(x)$ – where x is, as always, simply a number, and not a unit (degrees or radians[4]). Two different functions, not two different notations. And while the derivative of $\sin^{r}(x)$ is the function $\cos^{r}(x)$, the derivative of $\sin^{\circ}(x)$ is not $\cos^{\circ}(x)$, even though it is just a multiplicative constant away. So, do we again need to change different units into varied functions, this time in trigonometry?

In conclusion, I'm still pondering on what it is that makes a mathematical challenge challenging.

[4] And it is curious there is no actual notation for radians, other than the truncation 'rad', as there is for degrees. And there would need to be a third set of trigonometric functions if gradians (truncated 'grad') were used (90 degrees are equivalent to 100 gradians), a third arbitrary unit of angles. Yet a full rotation is, of course, not arbitrary at all.

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Part III
Collections of Mathematical Problems

Editor
Alexander Karp

Chapter 20
Introduction to Part III of Mathematical Challenges For All: In Search of Effectiveness and Meaningfulness

Alexander Karp

This part of the book is devoted to collections of problems, their meaning, role, place, and use in the process of problem solving. Today, problem solving is probably one of the most widely used expressions – the literature about this topic is vast, ranging from the works of Polya (1973) or Schoenfeld (1985), which have become classics, to such recent publications as Liljedahl et al. (2016), Liljedahl and Santos-Trigo (2019), and many others, including ones that will undoubtedly appear in print by the time this book is published. Moreover, and indeed, more importantly, each schoolteacher today knows about the importance of challenges and problem solving in the process of teaching and learning, and no one needs to be persuaded of this importance: the difficulties begin when educators begin to discuss what exactly this term means, and above all, what concretely is to be done. Without attempting in this brief introduction to examine all or even many of the possible interpretations of the concept of problem solving and consequently the concrete actions performed under this banner in class, we will cite just one real-world example, which the author has recently had occasion to encounter.

I received a call from friends whose son, a sixth grader – who, like his classmates, is still interested mainly in new episodes of *Star Wars* and computer games – was assigned the following problem:

Find the positive integers a and b such that $a > b$ and

$$\frac{1}{a+b}+\frac{1}{a-b}=\frac{1}{3}$$

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Not only the child, but his parents as well, did not know what to do. The boy's uncle – more experienced in computer possibilities than my friends – put together a table in Excel and came to the conclusion that the unknown numbers were 8 and 4, but this approach seemed dubious to the parents and so they turned to me.

I must admit that the equality $\frac{1}{12}+\frac{1}{4}=\frac{1}{3}$ was one that I simply remembered(probably because I spent a lot of time discussing so-called aliquot fractions with students, which come up in lectures on ancient Egyptian mathematics). It is not difficult to find the numbers a and b that correspond to these fractions: one must solve a system of linear equations, which will indeed give the solutions $a = 8$ and $b = 4$ This is in fact the only solution.

Indeed, suppose that $\frac{1}{a-b} \neq \frac{1}{4}$; but then $\frac{1}{a-b} \leq \frac{1}{5}$ (it cannot be otherwise,

since $\frac{1}{a-b} < \frac{1}{3}$).

But then

$$\frac{1}{a+b} = \frac{1}{3} - \frac{1}{a-b} \geq \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

It remains to remark that this gives us

$$\begin{cases} a-b \geq 5 \\ a+b \leq \frac{15}{2} \end{cases}, \text{ from which we see that} \begin{cases} b-a \leq -5 \\ a+b \leq \frac{15}{2} \end{cases} \text{and } 2b \leq 2.5$$

from which in turn it becomes clear that the only pair of positive integers satisfying the obtained system of inequalities is

$$\begin{cases} b=1 \\ a=6 \end{cases}, \text{ but this system does not satisfy the given equality.}$$

It is not difficult to come up with other solutions that in one way or another make use of considerations of divisibility, but it is far more interesting to ask: what exactly did the teacher have in mind (since solutions such as the one offered above or that make use of divisibility can hardly be expected of ordinary sixth-graders)? As might have been expected, the next day showed that the teacher was quite satisfied by the answer 8 and 4, which was expected to be obtained by trial and error.

At this point, we might lament about the insufficient mathematical preparedness of teachers, who do not understand what it means to solve a problem; but it is no less important to tell about how the teacher imagined problem solving. It was clearly conceived by the teacher as an activity isolated from any prior experience: the student approaches the question (the challenge!) creatively, that is, tries to do something, and as a result of his or her creative activity, comes upon the solution. Consequently, the teaching process itself (sometimes not just implicitly, but explicitly as well) turns out to consist in making students understand that they should not just sit and do nothing (which is undoubtedly true), but that they should try to plug something in somewhere. Creativity is in some measure contrasted with knowledge (in certain problems, this harmonizes quite well with the recommendation that these problems be solved in a particular way, because this is precisely how they are solved, which is something that has to be known) and turns out to be a mysterious process, which is not taught at all (by contrast with what goes on in English classes, it must be said, in which, despite whatever critical remarks might be made about what students are usually invited to do in classes on creative writing, teachers nonetheless attempt to teach them certain useful techniques).

It is not difficult to see that the reasoning given above in fact is by no means unique. Although the solver did not know the given problem, this was not his first encounter with a situation in which he was given the sum of fractions each of which sooner or later becomes small (readers might recall examples of such situations for themselves – we might mention, for example, the analysis of the equation that is formulated when proving that there are only five Platonic solids[1]). The given problem was not isolated, but similar to others, which is why it was easy to solve.

The meaning of heuristics offered by Polya in large part consists precisely in examining a problem not in isolation, but as part of some set. The classic recommendations:

> You may be obliged to consider auxiliary problems
>
> Have you seen it before? Or have you seen the same problem in a slightly different form?
>
> Do you know a related problem? Do you know a theorem that could be useful? (Polya, 1973)

in fact, encourage students to contextualize a given problem within some set of other problems. Meanwhile, Polya's subsequent recommendations – "Can you use the result, or the method, for some other problem?" – encourage them to extend the set of problems.

This does not mean, of course, that problem solving is merely the combination of several (even many) existing methods: it may turn out that an approach is needed that is completely new for the solver, and in general, success is not guaranteed to anyone (including all of humanity) by anything. The only claim being made here is that the experience of thinking and working with problem sets helps – and that the acquisition of such experience is indeed what is called learning. The teacher in the instance cited above in no way helped the students, although even help with organizing trials (if only by proposing a separate problem – filling in a reasonably organized table in Excel) might have been of some help. But to repeat, the teacher believed precisely that he was giving students an opportunity to engage in genuine problem solving, which students carry out on their own.

In fact, schoolchildren do not very often encounter isolated problems that they must in one way or another extrapolate into a set: usual school practice consists in giving schoolchildren what is a set already – a set of problems in class or a set of problems in the textbook. The set can be a more or less accidentally gathered

[1] Hyman Bass writes about such problems in his paper in this collection.

assemblage, in which case each problem indeed must be solved as if in isolation, as well as a preconceived system, which makes it possible not only to make the process of solving individual problems easier or even to show how the generalization of a result or its transference onto other objects is accomplished but also to show the unfolding of mathematical thinking, thus giving more than all of the problems would have given if they had been considered in isolation from one another.

Moreover, the meaningfulness of mathematical activity becomes clear only when we consider it within the framework of some system. The favorite question of many schoolchildren – "Why do we need to solve this or that problem?" – can rarely be answered without recalling other problems. Naturally, applications of mathematics to real-world problems make mathematics necessary and useful. The answer that one needs to know how to solve some problem because knowing how to do this will make it possible, for example, to determine the coordinates of a ship in the ocean or the height of a launched rocket after some interval of time, is meaningful and even persuasive (although stubborn schoolchildren might point out that no one searches for coordinates in the offered way anymore, and that they have no intention of launching any rockets). But this is not yet the end of the matter: the meaningfulness of mathematical activity is attested to by the fact that, as a result of it, certain existing problems are solved, or it proves possible to pose and solve certain new problems, ones that by no means necessarily have any direct practical application. In mathematics, people do not solve isolated brainteasers but think about complexes of problems. Constructing and solving such complexes is certainly no less exciting than moving from one level of a game to another, which is something that millions of schoolchildren are thrilled to do without asking how this benefits them or why they are playing the game to begin with.

The value of an assertion (problem) becomes clear only in juxtaposition with other problems. In a textbook coauthored with others, the outstanding Russian geometrician Alexander Alexandrov asked why the Pythagorean Theorem is considered so important (Alexandrov et al., 1992, p. 139). One of the reasons, of course, is the enormous number of its direct applications in various problems. But there are also more complex connections with other problems. Alexandrov wrote as follows:

> The Pythagorean theorem is also remarkable because in itself it is not at all obvious. If you look closely, for example, at an isosceles triangle with an added median, then you will be able to see directly all of the properties that are formulated in the theorem that deals with it. But no matter how long you look at a right triangle, you will never see that its sides stand in this simple relation to one another $a^2 + b^2 = c^2$.

By comparing and contrasting problems and assertions, we not only learn to solve them, but also experience a complex combination of aesthetic and emotional sensations (which absolutely should not be concealed from students). Two other Russian mathematicians, Glazman and Lyubich (1969), who wrote what is effectively a course in mathematics in the form of a collection of problems, compared their problem book to music lessons, "each of which is devoted to a specific aspect of musical preparation and which together form the foundation for the performance skills of the future musician" (p. 7).

The question of how problem sets are constructed and how they can be constructed (and the author of this introduction believes that the ability to construct them is, if not the most important, then certainly one of the most important aspects of the professional competence of the mathematics teacher) has evidently not been sufficiently studied – even though one can point to wonderful examples of such constructions, beginning with the famous book by Pólya and Szegő (1998). The present part is devoted to these questions.

To repeat once again, problem sets that a schoolchild encounters may be completely meaningless – as may be the case, to be sure, with all of the work done in class (recall the descriptions of a class as a kind of fixed ritual without any content given by Schoenfeld (1985) or the recent remarks by Liljedahl (2019) about the "non-thinking classroom"). Conversely, a problem set may also be a means to make a class meaningful and teaching more effective.

In speaking about problem sets, as has already been noted, we find ourselves at the intersection of various topics. It is natural to think about how problem sets might be used to solve various pedagogical problems (Karp, 2007) – for example, how to arouse students' interest, or how to lead them toward an understanding of various assertions, or how to help them to review what they have learned, and so on. In this respect, teachers (or authors of problem books) act as the engineers of mathematics education, as it were (to use the felicitous expression of Burkhardt (2006)): they give thought to how what is desirable might be realized in practice. Many specific questions arise in this connection because children might vary greatly – in terms of strength, preparedness, properties of perception, interests, and so on; because different forms of problems might be desirable – sometimes oral problems are needed, sometimes written ones, and so on; and because the conditions under which these problems might be posed also vary (in terms of how much time students have to solve an assigned problem, for example, or whether the problems are meant to be posed for individual or group solving, and so on).

There is a psychological side, as well. The composition of a classic problem set – from the simple to the difficult – immediately runs up against the question of how the difficulty of a problem is determined. Naturally, everything is clear enough in this respect when one is dealing with growing technical difficulty, understood as an increase in the number of operations that must be performed in order to solve a problem – it is more difficult to multiply three-digit numbers than two-digit ones. As soon as we bring up a different approach, however, measuring the difficulty of an assignment (or using another term, its complexity) by how many people complete it (under identical conditions, of course), everything immediately becomes more challenging. This effectively is the question about the problems in a set that make it easier to solve a difficult problem which Polya urged each solver to look for. Another psychological aspect of working with problem sets is familiar to many working teachers: after successfully solving the quadratic equation $x^2 - 4x + 3 = 0$, a student incorrectly solves a problem about finding the coordinates of the x–intercepts of the parabola $y = x^2 - 4x + 3$. The paper by Karp and Marcantonio (2010) examined the question of how differently schoolchildren solve what is in essence the same equation with absolute value, depending on what they are asked to do – to guess the solutions, to solve the equation algebraically, or to solve it graphically. How students perceive a problem set in which similar problems are formulated somewhat differently, what the role of such problem sets is, and how to write them – all of this is both useful to working teachers and interesting from the point of view of theory.

A theoretical question connected with this is that of the *morphology* of problem sets – the role of each problem within a set, its position in it, and the mental processes that take place during the transition from one problem to another (Karp, 2002). We will confine ourselves to one simple example. The mini-problem set

1. Determine whether the number 1 is a solution of the equation $x^2 - 2021x + 2020 = 0$,
2. Solve the equation $x^2 - 2021x + 2020 = 0$.

will be solved completely differently than the problem set

1. Solve the equation $x^2 - 2021x + 2020 = 0$,
2. Determine whether the number 1 is a solution of the equation $x^2 - 2021x + 2020 = 0$.

In the first case, we might suppose that the student will plug in 1 and ascertain that it turns the left-hand side of the equation to zero. And only then, knowing one solution, will the student look for the other (a bright student will immediately deduce that the second root is 2020 since the product of the roots must be 2020) – that is, the first problem serves as a hint for the second. In the second mini-set, however, the same problem will lead the student either to re-check the solution or simply to observe that such a number does in fact exist among the roots that were found (or not, if the solver has made a mistake). The functions of the problems turn out to be different.

In speaking of the morphology of a problem set, we are deliberately making reference to Propp's (1975) book on the morphology of the folk tale. Analyzing Russian folk tales, Propp shows that their episodes, which appear at first glance to be varied and dramatic, in fact carry identical functions: the point is not that the hero is forced to fetch the Firebird's feather or to obtain a ring from the finger of the Princess Who Would Not Laugh, but that a delay arises in the development of the main story. Taking this classic study in some measure as our inspiration, we might say that problem sets might be constructed based on their functions – the goal might be not to teach quadratic equations, but to teach the ability to re-check what has been found, to choose from what has been found, to make use of hints, and so on (which, of course, by no means negates the importance of studying quadratic equations). As has already been said, a problem set conveys more than each of its problems individually.

In speaking of problem sets, we should also not forget about social aspects, such as are vividly manifested, for example, in problem sets on exams. The classic exam with 30 multiple-choice problems differs radically from an exam that contains five problems for which detailed solutions must be given. One can point to countries and periods in which both types of exams were given (naturally, an enormous variety of exam types that lie somewhere between these two extremes exist as well). The form of an exam reflects a conception of what should and what should not be tested, and this conception in turn reflects an understanding of what it is that educators should aspire to make their students learn. Of course, there is no direct mechanism by which society or the government determines what exactly a set of exam problems should look like, but it can hardly be supposed that social factors do not play a role in this respect. Suffice it to mention that relatively recently in Russia the issue of multiple-choice problems on exams became practically politicized when such problems were purported to be a means of destroying Russian traditions and national identity (Karp & Shkolnyi, 2021). How various general principles, conceptions, and traditions that are widespread and preserved in a country and in society become translated into concrete forms and problems in reality is a topic that clearly needs further study.

Let us also mention the historical aspect. Thinking in terms of problem sets, and even more broadly, caring about solving problems, is a comparatively recent phenomenon. Ancient manuscripts containing problem sets have come down to us, but we would search in vain for any special meaning in the arrangement of problems in ancient papyri. Usually, what we encounter is a collection of separate problems, which are at best grouped together based on their content. The situation changed over time, however, above all during the last century and a half. How and why this happened is a topic that, once again, deserves study.

The topics and lines of research pertaining to the study of problem sets have hardly been exhausted. It is evident, however, that no exhaustive answers to the questions raised above can be given in this part of the book. Nonetheless, the papers here submitted to the reader's attention allow us to advance in the investigation of the questions raised and to pose new ones.

These papers are devoted to very different questions and examine collections of problems from different perspectives (nor do their authors' views by any means always coincide with one another). The mathematician's perspective is represented by Hyman Bass's paper, which demonstrates the wealth of mathematical ideas revealed by various collections of problems, which show the unexpected unity of mathematics. Rita Borromeo Ferri, Gabriele Kaiser, and Melanie Paquet discuss theoretical and empirical findings that demonstrate that sets of modeling problems possess a "self-differentiation potential," that is, are genuine "challenges for all," in the sense that different schoolchildren can find different things in the same problem depending on their own possibilities. The Cognitive Complexity Perspective is addressed in the paper by Hui-Yu Hsu, which analyzes sets of problems in geometry assigned by Taiwanese teachers. To a certain extent related to this topic is the paper by Ilana Waisman, Hui-Yu Hsu, and Roza Leikin, which investigates the Complexity of Geometry Problems in connection with parameters associated with problems' diagrams. Another related article is by Roza Leikin and Haim Elgrably – it, too, addresses the complexity of geometry problems and investigates the connection between certain tasks. The papers by Alexander Karp and Albina Marushina possess largely a historical character: the first of them is devoted to the organization of problem sets in American textbooks, while the second addresses differences among sets of Russian exam problems. Peter Liljedahl's paper continues his previous studies, describing the experience of working with collections of problems in the context of transformations in the teaching process proposed by the author.

To repeat, the orientations and methodologies of the studies submitted to the reader's attention in this part of the book, as well as the general theoretical and practical positions of their authors, vary greatly. Collections of problems may and should be investigated from different angles. One would like to hope that such investigations will be continued in the future.

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Chapter 21
Problem Collections and "The Unity of Mathematics"

Hyman Bass

> Science is built up of facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house. –Henri Poincaré

21.1 Problem Collections: To What End?

Problem-solving is normally framed in terms of solving an individual problem, sometimes invoking such heuristics as "looking at simpler versions of the problem," or "thinking of a related problem."

We consider here the idea of deliberately producing a *collection*, C, of distinct problems, as itself being a significant construct, more than the disjoint union of its parts. And as more than a set of practice exercises of an established problem-solving technique. What meaning and purpose could C, *as a collection*, have, and what features and relations among the constituents of C could arguably confer that meaning, and support that purpose? There are various productive answers to this question. I will discuss the following five types, of which examples will be provided below.

I. Problematized curriculum development.
II. Problems that collectively explore diverse aspects, representations, and applications of a focal mathematical context/space/phenomenon.
III. Problems that are "isomorphic," i.e. structurally the "same" problem, even though their contexts, even their mathematical domains, may be quite distinct.
IV. Mathematically distinct problems that reduce to a common mathematical model.
V. Mathematical problems with the same answer, unexpectedly, and for deep structural reasons.

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I see problem collections as an excellent instrument to give students rich opportunities to experience some of the larger-scale coherence and unity of mathematics as a discipline. And even to experience some simulations of the generation of new knowledge. Design of such problem collections offers perhaps a way to disrupt the compartmentalized sense of what mathematics is, a fragmentation conveyed by the siloed curricular geography of the school (including undergraduate) curriculum.

For each of the above collection types, there is some network of mathematical relations that *connect* the different problems and give the collection some mathematical coherence. And these relations/connections can be of a different nature for each type. In type I the relations tend to be somewhat sequential, making the problems into different stages in a curricular storyline, somewhat analogous to a learning trajectory. The students in this case simulate the process of knowledge development. In type II, the problems, though diverse, are all probes/explorations of the same focal mathematical context, and that common focus intrinsically mediates a network of problem connections. In type III, in contrast, the problem connections derive not from the external mathematical contexts, which may well appear to be unrelated, but rather from the internal mathematical structure of the problems themselves. This is the most subtle of the kinds of connections considered here, and it can be the most challenging for students. What does it mean for two seemingly very different mathematical problems to be (structurally) the *same*? And how can this sameness be articulated? Work on such problem sets is related to the issue of "transfer" in cognitive psychology. Type IV represents a different kind of relation among problems. The idea is that there may be some model (problem) such that a variety of genuinely quite different problems are all reducible (after perhaps substantial, and different kinds of mathematical work) to the model. This shows the versatility of the model, and exposes students to another interesting, and sometimes surprising, kind of mathematical connection.

Having the same answer is, by itself, a very weak indicator that two problems are significantly mathematically related. It may happen, nonetheless, that a common answer is both unexpected, and that it happens for structural reasons, but these reasons themselves are somewhat deep and unexpected. This is the situation of type V above. We report on a mathematical example of this below, but it is perhaps too advanced mathematically to submit to a reasonably accessible problem collection for any but advanced undergraduates.

I now turn to a more detailed discussion of the five types, grounded in some concrete example collections.

21.2 Problematized Curriculum Development

This refers to the coherent development of a mathematical theme, or theory, but is framed as a sequence of problems for which the students, perhaps collaboratively, are made responsible.

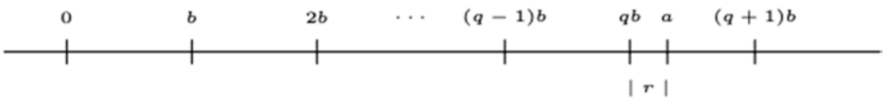
The iconic example of this was the doctoral course in general topology taught by R. L. Moore at the University of Texas (1920-, and earlier at the University of Pennsylvania). Students were presented with a list of precise definitions and unproved theorem statements and otherwise deprived of access to texts on the subject. The theorems were the problems for the students to solve/prove. Much more flexible versions of this "Moore Method" remain popular to this day, under the name, *Inquiry Based Learning* (IBL). This method, while obviously challenging for students, is powerful, not only to experience knowledge generation but also for learning to reason from definitions (plus developing intuition). This format may seem too advanced for the present discussion, but let me sketch an example that could be accessible to secondary or undergraduate mathematics students.

21.2.1 Real Additive Groups

The certification of secondary mathematics teachers commonly requires completion of a mathematics major, and that typically includes a course in abstract algebra. Recent attention has been given to possible ways to make the ideas of abstract algebra more visibly relevant to the secondary curriculum. Group theory is a natural topic to try, and some efforts in this direction, even going back to the "New Math," have been made. But the axiomatic approach is a difficult bridge to cross. And the first candidate examples in this approach, beyond dihedral groups (symmetries of regular n-gons, $n = 3, 4, 6$), are permutation groups, with which students have little experience (as groups), and in which group composition is notoriously difficult.

What I sketch here is an approach I developed in a capstone course for secondary teachers, that begins with the many important groups with which the students are already familiar but not by the name "group." I frame it here as a possible problematized curriculum. A cornerstone of this approach is:

Real Division with Remainder (DwR) Given $a, b \in R$, $b > 0$, there exist unique $q \in Z$, and r, $0 \leq r < b$, such that $a = qb + r$.



Note that DwR here is framed in terms of (continuous) linear measure, rather than integer arithmetic, consistent with the ideas of V. Davydov (1990). The problems below are labeled (DAn), $1 \leq n \leq 6$.

21.2.2 Discrete Real Additive Groups (DA)

Definitions Let $A \subseteq R$. Call A a (**real**) **additive group** if A contains 0, and is closed under addition and subtraction. Call A **discrete** if, for some $e > 0$, $|a| \geq e$ for all $a \neq 0$ in A. It follows directly then that $|a - b| \geq e$ for all $a \neq b$ in A.

(DA1)	Show that, if A is a discrete real additive group, then any bounded subset of A is finite.
(DA2)	(The **discrete-dense-dichotomy**) Let A be a real additive group. Either
	(a) A is discrete, or (b) A is dense in R.

Proof Suppose that A is not discrete. Given $x \in R$ and $e > 0$, we seek some $a \in A$ with $|x - a| < e$. Since A is not discrete, there is some $b \in A$ with $0 < b < e$. Using DwR, $x = qb + r$. Then $a = qb \in A$, and $|x - a| = r < b < e$.

(DA3)	(Discrete groups are "cyclic"). If A is a discrete real additive group, then $A = Za$ for a unique $a \geq 0$.

Proof If $A = \{0\}$ take $a = 0$. Otherwise, choose $b > 0$ in A. By (DA1), $A \cap (0, b]$ is finite, and so has a least element, a. If $c \in A$ then (DwR), $c = qa + r$ with $q \in Z$ and $0 \leq r < a$. Then $r = c - qa \in A \cap [0, a) = \{0\}$, so $c = qa \in Za$.

(DA4)	(**Commensurability Theorem**). For $a, b \in R$, $Za + Zb$ is discrete iff a and b are commensurable, i.e. $b = 0$ or a/b is rational. In this case, $Za + Zb = Zd$, $Za \cap Zb = Zm$ ($d, m \geq 0$), and we write $d = \gcd(a, b)$ and $m = \operatorname{lcm}(a, b)$.

Proof If $Za + Zb$ is discrete, then (DA3) $Za + Zb = Zd$ for some $d \geq 0$. Say $a = md$ and $b = nd$, with $m, n \in Z$. If $b \neq 0$, then $a/b = m/n \in Q$.

Suppose that a and b are commensurable. If $ab = 0$, say $b = 0$, then $Za + Zb = Za$ is discrete. So, assume that $ab \neq 0$, and write $a/b = m/n$ with $m, \in Z$. Put $d = a/m = b/n$. Then $a = md$ and $b = nd$, so, $Za + Zb \subseteq Zd$ is discrete.

Note that the definition of gcd in (DA4) is independent of prime factorization, and more general than the definition for integers. For example, it is immediate that $d = ua + vb$ for some $u, v \in Z$ (Bezout's Theorem), and $a, b \in Z$ iff $\in Z$. Further, for $c \in R$, $\gcd(ca, cb) = |c| \times \gcd(a, b)$, and one can choose $c \neq 0$ so that ca, $cb \in Z$. Moreover, for rational numbers, it can be shown that $\gcd(a/b, c/d) = \gcd(a, c) / \operatorname{lcm}(b, d)$.

(DA5)	Let $f: R \to R$. Call $p \in R$ a *period* of f if: $f(x + p) = f(x)$ for all x. Let Per(f) be the set of all periods of f. Show that Per(f) is a real additive group.

Proof Clearly $0 \in \mathrm{Per}(f)$. Say $p, q \in \mathrm{Per}(f)$. Then, for all x, $f(x+p+q) = f(x+p) = f(x)$, and $f(x-p) = f(x-p+p) = f(x)$. Thus, $p+q, -p \in \mathrm{Per}(f)$.

(DA6)	Let $f: R \to R$. be continuous and non-constant. Let p, q be periods of f.
	(a) What can you say about the relation of p and q? (b) What can you say about Per(f)?

Answers Note that f takes the constant value $f(0)$ on Per(f). If Per(f) is dense in R then, by continuity, f is constant, contrary to assumption. Then, by (DA2) and (DA5), Per(f) is discrete. Then $Zp + Zq \subseteq \mathrm{Per}(f)$ is discrete, so, by (DA4), p and q are commensurable.

21.2.3 Extensions

Analogues of these results for **multiplicative groups** can be developed using the monotone inverse group isomorphisms exp. and log, between the additive group R and the multiplicative group $(0, \infty)$ of positive real numbers.

Once modular arithmetic has been developed, including the construction of the **modular rings** Z/Zm, there is the analogous agenda of studying the additive group (Chinese Remainder Theorem) and multiplicative group of Z/Zm. This is more elaborate and complex. (See Bass, 2022, Ch. 5).

21.2.4 Discussion

The material on real additive groups strikes many people as too advanced for secondary teachers (and their students). I suspect that this reaction stems in part from the fact that, substantively, the material does not, on the surface, resemble things in the current secondary curriculum. That, in and of itself, does not make it “advanced” or inaccessible. In fact, the approach here offers and exploits the power of a linear measure treatment of division with remainder, a fundamental idea. Beyond that, the group theory rests on core properties of addition and multiplication, and it powerfully exposes the duality, not just distinction, between discreteness and continuity, and the natural appearance of commensurability at their boundary. Once the few basic concepts are understood, the proofs of several striking results are surprisingly short, simple, and intuitive. This treatment can afford a depth of understanding at least as robust as assimilation of the much more complex ideas of AP Calculus, for example. Indeed, the ideas featured here provide a helpful foundation for the understanding of AP calculus.

Further, I remind the reader that this package was framed as a problematized curriculum, and so it is intended to be enacted, over time, with students collaborating, and with mediation (including modest scaffolding) by the instructor. I have taught it this way to pre-service secondary teachers.

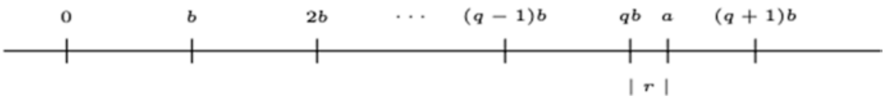
Finally, I note that the study of the additive and multiplicative groups of the basic rings of school mathematics ($Z, Q, R, C, Z/Zm$) is a powerful unifying theme to connect a wide range of important and diverse mathematics.

21.3 Problems that Collectively Explore Diverse Aspects, Representations, and Applications of a Focal Mathematical Context/Space/Phenomenon

Here we will discuss the Euclidean Algorithm (EA), and some surprisingly diverse problems to which it is intimately related. The problems are labeled (EA_n) $1 \leq n \leq 7$.

As with the preceding example, this is grounded on **Real Division with Remainder (DwR)**: Let $a \geq b > 0$ be *real numbers*. Then we can write, (DwR)

$$a = qb + r, \text{ with } q \ni \mathbf{Z}, \text{ and } 0 \leq r < b.$$



Moreover, q and r here are unique, so we can write

$$q = q_b(a) \text{ and } r = rb(a).$$

21.3.1 *The (Real) Euclidean Algorithm (EA)*

Using (DwR), we can then inductively construct, $EA_n(a, b)$

$$a_0 \geq a_1 > a_2 > \ldots > a_n > a_{n+1} \geq 0, \text{ with}$$
$$a_0 = a, \ a_1 = b, \text{ and, for } n \geq 1,$$
$$a_{n-1} = q_n a_n + a_{n+1}, \text{ with } qn = q_{a_n}(a_{n-1}) \text{ and } an + 1 = r_{a_n}(a_{n-1})$$

If $a_{n+1} > 0$, we can continue this process to $EA_{n+1}(a, b)$. If, on the other hand, $a_{n+1} = 0$, then we say that $EA(a, b)$ *terminates* at stage n, and we *define* the *greatest common divisor* of (a, b) by, $a_n =$ gcd (a, b).

Moreover, we then have the system of equations:

$$(\text{Eqs}) \qquad \begin{cases} a_0 = q_1 a_1 + a_2 \\ a_1 = q_2 a_2 + a_3 \\ a_2 = q_3 a_3 + a_4 \\ \qquad \cdots \\ a_{n-2} = q_{n-1} a_{n-1} + a_n \\ a_{n-1} = q_n a_n + 0 \end{cases} .$$

(EA1)	Under what conditions on (a, b) does EA(a, b) terminate?
	Note that this is assured if a and b are integers, since then (a_j) is a strictly decreasing (for $j \geq 1$) sequence of integers ≥ 0.
(EA2)	If EA(a, b) terminates, in what sense is it reasonable to call a_n the "greatest common divisor" of (a, b)?

Call d a "divisor" of a if $a = qd$ for some *integer* q. Then, in fact, a_n is a common divisor of a_h and a_{h+1}, for $0 \leq h < n$, and any common divisor of a and b is a divisor of a_n. This can be proved by reverse induction on $h < n$.

21.3.2 The Euclidean Square-Tiling of an (a × b)*-Rectangle*

Let $a \geq b > 0$ be *real numbers*, as above. Suppose that we want to tile the $(a \times b)$–rectangle $R = R(a, b)$ with square tiles (of variable size). The **Euclidean ("greedy") algorithm for square tiling** is to first fill as much of R as possible with the largest possible $(b \times b)$ square tiles.

Writing $a = qb + r$ (DwR), we see that we can fit q $(b \times b)$-tiles, and what remains is a rectangle $R(b, r)$. Continuing in the same way with $R(b, r)$, etc., we end up producing what we call the **Euclidean tiling** T_{E} of $R(a, b)$.

This provides a geometric picture of the Euclidean Algorithm (EA(a, b)). T_{E} is a finite tiling if and only if EA(a, b) terminates (Fig. 21.1).[1]

$$S_0(a,b) = \sum_{1 \leq j} q_j = \text{the number of tiles of } T_{\text{E}}, \text{and}$$

$$S_2(a,b) = \sum_{1 \leq j} q_j a_j^2 = ab = \text{Area}\big(R(a,b)\big)$$

What about $S_1(a, b)$?

[1] It is a classical theorem of Max Dehn (1903, See Bass, 2011) that a rectangle admits a finite square tiling if and only if its side lengths are commensurable.

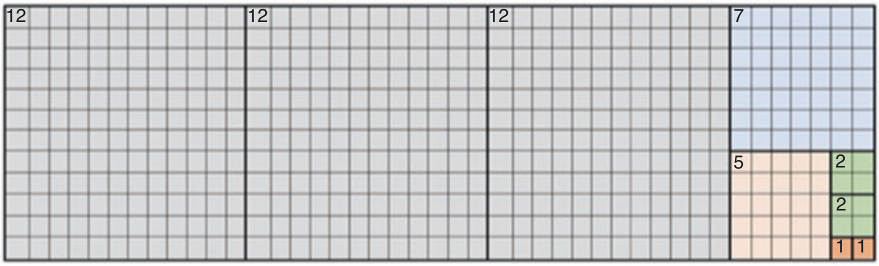


Fig. 21.1 The Euclidean tiling of $R(43, 12)$

(EA3)	**For e = 0, 1, 2, define** $S_e(a,b) = \sum_{1 \le j} q_j \; a_j^e$ (Notation as in the system of equations (Eqs) above.)
	(a) Interpret $S_e(a, b)$ geometrically (e = 0, 1, 2). (b) Evaluate $S_e(a, \mathrm{b})$ for $e = 1, 2$.

We claim that, $S_1(a,b) = \sum_{1 \le j} q_j \; a_j =$ the sum of the side-lengths of the tiles in $T_E = a + b -$ gcd (a,b) if EA(a,b) terminates.

More generally, for any square tiling T of a rectangle R, we shall write $p(T)$ for the sum of the side lengths of the tiles in T, and call this "the *perimeter* of T." Thus, $S_1(a,b) = \sum_{1 \le j} q_j \; a_j = p(T_E)$.

To prove that this equals, $a + b -$ gcd (a,b) if EA(a,b) terminates, add the equations in the system of equations (Eqs), and simplify.

21.3.3 Fair Distribution

Suppose that we want to distribute c cakes equally among s students ($c <$ s). Then each student will receive c/s of a cake. We shall write $p(s,c)$ for the minimum number of *cake pieces* needed to make this distribution. For any distribution D, we'll write (D) ($\ge p(s,c)$) for the number of cake pieces in that distribution.

(EA4)	The *Euclidean distribution,* D_E, proceeds as follows: First cut a (c/s)-size piece from each of the c cakes, and distribute these pieces to c of the students. Then there remain c partial cakes, of size $(1 - c/s)$ to be equally shared among $(s - c)$ students. Continue in the same manner with this (reduced) distribution, etc. Show that $p(D_E) = p(T_E)$, where T_E is the Euclidean tiling of $R(s,c)$.

Suggestion: Imagine that the cakes are $(1 \times s)$ rectangles. Stack them in a pile to form a $(c \times s)$ rectangle R. Then relate the Euclidean distribution to the Euclidean square tiling of R.

(EA5)	The *linear distribution*, D_L, proceeds as follows: Imagine that the cakes are thin rectangles in shape. Line them up end to end, treat this as one long "mega-cake," and cut it into s equal pieces, which are then the student shares. Show that $(D_L) = p(D_E)$.[2]

Choose linear units so that each rectangular cake has a length s. Then, end-to-end they have a total length $c \cdot s$. The cake separations occur at multiples of c, so there are $s - 1$ of these. The student share separations (cuts) are at multiples of s; there are $c - 1$ of these. The cuts common to these two sets occur at multiples of $m = \operatorname{lcm}(s, c)$. We have

$$c \cdot s = d \cdot m, d = \gcd(s, c),$$

so there are $d - 1$ common cuts. Hence the total number of cuts is,

$$(s-1)+(c-1)-(d-1)=(s+c-d)-1$$

And so, the number of cake pieces is $s + c - \gcd(s, c) = S_1(s, c)$ as in (EA3).

21.3.4 *The Diagonal of a* (c × s)*-Rectangle*

(EA6)	Consider the rectangle $R = R(c \times s)$ tiled by $s \cdot c$ unit squares. Let $\triangle$ be a diagonal of R. Show that the number of unit tiles that $\triangle$ enters is $p(D_L)$.

Say R has vertices $(0,0)$, $(0,c)$, $(s,0)$ and (s,c), and $\triangle$ joins $(0,0)$ to (c,s). Apart from the top edge and right edge $\triangle$ crosses c horizontal grid lines and s vertical grid lines. At each of these crossings, $\triangle$ enters a new unit tile, for altogether $c + s$ unit tiles. However, if the crossing occurs at a grid vertex, the intersection of a horizontal and a vertical grid line, then the new unit tile entered is counted twice. Thus, the number of unit tiles that $\triangle$ enters is $c + s - d$. where d is the number of grid vertices other than (s, c) through which $\triangle$ passes. Let $d = \gcd(s, c)$, and write $(s, c) = d \cdot (s', c')$, with $\gcd(s', c') = 1$. Then the set of grid vertices, other than (s, c), through which $\triangle$ passes is $\{t(s', c') \mid t = 0, 1, 2, \ldots, d - 1\}$. Hence, the number of unit tiles through which $\triangle$ passes is $c + s - \gcd(c, s)$.

This general line of investigation can be interestingly (and accessibly) pursued toward a discussion of the continued fraction representation of $a/b = a_0/a_1$ in terms of $(q_1, q_2, \ldots, q_n)$.

We close this section with a nice puzzle.

[2] It is shown in Bass (2011) that $p(D_L) = p(D_E) = s + c - \gcd(s, c)$ is, in fact, the minimum number, $p(s, c)$, of cake pieces for any cake distribution. Further, for any square tiling T of a $(c \times s)$-rectangle, $p(T) \geq p(T_E) = p(s, c) = s + c - \gcd(s, c)$. ("Isoperimetric Theorem").

21.3.5 Tony Gardener's Game of Euclid (Gardiner, 2002)

Each player chooses a positive whole number and records it secretly. The two players then toss to decide who should start, before revealing their chosen numbers – say a and b. Player I then changes the pair a, b by subtracting any positive multiple of the smaller number from the larger to produce a new pair a', b'. Negative numbers are forbidden. Player II can then transform the new pair a', b' in the same way, and so on. The first player to produce a pair in which one of the two numbers is zero is the winner.

(EA7)	When can the first player force a win? How then should she play in order to win?

This is a challenging problem to formulate a precise answer, but the idea for constructing a suitable strategy is rather simple. At each stage of the game say player ***P*** is faced with a pair (a, b), with $a \geq b > 0$. Then division with remainder gives, $a = qb + r$ $(0 < q, 0 \leq r < b)$ and ***P*** must choose a q', $0 < q' \leq q$, and replace (a, b) by (a', b), with $a' = a - q'b$.

We call this the **forced case** if $q = 1$. In that case ***P*** must choose $q' = q = 1$. If $q > 1$, we call this the **control case**, and the strategy calls for ***P*** to choose

Option 0: $q' = q$ or
Option 1: $q' = q - 1$

The point is to never, if avoidable, cede control to the opposing player ***P***$'$.

If $r = 0$, ***P*** chooses $q' = q$ (option 0) and wins. So assume that $r > 0$. If ***P*** chooses $q' = q - 1$ (option 1) then it presents ***P***$'$ with a forced case. To see when this is strategic, let's examine the (EA) more closely. Recall the system of equations that record the successive steps of the Euclidean Algorithm:

$$(\text{Eqs}) \qquad \begin{cases} a_0 = q_1 a_1 + a_2 \\ a_1 = q_2 a_2 + a_3 \\ a_2 = q_3 a_3 + a_4 \\ \qquad \cdots \\ a_{n-2} = q_{n-1} a_{n-1} + a_n \\ a_{n-1} = q_n a_n + 0 \end{cases}$$

And consider the sequence $(q_1, q_2, \ldots, q_n)$.

When all $q_j = 1$, we call this the "*Fibonacci case,*" because then

$$(0, a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$$

is just d times a segment of the Fibonacci sequence, $d = a_n = \gcd(a_0, a_1)$. In this case, all player moves are forced, and player one, **O**, wins if n is odd, and player two, **T**, wins if n is even.

If we are not in the Fibonacci case,

Let t = the least integer ≥ 0 so that $q_{t+1} > 1$.

Then $(q_1, q_2, \ldots, q_n) = (1, 1, 1, \quad q_{t+1}, q_{t+2}, \ldots, q_n)$

After t (forced) moves $(q_1, q_2, \ldots, q_n)$ is reduced to $(q_{t+1}, q_{t+2}, \ldots, q_n)$ with now **O** is the next player if t is even, and **T** is the next player if t is odd.

Claim **O** (resp., **T**) can force a win if t is even (resp. odd).

In light of the analysis above, it suffices to show that,

O can force a win if $q_1 > 1$.

Suppose that,

$$\left(q_1, q_2, \ldots, q_n\right) = \left(q_1, q_2, \ldots, q_s, 1, 1, 1, q_{s+t+1}, \ldots, q_n\right)$$

with $q_j > 1$ for $1 \leq j \leq s$ and $j = s + t + 1$.

If $s = 1$, **O** plays option 0 ($q' = q_1$) if t is even, and option 1 ($q' = q_1 - 1$) if t is odd. If $s > 1$, **O** keeps playing option 1 until arriving, as the first player, at (q_s, 1, 1, 1, q_{s+t+1}, …, q_n), which is the case just considered.

In all cases then, **O** arrives, as the first player, at (q_{s+t+1}, …, q_n), and the strategy continues, inductively, to a win.

This argument is too elaborate to expect students to handle, but there are many special cases worthy of student exploration, for example when all $q_j > 1$.

21.4 Problems that Are "Isomorphic," i.e. Structurally the "Same" Problem, Even Though Their Contexts, Even Their Mathematical Domains, May Be Quite Distinct (See Bass, 2017)

21.4.1 Isomorphic Problems

What does it mean when we say that two things are the *same*? It generally means that the two things share certain features that capture the essence of what we have in mind, while ignoring (or treating as irrelevant, or superficial) unrelated features. Of course, this is rather vague, especially since "what we have in mind" is often tacit, or imprecisely articulated (APA, 2015).

In mathematics, the sameness of two mathematical entities is often denoted by some variant of the equal sign, "=." For example, "4/6 = 6/9" signifies that the fractions 4/6 and 6/9 represent the same rational number. Or, an "algebraic identity" like, "$a^2 - b^2 = (a - b)(a + b)$," signifies that the relation follows formally from the "Rules of Arithmetic" (i.e. the axioms of a commutative ring). Other such relations may express congruence, or similarity, of geometric figures. Or isomorphism of groups in abstract algebra. More generally, an "isomorphism" of two mathematical objects A and B is understood to be an *invertible structure-preserving* *transformation* $f : A \rightarrow B$. Here A and B are typically sets, on which there is given some common specie of mathematical structure (a group, a polyhedron, etc).

What can we mean when we say that two *mathematical problems* A and B are the "same?" This question seems to have first been formally discussed in cognitive psychology, which introduced the notion of "problem isomorph," in connection with the theory of transfer of learning. Transfer asks whether knowing how to solve problem A transfers to aiding the effort to solve an isomorphic problem B.

But problem isomorphism was rather vaguely specified. From the APA dictionary, problem isomorphs are

> problems that have the same *underlying structure*, so that they require essentially the same operations to achieve a solution. Such problems may vary enormously in their *surface structure* and in the degree of difficulty experienced by solvers. (Emphasis added.)

Or, from IGI Global,

> A single problem can be stated in various ways, and often therefore can be variously represented. The particular representation, or **problem isomorph**, can influence the difficulty of solving the problem.

21.4.2 An Example, and a Question

Rather than try to resolve this conceptual imprecision, I will show an example, in our context, that illustrates the subtlety in trying to craft a precise definition of the notion of problem isomorphism. We are interested in problems that have the "same underlying mathematical structure," even though their contexts may be (even mathematically) quite different. Here we present a case in point (Problems (A), (B), and (C) below).

(A) Arisha, Brianna, and Carmen run a race. Assuming no ties, what are all possible outcomes, 1st, 2nd, 3rd?

(B) In a 3×3 grid square, shade three of the nine (unit) squares in such a way that there is exactly one shaded square in each row and in each column. What are all ways of doing this?

One of the solutions of (B) is (Fig. 21.2):

A way to map (B) to (A) is to imagine a solution of (B) as a picture of the finish of the race, the finish line being at the top, with the columns, left-to-right, being the running lanes of Arisha, Brianna, Carmen, respectively, and the shaded squares the finishing positions of the runners. This gives a structure-preserving correspondence between the solution spaces of (B) and (A), and it is a priori clear that both exhibit

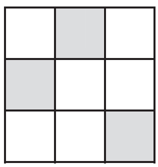


Fig. 21.2 Shading a 3×3 grid square

all 6 (=3!) permutations of three objects. On this basis, I would say that Problems (A) and (B) have the same underlying mathematical structure. But I further say that It does not suffice to say that the solution space to both problems is the set of permutations of three objects. It depends also on the *reasoning* to construct a correspondence and to show that the latter is an isomorphism.

For example, consider the problem,

(C) Find all symmetries of an equilateral triangle.

Each symmetry s of the triangle is a self-congruence of the triangle. It permutes the three vertices, and that permutation clearly determines s. Thus, we have an *injection* of the solution space of (C) into the set of permutations of three objects (the vertices). While this injection is in fact bijective, that fact is not a priori clear from the nature of the problem, as is the case in (A) and (B). On this ground, I will argue that (C) is *not* isomorphic to (A) and (B).

To elucidate the significance of this distinction, consider the analogues of these problems with $n \geq 4$ in place of $n = 3$. Thus, for $n = 4$, we have:

(A_4) Arisha, Brianna, Carmen, and Diana run a race. …
(B_4) Shade four of the unit squares of a 4×4 grid square…
(C_4) Find all symmetries of a square.

Then the solution spaces of (A_n) and (B_n) manifestly remain all $n!$ permutations of n objects, whereas the solution of (C_n) is the (dihedral) group of $2n$ symmetries of a regular $n -$ gon. ($2n = n!$ only for $n = 3$.)

21.4.3 Degrees of Sameness; Discernment Tasks

Thus, problems may be significantly related without being isomorphic in the above sense. Of course, these connections can have a variety of forms and different levels of strength. This is something worth noticing, but may be hard to measure objectively or precisely. The important thing is to identify and give a mathematically explicit articulation of the nature of the connection. For example, with problems (A), (B), and (C) above, though (C) is structurally distinguished from (A) and (B), all three problems can be described as determining some set of permutations of three objects, and that itself is a mathematical connection worth noting.

To provide students with opportunities for such discernment I used the following:

The Discernment Format

What I describe here is a template that I have used for presenting students with a list of problems and asking them to qualitatively identify, describe, and intuitively measure the strength of relations among the different problems.

The Problem Discernment Template

Below is a list of problems, labeled A,B,C,D,E, …

1. The first task is to place the letter of each problem in one of the boxes below.
2. Put letters in the same box if they represent problems that are mathematically the "same" problem, apart from superficial differences in context or presentation.
3. If problems in different boxes are significantly related mathematically, connect their boxes by a line, or by a double line if the connection is very strong.

   (Note, you need not use all of the boxes, and you may reasonably answer this question even if you have not completely solved the individual problems.)
4. Work first individually. Then compare and discuss answers with your group/partner.
5. With or without consensus, explain (to the whole group) your choices, in particular the nature of the connections (Fig. 21.3).

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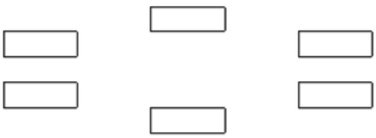


Fig. 21.3 The discernment format

The student products in this format will be "connection networks," with explanations, and these can be quite variable. The processes of explaining them, and the efforts to reconcile differences across different groups can help to develop a probing discourse about structural relations among problems.

Following are two discernment problem sets I have used in this way.

The 3-permutation Discernment Problem Set

(A) What are all three-digit numbers that you can make using each of the digits 1, 2, 3, and using each digit only once?
(B) In a group of five students, how many ways are there to pick a team of three students?
(C) You are watching Arisha, Brianna, and Carmen on a merry-go-round. At each moment you see them in some order – left, middle, right. As the merry-go-round turns, what are all the different orders in which you see them?
(D) If Arisha, Brianna, and Carmen have a race, and there are no ties, what are all possible outcomes: first, second, third?
(E) from a bag full of many pennies, nickels, and quarters, I randomly choose three coins. What are all possible amounts of money that I might have?
(F) In a 3 × 3 grid square, color three of the nine (unit) squares blue, in such a way that there is exactly one blue square in each row and in each column. What are all ways of doing this
(G) What are all the symmetries of an equilateral triangle?

Comments In this set, B and E are outliers, not deeply related to the other problems, or even to each other, though they both have ten solutions. A structure involved in each of A, C, D, F, G, is the set of permutations of three objects (for example, the vertices of the triangle in G), and the solution to each of these five problems is in fact the full set of six permutations. In A, D, and F this outcome is demonstrably inherent in the problem, though this fact is least obvious for F. On the other hand, it can be argued that in C and G there is no a priori guarantee that all permutations will be achieved.

In piloting this example, with pre-service secondary teachers, all senior mathematics majors, solved all of the individual problems without great difficulty, but they did not perceive the mathematical distinction of C and G from A, D, and F. To awaken their awareness of this, I asked them to formulate a parallel problem set with the number three replaced by four. For example, problem A became:

A′	What are all four-digit numbers that you can make using each of the digits 1, *2, 3, 4, and using each digit only once?*

They easily constructed a four-based parallel problem set, A′, B′, …, G′, for example with G′ about symmetries of a square. I then asked them to repeat the connection network activity with this modified problem set. In this case, A′, D′, and F′ still yielded the full set of 4 ! (=24) permutations, whereas G′ led to only 4 × 2 = 8 of them. Moreover C′ leads to a surprisingly complex problem in combinatorial geometry that, suitably interpreted, leads to 4 × 3 = 12 solutions. They were impressed that the seemingly modest change from three to four made such a profound difference in the nature of the relations among the problems.

The 8-Choose-3 Discernment Problem Set

(A) A taxi wants to drive from one corner to another that is 5 blocks north, and 3 blocks east. How many possible efficient routes are there to do this?
(B) On the number line, starting at 0, you are to take 8 steps, each of which is either distance 1 to the right, or distance 1 to the left, and in such a way that you end up at −2. How many different such walks are there?
(C) The home team won a soccer game 5 to 3. How many possible sequences of scoring were there as the game progressed?
(D) You have coins worth 3¢ and 5¢. With 8 such coins, how many different values can you obtain?
(E) From a group of 8 students, you need to select a (5-person) basketball team. How many different ways are there to do this?
(F) You are to cut a 9-inch ribbon into six pieces, each of length a whole number of inches. How many ways are there to do this?
(G) In the expansion of $(1 + x)^8$, what is the coefficient of x^3?

Comments Here there is one outlier, (D), with 9 solutions, $(8 \times 3 + 2n)$¢ $(0 \leq n \leq 8)$. The problem solution space of all of the other problems can be represented by the structure consisting of the set of all "binary sequences of length 8, with 3 terms of one type." These are sequences $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ in which each x_j takes one of two possible values, say 0 or 1, and exactly three of them take the value 1. In (A), the values would be N (north) and E (east). In (B) they would be −1 and +1. In (C) they would be H (home) and V (visitors). In (E), $x_j = 1$ if student j is on the team, and 0 otherwise. In (F), $x_j = 0$ if you cut at inch j, $(1 \leq j \leq 8)$, and 1 otherwise. In (G), $x_j = 1$ or x, and the product of the x_js is x^3.

21.4.4 A More Subtle Example of Common Structure

Consider the following two problems.

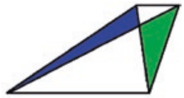
(A) The wine and tea problem: I have a barrel of red wine, and you have a cup of green tea. I put a teaspoon of my wine into your cup of tea. Then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel.

Question: Which is more now: the wine in the teacup or the tea in the wine barrel?

(B) Trapezoid diagonals problem: The two diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non-parallel sides) of the trapezoid? (Fig. 21.4)

Consider the following two problems.

(A) The wine and tea problem: I have a barrel of red wine, and you have a cup of green tea. I put a teaspoon of my wine into your cup of tea. Then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel.
Question: Which is more now: the wine in the teacup or the tea in the wine barrel?



(E) Trapezoid diagonals problem: The two diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non-parallel sides) of the trapezoid?

Fig. 21.4 Two isomorphic problems

At first sight, most observers see little, if any, mathematical connection between these two problems. I learned of (A) from Vladimir Arnold, who described it as a problem Russian parents give to very young, pre-mathematics-education children, who, according to Arnold, solve it more quickly and simply than mathematicians. I found problem (E) when trying to construct a geometric model of (A). Here is a solution to (E). Let **T** denote the trapezoid, and s and S its parallel side-lengths, say at a distance h apart. A diagonal d of **T** divides **T** into two triangles: t, with base s; and T, with base S, and both with height h. The other diagonal d′ similarly divides **T** into triangles t' and T', and clearly Area(T) = Area(T'). It follows that

$$\text{Area}\left(T / \left(T \cap T'\right)\right) = \text{Area}\left(T' / \left(T \cap T'\right)\right).$$

Notice that $T/(T \cap T')$ and $T'/(T \cap T')$ are the two (colored) triangles above containing the legs of **T**.

Now suppose that Area(**T**) represents the total volume of wine and tea in problem (A), and that t represents the tea in the teacup, and T represents the wine in the wine barrel. After the exchange, say t' represents the mixture in the teacup, and T' the mixture in the wine barrel. This makes sense since t and t', and T and T', have equal areas, respectively. Then $T\backslash(T \cap T')$ represents the wine in the teacup, and $T'\backslash(T \cap T')$ represents the tea in the wine barrel. So, the amounts are the same.

This analysis reveals a (measurement) structure common to problems (A) and (E). Either problem can easily be solved independently, without this observation. But this *structural connection* has a mathematical significance beyond the separate solutions. I thus say that these two problems have a *common structure*.

In fact, (A) and (E) are two of the following set of five common structure problems (Bass, 2017).

21.4.5 The Measure Exchange Common Structure Problem Set

(A) (Wine & Tea) I have a barrel of wine, and you have a cup of green tea. I put a teaspoon of my wine into your cup of tea. Then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel. *Question*: Is there now more wine in the teacup than there is tea in the wine barrel, or is it the other way around?

(B) (Heads Up) I place on the table a collection of pennies. I invite you to randomly select a set of these coins, as many as there were heads showing in the whole group. Next I ask you to turn over each coin in the set that you have chosen. *Then I tell you*: The number of heads now showing in your group is the same as the number of heads in the complementary group. *Question*: How do I know this?

(C) (Faces Up) I blindfold you and then place in front of you a standard deck of 52 playing cards in a single stack. I have placed exactly 13 of the cards face up, wherever I like in the deck. *Your challenge*, *while still blindfolded*, is to arrange the cards into two stacks so that each stack has the same number of face-up cards.

(D) (Triangle Medians) In a triangle, the medians from two vertices form two triangles that meet only at the intersection of the medians. How are the areas of these two triangles related? More precisely, let ABC be a triangle. Let A′ be the mid-point of AC, B′ the mid-point of BC, and D the intersection of AB′ and BA′. How are the areas of AA′D and BB′D related?

(E) (Trapezoid Diagonals) The diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non-parallel sides) of the trapezoid?

21.5 Mathematically Distinct Problems Reducible to a Common Mathematical Model

In 1968 Zalman Usisikin published, in *The Mathematics Teacher*, a brief paper, “Six nontrivial equivalent problems (Usiskin, 1968).” Based in part on conversations with Usiskin, I have expanded his list to the thirteen problems displayed below. They are organized into four groups, belonging to different mathematics domains: arithmetic; rates; geometry; and algebra. They do not all have a common structure in the sense of the previous section. Nonetheless, their diversity and dissimilarity notwithstanding, they all have a significant mathematical commonality. They can all, after some reduction, be modeled by the following Diophantine equation:

$$\text{Find all positive integer solutions } (m,n) \text{ of the equation,} \quad 1/m + 1/n = 1/2. \tag{21.1}$$

The reduction of the problems to (21.1) can be found below, where I also discuss how this problem set was enacted pedagogically.

21.5.1 The Expanded Usiskin Problem Set

These problems appear to be quite diverse. Usiskin's original list of six problems is Ar1, Ar2, Ar3, Ar4, R1, and G1 (Fig. 21.5). I thank Zal for the discussion of some of the other problems as well. Here is a calculus problem that could be added:

$$\text{For which positive integers } (m,n) \text{ is } 2\int_0^1 \left(x^{m-1} + x^{n-1}\right)dx = 1?$$

21.5.2 Presentation to the Students

After distributing the problem set, I asked each student to choose one of the four topic themes, and so formed the class into four student groups (arithmetic group, rates group, geometry group, and algebra group), each group to work collaboratively on its chosen thematic problem set, but the groups worked independently. Student choices were based mainly on things like the parts of mathematics they liked best or felt most confident with, or on which problems seemed, at first appearance, easiest for them to solve. Each group was assigned to solve its problem set and to present its solutions to the class the following week. They were free to consult outside resources, including myself, but to prepare a presentation that would instruct the rest of the class about what they found, and what they found to be difficult.

I deliberately said little about the overall mathematical point of this assignment, other than to solve and see relations among, each group's interesting set of problems. In particular, there was no suggestion of why these specific problem sets were collected together, in particular, that they might all be mathematically related in some way.[3]

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[3] This design of the instruction was based on a model that Davida Fischman used for a professional development session when I gave her this problem set.

ARITHMETIC

Ar1. Find all ways to express ½ as the sum of two unit fractions (i.e. fractions of the form $1/n$, n a positive integer).
Ar2. Find all rectangles with integer side lengths whose area and perimeter are numerically equal.
Ar3. The product of two integers is positive and twice their sum. What could these integers be?
Ar4. For which integers $n > 1$ does $n - 2$ divide $2n$?

RATES

R1. Which pairs of positive integers have harmonic mean equal to 4?(*)

(*) The harmonic mean h of n numbers $a_1, a_2, \ldots, a_n$:
1/h is the average of $1/a_1, 1/a_2, \ldots, 1/a_n$

R2. Nan can paint a house in n days, and her Mom can paint it in m days (n and m positive integers). Working together they can paint the house in 2 days. What are the possible values of (n, m)?
R3. A turtle travels up a hill at n miles per hour, and returns down the hill at m miles per hour (n ≤ m integers). Its average speed for the round trip is 4 miles per hour. What are the possible values of (n, m)?

GEOMETRY

G1. Given a point P in the plane, find all integers n such that a small circular disk centered at P can by covered by non-overlapping congruent tiles shaped like regular n-gons that have P as a common vertex.
G2. Two vertical poles, N and M, have heights n meters and m meters, respectively, with n and m integers. A wire is stretched from the top of pole N to the base of pole M, and another wire is stretched from the top of pole M to the base of pole N. These wires cross at a point 2 meters above the ground. What are the possible values of (n, m)?
G3. The base b and corresponding height h of a triangle are integers. A 2 x 2 square is inscribed in the triangle with one side on the given base, and other vertices on the other two sides. What are the possible values of the pair (b, h)?

ALGEBRA

Al1. For which positive numbers s does $p(x) = x^2 - sx + 2s$ have integer roots?
Al2. Let u be a positive real number. Find all solutions (n, m, v) with n and m positive integers, and v > 0, of the equations:

$$(uv)^2 = u^n = v^m$$

Al3. For which positive integers $(\boldsymbol{m}, \boldsymbol{n})$ **is** $\mathbf{2} \int_0^1 (x^{m-1} + x^{n-1}) dx = \mathbf{1}$?

Fig. 21.5 The expanded Usiskin problem set

21.5.3 *The Student Presentations: Slowly Raising the Curtain*

I had the groups make their presentations in the order of the list, the arithmetic group first. Problems **Ar1**, **Ar2**, **Ar3**, **Ar4**, in order, lead directly to the following Diophantine equations ("Diophantine" because one seeks (positive) *integer* solutions).

$$1/n + 1/m = 1/2 \tag{21.1}$$

$$2(n+m) = nm \tag{21.2}$$

$$nm = 2(n+m) > 0 \tag{21.3}$$

$$\text{For which } n > 1 \text{ is } 2n = (n-2)m \text{ for some integer } m? \tag{21.4}$$

Moreover, it is not difficult to see how Eqs. (21.1)–(21.4), are algebraically equivalent. Hence, solving any one of them solves the others. My students generally preferred to use (21.3) to express m in terms of n:

$$m = 2n/n - 2 \tag{21.5}$$

They then did numerical experiments to find those n for which $2n/(n-2)$ is an integer. (Some students even graphed m in (21.5) as a function of $n > 0$, and highlighted the integer points on the graph.) The solutions they found were:

$$\boldsymbol{(n,m) = (4,4), (3,6), \text{or } (6,3)} \tag{21.6}$$

None of the students tried to work directly with (21.1), which is my preferred approach. Using the symmetric roles of m and n, we can assume that $n \leq m$. Then $n \geq 3$; otherwise $1/n \geq 1/2$. Also, $n \leq 4$; otherwise $1/n + 1/m < 1/2$. Thus, either $n = 3$ (and so $m = 6$) or $n = 4$ (and $m = 4$).

The rate group gave an excellent survey of problems in which the harmonic mean (the concept was new to them) arises. Problem R1 corresponds to the equation.

$$1/4 = (1/2)(1/n + 1/m) \text{ which is } (1) \text{ multiplied by } \tfrac{1}{2}. \tag{21.7}$$

For **Problem R3**: If one travels distance d at speed v in time t, then: $d = vt$ and $t = d/v$. Now suppose that one travels a distance d at a speed v_1 in time t_1, and then returns at speed v_2 in time t_2. What is the average speed for the whole trip? It is

$$\begin{aligned} V_{\text{ave}} &= (\text{total distance})/(\text{total time}) \\ &= 2d/(t_1 + t_2) = 2d/(d/v_1 + d/v_2) = 2/(1/v_1 + 1/v_2) \end{aligned}$$

Thus, $1/V_{\text{ave}} = \left(\tfrac{1}{2}\right)/(1/v_1 + 1/v_2)$

In other words, V_{ave} is the harmonic mean of v_1 and v_2. In **Problem R2,** d would be the work of painting the house, and n and m describe the rates at which Nan and her Mom do that job. The rate of doing it together (analogous to average speed) is the harmonic mean of the two rates. Of course, the rate group sees that its work to solve (21.7), and hence each of the rate problems, is the same as the work already shown by the arithmetic group.

The geometry problems were less obviously related, but they too led to the same Diophantine equations.

In **Problem G1**, let $\alpha(n)$ denote the (equal) interior angle(s) of a regular n-gon: then it is known that $\alpha(n) = [(n - 2)/n] \times 180^\circ$. For some number, say m, of these regular n-gons to fit together to cover the area around a point P, we would need:

$$m \times \left[\frac{n-2}{n}\right] \times 180^\circ = 360^\circ$$

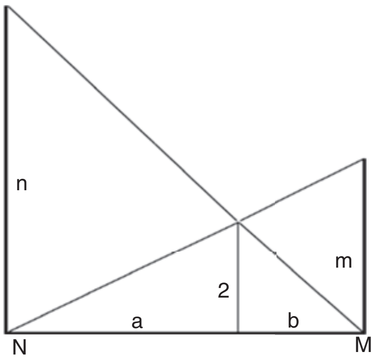
$$2n = m(n-2), \tag{21.4}$$

which is the same equation treated by the arithmetic group.

For **Problem G2** (also framed as the "crossing ladders problem"), consider the diagram (Fig. 21.6):

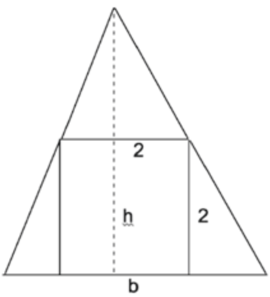
For G3, consider the diagram (Fig. 21.7):

The algebra group was the most challenged, it seems because the relevant algebraic methods were less familiar. In Al 1, they at first tried using the quadratic formula, which did not conveniently make available the information that the roots are integers.



Using similar triangles, we have:
$(a + b)/n = b/2$
and $(a = b)/m = a/2$
Adding these equations, and then dividing by $(a + b)$,
gives $1/n + 1/m = 1$

Fig. 21.6 Problem G2 Diagram



The big triangle and the one above the square are similar (corresponding sides are parallel), and so $h/b = (h-2)/2$,
whence, multiplying this by $2b$, the equation

$$2h = b(h-2)$$

already treated by the arithmetic group.

Fig. 21.7 Problem G3 Diagram

If instead, we formally factor p:

$$p(x) = x^2 - sx + 2s = (x - n)(x - m), n, m - \text{integers.}$$

We find that $n + m = s$ and $nm = 2s$ whence n and m are positive, since s is, and so we have the Eq. (21.2) $nm = 2n + m$ already treated by the arithmetic group.

Students needed the most help with **Al2**, where the mathematics is mainly happening in the exponents:

$$(uv)^2 = u^n = v^m$$

We first get, from $(uv)^2 = v^m$, $u^2 = v^{m-2}$ so $v = u^{2(m-2)}$

Then, substituting for v in $(uv)^2 = u^n$ gives: $(uu^{2/(m-2)})^2 = u^n$

Equating exponents then gives:

$n = 2\left[1 + \frac{2}{m-2}\right] = 2m/(m-2)$ whence, again, Eq. (21.4) $2m = n(m - 2)$

In **Al3**, the conditions on (r, b) are that $r + b = rb/2$.

Dividing this by rb gives,

$1/b + 1/r = 1/2$ already treated by the arithmetic group.

Some Student Reflections and Further Connections

Here is a sample of student reflections on this activity.

- The students were all surprised, and intrigued, to see that their diverse problems all led to essentially the same (Diophantine) equation: (21.1) $1/n + 1/m = 1/2$ and its variants.
- Many of them wondered whether there was some way that they could have anticipated this commonality, but they saw no simple way they could have done this.
- Some of them researched the web to see if they could find some standard discussion or identification of this basic equation. The closest thing to this was the connection with the harmonic mean, found by the rates group. Making worthwhile problems that are "internet proof" is something I regularly seek since I like to give take-home exams relieved of restrictive time pressure.
- The above outcome led many of my students to ask me how I found all these different problems with the "same" solutions, thinking that I was somehow being "sneaky."
- So, at this stage, the phenomenon seemed more like an intriguing coincidence, or uncanny craftiness – that so many different-looking problems could be modeled by the same equation (and its variants).
- Apart from that, it did not seem to provide any more general kind of mathematical insight beyond the immediate observation that a single equation could mathematically model an extraordinary variety of mathematical problems.
- Nonetheless, everyone found the activity to be interesting and worthwhile and looked forward to more of this kind of activity.

21.5.4 Relation to the Classification of Platonic Solids

Partly in response to this student reaction, and to further emphasize mathematical connections, I went on to engage the class in a quick discussion of Platonic Solids, and to illustrate how their (combinatorial) classification could be reduced to solving the following Diophantine equation, that is a slight perturbation of Eq. (21.1) above.

$$(1+)\ 1/n+1/m=1/2+1/E$$

The reduction to (1^{+}) uses Euler's formula,

$$V–E+F=2$$

where V, E, F are the numbers of vertices, edges, and faces of a convex polyhedron, and regularity is expressed by the fact that each face is an n-gon, and m faces meet at each vertex.

The point of this activity was to further emphasize that such Diophantine equations arise usefully in still more diverse contexts.

21.6 Problems Unexpectedly with the Same Answer, for Deep Structural Reasons

A mathematical example with a spectacular range of significantly different incarnations is the sequence of **Catalan numbers**, Cat(n). This sequence is the answer to a remarkable variety of questions. We provide below (Sects. 21.6.1, 21.6.2, 21.6.3, and 21.6.4) the statements of some of these questions, but without proof that they are answered by the Catalan numbers. These results provide a meaningful and accessible illustration of its surprising mathematical connections among these questions. Though this is a somewhat specialized combinatorial topic, it is technically within reach, with some instructional investment, of secondary mathematics. A more detailed discussion can be found in (Bass, 2022). A comprehensive treatment can be found in Richard Stanley's book (Stanley, 2015).

Here are some interesting, mathematically significant, and strikingly different incarnations of the Catalan numbers.

21.6.1 Walks on the Positive Half-Line

A sequence $x_0, x_1, . . . , x_n$ of real numbers such that $|x_j - x_{j-1}| = 1$ for $1 \le j \le n$, is called an *n-step walk on the number line, from* x_0 *to* x_n. It is a walk on the positive half-line if all $x_j \ge 0$. Let $W(n)$= the number of n-step walks on the positive half-line from 0 to 0.

21.6.2 Binary Rooted Trees

A binary rooted tree is a tree T with a specified vertex (the "root") such that each vertex is either terminal (a "leaf") or has two edges leading away from the root. We call T planar if T is embedded in the plane with the root on top, and edge paths from the root directed downward. Then the vertices at a given distance from the root have a well-defined left-to-right order. Thus, trees in Fig. 21.8 are isomorphic as planar rooted trees, but the isomorphism is not order-preserving.

Let BRT(n)=the number of order-preserving isomorphism classes of finite planar binary rooted trees with $n + 1$ leaves.

21.6.3 Associations

Let A be a set with a possibly non-associative binary operation, $a{*}b$. Given a sequence $a_0, a_1, . \; . . \; , a_n$ in A, let $\mathbf{As}(a_0, a_1, . \; . . \; , a_n)$ = the set of ways (using parentheses and $*$) to form an order-preserving product $a_1 * a_2 * \; * \; * a_n$. For example,

$$\text{As (a,b,c,d)} = \{((a * b) *c) *d, (a *(b * c)) *d, (a * b) *(c * d), a *(b *(c *d)), a * ((b *c) *d)\}$$

Let $\mathrm{As}(n) = |\,\mathbf{As}(a_1, \ldots, a_{n+1})|$

21.6.4 Triangulations

Let P be a convex n-gon, $n \geq 3$. By a *triangulation* T of P we mean a decomposition of P into $n - 2$ triangles, using $n - 3$ non-intersecting diagonals of P. Let Tr(n)= the number of triangulations of a convex ($n + 2$)-gon.

We agree that Tr(0) = 1.

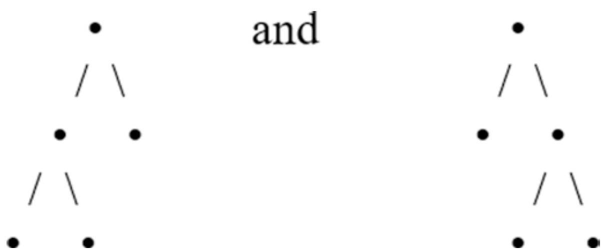


Fig. 21.8 Binary rooted trees

21.6.5 Multi-Theorem

(a) $W(n) = \mathrm{BTR}(n) = \mathrm{As}(n) = \mathrm{Tr}(n)$.

Call this common number the **Catalan number**, denoted Cat(n).

(b) $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$ (Formulas)

(c) Cat(0) = 1, and, for $n > 0$,

$\mathrm{Cat}(n+1) = \sum_{0 \le j \le n} \mathrm{Cat}(j)\mathrm{Cat}(n-j)$ (Recursion)

Of course, this (multi-)theorem could be decomposed as a collection of problems, a mix of our types I and II, but they would likely be too ambitious, in both technical terms, and the need for inventiveness, for general secondary and tertiary students.

21.7 Concluding Discussion

The problem collections presented here involve a variety of perhaps unfamiliar problem types. I have provided solutions in most cases, so as to give the reader a measure of both the complexity and the accessibility of the problems. Some of these problem collections may be judged (mistakenly, I believe, in some cases) to be too advanced, or too difficult, for school students, perhaps because they have no familiar expression in current school curricula.

Consider, for example, Part 1 (on discrete real additive groups) and Part 2 (variations on the Euclidean Algorithm). These are related in that they both rest on the intuitively clear linear measure treatment of division with remainder. But the two parts have contrasting pedagogical purposes. Part 1 develops the general theory of discrete additive groups, as a coherent body of fundamental theorems. Part 2, on the other hand, illustrates the diverse (perhaps surprising) and multi-domain interpretations and applications of the Euclidean Algorithm.

As stated in the discussion (1.3) of Part 1:

The group theory rests on the core properties of addition and multiplication, and it powerfully exposes the duality, not just distinction, between discreteness and continuity, and the natural appearance of commensurability at their boundary. Once the few basic concepts are understood, the proofs of several striking fundamental results are surprisingly short, simple, and intuitive. This treatment can afford a depth of understanding at least as robust as assimilation of the much more complex ideas of AP Calculus, for example. Indeed, the ideas featured here can provide a helpful foundation for the understanding of AP calculus.

So, I feel that this challenges the judgment that the material of Part 1 is “too advanced” or “not accessible.” If, on the other hand, this material is judged inappropriate because it does not find expression in the current school curriculum, I argue, based on what is presented here, that it does, in fact, merit a place in the secondary curriculum.

Also as stated in the introduction, a central theme in these collections is to disrupt the fragmentation about the nature of mathematics, induced in the minds of many students, by the siloed school curricular geography. This agenda has a built-in dilemma, a kind of "catch 22." In order to discuss connections across different mathematical domains, one must adopt a standpoint that is, in some sense "above or beyond" these domains, since they are the constituents of the discourse. This, in turn, compels the examples to be situated similarly beyond the standard curriculum, and so, in turn, they may have no natural home within that curriculum. The reader may well have detected this tension. I expose it without apology. My view is that the curriculum should be adapted to create a home for such rich mathematical connections, and the unmet learning opportunities that they afford.

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Chapter 22
Meeting the Challenge of Heterogeneity Through the Self-Differentiation Potential of Mathematical Modeling Problems

Rita Borromeo Ferri, Gabriele Kaiser, and Melanie Paquet

22.1 Introduction

Mathematical modeling has been established as a mandatory component of mathematics curricula from primary school to high school in many countries around the world, such as Germany and the United States, in the past few decades. More recently, other countries, such as China and South Korea, have started to include modeling in their school curricula (Borromeo Ferri, 2021; Schukajlow et al., 2018). Mathematical modeling is, briefly, the solution of real-world problems with the help of mathematical models (Borromeo Ferri, 2018; Niss & Blum, 2020). The modeling process (i.e., the solution steps for the real problem) can be visualized using modeling cycles, and different types of modeling cycles can be distinguished and employed depending on the learning objectives (for an overview, see Kaiser, 2017). The teaching and learning of mathematical modeling are very challenging for students and teachers; however, empirical studies have shown the value of teaching modeling in classrooms for all students from a cognitive and an affective perspective (Blum, 2015; Niss & Blum, 2020). Nevertheless, previous studies have demonstrated certain obstacles teachers face, such as lack of time, limited access to teaching materials, the difficulty of evaluating modeling problems, and (unfortunately) a lack of experience due to inadequate training and further education, make the everyday implementation of modeling in classrooms difficult (Schmidt, 2010). Good

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examples of evaluated learning environments for modeling have been developed in the last three decades, for both schools and universities, based on empirical evidence, to which teachers from all institutions can refer (Borromeo Ferri, 2018; Doerr & Lesh, 2003; Schukajlow & Blum, 2018). In addition, a wide range of materials are now available—above all, suitable and effective collections of modeling problems for all school levels, such as COMAP's Mathematical Modeling Handbook II (2013)—which teachers can use in their lessons. For teacher education on mathematical modeling, Borromeo Ferri and Blum (2010) developed a model for mathematical modeling competencies of teachers. This model conceptualizes and operationalizes teachers' modeling competencies according to four dimensions: (1) theoretical, (2) task, (3) instruction, and (4) diagnostic dimensions. It is evident that teachers need both a theoretical background and practical knowledge about collections of modeling problems and their many facets, especially for the task dimension of the model.

According to Maaß (2007), a good modeling problem is open, authentic, reality-based, and solvable through multiple approaches during the phases of the modeling cycle. An effective modeling problem also has a self-differentiating problem format (Maaß, 2007; Borromeo Ferri, 2018), because such problems allow different levels of processing through their problem setting, meaning that teachers do not need to create, for example, three different types of problems, but can use one modeling problem to cater to the different needs of learners. Knowledge of how modeling problems can be used in a self-differentiating way and what advantages this offers compared to elaborate preparation of different working materials for learners can support quality mathematics teaching in heterogeneous classes.

The daily handling of heterogeneity, studied by the Programme for International Student Assessment (PISA) organized by the Organisation for Economic Co-operation and Development (OECD) in Germany (OECD, 2010), which concerns the question of finding the best possible differentiation, is not a new problem (Krauthausen, 2018). So far, however, it has been challenging due to the lack of empirical studies on what kinds of tasks are suitable and whether and how modeling problems can actually be used in a self-differentiating way. Furthermore, the criteria by which modeling tasks can be defined as self-differentiating and thus can be successfully used in heterogeneous classes remain unclear.

Therefore, the aim of this study described in the paper was to develop a theoretical model and conduct a qualitative study to meet this challenge and identify criteria according to which the self-differentiation potential of modeling problems can be evaluated in terms of content and methodology. Overall, this paper aims to contribute to the collection of modeling problems, which permit self-differentiation. Although the process of the creation of this collection of papers as well as the empirical validation of the examples are described based on one problem the principles are general.

The intention was thus to gain more knowledge about facing the challenge of heterogeneity through the self-differentiation potential of mathematical modeling problems. First, this chapter presents the theoretical background to mathematical modeling, heterogeneity, and the concept of self-differentiation. It then discusses the research questions and the potential for self-differentiation of certain modeling examples, focusing on the differentiation of performance. Using a well-known example from the literature, a method is developed that allows the self-differentiating potential of modeling problems to be analyzed. Based on this theoretical framework, the actual potential of different modeling tasks is examined through a qualitative study of tenth-grade learners. The paper closes with a summary and discussion.

22.2 Theoretical Background

22.2.1 Mathematical Modeling—Modeling Problems and Modeling Processes

According to Pollak (2007), mathematical modeling can be described as translation processes that move back and forth between reality and mathematics, thus, real problems are fundamental to mathematical modeling and are concretized in modeling problems. An example of a modeling problem that the authors have often used in schools with learners during longer modeling activities from Grade 9 onward is: "Which is best: using traffic lights or a roundabout to control traffic flow in a city?" (Kaiser & Stender, 2013). Maaß (2008) characterized modeling problems using the following criteria: open, complex, realistic, authentic, and possible to solve via a modeling process. Modeling problems, therefore, differ from common word problems or pseudo-realistic problems, where the mathematics to be used is already obvious and well-known to students.

The modeling process is best illustrated using a modeling cycle and, as there are different types of cycles, Kaiser and Stender's (2013) cycle is referred to herein, as shown in Fig. 22.1 (for an overview see, among others, Kaiser, 2017). This chapter

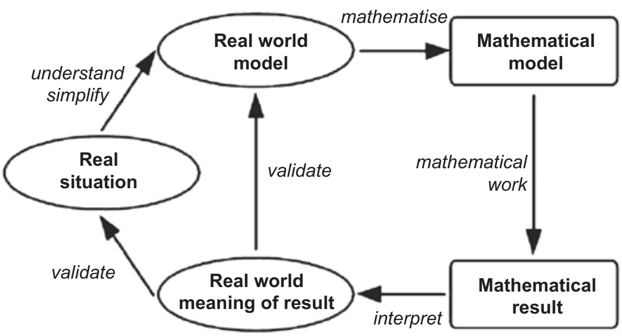


Fig. 22.1 Mathematical modeling cycle (Kaiser & Stender, 2013, p. 279)

briefly describes the phases of the modeling cycle, since these are fundamental to the analysis of the self-differentiating potential of modeling problems.

The starting point for modeling is a real situation, which always constitutes the real problem described in the mathematical problem. The real situation is then simplified and idealized with the help of retrieved and/or personal extra-mathematical knowledge, and a real model is developed for which learners must make assumptions. Subsequently, the mathematization phase follows (i.e. the translation of the real situation into the language of mathematics to create a mathematical model of the situation). Through the processing of the mathematical model within mathematics, mathematical results are formulated, which are then interpreted back into reality as real-world solutions and finally validated. If the problem's solution proves not to be appropriate, the cycle must be repeated. In general, several iterations are needed to obtain different versions of the model, for which mathematical modeling competence is necessary. According to Blomhøj and Jensen Højgaard (2007), modeling competence is the ability to construct and use mathematical models by performing appropriate steps (i.e. those mentioned above) and to analyze or compare certain models. The performance of the individual steps or phases requires modeling sub-competencies as well as global and social competencies (Kaiser, 2007), which must be developed through training (Blum, 2015).

Since this empirical study focused on the basic differentiation possibilities of modeling, the following sub-competencies were of specific interest: (1) constructing a real model, (2) constructing a mathematical model, (3) developing mathematical results, (4) interpreting the mathematical results, and (5) validating the real results (Kaiser & Brand, 2015). These sub-competencies provided the overall basis for developing the instrument for analyzing the self-differentiation potential of a modeling problem in the coding manual, which was then concretized in the developed instrument to analyze problems' potential for self-differentiation (see Fig. 22.2 in the next section).

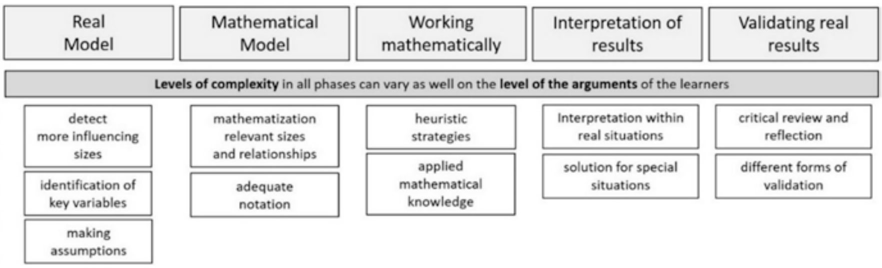


Fig. 22.2 Instrument for analyzing the self-differentiation potential of a modeling problem

22.2.2 Heterogeneity, Differentiation, and Self-Differentiation

The discussion about dealing with heterogeneity is not new. As early as 1976, the renowned German educational scientist Klafki argued that, through various differentiation possibilities, each child in a class can be individually supported (Klafki & Stöcker, 1976). In the context of school learning, heterogeneity first denotes, pragmatically, the inequality of students in a learning group with regard to the basic prerequisites and conditions of their learning, rather than being defined in terms of individual characteristics (Wischer, 2009). According to the research (Bräu & Schwärdt, 2005), heterogeneity comprises many dimensions, but this can result in unmanageable lists of characteristics that need to be considered by teachers (e.g., performance, age, gender, social status, family background, economic and cultural background, etc.). Accordingly, teachers must consider how to carry out methodological-didactical differentiation, which can be done in various ways. For example, taking heterogeneity of performance as an example; traditional methods such as social differentiation (i.e. single work, partner work, group work, etc.), or differentiation with media such as textbooks, worksheets, or digital media, are often used to enable learners to learn at their individual cognitive levels. Also, quantitative differentiation can be used (e.g., giving the same amount of time for different workloads/amounts of content). Finally, qualitative differentiation involves objectives and problems with different levels of difficulty (Krauthausen, 2018; Winkeler, 1978). These are important everyday methods of differentiation used by teachers, but they require appropriate preparation if the aim is to encourage and challenge learners.

So how can differentiation be achieved at the same time as minimizing the heavy workload of teachers and still providing quality mathematics instruction for all students? Initial approaches in the field of mathematics education have, in this respect, emerged strongly from German-language research for primary and secondary schools, especially that conducted by Krauthausen and Scherer (2010) and Büchter and Leuders (2014). The concept of natural differentiation (Wittmann, 2001) is particularly noteworthy in this context because, according to the mentioned researchers, this concept fills the gaps in traditional (internal) differentiation regarding mathematics teaching, and does so by

- Orientating actions of differentiation explicitly towards the *specifics of mathematics*
- Doing justice to the different areas of *responsibility* for teachers and for students
- Ensuring degrees of *freedom* for individual learning processes
- Laying great emphasis on guaranteeing common *social learning* (Krauthausen, 2018, p. 330).

According to Krauthausen (2018), the above aspects do not constitute a definition but rather a description of natural differentiation, and they are fundamental for the characterization of the concept of self-differentiation. Problem setting for a self-differentiated problem facilitates different levels of processing and can be understood as a format that is already itself differentiating. Modeling problems in particular, due to their openness and potential for multiple solutions on different levels (Schukajlow & Krug, 2014), offer rich learning opportunities for all students in a classroom (Vorhölter & Kaiser, 2019); hence, modeling problems are likely to contribute to the goal of inclusiveness better than distributing different worksheets, with easier problems for weaker students and difficult problems for stronger students, since a single modeling problem can be used for all students. A literature review found only a few studies that explicitly dealt with the potential for self-differentiation of modeling problems on a theoretical level, indicating that theoretical analysis of modeling problems on an empirical level is lacking (Borromeo Ferri, 2018; Ostkirchen & Wess, 2019). Accordingly, the following research questions were addressed for this chapter:

- Which key aspects should a theoretical analysis method include to characterize the self-differentiation potential of modeling problems, and what would such a method look like?
- What empirical evidence exists regarding the self-differentiation potential of modeling problems with regard to students' performance?

22.3 Theoretical Analysis of the Self-Differentiation Potential of a Modeling Problem

The previously presented detailed steps of the modeling cycle are analyzed below regarding their self-differentiation potential. The aim was to develop a general analysis method, and thus a theoretical model, and apply it to modeling problems. Some aspects of differentiation have already been mentioned, but further theoretical aspects were actually used in the model, which will be discussed in the following section.

Modeling problems are based on certain mathematical topics, providing categories for differentiation (Saalfrank, 2008) that were included in the developed model, because content can be differentiated either according to different topics or according to different foci within the same topic (Klafki & Stöcker, 1976). Different foci within a topic can also result from the different levels of difficulty at which a problem can be worked on in mathematics lessons. According to Prediger (2008), the difficulty-generating characteristics of mathematics problems, such as the type of cognitive activities, the degree of formalization of the problem, the technical complexity of the execution of the solution plan, or the degree of complexity, are suitable for generating different levels of difficulty in mathematics problems. In particular, the degree of complexity was included in the analysis method because—according to Prediger (2008)—the clarity of the situation and thus the obviousness of the solution and the number of necessary solution steps can vary when working on a problem. This is particularly evident in the case of modeling problems, as already explained. With regard to modeling, the focus is on the specific modeling activities carried out during the modeling cycle and on the levels of argumentation and complexity. These aspects were essential points for the study and observable in learners' interactions. In view of the many ways of differentiating between students, the focus was on performance differentiation, which is still of great relevance in school practice today. Figure 22.2 shows an overview of the general analysis categories. The individual categories of the model were used for the theoretical analysis of the modeling task on the one hand, and for the analysis of the students' modeling activities in the qualitative study on the other hand.

Instrument for Analyzing the Self-Differentiation Potential of a Modeling Problem: A Theoretical Analysis Method

The following section explains the model in Fig. 22.2 in more detail and concretizes it based on the modeling "hot air balloon" problem in Fig. 22.3, which was used in the present qualitative study. Based on the above model described and this study, the overall goal of the study is also to describe the development of a collection of modeling problems which permit self-differentiation.

Actually, the self-differentiation potential of the problem can be identified by analyzing this potential at each step of the work with the problem. Therefore below we will talk about the self-differentiation potential when building the real model self-differentiation potential when setting up the mathematical model, self-differentiation potential in mathematical work, self-differentiation potential in interpretation and self-differentiation potential in the validation of solutions. In detail:

Self-differentiation potential when building the real model: The identification of the influencing variables relates to the category of content. Two different types of influencing variables can be determined for this problem: on the one hand, influencing variables concerning the height of the man; on the other hand, variables concerning the balloon itself. Concerning the man, especially for the determination of the scale as an auxiliary variable, the real size of the man and the size of the man in the picture must be determined. Theoretically, many sizes are conceivable for the balloon, which results from the different sizes and shapes into which the balloon can be divided for simplification. Figure 22.4 illustrates different ways of building a real model. At a low level of complexity, the balloon can be roughly approximated as a

Stunt on the hot air balloon

A lot of hot air brings you up for sure. Nobody knows that better than Ian Ashpole. The 43-year-old was in England on the top of a hot air balloon. The stunt in 1500 meters height was still the safest part of the action. More critical was the start: secured only by a rope, Ashpole had to hold onto the filling balloon. During the landing, the hot air flowed out from a valve next to his legs. But except for minor burns, the balloonist received no injuries.

How many liters of air the balloon holds?

Fig. 22.3 Hot air balloon (Herget et al., 2009; translation by the first author)

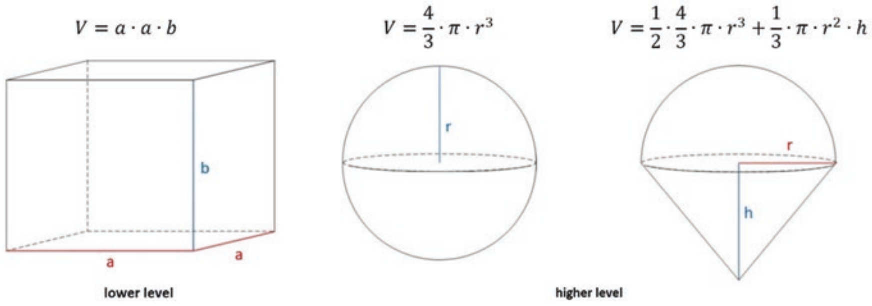


Fig. 22.4 Different ways of building a real model

cuboid block with a maximum width and maximum height (marked in color). At a medium level of complexity, modeling can be based on a ball (using the maximum height), with the excess and neglected volume approximately equalizing each other. This constitutes good argumentation. However, efficient learners at a high level can divide the hot air balloon into two simplified parts according to the principle of exhaustion, separated, for example, at the maximum width and individually understood as egg-shaped and funnel-shaped. Of course, other models are possible too.

With regard to the assumptions concerning the form, however, it is probable that German tenth-grade learners (aged 16 years) will already be familiar with the terms sphere, cone, and similar so that they may use these terms and thus proceed directly to the mathematical model. Consequently, it is possible to determine meaningful different numbers for the influencing variables. This modeling problem thus (theoretically) has self-differentiation potential according to the complexity of the real model. The question regarding which of the influencing variables should be considered more meaningful than others must then be discussed when the key variables have been identified: Here, all learners must realize that many fine subdivisions of the dimensions, and thus changes in the shape of the balloon, are less important than a few meaningful subdivisions, such as the maximum diameter. Stronger learners are likely to recognize that more than three different widths would result in an inadequate accuracy of the total volume, and they could use more appropriate measurements, such as the average width. These possibilities lead to a theoretical self-differentiation potential of content according to complexity.

The degree to which information can be classified as irrelevant is subjective, but must always be appropriate to the initial situation. Furthermore, irrelevant information should be recognized and excluded from the text describing the problem. The text for this problem contains a lot of information that is irrelevant to the solution of the problem. Weaker students, for example, have the option of classifying all other variables relating to the shape and dimensions of the balloon as irrelevant, apart from the maximum diameter and a cuboid outline. Smaller changes in shape, such as the flattening of the balloon on the upper side, are not considered further. Stronger learners, however, may recognize less influential factors as irrelevant and calculate the total volume using the maximum diameter, the approximate maximum height, an egg-shaped upper part, and a conical lower part (Fig. 22.4—real model of a hot air balloon at a higher level). Only simple relationships can be established between the single variables; that is, the influencing sizes (e.g., the size of the diameter influences the volume, so the larger the real size of the man, the larger the volume of the balloon, etc.).

Self-differentiation potential when setting up the mathematical model: Before the actual calculation of the balloon volume, the scale of the drawing must first be determined, based on the estimated real and the measured illustrated size of the man (1.80 m: 0.007 m on the learners' worksheet). Using this calculation, the dimensions of the balloon are then converted into reality, but in this step, no alternative processes are possible, so there is no self-differentiation potential. As already mentioned regarding the real model, it is probable that learners, when simplifying the form, will mathematize immediately and determine geometric shapes as substitutes for defined parts of the balloon; thus, geometric shapes of varying complexity can be used, as shown in Fig. 22.5.

Weaker students (lower level) could mathematize their simpler real model of the comprehensive cuboid block as a cuboid, with the length and width corresponding to the maximum diameter (marked in red) and the height to the maximum height of the balloon (marked in blue). Stronger students could mathematically advance their real model with mathematically more ambitious concepts, such as using a sphere or separating the balloon into a hemisphere and a circular cone (higher level). A further difficulty could arise from the learners' levels of mathematical knowledge. If learners only have knowledge of volume calculations for cuboids, but not for spheres and cones, this could lead to more complex solution processes; however, this also offers a methodical self-differentiation possibility, since the unknown content must be researched beforehand (e.g. on the Internet or in formulae collection). Consequently, the problem contains content-related self-differentiation potential relating to complexity, since the balloon can be approximated with varying degrees of accuracy and complexity, and the formulae for the geometric shapes are differently complicated. The calculation of the volume of a cone is technically (i.e. assuming adequate mathematical competence) more difficult than that of a cuboid, due to the different mathematical possibilities and the degree of familiarity with the means used. Learners

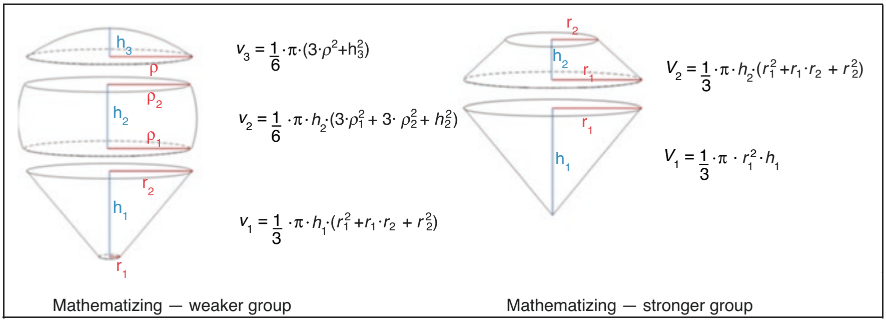


Fig. 22.5 Levels of building a mathematical model

can argue why the selected geometric form was chosen, deriving the argumentation from the arguments for the real model for simplification.

Self-differentiation potential in mathematical work: Here, too, the mathematical procedure can be analogized with the solution process for a similar problem. The division into sub-problems may have already taken place for the real model, but can also be done later when solving the mathematical model, by first calculating the volumes of the individual parts separately and then adding them to obtain the total volume. Consequently, the problem has methodical self-differentiation potential through the application of heuristic strategies (e.g., Pólya, 2010) to the mathematical model, and content-related self-differentiation potential according to complexity and intricacy, depending on the respective mathematization process. Based on the degree of complexity and intricacy used in the mathematization process (e.g., using a cube rather than a hemisphere and cone), the mathematical solution can vary in complexity as determined by the mathematization. Applied mathematical knowledge, such as estimating, calculating from and with scales, converting centimeters to millimeters and cubic centimeters to cubic millimeters, and solving equations, will be involved in any adequate modeling.

Potential for self-differentiation in interpretation: The mathematical solutions can be interpreted with varying degrees of accuracy in reality, either by translating only the final result (e.g., “Approx. 5–12 million liters of air will fit into the hot air balloon”) or by also considering the intermediate steps for justification (e.g., “Since the balloon has a maximum diameter of ... meters and a height of ... meters, the total volume is ... In reality, fewer liters fit into the balloon”). One possible solution based on the real and mathematical model at a higher level is about 6 million liters of air (see Herget et al., 2009).

As an argument, this more precise justification of the accuracy of the real result is conceivable: ‘In reality, fewer liters of air will fit into the balloon, since the assumed simplified shape of the balloon as a hemisphere and cone is slightly larger than that of the actual balloon.’ However, the exact number of liters cannot be determined due to the fuzzy image and the estimated reference size for the scale. This problem, therefore, through its various complex interpretation and argumentation possibilities, affords content-related self-differentiation potential according to complexity. The problem does allow for the generalization of the solution.

Self-differentiation potential in the validation of solutions: The real result can be critically reviewed and applied with varying degrees of accuracy. On the one hand, it is possible to check intuitively based on one’s inner logic; for example, a huge or minuscule number of liters may seem unfeasible to the learners. On the other hand, the real result can be checked at a higher level using other solution methods (e.g., by mathematization of a sphere, cuboid, etc.). Stronger learners can also validate their solutions on an external scale by using comparison values; for example, by researching on the Internet how much air fits into a standard hot air balloon, the size of which may vary between 400 and 12,000 m^3 (or 4–12 million liters). This search for comparative values again offers methodical self-differentiation potential; hence, the problem has content-related self-differentiation potential according to complexity, and methodical self-differentiation potential according to the search for missing data. Arguably, one could use the following: The comparative values can be included to varying degrees to justify the validation. Weaker learners may draw the simpler conclusion that their result is correct because their result lies within the researched solution range, while stronger learners may argue that their solution is probably correct because the balloon shown is a rather small model according to the measured sizes, and it, therefore, makes sense that their result lies in the lower range of the solution range. Thus, the argumentation itself offers content-related self-differentiation potential according to the complexity involved. The mathematical result can also be validated, as with any modeling problem, by reviewing calculation errors; recalculating values (for lower levels); calculating and comparing alternative solutions; or checking the magnitude of the result using rough calculations, estimates, and comparisons with the scale of other mathematical results. If a similar order of magnitude is obtained using these approaches, checked in different ways and supported by effective argument, this leads to higher-level results.

22.4 Methodology and Design of the Study

Since to the best of our knowledge no systematic research has been conducted on the self-differentiating potential of modeling problems, the aim of this study was to obtain empirical evidence of the self-differentiation potential of modeling problems relative to students' performance. The sample consisted of 37 Grade 10 students attending a high school (a so-called gymnasium) in a neighborhood with high socio-economic status in the northern part of Hamburg. The mathematics teachers in this school were interested in taking part in the study.

For the investigation, three modeling problems—hot air balloon, rainforest (Fig. 22.6), and cable drum (Fig. 22.7) problems—were selected because they differed in their structure and complexity. Furthermore, the tasks provided a good basis for a collection of modeling problems that prototypically demonstrate the self-differentiation potential and which have been used in many recent empirical studies. From the design of the study (Fig. 22.8), it can be seen that the students were initially and randomly divided into two to four groups according to each modeling problem. Each group worked on only one task at a time, and all groups worked at the same time and were videotaped. The modeling problems were assigned to the students by the project leaders who organized the study so that solutions for each task were available and comparisons could be made between the tasks carried out by high-performing and low-performing students.

After each modeling problem, the students were given a short feedback sheet to indicate their subjective assessment of the difficulty of the modeling problem using a five-point rating scale (i.e., very easy, easy, neither difficult nor easy, difficult, very difficult).

The participating learners had little experience of modeling problems and did not know either the modeling cycle on a meta-level or specific strategies for solving modeling problems. To investigate performance differentiation, the learners were

Initiative for the Rainforest

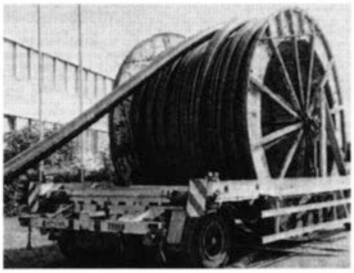
In 2002, 2003, 2005, 2006 and 2008 the brewery "Krombacher" carried out the following initiative in cooperation with the WWF (World Wildlife Foundation) for 3 months each year:

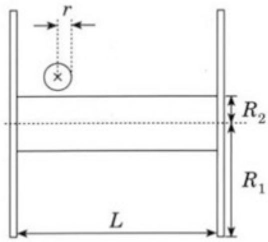
For every crate of Krombacher beer sold, one square metre of rainforest in Dzanga Sangha (Central African Republic) is sustainably protected.

Study the impact of this action on global rainforest deforestation.

For your information: Every day approx. 356 km² of rainforest are cleared or burned down worldwide, of which approximately 93 km² alone are in Africa. Germans drink an average of 107 litres of beer per person per year.

Fig. 22.6 Rainforest problem (Leiss et al., 2006)





How many meters of cable fit on this cable drum?

Measures: Layer width: L = 4.4 m Flange radius: $R_1 = 2.5$ m Hub radius: $R_2 = 60$ cm Cable radius: r = 15 cm	In the diagram above, the cable drum is correctly proportioned but on a smaller scale, whereas the cable is drawn twice the size it should be compared to the cable drum.

Fig. 22.7 Cable drum problem (Förster & Kaiser, 2010)

divided into homogeneous groups of four or five students according to their mathematical performance.

The teachers provided the study leaders with the learners' marks in mathematics and also advised on the composition of the groups for the study. The reason for dividing the students into homogeneous groups was the possibility of analyzing the learners' approaches more effectively, without having to constantly consider

One day: 90 minutes			
Four groups	Two groups	Three groups	
Modeling Problem 1 *Hot air balloon* Working in homogeneous groups of four students	Modeling Problem 2 *Rainforest* Working in homogeneous groups of four students	Modeling Problem 3 *Cable drum* Working in homogeneous groups of four or five students	Questionnaire regarding subjective feelings and level of task difficulty
Videotaping of each group			

Fig. 22.8 Design of the study

whether, for example, stronger learners might be steering or influencing the solution process too much. All groups were videotaped in separate rooms and, if questions arose, the project leaders answered them. Care was taken to ensure that all groups received the same level of assistance so as not to influence the results of the analysis.

The video recordings were transcribed. Based on the theoretical model (Fig. 22.2), data were analyzed according to grounded theory principles (Strauss & Corbin, 2003). The analysis process was thus guided by the existing categories of the theoretical model, meaning that the theoretical model formed the basis for the theoretical analysis method. The empirically confirmed self-differentiation potentials, which we describe as results in the following section, are presented in tables to illustrate the data analysis procedure.

22.5 Results of the Study

As can be seen from the design of the study, the empirical data were very extensive; therefore, prototypical results will be described and discussed of a high-performance group of learners with good grades in mathematics and a lower-performance group of learners with satisfactory grades in mathematics. Both groups modeled the hot air balloon problem. Since the approaches of the two groups were compared based on the central aspects in the presented tables, differences will become clear regarding their modeling processes. This perspective should offer insights regarding the extent to which the theoretical self-differentiation potentials were empirically justified.

22.5.1 Analysis of the Groups

No learner in either group initially chose to work alone and then compare their results with the others; by contrast, all learners discussed a mutual approach from the beginning. The common solution was then written down by one or more learners and illustrated on a poster at the end of the session; thus, the modeling process could not be traced back to individual students but was associated with an overall result for the respective group of students and their performance level.

The group processes during the phases of the modeling process and the analyzed self-differentiation potentials were very complex; hence, group differences are presented transparently and succinctly in Tables 22.1, 22.2, 22.3, 22.4, and 22.5. On the basis of the categories, the activities are summarized so that, on the one hand, it is possible to see whether and how the hot air balloon problem had an empirical self-differentiating effect and, on the other hand, to observe differences in the processing in terms of performance.

22.5.2 *Empirically Confirmed Self-Differentiation Potential in Setting Up the Real Models*

Table 22.1 Empirically confirmed self-differentiation potentials when building the real models

Categories	Stronger group	Weaker group
Influencing variables	Good ideas for determining the size of the man by measuring/comparing the picture with reality. Balloon dimensions based on the average width of the balloon. Principle of exhaustion used to approximate the shape of the balloon. The group chose to divide the balloon into two sections, striving for more complexity for later mathematization	Good ideas for determining the size of the man by measuring/comparing the picture with reality. Balloon dimensions based on the maximum width of the balloon. Principle of exhaustion used to approximate the shape of the balloon. The group chose to divide the balloon into three sections and measure quantities, striving for less complexity for later mathematization.
Identification of key variables	Learners were explicitly aware that no exact approximation to the shape of the balloon was possible. Learners noted and excluded irrelevant information in the text for the task.	Learners were explicitly aware that no exact approximation to the shape of the balloon was possible. Learners discussed the irrelevant information in the text for the task.
Assumptions/ Simplifications	Adequate simplification by reducing the balloon width to the average width.	Reduced simplification due to the three-part balloon.
Relationships between variables	Learners established a causal relationship between the variables: the age of the man and the real height of the man.	Learners discussed various relationships between variables.
Search for information	No further search	No further search
Arguments	High degree of complexity	Average degree of complexity

22.5.3 Empirically Confirmed Self-Differentiation Potential When Building the Mathematical Model

Table 22.2 Empirically confirmed self-differentiation potentials when building the mathematical model

Categories	Stronger group	Weaker group
Mathematization	A total of eleven mathematizations, as follows: Of different complexity. Selected mathematization was complex and contained two solutions. Decomposition 1: into a truncated circular cone and a circular cone. Decomposition 2: into a spherical section and a circular cone.	A total of four mathematizations, as follows: The first three are greatly simplified. Selected variant was more complex and accurate, but was slightly inappropriate: Decomposition of the balloon into a main spherical section, a spherical layer, and a truncated circular cone
Argumentation	High degree of complexity due to two mathematizations. Boy's utterance: "Because this is almost exactly a truncated circle, the top becomes blunt and it goes steeply upward. That's exact!" (Fig. 22.9).	Average degree of complexity due to the goal of accuracy. Girl's utterance: "Yes, and the cone section [here, the lower part the truncated circular cone is meant], I'd really start here under the dark area, because that's really an angular shape, there's hardly any rounding in it" (see Fig. 22.9).
Appropriate notation	Searched for new mathematical formulae and wrote them down correctly. Girl's utterance: "But now we'll figure out how to calculate it, which we haven't done yet."	Searched for new mathematical formulae and wrote them down correctly. Boy's utterance: "I'll check the formulae collection for the area of one cone."

Mathematizations of the Two Groups

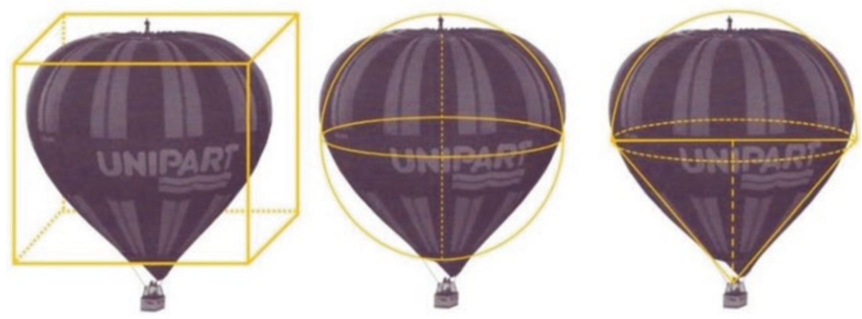


Fig. 22.9 Mathematizations of the two groups

22.5.4 Empirically Confirmed Self-Differentiation Potential in Mathematical Work

Table 22.3 Empirically confirmed self-differentiation potentials in mathematical work

Categories	Stronger group	Weaker group
Heuristic strategies	Decomposition of the problem into sub-problems (by dividing the shape of the balloon).	Decomposition of the problem into sub-problems (by dividing the shape of the balloon).
Applied mathematical knowledge	Calculation of scale, conversion of units (thereby faulty execution), rounding, estimating, volume calculation of various geometric bodies. Additional calculation of the mean values of the widths of balloons.	Calculation of scale, conversion of units (thereby faulty execution), rounding, estimating, volume calculation of various geometric bodies.
	Differences in the volume calculations resulted from the different mathematizations; however, they did not differ in the degree of complexity between the two groups.	
Argumentation	Since the mathematical models were finalized, no arguments for a higher level of modeling were presented	

22.5.5 Empirically Confirmed Self-Differentiation Potential When Interpreting the Mathematical Results

Table 22.4 Empirically confirmed self-differentiation potentials when interpreting the mathematical results

Categories	Stronger group	Weaker group
Interpretation	Group did not address prior intermediate steps in the interpretation to evaluate accuracy.	Group estimated the accuracy of the result and justified this using the reference value for the scale (the size of the man), which was only estimated. Boy's utterance: "We don't have to make a correct equal sign at the end, by the way, but a sign for 'approximately', because it is actually only an approximate value, because we have estimated the man."
Potential	Content-related self-diferentiation of both groups according to complexity and argumentation.	
Solution for special situations	No generalization.	No generalization.

22.5.6 Empirically Confirmed Self-Differentiation Potential in the Validation of the Solutions

Table 22.5 Empirically confirmed self-differentiation potentials in the validation of the solutions

Categories	Stronger group	Weaker group
Critical review	More complex, multiple ways of validation. The group validated their mathematical result for the volume of the upper part of the balloon by comparing the order of magnitude of their result with the magnitude of the mathematical solution for the volume of the lower part. They justified this by the fact that they expect a similar order of magnitude and carried out - further validations.	Less complex validation activities. The group validated their mathematical result for the total volume; after including the scale to calculate the real-world sizes, they decided this could not be correct and justified this mathematically. Boy's utterance: "1.26057778, which is not good. It's gotta be bigger." Re-validation through appraisal. Girl's utterance: "6308, no … but 63,816.75, that fits!"
Forms of validation	Estimation and comparison of orders of magnitude.	Estimation.
Solutions	Several solutions after mathematization.	One overall solution after mathematization.

22.5.7 Reflection

Regarding the second research question, the analysis of the modeling processes showed that, across groups, both the high- and the lower-performing learners were able to participate in finding the solutions for all three modeling problems. The self-differentiation potential of the problems, which was initially analyzed normatively, was also confirmed empirically.

The "hot air balloon" modeling problem, shown as an illustration, therefore offered great potential for self-differentiation on different levels, both theoretically and empirically, with regard to the promotion of modeling competencies on the one hand and personal mathematical competencies on the other hand. Both groups, in fact, incorrectly converted sizes; however, the modeling problem enabled the learners to achieve different degrees of complexity in all phases of the modeling process and on the level of argumentation without having to formulate further sub-problems, as they would have to do for many textbook problems. The stronger learners were able to generate multiple solution paths (Schukajlow & Krug, 2014), and the weaker learners found at least one solution path. The real context was also helpful in that all learners were able to make a contribution to the discussion, which initially had nothing to do with mathematics performance because it concerned everyday experiences, such as the size of a man or whether anyone had ever seen such a balloon. In particular, the weaker learners showed a higher level of mathematical modelling competencies when building a mathematical model than the stronger learners, because their selected model variation was more complex and adequate.

22.6 Summary and Discussion

Teachers deal with different dimensions of heterogeneity daily in schools, and there are many opportunities for mathematics teaching to encourage and challenge all learners. In particular, this contribution with the collection of modeling problems, which support self-differentiation, should clarify, based on the new approach of a theoretically derived analysis method and a qualitative study, the criteria that can be used to determine the self-differentiation potential of modeling problems in terms of content and methods; thus, this chapter should help to clarify the self-differentiating potential of modeling problems for facing the challenge of heterogeneity. The theoretical analysis and the empirical study answered the research questions: (1) whether a theoretical analysis method of the self-differentiation potential of modeling problems could be developed and how it should be structured, and (2) how the self-differentiation potential of modeling problems relates to the performance of learners. Based on the theoretical background regarding differentiation and modeling, it was first possible to develop an analysis method for the self-differentiation potential of modeling problems. For this purpose, each phase of the modeling process was examined for possible level differences in terms of content and method according to complexity, intricacy, argumentation, inquiry techniques, and problem-solving strategies.

These differences in level of the content probably resulted from the differentiation of performance. The theoretical and theoretical descriptions (Fig. 22.2) were specifically applied to the modeling problems in the empirical study (to the "hot air balloon" problem). This analysis showed that modeling problems have great self-differentiation potential. For all the problems presented to learners in the study, it was empirically confirmed that content-related self-differentiation in all phases of the modeling cycle was possible.

Since potentials could be reconstructed in the studied modeling problems, which were quite different, it can be hypothesized that modeling problems in general possess these content-related self-differentiation characteristics. A further important finding was that the different modeling problems in the study also had different self-differentiation potentials, which could partly be attributed to the special characteristics of the individual problems. Based on these results, the first research question was answered by developing a theoretical analysis method with corresponding categories.

This model could be validated mainly by empirical investigation and was prototypically illustrated by application to a high-performance and a lower-performance group of tenth-grade learners in the empirical study. In particular, the analysis highlighted the different degrees of complexity in the learners' approaches according to their performance levels, which led to more mathematization approaches and solutions being developed by the stronger learners than the weaker learners. Nevertheless, it could be shown (thus answering the second research question) that all learners were able to participate in the modeling problems, illustrating the power and diverse characteristics of modeling problems.

The students' solutions also showed some differentiations in the opposite direction, meaning that the supposedly weaker learners achieved a higher level of performance in some phases of the modeling cycle, particularly when building the real model. This may relate to the fact that some learners may be more proficient in some sub-competencies of modeling than in others, although the groups were divided according to their overall mathematics performance. There may be further opportunity to investigate this finding by classifying learners according to different levels of modeling competencies rather than their overall performance in mathematics. This result ultimately showed that modeling problems offer weaker students new possibilities for engaging with and applying mathematics.

A further future aim is to operationalize the theoretical analysis method with a larger sample, which will require new survey instruments. Besides the extension of the sample, further studies with learners of other age groups, using different modeling problems, would be interesting. Overall, our study could demonstrate that with modeling problems, teachers have an effective teaching tool that meets the challenge of dealing with heterogeneity in a quality, motivating, and work-relieving way.

Previous findings illustrating the challenging power of modeling problems have already been incorporated into courses on modeling in teacher training, but more activities and collections of problems are needed to unfold the pedagogical potential of mathematical modeling problems.

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Chapter 23
Taiwanese Teachers' Collection of Geometry Tasks for Classroom Teaching: A Cognitive Complexity Perspective

Hui-Yu Hsu

23.1 Introduction

Many researchers have pointed out the crucial role of mathematical instructional tasks in student learning outcomes (Boston & Smith, 2009; Henningsen & Stein, 1997; Silver & Stein, 1996; Stein et al., 1996). Mathematical tasks can direct students' attention to particular aspects of mathematics and structure their ways of thinking about mathematics (Doyle, 1983, 1988). The work students do determines how they think about a curricular domain and understand the meaning of mathematics. The types of tasks may also influence instruction, subsequently leading to different opportunities for students to learn mathematics (Doyle, 1988; Stein et al., 2000).

Of particular research interest is the relationship between mathematical tasks and the levels of cognitive demand, as this dramatically influences student learning outcomes (Boston & Smith, 2009; Henningsen & Stein, 1997; Silver & Stein, 1996; Stein et al., 1996). Leikin (2014) further proposed a more comprehensive conception of mathematical tasks, namely *mathematical challenge*, which highlights the importance of students thinking of tasks as interesting, thus motivating them to engage with mathematically difficult tasks. One key to determining mathematical challenge is the cognitive complexity[1] that a task entails. During instruction, teachers

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[1] Cognitive complexity and cognitive demand share a similar construct that denotes task features entailed, which influence the kinds of cognitive processes students may need to perform to solve the task (Stein et al., 1996). Cognitive complexity particularly refers to cognitively demanding or cognitively complex tasks. It possesses features that appear to require students to engage in high-level cognitive processes such as making connections or mathematical reasoning (Magone et al., 1994).

R. Leikin (ed.), *Mathematical Challenges For All*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-031-18868-8_23

have to maintain or increase the cognitive complexity of tasks to challenge students to move to a higher level of thinking (Leikin, 2009; Stein & Lane, 1996).

This study focuses on Taiwan mathematics instruction, as Taiwanese students are consistently in the top group in cross-national assessments (e.g., Mullis et al., 2012; OECD, 2014). One of the main reasons for these students' out-performance could be the mathematical tasks that Taiwanese mathematics teachers collect for classroom teaching. Hsu and Silver (2014) examined the type of geometry tasks used by Taiwanese mathematics teachers. They reported several significant findings concerning the collection of tasks and the cognitive complexity those tasks entail. The type of geometry task examined by Hsu and Silver was geometric calculation with numbers (GCN), which refers to tasks that involve numerical calculations done based on geometric properties or formulas in a geometric diagram environment. GCN tasks often require cognitive complexity as problem-solving requires high-level thought and reasoning processes (Magone et al., 1994). Hsu and Silver (2014) reported that Taiwanese teachers used tasks not just from textbooks but from other sources as well, and GCN tasks from non-textbook sources tended to be more cognitively challenging than those found in textbooks. This finding implies that the opportunity to practice tasks from non-textbook sources may be one of the critical factors in the superb mathematics achievements of East Asian students. Hsu and Silver's study also anchored a study by Silver et al. (2009) that showed that tasks used by teachers for the assessment of mathematical understanding tended to have higher cognitive demand characteristics than tasks used to develop mathematical understanding.

The study reported here is a follow-up to Hsu and Silver (2014), with an attempt to further examining different Taiwanese teachers' collections of sources of instructional/curricular materials from a cognitive complexity perspective. In particular, we intended to learn if students' mathematics performance influences teachers' collection of tasks. The research question for the study was as follows:

> What is the cognitive complexity of geometry tasks collected by Taiwanese mathematics teachers, and does the cognitive complexity of geometry tasks differ between schools with different mathematics performance levels?

23.2 Analytical Framework

Hsu and Silver (2014) extended the construct of cognitive complexity and proposed an analytical framework that can examine the cognitive complexity of geometry tasks. As shown in Fig. 23.1, the analytical framework includes two dimensions—diagram complexity and problem-solving complexity—each of which describes the kind of cognitive activity involved in geometry problem-solving. Diagram complexity refers to the segments and lines comprising a geometric diagram, which can influence cognitive complexity in solving geometry problems. Problem-solving complexity identifies four kinds of cognitive activity involved in geometry problem-solving processes. The details of each category of the dimensions are as follows.

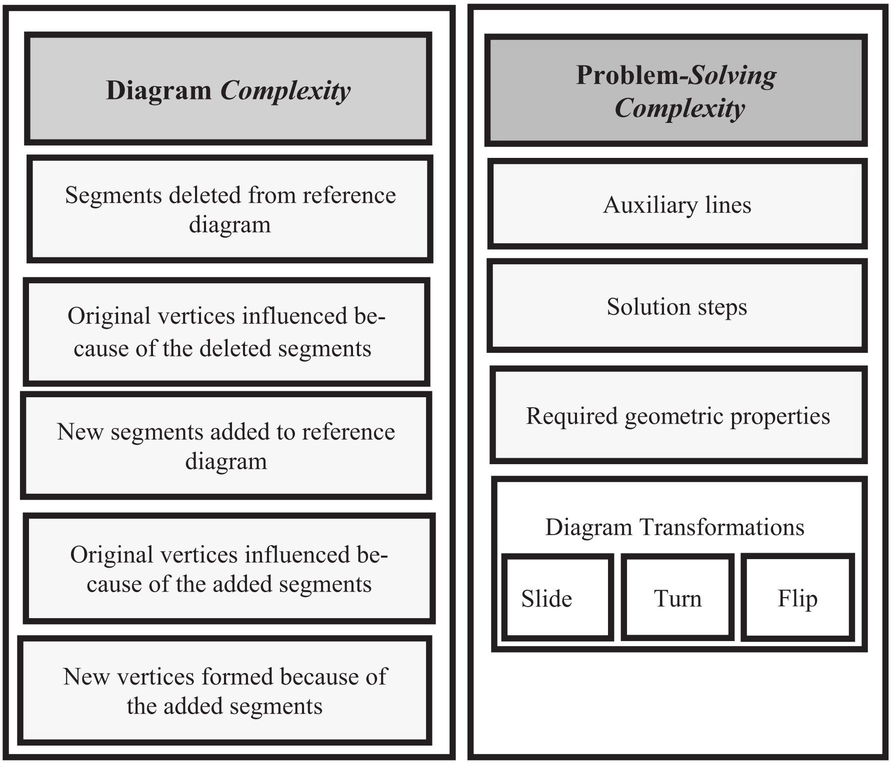


Fig. 23.1 The cognitive-complexity framework (Hsu & Silver, 2014)

23.2.1 Diagram Complexity Dimension

Hsu and Silver (2014) recognized the cognitive complexity that a geometric diagram might cause. The psychology literature confirms that the schemas used in problem-solving processes are strongly tied to diagrams, especially when dealing with high-level cognitive activities (Carlson et al., 2003; Greeno, 1978; Koedinger & Anderson, 1990; Larkin & Simon, 1987; Lovett & Anderson, 1994; Mousavi et al., 1995). Here, schemas refer to a "cluster of knowledge that contains information about the core concepts, the relations between concepts and knowledge about how and when to use these concepts" (Chinnappan, 1998, p. 202). To this end, geometry problem solving requires diagram parsing to identify familiar configurations with corresponding schemas in the diagram, which can be used to formulate a solution plan by reasoning forward and backward between the givens and the goals (Koedinger & Anderson, 1990).

Hsu and Silver proposed the construct of reference diagrams to analyze cognitive complexity concerning schema searching in geometry diagrams. They used it as the basis to analyze diagram complexity in a geometry task. They defined a *reference* *diagram* as a geometric diagram shown with a geometric property that is formally introduced in a textbook. A reference diagram provides a common point of contact through which many geometric concepts and properties are linked.

Hsu and Silver (2014) further explained the reasons for using a reference diagram as the basis for examining the cognitive complexity embedded in geometry diagrams. First, analyzing a geometry task diagram by comparing it to a reference diagram provides information regarding possible visual obstacles that students may encounter when identifying the reference diagram and its corresponding geometric properties in the given geometry task. Second, a reference diagram is an *external representation* (Laborde, 2005) presented in textbooks, thereby preventing coding inconsistencies that can arise when making inferences about the mental images of a diagram as processed internally by individuals.

Figure 23.2 shows a reference diagram of an isosceles triangle that is usually shown in textbooks. The reference diagram conveys not only the definition (e.g., that two of the three sides in the triangle are congruent) but also other related geometric properties (e.g., the sum of the interior angles of the triangle is 180°). A reference diagram also possesses visual features that can help draw attention to the salient geometric properties. For instance, one can easily recognize the congruence of the segments in an isosceles triangle. Figure 23.2 presents its reference diagram that has the lengths of the two congruent legs standing symmetrically on the two sides with the base side on the bottom parallel to the horizontal axis.

The categories of diagram complexity describe how a diagram given in a geometry task is altered compared to a reference diagram. Diagram complexity includes five categories used to describe the changes in terms of segments and vertices in a geometric diagram. Those five categories are the number of segments deleted from the reference diagram (category 1), the number of original vertices influenced by the deleted segments (category 2), the number of segments added to the reference diagram (category 3), the number of original vertices influenced by the added segments (category 4), and the number of new vertices created because of the added segments (category 5). One can see the details of analyzing diagram complexity for a geometry task along with the five categories in 2.3 in the session.

23.2.2 Problem-Solving Complexity Dimension

The problem-solving complexity in the analytical framework includes four categories, each of which refers to the cognitive processing that appears to be essential in geometry problem-solving. The categories are auxiliary lines, solution steps, required geometric properties, and diagram transformations.

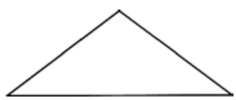


Fig. 23.2 Reference diagram for an isosceles triangle

The auxiliary lines category concerns cognitive complexity in analyzing geometry tasks to determine if drawing auxiliary lines is needed and, if so, where to draw the lines such that new subconfigurations and new geometric properties can be created and used to generate a solution. Drawing the lines requires recalling prior knowledge and previous problem-solving experiences (Pólya, 1945; 2nd edition 1957). Drawing auxiliary lines on a diagram is often cognitively demanding. It forces one to anticipate creating subconfigurations associated with corresponding geometric properties that can be used to generate a solution plan.

The solution steps category involves the analysis of the reasoning steps required to obtain a solution. A reasoning step is defined as a problem-solving action taken based on a geometric property. Hsu and Silver (2014) counted the number of reasoning steps required to solve a geometry task, as the number can significantly influence cognitive demand. Researchers have indicated that generating a multi-step solution is cognitively demanding as it requires students to identify geometric properties for each reasoning step and chain the reasoning steps into a logic sequence (Ayres & Sweller, 1990; Heinze et al., 2005). Thus, the number can be an indicator used to describe cognitive complexity in reasoning a geometry task. It is also recognized that a geometry task can be solved in multiple ways, which might lead to different numbers of reasoning steps. Hsu and Silver (2014) stipulated that the solution used to classify the number of steps for a task should require the minimum number of reasoning steps to obtain the correct answer. They noted that each reasoning step in the solution should be supported by a geometric property that students have learned or will learn in the current instructional unit. Thus, classifying geometry tasks based on the minimum number of reasoning steps provides information regarding what prior geometric knowledge students have to access to successfully solve the tasks.

In addition to using the solution steps to describe the cognitive complexity of a geometry task, Hsu and Silver (2014) also considered the analysis of the number of geometric properties needed for a solution. They included this category because the number of geometric properties needed to solve a geometry task may not be the same as the number of solution steps. The reason is that different reasoning steps in a solution may require using the same geometric property. Analyzing the number of geometric properties required in a solution could provide richer information for describing the cognitive complexity of a geometry task. Geometric properties are those geometric statements or definitions that have been formally introduced in textbooks.

The diagram transformation category focuses on analyzing the cognitive complexity involved in diagram transformations (e.g., rotating). When solving a geometry task, one may need to perform a diagram transformation to map reference diagrams onto the task diagram. The performance enables recognizing and retrieving the geometric properties embedded in subconfigurations in the diagram. The mapping process requires mentally or physically transforming the reference diagrams to check if they resemble a diagram configuration in the geometry task. Operations on diagrams cause cognitive challenges for students as the orientation and position of a geometry task diagram may influence the identification of the corresponding reference diagrams (Fischbein & Nachlieli, 1998). Hsu and Silver (2014) included three types of diagram transformation in this category: slide (translation), turn (rotation), and flip (reflection).

The analytical framework proposed by Hsu and Silver allows one to systematically and scientifically analyze geometry tasks without constraints caused by the diversity of students' prior knowledge and learning experiences. This is because the basis for analysis is the geometric properties and diagrams presented in textbooks. As the properties and reference diagrams offered in textbooks are the materials used by students to learn, analysis based on the proposed framework is still closely tied to student cognition.

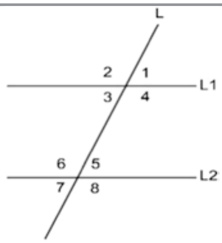
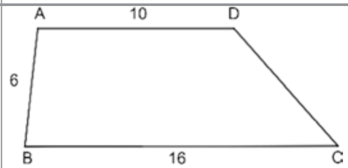
23.2.3 Analysis Examples of GCN Tasks

To unpack the cognitive complexity embedded in a GCN task, we provide two analysis examples based on the analytical framework (see Table 23.1).

Task A is considered as a low cognitive-complexity task, whereas Task B is a high cognitive-complexity task. Details of the analysis of Task B can be seen in the appendix as an external link to Hsu and Silver (2014). The elaboration of the cognitive complexity of the two tasks begins from the problem-solving dimension. The problem-solving processes influence the cognitive complexity related to decomposing and recomposing diagram configurations into subconfigurations in order to retrieve the geometric properties for a solution (Gal & Linchevski, 2010; Hsu & Silver, 2014).

Analysis Based on the Categories in Problem-Solving Complexity Dimension Solving Task A does not require the cognitive work of drawing the auxiliary line as the given information is enough to generate a solution. However, Task B cannot be solved unless an auxiliary line is drawn. Figure 23.3 shows a strategy to draw an

Table 23.1 Descriptions of two GCN tasks

	Task A	Task B
The given diagram		
The written task	Given that L1‖L2 and L intersects L1 and L2, and $m\angle 1 = 66°$. Find $m\angle 5$.	In a quadrilateral ABCD, given that AD‖BC and AD = 10, BC = 16, AB = 6, $m\angle DCB = 48°$. Find $m\angle BAD$.

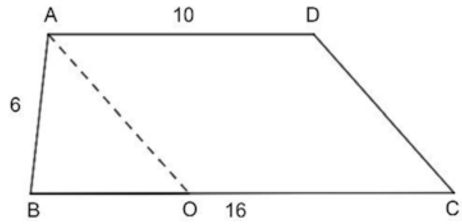


Fig. 23.3 The drawing of auxiliary line AO

auxiliary line AO such that AO is parallel to DC. The drawing of the auxiliary line allows one to reason that ABCD is a parallelogram as well as OC = 10 and BO = 6.

Table 23.2 shows the minimum solution steps and the geometric properties required as supportive reasons for Task A and Task B. As can be seen, Task A can be solved in one reasoning step and with one geometry property. In contrast, Task B involves higher cognitive complexity as it requires five reasoning steps and five geometric properties to find the answer.

Concerning the analysis of diagram transformation, it has to identify reference diagrams corresponding to each geometric property required in the solution (see Table 23.3). Identifying individual reference diagrams forms the basis for analyzing what transformation actions are needed to map the reference diagrams onto the GCN task diagram. After checking the reference diagrams for the geometric properties required in the solution, diagram transformations are examined. For Task A, as its task diagram structure is the same as that of the reference diagram (e.g., a pair of parallel lines and a transversal), no diagram transformation action is needed. However, Task B necessitates diagram transformations to map the reference diagrams onto the GCN task diagram (see Table 23.4). As a result, five diagram transformation actions are required for Task B.

Analysis Based on the Categories in Diagram Complexity Dimension

For Task A, as its task diagram shares the same structure as the reference diagram, the analysis of diagram complexity is denoted as 0 because no changes in terms of the segments and vertices can be identified. For Task B, one of the reference diagrams shown in Table 23.3 is used as the basis for the analysis of its diagram complexity. The reference diagram for the corresponding angles property is determined as the basis for the analysis of diagram complexity because the geometric property is one of the main contents to be learned in the lessons. Figure 23.4 shows how the reference diagram for the corresponding angles property resembles part of the Task B diagram. The diagram shown on the left side in Fig. 23.4 is the reference diagram for the corresponding angles property. The diagram shown on the right side is how the reference diagram resembles the GCN task diagram.

The analysis of diagram complexity along with the five categories is used to describe the changes to the reference diagram so that it becomes the Task B diagram. As shown in Table 23.5, the analysis of diagram complexity for Task B based on the analytical framework shows ten changes.

Table 23.6 summarizes the analysis results for Task A and Task B based on the categories of problem-solving complexity and diagram complexity in the analytical

Table 23.2 Solution steps and required geometric properties for Task A and Task B

Steps	Calculating sentences	Supportive reasons	Calculating sentences	Supportive reasons
Step one	As L1 and L2 are parallel, $m\angle 5 = m\angle 1 = 66°$	The corresponding angles property	As AD‖OC, AD = OC = 10, AOCD is a parallelogram	If one pair of the opposite sides is parallel and congruent, the quadrilateral is a parallelogram.
Step two			$m\angle DCO = m\angle DAO = 48°$	Opposite angles of a parallelogram are congruent
Step three			AO‖CD	The opposite sides of a parallelogram are parallel
Step four			$m\angle DCO = m\angle AOB = 48°$	The corresponding angles property
Step five			BA = BO = 6 $m\angle AOB = m\angle BAO = 48°$ So that $m\angle BAD = 48° + 48° = 96°$	Properties of an isosceles triangle

Table 23.3 Reference diagrams corresponding to the geometric properties required

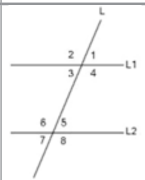
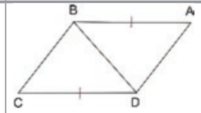
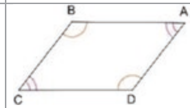
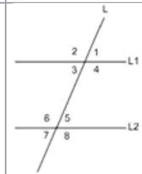
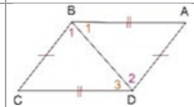
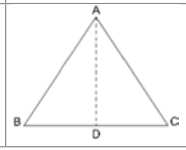
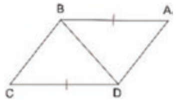
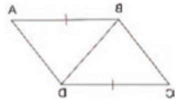
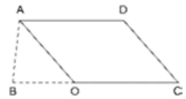
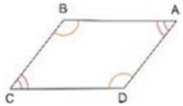
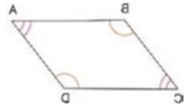
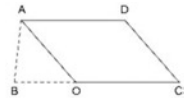
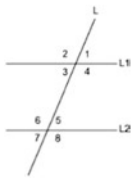
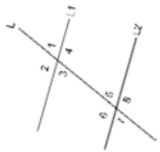
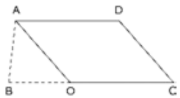
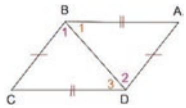
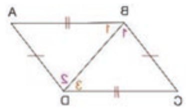
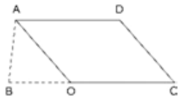
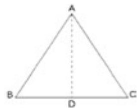
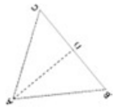
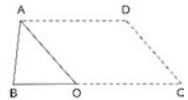
	Task A		Task B	
Steps	Geometric properties	Corresponding reference diagrams	Geometric properties	Corresponding reference diagrams
Step one	The corresponding angles property		If one pair of the opposite sides is parallel and congruent, the quadrilateral is a parallelogram	
Step two			Opposite angles of a parallelogram are congruent	
Step three			The corresponding angles property	
Step four			Opposite sides of a parallelogram are congruent	
Step five			If two sides of a triangle are congruent, then their corresponding angles are congruent	

Table 23.4 Diagram transformations for Task B

Solution steps	Transformation actions	Reference diagrams appearing in the textbooks	Reference diagrams after transformations	Subconfigurations in GCN task diagram
Step one	Flip			
Step two	Flip			
Step three	Turn			
Step four	Flip			
Step Five	Turn			

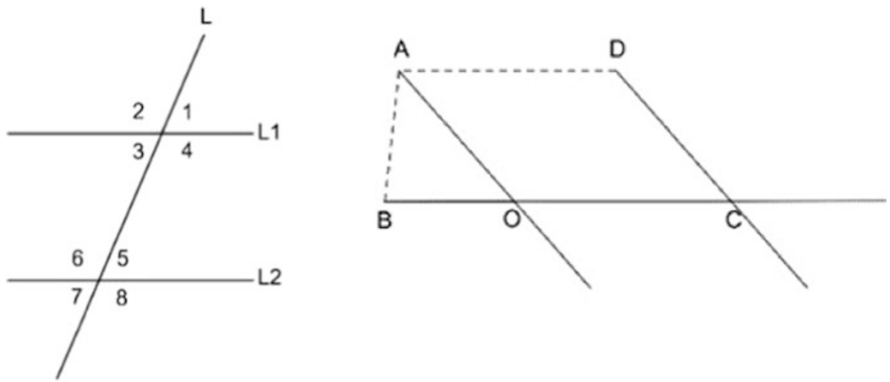


Fig. 23.4 Reference diagram (left) and how it resembles the Task B diagram (right)

Table 23.5 Analysis of diagram complexity for Task B

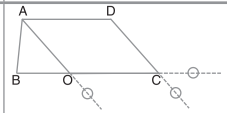
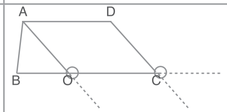
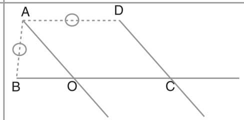
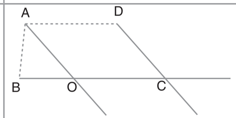
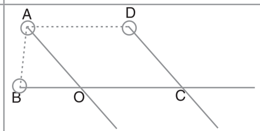
Categories of diagram complexity	Changes in diagram	Coding results
Segments deleted from reference diagram		Three segments are deleted
Original vertices influenced because of the deleted segments		Two original vertices are influenced because of the deleted segments
New segments added to reference diagram		Two new segments are added to the reference diagram
Original vertices influenced because of the added segments		No original vertex is influenced because of the added segments
New vertices formed because of the added segments		Three new vertices are formed because of the added segments
Sum of the changes		10

Table 23.6 Summary of the analysis of cognitive complexity for Task A and Task B

Coding		Task A	Task B
Problem-solving complexity	Auxiliary line	0	1
	Solution steps	1	5
	Required geometric properties	1	5
	Diagram transformation	0	5
Diagram complexity		0	10
Cognitive complexity		2	26

framework. As can be seen, the coding for Task A is denoted as 2, whereas that for Task B is 26. The numbers allow one to understand how geometry tasks are made cognitively complex based on the requirements with respect to diagram complexity and problem-solving complexity. The bigger the coding number, the more cognitive complexity the GCN task entails.

23.3 Methodology

23.3.1 Selection of Teachers

To select subjects for this study, we first searched for experienced Taiwanese mathematics teachers with more than 5 years of teaching experience who were willing to participate in the study. We then checked if those teachers taught eighth-grade students because a geometry topic designed for those students was the focus of the analysis. We then identified students' overall mathematics performance in the schools those teachers taught. In Taiwan, student mathematics performance varies between schools, which can be due to factors such as school reputation, the socio-economic status of the students' parents, and residential areas (e.g., remote areas) (Huang, 2017). In general, overall student mathematics performance in a school does not change much over the years. We decided to use overall school mathematics performance as an indicator for the students taught by the teachers who participated in the study. The first reason was that students in the same school have to be randomly grouped into classes. Thus, overall school mathematics performance can represent the students in classes due to the random assignment process.

The second reason was that overall school mathematics performance on high school entrance examinations could be obtained, which allowed a fair comparison among the classes the participating teachers taught. Taiwan's high school entrance examination is a nationwide examination that has to be taken by Taiwanese middle school students as they need an examination score to apply to high schools. The examination ranks students as level A, B, or C, where A represents a high-performance level, and C indicates a low-performance level. According to a report on the mathematics subject in the high school entrance examination (Comprehensive Assessment Program for Junior High School Students, 2020), 22% of students who took the examination were identified as level A, 50% were identified as level B, and about 28% were identified as Level C. We considered a school as having high mathematics performance if more than 50% of its students achieved level A, middle performance if more than 50% of its students achieved level B. Low performance if more than 50% of its students identified as level C. In the end, a total of six Taiwanese mathematics teachers participated in the study. Two of them represented high mathematics performance schools, two middle mathematics performance schools, and two low mathematics performance schools.

As shown in Table 23.7, the highest number of years of teaching was 27, and the lowest was 5. The mathematics teachers' majors were either in mathematics or mathematics-related areas (e.g., mathematics education). Teacher Jyu and Teacher Ing taught at high math performance schools; Teacher Sheng and Teacher Yao taught at middle math performance schools; Teacher May and Teacher Wen taught at the same school identified as a low math performance school.

23.3.2 Data Collection

The teaching of a geometric topic—properties related to parallel lines—in eighth-grade textbooks in Taiwan was the data collection context. The instructional goals set up with the geometric topic included helping students become familiar with geometric properties related to parallel lines and the concept of geometric proofs. In this regard, a high percentage of GCN tasks were included in the textbooks. All mathematical tasks situated in sources of instructional/curricular materials collected by the six teachers when they taught the geometric topic were the data for the analysis. The six teachers' teaching of the geometric topic was videotaped and analyzed. As a result, four sources of instructional materials were identified, including textbooks, supplementary materials, tests, and tasks created by the teacher during classroom teaching. The textbooks included both student textbooks and student workbooks. Different textbooks published in Taiwan all have to be evaluated based on the national mathematics curriculum, but the mathematical tasks included in the textbooks can be slightly different. The student textbook contained both instructional blocks comprised of diverse mathematical activities (e.g., diagram construction and proving) and exercise blocks. The student workbook included additional exercises for students to practice.

Teachers may feel that textbooks and workbooks are not enough for their students and decide to include supplementary materials for classroom teaching. Supplementary materials are either designed by mathematics teachers themselves or are published by textbook companies. No matter whether designed by mathematics teachers or by textbook publishers, supplementary materials usually have several characteristics. First, they are often arranged in the same sequence as textbooks. Second, they often include a large number of tasks. Third, they usually summarize the main mathematical content (e.g., definitions and geometry properties). Four out of the six teachers in our sample used supplementary materials for their teaching. However, the underlying reasons for the use of supplementary materials were different. Teacher Jyu indicated that she thinks textbooks are too easy to prepare students to obtain high scores on examinations. Thus, she decided to include supplementary materials in her teaching. Teacher Wen pointed out that she uses supplementary materials as they can provide students with extra opportunities to practice tasks and learn mathematical concepts. In particular, she indicated that she often chooses supplementary materials that include tasks with similar cognitive complexity to those included in textbooks.

Table 23.7 Teachers' background and information on instructional/curricular materials

Teacher Pseudonym	Years of teaching	School math performance	Sources of instructional/curricular materials used	Number of pages and tasks	Number of lessons
Jyu	27	High	Textbooks	21 pages with 77 tasks	9
			Supplementary materials	7 pages with 111 tasks	
			Tests	7 tests with 173 tasks	
			Problems created by the teacher	10 tasks	
Ing	6	High	Textbooks	23 pages with 59 tasks	11
			Supplementary materials	7 pages with 91 tasks	
			Tests	3 tests with 62 tasks	
			Problems created by the teacher	2 tasks	
Sheng	9	Middle	Textbooks	22 pages with 63 tasks	7
			Supplementary materials	Not used	
			Tests	2 tests with 38 tasks	
			Problems created by the teacher	9 tasks	
Yao	5	Middle	Textbooks	21 pages with 77 tasks	8
			Supplementary materials	Not used	
			Tests	2 tests with 31 tasks	
			Problems created by the teacher	2 tasks	
Teacher May	17	Low	Textbooks	22 pages with 63 tasks	7
			Supplementary materials	9 pages with 52 tasks	
			Tests	1 test with 18 tasks	
			Problems created by the teacher	2 tasks	
Teacher Wen	14	Low	Textbooks	22 pages with 63 tasks	7
			Supplementary materials	9 pages with 52 tasks	
			Tests	1 test with 18 tasks	
			Problems created by the teacher	Not used	

Another primary source of curricular/instructional material collected during classroom teaching was the tests often used for formative or summative purposes. Teachers may use tests to evaluate students' learning outcomes or assign test sheets as homework. All six teachers used tests in their teaching. In addition, they all created tasks as they thought those tasks would benefit student learning during classroom teaching. Table 23.7 shows the number of pages and tasks for each source of curricular/instructional materials collected by the six teachers. Table 23.7 also presents the number of lessons each teacher spent on teaching the geometric topic. In Taiwan, a lesson at the middle school level lasts for 45 minutes.

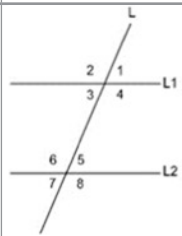
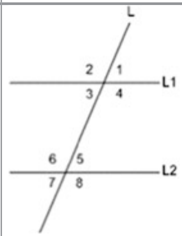
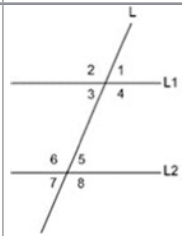
Interviews with the teachers were also implemented to better understand the reasons underlying their collection of tasks situated in sources of instructional/curricular materials.

23.3.3 Data Analysis

The data analysis started by identifying the types of tasks from the different kinds of instructional/curricular material collected from the six Taiwanese mathematics teachers. Different task types were identified, including exploration activities, geometric proof (GP) tasks, GCN tasks, geometric algebra (GA) tasks, and diagram construction tasks. Exploration refers to those activities that aim to help students understand geometric concepts through manipulation work. Construction refers to the work of drawing a geometric diagram using a compass and straightedge. Regarding the similarities and differences among GCN, GA, and GP, Table 23.8 shows examples of the three kinds of tasks. As can be seen, the three tasks use the same diagrams and given information to describe the diagram. The only difference among the three tasks is that GCN includes numerical information that can be used to reason unknown measures. GA requires applying algebraic skills in order to obtain a solution. GP involves finding reasons based on geometric properties that can be used to prove that a statement is always true. We counted the number of each type of task situated in the sources of instructional/curricular materials, where a task was defined as a problem asking for an answer (Charalambous et al., 2010).

After identifying a GCN task from sources of curricular/instructional materials, the problem-solving complexity dimension with its four analysis categories (auxiliary lines, solution steps, required geometric properties, and rigid transformation) was performed. The number of minimum solution steps was determined, which consequently became the basis for checking if drawing auxiliary lines was necessary. We also counted the number of geometric properties used to support the reasoning steps in the identified solution. In particular, we checked if those geometric properties had been formally introduced in the textbooks or had been learned previously by the students. Any geometric properties that 8th grade students have not learned were excluded, even if they could be used to generate a solution to a GCN task. The task analysis then focused on identifying diagram transformations with sliding, turning, and flipping actions. The diagram transformation analysis required

Table 23.8 Examples of GCN, GP, and GA tasks

	GCN	GP	GA
The given diagram			
The written task	*Given*: that L1‖L2 and L intersects L1 and L2,		
	$m \angle 1 = 56°$	$m \angle 1 = m \angle 5$	$m \angle 1 = (3X + 5)°$ $m \angle 5 = (23 - 6X)°$
	Find: $m \angle 5$.	*Prove*: $m \angle 4 + m \angle 5 = 180°$.	*Find*: value of X.

Note: *GA* geometric algebra, *GP* geometric proof the presence of given diagrams in the GCN tasks. Tasks in which a diagram was not provided were excluded. The minimum number of transformation actions necessary to map the reference diagrams representing the identified solution's geometric properties onto the given GCN task diagram was determined.

The next step was to analyze the diagram complexity of the GCN task. We determined the reference diagram as the basis for the examination of diagram complexity in a GCN task. As a solution often is generated by more than one geometric property, the reference diagram was decided based on two criteria. The first was that the reference diagram identified had to correspond to the geometric properties needed to obtain the minimum number of solution steps in a GCN task. The second was that the identified reference diagram represented one of the to-be-learned geometric properties in the current teaching topic. Once a reference diagram was determined for a GCN task, diagram complexity and its five coding categories were analyzed. Finally, we counted the number of changes needed to transform a reference diagram into a GCN task diagram and used the number to describe the GCN task's diagram complexity.

The author and a coder were responsible for the data analysis. To ensure the consistency of the coding results, tasks that were difficult to classify were selected for checking their reliability. Two coders analyzed those complex tasks individually and then discussed the coding results together. If an inconsistency occurred, both coders discussed the inconsistency until an agreement was reached. Regarding the interviews, we used a back-and-forth analysis process. Once we found something interesting from the data analysis, we showed those findings to the teachers to learn the reasons. It was also possible for the teachers' responses from the interviews to inform how we analyzed the collected data.

23.4 Results

23.4.1 Collections of Types of Mathematical Instructional Tasks

Among the multiple sources of instructional/curricular materials collected from the six teachers, we first identified the types of tasks and activities. Table 23.9 shows the types of tasks and the number that the six teachers collected. As can be seen, the types of tasks collected by the teachers included exploration activities, diagram construction activities, GCN tasks, GP tasks, and GA tasks. It is worth noting that teachers who taught in high mathematics performance schools collected more tasks than those teaching in middle and low mathematics performance schools (Teacher Jyu: 371 tasks; Teacher Ing: 214 tasks; Teacher Seng: 110 tasks; Teacher Yao: 110 tasks; Teacher May: 135 tasks; Teacher Wen: 140 tasks). It is recognized that tasks may entail different cognitive complexity and may be used in different ways (e.g., worked example vs. exercise) and for different instructional purposes (e.g.,

Table 23.9 Types of mathematical tasks collected by the teachers

Teacher	Math performance	Lessons	Mathematical tasks					
			Exploration	Construction	GCN	GP	GA	Total
Jyu	High	9	0 (0%)[a]	10 (3%)	229 (62%)	72 (19%)	60 (16%)	371 (100%)
Ing	High	11	0 (0%)	9 (4%)	132 (62%)	44 (21%)	29 (14%)	214 (100%)
Sheng	Middle	7	0 (0%)	7 (6%)	84 (76%)	10 (9%)	9 (8%)	110 (100%)
Yao	Middle	8	0 (0%)	2 (2%)	76 (69%)	22 (20%)	10 (9%)	110 (100%)
May	Low	7	0 (0%)	7 (5%)	94 (70%)	15 (11%)	19 (14%)	135 (100%)
Wen	Low	7	7 (5%)	7 (5%)	92 (66%)	15 (11%)	19 (14%)	140 (100%)
Total			7	42	707	178	146	1080

[a](number of tasks)/(total number of tasks collected by the teacher) × 100%

understanding the mathematical concept vs. applying the concept to a more complex task context). We found that high mathematics performance school teachers were inclined to collect more tasks for their students. Teacher Jyu said the following:

> I intended to include a high amount of tasks for my students as they can learn mathematics from a variety of tasks….By practicing the tasks, they can correct their misconceptions and understand what they did not understand previously….This is a very useful strategy to prepare students for the high school entrance examination." (Transcript of interview data, 20200302)

Among the types of tasks, GCN tasks occupied the highest percentage of tasks collected by the six teachers (Teacher Jyu: 62%; Teacher Ing: 62%; Teacher Sheng: 76%; Teacher Yao: 69%; Teacher May: 70%; Teacher Wen: 66%). The result made it reasonable to compare the cognitive complexity of the tasks collected for classroom teaching among the teachers. In addition, for the high and middle mathematics performance schools, the second-highest percentage of tasks collected by the teachers was GP tasks (Teacher Jyu: 19%; Teacher Ing: 21%; Teacher Sheng: 9%; Teacher: Yao 20%). Interviews with those teachers showed that they think their students can learn proofs even though textbooks do not include tasks that require students to construct geometric proofs themselves. For the low mathematics performance schools, the second-highest percentage of tasks collected for classroom teaching was GA (14% for both Teacher May and Teacher Wen). High-performance schools also used many GA tasks for teaching (Teacher Jyu: 16%; Teacher Ing: 14%). Only Teacher Wen included exploration tasks in her classroom teaching (5%).

23.4.2 Cognitive Complexity of the Tasks Collected by Taiwanese Teachers

We further examined the cognitive complexity of the GCN tasks collected by the six teachers. Table 23.10 shows the number of GCN tasks analyzed and the average diagram complexity, problem-solving complexity, and cognitive complexity for each teacher. Only GCN tasks accompanied by diagrams collected by the teachers were surveyed. As shown in Table 23.10, the tasks collected by the six teachers tended to entail both diagram complexity and problem-solving complexity, no matter if they taught at high mathematics performance or low mathematics performance schools. The average diagram complexity for the GCN tasks collected by all six teachers was 6.52, indicating that a GCN task was made about seven changes on average. The average problem-solving complexity was 5.39, implying that the GCN tasks were inclined to require multiple reasoning steps, multiple geometric properties for a solution, and the performance of diagram transformation. It is also likely that those GCN tasks asked students to draw auxiliary lines to obtain enough geometric properties to generate a solution. The average cognitive complexity for the GCN tasks collected by all six teachers was 11.90.

Table 23.10 Cognitive complexity of the GCN tasks collected by the teachers

Teacher	Math performance	Number of tasks	Diagram complexity	Problem-solving complexity	Cognitive complexity	Average
Jyu	High	216[a]	6.95	5.44	12.40	13.23
Ing	High	129	8.87	5.76	14.63	
Sheng	Middle	80	5.65	5.61	11.26	11.08
Yao	Middle	74	6.54	4.35	10.89	
May	Low	94	5.63	4.88	10.50	10.44
Wen	Low	92	5.53	4.85	10.37	
Average		114.17	6.52	5.39	11.90	11.90

[a]The number of GCN tasks refers to those accompanying a diagram, which may be inconsistent with the number of GCN tasks reported in Table 23.9

Of interest is the relationship between school mathematics performance and the cognitive complexity of GCN tasks. Table 23.10 shows that the better the mathematics performance of a school, the higher the cognitive complexity of the tasks that the teachers tended to collect for their students. The average cognitive complexity for Teacher Jyu and Teacher Ing, who taught high mathematics performance students, was the highest (13.23). The average cognitive complexity for Teacher May and Teacher Wen, who taught lower mathematics performance students, was the lowest (10.44). The average cognitive complexity for Teacher Sheng and Teacher Yao was in between, at 11.08. This finding shows that Taiwanese mathematics teachers consider cognitive complexity when collecting tasks from sources of instructional/curricular materials for classroom teaching.

23.4.3 Cognitive Complexity of Tasks Situated in Sources of Curricular/Instructional Materials

Four sources of instructional/curricular materials were identified from the six Taiwanese teachers, including textbooks, supplementary materials, tests, and tasks created by the teachers during classroom teaching. Table 23.11 shows the analysis of the cognitive complexity specific to each source of instructional/curricular materials collected by the six teachers. As can be seen, textbooks collected by the teachers possessed lower cognitive complexity than non-textbook sources. The average cognitive complexity of tasks situated in textbooks was 9.15. For the six teachers, the cognitive complexity of the tasks situated in the textbooks they used was similar. Cognitive complexity was 9.21 for Teacher Jyu, 9.82 for Teacher Ing, 9.12 for Teacher Sheng, 9.74 for Teacher Yao, and 9.12 for Teacher May and Teacher Wen.

The average cognitive complexity of the tasks situated in supplementary materials was slightly higher than that in textbooks, which was 10.7. Four out of the six teachers used supplementary materials in their teaching. An analysis of the supplementary materials showed that the cognitive complexity of the tasks collected by Teacher Ing (14.68) was much higher than those managed by Teacher Jyu (9.68), Teacher May (11.65), and Teacher Wen (11.65). Of interest is that the cognitive complexity of the tasks situated in supplementary materials used by Teacher Jyu, who taught high mathematics performance students, was lower than those used by Teacher May and Teacher Wen, who taught low mathematics performance students.

The interviews with Teacher Jyu and Teacher Wen revealed the underlying reason. Teacher Jyu said

> ...The supplementary materials used in my classes were designed by my colleagues and me….I use it [supplementary materials] for my teaching…but not the textbooks…because we have our own ideas on selecting tasks and sequencing them for our students….We use the materials to develop students mathematics concepts. (Transcript of interview data, 20190810)

Table 23.11 Cognitive complexity of tasks from sources of instructional/curricular materials collected by the teachers

Teacher Name	Math performance	Cognitive complexity	Sources of instructional/curricular materials				Average
			Textbooks	Supplementary materials	Tests	Tasks created by teacher	
Jyu	High	Number of tasks	57	62	94	3	216
		Diagram complexity	5.51	5.23	8.85	10.67	6.95
		Problem-solving complexity	3.7	4.45	7.04	9.00	5.44
		Cognitive complexity	9.21	9.68	15.89	19.67	12.40
Ing	High	Number of tasks	35	44	49	1	129
		Diagram complexity	6.13	9.32	10.04	27.00	8.87
		Problem-solving complexity	3.69	5.36	7.41	15.00	5.76
		Cognitive complexity	9.82	14.68	17.45	42.00	14.63
Sheng	Middle	Number of tasks	49	0	28	3	80
		Diagram complexity	4.31	0	7.43	11.00	5.65
		Problem-solving complexity	4.82	0	7.00	5.50	5.61
		Cognitive complexity	9.12	0	14.43	16.50	11.26
Yao	Middle	Number of tasks	57	0	17	0	74
		Diagram complexity	5.82	0	8.94	0	6.54
		Problem-solving complexity	3.91	0	5.82	0	4.35
		Cognitive complexity	9.74	0	14.76	0	10.89
May	Low	Number of tasks	49	33	10	2	94
		Diagram complexity	4.31	6.89	7.03	10.00	5.63
		Problem-solving complexity	4.82	4.76	5.26	6.50	4.88
		Cognitive complexity	9.12	11.65	12.29	16.50	10.50
Wen	Low	Number of tasks	49	33	10	0	92
		Diagram complexity	4.31	6.89	7.03	0	5.53
		Problem-solving complexity	4.82	4.76	5.26	0	4.85
		Cognitive complexity	9.12	11.65	12.29	0	10.37
Average		Number of tasks	296	172	208	9	685
		Diagram complexity	4.65	6.09	8.31	10.63	6.52
		Problem-solving complexity	4.51	4.61	6.78	7.06	5.39
		Cognitive complexity	9.15	10.70	15.10	17.69	11.90

Teacher Jyu indicated that she and her school colleagues write supplementary materials themselves and use them for classroom teaching. They use supplementary materials to scaffold students in building up new mathematical concepts. For the use of supplementary materials, Teacher Wen said

> ….We often use textbooks to teach our students as our students' mathematics is not very good….However, sometimes we select more challenging tasks from supplementary materials and discuss the tasks with our students….Even our students do not perform mathematics very well, they can learn from practicing those tasks from the supplementary materials (Transcript of interview data, 20190810)

Teacher Wen expressed a different way of using supplementary materials. She thinks textbooks are the appropriate instructional materials that fit her students' mathematical competence. Thus, Teacher Wen often teaches students mainly based on textbooks. Concerning the supplementary materials, she thinks they can provide her students with more opportunities to practice mathematics. In this regard, she collects tasks from supplementary materials to challenge her students. Different ways of using supplementary materials also influence the design of tasks concerning cognitive complexity. If the materials are used to help students build up mathematical concepts, the tasks included in the materials cannot be too cognitively demanding. If the materials are used to create more opportunities to practice mathematics, the tasks' cognitive complexity will increase.

The cognitive complexity of tests and tasks created by the teachers was much higher than those in textbooks and supplementary materials (cognitive complexity in tests: 15.10; cognitive complexity in tasks created by teachers: 17.69). The data shows that teachers intended to collect more cognitively complex tasks for formative and summative assessment purposes. They were also inclined to use very cognitively demanding tasks created by themselves during classroom teaching. The cognitive complexity of the tasks situated in the tests for Teacher Jyu was 15.89, for Teacher Ing was 17.45, for Teacher Sheng was 14.43, for Teacher Yao was 14.76, and for both Teacher May and Teacher Wen was 12.29. This finding reveals that the better the student's mathematics performance, the higher the cognitive complexity of the tasks in tests the teachers collected. Regarding the cognitive complexity of tasks created by the teachers, it was also higher than that of tasks in textbooks and supplementary materials (19.67 for Teacher Jyu, 42 for Teacher Ing, 16.5 for Teacher Sheng, and 16.5 for Teacher May). This finding suggests that the teachers tended to create more cognitively complex tasks during their classroom teaching.

23.5 Discussion

As Taiwanese students consistently perform at the top in cross-national mathematics assessments, this study investigated how Taiwanese mathematics teachers collect mathematical instructional tasks for their students. In particular, we examined if the mathematics performance of schools influences teachers in collecting tasks for their students. Based on the cognitive-complexity framework developed by Hsu and

Silver (2014), we analyzed six Taiwanese mathematics teachers who represented schools with different levels of mathematics performance.

The empirical analysis revealed that Taiwanese mathematics teachers tended to collect geometry tasks that entailed diagram complexity and problem-solving complexity, no matter the level of mathematics performance at the school where they taught. The diagrams accompanying geometry tasks are made complex, so they may not look like the reference diagrams accompanying geometric properties. The complex diagrams may consequently cause visual obstacles in identifying geometric properties that can be used to generate a solution to a geometry task. The geometry tasks collected by Taiwanese mathematics teachers also tended to require multiple reasoning steps and multiple geometric properties for a solution. Such tasks may also require the cognitive work of performing diagram transformations to identify the geometric properties embedded in the task diagrams successfully. The cognitive work required of solving various cognitive-complexity and non-routine tasks may subsequently equip Taiwanese students with abilities to attack high-level problems found in cross-national mathematics assessments (e.g., PISA and TIMSS).

The analysis also showed that the mathematics performance in the schools where the teachers taught did influence their collection of geometry tasks. The better the mathematics performance of the school, the higher the cognitive complexity of the geometry tasks the teacher collected. In addition, the cognitive complexity of tasks collected from non-textbook sources was higher than those from textbooks. Tasks situated in tests and those created by the teachers possessed the most increased cognitive complexity compared to textbooks and supplementary materials. Hsu and Silver (2014) reported a case study of a Taiwanese mathematics teacher. They indicated a tendency to include multiple sources of instructional/curricular materials with high cognitive-complexity tasks for classroom teaching. This study further confirmed this tendency by examining six Taiwanese mathematics teachers who taught students with different levels of mathematics performance.

Although the tasks situated in multiple instructional/curricular materials entail cognitive complexity, Taiwanese teachers consider students' mathematics performance when collecting tasks for classroom teaching. This finding implies a cultural script (Stigler & Hiebert, 1998) for teaching in East Asian countries, as teachers tend to increase the cognitive complexity as much as they can through the collection of tasks. Meanwhile, they also have to consider students' mathematics competence when collecting the tasks. The finding also brings several follow-up research questions. For example, researchers have indicated that challenging students by maintaining or increasing the cognitive complexity of tasks is vital for high-quality instruction (Leikin, 2009; Stein & Lane, 1996). In this regard, it is important to know how Taiwanese teachers manage to teach with those cognitive complexity tasks, especially when teaching in Taiwan is often described as teacher-centered (Lin & Tsao, 1999). Researchers from other countries may also expect to know the keys to determining the high quality of student learning outcomes. Can it be the case that collecting the cognitive complexity of tasks for classroom teaching ensures the high quality of student learning outcomes? Those questions require further investigations.

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Chapter 24
Problem Sets in School Textbooks: Examples from the United States

Alexander Karp

24.1 Introduction

The mathematical challenges that schoolchildren encounter most frequently are probably problems from textbooks, if only because it is with them that schoolchildren in most cases have to deal. Some authors of textbooks even include "challenge" subheadings to show the high level at which they are conducting instruction (meanwhile, the really challenging problems might turn out to be not the ones thus highlighted). Although textbook authors do usually try to make their textbooks a little easier, the word "challenge," and even more so the words "challenging education," have become in a way commendatory. This, of course, has not always been the case. Behind the change in attitude toward the words is the desire (even if often it is no more than rhetorical) to make the student an active participant in the process of instruction and to teach through problem solving. Let us repeat: this has not always been the case. Problems for students to solve on their own, even when they have appeared in textbooks, have played the most varied roles and consequently have been organized in different ways. By studying the history of problem sets in textbooks, we can better understand both how the process of mathematics teaching has developed, including teaching through problems, and how and to what degree students have been offered difficult and challenging assignments.

This article in some sense represents a continuation of the article Karp (2015), which is devoted to problem sets in old Russian mathematics textbooks. Here, as there, the discussion will mainly focus on problem sets from school textbooks. The process of problem solving in schools has been studied many times (we might mention, for example, the now classic work by Schoenfeld, 1985, in which not a few pages are devoted to what takes place in schools). On the other hand, the study of

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textbooks is gradually growing into a separate area of research, with its own conferences and published collections of articles (see, for example, Schubring et al., 2018). Strange as it may seem, problem sets from textbooks have not been studied sufficiently, in our opinion (for example, Donoghue, 2003 or Michailovicz & Howard, 2003 write about them, but their main focus is drawn to other aspects; one of the few exceptions is the paper by Ponte, 2014, devoted to Portuguese textbooks). Meanwhile, the principles on which they are constructed – whether recognized by the authors or applied unconsciously – constitute an important characteristic of the process of mathematics instruction as it actually exists in practice.

As a rule, in the process of instruction, schoolchildren usually encounter not separate problems, but problem sets (although, to be sure, sometimes the principle on which such sets are constructed consists precisely in the absence of any strategy). Mathematics educators who are concerned with what is being done in schools must inevitably think about how problem sets should be constructed and used, and how future teachers are taught to do this. As a step in this direction, we believe it is important to investigate how problem sets are actually constructed in practice, how their construction has changed in history, and what theoretical possibilities for their construction exist.

One can talk as long as one likes about the importance of reasoning and proving, creative or critical thinking, the ability to generalize and apply, and other fine things, but if a student never has to do this in practice, then all of these beautiful pronouncements remain empty and, if anything, merely confuse the working teacher, which is in fact what we see happening in some measure with the use of the expression “problem solving” – everyone knows that this is something good, but *what* it is exactly is understood in utterly different fashions.

It should be said at once that the author of this article seeks as much as possible to eschew an evaluative approach: the article’s purpose is not to say, “Would you look at what kind of garbage is being published!” but to show what processes are taking place – the article is largely historical in nature (which does not mean, however, that the author sees anything objectionable in methodological criticism aimed specifically at reviewing today’s production and analyzing what works, in the reviewer’s opinion, and what doesn’t. Methodological criticism must exist just as theatrical or literary criticism exist, helping to understand and to assess what is being put before us, which are necessarily more subjective than, say, scholarly studies in philology or theater – although the borderline here is not always rigid). This remark necessitates a methodological discussion, which will be undertaken below.

We should say at once that ideally the analysis of problem sets must be carried out with due consideration for context – general social-historical, cultural, and educational – if only because a textbook may be used in very many different ways. The present article must be considered to a certain degree preliminary because the analysis of textbooks in it is accompanied by the analysis of other sources (for example, those that shed light on how exactly textbooks were used in schools) only to a very limited degree. At the same time, a set of problems already by itself creates a certain context – and from this stems the key difference between analyzing an individual problem and analyzing a problem set.

For example, a problem's level of difficulty, understood not in terms of the number of operations that must be performed to solve it or some similar characteristic, but as a psychological or statistical characteristic (roughly speaking, how many students can solve the problem under given conditions), is practically impossible to determine based on a single isolated problem that is substantive to any degree – we simply do not know the conditions: it is one thing if the problem is simply given as is, but quite another if it was prefaced by the teacher's explanations or a series of preparatory problems (therefore, for example, assessments of the level of difficulty of problems on exams can hardly be made without additional sources of information). Discussing problem sets from textbooks is in this respect safer: the reader, as well as the researcher, sees, if not everything that the student using this textbook sees, then still a great part of it.

Let us repeat one more time that, naturally, there are differences between the textbook and what takes place in class. A set of problems in a textbook usually does not prescribe the order in which they should be solved rigidly (although there are techniques to emphasize it, for example, by regarding problems not in isolation from one another, but as items within the same problem). The sequence of the problems to be solved is usually prescribed by the teacher and different ways of sequencing can lead to different results. An effectively sequenced set of problems can make it easier to solve them by first solving a simple version of the problem and then moving on to a more difficult version. The effect achieved by the teacher who has effectively sequenced the problems to be solved can, however, be far more powerful and complex than that (Karp, 2004; see also the introduction to this section of the book). The role of emotions in mathematics education is now often discussed (Schukajlow et al., 2017), but often it somehow turns out that students' positive emotions can be triggered only by nonmathematical details – this, of course, is by no means the case. The sequencing of problems can be based on different principles and produce different effects – including emotional ones.

Textbook problems can be used in different ways; therefore, let us repeat, their analysis here is by definition incomplete. It is important, however, to ask whether the textbook offers problems in a sequence that is meaningful, whether such a sequence can be constructed on the basis of what is given in the textbook, and whether constructing such a sequence is easy.

This article begins with a brief historical note, proceeds to a discussion of methodology, goes on to analyze several problem sets from textbooks, and in conclusion relates the observations that have been made to one another and to certain theoretical and practical problems and theses.

24.2 On Problem Solving in Schools: A Historical Observation

Contrary to conventional wisdom, problem solving in schools is a relatively recent phenomenon. The frequently cited stories about various tournaments between solvers (the most popular of which seem to be about Fibonacci, who defeated all rivals in front of Emperor Frederick II) have nothing to do with this matter – we are talking specifically about schools. Students in schools usually solved a few problems, and in saying this, we use the word "problems" to mean not only assignments that required a nonstandard approach, but all assignments. This was noted explicitly in a report by the American subcommission of ICMI:

> In the textbooks on algebra 50 years ago much more stress was placed on logical exposition than on the solution of problems. The development of arithmetic, as followed in the textbooks of the elementary school, was faithfully imitated in algebra, and various "operations" (some, like the square and cube root of polynomials, having no conceivable use, and other mere pedantic elaborations of methods that in simpler form were well worthwhile) were laboriously discussed and exploited before the use of equations in discussing problems was entered upon. (Committee III and IV, 1911, p. 23)

The United States was no exception. For example, many surviving materials demonstrate how mathematics was taught to future Russian tsars and grand dukes (Karp, 2020) – at the beginning of the nineteenth century, they did solve problems, although not very many, but this was clearly seen as a direct continuation or even part of theoretical knowledge. Certain assertions in geometry were called theorems; others, which required students to do something – usually, to construct something – were called problems. In algebra, that which was solved usually served as an example of a learned rule or algorithm. A story has survived about Grand Duke Konstantin (the son of Nicholas I) taking an exam in mathematics: "The questions, as always, were posed by the Sovereign, as tests of ingenuity. In algebra, there were two: one, a first-degree equation; the other, a quadratic equation. In geometry, the construction of a regular 15-gon on a given line." (Golovnin, n.d., p. 22). Konstantin (1827–1892) was undoubtedly a remarkable individual; nonetheless, there can be no doubt that he was not required to invent an algorithm for constructing a regular 15-gon on the exam, nor does solving linear and quadratic equations offer very many opportunities for displays of ingenuity.

"Good" teaching (that is, teaching in a school of a high level) consisted precisely in the "pedantic elaboration" of various propositions (which were, to be sure, not always very precise by the standards of today's higher mathematics).

24.3 Analysis of Problem Sets: Certain Methodological Considerations

We mentioned above that sometimes the principle on which a problem set is built is precisely the absence of any principle. In reality, this is, of course, a simplification – the most superficial arranger of a problem set must make some choices in any event, in the first place by determining how many problems should be included. In fact, this simple and almost indisputable characteristic of a problem set already carries considerable information – in certain textbooks, there are almost no problems – they were unnecessary, in the writer's opinion (naturally, such cases must be distinguished from situations in which a textbook is accompanied by a separate problem book). The goal of instruction was seen to consist of something else, and a simply determinable number helps us to understand this.

There are certain other characteristics that it is desirable to determine in analyzing problem sets; for example, thematic distribution within a problem set: how many of the problems given involve the application of a formula directly, and how many of them involve applying it in one or another version – independently of what the arranger of the problem set planned to achieve (or of whether anything was planned at all), this is an important characteristic of the activity expected of the students.

When talking about modern textbooks, it is natural to consider a breakdown of their use of different forms, for example, how many multiple choice tasks, short answer tasks, and essay questions they contain (in old textbooks, such diversity was practically unknown). Moreover, it is natural to consider the type of activity suggested by a problem. Of course, to make judgements about this on the basis of a keyword in a problem – solve, draw, check, prove, compare, etc. – is somewhat of a simplification; nonetheless, analyzing problems from this (and even more so, from a more detail-oriented) perspective tells us about the conceptions of the learning process espoused by the problem set's arranger (and let us repeat one more time: whether these conceptions were conscious or not).

But such data alone are not enough. We are confronted with having to analyze a text, and just as in the case of a literary text – in particular, a poetic text – not everything here is expressed in numbers. As we have already noted, in the history of mathematics education, as in other historical disciplines, often (and even usually) use is made of what is known as the historical-philological method, which is based on the attentive reading of texts and their comparison and juxtaposition with one another. With one additional feature, however, the analyzed texts can be mathematical ones (Karp, 2014). In different cases, the historical-philological method may be applied in different ways. (Note that proposals to use various elements of the methodology used in the philological disciplines have been made for a long time – see Schubring, 1987; Karp, 2004 – just as methodological parallels between pedagogical studies and art have also been drawn – see Lawrence-Lightfoot, 1997).

We will confine ourselves to citing a researcher whose name is recognized across different fields: Vygotsky (1971) analyzed classic Russian works of literature, in the vein of the studies of the so-called Formalist School, which was contemporary to him. Comparing the content and meaning of a work that are seemingly straightforwardly and openly communicated by its author, on the one hand, and its structure, the order in which events in the plot are presented, and the literary techniques it uses (first and foremost, verbal peculiarities), on the other, he showed that in reality, the author tells us far more than he or she seemingly promised (or sometimes something completely different). It is precisely the structure, organization, and language of a work of art that make it truly deep and substantive.

Of course, we would hardly seek for hidden depths on every page of every problem book – or claim that something will be inevitably revealed to those who solve problem No. 23 immediately after solving problem No. 22 – but the sequencing of problems and the structure of a problem set can carry a meaning that cannot be reduced to the meaning and content of each of the individual problems in it.

Here, the question arises: how did Vygotsky (and many other researchers before and after him) achieve what Schoenfeld (2007) calls *trustworthiness,* and what is it exactly that makes a work scientific? In effect, the practice of these researchers has been briefly to paraphrase the text, in a specially organized fashion, with quotations and examples, all intended to underscore those distinctive features of the text to which the researcher wished to direct the reader's attention.

Just as any other method, this method can be criticized for a possible lack of objectivity – it is easy to select some text, and pick and choose quotes from it, to completely distort its meaning if one wishes to do so. Such distortions, however, can be disproven with other quotes, and the main thing that must be noted is that the described "philological" method, contrary to conceptions formed under the influence of generally justified respect for quantitative methods, may be every bit as fruitful as any "mathematical" (let's call it that) method, in which some mathematical model is somehow constructed.[1]

In what follows, we will describe the content of each analyzed problem set and offer examples.

Let us make a few more observations – in analyzing, one must distinguish between identical and nonidentical problems. The decision here depends on the reader's viewpoint. Unquestionably, for a mathematician the equations.

$$x^2 - 4x + 3 = 0 \text{ and } x^2 + 3 = 4x.$$

are identical.

That is not the case for the schoolchild who is just beginning to study quadratic equations. We strive to pick up on such differences.

In conclusion, it must be noted that the selection of textbooks which will be discussed, and the sections in them that will be analyzed, is quite arbitrary – those textbooks were selected which were widely used in the United States, while the section chosen – in cases where there were different sections to choose from – was always the same, quadratic equations, unquestionably an important section. However, there were quite many widely used textbooks, and there were quite many important sections of the course as well. It should be pointed out that the author of

[1] Here, the author of this article cannot help recalling a methodological debate that raged in the 1980s in Russia, when public opinion was drawn to the alleged discovery thirty years earlier of the tenth chapter of Pushkin's "Eugene Onegin", which had been considered lost. Two respected literary critics, deciding to use modern methods, compared the distribution of various indicators in the "discovered" manuscript and in the main text, and came to the conclusion that since the distributions differed, the manuscript was a fake. The critic Benedikt Sarnov (1987) wrote with not a little irony that this had hardly been necessary, since one could see immediately that the poetry in the manuscript was absolutely worthless, and that Pushkin could not have written this way. But "objective" proofs, which seemingly do not require human involvement, obviating the need to go into the substance of an issue, or even to read at all, have their attractiveness. The debate had a sequel when in 2006 a certain mathematician presented a new model according to which the text of the manuscript did not differ from Pushkin's (Esipov, 2010). Its worthlessness, however, did not disappear. Reading and understanding proved impossible to do without.

this paper assumes a certain homogeneity both individually within each textbook and in general among the totality of textbooks circulating at any one time. Differences exist, of course, but radical differences of any kind (some of the textbooks were printed in color, the rest in black-and-white, or some of the textbooks contain many problems, the rest very few, and so on) seem not very likely, especially if these differences were not mentioned in the press or other literature. At the same time, it needs to be said that the author does not aspire to any all-encompassing generalizations: the aim of this paper is not to formulate general assertions concerning all textbooks of a certain time, but merely to note what happened at different times; whether it happened always, and why it happened, are important questions, which the author hopes to answer in other studies.

24.4 Robinson's Geometry Textbook

Horatio Nelson Robinson (1806–1867) was a prominent writer whose textbooks were widely used for a long time. We will focus on a book with the long title – as was customary at the time – *Elements of Geometry and Plane and Spherical Trigonometry with Numerous Practical Problems* (Robinson, 1867). In his preface, the author again boasts that his book contains a "full collection of carefully selected Practical Problems" "given to exercise the powers and test the proficiency of the pupil." The word "problem" is found several times in the table of contents of the geometrical sections – it appears in the title "Book IV: Problems in the construction of figures in plane geometry," as well as in Book V and Book VII. Book IV provides solutions to a number of problems (the vast majority of them construction problems), beginning with constructing a perpendicular bisector and the bisector of an angle. The style of exposition here is practically identical to that of the other, "theoretical" sections. In the two other cases, judging by their subheadings, readers are offered practical problems specifically, just as promised – in Book V, in plane geometry; and in Book VII, in solid geometry. We will focus on the problems in plane geometry.

Thirty-nine problems are offered in all. Let us note at once that the word "practical" merely means that students will practice, or as the author writes, that these problems will "exercise students' power": one can detect in them no particular connection with real-world questions – they are ordinary geometrical problems about triangles, circles, trapezoids, and so on. From today's point of view, for all of plane geometry, the number of problems is very small, but as has already been said, the gist of a subject was not seen to consist in solving problems. From today's point of view, other criticisms can be voiced as well – for example, in problem 16 students are asked to find the area of a triangle with a given base and two adjacent angles of 80^0 and 70^0. In the answer, however, the author gives the sides of the constructed triangle, further noting that they cannot be determined exactly without using trigonometry, and trigonometry has not been covered yet, for which reason it is necessary to proceed by approximating – "we must be content with the approximate solutions obtained by the constructions and measurements" (p. 146). What we are interested in, however, is not this, but how the given problem set is organized.

The author himself says nothing about this. And he does so, even though certain problems are accompanied by solutions (so that it is possible to "exercise students' power" only if the students do not have the textbook) and even methodological comments of sorts, which confirms that the book often functioned not merely as a textbook, but at least to some extent also as a teacher's manual. It is not even entirely clear how the author envisioned the use of the problems that were offered – as a unified text, or one by one, as the corresponding topics were covered. The former seems far more likely – as attested to by the remarks with which the author opens the section, observing that what has been covered has been covered, and certain other topics still lie ahead, but before proceeding further, it is necessary to do some problem solving. However, it is still impossible altogether to rule out the possibility that students were given certain problems one by one as they progressed through the book.

It can be seen that certain problems directly support the course presented in the book – for example, in problem 14 students are shown a triangle with three given sides and asked to find the lengths of the segments into which the bisector of one of the angles will divide the opposite side. So that the reader (teacher?) should have no doubt as to how this problem should be solved, it is immediately accompanied by the indication, "see Theorem 24, Book II" (p, 145) – and this theorem answers precisely the question posed, in a general way. Far from all, problems are connected with the covered material in such a direct way, however, and conversely, by no means every theorem comes accompanied with a numerical example.

We can see several mini-sets in which problems are interconnected, for example, by focusing on the same geometrical situation – thus, problems 4, 5, and 6 are devoted to two parallel lines, the distances between them, and related questions. However, problem 15, in which three parallel lines are examined, and for which the ideas examined in problems 4–6 are useful, stands alone. This is not the only case. In problem 2, students are given a right triangle with a 30^0 angle and a shortest side of length 12 and asked to find the length of the hypotenuse. In problem 22, they are given a right triangle with a leg whose length is 320 and an adjacent angle of 60°, and asked to find the lengths of the remaining two sides (the Pythagorean Theorem was used in many problems at the beginning of the set). Practically identical problems have been placed at different ends.

From a didactic point of view, the problem set, of course, is not diverse. There are no problems aimed at reinforcement – the idea that a student who couldn't do something the first time might be able to do it the second or third time around is not made use of in any way, and there is no evidence that it occurred to the author. The author distinguishes between difficult and easy problems (for the difficult ones, he provides solutions), but he does not prepare students for solving the difficult ones in any way – there is no movement from the simple to the complicated. The set begins with a problem that indicates that the base of an isosceles triangle has a length of 6, while the angle opposite the base is equal to 60^0; students are then asked to find the sides' lengths. The problem is not difficult, although its solution does involve several steps. But, for example, problem 33, in which students are given an isosceles triangle with sides whose lengths are 20, 20 and 12, and asked to find the length of the altitude drawn to the base, is followed by problem 34, in which students are asked to construct a right triangle, given the length of its hypotenuse and the difference between the lengths of its legs, that is, something noticeably more difficult and not prepared in any way.

We have noted similar things in Russian textbooks also. Thus, Davidov's geometry textbook (1864) contained far more problems, but, for example, problems accompanying the section on the areas of polygons began with a problem in which students were asked to find the locus of the vertices of all triangles that are equal in area and have a common base – a problem that one could hardly expect to be solved by a student who did not possess that which Schoenfeld (1985) has referred to as "resources." How were these problems solved, then? We possess a sufficiently large amount of information about Russian gymnasium students of that time to be able to assert that at least one way was to solve problems at home with a tutor, and then to commit the solution to memory (Karp, 2018). From a certain point on, collections of solutions to problems from Davidov's textbooks began to be sold. The social background and consequently the habits and demeanor of students who were taught using Robinson's text and students at Russian gymnasia differed; identifying historical evidence indicating how problems were solved in American schools at that time appears to us an interesting question for research.

In any case, we can assert that an analysis of Robinson's text confirms that little attention was paid to problem solving at that time, and that problem sets were characterized by their didactic and methodological poverty; at the same time, the book contains problems that are grouped on the basis of some connection between them – for example, problems that pertain to similar geometrical situations. To what extent this was done deliberately, however, is not clear.

24.5 Algebra by Robinson

Both in terms of the history of its formation as a school subject, and in terms of its nature, algebra is, of course, different from geometry. Therefore, it is useful to examine an algebra textbook from that time as well – we will choose a textbook by the same Robinson for this purpose, *A Traditional and Practical Treatise on Algebra* (1848). It contains more problems to solve independently than the geometry textbook examined above; the textbook is divided into "Sections," and each of them contains problems. We will focus briefly on Chapter 1 of Section IV *Quadratic equations*, which is devoted to quadratic equations in one variable.

Here, we will also confine ourselves to two sets of "Examples for Practice" or simply "Examples." In addition to them, there are several other sets: a set devoted not to solving quadratic equations, but to the concept of the perfect square; a set devoted to equations of higher powers, which are solved by using quadratic equations; a set in which the author advocates solving certain problems with numerical coefficients by replacing them with letter coefficients, which allegedly makes calculations easier; and finally, a set that represents a kind of conclusion. Although all of this, along with the author's mathematical inaccuracies, is of interest, we will not dwell on it here (although it should be noted that some of these sets are evidently preparatory to some extent, while others conversely offer something like the application of what will be discussed later on).

The first set we will examine contains 16 problems. All of them are more or less of the same type: students are asked to solve a quadratic equation. The first 10 are devoted to equations with a leading coefficient of 1, and the last three of these have irrational roots. The remaining six problems have different coefficients, which the author deliberately emphasizes, and by analyzing problem No. 11, demonstrates how to reduce such equations to an equation with a leading coefficient of 1 (it is noteworthy that this is done not by means of simple division, but by means of a substitution, which the author deliberately thanks a certain professor for pointing out – the author avoids fractions). Problem No. 16 is again analyzed since in it the second coefficient is odd – students are advised to multiply the equation by two.

The second set contains 17 problems, the first of which is the following equation:

$$(x+12)^{1/2}+(x+12)^{1/4}=6.$$

In the preceding theoretical paragraph, the author developed the idea that if one exponent is twice as large as another while their bases are equal, then it is possible to transform the equation into a quadratic equation. This problem is solved following this model, as are the subsequent 11 problems. Probably the most difficult of them is problem No. 12, which requires students to solve the equation

$$x^2-2x+6\left(x^2-2x+5\right)^2=11,$$

in which the technique is slightly disguised. It is noteworthy that No. 13 is far easier – here, students must solve the equation

$$\frac{x^2}{361}-\frac{12x}{19}=-32,$$

The author, however, apparently does not notice that this equation can be solved without any special stratagems, without introducing a new variable $y = x/19$ (or else does notice it, but fears computational difficulties). Moreover, he makes the following observation: "If much difficulty is found in resolving this 13th example, the pupil can observe the 9th example" (p. 167). This ninth equation is as follows:

$$x^{6/5}+x^{3/5}=756,$$

and how exactly it is supposed to help (apart from increasing students' experience with working with comparatively large numbers) is not entirely clear. In the remaining cases, such observations are not made, but this observation serves as proof of the fact that the author recognized the importance of sequencing in solving problems. Nos. 14–16 do not resemble the problems that precede them at all; for example, in No. 14 students are offered the following equation:

$$81x^2 + 17 + \frac{1}{x^2} = 99.$$

The author gives a hint here, pointing out that the first and third terms of this expression are squares and referring students to the section on perfect squares. Indeed,

$$81x^2 + 18 + \frac{1}{x^2} = \left(9x + \frac{1}{x}\right)^2,$$

and using this fact, it is not difficult to solve the equation. But it is difficult to understand why these problems are offered in this set rather than the preceding one. As for the last problem, No. 17, this is an equation,

$$\frac{4x^2}{49} + \frac{8x}{21} = 6\frac{2}{3},$$

that may be converted into the form:

$$\left(\frac{2x}{7} + \frac{2}{3}\right)^2 = \frac{64}{9},$$

but which can also be solved without any special tricks.

Summing up, we would say that in the algebra textbook the connection between problems is felt by the author more strongly than in the geometry textbook. But this connection usually amounts to the author offering several problems in a row that focus on the same rule, even if he makes some attempt to group together problems that resemble each other in other ways. Also, the number of problems given is clearly too small, if one considers their use from today's perspective.

24.6 Textbook by William Hart

Let us examine a more recent textbook by Hart (1934). This is a textbook in algebra that is radically different from the one examined above in terms of the number of problems in it – there are very many of them. As the author himself explains in his introduction, "The practice examples conform to the principle that learners profit more from doing successfully many easy examples than from relatively futile efforts to solve complicated examples. Each topic is accompanied by an unusually large number of easy examples" (p. V). We will confine ourselves to one section, XIV Quadratics. This section includes theoretical subsections 263 through 281 (which contain many examples), numerous exercises, on which we will focus, as well as sections entitled "Chapter Mastery Test" and "Written Review," which also offer problem sets. It is important to note at once that both in the theoretical subsections and in the problems, the author uses the signs *X* and *Y* to distinguish certain sections, with the recommendation that "the study of such material be required only of the abler pupils" (p. V).

The material is organized as follows. The author begins with the theoretical subsections, which present quadratic equations and incomplete quadratic equations, followed by subsection 265, "Solving an incomplete quadratic equation," which provides examples of solving such equations, and as a conclusion, he formulates a rule about what needs to be done to solve such an equation. This is followed by Exercise 201, a set of problems devoted precisely to these equations. It contains 48 problems. Only the first 18 may be considered repetitions of what was required in the previously examined examples (and even this is a stretch – problem 2 already asks students to solve the equation

$$y^2 - 4 = 21,$$

which is not actually an incomplete quadratic equation, but one that can be brought into the standard form). In problems 19 through 32, students must convert equations to incomplete quadratic equations in more and more intricate and technically complicated ways (this should not be taken to mean that each successive problem is invariably more difficult than the preceding one in this respect, but the tendency is obvious – problem 19 asks students to solve the equation

$$5a^2 - 125 = 3a^2 - 27,$$

while problem 32 asks them to solve the equation

$$\frac{y+3}{y-3} + \frac{y-3}{y+3} = \frac{17}{4},$$

which is obviously more technically complicated, if only in terms of the number of operations that must be carried out).

This is followed by problems 33–41, in which students are asked to solve equations with letter coefficients, for example, No. 41:

$$\frac{ax^2}{b} - \frac{c}{b} = 1.$$

Lastly, problems 42–48 are equations with letter coefficients that represent relationships with which the students are familiar, for example, No. 43:

$$A = \pi r^2.$$

The next subsection is devoted to the Pythagorean Theorem. It is followed by Exercise 202, which contains 10 problems – the first five are purely algebraic. For example, No. 5:

Solve the formula [$a^2 + b^2 = c^2$] for c in terms of a and b.

The geometric problems that follow are more or less of the same type, for example, problem No. 10, the last and most difficult of them because magnitudes are given in letter form: "If the equal sides of an isosceles triangle are each m in long, and the base is 2n in long, find the length of the altitude."

Then follows a subsection about solving "a complete quadratic graphically," which introduces the parabola (constructed based on points), and then in Exercise 203 offers 12 problems that in effect repeat the example analyzed in the theoretical subsection, but in which different parabolas must be constructed.

The next subsection is entitled "Solution by completing the square" (in the case of an equation with a leading coefficient equal to 1). Here, we find preparatory exercises that help to elucidate the idea, after which a numerical example is employed to show how to solve an equation by using the explained method. Exercise 204 is a set of 30 problems, the first 19 of which are equations with integer roots, with No. 11 being practically fully solved in the text of the textbook. Problems No. 20–30 contain equations with irrational roots which must be found approximately. Problem No. 20 is again fully solved.

Then the text examines the case of a quadratic equation with a leading coefficient not equal to 1, and the set Exercise 205 follows, which repeats the structure of the set Exercise 204, but now for such equations. The theoretical subsection here is marked with an *X* – indicating heightened difficulty – while the set of exercises is not marked with such a sign. Finally, the formula for the roots of a quadratic equation is derived, about which the students are told: "take three minutes now and memorize this formula" (p. 345); and then follow two sets, Exercise 206 and Exercise 207 (both they and the corresponding theoretical paragraphs are marked with an *X*), in which students are required to solve equations using the formula and by factoring, if possible. It is easy to distinguish groups of problems: those in standard form and easily solvable by factoring; those with expressions on both sides of the equation; those with fractions; those with irrational roots, which must be solved approximately; those with quite complicated expressions that must be transformed in order to convert them into standard form. For example, these exercises are concluded by No. 34:

$$\frac{3r-1}{7-r} - \frac{5-4r}{2r+1} = 3.$$

Exercise 208 is devoted to word problems. The first three problems repeat almost verbatim the example analyzed in the theoretical part. Then come certain changes – in the theoretical part, a problem was analyzed in which students were asked to find two consecutive integers whose product was equal to 20. No. 1 is the same problem, but the product is equal to 72; No. 3 is about the sum of the squares of two consecutive integers; No. 8 gives the sum and product. Problems Nos. 10–14 have a geometrical content – for example, students are given the perimeter and area of a rectangle and asked to find its dimensions. Then come various problems both about numbers and geometrical questions, which are slightly more difficult than the ones previously analyzed. Exercise 209 is once again devoted to word problems, but now students are advised in certain cases to "draw a figure," and most importantly the whole set is marked with an *X*. And indeed, the problems here are somewhat more difficult – they also come in groups. A typical problem from the first group is No. 5: "A picture is 10 inches long and 5 inches wide. The area of the picture and its frame is 84 square inches. How wide is the frame?" Then come problems about motion – for example, No. 14: "An airplane flew 90 miles and returned in a total time of $2\frac{1}{8}$ h. The rate of the wind was 5 miles per hour. At what rate in calm air was the airplane flying?"

This section continues, and various other exercises are given, but we will stop our description here.

The difference from the previously analyzed textbooks consists not only in the quantity of problems but also in the clear understanding of the role of grouping and sequencing problems. The classic phrase that a problem can be reduced to the previous problem speaks specifically of the previous problem, not of one given 20 problems ago. The author proceeds in small and carefully thought out steps. Furthermore, the problems in Hart's textbooks are in general easier than many of those in Robinson's textbook – or more precisely, would have been easier if they had to be solved in isolation, but the problems in Robinson's textbook are not solved in isolation – the idea is presented in the theoretical section, and then in effect must be memorized by rote by being applied several times in a row. Hart's textbook contains relatively complicated problems, which students come to solve on their own by solving problems that precede them, and not simply by applying what was done in the theoretical section to a problem with different numerical values. But usually, the increase in the level of difficulty is technical – first, we apply the formula directly; next, we perform some algebraic operation, arrive at the standard form, and then apply the formula; and after that, we perform five algebraic operations, some of which are themselves not simple, and only at the very end apply the formula.

One can ask what spurred the authors of textbooks to organize problem sets better – and one can try to understand to what extent they were acting consciously, and to what extent they were, for example, copying other textbooks. It appears natural to think that a large role was played by the increase in the number of those taught and those teaching. It was precisely during the years when Hart's textbooks were being used, and those preceding them, that rapid growth in these numbers occurred – the textbook that won in competition with other textbooks was the one that taught students quadratic equations in a way that was more simple and effective, and hence it came about that more attention began to be paid to didactic principles.

The Progressive Era brought not only an increase in the number of high schools, but also a change in the understanding of the purposes and goals of mathematics education, including an increase in attention paid to the practical applications of mathematics, developing social efficacy, and so forth (Kilpatrick, 2009). In those parts of Hart's textbook which were analyzed above, the demand to make the course more practically oriented is difficult to discern – the word problems examined above were clearly formulated with other aims in mind. This does not mean, of course, that demands for a practical orientation exerted no influence on the teaching of mathematics in schools – they exerted such an influence if only because many students were left without a course in algebra – but even with all the will in the world (which was, of course, by no means necessarily shared by all authors of textbooks at that time), putting the exhortation to increase the practical orientation of the course into practice was far more difficult than merely formulating it.

On the other hand, what is clearly on display in the textbook is an individualization of the approach, which was also in line with the demand for social efficacy – teaching every intricacy to those who were "incapable" seemed unnecessary, and among other things, it was recommended not to insist that everyone learn to solve quadratic equations using the formula or to solve word problems of any degree of difficulty, which were undoubtedly identified and separated out by the experienced teacher.

The section from Hart's textbook analyzed above unquestionably contains more graphical and geometrical content than Robinson's algebra, but also not a great deal. In general, despite the quantity of problems, one does not see very much diversity among them: as we have seen, the absolute majority of them are "solve the equation" problems. Problems that require students to prove something, to verify something, to compare something, to invent something, and so on, are completely absent.

24.7 Textbook by Larson et al.

Let us look at problem sets about quadratic equations from a more modern textbook. The textbook by Larson et al. (2001) contains a chapter with the title "Quadratic Equations and Functions," which will be discussed below. Its first section is titled "Solving quadratic equations by finding square roots." This section provides the definitions of a root and of a quadratic equation and demonstrates how to solve the simplest quadratic equations. The authors provide an example that offers an algorithm for solving the following equation:

$$3x^2 - 48 = 0.$$

Numerous other examples are followed by a section with the title "Guided Practice," which practically repeats the definitions given previously and lists the basic problems that students must know how to solve (like the equation formulated above).

Then finally comes a section entitled “Practice and application,” which we will discuss in greater detail.

This section contains groups of problems unified under headings. The first four of these groups are devoted to square roots and expressions with them (each group contains eight or nine problems that are similar to one another). The next group is titled simply “Quadratic Equations”: there are 15 of them. There are differences among them: the equations involve different variables, they sometimes do and sometimes do not have a solution, sometimes they are given in standard form, sometimes certain operations are required in order to convert them to standard form. The group begins with the equation

$$x^2 = 36,$$

and ends with the equation

$$7x^2 - 63 = 0.$$

The next group contains nine problems and requires the use of a calculator for finding solutions to the nearest hundredths. The final equation is as follows:

$$5a^2 + 10 = 20.$$

The next group has the title “Critical Thinking” and contains three problems: write an equation of the form $x^2 = d$ so that it has one solution, two solutions, and no solutions.

Finally, the next seven headings (from one to six problems below each of them) are devoted to applying what has been learned – the authors give a quadratic formula that describes some real-world process and pose questions about this process (one of them has the subheading: “Challenge”; another: “Critical Thinking”). For example, already in the theoretical sections, the authors presented the so-called *failing object model*, according to which the height h of an object falling from height s at time t is equal to:

$$h = -16t^2 + s.$$

Subsequently, in both the examples analyzed and in the problems for independent work, the following question, for example, is discussed: when is the height $h = 0$, given that a value for s *i*s specified?

The sections that follow are devoted to radicals, the graph of a quadratic function, and solving quadratic equations by graphing. We will skip over them and proceed to section 9.5, “Solving quadratic equations by the quadratic formula.” This section gives the formula and offers examples of how it may be used to solve equations. The section “Practice and Applications” is organized in the same way as the section examined above. It begins with a subsection in which, in nine given equations, students must find the discriminant (this word, however, is introduced only in the next section – here, students are simply given the formula). There are certain technical differences among the problems – in some, the coefficients are fractions; in others, they are negative. The next set gives 12 quadratic equations in standard form – they must be solved using the formula. The differences among the problems again come from what kinds of coefficients they have. In the next set, which has nine problems, students must first convert equations into standard form and then solve them using the formula (all of them are polynomials, so all that is required is to move all terms over to the same side). The next group is devoted to finding "x-intercepts of the graph of the equation," that is, again solving a quadratic equation (it is not stipulated, however, that this must be done using the formula). In the problems of the next group, students are given a choice between solving simply "by square roots" or by "using the quadratic formula" (unfortunately, the question of whether it is possible to use the quadratic formula in those cases where solutions can be found by square roots is not raised). Next, students are given several more problems oriented around applications, that is, problems in which students are asked to solve quadratic equations using given quadratic functions that model various real-world processes. The last problem (which appears under the heading "Challenge") is of interest. It contains two parts – in the first, entitled "Visual thinking," students are asked to use a graph to find the equation for the axis of symmetry of a quadratic function and notice that it is "halfway between the two x-intercepts." In the second part, entitled "Writing," students are asked to make sure that their answer to the preceding part is correct, using the formula for the roots of a quadratic equation.

24.8 Discussion

An obvious distinction of the newer textbook consists in the fact that students are offered problems that are far more simple technically than those in Hart's textbook, let alone Robinson's textbook. The authors see no need for technical intricacy. Hart clearly believed that it was pointless to make students memorize several types of difficult problems, which it would have been impossible to teach them to solve in a meaningful and independent way in any case (let us recall his words about the "relatively futile efforts to solve complicated examples"). But his textbook retained many-step problems, which even students who were not considered very capable could, in Hart's opinion, be led up to solving. In the newer textbook, everything is simpler.

At the same time, connections between problems are undoubtedly acknowledged – the textbook includes material for reinforcing what has been learned, and as we have noted, there are certain differentiations within each group. On the other hand – without going into a discussion of how much technically difficult algebraic problems are needed in our computer age – we should note that what is disappearing (or at least noticeably decreasing) along with such problems is the incremental or many-step aspect of learning and reasoning in general. Broadly speaking, Robinson's textbooks (this is particularly evident in his geometry) taught students problems each time as a kind of isolated phenomenon. With Hart, we see a certain movement, even if it is limited to increasing technical difficulty. Hart clearly gave some thought to these matters – in his introduction, he writes about the "spiral organization" of his book, in which problems go back to ideas that had already come up, but now on a new level. The content is thus richer than simply the sum of the separate problems – there is, additionally, movement from problem to problem (or at least the possibility of achieving such movement in class or in teaching practice). In the newer textbook, the technical simplification of the problems as a whole has led to a simplification of the whole structure of the set as well.

Here, however, there are certain exceptions. We noted above that the textbook includes a group of problems, following the solution of a quadratic equation, that require students to find the x-intercept of a quadratic trinomial. Before us is the development of the examined problem, yet not in the direction of increasing technical difficulty, but in the direction of a kind of translation or transfer to a different object. The same thing, but in a different way – about different concepts. We have not come across such problems in other textbooks. In general, the role of graphs has noticeably increased, and therefore a space has been opened up for interactions between the purely formula-based and the graphical.

Let us also note the appearance in the newer textbook of several assignments in which students are asked not only to solve something, but also to provide examples, draw a conclusion based on a figure, and even engage in reasoning. This is another way in which this textbook differs from the older ones – although, to be sure, very few such problems are given, and in a number of cases they are given under the heading "Challenge," which hints that these problems must be assigned only in exceptional cases – and they are indeed difficult to prepare for by using the textbook.

Another obvious distinction of the textbook by Larson et al. (2001), which its authors themselves point out, is that it contains a large number of problems with content taken from the real world. These problems are, of course, altogether different from the classic ones about motion, areas, or finding numbers, which we mentioned when discussing the textbook by Hart. At the same time, they appear rather to advertise the importance of mathematics in the real world than to demonstrate this mathematics itself (in other places in the textbook, too, the authors never tire of repeating that mathematics is very necessary for people in various professions, with which, of course, one can only agree). The problems state that one or another mathematical model is being made use of, but the various models are discussed only later in a special section – in real life, the logic of reasoning is reversed. Although there are relatively many problems and they are devoted to a variety of objects, they are relatively unvarying in character.

Finally, let us touch on another aspect: Robinson did not worry about the individual approach in his textbooks – very few people used these textbooks in school, and the author clearly believed that if some of them failed to learn what they were taught, this was not anything to worry about. For Hart, differences in perception are important, and he methodically offers material for differentiation, emphasizing that he identifies the minimum necessary for everyone, and provides additional material for the gifted. For the more modern textbook, such an approach is unacceptable – if only due to the danger that the proclaimed minimum might become the maximum of what is possible for certain underserved groups of population. As a result, differentiation is provided for only by rare problems under the heading “Challenge,” plus recommendations to consult a website or some other source for additional problems.

24.9 Conclusion

As we stated at the very beginning, this article is preliminary in character: the examination of a greater number of American textbooks from different periods will help better to understand what was taking place in the country. The processes taking place in the country were connected with what was happening in the world, and therefore an analysis of textbooks from other countries from the same point of view would also be useful. In general, analyzing problem sets in textbooks is no less useful and informative than analyzing what might be called their presentation of theory. As has already been remarked, it is desirable to supplement the direct analysis of textbooks from this point of view with the analysis and collection of evidence indicating how problems from textbooks were solved in practice – in school, at home, with a private teacher, and so on. An understanding of the methodological changes taking place in the teaching of mathematics as part of broader social changes is precisely what we regard as the objective to be achieved, and it can be achieved only by combining the analysis of mathematical and methodological-mathematical texts with the analysis of all sorts of historical documents pertaining to everyday life.

The most recent of the textbooks examined by us above was published 20 years ago, that is, it, too, belongs to history. In the time that has passed since then, many textbooks have been published, and the textbook of these authors itself has gone through many changes, including changes touching on aspects that were discussed above. We should repeat that the present article is consciously historical. It would be interesting to juxtapose problem sets in textbooks published during the last decade. But historical analysis in itself compels us to think about the present period.

In the article Karp (2015), we reached the conclusion that approximately during the years 1880–1900, something like a methodological revolution took place in Russia – the system for and practice of working with problem sets in textbooks underwent a significant change. Among its causes, one might point to the growing scale of education, to foreign influences, and to the development of pedagogical thinking in general – further studies are unquestionably needed here. The present article confirms that such a methodological revolution occurred in the United States as well, even though no attempt is made here to assign to it a precise date.

A.R. Maizelis, an outstanding St. Petersburg teacher of mathematics, once jokingly told the author of this article that children somehow recognize the problems that were not in the famous problem book by Nikolay Rybkin (1861–1919) and reject them. A very high methodological art for working with problems was indeed attained at different periods in different countries, which made it possible to move very gradually and smoothly from problem to problem, increasing the technical as well as, sometimes, the conceptual level of difficulty and the substantive heft of each problem, while remaining accessible and comprehensible to children – developing them and coming up to the very limits of what is possible for them at each stage, but nonetheless staying within these limits.

It may be argued that we are living in a period of a new methodological revolution, or at least, a period when such a revolution is needed. The sensible and dynamic character of old textbooks is disappearing before our eyes, if only because many problems in today's computerized world are losing their value, just as many technical skills are losing theirs. However, intellectual skills, including the ability to move from one problem to another, are not going anywhere. This is what gives rise to the need to develop these skills under new conditions in schools and textbooks for the general population.

It may be said, based among other things on what has been said above about the types of problems found in the relatively new textbook, that today it is relatively widely recognized that school mathematics does not amount to simply "solve," "compute," and even "prove," but much else besides. This recognition must find expression in the problem sets in textbooks.

Simplicity in a textbook, which is worth striving for, does not preclude complexity in the organization of problem sets that are gripping to work on, in which each new success brings with it new feelings, giving meaning to new lines of reasoning. The experience of working with such sets will also help to develop teachers who will themselves take delight in mathematics, who will not treat it as medicine – useful, but unpleasant – and who will be able to transmit their feelings to students in class.

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Chapter 25
Exams in Russia as an Example of Problem Set Organization

Albina Marushina

25.1 Introduction

Probably historically the oldest denotation of the word “challenge” is “invitation to a contest or duel.” When this word is used today, what is usually meant is neither bloody combat nor even peaceful competition: what is usually meant is a competition against oneself, during which one must gather one’s strength and do what one has not had to do previously. It is quite correct to speak of challenges that a student might encounter (or not encounter) during an ordinary lesson. Yet it is still natural to devote special thought to situations in which students encounter a challenge in the original sense of the word when they are officially invited to overcome officially prescribed difficulties – to take exams. What is at stake here is not the actual difficulty of the problems that are given on exams, but the special role that these problems play, and not individually, but in their totality, as a whole set. In view of the fact that in many countries exams in mathematics are in one way or another taken practically by all students, it would be no mistake to say that today sets of exam problems are precisely “challenges for all,” although of course historically this has not always been the case and exams were for various reasons taken by far from all students. How and why specifically these challenges – exam problem sets – have been written in the past and are written now is an interesting and unstudied question.

The present article will address problem sets given on examinations in Russia at different times. It is evident that an exam is conceived of as a specific construction: it is impossible to give simply an arbitrary set of problems on some topic or topics on an exam. An exam must test to what extent what was required has been learned, and this automatically defines some structure for the set of problems on it. On the other hand, the understanding of what should be tested and how exactly it should be

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tested has varied at different times, in different countries, and under different circumstances. Consequently, exam problem sets are constructed in different ways.

It should be noted at once that literature about exams and exam problem sets is quite limited. The author knows only one book devoted to exit examinations in different countries (Britton & Raizen, 1996). Large-scale exams and various kinds of tests are often and extensively talked about as a kind of political and social phenomenon, for example, when discussing their objectivity or conversely their unobjectivity, in the sense that various categories of students find themselves in a worse position on exams than others (for example, Dyrvold et al., 2015; Levin & Shohamy, 2008). Less is said about problems and problem sets. Meanwhile, as may be argued, the analysis of concrete exams helps to elucidate the social significance of what is happening as well.

Suurtamm et al. (2016) rightly note that "Assessment should reflect the mathematics that is important to learn and the mathematics that is valued. This means that both large-scale and classroom assessment should take into account not only content but also mathematical practices, processes, proficiencies, or competencies" (p. 5). Exams are undoubtedly connected with the practices of schools, reflecting these practices, and sometimes dictating them. This interconnection, however, is far from fully studied.

It is noteworthy that the survey of the literature on the assessment by Suurtamm et al. (2016) confines itself to an exposition of Design Principles in Large-Scale Assessments, without going into (in part, of course, due to lack of space) the problematics of how these principles are embodied in practice under different conditions. The article by Swan & Burkhardt (2012) cited in that publication goes further, effectively formulating certain principles specifically for composing exams, rather than separate problems. For example, among the principles of the assessment of high quality, the first one listed is the following: "Reflect the curriculum in a balanced way. – Assessment should be based on a balanced set of tasks that, together, provide students with opportunities to show all types of performance that the curriculum goals set out or imply."

Moreover, Swan and Burkhardt (2012) discuss the following crucial contradiction. Noting that curricular documents in different countries "typically emphasise the societal, personal and intrinsic value of studying mathematics, describe the fundamental processes (or practices) that need to be developed and list the content domains that should be covered," they remark that "These aspirations are rarely reflected, however, in high-stakes assessments, which almost universally focus on assessing specific concepts and technical skills in isolation from each other and their use in 'doing mathematics.'"

The author of the present article believes that it would be more accurate to say that these aspirations are typically reflected in a very specific way in high-stakes assessments. Exam requirements, sometimes following traditions, sometimes innovatively going against them, have in any event arisen in connection with general societal, personal and intrinsic values, which with the passage of time might be no longer perceived.

Nabokov (1964) wrote in one of his books:

> There was mysterious sweetness in the fact that a long number, arrived at with difficulty, would at the decisive moment, after numerous adventures, be divided by nineteen without any remainder. (p. 17)

It can be said with confidence that the mathematics educators who wrote and gave such problems saw in them a foundation for the formation of a human being capable of carefully following instructions, and who moreover would relish the triumph of order and organization over chaos: they had far more in mind that merely the narrow technical problem of inculcating various computational skills. On the other hand, it is true that the point of view of today's educator would most likely be different: in life, nothing is usually divisible by 19 without a remainder, and to inculcate the expectation that such divisibility exists is harmful, for which reason such a problem (from this point of view) has no connection with the lofty goals described in the curriculum.

To trace behind this "very specific," or more precisely, historically determined way in which general requirements become transformed into concrete exam requirements would be ideally the ambition of the author of this article. Understanding how general societal, political, philosophical, and other requirements and conceptions were and are embodied in one or another concrete examination method and concrete exams is a difficult problem, and the present article does not in any way claim to present a complete solution. Nonetheless, thoughts about this topic seem useful, if only because they will help to see different principles of problem set organization and the general principles of mathematics education that underlie them.

Below, various Russian exit exams will be discussed, and sometimes entrance exams as well. This article is above all historical in character, and as is customary in historical studies, it will rely on the analysis of documents – readers are invited to read them carefully, attempting to establish their general and particular traits. The special characteristic of this analysis is that the documents in question are sets of mathematical problems. Their analysis will be carried out against the background of an analysis of relevant literature, which is more traditional for historical studies. Note that the present article to a large extent relies on studies by Karp (1998, 2003, 2007) and to a certain extent continues them.

25.2 Certain General Considerations

Before discussing historical evidence, we will relate certain general considerations pertaining to the composition of exams. Swan and Burkhardt (2012) are, of course, correct: the set of problems on an exam must be balanced from different angles. If the course in which the exam is given includes, say, basic trigonometry, exponential and logarithmic functions, and elementary calculus, as was the case at a certain stage on exit exams in the USSR, then it is natural to expect that an exam would contain problems in each of these sections. At this point, however, the question of the number of problems devoted to each topic immediately arises. Moreover, within each topic there are always numerous subtopics – we might note, for example, that the Soviet school textbook at a certain stage taught how to solve six types of exponential equations (Alimov et al., 1996). Their solutions were explained in the text of the textbook (under the heading "Examples"), and then reinforced through accompanying exercises, which were accordingly organized in groups. One might pose the question of whether it is sufficient to test the ability to solve one type of equation (for example, by giving the equation $25^x - 6 \bullet 5^x + 5 = 0$) or whether it is necessary to test the ability to solve all six, including, for example, the following: $3^x = 7^x$.

The issue, of course, is not about which approach is correct: it is easy to provide examples of exams whose writers took the first approach, and examples that take the second approach. This, however, is not a minor detail: it reveals ideas both about how a child learns and about what we would like to teach at a given stage. The already repeatedly quoted survey by Suurtamm et al. (2016) mentions the influence of the psychometric tradition and, citing Osterlind (1998), formulates the corresponding requirement as follows: "Unidimensionality: Each test item should focus on assessing a single objective or ability." (p. 8). "Objective or ability" is here understood as something basic, as far as possible, and not splittable. Another approach conversely understands what is being tested as a certain relatively broad field, presupposing (even if not explicitly) the existence of a kind of transference – if a student can solve one equation, then this student can also solve another equation.

The school curriculum cannot be reduced to a list of topics. It is easy to identify oppositions that are present in the course (at least, from a certain point of view): the computational vs. the logical or that which requires reasoning; the algebraic vs. the graphic; and algorithmic vs. that which requires an independent and innovative solution; and so forth. Ideally, an exam should achieve a balance from this point of view as well (providing "students with opportunities to show all types of performance that the curriculum goals set out or imply," as Swan and Burkhardt wrote). In other words, it would be undesirable for an exam to consist entirely of problems that require only a good technical ability to carry out algebraic transformations or only good skills at working with graphs.

Connected with the question of balancing problems that represent different parts of the curriculum is the question of balancing problems in terms of their difficulty. This notion in itself requires a clarification. One can speak about a certain objective characteristic of a problem – for example, the number of steps (elementary operations) involved in solving it. But what people usually have in mind is something different: the proportion of students, out of a standard group, for the situation being examined, who are capable of solving the problem (Stolyar, 1974). This proportion is usually determined either experimentally or by "eyeballing" it, relying on experience. A problem's difficulty thus turns out to be connected with social and educational factors: generally speaking, when looking at an exam from another country, it is impossible to say whether the problems on it are difficult or not, since we do not know what the students have solved as part of their preparation. Stolyar (1974) even proposed a formula: the real difficulty of a problem is equal to the difference between its objective difficulty and the objective difficulty of all of the component problems of this problem that have been solved by the students. Without agreeing with this formula (it was not grounded experimentally in any way, for example), we acknowledge the difficulty of the very concept of difficulty, and therefore also of its balancing.

One can readily imagine a set of problems that test altogether elementary skills that are generally identical in terms of their difficulty (defined, for example, as the number of operations needed to solve a problem – all of them require one operation). But usually, it is precisely the social functions of exams that allow for the possibility of a stratification among the results – some students manage to solve the difficult problems, others do not.

Let us repeat that easiness and difficulty depend on the conditions in which those taking the exam are taught. The inequality

$$\log_{x+1}\frac{4x-2}{x+1}\leq 1,$$

which was given in 2005 on the entrance exam to the St. Petersburg University International Relations Faculty, is easy only for those who have gone through special preparation. But nonetheless, it was clearly conceived of as easy by those who wrote the exam, against the background, for example, of the following problem from the same exam, which contained five problems in all:

> For which values of the coefficient a do the polynomials $P(\mathrm{x}) = -x^2 + ax + 3$ and $Q(x) = 3x^2 + (4- 5a)x + 1$ have a common root? (Semenov, 2006, p. 7)

While in the first case students were merely required carefully to apply certain techniques for solving logarithmic inequalities (even if in a situation that was much more technically cumbersome than in problems from the textbook, and therefore requiring additional preparation), in the second case technical proficiency is not sufficient and students must invent a special line of reasoning (or else to study completely outside the textbook and to learn in advance how to construct such lines of reasoning).

Among the important principles and foundations of exam composition, we cannot omit to mention technical conditions. Writing about the traditions of mathematical testing in Russia, Saul and Fomin (2010) recall even Dostoevsky and others who suffered for attempting to acquire a printing or copying apparatus. Indeed, in the USSR and in pre-revolutionary Russia, copying a text in any other way than by rewriting it was difficult. This automatically had an impact on the exam form: an exam could not have several pages. An exam had to be short enough to be written down on a blackboard. One can discuss to what extent this restriction influenced the rejection in the USSR, for example, of sets of problems with multiple-choice answers – this rejection was connected with other reasons as well, but the fact that external technical restrictions influenced the composition of exam problem sets is evident.

25.3 Russian Exit Exams in Algebra Before 1917

Consider the following exam from 1910 (Karp, 1998, p. 5):

> Divide the number m into two parts such that the ratio of their product to the sum of their squares is equal to the ratio of the sum of the roots of the equation $\sqrt{x+\frac{27}{x}}=2\sqrt{3}$ to the number n, where m is defined by the equality $\lg(m) = 2\,lg\,2 + lg\,3 + lg\,5$ and n is equal to double the coefficient of the fifth term of the expansion of $(x + a)^6$.

This exam was given in Voronezh, but similar exams were given across the whole country at the time. As we can see, the exam consists of a single problem, but this one problem consists of several, extremely artificially interconnected ones. If we were to rewrite this problem in a form that is familiar to us, we would obtain the following:

1. Find the number m from the equality $\lg(m) = 2\,lg\,2 + lg\,3 + lg\,5$
2. Find the number n equal to double the coefficient of the fifth term of the expansion of $(x + a)^6$.
3. Solve the equation $\sqrt{x+\frac{27}{x}}=2\sqrt{3}$ and find the sum of its roots, the number a.
4. Find the numbers p and q such that

$$p+q=m \text{ and } \frac{pq}{p^2+q^2}=\frac{a}{n}.$$

However, the exam obtained in this way is absolutely not equivalent to the original: perhaps the most difficult part of the problem – its "untangling," the determination of the order in which its component problems must be solved – has disappeared. There is also another significant distinction: a student who has failed to solve one of the four problems of the exam we have constructed can nonetheless receive credit for two or even three others. That is not the case in the original exam: here, any mistake becomes fatal.

The principle behind the construction of this exam is directly opposed to the psychometric principle mentioned earlier: what is being assessed is by no means a single objective or ability, but the ability to work with a difficult text. The student's mental development, which was, judging by the surviving documents and articles in the periodical press, the examiners' primary concern, was expected to be tested precisely on the basis of the student's ability to work with this text, as well as on the basis of the mathematical composition produced by the student – the written form of the solution (Karp, 1998). Actual equation solving or knowledge of the binomial theorem were also important, of course: there was no flexibility, which might have taken into account the fact that the student knew one thing even if he did not know another, but the exam was in a certain sense a humanities-style exam, as indeed the gymnasium was first and foremost a humanities-oriented educational institution. A graduate in 1910 could, of course, also go on to study to become an engineer, to which end he would usually spend a long time preparing for entrance exams in mathematics (Karp, 2022), but it was far more natural for him to enroll at the university, obtain an education in law, and become a civil servant analyzing tangled official correspondence. One might say that students were being prepared for this in mathematics classes as well. The humanities-style character of the exam was likewise underscored by the attention paid to the language of the solutions, including spelling and the choice of expressions – for example, exam-checkers' reports identified and discussed such students' expressions as "solve the binomial," instead of, say, "find the required coefficients of the expansion of the binomial" (Karp, 1998).

It is natural to connect the structure of exams of that time, and indeed the system of gymnasium education in general, with the theories of mental discipline (Stanic, 1986), which were, however, already somewhat losing their popularity. In the everyday use of this theory, the brain was compared with muscles, which must be developed through exercise. Consequently, just as few people in gym class thought about whether pushups were something necessary in real life, it was not always necessary to think about connections with life in mathematics classes either. However, the artificiality of the resulting composite problems, as they were called, was becoming increasingly more recognized. Calls for a new structure for exams were becoming increasingly louder. After the revolution of 1917, this change took place.

25.4 Soviet Exit Exams in Algebra

It should be noted that the new structure of the algebra exam did not instantly triumph. Immediately after the revolution of 1917, exit exams – and any other exams, for that matter – ceased to be conducted. They were seen as a throwback to the former regime, which had oppressed students and even subjected their health to danger through exams (Karp, 1998). When it was understood that exams could not be dispensed with, however, composite problems again appeared for a brief moment, simply because they were familiar to many teachers; but soon a new type of exam evolved, which remained largely unchanged from the 1930s to the 1980s. More precisely, the problems changed: over time, problems involving the binomial theorem disappeared, being considered obsolete; word problems shifted from exit exams to exams conducted earlier, and calculus problems appeared at a certain stage. The number of problems changed as well: there were sometimes three problems, sometimes four, and then there were five. But the principle behind the formation of the problem set remained the same: problems were given to test how well the main topics of the course had been assimilated, indeed, usually one problem represented a large topic without attempting to test everything that had been studied in connection with this topic. There was no flexibility whatsoever in the matter of assessment: students were offered no choice; in order to obtain the highest grade, everything had to be solved, without mistakes and with no more than two demerits (Chudovsky et al., 1986, p. 5) – the definition of a "demerit" being quite amorphous.

It should be noted here that the impeccable execution of an assignment included a flawless mathematical composition, which contained not only a detailed exposition of all the steps leading to the solution but the justification for these steps as well. The type of problem given was thus not simply an “essay task,” as problems requiring students to provide a full solution are often called, being contrasted with “short answer tasks” or “multiple choice tasks,” but an elaborate composition.

For example, the “governing article” (Gurvits & Filichev, 1947), as it was then said, recommended that the answer to the solution of the inequality $4x^2+16x+7>0$ is given as follows (and it should be emphasized that the authors were talking only about the answer here—it was supposed to be preceded by a detailed solution):

> Given the expression $4x^2 + 16x + 7 > 0$, if we replace x with any number lesser than $-\frac{7}{2}$ (for example, –4, –5, etc.) or any number greater than $-\frac{1}{2}$ (for example, $1, 4, -\frac{1}{3}$, etc.),we will obtain positive values. (p. 46) (cited in Karp, 2007)

Still, the cited article came under a certain criticism; but something similar, even if in a somewhat softer form, prevailed on the whole for a long time, which explains, for example, what a student had to work on during an exam that lasted many hours (three, and later four), and which contained, for example, the following problems (1977 exam, Chudovsky et al., 1986, pp. 97–98):

1. Prove the equality
$$\frac{\sin 2x}{1-\cos 2x} = \frac{1+\cos 2x}{\sin 2x}.$$
2. Find the function $F(x)$, given that $F'(x) = 4x^3 - 1$.
3. Solve the inequality $2x - x^2 - 5 \leq 0$.
4. Solve the system
$$\begin{cases} 2^{\log_2 y} - \log_3 x = 1 \\ y \log_3 x = 2 \end{cases}$$
5. The base of the pyramid $MABCD$ is a rectangle, whose perimeter is equal to 12 m. The edge $\overline{MB}$ is perpendicular to the plane ABC. The angle between the planes (MAD) and (BAD) is equal to 45°. What must be the height of the pyramid in order for the volume of the pyramid to be as large as possible?

The exam fully conforms to the principles that Chudovsky et al. (1986) formulate earlier:

> Of these five problems, one is connected with trigonometry: simplify a trigonometric expression, or prove an equality, or solve an equation, or solve an inequality. Another problem is connected with the antiderivative or the integral: find an antiderivative of the given function, or determine the integral, or determine the area of a figure bounded by the given lines. And another problem involves finding the greatest or least value of a function. The content of two other problems may be a problem from the curriculum mainly of grades nine and ten.[1] The fourth and fifth problems on the exam are somewhat more difficult than the first three. (p. 5)

In this way, the exam was rather rigidly determined. Of course, the fifth problem (that is, the most difficult) could theoretically turn out to be a trigonometry problem, rather than a problem to find the greatest value, but in those years this never happened. Note that the cited exam maintains another balance: there is an inequality, a proof (which involves an algebraic transformation), and equations (a system). A different exam could have been different from the one cited, of course, for example, it could include an exponential inequality, a trigonometric equation, and the transformation of a logarithmic expression.

Practically all of the problems on exams were more than one-step problems: even the first problem in the exam reproduced above required certain logical skills – the ability to construct the logical chain in the proof correctly – as well as the skill of carrying out algebraic transformations, as well as the knowledge of formulas – both algebraic and trigonometric ones. At the same time, the problems corresponded to what had been studied, at least in the sense that problems similar to the ones given could be found in the textbooks of the time (It is noteworthy that the cited handbook by Chudovsky et al. (1986) contains advanced problems, which are explicitly stated to be taken from the materials of entrance exams, while as the authors explain, such problems are rarely encountered in classes in school. In other words, it was openly stated that, for entrance exams, correspondence to what was studied in school was not necessary).

Despite the requirements that students had to meet in providing explanations and formulating solutions, which were put forth in the cited article by Gurvits and Filichev (1947) and others similar to it as testing students' understanding of what they had learned, their ability to reason, and so forth, the exams possess a certain technocratic character: there is a limited set of problems, which students are taught to solve in school, and the assimilation of those which are most important in the opinion of the examining agency – the Ministry of Education – is what is tested. It is noteworthy that during a discussion of exit exam requirements in 1947, Yakov Dubnov, a Moscow mathematician, put the question directly: what kind of person do we want our educational system to produce—someone who carries out instructions to the letter or someone capable of critically making sense of the work assigned to him? (O trebovaniyakh.., 1947, quoted in Karp, 2007). Attempts to determine the social significance of what was taking place thus existed at that time as well.

Let us say a few words about problem sets given on entrance exams (recall that two systems of exams existed in parallel). Usually, the written exam (there were

[1] Schools had ten grades at that time. Thus, this refers to the final two years of schooling.

also oral ones) during the last decades of the Soviet Union consisted of five problems also, but these included a geometry problem (or sometimes two – in plane geometry and in solid geometry). Since the problem of selection was more important in these exams than in exit exams, usually stratification into levels of difficulty was more noticeable in them, and in general, as has already been said, the problems were more difficult. There existed (in parallel with the school system) a kind of industry of preparation for entrance exams, a part of which was the publication of problem books for applicants (for example, the famous problem book by Skanavi, 1969). The problems in them were grouped into sections (say, “algebraic transformations” or “exponential and logarithmic equations”), and entrance exam problems were sometimes taken directly from such problem books and arranged into sets in a special manner.

25.5 The Age of Changes and Experiments

The process of Perestroika and the liberalization of the Soviet Union, initiated in 1985 by Mikhail Gorbachev, led to changes in education as well, including changes in the composition of exit exams. This sphere was by no means independent of the social and political atmosphere. The process of changes in it was slow, and this section will focus mainly on what happened already after the collapse of the USSR in 1991.

25.5.1 Changes in Ministry of Education Exams

Gradually, it became universally recognized that people are different and that consequently requiring the exact same thing of everyone was not right (Karp, 2020). The uniformity of Soviet exams had in fact already been shaken by that time, since along with ordinary exit exams, special exams had long since begun to be offered – for students of schools and classes with an advanced course of study in mathematics. But now a certain flexibility began to be permitted even within the bounds of the same exam. Consider the following exam from a general education school in 1994 (Zvavich, & Shlyapochnik, 1994, p. 13):

1. Solve the equation $\sqrt{10-x}=4-x$.
2. Solve the inequality $3\log_8(3x+2)<2$.
3. Indicate all roots of the equation $\sin 2x+\sqrt{2}\sin x=0$ that lie within the segment $[-3\pi/2,\ 3\pi/2]$.
4. Write down the equation for the tangent to the graph of the function $y = 4^x + 2^{x+1}$ at the point where it attains its minimum value.
5. Find the area of the figure bounded by the lines $y - x^2 = 0$ and $y^2 - x = 0$.
6. For which values of the parameter a a does the equation $x^2-(3a-1)|x|+2a^2-a=0$ have 4 different solutions?

The exam has six problems, while in order to obtain the highest grade, it is necessary to solve five. It may be noted that the revolutionary Perestroika, as was said at the time, was not all that radical when it came to exams. It may likewise be noted that the sixth problem was noticeably more difficult than the others, although formally it belonged to the curriculum not of the two oldest grades, but the earlier grades. In this way, the students were given a relatively standard exam (even so, however, one that seems to us more difficult than the exam from 1977, reproduced above), but those who were able to solve the difficult last problem received the highest grade, even if they had made a mistake somewhere else before it.

In reality, the step thus taken was significant. First, in this way truly difficult problems could appear on the exam – this immediately expanded the horizon both of the students and, above all, of the teachers. Second, the psychology of grading was changing: previously, whoever made no mistake was a hero; now, mistakes became permissible, as long as sufficiently many problems were solved without any mistake. Grading was now based on achievements, rather than on failures.

In exams prepared by the Ministry of Education, the difference between the total number of problems and the number necessary to obtain the highest grade was not large. The liberalism of that time, however, allowed exams to be prepared not only in one central location, but locally as well (on the condition that they would further be approved in Moscow). The exams prepared at the time in St. Petersburg went much farther (Karp, 2003).

25.5.2 St. Petersburg Exams

An exam for a general education school prepared at the time in St. Petersburg contained four blocks of problems, with each block containing four tasks. The student selected one of the two last blocks, obtaining in this way a set of three blocks and twelve problems, and to obtain the highest grade, it was sufficient to solve ten of them. Consequently, students could to a certain extent select topics with which they had greater affinity – one of the last blocks was usually devoted to elementary calculus, while the second dealt with irrational functions and equations. On the other hand, the possibility of mistakes was also permitted.

Another distinctive characteristic of these exams was the already mentioned organization of problems into blocks, and in such a way that the problems became interconnected, making it possible to some extent to check one against the other or to use one in some other way in solving another. Consider one block from the exam from 1994, in which these interconnections are sufficiently simple:

Given the function $f(x) = (x + 1)^2(x - 2)$

(a) Solve the equation $f(x) = (x - 2)$.
(b) Construct the graph of the function f.
(c) Find the greatest and least values of the function f on the segment $[-2, 1.5]$.
(d) Find the area of the figure located in the third quadrant of the coordinate plane and bounded by the graph of the function f and the straight line $y = x - 2$.

It is easy to note that the solution of problem a is used in problem d; and that the solution of problem b is used in problems c and d. Such connections among problems help the solver: sometimes, one of the problems constitutes a step in the solving of another problem; sometimes, using a later problem, it is possible to a certain extent to check the preceding one (for example, an incorrect solution of problem a might be identified in solving problem d). But what is even more important is that such problems, as we would argue, orient the student toward searching for such connections, and the ability to see them may be no less important than the ability to solve separate problems.

25.5.3 Centralized Testing

There were completely different experiments as well. One trajectory found expression not in official exit exams, but in so-called centralized testing, which was conducted by the Ministry of Education, and whose results were permitted to be counted in lieu of the results of both exit and entrance exams.

These exams (Tsentr testirovaniya, 2001) could, for example, consist of 12 multiple-choice problems and 10 short-answer problems – formats to which Russian schoolchildren were unaccustomed. It is not difficult to conclude that such exams did not develop without the influence of foreign – first and foremost, American – methodologies. Nor do the authors of the collection to which we have referred conceal that they wish to do everything properly, writing as follows:

> A properly written text consists of a unified whole of mutually balanced test problems. The number of problems on the test that pertain to different topics must be such as to proportionally reflect the basic content of the subject. Combinations of test problems of different levels of difficulty must provide for the same level of difficulty in different versions of the test. The differentiating powers of the test problems, in their turn, must provide for differentiation among the levels of preparedness of different students. (p. 3)

How this balancing was achieved, however, is not explained. Meanwhile, it can be noted that, for example, the number of problems in geometry was only 4 out of 22 (the test covered the entire course in mathematics), which can in no way be considered to be a proportional reflection of the content of the Russian school course in mathematics. Let us note, too, that the "American" form of the test was conjoined with problems that would not usually be encountered on American tests. Consider just one example:

The product of the roots of the equation $(x^2 + x + 1)(x^2 + x - 1) = 3$ is equal to

1. $\sqrt{10}$

2. −2
3. 8
4. −8
5. 10 (Tsentr testirovaniya, 2001, p. 5).

In the future, such testing was discontinued, giving way to the Uniform State Exam, which will be discussed below.

25.5.4 Collection of Problems

In the late 1990s, exit exams started to be conducted using so-called *open exams*. Special collections were published, which were bought by students in advance. Problems from these collections were solved and analyzed both at home and in class. On the day of the exam, it was announced which specific problems would be officially given as the problems on the exam. This idea, by this time, had long become a reality in ninth-grade exams, which marked the completion of lower secondary education. On exit exams, however, it was used in a somewhat new manner.

While collections for the ninth grade simply contained problems, the new collection (Dorofeev et al., 1999) provided not only separate problems but also whole problem sets. The result was as follows: 10 problems were given in all; the first five belonged to one of the numerous problem sets contained in the book; three more, according to a description by the collection's authors, were traditional exit exam problems (p. 5); and finally, two more problems were more difficult and were, again according to the authors' characterization, similar to those given on entrance exams. It was proposed that the highest grade be given to those who solved nine problems.

In order to demonstrate differences among the levels of difficulty, we will reproduce one problem from each of the three parts, with all three problems being devoted to trigonometric equations (each part contained problems from different sections of the course).

Problem from the first set: Find the roots of the equation $2\cos 3x + \sqrt{3} = 0$ (p. 18).
Problem from the second set (for ## 6–8): $\cos^2 x + 6 \sin x - 6 = 0$
Problem from the third set (for ## 9–10): $\sin x - \sin 2x + \sin 5x + \sin 8x = 0$ (p. 148).

It is clear that, in reality, there could not have been three problems in trigonometry out of ten problems on the exam. The authors, not of the collection, but of the actual exam, had to keep track of this. The procedure through which the exam was composed was by no means automatic, but nonetheless, a certain step in this direction had been made: the composition of the exam consisted mainly in assembling it out of certain components, rather than in selecting problems.

25.6 The Uniform State Exam

Beginning in 2009, the only form of the exit exam in Russia became the so-called Uniform State Exam. This exam is uniform for the whole country. Until 2015, it was both an exit and an entrance exam. Starting in 2015, the exam began to be conducted on two levels, basic and specialized. The specialized exam is counted as an entrance exam, also. Experiments with its organization began long before this, and its form has repeatedly changed (Karp, 2020). In a certain sense, centralized testing, which was discussed above, was also its forerunner. The literature pertaining to this exam is vast, although a large share of it consists of what may be described as opinion journalism rather than scholarly studies. Let us mention the study by Marushina (2012), which analyzes a specific variant of this exam.

We will examine one specialized exam in greater detail, using an officially given variant, the so-called demonstration exam (https://4ege.ru/matematika/60059-demoversii-ege-2021-po-matematike.html). The exam consists of two parts. The first contains eight short-answer problems. They are considered basic-level problems. In the second part, which contains 11 more problems, ## 9–17 are considered intermediate-level problems, and problems ## 18–19, are advanced-level problems. Problems 9–12 are also short-answer problems. The topics of the first 12 problems, and their types as well, are known in advance. For example, in the third problem, students are given a figure on graph paper and asked to find its area, while in problem 12, they must find the maximum or minimum of a given function using differential calculus. In problem 5, however, the choice is somewhat broader – it may involve an exponential equation or a logarithmic or irrational one.

The last seven problems require complete and well-grounded solutions. What a "well-grounded" solution means can be interpreted in various ways. The demonstration exam contains the following explanation:

> General requirements for completing problems with a detailed answer: the solution must be mathematically literate, complete; all possible cases must be considered. The methods used to arrive at a solution, the form in which the solution is written down, and the form in which the answer is written down may vary. A solution that arrives at the correct answer in a well-grounded manner will receive the maximum number of points. A right answer without the text of the solution will be given 0 points. Experts check only the mathematical content of the presented solution, without taking into account the particular characteristics of the way in which it is written.

The topics of the problems ## 13–19 are also known in advance (at list in some sense). The thirteenth problem is in trigonometry, the fourteenth is in solid geometry, and so on. Also known in advance are the points given for each problem. The first 12 problems receive one point each, problems ##13–15 receive two points each, problems ## 16–17 receive three points each, and problems ## 18–19 receive four points each.

Consider as an example problems 13 and 18, included in the demonstration exam cited here:

13. Solve the equation $2\sin\left(x+\pi/3\right)+\cos 2x=\sqrt{3}\cos x$.

18. Find all positive values of a for each of which the system

$$\begin{cases} (|x|-5)^2 + (y-4)^2 = 9 \\ (x+2)^2 + y^2 = a^2 \end{cases} \text{has a unique solution.}$$

It remains to note that a pool of exam problems has been created and continues to be replenished, which may in some measure become known before an exam. For example, the article by Marushina (2012) analyzes the following problem, which was recommended for the exam at the time:

Find all values of the parameter a for each of which the system of equations

$$\begin{cases} a(x^4+1) = y + 2 - |x| \\ x^2 + y^2 = 4 \end{cases} \text{has a unique solution.}$$

Of course, familiarity with this problem does not make the solution to problem 18 cited above easy, as it contains very many substantive differences, but probably no one would disagree that it makes it easier.

25.7 Discussion and Conclusion

Most likely, already at the beginning of this article, when we cited problems from a variant of the entrance exam to the Faculty of International Relations, the contemporary Western reader wondered about the usefulness of such problems for selecting future students for this faculty (even if it was for the department of applied computer science for the humanities, for which students were selected using the exam discussed above). Not having any evidence indicating the intentions of those who authored the exam, we can only surmise the following train of thought on their part: the Faculty is a highly prestigious one; the number of applicants is substantially higher than the number of available spots; it would be good for many applicants to be eliminated precisely through mathematics since a person's knowledge of mathematics indicates that person's level of intellectual development; therefore, let us see how well the applicants have assimilated school-level techniques, how precise they are in applying them, and how capable they are of problem solving, that is, how capable they are of thinking in an unfamiliar situation.

It may be said that this is another version of the old theory about mental discipline, which in this case is supported by the relative technical ease of eliminating applicants specifically through mathematics exams. Whatever our attitude toward such a philosophy, it is unlikely easy to say precisely which mathematics problems and problem sets should be used for selecting future employees of diplomatic missions. Cumbersome and technically intricate problems may seem strange, if only because no one solves them in real life, but to find problems for exams that specific applicants – rather than mankind in general – have happened or will happen to solve in real life is not easy in general. In any event, in this article, we are not interested so much in separate problems (although these inevitably yet have to be addressed) as in the construction of the whole set.

But the principles of construction have gone through multiple changes and very substantial ones. The study of the history of exams appears to us to be useful not least because it immediately disproves the opinion that exams are always conducted in more or less the same way. Different times have different exams. One can identify opposite possibilities in the trajectories of change in the composition of exams.

One of these oppositions is rigidness vs. flexibility in assessment, or – which is not, however, the same thing – grades based on positives vs. grades based on negatives. In the pre-Revolution exam discussed above, there was no flexibility at all: students had to complete everything (even if with "demerits"). Gradually, the situation changed: in Soviet times, students could pass an exam without solving several problems. Later still, students could not solve several problems and still get the highest score. The Uniform State Exam is also a competitive entrance exam – students receive higher scores for solving a greater number of problems (taking the "value" of each problem into account) – but as an exit exam, it is very flexible: students might do almost nothing and still receive a score sufficient for their school.

Evidently, the number of problems on the exam has increased, and this has occurred not only during the transition from composite problems to separate ones. "Open" exams (exams based on problem books) contained three problems, which were equivalent, in the authors' opinion, to "traditional" problems, that is, to problems on exams during Soviet times. Five problems on the open exam evidently involved no less work than the two remaining problems on the Soviet exam. But the open exam also had two "difficult" problems. As for the USE exams, there is no need to point out how enormous they are by Soviet standards.

What are these changes connected with? One possible explanation points to the reduction in requirements pertaining to the writing of the solution. It became possible to give even multiple-choice problems (admittedly, in the USE, on which such problems were also given at one time, they did not last long: it was found that they do not conform to Russian traditions and are in general harmful, see Karp, 2020). As for short-answer problems, these have been fully legitimized, while regarding problems that require full answers, we are told that the answer can be given in various different ways, and exam-checkers are clearly exhorted to be tolerant.

One can, however, connect the increase in the number of problems with the change in orientation that has occurred: in the Soviet exam, it was important that no one (or almost no one) fail. Later, the focus of concern shifted to comparatively strong students – on the USE, which serves as an entrance exam, this is particularly noticeable. A large number of failing grades on exit exams, which might lead to social problems, was something that no one wanted, naturally. But it turned out to be possible to control the number of failing students, by establishing a different passing score each year, and later also by creating a special version of the exam (in 2015) for those who do not intend to enroll at a higher educational institution (Karp, 2020).

Two more opposed tendencies, partly associated with changes in the style of presenting the solution, involve that which is algorithmic vs. that which requires reasoning. Verbose "humanities-style" arguments used to be considered, as has already been said, the basis of a reasoning mindset and a culture of reasoning: this is clearly disappearing, but at the same time the absence of reasoning altogether – only short answers – also turns out to be unacceptable. The ability to reason is tested on the USE only by problems that are declared in advance to be the most difficult, hence this turns out to be important only for the strongest students. And here again, a question arises about the meaning of what is taking place.

Schoenfeld (2013), contemplating the connection between problem solving and mathematical modeling, unified them by characterizing both as examples of "sense making." "Sense" and "meaningfulness" are defined not so much by the problem itself, as by the context in which it appears. Solving a logarithmic inequality can be a "sense making activity" in class, for example, because it figures in the solution of some general physical problem, say, or because it comes up in the course of discussing and comparing the solutions of other inequalities and so forth (of course, it is not in every class and not in every teacher's classroom that solving such inequalities becomes a "meaningful and sense making activity"). On exams, the problem is usually torn out of its context, as a result of which its meaningfulness unquestionably diminishes.

Schoenfeld (2013) rightly cites the formula, "What You Test Is What You Get." Exams not only test what is studied: but they also determine it. The appearance at one time of problems with parameters on exams at ordinary schools, discussed above, even if it was expected that they would be solved only by the strongest students, nonetheless was very important, at the very least because it broadened teachers' horizons. In our view, the appearance on exams of problems that create a kind of environment or situation, which is studied from different sides, so that the questions turn out to be interconnected and meaningful, likewise exerts and would exert an influence on what goes on in schools. In a number of countries, educators are working on including real-world problems, which make it possible to increase meaningfulness on exams and tests; but it is feasible within the framework of traditional school mathematics as well. In this respect, the exams in St. Petersburg, which were discussed above, merit attention. In general, the inclusion of problems that require thinking and reasoning, and which at the same time are addressed not only to the strongest students, is hardly feasible, in our view, without changing the structure of the problem set. Note that the creation of a kind of context, a system of interconnections, in the problem set on an exam stands in opposition to the mechanical inclusion of multiple separate problems in the set, which check separate skills and the assimilation of algorithms for solving separate types of problems.

Another opposition observed by the historian of exams is the opposition between the predictable and the new. The new may turn out to be too arduous, but to give students the same thing over and over again is also hardly appropriate. Students are given various substitutes: "the problem is not at all the same as last year – last year's problem had log base 2, this one has log base 3." Another approach is to introduce open exams, open problem pools, and so forth. In this case, nothing completely new is permitted at all; but on the other hand, there is a significant increase in variety.

The author, of course, does not claim to know how ideally to resolve all of the noted oppositions and contradictions. The objective of this article is far more modest: to describe the changes that have taken place, to attempt to identify the main tendencies of what is taking place, and also as far as possible to connect them with what has taken place in the country and in education. The seeming or actual meaninglessness of one or another problem for the educational process does not at all mean that there would be no sense in or reason for its appearance on an exam. It should be noted in conclusion that it would be interesting to compare the changes that have taken place in constructing exams in Russia with analogous processes in other countries.

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Chapter 26
Complexity of Geometry Problems as a Function of Field-Dependency and Asymmetry of a Diagram

Ilana Waisman, Hui-Yu Hsu, and Roza Leikin

26.1 Introduction

In this chapter, we discuss the complexity of geometry problems with different types of diagrams. First, we draw a distinction between field-independent (FID) diagrams that involve elements essential to the problem givens only, and field-dependent (FD) diagrams that include surplus information (in terms of Krutetskii, 1976). Field-dependent (FD) diagrams require the solvers to extract the information which is essential for answering the question. Second, we distinguish between symmetric and asymmetric diagrams. The comparison of the complexity of geometry problems with different types of diagrams is based on the phenomena described in the psychological literature: field dependency and asymmetry of geometric diagrams increase the complexity of visual comprehension (e.g., Adams & McLeod, 1979; Bornstein & Stiles-Davis, 1984; Evans et al., 2012; McLeod & Briggs, 1980). Connections between these conditions and the complexity of mathematical problems are often overlooked in mathematics education research. Examining Israeli geometry textbooks, one sees that during the secondary school grades, field independent situations mostly disappear from geometry textbooks. The 4 geometrical problems in Fig. 26.1 illustrate these characteristics of diagrams in geometry problems.

P1 and P3 are equivalent problems requiring proof based on the midline–in–triangle property: $EF = \frac{1}{2}AC, GH = \frac{1}{2}AC \Rightarrow EF = GH$. P2 and P4 are also

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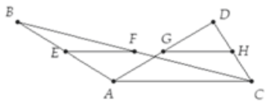
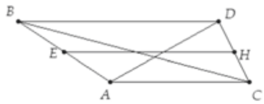
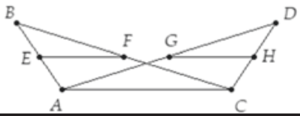
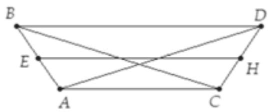
Diagram type		Field independent	Field dependent
	Given	P1 and P3 Triangles ABC and ADC with a common base	P2 and P4 EH is the midline in the trapezoid ABDC.
	Prove:	The midlines EF and GH in the triangles ABC and ADC are equal.	The parts of the midline in the trapezoid between lateral sides and the diagonals are equal
Asymetric		P1: Fig. 1.1 	P2: Fig 1.2 
Symetric diagram	Added given	P3: Triangles are congruent. Fig. 1.3 	P4: Trapezoid ABDC is equlateral. Fig. 1.4 

Fig. 26.1 Illustration of field (in)dependent and (a) symmetrical conditions in the given problems

equivalent problems that can be solved in the same way as P1 and P3 but that require "extracting" the triangles from the trapezoid. This distinction between the problems exemplifies the contrast between the field-dependent structure of figures 1.2 and 1.4 in problems P4 and P2 and the field-independent structure of figures 1.1 and 1.3 in problems P1 and P3. Symmetrical diagrams have one of the types of symmetry. P3 and P4 include symmetrical diagrams, in which an additional solution by mental reflection can be used – arguing that the segments are equal because of symmetry. This solution is not applicable to P1 and P2.

Problem P5 (Fig. 26.2) helps us to illustrate the role of field dependency and symmetry in solving geometry problems in different ways by "extracting" different elements of the given figure. Figure 26.2 outlines five proofs of P5 that are based on different elements of the given figure. Figure 26.3 illustrates P6 which is a symmetrical version of P5 accompanied by 2 additional proofs for P6 that are specific to symmetrical conditions. Due to symmetrical conditions, P6 also allows for performing a calculation instead of a proof.

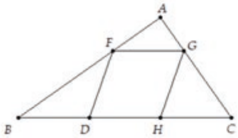
The complexity of solving geometry problems is broadly discussed in the educational literature (Battista, 2007; Weber, 2001). It is of a multidimensional nature, linked to visual abilities (Clements & Battista, 1992; Gal & Linchevski, 2010), auxiliary constructions required for solutions (Herbst & Brach, 2006; Palatnik & Dreyfus, 2019) and computational and proof skills (Mariotti, 2006; Hanna & DeVillers, 2012). Problem-solving competencies in geometry require from solvers a deep and robust knowledge of geometry concepts and their properties, i.e., definitions, axioms, and theorems (Herbst, 2002).

The psychological literature points out that both field dependency (Evans et al., 2013) and symmetry (Wagemans, 1997; Huang et al., 2018) determine the

P5

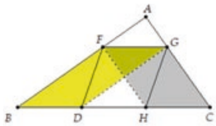
Given: $DFGH$ is a rhombus with vertices on the sides of the triangle ADG as shown. $BD = DH = HC$

Prove: ADG is a right triangle.



Proofs:

Fig. 2. 1



Fi .2. 2

The common observation for proofs 5.1, 5.2, 5.3 and 5.4
FG is equal and parallel to DH, as opposite sides of the rombus, and $BD = DH = HC$ the quadrilaterals $BDGF$ and $HCGF$ are parallelograms. In other words: *the diagonals of the rombus are parallel to the sides of the triangle.*

Proof 5.1: This proof requires knowledge of the theorem -- If two lines in plane are parallel to perpendicular lines then they are perpendicular.
Thus BA is perpendicular to CA.

Proof 5.2: *FAGC* is a rectangle

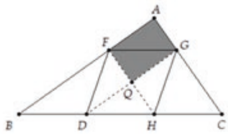


Fig. 2. 3

Proof 5.3: ΔDOH is a right triangle that can be translated* to ΔFAG

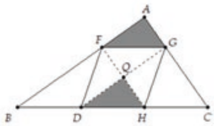


Fig 2. 4

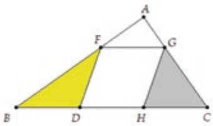


Fig. 2.5

The common observations for proofs 5.4 and 5.5

$\angle BDF + \angle GHC = 180°$ (*FD*||=*GH*).

Translate* ΔHGC to ΔDFH. $\angle AFH = 90°$ as a corresponding to $\angle DOH$ formed by parallel lines DG and BA.

Proof 5.4: $\angle BAC = 90°$ as a corresponding to $\angle BFH$ near parallel lines FH and BA (Fig. 2.6).

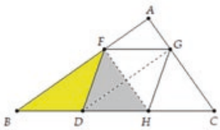


Fig 2. 6

Proof 5.5 Translate* ΔBFH along BA to ΔPAS. $\angle BAC = 90°$ (Fig 2.7)

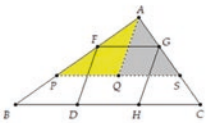


Fig 2. 7

*Translation can be replaced by proof of congruence

Fig. 26.2 Illustration of field dependency elements in proving

complexity of cognitive processing related to visual stimuli. Based on this literature, we raised two hypotheses:

H1: Field dependency affects the complexity of solving geometry problems:field-dependent (FD) diagrams are more complex than field-independent (FID) ones when solving equivalent problems (those that have identical solutions).

H2: Symmetry of geometric figures given in the problem influences problem-solving success: Problems that include symmetrical diagrams are less complex than equivalent problems that include asymmetrical diagrams.

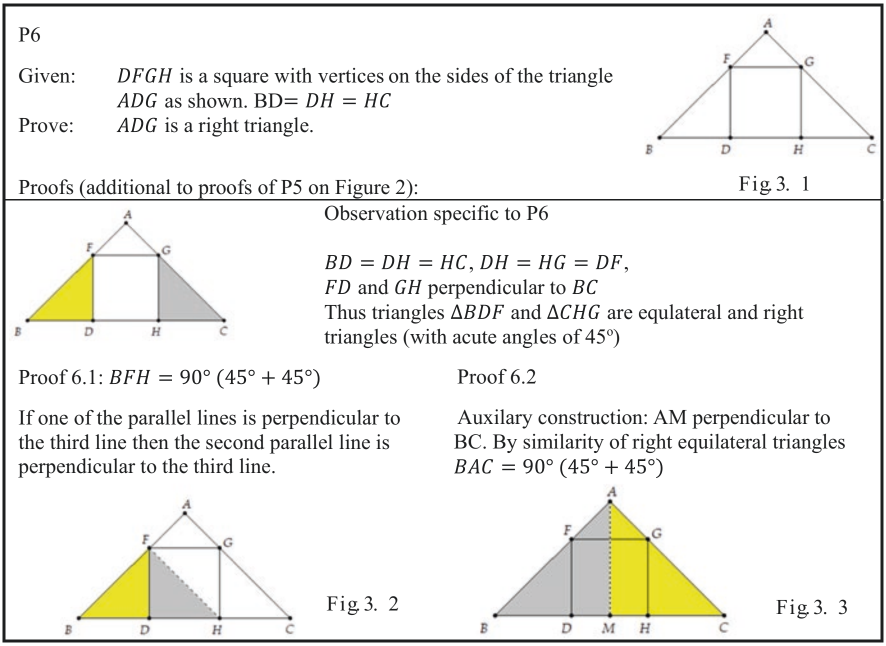


Fig. 26.3 Illustration of symmetrical diagrams in proof problems

In what follows we review related literature and introduce the design of the newly developed Geometry Field-Dependence-Symmetry (GFDS) test instrument that integrates both FID theory and Symmetry in psychology and in mathematics. We present findings that examine our hypotheses and suggest some recommendations for the task design principles in geometry.

26.2 Background

26.2.1 Field-Independence-Dependency (FID)

In the 60's and 70's, Witkin and his colleagues proposed a construct of Field-Independence-Dependence (FID) and used it to describe one's cognitive styles and behaviors. According to Witkin and his colleagues, FID generally refers to the extent to which one can perceive part of a field from the whole field (Witkin et al., 1977). The field-independent person is capable of breaking up the total field into parts, attending to the relevant parts while withholding attention from the irrelevant parts. In this regard, field-independent individuals are more likely to rely on internal frames of reference, whereas field-dependent individuals are more inclined to rely on external frames of reference (Davis & Frank, 1979). Thus, looking at individuals, field-independent persons are more capable of breaking up a complex diagram as they can mentally or physically decompose and recompose the diagram in order to recognize sub-configurations embedded in the given diagram. At the same time, when we consider geometry problems, FID diagrams are often simple and do not involve much complex surplus information, which allows individuals to easily identify subconfigurations and retrieve geometric properties.

Researchers in psychology have indicated that children at early ages are more oriented to global perception, and cannot articulate the differences among objects and activities (e.g., Witkin et al., 1979). When individuals' cognition develops, they become capable of differentiating one from the other. This is a common psychological phenomenon in human cognitive development (e.g., perceiving, thinking, learning) (Bloomberg, 1967). Davis and Frank (1979) argued that differentiation involves cognitive restructuring, which includes three separate but related operations. The first operation is to break up the organization of a stimulus complex so that its elements can be operated upon separately or in new combinations. The second operation is to provide structure for an ambiguous stimulus complex, and the third is to provide a structure different from that implied by the inherent structure of the stimulus complex.

Based on the construct of FID, Witkin and his colleagues developed several instruments to examine the ability of field-independent thinking that one possesses. These instruments allow researchers to understand how individuals perceive units of the field as discrete across different situations. One main type of test requires subjects to dis-embed and locate a previously seen figure within a complex figure designed to hide it, such as the Embedded Figures Test (EFT), the Group Embedded Figures Test (GEFT), and the Hidden Figures Test (HFT). Linn and Petersen (1985) identified HFT as a spatial visualization test with respect to gestalt rules.

Researchers have investigated FID in different ways. One is to explore the essence of FID as a general cognitive style or cognitive ability, and examine it in different groups or among individuals with different backgrounds (Coates et al., 1975; Goodenough & Witkin, 1977; Sternberg, 1997; Witkin, 1965; Witkin et al., 1977; Zhang, 2004). For example, Sternberg (1997) concluded that FID is a kind of spatial ability. Those studies nurture the development of FID theory and enable the revision of instruments in accordance with different examination purposes.

Adams and McLeod (1979) examined the interactions between FID and instructional treatments, concluding that there is no interaction between the two variables on the post-test; however, significant interaction with crystallized ability on the retention test was found. McLeod and Briggs (1980) examined prospective elementary school teachers and confirmed that those with a field-independent cognitive style can learn about numeration systems significantly better when provided with minimum guidance and maximum opportunities for discovery through the use of manipulative materials. On the other hand, field-dependent individuals learned better with maximum guidance and symbolic treatment. Tartre (1990) found that FID influences students' understanding of mathematical problems.

26.2.2 The Need to Develop a New FID Instrument Specific to Geometry

Different researchers have achieved different conclusions with respect to the relationship between FID and mathematics. On the one hand, some researchers (i.e., Dubois & Cohen, 1970; Tinajero & Páramo, 1997) stated a strong relationship between FID and various measures of academic achievement. On the other hand, others (i.e., Nappo et al., 2019; Zhang, 2004) asserted FID is not related to overall achievement in mathematics but only to achievement in geometry specifically. The inconsistency in the research findings indicates the need to develop a new FID instrument specific to geometry which enables the articulation of the influence that FID has on students' geometry learning. Our newly developed FID instrument particularly focuses on the students ability to tackle geometry problems with field-dependent and asymmetrical diagrams in a complex mathematics situation. This is in line with the Embedded Figures Tests (EFT), the Group Embedded Figures Tests (GEFT), and the Hidden Figures Test (HFT).

Figure 26.4 shows the HFT test items, which provide a number of referent shapes (on the top in Fig. 26.4) and require solvers to identify the referent shapes in complex figure environments. The cognitive work needed is to differentiate the complex figure environments by embedding and disembedding, and to perceive the referent shapes in the environments. Two characteristics can be noticed in the HFT items. First is that the referent shapes (top of Fig. 26.4) can be viewed as simple ones when compared to those figures shown on the bottom. However, those referent shapes are not simple and common as seen from a mathematics perspective. Taking the referent shape labeled by letter A shown on the top left in Fig. 26.4 as an example, it is not a commonly-seen hexagon such as a regular hexagon or convex hexagon. A complex and not-commonly-seen shape requires one to observe the characteristics of the shape (e.g., pointy angle) or to decompose and recompose the shape into familiar ones. For example, one may see the referent shape labeled by letter A as a

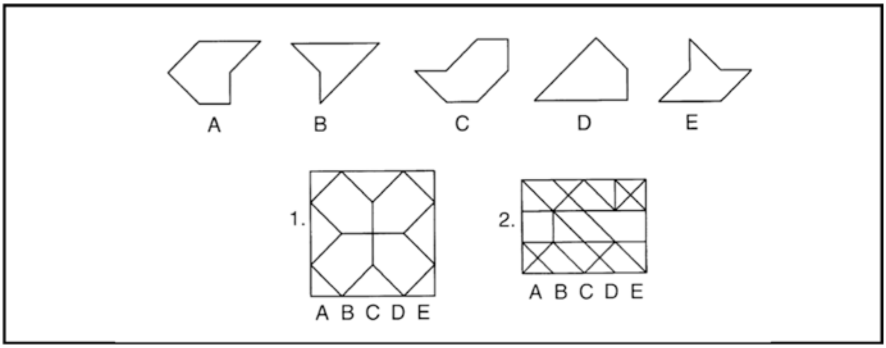


Fig. 26.4 Example of referent shapes (top) and complex figures (Bottom) included in the Hidden Figures Test (HFT) (Ekstrom et al., 1976)

combination of a home-plate shape and a triangle shape. Another may view it as a parallelogram with a trapezoid. Those cognitive activities may increase the demand on students in perceiving the referent shapes in the test. Second, the figures (see bottom in Fig. 26.4) are cognitively complex as they include a number of sub-configurations with different shapes, lengths and orientations. The complexity in the figure's environment increases the cognitive demand for decomposing and recomposing (Gal & Linchevski, 2010) and prevents one from easily perceiving the referent shapes in it.

To better understand how FID may influence geometry problem solving, the following two geometry problems, along with an analysis from (Hsu, 2010), can elaborate the cognitive complexity caused due to the changes of geometry diagrams.

As can be seen in Fig. 26.5, Problem 1 and Problem 2 have the same written information (e.g., BD=BC) and goal, which is to prove that $\angle ABD = 2 \angle BDC$. The only difference between the two problems is in the given diagrams. The one accompanying Problem 1 involves less complexity as the diagram more resembles images with respect to the external angle property. The diagram in Problem 2 involves much more complexity in terms of the segments and vertices so that different sub-configurations other than the one in association with the external angle property can be identified. Figure 26.6 illustrates some of the sub-configurations with corresponding geometric properties embedded in the Problem 2 diagram. When a geometry diagram is made complex, it not only changes the look of the diagram but may also increase cognitive complexity in terms of identifying sub-configurations and their corresponding geometric properties. For example, in the Problem 2 diagram, three different sub-configurations can be found even in association with the same geometric property, the external angle property. Each sub-configuration requires students to extract certain diagram elements – angles and segments – and to reassemble them according to the mental images of the external angle property that they may have in mind. Researchers have found that identifying sub-configurations in a

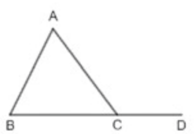
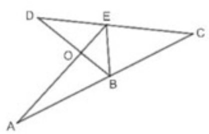
	Given diagrams	Given information
Problem 1		In $\Delta ABC, AC = BC, and\ BCD$ is collinear. Prove $\angle ACD = 2\angle ABC$.
Problem 2		In $\Delta BDC, BD = BC$, and A, B, C are collinear. Prove $\angle ABD = 2\angle BDC$

Fig. 26.5 Complexity in geometric problems linked to the given diagrams

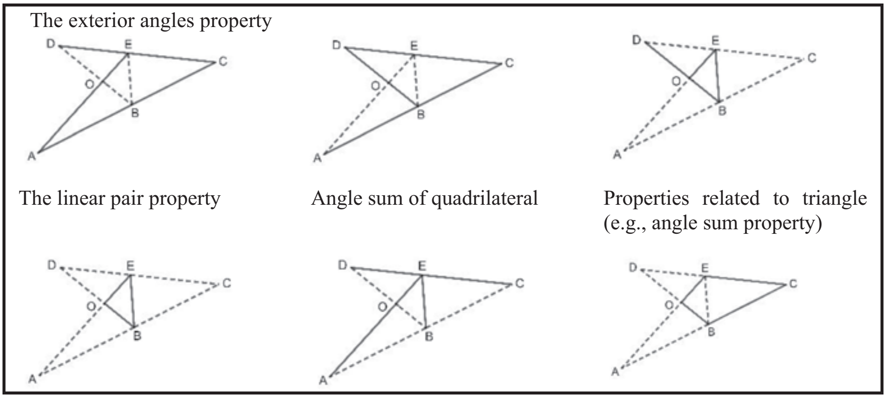


Fig. 26.6 List of sub-configurations and corresponding geometric properties

geometry diagram with a corresponding geometric property can be highly cognitively demanding (Kabanova-Meller, 1970; Kantowsk, 1975) (Fig. 26.6).

Based on a sample of 502 8th-grade students and 413 9th-grade students, Hsu showed that 8th-grade and 9th-grade students all performed significantly better on Problem 1 than Problem 2 (mean difference = 0.263, $p = 0.000$ for 8th grade; mean difference = 0.136, $p = 0.006$ for 9th grade). The effect sizes for the comparisons of the two problems for 8th grade (Cohen's d = 0.19) and for 9th grade (Cohen's $d = 0.1$) were below a small effect, indicating that the difference in performance for students in both grades between the two problems was not large. The analysis revealed that diagram complexity can be a factor in determining cognitive complexity. The analysis also draws attention to a well-designed study that can carefully examine the extent to which diagram complexity influences student problem-solving performance.

The analysis result shown above confirms the possibility that FID influences the cognitive complexity of geometry problem solving, and indicates the need to closely examine the systematic variations of geometry diagrams and their influence on geometry problem-solving.

26.2.3 Symmetry

Symmetry is a component of the natural world around us. It is an essential element of mathematical and scientific thinking and constitutes a fundamental aspect of spatial reasoning (e.g., Hargittai, 1986; Livio, 2006; Weyl, 1952). Symmetry makes it possible to observe, conceive, and sometimes even prove specific laws (Weyl, 1952). Moreover, symmetry is strongly linked to art and design (Arnheim, 1974; Ramachandran & Hirstein, 1999). The significance of symmetry is emphasized in painting, sculpture, architecture, literature, and music (Hodgson, 2011; Weyl, 1952).

For example, in fine art, symmetry is manifested in proportions, perspective, and harmony of an object's proportion and color (Arnheim, 1974). Various definitions of symmetry are available in the literature (e.g. Weyl, 1952). According to Leikin, Berman, and Zaslavsky (2000), symmetry is a triad of transformation, object and, property, such that the property of the figure is immune to the transformation. This definition covers different symmetry types, such as geometric symmetry (Weyl, 1952) and role symmetry in algebra and in proofs.

Research in mathematics education attaches importance to the study of symmetry in the classroom. For example, NCTM standards (1989) encourage students to "apply transformations and use symmetry to analyze mathematical situations." The incorporation of symmetry in mathematical classes makes it possible to develop mathematical thinking and spatial ability in learners, and promotes a scientific understanding of the physical world (Clements & Battista, 1992; Lowrey, 1989). Therefore, symmetry affects both comprehension processes and knowledge development processes (Lowrey, 1989). Using symmetry is enjoyable for students of all ages and abilities. Symmetry makes it possible to link mathematics with art, thus developing an assessment of mathematics' aesthetic aspects (Stylianou & Grzegorczyk, 2005; Dreyfus & Eisenberg, 1998). Students can observe and discover different geometric features of symmetric shapes or use symmetry as a basis for learning high-level mathematical topics (Geddes & Fortunato, 1992). Moreover, learning mathematics through symmetry invites exploration and discovery in mathematics lessons, integrates collaborative learning, promotes active learning for students, and as a result, allows students to believe in their ability to learn mathematics (Clements et al., 2001; Ng & Sinclair, 2015).

Symmetry also makes it possible to connect mathematics and other subjects, geometry and other mathematical fields (Bennett, 1989; Weyl, 1952; Leikin et al., 2000). Symmetry has an essential role in problem-solving, as it connects various branches of mathematics (Applebaum & Leikin, 2010) and can provide a more elegant solution (Dreyfus & Eisenberg, 1998; Leikin et al., 2000; Schoenfeld, 1985). For example, some optimization problems can be easily solved using line symmetry and not calculus methods (e.g., Leikin et al., 2000; Polya, 1981). However, despite the elegance of solutions using symmetry, students usually avoided using it (Vinner & Kopelman, 1998). For example, there is still insufficient use of symmetry in classifying geometric objects and proving specific properties (De Villiers, 2011; Sinclair et al., 2016).

Cognitive research focusing on the perception of symmetrical objects has mostly focused on its effect on perception and memory (Boswell, 1976; Pashler, 1990) or developing symmetry concepts through childhood (Hu & Zhang, 2019). Symmetric figures and objects are recognized faster and more accurately (Bornstein & Stiles-Davis, 1984; Evans et al., 2012; Wagemans, 1997) and often remembered better than asymmetrical ones, mainly when the axis of reflection is vertical (Howe & Jung, 1986; Rossi-Arnaud et al., 2006). For example, habituation is faster in symmetrical polygons than in asymmetrical ones (Bornstein et al., 1981). However, studies on distinctions between different types of symmetry found that children and adults more easily recognize and remember reflectional symmetry than rotational or

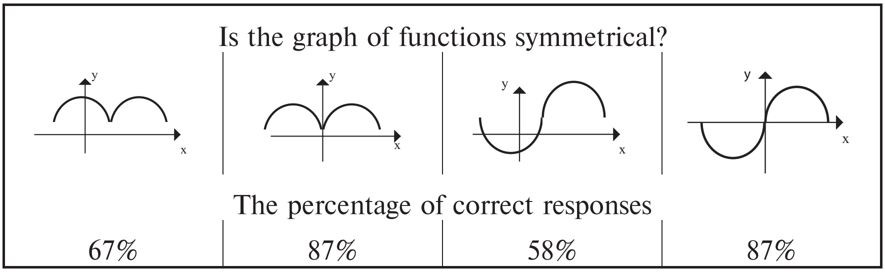


Fig. 26.7 Students' performance on identifying symmetrical graphs

translational (Dillon et al., 2013; Rossi-Arnaud et al., 2006; Wagemans, 1997). A possible explanation of such findings is that vertical reflectional symmetry demands less working memory (WM) resources, especially visuospatial and executive resources, compared to other types of symmetry (Wagemans, 1997). In addition, several studies have shown that participants tend to concentrate on 'visually-salient' features of objects, such as center points or diagonals, in performing reflection (Leikin et al., 2000; Edwards & Zazkis, 1993).

Leikin (1997) provided empirical evidence that the identification of symmetrical graphs of functions depends on the placement of the graphs with respect to the axes (see Fig. 26.7).

So far, most studies have concentrated on the recognition of various objects based on symmetry or manipulations such as reflecting, dragging or folding. There has been limited research concerning the effect of symmetry of the figures given in problems on the complexity of the problems. The current study was designed to examine the effect of a symmetry property of a geometric figure on performance in the verification of a geometric statement.

26.3 Development of Geometry Field-Dependency-Symmetry (GFDS) Instrument

When developing the GFDS instrument, we considered both FID construct and symmetry property. In GFDS, symmetry problems were created using the definition proposed by Leikin, Berman, and Zaslavsky (2000), that is, symmetry as a transformation of an object without any alteration of its features. The design of symmetry problems stems from the hypothesis that students' responses to geometric problems could be affected by perceptual features of the problems that interfere with logical reasoning (Stavy & Tirosh, 2000). For example, the perceptual effect of the length of the angle arms was reported to interfere with comparisons in different situations, such as vertical angles (Galili et al., 2020; Foxman & Ruddock, 1984). This study employs symmetric and asymmetric situations to reveal the reasoning processes associated with the relationship between symmetry and geometric properties. The symmetric items created for the GFDS test had reflectional or rotational symmetry, while the asymmetric figures kept the shape and size of the figure but violated the symmetry configuration. Symmetry-asymmetry can therefore be considered a "visually-salient" feature that interferes with reasoning about the relationships between the sides of the shape, probably due to attentional resources (Galili et al., 2020; Edwards & Zazkis, 1993).

The GFDS test includes four types of geometry problems: Symmetrical-field-independent (Sym-FID), Symmetry-field-dependent (Sym-FD), Asymmetry-field-independent (Asym-FID), and Asymmetry-field-dependent (Asym-FD). The difference between FID and FD is that FD involves cognitive complexity in terms of identifying the target diagrams from a complex one. Thus, an FID problem accompanies a simple geometry diagram whereas FD items include a more complex one, which requires extra cognitive work in the identification of the target diagrams. The difference between symmetry and asymmetry is in the diagram presentation. A symmetry diagram possesses reflectional or rotational symmetry properties whereas an asymmetry one does not.

The first examples with respect to the four types of geometry problems included in the GFDS test (see Fig. 26.8) are used to illustrate the differences between FID and FD as well as symmetry and asymmetry in the design of the GFDS test. The geometric problems shown in Fig. 26.8 are to determine the relationship of the lengths in the given diagrams. The difference between the FID problem and the FD problem is that the FID item involves a quadrilateral, whereas the FD accompanies a diagram of the same quadrilateral that is circumscribed by a circle. Because of the differences in the accompanying diagrams, the cognitive work of parsing the given diagram into sub-configurations with corresponding schemas can be quite different. For example, the FID problem requires parsing the given diagram into two triangles. FD requires this form of diagram parsing as well, and in addition, needs one to identify sub-configurations such as different circle arcs in the given diagram.

With respect to the statement encoding stage, because FD requires higher demand in diagram parsing, it also requires greater effort from students when it comes to statement encoding. As FID only requires the diagram parsing from a quadrilateral into two triangles, its statement encoding is tied to the properties related to the quadrilateral (e.g., the angle sum of a polygon) and the triangles (e.g., the Pythagorean Theorem).

For FD, other than the statement encoding with respect to the quadrilateral and triangles, it also requires statement encoding with respect to the circle. For example, the diagonal of the quadrilateral is also the diameter of the circle. The statement encoding in a complex diagram often requires the encoding of a diagram subject using different perspectives.

Krutetskii (1976) highlighted the key to experimental studies, that is, to systematically design experimental problems in accordance with the set-up hypotheses. The design of experimental problems can be complicated as it involves an evolution during the trial experiments by selecting and abandoning problems due to the consideration of the sufficiency of the problems in examining the identified cognitive characteristics. In this regard, designing a system that well considers

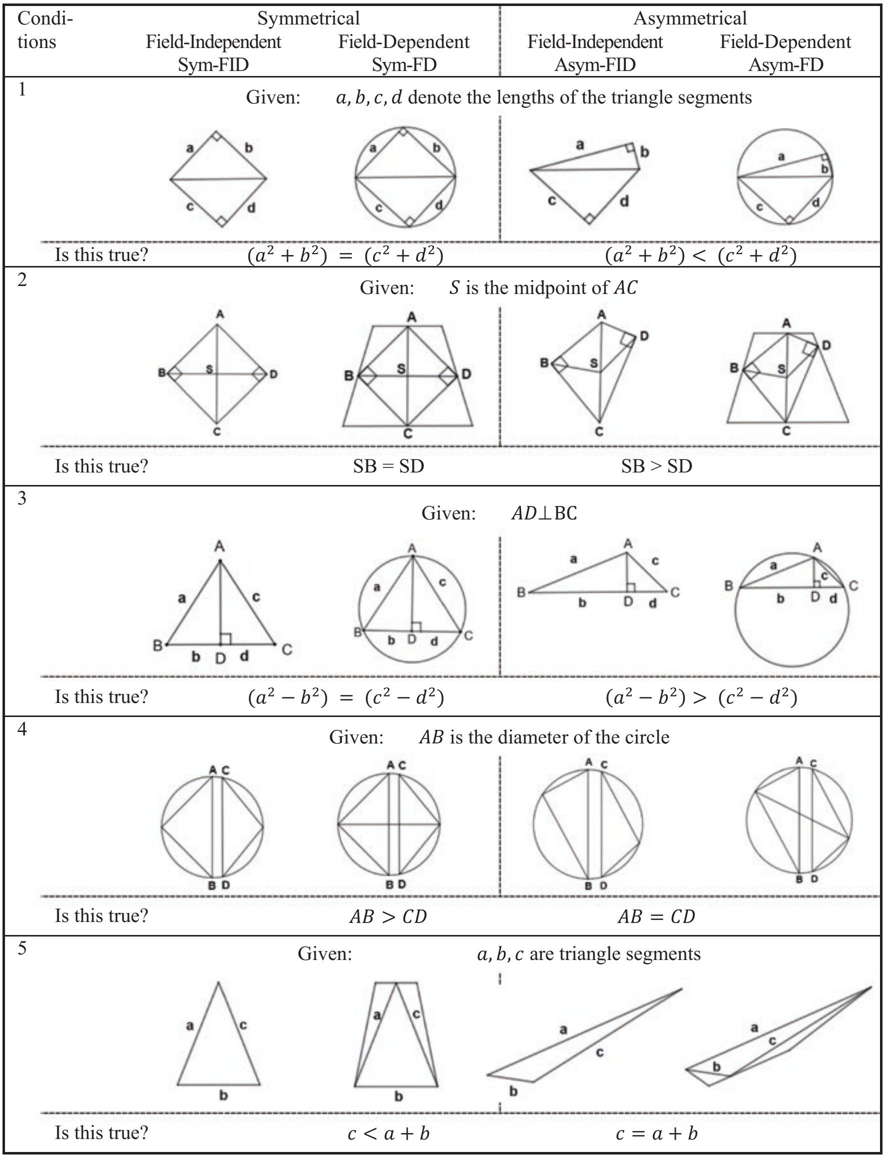


Fig. 26.8 Examples of tasks of four types of problems included in the GFDS instrument

variations of experimental problems becomes crucial. The GFDS instrument focuses on information gathering in terms of how geometry diagrams are represented and how the presentations consequently influence students' perception and interpretation. Figure 26.8 shows five sets of problems included in the GFDS instrument, which demonstrate the systematic variations of geometry diagrams in terms of the shapes (e.g., triangle, quadrilateral, circle), the surplus information, and the geometric properties. The systematic variations enable us to examine the effects of FID and symmetry on students' problem-solving.

The GFDS instrument includes 21 sets of geometry problems, each including Sym-FID, Sym-FD, Asym-FID, and Asym-FD diagrams. Each set was developed based on the same geometry property, with identical givens and an identical goal. Each individual geometry problem requires students to verify the correctness of two statements, one of which is true and the other, false. All the geometric problems included in the test instrument are specific to the comparison of segments as shown in Fig. 26.8. As a result, the GFDS test contains 21 sets and 84 geometry problems.

26.4 Examining the GFDS Instruments and the Research Hypotheses

26.4.1 Subjects, Setting, and Data Analysis

Forty-five undergraduate and graduate students (Mean age = 22.37; Std. of age = 2.08) from a Taiwan national university participated in the study. Those students are majoring either in social science (e.g., education) or in other fields that are not related to mathematics (e.g., landscape architecture).

We used E-prime software ((Schneider et al., 2002) to implement the test with students. In each one of the four conditions (Sym-FID, Sym-FD, Asym-FID and Asym-FD), 21 trials (tasks) depicted a true statement, while the other 21 trials depicted a false statement for the task given. Therefore, each subject had to complete the GFDS test with 168 tasks in the E-prime environment. Before working with the geometry problems, subjects had to practice with two extra simple geometry problems in order to become familiar with the tool.

Problem-solving in the E-prime environment starts with a 500 ms fixation and then presents the given and the diagram of a geometry problem for a time period of 2000 ms. After a 1000 ms time break, the goal of the problem along with the diagram is presented for 3500 ms. After another 1000 ms time break, a statement along with the diagram that requires students to evaluate is presented. There is no time limit for evaluating the statement. Once a student presses the bottom for an answer, another 1000 ms break was designed for the student to rest. All 168 trials were randomly presented in the E-prime environment in order to prevent learning transfer among the four types of geometry problems. Figure 26.9 depicts the sequence of events for each problem (item).

We analyzed subjects' responses to each trial in terms of accuracy of responses (Acc) and reaction time for correct responses (RTc). Acc was determined by students' percentage of correct responses to the 42 trials in each type of geometry problems. RTc associated with each type of geometry problems was calculated as the mean time spent for verification of an answer on stage 3 (S3) in all correctly solved trials.

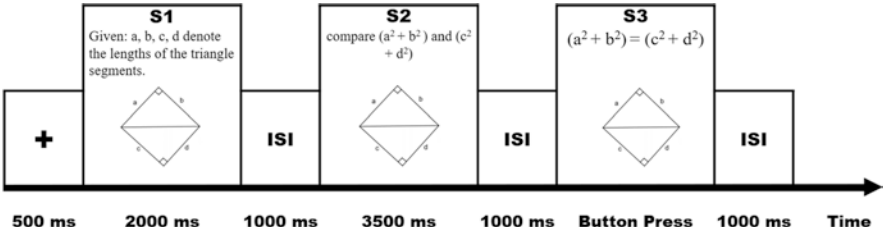


Fig. 26.9 Testing model for GFDS designed in E-prime software

Table 26.1 Cronbach's alpha for the four types of items in the research tool

	FID	FD
Sym	0.849	0.845
Asym	0.735	0.822

It has been frequently assumed that the mental effort involved in cognitive processes is manifested through certain behavioral measures, such as a subject's RT and response accuracy (Pachella, 1974). Moreover, RT and response accuracy can indicate problems'complexity (e.g., Goldhammer et al., 2014; Allaire-Duquette et al., 2019). It should be mentioned that most RT problems have an important working memory (WM) component, namely, the need to hold in mind newly instructed relations between stimuli and responses (Wilhelm & Oberauer, 2006). Using the findings showing that longer RTs and lower Acc indicate a more complex mental process, we examined whether asymmetrical or field-dependent diagrams affect the degree of complexity and, if so, in which way.

Internal validity of the instrument was assessed using Cronbach alpha for accuracy of the responses to the report. Table 26.1 presents Cronbach's alpha coefficients for the four types of items (Sym-FID, Sym-FD, Asym-FID, and Asym-FD). Cronbach's alpha was found to be high enough for the implementation of the test.

We analyzed differences in Acc and RTc using repeated measures ANOVAs taking Sym (symmetric vs. asymmetric) and FD (field-dependent vs. field-independent) as within-subject factors. Following a significant interaction, pairwise comparisons were performed.

26.4.2 Findings

Hypothesis 1 was confirmed by the research experiment: problems with field-dependent (FD) diagrams are more complex as compared to problems that include field-independent (FID) diagrams. Reaction time for correct responses only

Table 26.2 Acc and RTc associated with S and FID factors

Measure		Mean (SD)			FID factor
		Sym	Asym	**Overall**	*F* (1,44)
Acc (%)	FID	94.0(8.8)	92.5(7.4)	93.3(7.7)	0.003
	FD	94.0(8.7)	92.6(8.9)	93.3(8.4)	
	Overall	94.0(8.5)	92.6(7.9)		
	Sym factor *F* (1,44)	4.998*, η_ρ^2 = 0.102			
RTc (ms)	FID	2865.3 (1865.9)	4172.0 (4013.5)	3518.6 (2878.0)	6.437*, η_ρ^2 = 0.128
	FD	3611.8 (3341.3)	4154.5 (3316.5)	3883.2 (3208.8)	
	Overall	3238.6 (2542.2)	4163.3 (3.613.6)		
	Sym factor *F* (1,44)	13.685***, η_ρ^2 = 0.237			Interaction S × FID 3.683+, η_ρ^2 = 0.077

Acc accuracy, *RTc* reaction time for correct responses
*$p \leq 0.05$, ** $p \leq 0.01$, ***$p \leq 0.001$, +$p = 0.061$

supported this hypothesis. Participants responded significantly quicker on problems with FID diagrams as compared to the tasks with FD diagrams [F(1,44) =6.437, $p < 0.05$, η_ρ^2 = 0.128] (Table 26.2).

Hypothesis 2 also was confirmed by the research experiment: problems with asymmetric diagrams are significantly more complex as compared to problems that include symmetric diagrams. The complexity of the tasks is reflected both in the accuracy of responses and in the reaction time for correct responses. We found a significant effect of the Symmetry factor on Acc and on RTc (Table 26.2). Acc on the problems with symmetric diagrams was significantly higher as compared to the Acc associated with problems that included asymmetric diagrams [F(1,44) = 4.998, $p < 0.05$, η_ρ^2 = 0.102]. Correspondingly, RTc related to solving the problems with symmetric diagrams was significantly lower as compared to RTc associated with solving problems that included asymmetric diagrams [F(1,44) = 13.685, $p < 0.001$, η_ρ^2 = 0.237].

A marginally significant interaction of symmetry factor with FID factor on RTc was found: field-dependency significantly influenced solving problems with symmetric diagrams, while symmetry significantly affected solving problems with field-independent diagrams. Participants solved Sym-FID problems significantly more quickly compared to Sym-FD tasks [F(1,44) = 7.288, $p < 0.01$, η_ρ^2 = 0.142], and demonstrated similar RTc on the Asym-FID and on the Asym-FD problems. Additionally, participants exhibited significantly lower RTc on the Sym-FID problems as compared to RTc on the Asym-FID [F(1,44) = 12.704, $p < 0.001$, η_ρ^2 = 0.224], while there was no significant difference in RTc when solving Sym-FD and Asym-FD problems (Fig. 26.10).

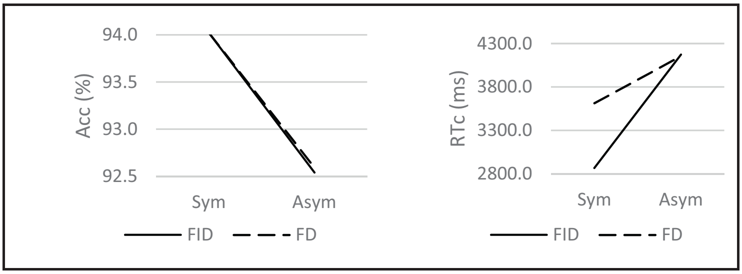


Fig. 26.10 Acc and RTc with Sym and FID factors

26.5 Discussion

Solving and proving geometry problems often involves cognitive complexity as it requires various kinds of cognitive work. As the proof problem shown in Fig. 26.2 illustrates, it asks for performing different cognitive processes such as identifying sub-configurations and their corresponding geometric properties (e.g., theorems, axiom, definition), drawing auxiliary lines, making rigid transformations and changing diagrams from asymmetric to symmetric. The complexity in geometry problem-solving calls for investigations that can examine each individual cognitive aspect in depth, as well as its relation to student learning of geometry.

In recognition of the significance of FID and symmetry in mathematics and other fields, we designed a new instrument, the Geometry-Field-Independence-Symmetry (GFDS) test, to systematically investigate how the variations of geometry problems influence students' problem-solving.

The empirical data collected from 45 undergraduate and graduate students confirmed that both FID and symmetry significantly influence students in solving geometry problems. The FID factor significantly influences students' reaction time to geometry problems, whereas the symmetry factor determines both the accuracy of students' responses and their reaction time. An interaction between the two factors was also found. Students responded significantly faster on Sym-FID problems than on Sym-FD problems. The data analyses based on undergraduate and graduate students indicate that cognitive complexity is associated with FID and symmetry factors. With respect to FID, the significant difference in reaction time and insignificant difference in accuracy between FID and FD show that FID does influence students in answering geometry problems. However, students still can figure out the correct answer if provided with enough time. Regarding symmetry, the significant difference in both reaction time and accuracy suggests that students are likely not able to figure out an answer to asymmetry problems even when provided with sufficient time.

The newly developed test instrument can be used to systematically compare FID and symmetry ability within a group of subjects and across different groups of subjects. The FID instruments developed previously in psychology have limitations in interpretation as they only can identify field independence and field dependence in a "more or less" way within a group of subjects (Evans et al., 2013). Those FID instruments do not allow for comparisons among groups of subjects, especially when those subjects may have different learning experiences. The empirical results based on 45 undergraduate and graduate students also raise questions with respect to the extent to which FID and symmetry influence students' geometry problem-solving. In particular, as we consider FID and symmetry to be cognitive and mathematical abilities, it is of interest to know if FID and symmetry are innate or whether it involves cognitive development over time and learning. The relationship between FID and symmetry and geometry competencies requires further investigation.

Another crucial issue is the question of how task designers consider FID and symmetry in the design of geometry problems integrated in instructional materials. The design of geometry tasks consequently influences how teachers provide opportunities for students to develop FID ability and to use symmetry as an efficient tool in geometry problem-solving. The principles for designing geometry problems that take FID and symmetry into account also need further investigation. For example, might it be a principle of task design that the complexity of geometry problems can be altered by increasing or deleting segments or angles in a given diagram?

Researchers associated with the fields of psychology and mathematics education have devoted themselves to understanding the cognitive mechanisms that are the keys to triggering the learning of mathematics. They also have worked on building theories and principles of students' cognitive development specific to mathematics learning. The process of establishing these theories and principles requires bridging the arguments in both fields. For example, the cognitive classification of conscious representation framework proposed by Duval (1995) considers the theory of perception, characteristics of mathematics, and its coordination. Another example is the cognitive development theory of geometry proposed by van Hiele (Fuys et al., 1988), which coordinates the cognitive theory from Piaget in general and geometry as a rigorous and abstract system in specific. Effort has been made in this chapter to coordinate the constructs of FID in psychology and symmetry in mathematics in order to better explain challenges in learning geometry.

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Chapter 27
Structuring Complexity of Mathematical Problems: Drawing Connections Between Stepped Tasks and Problem Posing Through Investigations

Roza Leikin and Haim Elgrably

27.1 Introduction

We consider the ability to engage with open questions and a deep understanding of mathematical structure to be two core elements of mathematical activity at an advanced level (Pehkonen, 1995; Silver, 1995). Openness and structuring of mathematical tasks are two sides of one coin in the construction of mathematical knowledge and skills: open tasks are associated with creative mathematical processing, and structuring, with the advancement of strategic mathematical reasoning (Leikin, 2019). We discuss engagement with open tasks using the example of a Problem Posing through Investigation (PPI) Task, and engagement with mathematical structuring using examples of Stepped Tasks designed based on the outcomes of PPI Tasks.

Before you continue reading the paper, we suggest solving Task 1 presented in Fig. 27.1.

27.2 Problem Posing Through Investigations

27.2.1 Characterization of PPI

Mathematical investigations and proofs are central to the work of professional mathematicians whose activity is devoted to mathematical discovery (Usiskin, 2000; Leikin, 2015), searching for questions that remain open and proving as yet unproved mathematical facts. Based on Usiskin's (2000) taxonomy of mathematical

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Task 1: Pose problems through investigations

Given: Segments AD, BC and DC are tangent to the circle with diameter AB and center O.

Pose as many problems as possible related to the given figure based on investigation in DGE.

* The Task was also used in Leikin & Elgrably (2015) and Leikin (2019).

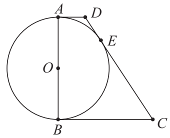


Fig. 27.1 Task 1

giftedness, Sriraman (2005) analyzed distinctions between the levels from the point of view of mathematical creativity and argued that creativity implies mathematical talent. Mathematical investigations are open since they are aimed at discovering what is yet unknown, and once the discovery ensues, proving comes in turn. The search (investigation) for mathematical discovery, and proving it, is based on natural curiosity, courage and enjoyment in one's accomplishments (Goldin, 2009).

We believe that feelings associated with mathematical investigations and discoveries in research mathematicians should become familiar to teachers and students and that creativity should be developed along with the advancement of mathematical knowledge and skills. Thus, mathematical investigations must become a common instructional practice directed at the development of mathematical knowledge, skills and creativity in both mathematics students and teachers (Da Ponte, 2007; Da Ponte & Henriques, 2013; Leikin, 2015).

This chapter is one in a series of papers that characterize Problem Posing through Investigation (PPI) as a research and didactical tool (Leikin, 2015; Leikin & Elgrably, 2015, 2020, 2022). As we previously demonstrated, PPI is a fundamentally challenging creative activity that requires participants to pose geometry problems through the investigation of a given geometric figure in a Dynamic Geometry Environment (DGE) (Leikin, 2015). Solving (proving) the posed problems is an integral requirement for PPI. PPI Task 1 (Fig. 27.1) helps us to exemplify the main ideas presented in this chapter. In terms of openness of mathematical problems: PPI tasks are *open-start* problems, since solvers are required to perform investigations in multiple directions, and *open-end* problems since participants are asked to pose several new mathematical problems, which are the products of investigation in DGE (Chap. 6 in this book, Leikin & Elgrably, 2022). If a PPI is performed successfully, the participant poses two new problems and performs at least two proofs. This determines both the complexity and the power of PPIs in terms of the development of mathematical expertise associated with proving skills as well as of creativity skills.

The level of mathematical challenge embedded in a PPI task is associated with multiple mathematical actions that participants are encouraged to perform, including auxiliary constructions related to the given figure, search for new properties that are immune to dragging through observing, and measuring. These properties serve as a basis for the discovered problems, which (as required by the task) have to be new for the participants. This means that PPI tasks require solvers to avoid discovering already learned properties and formulating theorems that solvers are required to know. For example, students in high school or preservice teachers are expected not to report discovering the Pythagorean Theorem or the ratio (1:2) of segments determined by the point of intersection of the medians in a triangle. For example, the discovery $\measuredangle AEB = 90°$ in Task 1 is trivial.

In previous publications, we demonstrated that PPI is an effective didactical and professional development tool. In Leikin and Elgrably (2020) we demonstrated that integrating PPI activities in courses for prospective high school mathematics teachers significantly improves their problem-solving proficiency and creativity. Elgrably and Leikin (2021) found that different types of problem-solving expertise (preparation for or participation in IMO or excellence in university mathematics courses) lead to different levels of creativity as well as to different levels of complexity of the posed problems. Furthermore, focusing on IMO problem-solving experts, we demonstrated differences between product (outcome) – related creativity and process (strategy) – related creativity (Leikin & Elgrably, 2022). For the purpose of these investigations, we developed a model for the evaluation of proof-related skills and creativity components associated with PPI. Here we describe characteristics related to the complexity of the posed problems.

27.2.2 Example of the Space of Posed Problems

Task 1 was presented to a group of prospective mathematics teachers and was also investigated using thought experiment methodology by the second author of this paper, who is an expert in mathematical problem-solving. In Leikin and Elgrably (2015) we compared problems posed by prospective mathematics teachers with problems posed by an expert. In this chapter, we analyze the collective space of the 29 posed problems. The criteria for evaluation of the complexity of the posed problems and PPI strategies suggested in Leikin and Elgrably (2015) were further developed and validated. Here we present simplified criteria for the evaluation of these components of PPI process and product.

In the context of this study, the suggested construct of *spaces of discovered properties* is equivalent to spaces of newly posed problems and is analogous to the notions of example spaces (Watson & Mason, 2006) and solution spaces (Leikin, 2007). As per this analogy, we suggest distinguishing between *individual spaces of discovered properties*, which are collections of properties discovered by an individual based on a particular problem, and *collective spaces of discovered properties*, which are a combination of the properties discovered by a group of individuals.

Figure 27.2, below, presents the collective space of the posed problems. The problems are presented in the order in which they were posed by the expert. For each posed problem we depict a figure that includes auxiliary constructions, proof outline and the discovery strategy, corresponding use of DGE and the evaluation of the complexity of the PPI processes and products. The processes are characterized by the PPI strategies used by the participants and the complexity of auxiliary

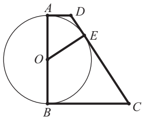
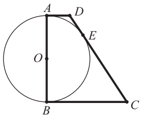
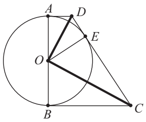
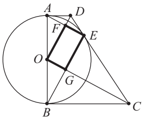
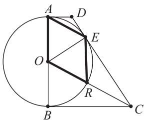
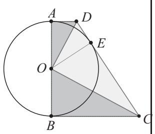
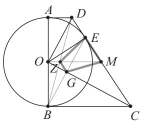
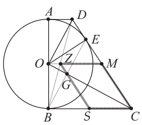
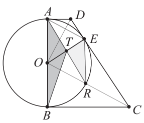
# of PP	Figure received after auxiliary constructions	Posed problem (PP)	The discovery strategy Proof outline	Use of DGE CAC, CPP
1.1		$BCEO, ADEO$ kites	Using a theorem $AD = DE$, $BC = CE$ t*angent segments* $AO = EO = BO$ *radiuses*	*No use of DGE* $CAC = 1$ $CPP = 1$
1.2		$ABCD$ trapezoid or rectangle	Using a theorem $CB \perp BO$ *and* $DA \perp AO$ *(tangent points)* $\Rightarrow AD \| \| BC$	*No use of DGE* $CAC = 1$ $CPP = 1$
1.3		$\sphericalangle COD = 90°$	Trial and error *Diagonals in the kites (1.1) that bisect the angles*	*Measurement in DGE* $CAC = 1$ $CPP = 1$
1.4		$EFOG$ *rectangle*	Association with a known problem Diagonals in the kites (1.1) are perpendicular	*Observation with dragging in DGE* $CAC = 2$ $CPP = 2$
1.5		$EROA$ trapezoid	Logical inference (1.4) Extension of the sides in parallelogram	*No use of DGE* $CAC = 2$ $CPP = 1$
1.6		$A(OAD)$ $+ A(BOC)$ $= A(DOC)$	Trial and error The main diagonal in a kite cross the area	*Measurement in DGE* $CAC = 1$ $CPP = 1$
1.7		M*-midpoint of* DC $Z = BD \cap OM$ $ZEMG$ trapezoid	Trial and error OM – midline in ABCD $\Rightarrow ZG$ midline in ΔBDE	*Observation in DGE* $CAC = 3$ $CPP = 2$
1.8		$ZMCS$ parallelogram	Logical inference (1.7) $ZM \| \| SC$ midline in ΔBCD ZS continuation of ZG	*No use of DGE* $CAC = 3$ $CPP = 2$
1.9		$A(ATB)$ $= 2A(ETR)$	Trial and error $A(\Delta ATO) = A(\Delta ETR)$ $A(\Delta ATO) = A(\Delta TOB)$	*Measurement in DGE* $CAC = 3$ $CPP = 2$

Fig. 27.2 The set of the posed problems for Task 1

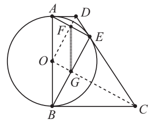
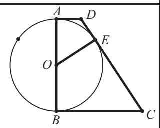
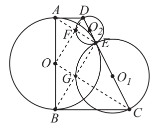
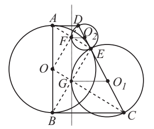
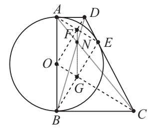
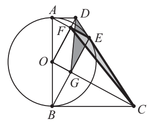
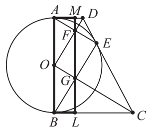
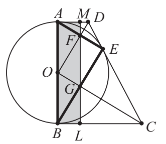
# of PP	Figure received *after auxiliary constructions*	Posed problem (PP)	*The discovery strategy* Proof outline	*Use of DGE* CAC, CPP
1.10		$F = OD \cap AE$ $G = OC \cap BE$ $FG \parallel AB$ $FG = \frac{1}{2}AB$	*Logical inference from 1.4* $AF = FE, BG = GE$ (intersection of diagonals in kites) $\Rightarrow FG$ midline in ΔAEB	*No use of DGE* $CAC = 3$ $CPP = 2$
1.11		$BCEO$, $ADEO$ cyclic quadrilaterals	*Using a theorem* Angles: OAD, DEO, OBC, CEO (tangent points)	*No use of DGE* $CAC = 1$ $CPP = 1$
1.12		Circles (O_1, CE) and (O_2, ED) pass through F and G respectively	*Association with a textbook problem* Angles CGE and EFD are right angles	*Verifying in DGE* $CAC = 3$ $CPP = 1$
1.13		FG is tangent to circles O_1 and O_2.	*Logical inference from discoveries 1.10, 1.12* GO_1 FO_2 are midlines in ΔBEC and ΔADE $FO_2 \| AD$ $1.10 \Rightarrow FG \| AB$	*Verifying in DGE* $CAC = 3$ $CPP = 3$
1.14		$N = AC \cap BD$ N is on FG	*Trial and error* ***There was a difficulty in finding a proof during the investigation***	*Observation in DGE* $CAC = 3$ $CPP = 3$
1.15		$A(CEF) =$ $= A(DEG) =$ $= \frac{1}{2}A(EFOG)$	*Searching for a proof (1.14)* $A(CEF) = \frac{1}{2}A(EFOG)$ $A(DEG) = \frac{1}{2}A(EFOG)$	*Measurement in DGE* $CAC = 3$ $CPP = 3$
1.16		$ABLM$ rectangle	*Searching for a proof (1.14)* $ABLM$ rectangle (based on 1.10)	*Observation in DGE* $CAC = 3$ $CPP = 3$
1.17		$A(ABLM) =$ $A(ABE) =$ $2A(EFOG)$	*Trial and error* $A(ABE) = A(ABLM)$ $A(ABE) = 4A(FEG) = 2A(EFOG)$	*Measurement in DGE* $CAC = 3$ $CPP = 3$

Fig. 27.2 (continued)

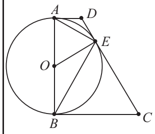
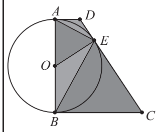
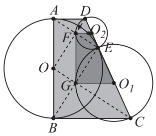
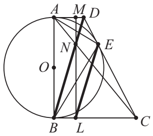
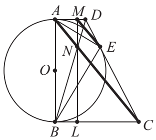
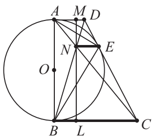
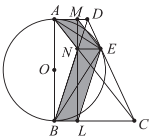
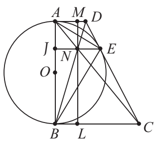
# of PP	Figure received *after auxiliary constructions*	Posed problem (PP)	*The discovery strategy* Proof outline	*Use of DGE* CAC, CPP
1.18		$\Delta ADE \sim \Delta BOE$ $\Delta AOE \sim \Delta BCE$	*Using a theorem* $\sphericalangle DAE = \sphericalangle OBE$ $\sphericalangle CBE = \sphericalangle BAE$	*No use of DGE* $CAC = 3$ $CPP = 1$
1.19		$\frac{A(BOE)}{A(ADE)} = \frac{A(BCE)}{A(AOE)}$	*Using a theorem* Based on the similarity of the triangles in 1.18	*No use of DGE* $CAC = 3$ $CPP = 2$
1.20		$FGO_1O_2 \sim ABCD$ $\frac{A(ABCD)}{A(FGO_1O_2)} = 4$	*Searching for a proof (1.14)* Poof based on 1.13 and 1.10	*Verifying in DGE* $CAC = 3$ $CPP = 3$
1.21		$N = AC \cap DB$ $ML \perp BC.$ $EL \parallel BD$	*Searching for a proof (1.14)* $\Delta ADN \sim \Delta CBN \Rightarrow \frac{AD}{BC} = \frac{AN}{NC}$ Thales Theorem $\frac{AN}{NC} = \frac{BL}{LC}$ $\Rightarrow \frac{AD}{BC} = \frac{DE}{CE} = \frac{BL}{LC} \Rightarrow EL \| BD$	*Observation in DGE* $CAC = 3$ $CPP = 3$
1.22		$ME \parallel AC$	*Symmetrical considerations (1.21)* $\Delta ADN \sim \Delta CBN \Rightarrow \frac{AD}{BC} = \frac{DN}{BN}$	*Observation in DGE* $CAC = 3$ $CPP = 3$
1.23		$NE \parallel BC$	*An association with figure in a textbook* $1.16 \Rightarrow \frac{DM}{AM} = \frac{DN}{NB}.$ $1.22 \Rightarrow \frac{DM}{AM} = \frac{DE}{EC} \Rightarrow \frac{DE}{EC} = \frac{DN}{NB}$	*No use of DGE* $CAC = 3$ $CPP = 3$
1.24		$BLEN$, $AMEN$ parallelograms	*Logical Inference from discoveries* *1.21; 1.22; 1.23*	*No use of DGE* $CAC = 3$ $CPP = 3$
1.25		$JN = NE$	*Using a theorem* Follows 1.24 and the property of the point of intersection of trapezoid	*No use of DGE* $CAC = 3$ $CPP = 3$

Fig. 27.2 (continued)

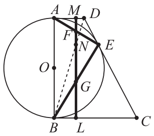
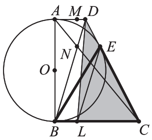
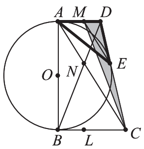
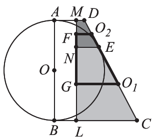
# of PP	Figure received *after auxiliary constructions*	Posed problem (PP)	*The discovery strategy* Proof outline	*Use of DGE* CAC, CPP
1.26		$F, G \in ML$ (= 1.14) $MF = FN$ $NG = GL$	*Logical inference from the discovery 1.24* F and G are points of intersection of the diagonals in parallelograms $BLEN$, $AMEN$	*No use of DGE* $CAC = 3$ $CPP = 3$
1.27		$A(CDL)$ $= A(BCE)$	*Trial and error* CEL is the intersection of DCL and BEC DEL and BEL has common basis and equal heights (1.21)	*Measurement in DGE* $CAC = 3$ $CPP = 3$
1.28		$A(CDM)$ $= A(ADE)$	*Symmetrical considerations (1.27)*	*Verifying discovery in DGE* $CAC = 3$ $CPP = 3$
1.29		$FGO_1O_2 \sim$ $ABCD \sim$ $MNED \sim NLCE$	*Intuition* Based on 1.20 There are equal ratios between the segments	*Verifying discovery in DGE* $CAC = 3$ $CPP = 3$

Fig. 27.2 (continued)

constrictions performed. The complexity of the PPI outcomes is evaluated based on the complexity of the proofs of the posed problems. The figure is followed by a detailed analysis of the complexity of the PPI process and products and of the connections between PPI strategies, the use of DGE and PPI complexity.

27.2.3 Complexity of PPI Processes and Outcomes

The complexity of the PPI processes and outcomes differ meaningfully depending on the problem-solving expertise of the participants (Elgrably & Leikin, 2021; Leikin & Elgrably, 2015) and within groups of participants with a similar level of expertise (Leikin & Elgrably, 2020). We evaluate the complexity of the posed problems with respect to (a) auxiliary constructions preceding the discovery and (b) the complexity of the proof required for the discovered property. Additionally, we analyze PPI strategies and the ways dynamic geometry is used in order to discover properties. We illustrate the analysis of complexity using a collection of properties discovered for Task 1 (Fig. 27.2).

The complexity of the auxiliary constructions performed in the course of the investigation preceding the discovery of the property was determined by the number

Table 27.1 Criteria for the evaluation of complexity of PPI

Complexity	High	Medium	Low
Complexity of auxiliary constructions	Nci ≥ 3 or Nco ≥ 2	Nci = 2 or Nco = 1	Nci ≤ 1
CAC	3	2	1
Complexity of a posed problem	Nps ≥ 7 and Ntc ≥ 5	4 ≤ Nps ≤ 6 and Ntc = 3 or 4	Nps = 1 or 2 or 3 and Ntc = 1 or 2
CPP	3	2	1

Nci no of constructions within the shape, *Nco* of constructions outside the shape, *Nps* no of proof stages, *Ntc* no of required theorems and/or concepts

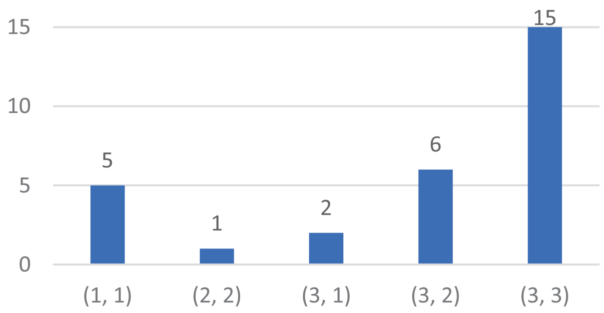


Fig. 27.3 Frequencies of the levels of complexity of the auxiliary constructions and of the posed problems

and the location of the auxiliary constructions. A property can be discovered without any auxiliary construction or using different numbers of constructions. The number of auxiliary constructions preceding the formulation of a posed problem determines the level of complexity of the auxiliary construction for the problem. Additionally, we distinguish between auxiliary construction "within the given figure" and auxiliary construction "outside the given figure", with constructions "outside" the figure considered more complex than those "within" the figure. Constructions "within/in the given figure" include (but are not restricted to): marking points on the border or in the interior part of the figure, construction of segments within the figure by connecting existing points or new points that belong to the figure, construction of special lines (medians, bisectors, altitudes, inscribed circles). The complexity of the posed problem was determined by its conceptual density, which comprises the number of concepts and properties essential for solving the problem (Silver & Zawodjewsky, 1997) combined with the length of the required proof. Table 27.1 depicts the levels of complexity of the auxiliary construction and the level of complexity of the posed problems.

Figure 27.3 summarizes the frequencies of the levels of complexity of the auxiliary constructions and of the posed problems for the 29 problems in the PPI space (Fig. 27.2). Interestingly, for 21 of 29 (72%) PPs the level of complexity of the posed problems equals the level of complexity of the auxiliary constructions.

For the other 8 PPs, auxiliary constructions at a high level led to low (in 2 PPs) and medium (6 PPs) levels of PP complexity. All 15 PPs of a high level of complexity involved auxiliary constructions at a high level.

27.2.4 Investigation Strategies

The experts discover properties using discovery strategies. A previous study (Leikin & Elgrably, 2015; Leikin & Elgrably, 2022) identified eight different types of such strategies. This study adds one additional strategy, which was created by motivating the participant to find an investigation task for his fellow students. We identified the following 8 types of investigation strategies.

Trial and Error strategy includes discovering a property that results from auxiliary constructions (if performed), checking values and ratios or identification of a special geometric figure by observation and measurement. In most of these cases dragging was used to verify the measurement. Seven posed problems (PPs) [1.3, 1.6, 1.7, 1.9, 1.14, 1.17 and 1.27 – Fig. 27.2] were discovered using the trial and error strategy.

Using a Theorem strategy involves discovering a property that can be inferred directly from a theorem or using construction to fit the theorem. The theorems could be curricula-based (ex. Ratio of segments based on the theorem about the intersection of median lines in a triangle) or be extracurricular (e.g. Ceva's Theorem). Six PPs [1.1, 1.2, 1.11, 1.18, 1.19, 1.25] were formulated all based on curricular theorems.

Building Logical Inference strategy refers to discovering a property through inference from a previous discovered property(s) or by applying several geometry theorems. In this study, 6 PPs [1.5, 1.8, 1.10, 1.13, 1.24, 1.26] were inferred from previously discovered properties.

Searching for a proof strategy includes discovering a property while searching for a proof for a different discovery, or during one of the stages of proving a different discovery. In this research, PP14 was difficult to prove and 4 PPs – 1.15, 1.16, 1.20 and 1.21 – were found in the attempt to prove PP14 using DGE.

Making an association to a theorem or a familiar problem involves discovering a property through analogy to a previously solved problem, or through association with a theorem proof. 3 PPs-1.4, 1.12, 1.23- were formulated in this way.

Using Symmetry Considerations refers to discovering a property based on symmetry of the image, or a symmetric auxiliary construction. PPs 1.22, 1.28 include properties discovered using symmetrical considerations.

Searching for a hard problem refers to discovering a property while aiming to formulate a challenging problem (for example for fellow math experts). Leikin and Elgrably (2022).

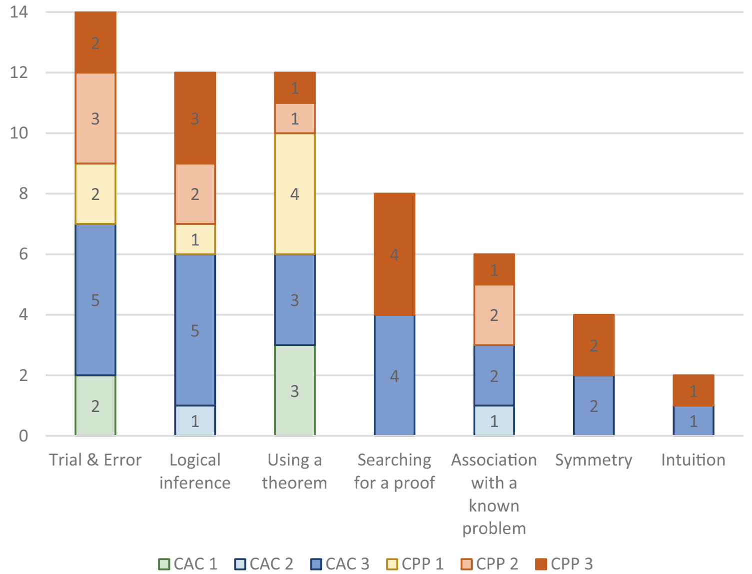


Fig. 27.4 Complexity of auxiliary constrictions (CAC) and the complexity of the posed problems (CPP) using different PPI strategies

Intuitive conjecturing is associated with conjecturing intuitively about a property and then tested using measurement and dragging. Only PP 1.29 was posed intuitively.

Different PPI strategies were associated with different levels of auxiliary constructions and led to PP of different levels of complexity. As depicted in Fig. 27.4, searching for proof, symmetrical considerations, and intuitive conclusions involve a high level of complexity of both the constructions and the problems. The other strategies lead to varying levels of complexity of auxiliary constructions and different numbers of posed problems.

27.2.5 Using DGE for Investigation and Discovery

When performing PPI tasks in DGE the participants, as usual, perform auxiliary constructions. The process of searching for a new property could be performed with or without the use of DGE. We identified 3 main ways of using the DGE in the investigation process: *observing, verifying with dragging, and measuring*. Thirteen of 29 PPs were formulated without using the DGE to discover the property. This was mainly when the new property was described using a theorem or through logical inference from one of the previous discoveries. These strategies are similar to 'proving as a source for discovery' while "a deductive argument can provide additional insight, and some form of novel discovery" (De Villiers, 2012, p. 1133).

Dragging is a critical feature of DGEs which makes investigation possible. The two main functions of dragging are *testing* and *searching* (Hölzl, 2001). Problem posing through investigation is usually associated with dragging and construction in a DGE. In line with Hölzl (1996, 2001), we identified 3 main ways of using DGE in the investigation process: observing, verifying with dragging, and measuring. By observing a figure after performing auxiliary construction a participant could hypothesize that a particular figure is of a specific type (e.g. PPs 1.8, 1.11), or draw a conclusion regarding mutual relations between elements of the figure (e.g.. the lines are parallel (PP 1.22) or the three points are on one line (PP 1.14)). Verifying with dragging followed intuitive discoveries (PP #29), discoveries based on symmetrical considerations (PP #28) or discoveries made while searching for a proof. Measuring was used mainly when using the trial and error strategy; this included construction and measuring the lengths of newly created segments.

Figure 27.5 depicts the distribution of different methods of DGE use among different PPI strategies and the frequencies of the PPI strategies. It clearly shows that logically developed PPs formulated using logical inference or using a theorem were based on auxiliary constructions without additional use of DGE. Trial and error included mainly measuring and verifying with dragging, and searching for proof included observing and verifying with dragging.

27.2.6 *On the Structure of the Set of Posed Problems*

As mentioned above, the complexity of a posed problem was determined by its conceptual density, which comprises the number of concepts and properties essential for solving the problem (Silver & Zawodjewsky, 1997) combined with the

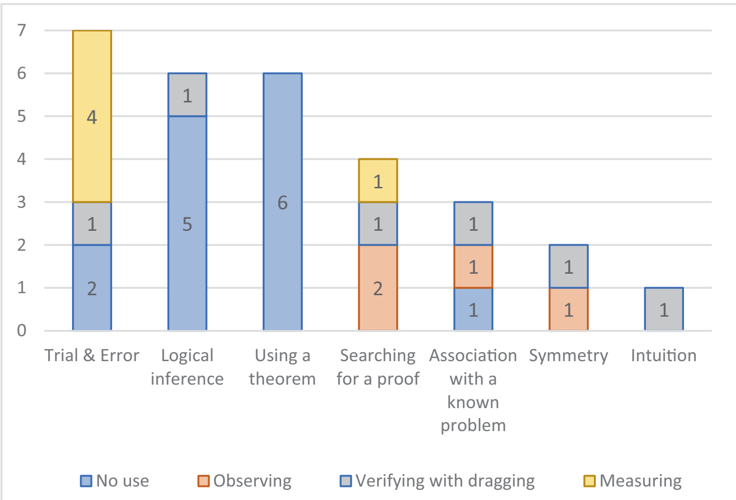


Fig. 27.5 Frequencies of use of PPI strategies and distribution of use of DGE per PPI strategy

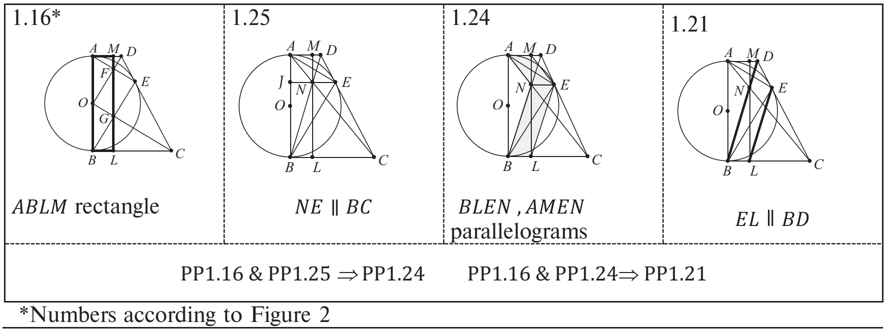


Fig. 27.6 Logical sequences of the posed problems

length of the required proof. According to this definition, since proof of PP1.24 (Fig. 27.6) can be performed using properties in problems PP1.16, and PP1.25, PP1.24 is more complex than PP1.16 and PP1.25. Similarly, PP1.21 follows from PP1.24 and thus PP1.21 is more complex than P1.24. The logical relationships in the set of the posed problems and the hierarchy of the conceptual density of the tasks do not correspond to the order in which the problems were posed. Thus we argue that the set of the posed problems is semi-structured. This structure creates an opportunity for the activity of structuring the set of tasks with the participants in the PPI activity. We consider sets of PPs to be a powerful source for constructing Stepped Tasks, which are described in the next section of this chapter.

27.3 Stepped Tasks

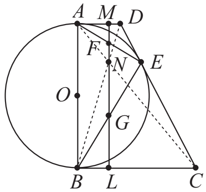
We suggest that readers solve the two following problems, Problems 1 and 2, which are used later in this chapter to illustrate Stepped Tasks (Fig. 27.7).

Leron (1983) contrasted between the linear (bottom-up) method of proving and the structural method of proving. The structural method of proving is based on "arranging the proof in levels, processing from the top down while the levels themselves consist of short autonomous "modules", each embodying one major idea of the proof" (p. 174). In this sense, in the problem chain in Fig. 27.6, problems 1.16 and 1.25 are modules in the proof of problem 1.24. Such structuring can be performed either in a bottom-up or top-down manner. We go one step further when structuring complex problem-solving using Stepped Tasks. We demonstrate that sets of problems created using PPI activity can be used in designing Stepped Tasks. In what follows, Stepped Tasks 1 and 2 illustrate Stepped Tasks constructed using the set of posed problems presented in Fig. 27.2.

Problem 1 (PP 1.26 in Figure 2)

Given (I) Segments AD, BC and DC are tangent to the circle with diameter AB and center O.
E is the tangent point of DC with the circle O.
$N = AC \cap BD$
(II) M is on BC and L is on AD, $N \epsilon ML$,
$ML || AB = AC \cap BD$
$F = AE \cap ML, G = BE \cap ML$

Prove: $(a)\ MF = FN,\ (b)\ NG = GL$



Problem 2 (based on PP1.13 in Figure 2)

Given (I) see in Problem 1
(II) Circle O_1 with diameter EC,
Circle O_2 with diameter ED,
XY is tangent to cicles O_1, O_2

Prove: (*a*) XY parallel to AB
(*b*) $A(ABCD) = 4 \cdot A(XYO_1O_2)$

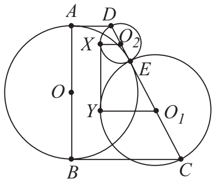


Fig. 27.7 Problems 1 and 2

27.3.1 Characterization of Stepped Tasks

A Stepped Task first asks students to solve a complex mathematical problem called a "target problem". The target problem is followed by a top-down structured set of problems organized in steps of decreasing levels of complexity. (Leikin, 2019; Leikin & Ovodenko, 2021). The variations in the problem's complexity are based on (a) the conceptual density of the problem, as determined by the number of concepts and theorems required for solving, (2) the length of the solution/proof chain and (c) the level of advanced mathematical knowledge required to solve the problem.

Stepped Tasks are intended to advance students' conceptual understanding, conceptual connections, strategical reasoning and self-regulated learning. Stepped Tasks provide multiple opportunities to match the challenge embedded in a task to students' mathematical knowledge and skills. Students are allowed to solve this target problem with or without using a number of steps that include other – less challenging – problems, the solutions to which can lead to solving the target problem.

Stepped Task 1

Construction of Stepped Tasks is not simple. In Leikin and Ovodenko (2021) we demonstrate the construction of Stepped Tasks through variation of the level of mathematical challenge by decreasing the level of conceptual density of the target problem. In this chapter we describe how a collection of posed problems can be used to construct Stepped Tasks by developing logical chains of Posed Problems. A teacher can lead the construction of Stepped Tasks with students following PPI activity.

Stepped Task 2

A Stepped Task starts with a target problem P. If P is too difficult for the students, they are provided with an opportunity to solve problems at Step-1, each of which is less complex than P. The problems at Step-1 are not necessarily sub-problems of P. However, solving problems at Step-1 evokes the use of concepts and tools relevant to P. After solving problems at Step-1, students are presumed to be able to solve the target problem. If the problems at Step-1are still too difficult, students can solve problems at Step-2 and then solve either the target problem or Step-1 problems.

Figures 27.8 and 27.10 depict Stepped Tasks designed for Problems 1 and 2 using the collection of posed problems depicted in Fig. 27.2. Figures 27.9 and 27.11

Solve target problem *P1*

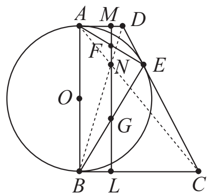
P1 (PP-1.26)

Given (I) Segments AD, BC and DC are tangent to the circle with diameter AB and center O.
E is the tangent point of DC with the circle O.
$N = AC \cap BD$

(II) M *is on* BC and L *is on* AD

(III) $N = AC \cap BD$ $N \in ML$, $ML || AB$
$F = AE \cap ML, G = BE \cap ML$

Prove: (a) $MF = FN$, (b) $NG = GL$



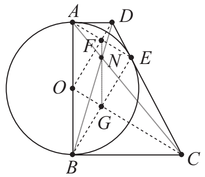
If proof is completed, find an additional proof
or *If needed go to Step 1*

Step 1 – Solve problem P1-1.1 and P1-1.2 and then P1

P1-1.1 (PP-1.14)

Given: see givens (I and II) in P1
(III) $F = AE \cap OD, G = BE \cap OC$

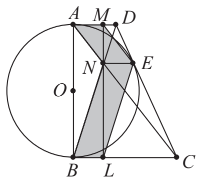
Prove: $N \in FG$



P1-1.2 (PP-1.24)

Given: see givens (I and II) in P1

Prove: $AMEN$ and $BLEN$ are parallelograms



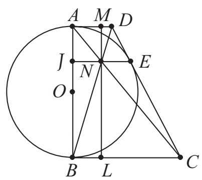
When proof is completed return to target problem P1
or *If needed go to Step 2*

Step 2 - Solve problems P1-2 and then P1 or go to Step 1

P1-2 (PP-1.25)

Given: see givens I and II in P1
$NE \cap AB = J$
$M \in AD$, $L \in BC$
$N \in ML$, $ML || AB$,

Prove: $(\boldsymbol{a})$ $NE || BC$, $(\boldsymbol{b})$ $NE = BL$



When proof is completed return to problem P1

Fig. 27.8 Stepped Task 1

PP1.16 & PP1.25 ⇒ PP1.24   PP1.16 & PP1.24 ⇒ PP1.26

Fig. 27.9 Logical chain of the posed problems constituting the basis of Stepped Task 1

Target problem: *Solve P2*

P2

Given (I) Segments AD, BC and DC are tangent to the circle with diameter AB and center O.
E is the tangent point of DC with the circle O.
Circle O_1 with diameter EC,
Circle O_2 with diameter ED,
(II) XY is tangent to cicles O_1, O_2

Prove: (*a*) XY parallel to AB
(*b*) $A(ABCD) = 4 \cdot A(XYO_1O_2)$

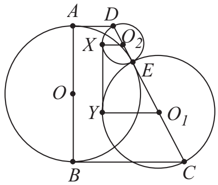


fig X.a

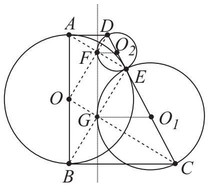
If proof is completed, find an additional proof
or *If needed go to Step 1*

Step 1 – Solve problem P2-1 and then P2

P2-1:
Given: see givens I in P2
(II) $F = AE \cap OD, G = BE \cap OC$

Prove: FG is tangent to circles O_1 and O_2

Return to target problem P1
or *If needed go to Step 2*



Step 2 - Solve problems P2-2.1, P2-2. 2 and then P2

P2-2:
Given: **P2-1**

Prove: (a) G is on the Circle O_1,F is on the Circle O_2
(b) $GO_1 \,||\, FO_2 \,||\, BC$

Return to the target problem P2

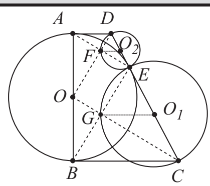


Fig. 27.10 Stepped Task 2

PP1.10 & PP1.12 ⇒ PP1.13   PP1.13 & PP1.10 ⇒ PP1.20

Fig. 27.11 Logical chain of the posed problems constituting the basis of Stepped Task 2 demonstrate logical changes to the posed problems which are at the base of the Stepped Tasks.

27.4 Concluding Notes

Stepped Tasks are connected to Schwarz and Skurnik's (2003) argument that "people are more likely to rely on their preexisting knowledge structures and routines, which have served them well in the past when things go smoothly and they do not face any hurdles " (p. 264). Successful (positive) problem solving is linked to top-down processing, whereas when experiencing difficulty, bottom-up processing is used, by paying increased attention to the problems (Schwarz & Skurnik, 2003; Wegner & Vallacher, 1986). The major difference between bottom-up and top-down strategies is that top-down processing is linked to use of knowledge-driven strategies while bottom-up processing is associated with data-driven strategies of information processing (e.g., Bless & Schwarz, 1999). Leron (1983) argued that a structured approach to proof and proving can be more effective than a linear - bottom-up - approach, which is used more often. The steps in Stepped Tasks are aimed at the development of learners' feeling of mathematical structure, in order to develop the skills needed for structured proving.

From a didactical point of view, Stepped Tasks are directed at the promotion of self-regulated learning and evaluation as well as the advancement of strategical reasoning. We describe how spaces of problems posed through PPI activity can serve as a basis for designing Stepped Tasks (see the next section of this chapter). The top-down approach to problem-solving implemented in Stepped Tasks is considered a goal-oriented one since the goal of the problem-solving process is explicitly presented to the participants at the beginning. Students can be engaged with the Stepped Tasks individually and decide on the problem-solving path appropriate to them. Alternatively, students can work with Stepped Tasks in a collaborative learning setting in which a group of students makes a joint decision about moving among the steps. In collaborative learning settings, the Stepped Task can be used by more knowledgeable students who can provide their own scaffolding for students who are struggling. Additionally, Stepped Tasks allow bottom-up implementation if teachers prefer to use this mode of implementation.

Top-down and bottom-up approaches are directed at similar goals but are different in terms of the ways in which they achieve them. Top-down teaching starts with the target task, which is the main goal of the mathematical activity. The solver has to uncover the necessary problem-solving strategies and mathematical concepts, which are not presented explicitly, and find the meaning of the problem by applying his/her own knowledge and skills. That is why the top-down approach is mostly student-regulated. Bottom-up teaching is more teacher-directed and focuses on ways of decoding and simplifying each component of a problem. The bottom-up teaching approach lacks an emphasis on learning the complete picture. An analogy can be made to a situation in which a person putting together a puzzle has to complete a picture that is known to him by searching independently for pieces of the picture. This is in contrast to completing a puzzle with guidance from a parent or a friend who knows the picture, with the picture appearing as if by magic at the hand of a more experienced individual.

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Chapter 28
Flow and Variation Theory: Powerful Allies in Creating and Maintaining Thinking in the Classroom

Peter Liljedahl

28.1 Introduction

I think that we can all agree that *engagement* is important to the process of learning mathematics. Likewise, I think we can all agree that *thinking* is also important to the process of learning mathematics. But what, exactly, is it that we mean when say that we want our students to be engaged and that we want our students to be thinking and what role do they play in the teaching and learning of mathematics? In this chapter, I take a closer look at both of these constructs and explore what happens when students encounter mathematics curriculum in classrooms specifically designed to foster and sustain engagement and thinking. Given the focus of the book, I begin by offering the reader two tasks to try to solve.

28.1.1 Diamonds

Take a deck of cards and pull out all the diamonds. Now, arrange the first four cards of this suit as in Fig. 28.1 and hold them in your hand.

Now take the top card—ace of diamonds—and place it face up on a table. Take the next card—three of diamonds—and move it to the back of the stack of cards in your hand. Take the top card—which is now the two of diamonds—and place it face up on the table on top of the ace of diamonds. Place the next card at the back of the cards in your hand. Continue this process of alternating between placing a card on

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Fig. 28.1 Starting configuration of cards

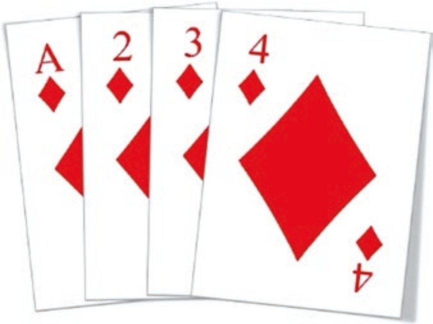

Fig. 28.2 Finished configuration of cards

the table and placing a card at the back of the cards in your hand until all the cards are on the table. When you are done you will have the cards arranged in rank order from ace to four (see Fig. 28.2).

This alternating process of placing a card on the table and placing a card at the back of the cards in your hand has transformed the seemingly random arrangement of cards in Figure 28.1 into the orderly sequence in Figure 28.2. Now try it with five cards. What does the order of the cards need to be to begin with such that after you have completed the aforementioned alternating process the cards on the table will be ordered ace through five? Take a moment and see if you can do it. When you can do five cards, try six, then seven, and so on until you can do all 13 cards. What did you notice about your experience as you worked on this task?

28.1.2 Answers

Now let's try a different task. Below is a list of five answers (see Fig. 28.3). These are the answers to five arithmetic expressions each consisting of two numbers and an operation (Fig. 28.3).

__ □ __ = 17

__ □ __ = 2

__ □ __ = 21

__ □ __ = 3

__ □ __ = 2

Fig. 28.3 Five answers

Using each of the numbers from 1 to 10 exactly once, and each of the operations (+ − × ÷) at least once take a moment to find what the five expressions are such that the answers are 17, 2, 21, 3, 2.

When you have found these expressions, try it for the five answers 10, 14, 1, 20, 16. When you have solved that one, try it for the following sequences of answers:

3, 3, 3, 3, 24
2, 2, 2, 2, 9
2, 3, 7, 7, 7
1, 2, 3, 4, 5

What did you notice about your experience as you worked on this task?

28.2 Engagement and the Optimal Experience

In the 1970's Mihály Csíkszentmihályi, a Hungarian born psychologist working at the University of Chicago became very interested in trying to understand a specific form of engagement that he referred to as the *optimal experience* (Csíkszentmihályi, 1990, 1996, 1998).

> a state in which people are so involved in an activity that nothing else seems to matter; the experience is so enjoyable that people will continue to do it even at great cost, for the sheer sake of doing it. (Csíkszentmihályi, 1990, p.4)

If you worked to solve either, or both, of the aforementioned sequences of tasks it is likely that you just had an optimal experience. Wanting to better understand this rare and powerful phenomenon, Csíkszentmihályi gathered cases from people he thought were most likely to have optimal experiences—musicians, artists, athletes, scientists, and mathematicians. Over time he gathered enough cases that he could begin to analyze them (Csíkszentmihályi, 1990). What emerged was a set of nine characteristics that accompanied the optimal experience, the first six of which were:

1. Action and awareness are merged
2. Distractions are excluded from consciousness
3. There is no worry of failure
4. Self-consciousness disappears
5. The sense of time becomes distorted
6. The activity was autotelic – a reward unto itself

That is, he noticed that whenever someone had an optimal experience, they lost track of time and much more time passed than the person realized (5). He noticed that when someone was having an optimal experience, they were un-distractible and unaware of things in their environment that would otherwise interfere with their focus (2). He noticed that their actions became a seamless and efficient extension of their will (1). And he noticed that they became less self-conscious (4), stopped worrying about failure (3), and that they were doing the activity for the sake of doing it and not for the sake of getting it done (6).

These six characteristics are all internal to the doer—they are how someone experienced the phenomenon of an optimal experience. If you engaged in the two tasks above you may have experienced some or all of these characteristics. If you did not yet try the tasks, I suggest you go back and do them now and make note of your experience vis-à-vis Csíkszentmihályi's six internal characteristics.

Csíkszentmihályi also noticed that whenever there was an optimal experience there were three additional characteristics that were external to the doer and present in the environment in which the optimal experience was taking place.

7. Clear goals every step of the way
8. Immediate feedback on one's actions
9. A balance between the ability of the doer and the challenge of the task

The two aforementioned tasks were designed specifically to have these three characteristics. They each have a clear goal. For the card task, the goal is to find the initial starting sequence for *N* cards such that when you follow the alternating procedure described above the cards will end in rank order from ace to *N*. For the answers task, the goal is to figure out how to organize the ten numbers (1–10) and the four operations (+ − × ÷) into five expressions such that they produce the five intended answers.

These two tasks also provide immediate feedback on your actions. You will notice that I did not provide any solutions for these tasks—the solutions are unnecessary. You will know immediately if you have solved the task correctly or not. In the case of the cards, you can run the process and see if it results in a rank order. For the answers, you can perform the arithmetic and see if you arrive at the five intended answers and you can count to see that each number (1–10) is used exactly once and each operation (+ − × ÷) is used at least once. The tasks, themselves, tell you if you are right or wrong—the tasks provide immediate feedback.

And both of the tasks maintained a perfect balance between your ability, as the doer, and the challenge of the task—something that comes into sharper focus when we consider the states that, I am guessing, you were not in when solving the two

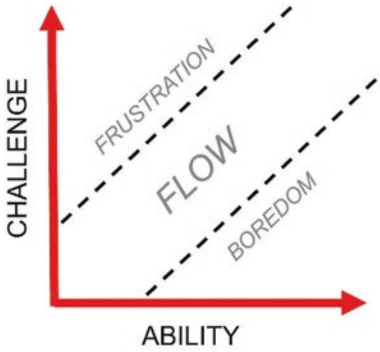

Fig. 28.4 Balance and imbalance between ability and challenge

tasks at the beginning of the chapter. For example, had the challenge of the activity far exceeded your ability, then you would have experienced a feeling of frustration (see Fig. 28.4). Conversely, if your ability far exceeds the challenge of the task, then you would have experienced boredom (see Fig. 28.4). If you were neither frustrated nor bored then it means that there existed a balance between your ability and the challenge of the task and you would have been in a state of, what Csíkszentmihályi refers to as *flow* (see Fig. 28.4). Flow is where the deep engagement of the optimal experience occurs.

28.3 Maintaining Flow

But flow is not just a collection of static ordered pairs within a well-defined region on the ability-challenge graph. Flow is a dynamic space wherein the balance between ability and challenge is not only created but also maintained. That is, as a person's ability increases, so too must the challenge of the task (Liljedahl, 2018, 2020). For example, solving the card task for five cards is not that challenging. But in doing so you learn something that is necessary, but not sufficient, to solve the task for six cards. Hence, six cards are more challenging than five cards. And seven cards are more challenging than six. And so on. As your ability increases with each stage of the task, the challenge of the task evolves to maintain the state of flow (see Fig. 28.5).

The same is true for the answers task. The first set of answers (17, 2, 21, 3, 2) has some obvious arithmetic expressions associated with them. For example, 17 must be added (8 + 9 *or* 7 + 10) and 21 must be multiplied (3 × 7). Given that you can only use each number once, this then means that 17 must be 8 + 9. You have learned something about how to reason your way through these tasks. And what you have learned will be necessary, but not sufficient, for the next set of five answers—as your ability increases the challenge of the task, likewise, increases (see Fig. 28.5).

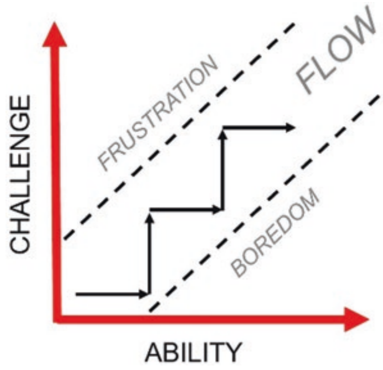

Fig. 28.5 Increases in challenge as ability increases

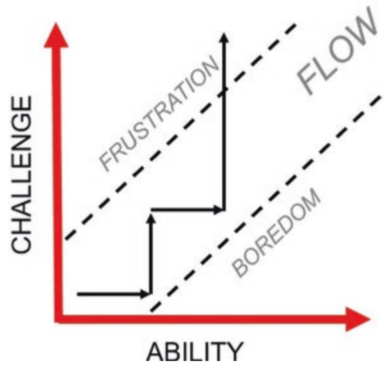

Fig. 28.6 Too big an increase in challenge

The consequences of not maintaining this balance are that the doer could be pushed into a state of frustration (see Fig. 28.6). For example, if you solved the card problem for five cards and six cards and thought that you were now ready to jump to 13 cards, this increase in the challenge would likely have been too great for the abilities you would have developed thus far. An imbalance would have been created that could put you into a state of frustration.

Should this occur, the relationship between ability and challenge can be re-balanced, and the state of frustration can be mitigated, if the doer were to receive a hint (Liljedahl, 2018, 2020). However, there are two types of hints (see Fig. 28.7)—hints that decrease challenge and hints that increase ability (Liljedahl, 2018, 2020). The first of these is quicker to give and can be achieved by shifting the doer to a simpler task or providing a partial answer. So, for example, if you were frustrated that you could not solve the card task for 13 cards may tell you that the eight, nine, and ten of diamonds will need to be in the fourth, eighth, and twelfth positions, respectively.

The second type of hint—increase ability—takes longer to give and requires the doer to either be reminded of a strategy or to receive a strategy. So, for example, I may remind you that placing the first seven cards into every second position to begin with—which I am sure you have already figured out—is actually a strategy of

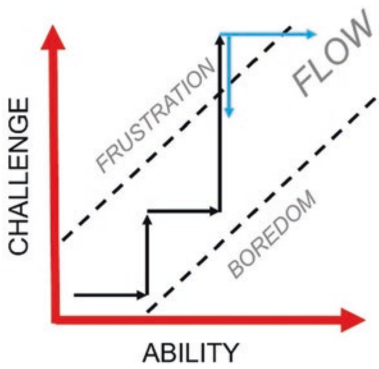

Fig. 28.7 Giving a hint to mitigate frustration

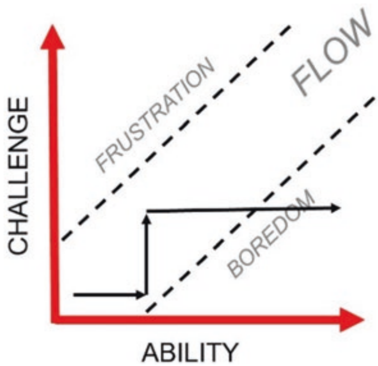

Fig. 28.8 Too big an increase in ability

placing approximately *half* of the cards you have yet to place into every second of the *available* positions. Other than how long it takes to give these hints, the main difference is that hints that decrease challenge are only useful in that moment whereas a hint that increases ability continues to be useful even as they move on to the next task in the sequence.

A state of imbalance between the doer's ability and the challenge of the task could also result in the doer being pushed into a state of boredom (see Fig. 28.8). For example, if for the answers task I had provided you with a different progression of answers wherein after you had solved the first set (17, 2, 21, 3, 2), I asked you to solve (56, 17, 7, 3, 4) and (90, 15, 30, 2, 2)—both of which are built on the exact same principles as the first set of answers—you would likely have become bored. Your ability would have increased without a commensurate increase in challenge (Fig. 28.8).

This too can be mitigated through two different strategies—increasing the challenge of the task (Fig. 28.9) and shifting the mode of engagement of the task (Fig. 28.10). Increasing the challenge is a simple matter of giving a more difficult task to solve. However, depending on how bored the doer is, or for how long they have been bored, giving the next task in the sequence may not increase the challenge enough (Fig. 28.9). For example, if after having solved the sequence up to (10, 14, 1, 20, 16), I sensed that you were getting bored, I may jump you up to the task (2, 3, 7, 7, 7) to reengage you.

Fig. 28.9 Increasing the challenge to mitigate boredom

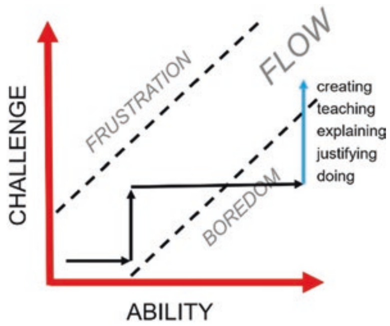

Fig. 28.10 Shifts in mode of engagement

Alternatively, I may increase the challenge by shifting your mode of engagement with the task (Liljedahl, 2020) (see Fig. 28.10). For example, when you are solving a task, your mode of engagement is *doing*—you are *doing* the task. This is the least challenging way to engage with a task. If I ask you if you think the solution you have found for (10, 14, 1, 20, 16) is unique, I just shifted your mode of engagement from doing to *justifying*. Justifying is more challenging and involves you trying to convince yourself that you are correct. When you are convinced that the solution is unique, I may ask you to *explain* to me how you know that the solution is unique. Explaining is more challenging than justifying as it requires the articulation of thought for an audience outside of yourself. When you are done explaining to me how you know that (10, 14, 1, 20, 16) is unique, I may direct you to someone who is struggling with knowing if their solution is unique or not and ask you to *teach* them something that could help. If we subscribe to the notion that teaching is different from telling or explaining, then this is another increase in challenge. Once you have finished teaching this person, I may finally ask you to *create* the next set of answers for that person to engage in—a set of answers that is more challenging than the one they have just finished. Creating is the most challenging mode of engagement as it requires you to not only see the didactics of the situation, but also the pedagogical needs and affordances of the next task in the sequence.

Taken together, maintaining flow, then, becomes a series of actions in and around how we manage hints and extensions to increase or decrease challenge and increase ability (see Figs. 28.7 and 28.9).

Flow is where the optimal experience happens and, as such, flow is where deep engagement happens. More than this, however, flow is where thinking happens. This is not to say that this is the only place that thinking happens, but it is a state in which we know thinking takes place—and, as such, flow is a central part of the Building Thinking Classroom framework (Liljedahl, 2018, 2020).

28.4 Building Thinking Classrooms

Building Thinking Classrooms (Liljedahl, 2020) is a teaching framework that was developed in response to the realization that much of what happens during a mathematics lesson is not thinking. In fact, the baseline data showed that in a typical lesson about 20% of students spend approximately 20% of the time thinking—8–12 minutes per hour—while the other 80% of students spend no time thinking (Liljedahl, 2020). This is a problem. If studetns are not thinking, they are not learning (Lucariello et al., 2017; Seel, 2012).

Research has shown that the normative practices present in many classrooms are promoting, in both explicit and implicit ways, non-thinking behaviors such as mimicking among students (Liljedahl & Allan, 2013a, b). These normative structures permeate classrooms around the world and are so entrenched that they transcend the idea of classroom norms (Cobb et al., 1991; Yackel & Cobb, 1996) and can only be described as institutional norms (Liu & Liljedahl, 2012)—norms that have extended beyond the classroom, even the school building, and have become ensconced in the very institution of school and fabric of what it means to teach

Much of how classrooms look and much of what happens in them today is guided by these institutional norms—norms which have not changed since the inception of an industrial-age model of public education. Yes, desks look different now, and we have gone from blackboards to greenboards to whiteboards to smartboards, but students are still sitting, and teachers are still standing. Although there have been many innovations in assessment, technology, and pedagogy, much of the foundational structure of school remains the same. If we want to promote and sustain thinking in the classroom, these norms are going to have to change (Liljedahl, 2020).

Over the course of 15 years, and through the conducting of thousands of micro-experiments with over 400 practicing teachers, a series of 14 practices emerged that break away from the aforementioned institutional normative ways of teaching and have been proven to get more students thinking and thinking for longer (Liljedahl, 2020). Each of these 14 practices is a response to one of the following 14 questions:

1. What are the types of tasks used?
2. How are collaborative groups formed?
3. Where do students work?
4. How is the furniture arranged?
5. How are questions answered?
6. When, where, and how are tasks given?
7. What does homework look like?
8. How is student autonomy fostered?
9. How are hints and extensions used?
10. How is a lesson consolidated?
11. How do students take notes?
12. What is chosen to evaluate?
13. How is formative assessment used?
14. How is grading done?

Although each of these 14 practices, on their own and in concert, have been empirically shown to increase student thinking in the classroom (Liljedahl, 2020) the visually defining qualities of a thinking classroom is that (1) students solve thinking tasks (2) in visibly random groups (3) on vertical non-permanent surfaces and that hints and extensions (9) are used to maintain high levels of thinking and engagement by maintaining flow.

28.4.1 Thinking Tasks

If we want our students to think, we need to give them something to think about—something that will not only require thinking but will also encourage thinking. In mathematics, this comes in the form of a problem-solving task, and having the right task is important. The research (Liljedahl, 2020) revealed that when first starting to build a thinking classroom it is important that these tasks are highly engaging non-curricular problem solving tasks—such as the two tasks offered at the beginning of this chapter. As the culture of thinking begins to develop, there needs to be a transition to using curriculum tasks. The goal of thinking classrooms is not to get students to engaging with non-curricular problem solving tasks day in and day out—that turned out to be rather easy (Liljedahl, 2020). Rather, the goal is to get more students thinking, and thinking for longer periods of time, within the context of curriculum.

28.4.2 Visibly Random Groups (VRG)

Once we have the thinking task students need someone to think with. We know from research that student collaboration is an important aspect of classroom practice because when it functions as intended, it has a powerful impact on learning (Edwards & Jones, 2003; Hattie, 2009; Slavin, 1996). How groups have traditionally been formed, however, makes it very difficult to achieve the powerful learning we know is possible. Whether students are grouped strategically (Dweck & Leggett, 1988; Hatano, 1988; Jansen, 2006) or students are allowed to form their own groups (Urdan & Maehr, 1995), 80% of students enter these groups with the mindset that, within this group, their job is not to think (Liljedahl, 2020). However, when frequent and visibly random groupings were formed, within 6 weeks, 100% of students entered their group with the mindset that they were not only going to think but that they were going to contribute. In addition, frequent and visible random groupings were shown to break down social barriers within the room, increase knowledge mobility, reduce stress, and increase enthusiasm for mathematics (Liljedahl, 2014).

28.4.3 Vertical Non-permanent Surfaces (VNPS)

Once students have a thinking task and collaborators to think with, they need somewhere to do their thinking. One of the most enduring institutional norms that exist in mathematics classrooms is students sitting at their desks—or tables—and writing in their notebooks. This turned out to be the workspace least conducive to thinking. What emerged as optimal from this research (Liljedahl, 2019, 2020) was to have the students standing and working on vertical non-permanent surfaces such as whiteboards, blackboards, or windows. It did not matter what the surface was, as long as it was vertical and erasable (non-permanent). The fact that it was non-permanent promoted more risk-taking and the fact that it was vertical prevented students from disengaging. Taken together, having students work, in their random groups, on VNPS had a massive impact on transforming previously passive learning spaces into active thinking spaces where students think, and keep thinking, for upwards of 60 minutes.

28.4.4 Using Hints and Extensions to Maintain Flow

As mentioned, getting students to think was not the challenge. An engaging non-curricular task is easy to find (Liljedahl, 2020)—take the two tasks at the beginning of this chapter, for example. The challenge is to not only get them to think about curricular tasks but also to maintain that thinking for extended periods of time. Using ideas from Csíkszentmihályi's theory of flow and the optimal experience (Csíkszentmihályi, 1998, 1996, 1990), coupled with the research on maintaining flow (Liljedahl, 2018, 2020), offered a path forward. The key, it turned out, was to use hints and extensions to manage flow as students worked through a sequence of curricular tasks that gradually increased in challenge. For example, consider the following sequence of tasks on the curricular topic of factoring quadratics:

1. $x^2 + 8x + 7$
2. $x^2 + 5x + 6$
3. $x^2 + 7x + 12$
4. $x^2 + 14x + 24$
5. $x^2 + 10x - 24$
6. $x^2 + 4x - 12$
7. $x^2 - x - 12$
8. $x^2 - 2x - 12$
9. …

This sequence gets progressively more challenging with each task. The first task is simple in that there are only two factors of 7 that students need to consider. The second task has more factors to consider, the third task has more, and the fourth task has even more. The fourth task requires students to now think about both the positive and negative factors of −24, and so on.

In addition to a consideration for progressive increase in challenge, this sequence was created using the two main principles of variation theory (Marton & Tsui, 2004). The first principle is that we can only see variation against a backdrop of non-variation. That is, before something changes, it has to stay the same. We see this in the transition from task 4 to 5. Prior to making the third coefficient negative, it has been positive for four tasks. The second principle is that only one thing is varied at a time. So, although the last task is very different from the first, at every stage only one thing was varied. First, we varied the number of factors that the third coefficient could provide. Then we made the third coefficient negative, then the second coefficient became negative. And so on.

28.5 Research Questions and Methodology

The goal of the Building Thinking Classrooms research has been to increase the number of students who were thinking and increase the percentage of time that they spent thinking. In this chapter, I present some of the results of this research vis-à-vis the specific practice of using hints and extension to manage flow as students work through sequences of curricular tasks. I exemplify these results through the presentation of three cases:

Case I: grade 10 students (ages 15–16) working through the aforementioned sequence of factoring quadratic tasks.
Case II: grade 5 students (ages 10–11) working through a sequence of one and two-step algebra tasks.
Case III: grade 11 students (ages 16–17) as they work through a sequence of radical expression tasks.

When conducting the thinking classroom research, I would, as much as possible, try to isolate a practice and study whether it increased the number of students thinking as well as the number of minutes they were thinking for (Liljedahl, 2020). For practices such as thinking tasks (1), visibly random groups (2), and vertical non-permanent surfaces (3) this worked quite well and revealed how each practice, on their own or in concert with each other or other thinking practices, produced drastic improvements in student thinking (Liljedahl, 2014, 2018, 2020). For the research into the use of hints and extensions to manage flow, however, this research was always done with thinking tasks, VRG, and VNPS having already been established within the classrooms. This created noise in the data as did the fact that many of the non-curricular thinking tasks used to introduce VRG and VNPS—including the two presented at the beginning of this chapter—were already designed with flow in mind. As such, in each of the aforementioned cases, students already had some experiences working within settings that were designed to increase thinking in general, as well as to induce and maintain flow in particular. Having said that, in cases I and II, the students had had no experience working in such an environment on sequences of curricular tasks and in all three cases, the students were seeing a new curricular topic for the first time.

Taken together, the research questions that I was able to answer, and that I report on for each case, are:

1. How much thinking, as measured in minutes per student, was visible for each case?
2. How far does each group get into the sequence of curricular tasks for each case?

Thinking is an internal and largely invisible process—I cannot see what a student is thinking or even if they are thinking. The deep engagement that occurs during flow, although manifesting through internal experiential components, is also an embodied experience that manifests itself in a physical display—through the look in a student's eyes, the way they lean into a task, the way they speak and gesture. That is, engagement is easily seen. And given that thinking is happening during flow, we can use the embodied physical display of engagement to code for thinking. This is how thinking was accounted for. This is not to say that they could not be thinking when not exhibiting engagement. But this is more difficult to deduce and, as such, I only coded for the thinking that happened in the company of engagement.

So, to gather data for the first research question, I simply ran a constant inventory of every student in the classroom and recorded on a map of the classroom whether a particular student, in a particular place, was engaged or not. If they were engaged, as demonstrated through their embodied behaviors, I assumed they were thinking.

Data for the second research question were likewise collected through the same constant inventory by noting on the map which task in the sequence each group was working on and at what time that observation was made. Although this did not always capture the exact time when they began or finished a task in the sequence, it did give an approximate time stamp and an approximate length of time they spent on each task. Taken together, the constantly running inventory for the two types of data allowed me to capture data for each student and each group every 3–5 minutes depending on the number of students and the number of groups.

In all cases, there were brief follow-up interviews with 3–5 students and a lengthier interview with the teacher. The students were interviewed immediately after the activity finished, which sometimes occurred in the dying minutes of the lesson and sometimes in the first few moments after the lesson had ended. The teacher was interviewed either at lunch or after school depending on what time of the day the lesson occurred. The goal for these interviews was twofold—first to triangulate some of the observations I had made during the constantly running inventory while at the same time probing deeper to find out their experiences and thoughts behind what I was observing.

In what follows I first present the results and analysis for each case followed by a discussion of some themes that emerged across the three cases.

28.6 Case I: Factoring Quadratics

As mentioned, the data from this case comes from a grade 10 classroom (ages 15–16). The class had 29 students (17 females, 12 males) and was situated in a school in a working-class neighborhood. The course was called *Foundations of Mathematics and Precalculus 10* (FMPC 10) which is designed for students on a pathway towards some form of post-secondary education (university, college, technical or vocational school). FMPC 10 is one of two grade 10 math courses available to students—the other being called *Applications and Workplace 10* (A&W 10), which is designed for students going into trades. All grade 10 students must take one of these two grade 10 mathematics courses to eventually graduate high school. In the school wherein the study was taking place there were approximately 300 grade 10 students, with 80% enrolled in FMPC 10. This is all to say that the data was gathered in a typical school in a typical grade 10 mathematics classroom with the full diversity that comes with such settings.

As mentioned, the sequence of tasks they were worked through was on the topic of factoring quadratics and was, in essence, an extension of the sequence presented above. In keeping with practice 6 of the Building Thinking Classroom framework, the teacher presented the task right at the beginning of the lesson and with the students gathered around him at a vertical whiteboard.

Teacher Let's start with a bit of review.
How would I distribute $(x + 2)(x + 5)$?
[Teacher writes on the board $(x + 2)(x + 5)=$*]*

Students $x^2 + 7x + 10$.
[Teacher writes on the board $(x + 2)(x + 5) = x^2 + 7x + 10$*]*

Teacher Ok. So what if my answer was $x^2 + 8x + 7$? What would the question be?
[Teacher writes on the board ( )( ) $= x^2 + 8x + 7$*]*

The teacher then put the students into random groups of three and sent them to their VNPS stations. Over the course of the next 55 minutes the teacher then gave extensions to groups when they were done a task, gradually working through the following sequence of factoring quadratic tasks:

1. $(x + 2)(x + 3) = x^2 + 8x + 7$
2. $(\quad)(\quad) = x^2 + 5x + 6$
3. $(\quad)(\quad) = x^2 + 7x + 12$
4. $(\quad)(\quad) = x^2 + 14x + 24$
5. $(\quad)(\quad) = x^2 + 10x - 24$
6. $(\quad)(\quad) = x^2 + 4x - 12$
7. $(\quad)(\quad) = x^2 - x - 12$
8. $(\quad)(\quad) = x^2 - 2x - 24$
9. $(\quad)(\quad) = x^2 - 6x - 16$
10. $(\quad)(\quad) = x^2 - 0x - 16$
11. $(\quad)(\quad) = x^2 - 25$
12. $(\quad)(\quad) = x^2 - 49$
13. $(\quad)(\quad) = x^2 - 10x + 24$
14. $(\quad)(\quad) = x^2 - 13x + 12$
15. $(\quad)(\quad) = 5x^2 + 36x + 7$
16. $(\quad)(\quad) = 3x^2 + 8x + 5$
17. $(\quad)(\quad) = 6x^2 + 9x + 3$
18. $(\quad)(\quad) = 8x^2 + 28x + 24$
19. $(\quad)(\quad) = 6x^2 + 7x - 3$
20. $(\quad)(\quad) = 15x^2 - 4x - 4$
21. $(\quad)(\quad) = 6x^2 - 26x + 8$
22. $(\quad)(\quad) = 9x^2 - 25$
23. $(\quad)(\quad) = 4x^2 + 14xy + 12y^2$
24. …

This sequence of tasks adheres to the three environmental characteristics necessary to for the optimal experience to occur. There is a clear goal—find the binomials that, when distributed, produces the specified trinomials. There is immediate feedback on action—when the binomials are found they can be multiplied to see if it produces the correct trinomial.

And, as discussed previously, this sequence was designed on the principles of variation theory to allow flow to be maintained through a gradual increase in challenge as the ability increases. The first task is very simple in that the constant term, 7, has only one pair of factors for the students to consider. The second task has two pairs of factors, the third task has three pairs, and the fourth task has four pairs of factors for the students to consider. The fifth task introduces the notion of a negative constant term. Although this amplifies the number of pairs of factors to consider, notice that the pair that is relevant for this task (±2, ±12) is the same pair that was used in the previous task (2, 12). This is in keeping with the principle of variation theory that only one thing is varied at a time.

From here, students move through tasks that have the second and third term as a negative (tasks 7–9), has the second term as a zero (task 10), is absent the second term (tasks 11–12, and has only the second term as negative (tasks 13–14), before introducing a leading coefficient greater than one (tasks 15–23). Each of these transitions marks an increase in challenge for the students to take on as their ability grows. Even this last set of tasks (tasks 15–23) have within it a progression with tasks 15 and 16 being bookended by prime numbers, before introducing a leading coefficient that is not prime (task 17), and so on.

28.6.1 Thinking

During the 55 minutes that the students worked on this sequence of tasks, I was able to capture 13–15 data points[1] per student and 15 data points per group. These data revealed that 22 of the students were thinking for 48–55 minutes (13–15 out 15 data points), 5 students were thinking for 37–40 minutes, and 2 students were thinking for 22–26 minutes (see Fig. 28.11). No student was thinking for less than 30 minutes.

This is significantly higher than the baseline data wherein 20% of the students spent 20% of the time thinking and 80% spent no time thinking.

28.6.2 Tasks

Within the 55 minutes that the class worked on the aforementioned sequence of tasks all 10 groups got through the first 14 tasks. In fact, two groups got through the sequence in less than 20 minutes, six groups got through it in less than 30 minutes, and two groups got through it in less than 40 minutes.

All of the groups got to at least task 19 with five groups going beyond task 24 and were working on tasks that involved factoring cubics. What is remarkable about this is that the students were not taught how to factor quadratics either in this lesson or a previous lesson. They were simply asked to think about these tasks as distribution in reverse. Even more remarkable, the curriculum and textbook for the FMPC 10 course indicate that 5–8 lessons be dedicated to this topic. Yet, the vast majority of groups and students demonstrated mastery of 80–100% of the curriculum outcomes

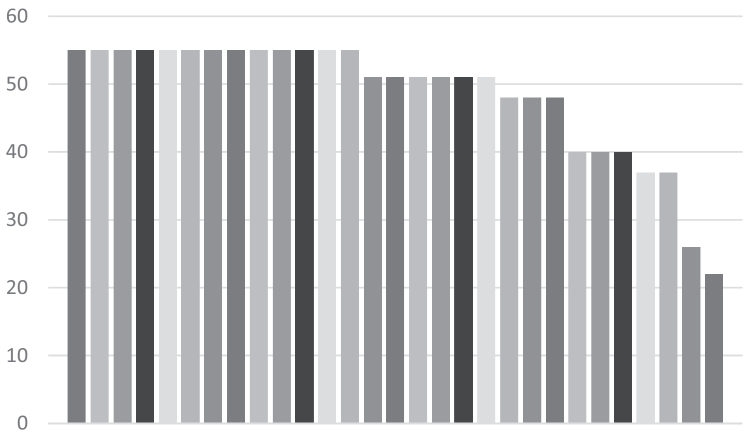

Fig. 28.11 Number of minutes of thinking per student

[1] The number of data points varied because students would sometime leave the room to go to the bathroom.

in less than 55 minutes. Within the experience of a sequence of tasks delivered and managed through the use of hints and extensions to maintain flow, these students moved through a tremendous amount of content in a very short period of time.

I want to emphasize how remarkable this is. This class of students moved through 5–8 hours of content in 55 minutes. This was not an enriched or accelerated class. This was a mainstream class in a typical school in a working-class neighborhood. This is remarkable.

This is not to say that 100% of students achieved success. When I say that a group got to a certain task, for seven of the groups this meant that every member of the group got this task and there was clear evidence that every member of the group was not only following along but also contributing ideas and taking turns writing on the VNPS. For three of the groups, however, there was one member of the group who eventually stopped contributing—two of whom eventually started to disengage.

28.7 Case II: Solving One and Two-Step Equations

The data for this case came from a grade 5 class (ages 10–11) in a school also situated in a working-class neighborhood. There were 22 students (12 males and 10 females). At the grade 5 level all students take the same math course. That is to say that, like with Case I, the data were gathered in a typical school within a typical grade 5 classroom with all the diversity that this entails. And like with Case I, the initial task was introduced right at the beginning of class, and with the students gathered around the teacher at a VNPS.

Teacher	We are going to play a game of guess what's in my head. I'm going to think of a number, and you are going to guess what it is. To help you make the guess I will give you exactly one clue.
Teacher	Ok—I have my number. Here is your clue—if I add three to my number the answer is 12. Thumbs up if you know my number.
Students	[Students put up their thumbs.]
Teacher	*[When enough are up the teacher calls on them.]* Ok—what is my number?
Students	9.
Teacher	Great. Ok—I have a new number. Here is your clue—if I double it and add three my answer is 15. Thumbs up if you know my number.
Students	*[Students put up their thumbs.]*
Teacher	*[When enough are up the teacher calls on them.]* Ok—what is my number?
Students	6.
Teacher	Ok. Before I give you the next one, we have to learn how to write what I just said. *[Teacher writes on the board:* $\square \times 2 + 3 = 15$*.]*
Teacher	And before I give you the next one there are three rules to this task.

You can use a calculator.
If you use a calculator, you must write down on the whiteboard what you type into the calculator.
You have to check your answer by putting it back into the calculator.
Teacher Here is your next one: *[Teacher writes on the board* □ *+ 3.014 = 7.22]*

The teacher than put students into random groups of three and sent them to their VNPS stations. When a group was done a task and they had checked their answer they were given an extension—gradually working their way through the following sequence of tasks.

1. □ + 3.014 = 7.22
2. □ − 15.1 = 7.88
3. □ × 4.25 = 24.8
4. □ ÷ 1.356 = 4.02
5. □ × 2.5 + 3.67 = 18.3
6. …

This sequence of tasks, like the one in Case I, has the three environmental characteristics necessary for the optimal experience to occur. There is a clear goal—find the number such that when the clue is applied it gives the correct answer. There is immediate feedback on action—*check your answer to see if it works*.

And the sequence was built on the principles of variation theory to maintain flow. The first task focused on addition, the next one on subtraction, then they saw multiplication and division. In task 5 they saw a combination of addition and multiplication, and so on. Each of these marks, not only variation of a single element at one time, but also a gradual increase in challenge.

The choice to use decimal numbers for this task was made in order to force the inverse operations to be considered by the students. When they were given the clue that *if I add three to my number the answer is 12*, students did not necessarily do subtraction to calculate *my number*. As evidenced by what was written on their VNPS, this was not the case for □ + 3.014 = 7.22. For this task each group found the answer by computing 7.22 − 3.014. This is why rules number one and two existed. I wanted the focus on the inverse operation, and not on performing manual calculations. Hence, they were afforded the use of a calculator (rule 1). However, I did not want the inverse operation to be lost in a jumble of button pushing. Hence, they had to write down what they entered into the calculator (rule 2).

28.7.1 Thinking

This activity ran for 35 minutes. In that time, I was able to collect 12 data points on every student and every group. These data showed that all 22 students were thinking for the full 35 minutes. That is, every single student I looked at was showing indicators of being fully engaged every time I looked at them. This is way above the baseline data.

28.7.2 Tasks

In the 35 minutes that the activity ran for every group got through at least 8 tasks. Six groups got through at least 12 tasks with one getting through 16 tasks. This means that every group solved at a minimum four tasks of the form $\square \div 7.33 + 3.1 = 23.005$. That is, every group was solving one and two-step algebraic equations. This is significant for two reasons.

First, like with Case I, the students were not taught how to solve one and two-step equations. They were simply asked to think about a way to figure out what the secret number was given a specific clue. This thinking was then supported through hints and extensions as well as the affordances of a calculator to get feedback on their answers. Second, solving one and two-step equations is not part of the grade 5 curriculum. In the jurisdiction in which this activity took place, this is a grade 8 (ages 13–14) topic. And within the curriculum documents and resources it is suggested that teachers spend 3–5 weeks on this topic. These grade 5 students covered all of the outcomes in 35 minutes. Remarkable!

28.8 Case III: Radical Expressions

The data for this case was collected in a Pre-Calculus 11 class (ages 16–17). This course, like the FMPC 10, is on a pathway for students who intend to continue their education at some form of a post-secondary institution. The school in which the class was situated was in a working-class neighborhood and had approximately 200 grade 11 students—75% of whom were enrolled in Pre-Calculus 11. There were 27 students (15 males and 12 females). In short, it was a typical classroom in a typical school.

Like with Case I and Case II, this case deals with the students' first exposure to a new topic—radical expressions—and, again following a specific script for introduction with the students gathered around the teacher at a VNPS right at the beginning of the lesson.

Teacher	[Without speaking, writes on the board: $\sqrt{50} = \sqrt{\_\times\_}$]
Students	5×10, 25×2.
Teacher	[Without speaking, writes on the board: $\sqrt{50} = \sqrt{25\times2} = \sqrt{25}\times\sqrt{2} = \_$]
Students	$5 \times \sqrt{2}$.
Teacher	[Without speaking, writes on the board: $\sqrt{50} = \sqrt{25\times2} = \sqrt{25}\times\sqrt{2} = 5\sqrt{2}$]
Teacher	[Without speaking, writes on the board: $\sqrt{18} = \_$]

The teacher then put students into random groups of three and sent them to their VNPS stations. As groups finished a specific task they were given extensions, gradually working their way through the following sequence of tasks:

1. $\sqrt{18} = \_$
2. $\sqrt{75} = \_$
3. $\sqrt{20} = \_$
4. $\sqrt{800} = \_$
5. $\sqrt{98 \times 16} = \_$
6. $\sqrt{72x^2} = \_$[2]
7. $\sqrt{16a^3} = \_$
8. $\sqrt{8a^5b^2} = \_$[2]
9. $\sqrt{\frac{x^4y^5}{4}} = \_$
10. $5\sqrt{27} = \_$
11. $6\sqrt{72} + \sqrt{98} = \_$
12. $6\sqrt{72} + 7\sqrt{98} - \sqrt{50} = \_$
13. $3\sqrt{x^3} + \sqrt{36x} = \_$
14. $3\sqrt{a} - \sqrt{\frac{a}{4}} = \_$
15. $\frac{1}{\sqrt{98}} = \_$
16. $\sqrt[3]{54} = \_$
17. …

This sequence, like the other sequences that we have seen, was designed on the principles of variation theory to maintain flow by maintaining a balance between the ability of the doer and the challenge of the activity. The first three tasks are similar in nature to the one that was demonstrated during the introduction to the activity. The fourth task is not, however. This task asks students to simplify $\sqrt{800}$. Although this could follow the same pattern of breaking the 800 into 400 × 2, this is not what we anticipated students would do. And, as anticipated, this is not what they did. Instead, they broke 800 into 100 × 8, which resulted in an answer of $10\sqrt{8}$. When the teacher drew to their attention that they were not done they immediately saw that $\sqrt{8}$ could be further simplified to $2\sqrt{2}$. This was an important task for the students to grapple with as it set them up for the next five tasks (tasks 5–9) where they may need to think about simplification as a multistep process. In addition, task 9 gives the students their first look at a fraction inside the radical.

From here, the sequence of tasks introduces them to the idea of a pre-existing coefficient (task 10), which is a concept they will need to use for the next four tasks (tasks 11–14). In addition, task 11 presents the first situation where to radical

[2] For the purposes of this lesson, the teacher allowed students to believe that $\sqrt{x^2} = x$ or $\sqrt{b^2} = b$. In the next lesson the teacher had the students graph $y = \sqrt{x^2}$ so they could see that their original assumption was only true for $x \geq 0$. They then simplified several radical expressions wherein the exponent under the radical was an odd multiple of 2.

expressions are added (or subtracted), which behaves very differently from the multiplicative reasoning they have been doing thus far. Task 15 puts the radical only in the denominator forcing the students to confront the question as to whether or not this is fundamentally different than the fractional radical in task 14. Incidentally, the students were not being asked to rationalize the denominator in task 15. Eventually, the students are introduced to radicals other than square root and a sequence of tasks using the ideas from tasks 1–15 is explored with these non-square root radicals.

As with the other sequences, this sequence of tasks presents students with a clear goal. However, unlike the tasks in Case I and Case II, this sequence of tasks do not provide feedback on the action—the students cannot easily check their answer. As such, they were provided with a sheet of answers that were not in order and not numbered. But they were told that for every task there was a unique answer and that the correct answer was somewhere on the answer sheet. This feedback, plus the feedback they were able to glean from within their own group (as well as other groups), coupled with the clear goal and balance between challenge and ability were enough to provide the environmental conditions necessary for an optimal experience to occur.

28.8.1 Thinking

The students worked in their groups on this sequence of tasks for 45 minutes during which time I was able to gather 10–12 data points on each student and each group. Based on these data it was clear that all but four students were fully engaged, and thinking, for the entirety of the 45 minutes. That is, 23 students showed they were engaged every time I looked at them. Of the four students who were not fully engaged for the entire time of the activity, two were fully engaged for the first 30 minutes before falling out of flow—one because the challenge began to exceed their ability and one because they "just ran out of energy". The remaining two students showed a clear lack of engagement, and lack of thinking, from the very beginning of the activity, yet stayed in their groups and pretended to be participating so as not to "get any hassle from the teacher". Again, this is well above the baseline data.

28.8.2 Tasks

Every one of the 10 groups got through the first 18 tasks in the 45 minutes with eight of the groups getting as far as solving equations of the type: $\sqrt{4x^3} + x\sqrt{x} = 9x$. Again, given the fact that the students had had no prior teaching of simplifying and solving radical expressions, this is a remarkable achievement. In 45 minutes even the slowest groups showed competency with the majority of curricular content outcomes for this unit of study, with eight of the ten groups showing competency on all of the outcomes.

28.9 Cross Case Analysis

Looking across all three cases a few common themes emerged. Obvious commonalities are the amount of think students exhibited in comparison to the baseline data as well as how effective groups were at moving through the sequences of tasks. These will be discussed in response to the research questions in the *Conclusions*. However, other common themes also emerged—two of which are discussed here.

28.9.1 Brevity of Introduction

In all three cases, the introduction to the sequence of tasks was extremely brief—all taking less than 5 minutes. In fact, for Cases I and II the introduction was less than 2 minutes. And although coincidental, this aligns with the thinking classroom research which showed that students are much more likely to engage in a task if it is given in the first 5 minutes of a lesson. This, we learned, is because the longer students spend sitting and listening to the teacher, the more passive they become and the more difficult it becomes to move them into an active state of thinking.

It is also worth noting that only in Case III did the introduction include what can be considered content instruction. When the teacher wrote $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$ they were providing direct instruction about mathematical conventions—something the students could not have discovered on their own (Hewitt, 1999). But even these were very brief in comparison to the 30–45 minute lectures I have observed on this topic in more conventional classrooms. It could also be argued that there was some direct instruction in Case II with respect to the notation for writing out the clues. I argue that this was a form of direct instruction. But I would also argue that instructions about how to write something is substantively different from instructions on how to solve something.

28.9.2 Maintaining Flow

Although not coded for during the continuous inventory of student thinking and progress through the sequences of tasks, it was clear from being in these rooms that the teachers were maintaining flow in all of the ways described in Sect. 28.3. What did come through in the inventory, however, was that even more so than the teachers, the groups were maintaining their own flow. When they were done a task, rather than wait for the teacher to give them a new one, they just stole a task from a group nearby. When they were stuck they would either passively look for hints around them or actively discuss with groups in close proximity. This was so prevalent in Case II that it was noted by another observer in the room that in many cases, the teacher only gave a particular task in the sequence to one or two groups. The rest of the groups simply stole the tasks.

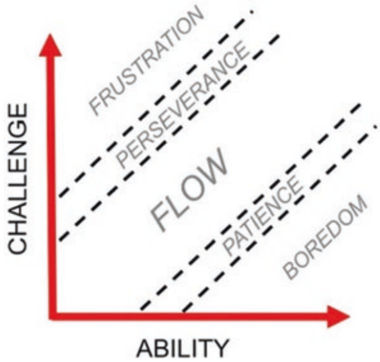

Fig. 28.12 Perseverance and patience

These behaviors align with observations in other thinking classrooms (Liljedahl, 2018) where it was concluded there are buffers on either side of flow—perseverance and patience (see Fig. 28.12)—that slows the transition from flow to either frustration or boredom. Perseverance is the willingness of a group to continue working on a task wherein the challenge of the task exceeds the ability of the group. Likewise, patience is the willingness of a group to continue to work on a task for which their ability exceeds the challenge of the task. These buffers can be very thin for inexperienced problem solvers but build up over time and exposure to problem-solving wherein there are temporary imbalances between their ability and the challenge of the task.

The fact that groups were self-maintaining their flow contributed to another interesting observation. Although the teachers also were very active in managing flow, it was interesting to note that increasing challenge through shifts in the mode of engagement (see Fig. 28.10) only happened later into the sequence, when "everything settles down" or "when the groups can take care of themselves". This, it turns out, is because shifting the mode of engagement requires a higher level of attention to what is happening in a group at the time and, as such, takes more time. Increasing challenge by moving a group to the next task in the sequence was seen as easier and faster by all three teachers.

Another feature of maintaining flow that was common across the three cases was that each of the teachers had a parallel set of tasks to the ones presented above. They used these tasks in cases where a group, although able to solve a specific task, struggled to do so. The idea behind this is that, although the ability has increased, it has not increased enough to warrant an increase in challenge—the ability still needed to increase more (see Fig. 28.13).

28.10 Conclusion

The goals of the research presented here were twofold. First, I was interested in seeing the degree to which a focus on creating and maintaining flow coupled with the use of sequences of curricular tasks designed on the principles of variation theory

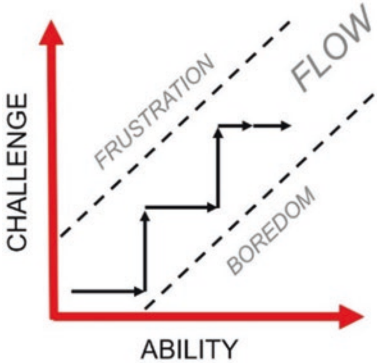

Fig. 28.13 Continuing to increase the ability

increased thinking in the classroom—were more students thinking and were they thinking for longer? The baseline data that was gathered in conventional mathematics classrooms showed that in a typical classroom, over the course of a typical lesson, 20% of students spend 20% of the time thinking while 80% spend zero time thinking. The students in all three case studies eclipsed these baseline data. In Case I, 76% of the students spent 87% of the time thinking and no student spent less than 40% of the time thinking. In Case II, 100% of students spent 100% of the time thinking. And in Case III, 85% of the students spent 100% of the time thinking.

In a way, this is not surprising. All the research into building thinking classrooms was vectored towards achieving these ends. The thinking practices that were in force in all of these classrooms (thinking tasks, VRG, and VNPS), among others, have been shown to be particularly effective at creating and sustaining a culture of thinking (Liljedahl, 2014, 2018, 2019, 2020). These practices, coupled with creating and maintaining flow on carefully designed sequences of tasks create an environment where thinking is not only encouraged, but also enabled.

Second, I was interested in seeing the degree to which this increase in thinking allowed students to move through curricular content. In this regard, the data showed that groups in all three cases covered huge amounts of content in very short amounts of time. In many cases, groups covered an entire unit of study in a single sequence of tasks—units of study that would normally take 6–15 lessons to cover using more conventional teaching methods. Again, this is not surprising. Thinking is a necessary precursor to learning. If students are not thinking they are not learning and everything is difficult and takes time to *teach*. When students are thinking, on the other hand, students take on new content as tasks to think about and *learning* speeds up. The research presented here shows that when that thinking is embedded within a structure designed to create and maintain flow coupled with carefully designed sequences of tasks, powerful things happen.

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Chapter 29
Commentary on Part III of Mathematical Challenges For All: On Problems, Problem-Solving, and Thinking Mathematically

Alan H. Schoenfeld

Eugene Wigner's (1960) essay "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" begins with part of a quote from Bertrand Russell's (1917) essay "The study of mathematics," which I provide here in slightly extended form:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not merely to be learnt as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

It's hard to think of a better introduction to this set of chapters. Indeed, as I faced them, I had the feeling of being like a child in a candy shop – there are so many sweet confections on offer! The chapters suggest the breadth and depth of mathematics; its coherence and connections; in modeling, a bit of its unreasonable effectiveness; and the psychological pleasures of deep engagement. And yet, there is more.

This year I am once again teaching my problem-solving course. In addition to the normal variations – the course is always different because we become a mathematical community, and the people in it are different – I find that the course itself has evolved. For a number of reasons, I am less focused on problem-solving strategies per se than I once was, although heuristic strategies (and Pólya) still receive significant attention. I also cover less, because I want my students to uncover/discover more. For reasons elaborated below, I focus more on the generative nature of

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mathematics and what it means to see mathematics as a field that is rich, deeply connected, and coherent.

In what follows I revisit some of the key ideas that have shaped my problem-solving courses through the years. I then discuss the specifics of how my students and I worked through some problems this year, and why. Ultimately, problems are the raw materials for mathematical construction; if they are rich in potential, then like fine wood, metal or gems, many different things can be made of them. The question is what might be made – and how, and why.

By way of preface, I note that all of my work on problem-solving has involved an ongoing dialectic between research and development. When I have had ideas regarding ways to help students become more effective mathematical thinkers, those ideas have been tested (whether formally or informally) in my problem-solving courses. In turn, my attempts to teach problem-solving have often caused me to re-think the implementation of those ideas, or to notice hitherto unnoticed aspects of mathematical thinking and problem-solving. There has, thus, been a natural evolution of focus in my problem-solving courses – at first, as I fleshed out a framework characterizing what matters in success in problem-solving, and later as I considered the goals of mathematics instruction more broadly.

29.1 Framing a Major Point of This Chapter

A slight historical detour by way of commentary: Pólya's written work on problem-solving strategies began with *How to Solve it* (1945) and continued with the two (1954) volumes of *Mathematics and plausible reasoning* and the two (1962, 1965/1981) volumes of *Mathematical Discovery.* Before producing these volumes, all of which emphasized strategy and, in the latter volumes, pedagogy (Pólya developed *Mathematical Discovery* for sessions for teachers), Pólya produced collections of problems that, in some ways, echo the problem collections in this section of this volume. The best-known versions date all the way back to 1925: Volumes I and II of *Aufgaben und Lehrsätze aus der Analysis* (Pólya & Szegö, 1925a, b) consist of extraordinarily rich sets of problems in calculus, the theory of functions, number theory, geometry, and more. The idea behind such thematically organized collections is that any student who works through those problems will develop a deep understanding of the content. Other notable mathematicians have done the same, e.g., Halmos (1991).

This historical fact introduces a major theme of this chapter. Pólya's earlier problem collections, and Halmos's volume, had as their primary purpose the teaching of *content*. If you manage to work the problems contained in them you will have learned a substantial amount of mathematics. While it is the case that the mathematical content of *Mathematics and Plausible Reasoning* and *Mathematical Discovery* was extremely rich, Pólya's use of the problems in those volumes was different. In those volumes, the problems were organized in such a way as to highlight aspects of mathematical thinking and problem-solving. This points to the fact that collections of problems can have different purposes. They may be used to teach mathematical content, to teach heuristic strategies, to focus on mathematical thinking more broadly, to support the development of mathematical practices, and to help students develop a sense of mathematical initiative and agency as part of their mathematical identities. Hence what matters is both the richness of the problems and the uses to which they are put. In this context, it is essential to consider pedagogical issues.

What follows is a chronological narrative, with the following leitmotif: What I emphasize is a function of what is known about mathematical thinking, and what seems to be missing.

29.2 Learning to Implement Heuristic Strategies

I began researching problem-solving in the mid-1970s. Pólya's ideas about problem-solving strategies felt right to me, although I had never been explicitly taught them. They also felt right to many mathematicians; there was no question that we used the heuristic strategies Pólya described. Yet, despite a fair amount of effort, students had not been successfully taught to do so. My early research focused on making heuristic strategies implementable.

That research consisted of a mix of experimental and observational studies aimed at understanding how to implement heuristic strategies successfully. Those first studies, motivated by contemporary work in artificial intelligence, revealed that the strategies Pólya described, such as "examining special cases" and "establishing subgoals," were far more complex than they appeared to be. Specifically, each of these strategies encompassed numerous sub-strategies. For example, as described in Chapter 3 of Schoenfeld (1985), a close analysis of problem-solving attempts showed that the general description "examining special cases" applied to situations such as the following:

- The presence of a tacit or explicit integer parameter, even in a problem as simple as "what is the sum of the first 97 odd numbers?", (where "97," tacitly, is an "n") may suggest trying values of $n = 1, 2, 3, 4, \ldots$, looking for a pattern, and verifying the pattern by induction or some other means.
- It may be possible to gain insight into the nature of problems that ask about specific features of classes of algebraic functions by focusing on examples that are easy to work with. For example, if asked about the roots of polynomials in general, one might examine easily factored polynomials - or even sets of "prefactored" polynomials such as (x), $(x)(x - 1)$, $(x)(x - 1)(x - 2)$, and so on. Doing so allows you to focus on the roots, and not get lost in algebraic manipulations.
- In computations that call for finding the limit of iterated or recursively defined terms, it may be useful to choose initial values for the terms such as 0 or 1 (if that can be done without significant loss of generality). Manipulating numbers rather than symbols may make it easier to find the underlying pattern.

- In problems that involve geometric figures, it may be useful to see if “nice” figures help – as long as one isn’t seduced into believing results that aren’t general. Given a problem with quadrilaterals, why not look at squares, rectangles, parallelograms, and trapezoids first? Given problems in various orientations, why not orient them conveniently?
- … and many more.

Essentially all of the heuristic strategies described in Pólya’s (1945) *How to Solve It* turned out to be characterizable in this way. The top-level name of the strategy, be it “establish subgoals, “draw a diagram,” or “work backwards,” was accurate, and mathematicians would typically recognize a strategy when they used it. But how the mathematicians learned to use “the strategy” was something else. What had happened, most likely, is that over time they had learned many of the relevant (unnamed) substrategies, including the contextual cues that might suggest each substrategy’s use – an explicit or tacit integer parameter for the first example given above, the wish to obtain roots of carefully chosen polynomials in the second example, and so on. Most of this happened without explicit labeling, as the result of repeated experiences (To paraphrase Pólya, a device used twice becomes a method). And, once one has such methods at one’s disposal, the name of the strategy makes sense.

My primary goal for instruction at that time was to provide students with the experiences that would allow them to do the same – to learn the substrategies and, cumulatively, the strategy. Thus, the initial collections of problems I gave students included sets of tasks that could be worked by the same substrategy. On the first task, I might need to tell them about the strategy, or (more typically) revoice or reframe a productive move a student had made, identifying the strategy as being important. When we discussed the second task, I might ask if they’d worked a problem using a method that had helped; in debriefing a solution once they had it, I would mention not only the strategy but some of the task features that might lead them to see commonalities between the two problems (despite some obvious differences in surface features). Over the first few weeks of encountering any substrategy the students would build up their skills both in recognizing when it might be used and in using it. In subsequent weeks I would decrease the use of problems for which that particular substrategy was useful until such tasks appeared only rarely. The idea was for students to be able to identify the relevance of the substrategy when they encountered relevant problems at random, not simply when they were practicing the substrategy itself.

I won’t go into detail here (See Schoenfeld, 1985), but there is clear evidence that students learned to use the substrategies, and thus the strategies themselves. Students’ problem-solving performance improved significantly on three classes of problems: problems that resembled ones we’d practiced in class, problems I knew could be solved by similar methods but that did not (on the surface) resemble problems we’d worked in class, and problems chosen from collections that did not “line up” with Pólya-like methods in any obvious ways.

Another caveat before I proceed. I emphasize the introductory phrase two paragraphs above: teaching heuristic strategies was my primary goal for instruction at the time. I was looking for an existence proof, clear evidence that students could learn heuristic strategies. At the same time, I must emphasize that my course was not taught in a laboratory; it was taught in a classroom. As a result, everything I knew as a teacher and as a mathematician came into play during instruction. In particular, I chose problems that I thought led to interesting mathematics; I induced my students to explore and to generate new problems, and so on. But a primary focus was on showing that students could learn to implement problem-solving strategies.

29.3 Metacognition: Monitoring and Self-Regulation

My early problem-solving courses were largely prescriptive in nature: I identified the problem-solving methods employed by proficient problem solvers and taught students to use them. The techniques I used to uncover such moves were largely drawn from artificial intelligence research, in that I looked for systematicity in the actions of people engaged problem-solving. Beyond skill at the level of implementing heuristics, there was the question of which strategies to try, and when – issues that I called "managerial" or "executive" strategies. A one-line encapsulation of the idea is, that one should try relatively simple (but relevant) methods before spending time on more complex methods. I had built and taught an executive strategy of this kind for techniques of integration (e.g., one should look for simple substitutions before trying integration by parts or partial fractions, and try those before using complex trigonometric substitutions), and it had proven effective. So, I built a comparable executive strategy for using heuristics. The truth is that it never felt right; it was too mechanistic. Although I showed it to my students, I never emphasized it; in discussing our work in general, what I emphasized was in line with the one-line summary given above.

A grant from the National Science Foundation in the late 1970s provided me with videotape equipment. I brought students into my office space to videotape problem-solving sessions before and after my problem-solving course, to see what differences I might find. In studying the "before" tapes, it became clear that the wrong choice of direction for a solution, unreversed, could doom students to failure. This kind of event – an inappropriate choice of initial direction that was never reconsidered – happened with astonishing regularity.

Simply put, knowledge doesn't do you any good if you don't think to use it. In Schoenfeld (1987) I describe a wide range of techniques I've used in my problem-solving courses to help students become more effective at monitoring and self-regulation. But here I want to focus on a sample problem and the way my use of it evolved.

Perhaps as early as the first iteration of my problem-solving course, I have given students a slightly modified version of “a problem of construction” from Pólya’s *How to Solve It:*

> You are given the triangle on the left in Fig. 29.1, below. A friend of mine claims that she can inscribe a square in the triangle – that is, that she can find a construction, using straight-edge and compass, that results in a square, all four of whose corners lie on the sides of the triangle. Is there such a construction – or might it be impossible? Do you know for certain there’s an inscribed square? Do you know for certain there’s a construction that will produce it?
>
> Is there anything special about the triangle you were given? That is, suppose you did find a construction. Will it work for all triangles or only some?

Pólya uses this problem to demonstrate the use of the following strategy: “If you cannot solve the given problem, try to solve first some related problem.” In this case, a key goal for the desired square is that all four of its vertices lie on the triangle. Asking for less – for three vertices to lie on the triangle – might be doable; in fact, it is easy to find more than one solution (Fig. 29.2).

There are, in fact, infinitely many solutions; and one of them must pass through the 3rd side of the triangle. See Fig. 29.3.

For Pólya, this is an introductory problem, to demonstrate how the technique of using an easier related problem can help solve the original, more challenging problem. “If the student is able to guess that the locus of the fourth corner is a straight line, he has got it” (*How to Solve it*, p. 25).

For me (and, I suspect for Pólya, when he actually taught with the problem) the problem serves multiple functions. Here are a few of the mathematical ideas that arise.

Pólya’s solution focuses on a particular way of finding an easier related problem, by relaxing the condition requiring that all four corners of the square lie on the triangle. That approach already narrows the solution path quite a bit. When I ask students if they can think of an easier related problem, their first response is usually to think about altering the given figure. Might it be easier to inscribe a square in an equilateral triangle? A right triangle? An isosceles triangle? Or, what about inscribing a circle in the given triangle? Occasionally, a student will suggest turning the problem inside–out – starting with a square and putting a triangle around it.

All of these suggestions raise interesting issues, among them: do you think you can solve the easier problem? If so, how might your solution lead to a solution to the original problem? And, since there are a fair number of possible approaches to take,

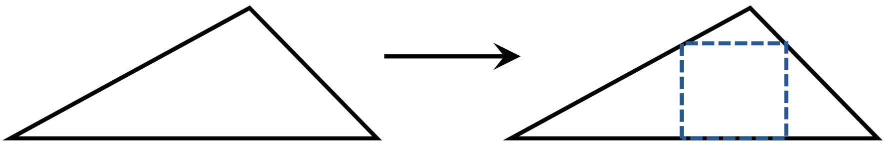


Fig. 29.1 Pólya’s “problem of construction”

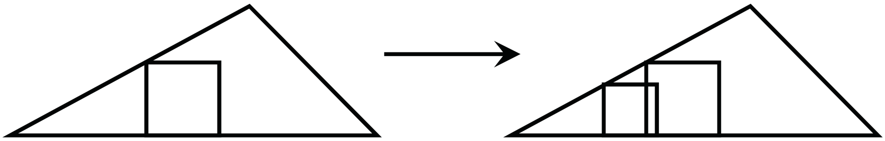


Fig. 29.2 There is more than 1 solution to the easier related problem

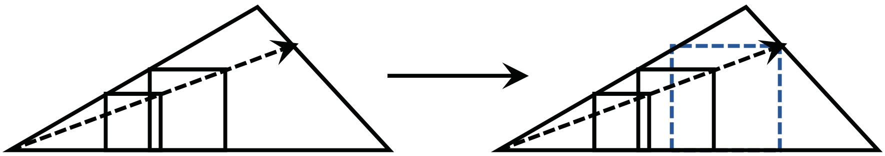


Fig. 29.3 Envisioning the solution to the original problem

how do you decide which one to try? This is our introduction to issues of metacognition.

I often let the problem "sit" until our next class meeting – has anyone made progress? Are there other ideas? If so, we pursue them. If not, I mention the generic version of Polya's suggestion: Consider the conditions of the problem and relax one of those conditions (that is, replace it with a condition that is easier to satisfy). The desired solution will be a square with four corners on the triangle. You can relax the condition of squareness to ask for a rectangle – and there are many, see Fig. 29.4 – or you can ask for fewer corners of the square to be on the triangle. Both approaches lead to existence proofs. In Fig. 29.4, one can start with short-and-wide rectangles and wind up with tall-and-thin rectangles; thus the progression must pass through a square. In Fig. 29.5, the squares with three corners on the triangle start growing inside the triangle and wind up outside it, so one of them must hit the opposite side of the triangle. In both cases, we know such a square exists. But, is there a construction that produces it? Which path should we pursue – one of these or one of the others? This makes the metacognitive challenge even more complex.

We've never managed to solve the problem for "special" triangles in a way that could be generalized; nor have we found a way to exploit the well-known method for inscribing a circle in a given triangle. And, we have yet to find a way to convert the existence proof in Fig. 29.4 into constructive proof. But, there is a lovely solution to the construction problem (first produced by my students, I hadn't known it) based on the idea of building a triangle around a square. In the tradition of classical exposition, the solution is left to the reader.

It is also worth noting that there are two very different solutions to this problem, the one in Fig. 29.5 (Pólya's solution) and the one that can be obtained by constructing a triangle similar to the original around a square. The purpose and value of problems with multiple solutions will be elaborated below.

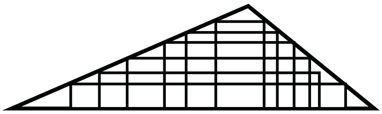


Fig. 29.4 Solutions to a different "easier related problem" lead to an existence proof

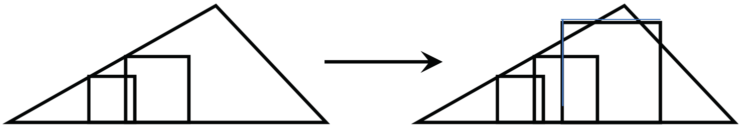


Fig. 29.5 The approach in Fig. 29.3 also leads to an existence proof

29.4 Problems as a Mechanism for Countering Unproductive Student Beliefs

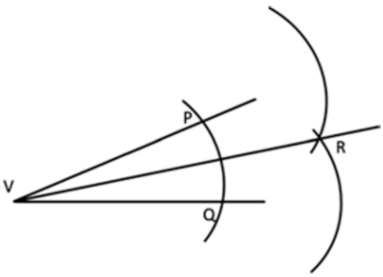
Early in my problem-solving work (See Schoenfeld, 1985) I found that many students, believing that the purpose of proofs in mathematics is to confirm in formal terms what is already understood to be true, ignored results that they had proved and made conjectures that contradicted those results. For that reason, I added a collection of (sometimes explicitly, sometimes tacitly) proof-related problems to the problem course. One year I began with a simple question: "Can anyone tell me how to bisect an angle, using a straightedge and compass?" A student quickly responded with the standard construction: (1) draw an arc from the vertex V that crosses both sides of the angle, at points P and Q; and (2) draw intersecting arcs of equal length from P and Q. Call that point of intersection R. The line from V to R bisects angle PVQ. See Fig. 29.6.

I then asked why the construction worked. There was silence at first, and then they got to work. A minute or two later, a student announced that if you drew in the line segments PR and QR, you could argue the following: PV and QV are equal because they are radii of the circle with V at the center; the compass had been kept at the same setting when creating the arcs with centers at P and Q, so PR = QR; and VR equals itself. Thus triangle PVR is congruent to triangle QVR. As corresponding parts of congruent triangles, angle PVR equals angle QVR. See Fig. 29.7.

I next asked if the students knew how to inscribe a circle in a triangle. Here too, someone remembered that the center of the desired circle lay at the intersection of the three angle bisectors. I asked why that construction worked. It took a bit longer, but before long a student produced a proof that the altitudes of the triangles drawn from the point of intersection of the three angle bisectors (the points of tangency of the inscribed circle) were all equal. At that point, a student asked, "Are you trying to tell us that proof is actually good for something?"

I said "yes, but telling you isn't enough. You have to experience it." At that point, we began to work the construction problems in Chapter 1 of Pólya's (1962/65) *Mathematical Discovery.* Once the students found, repeatedly, that deriving an

Fig. 29.6 The standard construction for bisecting an angle



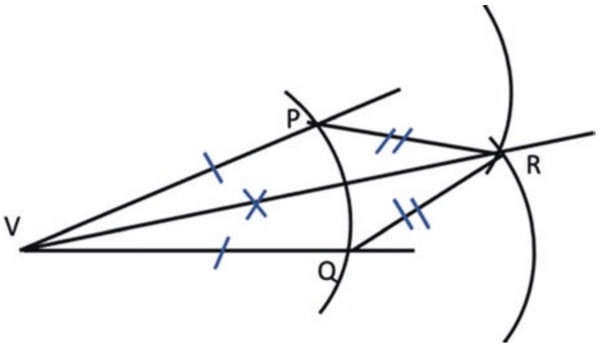


Fig. 29.7 A proof that the construction in Fig. 29.6 bisects angle V

intermediary result helped them solve a construction problem, they came to see proof as a generative tool.

The literature is replete with descriptions of counterproductive student beliefs. As Lampert wrote,

> Commonly, mathematics is associated with certainty; knowing it, with being able to get the right answer, quickly. These cultural assumptions are shaped by school experience, in which *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing. (Lampert, 1990, p. 31)

As in the case of proving described above, my problem sets and my pedagogy are aimed explicitly at countering such beliefs. They do so first by providing students with enough lived experiences to provide the underpinnings of belief change. (Beliefs develop over time, as a function of experience; they must be modified in the same way.) In addition, I make my intentions and reflections explicit, because the lessons learned from experience are more likely to take hold if they are made explicit and reflected upon. Here are some sample beliefs and the problems/actions I take to address them:

- "All problems can be solved in 5 min or less." We work on problems for days and weeks.

"Mathematics is about learning to solve problems by using methods you have just been taught." We work on problems in a variety of ways – and we generate new problems. The goal is for the students to experience mathematics as a generative activity. (See the two sections that follow.)

- "Compelling patterns are enough to be convincing; the proof is just a game mathematicians play." We build up the habit of looking for patterns as a heuristic activity (à la Pólya, "let us teach guessing") … but then I throw them a curve. After a bunch of problems for which the (provable) answer is 2^n, I give them this problem:

Suppose you pick 21 points on the boundary of a circle. You then draw all of the line segments that connect pairs of those points. If the points have been chosen so that no three of the segments intersect at the same point (that is, the circle is divided into the maximum possible number of regions), how many regions is the circle divided?

Additional examples are given in the following sections. The point is that, above and beyond "problem solving," the problems, their discussions, and the norms we cultivate in the classroom are all in the service of thinking mathematically.

29.5 Problems that Invite Multiple Solutions, an Antidote to "Answer Getting"

One of the problems I offer for discussion early in the course is this:

Take any three-digit number and write it down twice, to make a six-digit number. (For example, the three-digit number 789 gives us the six-digit number 789,789.) I'll bet you $1.00 that the six-digit number you've just written down can be divided by 7, without leaving a remainder.

OK, so I was lucky. Here's a chance to make your money back, and then some. Take the quotient that resulted from the division you just performed. I'll bet you $5.00 that quotient can be divided by 11, without leaving a remainder.

OK, OK, so I was very lucky. Now you can clean up. I'll bet you $25.00 that the quotient of the division by 11 can be divided by 13, without leaving a remainder.

Well, you can't win 'em all. But, you don't have to pay me if you can explain why this works.

One way to approach the problem is to note that dividing sequentially by 7, 11, and 13 is equivalent to dividing by their product – and

$$7 \times 11 \times 13 = 1001.$$

If you multiply the 3-digit number abc by 1001, you get *abc*, *abc*. Working backwards,

$$abc, abc / 1001 = abc.$$

An alternative route is to notice that the final quotient, after dividing by 7, 11, and 13,is the original number, 789. So,

$$789{,}789 / x = 789; x = 789{,}789 / 789 = 1001.$$

Another way is to say the number out loud:

$$\text{seven hundred eighty} - \text{nine thousand and seven hundred eighty} - \text{nine}$$
$$= 789(1000) + 789 = 789(1000 + 1).$$

A fourth is to ask what the number *abc*, *abc* actually means. When students are reminded to think about what base 10 notation stands for, they write out the expression

$$abc, abc = 100{,}000a + 10{,}000b + 1000c + 100a + 10b + c,$$

which they can factor as $1001(100a + 10b + c) = 1001(abc)$.

Each of these methods can be abstracted as a heuristic strategy. We discuss working forward, working backward, exploring representations, looking for patterns, and so on as various solutions emerge. In that sense, this problem fits my heuristic agenda. My favorite solution to the problem came from a graduating senior. An English major, she told me at the beginning of the course she told me that she had never liked or done well at math, and she was "giving it one last chance." After the class had generated the solutions discussed above, she raised her hand and said "I found a different solution." When I asked her, she went to the board and said,

> I didn't know what to do when I first looked at the problem, but I remembered a strategy we'd used on some other problems – if you don't know what to do, try some simple numbers and look for a pattern. The simplest 3-digit number is 001, and when I wrote 001,001 and did the divisions I saw that

$$001{,}001 = 7 \times 11 \times 13. \text{Then}$$

$$002{,}002 = 2 \times 7 \times 11 \times 13, \text{and}$$

$$003{,}003 = 3 \times 7 \times 11 \times 13, \text{so I got the pattern.}$$

She was proud of herself, even more so when I told her that I'd never seen that particular approach to the problem. She was energized and did well in the course.

Why work this problem so many ways? Because the goal is not to get an answer or solve a particular problem, but to perceive mathematical structure, and to make connections. Insights into the underlying structures might help develop deeper understanding of other mathematical situations. Moreover, one never knows which of the many approaches that unlock a particular problem may turn out to be useful in other situations (Problems we work on later in the semester use some of the multiple methods developed when solving earlier problems).

29.6 Transfer of Authority: Who Determines What's True?

In most mathematics courses the teacher is the sole arbiter of mathematical correctness. As Hugh Burkhardt summarizes it, "the students propose; the teacher disposes." At some point in the development of a mathematical career that has to change. Budding mathematicians come to internalize the standards of the discipline, learning to judge the correctness of arguments before they submit them for publication (A mathematician who submitted manuscripts in the hope that reviewers would determine their correctness wouldn't last very long!). Thus, the pedagogy of learning to think mathematically includes helping students come to understand that they themselves can, most of the time, determine whether or not their arguments are correct. That means helping students learn the sequence described by Mason et al. (1982) as "convince yourself, convince a friend, convince a skeptic." The problems do make a difference, however. Mathematically rich problems offer many pathways toward solutions and many ways to go astray.

Early in my problem-solving courses students will come to the board to present their work on a problem and look directly at me for affirmation. I deflect them, saying that it's the class's responsibility to question what they've done. After some time, this becomes a ritual: after finishing up at the board a student will turn to the class and say "OK, do you buy it?" With some training, the class becomes pretty good at determining whether an argument holds water. (I'm always there to do extra problematizing if need be.)

My favorite example, described in Schoenfeld (2012), is the concrete wheel problem:

> You are sitting in a room at ground level, facing a floor-to-ceiling window which is twenty feet square. A solid concrete wheel, 100 miles in diameter, is rolling down the street and is about to pass right in front of the window, from left to right. The center of the wheel is moving right at 100 miles per hour. What does the view look like from inside the room as the wheel passes by?

I will leave the solution to the reader – it's too good a problem to spoil. One of the things that's nice about the problem is that intuitions about what one would see under these circumstances vary. The wheel is moving really fast. Will the room darken almost instantaneously? Or, will it darken slowly? How long will it stay dark? What will the darkening look like – will it be like a curtain being pulled down, or will the darkness proceed from the upper left corner to the lower right corner, followed by lightness from the lower left to the upper right?

One year one group of students argued for a particular conjecture, while another group argued for a different conjecture. The argument got somewhat heated, with the rest of the class actively following the discussion. When one group prevailed, I moved to tie things up: "OK, shall I try to pull things together?" A student said "Don't bother. We got it." This was, I think, an important sign of the students' developing mathematical authority.

29.7 Problems that Are Generative: And a Bonus, for Developing a Sense of Mathematical Initiative

One of my first-day problems asks students to fill in a 3 × 3 magic square – a task that can be done by trial and error in five minutes or so. This "easy on-ramp" is one of the reasons I like the problem – all students experience success. I typically ask for a volunteer to present a solution. After they do, I ask if we're done. The class always says "yes." I respond, "No, we're not. We've only found one solution." We work through various approaches. Considering subgoals, for example, what number goes in the middle square? Or, what might the sum of each row, column, and diagonal, which we call the "magic number," be? We find solutions by working backwards (a method that allows you to find the magic number, which turns out to be 15), working forwards (listing all combinations that add up to 15), and exploiting symmetry. At that point, we have a fair number of different solutions. We've shown that there is no need for guesswork and that, save for symmetry, there is only one solution. At that point, I ask if we're done. Once again, the class says "yes." My answer, once again, is "No, we're not. To this point, you've only solved the problem I gave you to solve. If that's all mathematicians did, the field would never progress. The question now is, can we do something new and interesting grounded in what we've done? What kinds of questions can we ask?

In years gone by I've seeded the conversation by asking, "what about a magic square with the numbers 2 through 10? Or 2, 4, 6, 8, …, 18?" The class has noted that adding any constant to each cell of the 3 × 3 magic square leads to a magic square, as does multiplying each cell by a constant. So, if M is the original 3 × 3 magic square (considered as a matrix), then aM + b is also a magic square. That leads to the question, is *every* 3 × 3 magic square of the form aM+b (modulo symmetry)? When we first pondered that question, we were no longer "problem solving" or "problem posing"; we were simply doing mathematics.

Indeed, students come to recognize this as a design feature of the course. A few weeks into the course one year, I once again asked "are we done" after we had solved a problem. In mock dismay, a student threw his hands up in the air and cried "we're *never* done!" Some weeks later he asked in all seriousness why I wasn't asking "Are we done?" anymore. I answered that I didn't need to. The norms of inquiry had been established, and we were acting as a mathematical community.

29.8 This Year and the Years to Come

Through the years my problem-solving course has evolved as a function of who the students are and my perceptions of what would serve them best. As has been clear from this narrative, those perceptions evolve as my understandings of mathematical thinking and of the understandings that my students bring to the course grow and change.

Recent events have caused me, once again, to reflect on the goals of mathematics instruction. These thoughts are still in the formulation; I have been able to act on some of them, and some ideas are still prospective. The issue for me at the moment is, in what ways should people be able to use their mathematical understandings? In the context of my problem-solving courses, what are the implications for problems, problem-solving, and my pedagogy? I understand that these questions do not have unique answers, given students' varied desires and needs. I do think, however, that what follows applies to all students.

Both in my life as a private citizen and in my capacity as a member of a committee setting COVID-related safety policies for a residential program (see Schoenfeld, 2020, 2021), I have found myself wrestling with consequential "real world" problems. What policies with regard to masking, vaccination, social distancing and travel seem appropriate for our resident population? In another more personal set of issues, as someone with adult onset diabetes, I have been keeping track of my blood sugar levels for more than 20 years. Different foods affect one's blood sugar levels in different ways; the goal is to have a regimen of medicines, diet, and exercise that keeps blood sugar levels within safe bounds.

Without going into detail, I'll note that there was a practical need to take on the challenge of making COVID policy decisions. The policy climate in the US was such that government recommendations were not necessarily trustworthy or consistent (there were examples of federal policy changes within a few weeks' time when no new data had emerged to warrant a policy shift) and the residential program fell into a regulatory gap. We were on our own.

One might ask what positions me to be making COVID policy (as part of a team that includes a physician)? I'm not biologically savvy, so I certainly can't be working at a level of biological mechanism. But I can build simple mathematical models. And, while policy recommendations from federal agencies may be questionable at times, those recommendations cite the papers from which the recommendations were developed. Thus I can get information for those models without knowing the details of the biology. To give you one concrete example, here is a question I posed as I was trying to understand the aerosol dispersion of COVID particles.

> Recently, I found my nose irritated by the cigarette smoke produced by a smoker who was across the street. If that aerosol irritant could bother me at a distance of 30 feet, why is 6 feet of physical distancing considered safe for COVID?

You can find my resolution of the dilemma in Schoenfeld (2021). The resolution depends on determining two things: particle size and the density of particles expelled into the air. Once you have this information for cigarette smoke and COVID-infected molecules, the rest follows. Similarly, the same considerations explain the efficacy of masking. Vaccination data are compelling; you don't need to know the underlying science to draw conclusions about their efficacy.

Similarly, dietitians' recommendations with regard to food intake tend to be categorical ("avoid or limit rice and pasta intake") while one's reactions to white or brown rice, or different pasta dishes, can differ substantially. More consequentially, different medications for diabetes treatments have limited dosage options (10 or

25 mg in one case, multiples of 10 mg in another) and there are no impact data (that is, what dosages are likely to produce what reductions in blood sugar levels). So, when doctors and patients begin a new dietary regimen, they have to proceed empirically. At a coarse-grained level, doctors rely on a test called Hb1Ac (glycated hemoglobin) which, in rough terms, indicates a person's average blood sugar level over the past month – it's the average levels that turn out to be problematic. HbA1c results are supplemented by daily blood sugar readings, which indicate immediate sugar levels and point to possible problems. When I was diagnosed with diabetes I started simple logs of my daily sugar levels. I quickly learned to distinguish the impacts of different rice and pasta dishes (more detail than dietitians' generic information could provide), and following the data allowed me to select foods that enabled me to eat happily and well. I won a bet with my doctor, with the data showing that moderate wine consumption was good for my blood sugar levels. She won a return challenge, which showed this reluctant exerciser that a daily walk improved his blood sugar levels. I now enjoy my daily walk up the Berkeley hills, my wine with dinner, and the knowledge that they're both good for me. And, my doctor and I navigated the change of medicines smoothly, learning in the process that the impact of one medicine was not proportional to dosage (see Schoenfeld, 2021 for more detail).

What positioned me to do these things? Not my scientific knowledge; I relied on my ability to search the web and triangulate results to find the information I needed. Not my mathematical knowledge, beyond the basics that any high school graduate would have. What made the difference was my sense of initiative – the sense of personal agency that enabled me to say "let me see if I can build a simple mathematical model of the situation." Where the vast majority of people would defer to external expertise (if it existed) and its limitations, I felt personally empowered enough to look into these issues on my own, using only the web and some simple mathematical tools.

Why is it that the vast majority of people would not step outside the bounds of their school knowledge to address such consequential matters? I would argue that this happens because of a particular form of learned helplessness – one that is learned in school. In traditional schooling, students are taught content and methods that are organized to solve classes of problems. The unspoken didactical contract between teacher and students is that the tasks students will be asked to work on will closely resemble the tasks students have been taught to deal with and that when the students perform adequately on those tasks, they will be declared proficient. This is, for all intents and purposes, an inward-looking and essentially closed system. The lesson students learn from it is that their knowledge is limited to what they have been taught.

My problem-solving courses have always tried to address this issue, at least in part. The very idea of heuristic strategies is that they help one approach problems that one has not been taught to solve. More broadly, as I have outlined in this chapter, my goal has consistently been to provide my students with opportunities to *do* mathematics and to develop productive mathematical habits of mind. The very notion that "we're *never* done" is deeply embedded into the pedagogy of my problem-solving courses. But my concern goes beyond this, in that the goal is to have students feel that it is *natural* to be asking questions, and to be pursuing mathematical leads that seem interesting – whether or not those leads pan out in the end. Those thoughts have been in my mind, and they did make their way into this year's instruction.

The immediate context is that I have been teaching my graduate course on mathematical thinking and problem-solving this semester. The students all have solid mathematical backgrounds, although their interests vary: some intend to be mathematics education researchers, some science educators, and some specializing in policy and measurement. On average the class spends an hour or so each week working on problems. The rest of the time is spent reading, discussing the literature, and working on course projects.

I am about to describe a somewhat meandering classroom conversation that occurred a few weeks into the class.

As in previous years, the class and I worked through the problem of the 3×3 magic square. But, mindful of the need to be more exploratory, I looked for opportunities to highlight and support opportunities for branching out. At the beginning of the class following the class in which we had obtained a number of solutions to the problem, I provided a recap to lay the groundwork for our continued discussions. The recap described the highlights of the previous discussion, including the fact that the "magic number," the sum of each row, column, and diagonal, must be divisible by 3 (This was part of our derivation of the magic number for the original magic square, which is $(1/3)(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 15$). This was followed by an open invitation: "So now the question is, what do we do that's mathematically interesting? There's the question of generativity. What kinds of interesting questions can we ask at this point?" The conversation described below lasted about a half hour.

Two suggestions were offered. One was that we might explore 4×4, 5×5, or $n \times n$ magic squares. A second was "Can we use different numbers? Maybe higher numbers, or even numbers, or something?" I suggested we look at 2 through 10, after which one student said "the sum has to be divisible by 3, right?" I check that the sum from 2 through 10 is 54, which is divisible by 3. At that point, I say, "But suppose we wanted to cheat, what's the easiest way to use the digits 2 through 10?" A student responds, "Oh yeah, you can add … oh you can add a constant to the whole thing and it works." I elaborate, noting that if you add C to each cell in the original 3 x 3 magic square, the sum of each row, column, and diagonal is 15 + 3C.

A student then asks, "Is this kind of a proof that the sum of 9 consecutive numbers is divisible by 3?" I point out that that's a separate conjecture, and write this on the board:

Is the sum of 9 consecutive numbers divisible by 3?

One student says "I think the sum of every 3 consecutive integers is divisible by 3, so the sum of 9 numbers would actually be divisible by 3." I move to unpack the argument: The straightforward way to do the sum of 3 numbers is to write them as

a, $(a + 1)$,and $(a + 2)$; the sum is $3a + 3$, which has a factor of 3. What's the clever way?" The same student responds, "$a - 1, a, a + 1 = 3a$."

I elaborate: if you "start with the number in the middle being called a, then you get the three numbers are $a - 1$, a, and $a + 1$, and when you add them together you get the 3a directly. Now that's just a little bit of tweaking that makes a difference in terms of representation that makes where you want to go a little bit more easy." A student completes the argument by noting that each triad of the 9 consecutive integers is thus divisible by 3, so the sum is. As usual (the class is accustomed to doing things algebraically), I proceed by noting that we could write the nine numbers as a through a + 8, whose sum is $9a + 36$. I continue, "Or, you get sneaky, and you don't have to figure out the sum from 1 to 8, because if you call the middle one a, the numbers go from [at this point there is some choral support from the class] $a - 4$ to $a + 4$. Those all balance out… and you get $9a$."

The student who originally conjectured divisibility by 3 notes that if the number of consecutive terms is divisible by 3, then the sum will be – each set of 3 terms is divisible by 3. I then noted, "OK, but you're moving towards another generalization. The sum of 3 consecutive numbers turned out to be divisible by 3. The sum of 9 consecutive numbers turned out to be divisible not just by 3 but by 9. Hmmm…".

A slightly jumbled exchange ensued. The student who had conjectured divisibility by 3 earlier said, "Whenever you have any consecutive sum, where it's divided like that (pointing to the symmetric distribution on the board) then everything will cancel out, so if the number of summands has a factor divisible by 3 then the whole thing will always be divisible by 3." I say, "By 3?" and the student says "By N." I start writing on the board as I say,

Yeah, say you've got 5 consecutive integers, just call the middle one x, then they balance out, there's an $x - 1$ and an $x - 2$, and an $x + 1$ and an $x + 2$… (see Fig. 29.8)

And *that* (I point to the $x - 2$ and $x + 2$) gives you $2x$; *that* (I point to the $x - 1$ and $x + 1$) gives you $2x$; *that* (I point to the x in the middle) gives you an x, so it's $5x$.

A second student says, "Does it always work for odd numbers?"

I respond, "Well, that's what we have so far. The sum of N consecutive integers, where N is an odd number, is always divisible by N. But what does that say about even numbers?"

A number of students start to respond, but they wind up stopping half-way through what they were saying when they realize there is no "middle number" from which to make the same symmetry argument they were able to make when there was an odd number of summands.

I editorialize, "Have you noticed, by the way, that we've shifted to doing mathematics? You're no longer doing the problem that I gave you but we're doing exactly what we're supposed to do, which is that interesting thoughts lead to interesting

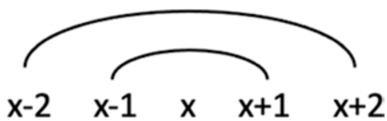


Fig. 29.8 Any five consecutive integers are "balanced" around the number in the middle

thoughts lead to… doing real mathematics. We're off in the space of conjecture, that builds off the thing that we started with."

Another student says, "I'm starting to think about even numbers. Would it be divisible by $n + 1$ if n is even?" I ask, "What do we do in the case of a conjecture like that?" and she responds, "Try a few cases."

We look at 1 + 2 = 3, which is divisible by 3. But 2 + 3 = 5, which is not. So the student's is not true. Another student says, "but it will always be odd."

After a long pause, I say, "well, a straightforward formula works for odd, but not for even… This sounds like something to leave for next week." The students laugh.

The following week I recapped what I described above and invited the students to jump back into exploring. I noted that if N was odd, the sum of N consecutive numbers was divisible by N, but that the same did not hold when n was even; in fact, the sum of two consecutive integers was always odd. One student mused that we might not be able to get any even numbers as a sum; we certainly couldn't get 2, and 4 didn't work. That was quickly put to the empirical test, and it failed:

$$1 + 2 + 3 = 6, \text{and} 1 + 2 + 3 + 4 = 10$$

But we were unable to obtain 8; the conjecture was modified to, "no power of 2 can be obtained as a sum of 2 or more consecutive integers." That begged to be proven but was also incomplete. We were able to get 6 = 2 × 3 and 10 = 2 × 5; what could we get? With this set of questions, the class was off and running. Someone conjectured that we could get every integer that was twice an odd number. I called a break, but the students worked right through the break. We turned to something else, but the class's e-conversation between in-person meetings was especially animated, and the class continued on its own initiative until we had a complete solution to the problem, "which integers and be expressed as the sum of two or more consecutive integers, and in how many ways can they be expressed as such a sum?"

That was a long and meandering example because our conversations were long and meandering. Let me take stock and bring things to a close.

The "consecutive sums" problem is well known. In fact, it's one of the problems I planned to assign the students later in the semester. It's fair to ask, "why to spend all that time wandering in the mathematical wilderness when you could have simply posed the problem, and perhaps even led students directly to a solution?" My response has to do with the issues of students taking mathematical initiative. There is no question that I helped to steer the conversations into mathematically productive directions, but I did so with an exceptionally light touch. Students made conjectures; they tested them; they built on what they did; they engaged in the messiness of mathematical creation, with all of the false starts that bedevil professional mathematicians when they engage in mathematics. And, they cared. They persevered because they cared, and they owned the mathematics that they produced. Moreover, they learned that they can think outside the curricular box. They learned that when issues are of interest to them, they can pursue those issues using what they know, even if they haven't been taught how to address them.

29.9 Concluding Thoughts

Let me return to the metaphor of problems as raw materials. Good problems and good problem collections are rare and wonderful things. I spent a year in the Berkeley library reading problem books before I taught the first version of my problem-solving course in the late 1970s. Of the tens of thousands of problems I examined, I found perhaps 100 that I thought merited students' attention. (By that I mean problems that are really worth working on and that students can learn valuable things from. My "problem aesthetic" is described in Schoenfeld, 2020). So, I value good problems immensely.

At the same time, problems can be used or abused, just as a piece of music can be played beautifully or badly. Wonderful raw materials can be put to good use, or they can be poorly used. What matters is how students engage in the problems. That's where pedagogy – more broadly, the creation of a learning environment in which students engage in powerful ways with mathematics – really matters.

As I hope this chapter makes clear, my thoughts on what makes for productive learning environments are very much a work in progress. In general terms, I have described the attributes of such learning environments in the Teaching for Robust Understanding (TRU) framework (see, e.g., Schoenfeld & the Teaching for Robust Understanding Project, 2016, and https://truframework.org/. In "real time," my problem-solving courses evolve as my understanding of what it means to think productively with mathematics evolves. Through the years the goals of the course have expanded to include various aspects of mathematical thinking such as heuristic strategies, metacognition, the development of productive mathematical belief systems, and powerful mathematical practices and habits of mind. They include students' development of a sense of mathematical initiative and agency and more generally powerful mathematical identities. I will, in the coming years, be grappling with questions of how to make such competencies more outward-facing, so that students will have the predilections and understandings that will enable them to use what they know more readily in contexts that matter in their personal lives.

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