Chapter 9
Periodic Contact Problem for a Two-level System of Punches and a Viscoelastic Half-space

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Abstract The paper presents a solution of the contact problem for a periodic two-level system of axisymmetric punches and a viscoelastic half-space. The case of a constant nominal pressure applied to the punch system is considered. The variation of the real contact area in time and the conditions that provide the contact of the punches of both levels with the half-space are investigated. The influence of the geometric parameters of the punch system and the mechanical properties of the viscoelastic half-space on the contact characteristics are analyzed. Numerical results are presented for the system of punches located in the nodes of a square lattice and penetrated into a viscoelastic half-space modeled by a standard linear solid. It is shown that the real contact area may increase greatly if the second level punches come into contact with the half-space at some instant of time.

Keywords Periodic contact · Linear viscoelasticity · Multi-level system of punches · Time dependent contact characteristics

9.1 Introduction

Discrete contact problems are of great theoretical and practical importance. These problems mainly arise in study of the contact of bodies taking into account their surface microrelief, which is formed by a surface roughness. Solutions of these problems differ considerably from the classical solutions of the contact problems for absolutely smooth bodies. The most significant difference is that the real contact area consists of the system of contact spots may be several times smaller than the nominal one. This fact is essential in analysis of many important operational properties of tribounits.

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The characteristics of the contact interaction are determined not only by the geometry of the contact surfaces of the bodies, but also by their mechanical properties. Many materials are characterized by viscoelastic properties, that is, their stress-strain state is time-dependent. For example, soft polymers belong to this type of materials. Polymers are widely used in many industries such as automotive, aerospace, construction, textile, medicine and others [1]. They are applied to produce food wraps, containers, adhesives, electric- and thermal insulation, lenses, windows, clothing, etc. Hence, the problems of discrete contact of viscoelastic bodies are of great relevance. An extensive study of the stress state of viscoelastic bodies is presented in the book of Arutunyan [2].

The roughness of surfaces has often a statistical nature; however, it can also be regular, for example, in the case when it is produced by artificial methods. Currently, numerical methods are widely used to solve the problems of contact of rough surfaces. Such approach makes it possible to consider any contact geometry of bodies and their mechanical properties. For example, numerical calculations were used in [3] to analyze the viscoelastic contact of tires with the road. In this study, the authors proposed to consider the contact at the macrolevel with limited number of asperities. The conjugate gradient method was used in [4] to solve the problem of the contact of a rigid smooth spherical indenter with a viscoelastic rough half-space. The problem of the rough viscoelastic contact is also numerically solved in [5] using both spatial and time discretization.

Despite a huge variety of problems that can be solved by numerical methods, analytical approaches to solving the discrete contact problems also do not lose their relevance. There are a number of analytical methods of solving the contact problems for viscoelastic bodies with regular and irregular microrelief. In the case of irregular roughness, the contact model is in general based on the Greenwood–Williamson approach [6], which respects the height distribution of asperities. The first attempt to extend the application of this model to the viscoelastic case has been performed in [7]. To take into account the viscoelasticity of bodies, the authors simply replaced the Young's modulus with the time-dependent relaxation function. However, this procedure is incorrect, as was shown in [8], where the accurate solution was derived. The solution obtained in [8] allows us to take into account the fact that asperities of different heights come into contact at various times. In addition to the Greenwood–Williamson statistical model, fractal geometry is also used to describe the contact of rough surfaces in the viscoelastic case (see, for example, [9]). Another approach to solving the contact problems for rough bodies based on the probabilistic method and the diffusion equation can be found in the Persson's works. In [10], this approach was used to solve the contact problem in the viscoelastic case for fixed nominal pressure. Later, the Persson's theory was extended to the viscoelastic contact under an arbitrarily time-varying applied load [11]. Despite the fact that the Persson's theory for the rough contact gives quite simple results, its justification is not strict enough that is noted, for example, in [12].

Analytical methods have also been developed to study the contact of bodies with periodic roughness. In [13], the localization method was suggested to solve the contact problems for a periodic system of axisymmetric punches and an elastic half-space. This method allows us to consider both single-level periodic systems of punches and systems of punches with different heights. Application of the localization method to solve the periodic problem with a viscoelastic half-space is carried out in [14], where the analytical solution for a single-level system of spherical punches which is in contact with the viscoelastic base was reduced. However, the surface roughness is often uneven in height, which is essential for studying the contact of bodies with rheological properties. In this study, using the localization principle, the contact of a two-level periodic system of axisymmetric punches with a viscoelastic half-space is analyzed.

The article is structured as follows. In Sect. 9.2, the formulation of the contact problem for the two-level periodic system of punches indenting into the viscoelastic half-space is presented. In Sect. 9.3, the one-level contact of the system of punches with the half-space is studied and the conditions of the second level punches coming into contact are analyzed. In Sect. 9.4, the two-level contact is investigated and the dependence of the real contact area on time is analyzed. Section 9.5 provides some main conclusions.

9.2 Statement of the Contact Problem

The indentation of a two-level periodic system of axisymmetric punches into a viscoelastic half-space is considered. The difference in the heights of the punches of the two levels is given and equal to Δh. The shape of the contact surface of the punches is described by the function $f(r) = Cr^s \big/ R^{s-1}$, where $s = 1, 2, \ldots$, R is the characteristic punch size, C is the dimensionless constant. The axes of symmetry of the punches are perpendicular to the boundary of the half-space. We connect the coordinate system with some fixed punch of the ith level ($i = 1, 2, \ldots$) in such a way as to the axis Oz coincides with the axis of rotation of the fixed punch, and the plane $Or\theta$ coincides with the undeformed surface of the half-space. We also assume that the each contact spot of the ith level is bounded by a circle of radius $a_i(t)$ (Fig. 9.1) that is valid for not very tight contact.

The boundary conditions of the problem are of a mixed type, since normal displacements of the half-space boundary are known at all contact spots, and outside the contact area, we have the condition of zero normal stresses. It is also assumed that the shear stresses are zero on the entire boundary of the half-space. In addition, as initial conditions, we suppose that before the interaction process begins, the viscoelastic half-space is not stressed and is at rest.

For certainty, we consider a system of punches located at each level in nodes of a square lattice, that is, at vertices of squares of one size $l \times l$ (Fig. 9.1). The punch system is loaded with a nominal pressure $\bar{p}(t) = \bar{p}_0 H(t)$ acting within one period. Here $\bar{p}_0$ is a given constant and $H(t)$ is the Heaviside function. We assume that the material of the half-space is homogeneous, isotropic, linearly viscoelastic, and has the constant Poisson's ratio ν. In this case, one relaxation function is sufficient

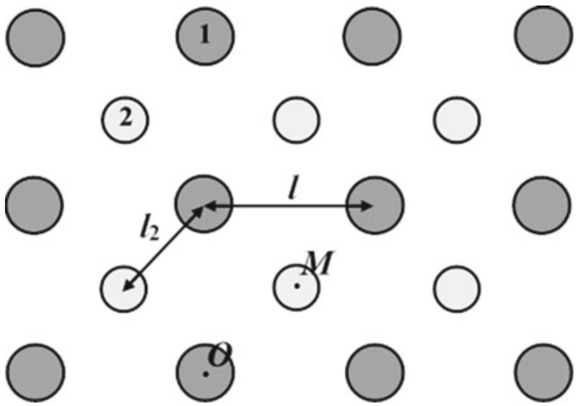


Fig. 9.1 Scheme of the location of punches in a two-level periodic system

to describe mechanical behavior of the material [15]. For example, we can use the relaxation function $E(t)$ corresponding to uniaxial tension/compression or the creep function $J(t)$ related to it.

9.3 Contact Problem Solution for the First Level Punches

If the value of the nominal pressure $\bar{p}(t)$ is not sufficient to immediately provide the two- level contact, only the first level punches come into contact with the half-space at the beginning of the contact interaction. In this part, the contact problem analysis for the one-level system of punches is presented and the condition when the second level of punches comes into contact is derived.

9.3.1 Derivation of the Main Equations for Calculation of the First Level Contact Characteristics

The solution of the problem of the indentation of a one-level punch system into a viscoelastic half-space is obtained in [16]. This solution is constructed using the extended correspondence principle [17] and the localization method [13]. In [14, 16], the one-level periodic contact problem is solved in the simplest case, when the real pressure distribution is taken into account only under the fixed punch, and the action of all the others is replaced by the nominal pressure. In this research, in order to improve the calculation accuracy, one more series of punches is added to consideration, and their action is replaced by the load of intensity $4P_1(t)$ distributed over a circle of radius l. In this case, the solution of the single-level problem takes the following form

$$p_1(r,t)=\int\limits_{0^-}^{t}E(t-\tau)\frac{\partial}{\partial\tau}\left(\frac{Cp_{(k,m)}(r,\tau)}{\pi\left(1-\nu^2\right)R^{s-1}}+\frac{2}{\pi^2}\left(\int\limits_{0^-}^{\tau}J(\tau-\tau')\frac{\mathrm{d}P_1(\tau')}{\mathrm{d}\tau'}\mathrm{d}\tau'\right)\right.$$
$$\left.\cdot\left(\frac{2}{l^2-r^2}\sqrt{\frac{a_1^2(\tau)-r^2}{l^2-a_1^2(\tau)}}+\frac{5}{A_1^2}\arctan\sqrt{\frac{a_1^2(\tau)-r^2}{A_1^2-a_1^2(\tau)}}\right)\right)\mathrm{d}\tau, \tag{9.1}$$

$$\frac{a_1^{s+1}(t)}{10\left(\arccos\left(\frac{a_1(t)}{A_1}\right)+\frac{a_1(t)}{A_1^2}\sqrt{A_1^2-a_1^2(t)}\right)-8\left(\arccos\left(\frac{a_1(t)}{l}\right)+\frac{a_1(t)}{\sqrt{l^2-a_1^2(t)}}\right)}$$
$$=\frac{\left(1-\nu^2\right)(s+1)\Gamma(s)R^{s-1}}{\pi 2^{s-1}s^2C\Gamma^2\left(\frac{s}{2}\right)}\int\limits_{0^-}^{t}J(t-\tau)\frac{\mathrm{d}P_1(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau, \tag{9.2}$$

where $\Gamma(x)$ is the gamma function, $p_1(r,t)$ is the contact pressure under the first level punch, $P_1(t)$ is the load applied to a single first level punch, and A_1 is the radius of the circle outside of which the nominal pressure is distributed, which replaces the action of all punches except the fixed one and four nearby punches. The functions $p_{(k)}(r,t)$ and $p_{(m)}(r,t)$ in Eq. (9.1) are determined by the parity of the exponent s of the function $f(r)$ and have the following form

$$p_{(k)}(r,t)=\left(\frac{(2k+1)!!}{(2k)!!}\right)^2 r^{2k}$$
$$\cdot\left(\cosh^{-1}\left(\frac{a_1(t)}{r}\right)+\sqrt{1-\left(\frac{r}{a_1(t)}\right)^2}\sum_{i=1}^{k}\frac{(2i-2)!!}{(2i-1)!!}\left(\frac{a_1(t)}{r}\right)^{2i}\right),$$

$$p_{(m)}(r,t)=\left(\frac{(2m)!!}{(2m-1)!!}\right)^2 a_1^{2m-1}(t)\sqrt{1-\left(\frac{r}{a_1(t)}\right)^2}\sum_{i=1}^{m}\frac{(2i-3)!!}{(2i-2)!!}\left(\frac{r}{a_1(t)}\right)^{2(m-i)},$$

where k are m are integers. The function with index k corresponds to the odd exponent s, that is, $s=2k+1$, and the function with index m corresponds to an even exponent s, that is, $s=2m$. According to the localization principle, the radius A_1 of the circle is determined by the average number $\bar{N}_1$ of contact spots per unit area and the number M_1 of punches located inside this circle, namely

$$\pi A_1^2=\frac{M_1}{\bar{N}_1}. \tag{9.3}$$

During the single-level contact $\bar{p}(t) = \bar{N}_1 P_1(t)$, so Eqs. (9.1) and (9.2) can be written in terms of the nominal pressure. Provided the constant value of the applied nominal pressure, we obtain

$$p_1(r,t) = \int\limits_{0^-}^{t} E(t-\tau)\frac{\partial}{\partial\tau}\left(\frac{Cp_{k,m}(r,\tau)}{\pi\left(1-\nu^2\right)R^{s-1}} + \frac{2}{5\pi}\bar{p}_0 J(\tau)\right.$$
$$\left.\cdot\left(\frac{2A_1^2}{l^2-r^2}\sqrt{\frac{a_1^2(\tau)-r^2}{l^2-a_1^2(\tau)}} + 5\arctan\sqrt{\frac{a_1^2(\tau)-r^2}{A_1^2-a_1^2(\tau)}}\right)\right)\mathrm{d}\tau, \tag{9.4}$$

$$\frac{a_1^{s+1}(t)}{10\left(\arccos\left(\frac{a_1(t)}{A_1}\right)+\frac{a_1(t)}{A_1}\sqrt{A_1^2-a_1^2(t)}\right)-8\left(\arccos\left(\frac{a_1(t)}{l}\right)+\frac{a_1(t)}{\sqrt{l^2-a_1^2(t)}}\right)}$$
$$= \frac{\left(1-\nu^2\right)(s+1)\Gamma(s)R^{s-1}A_1^2\bar{p}_0 J(t)}{2^{s-1}5s^2C\Gamma^2\left(\frac{s}{2}\right)}. \tag{9.5}$$

Equations (9.4) and (9.5) are used to calculate the pressure distribution at the initial stage of indentation process when only the first level punches come into contact.

9.3.2 Determination of the Instant of Time When the Second Level of Punches Comes into Contact with the Half-space

Let us find the time instant t_* when the second level punches come into the contact with the viscoelastic half-space. For this purpose, first it needs to investigate the variation in time of the vertical displacement of the point M of the half-space boundary (Fig. 9.1). The point M is located at the center of the square and the vertices of which are centers of the contact spots of the first level punches. For simplicity, the action of these four punches is replaced by the load of intensity $4P_1(t)$ distributed along the circumference of radius $l_2 = l\big/\sqrt{2}$, and the action of other first level punches is replaced by the nominal pressure distributed outside the circle of radius A. Based on (9.3), the radius A is determined from the condition $\pi A^2 = 4\big/\bar{N}_1$.

The vertical displacement $u_z(r,t)$ of the viscoelastic half-space (characterized by the constant Poisson ratio) due to the action of the axisymmetric normal pressure $p(r,t)$ applied over a circular area of radius $a(t)$ is determined by the following expression [18]

$$u_z(r,t) = \frac{4\left(1-\nu^2\right)}{\pi} \int\limits_{0^-}^{t} J(t-\tau) \frac{\partial}{\partial \tau} \left(\int\limits_{0}^{a(\tau)} p(\rho,\tau) \mathbf{K}\left(\frac{2\sqrt{\rho r}}{\rho + r}\right) \frac{\rho \mathrm{d}\rho}{\rho + r} \right) \mathrm{d}\tau, \quad (9.6)$$

where $\mathbf{K}(x)$ is the complete elliptic integral of the first kind. It is convenient to put the origin at the point M (Fig. 9.1). Based on the proposed replacement of the action of punches with the circumferentially distributed load $2P_1(t)\delta(r-l_2)\big/(\pi r)$ ($\delta(x)$ is the Dirac delta function) and the nominal pressure $\bar{p}(t)$, we obtain from Eq. (9.6) the following expression for the half-space boundary displacement for $r < l_2$

$$u_z(r,t) = \frac{8\left(1-\nu^2\right)}{\pi^2 l_2} \mathbf{K}\left(\frac{r}{l_2}\right) \int\limits_{0^-}^{t} J(t-\tau) \frac{\mathrm{d}P_1(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau + D_\infty$$
$$- \frac{4\left(1-\nu^2\right)}{\pi} A \mathbf{E}\left(\frac{r}{A}\right) \int\limits_{0^-}^{t} J(t-\tau) \frac{\mathrm{d}\bar{p}(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau, \quad (9.7)$$

where $\mathbf{E}(x)$ is the complete elliptic integral of the second kind and D_∞ is the displacement of the half-space boundary loaded everywhere with the nominal pressure. Hence, the displacement of the point M, i.e., $r = 0$, provided the constant nominal pressure $\bar{p}_0$ applied to the system, as follows from (9.7) and the equilibrium condition $\pi A_1^2 \bar{p}(t) = 5P_1(t)$, is

$$u_z(0,t) = 2\left(1-\nu^2\right)\left(\frac{2A_1^2}{5l_2} - A\right) \bar{p}_0 J(t) + D_\infty. \quad (9.8)$$

The vertical displacement of any fixed first level punch is determined by the magnitude of the indentation depth $D(t)$ of the periodic system of the first level punches under the given load $\bar{p}(t)$. By analogy with [14], we find the function of the additional displacement $d(t)$, which is

$$d(t) = D(t) - D_\infty = \frac{s\Gamma^2\left(\frac{s}{2}\right) C a_1^s(t)}{2^{2-s}\Gamma(s) R^{s-1}}$$
$$-2\left(1-\nu^2\right)\left(\sqrt{A_1^2 - a_1^2(t)} - \frac{2A_1^2}{5\sqrt{l^2 - a_1^2(t)}}\right) \bar{p}_0 J(t). \quad (9.9)$$

For further investigation, we introduce a function $h(t)$ equal to the difference between the displacements of the points O and M (Fig. 9.1). Based on Eqs. (9.8) and (9.9), this function is calculated by the following expression

$$h(t) = D(t) - u_z(0,t) = \frac{s\Gamma^2\left(\frac{s}{2}\right)Ca_1^s(t)}{2^{2-s}\Gamma(s)R^{s-1}} \\ -2\left(1-\nu^2\right)\left(\sqrt{A_1^2 - a_1^2(t)} - \frac{2A_1^2}{5\sqrt{l^2 - a_1^2(t)}} + \frac{2A_1^2}{5l_2} - A\right)\bar{p}_0 J(t). \tag{9.10}$$

Together with Eq. (9.5), which determines the dependence of the radius of a single contact spot on time, Eq. (9.10) allows us to calculate the value of the function $h(t)$ at each time. The time instant t_* when the value of the function $h(t)$ becomes equal Δh, i.e., $\Delta h = h(t_*)$, determines the moment when the second level punches come into contact with the half-space. It also follows that if $h(0) \geq \Delta h$, then the two-level contact occurs immediately from the beginning of the interaction process.

9.3.3 *Analysis of the Indentation of the First Level Punches into the Half-space*

Let us first analyze the dependence $h(a_1)$. Based on Eqs. (9.5) and (9.10), we get

$$h(a_1) = \frac{s\Gamma^2\left(\frac{s}{2}\right)Ca_1^s}{2^{2-s}\Gamma(s)R^{s-1}}\left(1 - \frac{20sa_1\left(\sqrt{A_1^2 - a_1^2} - \frac{2A_1^2}{5\sqrt{l^2-a_1^2}} + \frac{2A_1^2}{5l_2} - A\right)}{(s+1)A_1^2\left(10\left(\arccos\left(\frac{a_1}{A_1}\right) + \frac{a_1}{A_1}\sqrt{A_1^2 - a_1^2}\right) - 8\left(\arccos\left(\frac{a_1}{l}\right) + \frac{a_1}{\sqrt{l^2-a_1^2}}\right)\right)}\right).$$

Figure 9.2 illustrates the function $h(a_1)$ and its derivative for three values of the exponent s of the shape function $f(r)$ for the quadratic lattice (in this case, the average number of the contact spots per unit area is defined by the expression $\bar{N}_1 = 1/l^2$). As follows from the results of calculations, the function $h(a_1)$ is monotonically increasing (Fig. 9.2b shows that the derivative of the function $h(a_1)$ is positive everywhere). As shown in [14], at a constant nominal pressure, the dependence of the radius of the contact spot of the single first level punch with the viscoelastic half-space is described by a monotonically increasing function. Consequently, the difference in the displacements of the points O (the center of the contact area of the first level punch with the half-space) and M (the square center where the contact of the second level punch with the half-space should begin) also increases with time.

Let us analyze the dependence of the difference in the displacements of the points O and M (Fig. 9.1) on time for certain types of the creep function $J(t)$. We consider the viscoelastic model of the standard linear solid and the creep function of which has the following form [19]

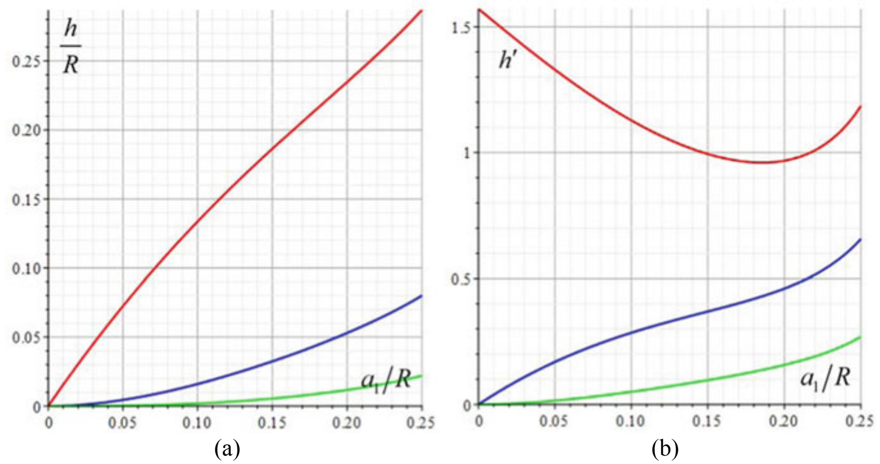
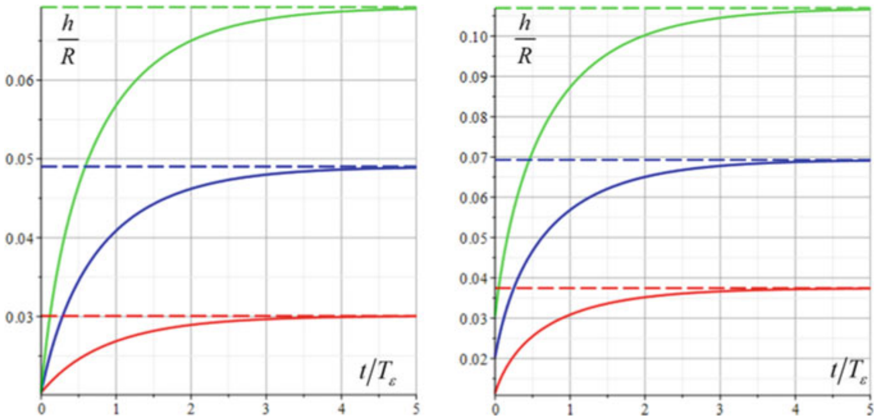


Fig. 9.2 Dependence of the function h (**a**) and its derivative (**b**) on the radius of the single contact spot a_1 for different punch shapes ($s = 1$ for the red lines, $s = 2$ for the blue lines, $s = 3$ for the green lines), and $C = 1$, $l = 0.5R$

$$J(t) = \frac{T_\varepsilon}{E_0 T_\sigma}\left(1 - \left(1 - \frac{T_\sigma}{T_\varepsilon}\right)\exp\left(-\frac{t}{T_\varepsilon}\right)\right), \tag{9.11}$$

where E_0 is the instantaneous elastic modulus, T_σ is the relaxation time and T_ε is the creep (retardation) time. Substituting Eq. (9.11) into Eq. (9.10), we obtain an expression that with Eq. (9.5) allows us to determine the moment t_* when the second level punches come into contact with the half-space. This moment is determined from the condition $h(t_*) = \Delta h$.

Figure 9.3 illustrates the dependence $h(t)$ for spherical punches with $f(r) = r^2/(2R)$ for different values of the ratio of creep and relaxation times, as well as for different densities of location of punches in the system (the different pitch l of the quadratic lattice). As follows from the calculation results, the value h for a fixed instant of time grows with an increase in the parameter $T = T_\varepsilon/T_\sigma$ (for a fixed instantaneous elastic modulus E_0), that is, in the viscosity of the half-space material, and with an increase in the pitch l of the quadratic lattice. Therefore, these parameters, as well as the values of the height difference Δh and the applied nominal pressure $\bar{p}_0$, influence the fact whether the second level punches will come into contact with the viscoelastic half-space. So, if $\Delta h = 0.04R$, then with the values of the parameters under consideration, the two-level contact does not occur at $T = 2$ or at $l = 0.5R$. Note that in the graphs of Fig. 9.3, the dotted lines correspond to the elastic case with the long-term (equilibrium) elastic modulus $E_\infty = E_0 T_\sigma / T_\varepsilon$.



(a) (b)

Fig. 9.3 Dependencies of the difference of the displacements of the points O and M on time (**a**) for different values of T ($T = 2$ for the red line, $T = 5$ for the blue line, $T = 10$ for the green line) at $l = 0.75R$; and (**b**) for different values of the pitch l ($l = 0.5R$ for the red line, $l = 0.75R$ for the blue line, $l = R$ for the green line) at $T = 10$; $\bar{p}_0 = 0.01E_0/\left(1-\nu^2\right)$

9.4 Two-level Contact of the System of Punches and the Half-space

If the conditions for the second level punches coming into contact are fulfilled, the two-level contact occurs. For the asymptotic analysis of the contact characteristics in the two-level contact, we use the correspondence elastic solution with instantaneous and long-term elastic modules of the viscoelastic material under consideration.

9.4.1 Asymptotic Analysis of the Contact Characteristics

A general approach to solving the problems of indentation of a multi-level periodic system of punches into an elastic half-space is presented in [20]. Fixing the punch of one of the two levels and replacing the action of the nearby four punches of another level, as well as four punches of the same level with loads distributed over the circles of radii l_2 and l, respectively, and all other punches of both levels with the nominal pressure, we obtain the following system of equations ($i = 1, 2$)

$$\bar{p} = \frac{1}{l^2}\left(P_1 + P_2\right), \tag{9.12}$$

$$\pi A_i^2 \bar{p} = 5P_i + 4P_j, \tag{9.13}$$

$$P_i = \frac{2^{s-1}s^2\Gamma^2\left(\frac{s}{2}\right)ECa_i^{s+1}}{(s+1)\Gamma(s)\left(1-\nu^2\right)R^{s-1}} + \frac{8P_j}{\pi}\left(\frac{a_i}{\sqrt{l_2^2-a_i^2}} - \arcsin\left(\frac{a_i}{l_2}\right)\right)$$
$$+\frac{8P_i}{\pi}\left(\frac{a_i}{\sqrt{l^2-a_i^2}} - \arcsin\left(\frac{a_i}{l}\right)\right) + 2\bar{p}\left(A_i^2\arcsin\left(\frac{a_i}{A_i}\right) - a_i\sqrt{A_i^2-a_i^2}\right), \tag{9.14}$$

$$\Delta h = \frac{1-\nu^2}{\pi E}\left(\frac{\pi s\Gamma^2\left(\frac{s}{2}\right)CE\left(a_1^s-a_2^s\right)}{2^{2-s}\Gamma(s)R^{s-1}\left(1-\nu^2\right)} - \left(\frac{4P_1}{\sqrt{l_2^2-a_2^2}} - \frac{4P_2}{\sqrt{l_2^2-a_1^2}}\right)\right.$$
$$\left. - \left(\frac{4P_2}{\sqrt{l^2-a_2^2}} - \frac{4P_1}{\sqrt{l^2-a_1^2}}\right) - 2\pi\bar{p}\left(\sqrt{A_1^2-a_1^2} - \sqrt{A_2^2-a_2^2}\right)\right). \tag{9.15}$$

Note that in Eqs. (9.13) and (9.14) $i \neq j$.

Table 9.1 gives the values P_i, a_i, A_i for spherical punches of each level ($i = 1, 2$) calculated by Eqs. (9.12)–(9.15) for the instantaneous and long-term elastic modulus presented in the table at $\bar{p}_0 = 0.06E_0\big/\left(1-\nu^2\right), l = 0.75R, \Delta h = 0.05R$. It follows from the results that the load applied to the single punch of the first level decreases, and the load applied to the single punch of the second level increases with time. The contact spot's radii a_i for punches of both levels increase over time. Note that the growth of the contact radius is limited by the condition $a_1 + a_2 \leq l_2$, that is, the sum of the radii of the contact spots does not exceed a half of the length of the diagonal of the lattice square.

The radii A_i ($i = 1, 2$) of the areas ($r > A_i$) in which, according to the used model, the nominal pressure acts are practically constant as follows from the calculation results presented in Table 9.1. Therefore, for the correct application of the localization principle for investigating the two-level contact in the viscoelastic case, we assume that the radii A_1 and A_2 do not change in time: $A_1(t) = A_1(t_*)$ and $A_2(t) = A_2(t_*)$, where $t_* \geq 0$.

Table 9.1 Instantaneous and long-term values of the contact characteristics of the two-level contact

E	$\frac{P_1(1-\nu^2)}{R^2E_0}$	$\frac{P_2(1-\nu^2)}{R^2E_0}$	$\frac{a_1}{R}$	$\frac{a_2}{R}$	$\frac{A_1}{R}$	$\frac{A_2}{R}$
E_0	0.0335	0.0003	0.2815	0.0567	0.9454	0.8472
$E_\infty = 0.8E_0$	0.0327	0.0011	0.2976	0.0926	0.9432	0.8497
$E_\infty = 0.5E_0$	0.0308	0.0029	0.3302	0.1455	0.9379	0.8555
$E_\infty = 0.4E_0$	0.03	0.0038	0.3447	0.1662	0.9355	0.8581

9.4.2 Solution of the Viscoelastic Two-level Periodic Problem

As it was shown in Sect. 9.4.1, if the condition $a_1 + a_2 \leq l_2$ is precisely satisfied, the radii a_1 and a_2 increase over time. This fact makes it possible to derive the viscoelastic solution based on the solution of the similar elastic problem using the extended correspondence principle [15]. According to this principle, by replacing p_i/E with

$$\int\limits_{0^-}^{t} J(t-\tau)\,(\partial p_i/\partial \tau)\,\mathrm{d}\tau$$

and, consequently, $\bar{p}/E$ and P_i/E with

$$\int\limits_{0^-}^{t} J(t-\tau)\,(\mathrm{d}\bar{p}/\mathrm{d}\tau)\,\mathrm{d}\tau$$

and

$$\int\limits_{0^-}^{t} J(t-\tau)\,(\mathrm{d}P_i/\mathrm{d}\tau)\,\mathrm{d}\tau,$$

respectively, and taking into account Eqs. (9.14)–(9.15) and the assumption that the radii A_1 and A_2 do not depend on time, we obtain the following system of equations for calculation of the contact characteristics in the two-level periodic contact problem for the viscoelastic half-space

$$Q_i(t) = \frac{2^{s-1}s^2\Gamma^2\left(\frac{s}{2}\right)Ca_i^{s+1}(t)}{(s+1)\Gamma(s)\left(1-\nu^2\right)R^{s-1}} + \frac{8Q_j(t)}{\pi}\left(\frac{a_i(t)}{\sqrt{l_2^2-a_i^2(t)}} - \arcsin\left(\frac{a_i(t)}{l_2}\right)\right)$$
$$+\frac{8Q_i(t)}{\pi}\left(\frac{a_i(t)}{\sqrt{l^2-a_i^2(t)}} - \arcsin\left(\frac{a_i(t)}{l}\right)\right) + 2\bar{p}_0 J(t)\left(A_i^2 \arcsin\left(\frac{a_i(t)}{A_i}\right) - \right.$$
$$\left. - a_i(t)\sqrt{A_i^2-a_i^2(t)}\right), \tag{9.16}$$

$$\Delta h = \frac{s\Gamma^2\left(\frac{s}{2}\right)C\left(a_1^s(t)-a_2^s(t)\right)}{2^{2-s}\Gamma(s)R^{s-1}} - \frac{4\left(1-\nu^2\right)}{\pi}\left(\frac{Q_1(t)}{\sqrt{l_2^2-a_2^2(t)}} - \frac{Q_2(t)}{\sqrt{l_2^2-a_1^2(t)}}\right)$$
$$-\frac{4\left(1-\nu^2\right)}{\pi}\left(\frac{Q_2(t)}{\sqrt{l^2-a_2^2(t)}} - \frac{Q_1(t)}{\sqrt{l^2-a_1^2(t)}}\right) - 2\left(1-\nu^2\right)\bar{p}_0 J(t)\left(\sqrt{A_1^2-a_1^2(t)}\right.$$
$$\left. - \sqrt{A_2^2-a_2^2(t)}\right), \tag{9.17}$$

where $Q_i(t) = \int_{0^-}^{t} J(t-\tau)\,(\mathrm{d}P_i/\mathrm{d}\tau)\,\mathrm{d}\tau$ $(i = 1, 2)$. It is also necessary to add the equilibrium condition (9.12) to (9.16)–(9.17). The values of the radii A_1 and A_2 are taken as follows

$$\pi A_i^2(t_*)\bar{p}_0 = 5P_i(t_*) + 4P_j(t_*). \tag{9.18}$$

In particular, if $t_* \neq 0$, that is, when the value of the nominal pressure is not sufficient for the two-level contact to occur immediately, the values $A_1(t) = A_1(t_*)$ and $A_2(t) = A_2(t_*)$ are calculated from the following expressions

$$\pi A_1^2(t_*)\bar{p}_0 = 5P_1(t_*) + 4P_2(t_*) = 5l^2\bar{p}_0 \implies A_1 = \sqrt{\frac{5}{\pi}}l,$$

$$\pi A_2^2(t_*)\bar{p}_0 = 5P_2(t_*) + 4P_1(t_*) = 4l^2\bar{p}_0 \implies A_2 = \frac{2l}{\sqrt{\pi}}.$$

Equation (9.12) for this case takes the following form

$$\int_{0^-}^{t} J(t-\tau)\frac{\mathrm{d}\bar{p}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = \frac{1}{l^2}\left(Q_1(t) + Q_2(t)\right). \tag{9.19}$$

The resulting system of Eqs. (9.16)–(9.19) enables to determine the dependencies on time of the radii of the contact spots of the punches of both levels, as well as the functions $Q_1(t)$ and $Q_2(t)$, which are then used to calculate the dependencies on time of the load distribution between the punches of both levels, that is, the functions $P_1(t)$ and $P_2(t)$.

Figure 9.4 illustrates the dependencies $a_1(t)$ and $a_2(t)$ for two values of the nominal pressure $\bar{p}_0$, one of which immediately provides the two-level contact, and the other provides it after some time. The results are calculated for the system of spherical punches ($f(r) = r^2\big/(2R)$). As follows from the results, the radii of the contact spots increase in time, tending to the constant values that correspond to the elastic solutions with the long-term elastic modulus. Note that the radii of the contact spots of the second level punches (red lines) increase significantly compared with the punches of the first level. For example, for the parameters used in calculations, in the case when the two-level contact occurs from the beginning of the indentation process (Fig. 9.4a), the radius a_1 increases about 1.2 times, while the radius a_2 increases 2.7 times to the time instant $t = 5T_\varepsilon$. Figure 9.4b illustrates the case when the second level punches come into contact with the half-space only some time after the beginning of the indentation process.

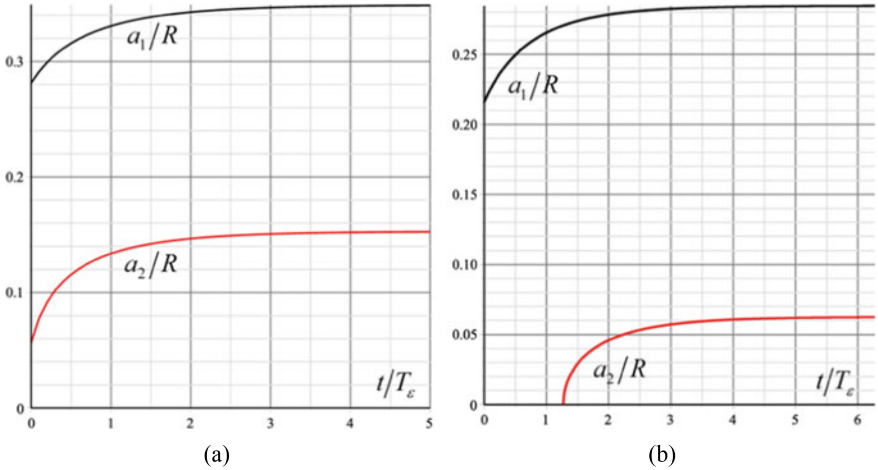


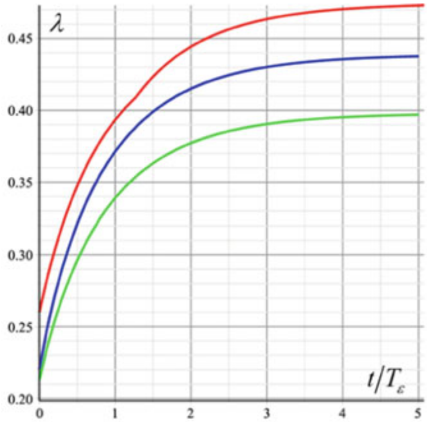
Fig. 9.4 Dependencies of the radii a_i $(i = 1, 2)$ of the contact spots of each level on time at $l = 0.75R$, $T = 2.5$, $\Delta h = 0.05R$ and $\bar{p}_0 = 0.06E_0 / \left(1 - \nu^2\right)$ (**a**); $\bar{p}_0 = 0.025E_0 / \left(1 - \nu^2\right)$ (**b**)

9.4.3 Analysis of the Dependence of the Real Contact Area on Time

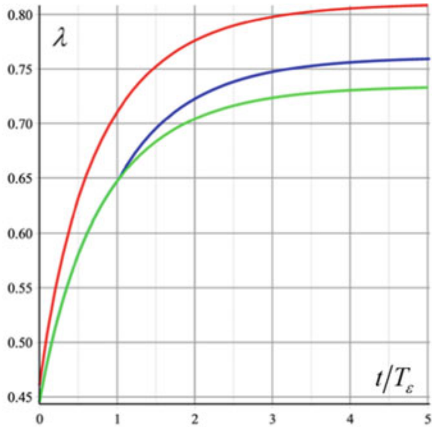
Let us introduce the following function that characterizes the evolution in time of the relative contact area for the two-level system of punches located in the nodes of quadratic lattice (Fig. 9.1)

$$\lambda(t) = \pi a_1^2(t)\bar{N}_1 + \pi a_2^2(t)\bar{N}_2 = \frac{\pi}{l^2}\left(a_1^2(t) + a_2^2(t)\right). \tag{9.20}$$

Figure 9.5 illustrates the dependencies of the relative contact area on time at different values of the pitch l of the quadratic lattice and different values of the height difference of the spherical punches of the two levels. The results are obtained for the viscoelastic model of the standard linear solid with $T = 2.5$. The results indicate that a decrease in the punch height difference and the distance between them leads to an increase in the relative contact area. Depending on the distance between the punches and their height difference, the contact area forms from interaction with the half-space of the only first level punches or both levels. So, if $\bar{p}_0 = 0.025E_0 / \left(1 - \nu^2\right)$, the transition from the single-level contact to the two-level contact occurs only for the square lattice with $l = 0.75R$ as follows from the results presented in Fig. 9.5a (red line). In the other two cases, the two-level contact is observed from the beginning of the interaction process. For the higher nominal pressure $\bar{p}_0 = 0.06E_0 / \left(1 - \nu^2\right)$, in the case of the system with $\Delta h = 0.05R$, the punches of the second level are in contact with the half-space from the beginning of the interaction process, for the system with $\Delta h = 0.075R$ the second level of punches comes into contact at $t = 1.03T_\varepsilon$



(a)



(b)

Fig. 9.5 Dependencies of the relative contact area on time **a** at $\Delta h = 0.05R$ and $\bar{p}_0 = 0.025E_0 / (1 - \nu^2)$ ($l = 0.75R$ for the red line, $l = R$ for the blue line, $l = 1.25R$ for the green line); **b** at $l = 0.75R$ and $\bar{p}_0 = 0.06E_0 / (1 - \nu^2)$ ($\Delta h = 0.05R$ for the red line, $\Delta h = 0.075R$ for the blue line, $\Delta h = 0.1R$ for the green line)

(blue line), and in the case of the system with $\Delta h = 0.1R$, the two-level contact is not possible (see Fig. 9.5b).

9.5 Conclusions

This study investigates the contact of the two-level periodic system of axisymmetric punches with the viscoelastic half-space under the action of the constant nominal pressure. Application of the localization method and the extended correspondence principle make it possible to analyze the dependence on time of the real contact area of the punch system with the half-space, and to study the conditions provided the contact of the punches of both levels with the half-space.

It is shown that the contact of the second level punches with the viscoelastic half-space is guaranteed not only by the specific geometric parameters of the system (the pitch of the periodic lattice and the height difference of the punches of the two levels), but also by the defined values of the viscoelastic properties of the half-space material. In this connection, there are three possible cases: the two-level contact occurs immediately, the two-level contact occurs after some time, or the two-level contact will never occur. The latter is valid only if the material of the viscoelastic half-space has a non-zero long-term elastic modulus, that is, the material is characterized by the limited creep.

Analysis of the real contact area evolution showed that the radii of contact spots of the punches of both levels increase in time at least for not tight contact. If the second level punches come into contact (note that this condition is realized for the certain geometric characteristics of the periodic punch system and the certain viscoelastic properties of the half-space), the relative real contact area may grow considerably over time. This fact must be taken into account for analysis of the contact characteristics of various tribounits.

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