A Non-standard Bezout Theorem for Curves

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Abstract This paper provides a non-standard analogue of Bezout's theorem for algebraic curves. We achieve this by showing that, in all characteristics, the notion of Zariski multiplicity coincides with intersection multiplicity when we consider the full families of projective degree d and degree e curves in $P^2(L)$. The result is particularly interesting in that it holds even when we consider intersections at singular points of curves or when the curves contain non-reduced components.

Keywords Bezout theorem · Zariski multiplicity · Intersection multiplicity

The techniques of non-standard analysis, originally developed for the real numbers, were recently introduced by Zilber in the context of Zariski structures. In [17], he gives a rigorous notion of Zariski multiplicity, which, in the case of 2 curves C_1 and C_2, intersecting in a point a, can count the number of intersections of the 2 curves in an infinitely small neighborhood of a after moving one of the curves. This idea was used intuitively in the work of the Italian school of algebraic geometry, in particular by Severi. One advantage of this approach is that it avoids an over reliance on algebra, in favour of a more geometric approach. The successes of their work are well known; the development of the notion of genus for algebraic curves, building on the original ideas of Plucker, and the classification of algebraic surfaces. This paper sets out to show that this non-standard analysis can be useful in algebraic geometry, by providing a more geometric framework for understanding intersections of algebraic curves in the plane. In particular, the main result of the paper, a geometric proof of Bezout's theorem, enhances an important idea in the foundational work of the Italian school. We assume some familiarity with certain notions from algebraic and analytic geometry, as well as the material from Sects. 1–5 of [7]. We summarise the relevant facts, for the proofs of the paper, in the following three sections.

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1 Etale Morphisms and Algebraic Multiplicity

Definition 1.1 A morphism f of finite type between varieties X and Y is said to be etale if for all $x \in X$ there are open affine neighborhoods U of x and V of $f(x)$ with $f(U) \subset V$ such that restricted to these neighborhoods the pull back on functions is given by the inclusion;

$$f^* : L[V] \to L[V]\frac{[x_1, \ldots, x_n]}{< f_1, \ldots, f_n >}$$

and

$$det(\frac{\partial f_i}{\partial x_j})(x) \neq 0 \text{ , } (*)$$

The coordinate free definition of etale is that f should be flat and unramified, where a morphism f is unramified if the sheaf of relative differentials $\Omega_{X/Y} = 0$, clearly this last confition is satisfied using the condition $(*)$. If we tensor the exact sequence,

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

with $L(x)$ the residue field of x, we obtain an isomorphism

$$f^*\Omega_Y \otimes L(x) \to \Omega_X \otimes L(x).$$

Identifying $\Omega_X \otimes L(x)$ with $T^*_{x,X}$ gives that

$$df : (m_x/m_x^2)^* \to (m_{f(x)}/m^x_{f(x)})^*$$

is an isomorphism of tangent spaces or dually $f^*(m_{f(x)}) = m_x$. Call this property of etale morphisms $(**)$.

We will also require some facts about the etale topology on an algebraic variety Y, see [14] for more details. We consider a category Y_{et} whose objects are etale morphisms $U \to Y$ and whose arrows are Y-morphisms from $U \to V$. This category has the following 2 desirable properties. First given $y \in Y$, the set of objects of the form $(U, x) \to (Y, y)$ form a directed system, namely $(U, x) \subset (U', x')$ if there exists an etale morphism $U \to U'$ taking x to x'. Secondly, we can take "intersections" of open sets U_i and U_j by considering $U_{ij} = U_i \times_Y U_j$; the projection maps are easily show to be etale and the composition of etale maps is etale, so $U_{ij} \to Y$ still lies in Y_{et}. If Y is an irreducible variety over L, then all etale morphisms into Y must come from reduced schemes of finite type over L, though they may well fail to be irreducible considered as algebraic varieties. Now we can define the local ring of Y in the etale toplogy to be;

$$O^{\wedge}_{y,Y} = lim_{\to, y \in U} O_U(U)$$

As any open set U of Y clearly induces an etale morphism $U \rightarrow_i Y$ of inclusion, we have that $O_{y,Y} \subset O^{\wedge}_{y,Y}$. We want to prove that $O^{\wedge}_{y,Y}$ is a Henselian ring and in fact the smallest Henselian ring containing $O_{y,Y}$. We need the following lemma about Henselian rings, which can be found in [15].

Lemma 1.2 *Let R be a local ring with residue field L, and maximal ideal* $\mathfrak{m}$. *Suppose that R satisfies the following condition.*

If $f_1, \ldots f_n \in R[x_1, \ldots x_n]$ *and the reductions modulo the maximal ideal* $\mathfrak{m}$, $\bar{f}_1 \ldots \bar{f}_n$ *have a common root* $\bar{a}$ *in* L^n, *for which* $Jac(\bar{f})(\bar{a}) = (\frac{\partial \bar{f}_i}{\partial x_j})_{ij}(\bar{a}) \neq 0$, *then* $\bar{a}$ *lifts to a common root in* R^n *(*).*

Then R is Henselian.

It remains to show that $O^{\wedge}_{y,Y}$ satisfies $(*)$.

Proof Given $f_{1,} \ldots f_n$ satisfying the condition of $(*)$, we can assume the coefficients of the f_i belong to $O_{U_i}(U_i)$ for covers $U_i \rightarrow Y$; taking the intersection $U_{1 \ldots i \ldots n}$ we may even assume the coefficients define functions on a single etale cover U of Y. By the remarks above we can consider U as an algebraic variety over L, and even an affine algebraic variety after taking the corresponding inclusion. We then consider the variety $V \subset U \times A^n$ defined by $Spec(\frac{R(U)[x_1, \ldots, x_n]}{f_1, \ldots f_n})$. Letting $u \in U$ denote the point in U lying over $y \in Y$, the residue of the coefficients of the f_i at u corresponds to the residue in the local ring R, which tells us exactly that the point $(u, \bar{a})$ lies in V. By the Jacobian condition, we have that the projection $\pi : V \rightarrow U$ is etale at the point $(u, \bar{a})$, and hence on some open neighborhood of $(u, \bar{a})$, using Nakayama's Lemma applied to $\Omega_{V/U}$. Therefore, replacing V by the open subset $U' \subset V$ gives an etale cover of U and therefore of Y, lying over y. Now clearly the coordinate functions $x_1, \ldots x_n$ restricted to U' lie in $O^{\wedge}_{y,Y}$ and lift the root $\bar{a}$ to a root in $O^{\wedge}_{y,Y}$ $\square$

We define the Henselization of a local ring R to be the smallest Henselian ring $R' \supset R$, with $R' \subset Frac(R)^{alg}$. We have in fact, see [14], that;

Theorem 1.3 *Given an algebraic variety Y,* $O^{\wedge}_{y,Y}$ *is the Henselization of* $O_{y,Y}$

We recall the following Definition 3.6.7 from [17];

Definition 1.4 Let $F \subset D \times \mathcal{M}^k$ be a finite covering of D and $(a, b) \in F$, then;

$$Mult_b(a, F/D) = Card(F(a', {}^*\mathcal{M}^k)) \cap \mathcal{V}_b$$

for $a' \in \mathcal{V}_a$ generic in D over $\mathcal{M}$, where;

$$\mathcal{V}_a = \{a' \in {}^*\mathcal{D} : \pi(a') = a\}$$

$\mathcal{M} \prec {}^*\mathcal{M}$ and $\pi : {}^*\mathcal{M} \rightarrow \mathcal{M}$ is a universal specialisation.

Definition 1.5 If F is a finite covering of D, we say that F is unramified in the sense of Zariski structures if for all $(a, b) \in F$, $mult_b(a, F/D) = 1$.

The following theorem requires some knowledge of Zariski structures, see Sects. 1–4 of [7], or Sect. 2 of this paper.

Theorem 1.6 *Zariski multiplcity is preserved by etale morphisms Let* $\pi : X \rightarrow Y$ *be an etale morphism with* Y *smooth, then any* $(ab) \in graph(\pi) \subset X \times Y$ *is unramified in the sense of Zariski structures.*

For this we need the following fact whose algebraic proof relies on the fact that etale morphisms are flat, see [13];

Fact 1.7 *Any etale morphism can be locally presented in the form*

$$\begin{array}{ccc} V & \xrightarrow{g} & Spec((A[T]/f(T))_d) \\ \downarrow{\pi} & & \downarrow{\pi'} \\ U & \xrightarrow{h} & Spec(A) \end{array}$$

where $f(T)$ *is a monic polynomial in* $A[T]$*, the derivative* $f'(T)$ *is invertible in* $(A[T]/f(T))_d$*,* g, h *are isomorphisms and* $(A[T]/f(T))_d = \{\frac{h}{d^n} : h \in A[T], n \in \mathcal{Z}_{\geq 0}\}$.

Using Lemma 4.6 of [7] and the fact that the open set V is smooth, we may safely replace $graph(\pi)$ by $\overline{graph(\pi')} \subset F'' \times F$ where F'' is the projective closure of $Spec((A[T]/f(T))$, F is the projective closure of $Spec(A)$ and $\overline{graph(\pi')}$ is the projective closure of $graph(\pi')$ and show that $(g(b)a)$ is Zariski unramified. Note that over the open subset $U = Spec(A) \subset F$, $\overline{graph(\pi')} = Spec(A[T]/f(T))$ as this is closed in $U \times F''$. For ease of notation, we replace $(g(b)a)$ by (ba).

Suppose that f has degree n. Let $\sigma_1 \ldots \sigma_n$ be the elementary symmetric functions in n variables $T_1, \ldots T_n$. Consider the equations

$$\sigma_1(T_1, \ldots, T_n) = a_1$$

$$\ldots$$

$$\sigma_n(T_1, \ldots, T_n) = a_n \; (*)$$

where $a_1, \ldots a_n$ are the coefficients of f with appropriate sign. These cut out a closed subscheme $C \subset Spec(A[T_1 \ldots T_N])$. Suppose $(ba) \in graph(\pi') = Spec(A[T]/f(T))$ is ramified in the sense of Zariski structures, then I can find $(a'b_1b_2) \in \mathcal{V}_{abb}$ with $(a'b_1)$,$(a'b_2) \in Spec(A(T)/f(T))$ and b_1, b_2 distinct. Then complete (b_1b_2) to an n-tuple $(b_1b_2c'_1 \ldots c'_{n-2})$ corresponding to the roots of f over a'. The tuple $(a'b_1b_2c'_1 \ldots c'_{n-2})$ satisfies C, hence so does the specialisation $(abbc_1 \ldots c_{n-2})$. Then the tuple $(bbc_1 \ldots c_{n-2})$ satisfies $(*)$ with the coefficients evaluated at a. However such a solution is unique up to permutation and corresponds to the roots of f over a. This shows that f has a double root at (ab) and therefore $f'(T)|_{ab} = 0$. As (ab) lies inside $Spec(A[T]/f(T))_d$, this contradicts the fact that f' is invertible in $A[T]/f(T))_d$.

We also review some facts about algebraic multiplicity and show that algebraic multiplicity is preserved by etale morphisms.

Definition 1.8 Given projective varieties X_1, X_2 and a finite morphism $f : X_1 \to X_2$, the algebraic multiplicity $mult^{alg}_{af(a)}(X_1/X_2)$ of f at $a \in X_1$ is $length(O_{a,X_1}/f^*m_{f(a)})$ where $m_{f(a)}$ is the maximal ideal of the local ring $O_{f(a)}$.

Remark 1.9 Note that this is finite, by the fact that finite morphisms have finite fibres and the ring $O_{a,X_1}/f^*m_{f(a)}$ is a localisation of the fibre $f^{-1}(f(a)) \cong R(f^{-1}(U)) \otimes_{R(U)} L \cong R(f^{-1}(U))/m_{f(a)}$ where U is an affine subset of X_2 containing $f(a)$.

We now have the following.

Theorem 1.10 (Algebraic multiplicity is preserved by etale morphisms) *Given finite morphisms $f : X_3 \to X_2$ and $g : X_2 \to X_1$ with f etale. If $a \in X_3$, then $mult^{alg}_{a,gf(a)}(X_3/X_1) = mult^{alg}_{f(a),gf(a)}(X_2/X_1)$.*

Proof This result is essentially given in [15]. Let $O^{\wedge}_{f(a),X_2}$ be the Henselisation of the local ring at $f(a)$. By base change, we have an etale morphism $f' : X' = X_3 \times_{X_2} Spec(O^{\wedge}_{f(a)}) \to Spec(O^{\wedge}_{f(a)})$. By the definition of an etale morphism given above, we may write this cover locally in the form $Spec(O^{\wedge}_{f(a)} \frac{[x_1,\ldots,x_n]}{f_1,\ldots,f_n})$, with $det(\frac{\partial f_i}{\partial x_j}) \neq 0$ at each closed point in the fibre over $f(a)$. At the closed point a, let a_i be the residues of the x_i in L, then we have that $(a_1, \ldots a_n)$ is a common root for $\{\bar{f}_1, \ldots, \bar{f}_n\}$ where $\bar{f}_i$ is obtained by reducing f_i with respect to the maximal ideal $m_{f(a),X_2}$ of $O^{\wedge}_{f(a)}$. As $O^{\wedge}_{f(a)}$ is Henselian, by the above, and the determinant condition, we can lift the roots a_i to roots α_i of the f_i in $O^{\wedge}_{f(a)}$. We therefore obtain a subscheme $Z = Spec(O^{\wedge}_{f(a)} \frac{[x_1,\ldots,x_n]}{<x_1-\alpha_1,\ldots,x_n-\alpha_n} >)$ of X' which is isomorphic to $Spec(O^{\wedge}_{f(a)})$ under the restriction of f. Let Q be the $O_{X'}$ ideal defining Z, we then have that $m_{a,X'} = f^*m_{f(a),X_2} \oplus Q_a$. As f is etale, by $(**)$ after Definition 1.1 above, $m_{a,X'} = f^*m_{f(a),X_2}$, therefore $Q_a = 0$ and by Nakayama's lemma $Q = 0$ in an open neighborhood of a in X'. This gives that $Z = X'$ in an open neighborhood of a. Hence we obtain the sequence $O_{f(a),X_2} \to_{f^*} O_{a,X_3} \to_{i^*} O_{a,X'}$ (***) where the map i^*f^* is the inclusion of $O_{f(a),X_2}$ inside $O^{\wedge}_{f(a),X_2}$. Now if $n \subset m_{f(a),X_2}$ is the pullback $g^*m_{gf(a),X_1}$, we have that $length(O_{f(a),X_2}/n) = length(O^{\wedge}_{f(a),X_2}/n)$, hence the result follows by $(***)$ as required. □

2 Zariski Multiplicity

We work in the context of Theorem 3.3 in [7]. Namely, W (we used the notation V in [7]) will denote a smooth projective variety defined over an algebraically closed field L, considered as a Zariski structure with closed sets given by algebraic subvarieties defined over L. All notions connected to the definition of Zariski multiplicity will come from a fixed specialisation map $\pi : W(K_\omega) \to W(L)$ where K_ω denotes a "universal" algebraically closed field containing $L = K_0$. We consider D a smooth subvariety of some cartesian power W^m and a finite cover, with respect to projection onto the first coordinate, $F \subset D \times W^k$, all defined over L (*). This allows us to make sense of Zariski multiplicity. In general, we can move freely between Zariski structure notation and algebraic geometry notation. Clearly $(*)$ makes sense algebraically. Conversely, if X and Y denote fixed projective varieties defined over L with Y smooth and a finite morphism $f : X \to Y$ over L is given , then we can reduce to the situation of $(*)$ by taking F to be $graph(f) \subset X \times Y$ with the projection map onto the second factor and W to be the corresponding projective space $P^n(L)$ where $X, Y \subset P^n(L)$. We can even take W to be the 1-dimensional Zariski structure $P^1(L)$ by using the embedding of $P^n(L)$ into the N'th Cartesian power of $P^1(L)$ for sufficiently large N.

We use the definition of Zariski multiplicity for irreducible finite covers, see Definition 1.4 and also given in 4.1 of [7]. We will also require the following generalisation.

Definition 2.1 Let $F \subset D \times W^k$ be an equidimensional, finite cover of smooth D, with irreducible components $C_1, \ldots, C_n$. Then for $(ab) \in F$, we define $Mult_{ab}(F/D) = \sum_{(ab)\in C_i} Mult_{ab}(C_i/D)$.

Clearly this is well defined using the definition of Zariski multiplicity for irreducible covers. However, until Lemma 2.10, the assumption that F is irreducible will be in force.

Lemma 2.2 (Zariski multiplicity is multiplicative over composition) *Suppose that F_1, F_2 and F_3 are smooth, irreducible, with $F_2 \subset F_1 \times W^k$ and $F_3 \subset F_2 \times W^l$ finite covers. Let $(abc) \in F_3 \subset F_1 \times W^k \times W^l$. Then $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$ $mult_{abc}(F_3/F_2)$.*

Proof To see this, let $m = mult_{ab}(F_2/F_1)$ and $n = mult_{abc}(F_3/F_2)$. Choose $a' \in \mathcal{V}_a \cap F_1(K_\omega)$ generic over L. By definition, we can find distinct $b_1 \ldots b_m$ in $W^k(K_\omega) \cap \mathcal{V}_b$ such that $F_2(a', b_i)$ holds. As F_2 is a finite cover of F_1, we have that $dim(a'b_i/L) = dim(a'/L) = dim(F_1) = dim(F_2)$, so each $(a'b_i) \in \mathcal{V}_{ab} \cap F_2$ is generic over L. Again by definition, we can find distinct $c_{i1} \ldots c_{in}$ in $W^l(K_\omega) \cap \mathcal{V}_c$ such that $F_3(a'b_ic_{ij})$ holds. Then the mn distinct elements $(a'b_ic_{ij})$ are in $\mathcal{V}_{abc}$, so by definition of multiplicity $mult_{abc}(F_3/F_1) = mn$ as required. □

Lemma 2.3 *Let hypotheses be as in the above lemma with the extra condition that the cover F_3/F_2 is etale. Then for $(abc) \in F_3$, $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$*

Proof This is an immediate consequence of Lemma 2.2 and Theorem 1.6. □

Lemma 2.4 (Zariski multiplicity is summable over specialisation) *Suppose that $F \subset D \times W^k$ is a finite irreducible cover with D smooth. Suppose $(ab) \in F$, $a' \in \mathcal{V}_a \cap D$ and $a'' \in \mathcal{V}_{a'} \cap D$ with a'' generic over L. Then*

$$Mult_{ab}(F/D) = \Sigma_{b' \in \mathcal{V}_b \cap F(a')} Mult_{a'b'}(F/D)$$

where $F(a') = \{y \in F : pr(y) = a'\}$ and $pr : F \to D$ is a projection.

Proof Suppose $F(a''b_1), \ldots F(a''b_n)$ hold with $b_i \in \mathcal{V}_b$, so $\{b_1, \ldots, b_n\}$ witness the fact that $Mult_{ab}(F/D) = n$. Write $\{b_1, \ldots b_n\}$ as $\{b_{11}, \ldots, b_{1m_1}, b_{21}, \ldots, b_{2m_2}, \ldots, b_{i1}, \ldots b_{ij}, \ldots, b_{im_i}, \ldots, b_{nm_n}\}$ (*), where b_{ij} maps to a_i in the specialisation taking a'' to a'. To prove the lemma, it is sufficient to show that $F(a'y) \cap \mathcal{V}_b = \{a_1, \ldots, a_n\}$ and $Mult_{(a'a_i)}(F/D) = m_i$. The second statement just follows from the fact that a'' is generic in D over L in $\mathcal{V}_{a'}$. To prove the first statement, suppose we can find a_{n+1} with $F(a'a_{n+1})$ and $a_{n+1} \in \mathcal{V}_b$ but $a_{n+1} \notin \{a_1, \ldots a_n\}$. By Theorem 3.3 in [7], we can find c with $F(a''c)$ and $(a''c)$ specialising to $(a'a_{n+1})$. As $a_{n+1} \in \mathcal{V}_b$, $(a'a_{n+1})$ specialises to (ab), hence so does $(a''c)$. Therefore, c must witness the fact that $Mult_{ab}(F/D) = n$ and appear in the set $\{b_1, \ldots, b_n\}$. This clearly contradicts the arrangement of $\{b_1, \ldots, b_n\}$ given in $(*)$. □

Definition 2.5 Let $F \subset U \times V \times W^k$ be an irreducible finite cover of $U \times V$ with U and V smooth.

Given $(u, v, x) \in F$ we define;

$LeftMult_{u,v,x}(F/D) = Card(\mathcal{V}_x \cap F(u', v))$ for $u' \in \mathcal{V}_u \cap U$ generic over L.

$RightMult_{u,v,x}(F/D) = Card(\mathcal{V}_x \cap F(u, v'))$ for $v' \in \mathcal{V}_v \cap V$ generic over L.

We first show that both left and right multiplicity are well defined. In order to see this, observe that the fibres $F(u, V)$ and $F(U, v)$ are finite covers of V and U respectively with U and V smooth. Moreover, the fibres $F(u, V)$ and $F(U, v)$ are equidimensional covers of V and U respectively. In order to see this, as U is smooth, it satisfies the presmoothness axiom with the smooth projective variety W^k given in Definition 1.1 of [7]. The fibre $F(u, V) = F \cap (W^k \times \{u\} \times V)$. By presmoothness, each irreducible component of the intersection has dimension at least $dim(F) + dim(W^k \times V) - dim(U \times V \times W^k) = dim(F) - dim(U) = dim(V)$. As $F(u, V)$ is a finite cover of V, it has exactly this dimension. Now we can use the definition of Zariski multiplicity given in Definition 2.1.

We then claim the following.

Lemma 2.6 (Factoring Multiplicity) *In the situation of the above definition, we have that;*

$Mult_{u,v,x}(F/U \times V) = \Sigma_{x' \in (\mathcal{V}_x \cap F(y,u',v))} RightMult_{x',u',v}(F/U \times V)$ for u' generic in U over L.

$Mult_{u,v,x}(F/U \times V) = \Sigma_{x' \in (\mathcal{V}_x \cap F(y,u,v'))} LeftMult_{x',u,v'}(F/U \times V)$ for v' generic in V over L.

Proof We just prove the first statement, the proof of the second is apart from notation identical. By the construction in Sect. 2 and Lemma 3.2 of [7], we can choose algebraically closed fields $L = K_0 \subset K_{n_1} \subset K_{n_2} \subset K_\omega$, and tuples $u' \in K_{n_1}$, $v' \in K_{n_2}$ such that u' is generic in U over L, v' is generic in V over K_{n_1} with specialisations $\pi_1 : P^n(K_{n_1}) \to P^n(L)$ and $\pi_2 : P^n(K_{n_2}) \to P^n(K_1)$ such that $\pi_2(u'v') = (u'v)$ and $\pi_1(u'v) = (uv)$. Now $dim(u'v'/L) = dim(v'/L(u')) + dim(u'/L) = dim(V) + dim(U)$, hence $u'v'$ is generic in $U \times V$ over L. Therefore $Mult_{u,v,x} = Card(\mathcal{V}_x \cap F(u'v'))$. Let $S = \{y_{11}, \dots, y_{1m_1}, \dots, y_{ij_i}, \dots, y_{n1}, \dots, y_{nm_n}\}$ be distinct elements in $\mathcal{V}_x \cap W^k$ witnessing this multiplicity such that for $1 \leq j_i \leq m_i, \pi_2(y_{ij_i}) = z_i \in \mathcal{V}_x \cap W^k$. It is sufficient to show that $RightMult_{u'v,z_i}(F/U \times V) = m_i$ and $\{z_1, \dots z_n\}$ enumerates $\mathcal{V}_x \cap F(y, u', v)$. The first statement follows as $v' \in \mathcal{V}_v \cap V$ is generic in V over $L(u')$. For the second statement, suppose that we can find $z_{n+1} \in \mathcal{V}_x \cap F(y, u', v)$ with $z_{n+1} \notin \{z_1, \dots z_n\}$. Consider $F(u', V)$ as a finite cover of V, defined over $L(u')$, so by the above $F(u', V)$ is an equidimensional, see Definition 2.9 finite cover of V. Then, as v' was chosen to be generic in V over $L(u')$, choosing an irreducible component of $F(u', V)$ passing through $(z_{n+1}, u'v)$, by the lifting result of Theorem 3.3 in [7], we can find $y_{n+1} \in \mathcal{V}_{z_{n+1}} \cap W^k$ such that $F(y_{n+1}, u', v')$. Clearly, $y_{n+1} \in S$ which contradicts the definition of S. □

Theorem 3.3 of [7] does not hold in the case when D fails to be smooth. However, in the case of etale covers, we still have the following result;

Lemma 2.7 *Lifting Lemma for Etale Covers*

Let $F \subset D \times W^k$ be an etale cover of D defined over L, with the projection map denoted by f. Then given $a \in D$, $(ab) \in F$ and $a' \in \mathcal{V}_a \cap D$ generic over L, we can find $b' \in \mathcal{V}_b$ such that $F(a', b')$ holds. Moreover b' is unique, hence $Mult_{ab}(F/D) = 1$. Moreover, in the situation of Lemma 2.3, without requiring that F_2 is smooth, we have that for $(abc) \in F_3$, $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$.

Proof Using the definition of etale given in Sect. 1 above, we can assume that the cover is given algebraically in the form $f^* : L[D] \to L[D]\frac{[x_1,\dots,x_n]}{f_1,\dots,f_n}$ with $det(\frac{\partial f_i}{\partial x_j})_{ij}(x) \neq 0$ for all $x \in F$. So we can present the cover in the form $f_1(x, y) = 0$, $f_2(x, y) = 0, \dots, f_n(x, y) = 0$, with y in D and x in $A^n(L)$. Let L_m be the algebraic closure of the field generated by L and $\bar{g}(a)$ where $\bar{g}$ is a tuple of functions defining D locally. Consider the system of equations $f_1(x, a) = f_2(x, a) = \dots = f_n(x, a) = 0$ defined over L_m. Then this system is solved by b in L_m with the property that $det(\frac{\partial f_i}{\partial x_j})_{ij}(b) \neq 0$ (*). Now suppose that $a' \in \mathcal{V}_a \cap D$ is chosen to be generic over L. By the construction given in Lemma 2.2 of [7], we may assume that a' lies in $L_s[[t^{1/r}]]$, the formal power series in the variable $t^{1/r}$ for some algebraically closed field L_s extending L_m. This is a henselian ring, hence if we consider the system of equations $f_1(x, a') = f_2(x, a') = \ldots = f_n(x, a') = 0$ with coefficients in $L_s[[t^{1/r}]]$, by the fact that the system specialises to a solution in L_s with the condition (*) we can find a solution b' in $L_s[[t^{1/r}]]$. Then $(a'b')$ lies in F and by construction $b' \in \mathcal{V}_b$. The uniqueness result follows from the proof of Theorem 1.6. For the last part, suppose that $mult_{ab}(F_2/F_1) = n$, then we can find $a' \in \mathcal{V}_a \cap F_1$ generic over L and $\{b_1, \ldots b_n\} \in \mathcal{V}_b \cap W^k$ distinct such that $F(a', b_i)$ holds. Each $(a'b_i)$ is generic in F_2 over L, hence by the previous part of the lemma, we can find a unique $c_i \in \mathcal{V}_c \cap W^l$ such that $F_3(a'b_ic_i)$ holds. This show that $mult_{abc}(F_3/F_1) = n$ as required. □

Lemma 2.8 (Lifting Lemma for Etale Covers with Right(Left) Multiplicity) *Let hypotheses be as in Lemma 2.2, with the additional assumption that $F_1 = U \times V$, F_2 is a smooth irreducible cover of F_1 and F_3 is an irreducible etale cover of F_2. Then, with notion as in Definition 2.5, given $(uvbc) \in F_3$, $RightMult_{uvbc}(F_3/F_1) = RightMult_{uvb}(F_2/F_1)$. Similarly for left multiplicity.*

Proof Suppose that $RightMult_{uvb}(F_2/F_1) = n$, then for $v' \in \mathcal{V}_b$ generic in V over L, we can find $\{b_1, \ldots, b_i, \ldots b_n\} \in \mathcal{V}_b$ with $F_2(uv'b_i)$ holding. For each b_i we claim that there exists a unique $c_i \in \mathcal{V}_c$ such that $F_3(uv'b_ic_i)$ holds. For the existence, we can use Lemma 2.7, with the simple modification that, with the notation there, if L_m is the algebraic closure of the field generated by $\bar{g}(uv)$, then provided $dim(V) \geq 1$, we can find $v' \in \mathcal{V}_v \cap V$ generic over L with $uv' \in L_s[[t^{1/r}]]$ for some algebraically closed field L_s containing L_m. For the uniqueness, we can use the fact that Zariski multiplicity is summable over specialisation, see Lemma 2.4, and the fact that for generic $(u'v'b_i') \in \mathcal{V}_{uvb} \cap F_2$, we can find a unique $c_i' \in \mathcal{V}_c$ such that $F_3(u'v'b_i'c_i')$ holds. Finally, we claim that $\{b_1c_1, \ldots, b_nc_n\}$ enumerate $F_3(uv'xy) \cap \mathcal{V}_{bc}$. This is clear by the above proof and the fact that $\{b_1, \ldots, b_n\}$ enumerates $F_2(uv'x) \cap \mathcal{V}_b$.□

Definition 2.9 We say that $g : F \to D$ is an equidimensional finite cover of D if $F = \bigcup_{1 \leq i \leq k} F_i$ with F_i irreducible, $dim(F) = dim(F_i)$, and $g : F_i \to D$ finite.

Lemma 2.10 *The following versions of the above properties hold when we consider finite equidimensional covers, possibly with components, with the definition of Zariski multiplicity given in Definition 2.1.*

Proof For Lemma 2.3, we replace the hypotheses with F_1 is smooth irreducible, F_2 is an equidimensional finite cover of F_1 and F_3 is an etale cover of F_2. We then claim, using notation as in Lemma 2.2, that $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$. By definition $mult_{abc}(F_3/F_1) = \sum_{(abc) \in C_i}(mult_{abc}(C_i/F_1))$, where C_i are the irreducible components of F_3 passing through (abc). As F_3 is an etale cover of F_2, the images of the C_i are precisely the irreducible components D_i of F_2 passing through (ab), each C_i is an etale cover of D_i and $mult_{ab}(F_2/F_1) = \sum_{(ab) \in D_i}(mult_{ab}(D_i/F_1))$. Hence, it is sufficient to prove the result in the case when F_2 and F_3 are irreducible. This is just Lemma 2.3.

For Lemma 2.4, we replace the hypothesis with F is an equidimensional finite cover of D. The proof then goes through exactly as in the lemma with the observation that if we find $a_{n+1} \in \mathcal{V}_b$ and $F(a'a_{n+1})$ then we can find an irreducible component C passing through $(a'a_{n+1})$ which allows us to apply Theorem 3.3 in [7] to obtain c with $C(a''c)$ and $(a''c)$ specialising to $(a'a_{n+1})$.

For Definition 2.5, we alter the hypothesis to F is an equidimensional finite cover of $U \times V$. Again, we can use an identical proof to show that left multiplicity and right multiplicity are well defined. The proof of Lemma 2.6 with the new hypothesis on F is identical.

We don't require a modified version of Lemma 2.7, the result we need is contained in the modified proof of Lemma 2.3.

For Lemma 2.8, we alter the hypotheses to F_2 is an equidimensional cover of F_1 and F_3 is an etale cover of F_2. We then claim that for (uvb) a non-singular point of F_2 and $(uvbc) \in F_3$, necessarily non-singular as well, that $RightMult_{uvbc}(F_3/F_1) = RightMult_{uvb}(F_2/F_1)$ and similarily for left multiplicity. To prove this, note that as (uvb) and $(uvbc)$ are non-singular points, there exist unique components C and D passing through (uvb) and $(uvbc)$ respectively. Now replacing C and D by the open subsets C' and D' of smooth points, we can apply the definition of Right Multiplicity and the proof of Lemma 2.8. □

3 Analytic Methods

In order to use the method of etale morphisms, which preserve Zariski multiplicity, we need to work inside the Henselisation of local rings $L[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$. In the next section, we will only need the result for the local ring in 2 variables $L[x, y]_{(x,y)}$.

We let $L[[x_1, \ldots, x_n]]$ denote the ring of formal power series in n variables, which is the formal completion of $L[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ with respect to the canonical order valuation, see for example Sect. 2 of [7]. The following is a classical result, requiring the fact that etale morphisms are flat, used in the proof of the Artin approximation theorem. This relates the henselisation of the ring $L\{x_1, \ldots, x_n\}$ of strictly convergent power series in several variables with its formal completion $L[[x_1, \ldots, x_n]]$, see [3] or [16]. Namely, that the henselisation of $(L[x_1, \ldots x_n]_{(x_1,\ldots x_n)})$ is equal to $L[[x_1, \ldots x_n]] \cap L(x_1, \ldots x_n)^{alg}$, where $L(x_1, \ldots x_n)^{alg}$ is the algebraic closure of the function field $L(x_1, \ldots x_n)$.

This implies that

$$O^{\wedge}_{\bar{0},A^n} \cong L[[x_1, \ldots x_n]] \cap L(x_1, \ldots x_n)^{alg}$$

The following result, which can be found in [4], is essential for the next section.

Lemma 3.1 (Weierstrass Preparation) *Let $F(x_1,\dots x_n)$ be a polynomial in $L[x_1,\dots,x_n]$ which is regular in the variable x_n. Then we have $F(x_1,\dots,x_n)=U(x_1,\dots,x_n)G(x_1,\dots,x_n)$ where $U(x_1,\dots,x_n)$ is a unit in the local ring $L[[x_1,\dots,x_n]]$ and*
$G(x_1,\dots,\dots x_n)$ is a Weierstrass polynomial in x_n with coefficients in $L[[x_1,\dots,x_{n-1}]]$

We will require the Weierstrass decomposition to hold inside the henselisation of $(L[x_1,\dots,x_n])$, therefore we need to show that the Weierstass data can be found inside $L(x_1,\dots,x_n)^{alg}$. This is achieved by the following lemma.

Lemma 3.2 (Definability of Weierstrass data) *Let $F(x_1,\dots,x_n)$ be a polynomial with coefficients in L such that F is regular in x_n, then if $F(x_1,\dots,x_n)=U(x_1,\dots,x_n)G(x_1,\dots,x_n)$ is the Weierstrass decomposition of F with $G(x_1,\dots,x_n)=x_n^m+a_1(x_1,\dots,x_{n-1})x_n^{m-1}+\dots+a_m(x_1,\dots,x_{n-1})$, and $a_i\in L[[x_1,\dots,x_{n-1}]]$, $U(x_1,\dots,x_n)\in L[[x_1,\dots,x_n]]$, then $a_i(x_1,\dots,x_{n-1})\in L(x_1,\dots,x_{n-1})^{alg}$ and $U(x_1,\dots,x_n)\in L(x_1,\dots,x_n)^{alg}$.*

Proof This can be proved by rigid analytic methods. Equip L with a complete non-trivial non-archimedean valuation v and corresponding norm $||.||_v$, this can be done for example by assuming that L is a power series field of large transcendence degree with a non-archimidean valuation, see [4, 6]. Let $T_{n-1}(L)$ be the free Tate algebra in the indeterminate variables $x_1,\dots,x_{n-1}$ over L, that is the subalgebra of strictly convergent power series in $L[[x_1,\dots,x_{n-1}]]$. By the proof of Weierstrass preparation in [4], as $F\in T_{n-1}(L)[x_n]$, the coefficients a_i lie in $T_{n-1}(L)$ and $U(x_1,\dots,x_n)\in T_{n-1}(L)[x_n]$. Now choose $(u_1,\dots u_{n-1})\subset L$ transcendental over the coefficients of F with $max(\{||u_i||\})\le 1$. Then if $s_1(\bar{u}),\dots,s_m(\bar{u})$ denote the roots of $F(\bar{u},x_n)$ with $||s_i(\bar{u})||\le 1$, then both $U(\bar{u},s_i(\bar{u}))$ and $G(\bar{u},s_i(\bar{u}))$ define elements of L and moreover, by a theorem in [16], we have that the coefficients $a_i(\bar{u})$ are symmetric functions of the $s_i(\bar{u})$. Hence the $a_i(\bar{u})$ belong to $L(\bar{u})^{alg}$. As $\bar{u}$ was transcendental, we have that each $a_i\in L[x_1,\dots,x_{n-1}]^{alg}$. As $U(x_1,\dots x_n)=F/G(x_1,\dots,x_n)$, we clearly have that $U(x_1,\dots,x_n)\in L[x_1,\dots,x_n]^{alg}$ as well. □

4 Families of Curves in $P^2(L)$

We consider the family Q_d of projective curves in $P^2(L)$ with degree d. An element of Q_d may be written;

$$\sum_{0\le i+j\le d} a_{ij}(X/Z)^i(Y/Z)^j=0$$

which, rewriting in homogenous form, becomes;

$$\sum_{0\le i+j\le d} a_{ij}X^iY^jZ^{d-(i+j)}=0$$

For ease of notation, we will use affine coordinates $x = X/Z$ and $y = Y/Z$. More generally, if we give an affine cover, we implicitly assume that it can be projectivized by taking $\bar{y} = (y_1, \ldots, y_n) = (Y_1/Z, \ldots, Y_n/Z)$. As the notion of Zariski multiplicity is local, this will not effect our calculations.

Now consider two such families Q_d and Q_e. Then we have the cover obtained by intersecting degree d and degree e curves

$$Spec(L[x, y, u_{ij}, v_{ij}]/ < s(u_{ij}, x, y), t(v_{ij}, x, y) >) \to Spec(L[u_{ij}, v_{ij}]).(*)$$

where

$$s(u_{ij}, x, y) = \sum_{0 \le i+j \le d} u_{ij} x^i y^j$$

$$t(v_{ij}, x, y) = \sum_{0 \le i+j \le e} v_{ij} x^i y^j$$

We denote the parameter space for degree d curves by U and the parameter space for degree e curves by V. These are affine spaces of dimension $(d + 1)(d + 2)/2$ and $(e + 1)(e + 2)/2$ respectively. Both Q_d and Q_e are irreducible. The cover (*) is generically finite, that is there exists an open subset $U' \subset Sp(L[u_{ij}, v_{ij}])$ for which the restricted cover has finite fibres. Throughout this section, we will denote the base space of the cover by $U \times V$, bearing in mind that we implicitly mean by this $(U \times V) \cap U'$. Now, given 2 fixed parameters sets $\bar{u}$ and $\bar{v}$, with $(\bar{u}, \bar{v}) \in U'$, corresponding to curves $C_{\bar{u}}$ and $C_{\bar{v}}$, the algebraic multiplicity of the cover $(*)$ at $(00, \bar{u}, \bar{v})$ is exactly the intersection multiplicity $I(C_{\bar{u}}, C_{\bar{v}}, 00)$ of the curves at (00). The cover (*) is equidimensional as $U \times V$ satisfies the presmoothness axiom with the smooth projective variety $P^2(L)$. Restricting to a finite cover over U', by definition 2.1 we can also define the Zariski multiplicity of the cover at the point $(00, \bar{u}, \bar{v})$. The main result that we shall prove in this paper is the following, which generalises an observation given in [12].

Theorem 4.1 *In all characteristics, the algebraic multiplicity and Zariski multiplicity of the cover* $(*)$ *coincide at* $(00, \bar{u}, \bar{v})$.

Definition 4.2 We say that a monic polynomial $p(x, \bar{y})$ is Weierstrass in x if $p(x, \bar{y}) = x^n + \cdots + q_j(\bar{y})x^{n-j} + \cdots + q_n(\bar{y})$ with $q_j(\bar{0}) = 0$.

Definition 4.3 Let $F(x, \bar{y})$ be a polynomial in x with coefficients in $L[\bar{y}]$. We say the cover

$$Spec(L[x\bar{y}]/ < F(x, \bar{y}) >) \to Spec(L[\bar{y}])$$

is generically reduced if for generic $\bar{u} \in Spec(L[\bar{y}])$, $F(x, \bar{u})$ has no repeated roots.

Definition 4.4 Let $F \rightarrow U \times V$ be a finite cover with U and V smooth, such that for $(\bar{u}, \bar{v}) \in U \times V$ the fibre $F(\bar{u}, \bar{v})$ consists of the intersection of algebraic curves $F_{\bar{u}}, F_{\bar{v}}$. We call the family sufficiently deformable at $(\bar{u}_0, \bar{v}_0)$ if there exists $\bar{u}' \in U$ generic over L such that $F_{\bar{u}'}$ intersects $F_{\bar{v}_0}$ transversely at simple points.

We now require a series of lemmas.

Lemma 4.5 *Let $F(x, \bar{y})$ be a Weierstrass polynomial in x with $F(0, \bar{0}) = 0$ then algebraic multiplicity and Zariski multiplicity coincide at $(0, \bar{0})$ if the cover*

$$Spec(L[x\bar{y}]/ < F(x, \bar{y}) >) \rightarrow Spec(L[\bar{y}])$$

is generically reduced.

Proof We have that $F(x, \bar{y}) = x^n + q_1(\bar{y})x^{n-1} + \ldots + q_n(\bar{y})$ where $q_i(\bar{0}) = 0$. The algebraic multiplicity is given by $length(L[x]/F(x, \bar{0})) = ord(F(x, \bar{0}) = n$ in the ring $L[x]$ with the canonical valuation. We first claim that the Zariski multiplicity is the number of solutions to $x^n + q_1(\bar{\epsilon})x^{n-1} + \ldots + q_n(\bar{\epsilon}) = 0$ (†), where $\bar{\epsilon}$ is generic in $\mathcal{V}_{\bar{0}}$. For suppose that $(a, \bar{\epsilon})$ is such a solution, then $F(a, \bar{\epsilon}) = 0$ and by specialisation $F(\pi(a), \bar{0}) = 0$. As F is a Weierstrass polynomial in x, $\pi(a) = 0$, hence $a \in \mathcal{V}_0$, giving the claim. We have that $Disc(F(x, \bar{y})) = Res_{\bar{y}}(F, \frac{\partial F}{\partial x})$ is a regular polynomial in $\bar{y}$ defined over L. By the fact that the cover is generically reduced, this defines a proper closed subset of $Spec(L[\bar{y}])$. Therefore, $Disc(F(x, \bar{y}))|\bar{\epsilon} \neq 0$, hence (†) has no repeated roots. This gives the lemma. □

Lemma 4.6 *Let $F(x, \bar{y})$ be any polynomial with $F(x, \bar{0}) \neq 0$ and $F(0, \bar{0}) = 0$. Then if the cover $Spec(L[x, \bar{y}]/ < F(x, \bar{y}) >) \rightarrow Spec(L[\bar{y}])$ is generically reduced, the Zariski multiplicity at $(0, \bar{0})$ equals $ord(F(x, \bar{0}))$ in $L[x]$.*

Proof By the Weierstrass Preparation Theorem, Lemma 3.1, we can write $F(x, \bar{y}) = U(x, \bar{y})G(x, \bar{y})$ with $U(x, \bar{y}), G(x, \bar{y}) \in L[[x, \bar{y}]]$, $G(x, \bar{y})$ a Weierstrass polynomial in x and $deg(G) = ord(F(x, \bar{0}))$, see also the more closely related statement given in [2]. By Lemma 3.2, we may take the new coefficients to lie inside the Henselized ring $L[x, \bar{y}]^{\wedge}_{\bar{0}}$, hence inside some finite etale extension $L[x, \bar{y}]^{ext}$ of $L[x, \bar{y}]$ (possibly after localising $L[x, \bar{y}]$ corresponding to an open subset of $Spec(L[x, \bar{y}])$ containing $(0, \bar{0})$). Now we have the sequence of morphisms;

$$Sp(L[x, \bar{y}]^{ext}/UG) \rightarrow Spec(L[x, \bar{y}]/F) \rightarrow Spec(L[\bar{y}])$$

The left hand morphism is etale at $\bar{0}$, hence by Lemma 2.3 or Lemma 2.7, to compute the Zariski multiplicity of the right hand morphism, we need to compute the Zariski multiplicity of the cover

$$Spec(L[x, \bar{y}]^{ext}/UG) \rightarrow Spec(L[\bar{y}])$$

at $(0, \bar{0})^{lift}$, the marked point in the cover above $(0, \bar{0})$. Choose $\bar{\epsilon} \in \mathcal{V}_{\bar{0}}$, the fibre of the cover is given formally analytically by $L[[x, \bar{y}]]/ < UG > \otimes_{L[\bar{y}], \bar{y} \mapsto \bar{\epsilon}} L$, hence by solutions to $U(x,\bar{\epsilon})G(x,\epsilon)$. By definition of Zariski multiplicity, we consider only solutions $(x\bar{\epsilon})$ in $\mathcal{V}_{(0,\bar{0})^{lift}}$. As $U(x,\bar{y})$ is a unit in the local ring $L[x,\bar{y}]^{ext}_{(0,\bar{0})^{lift}}$, we must have $U(x,\bar{\epsilon})\neq 0$ for such solutions, otherwise by specialisation $U((0,\bar{0})^{lift})=0$. Hence, the solutions are given by $G(x,\bar{\epsilon})=0$. Now, we use the previous lemma to give that the Zariski multiplicity is exactly $deg(G)$ as required. □

Now return to the cover

$$Sp(L[x,y,u_{ij},v_{ij}]/<s(u_{ij},x,y),t(v_{ij},x,y)>)\rightarrow Sp(L[u_{ij},v_{ij}])\ (*)$$

We will show below, Lemma 4.12, that this is a sufficiently deformable family at $(\bar{u}_0,\bar{v}_0)$ when $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ define reduced curves. We claim the following.

Lemma 4.7 *Suppose parameters $\bar{u}^0$ and $\bar{v}^0$ are chosen such that $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are reduced Weierstrass polynomials in x. Then the Zariski multiplicity of the cover $(*)$ at $(0,0,\bar{u}^0,\bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0},C_{\bar{v}^0},(0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

Proof Introduce new parameters $\bar{u}'$ and $\bar{v}'$. Let $C^{\bar{u}'}_{\bar{u}^0}$ and $C^{\bar{v}'}_{\bar{v}^0}$ denote the curves $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ deformed by the parameters $\bar{u}'$ and $\bar{v}'$ respectively. That is $C^{\bar{u}'}_{\bar{u}^0}$ is given by the new equation $\Sigma_{1\leq i+j\leq d}(u^0_{ij}+u'_{ij})x^iy^j$. Let $F(y,\bar{u}',\bar{v}')=Res(C^{\bar{u}'}_{\bar{u}^0},C^{\bar{v}'}_{\bar{v}^0})$. Then,

$$F(0,\bar{0},\bar{0})=Res(s(u^0_{ij},x,0),t(v^0_{ij},x,0))=0$$

as $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are Weierstrass in x and share a common solution at $(0,0)$. By a result due to Abhyankar, see for example [1], $ord_y(F(y,\bar{0},\bar{0}))=\Sigma_x I(C_{\bar{u}^0},C_{\bar{v}^0},(x0))$ at common solutions $(x,0)$ to $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ over $y=0$. As $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are Weierstrass polynomials in x, this is just $I(C_{\bar{u}^0},C_{\bar{v}^0},(0,0))$. By the previous lemma and the fact that $F(y,\bar{u},\bar{v})$ is generically reduced (see argument (†) below), it is therefore sufficient to prove that the Zariski multiplicity of the cover $(*)$ at $(00,\bar{u}^0,\bar{v}^0)$ equals the Zariski multiplicity of the cover $Spec(L[y,\bar{u}',\bar{v}']/<F(y,\bar{u}',\bar{v}')>)\rightarrow Spec(L[\bar{u}',\bar{v}'])\ (**)$ at $(0,\bar{0},\bar{0})$. Suppose the Zariski multiplity of $(**)$ equals n. Then there exist distinct $y_1,\ldots,y_n\in\mathcal{V}_0$ and $(\bar{\delta},\bar{\epsilon})$ generic in $\mathcal{V}_{(\bar{0},\bar{0})}\cap U\times V$ such that $F(y_i,\bar{\delta},\bar{\epsilon})$ holds. Consider $Q(\bar{u}',\bar{v}')=Res(F(y,\bar{u}',\bar{v}'),\partial F/\partial y(y,\bar{u}',\bar{v}'))$. This defines a closed subset of $U\times V$ defined over L, we claim that this in fact proper closed (†). By the fact that the family is sufficently deformable at $(\bar{u}_0,\bar{v}_0)$, we can find $(\bar{u},\bar{v}_0)$ such that $C_{\bar{u}}$ intersects $C_{\bar{v}_0}$ transversely at simple points. Without loss of generality, making a linear change of coordinates, we may suppose that for there do not exists points of intersection of the form (x_1y) and (x_2y) for $x_1\neq x_2$. By Abhyankar's result, this implies that $F(y,\bar{u}',\bar{v}_0)$ has no repeated roots. Then, by genericity of $(\bar{\delta},\bar{\epsilon})$, we have that $Q(\bar{\delta},\bar{\epsilon})\neq 0$. Hence $F(y_i,\bar{\delta},\bar{\epsilon})$ is a non-repeated root. By Abhyankar's result, we can find a unique x_i with (x_iy_i) a common solution to the deformed curves $C^{\bar{\delta}}_{\bar{u}^0}$ and $C^{\bar{\epsilon}}_{\bar{v}^0}$. We claim that each $(x_iy_i)\in\mathcal{V}_{00}$. As $C^{\bar{\delta}}_{\bar{u}^0}(x_iy_i)=0$, by the fact $(\bar{u}^0,\bar{\delta},y_i)$ specialises to $(\bar{u}^0,\bar{0},0)$ and $C_{\bar{u}^0}$ is a Weierstrass polynomial in x, we have that $\pi(x_i)=0$ as well. This shows that the Zariski multiplicity of the cover $(*)$, referred to at the beginning of Sect. 4, in this case, is at least n. Suppose the Zariski multiplicity of the cover $(*)$ is strictly bigger than n, then we can find generic parameters $\{\overline{u}', \overline{v}'\} \subset \mathcal{V}_{\bar{0},\bar{0}}$ and distinct $\{(x_1y_1), \ldots, (x_{n+1}y_{n+1})\} \subset \mathcal{V}_{(0,0)} \cap C^{\overline{u}}_{\overline{u}^0} \cap C^{\overline{v}}_{\overline{v}^0}$. If, for some y_i, there exist at least 2 distinct x_{j_i}, with $(x_{j_i}, y_i) \in \{(x_1y_1), \ldots, (x_{n+1}y_{n+1})\}$, then $ord_{y_i}F(y, \overline{u}', \overline{v}') \geq 2$, contradicting the fact that F is generically reduced. Otherwise, there exist at least $n+1$ distinct y_i, corresponding to solutions $F(y_i, \overline{u}', \overline{v}') = 0$, with $y_i \in \mathcal{V}_0$, $(\dagger\dagger)$. Using the fact that $ord_y F(y, \overline{0}, \overline{0}) = n$, and using Lemma 4.6, the Zariski multiplicity of the cover $Spec(L[\bar{u}, \bar{v}, y]/ < F(y, \bar{u}, \bar{v}) >) \rightarrow Spec(L[\bar{u}\bar{v}])$ at $(0, \bar{0}\bar{0})$ is n, contradicting $(\dagger\dagger)$. □

We now have the following result;

Lemma 4.8 *Let $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ be reduced curves, having finite intersection, then the Zariski multiplicity, see Definition 1.4, of the cover $(*)$ at $((0,0), \bar{u}^0, \bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0}, C_{\bar{v}^0}, (0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

Proof We have $C_{\bar{u}^0} = s(u^0_{ij}, x, y)$ and $C_{\bar{v}^0} = t(v^0_{ij}, x, y)$. By making the substitutions $\bar{U} = \bar{u}^0 + \bar{u}$ and $\bar{V} = \bar{v}^0 + \bar{v}$, we may assume that $\bar{u}^0 = \bar{v}^0 = \bar{0}$. Moreover, we can suppose that;

$s(\bar{0}_{ij}, x, 0) \neq 0$ and

$$t(\bar{0}_{ij}, x, 0) \neq 0.(**)$$

This can be achieved by making the invertible linear change of variables $(x' = x, y' = \lambda x + \mu y)$ with $(\lambda, \mu) \in L^2$ and $\mu \neq 0$, noting that as $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ are curves, for some choice of (λ, μ), the corresponding polynomials $s(u^0_{ij}, x, y)$ and $t(v^0_{ij}, x, y)$ do not vanish identically on the line $\lambda x + \mu y = 0$. It is trivial to check that the transformation preserves both Zariski multiplicity and intersection multiplicity, so our calculations are not effected.

We may then apply the Weierstrass preparation theorem, Lemma 3.1, in the ring $L[[u_{ij}, v_{ij}, x, y]]$, obtaining factorisations $s(u_{ij}, x, y) = U_1(u_{ij}, x, y)S(u_{ij}, x, y)$ and $t(v_{ij}, x, y) = U_2(v_{ij}, x, y)T(v_{ij}, x, y)$ where U_1 and U_2 are units in the local rings $L[[u_{ij}, x, y]]$ and $L[[v_{ij}, x, y]]$, S, T are Weierstrass polynomials in x with coefficients in $L[[u_{ij}, y]]$ and $L[[v_{ij}, y]]$ respectively. A close inspection of the Weierstrass preparation theorem, see [2], shows that we can obtain the following uniformity in the parameters $\bar{u}$ and $\bar{v}$.

Namely, if $U = \{u_{ij} : s(u_{ij}, x, 0) \neq 0\}$ and $V = \{v_{ij} : t(v_{ij}, x, 0) \neq 0\}$, are the constructible sets for which $(*)$ holds, then if we let R_U and R_V denote the coordinate rings of U and V, we may assume U_1, U_2 lie in $R_U[[x, y]]$ and the coefficients of S, T lie in $R_U[[y]]$ and $R_V[[y]]$ respectively. By Lemma 3.2, we may assume that U_1, U_2, S and T lie in a finite etale extension $R_{U\times V}[x, y]^{ext}$ of the algebra $A = R_{U\times V}[x, y]$ (again, possibly after localisation corresponding to an open subvariety of $Spec(A)$. Now we have the sequence of morphisms.

$Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1S, U_2T>}) \rightarrow Spec(\frac{R_{U\times V}[x,y]}{<s,t>}) \rightarrow Spec(R_{U\times V})$.

We claim that the left hand morphism is etale at the point $(\bar{0}, \bar{0}, (00)^{lift})$. This follows from the fact that $R_{U\times V}[x, y]^{ext}$ is an etale extension of $R_{U\times V}[x, y]$ and the maximal ideal given by $(\bar{0}, \bar{0}, (00)^{lift})$ contains $< U_1 S, U_2 T >$. Now consider the cover;

$$Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1 S, U_2 T>}) \rightarrow Spec(R_{U\times V}) \; (***)$$

For $\bar{u}, \bar{v}$ in $U \times V$, the fibre of this cover over $\bar{u}, \bar{v}$ corresponds exactly to the intersection of the reducible curves $C'_{\bar{u}}$ and $C'_{\bar{v}}$ which lift the original curves $C_{\bar{u}}$ and $C_{\bar{v}}$ to an etale cover of $Spec(L[xy])$. By Theorem 1.10 and Lemma 2.3, in the case when $C_{\bar{u}_0}, C_{\bar{v}_0}$ intersect at simple points, or Lemma 2.7, for singular points of intersection, and the corresponding Lemma 2.10 for reducible covers, it is sufficient to show that the Zariski multiplicity of the cover (***) at $(\bar{0}, \bar{0}, (00)^{lift})$ corresponds to the intersection multiplicity of the curves $C'_{\bar{u}_0}, C'_{\bar{v}_0}$ at $(00)^{lift}$. The idea now is to apply Lemma 4.7 to the Weierstrass factors of $C'_{\bar{u}}$ and $C'_{\bar{v}}$. This will be achieved by the "unit removal" lemma below, Lemma 4.15. □

In order to prove the "unit removal lemma", we first require some more definitions and a moving lemma for curves;

Definition 4.9 Let $X \rightarrow Spec(L[x, y])$ be an etale cover in a neighboorhood of $(0, 0)$, with distiguished point $(0, 0)^{lift}$. We call a curve C on X passing through $(0, 0)^{lift}$ Weierstrass if, in the power series ring $L[[x, y]]$, the defining equation of C may be written as a Weierstrass polynomial in x with coefficients in $L[[y]]$.

Definition 4.10 Let $F \rightarrow U \times V$ be a finite equidimensional cover of a smooth base of parameters $U \times V$ with a section $s : U \times V \rightarrow F$. We call the cover Weierstrass with units if the fibres $F(\bar{u}, \bar{v})$ can be written as the intersection of reducible curves $C'_{\bar{u}}$ and $C'_{\bar{v}}$ in an etale cover $A_{\bar{u},\bar{v}}$ of $U_{\bar{u},\bar{v}} \subset Spec(L[x, y])$ with the distinguished point $s(\bar{u}, \bar{v})$ lying above $(0, 0)$ and $C'_{\bar{u}}, C'_{\bar{v}}$ factoring as $U_{\bar{u}} F_{\bar{u}}$ and $U_{\bar{v}} F_{\bar{v}}$ with $U_{\bar{u}}, U_{\bar{v}}$ units in the local ring $O_{s(\bar{u},\bar{v}),A_{\bar{u},\bar{v}}}$ and $F_{\bar{u}}, F_{\bar{v}}$ Weierstrass curves in $A_{\bar{u},\bar{v}}$.

Let hypotheses on F, U and V be as above. We call the cover Weierstrass if the fibres $F(\bar{u}, \bar{v})$ can be written as above but with $C'_{\bar{u}}, C'_{\bar{v}}$ Weierstrass curves in $A_{\bar{u},\bar{v}}$.

We say that a Weierstrass cover (with units) factors through the family of projective degree d and degree e curves if the cover $F \rightarrow U \times V$ factors as $F \rightarrow F' \rightarrow U \times V$ where $F' \rightarrow U \times V$ is the finite equidimensional cover obtained by intersecting the families Q_d and Q_e restricted to U and V.

Lemma 4.11 *The cover (***) in Lemma 4.8 is a Weierstrass cover with units factoring through the family of projective degree d and degree e curves.*

Proof Clear by the above definitions. □

Lemma 4.12 *Moving Lemma for Reduced Curves*
Let Q_d and Q_e be the families of all *projective degree d and degree e curves. That is, with the usual coordinate convention $x = X/Z$, $y = Y/Z$, Q_d consists of all curves of the form $s(\bar{u}, x, y) = \sum_{0\leq i+j\leq d} u_{ij} x^i y^j$. Then, if $\bar{u}, \bar{v}$ are chosen in L, so that the*

reduced curves $C_{\bar{u}}$ and $C_{\bar{v}}$ are defined over L, if the tuple $\bar{u}'$ is chosen to be generic in U over L, the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}$ transversely at simple points.

Proof We can give an explicit calculation;

Let $C_{\bar{u}}^{\bar{u}'}$ be defined by the equation $s(\bar{u}', x, y) = \sum_{0 \leq i+j \leq d} u'_{ij} x^i y^j$ and $C_{\bar{v}}$ by $t(\bar{v}, x, y) = \sum_{0 \leq i+j \leq e} v_{ij} x^i y^j$ with $\{v_{ij} : 0 \leq i + j \leq e\} \subset L$ and $\{u'_{ij} : 0 \leq i + j \leq d\}$ algebraically independent over L. Let $(x_0 y_0)$ be a point of intersection, then $dim(x_0 y_0/L) = 1$, otherwise $dim(x_0 y_0/L) = 0$ and, as L is algebraically closed, we must have that $x_0, y_0 \in L$. Substituting $(x_0 y_0)$ into the equation $s(\bar{u}', x, y) = 0$, we get a non trivial linear dependence over L between u'_{00} and u'_{ij} for $1 \leq i + j \leq d$ which is impossible. Now, the locus of singular points for $C_{\bar{v}}$ is defined over L and hence $(x_0 y_0)$ is a simple point of $C_{\bar{v}}$. Now we further claim that $s(\bar{u}', x, y) = 0$ defines a non-singular curve in $P^2(K_\omega)$ with transverse intersection to $C_{\bar{v}}$ Consider the conditions Sing$(\bar{u})$ given by $\exists x_0 \exists y_0((\frac{\partial s}{\partial x}(x_0 y_0), \frac{\partial s}{\partial y}(x_0 y_0)) = (0, 0))$ and Non-Transverse$(\bar{u})$ by $\exists x_0 \exists y_0(\frac{\partial s}{\partial x}(x_0 y_0)\frac{\partial t}{\partial y}(x_0 y_0) - \frac{\partial s}{\partial y}(x_0 y_0)\frac{\partial t}{\partial x}(x_0 y_0) = 0)$ By the properness of $P^2(K_\omega)$, these conditions define closed subsets of the parameter space U defined over L. We claim that this in fact a proper closed subset. This can be proved in a number of ways. In the case where we restrict ourselves to affine curves, the result follows from a classical result of Kleiman, see [10], as affine space $A^2(K_\omega)$ is homogenous for the action of the additive group $(A^2(K_\omega), +)$. More generally, we can use the moving lemma, given in [9], by observing that the class of all degree d projective curves is closed under rational equivalence. We can also give an explicit proof using Bertini's theorem;

Observe that the curve $C_{\bar{u}}$ defines a complete linear system $|C_{\bar{u}}|$ corresponding exactly to the zero loci of sections σ of the bundle $\mathcal{O}_{P^2}(d)$. We claim the following;

(i). The system $|C_{\bar{u}}|$ is base point free.

(ii). The system $|C_{\bar{u}}|$ separates points.

Now we can define a morphism $\Phi_d : P^2(K) \to P^{d(d+3)/2}(K)$, by sending $x \in P^2$ to the hyperplane $H_x \subset U$ of curves of degree d, passing through x. By (i) and (ii), the restriction of Φ_d to $C_{\bar{v}}$ is injective. By arguments on Frobenius for curves, given in [7], we can assume that Φ_d is an immersion. Using Bertini's Theorem, a generic hyperplane $\mathcal{H}_{\bar{u}'}$ of $P^{d(d+3)/2}(K)$ will intersect $Im(C_{\bar{v}})$ transversely in simple points. By definition of the morphism Φ_d, and the fact that it is an immersion, the corresponding curve $C_{\bar{u}}^{\bar{u}'}$ also intersects $C_{\bar{v}}$ transversely in simple points.

One can also give an enumerative calculation, which was done in an older version of this paper, see [5], but it seems unnecessary. □

Remark 4.13 If we restrict the family of curves, the result in general fails. A simple example is given by the family of all projective degree 3 curves $Q_3^{0,0}$ passing through $(0, 0)$ with $x = X/Z$ and $y = Y/Z$. If we take $C_{\bar{v}}$ to be the cusp $x^2 - y^3$, then any curve in $Q_3^{0,0}$ will have a non-transverse intersection with $C_{\bar{v}}$ at the origin.

Lemma 4.14 *Moving Lemma for Curves with Finitely Many Marked Points*
Let hypotheses be as in the previous lemma with $C_{\bar{u}}$ and $C_{\bar{v}}$ defining reduced curves. Suppose also that there exists finitely many marked points $\{p_1, \ldots, p_n\}$ on $C_{\bar{v}}$ defined over L. Then for $\bar{u}' \in U$ generic over L the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}$ transversely at finitely many simple points excluding the set $\{p_1, \ldots, p_n\}$.

Proof As before, the condition that $\bar{u}'$ defines a curve $C_{\bar{u}}^{\bar{u}'}$ either with non-transverse intersection to $C_{\bar{v}}$ or passing through at least one of the points $\{p_1, \ldots, p_n\}$ is a closed subset of U defined over L. Using the above proof and the obvious fact that we can find a curve $C_{\bar{u}}^{\bar{u}'}$ not passing through any of the points $\{p_1, \ldots, p_n\}$, we see that it is proper closed. □

Lemma 4.15 *Unit Removal for Reduced Curves*
Let $(\pi, s) : F \to U \times V$ be a Weierstrass cover with units factoring through projective degree d and degree e curves. Let $(\bar{u}, \bar{v}) \in U \times V$, then there exists a Weierstrass cover $(\pi', s') : F^- \to U' \times V'$ with $U' \subset U$ and $V' \subset V$ open subsets, $(\bar{u}, \bar{v}) \in U' \times V'$, such that $Mult_{(\bar{u},\bar{v},s(\bar{u},\bar{v}))}(F/U \times V) = Mult_{\bar{u},\bar{v},s'(\bar{u},\bar{v}))}(F^-/U' \times V')$.

Proof Let $C'_{\bar{u}}$ and $C'_{\bar{v}}$ be the Weierstrass curves with units in $A_{\bar{u},\bar{v}}$ lifting the curves $C_{\bar{u}}$ and $C_{\bar{v}}$. Now suppose that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F/U \times V) = n$. Then we can find $(\bar{u}', \bar{v}') \in \mathcal{V}_{\bar{u}\bar{v}} \cap U \times V$ generic over L such that the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}^{\bar{v}'}$ at the n distinct points $x_1, \ldots, x_n$ in $\mathcal{V}_{s(\bar{u},\bar{v})}$. Now using the Weierstrass factorisations of $C_{\bar{u}}^{\bar{u}'}$ and $C_{\bar{v}}^{\bar{v}'}$, we claim that $U_{\bar{u}}^{\bar{u}'}(x_i) \neq 0$ and $U_{\bar{v}}^{\bar{v}'}(x_i) \neq 0$. Suppose not, then $U_{\bar{u}}^{\bar{u}'}(x_i) = U_{\bar{v}}^{\bar{v}'}(x_i) = 0$ and as $(\bar{u}', \bar{v}', x_i)$ specialises to $(\bar{u}, \bar{v}, s(\bar{u}, \bar{v}))$, then $U_{\bar{u}}(s(\bar{u}, \bar{v})) = U_{\bar{v}}(s(\bar{u}, \bar{v})) = 0$. This contradicts the fact that $U_{\bar{u}}$ and $U_{\bar{v}}$ are units in the local ring $O_{s(\bar{u},\bar{v}),A_{\bar{u},\bar{v}}}$. Therefore, we must have that $F_{\bar{u}}^{\bar{u}'}(x_i) = F_{\bar{v}}^{\bar{v}'}(x_i) = 0$. This shows that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F^-/U \times V) \geq n$ where $F^- \to U \times V$ is the cover of $U \times V$ obtained by taking as fibres $F^-(\bar{u}, \bar{v})$ the intersection of the Weierstrass factors $F_{\bar{u}}$ and $F_{\bar{v}}$. Formally, if F is defined by $Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1S,U_2T>})$ then F^- is defined by $Spec(\frac{R_{U\times V}[x,y]^{ext}}{<S,T>})$. Clearly as $F^- \subset F$ is a union of components of F, we have that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F^-/U \times V) \leq n$ as well. This proves the lemma. □

We now complete the proof of Lemma 4.8. By unit removal, it is sufficient to compute the Zariski multiplicity of the cover

$$Spec(\frac{R_{U\times V}[x, y]^{ext}}{< S, T >}) \to Spec(R_{U\times V})$$

The fibre over $(\bar{u}, \bar{v})$ of this cover corresponds exactly to the intersection of the Weierstrass curves $F_{\bar{u}}$ and $F_{\bar{v}}$ lifting $C_{\bar{u}}$ and $C_{\bar{v}}$. We then use Lemma 2.7, noting that the Weierstrass factors are still reduced, see [2], to finish the result, with the straightforward modification that we work in a uniform family of etale covers.

We now turn to the problem of non-reduced curves. We will show the following stronger version of Lemma 4.8.

Lemma 4.16 *Let $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ be non-reduced curves having finite intersection, then the Zariski multiplicity of the cover (*) at $((0,0), \bar{u}^0, \bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0}, C_{\bar{v}^0}, (0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

First, we will require some more lemmas.

Lemma 4.17 *Let $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ be reduced curves intersecting transversely at $(0,0)$. Then the Zariski multiplicity, left multiplicity and right multiplicity of the cover (*) at $((0,0), \bar{u}^0, \bar{v}^0)$ equals 1.*

Proof First note that by Lemma 2.6, and the corresponding Lemma 2.10, and the fact that a generic deformation $C_{\bar{v}_0}^{\bar{v}'}$ will still intersect $C_{\bar{u}_0}$ transversely by Lemma 4.12, it is sufficient to prove the result for right multiplicity.

In order to show this we require the following result, given for analytic curves in [2], we will only need the result for polynomials.

Implicit Function Theorem:

If $G(X,Y)$ is a power series with $G(0,0)=0$ then $G_Y(0,0)\neq 0$ implies there exists a power series $\eta(X)$ with $\eta(0)=0$ such that $G(X,\eta(X))=0$.

In order to show that $RightMult_{(0,0),\bar{u}^0,\bar{v}^0}(F'/U\times V)=1$, where F' is the family obtained by intersecting degree d and degree e curves, we apply the implicit function theorem to the curve $C_{\bar{u}^0}$ at the point $(0,0)$ of intersection with $C_{\bar{v}^0}$. Let $G(X,Y)$ and $H(X,Y)$ denote the polynomials defining the curves. We have that $G(0,0)=H(0,0)=0$. Moreover, as the first curve is non-singular at $(0,0)$, we may also assume that $G_Y(0,0)\neq 0$. Now let $\eta(X)$ be given by the theorem. As the intersection of the curves $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ is transverse, $ord_X H(X,\eta(X))=1$.Now we have the sequence of maps;

$$L[\bar{v}]\rightarrow \frac{L[X,Y][\bar{v}]}{<G(u^0,X,Y),H(\bar{v},X,Y)>}\rightarrow \frac{L[X]^{ext}[Y][\bar{v}]}{<Y-\eta(X),H(\bar{v},X,Y)>}.$$

where $L[X]^{ext}$ is an etale extension of $L[X]$ containing $\eta(X)$. Note that $\eta(X)$ is trivially algebraic over $L(X)$. This corresponds to a sequence of finite covers $F_1\rightarrow F'(u_0,V)\rightarrow Spec(L[\bar{v}])$. The left hand morphism is trivially etale at $(\bar{v}^0,(00)^{lift})$, hence it is sufficient to compute the Zariski multiplicity of $F'\rightarrow Spec(L[\bar{v}])$ at $(\bar{v}^0,(00)^{lift})$ by Lemma 2.3, or the corresponding Lemma 2.10. This is a straightforward calculation, the fibre over $\bar{v}^0$ consists of the scheme $Spec(\frac{L[X,\eta(X)]}{G(X,\eta(X))})=Spec(L)$ as $ord_X(H(X,\eta(X)))=1$, hence is etale at the point $(\bar{v}^0,(00)^{lift})$. By Theorem 1.6, the Zariski multiplicity is 1. □

Lemma 4.18 *Let hypotheses be as in Lemma 4.17, then for any $(\bar{u}',\bar{v}')\in\mathcal{V}_{(\bar{u}^0,\bar{v}^0)}$, we have that $Card(F'(\bar{u}',\bar{v}')\cap\mathcal{V}_{0,0})=1$*

Proof This follows immediately from Lemmas 4.17 and 2.4. □

Definition 4.19 For ease of notation, given curves $C_{\bar{u}}$ and $C_{\bar{v}}$ of degree d and degree e intersecting at $x \in P^2(K_\omega)$, we define $Mult_x(C_{\bar{u}}, C_{\bar{v}})$ to be the corresponding Zariski multiplicity of the cover $F' \to U \times V$ at the point $(x, \bar{u}, \bar{v})$. Similarly for left/right multiplicity.

We can now give the proof of Lemma 4.16.

Proof Case 1. $C_{\bar{v}_0}$ is a reduced curve (possibly having components). Write $C_{\bar{u}^0}$ as $G_1^{n_1}(X,Y)\dots G_m^{n_m}(X,Y)=0$ with G_i the reduced irreducible components of $C_{\bar{u}_0}$ with degree d_i passing through $(0,0)$. Choose $\bar{\epsilon}_1^1, \dots \bar{\epsilon}_1^{n_1}, \dots \bar{\epsilon}_i^j, \dots \bar{\epsilon}_m^1, \dots, \bar{\epsilon}_m^{n_m}$ independent generic in U_i, the parameter space for degree d_i projective curves with $\bar{\epsilon}_i^j \in \mathcal{V}_{\bar{u}_i^0}$, where $\bar{u}_i^0$ defines G_i. By repeated application of Lemma 4.14, the deformed curves $G_i^{\bar{\epsilon}_i^j}=0$ intersect $C_{\bar{v}_0}$ transversely at disjoint sets of points We denote by $Z_{\bar{\epsilon}_i^j}$ those points lying in $\mathcal{V}_{00}$. Now the curve defined by $\prod_{ij} G_i^{\bar{\epsilon}_i^j}=0$ is a deformation $C_{\bar{u}^0}^{\bar{\epsilon}}$ of $C_{\bar{u}^0}$. We let $Z_{\bar{\epsilon}}$ denote the points of intersection of $C_{\bar{u}^0}^{\bar{\epsilon}}$ with $C_{\bar{v}_0}$ in $\mathcal{V}_{00}$. Then we have;

$$Z_{\bar{\epsilon}} = \bigcup_{ij} Z_{\bar{\epsilon}_i^j}$$

$$Card(Z_{\bar{\epsilon}}) = \sum_{ij} Card(Z_{\bar{\epsilon}_i^j})$$

By Lemma 2.4, we have that

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{x \in Z_{\bar{\epsilon}}} LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0})$$

$$= \sum_{i,j} \sum_{x \in Z_{\bar{\epsilon}_i^j}} LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0})(*)$$

We now claim that for a point $x \in Z_{\bar{\epsilon}_i^j}$,

$$LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0}) = LeftMult_x(G_i^{\bar{\epsilon}_i^j}, C_{\bar{v}_0})(**)$$

This follows as both the reduced curves $C_{\bar{u}_0}^{\bar{\epsilon}}$ and $G_i^{\bar{\epsilon}_i^j}$ intersect $C_{\bar{v}_0}$ transversely at x. Hence, in both cases the left multiplicity is 1, by Lemma 4.17.

Combining $(*)$ and $(**)$, we obtain;

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i,j} \sum_{x \in Z_{\bar{\epsilon}_i^j}} LeftMult_x(G_i^{\bar{\epsilon}_i^j}, C_{\bar{v}^0})$$

Now using Lemma 2.4 again gives that;

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} n_i LeftMult_{(00)}(G_i, C_{\bar{v}^0})(***)$$

If we go through exactly the same calculation with Mult replacing Left Mult, we see as well that

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} n_i Mult_{(00)}(G_i, C_{\bar{v}^0})$$

By Lemma 4.8, this gives

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} n_i I(G_i, C_{\bar{v}^0}, (00))$$

By a straightforward algebraic calculation, see the references below at the end of the proof for the required more general result, this gives

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = I(C_{\bar{u}^0}, C_{\bar{v}^0}, (00))$$

as required.

Case 2. Both $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ define non-reduced curves. Write $C_{\bar{u}_0}$ as above and $C_{\bar{v}_0}$ as $H_1^{e_1} \ldots H_n^{e_n}$ with H_i the reduced compoments with degree c_i of $C_{\bar{v}_0}$ passing through (00). Then $H_1 \ldots H_n = 0$ defines a reduced curve passing through (00). Now repeat the argument in Case 1 for the curves $C_{\bar{u}_0}$ and $H_1 \ldots H_n = 0$. Again let $Z_{\bar{\epsilon}}$ be the intersection points of the deformed curve $C_{\bar{u}_0}^{\epsilon}$ with $H_1 \ldots H_n = 0$ in $\mathcal{V}_{(00)}$. By (***) of Case 1, Lemmas 2.4 and 4.18 with the fact that the intersection of $C_{\bar{u}^0}^{\bar{\epsilon}}$ with $H_1 \ldots H_n$ is transverse, we have;

$$Card(Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i Mult_{(00)}(G_i, H_1 \ldots H_n)$$

Now using the argument in Case 1 applied to the reduced curves G_i and $H_1 \ldots H_n$, we have;

$$Card(Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i \sum_{j=1}^{n} I(G_i, H_j, (00))(*)$$

We claim that for any component H_j

$$Card(H_j \cap Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i I(G_i, H_j, (00))$$

This follows as the deformed curve $C^{\bar{\epsilon}}_{\bar{u}_0}$ a fortiori intersects H_j transversely at simple points. Therefore, again by Case 1, gives the expected multiplicity. Now, using this together with (*), we write $Z_{\bar{\epsilon}}$ as $\cup_j Z^j_{\bar{\epsilon}}$ where $Z^j_{\bar{\epsilon}}$ are the disjoint sets consisting of the intersection of $C^{\bar{\epsilon}}_{\bar{u}_0}$ with H_j. Then by Lemma 2.6, we have that

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_j \sum_{x \in Z^j_{\bar{\epsilon}}} RightMult_x(C^{\bar{\epsilon}}_{\bar{u}^0}, C_{\bar{v}^0})$$

We can now calculate the Right Mult term by applying Case 1 to the intersection of $C_{\bar{v}_0}$ with the reduced curve $C^{\bar{\epsilon}}_{\bar{u}_0}$ at the points of intersection $x \in Z^j_{\bar{\epsilon}}$. At a point $x \in Z^j_{\bar{\epsilon}}$, we have that

$$RightMult_x(C^{\bar{\epsilon}}_{\bar{u}^0}, C_{\bar{v}^0}) = e_j I(C^{\bar{\epsilon}}_{\bar{u}^0}, H_j, x) = e_j$$

as the intersection is transverse. Finally this gives;

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} \sum_{j=1}^{n} n_i e_j I(G_i, H_j, (00))$$

By an algebraic result, see [11] for the case of complex algebraic curves, or [8] for its generalisation to algebraic curves in arbitrary characteristics, we have

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = I(C_{\bar{u}^0}, C_{\bar{v}^0}, (00))$$

as required. □

The following version of Bezout's theorem in all characteristics is now an easy generalisation from the above lemma. For curves C_1 and C_2 in $P^2(L)$, we let $M(C_1, C_2, x)$ denote the intersection multiplicity or the Zariski multiplicity, we know from the above that the two are equivalent.

Theorem 4.20 (Non-Standard Bezout)
Let C_1 and C_2 be projective curves of degree d and degree e in $P^2(L)$, possibly with non-reduced components, intersecting at finitely many points $\{x_1, \ldots, x_i, \ldots x_n\}$, then we have;

$$\sum_{i=1}^{n} M(C_1, C_2, x_i) = de$$

.

Of course, we could just quote the algebraic result given in [10] (though this in fact only holds for reduced curves). Instead we can give a non-standard proof, which in many ways is conceptually simpler and doesn't involve any algebra.

Proof Let Q_d and Q_e be the families of all projective degree d and degree e curves. Then we have the cover $F \to U \times V$ with $F \subset U \times V \times P^2(L)$ obtained by intersecting the families Q_d and Q_e. We have that

$$\sum_{i=1}^{n} M(C_1, C_2, x_i) = \sum_{i=1}^{n} Mult_{x_i \in F(\bar{u}_0, \bar{v}_0)}(F/U \times V)$$

where $(\bar{u}_0, \bar{v}_0)$ define C_1 and C_2. By Lemma 4.3 in [7], this equals

$$\sum_{x \in F(\bar{u}, \bar{v})} Mult_{x, \bar{u}, \bar{v}}(F/U \times V)$$

where $(\bar{u}, \bar{v})$ is generic in $U \times V$. Using, for example, the proof of Lemma 4.12, generically independent curves $C_{\bar{u}}$ and $C_{\bar{v}}$ intersect transversely at a finite number of simple points. Hence, by Lemma 4.17, the Zariski multiplicity calculated at these points is 1. As the cover F has degree de, there is a total number de of these points as required. □

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