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Birational Geometry, Kähler–Einstein Metrics and Degenerations

Moscow, Shanghai and Pohang, April–November 2019

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Preface

This volume originated as a proceedings of the series of three conferences on Algebraic Geometry titled

Birational Geometry, Kähler–Einstein Metrics and Degenerations

Organized by us before COVID-19 pandemic in the following cities:

1. Moscow in Russia (8–13 April 2019),
2. Shanghai in China (10–14 June 2019),
3. Pohang in Korea (18–22 November 2019).

The pilot conference in Moscow was hosted by *Laboratory for Mirror Symmetry and Automorphic Forms*, which was founded in 2017 to consolidate the achievements of Moscow school in Homological Mirror Symmetry and Birational geometry. The Soviet school of algebraic geometry, founded by Igor Shafarevich, played a very important role in modern Mathematics for decades until the collapse of the Soviet Union in 1991. Then the importance of research originating from Russia diminished due to the scientific emigration. To rectify this situation, High School of Economics (Moscow) established three research laboratories: *Laboratory of Algebraic Geometry and its Applications* directed by Fedor Bogomolov (New York University), *Laboratory of Representation Theory and Mathematical Physics* directed by Andrey Okounkov (Columbia University), and *Laboratory for Mirror Symmetry and Automorphic Forms* directed by Ludmil Katzarkov (University of Miami). These laboratories attracted top mathematicians who supervised young researchers and organized many international conferences and workshops, which invigorated Russian mathematics.

Following the successful launch of our conference in Russia, we continued the series in Shanghai. Known as the “Paris in the East”, Shanghai, in its heyday, was the most socially, culturally and economically advanced city in Asia. Embracing the reform and opening-up policies in the last 40 years, Shanghai has been the center of the country's commercial and cultural renaissance. The city has long charmed its visitors with tranquil tree-lined streets, designer boutiques and those majestic colonial buildings along the Bund, on the bank of the Huangpu River, the city's iconic river. Adding to the old charm is the famed skyline of Pudong on the other side of the river. Our conference site was located in this most urbanized part of the city, at *the Institute of Mathematical Science* (IMS) at ShanghaiTech University.

The modern mathematical research and education in China has been greatly influenced by Russia through the adaptation of its northern neighbor's higher education system and Russia-trained mathematicians in the 1950s. The impact is so profound that it is still felt today. Shanghai has long considered to be the most important mathematics research hub on the south side of the Yangtze River, with Beijing as its counterpart in the north. First rate results are routinely produced by the up-and-coming young researchers, as well as the more established ones, across all disciplines but more prominently in the fields of geometry, representation theory, dynamical systems and PDE. There is a quite vibrant community of seasoned algebraic geometers in Shanghai, notably those from East China Normal University, Fudan University and our host, IMS.

Founded in 2018, IMS, with Xiuxiong Chen as its founding director, has set its mission on nourishing a new generation of creative talents in all areas of mathematics. Thanks to its globe recruiting efforts, the institute has now over two dozens of tenured and tenure-track members on its faculty, with research areas covering Algebraic Geometry, Number Theory, Representation Theory, Geometric Analysis, Partial Differential Equations, Probability, Statistics, Numerical Analysis and Computational Mathematics. The institute will double its faculty size in the next decade and expand into other areas of mathematics, in particular in applied mathematics including data sciences. The institute has been enjoying its success with some high-profile research publications of the faculty in the last two years. Some of the recent achievements at the institute are featured in this volume.

The last conference of the series was held in Pohang, located in the south-eastern part of Korea. The city has been a powerhouse in the rapidly growing Korean economy for the last half century. The conference was hosted by *the Center for Geometry and Physics* (CGP) which was founded in July 2012 as one of the first research centers of the Institute for Basic Science (IBS). IBS was established in November 2011 as Korea's first dedicated basic science research institute. The center is located on the intellectually dynamic campus of Pohang University of Science and Technology (POSTECH) in Pohang, Korea. Members and visitors at the Center are immediately immersed in an intellectual network that reaches beyond the peaceful seaside city.

The CGP originated in a government funded award, via IBS, to the research program of its director, Yong-Geun Oh. This program aims to help establish and develop the emerging field of symplectic algebraic topology through a collaborative effort by experts in fields such as symplectic geometry, dynamical systems, algebraic geometry and mathematical physics. The center is now established as an international institution with a broad scope, focusing more generally on geometry and mathematical physics.

Even though Korea has a long and rich intellectual history, its participation in the modern scientific and mathematical communities is relatively new. In particular, institutes dedicated solely to mathematics are very rare, making the CGP a valuable institution with the potential to serve an important function within the larger Korean scientific community. The center's emphasis on international collaboration will offer a chance for scholars with similar passions to plant ideas together and watch them grow, no matter where they are on the globe, and will allow the center to serve as a bridge between Korean mathematicians and the international mathematical community. Pursuing this mission, the center hosted one of the three conferences in Pohang and contributed a couple of articles to this volume.

Fifty two mathematicians participated in our three conferences:

Valery Alexeev (Athens, Georgia), Harold Blum (Salt Lake City), Morgan Brown (Miami), Jacob Cable (Manchester), Minglian Cai (Shanghai), Paolo Cascini (London), Ivan Cheltsov (Edinburgh), Xiuxiong Chen (Stony Brook), Sung Rak Choi (Seoul), Giulio Codogni (Rome), Thibaut Delcroix (Montpellie), Ruadhai Dervan (Cambridge), Kento Fujita (Osaka), Alexei Golota (Moscow), Zhengyu Hu (Taipei), Jun-Muk Hwang (Seoul), Yosuke Imagi (Shanghai), Chen Jiang (Shanghai), Ludmil Katzarkov (Miami), Jonghae Keum (Seoul), Igor Krylov (Seoul), Kyoung-Seog Lee (Pohang), Chi Li (New Jersey), Qifeng Li (Seoul), Yan Li (Beijing), Yijia Liu (Montreal), Yuchen Liu (Chicago), Costya Loginov (Moscow), Dimitri Markushevich (Lille), Jesus Martinez-Garcia (Essex), David Witt Nystrom (Gothenburg), Yuji Odaka (Kyoto), Takuzo Okada (Saga), Jihun Park (Pohang), Jinhyung Park (Seoul), Alexander Petkov (Sofia), Yuri Prokhorov (Moscow), Victor Przyjalkowski (Moscow), Julius Ross (Chicago), Yanir Rubinstein (College Park), Taro Sano (Kobe), Costya Shramov (Moscow), Charlie Stibitz (Chicago), Hendrik Suess (Jena), Andrey Trepalin (Moscow), Junwu Tu (Shanghai), Nivedita Viswanathan (Loughborough), Joonyeong Won (Seoul), Kewei Zhang (Beijing), Chuyu Zhou (Beijing), Ziwen Zhu (Salt Lake City), Ziquan Zhuang (Princeton).

The conference cites, Moscow, Shanghai and Pohang, are united in the beautiful conference poster made by Elena Cheltsova, a Russian artist based in Edinburgh (Scotland).

Sixty nine mathematicians contributed forty three research and survey papers to this volume including 2018 Fields Medalist Caucher Birkar:

Terutake Abe, Edoardo Ballico, Grigory Belousov, Mohamed Benzerga, Caucher Birkar, Charles Boyer, Gavin Brown, Jaroslaw Buczynski, Igor Burban, Ivan Cheltsov, Giulio Codogni, Thibaut Delcroix, Adrien Dubouloz, Kento Fujita, Elizabeth Gasparim, Stanislav Grishin, Yoshinori Hashimoto, Zhengyu Hu, Yosuke Imagi, Kobina Jamieson, Dasol Jeong, Chen Jiang, Ming-Chang Kang, Ilya Karzhemanov, Alexander Kasprzyk, Ludmil Katzarkov, Julien Keller, Young-Hoon Kiem, In-kyun Kim, Sergey Kudryavtsev, Nikon Kurnosov, Antonio Laface, Kyoung-Seog Lee, Chi Li, Yan Li, Zhenye Li, Yuchen Liu, Yota Maeda, Leonid Makar-Limanov, Dimitri Markouchevitch, Jesus Martinez-Garcia, Anne Moreau, David Witt Nystrom, Yuji Odaka, Jihun Park, Jinhyung Park, Zsolt Patakfalvi, Jennifer Paulhus, Alexander

Petkov, Andrea Petracci, Tristram de Piro, Yuri Prokhorov, Rodrigo Quezada, Julius Ross, Francisco Rubilar, Yanir Rubinstein, Taro Sano, Cristiano Spotti, Hendrik Suess, Bruno Suzuki, Josef Svoboda, Matei Toma, Christina Tonnesen-Friedman, Nivedita Viswanathan, Joonyeong Won, Egor Yasinsky, Yuri Zarhin, Kewei Zhang, Ziquan Zhuang.

Many of them were participants of our Moscow—Shanghai—Pohang conferences, while the others helped to expand the research breadth of the volume—the diversity of their contributions reflects the vitality of modern Algebraic Geometry.

Traveling across borders has become much easier in recent decades. We have also benefited from contemporary mobility technologies, so that we can share and develop our ideas without any serious barriers. The series of three conferences in Moscow, Shanghai, and Pohang have been held under such circumstances. It is, however, an irony that the mobility enhanced by contemporary technologies boosted the COVID-19 pandemic. In the pandemic, we made a lot of efforts to find new ways of communicating and collaborating to contribute to the knowledge of mankind. Some of them were successful, and some of them were unsatisfactory. We have been struggling to step forward to a new world. It seems that we have been through the darkest part of the pandemic, which might be too optimistic at this moment. At any rate, we at least know that at the end of the pandemic, there will be another world waiting for us where the normal will be different from the previous ones. Even though many things will be changed, we believe that we will freely and continually meet and share our ideas sometime and somewhere in the real world or even in the virtual world, because our cultural world is one country, as addressed by David Hilbert.

Edinburgh, UK — Ivan Cheltsov
Stony Brook, USA — Xiuxiong Chen
Coral Gables, USA — Ludmil Katzarkov
Pohang, Korea (Republic of) — Jihun Park
June 2022

Contents

Classification of Exceptional Complements: Elliptic Curve Case

Terutake Abe

Abstract We classify the log del Pezzo surface (S, B) of rank 1 with no 1-,2-,3-,4-, or 6-complements with the additional condition that B has one irreducible component C which is an elliptic curve and C has the coefficient b in B with $\frac{1}{n}\lfloor(n+1)b\rfloor = 1$ for $n = 1, 2, 3, 4$, and 6.

Keywords Del Pezzo surfaces · Complements

1 Introduction

This paper is a part of the project to classify "log del Pezzo surfaces with no regular complements", that is, the pairs (S,B) of surface S and boundary B on S such that:

(EX1) $-(K + B)$ is nef ((S, B) is "quasi log del Pezzo"),
(EX2) $-(K + B)$ has no regular complements i.e. it has no n-complements for any of $n \in \{1, 2, 3, 4, 6\}$.

We assume throughout that coefficients of B are "standard", i.e. $B = \sum b_i C_i$ with $b_i = \frac{m-1}{m}$ where m natural number, or $b_i \geq \frac{6}{7}$. An invariant δ for such a pair is defined in [8, 5] by

$$\delta(S, B) = \sharp\{E | E \text{ is an exceptional or non-exceptional divisor} \\ \text{with log discrepancy } a(E) \leq \tfrac{1}{7} \text{ for } K + B\}$$

and it was proved there that $\delta \leq 2$ [8, Theorem 5.1]. We can assume, after crepant blow ups of exceptional E's with $a(E) \leq \frac{1}{7}$, that those E are all non-exceptional, and thus,

I would like to thank Professor Shokurov for setting the problem and for his valuable suggestions.

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(EX3) (S, B) is $\frac{1}{7}-$ log terminal.

Now define the divisor D by $D = \sum d_i C_i$ where $d_i = 1$ if $b_i \geq \frac{6}{7}$ and $d_i = b_i$ otherwise. And write $C = \lfloor D \rfloor = \sum_{a(C_i) \leq \frac{1}{7}} C_i$. We know by [8, Lemma 4.2] that if $\delta \geq 1$ then we can successively contract curves semi-negative with respect to $K + B$, but not components of C, and thereby assume

(EX4) $\rho(S) = 1$.

The conditions (EX1), (EX2) and (EX3), as well as the condition on the coefficients, are preserved under this reduction. We form a minimal resolution $f : (S^{\min}, B^{\min}) \to (S, B)$ where $B^{\min}$ is a crepant pullback, i.e. $K_{S^{\min}} + B^{\min} = f^*(K + B) = K + f^{-1}(B) + \Sigma e_j E_j$ satisfies $K + B^{\min} \cdot E_j = 0$ for all j. From $S^{\min}$ we contract (-1)-curves successively to get a smooth model S' which is either $\mathbb{P}^2$ or $\mathbb{F}_m$:

$$g : (S^{\min}, B^{\min}) \to (S', B').$$

If $\delta \geq 1$ we have $p_a(C) \leq 1$, and the same is true for the birational image of C on S' as well [8, Proposition 5.4]. In this paper we consider the case

- $\delta = 1$.

Thus, C is an irreducible curve of arithmetic genus ≤ 1. We write $C = C_1$, and $B = bC + \sum_{i=2}^{r} b_i C_i = bC + B_1$, with $b_i \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$ and $b \geq \frac{6}{7}$.

1.1 The Dual Graph of a Configuration

In the following we use the language of graphs to talk about the the configuration of curves.

The dual graph of a configuration of curves is a (weighted-multi) graph where we have a vertex for each curve and an n-ple edge for each intersection point with multiplicity n between two curves. Each vertex has a weight $\in \mathbb{Z}$ which is the self-intersection number of the curve.

Graphically, we use $\bullet$ ("b(lack)-vertex") to represent exceptional curves with self-intersection number ≤ -2, $\circ$ ("w(hite)-vertex") for (-1)-curves, and squares for curves with non-negative self-intersection. The weight of a vertex is shown by a number next to each vertex, and multiplicity of an edge by the number of lines joining the two end vertices.

"Blow up of an edge" means the transformation of the graph reflecting the blow up of the corresponding point, that is, introduce a new white vertex, decrease the multiplicity of edge by 1, decrease the weight of the both end vertices of the edge by 1, and join them to the new white vertex by a simple edge. "Blow up of a vertex" reflects the blow up of a point on the curve outside the intersection with neighboring curves: introduce a white vertex, decrease the weight of the vertex by 1, and join it to the new white vertex by a simple edge. Blow up of a complete subgraph of any cardinality k can be defined in the same way.

1.2 Types of Singularities on C

Lemma 1.1 *(i) The singularity of C is at worst a node, and it is outside $Sing(S) \cup Supp(B_1)$.*

(ii) At most one component C_i of B_1 passes through each point $P \in C$. If P in a smooth point of S, then the intersection is normal ,with one possible exception where C_i has coefficient $\frac{1}{2}$ and has a simple tangency with C at a smooth point P of S.

(iii) Singularity P of S on C is a cyclic quotient singularity, i.e. log terminal singularities with resolution graph $\mathbb{A}_n$ (a chain), where C meets one end curve of the chain normally. If another component C_i passes through P, then it meets the other end curve normally.

Proof Note that $K + D$, as defined above, is log canonical by the existence of local complements [7, Corollary 5.9.]. Then all the statements follow from the classification of surface log canonical singularities ([3], or [1]) and $\frac{1}{7}$-log terminal condition. For example, for (iii), if we had a type $\mathbb{D}_n$ singularity, (case (6) in [3, Theorem 9.6]) we would have a log discrepancy $\leq \frac{1}{7}$. Note also that the exception in (ii) is the only case where $K + D$ is not log terminal at P [8, Proposition 5.2]. □

As is well known, the singularities mentioned above are isomorphic, analytically, to the origin 0 in the quotient of $\mathbb{C}^2$ by the action of cyclic group μ_m of order m, where the generator $\varepsilon = e^{\frac{2\pi i}{m}}$ acts by

$$(z_1, z_2) \mapsto (\varepsilon^{-k} \cdot z_1, \varepsilon \cdot z_2), \text{ where } 1 \leq k \leq m \text{ and } gcd(m, k) = 1.$$

The minimal resolution of such a singularity has a chain of rational curves $E_1, E_2, \ldots, E_r$ as its exceptional locus, and the coefficients $(w_1, w_2, \ldots, w_r)$ in the continued fraction expansion

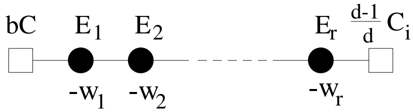
$$\frac{m}{m-k} = w_1 - \cfrac{1}{w_2 - \cfrac{1}{w_3 - \cdots}}$$

give their self intersection numbers (*cf.* for example, [2]). We call such a singularity P type $[m, k]$. We extend this correspondence to incorporate the information on the component C_i that passes through P (*cf.* [7, Corollary 3.10], [8, Lemma 2.22]).

Namely, if the component C_i has the standard coefficient $\frac{d-1}{d}$ and the singularity P has type (m', k'), we represent it by the pair $(m, k) = (dm', dk')$. The "dual graph" of the minimal resolution of this singularity is (Fig. 1).

Generalizing the notation of [4], we may denote the same singularity by $(\underline{w_1}, w_2, \ldots, w_r)_d$ with the underline indicating the curve meeting C.

Fig. 1 The dual graph of the minimal resolution of singularity of type (m′, k′)



This singularity has the minimal log discrepancy

$$mld(P, K+B) = a(E_1) = \frac{1+(m-k)(1-b)}{m} \; (\le \frac{1+\frac{1}{7}(m-k)}{m}),$$

where $m = d \cdot$ (index of P). Also we denote the co-discrepancy, or the coefficient, of P by $e(P, K+B) = 1 - mld(P, K+B)$.

Now the $\frac{1}{7}$-log terminal condition

$$\frac{1+\frac{1}{7}(m-k)}{m} > \frac{1}{7}$$

is equivalent to $k < 7$. Therefore the possible singularities on C are put into 21 $= 6(6+1)/2$ (infinite) series according to the pair $(m(\text{mod } k), k)$ with $1 \le k \le 6$. This will be convenient later on.

2 Elliptic Curve Case

Now we start the classification of the case $p_a(C) = 1$. Thus, $C \in S$ is a smooth curve of genus 1 or a rational curve with one node by Lemma 1.1. We call it the "elliptic curve case".

Lemma 2.1 *In the "elliptic curve case", the condition (EX2) is equivalent to the condition that (S, B) has log-singularities on C. That is, either S has singularities on C, or B has components other than C (which intersect C since $\rho(S) = 1$).*

Proof If (S, B) is smooth on C , then $(K + f^*(D)).C = (K + C).C = 0$ on $S^{\min}$, so $K + D = K + C \sim 0$ on S and (EX2) is not satisfied. In fact $K + B = K + bC$ has a 1-complement. On the other hand if (S, B) has a singularity on C, then $(K + f^*(D)).C > (K + C).C = 0$ so we have $K + D > 0$ on S, which implies (EX2). $\square$

The case when S is a cone ($\mathbb{P}^2$ or $\mathbb{Q}_m$) has been classified elsewhere and from it we have only one case with C=elliptic: $S = \mathbb{F}_2$, C =double section, $B_1 = \frac{1}{2}C_2$ where C_2 is a generator of the cone. Then $C \equiv 2H \equiv -K$, $C_2 \equiv \frac{1}{2}H$. So $K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0$ and $K + B$ has 7-complement $= 0$. It also has the trivial 8-complement: $K + \frac{7}{8}C + \frac{1}{2}C_2 \equiv 0$ (This is the entry ♯1 in the table at the end).

From now on we assume S is not a cone.

Lemma 2.2 *Inequality $C^2 \geq 3$ holds on S^{min}. If (S, B) has two singularities on C, then $C^2 \geq 6$. On the other hand, the minimum log discrepancy of the singularity P on C with respect to $K + B$ (hence also with respect to $K + bC$) is at least* $1 - (C^2/7)$.

Proof Because $-(K + B)$ is nef,

$$\begin{aligned} 0 \geq (K + B).C &= K_{S^{\min}} + bC + \Sigma b_i C_i + \Sigma d_j E_j \cdot C \\ &\geq -(1-b)C^2 + \Sigma b_i + \Sigma_P (1 - mld(P)) \\ &\geq -(1-b)C^2 + min\{b_i, 1 - mld(P)\} \\ &\geq -\tfrac{1}{7}C^2 + \tfrac{3}{7}. \end{aligned}$$

Note that, because of Lemma 1.1, $1 - mld(P) = d_j$ for the exceptional curve E_j meeting C. The last inequality holds because we have at least one nonzero b_i or $1 - mld(P)$ by Lemma 2.1 and the minimum nonzero value for b_i is $\frac{1}{2}$, that for $1 - mld(P)$ is $\frac{1}{2} \cdot \frac{6}{7} = \frac{3}{7}$, the latter being attained when P in duVal of type $\mathbb{A}_1$. Therefore, $C^2 \geq 3$. By the same calculation, if there are two singularities on C we have $0 \geq -\frac{1}{7}C^2 + \frac{6}{7}$. On the other hand, the second inequality in particular implies that $1 - mld(P) \leq (1-b)C^2 \leq \frac{1}{7}C^2$, whence the second assertion. □

2.1 Reduction to $\mathbb{F}_2$

We need the following

Lemma 2.3 *Let E be a (-1)-curve on S^{min}. Then on its image $f_*(E)$, S has either at least two singularities, or one singularity that is not log-terminal for $K + E$.*

Proof If , on E, S had at most one singularity P that is log-terminal for $K + E$, i.e. a cyclic quotient singularity such that E meets one end curve E_1 of the chain of the resolution, then we would have

$$(f_*(E))^2 = E.f^* f_*(E) = E.(E + (1 - a(E_1))E_1) = -1 + (1 - a(E_1)) < 0$$

which is absurd since $\rho(S) = 1$. □

Now we can prove

Lemma 2.4 *We can always obtain $\mathbb{F}_2$ as a smooth model of S (and C as a double section).*

Proof $p_a(C) = 1$ means that after reconstruction, C is either a cubic in $\mathbb{P}^2$, curve of bidegree $(2, 2)$ on $\mathbb{F}_0$, or a double section of $\mathbb{F}_2$. Suppose S' is $\mathbb{P}^2$ and C is a cubic, since there are no irreducible curves with arithmetic genus 1 on $\mathbb{F}_m$, with $m \geq 3$. If $g : S^{\min} \to \mathbb{P}^2$ contracts two or more exceptional curves to a point $P \in \mathbb{P}^2$, then we can choose different contractions to get $S' = \mathbb{F}_2$. Therefore we may assume that we have only one exceptional curve for g over each center $P \in \mathbb{P}^2$, and we shall derive a contradiction.

Since all the curves contracted by g are (-1)-curves on $S^{\min}$, no exceptional curve E_i for f are contracted and all of them are present on $\mathbb{P}^2$ as divisors. Thus we have an inequality

$$0 \geq deg(K_{\mathbb{P}^2} + bC + B_1') = -3 + \tfrac{6}{7} \cdot 3 + deg(B_1') = -\tfrac{3}{7} + deg(B_1')$$

Therefore, since the coefficients are standard, no component C_i other than C are present on $\mathbb{P}^2$. And we have

$$\Sigma d_j \leq \Sigma d_j \cdot deg(E_j) = deg(B_1') \leq \tfrac{3}{7} \tag{*}$$

We have the two possibilities:

(1) B has at least one component, say C_2, other than C. Then by the above, C_2 must be contracted on $\mathbb{P}^2$ and is a (-1)-curve on $S^{\min}$. Therefore, by Lemma 2.3, S must have either at least two singularities on C_2, or a singularity that is not log-terminal for $K + C_2$. In the former case, then, we would have $\Sigma d_j \geq (\frac{1}{2} + \frac{1}{2})b_2 \geq \frac{1}{2} > \frac{3}{7}$, contradicting $(*)$. In the latter case, we have an exceptional curve E with $a(E, K + C_2) \leq 0$. Then because $a(E, K + b_2C_2)$ is a linear function of b_2 and we also have $a(E, K + 0 \cdot C_2) = a(E, K) \leq 1$, we have $a(E, K + b_2C) \leq 1 - b_2$. Thus $d_2 = 1 - a(E, K + B) \geq 1 - a(E, K + b_2C_2) \geq b_2 \geq \frac{1}{2} > \frac{3}{7}$, again a contradiction to $(*)$.

(2) B has no other components than C, i.e. $B = bC$, and S has a singularity on C. Then $(*)$ implies that and we have $K + B > 0$ except in the following case: S has only one duVal singularity P of type $\mathbb{A}_1$ on C, the exceptional curve E_1 of the resolution of P is a line on $\mathbb{P}^2$, $b = \frac{6}{7}$, and B_1' has no other component than E_1, so that $K_{\mathbb{P}^2} + B' = K + \frac{6}{7}C + \frac{3}{7}E_1 \sim 0$. In particular all the singularities on S are duVal so

$$f^*(K + B) = K_{S^{\min}} + B^{\min} = K + \frac{6}{7}C + \frac{3}{7}E_1$$

Also, the triviality of $K + B$ means that pull back g^* is crepant so that the above is also equal to $g^*(K + B')$. On the other hand, since $E_1.C = 3$ on $\mathbb{P}^2$ and $E_1.C = 1$ on $S^{\min}$, two of the intersection points of E_1 and C has to be blown up on $S^{\min}$. The exceptional curve E for the first of such blowups would have the coefficient $\frac{6}{7} + \frac{3}{7} - 1 = \frac{2}{7}$ in $K_{S^{\min}} + B^{\min} = g^*(K + B')$. Contradicting the explicit form of $B^{\min}$ given above.

If we have a model $S' = \mathbb{F}_0$, then we have had at least one contraction of (-1)-curve so we can get $S' = \mathbb{P}^2$ by choosing other contractions, and we are reduced to the previous case. $\square$

Therefore we have a $\mathbb{P}^1$-fibration $p : S^{\min} \to \mathbb{F}_2 \to \mathbb{P}^1$. Now our strategy for the classification is to start from $\mathbb{F}_2 = S'$, make blow ups to construct $S^{\min}$, choose $B^{\min}$ on it so that resulting (S, B) would have singularities on C ($\Leftrightarrow$ (EX2) by Lemma 2.1) and would satisfy (EX1), (EX3), and (EX4).

The conditions (EX1) and (EX4) implies that the number of $(-n)$-curves, $n \geq 2$, on $S^{\min}$ must equal $\rho(S^{\min}) - 1$, and they are all exceptional for the resolution f. These curves are either in the fibres of p, or they are not, i.e. they are (multi-) sections of p. As for the number of curves of each type, we have the following:

Lemma 2.5 [9, Lemma 1.5] *We have*

$$
\begin{aligned}
r &= \sharp\{\textit{Exceptional curves } E_i\textit{'s of the resolution } f \textit{ that are not in the fibres of } p\} - 1 \\
&= \sharp\{(-1)\textit{-curves on } S^{min} \textit{ that are in the fibres of } p.\} \\
&\quad - \sharp\{\textit{Singular fibres of } p\}
\end{aligned}
$$

Proof Add $(2 + \sharp\{E_i$'s that are in the fibres of $p\})$ to both sides, and we get two expressions for $\rho(S^{\min})$. □

2.2 The Search for Exceptions

Case 1 $r = 0$, i.e. minimal section Σ on $\mathbb{F}_2$ is the only E_i with $p(E_i) = \mathbb{P}^1$.

Then there is only one (-1)-curve in each singular fibre of p. Therefore on each fibre F modified we have to have initially two blow ups at the same point P. Suppose $C^2 = w$ before the modification, then according as the intersection multiplicity $i = I(P; F \cap C) = 2, 1$, or 0, i.e. according as $P =$ tangency of F and C, normal intersection of F and C, or $P \in F \setminus (F \cap C)$, we get one of the three dual graphs in the Fig. 2 below.

In the figure the b-vertex at the bottom is the minimal section $\Sigma \in \mathbb{F}_2$. In the case (III), the curve C and neighboring (-2)-curve ($= F$) have either two normal intersections, one simple tangency, or C has a node on F.

Case(III) gives a non log canonical point (*cf.* Lemma 1.1(i)) and is excluded. Case(II) gives one example with trivial complement (entry $\sharp 2$ in the table at the end):

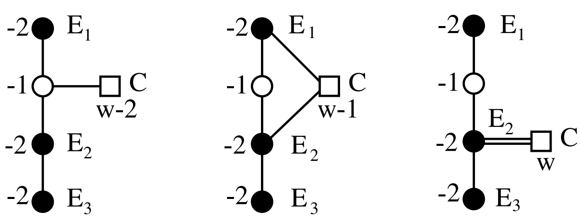


Fig. 2 Three dual graphs for case 1

$$S = \text{Gorenstein del Pezzo surface with singularities } \mathbb{A}_1 + \mathbb{A}_2,$$
$$C = \text{elliptic curve through } \mathbb{A}_1 \text{ and } \mathbb{A}_2 \text{ points},$$
$$K{+}B = K + \frac{6}{7}C \equiv 0$$
$$7(K_{S^{\min}}{+}B^{\min}) = 7(K + \frac{6}{7}C + \frac{3}{7}E_1 + \frac{4}{7}E_2 + \frac{2}{7}E_2) \sim 0$$

(Following [5], we denote the Gorenstein del Pezzo surfaces of rank 1 by its singularity type, for example, $S(A_1 + A_2)$ for the surface above, and their resolution by e.g. $\tilde{S}(A_1 + A_2)$.)

Since we already have $K + B \equiv 0$, if we make any more blow ups (which have to be on the unique (-1)-curve) or add other components to B, we would have $K + B > 0$ and (S, B) will violate (EX1). So we need not consider this case any longer. Thus we are left with case (I), i.e. two initial blow ups at the ramification point of $C \to \mathbb{P}^1$ (tangency of C and a fibre). In particular, in all the remaining cases, $C^2 \leq 6$, because $C^2 = 8$ on $\mathbb{F}_2$.

This implies that a smooth fibre F cannot be a component of B'_1, because if it were, we would have $0 \geq (K + bC + B_1).C \geq -(1 - b)C^2 + \frac{1}{2}F.C \geq -\frac{6}{7} + \frac{1}{2} \times 2 = \frac{1}{7}$, a contradiction. Therefore only singularities on C are those coming from the intersection of C and the singular fibres.

After (I), we can only blow up a point on the unique (-1)-curve on each fibre: otherwise we would introduce more than one (-1)-curves in a fibre, violating $r = 0$. There are two types of such blow ups. One is the blow ups of the intersection of C and the (-1)-curve, (blow up of the edge between the white vertex and C) which decrease C^2. The other is the blow ups of a point of (-1)-curve outside C.

We start from the first type of blow ups and get the resolutions of Gorenstein log del Pezzos of rank 1 with $K^2 = C^2 \geq 3$ (Lemma 2.2):

$$\begin{array}{ccccccc}
\tilde{S}(A_1 + A_2) & \longrightarrow & \tilde{S}(A_4) & \longrightarrow & \tilde{S}(D_5) & \longrightarrow & \tilde{S}(E_6) \\
 & \searrow & & \searrow & & & \\
 & & \tilde{S}(2A_1 + A_3) & \longrightarrow & \tilde{S}(A_1 + A_5). & &
\end{array}$$

Each "$\longrightarrow$" represents one blow up, and each "$\searrow$" two blow ups on a new fibre .

Then, starting from one of these, we make the second type of blow ups, which decrease the minimal log discrepancy of S, until either (EX1) or (EX3) is violated (see below). The Gorenstein rank 1 surfaces listed above are the image of $S^{\min}$ under the morphism $\phi_{|C|}$ defined by the linear system $|C|$ on it. We denote it by S_C, and its resolution (one of the above) by $\tilde{S_C}$.

Note that C meets every (-1)-curve E on S_C since $C \sim -K_{S_C}$ and $-K.E = 1$. Consider blow ups on one fibre starting at one such E. By Lemma 2.3, on E, S has either at least two singularity or one singularity that is not log-terminal for $K + E$. That is, on $\tilde{S_C}$, either E meets at least two trees T_1, T_2 of b-vertices, or one tree T_3 that gives non-log-terminal point for $K + E$.

Now consider the transformation of the subgraph consisting or C, E, and trees of b-vertices T_i meeting E on S_C. It should always contain a unique w-vertex.

If we blow up the vertex E, i.e. blow up a point on E other than the intersection points with neighboring exceptional curves, then after the transformation C would meet the b-vertex E in the black graph $T_1 - E - T_2$ or $E - T_3$. Either of these would contracts to a non-log-terminal point on S for $K + C$, contradicting Lemma 1.1. (For an example of the first situation, consider blow up of the white vertex in the configuration (I) in the Fig. 2 above. For the second, consider the same in the configuration of the table ♯9.) Therefore the first blow up has to be at the intersection point of E and one of the neighboring b-vertices, i.e. blow up of the edge joining E and one of its neighbors.

The same argument, repeated for the new white vertex E_1 at each stage, shows that successive blow ups also must be at the edge joining E_1 and a neighboring b-vertex, because the trees now meeting E_1 are even bigger than those that met E. Thus, by induction, we see that the full inverse image of E is of the form $E - T - E_1 - T'$, where T and T' are chains of b-vertices (T, or T' may be a part of a larger tree. And E may meet another tree T'' in which case T should be empty — Remember that C meets E), and E_1 is a w-vertex. The blow up described above either increases the weight of an end vertex of T next to E_1, or adds one (-2)-curve E_1 to it, depending on which side of E_1 we blow up. Either of such transformations (those which preserve log-terminal property), if repeated infinitely many times, make the log-discrepancy with respect to $K + bC$ of the resulting singularity on C monotonically decrease toward $1 - b \leq \frac{1}{7}$. Hence by Lemma 2.2, after finite number of steps, (EX3) will be violated (or perhaps, (EX1) may be violated first). Therefore this procedure of successive blowups must terminate.

We can now refine Lemma 2.2 as follows:

If $C^2 < 6$ then we have only one singularity by Lemma 2.2. But on the other hand, if $C^2 \geq 5$ we can have only one singular fibre, which means that in every case we have only one singularity of (S, B) on C. (EX1) restricts the possible types of singularities $[m, k]$ on C as follows:

$$\begin{aligned}
0 \geq K + bC + B' \cdot C &= -(1-b)C^2 + \tfrac{(k-1)+b(m-k)}{m} \\
&= \frac{1}{7}C^2 + \tfrac{(k-1)+\frac{6}{7}(m-k)}{m}, \\
&\mathit{or} \\
(6 - C^2)m &\leq 7 - k.
\end{aligned}$$

In this way, we find that there are 20 possible S's , with a few different B's for some of the S's. These are summarized in the table below.

Case 2 $r = 1$, i.e. we have one exceptional curve, say E, other than Σ that is a section of $\mathbb{P}^1$-fibration p.

Thus, exactly one fibre contains two (-1)- curves in it. If we modify at any other fibre it has to start like (I) of the Fig. 2 (two blow ups at the tangency with the fibre) because (II) and (III) have been eliminated. In particular each time we blow up on a new fibre we decrease C^2 by at least 2.

Claim Any exceptional curve E that is a (mutli-)section of p is in fact a 1-section that is disjoint from Σ.

Proof Let E be a (multi-)section, and $d = \mathrm{mult}_E(B'_1)$. Then if F is a fibre of p, we have

$$0 \geq (K + B').F \geq (K + bC + dE).F \leq -2 + \tfrac{6}{7} \cdot 2 + d.$$

Hence $d \leq \frac{2}{7} < \frac{3}{7}$. So E cannot intersect C on $S^{\min}$. Therefore all the intersection point of C and E have to be blown up on $S^{\min}$. If E is not a 1-section disjoint from Σ, then we have $C.E \geq 6$ on $\mathbb{F}_2$. So we would have $C^2 \leq 8 - 6 = 2$ on $S^{\min}$ which is impossible according to Lemma 2.2. This proves the claim.

So let E be a simple section disjoint from Σ. Then $E.C = 4$.

Suppose E intersects C at one point with multiplicity 4. Then after four blowups at this point we get $\tilde{S}(A_1 + A_3)$ (the configuration of the table ♯17, with a different choice of fibration), which has already been studied in the case 1 above.

If E intersects C at two points with multiplicity 3 and 1 respectively, then by the above observation we have at least $3 + 2 = 5$ blow ups on C, which gives $\tilde{S}(3A_2)$, with C passing through three (-1)-curves joining three $\mathbb{A}_2$ points. Since $C^2 = 3$ by Lemma 2.2 C can have at worst $\mathbb{A}_1$ (="type [2,1]") point on it, but that cannot be attained: We could at best choose $B_1 = \frac{1}{2}C_2$ where $C_2 =$ (image of one of the (-1)-curves meeting C) and thus get type [2,2] point on C, which is worse than $\mathbb{A}_1 = type[2, 1]$.

If E intersects C at more than 3 points, then we have at least six blow ups on C thus $C^2 \leq 2$, which is impossible by Lemma 2.2.

Finally, if E intersects C at two points with multiplicity 2 each, we would have two (-1)-curves in each fibre, and this violates (EX4).

Thus, we get no new examples from case 2.

Case 3 $r \geq 2$, i.e. we have at least two exceptional curves, say, E_1 and E_2, other than Σ, that are sections of p.

E_i are simple sections. Then because $C.E_i = 4$ and $E_1.E_2 = 2$, we must have at least $4 + 4 - 2 = 6$ blow ups on C in order to separate E_i's from C. Then $C^2 \leq 2$ and by Lemma 2.2, this is impossible. □

It turns out that in every case $K + B$ has a 7-complement. Moreover, we can choose g so that in every case B'_1 has only one component which is a fibre of $\mathbb{F}_2$.

Table

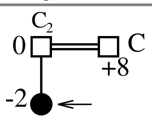
Thus we get the following table. Here,

- The first column shows the configuration on $S^{\min}$ of the exceptional curve E_i's, (-1)-curves, and the components of B. '$\circ$' denote (-1)-curve, '$\bullet$' are the E_i's with self intersection number (≤ -2) attached, with '$\leftarrow$' indicating (one possible) $\Sigma \subset \mathbb{F}_2$ after a suitable sequence of contractions of (-1)-curves. Squares are curves with non negative self intersection.
- The second column gives the fractional part B_1 of the boundary B, or rather, of D.
- The third column gives the number $(\frac{6}{7} \leq)$ $max\{b|K + bC + B_1 \leq 0\}$ (< 1)
- The fourth column gives an example of n-complements.
- The last column lists numerical relations between some relevant divisors on S, with H being the generator of $Pic(S)$.

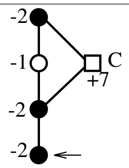
Note that we can compute intersection numbers on $S^{\min}$ using the crepant pullbacks, and a divisor on S is Cartier iff its crepant pullback is Cartier, i.e. iff it is integral (*cf.* [6]).

The table is organized according to S_C, the image of $S^{\min}$ under the morphism defined by the linear system $|C|$.

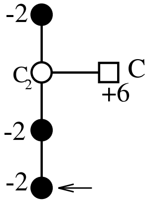
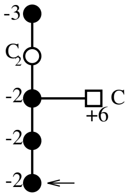
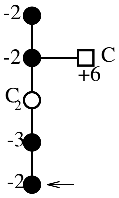
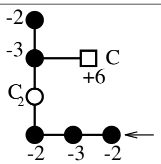
(1) $S = S(A-1) = \mathbb{Q}_2$ (=quadratic cone $\subset \mathbb{P}^3$, S_C = its Veronese image)

	configuration	B_1	max(b)	complements	divisors(Pic(S)=$\mathbb{Z}[H]$)
1		$\frac{1}{2}C_2$	$\frac{7}{8}$	• 7-compl.= 0 $(K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0)$ • trivial 8-compl.	$-K \equiv C \equiv 2H$ $C_2 \equiv \frac{1}{2}H$

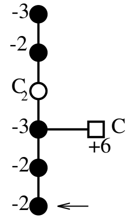
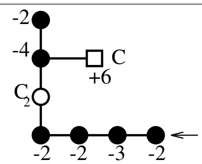
(2) $S_C = S_7$ (= a del Pezzo with degree 7)

2		0	$\frac{6}{7}$	trivial 7-compl. $(K + \frac{6}{7}C \equiv 0)$	$-K \equiv H$ $C \equiv \frac{7}{6}H$

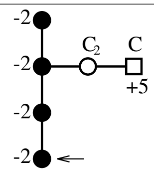
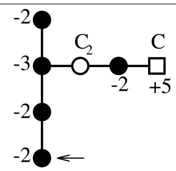
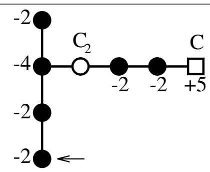
(3) $S_C = S(A_1 + A_2)$

3		$\frac{1}{2}C_2$	$\frac{9}{10}$	trivial 10-compl. $(K + \frac{9}{10}C + \frac{1}{2}C_2 \equiv 0)$	$-K \equiv C \equiv H$ $C_2 \equiv \frac{1}{6}H$
		$\frac{2}{3}C_2$	$\frac{8}{9}$	trivial 9-compl. $(K + \frac{8}{9}C + \frac{2}{3}C_2 \equiv 0)$	
		$\frac{3}{4}C_2$	$\frac{7}{8}$	trivial 8-compl. $(K + \frac{7}{8}C + \frac{3}{4}C_2 \equiv 0)$	
		$\frac{4}{5}C_2$	$\frac{13}{15}$	7-compl.=0 $(K + \frac{6}{7}C + \frac{6}{7}C_2 \equiv 0)$	
		$\frac{5}{6}C_2$	$\frac{31}{36}$	7-compl.=0 $(K + \frac{6}{7}C + \frac{6}{7}C_2 \equiv 0)$	
4		0	$\frac{8}{9}$	• 7-compl.=C_2 $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ • trivial 9-compl.	$-K \equiv \frac{8}{12}H$ $C \equiv \frac{9}{12}H$ $C_2 \equiv \frac{1}{12}H$
5		0	$\frac{9}{10}$	• 7-compl.=$3C_2$ $(K + \frac{6}{7}C + \frac{3}{7}C_2 \equiv 0)$ • trivial 10-compl.	$-K \equiv \frac{9}{15}H$ $C \equiv \frac{10}{15}H$ $C_2 \equiv \frac{1}{15}H$
6		0	$\frac{7}{8}$	• 7-compl.=$2C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ • trivial 8-compl.	$-K \equiv \frac{14}{40}H$ $C \equiv \frac{16}{40}H$ $C_2 \equiv \frac{1}{40}H$

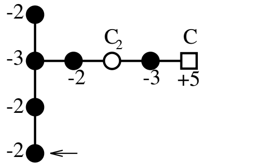
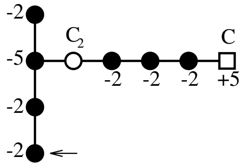
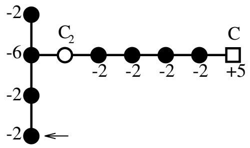
(continued)

7		0	$\frac{13}{15}$	7-compl.=C_2 $(K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)$	$-K \equiv \frac{13}{35}H$ $C \equiv \frac{15}{35}H$ $C_2 \equiv \frac{1}{35}H$
8		0	$\frac{19}{22}$	7-compl.=C_2 $(K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)$	$-K \equiv \frac{19}{77}H$ $C \equiv \frac{22}{77}H$ $C_2 \equiv \frac{1}{77}H$

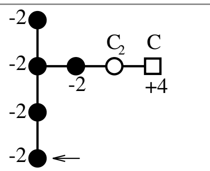
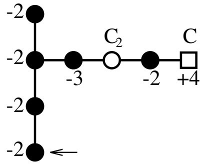
(4) $S_C = S(A_4)$

9		$\frac{1}{2}C_2$	$\frac{9}{10}$	• 7-compl.=C_2 $(K + \frac{6}{7}C + \frac{5}{7}C_2 \equiv 0)$ • trivial 10-compl.	$-K \equiv C \equiv H$ $C_2 \equiv \frac{1}{5}H$
		$\frac{2}{3}C_2$	$\frac{13}{15}$	7-compl.=0 $(K + \frac{6}{7}C + \frac{5}{7}C_2 \equiv 0)$	
10		0	$\frac{10}{11}$	• 7-compl.=$4C_2$ $(K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0)$ • trivial 11-compl.	$-K \equiv \frac{10}{22}H$ $C \equiv \frac{11}{22}H$ $C_2 \equiv \frac{1}{22}H$
		$\frac{1}{2}C_2$	$\frac{19}{22}$	7-compl.=0 $(K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0)$	
11		0	$\frac{15}{17}$	7-compl.=$3C_2$ $(K + \frac{6}{7}C + \frac{3}{7}C_2 \equiv 0)$	$-K \equiv \frac{15}{3\cdot 17}H$ $C \equiv \frac{17}{3\cdot 17}H$ $C_2 \equiv \frac{1}{3\cdot 17}H$

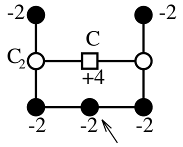
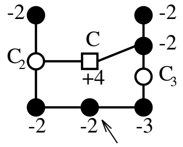
(continued)

12		0	$\frac{7}{8}$	• 7-compl.=$2C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ • trivial 8-compl.	$-K \equiv \frac{14}{48}H$ $C \equiv \frac{16}{48}H$ $C_2 \equiv \frac{1}{48}$
13		0	$\frac{20}{23}$	7-compl.=$2C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$	$-K \equiv \frac{20}{4{\cdot}23}H$ $C \equiv \frac{1}{4}H$ $C_2 \equiv \frac{1}{4{\cdot}23}H$
14		0	$\frac{25}{29}$	7-compl.=C_2 $(K + \frac{6}{7}C + \frac{1}{7}C_2 \equiv 0)$	$-K \equiv \frac{25}{5{\cdot}29}H$ $C \equiv \frac{1}{5}H$ $C_2 \equiv \frac{1}{5{\cdot}29}H$

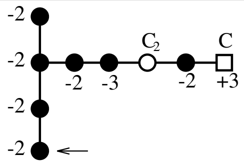
(5) $S_C = S(D_5)$

15		$\frac{1}{2}C_2$	$\frac{7}{8}$	• 7-compl.=0 $(K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0)$ • trivial 8-compl.	$-K \equiv C \equiv H$ $C_2 \equiv \frac{1}{4}H$
16		0	$\frac{8}{9}$	• 7-compl.=$2C_2$ $(K + \frac{6}{7}C + \frac{2}{7}C_2 \equiv 0)$ • trivial 9-compl.	$-K \equiv \frac{8}{18}H$ $C \equiv \frac{9}{18}H$ $C_2 \equiv \frac{1}{18}H$

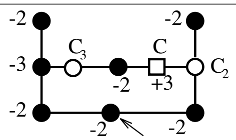
(6) $S_C = S(A_3 + 2A_1)$

17		$\frac{1}{2}C_2$	$\frac{7}{8}$	• 7-compl.=0) $(K + \frac{6}{7}C + \frac{4}{7}C_2 \equiv 0)$ • trivial 8-compl.	$-K \equiv C \equiv H$ $C_2 \equiv \frac{1}{4}H$
18		0	$\frac{6}{7}$	trivial 7-compl. $(K + \frac{6}{7}C \equiv 0)$	$-K \equiv \frac{12}{42}H$ $C \equiv \frac{14}{42}H$ $C_2 \equiv \frac{3}{42}H$ $C_3 \equiv \frac{2}{42}H$

(7) $S_C = S(E_6)$

19		0	$\frac{6}{7}$	trivial 7-compl. $(K + \frac{6}{7}C \equiv 0)$	$-K \equiv \frac{6}{14}H$ $C \equiv \frac{7}{14}H$ $C_2 \equiv \frac{1}{14}H$

(8) $S_C = S(A_5 + A_1)$

20		0	$\frac{6}{7}$	trivial 7-compl. $(K + \frac{6}{7}C \equiv 0)$	$-K \equiv \frac{6}{14}H$ $C \equiv \frac{7}{14}H$ $C_2 \equiv \frac{2}{14}H$ $C_3 \equiv \frac{1}{14}H$

References

1. Alexeev, V.: Classification of log canonincal surface singularities: arithmetical proof. Chap. 3 of [KCM]
2. Fulton, W.: Introduction to Toric Varieties. Princeton University Press, Princeton (1993)
3. Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degeneration of surfaces. Annals of Math. **127**, 93–163 (1988)
4. Keel, S., McKernan, J.: Rational curves on quasi-projective surfaces. Mem. Am. Math. Soc. **669** (1999)

5. Miyanishi, M., Zhang, D.Q.: Gorenstein log del Pezzo surfaces of rank 1. J. Algebra **118**, 63–84 (1988)
6. Sakai, F.: Weil divisors on normal surfaces. Duke Math. J. **51**, 877–887 (1984)
7. Shokurov, V.V.: 3-fold log flips. Rus. Acad. Sci. Izv. Math. **40**, 95–202 (1933)
8. Shokurov, V.V.: Complements on surfaces. J. Math. Sci. **102**, 3876–3932 (2000)
9. Zhang, D.Q.: Logarithmic del Pezzo surfaces of rank one with contractible boundaries. Osaka J. Math. **25**, 461–497 (1988)

Cylinders in Del Pezzo Surfaces of Degree Two

Grigory Belousov

Abstract We consider del Pezzo surfaces of degree two and ample divisors on these surfaces. For log pair (X, H), we define Fujita invariant of (X, H) by

$$\mu_H := \inf\{\lambda \in \mathbb{R}_{>0} \mid \text{the } \mathbb{R}\text{-divisor } K_X + \lambda H \text{ is pseudo-effective}\}.$$

The smallest extremal face Δ_H of the Mori cone that contains $K_X + \mu_H H$ is called the Fujita face of H. The Fujita rank of (X, H) is defined by $r_H := \dim \Delta_H$. Note that $r_H = 0$ if and only if $-K_X \equiv \mu_H H$. Let $f_H : X \to Z$ be the contraction given by the Fujita face Δ_H of the divisor H. Then either f_H is a birational morphism or a conic bundle with $Z \cong \mathbb{P}^1$. In the former case, the $\mathbb{R}$-divisor H is said to be of type $B(r_H)$ and in the latter case it is said to be of type $C(r_H)$. Assume that f_H is a birational morphism. Then f_H contracts r_H extremal rays. Suppose that f_H contracts r_H (-1)-curves. Then H is said to be of type smooth $B(r_H)$. We'll prove that a smooth general del Pezzo surface X of degree two has no H-polar cylinder, where H is an ample divisor of type $B(r_H)$ and $r_H = 2$. Also, we'll prove that a del Pezzo surface X of degree two with du Val singularities of types A_1 has an H-polar cylinder, where H is an ample divisor of type $B(r_H)$.

Keywords Cylinders · Del Pezzo surfaces

1 Introduction

A *log del Pezzo surface* is a projective algebraic surface X with only quotient singularities and ample anti-canonical divisor $-K_X$. In this paper we assume that X has only du Val singularities and we work over complex number field $\mathbb{C}$. Note that a del Pezzo surface with only du Val singularities is rational.

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Definition 1.1 (*see* [6]) Let M be an $\mathbb{R}$-divisor on a projective normal variety X. An *M-polar cylinder* in X is an open subset $U = X \setminus \operatorname{Supp}(D)$ defined by an effective $\mathbb{R}$-divisor D such that $D \equiv M$ and $U \cong Z \times \mathbb{A}^1$ for some affine variety Z.

Let H be an $\mathbb{R}$-divisor on a projective normal variety X. The existence of an H-polar cylinder in X is important due to the following fact.

Theorem 1.2 (see [7], Corollary 3.2) *Let Y be a normal algebraic variety over $\mathbb{C}$ projective over an affine variety S with $\dim_S Y \geq 1$. Let $H \in Div(Y)$ be an ample divisor on Y, and let $V = \operatorname{Spec} A(Y, H)$ be the associated affine quasicone over Y. Then V admits an effective G_a-action if and only if Y contains an H-polar cylinder.*

Also, the existence of an H-polar cylinder in X is important due the connection between the existence of a cylinder in a del Pezzo surface and tigers (see, [5]) on this surface (see [1, 3, 4]).

There exist a classification of smooth del Pezzo surfaces X such that X has a $-K_X$-polar cylinder (see [4, 6]). Also, there exist a classification of del Pezzo surfaces X with du Val singularities such that X has a $-K_X$-polar cylinder (see [3]).

Definition 1.3 (*see* [2]) For log pair (X, H), we define Fujita invariant of (X, H) by

$$\mu_H := \inf\{\lambda \in \mathbb{R}_{>0} \mid \text{the } \mathbb{R}\text{-divisor } K_X + \lambda H \text{ is pseudo-effective}\}.$$

The smallest extremal face Δ_H of the Mori cone that contains $K_X + \mu_H H$ is called the Fujita face of H. The Fujita rank of (X, H) is defined by $r_H := \dim \Delta_H$. Note that $r_H = 0$ if and only if $-K_X \equiv \mu_H H$.

Let $f_H : X \to Z$ be the contraction given by the Fujita face Δ_H of the divisor H. Then either f_H is a birational morphism or a conic bundle with $Z \cong \mathbb{P}^1$. In the former case, the $\mathbb{R}$-divisor H is said to be of type $B(r_H)$ and in the latter case it is said to be of type $C(r_H)$. Assume that f_H is a birational morphism. Then f_H contracts r_H extremal rays. Suppose that f_H contracts r_H (-1)-curves. Then H is said to be of type smooth $B(r_H)$. Let $E_1, E_2, \ldots, E_{r_H}$ be all the curves contracted by f_H. Then

$$K_X + \mu_H H \equiv \sum_{i=1}^{r_H} b_i E_i,$$

where $b_i > 0$. So, we may assume that

$$H = -K_X + \sum_{i=1}^{r_H} a_i E_i,$$

where $0 < a_i < 1$ if E_i is a (-1)-curve, $0 < a_i < l + 1$ if E_i contains a singular point of type A_l. Indeed, assume that E_i contains a singular point P of type A_l. Let $\pi : \bar{X} \to X$ be the minimal resolution of P, let $\bar{E}_i$ be the proper transform of

E_i. Then the exceptional divisor of π consists of (-2)-curves $D_1, D_2, \dots D_l$ with $D_i \cdot D_{i+1} = 1$ and $D_i \cdot D_j = 0$ when $j \neq i, i+1, i-1$. Moreover, $\bar{E}_i$ meets either D_1, either D_l and does not meet other component of the exceptional divisor. We may assume that $\bar{E}_i$ meets D_1. Then

$$\pi^* E_i = \bar{E}_i + \frac{l}{l+1} D_1 + \frac{l-1}{l+1} D_2 + \cdots + \frac{1}{l+1} D_l.$$

Hence, $E_i^2 = -\frac{1}{l+1}$. So, $0 < H \cdot E_i = 1 - a_i \frac{1}{l+1}$ if and only if $0 < a_i < l+1$.

The main result of Sect. 2 is the following.

Theorem 1.4 *Let X be a del Pezzo surface of degree two and let H be an ample divisor on X. Assume that X has singular points only of type A_1. Then X has an H-polar cylinder when*

- *X has only one singular point and H is of type smooth $B(r_H)$;*
- *X has only one singular point, H is of type $B(r_H)$ and $r_H \geq 2$;*
- *the number of singular points is at least two and H is of type $B(r_H)$;*
- *X has six singular points and $H \not\equiv -\mu K_X$.*

The main result of Sect. 3 is the following.

Theorem 1.5 *Let X be a smooth del Pezzo surface of degree two such that there do not exist two (-1)-curves that intersect tangentially. Let H be an ample divisor on X of type $B(r_H)$ and $r_H = 2$, i.e. $H \equiv -K_X + \lambda_1 E_1 + \lambda_2 E_2$. Put $\lambda = \lambda_1 + \lambda_2$. Assume that $\lambda < \frac{1}{7}$. Then X does not have an H-polar cylinder.*

Remark 1.6 In this paper we use the same symbols for curves on X and for the proper transforms of these curves.

Remark 1.7 In this paper, in every diagram, dotted line is a (-1)-curve, solid line is a $(-n)$-curve with $n \geq 2$. Every diagram gives all scheme of intersection among curves.

Note that the inequality $\lambda < \frac{1}{7}$ is essential, i.e. there exists an ample divisor H on X of type $B(r_H)$ and $r_H = 2$ such that X has an H-polar cylinder.

Example 1.8 Let X be a smooth del Pezzo surface, $\deg(X) = 2$. Put

$$H \equiv -K_X + \lambda_1 E_1 + \lambda_2 E_2,$$

where E_1, E_2 are (-1)-curves. Assume that $\lambda_1 > \frac{1}{2}$ and $\lambda_2 > \frac{1}{2}$. Note that there exists a (-1)-curve E and a smooth rational curve C such that C meets E in one point, say P and at this point $\mathrm{mult}_P(C \cdot E) = 4$. Moreover, $C^2 = 2$, $E_1 \cdot C = E_2 \cdot C = 1$ and $E_1 \cdot E = E_2 \cdot E = 0$. Note that there exist (-1)-curves $E_3, \dots, E_7$ such that $E_i \cdot E = 1$, $E_i \cdot C = E_i \cdot E_j = 0$ for every $i = 3, 4, 5, 6, 7$, $j = 1, 2$. Moreover, there exists a rational curve M such that M passes through intersection point of C and E and does not meet E_i for every $i = 1, \dots, 7$. We have

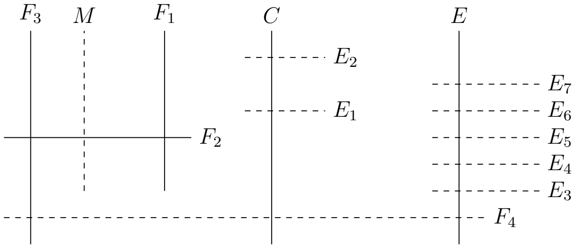


Fig. 1 Curves in the surface Y used in Example 1.8

$$H \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + (2c - 1)M + bE+$$
$$+(b - 1)(E_3 + E_4 + E_5 + E_6 + E_7),$$

where $a + b + c = 2$. Since $\lambda_1 > \frac{1}{2}$ and $\lambda_2 > \frac{1}{2}$, we see that there exist a, b, c such that L is an effective divisor. Let $g: Y \to X$ be sequence of blow ups with exceptional curves $F_1, \dots, F_4$ in this order. We have the following configuration on Y.

Note that there exists a $\mathbb{P}^1$-fibration $h: Y \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2 and C_3, where

$$C_1 = E + E_3 + E_4 + E_5 + E_6 + E_7, \quad C_2 = C + E_1 + E_2, \quad C_3 = 2M + 2F_2 + F_1 + F_3.$$

So,

$$X \setminus \mathrm{Supp}(L) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2, L_3 correspond to C_1, C_2, C_3 (Fig. 1).

The author is grateful to professor I. A. Cheltsov for suggesting me this problem and for his help.

2 Del Pezzo Surfaces with Singularities of Type A_1

Lemma 2.1 *Let X be a del Pezzo surface with du Val singularities,* $\deg(X) = 2$ *and let H be an ample divisor of type $B(r_H)$ and $r_H \geq 2$ or H is of smooth type $B(r_H)$ on X. Assume that X has a unique singular point of type A_1. Then X has an H-polar cylinder.*

Proof Let $\varphi: \tilde{X} \to X$ be the minimal resolution of singularity of X and let D be the exceptional divisor of φ. Note that there exists a divisor C such that C meets D in one point, say P and at this point $\mathrm{mult}_P(C \cdot D) = 2$. Moreover, C is either (0)-curve, either two (-1)-curves. Indeed, we see that $\tilde{X}$ can be obtained by blow ups $\mathbb{P}^2$ in seven points $P_1, P_2, \dots, P_7$, where P_1, P_2, P_3 lie on one line L. Note that D is the proper transform of L. Consider a linear system $|2T|$, where T is the class of a line. We have $\dim|2T| = 5$. So, there exists exactly one element $C' \in |2T|$ such that C' passes through P_4, P_5, P_6, P_7 and C' meets L in one point. Then C is the proper transform of C'.

2.1.1 Assume that $r_H = 1$ and $H = -K_X + \lambda E$, where E is a (-1)-curve, $1 > \lambda > 0$. Note that $C \cdot E = 1$. Moreover, there exist (-1)-curves $E_1, E_2, \ldots, E_6$ such that $E_i \cdot D = 1$, $E_i \cdot C = E_i \cdot E = 0$ for every i. Indeed, E_1, E_2, E_3 are the exceptional curves of blow ups of P_1, P_2, P_3, E_4, E_5, E_6 are the proper transforms of lines L_{74}, L_{75}, L_{76}, where line L_{ij} passes through P_i, P_j. Assume that C is a (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda)E + bD + (b - 1)(E_1 + E_2 + E_3 + E_4 + E_5 + E_6),$$

where $a + 2b = 3$. Since $\lambda > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 2).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = 2E + 2C + 2F_2 + F_1 + F_3, \quad C_2 = D + E_1 + E_2 + E_3 + E_4 + E_5 + E_6.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E meets R_1. We have

$$\varphi^*(H) \equiv L = aR_2 + (2a - 1)R_1 + (2a - 2 + \lambda)E + bD + (b - 1)(E_1 + E_2 + E_3 + E_4 + E_5),$$

where $a + b = 2$. Since $\lambda > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2, F_3 in this order. We have the following configuration on Y (Fig. 3).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = 2E + 2R_1 + 2F_1 + R_2 + F_2, \quad C_2 = D + E_1 + E_2 + E_3 + E_4 + E_5.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2 (Figs. 2 and 3).

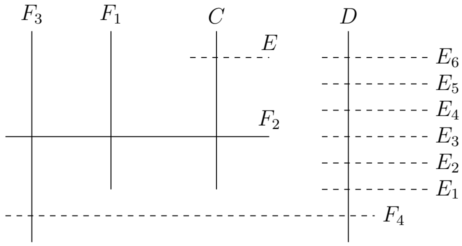


Fig. 2 Curves in the surface Y in case when C is a (0)-curve

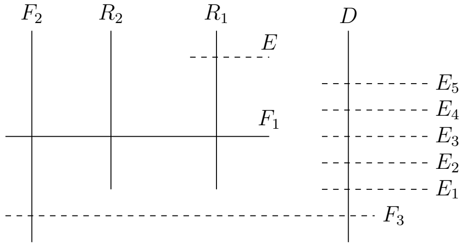


Fig. 3 Curves in the surface Y in case when C is reducible

2.1.2 Assume that $r_H = 2$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2$, where E_1, E_2 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 > 0$. Note that $C \cdot E_1 = C \cdot E_2 = 1$. Moreover, there exist (-1)-curves E_3, E_4, E_5, E_6 such that $E_i \cdot D = 1$, $E_i \cdot C = E_i \cdot E_j = 0$ for every $i = 3, 4, 5, 6$, $j = 1, 2$. Note that E_1, E_2, E_3, E_4, E_5 are the exceptional curves of blow ups of P_4, P_5, P_1, P_2, P_3 correspondingly, E_6 is the proper transform of line L_{76}, where line L_{ij} passes through P_i, P_j. Assume that C is a (0)-curve. Note that there exist (0)-curves M_1, M_2 such that M_1, M_2 pass through the intersection point of D and C, and $M_1 \cdot D = M_2 \cdot D = M_1 \cdot C = M_2 \cdot C = 1$, $M_j \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5, 6$, $j = 1, 2$. Note that M_1, M_2 are the proper transforms of lines L_6, L_7, where L_6 passes through P_6 and the intersection point of L and C', L_6 passes through P_7 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD+ \\ +(b - 1)(E_3 + E_4 + E_5 + E_6) + cM_1 + cM_2,$$

where $a + b + c = 2$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 4).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + E_1 + E_2, \quad C_2 = F_1 + M_1 + M_2, \quad C_3 = D + E_3 + E_4 + E_5 + E_6.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. Assume that E_1 and E_2 meet R_1. Note that there exist (-1)-curves M_1, M_2 such that M_1 and M_2 meet R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD + \\ +(b - 1)(E_3 + E_4 + E_5) + cR_2 + (c - 1)M_1 + (c - 1)M_2,$$

where $a + b + c = 3$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be the blow up of intersection point of C and D, F be the exceptional divisor of g. We have the following configuration on Y (Fig. 5).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = R_1 + E_1 + E_2, \quad C_2 = R_2 + M_1 + M_2, \quad C_3 = D + E_3 + E_4 + E_5.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Assume that E_1 meets R_1 and E_2 meets R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + aR_2 + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD + \\ +(b - 1)(E_3 + E_4 + E_5 + E_6),$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g_1\colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D, F_1 be the exceptional divisor of g_1. Let $g_2\colon Y_2 \to Y_1$ be the blow up of intersection point of D and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 6).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_3 \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = D + E_3 + E_4 + E_5 + E_6, \quad C_2 = F_1 + R_1 + R_2 + E_1 + E_2.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

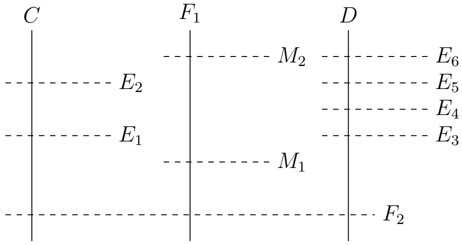


Fig. 4 Curves in the surface Y_2 used in 2.1.2 in case when C is a (0)-curve

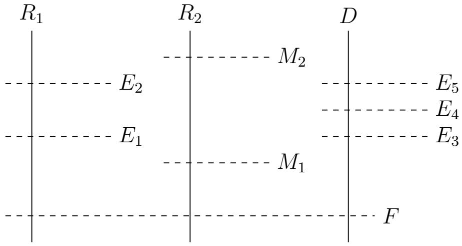


Fig. 5 Curves in the surface Y used in 2.1.2 in case when C is reducible

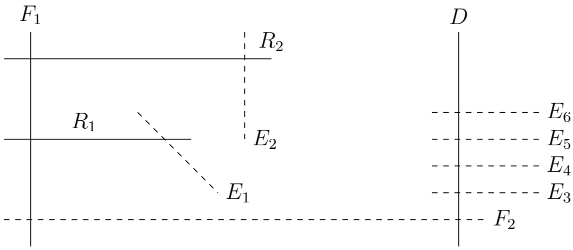


Fig. 6 Curves in the surface Y_2 used in 2.1.2 in case when C is reducible

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2 correspond to C_1, C_2 (Figs. 4, 5 and 6).

2.1.3 Assume that $r_H = 2$ and $H = -K_X + \lambda E + \lambda_1 \varphi(E_1)$, where E, E_1 are (-1)-curves such that E_1 meets D, $1 > \lambda > 0$, $2 > \lambda_1 > 0$. We have the same picture as in Case 2.1.1. So, X has an H-polar cylinder.

2.1.4 Assume that r_H=3 and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$, where E_1, E_2, E_3 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Note that $C \cdot E_1 = C \cdot E_2 = C \cdot E_3 = 1$. Moreover, there exist (-1)-curves E_4, E_5, E_6 such that $E_i \cdot D = 1$, $E_i \cdot C = E_i \cdot E_j = 0$ for every $i = 4, 5, 6$, $j = 1, 2, 3$. Note that $E_1, E_2, E_3, E_4, E_5, E_6$ are the exceptional curves of blow ups of $P_4, P_5, P_6, P_1, P_2, P_3$ correspondingly. Assume that C is a (0)-curve. Note that there exists a (0)-curve M such that M passes through the intersection point of D and C, and $M \cdot D = M \cdot C = 1$, $M \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5, 6$. Note that M is the proper transform of line L_7 that passes through P_7 and the intersection point of L and C'. We have

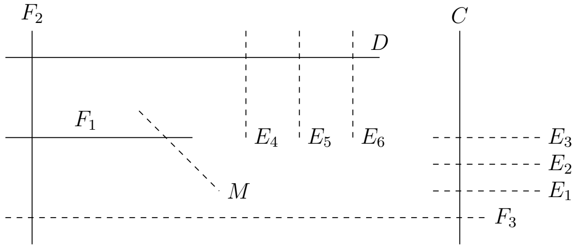


Fig. 7 Curves in the surface Y_3 used in 2.1.4

$$\varphi^*(H) \equiv L = aC + (a-1+\lambda_1)E_1 + (a-1+\lambda_2)E_2 + (a-1+\lambda_3)E_3 + bD+ \\ +(b-1)(E_4+E_5+E_6+M),$$

where $a+b=2$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g_1\colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D, F_1 be the exceptional divisor of g_1. Let $g_2\colon Y_2 \to Y_1$ be the blow up of intersection point of C, D and F_1, F_2 be the exceptional divisor of g_2. Let $g_3\colon Y_3 \to Y_2$ be the blow up of intersection point of C and F_2, F_3 be the exceptional divisor of g_3. We have the following configuration on Y_3 (Fig. 7).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_3 \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + E_3, \quad C_2 = M + F_1 + F_2 + D + E_4 + E_5 + E_6.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_1, E_2 meet R_1 and E_3 meets R_2. Then, we have the same picture as in Case 2.1.2 (see Fig. 5). So, X has an H-polar cylinder.

2.1.5 Assume that $r_H = 3$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 \varphi(E_3)$, where E_1, E_2, E_3 are (-1)-curves such that E_3 meets D, $1 > \lambda_1 \geq \lambda_2 > 0$, $2 > \lambda_3 > 0$. We have the same picture as in Case 2.1.2. So, X has an H-polar cylinder.

2.1.6 Assume that $r_H = 4$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4$, where E_1, E_2, E_3, E_4 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$. Note that $C \cdot E_i = 1$ for every $i = 1, 2, 3, 4$. Moreover, there exist (-1)-curves E_5, E_6, E_7 such that $E_i \cdot D = 1$, $E_i \cdot C = E_i \cdot E_j = 0$ for every $i = 5, 6, 7$, $j = 1, 2, 3, 4$. Note that E_1, E_2, E_3, E_4, E_5, E_6, E_7 are the exceptional curves of blow ups of P_4, P_5, P_6, P_7, P_1, P_2, P_3 correspondingly. Assume that C is a (0)-curve. We have

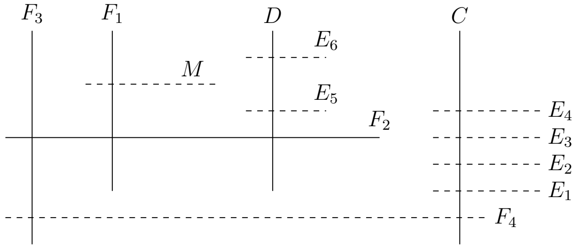


Fig. 8 Curves in the surface Y used in 2.1.6

$$\varphi^*(H) \equiv L = aC + (a-1+\lambda_1)E_1 + (a-1+\lambda_2)E_2 + (a-1+\lambda_3)E_3 + \\ +(a-1+\lambda_4)E_4 + (2b-1)D + (2b-2)(E_5+E_6+E_7),$$

where $a+b=2$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \dots, F_4$ in this order. We have the following configuration on Y (Fig. 8).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + E_3 + E_4, \quad C_2 = F_3 + F_1 + 2F_2 + 2D + 2E_5 + 2E_6 + 2E_7.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We see that E_1, E_2 meet R_1 and E_3, E_4 meet R_2. Then, we have the same picture as in Case 2.1.2 (see Fig. 5). So, X has an H-polar cylinder.

2.1.7 Assume that $r_H = 4$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 \varphi(E_4),$$

where E_1, E_2, E_3, E_4 are (-1)-curves such that E_4 meets D,

$$1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0,$$

$2 > \lambda_4 > 0$. We have the same picture as in Case 2.1.4. So, X has an H-polar cylinder.

2.1.8 Assume that $r_H = 5$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 + \lambda_5 E_5,$$

where E_1, E_2, E_3, E_4, E_5 are (-1)-curves, $1 > \lambda_i > 0$ for every i. Note that $C \cdot E_i = 1$ for every $i = 1, 2, 3, 4, 5$. Moreover, there exists a (-1)-curve E_6 such that $E_6 \cdot D = 1$, $E_6 \cdot C = E_6 \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5$. Note that E_6 is the exceptional curve of blow up of P_1, E_1, E_2 are the proper transforms of lines L_{24}, L_{34}, where line L_{ij} passes through P_i, P_j, E_3, E_4, E_5 are the proper transforms of conics C_{56}, C_{57}, C_{67}, where conic C_{ij} passes through P_2, P_3, P_4, P_i, P_j. Assume that C is a (0)-curve. Note that there exists a (0)-curve M such that M passes through the intersection point of D and C, and $M \cdot D = M \cdot C = 1$, $M \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5, 6$. Note that M is the proper transform of line that passes through P_4 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a-1+\lambda_1)E_1 + (a-1+\lambda_2)E_2 + (a-1+\lambda_3)E_3 + \\ +(a-1+\lambda_4)E_4 + (a-1+\lambda_5)E_5 + bD + (b-1)E_6 + (3b-3)M,$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g \colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_5$ in this order. We have the following configuration on Y (Fig. 9).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + E_3 + E_4 + E_5, \quad C_2 = 3M + 3F_1 + 3F_2 + 2F_3 + F_4 + D + E_6.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_5, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We see that E_1, E_2, E_3, E_4 meet R_1 and E_5 meets R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + (a-1+\lambda_1)E_1 + (a-1+\lambda_2)E_2 + (a-1+\lambda_3)E_3 + \\ +(a-1+\lambda_4)E_4 + (2b-1)R_2 + (2b-2+\lambda_5)E_5 + bD + (b-1)E_6,$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g \colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2, F_3 in this order. We have the following configuration on Y (Fig. 10).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = R_1 + E_1 + E_2 + E_3 + E_4, \quad C_2 = 2E_5 + 2R_2 + 2F_1 + F_2 + D + E_6.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

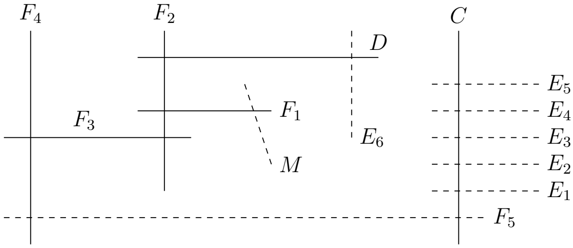


Fig. 9 Curves in the surface Y used in 2.1.8 in case when C is a (0)-curve

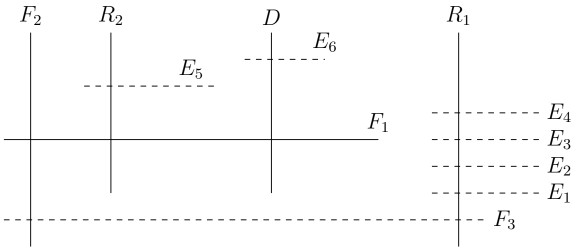


Fig. 10 Curves in the surface Y used in 2.1.8 in case when C is reducible

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2 (Figs. 9 and 10).

2.1.9 Assume that $r_H = 5$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 + \lambda_5 \varphi(E_5),$$

where E_1, E_2, E_3, E_4, E_5 are (-1)-curves such that E_5 meets D,

$$1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0,$$

$2 > \lambda_5 > 0$. We have the same picture as in Case 2.1.6. So, X has an H-polar cylinder.

2.1.10 Assume that $r_H = 6$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 + \lambda_5 E_5 + \lambda_6 E_6,$$

where E_1, E_2, E_3, E_4, E_5, E_6 are (-1)-curves,

$$1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq \lambda_6 > 0.$$

Put $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6$. Let $f_H \colon X \to Z$ be the contraction given by the Fujita face Δ_H of the divisor H. We see that Z is a cone. Then $\tilde{X}$ can be obtained by blow ups $\mathbb{F}_2$ in six points $P_1, P_2, \ldots, P_6$ and $E_1, E_2, \ldots, E_6$ are the exceptional curves. So, there exists a $\mathbb{P}^1$-fibration $\psi \colon \tilde{X} \to \mathbb{P}^1$ such that D is a section and every E_i is contained in singular fiber. So, there exist (-1)-curves E_1', E_2', E_3', E_4', E_5', E_6' such that $E_i + E_i'$ are singular fibers and $E_i' \cdot D = 1$, $E_i' \cdot C = 0$ for $i = 1, 2, 3, 4, 5, 6$. Assume that C is a (0)-curve. Note that there exists a fiber M such that M passes through the intersection points of D and C. So, $\lambda_i E_i \equiv -\lambda_i E_i' + \lambda_i M$, $i = 1, 2, 3, 4, 5, 6$. Assume that $\lambda > 2\lambda_1$. We have

$$\varphi^*(H) \equiv L = aC + (2a - 2 + \lambda)M + bD + (b - 1 - \lambda_1)E_1' + \\ +(b - 1 - \lambda_2)E_2' + (b - 1 - \lambda_3)E_3' + (b - 1 - \lambda_4)E_4' + (b - 1 - \lambda_5)E_5' + (b - 1 - \lambda_6)E_6,$$

where $a + b = 2$. Since $\lambda > 2\lambda_1$ and $\lambda_1 \geq \lambda_i$ for every $i = 2, 3, 4, 5, 6$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 11).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = D + E_1' + E_2' + E_3' + E_4' + E_5' + E_6', \quad C_2 = 2M + 2F_1 + 2F_2 + F_3 + C.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. So we may assume that $\lambda \leq 2\lambda_1$. Then $\lambda_1 > \lambda_2$. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda - \lambda_1)M + (a - 1 + \lambda_1)E_1 + bD + (b - 1 - \lambda_2)E_2' + \\ +(b - 1 - \lambda_3)E_3' + (b - 1 - \lambda_4)E_4' + (b - 1 - \lambda_5)E_5' + (b - 1 - \lambda_6)E_6',$$

where $a + b = 2$. Since $\lambda_1 > \lambda_2$ and $\lambda - \lambda_1 > \lambda_2$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2, F_3 in this order. We have the following configuration on Y (Fig. 12).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = D + E_2' + E_3' + E_4' + E_5' + E_6', \quad C_2 = M + F_1 + F_2 + C + E_1.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We see that either R_1 either R_2 is contained in a fiber of ψ. We may assume that R_2 is contained in a fiber of ψ. We may assume that E_1, E_2, E_3, E_4, E_5 meet R_1 and $E_6 + R_2$ is a fiber of ψ. We may assume that

$$1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 > 0.$$

As above, there exist (-1)-curves E_1', E_2', E_3', E_4', E_5' such that $E_i + E_i'$ are singular fibers and $E_i' \cdot D = 1$, $E_i' \cdot C = 0$ for $i = 1, 2, 3, 4, 5$. So, $\lambda_i E_i \equiv -\lambda_i E_i' + \lambda_i(E_6 + R_2)$. Assume that $\lambda_1 > \lambda_2$. We have

$$\varphi^*(H) \equiv L = aR_1 + (a + \lambda - \lambda_6)R_2 + (a - 1 + \lambda)E_6 + (a - 1 + \lambda_1)E_1 + bD+ \\ +(b - 1 - \lambda_2)E_2' + (b - 1 - \lambda_3)E_3' + (b - 1 - \lambda_4)E_4' + (b - 1 - \lambda_5)E_5',$$

where $a + b = 2$. Since $\lambda_1 > \lambda_2$ and $\lambda - \lambda_1 > \lambda_2$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2 in this order. We have the following configuration on Y (Fig. 13).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = D + E_2' + E_3' + E_4' + E_5', \quad C_2 = F_1 + R_1 + R_2 + E_1 + E_6.$$

So,

$$\tilde{X} \backslash \operatorname{Supp}(L) \cong X \backslash \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \backslash (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Assume that $\lambda_1 = \lambda_2$. We have

$$\varphi^*(H) \equiv L = aR_1 + (2a - 1 + \lambda - \lambda_6)R_2 + (2a - 2 + \lambda)E_6 + bD + (b - 1 - \lambda_1)E_1'+ \\ +(b - 1 - \lambda_2)E_2' + (b - 1 - \lambda_3)E_3' + (b - 1 - \lambda_4)E_4' + (b - 1 - \lambda_5)E_5',$$

where $a + b = 2$. Since $\lambda - \lambda_6 > 2\lambda_1$ and $\lambda_1 \geq \lambda_i$ for every $i = 2, 3, 4, 5$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2, F_3 in this order. We have the following configuration on Y (Fig. 14).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = D + E_1' + E_2' + E_3' + E_4' + E_5', \quad C_2 = 2E_6 + 2R_2 + 2F_1 + F_2 + R_1.$$

So,

$$\tilde{X} \backslash \operatorname{Supp}(L) \cong X \backslash \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \backslash (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2 (Figs. 11, 12, 13 and 14).

Fig. 11 Curves in the surface Y used in 2.1.10 in case when C is a (0)-curve and $\lambda > 2\lambda_1$

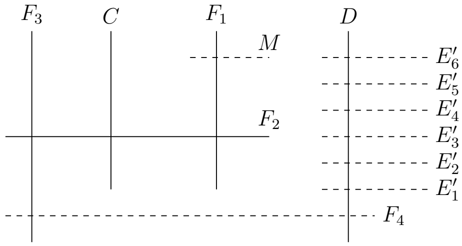


Fig. 12 Curves in the surface Y used in 2.1.10 in case when C is a (0)-curve and $\lambda \leq 2\lambda_1$

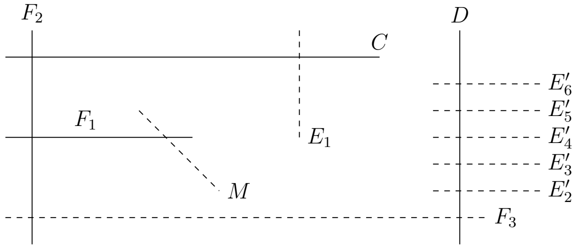


Fig. 13 Curves in the surface Y used in 2.1.10 in case when C is reducible and $\lambda_1 > \lambda_2$

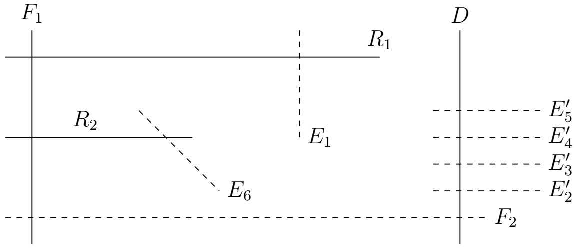
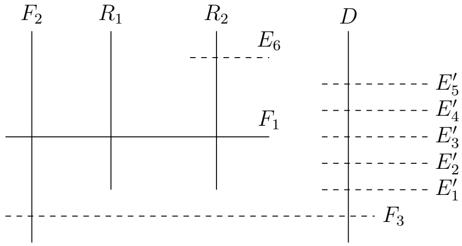


Fig. 14 Curves in the surface Y used in 2.1.10 in case when C is reducible and $\lambda_1 = \lambda_2$



2.1.11 Assume that $r_H = 6$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 + \lambda_5 E_5 + \lambda_6 \varphi(E_6),$$

where E_1, E_2, E_3, E_4, E_5, E_6 are (-1)-curves such that E_6 meets D,

$$1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 > 0,$$

$2 > \lambda_6 > 0$. We have the same picture as in Case 2.1.8. So, X has an H-polar cylinder.

□

Lemma 2.2 *Let X be a del Pezzo surface with du Val singularities,* $\deg(X) = 2$ *and let H be an ample divisor of type $B(r_H)$ on X. Assume that X has two singular points of type A_1. Then X has an H-polar cylinder.*

Proof Let $\varphi\colon \tilde{X} \to X$ be the minimal resolution of singularities of X and let $D = D_1 + D_2$ be the exceptional divisor of φ. Note that there exists a divisor C such that $C \cdot D_1 = 2$ and C meets D_1 in one point. Moreover, $D_2 \cdot C = 0$. Note that C is either (0)-curve, either two (-1)-curves. Indeed, we see that $\tilde{X}$ can be obtained by blow ups $\mathbb{P}^2$ in seven points $P_1, P_2, \ldots, P_7$, where P_1, P_2, P_3 lie on one line L and P_3, P_4, P_5 lie on one line L', i.e. P_3 is the intersection point of L and L'. Note that D_1 is the proper transform of L, D_2 is the proper transform of L'. Consider a linear system $|2T|$, where T is the class of a line. We have $\dim |2T| = 5$. So, there exists exactly one element $C' \in |2T|$ such that C' passes through P_4, P_5, P_6, P_7 and C' meets L in one point. Then C is the proper transform of C'.

2.2.1 Assume that $r_H = 1$ and $H = -K_X + \lambda E$, where E is a (-1)-curve, $1 > \lambda > 0$. Note that $C \cdot E = 1$. Moreover, there exist (-1)-curves E_1, E_2, E_3, E_4 such that $E_i \cdot D_1 = 1$, $E_i \cdot D_2 = 0$, $E_i \cdot C = E_i \cdot E = 0$ for every $i = 1, 2, 3, 4$, there exists a (-1)-curve E_5 such that $E_5 \cdot D_1 = E_5 \cdot D_2 = 1$, $E_i \cdot E_5 = E_5 \cdot C = E_5 \cdot E = 0$ for every $i = 1, 2, 3, 4$. Note that E, E_1, E_2, E_5 are the exceptional curves of blow ups of P_6, P_1, P_2, P_3 correspondingly, E_3, E_4 are the proper transforms of lines L_{47}, L_{57}, where line L_{ij} passes through P_i, P_j. Assume that C is a (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda)E + bD_1 + (b - 1)(E_1 + E_2 + E_3 + E_4) + (b - 1)D_2 + \\ + (2b - 2)E_5,$$

where $a + 2b = 3$. Since $\lambda > 0$, we see that there exist a, b such that L is an effective divisor. Let $g : Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 15).

Note that there exists a $\mathbb{P}^1$-fibration $h : Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = 2E + 2C + 2F_2 + F_1 + F_3, \quad C_2 = D_1 + E_1 + E_2 + E_3 + E_4 + 2E_5 + D_2.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We have

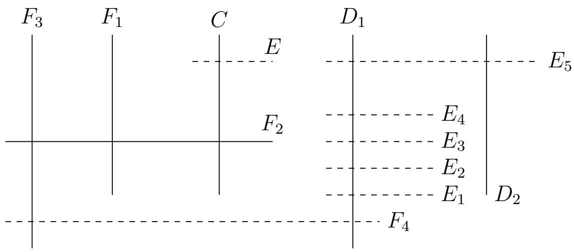


Fig. 15 Curves in the surface Y used in 2.2.1 in case when C is a (0)-curve

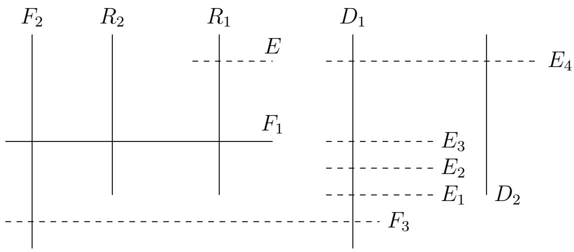


Fig. 16 Curves in the surface Y used in 2.2.1 in case when C is reducible

$$\varphi^*(H) \equiv L = aR_2 + (2a-1)R_1 + (2a-2+\lambda)E + bD_1 + (b-1)(E_1+E_2+E_3) + \\ +(b-1)D_2 + (2b-2)E_4,$$

where $a + b = 2$. Since $\lambda > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2, F_3 in this order. We have the following configuration on Y (Fig. 16).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = 2E + 2R_1 + 2F_1 + R_2 + F_2, \quad C_2 = D_1 + E_1 + E_2 + E_3 + 2E_4 + D_2.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2 (Figs. 15 and 16).

2.2.2 Assume that $r_H = 1$ and $H = -K_X + \lambda\varphi(E)$, where E is a (-1)-curve such that E meets D_2, $2 > \lambda > 0$. Note that $C \cdot E = 1$. Moreover, there exist (-1)-curves E_1, E_2, E_3, E_4 such that

$$E_i \cdot D_1 = 1, \quad E_i \cdot C = E_i \cdot D_2 = 0.$$

Note that E, E_1, E_2 are the exceptional curves of blow ups of P_4, P_1, P_2 correspondingly, E_3, E_4 are the proper transforms of lines L_{56}, L_{57}, where line L_{ij} passes through P_i, P_j. Assume that C is a (0)-curve. Note that there exist (0)-curves M_1, M_2 such that M_1, M_2 pass through the intersection point of D_1 and C, and

$$M_1 \cdot D_1 = M_2 \cdot D_1 = M_1 \cdot C = M_2 \cdot C = 1, \quad M_1 \cdot D_2 = M_2 \cdot D_2 = 0, \quad M_j \cdot E_i = 0$$

for every $i = 1, 2, 3, 4$, $j = 1, 2$. We see that M_1, M_2 are the proper transforms of lines that passes through P_6, P_7 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + \left(a - 1 + \frac{\lambda}{2}\right) D_2 + (2a - 2 + \lambda)E + bD_1 + \\ + (b-1)(E_1 + E_2 + E_3 + E_4) + cM_1 + cM_2,$$

where $a + b + c = 2$. Since $\lambda > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 17).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + 2E + D_2, \quad C_2 = F_1 + M_1 + M_2, \quad C_3 = D_1 + E_1 + E_2 + E_3 + E_4.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E meets R_1. Note that there exist (-1)-curves M_1, M_2 such that M_1 and M_2 meet R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + \left(a - 1 + \frac{\lambda}{2}\right) D_2 + (2a - 2 + \lambda)E + bD_1 + \\ + (b-1)(E_1 + E_2 + E_3) + cR_2 + (c-1)M_1 + (c-1)M_2,$$

where $a + b + c = 3$. Since $\lambda > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g \colon Y \to \tilde{X}$ be the blow up of intersection point of C and D_1, F be the exceptional divisor of g. We have the following configuration on Y (Fig. 18).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = R_1 + 2E + D_2, \quad C_2 = R_2 + M_1 + M_2, \quad C_3 = D_1 + E_1 + E_2 + E_3.$$

Fig. 17 Curves in the surface Y_2 used in 2.2.2 in case when C is a (0)-curve

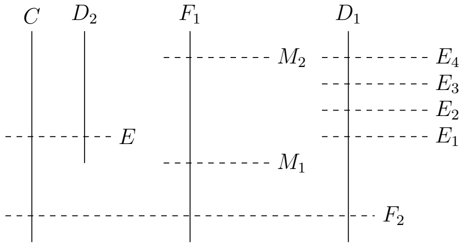
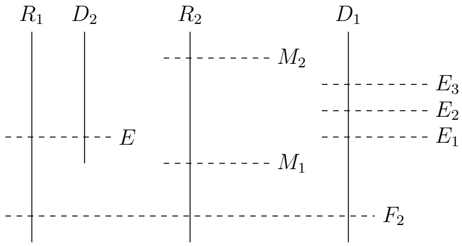


Fig. 18 Curves in the surface Y used in 2.2.2 in case when C is reducible



So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F, L_1, L_2, L_3 correspond to C_1, C_2, C_3 (Figs. 17 and 18).

2.2.3 Assume that $r_H = 2$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2$, where E_1, E_2 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 > 0$. Note that $C \cdot E_1 = C \cdot E_2 = 1$. Moreover, there exist (-1)-curves E_3, E_4 such that $E_i \cdot D_1 = 1$, $E_i \cdot D_2 = 0$, $E_i \cdot C = E_i \cdot E_j = 0$ for every $i = 3, 4$, $j = 1, 2$. Note that there exists a (-1)-curve E_5 such that $E_5 \cdot D_1 = E_5 \cdot D_2 = 1$, $E_i \cdot E_5 = E_5 \cdot C = 0$ for every $i = 1, 2, 3, 4$. Note that E_1, E_2, E_3, E_4, E_5 are the exceptional curves of blow ups of P_6, P_7, P_1, P_2, P_3 correspondingly. Assume that C is a (0)-curve. Note that there exist (0)-curves M_1, M_2 such that M_1, M_2 pass through the intersection point of D_1 and C, and

$$M_1 \cdot D_1 = M_2 \cdot D_1 = M_1 \cdot C = M_2 \cdot C = 1, \quad M_1 \cdot D_2 = M_2 \cdot D_2 = 0, \quad M_j \cdot E_i = 0$$

for every $i = 1, 2, 3, 4, 5$, $j = 1, 2$. We see that M_1, M_2 are the proper transforms of lines that passes through P_4, P_5 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD_1 + (b - 1)(E_3 + E_4) + \\ + (b - 1)D_2 + (2b - 2)E_5 + cM_1 + cM_2,$$

where $a + b + c = 2$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of

C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 19).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + E_1 + E_2, \quad C_2 = F_1 + M_1 + M_2, \quad C_3 = D_1 + E_3 + E_4 + 2E_5 + D_2.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. Assume that E_1 and E_2 meet R_1. Note that there exist (-1)-curves M_1, M_2 such that M_1 and M_2 meet R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD_1 + (b - 1)E_3 + \\ + (b - 1)D_2 + (2b - 2)E_4 + cR_2 + (c - 1)(M_1 + M_2),$$

where $a + b + c = 3$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g \colon Y \to \tilde{X}$ be the blow up of intersection point of C and D_1, F be the exceptional divisor of g. We have the following configuration on Y (Fig. 20).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = R_1 + E_1 + E_2, \quad C_2 = R_2 + M_1 + M_2, \quad C_3 = D_1 + E_3 + 2E_4 + D_2.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Assume that E_1 meets R_1 and E_2 meets R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + aR_2 + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD + \\ + (b - 1)(E_3 + E_4) + (b - 1)D_2 + (2b - 2)E_5,$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of D and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 21).

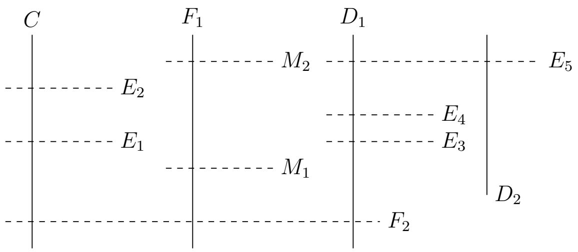


Fig. 19 Curves in the surface Y_2 used in 2.2.3 in case when C is a (0)-curve

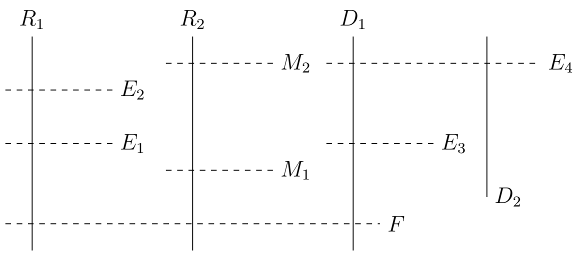


Fig. 20 Curves in the surface Y used in 2.2.3 in case when C is reducible

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_3 \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = D_1 + E_3 + E_4 + 2E_5 + D_2, \quad C_2 = F_1 + R_1 + R_2 + E_1 + E_2.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2 correspond to C_1, C_2, C_3 (Figs. 19, 20 and 21).

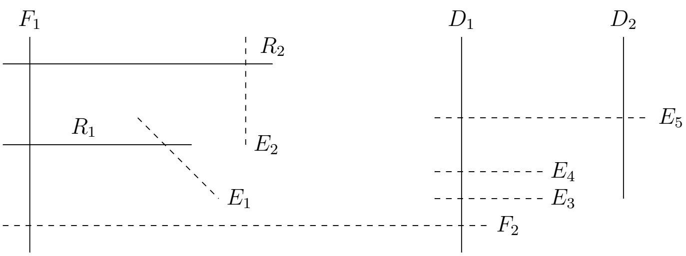


Fig. 21 Curves in the surface Y_2 used in 2.2.3 in case when C is reducible

2.2.4 Assume that $r_H = 2$ and $H = -K_X + \lambda E + \lambda_1\varphi(E_1)$, where E, E_1 are (-1)-curves such that E_1 meets D_1, $1 > \lambda > 0$, $2 > \lambda_1 > 0$. We have the same picture as in Case 2.2.1. So, X has an H-polar cylinder.

2.2.5 Assume that $r_H = 2$ and $H = -K_X + \lambda_1\varphi(E_1) + \lambda\varphi(E)$, where E_1, E are (-1)-curves such that E_1 meets D_1, E meets D_2, $2 > \lambda_1 \geq \lambda > 0$. We have the same picture as in Case 2.2.2. So, X has an H-polar cylinder.

2.2.6 Assume that r_H=3 and H=$-K_X$+$\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$, where E_1, E_2, E_3 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Note that $C \cdot E_1 = C \cdot E_2 = C \cdot E_3 = 1$. Moreover, there exists a (-1)-curves E_4 such that $E_4 \cdot D_1 = 1$, $E_4 \cdot C = E_4 \cdot D_2 = E_4 \cdot E_j = 0$ for every $j = 1, 2, 3$. Note that there exists a (-1)-curve E_5 such that $E_5 \cdot D_1 = E_5 \cdot D_2 = 1$, $E_5 \cdot C = E_5 \cdot E_j = 0$ for every $j = 1, 2, 3$, there exists a (0)-curve M such that M passes through the intersection point of D_1 and C, and $M \cdot D_1 = M \cdot C = 1$, $M \cdot E_i = M \cdot D_2 = 0$ for every $i = 1, 2, 3, 4, 5$. Note that E_1, E_2, E_4, E_5 are the exceptional curves of blow ups of P_6, P_7, P_1, P_3 correspondingly, E_3 is the proper transform of line that passes through P_2 and P_4, M is the proper transform of conic that passes through P_5 and the intersection point of L and C'. Assume that C is a (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (a-1+\lambda_1)E_1 + (a-1+\lambda_2)E_2 + (a-1+\lambda_3)E_3 + bD+ \\ +(b-1)(E_4 + D_2 + M) + (2b-2)E_5,$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g_1\colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2\colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. Let $g_3\colon Y_3 \to Y_2$ be the blow up of intersection point of C and F_2, F_3 be the exceptional divisor of g_3. We have the following configuration on Y_3 (Fig. 22).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_3 \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + E_3, \quad C_2 = M + F_1 + F_2 + D_1 + E_4 + 2E_5 + D_2.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_1, E_2 meet R_1, E_3 meets R_2. Then, we have the same picture as in Case 2.2.3 (see Fig. 20). So, X has an H-polar cylinder.

2.2.7 Assume that $r_H = 3$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3\varphi(E_3)$, where E_1, E_2, E_3 are (-1)-curves such that E_3 meets D_1, $2 > \lambda_3 > 0$, $1 > \lambda_1 \geq \lambda_2 > 0$. We have the same picture as in Case 2.2.3. So, X has an H-polar cylinder.

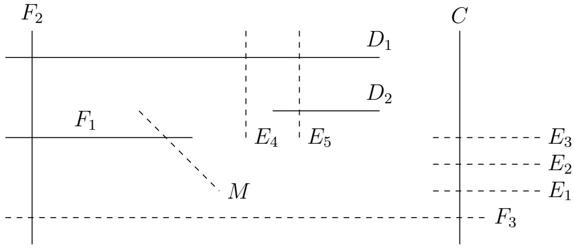


Fig. 22 Curves in the surface Y_3 used in 2.2.6

2.2.8 Assume that $r_H = 3$ and $H = -K_X + \lambda_1 E_1 + \lambda_2\varphi(E_2) + \lambda_3\varphi(E_3)$, where E_1, E_2, E_3 are (-1)-curves such that E_3 meets D_1, E_2 meets D_2, $1 > \lambda_1 > 0$, $2 > \lambda_2 \geq \lambda_3 > 0$. Note that $C \cdot E_1 = C \cdot E_2 = 1$, $C \cdot E_3 = 0$. Moreover, there exist (-1)-curves E_4, E_5 such that $E_i \cdot D_1 = 1$, $E_i \cdot D_2 = 0$, $E_i \cdot C = E_i \cdot E_1 = E_i \cdot E_2 = 0$ for every $i = 3, 4, 5$. Note that E_1, E_2, E_3, E_4 are the exceptional curves of blow ups of P_6, P_4, P_1, P_2 correspondingly, E_5 is the proper transform of line that passes through P_5 and P_7. Assume that C is a (0)-curve. Note that there exists a (0)-curve M such that M passes through the intersection point of D_1 and C, and $M \cdot D_1 = M \cdot C = 1$, $M \cdot D_2 = 0$, $M \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5$. We see that M is the proper transform of line that passes through P_7 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + \left(a - 1 + \frac{\lambda_2}{2}\right) D_2 + (2a - 2 + \lambda_2)E_2 + bD+$$
$$+(b - 1)(E_3 + E_4 + E_5 + M),$$

where $a + b = 2$. Since $\lambda_1 > 0$ and $\lambda_2 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g_1\colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2\colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. Let $g_3\colon Y_3 \to Y_2$ be the blow up of intersection point of C and F_2, F_3 be the exceptional divisor of g_3. We have the following configuration on Y_3 (Fig. 23).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_3 \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + D_2 + 2E_2, \quad C_2 = M + F_1 + F_2 + D + E_3 + E_4 + E_5.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_2 meets R_1, E_3 meets R_2. Then, we have the same picture as in Case 2.2.2 (see Fig. 18). So, X has an H-polar cylinder.

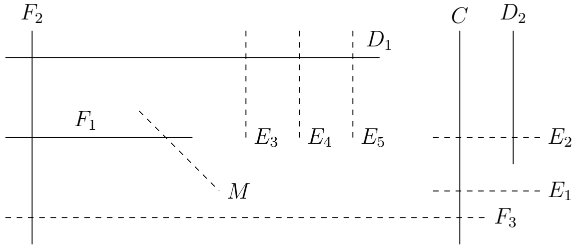


Fig. 23 Curves in the surface Y_3 used in 2.2.8

2.2.9 Assume that $r_H = 4$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4$, where E_1, E_2, E_3, E_4 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$. Note that $C \cdot E_i = 1$ for every $i = 1, 2, 3, 4$. Moreover, there exists a (-1)-curve E_5 such that $E_5 \cdot D_1 = E_5 \cdot D_2 = 1$, $E_5 \cdot C = E_5 \cdot E_j = 0$ for every $j = 1, 2, 3, 4$, there exists a (-1)-curve E_6 such that $E_6 \cdot D_1 = 1$ and $E_6 \cdot D_2 = E_6 \cdot C = E_6 \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5$. Note that E_1, E_2, E_5, E_6 are the exceptional curves of blow ups of P_6, P_7, P_3, P_1 correspondingly, E_3, E_4 are the proper transforms of lines L_{24}, L_{25}, where line L_{ij} passes through P_i, P_j. Assume that C is a (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + (a - 1 + \lambda_3)E_3 + \\ +(a - 1 + \lambda_4)E_4 + (2b - 1)D_1 + (2b - 2)D_2 + (4b - 4)E_5 + (2b - 2)E_6,$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g \colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 24).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + E_3 + E_4, \quad C_2 = 4E_5 + 2D_1 + 2D_2 + 2E_6 + 2F_2 + F_3 + F_1.$$

So,

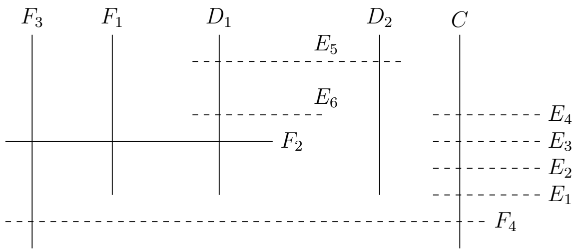
$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_1, E_2 meet R_1, E_3, E_4 meet R_2. Then, we have the same picture as in Case 2.2.3 (see Fig. 20). So, X has an H-polar cylinder.

2.2.10 Assume that $r_H = 4$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 \varphi(E_4)$, where E_1, E_2, E_3, E_4 are (-1)-curves such that E_4 meets D_1, $2 > \lambda_4 > 0$, $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. We have the same picture as in Case 2.2.6. So, X has an H-polar cylinder.

Fig. 24 Curves in the surface Y used in 2.2.9



2.2.11 Assume that $r_H = 4$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 \varphi(E_3) + \lambda_4 \varphi(E_4),$$

where E_1, E_2, E_3, E_4 are (-1)-curves such that E_4 meets D_1, E_3 meets D_2, $2 > \lambda_3 \geq \lambda_4 > 0$, $1 > \lambda_1 \geq \lambda_2 > 0$. Note that $C \cdot E_4 = 0$, $C \cdot E_i = 1$ for every $i = 1, 2, 3$. Moreover, there exists a (-1)-curve E_5 such that $E_5 \cdot D_1 = 1$, $E_5 \cdot C = E_5 \cdot E_j = 0$ for every $j = 1, 2, 3, 4$. Note that E_1, E_2, E_3, E_4, E_5 are the exceptional curves of blow ups P_6, P_7, P_4, P_1, P_2 correspondingly. Assume that C is a (0)-curve. Note that there exists a (0)-curve M such that M passes through the intersection point of D_1 and C, and $M \cdot D_1 = M \cdot C = 1$, $M \cdot D_2 = M \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5, 6$. We see that M is the proper transform of line that passes through P_5 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + (2a - 2 + \lambda_3)E_3 + \\ + \left(a - 1 + \frac{\lambda_3}{2}\right) D_2 + \left(b + \frac{\lambda_4}{2}\right) D_1 + (b - 1 + \lambda_4)E_4 + (b - 1)E_5 + (2b - 2)M,$$

where $a + b = 2$. Since $2 > \lambda_3 \geq \lambda_4 > 0$ and $1 > \lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g : Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 25).

Note that there exists a $\mathbb{P}^1$-fibration $h : Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + 2E_3 + D_2, \quad C_2 = 2M + 2F_1 + 2F_2 + D_1 + E_4 + E_5 + F_3.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

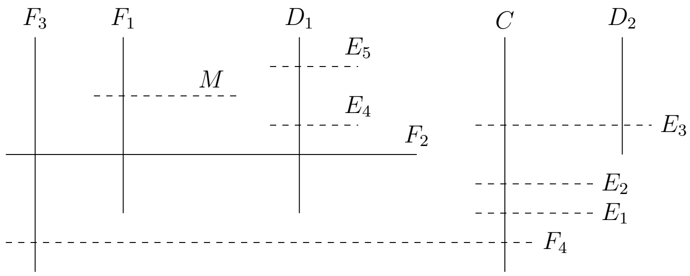


Fig. 25 Curves in the surface Y used in 2.2.11

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_3 meets R_1, E_1, E_2 meet R_2. Then, we have the same picture as in Case 2.2.2 (see Fig. 18). So, X has an H-polar cylinder.

2.2.12 Assume that $r_H = 5$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 + \lambda_5 \varphi(E_5),$$

where E_1, E_2, E_3, E_4, E_5 are (-1)-curves such that E_5 meets D_1. We have the same picture as in Case 2.2.9. So, X has an H-polar cylinder.

2.2.13 Assume that $r_H = 5$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 \varphi(E_4) + \lambda_5 \varphi(E_5),$$

where E_1, E_2, E_3, E_4, E_5 are (-1)-curves such that E_4 meets D_2, E_5 meets D_1,

$$2 > \lambda_4 \geq \lambda_5 > 0, \quad 1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0.$$

Note that $C \cdot E_i = 1$ for every $i = 1, 2, 3, 4$, $C \cdot E_5 = 0$. Note that E_1, E_2, E_4, E_5 are the exceptional curves of blow ups of P_6, P_7, P_4, P_1 correspondingly, E_3 is the proper transform of line that passes through P_2 and P_5. Assume that C is a (0)-curve. Note that there exists a (0)-curve M such that M passes through the intersection point of D_1 and C, and $M \cdot D_1 = M \cdot C = 1$, $M \cdot D_2 = M \cdot E_i = 0$ for every $i = 1, 2, 3, 4, 5$. We see that M is the proper transform of line that passes through P_5 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + (a - 1 + \lambda_3)E_3+$$
$$+(2a - 2 + \lambda_4)E_4 + \left(a - 1 + \frac{\lambda_4}{2}\right) D_2 + \left(b + \frac{\lambda_5}{2}\right) D_1 + (b - 1 + \lambda_5)E_5 + (3b - 3)M,$$

where $a + b = 2$. Since $\lambda_4 > 0$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \dots, F_5$ in this order. We have the following configuration on Y (Fig. 26).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + E_3 + 2E_4 + D_2, \quad C_2 = 3M + 3F_1 + 3F_2 + 2F_3 + F_4 + D_1 + E_5.$$

So,

$$\tilde{X}\setminus \mathrm{Supp}(L) \cong X\setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1\setminus(K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_5, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We see that E_1, E_2, E_4 meet R_1 and E_3 meets R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + (a-1+\lambda_1)E_1 + (a-1+\lambda_2)E_2 + (2a-2+\lambda_4)E_4 +$$
$$+\left(a-1+\frac{\lambda_4}{2}\right)D_2 + \left(b+\frac{\lambda_5}{2}\right)D_1 + (b-1+\lambda_5)E_5 + (2b-1)R_2 + (2b-2+\lambda_3)E_3,$$

where $a+b=2$. Since $\lambda_4 \geq \lambda_5 > 0$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2, F_3 in this order. We have the following configuration on Y (Fig. 27).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + 2E_4 + D_2, \quad C_2 = 2E_3 + 2R_2 + 2F_1 + F_2 + D_1 + E_3.$$

So,

$$\tilde{X}\setminus \mathrm{Supp}(L) \cong X\setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1\setminus(K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2 (Figs. 26 and 27).

□

Lemma 2.3 *Let X be a del Pezzo surface with du Val singularities,* $\deg(X) = 2$ *and let H be an ample divisor of type $B(r_H)$ on X. Assume that X has three singular points of type A_1. Then X has an H-polar cylinder.*

Proof Let $\varphi\colon \tilde{X} \to X$ be the minimal resolution of singularities of X and let

$$D = D_1 + D_2 + D_3$$

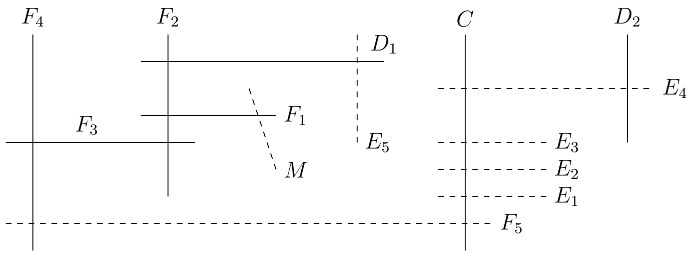


Fig. 26 Curves in the surface Y used in 2.2.13 in case when C is a (0)-curve

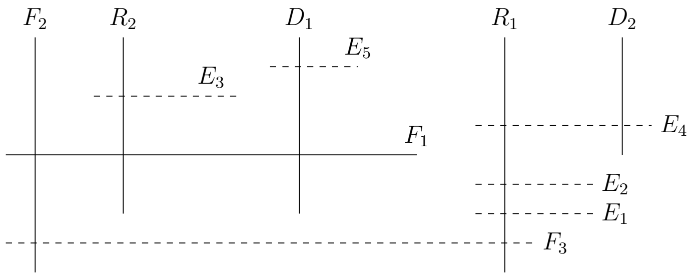


Fig. 27 Curves in the surface Y used in 2.2.13 in case when C is reducible

be the exceptional divisor of φ. Note that there exists a divisor C such that $C \cdot D_1 = 2$ and C meets D_1 in one point. Moreover, $D_2 \cdot C = D_3 \cdot C = 0$. Note that C is either (0)-curve, either two (-1)-curves. Indeed, we see that $\tilde{X}$ can be obtained by blow ups $\mathbb{P}^2$ in seven points $P_1, P_2, \ldots, P_7$, where P_1, P_2, P_3 lie on one line L, P_3, P_4, P_5 lie on one line L' and P_1, P_5, P_6 lie on one line L'', i.e. P_1, P_3, P_5 are the intersection points of lines L, L', L''. Note that D_1 is the proper transform of L, D_2 is the proper transform of L' and D_3 is the proper transform of L''. Consider a linear system $|2T|$, where T is the class of a line. We have $\dim |2T| = 5$. So, there exists exactly one element $C' \in |2T|$ such that C' passes through P_4, P_5, P_6, P_7 and C' meets L in one point. Then C is the proper transform of C'.

2.3.1 Assume that $r_H = 1$ and $H = -K_X + \lambda E$, where E is a (-1)-curve, $1 > \lambda > 0$. Note that $C \cdot E = 1$. Moreover, there exist (-1)-curves E_1, E_2 such that $E_1 \cdot D_1 = E_1 \cdot D_2 = 1$,

$$E_1 \cdot D_3 = 0, \quad E_2 \cdot D_1 = E_2 \cdot D_3 = 1, \quad E_2 \cdot D_2 = 0, \quad E_1 \cdot C = E_1 \cdot E = E_2 \cdot C = E_2 \cdot E = 0,$$

there exist (-1)-curves E_3, E_4 such that

$$E_3 \cdot D_1 = E_4 \cdot D_1 = 1, \quad E_i \cdot E_j = E_j \cdot C = E_j \cdot E = D_2 \cdot E_j = D_3 \cdot E_j = 0$$

for every $i = 1, 2$, $j = 3, 4$. Note that E, E_1, E_2, E_3 are the exceptional curves of blow ups of P_7, P_3, P_1, P_2 correspondingly, E_4 is the proper transform of line that passes through P_4 and P_6. Assume that C is a (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda)E + bD_1 + (b - 1)(D_2 + D_3 + E_3 + E_4) + (2b - 2)E_1+ \\ +(2b - 2)E_2,$$

where $a + 2b = 3$. Since $\lambda > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 28).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_4 \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = 2E + 2C + 2F_2 + F_1 + F_3, \quad C_2 = D_1 + E_3 + E_4 + 2E_1 + 2E_2 + D_2 + D_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E meets R_1. We have

$$\varphi^*(H) \equiv L = aR_2 + (2a - 1)R_1 + (2a - 2 + \lambda)E + bD_1 + (b - 1)(D_2 + D_3 + E_3)+ \\ +(2b - 2)E_1 + (2b - 2)E_2,$$

where $a + b = 2$. Since $\lambda > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves F_1, F_2, F_3 in this order. We have the following configuration on Y (Fig. 29).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y_4 \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = 2E + 2R_1 + 2F_1 + R_2 + F_2, \quad C_2 = D_1 + E_3 + 2E_1 + 2E_2 + D_2 + D_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_3, L_1, L_2 correspond to C_1, C_2 (Figs. 28 and 29).

2.3.2 Assume that $r_H = 1$ and $H = -K_X + \lambda\varphi(E)$, where E is a (-1)-curve such that E meets D_2, $2 > \lambda > 0$. Note that $C \cdot E = 1$. Moreover, there exists a (-1)-curve E_1 such that $E_1 \cdot D_1 = E_1 \cdot D_3 = 1$, $E_1 \cdot D_2 = E_1 \cdot C = E_1 \cdot E = 0$, there

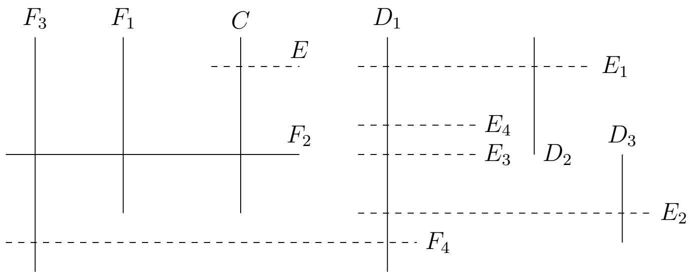


Fig. 28 Curves in the surface Y used in 2.3.1 in case when C is a (0)-curve

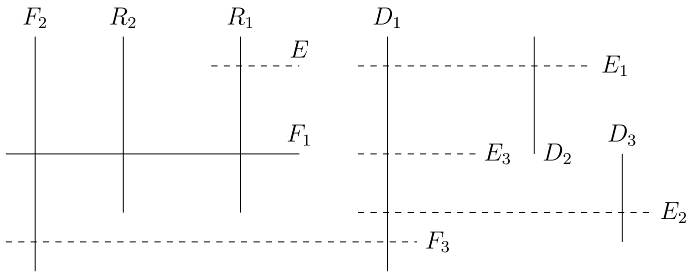


Fig. 29 Curves in the surface Y used in 2.3.1 in case when C is reducible

exist (-1)-curves E_2, E_3 such that $E_i \cdot D_1 = 1,\ E_i \cdot C = E_i \cdot D_2 = E_i \cdot D_3 = 0$ for every $i = 2, 3$. Note that E, E_1, E_2 are the exceptional curves of blow ups of P_4, P_1, P_2 correspondingly, E_3 is the proper transform of line that passes through P_5 and P_7. Assume that C is a (0)-curve. Note that there exist (0)-curves M_1, M_2 such that M_1, M_2 pass through the intersection point of D_1 and C, and $M_1 \cdot D_1 = M_2 \cdot D_1 = M_1 \cdot C = M_2 \cdot C = 1,\ M_1 \cdot D_2 = M_2 \cdot D_2 = 0,\ M_1 \cdot D_3 = M_2 \cdot D_3 = 0, M_j \cdot E_i = 0$ for every $i = 1, 2, 3,\ j = 1, 2$. We see that M_1 is the proper transform of line that passes through P_5 and the intersection point of L and C', M_2 is the proper transform of conic that passes through P_3, P_5, P_6, P_7 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + \left(a - 1 + \frac{\lambda}{2}\right) D_2 + (2a - 2 + \lambda)E + bD_1 + (b - 1)(E_2 + E_3 + D_3) + \\ + (2b - 2)E_1 + cM_1 + cM_2,$$

where $a + b + c = 2$. Since $\lambda > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 30).

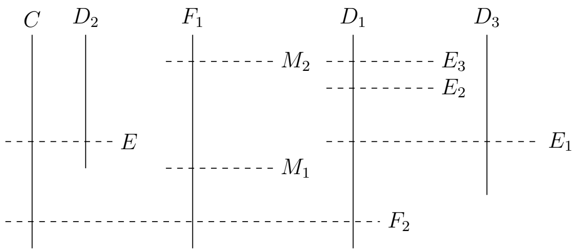


Fig. 30 Curves in the surface Y_2 used in 2.3.2 in case when C is a (0)-curve

Note that there exists a $\mathbb{P}^1$-fibration $h: Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + 2E + D_2, \quad C_2 = F_1 + M_1 + M_2, \quad C_3 = D_1 + 2E_1 + E_2 + E_3 + D_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E meets R_1. Note that there exist (-1)-curves M_1, M_2 such that M_1 and M_2 meet R_2. We have

$$\varphi^*(H) \equiv L = aR_1 + \left(a - 1 + \frac{\lambda}{2}\right) D_2 + (2a - 2 + \lambda)E + bD_1 + (b-1)E_2 + (b-1)D_3 + \\ +(2b-2)E_1 + cR_2 + (c-1)M_1 + (c-1)M_2,$$

where $a + b + c = 3$. Since $\lambda > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g: Y \to \tilde{X}$ be the blow up of intersection point of C and D_1, F be the exceptional divisor of g. We have the following configuration on Y (Fig. 31).

Note that there exists a $\mathbb{P}^1$-fibration $h: Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = R_1 + 2E + D_2, \quad C_2 = R_2 + M_1 + M_2, \quad C_3 = D_1 + 2E_1 + E_2 + D_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F, L_1, L_2, L_3 correspond to C_1, C_2, C_3 (Fig. 30).

2.3.3 Assume that $r_H = 2$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2$, where E_1, E_2 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 > 0$. Note that $C \cdot E_1 = C \cdot E_2 = 1$. Moreover, there exist (-1)-curves E_3, E_4 such that

$$E_3 \cdot D_1 = E_4 \cdot D_1 = E_3 \cdot D_3 = E_4 \cdot D_2 = 1,$$

$E_3 \cdot D_2 = E_4 \cdot D_3 = 0$, $E_i \cdot C = E_i \cdot E_j = 0$ for every $i = 3, 4$, $j = 1, 2$. Note that E_1, E_3, E_4 are the exceptional curves of blow ups of P_7, P_3, P_1 correspondingly, E_2 is the proper transform of line that passes through P_2 and P_5. Assume that C is a (0)-curve. Note that there exist (0)-curves M_1, M_2 such that M_1, M_2 pass through the intersection point of D_1 and C and $M_i \cdot D_j = M_i \cdot E_k = 0$ for every $i = 1, 2$, $j = 2, 3$, $k = 1, 2, 3, 4$. We see that M_1 is the proper transform of line that passes through P_5 and the intersection point of L and C', M_2 is the proper transform of conic that passes through P_2, P_4, P_5, P_6 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD_1 + (b - 1)(D_2 + D_3) + \\ +(2b - 2)(E_3 + E_4) + cM_1 + cM_2,$$

where $a + b + c = 2$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 32).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + E_1 + E_2, \quad C_2 = F_1 + M_1 + M_2, \quad C_3 = D_1 + 2E_3 + 2E_4 + D_2 + D_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_1, E_2 meet R_2. Let $g \colon Y \to \tilde{X}$ be the blow up of intersection point of C and D_1, F be the exceptional divisor of g. We obtain the same picture as in Fig. 31 ($E_1 = M_1$, $E_2 = M_2$). On the other hand, we have

$$\varphi^*(H) \equiv L = aR_1 + (a - 1)D_2 + (2a - 2)E + bD_1 + (b - 1)E_2 + (b - 1)D_3 + \\ +(2b - 2)E_1 + cR_2 + (c - 1 + \lambda_1)M_1 + (c - 1 + \lambda_2)M_2,$$

where $a + b + c = 3$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b, c such that L is an effective divisor. As in Case 2.3.2, X has an H-polar cylinder (Figs. 32).

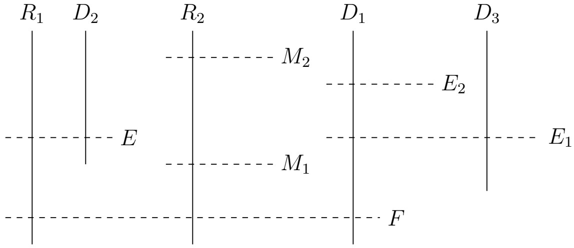


Fig. 31 Curves in the surface Y used in 2.3.2 in case when C is reducible

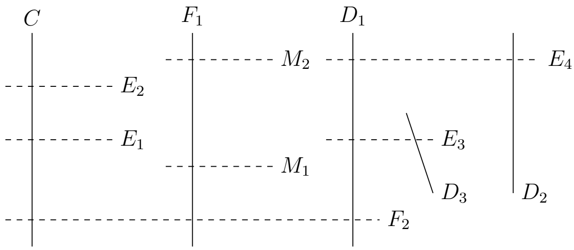


Fig. 32 Curves in the surface Y_2 used in 2.3.3

2.3.4 Assume that $r_H = 2$ and $H = -K_X + \lambda E + \lambda_3 \varphi(E_3)$, where E, E_3 are (-1)-curves such that E_3 meets D_1, $1 > \lambda > 0$, $2 > \lambda_3 > 0$. We have the same picture as in Case 2.3.1. So, X has an H-polar cylinder.

2.3.5 Assume that $r_H = 2$ and $H = -K_X + \lambda_1 \varphi(E_2) + \lambda_2 \varphi(E)$, where E_2, E are (-1)-curves such that E_2 meets D_1, E meets D_2, $2 > \lambda_1 \geq \lambda_2 > 0$. We have the same picture as in Case 2.3.2. So, X has an H-polar cylinder.

2.3.6 Assume that $r_H = 3$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 \varphi(E_3)$, where E_1, E_2, E_3 are (-1)-curves, $E_3 \cdot D_3 = 1$, $1 > \lambda_1 \geq \lambda_2 > 0$, $2 > \lambda_3 > 0$. Note that $C \cdot E_i = 1$ for every $i = 1, 2, 3$. Moreover, there exist (-1)-curves E_4, E_5 such that $E_4 \cdot D_1 = 1$, $E_4 \cdot D_2 = E_4 \cdot D_3 = E_4 \cdot C = E_4 \cdot E_i = 0$ for every $i = 1, 2, 3, 5$ and $E_5 \cdot D_1 = E_5 \cdot D_2 = 1$, $E_5 \cdot C = E_5 \cdot E_i = 0$ for every $i = 1, 2, 3, 5$. Note that E_1, E_3, E_4, E_5 are the exceptional curves of blow ups of P_7, P_6, P_2, P_3 correspondingly, E_2 is the proper transform of line that passes through P_1 and P_4. Assume that C is a (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + (2a - 2 + \lambda_3)E_3 + \\ + \left(a - 1 + \frac{\lambda_3}{2}\right) D_3 + (2b - 1)D_1 + (2b - 2)D_2 + (4b - 4)E_5 + (2b - 2)E_4,$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_3 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g \colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 33).

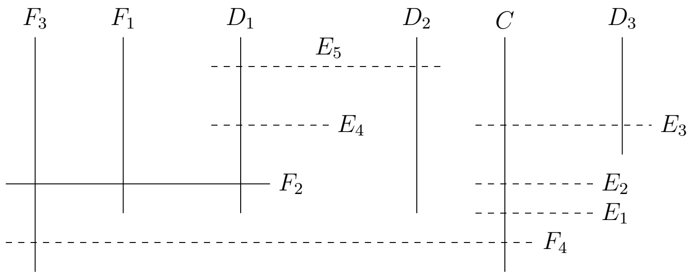


Fig. 33 Curves in the surface Y used in 2.3.6

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + E_2 + 2E_3 + D_3, \quad C_2 = 4E_5 + 2D_1 + 2D_2 + 2E_4 + 2F_2 + F_1 + F_3.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_3 meets R_1, E_1, E_2 meet R_2. Then, we have the same picture as in Case 2.3.2 (see Fig. 31). So, X has an H-polar cylinder.

2.3.7 Assume that $r_H = 3$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 \varphi(E_2) + \lambda_3 \varphi(E_3)$, where E_1, E_2, E_3 are (-1)-curves, $E_3 \cdot D_3 = 1$, $E_2 \cdot D_2 = 1$, $1 > \lambda_1 > 0$, $2 > \lambda_2 \geq \lambda_3 > 0$. Note that $C \cdot E_i = 1$ for every $i = 1, 2, 3$. Moreover, there exists a (-1)-curve E_4 such that $E_4 \cdot D_1 = 1$,

$$E_4 \cdot D_2 = E_4 \cdot D_3 = E_4 \cdot C = E_4 \cdot E_j = 0$$

for every $j = 1, 2, 3$. Note that E_1, E_2, E_3, E_4 are the exceptional curves of blow ups of P_7, P_4, P_6, P_2 correspondingly. Assume that C is a (0)-curve. Note that there exists a (0)-curve M such that M passes through the intersection point of D_1 and C, and

$$M \cdot D_1 = M \cdot C = 1, \quad M \cdot D_2 = M \cdot D_3 = M \cdot E_i = 0$$

for every $i = 1, 2, 3, 4$. We see that M is the proper transform of line that passes through P_5 and the intersection point of L and C'. We have

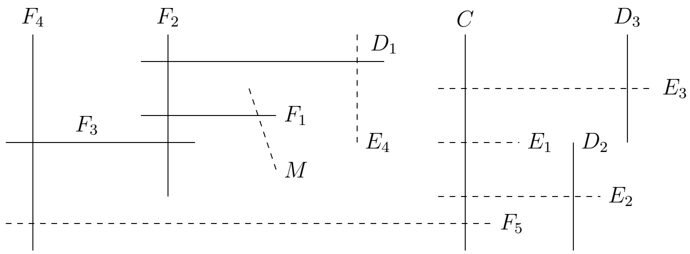


Fig. 34 Curves in the surface Y used in 2.3.7

$$\varphi^*(H) \equiv L = aC + (a-1+\lambda_1)E_1 + (2a-2+\lambda_2)E_2 + (2a-2+\lambda_3)E_3 + \\ + \left(a-1+\frac{\lambda_2}{2}\right)D_2 + \left(a-1+\frac{\lambda_3}{2}\right)D_3 + bD_1 + (b-1)E_4 + (3b-3)M,$$

where $a+b=2$. Since $\lambda_1 > 0$ and $\lambda_2 \geq \lambda_3 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_5$ in this order. We have the following configuration on Y (Fig. 34).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + E_1 + 2E_2 + 2E_3 + D_2 + D_3, \quad C_2 = 3M + 3F_1 + 3F_2 + 2F_3 + F_4 + D_1 + E_4.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_5, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. Note that we have the same picture as in Case 2.3.2 (see Fig. 31). So, X has an H-polar cylinder (Fig. 34).

2.3.8 Assume that r_H=3 and $H = -K_X + \lambda_1\varphi(E_1) + \lambda_2\varphi(E_2) + \lambda_3\varphi(E_3)$, where E_1, E_2, E_3 are (-1)-curves, $E_1 \cdot D_1$=$E_2 \cdot D_2 = E_3 \cdot D_3 = 1, 2 > \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Note that $C \cdot E_2 = C \cdot E_3 = 1$, $C \cdot E_1 = 0$. Moreover, there exists a (-1)-curve E_4 such that $E_4 \cdot D_1 = 1$,

$$E_4 \cdot D_2 = E_4 \cdot D_3 = E_4 \cdot C = E_4 \cdot E_j = 0$$

for every $j = 1, 2, 3$. Note that E_1, E_2, E_3 are the exceptional curves of blow ups of P_2, P_4, P_6 correspondingly, E_4 is the proper transform of line that passes through P_5 and P_7. Assume that C is a (0)-curve. Note that there exists a (0)-curve M such that M passes through the intersection point of D_1 and C, and $M \cdot D_1 = M \cdot C = 1$, $M \cdot D_2 = M \cdot D_3 = M \cdot E_i = 0$ for every $i = 1, 2, 3, 4$. We see that M is the proper transform of line that passes through P_5 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (2a-2+\lambda_2)E_2 + (2a-2+\lambda_3)E_3 + \left(a-1+\frac{\lambda_2}{2}\right)D_2 + \\ + \left(a-1+\frac{\lambda_3}{2}\right)D_3 + \left(b+\frac{\lambda_4}{2}\right)D_1 + (b-1+\lambda_4)E_1 + (b-1)E_4 + (2b-2)M,$$

where $a+b=2$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 35).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + 2E_2 + 2E_3 + D_2 + D_3, \quad C_2 = 2M + 2F_1 + 2F_2 + D + E_1 + E_4 + F_3.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_2 meets R_1, E_3 meets R_2. Note that there exists E_5 such that E_5 is a (-1)-curve and $E_5 \cdot D_1 = 1$, $E_5 \cdot D_2 = E_5 \cdot D_3 = E_5 \cdot E_i = 0$ for every $i = 1, 2, 3, 4$. We have

$$\varphi^*(H) \equiv L = aR_1 + \left(a-1+\frac{\lambda_2}{2}\right)D_2 + (2a-2+\lambda_2)E_2 + \left(b+\frac{\lambda_1}{2}\right)D_1 + (b-1+\lambda_1)E_1 + \\ + (b-1)E_4 + (b-1)E_5 + cR_2 + \left(c-1+\frac{\lambda_3}{2}\right)D_3 + (2c-2+\lambda_3)E_3,$$

where $a+b+c=3$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be the blow up of intersection point of C and D_1, F be the exceptional divisor of g. We have the following configuration on Y (Fig. 36).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = R_1 + 2E_2 + D_2, \quad C_2 = R_2 + 2E_3 + D_3, \quad C_3 = D_1 + E_1 + E_4 + E_5.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F, L_1, L_2, L_3 correspond to C_1, C_2, C_3 (Figs. 35 and 36).

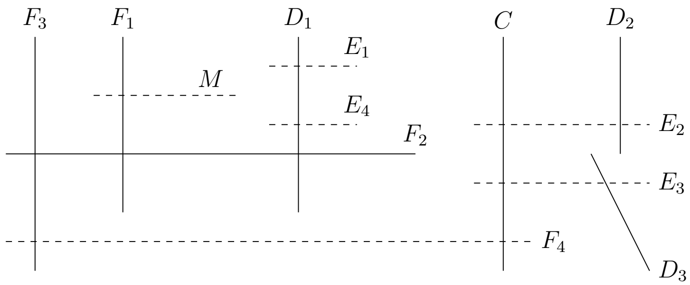
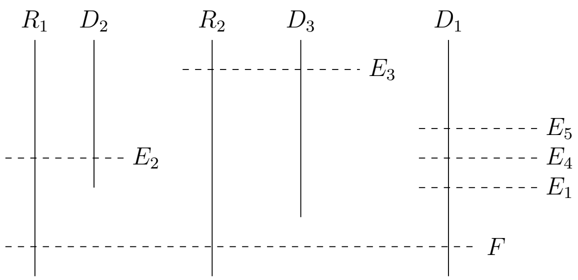


Fig. 35 Curves in the surface Y used in 2.3.8 in case when C is a (0)-curve

Fig. 36 Curves in the surface Y used in 2.3.8 in case when C is reducible



2.3.9 Assume that $r_H = 4$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 \varphi(E_2) + \lambda_3 \varphi(E_3) + \lambda_4 \varphi(E_4),$$

where E_1, E_2, E_3, E_4 are (-1)-curves such that E_4 meets D_1, $2 > \lambda > 0$, E_2 meets D_2, E_3 meets D_3, $1 > \lambda_1 > 0$, $2 > \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$. We have the same picture as in Case 2.3.7. So, X has an H-polar cylinder.

2.3.10 Assume that $r_H = 4$ and

$$H = -K_X + \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 \varphi(E_3) + \lambda_4 \varphi(E_4),$$

where E_1, E_2, E_3, E_4 are (-1)-curves such that E_3 meets D_3, E_4 meets D_1, $2 > \lambda_3 \geq \lambda_4 > 0$, $1 > \lambda_1 \geq \lambda_2 > 0$. We have the same picture as in Case 2.3.6. So, X has an H-polar cylinder.

$\square$

Lemma 2.4 *Let X be a del Pezzo surface with du Val singularities,* $\deg(X) = 2$ *and let H be an ample divisor of type $B(r_H)$ on X. Assume that X has four singular points of type A_1. Then X has an H-polar cylinder.*

Proof Let $\varphi\colon \tilde{X} \to X$ be the minimal resolution of singularities of X and let

$$D = D_1 + D_2 + D_3 + D_4$$

be the exceptional divisor of φ.

2.4.1 Assume that $r_H = 1$ and $H = -K_X + \lambda E$, where E is a (-1)-curve, $1 > \lambda > 0$. Note that we can obtain $\tilde{X}$ by the following way. Consider a conic B on $\mathbb{P}^2$. Choose three points P_1, P_2, P_3 on B and one point P_4 such that $P_4 \notin B$. Let $\psi_1\colon X' \to \mathbb{P}^2$ be the blow ups of P_1, P_2, P_3, P_4, and S_1, S_2, S_3, E be the exceptional curves of ψ_1 such that S_i corresponds to P_i, E corresponds to P_4. Put B' is the proper transform of B. Let $\psi_2\colon \tilde{X} \to X'$ be the blow ups of intersection points of B' and S_1, S_2, S_3, and E_1, E_2, E_3 be the exceptional curves of ψ_2. Then the proper transform of B is D_1, the proper transforms of S_1, S_2, S_3 are D_2, D_3, D_4. Note that there exists line C' on $\mathbb{P}^2$ such that C' meets B in one point. Put C is the proper transform of C' on $\tilde{X}$. We see that $C \cdot D_1 = 2$ and C meets D_1 in one point. Moreover, $D_2 \cdot C = D_3 \cdot C = D_4 \cdot C = 0$, $C \cdot E = 1$ and C is (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda)E + bD_1 + (b-1)(D_2 + D_3 + D_4) + \\ +(2b-2)(E_1 + E_2 + E_3),$$

where $a + 2b = 3$. Since $\lambda > 0$, we see that there exist a, b such that L is an effective divisor. Let $g\colon Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \dots, F_4$ in this order. We have the following configuration on Y (Fig. 37).

Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = 2E + 2C + 2F_2 + F_1 + F_3, \quad C_2 = D_1 + 2E_1 + 2E_2 + 2E_3 + D_2 + D_3 + D_4.$$

So,

$$\tilde{X} \backslash \operatorname{Supp}(L) \cong X \backslash \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \backslash (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2 (Fig. 37).

2.4.2 Assume that $r_H = 1$ and $H = -K_X + \lambda\varphi(E)$, where E is a (-1)-curve such that E meets D_4, $2 > \lambda > 0$. Note that we can obtain $\tilde{X}$ by the following way. Consider a conic L on $\mathbb{P}^2$. Choose two points P_1, P_2 on L and two points P_3, P_4 such that $P_3 \notin B$ and $P_4 \notin L$. Let $\psi_1\colon X' \to \mathbb{P}^2$ be the blow ups of P_1, P_2, P_3, P_4, and S_1, E_2, S_3, S_4 be the exceptional curves of ψ_1 such that S_i corresponds to P_i, E_2 corresponds to P_2. Put L' is the proper transform of L. Let $\psi_2\colon \tilde{X} \to X'$ be the blow ups of the intersection point P_1' of L' and S_1 and two points P_3', P_4' on S_3, S_4, and E_1, E', E be the exceptional curves of ψ_2. Then the proper transform of L is D_1, the proper transforms of S_1, S_3, S_4 are D_2, D_3, D_4 correspondingly. Consider a linear

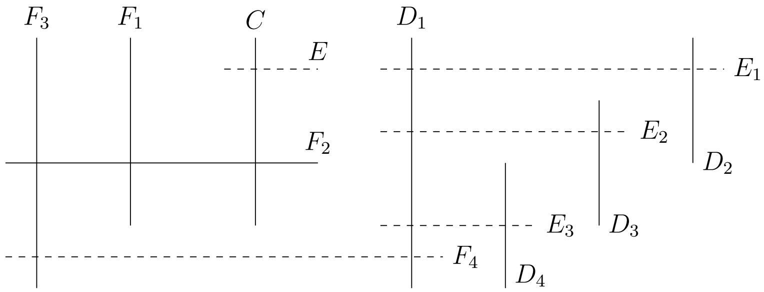


Fig. 37 Curves in the surface Y used in 2.4.1

system $|2T|$, where T is the class of a line. We have $\dim |2T| = 5$. So, there exists exactly one element $C' \in |2T|$ such that C' passes through P_4, P_5, P_6, P_7 and C' meets L in one point, C' passes through P_3, P_4 and the proper transform of C' on X' passes through P_3', P_4'. Put C is the proper transform of C'. Then C is either (0)-curve, either two (-1)-curves. Note that $C \cdot E = 1$. Moreover, there exist a (-1)-curve E_3 such that

$$E_3 \cdot D_1 = 1, \quad E_3 \cdot D_2 = E_3 \cdot D_3 = E_3 \cdot D_4 = E_3 \cdot E_1 = E_3 \cdot E_2 = E_3 \cdot E = 0.$$

We see that E_3 is the proper transform of line that passes through P_3 and the proper transform of this line passes through P_3'. Assume that C is a (0)-curve. Note that there exists a (0)-curves M such that M passes through the intersection point of D_1 and C and $M \cdot D_j = M \cdot E_k = 0$ for every $j = 2, 4$, $k = 1, 2, 3$, $M \cdot E = 0$, $M \cdot D_3 = 1$. Note that M is the proper transform of the line that passes through P_3 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (2a - 2 + \lambda)E + \left(a - 1 + \frac{\lambda}{2}\right) D_4 + bD_1 + (b-1)(D_2 + E_2 + E_3)$$
$$+ (2b - 2)E_1 + 2cM + cD_3,$$

where $a + b + c = 2$. Since $\lambda > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 38).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + 2E + D_4, \quad C_2 = F_1 + 2M + D_3, \quad C_3 = D_1 + 2E_1 + E_2 + E_3 + D_2.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

Fig. 38 Curves in the surface Y_2 used in 2.4.2 in case when C is a (0)-curve

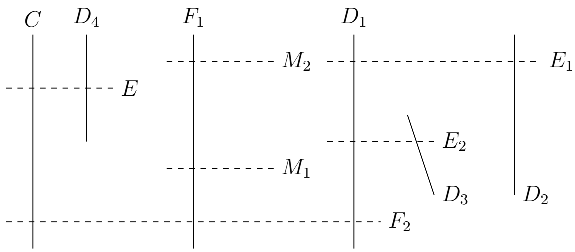
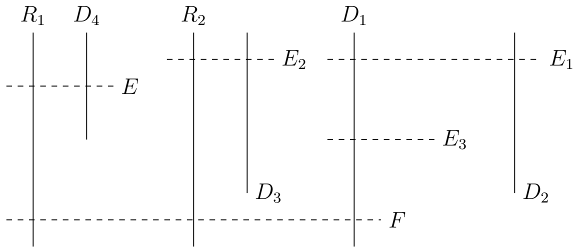


Fig. 39 Curves in the surface Y used in 2.4.2 in case when C is reducible



where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3.

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E meets R_1. Let $g\colon Y \to \tilde{X}$ be the blow up of intersection point of C and D_1, F be the exceptional divisor of g. We have the following configuration on Y (Fig. 39).

On the other hand,

$$\varphi^*(H) \equiv L = aR_1 + (2a - 2 + \lambda)E + \left(a - 1 + \frac{\lambda}{2}\right) D_4 + bD_1 + (b-1)D_2 + (2b-2)E_1 + \\ +(b-1)E_3 + cR_2 + (2c-2)E_2 + (c-1)D_3,$$

where $a + b + c = 3$. Since $\lambda > 0$, we see that there exist a, b, c such that L is an effective divisor. Note that there exists a $\mathbb{P}^1$-fibration $h\colon Y \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = R_1 + 2E + D_4, \quad C_2 = R_2 + 2E_2 + D_3, \quad C_3 = D_1 + 2E_1 + D_2 + E_3.$$

So,

$$\tilde{X} \setminus \operatorname{Supp}(L) \cong X \setminus \operatorname{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F, L_1, L_2, L_3 correspond to C_1, C_2, C_3 (Figs. 38 and 39).

2.4.3 Assume that $r_H = 2$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 E_2$, where E_1, E_2 are (-1)-curves, $1 > \lambda_1 \geq \lambda_2 > 0$. We can obtain $\tilde{X}$ by the same way as in Case 2.4.1. So, there

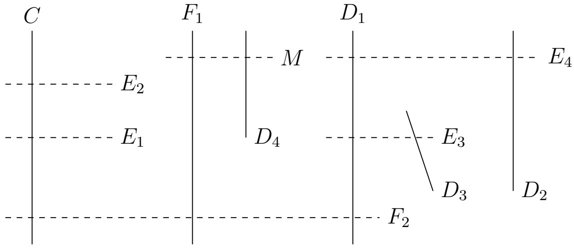


Fig. 40 Curves in the surface Y_2 used in 2.4.3

exists a (0)-curve C such that $C \cdot D_1 = 2$ and C meets D_1 in one point. Moreover, $D_2 \cdot C = D_3 \cdot C = D_4 \cdot C = 0$, $C \cdot E_1 = C \cdot E_2 = 1$. Moreover, there exist (-1)-curves E_3, E_4 such that

$$E_3 \cdot D_1 = E_4 \cdot D_1 = E_3 \cdot D_3 = E_4 \cdot D_2 = 1,$$

$E_3 \cdot D_2 = E_4 \cdot D_3 = 0, E_i \cdot C = E_i \cdot E_j = 0$ for every $i = 3, 4, j = 1, 2, E_i \cdot D_4 = 0$ for every $i = 1, 2, 3, 4$. We see that there exists a (0)-curve M such that M passes through the intersection point of D_1 and C and $M \cdot D_2 = M \cdot D_3 = M \cdot E_i = 0$ for every $i = 1, 2, 3, 4$, $M \cdot D_4 = 1$. Note that M is the proper transform of the line that passes through P_3 and the intersection point of L and C'. We have

$$\varphi^*(H) \equiv L = aC + (a - 1 + \lambda_1)E_1 + (a - 1 + \lambda_2)E_2 + bD_1 + (b - 1)(D_2 + D_3)+ \\ +(2b - 2)(E_3 + E_4) + 2cM + cD_4,$$

where $a + b + c = 2$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 40).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + E_1 + E_2, \quad C_2 = F_1 + 2M + D_4, \quad C_3 = D_1 + 2E_3 + 2E_4 + D_2 + D_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3 (Fig. 40).

2.4.4 Assume that $r_H = 2$ and $H = -K_X + \lambda_1 E_1 + \lambda_2 \varphi(E_2)$, where E_1, E_2 are (-1)-curves such that E_2 meets D_1, $1 > \lambda_1 > 0, 2 > \lambda_2 > 0$. This case is impossible.

Indeed, let $X \to Y$ be the contraction of $E_1, \varphi(E_2)$. We obtain a del Pezzo surface Y with $\rho(Y) = 2$ and Y has three singular points of type A_1. Let $f: Y \to Z$ be the contraction of extremal ray. We have two cases.

(1) $Z = \mathbb{P}^1$. Note that every singular fiber of f contains at most two singular points. So, there exists a singular fiber C that contains one singular point. We see that C consists of two curves. Then $\rho(Y) \geq 3$, a contradiction.
(2) Z is a del Pezzo surface with $\rho(Z) = 1$. Note that Z has three or two singular points of type A_1. This is impossible by classification (see, for example, [8]).

2.4.5 Assume that $r_H = 2$ and $H = -K_X + \lambda_1\varphi(E_1) + \lambda_2\varphi(E_2)$, where E_1, E_2 are (-1)-curves such that E_1 meets D_3, E_2 meets D_4, $2 > \lambda_1 \geq \lambda_2 > 0$. We can obtain $\tilde{X}$ by the same way as in R111S. So, there exists a divisor C such that $C \cdot D_1 = 2$ and C meets D_1 in one point. Moreover, $D_2 \cdot C = D_3 \cdot C = D_4 \cdot C = 0$. Note that C is either (0)-curve, either two (-1)-curves. Note that $C \cdot E_1 = C \cdot E_2 = 1$. Moreover, there exist (-1)-curves E_3, E_4 such that $E_3 \cdot D_1 = 1$,

$$E_3 \cdot D_2 = E_3 \cdot D_3 = E_3 \cdot D_4 = E_3 \cdot C = E_3 \cdot E_j = 0$$

for every $j = 1, 2$, $E_4 \cdot D_1 = E_4 \cdot D_2 = 1$,

$$E_4 \cdot D_3 = E_4 \cdot D_4 = E_4 \cdot C = E_4 \cdot E_j = 0$$

for every $j = 1, 2, 3$. Assume that C is a (0)-curve. We have

$$\varphi^*(H) \equiv L = aC + (2a - 2 + \lambda_1)E_1 + (2a - 2 + \lambda_2)E_2 + \left(a - 1 + \frac{\lambda_1}{2}\right) D_3 +$$
$$+ \left(a - 1 + \frac{\lambda_2}{2}\right) D_4 + (2b - 1)D_1 + (2b - 2)E_3 + (4b - 4)E_4 + (2b - 2)D_2,$$

where $a + b = 2$. Since $\lambda_1 \geq \lambda_2 > 0$, we see that there exist a, b such that L is an effective divisor. Let $g: Y \to \tilde{X}$ be a sequence of blow ups with exceptional curves $F_1, \ldots, F_4$ in this order. We have the following configuration on Y (Fig. 41).

Note that there exists a $\mathbb{P}^1$-fibration $h: Y \to \mathbb{P}^1$ such that h has only two singular fibers C_1 and C_2, where

$$C_1 = C + 2E_1 + 2E_2 + D_3 + D_4, \quad C_2 = 4E_4 + 2D_1 + 2D_2 + 2E_3 + 2F_2 + F_1 + F_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2),$$

where K is a unique (-1)-curve and L_1, L_2 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_4, L_1, L_2 correspond to C_1, C_2.

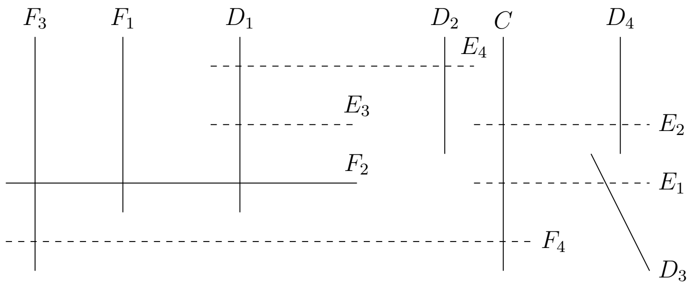


Fig. 41 Curves in the surface Y used in 2.4.5

Assume that $C = R_1 + R_2$, where R_1, R_2 are (-1)-curves. We may assume that E_2 meets R_1, E_1 meets R_2. Note that we have the same picture as in Case 2.4.2 (see Fig. 39). So, X has an H-polar cylinder (Fig. 41).

2.4.6 Assume that $r_H = 3$. Let $f_H : X \to Z$ be the contraction given by the Fujita face Δ_H of the divisor H. Note that Z is a del Pezzo surface with $\rho(Z) = 1$. By classification (see, for example, [8]), we see that Z has one singular point of type A_1. So,

$$H = -K_X + \lambda_1\varphi(E_1) + \lambda_2\varphi(E_2) + \lambda_3\varphi(E_3),$$

where E_1, E_2, E_3 are (-1)-curves such that E_1 meets D_3, E_2 meets D_4, E_3 meets D_1, $2 > \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. We have the same picture as in Case 2.4.5. So, X has an H-polar cylinder.

□

Lemma 2.5 *Let X be a del Pezzo surface with du Val singularities,* $\deg(X) = 2$ *and let H be an ample divisor of type $B(r_H)$ on X. Assume that X has five singular points of type A_1. Then X has an H-polar cylinder.*

Proof Let $\varphi : \tilde{X} \to X$ be the minimal resolution of singularities of X and let

$$D = D_1 + D_2 + D_3 + D_4 + D_5$$

be the exceptional divisor of φ. Let $f_H : X \to Y$ be the contraction given by the Fujita face Δ_H of the divisor H. Assume that $r_H = 2$. Then Y is a del Pezzo surface with $\rho(Y) = 1$. Note that Y has at least three singular point of type A_1, a contradiction (see, for example, [8]). So, $r_H = 1$. Let $g : Y \to Z$ be the contraction of extremal ray. As above, $Z = \mathbb{P}^1$. Assume that there exists a singular fiber C such that C contains only one singular point. Then C consists of two curves. Hence, $\rho(Y) \geq 3$, a contradiction. So, Y has four singular points of type A_1. We see that $H = -K_X + \lambda\varphi(E)$, where E is a (-1)-curve such that E meets D_5, $2 > \lambda > 0$. Note that we can obtain $\tilde{X}$ by the following way. Consider three lines L, $L'L''$ on $\mathbb{P}^2$ such that L, $L'L''$ pass through one point P. Let P_1, P_2, P_3, P_4 be points on $\mathbb{P}^2$ such that $P_1 \in L$, $P_2, \in L'$, P_3, $P_4 \in L''$ and $P_i \neq P$ for $i = 1, 2, 3, 4$. Let $\psi_1 \colon X' \to \mathbb{P}^2$ be the blow ups of P, P_1, P_2, P_3, P_4, and S, S_1, S_2, S_3, S_4 be the exceptional curves of ψ_1 such that S_i corresponds to P_i, S corresponds to P. Let $\psi_2 \colon \tilde{X} \to X'$ be the blow ups of the intersection points of L, L' and S_1, S_2, and S_1', S_2' be the exceptional curves of ψ_2. Then the proper transform of L, L', L'' are D_1, D_2, D_3, the proper transforms of S_1, S_2 are D_4, D_5. Note that there exists a conic C' on $\mathbb{P}^2$ such that C' meets L in one point, C' passes through P_2, P_3, P_4 and the proper transform of C' on X' passes through the intersection point of L' and S_2. Put C is the proper transform of C' on $\tilde{X}$. Then C is a (0)-curve such that $C \cdot D_1 = 2$ and C and D_1 intersect tangentially, $C \cdot D_i = 0$ for every $i = 2, 3, 4, 5$, $C \cdot E = 1$. Moreover, there exist (-1)-curves E_1, E_2 such that $E_1 \cdot D_1 = E_1 \cdot D_2 = 1$, $E_1 \cdot D_3 = E_1 \cdot D_4 = E_1 \cdot D_5 = 0$, $E_2 \cdot D_1 = E_2 \cdot D_3 = 1$, $E_2 \cdot D_2 = E_2 \cdot D_4 = E_2 \cdot D_5 = 0$, there exists a (0)-curve M such that M passes through the intersection point of D_1 and C, and $M \cdot D_4 = 1$, $M \cdot E = M \cdot D_j = M \cdot E_i = 0$ for every $i = 1, 2$, $j = 2, 3, 5$. We have

$$\varphi^*(H) \equiv L = aC + (2a - 2 + \lambda)E + \left(a - 1 + \frac{\lambda}{2}\right) D_5 + bD_1 + (b - 1)(D_2 + D_3) + \\ + (2b - 2)(E_1 + E_2) + 2cM + cD_4,$$

where $a + b + c = 2$. Since $\lambda > 0$, we see that there exist a, b, c such that L is an effective divisor. Let $g_1 \colon Y_1 \to \tilde{X}$ be the blow up of intersection point of C and D_1, F_1 be the exceptional divisor of g_1. Let $g_2 \colon Y_2 \to Y_1$ be the blow up of intersection point of C, D_1 and F_1, F_2 be the exceptional divisor of g_2. We have the following configuration on Y_2 (Fig. 42).

Note that there exists a $\mathbb{P}^1$-fibration $h \colon Y_2 \to \mathbb{P}^1$ such that h has only three singular fibers C_1, C_2, C_3, where

$$C_1 = C + 2E + D_5, \quad C_2 = F_1 + 2M + D_4, \quad C_3 = D_1 + 2E_1 + 2E_2 + D_2 + D_3.$$

So,

$$\tilde{X} \setminus \mathrm{Supp}(L) \cong X \setminus \mathrm{Supp}(\varphi(L)) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3),$$

where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. Note that K corresponds to F_2, L_1, L_2, L_3 correspond to C_1, C_2, C_3 (Fig. 42). □

Lemma 2.6 *Let X be a del Pezzo surface with du Val singularities,* $\deg(X) = 2$ *and let H be an ample divisor on X such that $H \not\equiv -\mu K_X$. Assume that X has six singular points of type A_1. Then X has an H-polar cylinder.*

Proof Let $f_H \colon X \to Y$ be the contraction given by the Fujita face Δ_H of the divisor H. Assume that Y is a del Pezzo surface. Since $\rho(X) = 2$, we see that $\rho(Y) = 1$. Moreover, Y has at least five singular points of type A_1, a contradiction (see, for example, [8]). So, H is of type $C(r_H)$. Since $\rho(X) = 2$, we see that every fiber of

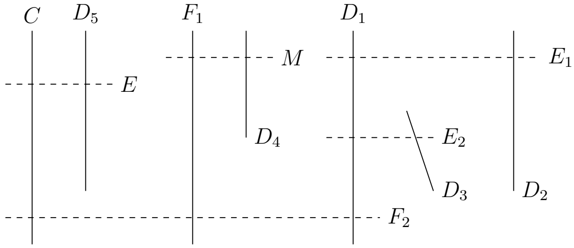


Fig. 42 Curves in the surface Y_2 used in Lemma 2.5

f_H consists of one curve. Then $H = -K_X + \mu C$, where C is a (0)-curve, $\mu > 0$. Let $f_H\colon X \to \mathbb{P}^1$ be the $\mathbb{P}^1$-fibration. Note that C is a fiber of f_H. We see that f_H has three singular fibers C_1, C_2, C_3 and every singular fiber contains two singular points. Note that there exists a section F such that F passes through three singular points and $-K_X \equiv 2F$. Then

$$H \equiv L = 2F + \frac{\mu}{3}C_1 + \frac{\mu}{3}C_2 + \frac{\mu}{3}C_3.$$

So, $X \setminus \operatorname{Supp}(L) \cong \mathbb{F}_1 \setminus (K + L_1 + L_2 + L_3)$, where K is a unique (-1)-curve and L_1, L_2, L_3 are fibers. Then X has an H-polar cylinder. □

So, Theorem 1.4 follows from Lemmas 2.1, 2.2, 2.3, 2.4, 2.5, 2.6.

3 Smooth Del Pezzo Surfaces

Let X be a smooth del Pezzo surface of degree two such that there does not exist two (-1)-curves that intersect tangentially. Assume that there exists an H-polar cylinder, where

$$H \equiv -K_X + \lambda_1 E_1 + \lambda_2 E_2,$$

where E_1, E_2 are (-1)-curves and $\lambda_1 + \lambda_2 < \frac{1}{7}$. Put $D \equiv H$ and $X \setminus \operatorname{Supp}(D) \cong Z \times \mathbb{A}^1$.

Theorem 3.1 ([2], Theorem 5.1.1) *Let X be a smooth del Pezzo surface,* $\deg(X) = 2$ *and let H be an ample divisor on X,*

$$H \equiv -K_X + \lambda_1 E_1 + \lambda_2 E_2.$$

Assume that X has an H-polar cylinder. Then there exists a point P such that (X, D) is not lc at P.

The natural projection $X \setminus \operatorname{Supp}(D) \cong Z \times \mathbb{A}^1 \to Z$ induces a rational map $\phi\colon X \dashrightarrow \mathbb{P}^1$. Resolving the base locus of ϕ we obtain a sequence of blow-ups $\varphi\colon W \to X$ such that there exists a $\mathbb{P}^1$-fibration $\pi\colon W \to \mathbb{P}^1$ and

$$K_W + \bar{D} + \sum b_i R_i = \varphi^*(K_X + D),$$

where $\bar{D}$ is the proper transform of D, R_i is the exceptional curves of the birational morphism π and $b_i > 0$. Moreover, we may assume that $\mathrm{Supp}(\bar{D} + \sum b_i R_i)$ consists of one section and every singular fiber. Let

$$W = W_n \to W_{n-1} \to \cdots \to W_1 \to X$$

be a sequence of blow-ups and R_i be the exceptional curve of φ_i. Put $D^{(k)} = \bar{D} + \sum_{i=1}^{k} b_i R_i$ where $\bar{D}$ is the proper transform of D on W_k.

Lemma 3.2 *$E_1 \not\subset \mathrm{Supp}\, D$ and $E_2 \not\subset \mathrm{Supp}\, D$.*

Proof Assume that E_1 is a component of D. Note that E_1 does not pass through P. So, E_1 meets only one component of D. Denote this component by D_1. Put $D = aE_1 + \sum a_i D_i$. We see that

$$1 - \lambda_1 = E_1 \cdot (-K_X + \lambda_1 E_1 + \lambda_2 E_2) = -a + a_1.$$

So, $a_1 > 1 - \lambda_1$. Assume that D_1 is a (-1)-curve. Note that X can be obtained by blow ups $\mathbb{P}^2$ in seven points $P_1, P_2, \ldots, P_7$ in general position. Put C is a conic that passes through $P_1, P_2, \ldots, P_5$, L is a line that passes through P_6, P_7. We see that the proper transform of C and L are (-1)-curves. So, for every (-1)-curve E there exists a (-1)-curve E' such that $E \cdot E' = 2$. Moreover, if E'' is a (-1)-curve that meets E then E'' does not meet E'. So, there exists a (-1)-curve E_3 on X such that $E_3 \cdot D_1 = 2$, $E_3 \cdot E_1 = 0$, $E_3 \cdot E_2 \leq 1$. Since E_3 is not a component of D, we see that

$$1 + \lambda_2 \geq E_3 \cdot (-K_X + \lambda_1 E_1 + \lambda_2 E_2) \geq 2a_1 > 2 - 2\lambda_1,$$

a contradiction. Assume that $-K_X \cdot D_1 = 2$. Then D_1 is a (0)-curve. Note that there exist at least three curves E_3, E_4, E_5 such that $E_3 \cdot D_1 = E_4 \cdot D_1 = E_5 \cdot D_1 = 2$ and $E_i \cdot E_2 \leq 1$ for every $i = 3, 4, 5$. We see that at least one of E_3, E_4, E_5 is not a component of D. We may assume that E_3 is not a component of D. Then

$$1 + \lambda_2 \geq E_3 \cdot (-K_X + \lambda_1 E_1 + \lambda_2 E_2) \geq 2a_1 > 2 - 2\lambda_1,$$

a contradiction. Assume that $-K_X \cdot D_1 \geq 3$. Then

$$2 + \lambda_1 + \lambda_2 = -K_X \cdot (-K_X + \lambda_1 E_1 + \lambda_2 E_2) \geq 3a_1 > 3 - 3\lambda_1,$$

a contradiction. □

So, we may assume that $E_1 \not\subset \mathrm{Supp}\, D$, $E_2 \not\subset \mathrm{Supp}\, D$. Let $g\colon X' \to X$ be the blow-up of P. Since X' has $-K_{X'} + \lambda_1 E_1 + \lambda_2 E_2$-polar cylinder, we see that there exists a (-2)-curve on X' (see [2]). So, there exist only two cases.

(1) There exists rational singular curve $C \in |-K_X|$ such that P is the double singular point of C.
(2) There exists a (-1)-curve that passes through P.

Assume that we have the first case. Note that E_1 and E_2 meet C. Let $f: X' \to Y$ be the contraction of E_1 and E_2. Since X' is a del Pezzo surface of degree one, we see that Y is smooth del Pezzo surface of degree three. As in [2], we have a contradiction. So, we may assume that there exists a (-1)-curve that passes through P.

Lemma 3.3 *Put $D = \sum a_i D_i$. Then $a_i < \frac{2+\lambda_1+\lambda_2}{3}$ for every i. Moreover, assume that D_i is a (-1)-curve or a (0)-curve. Then $a_i \leq \frac{1+\lambda_1+\lambda_2}{2}$.*

Proof Assume that D_i is a (-1)-curve. As above (see Lemma 3.2), we see that there exists a (-1)-curve E_3 on X such that $E_3 \cdot D_i = 2$, $E_3 \cdot E_1 \leq 1$, $E_3 \cdot E_2 \leq 1$. Then

$$1 + \lambda_1 + \lambda_2 \geq E_3 \cdot (-K_X + \lambda_1 E_1 + \lambda_2 E_2) \geq 2a_i.$$

Then $a_i \leq \frac{1+\lambda_1+\lambda_2}{2}$. Assume that D_i is a (0)-curve. Note that there exist at least three (-1)-curves E_3, E_4, E_5 on X such that

$$E_3 \cdot D_i = E_4 \cdot D_i = E_5 \cdot D_i = 2$$

and $E_i \cdot E_j \leq 1$ for every $i = 3, 4, 5$, $j = 1, 2$. Then at least one of these curves is not a component of D. Assume that this is E_3. Then

$$1 + \lambda_1 + \lambda_2 \geq E_3 \cdot (-K_X + \lambda_1 E_1 + \lambda_2 E_2) \geq 2a_i.$$

Then $a_i \leq \frac{1+\lambda_1+\lambda_2}{2}$. So, $-K_X \cdot D_i \geq 3$. Then

$$2 + \lambda_1 + \lambda_2 = -K_X \cdot D > 3a_i.$$

Hence, $a_i < \frac{2+\lambda_1+\lambda_2}{3}$. □

Lemma 3.4 *Every component of D passes through P.*

Proof Assume that there exists a component D_1 of D such that D_1 does not pass through P. Then D_1 is a (-1)-curve. Since D is connected, we see that there exists a component D_2 of D such that $D_2 \cdot D_1 = 1$. Moreover, $D_1 \cdot D_i = 0$ for all $i \geq 3$. Then

$$1 + r_1\lambda_1 + r_2\lambda_2 = D_1 \cdot (-K_X + \lambda_1 E_1 + \lambda_2 E_2) = -a_1 + a_2,$$

where $r_1 = E_1 \cdot D_1$, $r_2 = E_2 \cdot D_1$. Hence, $a_2 > 1$, a contradiction (see Lemma 3.3).

□

3.1 *P is an Intersection Point of Two* **(−1)**-*Curves*

Note that $D = aE + bE' + \sum a_i D_i$, where E, E' are (-1)-curves that pass through P, D_i are irreducible components. Consider W_1.

Lemma 3.5 *There does not exist a component D_i of D such that the proper transform of D_i on W_1 is a $(-n)$-curve ($n \geq 2$), D_i meets R_1 and D_i is not E or E'.*

Proof Assume that there exists a component D_i of D such that the proper transform of D_i on W_1 is a $(-n)$-curve ($n \geq 2$), D_i meets R_1 and D_i is not E or E'. Then D_i is a (-1)-curve on X. We may assume that $i = 1$. Since there exists a $\mathbb{P}^1$-fibration $\pi : W \to \mathbb{P}^1$, we see that there exists a component D_2 of D such that D_2 is a (-1)-curve and D_2 meets E or meets E' or meets D_1. Assume that D_2 meets E. Then $a > 1$, a contradiction (see Lemma 3.3). By the same reason D_2 does not meet E' and D_1. □

Lemma 3.6 *The birational morphism φ_2 blows up either the intersection point of R_1 and the proper transform of E, or the intersection point of R_1 and the proper transform of E'.*

Proof Assume that φ_2 blows up P' and $P' \notin E$, $P' \notin E'$. Since there exists a $\mathbb{P}^1$-fibration $\pi : W \to \mathbb{P}^1$, we see that there exists a component D_1 of D such that D_1 is a (-1)-curve and D_1 meets E or meets E'. Assume that D_1 meets E. Then $a > 1$, a contradiction (see Lemma 3.3). □

So, we may assume that φ_2 blows up the intersection point of R_1 and the proper transform of E. Then $b_1 \geq 2b$. Indeed, since every component of D passes through P (see Lemma 3.4), we see that R_1 is a unique component of $D^{(1)}$ that meets E'. Then

$$0 \leq E' \cdot (-K_{W_1} + \lambda_1 E_1 + \lambda_2 E_2) = E' \cdot D^{(1)} = -2b + b_1.$$

Lemma 3.7 $b \leq \frac{1+\lambda_1+\lambda_2}{3}$, $b_1 \leq \frac{2}{3}(1 + \lambda_1 + \lambda_2)$.

Proof We see that X can be obtained by blow ups $\mathbb{P}^2$ in seven points $P_1, P_2, \ldots, P_7$ in general position. Note that we may assume that E, E' are the proper transforms of lines L_1, L_2, E_1, E_2 are exception curves and $P_1, P_2 \in L_1$, $P_3, P_4 \in L_2$. We see that there exists a conic C that passes through P_1, P_5, P_6, P_7 and the intersection point of L_1 and L_2. The proper transform of C on W_1 is a (-1)-curve E_3. We have

$$E_3 \cdot R_1 = E_3 \cdot E' = 1.$$

Moreover, $E_3 \cdot E_1 \leq 1$, $E_3 \cdot E_2 \leq 1$. Then

$$1 + \lambda_1 + \lambda_2 \geq E_3 \cdot (-K_{W_1} + \lambda_1 E_1 + \lambda_2 E_2) \geq b_1 + b \geq 3b.$$

Hence, $b \leq \frac{1+\lambda_1+\lambda_2}{3}$. □

Lemma 3.8 *There does not exist a component D_i of D such that the proper transform of D_i on W_2 is a $(-n)$-curve ($n \geq 2$), D_i meets R_2 and D_i is not E.*

Proof Assume that there exists a component D_i of D such that the proper transform of D_i on W_2 is a $(-n)$-curve ($n \geq 2$), D_i meets R_2 and D_i is not E. Then D_i is a (0)-curve on X. We may assume that $i = 1$. As above, there exists a component D_2 of D such that D_2 is a (-1)-curve and D_2 meets one of E, E', R_1, D_1. Since $a \leq \frac{1+\lambda_1+\lambda_2}{2}$ and $b \leq \frac{1+\lambda_1+\lambda_2}{3}$, we see that D_2 meets D_1. Then $a_1 > 1$, a contradiction (see Lemma 3.3). □

Lemma 3.9 $b_2 < \frac{1+3\lambda_1+3\lambda_2}{2}$.

Proof We have

$$\lambda_1 + \lambda_2 = -K_{W_2} \cdot (-K_{W_2} + \lambda_1 E_1 + \lambda_2 E_2) = -K_{W_2} \cdot D^{(2)} > -a + b_2.$$

Then, by Lemma 3.3,

$$b_2 < a + \lambda_1 + \lambda_2 \leq \frac{1 + 3\lambda_1 + 3\lambda_2}{2}.$$

□

Lemma 3.10 *The birational morphism φ_3 blows up either the intersection point of R_2 and the proper transform of E, or the intersection point of R_2 and the proper transform of R_1.*

Proof Assume that φ_3 blows up P' and $P' \notin E$, $P' \notin R_1$. Since there exists a $\mathbb{P}^1$-fibration $\pi : W \to \mathbb{P}^1$, we see that there exists a component D_1 of D such that D_1 is a (-1)-curve and D_1 meets E or meets E' or meets R_1. Since $a \leq \frac{1+\lambda_1+\lambda_2}{2}$ and $b \leq \frac{1+\lambda_1+\lambda_2}{3}$, we have a contradiction. □

Consider the case when φ_3 blows up the intersection point of R_2 and the proper transform of E. Note that in this case $b_2 \geq 3b$. Indeed,

$$0 = R_1 \cdot (-K_{W_2} + \lambda_1 E_1 + \lambda_2 E_2) = R_1 \cdot D^{(2)} = b_2 - 2b_1 + b.$$

Then $b_2 = 2b_1 - b \geq 3b$. By Lemma 3.9, $b < \frac{1+3\lambda_1+3\lambda_2}{6}$.

Lemma 3.11 *There does not exist a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve ($n \geq 2$), D_i meets R_3 and D_i is not E.*

Proof Assume that there exists a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve ($n \geq 2$), D_i meets R_3 and D_i is not E. Then D_i is a (-2)-curve on W_3. Hence, $D_i \cdot (-K_X) = 3$. We may assume that $i = 1$. As above, there exists a component D_2 of D such that D_2 is a (-1)-curve and D_2 meets one of E, E', R_1, R_2, D_1. Since $a \leq \frac{1+\lambda_1+\lambda_2}{2}$ and $b \leq \frac{1+3\lambda_1+3\lambda_2}{6}$, we see that D_2 meets D_1. Then $a_1 > 1$, a contradiction (see Lemma 3.3). □

Lemma 3.12 *The birational morphism φ_4 blows up the intersection point of R_3 and the proper transform of R_2.*

Proof Assume that φ_4 blows up P' and $P' \notin E$, $P' \notin R_2$. Since there exists a $\mathbb{P}^1$-fibration $\pi : W \to \mathbb{P}^1$, we see that there exists a component D_1 of D such that D_1 is a (-1)-curve and D_1 meets one of E, E', R_1, R_2. As above, we have a contradiction. Assume that φ_4 blows up the intersection point of R_3 and the proper transform of E. Then E is a (-5)-curve on W_4 and there does not exist another component D_i of $D^{(4)}$ such that D_i is a $(-n)$-curves on W_4. Therefore,

$$-2 + \lambda_1 + \lambda_2 = -K_{W_4} \cdot D^{(4)} > -K_{W_4} \cdot aE = -3a.$$

Then $a > \frac{2-\lambda_1-\lambda_2}{3}$, a contradiction (see Lemma 3.3). □

Since φ_4 blows up the intersection point of R_3 and the proper transform of R_2, we see that $b_3 \geq 4a - 2$. Indeed,

$$-2 = E \cdot (-K_{W_3}) \leq E \cdot (-K_{W_3} + \lambda_1 E_1 + \lambda_2 E_2) = E \cdot D^{(3)} = -4a + b_3.$$

Note that $(W_3, D^{(3)})$ is not lc in P', where P' is the intersection point of R_3 and the proper transform of R_2. Put P'' is the intersection point of R_2 and the proper transform of E. By property of blow ups, we see that $\mathrm{mult}_{P''}(D_i) \geq \mathrm{mult}_{P'}(D_i)$ for every component D_i of D. Then $4a - 2 \geq a$. Hence, $a \geq \frac{2}{3}$, a contradiction (see Lemma 3.3).

Consider the case when φ_3 blows up the intersection point of R_2 and the proper transform of R_1. Then $b_2 \geq 3a - 1$. Indeed,

$$-1 = E \cdot (-K_{W_2}) \leq E \cdot (-K_{W_3} + \lambda_1 E_1 + \lambda_2 E_2) = E \cdot D^{(2)} = -3a + b_2.$$

As above, $3a - 1 \geq a$. So, $a \geq \frac{1}{2}$. By Lemma 3.9, we see that $b_2 < 1$. Note that a, b, b_1 are also less than one. As above, there does not exist a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve ($n \geq 2$), D_i meets R_3, and φ_4 blows up either the intersection point of R_3 and the proper transform of R_1, or the intersection point of R_3 and the proper transform of R_2.

Assume that φ_4 blows up the intersection point of R_3 and the proper transform of R_1. Then $b_3 \geq 5a - 2$. Indeed,

$$0 = R_2 \cdot (-K_{W_3} + \lambda_1 E_1 + \lambda_2 E_2) = -2b_2 + a + b_3.$$

So, $b_3 = 2b_2 - a \geq 5a - 2$. On the other hand,

$$-1 + \lambda_1 + \lambda_2 = -K_{W_3} \cdot D^{(3)} > -K_{W_3} \cdot (aE + b_1 R_1 + (5a - 2)R_3) = 4a - b_1 - 2.$$

We obtain $b_1 > 4a - 1 - \lambda_1 - \lambda_2 \geq 1 - \lambda_1 - \lambda_2$, a contradiction (see Lemma 3.7).

Assume that φ_4 blows up the intersection point of R_3 and the proper transform of R_2. Then $b_3 = 3b_1 - b - 1$. Indeed,

$$-1 = R_1 \cdot (-K_{W_3} + \lambda_1 E_1 + \lambda_2 E_2) = -3b_1 + b + b_3.$$

So, $b_3 = 3b_1 - b - 1 \geq 5b - 1$. As above, $5b - 1 \geq 2b$. Hence, $b \geq \frac{1}{3}$ On the other hand,

$$-1 + \lambda_1 + \lambda_2 = -K_{W_3} \cdot D^{(3)} > -K_{W_3} \cdot (aE + b_1 R_1 + b_3 R_3) = -a - b_1 + b_3 = -a - b + 2b_1 - 1.$$

We obtain $a > 3b - \lambda_1 - \lambda_2 \geq 1 - \lambda_1 - \lambda_2$, a contradiction (see Lemma 3.3). So, P is not an intersection point of two (-1)-curves.

3.2 P is Not an Intersection Point of Two (-1)*-Curves*

Let E be a unique (-1)-curve that passes through P. Note that E_1, E_2 do not meet E. Put $D = aE + \sum a_i D_i$, where D_i are irreducible components. Consider W_1.

Lemma 3.13 $b_1 \leq \frac{1+\lambda_1+\lambda_2}{2}$.

Proof We see that X can be obtained by blow ups $\mathbb{P}^2$ in seven points $P_1, P_2, \ldots, P_7$ in general position. Note that we may assume that E is the proper transform of line L, E_1, E_2 are exception curves and $P_6, P_7 \in L$. We see that there exists a rational cubic curve C such that C passes through $P_1, P_2, \ldots, P_6$ and C has a singularity in P. The proper transform of C on W_1 is a (-1)-curve E_3. We have

$$E_3 \cdot R_1 = 2, \quad E_3 \cdot E_1 = E_3 \cdot E_2 = 1.$$

Then

$$1 + \lambda_1 + \lambda_2 = E_3 \cdot (-K_{W_1} + \lambda_1 E_1 + \lambda_2 E_2) \geq 2b_1.$$

Then $b_1 \leq \frac{1+\lambda_1+\lambda_2}{2}$. □

Lemma 3.14 *The birational morphism φ_2 blows up the intersection point of R_1 and the proper transform of E.*

Proof Assume that φ_2 blows up a point $P' \in R_1$ and P' is not the intersection point of R_1 and the proper transform of E. Since

$$-K_{W_3} \cdot D^{(3)} = -1 + \lambda_1 + \lambda_2 < 0,$$

we see that there exists a component N of $D^{(3)}$ such that N is a (-3)-curve on W_3 and the coefficient of N in $D^{(3)}$ is at least $1 - \lambda_1 - \lambda_2$, a contradiction (see Lemmas 3.3 and 3.13). □

Lemma 3.15 *There does not exist a component D_i of D such that the proper transform of D_i on W_2 is a $(-n)$-curve ($n \geq 2$), D_i meets R_2 and D_i is not E.*

Proof Assume there exists a component D_i of D such that the proper transform of D_i on W_2 is a $(-n)$-curve ($n \geq 2$), D_i meets R_2 and D_i is not E. Note that $n = 2$ and D_i is a (0)-curve on X. We may assume that $i = 1$. Since there exists a $\mathbb{P}^1$-fibration $\pi : W \to \mathbb{P}^1$, we see that there exists a component D_2 of D such that D_2 is a (-1)-curve and D_2 meets E or meets R_1 or meets D_1. Assume that D_2 meets E. Then $a > 1$, a contradiction (see Lemma 3.3). Assume that D_2 meets R_1. Then $b_1 > 1$, a contradiction (see Lemma 3.13). Assume that D_2 meets D_1. Then $a_1 > 1$, a contradiction (see Lemma 3.3). □

Lemma 3.16 *The birational morphism φ_3 blows up either the intersection point of R_2 and the proper transform of E, or the intersection point of R_2 and the proper transform of R_1.*

Proof Assume that φ_3 blows up P' and $P' \notin E$, $P' \notin R_1$. Then W_3 has only one (-3)-curve E, and every component of $D^{(3)}$ has self-intersection at least -2. So,

$$-1 + \lambda_1 + \lambda_2 = -K_{W_3} \cdot D^{(3)} \geq -a.$$

Then $a \geq 1 - \lambda_1 - \lambda_2$, a contradiction. □

Lemma 3.17 *Assume that φ_3 blows up the intersection point of R_2 and the proper transform of R_1. Then there does not exist a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve ($n \geq 2$), D_i meets R_3.*

Proof Assume that there exists a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve ($n \geq 2$), D_i meets R_3. We may assume $i = 1$. As above, there exists a component D_2 of D such that D_2 is a (-1)-curve and D_2 meets one of the proper transform of D_1, R_1, R_2, E. Assume that D_2 meets the proper transform of E. Then $a > 1$, a contradiction. Assume that D_2 meets the proper transform of R_1. Then $b_1 > 1$, a contradiction. Assume that D_2 meets the proper transform of D_1. Then $a_1 > 1$, a contradiction. Assume that D_2 meets the proper transform of R_2. Then $b_2 > 1$. Consider W_2. We have

$$\lambda_1 + \lambda_2 = -K_{W_2} \cdot D^{(2)} \geq -K_{W_2} \cdot (b_2 R_2 + aE) = b_2 - a.$$

Then $a > 1 - \lambda_1 - \lambda_2$, a contradiction. □

Lemma 3.18 *The birational morphism φ_3 does not blow up the intersection point of R_2 and the proper transform of R_1.*

Proof Assume that φ_3 blows up the intersection point of R_2 and the proper transform of R_1. Then $b_2 = 3a - 1$. Consider W_4. Put $P' = \varphi_4(R_4)$. Assume that $P' \notin R_2$ and $P' \notin R_1$. As above, there exists a component D_1 of D such that D_1 is a (-1)-curve on W_3 and D_1 meets one of the proper transform of R_1, R_2, E. Assume that D_1 meets

E. Then $a > 1$, a contradiction (see Lemma 3.3). Assume that D_1 meets R_1. Then $b_1 > 1$, a contradiction (see Lemma 3.13). Assume that D_1 meets R_2. Then $b_2 > 1$. Hence, $3a - 1 > 1$. So, $a > \frac{2}{3}$, a contradiction (see Lemma 3.3). Assume that P' is the intersection point of R_3 and R_1. We have

$$-2 + \lambda_1 + \lambda_2 = -K_{W_4} \cdot D^{(4)} \geq -a - 2b_1.$$

Hence, $a > \frac{1}{3}(2 - \lambda_1 - \lambda_2)$ or $b_1 > \frac{1}{3}(2 - \lambda_1 - \lambda_2)$, a contradiction (see Lemmas 3.13, 3.3). Assume that P' is the intersection point of R_3 and R_2. Note that $b_2 = 3a - 1$. Since there does not exist a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve where $n \geq 2$ (see Lemma 3.17), we see that

$$-2 + \lambda_1 + \lambda_2 = -K_{W_4} \cdot D \geq -a - 3a + 1 - b_1 = -4a - b_1 + 1.$$

Then $4a + b_1 \geq 3 - \lambda_1 - \lambda_2$. Hence, $a > \frac{1}{5}(3 - \lambda_1 - \lambda_2)$ or $b_1 > \frac{1}{5}(3 - \lambda_1 - \lambda_2)$, a contradiction (see Lemmas 3.13, 3.3). □

By Lemmas 3.18 and 3.16, we see that φ_3 blows up the intersection point of R_2 and the proper transform of E. Then $b_2 = 2b_1$. Indeed,

$$0 = R_1 \cdot (-K_{W_2} + \lambda_1 E_1 + \lambda_2 E_2) = R_1 \cdot D^{(2)} = 2b_2 - b_1.$$

Lemma 3.19 *There does not exist a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve ($n \geq 2$), D_i meets R_3 and D_i is not E.*

Proof Assume that there exists a component D_i of D such that the proper transform of D_i on W_3 is a $(-n)$-curve ($n \geq 2$), D_i meets R_3 and D_i is not E. We may assume $i = 1$. As above, there exists a component D_2 of D such that D_2 is a (-1)-curve and D_2 meets one of the proper transform of D_1, R_1, R_2, E. Assume that D_2 meets the proper transform of E. Then $a > 1$, a contradiction. Assume that D_2 meets the proper transform of R_1. Then $b_1 > 1$, a contradiction. Assume that D_2 meets the proper transform of D_1. Then $a_1 > 1$, a contradiction. Assume that D_2 meets the proper transform of R_2. Then $b_2 > 1$. Consider W_2. We have

$$\lambda_1 + \lambda_2 = -K_{W_2} \cdot D^{(2)} \geq -K_{W_2} \cdot (b_2 R_2 + aE) = b_2 - a.$$

Then $a > 1 - \lambda_1 - \lambda_2$, a contradiction. □

Lemma 3.20 *The birational morphism φ_4 blows up the intersection point of R_3 and the proper transform of R_2.*

Proof Put $P' = \varphi_4(R_4)$. Assume that $P' \notin R_2$ and $P' \notin E$. As above, there exists a component D_1 of D such that D_1 is a (-1)-curve and D_1 meets one of the proper transform of R_1, R_2, E. Assume that D_1 meets E. Then $a > 1$, a contradiction (see Lemma 3.3). Assume that D_1 meets R_1. Then $b_1 > 1$, a contradiction (see Lemma 3.13). Assume that D_1 meets R_2. Then $b_2 > 1$. Consider W_2. We have

$$\lambda_1 + \lambda_2 = -K_{W_2} \cdot D^{(2)} > b_2 - a.$$

Then $a > 1 - \lambda_1 - \lambda_2$, a contradiction (see Lemma 3.3). Assume that P' is the intersection point of R_3 and the proper transform of E. We have

$$-2 + \lambda_1 + \lambda_2 = -K_{W_4} \cdot D^{(4)} \geq -aK_{W_4} \cdot E = -3a.$$

Hence, $a > \frac{1}{3}(2 - \lambda_1 - \lambda_2)$, a contradiction (see Lemma 3.3). □

By Lemma 3.20 φ_4 blows up the intersection point of R_3 and the proper transform of R_2. Then $b_3 = 4a - 2$. As above, $4a - 2 \geq a$. Hence, $a \geq \frac{2}{3}$, a contradiction (see Lemma 3.3). So, Theorem 1.5 is proved.

References

1. Belousov, G.: Cylinders in del Pezzo surfaces with du Val singularities. Bul. Kor. Math Soc. **54**(5), 1655–1667 (2017)
2. Cheltsov, I., Park, J., Won, J.: Cylinders in del Pezzo surfaces. Int. Math. Res. Not. **2017**, 1179–1230 (2017)
3. Cheltsov, I., Park, J., Won, J.: Cylinders in singular del Pezzo surfaces. Comp. Math. **152**, 1198–1224 (2016)
4. Cheltsov, I., Park, J., Won, J.: Affine cones over smooth cubic surfaces. J. Eour. Math. **18**, 1537–1564 (2016)
5. Keel, S., McKernan, J.: Rational curves on quasi-projective surfaces. Memoirs AMS **140**(669) (1999)
6. Kishimoto, T., Prokhorov, Yu., Zaidenberg, M.: Group actions on affine cones. Affine algebraic geometry. CRM Proceeding of the Lecture Notes, vol. 54, pp. 123–163. American Mathematical Society (2011)
7. Kishimoto, T., Prokhorov, Yu., Zaidenberg, M.: G_a-actions on affine cones. Transf. Groups **18**, 1137–1153
8. Miyanishi, M., Zhang, D.-Q.: Gorenstein log del Pezzo surfaces of rank one. J. Algebra. **118**, 63–84 (1988)

Finiteness of Real Structures on KLT Calabi–Yau Regular Smooth Pairs of Dimension 2

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Abstract In this article, we prove that a smooth projective complex surface X which is regular (i.e. such that $h^1(X, \mathcal{O}_X) = 0$) and which has a $\mathbb{R}$-divisor Δ such that (X, Δ) is a KLT Calabi–Yau pair has finitely many real forms up to isomorphism. For this purpose, we construct a complete CAT(0) metric space on which Aut X acts properly discontinuously and cocompactly by isometries, using Totaro's Cone Theorem. Then we give an example of a smooth rational surface with finitely many real forms but having a so large automorphism group that [2] does not predict this finiteness.

Keywords Rational surfaces · Real structures · Real forms · KLT Calabi–Yau pairs · CAT(0) metric spaces

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1 Introduction

A *real form* of a complex projective variety X is a scheme over $\mathbb{R}$ whose complexification is $\mathbb{C}$-isomorphic to X. A *real structure* on X is an antiregular (or antiholomorphic) involution $\sigma : X \to X$ (cf. [10, Chap. 2]). Two real structures σ and σ' are *equivalent* if there is a $\mathbb{C}$-automorphism φ of X such that $\sigma' = \varphi\sigma\varphi^{-1}$.

By Weil descent of the base field (cf. [12, III.§1.3]), there is a bijective correspondence between the set of $\mathbb{R}$-isomorphism classes of real forms of X and the set of equivalence classes of real structures on X. Moreover, if σ is a real structure on X, this set is parametrized by the *first Galois cohomology set* $H^1(G, \mathrm{Aut}_{\mathbb{C}} X)$, where $G = \langle\sigma\rangle$ acts on $\mathrm{Aut}_{\mathbb{C}} X$ by conjugation.

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The results of this paper are motivated by the study of the finiteness problem for real forms of rational surfaces. We already addressed this question in our previous article [2] whose main result, combined with [*loc. cit.* §3.2], states as follows:

Theorem 1 *Let X be a smooth complex rational surfaces and let* $\mathrm{Aut}^* X$ *be the image of the natural morphism* $\mathrm{Aut}\, X \to \mathrm{O}(\mathrm{Pic}\, X)$.

If $\mathrm{Aut}^* X$ *does not contain a non-abelian free group $\mathbb{Z} * \mathbb{Z}$ then X has finitely many real forms up to $\mathbb{R}$-isomorphism.*

However, this result does not completely solve the problem since there are rational surfaces whose automorphism group does contain a non-abelian free group (cf. Example 4.1). In fact, it is not known how $\mathrm{Aut}^* X$ can be large for a rational surface X. For example, up to our knowledge, the problem of the finite generation of the group $\mathrm{Aut}^* X$ is open (but Lesieutre constructed in [8] a six-dimensional variety X such that $\mathrm{Aut}^* X$ is not finitely generated and he showed that X is an example of a smooth projective variety having infinitely many non-isomorphic real forms[1]).

The aim of this article is to prove the following result:

Theorem 2 *Let X be a smooth projective complex surface which is regular (i.e. $q(X) := h^1(X, \mathcal{O}_X) = 0$).*

If there is a $\mathbb{R}$-divisor Δ on X such that (X, Δ) is a KLT Calabi–Yau pair,[2] then X has finitely many real forms up to $\mathbb{R}$-isomorphism.

The proof of this result uses different kind of tools than those we used in [2] and mostly geometric actions on complete CAT(0) metric spaces: roughly speaking, a metric space is CAT(0) if it has "nonpositive curvature". We will give precise definitions in Sect. 2. Then, we recall the definition of KLT Calabi–Yau pairs and we prove finiteness of real forms for them using Totaro's Cone Theorem 3.3. We give an example of a rational surface whose finiteness of real forms cannot be deduced from Theorem 1 but is obtained from Theorem 3.4. Finally, we present an example of a rational surface for which the finiteness problem remains open and which can be equipped of a $\mathbb{Q}$-divisor Δ such that (X, Δ) is log-canonical Calabi–Yau.

2 Preliminaries: Geometric Actions on CAT(0) Spaces

We begin this section by a brief explanation of the link between finiteness of real forms and geometric (i.e. proper and cocompact) actions on CAT(0) spaces, which are a generalization of manifolds with nonpositive curvature (see [3, I.1.3, II.1.1]): this will be our main tool in order to turn our finiteness problem into a problem of hyperbolic geometry.

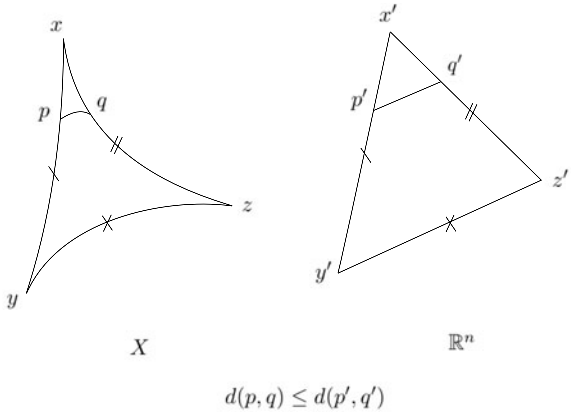
[1] To the best of our knowledge, it is the first known example.

[2] See Definition 3.1.

Definition 2.1 • A **geodesic** between two points a and b in a metric space (X, d) is a map $\gamma : [0, \ell] \to X$ such that $\gamma(0) = a$, $\gamma(\ell) = b$ and $\forall t, t' \in [0, \ell]$, $d(\gamma(t), \gamma(t')) = |t - t'|$ (in particular, γ is continuous and $\ell = d(a, b)$). A **geodesic triangle** Δ in X consists of three points $x, y, z \in X$ and three geodesic segments $[x, y], [y, z], [z, x]$.

- A metric space (X, d) is **geodesic** if every two points in X are joined by a geodesic (not necessarily unique).
- A geodesic metric space (X, d) is said to be a **CAT**(0) **space** if for every geodesic triangle Δ in X, there exists a triangle Δ' in $\mathbb{R}^n$ endowed with the euclidean metric, with sides of the same length as the sides of Δ, such that distances between points on Δ are less than or equal to the distances between corresponding points on Δ'.
- [3, I.8.2][3] Let Γ be a group acting by isometries on a metric space X. The action is said to be **proper** (or **properly discontinuous**) if[4]

$$\forall x \in X,\ \exists r > 0,\ \{\gamma \in \Gamma | \gamma.B(x, r) \cap B(x, r) \neq \varnothing\} \text{ is finite.}$$



Theorem 2.2 ([3, II.2.8]) *If a group Γ acts geometrically (i.e. properly discontinuously and cocompactly by isometries) on a complete CAT(0) space, then Γ contains only finitely many conjugacy classes of finite subgroups.* □

[3] As explained in *op.cit.*, I.8.3, if every closed ball of X is compact, then this definition is equivalent to the standard definition where the open balls are replaced by the compact subsets of X: for us, this is always the case.

[4] Denoting by $B(x, r) = \{y \in X | d(x, y) < r\}$ the open ball of center $x \in X$ and radius $r \geq 0$.

3 Finiteness Theorem

Firstly, let us introduce the surfaces we deal with (cf. [13], [14, 8.2, 8.4] for a slightly more general definition):

Definition 3.1 (*KLT Calabi–Yau pair*)
Let X be a *smooth* projective complex variety and Δ be a $\mathbb{R}$-divisor on X.

(X, Δ) is **a KLT (*resp.* log-canonical) Calabi–Yau pair** if there exists a resolution $\pi : (\widetilde{X}, \widetilde{\Delta}) \to (X, \Delta)$ satisfying the following conditions:

- $K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta)$;
- $\widetilde{\Delta}$ has simple normal crossings and his coefficients are < 1 (KLT condition), *resp.* ≤ 1 (log-canonical condition);
- Δ is an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is numerically trivial (Calabi–Yau condition).

Example 3.2 Let us present here some examples of KLT Calabi–Yau pairs. The reader may also look at Examples 4.1 and 4.2.

- Of course, there are irrational surfaces X having a $\mathbb{R}$-divisor Δ such that (X, Δ) is KLT Calabi–Yau (simply think of X being Calabi–Yau smooth and $\Delta = 0$). There are less trivial examples, like some $\mathbb{P}^1$-bundles over elliptic curves (cf. [1, 1.4]).
- If X is a *Halphen surface* of index $m \geq 2$ (for the definition, cf. [5, Sect. 2]) and F a reduced fibre of the elliptic fibration on X with simple normal crossings, then $\left(X, \frac{1}{m}F\right)$ is a KLT Calabi–Yau pair: for, the definition of a Halphen surface shows that $F \sim -mK_X$ and $\frac{1}{m} < 1$. If X is of index 1, then $\left(X, \frac{1}{2}(F + F')\right)$ is a KLT Calabi–Yau pair for two distinct reduced smooth fibers F and F' of $X \to \mathbb{P}^1$.
- Similarly, if X is a *Coble surface* and if the special fibre F (see [5, Proposition 3.1]) is reduced and has simple normal crossings, then $\left(X, \frac{1}{2}F\right)$ is a KLT Calabi–Yau pair.

For our purposes, we need the following finiteness theorem:

Theorem 3.3 (Cone theorem for KLT Calabi–Yau pairs—[14, 8.7]) *Let* (X, Δ) *be a KLT Calabi–Yau pair.*

If X is a surface, then the action of Aut X *on the nef cone has a rational polyhedral fundamental domain (i.e. it is the closed convex cone spanned by a* finite *set of Cartier divisors in* Pic $X \otimes_{\mathbb{Z}} \mathbb{R}$*).* □

The aim of this article is to prove the following result:

Theorem 3.4 *Let X be a smooth projective complex surface which is regular (i.e. $q(X) := h^1(X, \mathcal{O}_X) = 0$).*

If there is a $\mathbb{R}$-divisor Δ on X such that (X, Δ) is a KLT Calabi–Yau pair, then X has finitely many real forms up to $\mathbb{R}$-isomorphism.

Thus, Example 3.2 shows that our previous result about finiteness of real forms for Cremona special surfaces, i.e. (cf. [2, 3.2]) is a special case of this result when the fibre F we mentioned in Example 3.2 above is reduced with simple normal crossings.

Strategy of the proof. — Let σ be a real structure on X and let $\mathrm{Aut}^{\#}X$ (resp. $\mathrm{Aut}^{*}X$) be the kernel (resp. the image) of the natural morphism $p : \mathrm{Aut}\, X \to \mathrm{O}(\mathrm{Pic}\, X)$. If $G = \langle\sigma\rangle$ acts on $\mathrm{Aut}\, X$ by conjugation (i.e. $\forall\varphi \in \mathrm{Aut}\, X,\ \sigma.\varphi := \sigma\varphi\sigma^{-1}$), then the exact sequence

$$1 \longrightarrow \mathrm{Aut}^{\#}X \longrightarrow \mathrm{Aut}\, X \longrightarrow \mathrm{Aut}^{*}X \longrightarrow 1$$

is G-equivariant and induces an exact sequence in Galois cohomology. By [12, I.§5.5, Corollary 3], it suffices to prove that $H^1(G, \mathrm{Aut}^{*}X)$ is finite and that $\forall b \in Z^1(G, \mathrm{Aut}\, X),\ H^1(G, (\mathrm{Aut}^{\#}X)_b)$ is finite. But this last condition is true for every smooth irreducible projective complex variety by [12, III.§4.3] and [2, 1.2] (see also the proof of Theorem 2.5 in *loc. cit.*). Thus, we need only to show the finiteness of $H^1(G, \mathrm{Aut}^{*}X)$.

Now, for the special case of KLT Calabi–Yau pairs, the idea is the following: using Totaro's Cone Theorem, we will construct a complete CAT(0) space on which $\mathrm{Aut}^{*}X \rtimes \langle\sigma^*\rangle$ acts geometrically (where $\sigma^* \in \mathrm{O}(\mathrm{Pic}\, X)$ is the isometry induced by σ). Then, we will be able to conclude that $H^1(G, \mathrm{Aut}^{*}X)$ is finite using Theorem 2.2 together with the following result (which can be proved easily using the definitions as in the proof of [2, Theorem 2.4]) :

Lemma 3.5 *Let $G = \langle\sigma\rangle \simeq \mathbb{Z}/2\mathbb{Z}$, A be a G-group and $A \rtimes G$ the semidirect product defined by the action of G on A.*

If $A \rtimes G$ has a finite number of conjugacy classes of elements of order 2 (in particular, if it has finitely many conjugacy classes of finite subgroups), then $H^1(G, A)$ is finite.

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Before beginning the proof of this theorem, let us give some definitions to clarify the terms we use:

Definition 3.6 Let X be either $\mathcal{H}^n$ or $\mathbb{R}^n$ ([5]).

- A subset C of X is **convex** if $\forall x, y \in C$, the geodesic segment linking x and y is contained in C.
- A **side** of C is a maximal nonempty convex subset of the relative boundary ∂C (cf.[11, p. 195, 198]).
- A **(convex) polyhedron** of X is a nonempty closed convex subset of X whose collection of sides is locally finite. In what follows, we will always say "polyhedron" instead of "convex polyhedron".

[5] In what follows, when writing $\mathbb{R}^n$, it is understood as $\mathbb{R}^n$ equipped with its euclidean metric (which is denoted by E^n in [11]).

- Let X be a subset of either $\mathcal{H}^n$ or $\mathbb{R}^n$. A **fundamental polyhedron (or polyhedral fundamental domain)** for the action of a discrete group Γ of isometries of X is a polyhedron P whose interior $\mathring{P}$ is such that the elements of $\{g(\mathring{P}),\ g \in \Gamma\}$ are pairwise disjoint and $X = \bigcup_{g \in \Gamma} g(P)$. Moreover, P is a **locally finite fundamental polyhedron** if the set $\{g(P),\ g \in \Gamma\}$ is locally finite, i.e. if for all compact $K \subseteq X$, there are only finitely many elements of $\{g(P),\ g \in \Gamma\}$ which intersect K.

Proof of Theorem 3.4 We begin by explaining how we can turn our problem into a problem of hyperbolic geometry. Hodge index Theorem shows that the signature of the intersection form on $\mathrm{NS}(X) = \mathrm{Pic}\, X$ is $(1, n)$, where $\mathrm{rk}\,\mathrm{Pic}\, X = n + 1$. Note that this is the only place where we use the fact that $h^1(X, \mathcal{O}_X) = 0$. In fact, we could try to remove this hypothesis but we should replace $\mathrm{Aut}^{\#} X$ and $\mathrm{Aut}^{*} X$ with the analogous groups corresponding to the action of Aut X on $\mathrm{NS}(X)$ instead of Pic X but we do not have a general result of cohomological finiteness for the kernel of the action of Aut X on $\mathrm{NS}(X)$ whereas we gave such a result for the kernel of the action of Aut X on Pic X in the paragraph "Strategy of the proof" above.

Thus, we obtain the *hyperboloid model* of the hyperbolic space $\mathcal{H}^n := \{v \in \mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R} \mid v^2 = 1,\ v.H > 0\}$ equipped with the distance $d : (u, v) \mapsto \mathrm{argcosh}(u.v)$ (where $u.v$ is the intersection product of u and v and H is an ample divisor class on X.

The radial projection $\pi : \mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R}$ from the origin onto the hyperplane $\{v \in \mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R} \mid v.E_0 = 1\} \simeq \mathbb{R}^n$ maps the hyperboloid $\mathcal{H}^n$ onto the open unit ball D^n of this hyperplane: when endowed with the appropriate metric, this is the *Klein (projective) model* of $\mathcal{H}^n$ and π restricts to an isometry $\mathcal{H}^n \to D^n$. The geodesic lines of this model are straight line segments so that the convex subsets of D^n (for the hyperbolic metric) are exactly its convex subsets for the euclidean metric of $D^n \subseteq \mathbb{R}^n$. Note that π maps the isotropic half-cone $\{v \in \mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R} \mid v^2 = 0,\ v.E_0 > 0\}$ onto the boundary ∂D^n (which can be seen as the set of lines of this isotropic half-cone). We also want to mention the *Poincaré ball model* B^n of $\mathcal{H}^n$ which is obtained from the hyperboloid model by means of a stereographic projection from the south pole of the unit sphere of $\mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R}$ on the hyperplane $\{v \in \mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R} \mid v.E_0 = 0\}$.

Finally, if we denote by Nef X the nef effective cone of X and if $N := \pi(\mathrm{Nef}\, X \cap \mathcal{H}^n) \simeq (\mathrm{Nef}\, X \cap \mathcal{H}^n)/\mathbb{R}^*$, then we see easily that N is a closed convex subset of D^n and that, by Totaro Cone Theorem 3.3, Aut X (or $\mathrm{Aut}^* X$) acts on it with a *finitely sided* polyhedral fundamental domain (namely, the projection onto D^n of a polyhedral fundamental domain of the action on Nef X). Note that, as we said in the statement of Theorem 3.3, there is a fundamental domain $\mathcal{P}$ of the action of Aut X on Nef X which is the closed convex cone generated by finitely many points of $\mathrm{Pic}\, X \otimes_{\mathbb{Z}} \mathbb{R}$; hence, $\overline{P} := \pi(\mathcal{P}) \subseteq \overline{D^n}$ is the convex hull of finitely many points and classical results about convex polyhedra of $\mathbb{R}^n$ show that such a convex set is the intersection of finitely many half spaces and has finitely many sides (which are defined by the bounding hyperplanes of $\overline{P}$). Hence, this is also true for $P := \overline{P} \cap D^n$, the fundamental polyhedron of the action of Aut X on N.

In order to use Lemma 3.5 and Theorem 2.2, we have to prove that $\mathrm{Aut}^*X \rtimes \langle\sigma^*\rangle$ acts properly and cocompactly by isometries on a CAT(0) complete metric space. In fact, we are reduced to prove Lemma 3.9, which is the adaptation to our case of [11, 12.4.5, 1⇒ 2] (where we replaced a fundamental domain of the action on $\mathcal{H}^n$ by a fundamental domain on a closed convex subset, which is our N), and Lemma 3.10. *The proof ends on Sect. 2.* □

Lemma 3.7 *Any discrete subgroup* Γ *of* $\mathrm{Isom}\,(\mathcal{H}^n)$ *acts properly discontinuously on* $\mathcal{H}^n$.

Proof Note that the action of $\mathrm{Isom}\,(\mathcal{H}^n) \simeq \mathrm{O}^+(1, n)$ on $\mathcal{H}^n$ is transitive and that the stabilizer of a point $x \in \mathcal{H}^n$ (in the hyperboloid model) is the orthogonal group $\mathrm{O}(x^\perp) \simeq \mathrm{O}_n(\mathbb{R})$. Thus, this action induces a bijection $\mathcal{H}^n \simeq \mathrm{O}^+(1, n)/\mathrm{O}_n(\mathbb{R})$. Since Γ is discrete in the locally compact group $\mathrm{O}^+(1, n)$ and since $\mathrm{O}_n(\mathbb{R})$ is compact, the result follows from [15, 3.1.1]. □

Before stating our Lemmas 3.9 and 3.10, let us give some other definitions (cf. [11]):

Definition 3.8 Let Γ be a discrete subgroup of $\mathrm{Isom}\,(\mathcal{H}^n)$.

- A point $a \in \partial\mathcal{H}^n$ is a **limit point** of Γ if there is a point x of $\mathcal{H}^n$ and a sequence (g_i) of elements of Γ such that $(g_i(x))$ converges to a.
- In the Poincaré ball model B^n, an **horoball** based at a point $a \in \partial B^n$ is an Euclidean ball contained in $\overline{B^n}$ which is tangent to ∂B^n at the point a.
- Assume Γ contains a parabolic element (cf. [11, Sect. 4.7]) having $a \in \partial\mathcal{H}^n$ as its fixed point. **horocusp region** is an open horoball B based at a point $a \in \partial\mathcal{H}^n$ such that
$$\forall g \in \Gamma \setminus \mathrm{Stab}_\Gamma(a),\ \ g(B) \cap B = \varnothing.$$

The following Lemma develops and makes more precise an idea of Totaro in [14, Sect. 7]:

Lemma 3.9 *Let* Γ *be a discrete subgroup of* $\mathrm{Isom}\,(\mathcal{H}^n)$*,* $L(\Gamma)$ *the set of limit points of* Γ *in* $\overline{\mathcal{H}^n}$*,* $C(\Gamma)$ *the convex hull of* $L(\Gamma)$ *in* $\overline{\mathcal{H}^n}$ *and* N *a* Γ*-invariant closed convex subset of* $\mathcal{H}^n$.

If the action of Γ *on* N *has a finitely sided polyhedral fundamental domain* P*, then there exists a finite union (maybe empty)* V_0 *of horocusp regions with disjoint closures such that* $(P \cap C(\Gamma)) \setminus V_0$ *is compact.*

Sketch of proof The proof of this lemma is an adaptation of the proof of [11, 12.4.5, 1⇒ 2]: we replaced the fundamental domain of the action on $\mathcal{H}^n$ by a fundamental domain of the action on a closed convex subset, which is our N and we replaced geometrical finiteness hypothesis for Γ (which is more general than the existence of a finitely sided fundamental polyhedral domain of Γ on $\mathcal{H}^n$) by the hypothesis of the existence of a finitely sided fundamental polyhedral domain for the action of Γ on N. Thus, we have to check all the proofs of the results used by [11] in the proof of [11, 12.4.5, 1⇒ 2] in order to replace $\mathcal{H}^n$ by a closed convex subset N. The details of these verifications are in the Appendix. Here, we sum up the main points:

- if P_0 is a fundamental polyhedron of the action of Γ on $\mathcal{H}^n$, then $P = P_0 \cap N$ is a fundamental polyhedron of the action of Γ on $\mathcal{H}^n$
- by (6.6.10, 8.5.7), like $\mathcal{H}^n$, a closed convex subset N of $\mathcal{H}^n$ is a proper geodesically connected and geodesically complete metric space[6]: indeed, N is proper as a subspace of the proper metric space $\mathcal{H}^n$, N is geodesically complete, as it is complete, and it is geodesically connected, since it is convex. From this fact, we can deduce that the action of Γ on N has a (locally finite) exact convex fundamental polyhedron, e.g. a Dirichlet polyhedron, by (5.3.5, 6.6.13) and (6.7.4 (2)) since the group is discrete (and hence acts properly discontinuously, cf. Lemma 3.7) and since there is a point $a \in N$ whose stabilizer Γ_a is trivial (by (6.6.12)).
- for the other points, it is a question of replacing $\mathcal{H}^n$ by N and checking that everything remains true (sometimes by using convexity and/or closedness of N in $\mathcal{H}^n$). □

Lemma 3.10 *Let C be a closed convex subset of $\mathcal{H}^n$, Γ a discrete subgroup of* Isom $(\mathcal{H}^n)$ *stabilizing C, V_0 a finite family of open horoballs with disjoint closures and* $V_1 := \bigcup_{\gamma\in\Gamma} \gamma(V_0)$.

There is a family of open horoballs with disjoint closures, obtained by shrinking the horoballs of V_1, whose union U is such that $C \setminus U$ is a complete CAT(0) space.

Proof By [3, II.11.27], for every family U of *disjoint*[7] open horoballs, $\mathcal{H}^n \setminus U$ is a complete CAT(0) space for the induced length metric (this distance is defined between 2 points as the infimum of the lengths of rectifiable curves of $\mathcal{H}^n \setminus U$ between those two points ; it is different from the metric induced by the hyperbolic metric on $\mathcal{H}^n \setminus U$). Thus, $C \setminus U$ is complete as a closed subset of the complete space $\mathcal{H}^n \setminus U$.

It remains to study geodesic connectedness (term of [11, Sect. 1.4]) or convexity (term of [3, I.1.3]) of $C \setminus V_1$ in $\mathcal{H}^n \setminus V_1$ to conclude (using [3, II.1.15.(1)]) that $C \setminus V_1$ is CAT(0) for the metric induced by the distance of $\mathcal{H}^n \setminus V_1$ (which is itself the length metric induced by the metric of $\mathcal{H}^n$). So let $x, y \in C \setminus V_1$: if the geodesic γ of $\mathcal{H}^n$ joining x and y is contained in $\mathcal{H}^n \setminus V_1$, then it is also contained in $C \setminus V_1$ since C is convex. Otherwise, $\operatorname{Im}\gamma$ passes through at least one horoball and [3, II.11.33, II.11.34] shows that a geodesic δ of $\mathcal{H}^n \setminus V_1$ linking x and y is obtained by concatenation of the hyperbolic geodesics which are tangent to the bounding horospheres of the horoballs crossed by γ on the one hand and geodesics of these horo*spheres* on the other hand. A priori, it may happen that $\operatorname{Im}\delta$ is not contained in C. But we can shrink V_0 so that the antipodal point of the base point of each horosphere of V_0 belongs to C ([8]), which causes this effect on all the horoballs of V_1 under a finite number of operations. Hence, the geodesics of the horospheres are contained in C and the hyperbolic geodesics are contained in C because they join two points of C. Thus, if we denote by U the result of this shrinking of V_1, we showed that $C \setminus U$

[6] A metric space is **proper** or **finitely compact** if every bounded closed subset of it is compact.

[7] This is the key point which explains why we had to prove Lemma 3.9.

[8] More simply, the horospheres do not “get out” of C.

is geodesically connected in $\mathcal{H}^n \setminus U$ and this shows that $C \setminus U$ is a complete CAT(0) space. □

End of the proof of Theorem 3.4 We apply Lemma 3.9 with $\Gamma = \mathrm{Aut}^*X$, $N = \pi(\mathrm{Nef}(X) \cap \mathcal{H}^n) \simeq \mathrm{Nef}(X)/\mathbb{R}^*$ and P being a fundamental polyhedron of the action of Γ on N: this gives us a finite family V_0 of open horoballs *with disjoint closures* and a convex subset $P_C = P \cap C(\Gamma)$ of P such that $P_C \setminus V_0$ is *compact*. Thus, Γ acts properly (by Lemma 3.7) and cocompactly on $C \setminus V_1$ where $C := N \cap C(\Gamma)$ and $V_1 := \bigcup_{\gamma \in \Gamma} \gamma(V_0)$ (note that $C(\Gamma)$ is a Γ-invariant closed convex subset of $\mathcal{H}^n$).

Now, by Lemma 3.10, we can replace V_1 by another family U of open horoballs *with disjoint closures* such that $C \setminus U$ is a complete CAT(0) space. The compacity of a fundamental domain is preserved by this shrinking because $P \setminus (U \cap P)$ is a bounded closed subset of $\mathcal{H}^n$, so it is compact because $\mathcal{H}^n$ is a proper metric space. By the way, one can verify that the proof of [11, 12.4.5, 1⇒ 2] allows to shrink the horoballs without trouble.

Finally, we can conclude that Aut^*X acts properly and cocompactly on the complete CAT(0) space $C \setminus U$. It is not enough: in order to apply Lemma 3.5 and Theorem 2.2, we must obtain the same result for $\mathrm{Aut}^*X \rtimes \langle \sigma^* \rangle$, where σ is a real structure on X. By Lemma 3.7, the discrete isometry group $\mathrm{Aut}^*X \rtimes \langle \sigma^* \rangle$ acts properly on $\mathcal{H}^n$ (hence also on $\mathcal{H}^n \cap (C \setminus U)$). Since there is a fundamental domain of $\mathrm{Aut}^*X \rtimes \langle \sigma^* \rangle$ which is a closed subset of that of Aut^*X which is compact, we see that $\mathrm{Aut}^*X \rtimes \langle \sigma^* \rangle$ also acts cocompactly: this concludes the proof. □

Remark 3.11 In fact, we could give a much shorter proof of Theorem 3.4 if N were smooth complete.

First, note that N is a pinched Hadamard manifold[9] as a convex subset of $\mathcal{H}^n$: in particular, note that it is simply connected because of its convexity (which can be seen in the Klein model, where convexity is the same as Euclidean convexity and really implies simply connectedness).

Now, if we denote by P a fundamental domain of the action of Aut X on $N \subseteq D^n$, we remark that $\overline{P}$ is a fundamental domain of the action of Aut X on $\overline{N} \subseteq \overline{D^n}$ and that it is a convex polyhedron of the Klein model $\overline{D^n}$ of $\overline{\mathcal{H}^n}$ and also of the Euclidean space $\mathbb{R}^n$ (since convexity in the Klein model is the same as Euclidean convexity). Indeed, by Definition 3.6, we need to check that $\overline{P}$ has a locally finite collection of sides. But this is true since Totaro Cone Theorem 3.3 shows that it is finitely sided (as we have seen in the beginning of the proof of Theorem 3.4). Thus, by [11, 6.4.8], $\overline{P}$ has finite volume. Therefore, P is also of finite volume. Since there is a fundamental domain P' of Aut $X \rtimes \langle \sigma \rangle$ which is contained in P, we see that P' is also of finite volume. Thus, the quotient of N by $\Gamma :=$ Aut $X \rtimes \langle \sigma \rangle$ is a finite volume quotient of the pinched Hadamard manifold N and [4, 5.4.2, F5, 6.1, 5.5.2] shows that Γ has finitely many conjugacy classes of finite subgroups: thus X has a finite number of real forms by Lemma 3.5.

[9] A **pinched Hadamard manifold** is a complete simply connected Riemannian manifold whose all sectional curvatures lie between two negative constants.

4 Two Examples

Example 4.1 Here we study the example given by Totaro in [13]: it is a blow-up X of $\mathbb{P}^2$ at 12 points and we show that Aut X contains a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$. Since there exists a $\mathbb{R}$-divisor Δ such that (X, Δ) is a KLT Calabi–Yau pair, X has finitely many non-isomorphic real forms and this finiteness cannot be deduced from Theorem 1.

Let $\zeta = e^{2i\pi/3}$. We denote by X the blow-up of $\mathbb{P}^2$ at the 12 points of the set $\mathcal{P} = \{[1 : \zeta^i : \zeta^j],\ (i, j) \in [\![0; 2]\!]^2\} \cup \{[1 : 0 : 0],\ [0 : 1 : 0],\ [0 : 0 : 1]\}$. Let $C_1, \dots, C_9$ be the lines of $\mathbb{P}^2$ of equations $(y = x)$, $(y = \zeta x)$, $(y = \zeta^2 x)$, $(z = x)$, $(z = \zeta x)$, $(z = \zeta^2 x)$, $(z = y)$, $(z = \zeta y)$, $(z = \zeta^2 y)$. We can easily verify that each line C_i passes exactly through 4 of the points of $\mathcal{P}$ and that each point blown-up is the intersection point of exactly 3 of the C_i: this is called the *dual of Hesse configuration.*[10]

Note that $\left(X, \dfrac{1}{3}\displaystyle\sum_{i=1}^{9} \widehat{C_i}\right)$ is a KLT Calabi–Yau pair because:

- X is smooth;
- $\dfrac{1}{3}\displaystyle\sum_{i=1}^{9} \widehat{C_i}$ has simple normal crossings and its coefficients are < 1;
- $-K_X = \dfrac{1}{3}\displaystyle\sum_{i=1}^{9} \widehat{C_i}$.

Now, let us give some results about Aut X. Firstly, one can show that if a line D passes through one of the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ and one of the $[1 : \zeta^i : \zeta^j]$, then D is one of the C_i: for example, if D passes through $[1 : 0 : 0]$ and one of the $[1 : \zeta^i : \zeta^j]$, then, in the affine chart $(x \neq 0)$ of $\mathbb{P}^2$, we have $D = (z = \zeta^{j-i} y)$.

We claim that $\boxed{\mathrm{Aut}^{\#} X = \{\mathrm{Id}\}}$: since all the points blown-up belong to $\mathbb{P}^2$, it suffices to check that there does not exist any line passing through at least 11 of the 12 points of $\mathcal{P}$ (in fact, $\mathrm{Aut}^{\#} X$ is non-trivial if and only if $\mathcal{P}$ does not contain 4 points in general position, i.e. if and only if all the points of $\mathcal{P}$ are collinear except maybe only one of them). But if D was such a line, then it would necessarily pass through one of the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ and one of the $[1 : \zeta^i : \zeta^j]$: thus, D would be one of the C_i. Since none of the C_i passes through 11 of the 12 points blown-up, we see that D does not exist: this proves the claim.

By the end of the example of [13, Sect. 2], we have (denoting $E := \mathbb{C}/\mathbb{Z}[\zeta]$):

$$\boxed{\mathrm{Aut}\, X \simeq \mathrm{Aut}^{*} X = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \frac{GL_2(\mathbb{Z}[\zeta])}{\mathbb{Z}/3\mathbb{Z}} = \mathrm{Aut}\,((E \times E)/(\mathbb{Z}/3\mathbb{Z}))}$$

[10] Hesse configuration itself is not interesting for our purposes: since it contains exactly 9 points (and 12 lines), the surface obtained by blowing up these points has finitely many non-equivalent real structures by Theorem 1.

Finally, we want to show that Aut X contains a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$:

- it is well-known that $SL_2(\mathbb{Z})$ contains finite index subgroups isomorphic to $\mathbb{Z} * \mathbb{Z}$ (for example, $S := \left\langle \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} \right\rangle$ has index 12, cf. [6, II.25]) ;
- $GL_2(\mathbb{Z}[\zeta])$ acts on $E \times E = \mathbb{C}^2/(\mathbb{Z}[\zeta]^2)$ by matrix product and $\dfrac{GL_2(\mathbb{Z}[\zeta])}{\mathbb{Z}/3\mathbb{Z}}$ is the quotient by the subgroup generated by $\zeta.I_2$. Clearly, two elements of $SL_2(\mathbb{Z})$ (or even $GL_2(\mathbb{Z})$) are never equal modulo $\langle \zeta.I_2 \rangle$ so $SL_2(\mathbb{Z})$ injects into $\dfrac{GL_2(\mathbb{Z}[\zeta])}{\mathbb{Z}/3\mathbb{Z}}$: this concludes the proof.

Example 4.2 ([7, 6.10]) As promised, we now describe the example of a rational surface for which the finiteness problem for real forms remains open.

Let $L_1, \dots, L_5$ be five lines in general linear position in $\mathbb{P}^2$. For $i, j \in [\![1; 4]\!]$, we denote by p_{ij} the intersection point of L_i and L_j. Let us fix a cubic C_3 passing through the points p_{ij} and intersecting L_5 at three distinct points q_1, q_2, q_3. Finally, let a be another point of L_5.

We consider the blow-up X of $\mathbb{P}^2$ at the 10 points p_{ij}, q_k and a: it is a nodal Coble surface since $|-K_X| = \varnothing$ and $|-2K_X| = \{C_6 := R_1 + R_2 + R_3 + R_4 + 2R_5\}$, where the R_i's are the strict transform of the L_i's in X (note that X is nodal since $R_1,\dots,R_4$ are (-2)-curves).

In [7], it is claimed that Aut X has infinitely many orbits on the set of (-1)-curves of X but, in a private communication, Dolgachev explained me that there is a gap in the proof of this fact (more precisely, the elements of the group G constructed in *op. cit.* cannot be lifted to the double covering $S(A)$ of X ramified along $R_1 + \cdots + R_4$). Note that if it were true, this would show that X does not contain a divisor Δ such that (X, Δ) is a KLT Calabi–Yau pair. For, if such a divisor existed, then Cone Theorem would imply that Aut X has finitely many orbits on the extremal rays of the nef cone of X and this would be true also for its dual cone, which is the cone of curves of X (cf. [9, 4.1]): this is absurd because (-1)-curves form an Aut X-invariant subset of the set of extremal rays of $\overline{NE}(X)$.

However, note that $\left(X, \dfrac{1}{2}C_6\right)$ is a log-canonical Calabi–Yau pair since $\dfrac{1}{2}C_6 = \dfrac{1}{2}(R_1 + R_2 + R_3 + R_4) + R_5$ has clearly simple normal crossings, has coefficients ≤ 1 and satisfies the condition $K_X + \dfrac{1}{2}C_6 \equiv 0$.

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5 Appendix

In this appendix, we provide a detailed proof of Lemma 3.9, i.e. a detailed inspection of all the proofs of the results used by [11] in the proof of [11, 12.4.5] in order to replace $\mathcal{H}^n$ by a closed convex subset N. We recall here the statement of Lemma 3.9:

Lemma 3.9 *Let* Γ *be a discrete subgroup of* Isom $(\mathcal{H}^n)$*,* $L(\Gamma)$ *the set of limit points of* Γ *in* $\overline{\mathcal{H}^n}$*,* $C(\Gamma)$ *the convex hull of* $L(\Gamma)$ *in* $\overline{\mathcal{H}^n}$ *and* N *a* Γ*-invariant closed convex subset of* $\mathcal{H}^n$*.*

If the action of Γ *on* N *has a finitely sided polyhedral fundamental domain* P*, then there exists a finite union (maybe empty)* V_0 *of horocusp regions with disjoint closures such that* $(P \cap C(\Gamma)) \setminus V_0$ *is compact.*

In what follows, all numbers like (12.4.2) refer to [11]. Moreover, when some notations are undefined, please consider they are the same as in [11], *mutatis mutandis*. Finally, when some results cited in the diagram (Fig. 1) are not cited in the text below, then these are general results which apply to our case either without any change, or changing only $\mathcal{H}^n$ into N.

Some remarks are widely used below so we gather them here:

- if P_0 is a fundamental polyhedron of the action of Γ on $\mathcal{H}^n$, then $P = P_0 \cap N$ is a fundamental polyhedron of the action of Γ on $\mathcal{H}^n$
- by (6.6.10, 8.5.7), like $\mathcal{H}^n$, a closed convex subset N of $\mathcal{H}^n$ is a proper geodesically connected and geodesically complete metric space[11]: indeed, N is proper as a subspace of the proper metric space $\mathcal{H}^n$, N is geodesically complete, as it is complete, and it is geodesically connected, since it is convex. From this fact, we can deduce that the action of Γ on N has a (locally finite) exact convex fundamental polyhedron, e.g. a Dirichlet polyhedron, by (5.3.5, 6.6.13) and (6.7.4 (2)) since the group is discrete (and hence acts properly discontinuously, cf. Lemma 3.7) and since there is a point $a \in N$ whose stabilizer Γ_a is trivial (by (6.6.12)).
- if Γ is a discrete group of isometries of $\mathcal{H}^n$ (seen as Poincaré half-space), then the stabilizer Γ_∞ of the point at infinity induces a discrete subgroup of Isom $(\mathbb{R}^{n-1}) =$ Isom $(\partial\mathcal{H}^n \setminus \{\infty\})$. By (5.4.6), there is a Γ_∞-invariant affine subspace Q of $\mathbb{R}^{n-1}$ of dimension $m \leq n-1$ and Γ_∞ is a finite extension of a $\mathbb{Z}^m$. By (7.5.2), Γ_∞ is a crystallographic isometry group of $\mathbb{R}^m \simeq Q$, i.e. Q/Γ_∞ is compact
- for the other points, it is a question of replacing $\mathcal{H}^n$ by N and checking that everything remains true (sometimes by using convexity and/or closedness of N in $\mathcal{H}^n$).

(12.3.7): Γ is a discrete subgroup of Isom $(\mathcal{H}^n)$ so we can define "limit point", "bounded parabolic point"... with regard to its action on the whole space $\mathcal{H}^n$. Note that a is a limit point if and only if $\exists (g_i) \in \Gamma^{\mathbb{N}}, \forall x \in \mathcal{H}^n, g_i(x) \xrightarrow[i \to +\infty]{} a$: in particular, if $x \in N$, then $\forall i$, $g_i(x) \in N$. The rest of the proof can be followed, except that we can check that the geodesic ray R_i is contained in N.

[11] A metric space is **proper** or **finitely compact** if every bounded closed subset of it is compact.

(12.4.3) Firstly, note that (12.4.2) is not necessary for our purposes since P is finitely-sided. We can make the same reasoning with N instead of $\mathcal{H}^n$: if P is a fundamental polyhedron of Γ acting on N, then $\{g(P)|\ g \in \Gamma\}$ is an exact tessellation of N and $\{\nu g(P)|\ g \in \Gamma\} = \mathcal{T}$ is an exact tessellation of $\nu(N) \subseteq \mathbb{R}^{n-1}$. But $\bigcup_{g\in\Gamma} g(P) = N$ so $U \subseteq \nu(N)$. Since U is an open closed subset of $\mathbb{R}^{n-1}$ and $U \subseteq \nu(N)$, we see that U is open and closed in the non-empty connected space $\nu(N)$ so that $U = \nu(N)$.

(12.4.4) The beginning of the proof remains valid: it shows that if $x \in \overline{P} \cap L(\Gamma)$, where $\overline{P}$ is the closure of P in $\mathcal{H}^n$, then the stabilizer Γ_x is infinite and elementary of parabolic type (cf. [11, Sect. 5.5]). Of course, $\mathcal{T}$ is an exact tessellation of $\nu(N)$ instead of $\mathbb{R}^{n-1}$. If $c \in N$ is a cusp point of Γ, then $U(Q, r) \cap N \neq \varnothing$ because $U(Q, r)$ is a neighborhood of c: thus it suffices to replace $U(Q, r)$ by $N \cap U(Q, r)$ in the end of the proof to conclude.

(12.4.5) Firstly, we note that (12.4 Corollary 3) is a direct corollary of (12.3.7), (12.4.1) and (12.4.4) and that $\overline{P}$ is the closure of P in $\mathcal{H}^n$. It suffices to replace:

- "Γ is geometrically finite" by "the fundamental polyhedron of Γ on N is finitely-sided" (see (Sect. 12.4, Example 1));

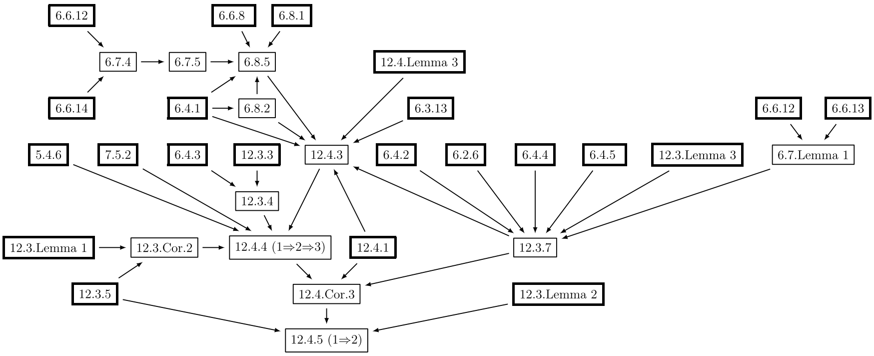

Fig. 1 We framed with bold lines the "initial" results, i.e. those whose proof does not require anything else that standard definitions and results (in topology, group theory, etc.) or those whose statement can be adapted to our case without examining their proof and the results used by it

- in view of the statement of our Lemma 3.9, all the statements made in the proof of (12.4.5) concerning π, V, M are useless for our purposes and all we need is $V_0 := \bigcup_{i=1}^{m} B_i$

and we have to note that K is a closed subset of B^n included in the closed subset N (since $P \subseteq N$) hence K is a closed subset of N.

References

1. Alexeev, V., Mori, S.: Bounding singular surfaces of general type. Algebra, Arithmetic and Geometry with Applications, vol. 2004, pp. 143–174. Springer, Berlin, West Lafayette, IN (2000)
2. Benzerga, M.: Real structures on rational surfaces and automorphisms acting trivially on Picard groups. Math. Z. **282**(3–4), 1127–1136 (2016)
3. Bridson, M.R., Haefliger, A.: Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319. Springer, Berlin (1999)
4. Bowditch, B.H.: Geometrical finiteness with variable negative curvature. Duke Math. J. **77**(1), 229–274 (1995)
5. Cantat, S., Dolgachev, I.: Rational surfaces with a large group of automorphisms. J. Amer. Math. Soc. **25**(3), 863–905 (2012)
6. de la Harpe, P.: Topics in geometric group theory. Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL (2000)
7. Dolgachev, I.V., Zhang, D.-Q.: Coble rational surfaces. Amer. J. Math. **123**(1), 79–114 (2001)
8. Lesieutre, J.: A projective variety with discrete, non-finitely generated automorphism group (2017). arXiv:1609.06391
9. Looijenga, E.: Discrete automorphism groups of convex cones of finite type. Compos. Math. **150**(11), 1939–1962 (2014)
10. Mangolte, F.: Variétés algébriques réelles. To appear, cf. http://math.univ-angers.fr/~mangolte/
11. Ratcliffe, J.G.: Foundations of Hyperbolic Manifolds. Graduate Texts in Mathematics, 2nd edn, vol. 149. Springer, New York (2006)
12. Serre, J.-P.: Galois cohomology, English edn. Springer Monographs in Mathematics. Springer, Berlin (2002). Translated from the French by Patrick Ion and revised by the author
13. Totaro, B.: The cone conjecture for Calabi-Yau pairs in dimension 2. Duke Math. J. **154**(2), 241–263 (2010)
14. Totaro, B.: Algebraic surfaces and hyperbolic geometry. Current developments in algebraic geometry. Mathematical Sciences Research Institute Publications, vol. 59, pp. 405–426. Cambridge University Press, Cambridge (2012)
15. Wolf, J.A.: Spaces of Constant Curvature, 5th edn. Publish or Perish (1984)

Anticanonical Volumes of Fano 4-Folds

Caucher Birkar

Abstract We find an explicit upper bound for the anticanonical volumes of Fano 4-folds with canonical singularities.

Keywords Fano varieties · Anticanonical volumes

2010 MSC 14J45 · 14E30

1 Introduction

We work over an algebraically closed field of characteristic zero.

Fano varieties constitute a fundamental class of algebraic varieties in algebraic geometry and many other fields. Due to their special features, it is more likely to have a detailed classification of Fano varieties compared to other classes such as Calabi–Yau varieties or varieties of general type.

An important step in classifying Fano varieties is to obtain an explicit upper bound for their anticanonical volume under mild conditions on the singularities. Bounding the anticanonical volume of smooth Fano 3-folds goes back to Fano himself and it is a crucial step in showing that smooth Fano 3-folds form a bounded family. Similarly boundedness of anticanonical volume of smooth Fano varieties of fixed dimension is used to show that such Fano varieties are bounded: see Nadel [14] for the Picard number one case and Kollár–Miyaoka–Mori [10] for the general case which also uses Mori's bend and break technique.

Not surprisingly everything gets more complicated when we allow singularities. The surface case is well-understood. An explicit upper bound for anticanonical volume of Fano 3-folds with canonical singularities is a quite recent result of Jiang–Zou [6]: the upper bound is 324. In the $\mathbb{Q}$-factorial Picard number one case, an explicit

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upper bound was earlier found by Lai [12] for ϵ-log canonical (ϵ-lc for short) Fano 3-folds. For more partial results in dimension 3, see [8, 11], and the references in [6].

In this note we find an explicit upper bound for Fano 4-folds with canonical singularities.

Theorem 1.1 *Any Fano variety X of dimension 4 with canonical singularities has*

$$\mathrm{vol}(-K_X) = (-K_X)^4 \leq (104\mu(3,1)+8)^4$$

where

$$\mu(3,1) = (840)^2 \left(\frac{6\left(\mu\left(2,\frac{1}{2}\right)+\frac{1}{2}\right)}{\frac{1}{2}}\right)^3$$

is given by Lemma 2.3 below and in turn $\mu(2,\frac{1}{2})$ is given by the formula

$$\mu(2,\delta) = \left(\frac{48}{\delta^2}\right) 2^{\frac{64}{\delta^3}}$$

appearing in Lemma 2.2 below.

To the best of our knowledge this is the first result of its kind in dimension 4 for singular Fano varieties. The proof closely follows the proof of [2, Theorem 1.6]. We did not aim to find an optimal bound but rather just an explicit bound. The upper bound in the theorem is unlikely to be anywhere close to the optimal bound. The number $\mu(3,1)$ is an explicit (not necessarily optimal) upper bound on cofficients of divisors $0 \leq B_V \sim_{\mathbb{R}} -K_V$ for Fano 3-folds V with canonical singularities. Here 3 stands for dimension and 1 stands for 1-log canonical which is the same as canonical. A similar notation is used below in dimension 2.

In dimension 3 we prove a more general result.

Theorem 1.2 *Let ϵ be a positive real number. Let X be a Fano variety of dimension 3 with ϵ-lc singularities. Then for any $0 < \delta < \epsilon$ we have*

$$\mathrm{vol}(-K_X) \leq \left(\frac{6\left(\mu\left(2,\delta\right)+\epsilon-\delta\right)}{\epsilon-\delta}\right)^3.$$

In particular, taking $\delta = \frac{\epsilon}{2}$, we have

$$\mathrm{vol}(-K_X) \leq v(3,\epsilon) := \left(\frac{6\left(\mu\left(2,\frac{\epsilon}{2}\right)+\frac{\epsilon}{2}\right)}{\frac{\epsilon}{2}}\right)^3.$$

Here the choice $\delta = \frac{\epsilon}{2}$ is arbitrary. When δ tends to ϵ, the right hand side of the first inequality tends to $+\infty$. Similarly when δ tends to 0 again the right hand side tends to $+\infty$. So the right hand side takes minimum for some value $\delta \in (0,\epsilon)$.

Chen Jiang informed us that he also has a proof of this theorem using different arguments.

Thanks to the referees for their helpful comments.

2 Proof of Results

We will use standard terminology in birational geometry regarding pairs, singularities, etc. Recall that a pair (X, B) has ϵ-log canonical (ϵ-lc) singularities if its log discrepancies are at least ϵ. When $B = 0$ we just say that X has ϵ-lc singularities.

Proposition 2.1 *Let $d \geq 2$ be a natural number and $\delta < \epsilon$ be positive real numbers. Assume that X is a Fano variety of dimension d with ϵ-lc singularities and with $\mathrm{vol}(-K_X) > (2d)^d$. Then for any real number $a > \frac{d}{\sqrt[d]{\mathrm{vol}(-K_X)}}$, there is a pair (V, Ω_V) where*

- $0 < \dim V < d$,
- *V is a δ-lc Fano variety,*
- *$K_V + \Omega_V \sim_{\mathbb{Q}} 0$, and*
- *there is a component of Ω_V with coefficient more than*

$$\frac{(1-2a)(\epsilon-\delta)}{2a}.$$

Proof *Step 1.* In this step we introduce some notation. Let $\alpha = \frac{d}{\sqrt[d]{\mathrm{vol}(-K_X)}}$. Then

$$\mathrm{vol}(-\alpha K_X) = \alpha^d \,\mathrm{vol}(-K_X) = d^d.$$

It is enough to prove the proposition for a rational number $a > \alpha$ sufficiently close to α. Then $\mathrm{vol}(-aK_X) > d^d$. Since $\mathrm{vol}(-K_X) > (2d)^d$ but

$$\mathrm{vol}(-2\alpha K_X) = 2^d \,\mathrm{vol}(-\alpha K_X) = (2d)^d,$$

we have $2\alpha < 1$, so we can assume that $2a < 1$.

Step 2. In this step we create a family of divisors and a covering family of subvarieties on X. Since $\mathrm{vol}(-aK_X) > d^d$, there exists $0 \leq B \sim_{\mathbb{Q}} -aK_X$ such that (X, B) is not klt (cf. [7, Lemma 3.2.2]). Pick a closed point $x \in X$ outside the non-klt locus of (X, B). Then again since $\mathrm{vol}(-aK_X) > d^d$, there exists $0 \leq C \sim_{\mathbb{Q}} -aK_X$ such that (X, C) is not klt at x. Changing C up to $\mathbb{Q}$-linear equivalence we can assume that (X, C) is not klt but lc at x. Perhaps increasing a slightly and changing C again we can assume that $(X, B + C)$ has a unique non-klt place whose centre, say G, contains x [1, Lemma 2.16] (also see [7, Lemma 3.2.3] and its proof). Now put $\Delta = B + C$.

In the above construction B is fixed but Δ depends on x. We have thus created a family of divisors Δ and a covering family of non-klt centres G on X.

By construction,

$$-(K_X + \Delta) = -(K_X + B + C) \sim_{\mathbb{Q}} -(K_X - aK_X - aK_X) = -(1 - 2a)K_X$$

is ample as $2a < 1$. Therefore, the non-klt locus of (X, Δ) is connected, by the connectedness principle [9, Theorem 17.4]. This locus contains G together with the non-klt locus of (X, B). Since x was chosen outside the non-klt locus of (X, B) and since G contains x, G is not contained in the non-klt locus of (X, B). Then G intersects another non-klt centre of (X, Δ). This in particular means $\dim G > 0$ because no other non-klt centre contains x.

Step 3. In this step we apply adjunction. From now on we assume that G is a general member of the above covering family. Let F be the normalisation of G. By [7, Theorem 4.2] (also see [2, Construction 3.9 and Theorem 3.10]), we can write an adjunction formula

$$(K_X + \Delta)|_F \sim_{\mathbb{Q}} K_F + \Delta_F := K_F + \Theta_F + P_F$$

where $\Theta_F \geq 0$ and P_F is pseudo-effective. Increasing a slightly and adding to Δ we can assume P_F is big and effective.

By assumption $\delta \in (0, \epsilon)$. Recall that G intersects another non-klt centre of (X, Δ). Then we can choose P_F such that we can assume (F, Δ_F) is not δ-lc by [2, Lemma 3.14(2)].

Step 4. In this step we define a boundary $\Pi_{F'}$ and a divisor $N_{F'}$. Let $F' \to F$ be a log resolution of (F, Δ_F). Let $K_{F'} + \Delta_{F'}$ be the pullback of $K_F + \Delta_F$. Define $\Pi_{F'}$ on F' as follows. For each prime divisor D on F' define the coefficient

$$\mu_D \Pi_{F'} := \begin{cases} 0 & \text{if } \mu_D \Delta_{F'} < 0, \\ \mu_D \Delta_{F'} & \text{if } 0 \leq \mu_D \Delta_{F'} \leq 1 - \delta, \\ 1 - \delta & \text{if } \mu_D \Delta_{F'} > 1 - \delta \end{cases}$$

Clearly $(F', \Pi_{F'})$ is a klt pair, in fact, it is δ-lc.

Put

$$N_{F'} := \Delta_{F'} - \Pi_{F'}.$$

Note that any component D of $N_{F'}$ with negative coefficient is also a component of $\Delta_{F'}$ with negative coefficient. Since the components of $\Delta_{F'}$ with negative coefficient are exceptional over F, we deduce that the pushdown of $N_{F'}$ to F is effective.

Step 5. In this step we consider a birational model F'' from which we obtain a Mori fibre space $F''' \to T$. Let $(F'', \Pi_{F''})$ be a log minimal model of $(F', \Pi_{F'})$ over F. We use $N_{F''}, \Delta_{F''}$ to denote the pushdowns of $N_{F'}, \Delta_{F'}$. We will use similar notation for other divisors and for pushdown to F''' defined below. By construction,

$$K_{F''} + \Pi_{F''} + N_{F''} = K_{F''} + \Delta_{F''} \sim_{\mathbb{Q}} 0/F,$$

so $N_{F''}$ is anti-nef over F. On the other hand, the pushdown of $N_{F''}$ to F is effective. So by the negativity lemma, $N_{F''} \geq 0$. In particular, $\Delta_{F''} \geq 0$. Moreover, since (F, Δ_F) is not δ-lc, $(F'', \Delta_{F''})$ is not δ-lc while $(F'', \Pi_{F''})$ is δ-lc. Therefore, $N_{F''} \neq 0$.

Since $-(K_X + \Delta)$ is ample, $-(K_F + \Delta_F)$ is ample, hence $-(K_{F''} + \Delta_{F''})$ is semi-ample and big. Pick a general

$$0 \leq L_{F''} \sim_{\mathbb{Q}} -(K_{F''} + \Delta_{F''})$$

so that $(F'', \Pi_{F''} + L_{F''})$ is δ-lc. Now running an MMP on $K_{F''} + \Pi_{F''} + L_{F''}$ ends with a Mori fibre space $F''' \to T$ because

$$K_{F''} + \Pi_{F''} + L_{F''} + N_{F''} = K_{F''} + \Delta_{F''} + L_{F''} \sim_{\mathbb{Q}} 0$$

and $N_{F''} \neq 0$.

Step 6. In this step we finish the proof. By [7, Theorem 4.2], [2, Theorem 3.12], we can write $K_X|_F = K_F + \Lambda_F$ where (F, Λ_F) is sub-ϵ-lc and $\Lambda_F \leq \Delta_F$ (Λ_F may have negative coefficients). Let $K_{F''} + \Lambda_{F''}$ be the pullback of $K_F + \Lambda_F$. Then $(F'', \Lambda_{F''})$ is also sub-ϵ-lc, so the coefficients of $\Lambda_{F''}$ are $\leq 1 - \epsilon$. Moreover, $\Lambda_{F''} \leq \Delta_{F''}$.

By construction $N_{F'''}$ is ample over T. Let D'' be a component of $N_{F''}$ so that D''' is ample over T. By the definition of $\Pi_{F'}$ and the fact $\Delta_{F''} \geq 0$, the components of $N_{F''}$ are exactly the components of $\Delta_{F''}$ with coefficient $> 1 - \delta$. So we have

$$\mu_{D''}(\Delta_{F''} - \Lambda_{F''}) > 1 - \delta - (1 - \epsilon) = \epsilon - \delta.$$

Note that

$$\Delta_{F''} - \Lambda_{F''} = (K_{F''} + \Delta_{F''}) - (K_{F''} + \Lambda_{F''})$$

$$\sim_{\mathbb{Q}} (K_X + \Delta)|_{F''} - (K_X|_{F''}) = \Delta|_{F''} \sim_{\mathbb{Q}} -2aK_X|_{F''}.$$

On the other hand,

$$L_{F''} \sim_{\mathbb{Q}} -(K_X + \Delta)|_{F''} \sim_{\mathbb{Q}} -(1 - 2a)K_X|_{F''}$$

$$= \frac{1-2a}{2a}(-2aK_X|_{F''}) \sim_{\mathbb{Q}} \frac{1-2a}{2a}(\Delta_{F''} - \Lambda_{F''}).$$

Now let V be a general fibre of $F''' \to T$. Then from

$$K_{F'''} + \Omega_{F'''} := K_{F'''} + \Pi_{F'''} + \frac{1-2a}{2a}(\Delta_{F'''} - \Lambda_{F'''}) + N_{F'''}$$

$$\sim_{\mathbb{Q}} K_{F'''} + \Pi_{F'''} + L_{F'''} + N_{F'''} = K_{F'''} + \Delta_{F'''} + L_{F'''} \sim_{\mathbb{Q}} 0$$

and its restriction to V we get (V, Ω_V) such that $K_V + \Omega_V \sim_{\mathbb{Q}} 0$. On the other hand, since $(F'', \Pi_{F''} + L_{F''})$ is δ-lc, $(F''', \Pi_{F'''} + L_{F'''})$ is δ-lc, hence we see that F''' is δ-lc, so V is a δ-lc Fano variety with $0 < \dim V < d$.

Since D''' is a component of $\Delta_{F'''} - \Lambda_{F'''}$ with coefficient $> \epsilon - \delta$ and since D''' intersects V, we deduce that Ω_V has a component with coefficient more than

$$\frac{(1-2a)(\epsilon-\delta)}{2a}.$$

□

Lemma 2.2 *Let ϵ be a positive real number. Let X be an ϵ-lc Fano surface and let $B \geq 0$ be an $\mathbb{R}$-divisor with $K_X + B \sim_{\mathbb{R}} 0$. Then the coefficient of each component of B is $\leq \mu(2, \epsilon)$ where*

$$\mu(2,\epsilon) := \left(\frac{48}{\epsilon^2}\right) 2^{\frac{64}{\epsilon^3}}.$$

Proof By the proof of [5, Theorem 2.8], any coefficient of B is at most

$$l(\epsilon) := \frac{(2+4\epsilon)(4F_{\lfloor 64/\epsilon^3 \rfloor + 2} - 4)}{\epsilon^2}$$

where F_n denotes the Fibonacci number defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Inductively we can easily see that $F_n \leq 2^{n-1}$. So

$$l(\epsilon) \leq \frac{24 F_{\lfloor 64/\epsilon^3 \rfloor + 2}}{\epsilon^2} \leq \mu(2,\epsilon) := \left(\frac{48}{\epsilon^2}\right) 2^{64/\epsilon^3}.$$

□

Proof (*of Theorem* 1.2) Let X be a Fano 3-fold with ϵ-lc singularities. The right hand side of the both inequalities in Theorem 1.2 are more than 6^3, so it is enough to treat the case when $\mathrm{vol}(-K_X) > 6^3$. Pick a positive real number $\delta < \epsilon$ and pick a real number $a > \frac{3}{\sqrt[3]{\mathrm{vol}(-K_X)}}$. Applying Proposition 2.1, there is a pair (V, Ω_V) where

- $0 < \dim V < 3$,
- V is a δ-lc Fano variety,
- $K_V + \Omega_V \sim_{\mathbb{Q}} 0$, and
- there is a component of Ω_V with coefficient more than

$$\frac{(1-2a)(\epsilon-\delta)}{2a}.$$

So $\dim V = 1$ or 2. If $\dim V = 1$, then $V \simeq \mathbb{P}^1$, so

$$\frac{(1-2a)(\epsilon-\delta)}{2a} < \mu(1,\delta) := 2.$$

On the other hand, if $\dim V = 2$, then by Lemma 2.2, we have

$$\frac{(1-2a)(\epsilon-\delta)}{2a} < \mu(2,\delta) = \left(\frac{48}{\delta^2}\right) 2^{\frac{64}{\delta^3}}.$$

Note that $\mu(1,\delta) < \mu(2,\delta)$. Thus we can calculate that

$$\frac{1}{a} < \frac{2(\mu(2,\delta)+\epsilon-\delta)}{\epsilon-\delta}.$$

Fixing δ and taking the limit when a approaches $\alpha := \frac{3}{\sqrt[3]{\mathrm{vol}(-K_X)}}$, we see that

$$\frac{1}{\alpha} \leq \frac{2(\mu(2,\delta)+\epsilon-\delta)}{\epsilon-\delta}.$$

This in turn gives

$$\mathrm{vol}(-K_X) \leq \left(\frac{2\left(\mu(2,\delta)+\epsilon-\delta\right)}{\epsilon-\delta}\right)^3 3^3.$$

Applying this to $\delta := \frac{\epsilon}{2}$, we have

$$\mathrm{vol}(-K_X) \leq \left(\frac{2\left(\mu\left(2,\frac{\epsilon}{2}\right)+\frac{\epsilon}{2}\right)}{\frac{\epsilon}{2}}\right)^3 3^3.$$

□

The next lemma is preparation for the proof of boundedness of volume in dimension 4.

Lemma 2.3 *Let X be a Fano 3-fold with canonical singularities and let $B \geq 0$ be an $\mathbb{R}$-divisor with $K_X + B \sim_{\mathbb{R}} 0$. Then the coefficient of each component of B is $\leq \mu(3,1)$ where*

$$\mu(3,1) := (840)^2 v(3,1) = (840)^2 \left(\frac{6\left(\mu\left(2,\frac{1}{2}\right)+\frac{1}{2}\right)}{\frac{1}{2}}\right)^3.$$

Proof By applying [3, Proposition 2.4] to a terminal crepant model of X, we deduce that IK_X is Cartier for some natural number $I \leq 840$. On the other hand, we need an upper bound for the $\mathrm{vol}(-K_X)$. Such a bound is given by Theorem 1.2 which is

$$v(3,1) = \left(\frac{6\left(\mu\left(2,\frac{1}{2}\right)+\frac{1}{2}\right)}{\frac{1}{2}}\right)^3.$$

One could also use the upper bound $\mathrm{vol}(-K_X) \leq 324$ by [6] but to make the theorem logically independent of [6] we will use $v(3, 1)$.

Let D be a component B. Then

$$\mu_D B \leq (\mu_D B)D \cdot (-IK_X)^2 \leq B \cdot (-IK_X)^2$$

$$= (I)^2(-K_X)^3 \leq (840)^2 v(3, 1).$$

□

Proof *(of Theorem 1.1)* Let X be a Fano 4-fold with canonical singularities. We can assume that $\mathrm{vol}(-K_X) > 8^4$. Pick a positive real number $\delta \in (\frac{12}{13}, 1)$ and pick a real number $a > \alpha := \frac{4}{\sqrt[4]{\mathrm{vol}(-K_X)}}$. Applying Proposition 2.1, there is a pair (V, Ω_V) where

- $0 < \dim V < 4$,
- V is a δ-lc Fano variety,
- $K_V + \Omega_V \sim_{\mathbb{Q}} 0$, and
- there is a component of Ω_V with coefficient more than

$$\frac{(1-2a)(1-\delta)}{2a}.$$

Since $\delta \in (\frac{12}{13}, 1)$ and since V is δ-lc of dimension at most 3, V actually has canonical singularities, by [4, 13]. Therefore, considering the cases $\dim V = 1, 2, 3$ separately, we have

$$\frac{(1-2a)(1-\delta)}{2a} < \max\{\mu(1, 1), \mu(2, 1), \mu(3, 1)\} = \mu(3, 1).$$

So we get

$$\frac{1}{a} < \frac{2(\mu(3, 1) + 1 - \delta)}{1-\delta}.$$

Taking limit as a approaches α we then have

$$\frac{1}{\alpha} = \frac{\sqrt[4]{\mathrm{vol}(-K_X)}}{4} \leq \frac{2(\mu(3, 1) + 1 - \delta)}{1-\delta}.$$

In turn taking limit when δ approaches $\frac{12}{13}$ we see that

$$\mathrm{vol}(-K_X) \leq \left(\frac{8\left(\mu(3, 1) + \frac{1}{13}\right)}{\frac{1}{13}}\right)^4 = (104\mu(3, 1) + 8)^4.$$

We can now apply Lemma 2.3 to get an explicit bound. □

References

1. Birkar, C.: Geometry of polarised varieties. arXiv:2006.11238v2
2. Birkar, C.: Anti-pluricanonical systems on Fano varieties. Ann. Math. (2) **190**(2), 345–463 (2019)
3. Chen, M., Jiang, C.: On the anti-canonical geometry of Q-Fano threefolds. J. Differ. Geom. **104**(1), 59–109 (2016)
4. Jiang, C.: A gap theorem for minimal log discrepancies of non-canonical singularities in dimension three. arXiv:1904.09642v1
5. Jiang, C.: Boundedness of anti-canonical volumes of singular log Fano threefolds. arXiv:1411.6728
6. Jiang, C., Zou, Y.: An effective upper bound for anti-canonical volumes of canonical Q-Fano threefolds. arXiv:2107.01037
7. Hacon, C.D., M^{c}Kernan, J., Xu, C.: ACC for log canonical thresholds. Ann. Math. (2) **180**(2), 523–571 (2014)
8. Kawamata, Y.: Boundedness of $\mathbb{Q}$-Fano threefolds. In: Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989). Contemporary Mathematics, vol. 131, pp. 439–445. American Mathematical Society, Providence, RI (1992)
9. Kollár, J.: Flips and Abundance for Algebraic Threefolds. Astérisque 211. Societe Mathematique de France, Paris (1992), (with 14 co-authors; papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991)
10. Kollár, J., Miyaoka, Y., Mori, S.: Rational connectedness and boundedness of Fano manifolds. J. Differ. Geom. **36**(3), 765–779 (1992)
11. Kollár, J., Miyaoka, Y., Mori, S., Takagi, H.: Boundedness of canonical $\mathbb{Q}$-Fano 3-folds. Proc. Jpn. Acad. Ser. A Math. Sci. **76**(5), 73–77 (2000)
12. Lai, C.-J.: Bounding volumes of singular Fano threefolds. Nagoya Math. J. **224**(1), 37–73 (2016)
13. Liu, J., Xiao, L.: An optimal gap of minimal log discrepancies of threefold non-canonical singularities. arXiv:1909.08759v2
14. Nadel, A.M.: The boundedness of degree of Fano varieties with Picard number one. J. Amer. Math. Soc. **4**(4), 681–692 (1991)

Constant Scalar Curvature Sasaki Metrics and Projective Bundles

Charles P. Boyer and Christina W. Tønnesen-Friedman

Abstract In this paper we consider the Boothby-Wang construction over twist 1 stage 3 Bott orbifolds given in terms of the log pair $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$. We give explicit constant scalar curvature (CSC) Sasaki metrics either directly from CSC Kähler orbifold metrics or by using the weighted extremal approach of Apostolov and Calderbank. The Sasaki 7-manifolds (orbifolds) are finitely covered by compact simply connected manifolds (orbifolds) with the rational homology of the 2-fold connected sum of $S^2 \times S^5$.

Keywords Admissible construction · Extremal sasaki metrics · CSC sasaki metrics

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Introduction

Arguably the foremost problem in Sasaki geometry is that of determining which isotopy classes of Sasakian structures admit Sasaki metrics of constant scalar curvature (CSCS). The Sasaki version of the Yau-Tian-Donaldson conjecture says roughly that CSCS metrics correspond to the affine variety satisfying some type of K-stability requirement. Collins and Székelyhidi [28] proved that a CSC Sasaki metric implies the K-semistablility of corresponding affine Kähler variety. Recently, following the

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important work in the Kähler category by Chen and Cheng [23, 24], He and Li [36, 37] have related the existence of CSC Sasaki metrics to the reduced properness of the Mabuchi K-energy.

Complementary to these existence results, it is important to explore explicit examples. Of particular interest to us are those Sasaki manifolds (orbifolds) that admit a transverse admissible construction [3]. A special case of such Sasaki orbifolds consist of principle S^1 orbibundles over Bott manifolds (orbifolds) [9, 33] to be described in the first Section. In this paper we discuss two fairly explicit methods for constructing constant scalar curvature Sasaki metrics on a class of 7-manifolds having the rational cohomology of the 2-fold connected sum of $S^2 \times S^5$. Specifically, these 7-manifolds, $M^7_{\mathbf{n},\mathbf{m}}$, arise by a Boothby-Wang construction over a polarized admissible orbifold, $S_{\mathbf{n},\mathbf{m}}$, given as a log pair $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$, where $S_{\mathbf{n}} = \mathbb{P}\big(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)\big) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\Delta_{\mathbf{m}} = \Big(1 - \frac{1}{m_0}\Big)e_0 + \Big(1 - \frac{1}{m_\infty}\Big)e_\infty$, with e_0 and e_∞ are the infinity sections of the bundle $S_{\mathbf{n}}$. The first method is the standard Boothby-Wang construction on twist one stage 3 Bott orbifolds (KS orbifolds). The second method applies the weighted extremal approach of Apostolov and Calderbank [1]. Both methods employ the admissible construction at the Kähler level (see [3]) either directly or indirectly. Specifically, in Sect. 4, with the main work horses being Propositions 4.16 and Proposition 4.14 (the latter with limitations as mentioned in Remark 4.15), we prove Theorem 4.17 which together with Remark 4.18 give.

Main Theorem. *Let $M^7_{\mathbf{n},\mathbf{m}}$ be a Boothby-Wang constructed manifold with maximal symmetry over a polarized KS orbifold. Then there always exists a positive constant scalar curvature Sasaki metric in the Sasaki cone.*

In terms of K-stability we have the following corollary of the Main Theorem and Corollary 1.1 of [28].

Corollary *Let $M^7_{\mathbf{n},\mathbf{m}}$ be a Boothby-Wang constructed Sasaki manifold over the KS orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$. Then there exists a Reeb vector field ξ in the Sasaki cone such that the corresponding polarized affine variety (Y, ξ) is K-semistable, equivalently the Sasakian structure (ξ, η, Φ, g) is K-semistable.*

In the Gorenstein case, a much stronger statement is known, namely the Sasaki version of the famous Chen-Donaldson-Sun [25–27] result by Collins and Székelyhidi [29]. In particular, the recent classification of all smooth Fano threefolds that admit Kähler-Einstein metrics [2] gives a classification of all regular Sasaki-Einstein manifolds over smooth KS threefolds.

It is interesting to compare the Sasaki-Einstein solutions obtained from the Boothby-Wang construction from the orbifold Kähler-Einstein solutions of Sect. 4.4 with those described in [18]. Both types live on simply connected 7-manifolds with the rational cohomology of the 2-fold connected sum of $S^2 \times S^5$ with torsion in H^4. However, they are of a somewhat different nature. Those in [18] have holomorphic twist 2, whereas, those in this paper have holomorphic twist 1. Assuming they have the same torsion group one may ask whether they are homotopy equivalent, have the same diffeomorphism type, and/or belong to inequivalent CR structures, contact structures, etc.

1 The Projective Bundles

We view our projective bundle as a special type of stage 3 Bott manifold, namely the Bott tower $M_3(0, n_1, n_2)$ where $n_1, n_2 \in \mathbb{Z}$. We recall the definition of a *stage k Bott manifold* of the *Bott tower of height n*:

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots \to M_2 \xrightarrow{\pi_2} M_1 = \mathbb{CP}^1 \xrightarrow{\pi_1} \{pt\}$$

where M_k is defined inductively as the total space of the projective bundle

$$\mathbb{P}(\mathbb{1} \oplus L_k) \xrightarrow{\pi_k} M_{k-1}$$

with fiber $\mathbb{CP}^1$, and some holomorphic line bundle L_k on M_{k-1}.We refer to [9, 33] for details. In this paper we treat twist 1 stage 3 Bott manifolds $M_3(0, n_1, n_2)$ and orbifolds. Explicitly these are ruled manifolds of the form

$$S_{\mathbf{n}} = \mathbb{P}\big(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)\big) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 \tag{1}$$

with $n_1, n_2 \in \mathbb{Z}^*$. To this we add an orbifold structure given in terms of the log pair $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ where

$$\Delta_{\mathbf{m}} = \Big(1 - \frac{1}{m_0}\Big)e_0 + \Big(1 - \frac{1}{m_\infty}\Big)e_\infty \tag{2}$$

and e_0, e_∞ are defined by the zero and infinity sections of the bundle. A motivation of our study comes from the 1986 paper of Koiso and Sakane [44] where they study Kähler-Einstein metrics on projectivizations of the form $\mathbb{P}(\mathbb{1} \oplus L_1 \otimes_{ext} L_2)$ over $N \times N$ where N has a Kähler-Einstein metric. For this reason we refer to our log pairs $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ as *KS orbifolds*. The underlying complex manifolds are smooth toric varieties whose Kähler geometry was studied in [9]. The positive integers m_0, m_∞ are called *ramification indices*. For each choice of orbifold Kähler form $\omega_{\mathbf{n},\mathbf{m}}$, defining an integer Kähler class on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ we have a principal S^1 orbibundle

$$S^1 \longrightarrow M^7 \longrightarrow (S_{\mathbf{n}}, \Delta_{\mathbf{m}})$$

with an induced Sasakian structure $\mathcal{S}_{\mathbf{n},\mathbf{m}}$; moreover, when the class $[\omega_{\mathbf{n},\mathbf{m}}]$ is primitive in $H^2_{orb}((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$, M^7 is simply connected. While there are only 2 equivalences classes of Fano KS manifolds, it is clear from Proposition 1.1 below that log Fano KS orbifolds are abundant.

In [16] we described a functor from the category of $S^3_{\mathbf{w}}$ Sasaki joins to the category of Bott orbifolds. However, as noted there, this functor is not surjective on objects. One purpose of this paper is to study this lack of surjectivity in the particular case of twist 1 stage 3 Bott orbifolds. A key notion of Sasaki joins is that of *cone decomposability* which can be thought of as the Sasaki version of de Rham decomposability in the Kähler case [14]. Cone decomposability is an invariant of the underlying Sasaki CR structure. We shall see how our KS orbifolds give rise to both cone decomposable Sasakian structures as well as Sasakian structures that are likely cone indecomposable. See Remark 3.10 below.

Note that the ordinary Bott manifold $M_3(0, n_1, n_2)$ corresponds to the trivial orbifold $(S_{\mathbf{n}}, \emptyset)$. We denote by KS^{orb} the set of manifolds with these orbifold structures, that is,

$$KS^{orb} = \{(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \mid n_1, n_2 \in \mathbb{Z}^*,\ m_0, m_\infty \in \mathbb{Z}^+\}.$$

Such orbifolds are the objects of a groupoid whose morphisms are biholomorphisms that intertwine the orbifold structures. Actually we are interested in *polarized* KS orbifolds that are polarized by a (primitive) orbifold Kähler class $[\omega_{\mathbf{n},\mathbf{m}}] \in H^2_{orb}(S_{\mathbf{n}}, \mathbb{Z})$. These form the objects of our groupoid $\mathcal{KS}$ whose morphisms are biholomorphisms that intertwine the orbifold structures and intertwine their orbifold Kähler classes. We remark that since the objects of $\mathcal{KS}$ are themselves orbifolds, we are working with a 2-category. The action of the Coexter group $\mathrm{Sym}_2 \ltimes \mathbb{Z}_2^3$ on KS_{orb} is generated by the transposition $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \mapsto (S_{n_2,n_1}, \Delta_{\mathbf{m}})$ and the fiber inversion sending $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \mapsto (S_{-\mathbf{n}}, \Delta_{m_\infty,m_0})$. Note the interchange of ramification indices m_0, m_∞ induced by the fiber inversion map.

Of special interest is the full subgroupoid $\mathcal{KS}^{mon}$ of monotone KS orbifolds which we now describe. Consider the orbifold canonical divisor or dually the orbifold first Chern class. In particular, when is the orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ log Fano, equivalently when does $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ lie in the Kähler cone? From Sect. 1.3 of [18] we have

$$\begin{aligned} c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) &= \Big(2 - \frac{n_1}{m_\infty}\Big)y_1 + \Big(2 - \frac{n_2}{m_\infty}\Big)y_2 + \Big(\frac{1}{m_0} + \frac{1}{m_\infty}\Big)y_3 \\ &= \Big(2 + \frac{n_1}{m_0}\Big)x_1 + \Big(2 + \frac{n_2}{m_0}\Big)x_2 + \Big(\frac{1}{m_0} + \frac{1}{m_\infty}\Big)x_3. \end{aligned} \tag{3}$$

with respect to the invariant bases $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$, respectively. For $i = 1, 2$ here $x_i = y_i$ is the class of the Fubini-Study metric on the ith factor of the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ pulled back to $S_{\mathbf{n}}$, while $x_3(y_3)$ is the Poincaré dual of $e_\infty(e_0)$, respectively. From this we arrive at a special case of Lemma 1.2 of [18].

Proposition 1.1 *The orbifold* $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ *in* KS^{orb} *is log Fano if and only if the inequalities*

$$\frac{n_1}{m_\infty} < 2, \quad \frac{n_2}{m_\infty} < 2, \quad -\frac{n_1}{m_0} < 2, \quad -\frac{n_2}{m_0} < 2$$

hold.

We obtain a primitive Kähler class $\frac{c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})}{\mathcal{I}_{\mathbf{n},\mathbf{m}}}$ in $H^2_{orb}((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$ with *Fano index* given by

$$\mathcal{I}_{\mathbf{n},\mathbf{m}} = \gcd(2m v_0 v_\infty - n_1 v_0, 2m v_0 v_\infty - n_2 v_0, v_0 + v_\infty) \tag{4}$$

where $\mathbf{m} = (m_0, m_\infty) = m(v_0, v_\infty) = m\mathbf{v}$ and $m = \gcd(m_0, m_\infty)$. Here we have used the identity $x_3 = y_3 - n_1x_1 - n_2x_2$. The orbifold $(S_\mathbf{n}, \Delta_\mathbf{m})$ is log Fano if and only if

$$c_1^{orb}(S_\mathbf{n}, \Delta_\mathbf{m}) = \frac{1}{mv_0v_\infty}\Big[\Big(2mv_0v_\infty - n_1v_0\Big)y_1 + \Big(2mv_0v_\infty - n_2v_0\Big)y_2 + \Big(v_0 + v_\infty\Big)y_3\Big]$$

is positive.

The Poincaré dual to $c_1^{orb}(S_\mathbf{n}, \Delta_\mathbf{m})$ is the orbifold anti-canonical divisor which is an ample $\mathbb{Q}$-divisor on $S_\mathbf{n}$ when $c_1^{orb}(S_\mathbf{n}, \Delta_\mathbf{m})$ is positive. The primitive class $\frac{c_1^{orb}(S_\mathbf{n}, \Delta_\mathbf{m})}{\mathcal{I}_{\mathbf{n},\mathbf{m}}}$ can be represented by an orbifold Kähler form $\omega_{\mathbf{n},\mathbf{m}}$ on $S_\mathbf{n}$, or equivalently an orbifold Kähler metric $g_{\mathbf{n},\mathbf{m}}$.

2 The Orbifold Boothby-Wang Construction

The well known Boothby-Wang Theorem [21] that associates regular contact structures to the total space of S^1 bundles over symplectic manifolds generalizes easily to the orbifold category. See for example Theorem 7.1.3 of [11]. Here as in this theorem we are interested in Sasakian structures over Kählerian structures. Given an integral Kähler class $[\omega_{\mathbf{n},\mathbf{m}}] \in H^{1,1}_{orb}((S_\mathbf{n}, \Delta_\mathbf{m}), \mathbb{Z})$ we shall always choose a Kähler form $\omega_{\mathbf{n},\mathbf{m}}$ with maximal symmetry. In this case the total space M of the corresponding principal S^1 orbibundle has a maximal family $\mathfrak{t}^+$ of Sasakian structures, called the *Sasaki cone*. Moreover, choosing a connection 1-form η on M such that $d\eta = \omega_{\mathbf{n},\mathbf{m}}$ determines a natural Sasakian structure $\mathcal{S}_{\mathbf{n},\mathbf{m}} = (\xi, \eta, \Phi, g)$ in $\mathfrak{t}^+$. We refer to this construction as the *orbifold Boothby-Wang construction*. Generally, unlike $(S_\mathbf{n}, \Delta_\mathbf{m})$, the total space M has an orbifold structure whose underlying topological space is singular with finite cyclic singularities. The general problem of finding conditions when M is smooth is quite subtle [40, 43, 49]: however, Kegel and Lange show that M is a smooth manifold if and only if $H^r_{orb}(M, \mathbb{Z}) = 0$ for all $r > 2n + 1$. More precisely M is smooth if and only if multiplication by the Euler class $H^r_{orb}(\mathcal{Z}, \mathbb{Z}) \xrightarrow{\cup_e} H^{r+2}_{orb}(\mathcal{Z}, \mathbb{Z})$ in the Gysin sequence of the orbibundle $M \longrightarrow \mathcal{Z}$ is an isomorphism for all $r > 2n + 1$. As we shall see in Sect. 3.5 precise conditions can be obtained in certain special cases (see also [11]).

The question arises as to when there exists a constant scalar curvature Sasaki metric in $\mathfrak{t}^+$ and how many. In the Gorenstein case this was answered by Futaki, Ono, and Wang [32], but the general CSC case remains open. Some partial results have been obtained by Legendre [45], and by the authors [17, 19] and references therein.

Remark 2.1 In categorical language we work with the groupoid $\mathcal{SKS}$ whose objects $\{M^7_{\mathbf{n},\mathbf{m}}\}$ are simply connected quasiregular Sasaki 7-orbifolds whose projective algebraic base is a Koiso-Sakane orbifold, and whose morphisms are orbifold biholomorphisms. Here the objects are classes of Sasakian or Koiso-Sakane structures, thus again, they are 2-categories. We are mainly interested in certain subgroupoids, namely, the full subgroupoid $\mathcal{SKS}^F$ of positive Sasakian structures whose base Koiso-Sakane orbifold is log Fano. Actually, we consider the full subgroupoid $\mathcal{SKS}^{MF}$ whose Kähler form $\omega_{\mathbf{n},\mathbf{m}}$ lies in $\frac{c_1^{orb}(S_{\mathbf{n}},\Delta_{\mathbf{m}})}{\mathcal{I}_{\mathbf{n},\mathbf{m}}}$, that is the Koiso-Sakane base orbifold is $(S_{\mathbf{n}},\Delta_{\mathbf{m}})$ is monotone log Fano. We also define the subgroupoids $\mathcal{SKS}_{\mathbf{m}}, \mathcal{SKS}^F_{\mathbf{m}}, \mathcal{SKS}^{MF}_{\mathbf{m}}$ consisting of the corresponding objects with fixed $\mathbf{m}$. The objects of the groupoid $\mathcal{SKS}_{\mathbf{m}}$ are Sasaki classes on the 7-manifolds $M^7_{\mathbf{m}}$ whose cohomology we describe in the next section. The morphisms are those induced by diffeomorphisms that intertwine the corresponding Sasakian structures.

3 The Topology of the Orbifolds

In this section we describe both the orbifold topology of $(S_{\mathbf{n}},\Delta_{\mathbf{m}})$ and topology of the total space of the principal S^1 orbibundles over the log Fano Koiso-Sakane orbifolds $(S_{\mathbf{n}},\Delta_{\mathbf{m}})$. First we recall the cohomology ring of the complex manifold $S_{\mathbf{n}}$ which is well known and can be found in [9] and references therein.

$$H^*(S_{\mathbf{n}},\mathbb{Z})=\mathbb{Z}[y_1,y_2,y_3]/\big(y_1^2,y_2^2,y_3(-n_1y_1-n_2y_2+y_3)\big) \tag{5}$$

We remark that the addition of orbifold structures on the invariant divisors does not effect the cohomology $H^*(S_{\mathbf{n}},\mathbb{Z})$; however, as we shall see it strongly effects the cohomology of the Sasakian 7-manifolds.

3.1 The Orbifold Cohomology Groups

Next we compute the orbifold cohomology groups.

Lemma 3.1

$$H^r_{orb}((S_{\mathbf{n}},\Delta_{\mathbf{m}}),\mathbb{Z})=\begin{cases}\mathbb{Z} & \textit{if } r=0\\ \mathbb{Z}^3 & \textit{if } r=2\\ \mathbb{Z}^3\oplus\mathbb{Z}^2_{m^0}\oplus\mathbb{Z}^2_{m^\infty} & \textit{if } r=4\\ \mathbb{Z}\oplus\mathbb{Z}^3_{m^0}\oplus\mathbb{Z}^3_{m^\infty} & \textit{if } r=6\\ \mathbb{Z}^3_{m^0}\oplus\mathbb{Z}^3_{m^\infty} & \textit{if } r=8,10,\cdots\\ 0 & \textit{if } r \textit{ is odd.}\end{cases}$$

Proof Using Lemma 4.3.7 of [11] we compute the Leray spectral sequence of the classifying map

$$p:\mathsf{B}(S_{\mathbf{n}},\Delta_{\mathbf{m}})\longrightarrow S_{\mathbf{n}}.$$

The Leray sheaf of p is the derived functor sheaf $R^s p_*\mathbb{Z}$, that is, the sheaf associated to the presheaf $U \mapsto H^s(p^{-1}(U), \mathbb{Z})$. For $s > 0$ the stalks of $R^s p_*\mathbb{Z}$ at points of U vanish if U lies in the regular locus of $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ which is the complement of the union of the zero e_0 and infinity e_∞ sections of the natural projection $S_{\mathbf{n}} \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$. At points of e_0 and e_∞ the fibers of p are the Eilenberg-MacLane spaces $K(\mathbb{Z}_{m^0}, 1)$ and $K(\mathbb{Z}_{m^\infty}, 1)$, respectively. So at points of $e_0(e_\infty)$ the stalks are the group cohomology $H^s(\mathbb{Z}_{m^0}, \mathbb{Z})\big(H^s(\mathbb{Z}_{m^\infty}, \mathbb{Z})\big)$. This is $\mathbb{Z}$ for $s = 0$ and $\mathbb{Z}_{m^0}(\mathbb{Z}_{m^\infty})$ at points of $e_0(e_\infty)$ when $s > 0$ is even; it vanishes when s is odd. The E_2 term of the Leray spectral sequence of the map p is

$$E_2^{r,s} = H^r(S_{\mathbf{n}}, R^s p_*\mathbb{Z})$$

and by Leray's theorem this converges to the orbifold cohomology $H^{r+s}_{orb}((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$. Now $E_2^{r,0} = H^r(S_{\mathbf{n}}, \mathbb{Z})$ and $E_2^{r,s} = 0$ for r or s odd. For $r = 0$ the only continuous section of $R^s p_*\mathbb{Z}$ is the 0 section which implies that $E_2^{0,s} = 0$ for all s. Now we have $E_2^{2r,2s} = 0$ for $r > 2$ and

$$\begin{aligned} E_2^{2,2s} &= H^2(S_{\mathbf{n}}, R^{2s}p) = H^2(e_0, \mathbb{Z}_{m^0}) \oplus H^2(e_\infty, \mathbb{Z}_{m^\infty}) = \mathbb{Z}^2_{m^0} \oplus \mathbb{Z}^2_{m^\infty}, \\ E_2^{4,2s} &= H^4(S_{\mathbf{n}}, R^{2s}p) = H^4(e_0, \mathbb{Z}_{m^0}) \oplus H^4(e_\infty, \mathbb{Z}_{m^\infty}) = \mathbb{Z}_{m^0} \oplus \mathbb{Z}_{m^\infty}. \end{aligned}$$

One easily sees that this spectral sequence collapses whose limit is the orbifold cohomology $H^r_{orb}((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$ which implies the result. □

In addition Lemma 3.1 implies

Lemma 3.2 $\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \mathbb{1}$.

Proof From the homotopy sequence of the orbibundle

$$\mathbb{CP}^1[\mathbf{v}]/\mathbb{Z}_m \longrightarrow (S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$$

one easily sees that $\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ is Abelian. But this implies

$$\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \approx H_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z}),$$

and this vanishes by Lemma 3.1 and universal coefficients. □

3.2 *The Cohomology of the Branched Covers*

We consider the total space $M^7_{\mathbf{n},\mathbf{m}}$ of an S^1 orbibundle over a KS orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ with an orbifold Kähler form $\omega_{\mathbf{n},\mathbf{m}}$. Note that in our case $S_{\mathbf{n}}$ is a smooth projective variety so $M^7_{\mathbf{n},\mathbf{m}}$ is a Seifert bundle as defined in the foundational paper [48] of Orlik and Wagreich (see also the more recent unpublished description by Kollár [42]). A fundamental result of Orlik and Wagreich says that every $\mathbb{C}^*$ Seifert bundle is a branched cover over a principal $\mathbb{C}^*$ bundle. In our case this branching occurs precisely along the orbifold divisor $\Delta_{\mathbf{m}}$. The goal is to understand the integral cohomology of $M^7_{\mathbf{n},\mathbf{m}}$. However, before treating the orbifold case, we need to consider the regular case.

3.3 The Regular Case

For simplicity we treat only the case where the Kähler class is primitive, i.e.

$$[\omega_{\mathbf{n}}] = c_1 y_1 + c_2 y_2 + c_3 y_3 \text{ with } c_i \in \mathbb{Z}^+, \qquad \gcd(c_1, c_2, c_3) = 1. \tag{6}$$

From Sect. 3.3 of [9] this is a Kähler class for all positive c_i if $n_1 n_2 > 0$, whereas, if $n_1 n_2 < 0$, it is a Kähler class for $c_1 > 0,\ c_2 > -n_2 c_3,\ c_3 > 0$.

One sees from the long exact homotopy sequence of the fibration

$$S^1 \longrightarrow M^7_{\mathbf{n}} \longrightarrow S_{\mathbf{n}} \tag{7}$$

that

Lemma 3.3 *Let $M^7_{\mathbf{n}}$ denote the total space of the principal S^1 bundle over a KS manifold $S_{\mathbf{n}}$ defined by a primitive Kähler class $[\omega_{\mathbf{n}}] \in H^2(S_{\mathbf{n}}, \mathbb{Z})$. Then $M^7_{\mathbf{n}}$ has the rational cohomology of the 2-fold connected sum $2\#(S^2 \times S^5)$, and we have*

1. $\pi_1(M^7_{\mathbf{n}}) = \mathbb{1}$,
2. $\pi_2(M^7_{\mathbf{n}}) = \mathbb{Z}^2$.

Thus, by the Hurewicz Theorem we also have $H_2(M^7_{\mathbf{n}}, \mathbb{Z}) = \mathbb{Z}^2$ which implies that $H^3(M^7_{\mathbf{n}}, \mathbb{Z})_{tor} = 0$. One can also see from the Leray-Serre spectral sequence of the fibration (7) that $M^7_{\mathbf{n}}$ has the rational cohomology of the 2-fold connected sum $2\#(S^2 \times S^5)$, and we also have $H^3(M^7_{\mathbf{n}}, \mathbb{Z}) = 0$. We now show that all such simply connected spaces have nonzero torsion in H^4. There will be zero torsion in H^4 if and only if the differential $d_2 : E_2^{2,1} \longrightarrow E_2^{4,0}$ is invertible over $\mathbb{Z}$. This differential is represented by the matrix

$$C = \begin{pmatrix} c_2 & c_3 & 0 \\ c_1 & 0 & c_3 \\ 0 & c_1 + n_1 c_3 & c_2 + n_2 c_3 \end{pmatrix} \tag{8}$$

with respect to the basis $\{\alpha \otimes y_1, \alpha \otimes y_2, \alpha \otimes y_3\}$ of $E_2^{2,1}$ and the basis $\{y_1 y_2, y_1 y_3, y_2 y_3\}$ of $E_2^{4,0}$. But then we have

$$-\det C = c_3[c_2(c_1 + n_1 c_3) + c_1(c_2 + n_2 c_3)] \tag{9}$$

which implies that the order of $H^4(M^7_{\mathbf{n}},\mathbb{Z})$ is $c_3[c_2(c_1+n_1c_3)+c_1(c_2+n_2c_3)]$. This is always greater than 1, so $H^4(M^7_{\mathbf{n}},\mathbb{Z})_{tor}\neq 0$. Summarizing we have

Theorem 3.4 *Let $M^7_{\mathbf{n}}$ denote the total space of the principal S^1 bundle over a KS manifold $S_{\mathbf{n}}$ defined by a primitive Kähler class $[\omega_{\mathbf{n}}]=c_1x_1+c_2x_2+c_3x_3$. Then*

$$H^r(M^7_{\mathbf{n}},\mathbb{Z})=\begin{cases}\mathbb{Z} & if\, r=0,7\\ \mathbb{Z}^2 & if\, r=2,5\\ G_{reg} & if\, r=4\\ 0 & otherwise.\end{cases}$$

where G_{reg} is an Abelian group of order $c_3[c_2(c_1+n_1c_3)+c_1(c_2+n_2c_3)]$. Moreover, G_{reg} is never the identity.

Example 3.1 The simplist case is the standard product $\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1$ with Kähler class

$$[\omega]=c_1x_1+c_2x_2+c_3x_3,\qquad \gcd(c_1,c_2,c_3)=1$$

and $n_1=n_2=0$ which gives $|G_{reg}|=2c_1c_2c_3$. Taking $c_1=c_2=c_3=1$ gives the homogeneous space

$$M^7=\big(SU(2)\times SU(2)\times SU(2)\big)/U(1)\times U(1)$$

with its Sasaki-Einstein metric (see for example [10]). Here $G_{reg}=\mathbb{Z}_2$.

Example 3.2 Monotone Fano case. Up to equivalence there are two monotone Fano cases polarized by the first Chern class, namely, $\mathbf{n}=(1,-1)$ and $\mathbf{n}=(1,1)$. From Eqs. (3) and (6) we have $c_1=1, c_2=3, c_3=2$ when $\mathbf{n}=(1,-1)$, and $c_1=1, c_2=1, c_3=2$ when $\mathbf{n}=(1,1)$. These give $|G_{reg}|=20$ and $|G_{reg}|=12$, respectively. Note that $M^7_{1,-1}$ is conjecturely[1] cone indecomposable; whereas, $M^7_{1,1}$ is cone decomposible. The latter is described by Theorem 2.5 in [16] with $\mathbf{l}_1=(1,1)$, $\mathbf{w}_1=(1,1)$, $\mathbf{l}_2=(1,2)$, and $\mathbf{w}_2=(3,1)$.

3.4 The General Case; Branched Covers

Returning to the orbifold case, we consider the Leray spectral sequence of the quotient map $M^7_{\mathbf{n},\mathbf{m}}\xrightarrow{\pi}(S_{\mathbf{n}},\Delta_{\mathbf{m}})$ viewed as an S^1-Seifert bundle over $(S_{\mathbf{n}},\Delta_{\mathbf{m}})$. Since the fibers are S^1 we only have the $R^1\pi_*\mathbb{Z}_M$ direct image sheaf, the E_2 term of the Leray spectral sequence

$$E_2^{p,q}=H^p(S_{\mathbf{n}},R^q\pi_*\mathbb{Z}) \tag{10}$$

[1] See Remark 3.10 below.

satisfies $E_2^{p,q} = 0$ when $q \neq 0, 1$, and this converges to $H^{p+q}(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z})$. So the only differentials are those induced by

$$d_2 : E_2^{0,1} = H^0(S_\mathbf{n}, R^1\pi_*\mathbb{Z}_M) \longrightarrow E_2^{2,0} = H^2(S_\mathbf{n}, \mathbb{Z}) \approx \mathbb{Z}^3. \tag{11}$$

We note that the sheaf injection

$$R^1\pi_*\mathbb{Z}_M \longrightarrow \mathbb{Z}_{S_\mathbf{n}} \tag{12}$$

is multiplication by 1 on the regular locus, multiplication by m_0 on e_0 and by m_∞ on e_∞. So the image of the map (11) is the order $\mu = \mathrm{lcm}(m_0, m_\infty)$ times a primitive orbifold Kähler class $[\omega_{\mathbf{n},\mathbf{m}}] \in H^2_{orb}((S_\mathbf{n}, \Delta_\mathbf{m}), \mathbb{Z})$. To compute the cohomology of such spaces we consider the $\mathbb{C}^*$ Seifert bundle over $(S_\mathbf{n}, \Delta_\mathbf{m})$ which is the cone $C(M^7_{\mathbf{n},\mathbf{m}}) = Y_{\mathbf{n},\mathbf{m}} = M^7_{\mathbf{n},\mathbf{m}} \times \mathbb{R}^+$ so that $Y_{\mathbf{n},\mathbf{m}}|_{r=1} = M^7_{\mathbf{n},\mathbf{m}}$. Now $Y_{\mathbf{n},\mathbf{m}}$ is a branched cover of $Y_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu$ with branching locus $\Delta_\mathbf{m}$. This follows from [42] by identifying the orbifold class $[\omega_{\mathbf{n},\mathbf{m}}]$ with the rational first Chern class $c_1(Y/X)$ in [42] with $X = S_\mathbf{n}$. Now generally the total space $M^7_{\mathbf{n},\mathbf{m}}$ is a Sasaki orbifold whose underlying topological space is a compactly generated Hausdorff space which allows us to apply the theory of ramified covers [8, 50]. The space $M^7_{\mathbf{n},\mathbf{m}}$ is simply connected, and a μ-fold cover of the total space of the ordinary S^1 bundle defined by the primitive integral class $\mu[\omega_{\mathbf{n},\mathbf{m}}]$. Thus, we have the commutative diagram of Seifert orbibundles:

$$\begin{array}{ccccc} S^1 & \longrightarrow & M^7_{\mathbf{n},\mathbf{m}} & \xrightarrow{f} & (S_\mathbf{n}, \Delta_\mathbf{m}) \\ \downarrow & & \downarrow \pi & & \downarrow \\ S^1/\mathbb{Z}_\mu & \longrightarrow & M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu & \xrightarrow{f_\mu} & (S_\mathbf{n}, \emptyset). \end{array} \tag{13}$$

Note that since $M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu$ is the total space of a principal S^1 bundle over the simply connected smooth projective algebraic variety $S_\mathbf{n}$ defined by a primitive integral class $\mu[\omega_{\mathbf{n},\mathbf{m}}]$, it is a simply connected smooth 7-manifold. Moreover, the covering map π induces a map of the corresponding Leray spectral sequences, and since the fiber is an S^1 the only nonzero higher direct image sheaf is $R^1\pi_*\mathbb{Z}$. This gives the natural isomorphism $R^1\pi_*\mathbb{Q}_M \approx \mathbb{Q}_{S_\mathbf{n}}$ which then implies

Lemma 3.5 *There is an isomorphism* $H^*(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Q}) \approx H^*(M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu, \mathbb{Q})$.

However, we would also like information about integral cohomology groups. For this we consider the **transfer homomorphism** for ramified covers [50]. Following [8] we apply this to our branched cover $\pi : M^7_{\mathbf{n},\mathbf{m}} \longrightarrow M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu$. Theorem 5.4 of [8] says that the transfer homomorphism

$$\tau : H^*(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z}) \longrightarrow H^*(M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu, \mathbb{Z})$$

induces multiplication by μ in $H^*(M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu, \mathbb{Z})$. We set $G_{reg} = H^4(M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu, \mathbb{Z})$ and apply this to $* = 4$. If $\gcd(|G|, \mu) = 1$ we see that

$$\tau \circ \pi^* : G_{reg} = H^4(M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu, \mathbb{Z}) \longrightarrow G_{reg} = H^4(M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu, \mathbb{Z})$$

is an isomorphism which implies that $\pi^* : H^4(M^7_{\mathbf{n},\mathbf{m}}/\mathbb{Z}_\mu, \mathbb{Z}) \longrightarrow H^4(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z})$ is injective. Thus, $\pi^*(G_{reg})$ is a subgroup of $H^4(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z})$ in this case. When $\gcd(|G_{reg}|, \mu) \neq 1$ the homomorphism $\tau \circ \pi^*$ has a non-trivial kernel. This kernel consists of all the prime factors $\mathbb{Z}_{p_i^{r_1}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$ of G_{reg} such that p_i is a prime in μ. Denote this group by G^μ_{reg} In this case $H^4(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z})$ contains the subgroup of G_{reg} containing only those prime factors whose primes p are not in μ, that is the factor group G_{reg}/G^μ_{reg}. Summarizing we have

Lemma 3.6 *$H^4(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z})$ contains G_{reg}/G^μ_{reg}.*

We also have

Lemma 3.7 *$H^3(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z}) = 0$.*

Proof The only non-zero term of total degree 3 in (10) is

$$E_2^{2,1} = H^2(S_{\mathbf{n}}, R^1\pi_*\mathbb{Z})$$

where we have a differential

$$d_2 : E_2^{2,1} = H^2(S_{\mathbf{n}}, R^1\pi_*\mathbb{Z}) \longrightarrow E_2^{4,0} = H^4(S_{\mathbf{n}}, \mathbb{Z}) = \mathbb{Z}^3.$$

This differential gives torsion in $H^4(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z})$ implying that $H^4(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Q})$ vanishes which in turn implies that $E_r^{2,1} = 0$ for $r > 2$ which proves the lemma. □

Theorem 3.8 *Let $M^7_{\mathbf{n},\mathbf{m}}$ denote the total space of the principal S^1 orbibundle over a KS orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ defined by a primitive orbifold Kähler class $[\omega_{\mathbf{n},\mathbf{m}}] = c_1x_1 + c_2x_2 + c_3x_3$ such that $\mu[\omega_{\mathbf{n},\mathbf{m}}]$ is a primitive class in $H^2(S_{\mathbf{n}}, \mathbb{Z})$. Then*

$$H^r(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{if } r = 0, 7 \\ \mathbb{Z}^2 & \textit{if } r = 2, 5 \\ G & \textit{if } r = 4 \\ 0 & \textit{otherwise.} \end{cases}$$

where G is an Abelian group that contains G_{reg}/G^μ_{reg}. In particular, if $\gcd(|G_{reg}|, \mu) = 1$ then G contains G_{reg}.

Proof Since $[\omega_{\mathbf{n},\mathbf{m}}]$ is a primitive orbifold class and $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ has $\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \mathbb{1}$, then the orbifold $M^7_{\mathbf{n},\mathbf{m}}$ has $\pi_1^{orb}(M^7_{\mathbf{n},\mathbf{m}}) = \mathbb{1}$ by the long exact homotopy sequence of the classifying spaces. The remainder of the theorem then follows from Lemmas 3.5–3.7. □

Remark 3.9 It follows from Theorem 3.4 that G_{reg} is never the identity; although, we can have $G_{reg} = G^{\mu}_{reg}$ so that $G_{reg}/G^{\mu} = \mathbb{1}$ in this case. However, this does not imply that $H^4(M^7_{\mathbf{n},\mathbf{m}}, \mathbb{Z})$ is the identity.

Remark 3.10 Let M be a Boothby-Wang Sasaki manifold over $\mathbb{CP}^1 \times \mathbb{CP}^1$. The set of KS orbifolds $\mathcal{KS}_0$ splits into a disjoint union two types; those that have a polarization such that the corresponding Boothby-Wang constructed Sasaki orbifold is represented by a quasi-regular ray in the **w**-cone of a join of the form $M \star_{\mathbf{l}} S^3_{\mathbf{w}}$, and those that do not have such a polarization. The former are characterized by the condition $n_1 n_2 > 0$ (see Theorem 4.3 below) and the corresponding Sasaki manifolds are called *cone decomposable* (see Definitions 3.1 and 4.1 in [14]). It remains an open question whether a Boothby-Wang constructed Sasaki orbifold over the latter type, with $n_1 n_2 < 0$, is necessarily *cone indecomposable* or simply not represented by a quasi-regular ray in the **w**-cone of a join of the form $M \star_{\mathbf{l}} S^3_{\mathbf{w}}$.

3.5 When M^7 is an $S^3_{\mathbf{w}}$-Join

Generally, we are interested in how G depends on **m** and **n**. Unfortunately, Theorem 3.8 does not give us much useful information about this dependence. However, we can determine this dependence in a particular cone decomposable case, namely when the space $M^7_{\mathbf{n},\mathbf{m}}$ can be represented as a join of the form $M_{\mathbf{l},\mathbf{w}} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$ where M is a principal S^1 bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Let M^5 denote the total space of the Boothby-Wang bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ determined by the Kähler class $k_1 y_1 + k_2 y_2$ with $k_1, k_2 \in \mathbb{Z}^+$. When k_1 and k_2 are relatively prime, M^5 is diffeomorphic to $S^2 \times S^3$. More generally $S^2 \times S^3$ is an k-fold cover of M^5, namely we have $M^5 \approx (S^2 \times S^3)/\mathbb{Z}_k$ where $k = \gcd(k_1, k_2)$. Moreover, if $\gcd(l_\infty, w_0) = \gcd(l_\infty, w_\infty) = 1$, $M_{\mathbf{l},\mathbf{w}}$ will be smooth, and we have an S^1 bundle

$$S^1 \longrightarrow M^5 \times S^3_{\mathbf{w}} \longrightarrow M^5 \star_{\mathbf{l}} S^3_{\mathbf{w}}$$

which gives the homotopy exact sequence

$$0 \longrightarrow \pi_2(M^5) \longrightarrow \pi_2(M^5 \star_{\mathbf{l}} S^3_{\mathbf{w}}) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_k \longrightarrow \pi_1(M^5 \star_{\mathbf{l}} S^3_{\mathbf{w}}) \longrightarrow 1. \tag{14}$$

So when $k = 1$ so $M^5 = S^2 \times S^3$, the join $M^7_{\mathbf{l},\mathbf{w}}$ will also be simply connected with $\pi_2(M^7_{\mathbf{l},\mathbf{w}}) = \mathbb{Z}^2$. More generally, the topological analysis proceeds as in Sect. 4 of [17], see also [18]. We consider the commutative diagram of fibrations

$$\begin{array}{ccccc}
(S^2\times S^3)/\mathbb{Z}_k\times S^3_{\mathbf{w}} & \longrightarrow & M_{\mathbf{l},\mathbf{w}} & \longrightarrow & \mathsf{B}S^1 \\
\Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\psi} \\
(S^2\times S^3)/\mathbb{Z}_k\times S^3_{\mathbf{w}} & \longrightarrow & \mathbb{CP}^1\times\mathbb{CP}^1\times\mathsf{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathsf{B}S^1\times\mathsf{B}S^1
\end{array} \tag{15}$$

where $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space [35] of an orbifold if G is an orbifold. Note that $H^2((S^2\times S^3)/\mathbb{Z}_k,\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}_k$. The diagram gives a map of spectral sequences, and we note that the lower fibration is a product of well understood fibrations. The E_2 term of the top fibration is

$$E_2^{p,q}=H^p(\mathsf{B}S^1,H^q(S^2\times S^3\times S^3_{\mathbf{w}},\mathbb{Z}))\approx\mathbb{Z}[s]\otimes\Lambda[\alpha,\beta,\gamma], \tag{16}$$

where α is a 2-class, β,γ are 3-classes, and s_1,s_2 are the positive generators of $H^*(\mathsf{B}S^1,\mathbb{Z})$. By the Leray-Serre Theorem this converges to $H^{p+q}(M_{\mathbf{l},\mathbf{w}},\mathbb{Z})$. The non-vanishing differentials of the bottom product fibration are, first $d_4(\beta)=s_1^2$ and second $d_4(\gamma)=w_0w_\infty s_2^2$ and those induced by naturality. As described in [17] we also have $\psi^*s_1=l_\infty s$ and $\psi^*s_2=-l_0s$. So by naturality the differentials of the top fibration are $d_4(\beta)=l_\infty^2s^2$ and $d_4(\gamma)=w_0w_\infty l_0^2s^2$. Now $M_{\mathbf{l},\mathbf{w}}$ is smooth if and only if $\gcd(l_\infty,w_0w_\infty)=1$. This gives $H^4(M_{\mathbf{l},\mathbf{w}},\mathbb{Z})=\mathbb{Z}_{l_\infty^2}\times\mathbb{Z}_{w_0w_\infty l_0^2}=\mathbb{Z}_{w_0w_\infty l_0^2l_\infty^2}$.

4 Admissible Projective Bundles

Admissible projective bundles were described in general [3]. Here we will restrict ourselves to a specific type of these, namely projective bundles of the form

$$S_{\mathbf{n}}=\mathbb{P}(\mathbb{1}\oplus\mathcal{O}(n_1,n_2))\longrightarrow\mathbb{CP}^1\times\mathbb{CP}^1$$

that satisfy the following conditions:

- For $i=1,2$ let $(\pm\omega_i,\pm g_i)$ be Kähler metrics with constant scalar curvature $\pm 2s_i=\pm 4/n_i$ on $\mathbb{CP}^1$. [It is assumed that $n_i\neq 0$.] The $\pm$ means that either $+g_i$ or $-g_i$ is positive definite.
- $\mathcal{O}(n_1,n_2)\longrightarrow\mathbb{CP}^1\times\mathbb{CP}^1$ is the holomorphic line bundle over $\mathbb{CP}^1\times\mathbb{CP}^1$ which satisfy
$$c_1(\mathcal{O}(n_1,n_2))=\frac{1}{2\pi}[\omega_1+\omega_2]=n_1x_1+n_2x_2$$

On admissible projective bundles such as the ones defined above we can now construct the *admissible metrics*, [3]. Here we recover the main points of this construction.

Consider the standard circle action on $S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$. It extends to a holomorphic $\mathbb{C}^*$ action. The open and dense set $S_{\mathbf{n}}^0$ of stable points with respect to the latter action has the structure of a principal circle bundle over the stable quotient. The hermitian norm on the fibers induces via a Legendre transform a function $\mathfrak{z} : S_{\mathbf{n}}^0 \to (-1, 1)$ whose extension to $S_{\mathbf{n}}$ consists of the critical manifolds $e_0 := \mathfrak{z}^{-1}(1) = \mathbb{P}(\mathbb{1} \oplus 0)$ and $e_\infty := \mathfrak{z}^{-1}(-1) = \mathbb{P}(0 \oplus \mathcal{O}(n_1, n_2))$. Letting θ be a connection one form for the Hermitian metric on $S_{\mathbf{n}}^0$, with curvature $d\theta = \omega_1 + \omega_2$, an admissible Kähler metric and form are given (up to scale) by the respective formulas

$$g = \frac{1 + r_1\mathfrak{z}}{r_1} g_1 + \frac{1 + r_2\mathfrak{z}}{r_2} g_2 + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \quad \omega = \frac{1 + r_1\mathfrak{z}}{r_1}\omega_1 + \frac{1 + r_2\mathfrak{z}}{r_2}\omega_2 + d\mathfrak{z} \wedge \theta, \tag{17}$$

valid on $S_{\mathbf{n}}^0$. Here Θ is a smooth function with domain containing $(-1, 1)$ and r_i, $i = 1, 2$ are real numbers of the same sign as g_i and satisfying $0 < |r_i| < 1$. The complex structure yielding this Kähler structure is given by the pullback of the base complex structure along with the requirement $Jd\mathfrak{z} = \Theta\theta$. The function $\mathfrak{z}$ is hamiltonian with $K = J\, grad\, \mathfrak{z}$ a Killing vector field. In fact, $\mathfrak{z}$ is the moment map on $S_{\mathbf{n}}$ for the circle action, decomposing M into the free orbits $S_{\mathbf{n}}^0 = \mathfrak{z}^{-1}((-1, 1))$ and the special orbits $\mathfrak{z}^{-1}(\pm 1)$. Finally, θ satisfies $\theta(K) = 1$.

Now g is a (positive definite) Kähler metric which extends smoothly to all of $S_{\mathbf{n}}$ if and only if Θ satisfies the following positivity and boundary conditions

$$(i)\ \Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \quad (ii)\ \Theta(\pm 1) = 0, \quad (iii)\ \Theta'(\pm 1) = \mp 2. \tag{18}$$

The Kähler class $\Omega_{\mathbf{r}} = [\omega]$ of an admissible metric as in (17) is also called *admissible* and is uniquely determined by the parameters r_1, r_2, once the data associated with M (i.e. $s_i = 2/n_i$, g_i etc.) is fixed. Indeed, we have

$$\Omega_{\mathbf{r}} = \frac{[\omega_1]}{r_1} + \frac{[\omega_2]}{r_2} + 2\pi\,\Xi,$$

where Ξ is the Poincare dual of $e_0 + e_\infty$. For a more thorough description of Ξ, please consult Sect. 1.3 of [3]. Note that on $S_{\mathbf{n}}$ any Kähler class is admissible up to scale. Using that $y_3 - x_3 = n_1x_1 + n_2x_2$, we can also write

$$\begin{aligned}\Omega_{\mathbf{r}}/2\pi &= \tfrac{n_1}{r_1}x_1 + \tfrac{n_2}{r_2}x_2 + x_3 + y_3 \\ &= \tfrac{n_1(1+r_1)}{r_1}x_1 + \tfrac{n_2(1+r_2)}{r_2}x_2 + 2x_3 \\ &= \tfrac{n_1(1-r_1)}{r_1}x_1 + \tfrac{n_2(1-r_2)}{r_2}x_2 + 2y_3.\end{aligned}$$

Define a function $F(\mathfrak{z})$ by the formula $\Theta(\mathfrak{z}) = F(\mathfrak{z})/p_c(\mathfrak{z})$, where $p_c(\mathfrak{z}) = (1 + r_1\mathfrak{z})(1 + r_2\mathfrak{z})$. Since $p_c(\mathfrak{z})$ is positive for $-1 \leq \mathfrak{z} \leq 1$, conditions (18) are equivalent to the following conditions on $F(\mathfrak{z})$.

$$(i)\ F(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \quad (ii)\ F(\pm 1) = 0, \quad (iii)\ F'(\pm 1) = \mp 2p_c(\pm 1). \tag{19}$$

4.1 Orbifolds

Now we allow our admissible metrics to compactify as orbifold metrics on the log pair

$$(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \big(\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2), \Delta_{\mathbf{m}}\big) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1,$$

where

$$\Delta_{\mathbf{m}} = (1 - 1/m_0)D_1 + (1 - 1/m_\infty)D_2 = (1 - 1/m_0)e_0 + (1 - 1/m_\infty)e_\infty,$$

and $m_0, m_\infty \in \mathbb{Z}^+$. Then (18) generalizes to

$$\begin{array}{ll} (i) & \Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \\ (ii) & \Theta(\pm 1) = 0, \\ (iii) & \Theta'(-1) = 2/m_\infty, \quad \text{and} \quad \Theta'(1) = -2/m_0. \end{array} \tag{20}$$

and, with $\Theta(\mathfrak{z}) = F(\mathfrak{z})/p_c(\mathfrak{z})$ as above, we get that this is equivalent to

$$\begin{array}{ll} (i) & F(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \\ (ii) & F(\pm 1) = 0, \\ (iii) & F'(-1) = 2p_c(-1)/m_\infty, \quad \text{and} \quad F'(1) = -2p_c(1)/m_0. \end{array} \tag{21}$$

Note that this does not change the expression for Ω_r above whereas from (3) we already know that the adjusted Chern class is

$$\begin{aligned} c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) &= \Big(2 - \tfrac{n_1}{m_\infty}\Big)x_1 + \Big(2 - \tfrac{n_2}{m_\infty}\Big)x_2 + \Big(\tfrac{1}{m_0} + \tfrac{1}{m_\infty}\Big)y_3 \\ &= \Big(2 + \tfrac{n_1}{m_0}\Big)x_1 + \Big(2 + \tfrac{n_2}{m_0}\Big)x_2 + \Big(\tfrac{1}{m_0} + \tfrac{1}{m_\infty}\Big)x_3. \end{aligned}.$$

4.2 Connection with $S^3_{\mathbf{w}}$-Joins

Consider $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \big(\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2), \Delta_{\mathbf{m}}\big) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$, polarized with a primitive orbifold Kähler class $[\omega_{\mathbf{n},\mathbf{m}}]$. As mentioned above, this will be the rescale of some admissible Kähler class $\Omega_{\mathbf{r}}$ determined by $\mathbf{r} = (r_1, r_2)$. If $n_1 n_2 > 0$, we will say that $[\omega_{\mathbf{n},\mathbf{m}}]$ is *diagonally admissible* if and only if $r_1 = r_2$. Note that if $n_1 n_2 < 0$, it is never possible to have $r_1 = r_2$, since the sign of r_i is equal to the sign of n_i. Note that diagonally admissible is equivalent to being admissible (up to scale) as defined (more narrowly) in Sect. 2.5.1 of [19] with $N = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $[\omega_N] = \frac{n_1}{n} x_1 + \frac{n_2}{n} x_2$, where $n = (\text{sign of}\, n_i)\gcd(|n_1|, |n_2|)$.

Indeed, suppose $n_1 n_2 > 0$ and $r_1 = r_2 = r$ in the above setting. Then the admissible metric simplifies to

$$g = \frac{1 + r\mathfrak{z}}{r} g_{N_n} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \quad \omega = \frac{1 + r\mathfrak{z}}{r}\omega_{N_n} + d\mathfrak{z} \wedge \theta, \tag{22}$$

with admissible Kähler class

$$\begin{aligned} \Omega_{\mathbf{r}}/2\pi &= \tfrac{n}{r}\left(\tfrac{n_1}{n} x_1 + \tfrac{n_2}{n} x_2\right) + x_3 + y_3 \\ &= \tfrac{n(1+r)}{r}\left(\tfrac{n_1}{n} x_1 + \tfrac{n_2}{n} x_2\right) + 2x_3 \\ &= \tfrac{n(1-r)}{r}\left(\tfrac{n_1}{n} x_1 + \tfrac{n_2}{n} x_2\right) + 2y_3, \end{aligned}$$

where $g_{N_n} = n_1 g_1 + n_2 g_2$. As such, we can view this metric as an admissible metric on $S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus L_n) \longrightarrow N$, where $N = \mathbb{CP}^1 \times \mathbb{CP}^1$ and

- $(\pm\omega_{N_n}, \pm g_{N_n})$ is a Kähler metric with constant scalar curvature $\pm 4s = \pm 4(\frac{1}{n_1} + \frac{1}{n_2})$. [The $\pm$ still means that either $+g_{N_n}$ or $-g_{N_n}$ is positive definite.]
- $L_n \longrightarrow N$ is the holomorphic line bundle over $N = \mathbb{CP}^1 \times \mathbb{CP}^1$ which satisfy

$$c_1(L_n) = \frac{1}{2\pi}[\omega_{N_n}] = n\left(\frac{n_1}{n} x_1 + \frac{n_2}{n} x_2\right).$$

This special case, is exactly of the type considered in e.g. Sect. 5 of [17] or more generally (including cyclic orbifolds N) in Sect. 2.5 of [19]. Note that the base metric $(\pm\omega_{N_n}, \pm g_{N_n})$ is only Kähler-Einstein if also $n_1 = n_2$.

Now consider the natural Boothby-Wang constructed Sasaki structure on the S^1-bundle $M \to N$ defined by the primitive $[\omega_N] = \left(\frac{n_1}{n} x_1 + \frac{n_2}{n} x_2\right)$. Following Proposition 4.22 of [14] (with the amendment of Lemma 2.12 and Corollary 2.14 of [19]) we then realize the following connection with $S^3_{\mathbf{w}}$-joins.

Proposition 4.1 *For $n_1 n_2 > 0$, $r_1 = r_2 = r$ rational, $n = (\text{sign of}\, n_i)\gcd(|n_1|, |n_2|)$, and $\gcd(m_1, m_2, |n|) = 1$, there is a choice of co-prime w_0, $w_\infty \in \mathbb{Z}^+$ and co-prime l_0, l_∞ such that, when we form the $S^3_{\mathbf{w}}$-join $M_{l_0,l_\infty,\mathbf{w}} := M \star_{l_0,l_\infty} S^3_{w_0,w_\infty} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$, the quasi-regular quotient of $M_{l_0,l_\infty,\mathbf{w}}$ by the flow of the Reeb vector field $\xi_{\mathbf{v}}$ determined by (v_0, v_∞) in the $\mathbf{w}$-cone (where $m = \gcd(m_0, m_\infty)$ and $\mathbf{m} = (m_0, m_\infty) = m\mathbf{v} = m(v_0, v_\infty)$), is the log pair $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ with induced (transverse) Kähler class $m\gcd(\mathfrak{s}, w_0 v_\infty)[\omega_{\mathbf{n},\mathbf{m}}]$ where $\mathfrak{s} = \gcd(l_\infty, |w_0 v_\infty - w_\infty v_0|)$ and $[\omega_{\mathbf{n},\mathbf{m}}]$ is a orbifold primitive class that is an appropriate rescale of the admissible class $\Omega_{\mathbf{r}}$*

on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$. *The join is smooth if and only if* $\gcd(w_0, l_\infty) = \gcd(w_\infty, l_\infty) = 1$. *In particular, the join is smooth if* $m_0 = m_\infty = 1$.

Proof We follow the proof of Proposition 4.22 in [14] (including the paragraphs leading up to the proposition), to identify the join by first picking $\mathbf{w} = (w_0, w_\infty)$ to be the unique positive, integer, and co-prime solution of

$$r = \frac{w_0 m_\infty - w_\infty m_0}{w_0 m_\infty + w_\infty m_0}$$

and then, as the next step, picking the pair (l_0, l_∞) as the unique, positive integers, and co-prime solution of

$$l_\infty n = l_0(w_0 m_\infty - w_\infty m_0).$$

Proposition 4.22 in [14] was slightly misleading in implying that the quasi-regular quotient of $\xi_{\mathbf{v}}$ would always produce a primitive orbifold Kähler class, but, with Lemma 2.12 and Corollary 2.14 of [19] in hand, we can say that the transverse Kähler class is $m \gcd(\mathfrak{s}\Upsilon_N, w_0 v_\infty l_0)[\omega_{\mathbf{n},\mathbf{m}}]$ where $[\omega_{\mathbf{n},\mathbf{m}}]$ is a orbifold primitive class that is an appropriate rescale of the admissible class $\Omega_{\mathbf{r}}$ on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ and Υ_N is the orbifold order of N. Since $\Upsilon_N = 1$ and $\gcd(\mathfrak{s}, l_0) = 1$ (recall $\gcd(l_0, l_\infty) = 1$) we get the desired Kähler class.

The rest of the claims follow straight from the proof Proposition 4.22 in [14]. In particular, note that $m = l_\infty/\mathfrak{s}$, as it should be according to Theorem 3.8 in [17]. □

Remark 4.2 Note that when the join is smooth, it is easy to see that $\gcd(\mathfrak{s}, w_0 v_\infty) = 1$ and so the transverse Kähler class is $m[\omega_{\mathbf{n},\mathbf{m}}]$. In particular, if $m_0 = m_\infty = 1$, we do have a primitive (and admissible) transverse Kähler class.

Combining Theorem 2.7 and Lemma 2.12 from [19] for the"only if" and Proposition 4.1 for the "if", we arrive at the following theorem.

Theorem 4.3 *Assume* $\gcd(m_1, m_2, |n_1|, |n_2|) = 1$ *and* $[\omega_{\mathbf{n},\mathbf{m}}]$ *is a orbifold primitive class on* $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$. *There exist a constant* $k \in \mathbb{Z}^+$ *such that the polarized orbifold* $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}, k[\omega_{\mathbf{n},\mathbf{m}}])$ *is the quotient with respect to (the canonical Reeb vector field in) a quasi-regular ray in the* $\mathbf{w}$*-cone of a (possibly non-smooth) join of the form* $M \star_{\mathbf{l}} S^3_{\mathbf{w}}$, *where* M *is a Boothby-Wang constructed Sasaki manifold over* $\mathbb{CP}^1 \times \mathbb{CP}^1$, *if and only if* $n_1 n_2 > 0$ *and* $[\omega_{\mathbf{n},\mathbf{m}}]$ *is a diagonally admissible Kähler class.*

Suppose now that $n_1 n_2 > 0$, we are in the log Fano case as in Proposition 1.1, and the polarization is chosen such that $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \mathcal{I}_{\mathbf{n},\mathbf{m}}[\omega_{\mathbf{n},\mathbf{m}}]$. Since $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \left(2 - \frac{n_1}{m_\infty}\right)x_1 + \left(2 - \frac{n_2}{m_\infty}\right)x_2 + \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)y_3$ and a diagonally admissible Kähler class is a rescale of $\Omega_{\mathbf{r}} = \frac{(1-r)}{r}(n_1 x_1 + n_2 x_2) + 2y_3$ for some $0 < |r| < 1$ with $r n_i > 0$, we see that $[\omega_{\mathbf{n},\mathbf{m}}] = c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})/\mathcal{I}_{\mathbf{n},\mathbf{m}}$ is diagonally admissible if and only if

$$2-\frac{n_1}{m_\infty}=\left(\frac{1}{m_0}+\frac{1}{m_\infty}\right)\frac{n_1(1-r)}{2r} \quad \text{and} \quad 2-\frac{n_2}{m_\infty}=\left(\frac{1}{m_0}+\frac{1}{m_\infty}\right)\frac{n_2(1-r)}{2r},$$

i.e.,

$$n_1\left(\frac{1}{m_\infty}+\left(\frac{1}{m_0}+\frac{1}{m_\infty}\right)\frac{(1-r)}{2r}\right)=2=n_2\left(\frac{1}{m_\infty}+\left(\frac{1}{m_0}+\frac{1}{m_\infty}\right)\frac{(1-r)}{2r}\right).$$

This clearly implies that we must have $n_1=n_2$. On the other hand, if $n_1=n_2$ (and still assuming log Fano) we can solve for an appropriate r. In conclusion,

Corollary 4.4 *In the log Fano case, there exists a constant $k\in\mathbb{Z}^+$ such that the polarized orbifold $(S_{\mathbf{n}},\Delta_{\mathbf{m}},kc_1^{orb}(S_{\mathbf{n}},\Delta_{\mathbf{m}}))$ is the quotient with respect to a quasi-regular ray in the $\mathbf{w}$-cone of a (possibly non-smooth) join of the form $M\star_{\mathbf{l}}S^3_{\mathbf{w}}$, where M is a Boothby-Wang constructed Sasaki manifold over $\mathbb{CP}^1\times\mathbb{CP}^1$, if and only if $n_1=n_2$.*

Returning to Proposition 4.1 and Theorem 4.3, we note that in the case where $(S_{\mathbf{n}},\Delta_{\mathbf{m}},[\omega_{\mathbf{n},\mathbf{m}}])$ is the quotient with respect to a quasi-regular ray in the $\mathbf{w}$-cone of a join of the form $M\star_{\mathbf{l}}S^3_{\mathbf{w}}$ (so assuming $k=1$ in Theorem 4.3, i.e., assuming the transverse Kähler class is primitive) we must have that $m=1$ and $\gcd(\mathfrak{s},w_0v_\infty)=1$. The first equality implies that $l_\infty=\mathfrak{s}=\gcd(l_\infty,|w_0v_\infty-w_\infty v_0|)$ and hence l_∞ is a factor of $|w_0v_\infty-w_\infty v_0|$. Since we also have that $\gcd(l_\infty,w_0v_\infty)=1$, we must have $\gcd(l_\infty,w_\infty v_0)=1$ and in particular $\gcd(w_0,l_\infty)=\gcd(w_\infty,l_\infty)=1$. This means that $M\star_{\mathbf{l}}S^3_{\mathbf{w}}$ is smooth. We therefore have the following companion to Theorem 4.3.

Proposition 4.5 *Consider the polarized orbifold $(S_{\mathbf{n}},\Delta_{\mathbf{m}},[\omega_{\mathbf{n},\mathbf{m}}])$, where $[\omega_{\mathbf{n},\mathbf{m}}]$ is a primitive integer orbifold Kähler class and* $\gcd(m_0,m_\infty,|n_1|,|n_2|)=1$.

If $(S_{\mathbf{n}},\Delta_{\mathbf{m}},[\omega_{\mathbf{n},\mathbf{m}}])$ is the quotient with respect to (the canonical Reeb vector field in) a quasi-regular ray in the $\mathbf{w}$-cone of a join of the form $M\star_{\mathbf{l}}S^3_{\mathbf{w}}$, then this join is smooth. Moreover, in this case $\gcd(m_0,m_\infty)=1$.

4.3 Connection with Yamazaki's Fiber Joins

In [54] T. Yamazaki introduced the fiber join for K-contact structures and in [20] this was extended to Sasaki structures. In the smooth case ($m_0=m_\infty=1$) of KS orbifolds, it follows from Sect. 5.3 of [20] that $S_{\mathbf{n}}$ polarized by a primitive Kähler class, which in turn is an appropriate rescale of $\Omega_{\mathbf{r}}$, is the quotient of the regular ray in the $\mathfrak{t}^+_{sphr}$ cone of a Yamazaki fiber join over $\mathbb{CP}^1\times\mathbb{CP}^1$, if an only if there exist $k_1^1,k_1^2,k_2^1,k_2^2\in\mathbb{Z}^+$ such that

$$\begin{aligned} k_1^1-k_2^1&=n_1 & \qquad k_1^2-k_2^2&=n_2\\ \frac{k_1^1-k_2^1}{k_1^1+k_2^1}&=r_1 & \qquad \frac{k_1^2-k_2^2}{k_1^2+k_2^2}&=r_2. \end{aligned} \tag{23}$$

In that case, the corresponding Yamazaki fiber join is given as follows[2]:

Let $\pi_j : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ denote the natural projection to the j^t factor of the product $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ and let $\mathcal{K}$ denote the canonical bundle on $\mathbb{CP}^1$. Let L_i be a holomorphic line bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ given by $L_i = k_i^1 \pi_1^* \mathcal{K}^{\frac{-1}{2}} + k_i^2 \pi_2^* \mathcal{K}^{\frac{-1}{2}}$, for $k_i^j \in \mathbb{Z}^+$. The choice of L_1 and L_2 can be given by the matrix $K = \begin{pmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{pmatrix}$. Note that $c_1(L_i)$ are both in the Kähler cone of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and hence L_i are positive line bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Each $c_1(L_i)$ also defines a principal S^1-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$, $M_i \to \mathbb{CP}^1 \times \mathbb{CP}^1$, and we identify L_i with $M_i \times_{S^1} \mathbb{C}$. Then $M_i \xrightarrow{\pi} \mathbb{CP}^1 \times \mathbb{CP}^1$ has a natural Sasaki structure defined by the Boothby-Wang construction. Consider now $L_1^* \oplus L_2^*$ and equip each L_i^* with a Hermitian metric giving us a norm $d_i : L_i^* \to \mathbb{R}^{\geq 0}$. Then, the fiber join, $M_K = M_1 \star_f M_2$ is defined as the S^3-bundle over S whose fibers are given by $d_1^2 + d_2^2 = 1$. Now M_K has a natural CR-structure $(\mathcal{D}, J)$ with a family, $\mathfrak{t}^+_{sphr}$, of compatible Sasaki structures $\mathcal{S}_{\mathbf{a}} = (\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$, where $\mathbf{a} = (a_1, a_2) \in (\mathbb{R}^+)^2$ and $(a_1, a_2) = (1, 1)$ corresponds to the regular Sasaki structure in $\mathfrak{t}^+_{sph}(\mathcal{D}, J)$. Note that $\mathfrak{t}^+_{sph}$ is a proper subcone of the unreduced Sasaki cone $\mathfrak{t}^+$ of $(M_K, \mathcal{D}, J)$.

Remark 4.6 In [20] we did not consider the more general question of determining the quotients of quasi-regular Sasaki structures in $\mathfrak{t}^+_{sph}$ and the transverse Kähler class. We conjecture that those would indeed be certain polarized KS orbifolds and will explore this in future studies.

4.4 Ricci Solitons and Kähler-Einstein

It is well-known that there exists a unique Kähler-Ricci soliton on any toric compact Fano complex orbifold [30, 51–53, 55]. The existence proofs by Wang-Zhu [53], Shi-Zhu [51] and Donaldson [30] all use a continuity method and thus do not provide an explicit expression of the Kähler-Ricci soliton. It is therefore interesting to explore cases where explicit descriptions of Kähler-Ricci solitons are possible. Explicit examples can be found in e.g.[31, 46].

Here we want to explore the Ricci Soliton and more specifically the Kähler-Einstein equations under the constraint of the orbifold endpoint conditions (21). This will yield explicit examples on Fano $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ and is essentially a mild orbifold extension of the work by Koiso and Sakane [41, 44]. We will follow in their footsteps using the notation of Sect. 3 of [4].

The admissible metrics are Ricci solitons with $V = (\frac{c}{2})\, grad_g \mathfrak{z}$ if and only if

$$\rho - \lambda\omega = \mathcal{L}_V \omega, \tag{24}$$

[2] We refer to [20] for details and a more general description of the Yamazaki fiber joins.

where the Ricci form is

$$\rho = s_1\omega_1 + s_2\omega_2 - \frac{1}{2}dd^c \log F = \left(s_1 - \frac{1}{2}\frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})}\right)\omega_1 + \left(s_2 - \frac{1}{2}\frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})}\right)\omega_2 - \frac{1}{2}\left(\frac{F'}{p_c}\right)'(\mathfrak{z})\, d\mathfrak{z}\wedge\theta$$

and $\lambda, c \in \mathbb{R}$. Obviously, $c = 0$ corresponds to Kähler-Einstein metrics. Since $\mathcal{L}_V\omega = (\frac{c}{2})\, dd^c\mathfrak{z}$ and $s_i = 2/n_i$, Eq. (24) becomes a pair of ODEs

$$\begin{aligned}\frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} + c\frac{F(\mathfrak{z})}{p_c(\mathfrak{z})} &= 4/n_1 - 2\lambda\left(\mathfrak{z} + \frac{1}{r_1}\right)\\ \frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} + c\frac{F(\mathfrak{z})}{p_c(\mathfrak{z})} &= 4/n_2 - 2\lambda\left(\mathfrak{z} + \frac{1}{r_2}\right),\end{aligned} \tag{25}$$

Note that (25) implies that $n_1 = n_2$ if and only if $r_1 = r_2$.

Note that (*ii*) and (*iii*) of (21) together with (25) implies the necessary conditions

$$2\lambda = \frac{1}{m_0} + \frac{1}{m_\infty} \tag{26}$$

and

$$r_1 = \frac{\frac{1}{m_0} + \frac{1}{m_\infty}}{\frac{4}{n_1} + \frac{1}{m_0} - \frac{1}{m_\infty}}, \qquad r_2 = \frac{\frac{1}{m_0} + \frac{1}{m_\infty}}{\frac{4}{n_2} + \frac{1}{m_0} - \frac{1}{m_\infty}}. \tag{27}$$

On the other hand, assuming (26) and (27), the ODEs of (25) are equivalent to the single ODE

$$\frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} + c\frac{F(\mathfrak{z})}{p_c(\mathfrak{z})} = \left(\frac{1}{m_\infty} - \frac{1}{m_0}\right) - \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)\mathfrak{z} \tag{28}$$

Further, (28) and (*ii*) of (21) together imply (*iii*) of (21). In summary, we will get an admissible Ricci soliton solution as in (24) exactly when (26) and (27) are satisfied and (28) has a solution $F(\mathfrak{z})$ that satisfies (*i*) and (*ii*) of (21).

Notice that since we require $0 < |r_i| < 1$ and $r_i n_i > 0$, (27) has an appropriate solution (r_1, r_2) iff

$$\frac{n_1}{m_\infty} < 2, \quad \frac{n_2}{m_\infty} < 2, \quad -\frac{n_1}{m_0} < 2, \quad -\frac{n_2}{m_0} < 2.$$

This is exactly the Fano condition in Proposition 1.1. In turn, under this condition, (26) and (27) correspond exactly to $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \lambda\Omega_{\mathbf{r}}/2\pi$.

Similarly to the smooth case, we shall see that in the Fano case, there is indeed always an admissible Ricci soliton in the appropriate Kähler class:

Using the integrating factor method, observe that

$$\begin{aligned}F(\mathfrak{z}) &= e^{-c\,\mathfrak{z}}\int_{-1}^{\mathfrak{z}} e^{c\,t}\left(\left(\frac{1}{m_\infty} - \frac{1}{m_0}\right) - \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)t\right)p_c(t)dt\\ &= e^{-c\,\mathfrak{z}}\int_{-1}^{\mathfrak{z}} e^{c\,t}(t - t_0)g(t)dt\end{aligned} \tag{29}$$

solves (28) and (*ii*) of (21) iff $G(c) = 0$, where

$$G(k) = e^{k\,t_0} \int_{-1}^{1} e^{k\,(t-t_0)}(t - t_0)g(t)dt, \tag{30}$$

$t_0 = \frac{m_0 - m_\infty}{m_0 + m_\infty}$, and $g(t) = -(\frac{1}{m_0} + \frac{1}{m_\infty})p_c(t)$.

Note that $t_0 \in (-1, 1)$ and $g(t) < 0$ for $t \in [-1, 1]$. Thus $e^{-kt_0}G(k)$ is a strictly decreasing function of k tending to $\mp\infty$ as $k \to \pm\infty$, and hence has a unique zero c (consistent with the uniqueness of Ricci solitons).

To check (*i*) of (21) we consider another auxiliary function

$$h(\mathfrak{z}) = e^{c\,\mathfrak{z}}F(\mathfrak{z}) = \int_{-1}^{\mathfrak{z}} e^{c\,t}(t - t_0)g(t)dt.$$

Note that the sign of $h(\mathfrak{z})$ equals the sign of $F(\mathfrak{z})$. In particular, $h(\pm 1) = 0$, and (due to (*iii*) of (21)) h is positive to the immediate right of $\mathfrak{z} = -1$ and immediate left of $\mathfrak{z} = +1$. Now, since h' clearly has exactly one zero (namely t_0) in $(-1, 1)$, it is positive on $(-1, 1)$. Therefore (*i*) of (21) is also satisfied.

Combining the arguments above with Proposition 1.1 we conclude the following.

Proposition 4.7 *For $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ the following conditions are equivalent:*

- *The inequalities $\frac{n_1}{m_\infty} < 2$, $\frac{n_2}{m_\infty} < 2$, $-\frac{n_1}{m_0} < 2$, $-\frac{n_2}{m_0} < 2$ are satisfied;*
- *$(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ is log Fano;*
- *There exist a Kähler-Ricci soliton on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$.*

In this case, the Kähler-Ricci soliton (g, ω) is admissible and satisfies $\rho = \mathcal{L}_V\omega$, with $\lambda = \frac{1}{2m_0} + \frac{1}{2m_\infty}$ and $V = (\frac{c}{2})\, grad_g\mathfrak{z}$ for a suitable real constant c. The Kähler-Ricci soliton is Kähler-Einstein iff this c is equal to zero.

Of course this can also be seen as an orbifold extension of Theorem 3.1 in [3] (which in turn is essentially due to Koiso [41]) in the case of $S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$.

Remark 4.8 Note that, assuming we are in the Fano case, c in the above proposition is a non-zero multiple of the Futaki invariant, $\mathcal{F}(V)$, of $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ applied to V. This underscores the well-known fact that the existence of a Kähler-Ricci Soliton with respect to a non-trivial holomorphic vector field is an obstruction to the existence of a Kähler-Einstein metric.

4.4.1 Kähler-Einstein

Now we turn our attention to Kähler-Einstein examples on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$. From the work above we see that under the assumption that the log Fano condition and (27) hold, $\Omega_{\mathbf{r}}$

contains an admissible Kähler-Einstein metric with $\rho = (\frac{1}{2m_0} + \frac{1}{2m_\infty})\omega$ if and only if G defined in (30) satisfies that $G(0) = 0$, i.e.,

$$\int_{-1}^{1} \left(\left(\frac{1}{m_\infty} - \frac{1}{m_0}\right) - \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right) t\right)(1 + r_1 t)(1 + r_2 t)dt = 0. \tag{31}$$

Carrying out the integration in (31) and substituting (27), we have the following

Proposition 4.9 *The Bott orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ admits a Kähler-Einstein metric (which happens to be admissible) iff the following two conditions are satisfied*

1. $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ *is log Fano (i.e.,* $\frac{n_1}{m_\infty} < 2$, $\frac{n_2}{m_\infty} < 2$, $-\frac{n_1}{m_0} < 2$, $-\frac{n_2}{m_0} < 2$*)*
2.

$$24(m_0^3 m_\infty^2 - m_0^2 m_\infty^3) - 8(n_1 + n_2)(m_0^3 m_\infty - m_0^2 m_\infty^2 + m_0 m_\infty^3) + 3n_1 n_2 (m_0^3 - m_0^2 m_\infty + m_0 m_\infty^2 - m_\infty^3) = 0$$

Example 4.1 If we assume $n_1 = 1$ and $n_2 = 2$, then the equation in Proposition 4.9 rewrites to

$$6(2m_0 m_\infty - m_0 + m_\infty)(m_0^2(-1 + 2m_\infty) - m_\infty^2(1 + 2m_0)) = 0.$$

Clearly $2m_0 m_\infty - m_0 + m_\infty = 0$ has no positive integer solutions (m_0, m_∞). Likewise, assume by contradiction that (m_0, m_∞) are positive integer solutions of

$$m_0^2(-1 + 2m_\infty) - m_\infty^2(1 + 2m_0) = 0,$$

i.e.,

$$m_0^2(-1 + 2m_\infty) = m_\infty^2(1 + 2m_0).$$

Since $gcd(2m_i \pm 1, m_i) = 1$ for $i = 0, \infty$ we see that this would imply that $m_0 = m_\infty$ (since they would have to have the same prime factorization). But $m_0 = m_\infty$ will clearly never solve the equation.

Remark 4.10 The diophantine nature of the equation in Proposition 4.9 makes it hard to spot solutions other than the classic smooth Koiso-Sakane example (see Example 4.2 below). Further, Example 4.1 and Proposition 4.9 tells us that there exist at least one pair $\mathbf{n} = (n_1, n_2)$ such that for all pairs $\mathbf{m} = (m_0, m_\infty) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, the Bott orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ admits no Kähler-Einstein metric. For these reasons we will instead view a choice of $(r_1, r_2) \in \mathbb{Q}^2$ such that $0 < |r_i| < 1$ as a pair of parameters that determine a unique KE example on the appropriate corresponding KS orbifold.

From this point of view, note that (31) is equivalent to the equation

$$\frac{m_\infty}{m_0} = \frac{3 + r_1 r_2 - r_1 - r_2}{3 + r_1 r_2 + r_1 + r_2}. \tag{32}$$

and, assuming (32), Eq. (27) is equivalent to

$$\frac{n_1}{m_0} = \frac{2r_1(3 + r_1r_2 - r_1 - r_2)}{3 + 2r_1r_2 + r_1^2}, \quad \frac{n_2}{m_0} = \frac{2r_2(3 + r_1r_2 - r_1 - r_2)}{3 + 2r_1r_2 + r_2^2}. \tag{33}$$

We notice that if we pick a rational pair (r_1, r_2) such that $0 < |r_i| < 1$, then there is a unique quadruple of appropriate integers $(n_1, n_2, m_0, m_\infty)$, solving (32) and (33), such that n_i has the same sign as r_i, $m_i > 0$, and $\gcd(|n_1|, |n_2|, m_0, m_\infty) = 1$. This yields a KE example on the corresponding KS orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ with fibers $\mathbb{CP}^1[v_0, v_\infty)]/\mathbb{Z}_m$. Here we have used our earlier notation; $m = \gcd(m_0, m_\infty)$ and $\mathbf{m} = (m_0, m_\infty) = m\mathbf{v} = m(v_0, v_\infty)$.

Example 4.2 Assume $r_1 = r$ and $r_2 = -r$ with $r \in (0, 1) \cap \mathbb{Q}$. Then (32) and (33) yields

$$\frac{m_\infty}{m_0} = 1, \quad \frac{n_1}{m_0} = 2r, \quad \frac{n_2}{m_0} = -2r.$$

When $r = 1/2$ this yields the Koiso-Sakane smooth Kähler-Einstein metric on $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1, -1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ ([44]). In general, we write $r = p/q$ in reduced form with co-prime integers $0 < p < q$. Then

$$(n_1, n_2, m_0, m_\infty) = \begin{cases} (2p, -2p, q, q) & \text{if } q \text{ is odd} \\ (p, -p, q/2, q/2) & \text{if } q \text{ is even} \end{cases}$$

This gives us Kähler-Einstein metrics on $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(2p, -2p))$ with fibers $\mathbb{CP}^1/\mathbb{Z}_q$ and on $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(p, -p))$ with fibers $\mathbb{CP}^1/\mathbb{Z}_{q/2}$, respectively. Using (4) we see that the index is equal to 2 if q is odd and equal to 1 if q is even.

Below we will explore more examples beyond the Koiso-Sakane examples. We set $r_i = p_i/q_i$ with $q_i \in \mathbb{Z}^{\geq 2}$ and $p_i \in \mathbb{Z}$ such that $0 < |p_i| < q_i$ and $\gcd(|p_i|, q_i) = 1$. Further, without loss we can assume that r_1, hence p_1, is positive. We then observe that (32) and (33) are equivalent to

$$\begin{aligned} \frac{m_\infty}{m_0} &= \frac{3q_1q_2+p_1p_2-p_1q_2-p_2q_1}{3q_1q_2+p_1p_2+p_1q_2+p_2q_1} \\ \frac{n_1}{m_0} &= \frac{2p_1(3q_1q_2+p_1p_2-p_1q_2-p_2q_1)}{3q_1^2q_2+2p_1p_2q_1+p_1^2q_2} \\ \frac{n_2}{m_0} &= \frac{2p_2(3q_1q_2+p_1p_2-p_1q_2-p_2q_1)}{3q_1q_2^2+2p_1p_2q_2+p_2^2q_1}. \end{aligned} \tag{34}$$

Note that, due to $0 < |p_i| < q_i$, the denominator of each the right hand fractions in (34) are all positive. Further, the numerator of the first two right hand fractions are positive while the numerator of the last right hand fraction has the same sign as p_2. Thus if we set

$$k = \gcd\begin{pmatrix} (3q_1q_2+p_1p_2+p_1q_2+p_2q_1)\left(3q_1^2q_2+2p_1p_2q_1+p_1^2q_2\right)\left(3q_1q_2^2+2p_1p_2q_2+p_2^2q_1\right), \\ (3q_1q_2+p_1p_2-p_1q_2-p_2q_1)\left(3q_1^2q_2+2p_1p_2q_1+p_1^2q_2\right)\left(3q_1q_2^2+2p_1p_2q_2+p_2^2q_1\right), \\ 2p_1(3q_1q_2+p_1p_2-p_1q_2-p_2q_1)(3q_1q_2+p_1p_2+p_1q_2+p_2q_1)\left(3q_1q_2^2+2p_1p_2q_2+p_2^2q_1\right), \\ 2|p_2|(3q_1q_2+p_1p_2-p_1q_2-p_2q_1)(3q_1q_2+p_1p_2+p_1q_2+p_2q_1)\left(3q_1^2q_2+2p_1p_2q_1+p_1^2q_2\right) \end{pmatrix}$$

we have an appropriate solution $(n_1, n_2, m_0, m_\infty)$ to (34) given by

$$\begin{aligned} n_1 &= \frac{2p_1(3q_1q_2+p_1p_2-p_1q_2-p_2q_1)(3q_1q_2+p_1p_2+p_1q_2+p_2q_1)\left(3q_1q_2^2+2p_1p_2q_2+p_2^2q_1\right)}{k} \\ n_2 &= \frac{2p_2(3q_1q_2+p_1p_2-p_1q_2-p_2q_1)(3q_1q_2+p_1p_2+p_1q_2+p_2q_1)\left(3q_1^2q_2+2p_1p_2q_1+p_1^2q_2\right)}{k} \\ m_0 &= \frac{(3q_1q_2+p_1p_2+p_1q_2+p_2q_1)\left(3q_1^2q_2+2p_1p_2q_1+p_1^2q_2\right)\left(3q_1q_2^2+2p_1p_2q_2+p_2^2q_1\right)}{k} \\ m_\infty &= \frac{(3q_1q_2+p_1p_2-p_1q_2-p_2q_1)\left(3q_1^2q_2+2p_1p_2q_1+p_1^2q_2\right)\left(3q_1q_2^2+2p_1p_2q_2+p_2^2q_1\right)}{k}. \end{aligned} \tag{35}$$

Each of these solutions then yields a KE example on the KS orbifold $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = (\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)), \Delta_{\mathbf{m}})$ with fibers $\mathbb{CP}^1[m_0/\gcd(m_0, m_\infty), m_\infty/\gcd(m_0, m_\infty)]/\mathbb{Z}_{\gcd(m_0,m_\infty)}$.

Proposition 4.11 *There exists a four-parameter family of KS orbifolds with KE orbifold metrics. The parameters* (p_1, p_2, q_1, q_2) *are integers such that* $0 < p_1 < q_1$, $0 < |p_2| < q_2$, *and* $\gcd(|p_i|, q_i) = 1$.

Remark 4.12 One might ask the following question. Suppose we have a coprime quadruple $(n_1, n_2, m_0, m_\infty)$ satisfying the conditions in Proposition 4.9, that is, $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ admits a KE metric with $\mathbf{n} = (n_1, n_2)$ and $\mathbf{m} = (m_0, m_\infty)$. Fixing $\mathbf{n} = (n_1, n_2)$, does there exist another pair $\tilde{\mathbf{m}} = (\tilde{m}^0, \tilde{m}^\infty)$ with $\tilde{\mathbf{m}} \neq \mathbf{m}$, such that $(n_1, n_2, \tilde{m}^0, \tilde{m}^\infty)$ is coprime and also satisfies the conditions in Proposition 4.9. In other words, if $\mathbf{n}$ has an appropriate choice of $\mathbf{m}$ such that $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ admits a KE metric, is this $\mathbf{m}$ then unique?

Example 4.3 If we pick $p_1 = p_2 = 1$, $q_1 = 2$, and $q_2 = q > 2$, then (35) becomes

$$\begin{aligned} n_1 &= \frac{4(5q-1)(7q+3)\left(3q^2+q+1\right)}{k} = \frac{2(5q-1)(7q+3)\left(3q^2+q+1\right)}{\hat{k}} \\ n_2 &= \frac{2(5q-1)(7q+3)(13q+4)}{k} = \frac{(5q-1)(7q+3)(13q+4)}{\hat{k}} \\ m_0 &= \frac{2(7q+3)(13q+4)\left(3q^2+q+1\right)}{k} = \frac{(7q+3)(13q+4)\left(3q^2+q+1\right)}{\hat{k}} \\ m_\infty &= \frac{2(5q-1)(13q+4)\left(3q^2+q+1\right)}{k} = \frac{(5q-1)(13q+4)\left(3q^2+q+1\right)}{\hat{k}}, \end{aligned}$$

where

$$\hat{k} = \gcd\begin{pmatrix} 2(5q-1)(7q+3)\left(3q^2+q+1\right), \\ (5q-1)(7q+3)(13q+4), \\ (7q+3)(13q+4)\left(3q^2+q+1\right), \\ (5q-1)(13q+4)\left(3q^2+q+1\right) \end{pmatrix}.$$

Likewise, if we pick $p_1 = 1$, $p_2 = -1$, $q_1 = 2$, and $q_2 = q > 2$, then (35) becomes

$$
\begin{aligned}
n_1 &= \tfrac{4(5q+1)(7q-3)\left(3q^2-q+1\right)}{k} = \tfrac{2(5q+1)(7q-3)\left(3q^2-q+1\right)}{\hat{k}} \\
n_2 &= \tfrac{-2(5q+1)(7q-3)(13q-4)}{k} = \tfrac{-(5q+1)(7q-3)(13q-4)}{\hat{k}} \\
m_0 &= \tfrac{2(7q-3)(13q-4)\left(3q^2-q+1\right)}{k} = \tfrac{(7q-3)(13q-4)\left(3q^2-q+1\right)}{\hat{k}} \\
m_\infty &= \tfrac{2(5q+1)(13q-4)\left(3q^2-q+1\right)}{k} = \tfrac{(5q+1)(13q-4)\left(3q^2-q+1\right)}{\hat{k}},
\end{aligned}
$$

where

$$
\hat{k} = \gcd \begin{pmatrix} 2(5q+1)(7q-3)\left(3q^2-q+1\right), \\ (5q+1)(7q-3)\,(13q-4)\,, \\ (7q-3)\,(13q-4)\left(3q^2-q+1\right), \\ (5q+1)\,(13q-4)\left(3q^2-q+1\right) \end{pmatrix}.
$$

The appendix contains a table representing a sample family of these solutions as well as the classic smooth Koiso-Sakane example from Example 4.2. Using (4), we calculated the (orbifold) index $\mathcal{I}_{\mathbf{n},\mathbf{m}}$ of each $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$.

4.5 Extremal and CSC Kähler Metrics

More generally, the admissible metrics are *extremal*, as defined by Calabi [22], if and only if the scalar curvature of g, which is given as a function of $\mathfrak{z}$ by

$$
Scal(g) = \frac{2s_1 r_1}{1 + r_1\mathfrak{z}} + \frac{2s_2 r_2}{1 + r_2\mathfrak{z}} - \frac{F''(\mathfrak{z})}{p_c(\mathfrak{z})}, \tag{36}
$$

is a holomorphic potential, i.e., a linear affine function of $\mathfrak{z}$. Following the arguments in [3] and considering the orbifold case at hand, it is easy to see that the proof of Proposition 11 together with Sect. 2.2 in [3] adapts to give us the following result.

Proposition 4.13 *Any Kähler class $\Omega_{\mathbf{r}}$ on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ admits an admissible extremal metric with scalar curvature equal to an affine linear function of $\mathfrak{z}$. Moreover, this metric is (positive) CSC (i.e., the function is constant) if and only if $\alpha_0\beta_1 - \alpha_1\beta_0 = 0$, where*

$$
\begin{aligned}
\alpha_r &= \int_{-1}^{1} t^r p_c(t)dt \\
\beta_r &= \int_{-1}^{1} \Big(r_1 s_1(1 + r_2 t) + r_2 s_2(1 + r_1 t)\Big) t^r dt \\
&\quad + (-1)^r p_c(-1)/m_\infty + p_c(1)/m_0.
\end{aligned} \tag{37}
$$

In the smooth case, this has been thoroughly explored in [34, 38, 39], as well as Sect. 3.4 of [3]. Here we will just mention that with $s_1 = 2/n_1$ and $s_2 = 2/n_2$, the equation $\alpha_0\beta_1 - \alpha_1\beta_0 = 0$ is equivalent to $f(r_1, r_2) = 0$ where

$$
\begin{aligned}
f(r_1, r_2) =&9(m_0 - m_\infty)n_1n_2 - 6(m_0 + m_\infty)n_1n_2(r_1 + r_2) + 6(m_0 - m_\infty)n_1n_2r_1r_2 \\
&+ 3n_2(4m_0m_\infty - n_1(m_0 - m_\infty))r_1^2 + 3n_1(4m_0m_\infty - n_2(m_0 - m_\infty))r_2^2 \\
&- (4m_0m_\infty(n_1 + n_2) - 3(m_0 - m_\infty)n_1n_2)r_1^2r_2^2.
\end{aligned}
\tag{38}
$$

Proposition 4.14 *For any value of $n_1, n_2 \neq 0$, there exist a choice of m_0 and m_∞ such that $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ admits Kähler classes $\Omega_{\mathbf{r}}$ with admissible constant scalar curvature Kähler metrics.*

Proof In the case of $n_1n_2 < 0$ we can choose $m_0 = m_\infty$ and the existence of CSC Kähler classes follows from [39] (see also [3] for details using the present notation). In the case of $n_1n_2 > 0$, notice that $f(0, 0) = 9(m_0 - m_\infty)n_1n_2$ while $f(1, 1) = 8m_\infty(m_0(n_1 + n_2) - 3n_1n_2)$. So, if we choose m_0 and m_∞ such that $m_\infty > m_0 > \frac{3n_1n_2}{n_1+n_2}$, we have $f(0, 0) < 0$ and $f(1, 1) > 0$. This means that any continuous curve going from $(0, 0)$ to $(1, 1)$ in the square $0 < r_1, r_2 < 1$ must contain at least one point (r_1, r_2) where f vanishes. This ensures the existence of solutions $0 < r_1, r_2 < 1$ to the equation $f(r_1, r_2) = 0$. In turn, this implies the existence of classes $\Omega_{\mathbf{r}}$ with admissible constant scalar curvature Kähler metrics. □

Example 4.4 When $m_0 = m_\infty = 1$, $n_1 = 1$, and $n_2 = -1$, we have $f(r_1, r_2) = -12(-1 + r_1 - r_2)(r_1 + r_2)$, reconfirming the smooth CSC Kähler metrics on $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1, -1) \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ as stated in Theorem 9 of [3].

Remark 4.15 Note that in general there is no guarantee the *CSC* Kähler classes $\Omega_{\mathbf{r}}$ from Proposition 4.14 are rational (i.e. both r_1 and r_2 are rational) and thus useful from the Sasakian geometry point of view. With the constraint that r_1, r_2 are rational, the equation $f(r_1, r_2) = 0$ is diophantine in nature. Suppose $f(r_1, r_2) \neq 0$ for some rational class $\Omega_{\mathbf{r}}$ and assume we have a Boothby-Wang constructed Sasaki manifold defined by an integer orbifold Kähler class obtained by an appropriate rescale of Ω_r. We shall see in the next subsection that then we will have a CSC Sasaki metric somewhere else in the Sasaki cone of this Sasaki manifold. The key to seeing this is to use the connection discovered by Apostolov and Calderbank in [1] between the so-called *weighted extremal metrics* and extremal metrics appearing as transverse structures for different rays in the same Sasaki cone.

4.6 Weighted Extremal Metrics

Let (M, g, ω) be a Kähler orbifold of complex dimension m, f a positive Killing potential on M, and (weight) $p \in \mathbb{R}$. Then the (f, p)-Scalar curvature of g is given by

$$Scal_{f,p}(g) = f^2 Scal(g) - 2(p-1) f \Delta_g f - p(p-1)|df|_g^2, \tag{39}$$

If $Scal_{f,p}(g)$ is a Killing potential, g is said to be a (f, p)*-extremal Kähler metric*. The case $p = 2m$ has been studied by several people and is interesting due to the fact that $Scal_{f,2m}(g)$ computes the scalar curvature of the Hermitian metric $h = f^{-2} g$.

However, the case of interest to us here is when $p = m + 2$. This case is related to the study of extremal Sasaki metrics [1, 5, 6]. Indeed, if we assume that the Kähler class $[\omega/2\pi]$ is an integer orbifold class giving a Boothby-Wang constructed (smooth) Sasaki manifold over (M, g, ω), then $\frac{Scal_{f,m+2}(g)}{f}$ is equal to the transverse scalar curvature of a certain Sasaki structure (determined by f) in the Sasaki cone. More precisely, if χ is the Reeb vector field of the Sasaki structure coming directly from the Boothby-Wang construction over (M, g, ω) and f is viewed as a pull-back to the Sasaki manifold, then (mod $\mathcal{D}$) $\xi := f\chi$ is a Reeb vector field in the Sasaki cone giving a new Sasaki structure. While the pull-back from M of $Scal(g)$ is the Tanaka-Webster scalar curvature of the Tanaka-Webster connection induced by χ, the expression $\frac{Scal_{f,m+2}(g)}{f}$ pulls back from M to be the Tanaka-Webster scalar curvature of the Tanaka-Webster connection induced by ξ. The latter is then also identified with the transverse scalar curvature of the Sasaki structure defined by ξ. This fact is seen from the details of the proof of Lemma 3 in [1]. As also follows from [1] (see their Theorem 1), the Sasaki structure determined by f is extremal if and only if g is $(f, m+2)$-extremal.

Now, returning to our orbifolds at hand we have that $m = 3$ and so $m + 2 = 5$. Note that $|\mathfrak{z} + b|$ for $b \in \mathbb{R}$ such that $|b| > 1$ defines a Killing potential on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$.

It is not hard to check that Sect. 2 and the existence result in the second half of Theorem 3.1. of [7] adapts to get us the next proposition. Note in particular, that whether we use $f = \mathfrak{z} + b$ (when $(b > 1)$ or $f = -(\mathfrak{z} + b)$ (when $b < -1$), the formula for $Scal_{f,p}$ in (39) will give us the same result, i.e., the right hand side of (19) in [7]. In general the assumption of $b > 1$ in [7] is merely practical and all the arguments are easily adapted to include the $b < -1$ case as well. Further, the root counting argument in the proof of Theorem 3.1.in [7] is not affected by our mild orbifold conditions.

Proposition 4.16 *Let $b \in \mathbb{R}$ such that $|b| > 1$. Any Kähler class $\Omega_{\mathbf{r}}$ on $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ admits a $(|\mathfrak{z} + b|, 5)$-extremal Kähler metric with weighted scalar curvature $Scal_{|\mathfrak{z}+b|,5} = A_1\mathfrak{z} + A_2$ for constants A_1, A_2 given as the unique solutions of the following linear system*

$$\begin{aligned} \alpha_{1,-6} A_1 + \alpha_{0,-6} A_2 &= 2\beta_{0,-4} \\ \alpha_{2,-6} A_1 + \alpha_{1,-6} A_2 &= 2\beta_{1,-4}, \end{aligned} \tag{40}$$

where

$$
\begin{aligned}
\alpha_{r,-6} &= \int_{-1}^{1} (t+b)^{-6} t^r p_c(t) dt \\
\beta_{r,-4} &= \int_{-1}^{1} \Big(\frac{2r_1}{n_1}(1+r_2 t) + \frac{2r_2}{n_2}(1+r_1 t) \Big) t^r (t+b)^{-4} dt \\
&\quad + (-1)^r (b-1)^{-4} p_c(-1)/m_\infty + (1+b)^{-4} p_c(1)/m_0.
\end{aligned}
\tag{41}
$$

Moreover, assuming that $\Omega_{\mathbf{r}}$ is rational and we can form a Boothby-Wang constructed Sasaki manifold with respect to an appropriate rescale of $\Omega_{\mathbf{r}}$, the extremal Sasaki structure determined by $f = |\mathfrak{z} + b|$ has constant scalar curvature if and only if $Scal_{|\mathfrak{z}+b|,5} = A_1 \mathfrak{z} + A_2$ is a constant multiple of $|\mathfrak{z} + b|$, i.e., if and only if $A_1 b - A_2 = 0$.

Suppose a rational $\Omega_{\mathbf{r}}$ is given on a specific $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$. Solving the linear system for A_1 and A_2 in Proposition 4.16 (note that $\alpha_{1,-6}^2 - \alpha_{0,-6}\alpha_{2,-6} \neq 0$) and simplifying a bit, the equation $A_1 b - A_2 = 0$ may be re-written as $h(b) = 0$ where

$$
h(b) = (b^2-1)^7 (b(\alpha_{1,-6}\beta_{0,-4} - \alpha_{0,-6}\beta_{1,-4}) - (\alpha_{1,-6}\beta_{1,-4} - \alpha_{2,-6}\beta_{0,-4})).
\tag{42}
$$

Using (41) we see that $h(b)$ simplifies as a polynomial of degree 5 in the variable b with leading coefficient equal to $\frac{2f(r_1,r_2)}{9m_0 m_\infty n_1 n_2}$, where $f(r_1, r_2)$ is given by (38). Since further $\lim_{b \to \pm 1^\mp} h(b) > 0$, we conclude that unless $f(r_1, r_2) = 0$ (in which case $\Omega_{\mathbf{r}}$ itself has an admissible CSC representative), there must exist at least one value $b \in (-\infty, -1) \cup (1, +\infty)$ such that $h(b) = 0$, hence $A_1 b - A_2 = 0$, and therefore, for this b value, the extremal Sasaki structure determined by $f = |\mathfrak{z} + b|$ has constant scalar curvature. If b is an irrational number, then the corresponding Sasaki structure is irregular.

Theorem 4.17 *Suppose $\Omega_{\mathbf{r}}$ is a rational admissible Kähler class on a KS orbifold of the form $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$. Let $\mathcal{S}$ be a Boothby-Wang constructed Sasaki manifold given by an appropriate rescale of $\Omega_{\mathbf{r}}$. Then the corresponding Sasaki cone will always have a (possibly irregular) CSC-ray (up to isotopy).*

Remark 4.18 It is not too difficult (using (36) and (21)) to produce some admissible Kähler metric of positive scalar curvature in $\Omega_{\mathbf{r}}$. Using the corresponding Kähler form of this metric for the Boothby-Wang construction, we can use Lemma 5.2 in [12]) to conclude that the CSC-ray alluded to in the theorem above has positive transverse scalar curvature. This together with Theorem 4.17 proves the main theorem in the introduction.

Remark 4.19 The observations in this section—in particular, Proposition 4.16 and Theorem 4.17—can be generalized to a larger group of admissible manifolds [13]. Indeed, in [13] we use a more general version of Proposition 4.16 to answer the open question in Problem 6.1.2 of [15] by providing a counter example where the Sasaki cone contains some extremal Sasaki structures, but no CSC Sasaki metric at all.

Example 4.5 Suppose $m_1 = m_2 = 1$ and suppose $n_1 > 4$. Now let $r_1 = \frac{5n_1^2-4}{n_1(n_1^2+4)}$, $r_2 = 2/n_1$, and n_2 be equal to any positive integer. Note that since $n_1 > 4$ we have ensured that $r_i \in (0, 1)$ for $i = 1, 2$ and $r_1 \neq r_2$. Then $\Omega_{\mathbf{r}}$ is a rational (non-diagonally) admissible Kähler class on the smooth KS manifold ($S_{\mathbf{n}}$). Note that since $\Omega_{\mathbf{r}}$ is non-diagonally admissible, we know from Theorem 4.3 that $S_{\mathbf{n}}$ (with any appropriate rescale of $\Omega_{\mathbf{r}}$) is not the quotient of a regular ray in the $\mathbf{w}$-cone of the type of $S^3_{\mathbf{w}}$-join described in Theorem 4.3. Further, it can be shown (as we will do near the end of this example) that $S_{\mathbf{n}}$ (with any appropriate rescale of $\Omega_{\mathbf{r}}$) is also not the quotient of the regular ray in the $\mathfrak{t}^+_{sphr}$ cone of a Yamazaki fiber join over $\mathbb{CP}^1 \times \mathbb{CP}^1$. The Boothby-Wang constructed Sasaki manifold over $S_{\mathbf{n}}$, given by an appropriate rescale of $\Omega_{\mathbf{r}}$ is thus in a certain sense a truly new example.

By Theorem 8 and Proposition 11 of [3] we know that the Kähler class $\Omega_{\mathbf{r}}$, while having an extremal admissible Kähler metric representative, does not admit a CSC Kähler metric. [This fact can also easily be checked by seeing that $f(r_1, r_2)$ from (38) cannot be zero in this case.] We now calculate that with the given choices of $(\mathbf{m}, \mathbf{n}, \mathbf{r})$, we have that $h(b)$ from (42) is equal to

$$h(b) = \frac{4(n_1 - 2b)p(b)}{9n_1^5\left(n_1^2+4\right)^2 n_2}$$

where $p(b)$ is given by

$$\begin{aligned} p(b) = {} & 3n_1^8n_2 - 4n_1^7 - 13n_1^6n_2 + 268n_1^5 + 1252n_1^4n_2 - 544n_1^3 - 1456n_1^2n_2 + 192n_1 + 192n_2 \\ & - 4n_1\left(9n_1^6n_2 + 20n_1^5 + 482n_1^4n_2 + 64n_1^3 - 464n_1^2n_2 - 64n_1 + 224n_2\right)b \\ & + 16\left(n_1^7 + 79n_1^6n_2 - 17n_1^5 - 35n_1^4n_2 + 56n_1^3 + 32n_1^2n_2 - 16n_1 - 16n_2\right)b^2 \\ & - 4n_1\left(63n_1^6n_2 - 20n_1^5 + 106n_1^4n_2 - 64n_1^3 - 16n_1^2n_2 + 64n_1 - 32n_2\right)b^3 \\ & + (21n_1^8n_2 - 12n_1^7 + 21n_1^6n_2 + 4n_1^5 + 268n_1^4n_2 - 352n_1^3 - 208n_1^2n_2 + 64n_1 + 64n_2)b^4. \end{aligned}$$

Clearly, $b = n_1/2$ is a rational root of $h(b)$. Note that since $n_1 > 4$ and $n_2 > 0$, we also have that

- The b^4 coefficient of $p(b)$ is positive
- $p(\pm 1) > 0$
- $p'(-1) < 0$ and $p'(1) > 0$
- $f(b) := p''(b)$ is a concave up 2nd order polynomial with $f(\pm 1) > 0$, $f'(-1) < 0$, and $f'(1) > 0$. Thus any roots of $f(b)$ would be inside the interval $(-1, 1)$.

Putting this together we have that $p(b)$ is positive and decreasing at $b = -1$, positive and increasing at $b = 1$, and concave up for $|b| \geq 1$. Hence $p(b)$ has no roots in $(-\infty, -1) \cup (1, +\infty)$ and hence $b = n_1/2$ is the ONLY root of $h(b)$.

All in all we conclude that while the original regular ray coming from the Boothby-Wang constructed Sasaki manifold over $S_{\mathbf{n}}$, given by an appropriate rescale of $\Omega_{\mathbf{r}}$ is not CSC, the CSC ray alluded to in Theorem 4.17 is quasi-regular. It would be interesting to explore what the transverse Kähler orbifold is for this ray. Conjecturely, it may be a KS orbifold.

Finally, let us finish this example by confirming that $S_{\mathbf{n}}$ (with any appropriate rescale of $\Omega_{\mathbf{r}}$) is also not the quotient of the regular ray in the $\mathfrak{t}^+_{sphr}$ cone of a Yamazaki fiber join over $\mathbb{CP}^1 \times \mathbb{CP}^1$:

For $S_{\mathbf{n}}$ (with an appropriate rescale of $\Omega_{\mathbf{r}}$) to be such a quotient, it follows from Sect. 4.3 and (23) that there should exist $k_1^1, k_1^2, k_2^1, k_2^2 \in \mathbb{Z}^+$ such that

$$k_1^1 - k_2^1 = n_1, \quad k_1^2 - k_2^2 = n_2, \quad \frac{k_1^1 - k_2^1}{k_1^1 + k_2^1} = \frac{5n_1^2 - 4}{n_1\left(n_1^2 + 4\right)}, \quad \frac{k_1^2 - k_2^2}{k_1^2 + k_2^2} = 2/n_1.$$

This would place further restrictions on n_1 and n_2 and, in particular, we would need n_1 to satisfy that $\frac{n_1^2(n_1^2+4)}{5n_1^2-4}$ is a positive integer (that positive integer is $k_1^1 + k_2^1$). By the following elementary lemma this is not possible and so $S_{\mathbf{n}}$ (with any appropriate rescale of $\Omega_{\mathbf{r}}$) is not the quotient of the regular ray in the $\mathfrak{t}^+_{sphr}$ cone of a Yamazaki fiber join.

Lemma 4.20 *For all integers $x \geq 3$, $\frac{x^2(x^2+4)}{5x^2-4}$ is NOT an integer.*

Proof Assume for contradiction that there exists an integer $x \geq 3$ such that $\frac{x^2(x^2+4)}{5x^2-4}$ is an integer. Let $d > 1$ be a prime divisor of $5x^2 - 4$. Then, due to our assumption, $d \mid x^2$ or $d \mid (x^2 + 4)$. If $d \mid x^2$, then, since also $d \mid (5x^2 - 4)$, we have that $d \mid 4$ and so $d = 2$. If $d \mid (x^2 + 4)$, then, since again $d \mid (5x^2 - 4)$, we have that $d \mid (5(x^2 + 4) - (5x^2 - 4))$, i.e., $d \mid 24$, and so $d = 2$ or $d = 3$.

By considering the three possible cases $x = 3k$, $x = 3k + 1$, and $x = 3k + 2$ with $k \in \mathbb{Z}$, we realize that $3 \nmid (5x^2 - 4)$. Therefore, $d = 2$ and $5x^2 - 4$ is even. This implies that x^2 is even and hence x is even. In particular, $x \neq 3$ and moving forward we may assume $x \geq 4$.

Since $5x^2 - 4$ is even, we have that $(5x^2 - 4) = 2^i$ for some non-negative integer i. We also know that $5x^2 - 4 \geq 76$ (since $x \geq 4$), so $i > 6$.

Now, writing $x = 2k$ for some $k \in \mathbb{Z}^+$ we see that $(5x^2 - 4) = 2^i$ implies that $20k^2 - 4 = 2^i$ and hence $5k^2 = 2^{i-2} + 1$. Thus $5k^2$ is odd, which gives us that k^2 is odd and hence k is odd. Let us write $k = 2l + 1$ for some $l \in \mathbb{Z}^+$. So $x = 2(2l + 1)$. Then

$$\begin{aligned} x^2\left(x^2 + 4\right) &= 2^2(4l^2 + 4l + 1)(2^2(4l^2 + 4l + 1) + 4) \\ &= 2^2(4l^2 + 4l + 1)(2^2(4l^2 + 4l + 2)) \\ &= 2^2(4l^2 + 4l + 1)(2^3(2l^2 + 2l + 1)) \\ &= 2^5(2(2l^2 + 2l) + 1)(2(l^2 + l) + 1). \end{aligned}$$

Since $(2(2l^2+2l)+1)(2(l^2+l)+1)$ is odd, we realize that $\frac{x^2(x^2+4)}{5x^2-4} = \frac{(2(2l^2+2l)+1)(2(l^2+l)+1)}{2^{i-5}}$. Since $i > 6$, this cannot possible be an integer and hence we have arrived at a contraction. This completes the proof of the lemma. □

As an immediate consequence of Theorem 4.17 we get the following special case of the existence result for toric Sasaki-Einstein metrics due to Futaki, Ono, and Wang [32].

Corollary 4.21 *Suppose a KS orbifold of the form $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ is log Fano and consider a Boothby-Wang constructed Sasaki manifold over $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ given by some Kähler form representing $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})/\mathcal{I}_{\mathbf{n},\mathbf{m}}$. Then the corresponding Sasaki cone will always have a (possibly irregular) Sasaki-Einstein structure (up to isotopy).*

Proof First, we note that $c_1(\mathcal{D}) = 0$ and so the CSC Sasaki metrics from Theorem 4.17 above are now η-Einstein. Second, we see from Proposition 5.3 in [12]) that since the basic first Chern class of the initial Sasaki metric is positive (as a pullback of the positive class $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})/\mathcal{I}_{\mathbf{n},\mathbf{m}}$), the average transverse scalar curvature of any Sasaki structure in the Sasaki cone must be positive. In particular, the transverse (constant) scalar curvature of any η-Einstein structure in the cone must be positive. This means that any η-Einstein ray admits a Sasaki-Einstein structure. □

Remark 4.22 Examples similar in spirit to Corollary 4.21, but also including non-toric cases have been given by Mabuchi and Nakagawa [47]. See also [6] and references therein.

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Appendix

p_1	q_1	p_2	q_2	n_1	n_2	m_0	m_∞	m	v_0	v_∞	$\mathcal{I}_{\mathbf{n},\mathbf{m}}$
1	2	−1	15	5124072	−740316	6438801	4797538	126251	51	38	89
1	2	−1	14	775675	−120061	972325	726685	10235	95	71	83
1	2	−1	13	48	−8	60	45	15	4	3	7
1	2	−1	12	2080161	−375516	2591676	1951756	31996	81	61	71
1	2	−1	11	1462832	−288008	1815479	1373876	49067	37	28	65
1	2	−1	10	110483	−23919	136479	103887	2037	67	51	59
1	2	−1	9	129720	−31188	159330	122153	5311	30	23	53
1	2	−1	8	80401	−21730	98050	75850	1850	53	41	47
1	2	−1	7	25944	−8004	31349	24534	1363	23	18	41
1	2	−1	6	124527	−44733	148629	118141	3811	39	31	35
1	2	−1	5	59072	−25376	69296	56303	4331	16	13	29
1	2	−1	4	525	−280	600	504	24	25	21	23
1	2	−1	3	1440	−1008	1575	1400	175	9	8	17
1	2	−1	2	1	−1	1	1	1	1	1	1
1	2	1	2	51	51	85	45	5	17	9	13
1	2	1	3	10416	7224	15996	9331	1333	12	7	19
1	2	1	4	31217	16492	46004	28196	1484	31	19	25
1	2	1	5	8208	3496	11799	7452	621	19	12	31
1	2	1	6	30015	10701	42435	27347	943	45	29	37
1	2	1	7	54808	16796	76570	50065	2945	26	17	43
1	2	1	8	51389	13806	71154	47034	1206	59	39	49
1	2	1	9	552	132	759	506	253	3	2	5
1	2	1	10	1112447	239659	1521101	1021013	20837	73	49	61
1	2	1	11	36000	7056	49000	33075	1225	40	27	67
1	2	1	12	456837	82128	619440	420080	7120	87	59	73
1	2	1	13	3134336	520384	4236251	2884256	90133	47	32	79
1	2	1	14	466923	72013	629331	429939	6231	101	69	85
1	2	1	15	5522472	795204	7425486	5087833	137509	54	37	91

TABLE: This gives a sample of Kähler−Einstein orbifold solutions. See Example 4.3.

References

1. Apostolov, V., Calderbank, D.M.J.: The CR geometry of weighted extremal Kähler and Sasaki metrics. Math. Ann. **379**(3–4), 1047–1088 (2021). **MR** 4238260
2. Araujo, C., Castravet, A.-M., Cheltsov, I., Fujita, K., Kaloghiros, A.-S., Martinez-Garcia, J., Shramov, C., Süss, H., Viswanathan, N.: The Calabi problem for Fano threefolds (2021)
3. Apostolov, V., Calderbank, D.M.J., Gauduchon, P., Tønnesen-Friedman, C.W.: Hamiltonian 2-forms in Kähler geometry. III. Extremal metrics and stability. Invent. Math. **173**(3), 547–601 (2008). **MR** 2425136 (2009m:32043)
4. Apostolov, V., Calderbank, D.M.J., Gauduchon, P., Tønnesen-Friedman, C.W.: Hamiltonian 2-forms in Kähler geometry. IV. Weakly Bochner-flat Kähler manifolds. Commun. Anal. Geom. **16**(1), 91–126 (2008). **MR** 2411469 (2010c:32043)

5. Apostolov, V., Calderbank, D.M.J., Legendre, E.: Weighted K-stability of polarized varieties and extremality of Sasaki manifolds. Adv. Math. **391** (2021). Paper No. 107969, 63. **MR** 4300921
6. Apostolov, V., Jubert, S., Lahdili, A.: Weighted K-Stability and coercivity with applications to Extremal Kähler and Sasaki Metrics (2021). arXiv: 2104.09709
7. Apostolov, V., Maschler, G., Tønnesen-Friedman, C.W.: Weighted Extremal Kähler metrics and the Einstein-Maxwell geometry of Projective Bundles. Comm. Anal. Geom. **30**(4), 689–744 (2022)
8. Aguilar, M.A., Prieto, C.: Transfers for ramified covering maps in homology and cohomology. Int. J. Math. Math. Sci. (2006). Art. ID 94651, 28. **MR** 2251716
9. Boyer, C.P., Calderbank, D.M.J., Tønnesen-Friedman, C.W.: The Kähler geometry of Bott manifolds. Adv. Math. **350**, 1–62 (2019). **MR** 3945589
10. Baum, H., Friedrich, T., Grunewald, R., Kath, I.: Twistors and Killing spinors on Riemannian manifolds, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 124. B.G. Teubner Verlagsgesellschaft mbH, Stuttgart (1991). With German, French and Russian summaries. **MR** 94a:53077
11. Boyer, C.P., Galicki, K.: Sasakian geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford (2008). **MR** MR2382957 (2009c:53058)
12. Boyer, C., Huang, H., Legendre, E.: An application of the Duistermaat–Heckman theorem and its extensions in Sasaki geometry. Geom. Topol. **22**(7), 4205–4234 (2018). MR 3890775
13. Boyer, C.P., Huang, H., Legendre, E., Tønnesen-Friedman, C.W: Existence and non-existence of constant scalar curvature and extremal Sasaki metrics. arXiv:2210.12203
14. Boyer, C.P., Huang, H., Legendre, E., Tønnesen-Friedman, C.W.: Reducibility in Sasakian geometry. Trans. Am. Math. Soc. **370**(10), 6825–6869 (2018). MR 3841834
15. Boyer, C.P., Huang, H., Legendre, E., Tønnesen-Friedman, C.W.: Some Open Problems in Sasaki Geometry. Differential Geometry in the Large, pp. 143–168. Cambridge University Press, Cambridge, UK (2021)
16. Boyer, C.P., Tønnesen-Friedman, C.W.: Iterated S^3 *Sasaki joins and Bott orbifolds*. Ann. Fac. Sci. Toulouse Math. **31**(3), 837–860 (2022) (6)
17. Boyer, C.P., Tønnesen-Friedman, C.W.: The Sasaki join, Hamiltonian 2-forms, and constant scalar curvature. J. Geom. Anal. **26**(2), 1023–1060 (2016). MR 3472828
18. Boyer, C.P., Tønnesen-Friedman, C.W.: Sasaki-Einstein metrics on a class of 7-manifolds. J. Geom. Phys. **140**, 111–124 (2019). MR 3923473
19. Boyer C.P., Tønnesen-Friedman, C.W.: The S^3 Sasaki join construction (2021). J. Math. Soc. Japan **74**(4), 1335–1371 (2022)
20. Boyer, C.P., Tønnesen-Friedman, C.W.: Sasakian geometry on sphere bundles. Differ. Geom. Appl. **77**, 101765, 37 (2021). MR 4253896
21. Boothby, W.M., Wang, H.C.: On contact manifolds. Ann. of Math. (2) **68**, 721–734 (1958). MR 22 #3015
22. Calabi, E.: Extremal Kähler metrics,. In: Seminar on Differential Geometry. Annals of Mathematics Studies, vol. 102, pp. 259–290. Princeton University Press, Princeton, N.J. (1982) MR 83i:53088
23. Chen, X., Cheng, J.: On the constant scalar curvature Kähler metrics (I)—A priori estimates. J. Am. Math. Soc. **34**(4), 909–936 (2021). MR 4301557
24. Chen, X., Cheng, J.: On the constant scalar curvature Kähler metrics (II)—Existence results. J. Am. Math. Soc. **34**(4), 937–1009 (2021). MR 4301558
25. Chen, X., Donaldson, S., Sun, S.: Kähler-Einstein metrics on Fano manifolds. I: approximation of metrics with cone singularities. J. Am. Math. Soc. **28**(1), 183–197 (2015). MR 3264766
26. Chen, X., Donaldson, S., Sun, S.: Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π. J. Am. Math. Soc. **28**(1), 199–234 (2015). MR 3264767
27. Chen, X., Donaldson, S., Sun, S.: Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof. J. Amer. Math. Soc. **28**(1), 235–278 (2015). MR 3264768

28. Collins, T.C., Székelyhidi, G.: K-semistability for irregular Sasakian manifolds. J. Differ. Geom. **109**(1), 81–109 (2018). MR 3798716
29. Collins, T.C., Székelyhidi, G.: Sasaki-Einstein metrics and K-stability. Geom. Topol. **23**(3), 1339–1413 (2019). MR 3956894
30. Donaldson, S.K.: Extremal metrics on toric surfaces: a continuity method. J. Differ. Geom. **79**(3), 389–432 (2008). MR MR2433928 (2009j:58018)
31. Dancer, A.S., Wang, M.Y.: On Ricci solitons of cohomogeneity one. Ann. Global Anal. Geom. **39**(3), 259–292 (2011). MR 2769300
32. Futaki, A., Ono, H., Wang, G.: Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds. J. Differ. Geom. **83**(3), 585–635 (2009). MR MR2581358
33. Grossberg, M., Karshon, Y.: Bott towers, complete integrability, and the extended character of representations. Duke Math. J **76**(1), 23–58 (1994). MR 1301185 (96i:22030)
34. Guan, D.: Existence of extremal metrics on compact almost homogeneous Kähler manifolds with two ends. Trans. Am. Math. Soc. **347**(6), 2255–2262 (1995). MR1285992 (96a:58059)
35. Haefliger, A.: Groupoïdes d'holonomie et classifiants. Astérisque **116**, 70–97 (1984). Transversal structure of foliations (Toulouse, 1982). MR 86c:57026a
36. He, W.: *Scalar curvature and properness on Sasaki manifolds* (2018). arXiv:1802.03841
37. He, W., Li, J.: Geometric pluripotential theory on Sasaki manifolds. J. Geom. Anal. **31**(2), 1093–1179 (2021). MR 4215260
38. Hwang, A.D., Singer, M.A.: A momentum construction for circle-invariant Kähler metrics. Trans. Am. Math. Soc. **354**(6), 2285–2325 (2002). (electronic). MR 1885653 (2002m:53057)
39. Hwang, A.D.: On existence of Kähler metrics with constant scalar curvature. Osaka J. Math. **31**(3), 561–595 (1994). MR 1309403 (96a:53061)
40. Kegel, M., Lange, C.: A Boothby-Wang theorem for Besse contact manifolds. Arnold Math. J. **7**(2), 225–241 (2021). textbfMR 4260074
41. Koiso, N.: On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics. In: Recent Topics in Differential and Analytic Geometry. Advance Studies Pure Mathematics, vol. 18, pp. 327–337. Academic, Boston, MA (1990) MR 1145263 (93d:53057)
42. Kollár, J.: Seifert G_m-bundles (2004). arXiv:math.AG/0404386
43. Kollár, J.: Einstein metrics on connected sums of $S^2 \times S^3$. J. Differential. Geom. **77**(2), 259–272 (2007). MR 2269781
44. Koiso, N., Sakane, Y.: Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds. In: Curvature and Topology of Riemannian Manifolds (Katata, 1985). Lecture Notes in Mathematics, vol. 1201, pp. 165–179. Springer, Berlin (1986) MR 859583 (88c:53047)
45. Legendre, E.: Existence and non-uniqueness of constant scalar curvature toric Sasaki metrics. Compos. Math. **147**(5), 1613–1634 (2011). MR 2834736
46. Legendre, E., Tønnesen-Friedman, C.W.: Toric generalized Kähler-Ricci solitons with Hamiltonian 2-form. Math. Z **274**(3–4), 1177–1209 (2013). MR 3078263
47. Mabuchi, T., Nakagawa, Y.: New examples of Sasaki-Einstein manifolds. Tohoku Math. J. (2) **65**(2), 243–252 (2013). MR 3079287
48. Orlik, P., Wagreich, P.: Seifert n-manifolds. Invent. Math. **28**, 137–159 (1975). MR 361150
49. Quillen, D.: he spectrum of an equivariant cohomology ring. I, II. Ann. Math. (2) **94**, 549–572 (1971). ibid. (2) 94 (1971), 573–602. MR 298694
50. Smith, L.: Transfer and ramified coverings. Math. Proc. Camb. Philos. Soc. **93**(3), 485–493 (1983). MR 698352
51. Shi, Y., Zhu, X.: Kähler-Ricci solitons on toric Fano orbifolds. Math. Z. **271**(3–4), 1241–1251 (2012). MR 3665915
52. Tian, G., Zhu, X.: Uniqueness of Kähler-Ricci solitons. Acta Math. **184**(2), 271–305 (2000). MR 1768112
53. Wang, X.-J., Zhu, X.: Kähler-Ricci solitons on toric manifolds with positive first Chern class. Adv. Math. **188**(1), 87–103 (2004). MR MR2084775 (2005d:53074)
54. Yamazaki, T.: A construction of K-contact manifolds by a fiber join. Tohoku Math. J. (2) **51**(4), 433–446 (1999). MR 2001e:53094
55. Zhu, X.: Kähler-Ricci soliton typed equations on compact complex manifolds with$C_1(M)$t0. J. Geom. Anal. **10**(4), 759–774 (2000). MR 1817785

Toric Sarkisov Links of Toric Fano Varieties

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Abstract We explain a web of Sarkisov links that overlies the classification of Fano weighted projective spaces in dimensions 3 and 4, extending results of Prokhorov.

Keywords Sarkisov links · Toric geometry

1 Introduction

A normal projective n-dimensional complex variety is called *Fano* if it has ample anticanonical class and canonical singularities. The construction and classification of Fano n-folds is a major concern of birational geometry. Dimensions $n = 3$ and 4 are at the cutting edge, where birational methods of construction play a central role. In various respects, toric Fano varieties arise at the extremes of classification (compare [5]). In this paper we consider the toric birational geometry of certain toric Fano 4-folds, and in particular the Sarkisov links between them; we review terminology in Sect. 2.1 and the results are surveyed in Sect. 5, with full details of over a million links relegated to a webpage [4]. Although as a study of the 4-dimensional Sarkisov program this is a baby case, it does provide a large number of examples and gives some indication of the 4-fold phenomena we can expect to encounter when we push beyond toric. It also describes classes of new toric Fano 4-folds, as we discuss in Sect. 2.3.

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A *Fano polytope* is a polytope $\Delta \subset \mathbb{R}^n = L \otimes \mathbb{R}$ whose vertices lie on a fixed lattice $L \cong \mathbb{Z}^n$ and whose only strictly interior lattice point $p \in \Delta^\circ \cap L$ is the origin. A Fano polytope is *terminal* if its vertices are the only lattice points lying on its boundary, and it is $\mathbb{Q}$-*factorial* (or *simplicial*) if each facet is an $(n-1)$-simplex. In 3 dimensions Fano polytopes are classified up to $\mathrm{Aut}(L)$ into 674, 688 cases [20, 21]: 12, 190 are simplicial; 634 are terminal; 233 are $\mathbb{Q}$-factorial terminal, and of these only 8 have the minimal number of four vertices, the terminal simplicial tetrahedra. In 4 dimensions, Fano polytopes are not yet classified, but the terminal simplicial polytopes (on five vertices) are, with 35, 947 cases [22].

Toric geometry connects these two realms. Given a Fano polytope Δ, the spanning fan Σ of Δ is the collection of lattice cones on the facets and smaller strata of Δ. This fan Σ determines a normal complex projective variety $X = X_\Sigma$ [14, 15], that has ample anticanonical class $-K_X$ (by the convexity of Δ) and canonical singularities (by the assumption on $\Delta^\circ \cap L$) [25]. In other words, X is a Fano n-fold with canonical singularities, and furthermore it is terminal or $\mathbb{Q}$-factorial exactly when Δ is. In the $\mathbb{Q}$-factorial case, the Picard rank ρ_X of X is the number of vertices of Δ minus n. A *Mori–Fano n-fold* is a $\mathbb{Q}$-factorial terminal Fano n-fold X with $\rho_X = 1$. The toric Mori–Fano n-folds are those weighted projective spaces and their quotients by the finite discriminant group $L/\langle \mathrm{vertices}(\Delta)\rangle$, often called *fake weighted projective spaces* (compare [8, 6.4]), that have terminal singularities. Thus there are precisely 8 toric Mori–Fano 3-folds (compare [1]), and 35, 947 toric Mori–Fano 4-folds (of which 24, 511 are weighted projective spaces).

Toric maps from X are controlled by the combinatorics of the fan Σ. Prokhorov [24, Sect. 10] works out a beautiful web of Sarkisov links between 7 of the weighted projective space 3-folds. In this paper we complete that web with all possible toric links. We then extend the programme into dimension 4, closely following [25], giving examples to illustrate the phenomena that arise, and providing full results online at [4] (with associated Magma code [2] that generates them). Guerreiro [16] recently computed all Sarkisov links from $\mathbb{P}^4$ that start with a weighted point blowup, which we discuss in Sect. 6.1.

Before starting, we return to the polytope point of view. Reid [25, (4.2)] defines the *shed* of a fan Σ to be

$$\operatorname{shed} \Sigma = \bigcup_{C \in \Sigma^{(n)}} \operatorname{shed} C$$

where the union is over the top-dimensional cones C of Σ, and shed C is the convex hull of the primitive points on each (extreme) ray of C together with the origin. When Σ is the spanning fan of a Fano polytope Δ, then shed $\Sigma = \Delta$; in contrast, if Σ is the fan of X with $-K_X$ not nef, then the shed is not convex; see [25, (4.3)] or the shed of Y_2 in Fig. 1. In these terms, the operations (3.1–3) below are lattice piecewise linear (LPL) surgeries on the shed, adding or removing vertices, or snapping LPL portions of the boundary into other LPL configurations, all the time preserving the property that the only lattice points in the shed are its vertices and the internal origin. A Sarkisov link then appears as a kind of LPL origami, moving from one Fano polytope to another by a series of LPL modifications, as we illustrate in Fig. 1.

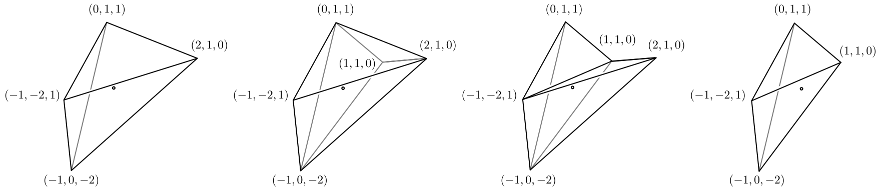

Fig. 1 The sheds of the toric fans of $X \leftarrow Y_1 \dashrightarrow Y_2 \rightarrow X'$ in the link (4.1)

2 The Context

2.1 Overview of the Sarkisov Program

A morphism $\varphi\colon V \to S$ of normal projective varieties is, by definition, a Mori fibre space if $\varphi_*\mathcal{O}_V = \mathcal{O}_S$, V has $\mathbb{Q}$-factorial terminal singularities, $\dim S < \dim V$, $\rho_V = \rho_S + 1$ and $-K_V$ is φ-ample. The Sarkisov program [10, 17] decomposes birational maps $f\colon V \dashrightarrow V'$ between Mori fibre spaces. It factorises any such birational map into a composition of very particular birational maps called *Sarkisov links*, which we explain in our context below, and so understanding individual Sarkisov links becomes crucial. Sarkisov links are built by patching together steps of the form

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ V_1 \arrow[dr] & & V_2 \arrow[dl] \\ & S & \end{tikzcd} \tag{2.1}$$

The key point is that $\rho_{U/S} = 2$ in the square (2.1), so that up to isomorphism U admits at most two S-morphisms of relative Picard rank 1 with connected fibres. The construction of a Sarkisov link is an inductive procedure in which the map $U \to V_1$ is given and we must solve for $U \to V_2$ (over S). The link is called *bad*, and we abandon it, if $U \to V_2$ does not exist, or if the properties of V_2 break any running hypotheses, for example on the singularities permitted. (See [11, 2.2] for full details in these terms.)

When X is a Mori–Fano n-fold, a Sarkisov link will always take the form

$$\begin{tikzcd} & Y_1 \arrow[dl] \arrow[dr] \arrow[rr, dashed] & & Y_2 \arrow[dl] \arrow[r, dashed] & \cdots \arrow[r, dashed] & Y_s \arrow[dr] & \\ X & & Z_1 & & & & Z_s = X' \end{tikzcd} \tag{2.2}$$

where $g_1\colon Y_1 \to X$ is an extremal extraction (that is, $g_{1*}\mathcal{O}_{Y_1} = \mathcal{O}_X$, Y_1 has terminal $\mathbb{Q}$-factorial singularities, $\rho_{Y_1/X} = 1$ and $-K_{Y_1}$ is relatively ample), each $Y_i \dashrightarrow Y_{i+1}$ is a generalised flip (that is, a flip or a flop or an antiflip) that we may factorise as a pair of birational morphisms $Y_i \to Z_i \leftarrow Y_{i+1}$. When $Y_i \dashrightarrow Y_{i+1}$ is an antiflip, the link is bad if Y_{i+1} does not have terminal singularities. The final extremal contraction $Y_s \to X'$ may be to another Fano n-fold X', or have $\dim Z_s < \dim Y_s$ so that $Y_s \to Z_s$ is a strict Mori fibre space; the link is referred to as being of Type I or Type II in these two cases respectively [11] (the opposite naming convention is used in [17]).

2.2 Hard Fano Varieties from Easy Ones

Sarkisov links (2.2) of Type I relate certain members of one family of Fano varieties with certain members of another. In Sect. 4, we calculate all Sarkisov links composed of toric maps between any two toric Fano 3-folds. The results are listed in Table 1, and our choice of presentation illustrates how we might treat this idea as passing from simpler varieties to more complicated ones. Corti–Mella [13] describe links from (general) quasismooth codimension 2 Fano 3-fold to a (particular) singular Fano hypersurface. Those links start with a projection. Reversing that step as an 'unprojection' is the key idea behind the constructions in codimension 4 of [6], which are extended to Sarkisov links in [9]. One may view this construction and classification method, and results of Takeuchi, Takagi and others [26, 27], in this same light: crudely speaking, they list the possible numerical data that could be realised by a Sarkisov link, and then construct or eliminate each case.

In three dimensions, the Graded Ring Database [4] presents a first overview of the numerical data of the classification of Fano 3-folds, again, crudely speaking, listing the possible numerical data that a Fano 3-fold might have, with the aim of constructing or eliminating each case; see [5, Fig. 1]. In that picture, families are connected by projection, but the numerical data of Sarkisov links, such as [27, (2.8)] or [26, 0.3], is missing. We regard our exercise here is a first (very baby) step in understanding how the possible numerical data of Sarkisov links between Fano varieties might be included in the Graded Ring Database.

2.3 The Midpoints of Sarkisov Links

The endpoints of a Sarkisov link are not the only places a Fano variety might appear. The Sarkisov link (2.2) may include a flop $Y_i \to Z_i \leftarrow Y_{i+1}$. In that case, Z_i is a toric Fano n-fold with terminal singularities and Picard rank $\rho_{Z_i} = 1$, but with Weil divisor class group of rank 2 so not $\mathbb{Q}$-factorial. Karzhemanov's famous example $Z_{70} \subset \mathbb{P}^{37}$ [19, 1.2], the unique Fano 3-fold of degree $-K_Z^3 = 70$, is of this nature, though it is not toric. Starting with $X = \mathbb{P}(1, 1, 4, 6)$, Karzhemanov makes a (weighted) blowup $X \leftarrow Y_1$ of a midpoint of the $\mathbb{P}(4, 6)$ stratum $L \subset X$ to give a weak Fano 3-fold Y_1 on which the proper transform of L is a flopping curve: Z_{70} is the anticanonical image of Y_1, namely the base of that flop.

Table 1 Sarkisov links between the 8 toric Mori–Fano 3-folds and some Mori fibre spaces (Mfs). We abbreviate a divisorial contraction $(-r, a, b, c)$ by $\frac{1}{r}(a, b, c)$, omitting the fraction when $r = 1$. Mfs are indicated only by F/B, where F is the fibre and B is the base

#	X	Blowup	Antiflip	Flop/Flip	Blowdown	End of link
1	$\mathbb{P}(1,1,1,1)$	(1, 1, 0)			Mfs	$\mathbb{P}^2/\mathbb{P}^1$
2		(1, 1, 1)			Mfs	$\mathbb{P}^1/\mathbb{P}^2$
3		(1, 1, 2)			(1, 1, 0)	$\mathbb{P}(1,1,1,2)$
4		(1, 2, 3)	$1, 1, -1, -2$		(1, 1, 2)	$\mathbb{P}(1,1,2,3)$
5		(1, 2, 5)	$1, 1, -1, -4$		$\frac{1}{3}(1,1,2)$	$\mathbb{P}(1,3,4,5)$
6	$\mathbb{P}(1,1,1,2)$	$\frac{1}{2}(1,1,1)$				$\mathbb{P}^1/\mathbb{P}^2$
7		(1, 1, 1)		$2, 1, -1, -1$	Mfs	$\mathbb{P}^2/\mathbb{P}^1$
8		(1, 1, 2)	$2, 1, -1, -3$		(1, 1, 1)	$\mathbb{P}(1,1,2,3)$
9		(1, 1, 2)			Mfs	$\mathbb{P}^1/\mathbb{P}(1,1,2)$
10		(1, 1, 3)			(1, 1, 0)	$\mathbb{P}(1,1,2,3)$
11		(1, 1, 3)	$2, 1, -1, -5$		$\frac{1}{2}(1,1,1)$	$\mathbb{P}(1,2,3,5)$
12		(1, 2, 3)		$1, 1, -1, -1$	(1, 2, 3)	itself
13		(1, 3, 4)	$1, 1, -1, -2$		$\frac{1}{5}(1,2,3)$	$\mathbb{P}(1,3,4,5)$
14		(1, 2, 5)	$1, 1, -1, -3$		(1, 1, 2)	$\mathbb{P}(1,2,3,5)$
15		(1, 2, 7)	$1, 1, -1, -5$		$\frac{1}{3}(1,1,2)$	$\mathbb{P}(2,3,5,7)$
	and the inverse of 3					
16	$\mathbb{P}(1,1,2,3)$	$\frac{1}{3}(1,1,2)$			Mfs	$\mathbb{P}^1/\mathbb{P}(1,1,2)$
17		$\frac{1}{2}(1,1,1)$		$3, 1, -1, -1$	Mfs	$\mathbb{P}^2/\mathbb{P}^1$
18		(1, 1, 2)	$3, 1, -1, -5$		$\frac{1}{3}(1,1,2)$	$\mathbb{P}(1,2,3,5)$
19		(1, 1, 3)	$2, 1, -1, -3$		Mfs	$\mathbb{P}^2/\mathbb{P}^1$
20		(1, 2, 3)			Mfs	$\mathbb{P}^1/\mathbb{P}(1,2,3)$
21		(1, 2, 3)	$3, 2, -1, -5$		(1, 1, 1)	$\mathbb{P}(1,2,3,5)$
22		(1, 1, 4)	$2, 1, -1, -5$		(1, 1, 1)	$\mathbb{P}(1,3,4,5)$
23		(1, 1, 5)	$2, 1, -1, -7$		$\frac{1}{2}(1,1,1)$	$\mathbb{P}(2,3,5,7)$
24		(1, 3, 4)		$1, 1, -1, -1$	(1, 3, 4)	itself
25		(1, 3, 5)	$1, 1, -1, -2$		(1, 2, 3)	$\mathbb{P}(1,2,3,5)$
26		(1, 4, 5)	$1, 1, -1, -3$		$\frac{1}{7}(1,3,4)$	$\mathbb{P}(2,3,5,7)$
27		(1, 3, 7)	$1, 1, -1, -4$		$\frac{1}{5}(1,2,3)$	$\mathbb{P}(3,4,5,7)$
	and the inverses of 4, 8, 10					
28	$\mathbb{P}(1,2,3,5)$	$\frac{1}{5}(1,2,3)$			Mfs	$\mathbb{P}^1/\mathbb{P}(1,2,3)$
29		(1, 1, 3)	$3, 1, -1, -4$		(1, 1, 2)	$\mathbb{P}(1,3,4,5)$
30		(1, 1, 4)	$3, 1, -1, -7$		$\frac{1}{3}(1,1,2)$	$\mathbb{P}(3,4,5,7)$
31		(1, 2, 5)	$2, 1, -1, -5$		$\frac{1}{5}(1,2,3)$	$\mathbb{P}^3/\frac{1}{5}(1,2,3,4)$
	and the inverses of 11, 14, 18, 21, 25					
32	$\mathbb{P}(1,3,4,5)$	$\frac{1}{4}(1,1,3)$		$5, 1, -1, -3$	Mfs	$\mathbb{P}^2/\mathbb{P}^1$
33		(1, 1, 2)	$5, 1, -1, -7$		$\frac{1}{5}(1,1,4)$	$\mathbb{P}(2,3,5,7)$
34		(1, 1, 3)	$4, 1, -1, -7$		$\frac{1}{4}(1,1,3)$	$\mathbb{P}(3,4,5,7)$
35		(1, 2, 3)	$3, 1, -1, -5$		$\frac{1}{7}(1,2,5)$	$\mathbb{P}(3,4,5,7)$
	and the inverses of 5, 13, 22, 29					
	$\mathbb{P}(2,3,5,7)$ the inverses of 15, 23, 26, 33					
	$\mathbb{P}(3,4,5,7)$ the inverses of 27, 30, 34, 35					
	$\mathbb{P}^3/\frac{1}{5}(1,2,3,4)$ the inverse of 31					

The links we construct for toric Fano 4-folds provide around 10,000 such flopping bases, each of which is a toric Fano 4-fold, the majority not known to us. Of these, around 2000 arise as Z_1 in (2.2), that is from a terminal weak Fano blowup of a (possibly fake) weighted projective space. For example, if $X = \mathbb{P}(1^4, 2)$ with orbifold point $Q = (0:0:0:0:1) \in X$, and $X \leftarrow Y_1$ is the blowup of a smooth point $P \in X$, then the proper transform $C \subset Y_1$ of the line $L \cong \mathbb{P}(1,2)$ through P and Q is a flopping curve (and a toric stratum, for suitable torus). After contracting C, the result is a Gorenstein Fano 4-fold $Z_1 \subset \mathbb{P}^{114}$ of degree 567 in its anticanonical embedding.

It is also possible for one of the $\mathbb{Q}$-factorial, Picard rank 2 varieties Y_i appearing in a link (2.2) to be a Fano variety. Indeed if the link starts with a flip $Y_1 \dashrightarrow Y_2$, then Y_1 is such a Fano variety, or if there is a sequence $Y_{i-1} \dashrightarrow Y_i \dashrightarrow Y_{i+1}$ comprising an antiflip followed by a flip, then again Y_i is a Fano variety; we give an example of this in Sect. 5.1 below.

3 Operations on a Simplicial Fan

We summarise the toric operations that arise in the links we construct, closely following Reid [25, Sect. 2–4], which also has guiding pictures. We start with a terminal Fano (fake) weighted projective n-space X determined by a fan Σ of cones $C_1, \dots, C_{n+1}$ on rays generated by vertices $v_1, \dots, v_{n+1}$ that are primitive (indivisible in L). In the notation of (2.2), so that, for example, each Y_i is $\mathbb{Q}$-factorial of Picard rank $\rho_{Y_i} = 2$, the steps of the link are:

(3.1) Terminal extractions.

The map $X \leftarrow Y_1$ arises from a fan refinement $\Sigma_1 \subset \Sigma$, where Σ_1 is the subdivision of Σ by the ray though a new primitive vertex $v \in L$.

In dimension 3, if $v \in C_i$ and C_i is a terminal quotient singularity $\frac{1}{r_i}(1, a, r_i - a)$ of index $r_i > 1$, then v is necessarily the vertex of height $r_i + 1$ inside C_i, the so-called *Kawamata blowup* [23] (denoted by $\frac{1}{r_i}(1, a, r_i - a)$ in Table 1). If C_i is a nonsingular cone (its vertices s_1, s_2, s_3 form a basis of L) then either $v = s_1 + as_2 + bs_3 \in C_i^\circ$ for coprime $a, b > 1$ (denoted by $(1, a, b)$ in Table 1) or $v = s_1 + s_2 \in \partial C_i$, up to permutations of the s_i (denoted by $(1, 1, 0)$ in Table 1). In dimension 4 there are no general results specifying those $v \in C_i$ that determine terminal extractions.

(3.2) Flips, flops and anti-flips [25, Sect. 3].

A map $\varphi_i : Y_i \to Z_i$ may arise by amalgamating a union of ≥ 2 cones C_j of Σ_i into a single convex cone D. If D is not simplicial, then in our context (2.2) D necessarily has $n+1$ vertices and (as $\rho_{Y_i} = 2$ so $\rho_{Y_i/Z_i} = 1$) there is a unique alternative way of subdividing D into simplicial cones on the same vertices. This forms a new simplicial fan Σ_{i+1}, giving the composition $Y_i \to Z_i \leftarrow Y_{i+1}$ which is an isomorphism in codimension 1 (see [25, (3.4)]).

Consider the vertices $\{s_1, \dots, s_{n+1}\}$ of D, ordered so that $s_1, \dots, s_n$ span one of the cones C_j of Σ_i and thus lie on an affine hyperplane $H \subset L$. If s_{n+1} also lies on H, then $Y_i \dashrightarrow Y_{i+1}$ is a flop (φ_i contracts only curves Γ with $K\Gamma = 0$) and Y_{i+1} again has $\mathbb{Q}$-factorial terminal singularities. If s_{n+1} lies on the same side of H as the origin, then $-K_{Y_i}$ is φ_i-ample, $Y_i \dashrightarrow Y_{i+1}$ is a flip, and again Y_{i+1} has $\mathbb{Q}$-factorial terminal singularities (see [25, Sect. 4]).

On the other hand, if s_{n+1} lies on the opposite side of H to the origin, then $Y_i \dashrightarrow Y_{i+1}$ is an anti-flip, and we lose control of the singularities of Y_{i+1}: the shed has grown and may now admit interior points. We must check: if the singularities are terminal, then we continue with Y_{i+1}; if not, then the link is bad.

In Table 1 each of these operations is denoted by a vector $(b_1, \dots, b_{n+1})$ for which $\sum b_k s_k = 0$.

(3.3) Divisorial contractions.

Continuing the notation of (3.2), if D is simplicial then φ_i is a divisorial contraction to a point and $Z_i = X'$ is the end of the link (again as $\rho_{Y_i} = 2$ so $\rho_{Y_i/Z_i} = 1$).

It may also happen that $\varphi_i\colon Y_i \to Z_i$ arises by a subset of cones of Σ_i combining to make a union of simplicial cones of a new fan, in which case φ_i is a divisorial contraction to a locus of dimension > 0, and again we have reached the end of the link.

(3.4) Mori fibre spaces [25, (2.5–6)].

When $\dim Z_i < \dim Y_i$, then $i = s$ is the end of the link and $Y_s \to Z_s$ is a Mori fibre space with (possibly fake) weighted projective spaces as fibres.

(3.5) Blowup and flip notation and the weights of $\mathbb{C}^*$ variations.

As indicated above, Table 1 abbreviates the data of each birational map by a sequence of numbers, namely the coefficients of the linear relations among rays of the relevant cones. For 4-fold output, including the much larger data set recorded at [4], we abbreviate the birational maps as follows.

Blowups (and blowdowns) are determined by their centre P and the primitive lattice point v on the the new subdividing ray that describes the blown-up fan. Thus we record blowups (and blowdowns) by two pieces of data, as follows: first, a sequence $[a_0, a_1, \dots, a_d]$ indicating the weights of coordinates along the blowup centre—that is, the d-dimensional toric stratum $P = \mathbb{P}(a_0, \dots, a_d)$ being blown up—and, second, a vector $(b_0, b_1, \dots, b_{4-d})$ that gives the coefficients of the minimal integral relation defining v in terms of the rays of P (in some order that is not recorded). The notation does not determine the (singularity) type of the centre being blown up, other than its dimension and lattice index, but that can be recovered unambiguously from the weighted projective space.

Some examples illustrate this notation for blowups of a toric 4-fold:

- $[1](-1, 1, 1, 1, 2)$ is the $(1, 1, 1, 2)$-weighted blowup of a nonsingular toric point-stratum. The blowup is determined by the fan subdivision at a ray through $v = s_1 + s_2 + s_3 + 2s_4 \in L$, where s_i are the vertices of a regular 4-dimensional cone corresponding to the point, while the sequence $[1]$ indicates that this is a cone of index 1, that is, a regular cone.

- [1, 1]$(-1, 1, 1, 1)$ is the ordinary blowup of a nonsingular toric curve stratum The blowup is determined by the subdivision at $v = s_1 + s_2 + s_3$, which lies on a regular 3-dimensional cone with vertices s_i.
- [3]$(-3, 1, 1, 1, 2)$ is the blowup by $v = \frac{1}{3}(s_1 + s_2 + s_3 + 2s_4)$ in a 4-dimensional cone of index 3.
- [4]$(-2, 1, 1, 1, 1)$ is the blowup by $v = \frac{1}{2}(s_1 + \cdots + s_4)$ in a 4-dimensional cone of index 4, for which $\sum s_i$ is divisible by 2 in the lattice L, but not by 4.
- [2, 2]$(-2, 1, 1, 1)$ is the Kawamata blowup along a curve of transverse $\frac{1}{2}(1, 1, 1)$ singularities, that is, the blowup by $v = \frac{1}{2}(s_1 + s_2 + s_3)$ in the corresponding 3-dimensional cone.
- [2, 4]$(-2, 1, 1, 1)$ is the unique extremal extraction along a curve of generically transverse $\frac{1}{2}(1, 1, 1)$ singularities that equals the Kawamata blowup at a general point. (This occurs, for example, as the start of a link from $\mathbb{P}(1, 1, 2, 3, 4)$.)

For flips and other isomorphisms in codimension 1, we record a sequence of integers that are the coefficients of the minimal integral relation among the rays of D, in the notation of (3.2). They may also be treated as weights for a $\mathbb{C}^*$ action; see [3, Sect. 1] for example.

- $(4, 1, -1, -1, -3)$ is a 4-fold flop contracting $\mathbb{P}(4, 1)$ and $\mathbb{P}(1, 1, 3)$ on the two sides respectively to a common point in the base.
- $(2, 1, 0, -1, -1)$ is a 4-fold flip over a curve in the base that makes a Francia flip (namely $(2, 1, -1, -1)$ in a 3-fold) over each point of the curve.

In each case, the rays involved may or may not generate the whole lattice, but we do not record this co-index.

4 Extending Prokhorov's Web

Prokhorov [24, Sect. 10] computes links from 3-fold weighted projective spaces that start with the Kawamata blowups of quotient singularities. We list all toric Sarkisov links between toric Mori–Fano 3-folds in Table 1, and discuss some particular cases here. Since we list the 3-folds X with smaller weights first (as one might if using Sarkisov links to construct 'complicated' Fano 3-folds from 'simple' ones), the typical behaviour is a relatively high discrepancy blowup followed by an antiflip; the picture would be reversed if we listed larger weights first. By [23, 5], it remains only to consider the blowups of 1-strata that do not pass through a quotient singularity and weighted blowups $(1, a, b)$ of a smooth 0-stratum. The latter is an infinite collection of blowups, and we discuss bounds in Sect. 4.4.

4.1 Links from a Smooth Point

Consider the weighted $(1, 1, 2)$ blowup of the smooth 0-stratum of $\mathbb{P}(1, 3, 4, 5)$. This extends to a Sarkisov link:

$$\begin{tikzcd}[column sep=small] & {}^{(-1,1,1,2)}\, Y_1 \arrow[dl] \arrow[dr] \arrow[rr, dashed, "{(4,1,-1,-3)}"] & & Y_2\, {}^{(3,1,1,-1)} \arrow[dl] \arrow[dr] & \\ \mathbb{P}(1,3,4,5) & & Z_1 & & \mathbb{P}(1,2,3,5) \end{tikzcd}$$

In terms of the rays of the fan, we start with $\{(1, 1, 0), (0, -1, 1), (1, -1, -1), (-2, 0, -1)\}$, and the blowup inserts

$$(0, 1, 0) = 1 \cdot (-2, 0, -1) + 1 \cdot (0, -1, 1) + 2 \cdot (1, 1, 0). \qquad (-1, 1, 1, 2)$$

The flip then expresses the two subdivisions

$$4 \cdot (0, 1, 0) + 1 \cdot (1, -1, -1) = 1 \cdot (-2, 0, -1) + 3 \cdot (1, 1, 0) \qquad (4, 1, -1, -3)$$

after which the ray $(1, 1, 0)$ can be contracted.

4.2 Links from the Eighth Toric Mori–Fano 3-Fold

We denote the action of $\varepsilon \in \mathbb{Z}/5$ act on $\mathbb{P}^3$ by $(\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4)$ by $\frac{1}{5}(1, 2, 3, 4)$. Then $\mathbb{P}^3/\frac{1}{5}(1, 2, 3, 4)$ is a well-known toric Mori–Fano 3-fold which is a fake weighted projective space (compare [1]). Its only toric extremal extraction is the $\frac{1}{5}(1, 2, 3)$ blowup of any of the four quotient singularities, and this extends to a Sarkisov link:

$$\begin{tikzcd} & {}^{(-5,1,2,3)}\, Y_1 \arrow[dl] \arrow[r, dashed, "{(5,1,-1,-2)}"] & Y_2\, {}^{(5,2,1,-1)} \arrow[dr] & \\ X = \mathbb{P}^3/\frac{1}{5}(1,2,3,4) & & & \mathbb{P}(1,2,3,5) = X' \end{tikzcd} \qquad (4.1)$$

In coordinates, X may be defined by the four cones on the vertices

$$v_1 = (0, 1, 1), \quad v_2 = (-1, 0, -2), \quad v_3 = (-1, -2, 1), \quad v_4 = (2, 1, 0)$$

(whose sum is the origin, but which generate only a sublattice of index 5), Y_1 is the $\frac{1}{5}(1, 2, 3)$ blowup at the new vertex $v_5 = (1, 1, 0)$ which satisfies

$$5 \cdot v_5 = 1 \cdot v_2 + 2 \cdot v_1 + 3 \cdot v_4 \qquad (-5, 1, 2, 3)$$

Y_2 is the flip by subdividing $\langle v_1, v_3, v_4, v_5\rangle$ the other way, corresponding to the linear relation

$$5 \cdot v_5 + 1 \cdot v_3 = 1 \cdot v_1 + 2 \cdot v_4 \qquad (5, 1, -1, -2)$$

and $X' = \mathbb{P}(1, 2, 3, 5)$ follows from blowing down

$$5 \cdot v_5 + 2 \cdot v_3 + 1 \cdot v_2 = v_4 \qquad (5, 2, 1, -1)$$

leaving

$$1 \cdot v_1 + 2 \cdot v_2 + 3 \cdot v_3 + 5 \cdot v_5 = 0. \qquad (1, 2, 3, 5)$$

4.3 Links from 1-Dimensional Centres

A toric 1-stratum $L \subset X$ is the centre of an extremal extraction only when X is smooth in a neighbourhood of L and $X \leftarrow Y_1$ is the blowup of L. In each case there is only a single $L \subset X$ up to symmetry, and its blowup extends to a link; the blowup (or blowdown) is denoted $(1, 1, 0)$ in Table 1.

4.4 Bounding Links from a Smooth Centre

Though the number of Sarkisov links between toric Fano 3-folds is finite, the number of terminal extractions from a smooth point is infinite, even in the toric case: the $(1, a, b)$ blowup is terminal for coprime $a, b \geq 1$.

Consider the main case of the blowup of a point on $\mathbb{P}^3$. We may define the fan of $\mathbb{P}^3$ on the four rays e_1, e_2, e_3 (the standard basis of L) and $e_4 = (-1, -1, -1)$, and without loss of generality we consider blowing up by the point $e_5 = (a, 1, b)$ with $a > b > 1$ coprime. (By symmetry, the only other cases are when $b = 1$, and these smaller cases may be handled similarly.) Firstly, the blowup is indeed terminal. After blowup, the edge e_1, e_3 must be antiflipped to e_4, e_5: indeed the equation

$$g_{134} = x - 3y + z$$

which supports the roof of the shed $(g_{134} = 1)$ of cone $\sigma_{134} = \langle e_1, e_3, e_4\rangle$ is strictly positive at e_5, as $a \geq 3$ and $b \geq 2$. This shows that $-K$ is positive on the toric 1-stratum corresponding to $\langle e_1, e_3\rangle$, but even without that observation the union of cones $\sigma_{134} \cup \sigma_{135}$ (which is strictly convex, as $g_{134} > 0$) may be differently subdivided into convex cones by the hyperplane through e_4 and e_5.

After this antiflip, the lattice point $v = (1, 0, 1)$ lies in the cone $\sigma_{345} = \langle e_3, e_4, e_5\rangle$. This cone has supporting equation $(g_{345} = 1)$ where

$$g_{345} = \left(\frac{3-b}{a-1}\right)x + \left(\frac{b-2a-1}{a-1}\right)y + z$$

so that, if $a > b \geq 3$, the blow up at v has discrepancy $g_{345}(v) - 1 = (3-b)/(a-1) \leq 0$, which violates terminality.

In the case $a > b = 2$, the link would end by contracting

$$e_1 = \tfrac{1}{a-2}(1 \cdot e_2 + 2 \cdot e_4 + 1 \cdot e_5)$$

but that contraction to a quotient singularity $\frac{1}{a-2}(1, 2, 1)$ has non-positive discrepancy unless $a - 2 \leq 3$. Thus we must be in the situation $a \leq 5$ and $b = 2$, and the possible blowups leading to a Sarkisov link are indeed bounded.

4.5 Higher Rank Fano 3-Folds, Links and Relations

Since each link $X \dashrightarrow X'$ in Table 1 identifies the big torus, it is easy to compose sequences of these links to give birational automorphims $X \dashrightarrow X$, describing relations in the Sarkisov program. Following [18], relations are derived from minimal model programs (MMPs) on certain varieties Z with $\rho_Z = 3$. Explicitly in the toric case, most links in Table 1 arise by patching together two different MMPs on toric Fano 3-folds Y with $\rho_Y = 2$ (since one of the Y_i in (2.2) is Fano, unless there is a flop). Such Y are classified (into 35 cases, coincidentally), and further MMPs from the 75 rank 3 toric Fano 3-folds describe relations. This can all be carried out by similar, but bigger, toric considerations.

5 The Web of 4-Dimensional Toric Sarkisov Links

The main result of this paper is an attempt to construct data analogous to Table 1 for toric Mori–Fano 4-folds. We do not attempt a complete classification of links; compare [16] for the complete classification in the case $X = \mathbb{P}^4$. Instead, for each of the 35, 947 varieties we compute all toric (point, curve and surface) blowups of discrepancy at most 5, and for each one either extend to a Sarkisov link or discard as a bad link. Even subject to this severe constraint on discrepancy (at one end of the link) there are over a million links, and we record them all in full detail online at [4]. We illustrate some general features of the results here by discussing links from three exemplary starting points X.

5.1 *Blowups of* $\mathbb{P}(\mathbf{1, 2, 3, 4, 5})$

Under the given restrictions on blowups and their discrepancies, there are 275 Sarkisov links from $X = \mathbb{P}(1, 2, 3, 4, 5)$.

For example, there are 4 ways of making a link from a $(-5, 1, 2, 3, 4)$ blowup of the index 5 point: after the blowup $X \leftarrow Y_1$, these links are completed by

(i) $Y_1 \to \mathbb{P}(1, 2, 3, 4)$ Mori fibre space with $\mathbb{P}^1$ fibre
(ii) a $(3, 1, -1, -1, -2)$ flop, then $(4, 3, 1, -1, -2)$ flip to $Y_3 \to \mathbb{P}^1$ Mori fibre space with $\mathbb{P}(1, 1, 1, 2)$ fibre
(iii) a $(2, 1, 0, -1, -1)$ flip, then $(1, 2, 3, 4)$-weighted blowdown $Y_2 \to \mathbb{P}(1, 1, 1, 2, 3)$ to a smooth point
(iv) a $(4, 1, -1, -2, -3)$ antiflip, then $(3, 2, 1, -1, -2)$ flip, then $(1, 2, 2, 3)$-weighted blowdown $Y_3 \to \mathbb{P}(1, 1, 2, 3, 4)$ to a smooth point.

The last of these exhibits antiflip–flip behaviour which are rare for 3-folds (the index 1 cases in [7, 9, 13] all involve a flop, but Guerreiro has examples for special members of a family in higher index). In particular, the variety Y_2 in the middle of the link is a Fano 4-fold: it has Picard rank 2, with both extremal rays flipping, and may be defined as the $(\mathbb{C}^\times)^2$ quotient

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{array}\right)$$

This phenomenon seems to be common in dimension 4.

5.2 *Links Between Fake Weighted Projective Spaces*

The web of Sarkisov links is of course connected, and so there are necessarily links between fake weighted projective spaces with different discriminant groups. For example there is a Sarkisov link

$$X = \mathbb{P}(2, 3, 5, 5, 13)/\tfrac{1}{5}(0, 1, 3, 4, 3) \dashrightarrow X' = \mathbb{P}(4, 5, 6, 7, 17)/\tfrac{1}{2}(1, 1, 1, 0, 0)$$

that factorises as

$$\begin{array}{ccccccc} & {}^{(-25,1,5,6,14)}\, Y_1 & \overset{(15,1,-1,-2,-8)}{\dashrightarrow} & Y_2 & \overset{(65,6,1,-7,-34)}{\dashrightarrow} & Y_3\, {}^{(25,7,5,2,-12)} & \\ \swarrow & & & & & & \searrow \\ X & & & & & & X' \end{array}$$

that is, a divisorial extraction from a point, two consecutive flips, followed by a divisorial contraction to a point. Again, this phenomenon is not something we have seen in 3 dimensions (Campo [9] finds cases with multiple flips, but always following a flop). In particular, the variety Y_1 is a Fano 4-fold with Picard rank 2.

5.3 Blowups of $\mathbb{P}^4$

Starting with $X = \mathbb{P}^4$, our bounds permit just one example in which the initial extremal extraction is followed by a flop

$$\begin{tikzcd}[column sep=small] & {}^{(-1,1,1,2,2)}\ Y_1 \arrow[dl] \arrow[rr, dashed, "{(1,1,0,-1,-1)}"] & & Y_2\ (\mathbb{P}(1,1,1,2)\text{-bundle}) \arrow[dr] & \\ \mathbb{P}^4 & & & & \mathbb{P}^1 \end{tikzcd}$$

However, allowing bigger discrepancies it is easy to construct cases of the form

$$\mathbb{P}^4 \leftarrow Y_1 \overset{\text{flop}}{\dashrightarrow} Y_2 \overset{\text{flip}}{\dashrightarrow} Y_3 \to X'$$

where the blowups have weights

$$(7, 8, 9, 12), \quad (10, 11, 13, 17), \quad (11, 12, 14, 19), \quad (13, 14, 17, 22) \tag{5.1}$$

and the resulting X' are respectively

$$\mathbb{P}(1, 3, 4, 5, 12), \quad \mathbb{P}(1, 4, 6, 7, 17), \quad \mathbb{P}(1, 5, 7, 8, 19), \quad \mathbb{P}(1, 5, 8, 9, 22).$$

(The 4-tuples in (5.1) are simply the solutions (d, a, b, c) of the equation $a + b + c = 4d + 1$ with $a, b, c \geq d \geq 1$, with any three coprime, for which the blowup (a, b, c, d) of the standard toric $\mathbb{P}^4$ terminal, subject to the bound $a, b, c, d \leq 100$. The equation guarantees the flop.)

Similarly, our bounds permit only a single case when the blowup is followed by a flip

$$\begin{tikzcd}[column sep=small] & {}^{(-1,3,3,4,5)}\ Y_1 \arrow[dl] \arrow[rr, dashed, "{(3,1,0,-1,-2)}"] & & Y_2\ {}^{(4,1,1,1,-1)} \arrow[dr] & \\ \mathbb{P}^4 & & & & \mathbb{P}(1,1,2,2,5) \end{tikzcd}$$

Again higher discrepancies allow a handful of other cases $\mathbb{P}^4 \leftarrow Y_1 \overset{\text{flip}}{\dashrightarrow} Y_2 \to X'$, namely weighted blowups $(4, 4, 5, 7)$, $(5, 5, 6, 8)$, $(12, 13, 15, 20)$ with X' respectively $\mathbb{P}(1, 2, 3, 3, 7)$, $\mathbb{P}(1, 2, 3, 3, 8)$ and $\mathbb{P}(1, 5, 7, 8, 20)$. (These are the solutions (d, a, b, c) to $a + b + c < 4d + 1$ with $a, b, c \leq 100$, $d \leq 50$ and the same additional conditions as above.) In each case, Y_1 is a Fano 4-fold of Picard rank 2.

The typical behaviour, though, is that the extremal extraction is followed by an antiflip. This can be followed by a flop or a flip, but a sequence of two antiflips is common, as in the following:

$$
\begin{tikzcd}
 & Y_1 \arrow[dl, "{(-1,1,2,5,9)}"'] \arrow[r, dashed, "{(1,1,-1,-4,-8)}"] & Y_2 \arrow[r, dashed, "{(2,1,-1,-3,-7)}"] & Y_3 \arrow[dr, "{(5,4,3,1,-4)}"] & \\
\mathbb{P}^4 & & & & \mathbb{P}(1,4,7,8,9)
\end{tikzcd}
$$

In this case, Y_3 is a Fano 4-fold of Picard rank 2.

6 Further Related Problems

6.1 High-Discrepancy Blowups

The list we describe above certainly misses some Sarkisov links, such as those in Sect. 5.3, since we restrict attention to discrepancy at most 5 at one of the ends. There are only finitely many Sarkisov links from any given centre, and with more work one can bound the terminal extractions that initiate these links and describe all the missing cases. In the case of $X = \mathbb{P}^4$, Guerreiro [16] computes all 421 Sarkisov links from $\mathbb{P}^4$ that start with the blowup of a point.

6.2 Non-terminal Singularities

The classification [21] includes a further 348, 930 Fano 3-folds X with strictly canonical singularities and $\rho_X = 1$. It makes sense to compute links of the form (2.2) between these, for example by permitting crepant extractions, even though they would be regarded as 'bad' links in the usual terminal situation.

6.3 Running the Sarkisov Program on Toric n-Folds

In the toric context, any two toric Mori–Fano n-folds X and Y have a common big torus, and any such identification extends to a birational map $X \dashrightarrow Y$. At first sight, given any two Fano n-fold polytopes P_1 and P_2 and elements $g_1, g_2 \in \mathrm{GL}(n, \mathbb{Z})$, the task is to compute a series of operations of the form (3.1–3) that takes the spanning fan of $g_1(P_1)$ to that of $g_2(P_2)$, via fans on at most $n + 1$ rays. But the Picard rank can increase when Mori fibre spaces appear, and the analysis may be rather subtle; the Sarkisov program applies to Mori fibre spaces, not only Mori–Fano n-folds. This is a much more substantial problem than the one we address in Sect. 5.

6.4 Higher Picard Rank

The secondary fan of a toric variety V (of arbitrary Picard rank) contains the data of all minimal model programs (MMPs). Magma [2] includes functions to run all toric MMPs from a given V, inductively contracting all extreme rays of the Mori cones that arise. The sets of all toric varieties and all their MMPs are infinite but could be enumerated up to a bound. This would contain all Sarkisov links that are dominated by such V and, following [18], relations in the toric Sarkisov program among those.

6.5 Fano 3-Folds and 4-Folds More Generally

The toric case considered here is a test case for overlaying other (partial) classifications of Fano varieties by webs of Sarkisov links. Analysis of the numerical data of possible links has had spectacular success contributing to the classification of particular types of Fano varieties (for example, in works of Takeuchi, Alexeev, Takagi and Prokhorov, among others) and a 'numerical' web in the spirit of the Graded Ring Database [4] may be possible.

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References

1. Borisov, A.A., Borisov, L.A.: Singular toric Fano three-folds. Mat. Sb. **183**(2), 134–141 (1992)
2. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. J. Symbolic Comput. **24**(3-4), 235–265 (1997). Computational algebra and number theory (London, 1993)
3. Brown, G.: Flips arising as quotients of hypersurfaces. Math. Proc. Camb. Philos. Soc. **127**(1), 13–31 (1999)
4. Brown, G., Kasprzyk, A.M.: The graded ring database webpage. http://www.grdb.co.uk, 2009–2022. Follow link to 'Downloads' for data and Magma code
5. Brown, G., Kasprzyk, A.M.: Kawamata boundedness for Fano threefolds and the graded ring database, 23 pp (2022). arXiv:2201.07178
6. Brown, G., Kerber, M., Reid, M.: Fano 3-folds in codimension 4, Tom and Jerry. Part I. Compos. Math. **148**(4), 1171–1194 (2012)
7. Brown, G., Zucconi, F.: Graded rings of rank 2 Sarkisov links. Nagoya Math. J. **197**, 1–44 (2010)

8. Buczyńska, W.: Fake weighted projective spaces, 20 pp (2002). arXiv:0805.1211
9. Campo, L.: Sarkisov links for index 1 Fano 3-folds in codimension 4, 21 pp. (2022). To appear in Math. Nachr. arXiv:2011.12209
10. Corti, A.: Factoring birational maps of threefolds after Sarkisov. J. Algebraic Geom. **4**(2), 223–254 (1995)
11. Corti, A.: Singularities of linear systems and 3-fold birational geometry. In: Explicit birational geometry of 3-folds. London Mathematical Society Lecture Note series, vol. 281, pp. 259–312. Cambridge University Press, Cambridge (2000)
12. Corti, A., Pukhlikov, A., Reid, M.: Fano 3-fold hypersurfaces. In: Explicit Birational Geometry of 3-Folds. London Mathematical Society Lecture Note Series, vol. 281, pp. 175–258. Cambridge University Press, Cambridge (2000)
13. Corti, A., Mella, M.: Birational geometry of terminal quartic 3-folds. I. Amer. J. Math. **126**(4), 739–761 (2004)
14. Danilov, V.I.: The geometry of toric varieties. Uspekhi Mat. Nauk **33**(2(200)), 85–134, 247 (1978)
15. Fulton, W.: Introduction to Toric Varieties. Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton, NJ (1993). The William H. Roever Lectures in Geometry
16. Guerreiro, T.D.: Sarkisov links from toric weighted blowups of $\mathbb{P}^3$ and $\mathbb{P}^4$ at a point, 21 pp (2022). arXiv:2204.09137
17. Hacon, C.D., McKernan, J.: The Sarkisov program. J. Algebraic Geom. **22**(2), 389–405 (2013)
18. Kaloghiros, A.-S.: Relations in the Sarkisov program. Compos. Math. **149**(10), 1685–1709 (2013)
19. Karzhemanov, I.: Fano threefolds with canonical Gorenstein singularities and big degree. Math. Ann. **362**(3–4), 1107–1142 (2015)
20. Kasprzyk, A.M.: Toric Fano three-folds with terminal singularities. Tohoku Math. J. (2) **58**(1), 101–121 (2006)
21. Kasprzyk, A.M.: Canonical toric Fano threefolds. Canad. J. Math. **62**(6), 1293–1309 (2010)
22. Kasprzyk, A.M.: Classifying terminal weighted projective space, 19pp (2013). arXiv:1304.3029
23. Kawamata, Y.: Divisorial contractions to 3-dimensional terminal quotient singularities. In: Higher-dimensional complex varieties (Trento, 1994), pp. 241–246. de Gruyter, Berlin (1996)
24. Prokhorov, Y.: $\mathbb{Q}$-Fano threefolds of large Fano index. I. Doc. Math. **15**, 843–872 (2010)
25. Reid, M.: Decomposition of toric morphisms. In: Arithmetic and Geometry, vol. II. Progress in Mathematics, vol. 36, pp. 395–418. Birkhäuser Boston, Boston, MA (1983)
26. Takagi, H.: On classification of $\mathbb{Q}$-Fano 3-folds of Gorenstein index 2. I, II. Nagoya Math. J. **167**, 117–155, 157–216 (2002)
27. Takeuchi, K.: Some birational maps of Fano 3-folds. Compositio Math. **71**(3), 265–283 (1989)

Du Val Singularities

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Abstract These are extended and updated notes of my lecture series on Du Val surface singularities, given at the University of Kaiserlautern in the summer term 2002.

Keywords Du Val singularities · Quotient singularities · Crepant resolutions

1 Introduction

We consider quotient singularities $\mathbb{C}^2/G$, where $G \subseteq \mathsf{SL}_2(\mathbb{C})$ is a finite subgroup. Let $\mathbb{C}[\![x, y]\!]$ be the ring of formal power series and $\mathbb{C}[\![x, y]\!]^G$ the corresponding ring of invariants. Since the ring extension $\mathbb{C}[\![x, y]\!]^G \subseteq \mathbb{C}[\![x, y]\!]$ is finite, the Krull dimension of $\mathbb{C}[\![x, y]\!]^G$ is two.

Note that $\mathbb{C}[\![x, y]\!]^{\Sigma_2} = \mathbb{C}[\![(x + y, xy)]\!] \cong \mathbb{C}[\![x, y]\!]$, where $\Sigma_2 = \langle \sigma \mid \sigma^2 = e \rangle$ is the symmetric group of order two and $\sigma(x) = y$. In order to get a more precise connection between finite subgroups of $G \subseteq \mathsf{SL}_2(\mathbb{C})$ and the corresponding quotient singularities, we need the following definition.

Definition 1.1 Let $G \subseteq \mathsf{GL}_n(\mathbb{C})$ be a finite subgroup. An element $g \in G$ is called *pseudo-reflection* if it is conjugated to $\mathrm{diag}(1, 1, \ldots, 1, \lambda)$, where $\lambda \neq 1$ (note that λ is a root of unity). A group G is called *small* if it contains no pseudo-reflections.

For the following results, we refer to [4, Sect. 4] and references therein.

Proposition 1.2 *Let $G \subseteq \mathsf{GL}_n(\mathbb{C})$ be a finite subgroup. Then the following results are true.*

(1) The ring of invariants $\mathbb{C}[\![x_1, x_2, \ldots, x_n]\!]^G$ is normal and Cohen–Macaulay.

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(2) There exists a small subgroup of $\mathsf{GL}_n(\mathbb{C})$ *such that*

$$\mathbb{C}[\![x_1, x_2, \dots, x_n]\!]^G \cong \mathbb{C}[\![x_1, x_2, \dots, x_n]\!]^{G'}.$$

(3) Let $G', G'' \subseteq \mathsf{GL}_n(\mathbb{C})$ *be two small subgroups. Then*

$$\mathbb{C}[\![x_1, x_2, \dots, x_n]\!]^{G'} \cong \mathbb{C}[\![x_1, x_2, \dots, x_n]\!]^{G''}$$

if and only if G' *and* G'' *are conjugated.*

(4) Let $G \subseteq \mathsf{GL}_n(\mathbb{C})$ *be a small finite subgroup. Then* $\mathbb{C}[\![x_1, x_2, \dots, x_n]\!]^G$ *is Gorenstein if and only if* $G \subseteq \mathsf{SL}_n(\mathbb{C})$*.*

Remark 1.3 Note that every subgroup of $\mathsf{SL}_n(\mathbb{C})$ is small.

Now we want to address the following question: what are finite subgroups of $\mathsf{SL}_2(\mathbb{C})$ up to conjugation?

2 Finite Subgroups of $\mathsf{SL}_2(\mathbb{C})$

Lemma 2.1 *Every finite subgroup of* $\mathsf{SL}_n(\mathbb{C})$ *(respectively, of* $\mathsf{GL}_n(\mathbb{C})$*) is conjugated to a subgroup of* $\mathsf{SU}_n(\mathbb{C})$ *(respectively, of* $\mathsf{U}_n(\mathbb{C})$*).*

Proof Let $(\ ,\ )$ be the standard Hermitian inner product on $\mathbb{C}^n$. For any $u, v \in \mathbb{C}^n$ we put:

$$\langle u, v\rangle := \frac{1}{|G|}\sum_{g\in G}(gu, gv).$$

Note that for $G \subseteq \mathsf{U}_n(\mathbb{C})$ we have: $\langle\ ,\ \rangle = (\ ,\ )$. Observe that $\langle\ ,\ \rangle$ is a new Hermitian inner product on $\mathbb{C}^n$. Indeed:

- $\langle u, u\rangle \geq 0$.
- $\langle u, u\rangle = 0$ implies $u = 0$.
- $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

Moreover, for any $h \in G$ we have:

$$\langle hu, hv\rangle = \frac{1}{|G|}\sum_{g\in G}(ghu, ghv) = \langle u, v\rangle.$$

Hence G is unitary with respect to $\langle\ ,\ \rangle$. Moreover, we can find an isometry $S : \big(\mathbb{C}^n, \langle\ ,\ \rangle\big) \longrightarrow \big(\mathbb{C}^n, (\ ,\ )\big)$, i.e. an isomorphism of vector spaces $\mathbb{C}^n \xrightarrow{S} \mathbb{C}^n$ such that $\langle u, v\rangle = (Su, Sv)$. We know that $\langle gu, gv\rangle = \langle u, v\rangle$ for all $g \in G$ and $u, v \in \mathbb{C}^n$. Hence we get $(Su, Sv) = (Sgu, Sgv)$. Replacing u and v by $S^{-1}u$ and $S^{-1}v$ we obtain:

$$(u, v) = (SgS^{-1}u, SgS^{-1}v)$$

for all $g \in G$ and $u, v \in \mathbb{C}^n$. □

Now, we describe all finite subgroups of $\mathsf{SU}_2(\mathbb{C})$. Recall that

$$\mathsf{SU}_2(\mathbb{C}) = \left\{A \in \mathsf{SL}_2(\mathbb{C}) \mid A^{-1} = A^*\right\} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Topologically, we have: $\mathsf{SU}_2(\mathbb{C}) \cong S^3$.

Theorem 2.2 *There is an exact sequence of groups*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathsf{SU}_2(\mathbb{C}) \xrightarrow{\pi} \mathsf{SO}_3(\mathbb{R}) \longrightarrow 1.$$

More precisely,

$$\mathsf{Ker}(\pi) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Topologically, π is the map $S^3 \xrightarrow{2:1} \mathbb{RP}^3$.

From this theorem (a proof of which can be for instance found in [6]) it follows that there is a close connection between finite subgroups of $\mathsf{SU}_2(\mathbb{C})$ and $\mathsf{SO}_3(\mathbb{R})$. A classification of finite rotation groups of $\mathbb{R}^3$ is a classical result of F. Klein (dating back to Plato).

Theorem 2.3 *Up to conjugation, there are the following finite subgroups of $\mathsf{SO}_3(\mathbb{R})$:*

(1) A cyclic subgroup $\mathbb{Z}_n$, generated, for instance, by

$$\begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) & 0 \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(2) Dihedral group D_n ($|D_n| = 2n$), which is the automorphism group of a prism. It is generated by a rotation a and a reflection b which satisfy the relations $a^n = e$, $b^2 = e$, $(ab)^2 = e$. Note that the last relation can be replaced by $ba = a^{n-1}b$ or $bab^{-1} = a^{-1}$.
(3) Group of automorphisms of a regular tetrahedron $T = A_4$. Note that $|T| = 12$.
(4) Group of automorphisms of a regular octahedron $O = S_4$, $|O| = 24$.
(5) Group of automorphisms of a regular icosahedron $I = A_5$, $|I| = 60$.

A proof of this result can be for instance found in [6]. Observe that all non-cyclic subgroups of $\mathsf{SO}_3(\mathbb{R})$ have even order.

Lemma 2.4 *The matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the only element of $\mathsf{SU}_2(\mathbb{C})$ of order two.*

Proof From

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}^2 = \begin{pmatrix} \alpha^2 - |\beta|^2 & \alpha\beta + \beta\bar{\alpha} \\ -\bar{\beta}\alpha - \bar{\beta}\bar{\alpha} & \bar{\alpha}^2 - |\beta|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

it follows that $\alpha^2 = \bar{\alpha}^2$, thus $\alpha = \pm\bar{\alpha}$. Hence $\alpha = x$ with $x \in \mathbb{R}$ or $\alpha = ix$ with $x \in \mathbb{R}$. Moreover $\alpha \neq 0$, since otherwise must hold $-|\beta|^2 = 1$. In the case $\alpha \in \mathbb{R}$ we have: $\alpha\beta + \beta\bar{\alpha} = 2\alpha\beta$, hence $\beta = 0$ and $\alpha = \pm 1$. If $\alpha = ix$ is purely imaginary then $\alpha^2 - |\beta|^2 = -|x|^2 - |\beta|^2 < 0$, contradiction. □

Let $G \subseteq \mathsf{SU}_2(\mathbb{C})$ be a finite subgroup and $\mathsf{SU}_2(\mathbb{C}) \xrightarrow{\pi} \mathsf{SO}_3(\mathbb{R})$ the covering from Theorem 2.2. Consider the following two cases.

(1) $|G|$ is odd. Then $G \cap \mathbb{Z}_2 = \{e\}$ (there are no elements of order 2 in G). So $\mathsf{Ker}(\pi) \cap G = \{e\}$ and $G \xrightarrow{\pi} \pi(G)$ is an isomorphism. Hence G is cyclic.
(2) $|G|$ is even. By Sylow's theorem, G contains a subgroup of order 2^k and, as a consequence, an element of order 2. But there is exactly one such element in $\mathsf{SU}_2(\mathbb{C})$ (see Lemma 2.4). Hence $\mathsf{Ker}(\pi) \subseteq G$ and $G = \pi^{-1}\big(\pi(G)\big)$. So, in this case G is the preimage of a finite subgroup of $\mathsf{SO}_3(\mathbb{R})$.

Classification of Finite Subgroups of $\mathsf{SL}_2(\mathbb{C})$

From what was said above, we obtain a full classification of finite subgroups of $\mathsf{SL}_2(\mathbb{C})$ up to conjugation.

(1) A cyclic subgroup $\mathbb{Z}_k$. Let g be its generator. From $g^k = e$ it follows that

$$g \sim \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},$$

where ε is some primitive root of 1 of kth order.
(2) Binary dihedral group $\mathbb{D}_n$, $|\mathbb{D}_n| = 4n$. To find the generators of $\mathbb{D}_n$ we have to know the explicit form of the map $\mathsf{SU}_2(\mathbb{C}) \xrightarrow{\pi} \mathsf{SO}_3(\mathbb{R})$. Skipping all details, we just write down the final answer: $\mathbb{D}_n = \langle a, b \rangle$ with relations

$$\begin{cases} a^n &= b^2 \\ b^4 &= e \\ bab^{-1} &= a^{-1}. \end{cases}$$

To be concrete,

$$a = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \text{ where } \varepsilon = \exp\left(\frac{\pi i}{n}\right) \text{ and } b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(3) Binary tetrahedral group $\mathbb{T}$, $|\mathbb{T}| = 24$. We have: $\mathbb{T} = \langle \sigma, \tau, \mu \rangle$, where

$$\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mu = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}, \ \varepsilon = \exp\left(\frac{2\pi i}{8}\right).$$

(4) Binary octahedral group $\mathbb{O}$, $|\mathbb{O}| = 48$. This group is generated by σ, τ, μ as in the case of $\mathbb{T}$ and by

$$\kappa = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}.$$

(5) Binary icosahedral subgroup $\mathbb{I}$, $|\mathbb{I}| = 120$. We have: $\mathbb{I} = \langle \sigma, \tau \rangle$, where

$$\sigma = -\begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}, \quad \varepsilon = \exp\left(\frac{2\pi i}{5}\right).$$

Remark 2.5 A classification of all finite subgroups of $\mathsf{GL}_2(\mathbb{C})$ can be for instance found in [3]. The problem of classification of finite subgroups of $\mathsf{SL}_3(\mathbb{C})$ is even more complicated; see [9]. Indeed, every finite subgroup $G \subseteq \mathsf{GL}_2(\mathbb{C})$ can be embedded into $\mathsf{SL}_3(\mathbb{C})$ via the group monomorphism

$$\mathsf{GL}_2(\mathbb{C}) \longrightarrow \mathsf{SL}_3(\mathbb{C}), \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & \frac{1}{\det(g)} \end{pmatrix}.$$

Finite subgroups of $\mathsf{GL}_2(\mathbb{C})$ provide main series of finite subgroups of $\mathsf{SL}_3(\mathbb{C})$; see [9] for details and proofs.

Now we can give a description of the corresponding invariant subrings.

3 Description of Du Val Singularities

Definition 3.1 Let X be a complex surface and $p \in X$ its singular point. Then p is a Du Val singularity if the completed ring $\widehat{\mathcal{O}}_{X,p}$ is isomorphic to $\mathbb{C}[\![x, y]\!]^G$, where $G \subset \mathsf{SL}_2(\mathbb{C})$ is a finite subgroup.

Now we provide a description of coordinate algebras of Du Val singularities.

(1) Consider a cyclic subgroup $\mathbb{Z}_n = \langle g \rangle$, $g : x \mapsto \varepsilon x$, $y \mapsto \varepsilon^{-1} y$, where $\varepsilon = \exp\left(\frac{2\pi i}{n}\right)$ for $n \geq 2$. It is not difficult to see that $X = x^n$, $Y = y^n$ and $Z = xy$ generate the whole ring of invariants. We get:

$$\mathbb{C}[\![x, y]\!]^{\mathbb{Z}_n} = \mathbb{C}[\![X, Y, Z]\!]/(XY - Z^n) \cong \mathbb{C}[\![u, v, w]\!]/(u^2 + v^2 + w^n)$$

is an A_{n-1}-singularity, where $n \geq 2$.

(2) Binary dihedral group $\mathbb{D}_n$, where $n \geq 2$. It acts on $\mathbb{C}[\![x, y]\!]$ by the rule

$$\sigma : \begin{cases} x \mapsto \varepsilon x \\ y \mapsto \varepsilon^{-1} y \end{cases} \qquad \tau : \begin{cases} x \mapsto -y \\ y \mapsto x, \end{cases}$$

where $\varepsilon = \exp\left(\frac{\pi i}{n}\right)$. The corresponding ring of invariants is generated by the polynomials

$$F = x^{2n} + y^{2n},\ H = xy(x^{2n} - y^{2n}) \text{ and } I = x^2y^2.$$

They satisfy the relation

$$H^2 = x^2y^2(x^{4n} + y^{4n} - 2x^{2n}y^{2n}) = IF^2 - 4I^{n+1}.$$

It follows that

$$\mathbb{C}[\![x, y]\!]^{\mathbb{D}_n} \cong \mathbb{C}[\![u, v, v]\!]/(u^{n+1} + uv^2 + w^2).$$

It is an equation of a D_{n+2}-singularity.

(3) It can be checked that for the groups $\mathbb{T}$, $\mathbb{O}$, $\mathbb{I}$ we get the following hypersurface singularities:

(a) E_6: $u^3 + v^4 + w^2 = 0$,
(b) E_7: $u^3v + v^3 + w^2 = 0$,
(c) E_8: $u^3 + v^5 + w^2 = 0$.

Summing up, we get the following result:

Theorem 3.2 *Du Val singularities are precisely simple hypersurface singularities of type* A_n (for $n \geq 1$)*,* D_n (for $n \geq 4$) *or* E_n (for $n = 6, 7, 8$).

We want now to answer our next question: what are minimal resolutions and dual graphs of Du Val singularities?

4 A_1-Singularity

In what follows, we refer to [8] for all definitions and basic notions, related with algebraic geometry of complex surface singularities. Consider the singular surface $S = V(x^2 + y^2 + z^2) \subset \mathbf{A}^3$. We describe its blow-up at the singular point $o = (0, 0, 0)$. Recall that

$$\tilde{\mathbf{A}}^3 = \left\{((x, y, z), (u : v : w)) \in \mathbf{A}^3 \times \mathbf{P}^2 \,\middle|\, xv = yu, xw = zu, yw = zv\right\}.$$

Take the chart $u \neq 0$, i.e. $u = 1$. We get

$$\begin{cases} x = & x \\ y = & xv \\ z = & xw. \end{cases}$$

What is $\tilde{S} = \overline{\pi^{-1}(S\backslash\{o\})}$? Consider first $x \neq 0$. We describe $\pi^{-1}(S\backslash\{o\})$ in this chart. From the constraints

$$x^2 + x^2v^2 + x^2w^2 = 0, \;\; x \neq 0,$$

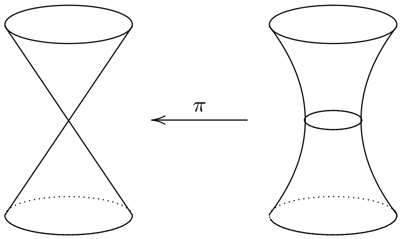
we get:

$$1 + v^2 + w^2 = 0, \;\; x \neq 0.$$

In order to get a description of $\tilde{S}$, we should allow x to be arbitrary. Hence, in this chart the surface $\tilde{S}$ is the cylinder $V(1 + v^2 + w^2) \subset \mathbf{A}^3$. What is $\pi^{-1}(o)$? Obviously it is the intersection of $\tilde{S}$ with the exceptional plane $\big((0, 0, 0), (u : v : w)\big)$. In this chart we just have to set $x = 0$ in addition to the equation of the surface $\tilde{S}$:

$$\pi^{-1}(o) = \begin{cases} 1 + v^2 + w^2 = 0 \\ \hfill x = 0. \end{cases}$$

We see that $E = \pi^{-1}(o)$ is rational and since all three charts of $\tilde{S}$ are symmetric, we conclude that E is smooth, so $E \cong \mathbf{P}^1$.



$x^2+y^2+z^2=0$

Now we have to compute the self-intersection number E^2. We do it using the following trick.

Let $\tilde{S} \xrightarrow{\pi} S$ be a minimal resolution of singularities of S. Then π induces an isomorphism of fields of rational functions $\mathbb{C}(S) \xrightarrow{\pi^*} \mathbb{C}(\tilde{S})$. Let $g \in \mathbb{C}(\tilde{S})$ be a rational function. Then we have:

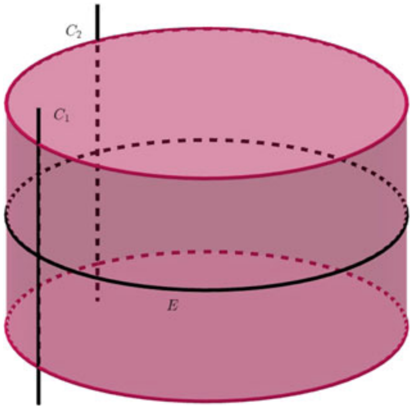
$$(g).E = \deg_E\big(\mathcal{O}_E \otimes \mathcal{O}_{\tilde{S}}(g)\big) = \deg_E(\mathcal{O}_E) = 0.$$

In particular, for any $f \in \mathfrak{m} \subseteq \mathcal{O}_{S,0}$ we have: $\big(\pi^*(f)\big).E = 0$.

Consider the function $y \in \mathcal{O}_{S,0}$ and put: $g = \pi^*(y)$. In the chart $u = 1$ we get $g = xv$. What is the vanishing set of g?

- $x = 0$ is an equation of E.
- $v = 0$ implies: $w^2 + 1 = 0$, i.e. $w = \pm i$. These equations define curves C_1 and C_2.

It follows that $(g) = E + C_1 + C_2$ and we have the following picture:



So, $(g).E = E^2 + C_1.E + C_2.E = E^2 + 2 = 0$. From it follows $E^2 = -2$.

5 E_6-Singularity

In this section we compute a minimal resolution of singularities $\tilde{X} \xrightarrow{\pi} X$ and the dual graph of the E_6-singularity $X = V(x^2 + y^3 + z^4) \subset \mathbf{A}^3$. Let $E = \cup_{i=1}^n E_i = \pi^{-1}(o)$ be the corresponding exceptional divisor, where $o = (0, 0, 0)$.

The strategy of computation of the self-intersection numbers E_i^2 will be the same as for the A_1-singularity: we consider the ring isomorphism $\mathbb{C}(X) \xrightarrow{\pi^*} \mathbb{C}(\tilde{X})$ and choose $f \in \mathfrak{m}_X \subset \mathcal{O}_X$. Then the equalities $\big(\pi^*(f)\big).E_i = 0$ for $1 \le i \le n$ will allow to compute the self-intersection number of each irreducible component E_i. In what follows, we take $f = x \in \mathbb{C}(X)$.

First step. Consider the blow-up of $\mathbf{A}^3$ at the point o:

$$\tilde{\mathbf{A}}^3 = \left\{((x, y, z), (u : v : w)) \in \mathbf{A}^3 \times \mathbf{P}^2 \,\middle|\, xv = yu, xw = zu, yw = zv\right\}.$$

Take first the chart $v \neq 0$ (i.e. $v = 1$). We get the equations

$$\begin{cases} x = yu \\ y = y \\ z = yw. \end{cases}$$

To get the equation of the strict transform $\tilde{X}_1$ of X, we assume that $y \neq 0$. From the conditions

$$y^2u^2 + y^3 + y^4w^4 = 0, \;\; y \neq 0$$

it follows that

$$u^2 + y + y^2w^4 = 0.$$

In this chart, the surface $\tilde{X}_1$ is smooth. Indeed, the system

$$\begin{cases} u = 0 \\ 1 + 2yw^4 = 0 \\ w^3y^2 = 0 \end{cases}$$

has no solutions, hence $\tilde{X}_1$ is smooth in this chart by Jacobi criterion.

A similar computation shows that in the chart $u \neq 0$ (i.e. $u = 1$), the strict transform $\tilde{X}_1$ is smooth as well.

Finally, consider the chart $w \neq 0$ (i.e. $w = 1$).

$$\begin{cases} x = zu \\ y = zv \\ z = z. \end{cases}$$

From $z^2u^2 + z^3v^3 + z^4 = 0$, $z \neq 0$ we get the equation of the strict transform $\tilde{X}_1$ of X in this chart:

$$u^2 + zv^3 + z^2 = 0.$$

Jacobi criterion implies that this surface has a unique singular point $u = 0, v = 0, z = 0$, or, in the global coordinates $((0, 0, 0), (0 : 0 : 1))$. We see that this point indeed belongs only to a one of three affine charts of $\tilde{X}_1$.

We now need a description of the exceptional fibre E_0 which is by definition the intersection $\tilde{X}_1 \cap \{((0, 0, 0), (u : v : w))\}$. To get its local equation in the chart $w = 1$, we just have to add $z = 0$ to the system of equations describing $\tilde{X}_1$. The condition $z = 0$ implies that $u = 0$. Hence, we get:

$$E_0 = \{((0, 0, 0), (0 : v : 1))\} \cong \mathbf{A}^1.$$

Going to the other charts shows that $E_0 \cong \mathbf{P}^1$.

Finally, the function f in this chart gets the form $f = zu$.

Convention. Since the number of indices would grow exponentially with the number of blow-ups, we shall always denote the local coordinates of all charts of all blowing-ups $\tilde{X}_i$ by the letters (x, y, z).

Second step. We have the following situation:

$$\begin{cases} \text{surface} & x^2 + zy^3 + z^2 = 0 \\ \text{function} & f = xz \\ \text{exceptional divisor } E_0 & x = 0, z = 0. \end{cases}$$

Consider again the blow-up of this surface at the point $(0, 0, 0)$. It is easy to see that the only interesting chart is

$$\begin{cases} x = & yu \\ y = & y \\ z = & yw. \end{cases}$$

From the constraints

$$y^2u^2 + ywy^3 + y^2w^2 = 0, \ \ y \neq 0,$$

we get the equation of the strict transform:

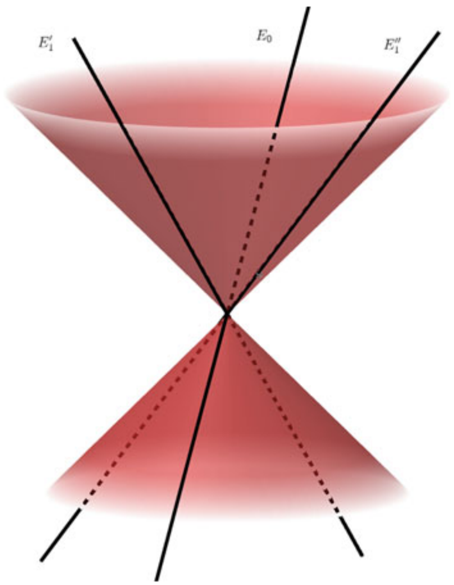
$$u^2 + y^2w + w^2 = 0.$$

Again $y = 0, u = 0, w = 0$ is the only singularity of the blown-up surface. The exceptional fibre E_1 of this blow-up has two irreducible components: $y = 0$ implies $u \pm iw = 0$ (we call these components E_1' and E_1'').

What is the preimage (under preimage we mean its strict transform) of E_0? $x = 0, z = 0$ implies $u = 0, w = 0$.
The function $f = xz$ gets in this chart the form $f = y^2uw$.

Third step. We have the following situation:

$$\begin{cases} \text{surface} & x^2 + y^2z + z^2 = 0 \\ \text{function} & f = xy^2z \\ \text{exceptional divisor } E_0 & x = 0, z = 0 \\ \text{exceptional divisor } E_1 & y = 0, x \pm iz = 0. \end{cases}$$



Let us consider the next blow-up at the point $(0, 0, 0)$. Consider the chart

$$\begin{cases} x = & yu \\ y = & y \\ z = & yw. \end{cases}$$

From the constraints

$$y^2u^2 + y^2yw + y^2w^2 = 0, \;\; y \neq 0,$$

we get the equation of the strict transform:

$$u^2 + yw + w^2 = 0.$$

This is a singular surface with a single A_1-singularity (what means that we are almost done). The exceptional fibre E_2 of this blow-up consists again of two irreducible components E_2' and E_2'' given by the local equations $y = 0$, $u \pm iw = 0$. The function f is uy^4w. It is easy to see that the preimage of E_0 is given by the equations $u = 0$, $w = 0$.

What about the preimage of E_1? We consider the affine chart $\mathbf{A}^3$ embedded into $\mathbf{A}^3 \times \mathbf{P}^2$ via the map $(y, u, w) \mapsto \big((yu, y, yw), (u : 1 : w)\big)$. But then the condition $y = 0$ would imply that the preimage of E_1 belongs to the exceptional plane $\big((0, 0, 0)(u : 1 : w)\big)$. But it can not be true! The solution of this paradox is that the preimage of the curve E_1 lies in another coordinate chart.

Consider

$$\begin{cases} x = x \\ y = xv \\ z = xw. \end{cases}$$

From

$$x^2 + x^2v^2xw + x^2w^2 = 0, \;\; x \neq 0$$

we get the equation of the strict transform

$$1 + xv^2w + w^2 = 0.$$

The exceptional fibre E_2 in this chart is the intersection of the strict transform with the plane $x = 0$, what implies $w = \pm i$. The preimage of E_1

$$\begin{cases} x \pm iz = 0 \\ \quad y \quad\;\, = 0 \end{cases}$$

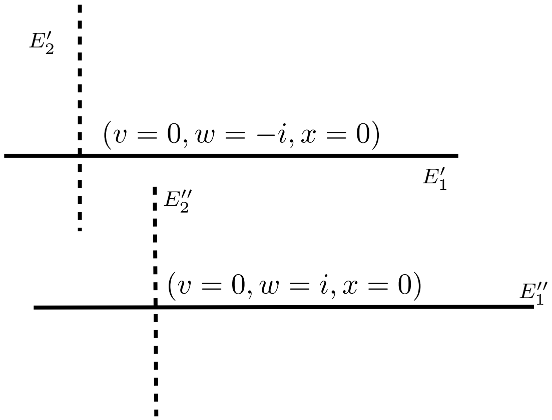
is given by the closure of

$$\begin{cases} x \pm ixw = 0 \\ xv \quad = 0 \\ x \quad \neq 0. \end{cases}$$

Hence, the preimage of E_1 is given by the equations

$$\begin{cases} w = & \pm i \\ v = & 0 \\ x & \text{arbitrary.} \end{cases}$$

In the picture it looks like:



It is easy to see that all intersections are transversal.

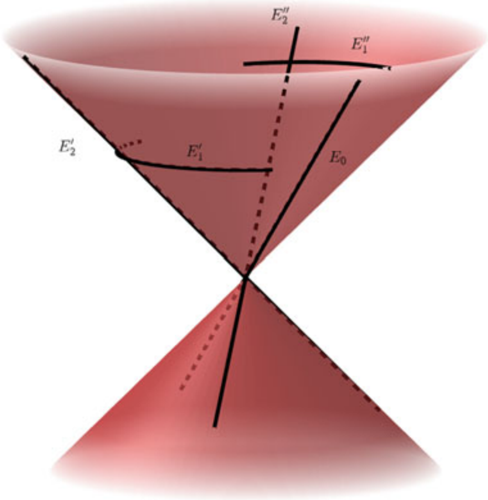
Fourth step. We have the following situation: there are two coordinate charts

$$\begin{cases} \text{surface} & x^2 + yz + z^2 = 0 \\ \text{function} & f = xy^4z \\ \text{exceptional divisor } E_0 & x = 0, z = 0 \\ \text{exceptional divisor } E_2 & y = 0, x \pm iz = 0 \end{cases}$$

and

$$\begin{cases} \text{surface} & 1 + xy^2z + z^2 = 0 \\ \text{function} & f = x^4y^2z \\ \text{exceptional divisor } E_1 & y = 0, z = \pm i \\ \text{exceptional divisor } E_2 & x = 0, z = \pm i. \end{cases}$$

In the picture it looks like as follows:



Our next step is the blowing-up at the point (0, 0, 0) in the first coordinate chart. Again, in order to get equations of the preimages of E_0, E_1 and E_2 we have to consider two coordinate charts. In the first chart we have:

$$\begin{cases} x = & yu \\ y = & y \\ z = & yw. \end{cases}$$

The strict transform is the cylinder

$$u^2 + w + w^2 = 0.$$

The preimage of E_0 is given by equations $u = 0$, $w = 0$, the exceptional fibre E_3 is given by $u^2 + w + w^2 = 0$, $y = 0$, our function $f = uy^6 w$. In another chart we have

$$\begin{cases} x = & x \\ y = & xv \\ z = & xw. \end{cases}$$

The strict transform is the closure of

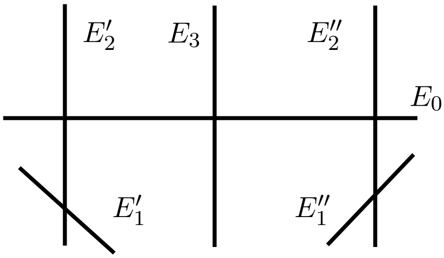
$$x^2 + x^2vw + x^2w^2 = 0, \;\; x \neq 0$$

i.e.

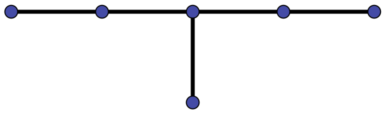
$$1 + vw + w^2 = 0.$$

The exceptional fibre E_3 is given by $1 + vw + w^2 = 0$, $x = 0$, the preimages of E_2' and E_2'' are given by $v = 0$, $w = \pm i$, $f = x^6 v^4 w$.

Summing up, the exceptional fibre E of the constructed resolution of singularities $\tilde{X} \xrightarrow{\pi} X$ is given by the following configuration of projective lines:



The dual graph of this configuration is



Fifth step. We have to take into account three coordinate charts of a minimal resolution.

$$\begin{cases} \begin{cases} \tilde{X} : \ x^2 + z + z^2 = 0 \\ f : \ xy^4z \\ E_0 : x = 0, z = 0 \\ E_3 : y = 0, x^2 + yz + z^2 = 0 \end{cases} \\ \begin{cases} \tilde{X} : \ 1 + yz + z^2 = 0 \\ f : \ x^6y^4z \\ E_3 : x = 0, x^2 + yz + z^2 = 0 \\ E_2 : y = 0, z = \pm i \end{cases} \\ \begin{cases} \tilde{X} : \ 1 + xy^2z + z^2 = 0 \\ f : \ x^6y^2z \\ E_1 : y = 0, z = \pm i \\ E_2 : x = 0, z = \pm i. \end{cases} \end{cases}$$

Now we compute the divisor (f).

Let $X \subset \mathbf{A}^3$ be a normal surface, $Y \subset X$ a closed curve, $f \in \mathbb{C}(X)$ a rational function. Suppose that $\mathfrak{p} \subset \mathbb{C}[X]$ is the prime ideal corresponding to Y. Then $\mathbb{C}[X]_{\mathfrak{p}}$ is a discrete valuation ring and

$$\mathrm{mult}_Y(f) = \mathrm{val}_{\mathbb{C}[X]_{\mathfrak{p}}}(f).$$

- Consider the first chart, in which $f = xy^6z = 0$. If $x = 0$ then $z = 0$ or $z = -1$. Note that $E_0 = V(\mathfrak{p})$ with $\mathfrak{p} = (x, z)$, whereas $x = 0, z = -1$ is the strict transform C of the curve $x = 0$ in the original singular surface $X = V(x^2 + y^3 + z^4) \subset \mathbf{A}^3$.

What is the multiplicity of E_0? The generator of the maximal ideal of the ring $(\mathbb{C}[x, y, z]/(x^2 + z + z^2))_{\mathfrak{p}}$ is $\bar{x}$ and $\bar{x}^2 \sim \bar{z}$. Therefore $\mathrm{mult}_{E_0}(f) = 3$.

Next, $y = 0$ gives an equation of E_3. It is easy to see that $\mathrm{mult}_{E_3}(f) = 6$. Note that the curve C has transversal intersection with E_3 at the point $x = 0$, $y = 0$, $z = -1$.

- Consider the second chart. We have: $f = x^6 y^4 z = 0$. The intersection of $\tilde{X}$ with $z = 0$ is empty, $x = 0$ cuts out the divisor E_3 and $y = 0$ the union of E_2' and E_2''. The same computation as above shows that $\mathrm{mult}_{E_3}(f) = 6$ (what is not surprise and makes us sure that we did not make a mistake in computations) and $\mathrm{mult}_{E_2'}(f) = \mathrm{mult}_{E_2''}(f) = 4$.
- In the same way we get: $\mathrm{mult}_{E_1'}(f) = \mathrm{mult}_{E_1''}(f) = 2$. Therefore we obtain:

$$(f) = 6E_3 + 4(E_2' + E_2'') + 2(E_1' + E_1'') + 3E_0 + C.$$

We have $C.E_3 = 1$, all other intersection numbers of C with irreducible components of E are zero. Intersection numbers of irreducible components are coded in the dual graph (which is has type E_6, see the picture above).

The entire job was done in order to compute self-intersections. For example,

$$(f).E_0 = 6 + 3E_0^2 = 0 \text{ implies that } E_0^2 = -2.$$

In the same way we conclude that the self-intersection numbers of the other irreducible components of E are -2 as well.

Remark 5.1 Let X be a normal surface singularity, $\tilde{X} \xrightarrow{\pi} X$ its minimal resolution and $E = \bigcup_{i=1}^{n} E_i = \pi^{-1}(o)$ the exceptional divisor. Suppose that $\tilde{X}$ is a good resolution and $E_i \cong \mathbf{P}^1$ with $E_i^2 = -2$ for all $1 \le i \le n$. Then X is a simple hypersurface singularity. Indeed we know that the intersection matrix $(E_i.E_j)_{i,j=1}^n$ is negatively definite; see [7]. Let Γ be the dual graph of X. Then the quadratic form given by intersection matrix coincide with the Tits form of Γ:

$$Q_\Gamma(x_1, x_2, \ldots, x_n) = -2\Big(\sum_{i=1}^{n} x_i^2 - \sum_{1=i<j=n} a_{ij}x_i x_j\Big),$$

where a_{ij} is the number of arrows connecting vertices i and j. From a theorem of Gabriel we know that Q is negatively definite (and quiver is representation finite) if and only if Γ has ADE type; see [5]. Since our singularity is rational, it is taut and hence is uniquely determined by its dual intersection graph; see [1].

6 Two-Dimensional McKay Correspondence

Recall that we defined Du Val singularities as the quotient singularities $\mathbb{C}[\![x, y]\!]^G$, where $G \subseteq \mathsf{SU}_2(\mathbb{C})$ is some finite subgroup. A natural question is: are there any connections between representation theory of G and geometry of the minimal resolution of the corresponding quotient singularity?

Let us recall some standard facts about representations of finite groups; see for instance [6].

Theorem 6.1 (Mashke) *Let G be a finite group. Then the category of $\mathbb{C}[G]$-modules is semi-simple.*

This theorem means that any exact sequence of $\mathbb{C}[G]$-modules splits. In particular, every finite-dimensional $\mathbb{C}[G]$-module is projective. It follows from Krull–Schmidt theorem that any indecomposable projective module is isomorphic to a direct summand of the regular module. Let

$$\mathbb{C}[G] \cong \bigoplus_{i=0}^{s} \Phi_i^{n_i}$$

be a direct sum decomposition of $\mathbb{C}[G]$. Then $\Phi_0, \Phi_1, \ldots, \Phi_s$ is the complete list of indecomposable $\mathbb{C}[G]$-modules.

Lemma 6.2 *Let $\mathbb{C}[G] \cong \bigoplus\limits_{i=0}^{s} \Phi_i^{n_i}$ be a decomposition of the regular module into a direct sum of indecomposable ones. Then we have:* $\dim_{\mathbb{C}}(\Phi_i) = n_i$. *In particular, the following identity is true:* $\sum\limits_{i=0}^{s} n_i^2 = |G|$.

Definition 6.3 Let G be a group and $\Phi = (V, \Phi)$ be its finite dimensional complex representation (i.e. V is a finite dimensional complex vector space and $G \xrightarrow{\Phi} \mathrm{GL}(V)$ is a group homomorphism). Then the character of $\Phi = (V, \Phi)$ is the function $G \xrightarrow{\chi_\Phi} \mathbb{C}$ defined by the rule $\chi_\Phi(g) = \mathsf{Tr}\big(\Phi(g)\big)$.

Remark 6.4 (1) It is easy to see that the character does not depend on the choice of a representative from the isomorphism class of a representation:

$$\mathsf{Tr}\big(\Phi(g)\big) = \mathsf{Tr}\big(S^{-1}\Phi(g)S\big).$$

(2) We have: $\chi_{\Phi\otimes\Psi} = \chi_\Phi\chi_\Psi$ and $\chi_{\Phi\oplus\Psi} = \chi_\Phi + \chi_\Psi$.
(3) Moreover,

$$\chi_\Phi(h^{-1}gh) = \mathsf{Tr}\big(\Phi(h^{-1}gh)\big) = \mathsf{Tr}\big(\Phi(h)^{-1}\Phi(g)\Phi(h)\big) = \mathsf{Tr}\big(\Phi(g)\big) = \chi_\Phi(g).$$

It means that χ_Φ is a *central function*, i.e. a function which takes the same value for each pair of conjugate elements of G.

Theorem 6.5 *Any finite dimensional representation of a finite group G is uniquely determined (up to an isomorphism) by its character.*

Idea of the proof. Let φ, ψ be two central functions on G. Set

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_{\Phi}(g) \overline{\chi_{\Psi}(h)}.$$

It defines a Hermitian inner product on the space of all central functions on G. The theorem follows from the fact that $\chi_{\Phi_0}, \chi_{\Phi_1}, \dots, \chi_{\Phi_s}$ is an orthonormal basis of this vector space. Indeed, let Φ be any finite-dimensional representation of G. Then we know that

$$\Phi \cong \bigoplus_{i=0}^{s} \Phi_i^{m_i}.$$

Then $m_i = \langle \chi_{\Phi}, \chi_{\Phi_i} \rangle$ is clear.

Corollary 6.6 *The number of indecomposable representations of a finite group G is equal to the number of its conjugacy classes.*

Definition 6.7 (*McKay quiver*) Let $G \subseteq \mathsf{SU}_2(\mathbb{C})$ be a finite subgroup, $\Phi_0, \Phi_1, \dots,$ Φ_s all indecomposable representations of G. Let Φ_0 be the trivial representation and Φ_{nat} its natural representation (i.e. the representation given by the inclusion $G \subset \mathsf{SU}_2(\mathbb{C})$). Define the McKay graph of G as the following:

- Its vertices are indexed by $\Phi_1, \dots, \Phi_s$ (note that we skip Φ_0).
- Let $\Phi_i \otimes \Phi_{\text{nat}} \cong \bigoplus\limits_{j=0}^{s} \Phi_j^{a_{ij}}$ (or, equivalently, $\chi_i \chi_{\text{nat}} = \sum\limits_{i=0}^{s} a_{ij} \chi_j$). Then we connect vertices Φ_i and Φ_j by a_{ij} vertices.

Remark 6.8 For all $1 \le i \ne j \le s$ we have: $a_{ij} = a_{ji}$. Indeed,

$$a_{ij} = \langle \chi_i \chi_{\text{nat}}, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g) \overline{\chi_j(g)} = \sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g) \chi_j(g^{-1}).$$

Here we use that $g^n = e$ for some $n \in \mathbb{N}$ implying that $\Phi(g)^n = \text{id}$. It follows that $\Phi(g) \sim \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ and

$$\Phi(g^{-1}) \sim \text{diag}(\varepsilon_1^{-1}, \varepsilon_2^{-1}, \dots, \varepsilon_k^{-1}) = \text{diag}(\bar{\varepsilon}_1, \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_k).$$

Since Φ_{nat} is the natural representation, for all $g \in G$ we have: $\Phi_{\text{nat}}(g) \in \mathsf{SU}_2(\mathbb{C})$.

Let $A \in \mathsf{SU}_2(\mathbb{C})$. If $A \sim \text{diag}(a, b)$ then $A^{-1} \sim \text{diag}(b, a)$ (as $ab = 1$). Therefore we have $\chi_{\text{nat}}(g) = \chi_{\text{nat}}(g^{-1})$. Then we can continue our equality:

$$\sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g) \chi_j(g^{-1}) = \sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g^{-1}) \chi_j(g^{-1}) = \langle \chi_i, \chi_{\text{nat}} \chi_j \rangle = a_{ji}.$$

Example 6.9 Let $G = \mathbb{D}_3$ be a binary dihedral group. As we already know, $|\mathbb{D}_3| = 12$. The group $\mathbb{D}_3$ has two generators a, b, which satisfy the following relations:

$$\begin{cases} a^3 &= b^2 \\ b^4 &= e \\ aba &= b^{-1}. \end{cases}$$

The group $\mathbb{D}_3$ has four one-dimensional representations $a = 1, b = 1$; $a = 1, b = -1$; $a = -1, b = i$ and $a = -1, b = -i$. The natural representation is also known: it is just

$$a = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

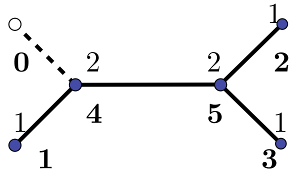
where $\varepsilon = \exp(\frac{\pi i}{6}) = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. There is also another irreducible two-dimensional representation:

$$a = \begin{pmatrix} \cos\frac{2\pi}{3} & i\sin\frac{2\pi}{3} \\ i\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have found all indecomposable representations of G (completeness of the list follows from the fact that $1 + 1 + 1 + 1 + 4 + 4 = 12 = |\mathbb{D}_3|$). We can collect the obtained information into the character table.

	$\chi(a)$	$\chi(b)$	dim	
0	1	1	1	trivial
1	1	-1	1	
2	-1	i	1	
3	-1	$-i$	1	
4	1	0	2	natural
5	-1	0	2	

From this table it is easy to deduce that $\chi_{\text{nat}}^2 = \chi_0 + \chi_1 + \chi_5$ and $\chi_5\chi_{\text{nat}} = \chi_2 + \chi_3 + \chi_4$. We get the McKay graph of $\mathbb{D}_3$:



Observe that it is also the dual graph of the D_5-singularity. Note that the fundamental cycle of the D_5-singularity is

$$Z_{\text{fund}} = E_1 + 2E_4 + 2E_5 + E_2 + E_3.$$

Observe that the coefficients in this decomposition are the same as the dimensions of the representations corresponding to the vertices of the McKay graph. Of course, it is not a coincidence.

The following result is due to Artin and Verdier [2].

Theorem 6.10 (McKay correspondence) *Let $G \subseteq \mathsf{SU}_2(\mathbb{C})$ be a finite subgroup and $\mathbb{C}[\![x, y]\!]^G$ be the corresponding invariant subring. Then the McKay graph of G coincides with the dual graph of $\mathbb{C}[\![x, y]\!]^G$. Furthermore, the dimension of the representation corresponding to a vertex of the McKay graph is equal to the multiplicity of the corresponding component of the exceptional fibre in the fundamental cycle.*

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References

1. Artin, M.: On isolated rational singularities of surfaces. Amer. J. Math. **88**, 129–136 (1966)
2. Artin, M., Verdier, J.-L.: Reflexive modules over rational double points. Math. Ann. **270**(1), 79–82 (1985)
3. Behnke, K., Riemenschneider, O.: Quotient surface singularities and their deformations. In: Singularity Theory, vol. 1995, pp. 1–54. World Scientific Publishing, River Edge, NJ, Trieste (1991)
4. Burban, I., Drozd, Yu.: Maximal Cohen–Macaulay modules over surface singularities. In: Trends in Representation Theory of Algebras and Related Topics. EMS Series of Congress Reports, pp. 101–166. European Mathematical Society, Zürich (2008)
5. Gabriel, P.: Unzerlegbare Darstellungen. I. Manuscripta Math. **6**, 71–103 (1972)
6. Kostrikin, A.: Introduction to Algebra. Springer, New York (1982)
7. Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Inst. Hautes Études Sci. Publ. Math. No. **9**, 5–22 (1961)
8. Reid, M.: Chapters on algebraic surfaces. Complex Algebraic Geometry. IAS/Park City Mathematics Series, vol. 3, pp. 3–159. American Mathematical Society, Providence, RI (1997)
9. Yau, S.-T., Yu, Y.: Gorenstein quotient singularities in dimension three. Mem. Amer. Math. Soc. **105**(505), viii+88 pp (1993)

K-Polystability of Two Smooth Fano Threefolds

Ivan Cheltsov and Hendrik Süß

Abstract We give new proofs of the K-polystability of two smooth Fano threefolds. One of them is a smooth divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of degree $(1, 1, 1)$, which is unique up to isomorphism. Another one is the blow up of the complete intersection $\Big\{x_0x_3 + x_1x_4 + x_2x_5 = x_0^2 + \omega x_1^2 + \omega^2 x_2^2 + \big(x_3^2 + \omega x_4^2 + \omega^2 x_5^2\big) + \big(x_0x_3 + \omega x_1x_4 + \omega^2 x_2 x_5\big)\Big\} \subset \mathbb{P}^5$ in the conic cut out by $x_0 = x_1 = x_2 = 0$, where ω is a primitive cube root of unity.

Keywords K-stability · Fano varieties · Alpha-invariant of Tian

1 Introduction

Let X be a smooth Fano threefold. Then X is contained in one of 105 families, which are explicitly described in [4], These families are labeled as No1.1, No1.2, . . ., No 9.1, No10.1, and members of each family can be parametrized by an irreducible rational variety.

Theorem 1.1 ([1]) *Suppose that X is a general member of the family No$\mathscr{N}$. Then*

$$X \textit{ is K-polystable} \iff \mathscr{N} \notin \left\{ \begin{array}{c} 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, 3.14, \\ 3.16, 3.18, 3.21, 3.22, 3.23, 3.24, 3.26, 3.28, 3.29, \\ 3.30, 3.31, 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, 5.2 \end{array} \right\}.$$

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In the proof of this theorem, many explicitly given smooth Fano threefolds has been proven to be K-polystable. Among them are the two threefolds described in the abstract.

Let G be a reductive subgroup in $\mathrm{Aut}(X)$, and let $f\colon \widetilde{X} \to X$ be a G-equivariant birational morphism with smooth $\widetilde{X}$, and let E be any G-invariant prime divisor in $\widetilde{X}$. We say that E is a G-invariant prime divisor *over* X, and we let $C_X(E) = f(E)$. Then

$$K_{\widetilde{X}} \sim f^*(K_X) + \sum_{i=1}^{n} a_i E_i$$

where $E_1, \dots, E_n$ are f-exceptional surfaces, and $a_1, \dots, a_n$ are strictly positive integers. If $E = E_i$ for some $i \in \{1, \dots, n\}$, we let $A_X(E) = a_i + 1$. Otherwise, we let $A_X(E) = 1$. The number $A_X(E)$ is known as the log discrepancy of the divisor E. Then we let

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^{\tau(E)} \mathrm{vol}\big(f^*(-K_X) - xE\big)dx$$

and $\beta(E) = A_X(E) - S_X(E)$, where $\tau(E)$ is the pseudoeffective threshold of the divisor E, i.e. $\tau(E) = \sup\{x \in \mathbb{R}_{>0} \mid \text{the divisor } f^*(-K_X) - xE \text{ is pseudoeffective}\}$. We have

Theorem 1.2 ([3, 6, 9]) *The smooth Fano threefold X is K-polystable if $\beta(F) > 0$ for every G-invariant prime divisor F over X.*

Now, we let

$$\alpha_G(X) = \sup\left\{\epsilon \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } \Big(X, \dfrac{\epsilon}{m}\mathcal{D}\Big) \text{ is log canonical for any } m \in \mathbb{Z}_{>0} \\ \text{and every } G\text{-invariant linear subsystem } \mathcal{D} \subset \big| -mK_X\big| \end{array}\right\}.$$

This number, known as the global log canonical threshold [2], has been defined in [8] in a different way. But both definitions agree by [2, Theorem A.3]. If G is finite, then

$$\alpha_G(X) = \sup\left\{\epsilon \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, \epsilon D) \text{ is log canonical for every} \\ G\text{-invariant effective } \mathbb{Q}-\text{divisor } D \sim_{\mathbb{Q}} -K_X \end{array}\right\}.$$

by [1, Lemma 1.4.1]. We have the following result:

Theorem 1.3 ([1, 8]) *If $\alpha_G(X) \geqslant \frac{3}{4}$, then X is K-polystable.*

In Sects. 2 and 3, we will use Theorems 1.2 and 1.3 to prove that the Fano threefolds described in the abstract are both K-polystable. The K-polystability of these threefolds has been proved in [1] using a different approach. We believe that our proof is deserved to be published, because our methods can be applied in to solve other relevant problems.

2 Smooth Divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of Degree (1, 1, 1)

Let X be the unique smooth Fano threefold in the family No 3.17. Then X is the divisor

$$\Big\{x_0y_0z_2 + x_1y_1z_0 = x_0y_1z_1 + x_1y_0z_1\Big\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2,$$

where $([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2])$ are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.

Let $G = \mathrm{Aut}(X)$. Then $G \cong \mathrm{PGL}_2(\mathbb{C}) \rtimes \boldsymbol{\mu}_2$, where $\boldsymbol{\mu}_2$ is generated by an involution ι that acts as

$$\big([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2]\big) \mapsto \big([y_0 : y_1], [x_0 : x_1], [z_0 : z_1 : z_2]\big),$$

and $\mathrm{PGL}_2(\mathbb{C})$ acts on each factor via an appropriate irreducible $\mathrm{SL}_2(\mathbb{C})$-representation. More precisely, an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C})$ acts as follows:

$$\begin{aligned}\big([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2]\big) \mapsto \big(&[ax_0 + cx_1 : bx_0 + dx_1], [ay_0 + cy_1 : by_0 + dy_1],\\ &[a^2z_0 + 2acz_1 + c^2z_2 : abz_0 + (ad + bc)z_1 + cdz_2 : b^2z_0 + 2bdz_1 + d^2z_2]\big)\end{aligned}$$

Let E_1 be the surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ that is given by

$$\begin{cases} x_0z_2 - x_1z_1 = 0, \\ x_1z_0 - x_0z_1 = 0, \end{cases} \tag{2.1}$$

and let E_2 be the surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ that is given by

$$\begin{cases} y_0z_2 - y_1z_1 = 0, \\ y_1z_0 - y_0z_1 = 0. \end{cases} \tag{2.2}$$

Then $E_1 \subset X$ and $E_2 \subset X$. Let $\pi_1 \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the morphism that is given by

$$\big([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2]\big) \mapsto \big([x_0 : x_1], [z_0 : z_1 : z_2]\big),$$

and let $\pi_2 \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the morphism that is given by

$$\big([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : z_2]\big) \mapsto \big([y_0 : y_1], [z_0 : z_1 : z_2]\big).$$

Then π_1 and π_2 are birational. Moreover, the morphism π_1 contracts E_1 to a smooth rational curve $C_1 \subset \mathbb{P}^1 \times \mathbb{P}^2$ that is given by (2.1), where we consider $([x_0 : x_1], [z_0 : z_1 : z_2])$ as coordinates on $\mathbb{P}^1 \times \mathbb{P}^2$. Similarly, the morphism π_2 contracts the surface E_2 to a smooth rational curve $C_2 \subset \mathbb{P}^1 \times \mathbb{P}^2$ that is given by (2.2), where we consider $([y_0 : y_1], [z_0 : z_1 : z_2])$ as coordinates on $\mathbb{P}^1 \times \mathbb{P}^2$.

Observe that both morphisms π_1 and π_2 are $\mathrm{PGL}_2(\mathbb{C})$-equivariant. Therefore, we have the following $\mathrm{PGL}_2(\mathbb{C})$-equivariant commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^1 \times \mathbb{P}^2 \arrow[dr, "\mathrm{pr}_2"'] & & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[dl, "\mathrm{pr}_2"] \\ & \mathbb{P}^2 & \end{tikzcd}$$

where pr_2 is the projection to the second factor, the $\mathrm{PGL}_2(\mathbb{C})$-action on $\mathbb{P}^2$ is faithful, and $\mathrm{pr}_2(C_1) = \mathrm{pr}_2(C_2)$ is the $\mathrm{PGL}_2(\mathbb{C})$-invariant conic given by $z_0z_2 - z_1^2 = 0$.

Note that it follows from [1, Lemma 4.2.6] that the Fano threefold X is K-polystable. Let us give an alternative proof of this assertion.

Let $\mathrm{pr}_1 \colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the projection to the first factor. Using $\mathrm{pr}_1 \circ \pi_1$ and $\mathrm{pr}_1 \circ \pi_2$, we obtain a $\mathrm{PGL}_2(\mathbb{C})$-equivariant $\mathbb{P}^1$-bundle $\phi \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$, where the $\mathrm{PGL}_2(\mathbb{C})$-action on the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is diagonal. Let $C = E_1 \cap E_2$. Then $\phi(C)$ is a diagonal curve. Denote its preimage on X by R. Then $C = R \cap E_1 \cap E_2$, the curve C and the surface R are the only proper closed G-invariant irreducible subvarieties in X, and $-K_X \sim E_1 + E_2 + R$.

Let $H_1 = (\mathrm{pr}_1 \circ \pi_1)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, let $H_2 = (\mathrm{pr}_1 \circ \pi_2)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, let $H_L = (\mathrm{pr}_2 \circ \pi_2)^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Then the group $\mathrm{Pic}(X)$ is generated by H_1, H_L, E_1. Moreover, it easily follows from the description of the birational morphisms π_1 and π_2 that

$$\begin{aligned} H_2 &\sim H_1 + H_L - E_1, \\ E_2 &\sim 2H_L - E_1, \\ R &\sim H_1 + H_2, \\ R &\sim 2H_1 + H_L - E_1, \\ -K_X &\sim 2H_1 + 3H_L - E_1. \end{aligned}$$

Note that $H_1^2 \cdot H_2 = 0$, $H_1^2 \cdot H_L = 0$, $H_1^3 = 0$, $H_2^2 \cdot H_1 = 0$, $H_1 \cdot H_2 \cdot H_L = 1$, $H_2^3 = 0$, $H_1 \cdot H_L^2 = 1$, $H_2 \cdot H_L^2 = 1$, $H_L^3 = 0$, $H_2^2 \cdot H_L = 0$.

By [2, Lemma 8.17], we have $\alpha(X) = \frac{1}{2}$. Since $-K_X \sim E_1 + E_2 + R$ and $C = R \cap E_1 \cap E_2$, we also have $\alpha_G(X) \leqslant \frac{2}{3}$. In particular, we cannot apply Theorem 1.3 to prove that the threefold X is K-polystable. Let us apply Theorem 1.2 instead.

Let $\eta \colon Y \to X$ be a G-equivariant birational morphism, let D be a prime G-invariant divisor in Y, let t be a non-negative real number, and let

$$S_X(D, t) = \frac{1}{-K_X^3} \int_0^t \mathrm{vol}\big(\eta^*(-K_X) - xD\big) dx.$$

Then we have $S_X(D) = S(D, \infty)$ and $\beta(D) = A_X(D) - S_X(D)$. By Theorem 1.2, to prove that X is K-polystable it is enough to show that $\beta(D) > 0$. Let us first show this in the case when η is an identify map:

Lemma 2.1 *One has $S_X(R) = \frac{4}{9}$ and $\beta(R) = \frac{5}{9}$.*

Proof Let x be a non-negative real number. Then

$$-K_X - xR \sim_{\mathbb{R}} E_1 + E_2 + (1-x)R.$$

On the other hand, the divisor $-K_X - R \sim E_1 + E_2 \sim 2H_L$ is not big. This implies that the divisor $-K_X - xR$ is pseudoeffective if and only if $x \leqslant 1$. i.e. we see that $\tau(R) = 1$, where $\tau(R)$ is the pseudoeffective threshold of the divisor R. Moreover, observe also that the divisor $-K_X - xR$ is nef for every $x \in [0, 1]$, so that

$$\begin{aligned} S_X(R) = \frac{1}{-K_X^3}\int_0^1 (-K_X - xR)^3 dx = \\ = \frac{1}{-K_X^3}\int_0^1 (-K_X)^3 - 3R(-K_X)^2x + 3R^2(-K_X)x^2 - R^3x^3\, dx = \\ = \frac{1}{36}\int_0^1 12x^2 - 48x + 36\, dx = \frac{4}{9}. \end{aligned}$$

Since $A_X(R) = 1$, we have $\beta(R) = \frac{5}{9}$. □

Now, let $f\colon \widetilde{X} \to X$ be the blow-up of the curve C, let E be the f-exceptional surface, let $\widetilde{R}$, $\widetilde{E}_1$, $\widetilde{E}_2$ be the proper transforms on $\widetilde{X}$ of the surfaces R, E_1, E_2, respectively. Then $E \cong \widetilde{R} \cong \widetilde{E}_1 \cong \widetilde{E}_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, we have

$$\begin{aligned} \widetilde{E}_1 &\sim f^*(E_1) - E, \\ \widetilde{E}_2 &\sim f^*(2H_L - E_1) - E, \\ \widetilde{R} &\sim f^*(2H_1 + H_L - E_1) - E. \end{aligned}$$

Note that $f^*(H_2) \cdot f^*(H_L) \cdot E = 0$, $f^*(H_1) \cdot f^*(H_L) \cdot E = 0$, $f^*(H_L) \cdot f^*(H_L) \cdot E = 0$, $f^*(H_1) \cdot f^*(H_2) \cdot E = 0$, $f^*(H_2) \cdot f^*(H_2) \cdot E = 0$, $E^2 \cdot f^*(H_1) = -1$, $E^2 \cdot f^*(H_2) = -1$, $E^2 \cdot f^*(H_L) = -2$, $E \cdot f^*(H_1) \cdot f^*(H_1) = 0$ and $E^3 = -4$.

Lemma 2.2 *One has $S_X(E) = \frac{11}{9}$ and $\beta(E) = \frac{7}{9}$. Moreover, if $0 \leqslant t \leqslant 1$, then*

$$S_X(E, t) = \frac{1}{36}\int_0^t (36 - 18x^2 + 4x^3)dx = \frac{1}{36}t^4 - \frac{1}{6}t^3 + t.$$

Proof Let x be a non-negative real number. Then

$$f^*(-K_X) - xE \sim_{\mathbb{R}} f^*(R + E_1 + E_2) - xE \sim_{\mathbb{R}} \widetilde{R} + \widetilde{E}_1 + \widetilde{E}_2 + (3-x)E.$$

which gives that $\tau(E) \geqslant 3$, where $\tau(E)$ is the pseudoeffective threshold of the divisor E. In fact, these equivalences also give $\tau(E) \leqslant 3$, because the divisor $\widetilde{R} + \widetilde{E}_1 + \widetilde{E}_2$ is not big, since the surfaces $\widetilde{R}, \widetilde{E}_1, \widetilde{E}_2$ are disjoint and contractible (see Remark 2.5 below).

One can easily check that all restrictions $(f^*(-K_X) - xE)|_{\widetilde{R}}$, $(f^*(-K_X) - xE)|_{\widetilde{E}_1}$, $(f^*(-K_X) - xE)|_{\widetilde{E}_2}$, $(f^*(-K_X) - xE)|_E$ are nef for $x \in [0, 1]$. Therefore, if $0 \leqslant x \leqslant 1$, then the divisor $f^*(-K_X) - xE$ is nef, which gives

$$\begin{aligned}\mathrm{vol}(f^*(-K_X) - xE) &= \Big(f^*(-K_X) - xE\Big)^3 = \\ &= f^*(-K_X)^3 + 3x^2 f^*(-K_X)E^2 - x^3E^3 = 36 - 18x^2 + 4x^3.\end{aligned}$$

If $3 > x > 1$, then both surfaces $\widetilde{E}_1$ and $\widetilde{E}_2$ lies in the asymptotic base locus of the big divisor $f^*(-K_X) - xE$, because the divisor $f^*(-K_X) - xE$ intersect negatively with the rulings of the natural projections $\widetilde{E}_1 \to C_1$ and $\widetilde{E}_2 \to C_2$. Moreover, if $x \in [1, 2]$, then the divisor

$$f^*(-K_X) - xE - \frac{1}{2}(x-1)\big(\widetilde{E}_1 + \widetilde{E}_2\big) \sim_{\mathbb{R}} \widetilde{R} + \frac{3-x}{2}\big(\widetilde{E}_1 + \widetilde{E}_2\big) + (3-x)E$$

intersects trivially with the rulings of the projections $\widetilde{E}_1 \to C_1$ and $\widetilde{E}_2 \to C_1$, and this divisor is nef. So, if $x \in [1, 2]$, the Zariski decomposition of the divisor $f^*(-K_X) - xE$ is

$$f^*(-K_X) - xE \sim_{\mathbb{R}} \frac{1}{2}(x-1)\big(\widetilde{E}_1 + \widetilde{E}_2\big) + \underbrace{\Big(f^*(-K_X) - xE - \frac{1}{2}(x-1)\big(\widetilde{E}_1 + \widetilde{E}_2\big)\Big)}_{\text{positive part}}.$$

Thus, if $x \in [1, 2]$, then we have

$$\mathrm{vol}(f^*(-K_X) - xE) = \Big(f^*(-K_X) - xE - \frac{1}{2}(x-1)\big(\widetilde{E}_1 + \widetilde{E}_2\big)\Big)^3 = 6x^2 - 36x + 52.$$

Similarly, if $x \in (2, 3)$, we see that the positive part of the Zariski decomposition of the big divisor $f^*(-K_X) - xE$ is

$$f^*(-K_X) - xE - \frac{1}{2}(x-1)\big(\widetilde{E}_1 + \widetilde{E}_2\big) - (x-2)\widetilde{R}.$$

Thus, if $x \in [1, 2]$, then

$$\mathrm{vol}(f^*(-K_X) - xE) = \Big(f^*(-K_X) - xE - \frac{1}{2}(x-1)\big(\widetilde{E}_1 + \widetilde{E}_2\big) - (x-2)\widetilde{R}\Big)^3 = 4(3-x)^3.$$

Summarizing and integrating, we see that

$$
\begin{aligned}
S_X(E) = &\frac{1}{36}\int_0^1 (36 - 18x^2 + 4x^3)dx + \frac{1}{36}\int_1^2 (6x^2 - 36x + 52)dx \\
&+ \frac{1}{36}\int_2^3 4(3-x)^3 dx = \frac{11}{9},
\end{aligned}
$$

which gives $\beta(E) = \frac{7}{9}$, because $A_X(E) = 2$. Similarly, we compute $S_X(E, t)$. $\square$

In the following, we will need one well-known result.

Lemma 2.3 ([7, Theorem 5.1]) *Let $S = \mathbb{F}_n$ for $n \in \mathbb{Z}_{\geqslant 0}$, let $\mathbf{s}$ be a section of the natural projection $S \to \mathbb{P}^1$ such that $\mathbf{s}^2 = -n$, and let $\mathbf{f}$ be a fiber of this projection. Fix a faithful action of the group $\mathrm{PGL}_2(\mathbb{C})$ on the surface S. If $n = 0$, then*

(1) either $\mathrm{PGL}_2(\mathbb{C})$ acts trivially on one of the factors of the surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$;
(2) or $\mathrm{PGL}_2(\mathbb{C})$ acts diagonally on S, and the only proper closed $\mathrm{PGL}_2(\mathbb{C})$-invariant subvariety in the surface S is its diagonal.

If $n \geqslant 1$, then S has exactly two proper closed irreducible $\mathrm{PGL}_2(\mathbb{C})$-invariant subvarieties: the curve $\mathbf{s}$ and a unique $\mathrm{PGL}_2(\mathbb{C})$-invariant curve in $|\mathbf{s} + n\mathbf{f}|$ disjoint from $\mathbf{s}$.

The action of the group G lift to the threefold $\widetilde{X}$, and $E \cap \widetilde{R}$ is a G-invariant irreducible curve, which is contained in the pencil $|\widetilde{R}|_E|$. Therefore, using Lemma 2.3, we see that the group $\mathrm{PGL}_2(\mathbb{C})$ must act trivially on the fibers of the natural projection $E \to C$. Since the curves $\widetilde{E}_1|_E$ and $\widetilde{E}_2|_E$ are swapped by the action of the group G, we conclude that the pencil $|\widetilde{R}|_E|$ contains exactly two G-invariant curves: $E \cap \widetilde{R}$ and another curve, which we denote by C'.

Now, let $g\colon \widehat{X} \to \widetilde{X}$ be the blow up of the curve C', let R' be the g-exceptional surface, let $\widehat{E}_1$, $\widehat{E}_2$, $\widehat{E}$, $\widehat{R}$ be the proper transforms on $\widehat{X}$ of the surfaces $\widetilde{E}_1$, $\widetilde{E}_2$, E, $\widetilde{R}$, respectively. Then we have

$$(f \circ g)^*(-K_X) \sim_{\mathbb{R}} \widehat{E}_1 + \widehat{E}_2 + \widehat{R} + 3\widehat{E} + 3R',$$

so the pseudoeffective threshold $\tau(R')$ is at least 3. In fact, we have $\tau(R') = 3$, because the divisor $\widehat{E}_1 + \widehat{E}_2 + \widehat{R} + 3\widehat{E}$ is not big. On the other hand, we have

Lemma 2.4 *One has $\beta(R') \geqslant \frac{5}{9}$.*

Proof Let x be a non-negative real number such that $x < 3$. Then $\widehat{E}$ lies in the stable base locus of the divisor $(f \circ g)^*(-K_X) - xR'$, and the positive part of the Zariski decomposition of this divisor has the following form:

$$(f \circ g)^*(-K_X) - xR' - \frac{x}{2}\widehat{E} - D$$

for an effective $\mathbb{R}$-divisor D. Indeed, if ℓ is a general fiber of the projection $\widehat{E} \to C$, then

$$\Big((f\circ g)^*(-K_X) - xR'\Big)\cdot \ell = -x$$

and $\widehat{E}\cdot\ell = -2$, which implies the required assertion. Thus, we have

$$S_X(R') \leqslant \frac{1}{36}\int_0^3 \operatorname{vol}\Big((f\circ g)^*(-K_X) - \frac{x}{2}\widehat{E}\Big)\,dx \leqslant 2S_X(E) = \frac{22}{9},$$

because $S_X(E) = \frac{11}{9}$ by Lemma 2.2. Then

$$\beta(R') = A_X(R') - S_X(R') = 3 - S_X(R') \geqslant 3 - \frac{22}{9} = \frac{5}{9}$$

as required. □

The action of the group G lifts to $\widehat{X}$, and the surfaces R', $\widehat{E}$ and $\widehat{R}$ are G-invariant.

Remark 2.5 There exists the following G-equivariant commutative diagram:

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[dl, "f"'] \arrow[dr, "h"] & \widehat{X} \arrow[l, "g"'] \arrow[d, "\upsilon"] & \\
R \arrow[r, hook] \arrow[d] & X \arrow[d, "\phi"] & & \overline{X} \arrow[r, "\vartheta"] \arrow[d, "\psi"] & W \\
C \arrow[r, hook] & \mathbb{P}^1\times\mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1\times\mathbb{P}^1 &
\end{tikzcd}$$

where h is the contraction of the surface $\widetilde{R}$, υ is the contraction of the surfaces R' and $\widehat{R}$, the map ψ is a $\mathbb{P}^1$-bundle, and ϑ is the birational contraction of the surfaces $\widehat{E}_1$ and $\widehat{E}_2$. Moreover, one can show that $\overline{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,2))$, and W is a G-equivariant quotient of the projective space $\mathbb{P}^3$ by an involution that pointwise fixes two skew lines. Using this, one can show that there exists an involution $\sigma\in\operatorname{Aut}(\overline{X})$ such that σ swaps the curves $\upsilon(R')$ and $\upsilon(\widehat{R})$. Then σ lifts to $\widehat{X}$ and swaps the divisors R' and $\widehat{R}$.

The threefold $\widetilde{X}$ contains two G-invariant irreducible curves: the curves $E\cap\widetilde{R}$ and C'. The threefold $\widehat{X}$ also contains just two G-invariant irreducible curves: $\widehat{E}\cap\widehat{R}$ and $\widehat{E}\cap R'$, which are swapped by the involution σ from Remark 2.5. Blowing up one of the curves, we obtain a new threefold that contains exactly three G-invariant irreducible curves that can be described in a very similar manner. Now, iterating this process, we obtain infinitely many G-invariant prime divisors over X, which can be described using weighted blow ups.

Definition 2.6 Let V be a smooth threefold that contains two smooth irreducible distinct surfaces A and B that intersect transversally along a smooth irreducible curve Z, and let $\theta\colon U\to V$ be the weighted blow up with weights (a,b) of the curve

Z with respect to the local coordinates along Z that are given by the equations of the surfaces A and B, and let F be the exceptional surface of the weighted blow up θ. Then

- the morphism θ is said to be an (a, b)-blow up between A and B,
- the surface F is said to be an (a, b)-divisor between A and B.

Observe that $(1, 1)$-blow up in this construction is the usual blow up of the intersection curve. To proceed, we need the following well-known result:

Lemma 2.7 *In the assumptions of Definition 2.6 and notations introduced in this definition, suppose that* $(a, b) = (1, 1)$ *and* $Z \cong \mathbb{P}^1$. *Let* $n = |\alpha - \beta|$, *where* α *and* β *be integers such that*

$$Z^2 = \begin{cases} \alpha \text{ on the surface } A, \\ \beta \text{ on the surface } B. \end{cases}$$

Denote by $\widetilde{A}$ *and* $\widetilde{B}$ *the proper transforms on* U *of the surfaces* A *and* B, *respectively. Then* $F \cong \mathbb{F}_n$, *the surfaces* $\widetilde{A}$ *and* $\widetilde{B}$ *are disjoint,* $\widetilde{A}|_F$ *and* $\widetilde{B}|_F$ *are sections of the natural projection* $F \to Z$ *such that* $(\widetilde{A}|_F)^2 = (\beta - \alpha)$ *and* $(\widetilde{B}|_F)^2 = (\alpha - \beta)$.

Proof Left to the reader. □

Now, we are ready to prove

Lemma 2.8 *All* G*-invariant prime divisors over* X *can be described as follows:*

(1) the surfaces R, E *or* R',
(2) an (a, b)*-divisor between* E *and* $\widetilde{R}$,
(3) an (a, b)*-divisor between* $\widehat{E}$ *and* R'.

Proof Let F be a G-invariant prime divisor over X that is different from R, E and R'. Then its center on $\widetilde{X}$ is either $E \cap \widetilde{R}$ or C', since R, E, R', $E \cap \widetilde{R}$, C' are the only proper closed G-invariant irreducible subvarieties in $\widehat{X}$, because C and R are the only proper closed G-invariant irreducible subvarieties in X. Keeping in mind Remark 2.5, we may assume that its center on $\widetilde{X}$ is $E \cap \widetilde{R}$. Let us show that F is an exceptional divisor of a weighted blow up between the surfaces E and $\widetilde{R}$,

Let $V_0 = X$ and $Z_0 = E \cap \widetilde{R}$. Then there exists a sequence of G-equivariant blow ups

$$V_m \xrightarrow{\theta_m} V_{m-1} \xrightarrow{\theta_{m-1}} \cdots \xrightarrow{\theta_2} V_1 \xrightarrow{\theta_1} V_0$$

such that θ_1 is the blow up of the curve Z_0, the surface F is the θ_m-exceptional surface, the morphism θ_k is a blow up of a G-invariant irreducible smooth curve $Z_{k-1} \subset V_{k-1}$ such that the curve Z_{k-1} is contained in the θ_{k-1}-exceptional surface provided that $k \geqslant 2$.

For every $k \in \{1, \dots, m\}$, let F_k be the θ_k-exceptional surface, so that we have $F = F_m$. To prove that $F = F_m$ is an exceptional divisor of a weighted blow up between E and $\widetilde{R}$, it sufficient to prove the following assertion for every k:

- the surface F_k contains exactly two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves,
- the two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves in F_k are disjoint,
- if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_k, then $\mathscr{C}$ is cut out by the strict transform of one of the following surfaces:
  - the surface F_r for some $r \in \{1, \dots, m\}$ such that $r \neq k$,
  - the surface E,
  - the surface $\widetilde{R}$.

Clearly, it is enough to prove this assertion only for $k = m$. Let us do this.

Let $F_0 = E$ and $F_{-1} = \widetilde{R}$. For every $k \in \{-1, 0, 1, \dots, m-1\}$, let $\overline{F}_k$ be the proper transform of the surface F_k on the threefold V_m. We claim that

(i) $F_m \cong \mathbb{F}_n$ for some $n > 0$;
(ii) the surface F_m contains exactly two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves,
(iii) the two $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves in F_m are disjoint,
(iv) if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_m, then $\mathscr{C}^2 \in \{-n, n\}$,
(v) if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_m, then

$$\mathscr{C} = F_m \cap \overline{F}_r$$

for some $r \in \{-1, 0, 1, \dots, m-1\}$ and the following assertions hold:

- if $\mathscr{C}^2 = n$ on the surface F_m, then $\mathscr{C}^2 \leqslant 0$ on the surface $\overline{F}_r$,
- if $\mathscr{C}^2 = -n$ on the surface F_m, then $\mathscr{C}^2 > 0$ on the surface $\overline{F}_r$.

Let us prove this (stronger than we need) statement by induction on m.

Suppose that $m = 1$. It follows from Lemma 2.7 that $F_0 = E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, we also know that $F_{-1} = \widetilde{R} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, we have

$$Z_0^2 = \big(\widetilde{R}|_E \cdot \widetilde{R}|_E\big)^2 = \widetilde{R}^2 \cdot E = \big(f^*(H_1 + H_2) - E\big)^2 \cdot E = 0$$

on the surface F_0, and we have

$$Z_0^2 = \big(E|_{\widetilde{R}} \cdot E|_{\widetilde{R}}\big)^2 = E^2 \cdot \widetilde{R} = E^2 \cdot \big(f^*(H_1 + H_2) - E\big) = 2$$

on the surface F_{-1}. Then $F_1 \cong \mathbb{F}_2$ by Lemma 2.7. Moreover, since $\mathrm{PGL}_2(\mathbb{C})$ acts faithfully on the curve Z_0, it acts faithfully on F_1. Hence, if $\mathscr{C}$ is a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_1, then it follows from Lemma 2.3 that either $\mathscr{C} = \overline{F}_0 \cap F_1$ or $\mathscr{C} = \overline{F}_{-1} \cap F_1$. Using Lemma 2.7 again, we see that

- if $\mathscr{C} = \overline{F}_0 \cap F_1$, then $\mathscr{C}^2 = 2$ on the surface F_1, and $\mathscr{C}^2 = 0$ on the surface $\overline{F}_0$,
- if $\mathscr{C} = \overline{F}_{-1} \cap F_1$, then $\mathscr{C}^2 = -2$ on the surface F_1, while $\mathscr{C}^2 = 2$ on the surface $\overline{F}_0$.

Thus, we conclude that our claim holds for $m = 1$. This is the base of induction.

Suppose that our claim holds for $m \geqslant 1$. Let us show that it holds for $m + 1$ blow ups. Let $\mathscr{C}$ be a $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curve in F_m, let $\Theta\colon \mathcal{V} \to V_m$

be its blow up, and let $\mathcal{F}$ be the Θ-exceptional surface. By induction, we know that $F_m \cong \mathbb{F}_n$ for $n > 0$. Moreover, we also know that

$$\mathscr{C} = F_m \cap \overline{F}_r$$

for some $r \in \{-1, 0, 1, \ldots, m-1\}$. Furthermore, one of the following two assertions holds:

- either $\mathscr{C}^2 = n > 0$ on the surface F_m, and $\mathscr{C}^2 \leqslant 0$ on the surface $\overline{F}_r$,
- or $\mathscr{C}^2 = -n < 0$ on the surface F_m, and $\mathscr{C}^2 > 0$ on the surface $\overline{F}_r$.

Let $\mathcal{F}_m$ and $\mathcal{F}_r$ be the strict transforms on $\mathcal{V}$ of the surfaces F_m and $\overline{F}_r$, respectively. Then $\mathcal{F} \cap \mathcal{F}_m$ and $\mathcal{F} \cap \mathcal{F}_r$ are disjoint $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves that are sections of the projection $\mathcal{F} \to \mathscr{C}$. Let γ be the self-intersection $\mathscr{C}^2$ on the surface $\overline{F}_r$. Then it follows from Lemma 2.7 that $F_{m+1} \cong \mathbb{F}_s$ for

$$s = n + |\gamma| > 0.$$

Thus, by Lemma 2.3, the curves $\mathcal{F} \cap \mathcal{F}_m$ and $\mathcal{F} \cap \mathcal{F}_r$ are the only $\mathrm{PGL}_2(\mathbb{C})$-invariant irreducible curves in the surface $\mathcal{F}$. Let $\mathcal{C}_1 = \mathcal{F} \cap \mathcal{F}_m$ and $\mathcal{C}_2 = \mathcal{F} \cap \mathcal{F}_r$.

Suppose that $\mathscr{C}^2 = n$ on the surface F_m. In this case, we have $\gamma \leqslant 0$ and $s = n - \gamma > 0$. By Lemma 2.7, we have $\mathcal{C}_1^2 = n > 0$ on the surface $\mathcal{F}_m$, and $\mathcal{C}_1^2 = -s$ on the surface $\mathcal{F}$. Similarly, we see that $\mathcal{C}_2^2 = \gamma \leqslant 0$ on the surface $\mathcal{F}_r$, and $\mathcal{C}_2^2 = s > 0$ on the surface $\mathcal{F}$. Thus, we see that the required claim holds for $m+1$ blow ups in this case.

Finally, we suppose that $\mathscr{C}^2 = -n$ on the surface F_m. Then $\gamma > 0$ and $s = n + \gamma > 0$. By Lemma 2.7, we have $\mathcal{C}_1^2 = -n < 0$ on the surface $\mathcal{F}_m$, and $\mathcal{C}_1^2 = s$ on the surface $\mathcal{F}$. Similarly, we have $\mathcal{C}_2^2 = \gamma > 0$ on the surface $\mathcal{F}_r$, and $\mathcal{C}_2^2 = -s < 0$ on the surface $\mathcal{F}$. Therefore, we proved that the required claim holds for $m+1$ blow up also in this case. Hence, it holds for any number of blow ups (by induction). $\square$

It follows from Lemmas 2.1, 2.2, 2.4 that $\beta(R) > 0, \beta(E) > 0, \beta(R') > 0$, respectively. So, to prove that X is K-polystable, it is enough to check that $\beta(F) > 0$ in the following cases:

(1) when F is the (a, b)-divisor between E and $\widetilde{R}$,
(2) when F is the (a, b)-divisor between $\widehat{E}$ and R'.

We start with the first case.

Proposition 2.9 *Let $\nu \colon Y \to \widetilde{X}$ be the (a, b)-blow up between the surfaces E and $\widetilde{R}$, and let F be the ν-exceptional surface. Then $\beta(F) > 0$.*

Proof Let $\overline{E}_1$, $\overline{E}_2$, $\overline{E}$, $\overline{R}$ be the proper transforms on Y of the surfaces $\widetilde{E}_1$, $\widetilde{E}_2$, E, $\widetilde{R}$, respectively. Take a non-negative real number x. Put $\eta = f \circ \nu$. Then

$$\eta^*(-K_X) - xF \sim_{\mathbb{R}} \overline{E}_1 + \overline{E}_2 + \overline{R} + 3\overline{E} + (a + 3b - x)F,$$

so that the pseudoeffective threshold $\tau = \tau(F)$ is at least $a + 3b$.

Suppose that $x < \tau$. Then $\overline{E}$ lies in the stable base locus of the divisor $\eta^*(-K_X) - xF$. Moreover, we claim that the positive part of the Zariski decomposition of this divisor has the following form:

$$\eta^*(-K_X) - \frac{x}{a+b}\overline{E} - xF - D$$

for an effective $\mathbb{R}$-divisor D. Indeed, if ℓ is a general fiber of the projection $\overline{E} \to C$, then

$$\Big(\eta^*(-K_X) - xF\Big) \cdot \ell = -\frac{x}{a},$$

because $\eta^*(-K_X) \cdot \ell = 0$ and $F \cdot \ell = \frac{1}{a}$. On the other hand, we have $\overline{E} \cdot \ell = -\frac{a+b}{a}$, which implies the required claim. Thus, if $7b > 2a$, then arguing as in the proof of Lemma 2.4, we get

$$S_X(F) \leqslant (a+b)S_X(E) = \frac{11}{9}(a+b),$$

because $S_X(E) = \frac{11}{9}$ by Lemma 2.2. Thus, if $\frac{b}{a} > \frac{2}{7}$, then

$$\beta(F) = A_X(F) - S_X(F) = a + 2b - S_X(F) \geqslant a + 2b - \frac{11}{9}(a+b) = \frac{7b-2a}{9} > 0$$

as required. Hence, we may assume that $\frac{b}{a} \leqslant \frac{2}{7}$.

If $x > 2b$, then the surface $\overline{R}$ lies in the stable base locus of the divisor $\eta^*(-K_X) - xF$. Moreover, in this case, the Zariski decomposition of this divisor has the following the form:

$$\eta^*(-K_X) - \frac{x}{a+b}\overline{E} - \frac{x-2b}{a+b}\overline{R} - xF - D$$

for some effective $\mathbb{R}$-divisor D (supported in $\overline{E}_1, \overline{E}_2, \overline{E}, \overline{R}, F$). Indeed, if ℓ is a general fiber of the natural projection $\overline{R} \to \phi(C)$. Then $\overline{R} \cdot \ell = -\frac{a+b}{b}$ and

$$\Big(\eta^*(-K_X) - xF\Big) \cdot \ell = 2 - \frac{x}{b},$$

which implies that the Zariski decomposition has the required form for $x > 2b$. Then

$$S_X(F) \leqslant \frac{1}{36}\int_0^{2b} \mathrm{vol}\left(\varphi^*(-K_X) - \frac{x}{a+b}E\right)dx + \frac{1}{36}\int_{2b}^{\infty} \mathrm{vol}\left(\varphi^*(-K_X) - \frac{x-2b}{a+b}R\right)dx =$$
$$= (a+b) \cdot S\left(E, \frac{2b}{a+b}\right) + (a+b) \cdot S(R) < \frac{5}{9}(a+b) + \frac{4}{9}(a+b) = a+b.$$

because we have $S(R) = \frac{4}{9}$ by Lemma 2.1, and we have $S\big(E, \frac{2b}{a+b}\big) < \frac{5}{9}$ by Lemma 2.2. This gives $\beta(F) > 0$, since $A_X(F) = a + 2b$. $\square$

Finally, we deal with (a, b)-divisors between $\widehat{E}$ and R'.

Proposition 2.10 *Let $\nu\colon Y \to \widehat{X}$ be the (a, b)-blow up between the surfaces $\widehat{E}$ and R', and let F be the ν-exceptional surface. Then $\beta(F) > 0$.*

Proof Let $\overline{E}_1$, $\overline{E}_2$, $\overline{E}$, $\overline{R}$, $\overline{R}'$ be the proper transforms on Y of E_1, E_2, E, $\widetilde{R}$, R', respectively. Take a non-negative real number x. Put $\eta = f \circ g \circ \nu$. Then

$$\eta^*(-K_X) - xF \sim_{\mathbb{R}} \overline{E}_1 + \overline{E}_2 + \overline{R} + 3\overline{E} + 3\overline{R}' + (3a + 3b - x)F,$$

so that the pseudoeffective threshold $\tau = \tau(F)$ is at least $3a + 3b$.

Suppose that $x < \tau$. Then $\overline{E}$ lies in the stable base locus of the divisor $\eta^*(-K_X) - xF$. Moreover, we claim that the positive part of the Zariski decomposition of this divisor has the following form:

$$\eta^*(-K_X) - \frac{x}{2a+b}\overline{E} - xF - D$$

for an effective $\mathbb{R}$-divisor D. Indeed, if ℓ is a general fiber of the projection $\overline{E} \to C$, then

$$\Big(\eta^*(-K_X) - xF\Big) \cdot \ell = -\frac{x}{a},$$

because $\eta^*(-K_X) \cdot \ell = 0$ and $F \cdot \ell = \frac{1}{a}$. On the other hand, we have $\overline{E} \cdot \ell = -\frac{2a+b}{a}$, which implies the required claim. Thus, we have

$$S_X(F) \leqslant (2a + b)S_X(E) = \frac{11}{9}(2a + b),$$

because $S_X(E) = \frac{11}{9}$ by Lemma 2.2. Then

$$\beta(F) = A_X(F) - S_X(F) = 3a + 2b - S_X(F) \geqslant 3a + 2b - \frac{11}{9}(2a + b) = \frac{5a + 7b}{9} > 0$$

as required. □

Thus, we see that $\beta(F) > 0$ for every G-invariant prime divisor F over the threefold X. Then X is K-polystable by Theorem 1.2.

3 Blow up of a Complete Intersection of Two Quadrics in a Conic

Let $Q_1 = \{f = 0\} \subset \mathbb{P}^5$, where

$$f = x_0x_3 + x_1x_4 + x_2x_5,$$

and let $Q_2 = \{g = 0\} \subset \mathbb{P}^5$, where

$$g = x_0^2 + \omega x_1^2 + \omega^2 x_2^2 + (x_3^2 + \omega x_4^2 + \omega^2 x_5^2) + (x_0 x_3 + \omega x_1 x_4 + \omega^2 x_2 x_5),$$

and ω is a primitive cubic root of unity. Let $V_4 = Q_1 \cap Q_2$. Then V_4 is smooth. Let G be a subgroup in $\mathrm{Aut}(\mathbb{P}^5)$ such that $G \cong \boldsymbol{\mu}_2^2 \rtimes \boldsymbol{\mu}_3$, where the generator of $\boldsymbol{\mu}_3$ acts by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[x_1 : x_2 : x_0 : x_4 : x_5 : x_3\big],$$

the generator of the first factor of $\boldsymbol{\mu}_2^2$ acts by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[-x_0 : x_1 : -x_2 : -x_3 : x_4 : -x_5\big],$$

and the generator of the second factor of $\boldsymbol{\mu}_2^2$ acts by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[-x_0 : -x_1 : x_2 : -x_3 : -x_4 : x_5\big].$$

Then $G \cong \mathfrak{A}_4$, and $\mathbb{P}^5 = \mathbb{P}(\mathbb{U}_3 \oplus \mathbb{U}_3)$, where $\mathbb{U}_3$ is the unique (unimodular) irreducible three-dimensional representation of the group G. Note that Q_1 and Q_2 are G-invariant, so that V_4 is also G-invariant. Thus, we may identify G with a subgroup in $\mathrm{Aut}(V_4)$.

Let τ be an involution in $\mathrm{Aut}(\mathbb{P}^5)$ that is given by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[x_3 : x_4 : x_5 : x_0 : x_1 : x_2\big].$$

Then Q_1 and Q_2 are τ-invariant, so that V_4 is also τ-invariant.

Using explicit description of the G-action on $\mathbb{P}^5$, one can check that G does not have fixed points in $\mathbb{P}^5$, and there are no G-invariant lines in $\mathbb{P}^5$. Moreover, every G-invariant plane in $\mathbb{P}^5$ is given by

$$\begin{cases} \lambda x_0 + \mu x_3 = 0, \\ \lambda x_1 + \mu x_4 = 0, \\ \lambda x_2 + \mu x_5 = 0, \end{cases}$$

where $[\lambda : \mu] \in \mathbb{P}^1$. Using this, we see that V_4 contains exactly four G-invariant conics. These conics are cut out on V_4 by the following G-invariant planes: the plane Π_1 given by $x_0 = x_1 = x_2 = 0$, the plane $\Pi_2 = \tau(\Pi_1)$, the plane Π_3 given by

$$\begin{cases} x_0 = \omega x_3, \\ x_1 = \omega x_4, \\ x_2 = \omega x_5, \end{cases}$$

and the plane $\Pi_4 = \tau(\Pi_3)$. We let $C_1 = V_4 \cap \Pi_1$, $C_2 = V_4 \cap \Pi_2$, $C_3 = V_4 \cap \Pi_3$, $C_4 = V_4 \cap \Pi_4$. Then the conics C_1, C_2, C_3, C_4 are pairwise disjoint, $C_2 = \tau(C_1)$ and $C_4 = \tau(C_3)$.

For every $i \in \{1, 2, 3, 4\}$, we let $\pi_i \colon X_i \to V_4$ be the blow up of the conic C_i, and we denote by E_i the exceptional surface of the blow up π_i. Then $X_1 \cong X_2$ and $X_3 \cong X_4$ are smooth Fano threefolds No 2.16, and the action of the group G lifts to its action on them.

For every $i \in \{1, 2, 3, 4\}$, we have the following G-equivariant diagram:

$$\begin{tikzcd} & X_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\ V_4 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd}$$

where the dashed arrow is a linear projection from the plane Π_i, and η_i is a conic bundle that is given by the linear system $|\pi_i^*(H) - E_i|$, where H is a hyperplane section of the threefold V_4. In each case, we have $\mathbb{P}^2 = \mathbb{P}(\mathbb{U}_3)$.

Lemma 3.1 ([1, Lemma 5.6.1]) *One has* $E_1 \cong E_2 \cong E_3 \cong E_4 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

For each $i \in \{1, 2, 3, 4\}$, let Δ_i be the discriminant curve in $\mathbb{P}^2$ of the conic bundle η_i. Then Δ_i is a (possibly reducible) quartic curve with at most ordinary double points.

Lemma 3.2 *The curves* Δ_1, Δ_2, Δ_3, Δ_4 *are smooth.*

Proof If $i = 1$, then the linear projection $V_4 \dashrightarrow \mathbb{P}^2$ from the plane Π_1 is given by

$$\big[x_0 : x_1 : x_2 : x_3 : x_4 : x_5\big] \mapsto \big[x_0 : x_1 : x_2\big].$$

Using this, one can deduce that Δ_1 is given by $4x_0^4 - x_0^2x_1^2 - x_0^2x_2^2 + 4x_1^4 - x_1^2x_2^2 + 4x_2^4 = 0$. This curve is smooth. Thus, the curve $\Delta_2 \cong \Delta_1$ is also smooth.

Let $y_0 = x_0 - \omega x_3$, $y_1 = x_1 - \omega x_4$, $y_2 = x_2 - \omega x_5$, $y_3 = x_3$, $y_4 = x_4$, $y_5 = x_5$. In new coordinates, the linear projection $V_4 \dashrightarrow \mathbb{P}^2$ from the plane Π_3 is given by

$$\big[y_0 : y_1 : y_2 : y_3 : y_4 : y_5\big] \mapsto \big[y_0 : y_1 : y_2\big].$$

Then Δ_3 is given by $4x_0^4 - \omega x_0^2x_1^2 + (\omega + 1)x_2^2x_0^2 - 4(\omega + 1)x_1^4 - x_1^2x_2^2 + 4\omega x_2^4$. This curve is smooth, so that $\Delta_4 \cong \Delta_3$ is also smooth. $\square$

Observe that $\mathbb{P}^2 = \mathbb{P}(\mathbb{U}_3)$ has three G-invariant conics. Denote them by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, and denote by $F_{1,i}$, $F_{2,i}$ and $F_{3,i}$ their preimages on X_i via η_i, respectively. Then

$$F_{1,i} \sim F_{2,i} \sim F_{3,i} \sim \pi_i^*(2H) - 2E_i.$$

For every $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$, let $\overline{F}_{j,i} = \pi_i(F_{j,i})$. Then $\overline{F}_{j,i}$ is an irreducible surface in $|2H|$ that is singular along the conic C_i. Without loss of generality, we may assume that $\overline{F}_{1,1}$ is cut out on V_4 by the equation $f_{1,1} = 0$ for $f_{1,1} = x_0^2 + x_1^2 + x_3^2$, and the surface $\overline{F}_{2,1}$ is cut out on V_4 by the equation $f_{2,1} = 0$ for $f_{2,1} = x_0^2 + \omega x_1^2 + \omega^2 x_3^2$. Then the surface $\overline{F}_{3,1}$ is cut out on V_4 by the equation $f_{3,1} = 0$, where $f_{3,1} = x_0^2 + \omega^2 x_1^2 + \omega x_3^2$. Using the involution τ, we also see that $\overline{F}_{1,2} = \tau(\overline{F}_{1,1})$, $\overline{F}_{2,2} = \tau(\overline{F}_{2,1})$ and $\overline{F}_{3,2} = \tau(\overline{F}_{3,1})$, so that we let $f_{1,2} = \tau^*(f_{1,1})$, $f_{2,2} = \tau^*(f_{2,1})$ and $f_{3,2} = \tau^*(f_{3,1})$. Then $\overline{F}_{1,3}$ is cut out by $f_{1,3} = 0$, where $f_{1,3} = (x_0 - \omega x_3)^2 + (x_1 - \omega x_4)^2 + (x_2 - \omega x_5)^2$. Likewise, the surface $\overline{F}_{2,3}$ is cut out on V_4 by the equation $f_{2,3} = 0$, where $f_{2,3} = (x_0 - \omega x_3)^2 + \omega(x_1 - \omega x_4)^2 + \omega^2(x_2 - \omega x_5)^2$, Similarly, $\overline{F}_{3,3}$ is cut out by $f_{3,3} = 0$, where $f_{3,3} = (x_0 - \omega x_3)^2 + \omega^2(x_1 - \omega x_4)^2 + \omega(x_2 - \omega x_5)^2$. Finally, we conclude that $\overline{F}_{1,4} = \tau(\overline{F}_{1,3})$, $\overline{F}_{2,4} = \tau(\overline{F}_{2,3})$ and $\overline{F}_{3,4} = \tau(\overline{F}_{3,3})$, so that we let $f_{1,4} = \tau^*(f_{1,3})$, $f_{2,4} = \tau^*(f_{2,3})$ and $f_{3,4} = \tau^*(f_{3,3})$.

Remark 3.3 Using the explicit equations of the surfaces $\overline{F}_{1,1}, \overline{F}_{2,1}, \overline{F}_{3,1}, \overline{F}_{1,2}, \overline{F}_{2,2}, \overline{F}_{3,2}, \overline{F}_{1,3}, \overline{F}_{2,3}, \overline{F}_{3,3}, \overline{F}_{1,4}, \overline{F}_{2,4}, \overline{F}_{3,4}$ given above, we can describe the incidence relation between the surfaces $\overline{F}_{1,1}, \overline{F}_{2,1}, \overline{F}_{3,1}, \overline{F}_{1,2}, \overline{F}_{2,2}, \overline{F}_{3,2}, \overline{F}_{1,3}, \overline{F}_{2,3}, \overline{F}_{3,3}, \overline{F}_{1,4}, \overline{F}_{2,4}, \overline{F}_{3,4}$ and the conics C_1, C_2, C_3, C_4. It is given in the following table: Here, No

	$\overline{F}_{1,1}$	$\overline{F}_{2,1}$	$\overline{F}_{3,1}$	$\overline{F}_{1,2}$	$\overline{F}_{2,2}$	$\overline{F}_{3,2}$	$\overline{F}_{1,3}$	$\overline{F}_{2,3}$	$\overline{F}_{3,3}$	$\overline{F}_{1,4}$	$\overline{F}_{2,4}$	$\overline{F}_{3,4}$
C_1	Node	Node	Cusp	No	Yes	No	No	Yes	No	No	Yes	No
C_2	No	Yes	No	Node	Node	Cusp	No	Yes	No	No	Yes	No
C_3	Yes	No	No	Yes	No	No	Node	Node	Cusp	Yes	No	No
C_4	Yes	No	No	Yes	No	No	Yes	No	No	Node	Node	Cusp

means that the surface does not contains the conic, and in all other cases the surface contains the conic. Likewise, Node means the the surface has an ordinary double point in general point of the conic, and Cusp means that the surface has an ordinary cusp in general point of the conic. In all remaining cases the surface is smooth at general point of the conic (we will see later that it is smooth along this conic).

Corollary 3.4 *For every $i \in \{1, 2, 3, 4\}$, one has $\alpha_G(X_i) \leqslant \frac{3}{4}$.*

Proof Observe that $F_{3,i} + E_i \sim -K_{X_i}$. Moreover, it follows from Remark 3.3 that the surface $F_{3,i}$ is tangent to E_i along a section of the projection $E_i \to C_i$. Thus, we conclude that $\alpha_G(X_i) \leqslant \mathrm{lct}(X_i, F_{3,i} + E_i) \leqslant \frac{3}{4}$ as required. $\square$

Recall that the group $G \cong \boldsymbol{\mu}_2^2 \rtimes \boldsymbol{\mu}_3$ has three different one-dimensional representations: the trivial representation with the character χ_0, the non-trivial representation with the character χ_1 that sends the generator of $\boldsymbol{\mu}_3$ to ω, and the non-trivial representation with the character χ_2 that sends the generator of $\boldsymbol{\mu}_3$ to ω^2. On the other hand, the polynomials f, g, $f_{1,1}$, $f_{2,1}$, $f_{3,1}$, $f_{1,2}$, $f_{2,2}$, $f_{3,2}$, $f_{1,3}$, $f_{2,3}$, $f_{3,3}$, $f_{1,4}$, $f_{2,4}$, $f_{3,4}$ are semi-invariants of the group G considered as a subgroup in $\mathrm{SL}_6(\mathbb{C})$. These polynomials split into three groups with respect to the characters χ_0, χ_1 and χ_2 as follows:

(χ_0) f, $f_{1,1}$, $f_{1,2}$, $f_{1,3}$, $f_{1,4}$ are G-invariants;
(χ_1) $f_{3,1}$, $f_{3,2}$, $f_{3,3}$, $f_{3,4}$ are G-semi-invariants with character χ_1;
(χ_2) g, $f_{2,1}$, $f_{2,2}$, $f_{2,3}$, $f_{2,4}$ are G-semi-invariants with character χ_2.

Note that $f_{1,4} = -(\omega+2)f_{1,1} + (\omega+2)f_{1,2} + f_{1,3}$ and $(\omega+1)f_{1,1} - \omega f_{1,2} - (\omega+1)f_{1,3} + 2f = 0$, which implies that $\overline{F}_{1,1}$, $\overline{F}_{1,2}$, $\overline{F}_{1,3}$, $\overline{F}_{1,4}$ generate a pencil on V_4, which we denote by $\mathcal{P}_0$. Similarly, we have $f_{3,4} = -(\omega+2)f_{3,1} + (\omega+2)f_{3,2} + f_{3,3}$, and the surfaces $\overline{F}_{3,1}$, $\overline{F}_{3,2}$, $\overline{F}_{3,3}$, $\overline{F}_{3,4}$ generate two-dimensional linear system (net), which we denote by $\mathcal{M}_1$. This linear system $\mathcal{M}_1$ contains four pencils, which we denote by $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,3}$ and $\mathcal{P}_{1,4}$, that consist of surfaces containing the conics C_1, C_2, C_3 and C_4, respectively. Likewise, we have $f_{2,4} = -(\omega+2)f_{2,1} + (\omega+2)f_{2,2} + f_{2,3}$ and $(\omega-1)f_{2,1} - (\omega+2)f_{2,2} - (\omega+1)f_{2,3} + 2g = 0$, so that $\overline{F}_{2,1}$, $\overline{F}_{2,2}$, $\overline{F}_{2,3}$, $\overline{F}_{2,4}$ generates a pencil on V_4, which we denote by $\mathcal{P}_2$.

For every $i \in \{1, 2, 3, 4\}$, denote by $\mathcal{P}_0^i$, $\mathcal{P}_{1,1}^i$, $\mathcal{P}_{1,2}^i$, $\mathcal{P}_{1,3}^i$, $\mathcal{P}_{1,4}^i$ and $\mathcal{P}_2^i$ the strict transforms on X_i of the pencils $\mathcal{P}_0$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,3}$, $\mathcal{P}_{1,4}$ and $\mathcal{P}_2$. Then

$$
\begin{aligned}
\mathcal{P}_{1,1}^1 &\sim \mathcal{P}_2^1 \sim -K_{X_1},\\
\mathcal{P}_{1,2}^2 &\sim \mathcal{P}_2^2 \sim -K_{X_2},\\
\mathcal{P}_{1,3}^3 &\sim \mathcal{P}_0^3 \sim -K_{X_3},\\
\mathcal{P}_{1,4}^4 &\sim \mathcal{P}_0^4 \sim -K_{X_4}.
\end{aligned}
$$

Moreover, we have $F_{3,1} + E_1 \in \mathcal{P}_{1,1}^1$, $F_{2,1} + E_1 \in \mathcal{P}_2^1$, $F_{3,2} + E_2 \in \mathcal{P}_{1,2}^2$, $F_{2,2} + E_2 \in \mathcal{P}_2^2$, $F_{3,3}+E_3 \in \mathcal{P}_{1,3}^3$, $F_{1,3} + E_3 \in \mathcal{P}_0^3$, $F_{3,4} + E_4 \in \mathcal{P}_{1,4}^4$, $F_{1,4} + E_3 \in \mathcal{P}_0^4$. This implies that $\mathcal{P}_{1,1}^1|_{E_1}$, $\mathcal{P}_2^1|_{E_1}$, $\mathcal{P}_{1,2}^2|_{E_2}$, $\mathcal{P}_2^2|_{E_2}$, $\mathcal{P}_{1,3}^3|_{E_3}$, $\mathcal{P}_0^3|_{E_3}$, $\mathcal{P}_{1,4}^4|_{E_4}$, $\mathcal{P}_0^4|_{E_4}$ are zero-dimensional linear systems (which can be considered as G-invariant curves) in E_1, E_2, E_3, E_4, respectively. Denote them by Z_1, Z_1', Z_2, Z_2', Z_3, Z_3', Z_4, Z_4', respectively. Observe that $Z_1 \neq Z_1'$, $Z_2 \neq Z_2'$, $Z_3 \neq Z_3'$ and $Z_4 \neq Z_4'$. This follows from the exact sequence of G-representations

$$
0 \to H^0\Big(\mathcal{O}_{X_i}\big(-K_{X_i} - E_i\big)\Big) \to H^0\Big(\mathcal{O}_{X_i}\big(-K_{X_i}\big)\Big) \twoheadrightarrow H^0\Big(\mathcal{O}_{E_i}\big(-K_{X_i}\big|_{E_i}\big)\Big),
$$

where the surjectivity of the last map follows from Kodaira vanishing. Alternatively, one can show this using the explicit equations of the pencils $\mathcal{P}_0$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{P}_{1,3}$, $\mathcal{P}_{1,4}$ and $\mathcal{P}_2$.

Recall that $E_1 \cong E_2 \cong E_3 \cong E_4 \cong \mathbb{P}^1 \times \mathbb{P}^1$ by Lemma 3.1. For every $i \in \{1, 2, 3, 4\}$, let s_{E_i} be a section of the projection $E_i \to C_i$ such that $s_{E_i}^2 = 0$, and let f_{E_i} be a fiber of this projection. Then $-E_i|_{E_i} = s_{E_i} - f_{E_i}$, so that $-K_{X_i}|_{E_i} \sim s_{E_i} + 3f_{E_i}$.

Hence, we see that $Z_i \sim Z_i' \sim s_{E_i} + 3f_{E_i}$, which immediately implies that both curve Z_i and Z_i' are irreducible, because C_i does not have G-orbits of lengths 1, 2 and 3.

For each $i \in \{1, 2, 3, 4\}$, the conic bundle η_i gives a double cover $E_i \to \mathbb{P}^2$, whose branching curve is $\mathcal{C}_3$. Indeed, one has $F_{3,i} \sim \pi_i^*(2H) - 2E_i$, and $\overline{F}_{3,i}$ has a cusp at general point of the conic C_i. Since $F_{3,i}|_{E_i} \sim 2s_{E_i} + 2f_{E_i}$, we have $\overline{F}_{3,i}|_{E_i} = 2C_i^i$ for some irreducible curve $C_i^i \in |s_{E_i} + f_{E_i}|$. Since the double cover $E_i \to \mathbb{P}^2$ is given by a linear subsystem in $|s_{E_i} + f_{E_i}|$, we conclude that $\eta_i(C_i^i)$ is the branching curve of this double cover. But $\eta_i(C_i^i) = \mathcal{C}_3$, since $F_{3,i}$ is the preimage of the curve $\mathcal{C}_3$ via η_i.

For every i and j in $\{1, 2, 3, 4\}$ such that $j \neq i$, denote by C_j^i the strict transform of the conic C_j on the threefold X_i. Then $-K_{X_i} \cdot C_1^i = -K_{X_i} \cdot C_2^i = -K_{X_i} \cdot C_3^i = -K_{X_i} \cdot C_4^i = 4$ and $-K_{X_i} \cdot Z_i = -K_{X_i} \cdot Z_i' = 6$. Observe also that C_1^i, C_2^i, C_3^i, C_4^i, Z_i, Z_i' are smooth rational curves. Moreover, we have the following result:

Lemma 3.5 *Let C be an irreducible G-invariant curve in X_i such that $C \cong \mathbb{P}^1$ and $-K_{X_i} \cdot C < 8$. Then C is one of the curves C_1^i, C_2^i, C_3^i, C_4^i, Z_i, Z_i'.*

Proof The proof is the same for every $i \in \{1, 2, 3, 4\}$. Thus, for simplicity of notations, we assume that $i = 1$. Suppose that C is not one of the curves C_1^1, C_2^1, C_3^1, C_4^1, Z_1, Z_1'. Let us seek for a contradiction.

First, we suppose that $C \subset E_1$. Then $C \sim as_{E_1} + bf_{E_1}$ for some non-negative integers a and b. Since $-K_{X_1}|_{E_1} \sim s_{E_1} + 3f_{E_1}$, we see that $3a + b = -K_{X_i} \cdot C < 8$. Moreover, since $C_1^1 \cdot C = a + b$, we conclude that $a + b \geqslant 4$ and $a + b \neq 5$, because C_1^1 does not have G-orbits of lengths 1, 2, 3 and 5. Thus, since C is irreducible, we conclude that $a = 1$ and $b = 3$.

Let us describe the action of G on the surface $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since G acts faithfully on $C_1 \cong \mathbb{P}^1$, this action is given by the unique (unimodular) irreducible two-dimensional representation of the central extension $2.G \cong \mathrm{SL}_2(\mathbb{F}_3)$ of the group G, which we denote by $\mathbb{W}_3$. Since $|s_{E_1} + f_{E_1}|$ contains a G-invariant curve, and the projection $E_1 \to C_1$ is G-equivariant, and we deduce that the action of G on the surface E_1 is given by the identification $E_1 = \mathbb{P}(\mathbb{W}_2) \times \mathbb{P}(\mathbb{W}_2)$. Thus, the G-invariant curves in $|s_{E_1} + 3f_{E_1}|$ corresponds to one-dimensional subrepresentations of the group $2.G$ in $\mathbb{W}_2 \otimes \mathrm{Sym}^3(\mathbb{W}_2)$. Using the following GAP script, we conclude that there are two such subrepresentations:

```
G:=Group("SL(2,3)");
R:=IrreducibleModules(G,CyclotomicField(3));
M:=TensorProduct(R[4],SymmetricPower(R[4],3));
IndecomposableSummands(M);
```

These subrepresentations corresponds to the curves Z_1 and Z_1', so that C must be one of them, which is impossible by assumption.

Thus, we see that C is not contained in E_1. Let $\overline{C} = \pi_1(C)$. Then $\pi_1^*(H) \cdot C = H \cdot \overline{C} \geqslant 2$. Moreover, if $H \cdot \overline{C} = 2$, then $\overline{C}$ is one of the conics C_1, C_2, C_3 or C_4, because these are the only G-invariant conics in V_4. Since $C \not\subset E_1$ and C is not one of the curves C_2^1, C_3^1, C_4^1, we see that $H \cdot \overline{C} \neq 2$, so that $\pi_1^*(H) \cdot C \geqslant 3$.

Note also that $\eta_1(C)$ is a curve, because G does not have fixed points in $\mathbb{P}^2$. Similarly, we see that $\eta_1(C)$ is not a line. Hence, we conclude that $(\pi_1^*(H) - E_1) \cdot C \geqslant \deg(\eta_1(C)) \geqslant 2$. One the other hand, we have $E_1 \cdot C$ must be even since C does not have G-orbits of odd length. Moreover, we have

$$7 \geqslant -K_{X_1} \cdot C = \big(\pi_1^*(2H) - E_1\big) \cdot C = \pi_1^*(H) \cdot C + \big(\pi_1^*(H) - E_1\big) \cdot C \geqslant 5,$$

so that $-K_{X_1} \cdot C = 6$, $\pi_1^*(H) \cdot C = 3$ and $(\pi_1^*(H) - E_1) \cdot C = 3$, which gives $E_1 \cdot C = 0$. Hence, we see that $\overline{C}$ is a smooth rational cubic curve, and $\eta_1(C)$ is a singular cubic curve. This is impossible, since G does not have fixed points in $\mathbb{P}^2$. □

Lemma 3.6 *Let S be a G-invariant surface such that $-K_{X_i} \sim_{\mathbb{Q}} aS + \Delta$ for a rational number a and an effective G-invariant $\mathbb{Q}$-divisor Δ on X_i. Then $a \leqslant 1$.*

Proof If $S = E_i$, then $2 = -K_{X_i} \cdot \mathscr{C} = aS \cdot \mathscr{C} + \Delta \cdot \mathscr{C} \geqslant aE_i \cdot \mathscr{C} = 2a$ for a general fiber $\mathscr{C}$ of the conic bundle ν_i. Therefore, we may assume that $S \neq E_i$. Then $\pi_i(S)$ is a surface, and $2H \sim_{\mathbb{Q}} a\pi_i(S) + \pi_i(\Delta)$. So, if $a > 1$, then $\pi_i(S) \sim H$, which is impossible, because we know that $\mathbb{P}^5$ does not contain G-invariant hyperplanes.□

Now we are ready to state the main technical result of this section:

Lemma 3.7 *Let a and λ be positive rational numbers such that $a \geqslant 1$ and $\lambda < \frac{3}{4}$, and let D be an effective G-invariant $\mathbb{Q}$-divisor on X_i such that $D \sim_{\mathbb{Q}} \pi_i^*(2H) - aE_i$. Then E_i, C_i^i, Z_i and Z_i' are not log canonical centers of the log pair $(X_i, \lambda D)$.*

Let us use this result to prove

Proposition 3.8 *One has $\alpha_G(X_1) = \alpha_G(X_2) = \alpha_G(X_3) = \alpha_G(X_4) = \frac{3}{4}$.*

Proof Suppose $\alpha_G(X_i) < \frac{3}{4}$. Let us seek for a contradiction. Since X_i has no G-fixed points, it follows from [1, Lemma A.4.8] and Lemma 3.6 that there is a G-invariant effective $\mathbb{Q}$-divisor D on the threefold X_i such that $D \sim_{\mathbb{Q}} -K_{X_i}$, the log pair $(X_i, \lambda D)$ is strictly log canonical for some positive rational number $\lambda < \frac{3}{4}$, and the only center of log canonical singularities of this log pair is an irreducible G-invariant smooth irreducible rational curve $Z \subset X_i$. Moreover, applying [1, Theorem 1.4.11(3.1)], we get $-K_{X_i} \cdot Z < 8$. Then it must be one of the curves C_1^i, C_2^i, C_3^i, C_4^i, Z_i, Z_i' by Lemma 3.5. On the other hand, it follows from Lemma 3.7 that Z is not one of the curves C_i^i, Z_i, Z_i', so that $Z = C_j^i$ for some $j \in \{1, 2, 3, 4\}$ such that $j \neq i$.

Let $\nu\colon V \to X_i$ be the blow up of the curve Z, let F be the ν-exceptional surface, let $\widetilde{D}$ be strict transform of the divisor D via ν, and let $m = \mathrm{mult}_Z(D)$. Then $m \geqslant \frac{1}{\lambda}$ and

$$K_V + \lambda\widetilde{D} + \big(\lambda m - 1\big)F \sim_{\mathbb{Q}} \nu^*\big(K_{X_i} + \lambda D\big).$$

Thus, either $\lambda m - 1 \geqslant 1$ or the surface F contains an irreducible G-invariant smooth rational curve $\widetilde{Z}$ such that $\nu(\widetilde{Z}) = Z$, the curve $\widetilde{Z}$ is a section of the projection $F \to Z$, and $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V, \lambda\widetilde{D} + (\lambda m - 1)F)$.

Let $\upsilon\colon V \to X_j$ be the birational contraction of the strict transform of the surface E_i, and let $\overline{D} = \upsilon(\widetilde{D})$. Then $\upsilon(F) = E_j$ and $\overline{D} \sim_{\mathbb{Q}} \pi_j(2H) - mE_j$, so that

$$\overline{D} + \Big(m - \frac{1}{\lambda}\Big)E_j \sim_{\mathbb{Q}} \pi_j(2H) - \frac{1}{\lambda}E_j.$$

Then the surface E_j and the curves C_j^j, Z_j and Z_j' are not log canonical centers of the log pair $(X_j, \lambda\overline{D} + (\lambda m - 1)E_j)$ by Lemma 3.7. In particular, we see that $\lambda m - 1 < 1$, so that the surface E_j contains an irreducible G-invariant smooth rational curve $\overline{Z}$ such that $\pi_j(\overline{Z}) = Z$, the curve $\overline{Z}$ is a section of the projection $E_j \to C_j$, and $\overline{Z}$ is a center of log canonical singularities of the log pair $(X_j, \lambda\overline{D} + (\lambda m - 1)E_j)$. Let us repeat that the curve $\overline{Z}$ is not one of the curves C_j^j, Z_j and Z_j' by Lemma 3.7.

Recall that $E_j \cong \mathbb{P}^1 \times \mathbb{P}^1$. Write $\overline{D}|_{E_j} = \delta Z + \Upsilon$. where δ is a non-negative rational number, and Υ is an effective $\mathbb{Q}$-divisor on E_j such that its support does not contain the curve Z. Then $\delta \geqslant \frac{1}{\lambda} > \frac{4}{3}$ by [5, Theorem 5.50]. But

$$\overline{D}\big|_{E_j} \sim_{\mathbb{Q}} \Big(\pi_j(2H) - mE_j\Big)\Big|_{E_j} \sim_{\mathbb{Q}} 4f_{E_j} + m(s_{E_j} - f_{E_j}) = ms_{E_j} + (4-m)f_{E_j},$$

and $Z \sim s_{E_j} + kf_{E_j}$ for some non-negative integer k. This gives

$$\Upsilon \sim_{\mathbb{Q}} ms_{E_j} + (4-m)f_{E_j} - \delta\big(s_{E_j} + kf_{E_j}\big) = (m-\delta)s_{E_j} + (4-m-\delta k)f_{E_j}.$$

Since $m \geqslant \frac{1}{\lambda} > \frac{4}{3}$ and $\delta > \frac{4}{3}$, we get $k = 0$ or $k = 1$, so that $Z = C_j^j$ by Lemma 3.5, which is impossible by Lemma 3.7. □

By Proposition 3.8 and Theorem 1.3, the smooth Fano threefolds $X_1 \cong X_2$ and $X_3 \cong X_4$ are K-polystable. However, to complete the proof of Proposition 3.8, we have to prove technical Lemma 3.7. Note that it is enough to prove this lemma for X_1 and X_3, so that we will assume in the following that either $i = 1$ or $i = 3$.

Fix rational numbers a and λ such that $a \geqslant 1$ and $0 < \lambda < \frac{3}{4}$. Let D be a G-invariant effective $\mathbb{Q}$-divisor on the threefold X_i such that $D \sim_{\mathbb{Q}} \pi_i^*(2H) - aE_i$. Then we must show that E_i, C_i^i, Z_i and Z_i' are also not log canonical centers of the pair $(X_i, \lambda D)$. Replacing D by $D + (a-1)E_i$, we may assume that $a = 1$, so that $D \sim_{\mathbb{Q}} -K_{X_i}$. Write $D = \varepsilon E_i + \Delta$, where $\varepsilon \in \mathbb{Q}_{\geqslant 0}$, and Δ is effective $\mathbb{Q}$-divisor on X_i whose support does not contain E_i. Then $\varepsilon \leqslant 1$ by Lemma 3.6, so that E_i is not a log canonical center of the log pair $(X_i, \lambda D)$.

Lemma 3.9 *Neither Z_i nor Z_i' is a log canonical center of the pair $(X_i, \lambda D)$.*

Proof Denote by Z one of the curves Z_i or Z_i'. Let $m_\Delta = \mathrm{mult}_Z(\Delta)$ and $m = \mathrm{mult}_Z(D)$. Then $m = m_\Delta + \varepsilon$. Let us bound m. To do this, write $\Delta\big|_{E_i} = \delta Z + \Upsilon$, where δ is a rational number such that $\delta \geqslant m_\Delta$, and Υ is an effective $\mathbb{Q}$-divisor on the surface $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that its support does not contain Z. Observe that

$$\Delta\big|_{E_i} \sim_{\mathbb{Q}} \Big(\pi_i(2H) - (1+\varepsilon)E_i\Big)\Big|_{E_i} \sim_{\mathbb{Q}} 4f_{E_i} + (1+\varepsilon)(s_{E_i} - f_{E_i}) = (1+\varepsilon)s_{E_i} + (3-\varepsilon)f_{E_i}$$

and $Z \sim s_{E_i} + 3f_{E_i}$. This gives $\Upsilon \sim_{\mathbb{Q}} (1+\varepsilon-\delta)s_{E_i} + (3-\varepsilon-3\delta)f_{E_i}$, which gives $\delta \leqslant 1-\frac{\varepsilon}{3}$. In particular, we get $m = m_\Delta + \varepsilon \leqslant \delta + \varepsilon \leqslant 1 + \frac{2\varepsilon}{3} \leqslant \frac{5}{3}$.

Let $\nu\colon V \to X_i$ be the blow up of the curve Z, and let F be the ν-exceptional surface. Then the action of the group G lifts to the threefold V, since Z is G-invariant.

Recall that Z is cut out on E_i by a G-invariant surface in $|-K_{X_i}|$. Since $Z \cong \mathbb{P}^1$, this gives $\mathcal{N}_{Z/X_i} \cong \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, because $-K_{X_i} \cdot Z = 6$, and $Z^2 = 6$ on the surface E_i. Thus, we have $F \cong \mathbb{F}_8$. Moreover, since $F^3 = -4$, we deduce that $-F\big|_F \sim s_F + 2f_F$, where s_F is a section of the projection $F \to Z$ such that $s_F^2 = -8$, and f_F is a fiber of this projection. Let $\widetilde{E}_i$ and $\widetilde{D}$ be the proper transforms of the divisors E_i and D on the threefold V, respectively. Then $\widetilde{E}_i|_F \sim (\nu^*(E_i) - F)|_F \sim s_F$, since $E \cdot Z = -2$. Thus, we see that $\widetilde{E}_i|_F = s_F$. Similarly, we get $\widetilde{D}\big|_F \sim_{\mathbb{Q}} ms_F + (2m+6)f_F$.

Now we suppose that Z is a log canonical center of the pair $(X_i, \lambda D)$. Let us seek for a contradiction. Since $\lambda m - 1 < 1$ and $K_V + \lambda\widetilde{D} + (\lambda m - 1)F \sim_{\mathbb{Q}} \nu^*(K_{X_i} + \lambda D)$, the surface F contains an irreducible G-invariant smooth rational curve $\widetilde{Z}$ such that $\nu(\widetilde{Z}) = Z$, the curve $\widetilde{Z}$ is a section of the projection $F \to Z$, and $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V, \lambda\widetilde{D} + (\lambda m - 1)F)$. Write $\widetilde{D}|_F = \theta\widetilde{Z} + \Omega$, where θ is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on F such that its support does not contain the curve $\widetilde{Z}$. Then using [5, Theorem 5.50], we get $\theta \geqslant \frac{1}{\lambda} > \frac{4}{3}$. On the other hand, we have $\widetilde{Z} \sim s_F + kf_F$ for some non-negative integer k such that either $k = 0$ or $k \geqslant 8$. Thus, we have $\Omega \sim_{\mathbb{Q}} (m-\theta)s_F + (2m + 6 - \theta k)f_F$. Hence, if $k \neq 0$, then $0 \leqslant 2m + 6 - \theta k \leqslant 2m + 6 - 8\theta < 2m + 6 - \frac{32}{3} = \frac{6m-14}{3}$, so that $m > \frac{7}{3}$, which is impossible, since $m \leqslant \frac{5}{3}$. Then $k = 0$, so that $\widetilde{Z} = s_F = \widetilde{E}_i \cap F$.

Recall that $D = \varepsilon E_i + \Delta$, where ε is a non-negative rational number such that $\varepsilon \leqslant 1$, and Δ is an effective $\mathbb{Q}$-divisor on the threefold X_i whose support does not contain E_i. Denote by $\widetilde{\Delta}$ the proper transform of this divisor on the threefold V. Then $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V, \lambda\varepsilon\widetilde{E}_i + \lambda\widetilde{\Delta} + (\lambda m_\Delta + \lambda\varepsilon - 1)F)$, where $m_\Delta = \mathrm{mult}_Z(\Delta)$. Using [5, Theorem 5.50] again, we see that $\widetilde{Z}$ is a center of log canonical singularities of the log pair $(\widetilde{E}_i, \lambda\widetilde{\Delta}|_{\widetilde{E}_i} + (\lambda m_\Delta + \lambda\varepsilon - 1)F|_{\widetilde{E}_i})$, where $F|_{\widetilde{E}_i} = \widetilde{Z}$. This simply means that $\lambda\widetilde{\Delta}|_{\widetilde{E}_i} + (\lambda m_\Delta + \lambda\varepsilon - 1)F|_{\widetilde{E}_i} = c\widetilde{Z} + \Xi$ for some rational number $c \geqslant 1$, where Ξ is an effective $\mathbb{Q}$-divisor on $\widetilde{E}_i$ whose support does not contain the curve $\widetilde{Z}$.

Now, let us compute the numerical class of the restriction $\widetilde{\Delta}\big|_{\widetilde{E}_i}$. Observe that $\widetilde{E}_i \cong E_i$. Denote by $s_{\widetilde{E}_i}$ and $f_{\widetilde{E}_i}$ the strict transforms on $\widetilde{E}_i$ of the curves s_{E_i} and f_{E_i}, respectively. Then $\widetilde{\Delta}|_{\widetilde{E}_i} \sim_{\mathbb{Q}} (1+\varepsilon)s_{\widetilde{E}_i} + (3-\varepsilon)f_{\widetilde{E}_i} - m_\Delta\widetilde{Z} = (1+\varepsilon-m_\Delta)s_{\widetilde{E}_i} + (3-\varepsilon-3m_\Delta)f_{\widetilde{E}_i}$. Thus, we see that

$$
\begin{aligned}
c\big(s_{\widetilde{E}_i} + 3f_{\widetilde{E}_i}\big) + \Xi &\sim_{\mathbb{Q}} \lambda\widetilde{\Delta}\big|_{\widetilde{E}_i} + \big(\lambda m_\Delta + \lambda\varepsilon - 1\big)F\big|_{\widetilde{E}_i} \sim_{\mathbb{Q}} \\
&\sim_{\mathbb{Q}} \lambda(1+\varepsilon-m_\Delta)s_{\widetilde{E}_i} + \lambda(3-\varepsilon-3m_\Delta)f_{\widetilde{E}_i} + \big(\lambda m_\Delta + \lambda\varepsilon - 1\big)\widetilde{Z} \sim_{\mathbb{Q}} \\
&\sim_{\mathbb{Q}} (\lambda + 2\lambda\varepsilon - 1)s_{\widetilde{E}_i} + (3\lambda + 2\lambda\varepsilon - 3)f_{\widetilde{E}_i},
\end{aligned}
$$

so that $\Xi \sim_{\mathbb{Q}} (\lambda + 2\lambda\varepsilon - 1 - c)s_{\widetilde{E}_i} + (3\lambda + 2\lambda\varepsilon - 3 - 3c)f_{\widetilde{E}_i}$, which gives $3\lambda + 2\lambda\varepsilon - 3 - 3c \geqslant 0$. Since $c \geqslant 1$ and $\lambda < \frac{3}{4}$, we deduce that $\varepsilon \geqslant \frac{3}{\lambda} - \frac{3}{2} > 4 - \frac{3}{2} = \frac{5}{2}$. But $\varepsilon \leqslant 1$. The obtained contradiction completes the proof of the lemma. □

To complete the proof of Lemma 3.7, we must show that C_i^i is not a log canonical center of the log pair $(X_i, \lambda D)$. Let $Z = C_i^i$. Suppose that Z is a log canonical center of the pair $(X_i, \lambda D)$. Let us seek for a contradiction. Observe that $\mathrm{mult}_Z\big(D\big) \geqslant \frac{1}{\lambda} > \frac{4}{3}$. Observe also that Z is not a log canonical center of the log pair $(X_i, \lambda(F_{3,i} + E_i))$ and $D \sim_{\mathbb{Q}} F_{3,i} + E_i$. Thus, replacing D by a divisor $(1 + \mu)D - \mu(F_{3,i} + E_i)$ for an appropriate non-negative rational number μ, we may assume that either the surface $F_{3,i}$ or the surface E_i is not contained in the support of the $\mathbb{Q}$-divisor D. Then we conclude that $F_{3,i}$ is not contained in the support of the $\mathbb{Q}$-divisor D, because

Lemma 3.10 *The surface E_i is contained in the support of the $\mathbb{Q}$-divisor D.*

Proof Let $\mathscr{C}$ be a general fiber of the projection $E_i \to Z$. If the surface E_i is contained in the support of the $\mathbb{Q}$-divisor D, then $1 = -K_{X_i} \cdot \mathscr{C} = D \cdot \mathscr{C} \geqslant \mathrm{mult}_Z(D) \geqslant \frac{1}{\lambda} > \frac{4}{3}$, which is absurd. □

Let $\nu\colon V \to X_i$ be the blow up of the curve Z, let F be the ν-exceptional surface, and let $\widetilde{E}_i$ be the strict transform of the surface F via ν. Then $F \cong \mathbb{F}_n$ for some integer $n \geqslant 0$, and $F|_F \sim -s_F + af_F$ for some integer a, where s_F is a section of the projection $F \to Z$ such that $s_F^2 = -n$, and f_F is a fiber of this projection. Since $-K_{X_i} \cdot Z = 4$, we conclude that $F^3 = -2$. Thus, we have $-2 = F^3 = (-s_F + af_F)^2 = -n - 2a$, so that $a = \frac{2-n}{2}$. On the other hand, we have $\widetilde{E}_i\big|_F \sim s_F + \frac{n-2}{2} f_F$, since $E_i \cdot Z = (-s_{E_i} + f_{E_i}) \cdot (s_{E_i} + f_{E_i}) = 0$. But $\widetilde{E}_i\big|_F$ is an irreducible curve, which implies that $n = 2$, since $\frac{n-2}{2} < n$. Thus, we see that $F \cong \mathbb{F}_2$ and $-F|_F \sim \widetilde{E}_i|_F = s_F$. Observe also that the action of the group G lifts to the threefold V, since Z is G-invariant.

Remark 3.11 The divisor $-K_V$ is nef and big. Indeed, the linear system $|\pi_i^*(2H) - 2E_i|$ is base point free. Let $\mathcal{M}$ be its strict transform on V. Then $\mathcal{M} + \widetilde{E}_i$ is a linear subsystem of the linear system $| - K_V|$, so that the base locus of the linear system $| - K_V|$ is contained in $\widetilde{E}_i$. But $\widetilde{E}_i \cong E_i$ and $-K_V|_{\widetilde{E}_i} \sim 2f_{\widetilde{E}_i}$, where $f_{\widetilde{E}_i}$ is a strict transform of the curve f_{E_i} on the surface $\widetilde{E}_i$. Then $-K_V|_{\widetilde{E}_i}$ is nef, so that $-K_V$ is also nef. Since $-K_V^3 = 12$, we see that $-K_V$ is big.

Let $m = \mathrm{mult}_Z(D)$, and let $\widetilde{D}$ be the proper transform of the divisor D via ν. Then

$$\widetilde{D}\big|_F \sim_{\mathbb{Q}} \big(\nu^*(-K_{X_i}) - mF\big)\big|_F \sim_{\mathbb{Q}} ms_F + 4f_F.$$

Let $\mathscr{C}$ be a sufficiently general fiber of the conic bundle ν_i that is contained in $F_{3,i}$, and let $\widetilde{\mathscr{C}}$ be its strict transform on the threefold V. Then $\mathscr{C}$ is an irreducible curve that is not contained in the support of the divisor D, because we assumed that $F_{3,i} \not\subset \mathrm{Supp}(D)$. Moreover, the curve $\mathscr{C}$ intersects the curve Z, because $F_{3,i}|_{E_i} = 2Z$. Thus, we have

$$2 - m = 2 - mF \cdot \widetilde{\mathscr{C}} = \big(\nu^*(-K_{X_i}) - mF\big) \cdot \widetilde{\mathscr{C}} = \widetilde{D} \cdot \widetilde{\mathscr{C}} \geqslant 0,$$

so that $m \leqslant 2$. Since $\lambda m - 1 < 1$ and $K_V + \lambda\widetilde{D} + (\lambda m - 1)F \sim_{\mathbb{Q}} \nu^*(K_{X_i} + \lambda D)$, the surface F contains an irreducible G-invariant smooth curve $\widetilde{Z}$ such that $\nu(\widetilde{Z}) = Z$, the curve $\widetilde{Z}$ is a section of the projection $F \to Z$, and $\widetilde{Z}$ is a center of log canonical singularities the log pair $(V, \lambda\widetilde{D} + (\lambda m - 1)F)$. Let $\widetilde{m} = \mathrm{mult}_{\widetilde{Z}}(\widetilde{D})$. Then

$$m + \widetilde{m} \geqslant \frac{2}{\lambda} > \frac{8}{3}, \tag{3.1}$$

because the multiplicity of the divisor $\lambda\widetilde{D} + (\lambda m - 1)F$ at the curve $\widetilde{Z}$ must be at least 1.

Lemma 3.12 *Either $\widetilde{Z} = s_F$ or $\widetilde{Z} \sim s_F + 2f_F$.*

Proof Write $\widetilde{D}|_F = \theta\widetilde{Z} + \Omega$, where θ is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor on F such that its support does not contain $\widetilde{Z}$. Using [5, Theorem 5.50], we get $\theta \geqslant \frac{1}{\lambda} > \frac{4}{3}$. But $\widetilde{Z} \sim s_F + kf_F$ for $k \in \mathbb{Z}$ such that $k = 0$ or $k \geqslant 2$. Thus, we have

$$\Omega \sim_{\mathbb{Q}} ms_F + 4f_F - \theta\widetilde{Z} \sim_{\mathbb{Q}} (m - \theta)s_F + (4 - \theta k)f_F.$$

Hence, if $k \neq 0$, then $0 \leqslant 4 - \theta k < 4 - \frac{4}{3}k$, so that $k = 2$. Then $\widetilde{Z} = s_F$ or $\widetilde{Z} \sim s_F + 2f_F$. □

Let $\widetilde{F}_{3,i}$ be the proper transform on V of the surface $F_{3,i}$. If $\widetilde{Z} = s_F$, then $\widetilde{Z} = \widetilde{E}_i \cap \widetilde{F}_{3,i}$, because $F_{3,i}$ is tangent to E_i along the curve Z and $\widetilde{E}_i|_F = \widetilde{Z}$. Using this, we get

Lemma 3.13 *One has $\widetilde{Z} \neq s_F$.*

Proof If $\widetilde{Z} = s_F$, then $\widetilde{\mathscr{C}}$ intersects the curve $\widetilde{Z}$, so that $2 - m \geqslant 2 - mF \cdot \widetilde{\mathscr{C}} = \widetilde{D} \cdot \widetilde{\mathscr{C}} \geqslant \widetilde{m}$, which contradicts (3.1). □

Thus, we see that $\widetilde{Z} \sim s_F + 2f_F$.

Remark 3.14 The curve $\widetilde{Z}$ is unique G-invariant curve in the linear system $|s_F + 2f_F|$, because $(s_F + 2f_F) \cdot \widetilde{Z} = 2$, and $\widetilde{Z}$ does not have G-orbits of length less than 4.

Let $\rho\colon Y \to V$ be the blow up of the curve $\widetilde{Z}$, and let R be the ρ-exceptional surface. Then $-K_Y^3 = 2$.

Lemma 3.15 *The divisor $-K_Y$ is nef.*

Proof Let $\widehat{F}_{3,i}, \widehat{E}_i, \widehat{F}$ be the strict transforms of the surfaces $F_{3,i}, E_i, F$, respectively. Then $|-K_Y|$ contains the divisor $\widehat{F}_{3,i} + \widehat{E}_i + \widehat{F}$. Therefore, to prove the required assertion, it is enough to prove that the restrictions $-K_Y|_{\widehat{F}_{3,i}}$, $-K_Y|_{\widehat{E}_i}$ and $-K_Y|_{\widehat{F}}$ are nef.

The nefness of the restriction $-K_Y|_{\widehat{E}_i}$ follows from the nefness of the restriction $-K_V|_{\widetilde{E}_i}$, because $\widetilde{Z}$ is disjoint from the surface $\widetilde{E}_i$. To check the nefness of the restriction $-K_Y|_{\widehat{F}}$, note that $\widetilde{Z} \sim s_F + 2f_F$ and $-K_V|_F \sim s_F + 4f_F$, so that $-K_Y|_{\widehat{F}}$ is rationally equivalent to the sum of two fibers of the projection $\widehat{F} \to \mathbb{P}^1$. Hence, the restriction $-K_Y|_{\widehat{F}}$ is nef.

Thus, we must prove that $-K_Y|_{\widehat{F}_{3,i}}$ is nef. To do this, recall that $F_{3,i}$ is a preimage via the conic bundle η_i of a G-invariant conic in $\mathbb{P}^2$, which we denoted earlier by $\mathcal{C}_3$. Using explicit equation of the surface $F_{3,i}$, one can check that this conic intersects the discriminant curve Δ_i by four points that form a G-orbit of length 4, so that $\mathcal{C}_3$ has simple tangency with Δ_i at every intersection point. Denote the points in $\mathcal{C}_3 \cap \Delta_i$ by P_1, P_2, P_3 and P_4. For each $k \in \{1, 2, 3, 4\}$, we have $\eta_i^{-1}(P_k) = \ell_k + \ell'_k$, where ℓ_k and ℓ'_k are smooth rational curve that intersect transversally at one point. Thus, in total we obtain eight smooth rational curves $\ell_1, \ell'_1, \ell_2, \ell'_2, \ell_3, \ell'_3, \ell_4, \ell'_4$. Denote their images in V_4 by $\overline{\ell}_1, \overline{\ell}'_1, \overline{\ell}_2, \overline{\ell}'_2, \overline{\ell}_3, \overline{\ell}'_3, \overline{\ell}_4, \overline{\ell}'_4$, respectively. Then these eight curves are lines, which we will describe later. Similarly, denote their strict transforms on V by $\widetilde{\ell}_1, \widetilde{\ell}'_1, \widetilde{\ell}_2, \widetilde{\ell}'_2, \widetilde{\ell}_3, \widetilde{\ell}'_3, \widetilde{\ell}_4, \widetilde{\ell}'_4$, respectively. Then, by construction, we have

$$-K_V \cdot \widetilde{\ell}_1 = -K_V \cdot \widetilde{\ell}'_1 = -K_V \cdot \widetilde{\ell}_2 = -K_V \cdot \widetilde{\ell}'_2 = -K_V \cdot \widetilde{\ell}_3 = -K_V \cdot \widetilde{\ell}'_3 = -K_V \cdot \widetilde{\ell}_4 = -K_V \cdot \widetilde{\ell}'_4 = 0.$$

Finally, let us denote the strict transforms on Y of these eight curves by $\widehat{\ell}_1, \widehat{\ell}'_1, \widehat{\ell}_2, \widehat{\ell}'_2, \widehat{\ell}_3, \widehat{\ell}'_3, \widehat{\ell}_4, \widehat{\ell}'_4$, respectively. For every $k \in \{1, 2, 3, 4\}$, we have $-K_Y \cdot \widehat{\ell}_k = -R \cdot \widehat{\ell}_k$ and $-K_Y \cdot \widehat{\ell}'_k = -R \cdot \widehat{\ell}'_k$. Therefore, if $\widetilde{Z}$ intersects a curve $\widehat{\ell}_k$ or $\widehat{\ell}'_k$, then $-K_Y$ is not nef, because in these case we have $-K_Y \cdot \widehat{\ell}_k < 0$ or $-K_Y \cdot \widehat{\ell}'_k < 0$, respectively.

First, let us show that the curves $\widehat{\ell}_1, \widehat{\ell}'_1, \widehat{\ell}_2, \widehat{\ell}'_2, \widehat{\ell}_3, \widehat{\ell}'_3, \widehat{\ell}_4, \widehat{\ell}'_4$ are the only curves in $\widehat{F}_{3,i}$ that a priori may have negative intersections with the divisor $-K_Y$. After this, we will explicitly check that $\widetilde{Z}$ does not intersects any of the curves $\widetilde{\ell}_1, \widetilde{\ell}'_1, \widetilde{\ell}_2, \widetilde{\ell}'_2, \widetilde{\ell}_3, \widetilde{\ell}'_3, \widetilde{\ell}_4, \widetilde{\ell}'_4$, which would imply that $-K_Y$ is indeed nef.

By construction, the curves $\ell_1, \ell'_1, \ell_2, \ell'_2, \ell_3, \ell'_3, \ell_4, \ell'_4$ form two G-irreducible curves (G-invariant curves such that the group G acts transitively on the set of their irreducible components) each consisting of four irreducible components. Without loss of generality, we may assume that $\ell_1 + \ell_2 + \ell_3 + \ell_4$ is one of these curves, and $\ell'_1 + \ell'_2 + \ell'_3 + \ell'_4$ is another curve.

Observe that $\widetilde{F}_{3,i}|_F \sim s_F + 4f_F$ and the intersection $\widetilde{F}_{3,i} \cap F$ contains the curve s_F. This implies that $\widetilde{F}_{3,i}|_F = s_F + e_1 + e_2 + e_3 + e_4$, where e_k is a fiber of the projection $F \to Z$ such that $\nu(e_k) = \ell_k \cap \ell'_k$. Since $\widetilde{F}_{3,i}|_{\widetilde{E}_i} = s_F$, we see that $\widetilde{F}_{3,i}$ is smooth. Moreover, we have $(s_F \cdot s_F)_{\widetilde{F}_{3,i}} = -2$, because $\widetilde{E}_i^2 \cdot \widetilde{F}_{3,i} = -2$. Now, using this and $F^2 \cdot \widetilde{F}_{3,i} = -2$, we conclude that $(e_1 \cdot e_1)_{\widetilde{F}_{3,i}} = (e_2 \cdot e_2)_{\widetilde{F}_{3,i}} = (e_3 \cdot e_3)_{\widetilde{F}_{3,i}} = (e_4 \cdot e_4)_{\widetilde{F}_{3,i}} = -2$. Thus, we conclude that $F_{3,i}$ has an ordinary double point at each point $\ell_k \cap \ell'_k$, and the birational morphism ν induces the minimal resolution of singularities $\widetilde{F}_{3,i} \to F_{3,i}$, which contracts the curve e_k to the point $\ell_k \cap \ell'_k$.

The composition $\eta_i \circ \nu$ induces a conic bundle $\widetilde{F}_{3,i} \to \mathcal{C}_3$. The curve s_F is its section, and its (scheme) fibers over the points P_1, P_2, P_3, P_4 are $e_1 + \widetilde{\ell}_1 + \widetilde{\ell}'_1$, $e_2 +$

$\widetilde{\ell}_1+\widetilde{\ell}'_2$, $e_3+\widetilde{\ell}_1+\widetilde{\ell}'_3$, $e_4+\widetilde{\ell}_1+\widetilde{\ell}'_4$, respectively. Thus, for every $k\in\{1,2,3,4\}$, the curves $\widetilde{\ell}_k$ and $\widetilde{\ell}'_k$ are disjoint (-1)-curves on the surface $\widetilde{F}_{3,i}$, which both do not intersect the section s_F, because s_F intersects the (-2)-curve e_k. Moreover, we have

$$-K_V\big|_{\widetilde{F}_{3,i}}\sim s_F+\sum_{k=1}^{4}\big(e_k+\widetilde{\ell}_k+\widetilde{\ell}'_k\big),$$

because $-K_V\sim\nu^*(F_{3,i})+\widetilde{E}_i$ and $\widetilde{E}_i|_{\widetilde{F}_{3,i}}=s_F$.

The curve $\widetilde{Z}$ intersects the surface $\widetilde{F}_{3,i}$ transversally by a G-orbit of length 4, because it intersects the (reducible) curve $s_F+e_1+e_2+e_3+e_4$ transversally by the points $\widetilde{Z}\cap e_1$, $\widetilde{Z}\cap e_2$, $\widetilde{Z}\cap e_3$, $\widetilde{Z}\cap e_4$, which form one G-orbit. Thus, the morphism ρ induces a birational morphism $\varrho\colon\widehat{F}_{3,i}\to\widetilde{F}_{3,i}$ that is a a blow up of this G-orbit. Using this, we see that

$$-K_Y\big|_{\widehat{F}_{3,i}}\sim\varrho^*\Big(s_F+\sum_{k=1}^{4}\big(e_k+\widetilde{\ell}_k+\widetilde{\ell}'_k\big)\Big)-r_1-r_2-r_3-r_4$$

where r_k is the exceptional curve of ϱ that is contracted to the point $\widetilde{Z}\cap e_k$. Observe that these four points $\widetilde{Z}\cap e_1$, $\widetilde{Z}\cap e_2$, $\widetilde{Z}\cap e_3$, $\widetilde{Z}\cap e_4$ are not contained in the curve s_F, because the curves $\widetilde{Z}$ and s_F are disjoint. Moreover, we have three mutually excluding options:

(1) the G-orbit $\widetilde{Z}\cap\widetilde{F}_{3,i}$ is contained in the curve $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$;
(2) the G-orbit $\widetilde{Z}\cap\widetilde{F}_{3,i}$ is contained in the curve $\widetilde{\ell}'_1+\widetilde{\ell}'_2+\widetilde{\ell}'_3+\widetilde{\ell}'_4$;
(3) the G-orbit $\widetilde{Z}\cap\widetilde{F}_{3,i}$ is contained in the curves $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$ and $\widetilde{\ell}'_1+\widetilde{\ell}'_2+\widetilde{\ell}'_3+\widetilde{\ell}'_4$.

As we already mentioned, the divisor $-K_Y$ is not nef in the first two cases. In the third case, we have

$$-K_Y\big|_{\widehat{F}_{3,i}}\sim\widehat{s}_F+\sum_{k=1}^{4}\big(\widehat{e}_k+\widehat{\ell}_k+\widehat{\ell}'_k\big),$$

where $\widehat{s}_F$ and $\widehat{e}_k$ are strict transforms of the curves s_F and e_k on the surface $\widehat{F}_{3,i}$. Moreover, in this case, we have $\widehat{s}_F\cdot\widehat{s}_F=-2$, $\widehat{s}_F\cdot\widehat{e}_k=1$, $\widehat{\ell}_k\cdot\widehat{\ell}_k=-1$, $\widehat{\ell}'_k\cdot\widehat{\ell}'_k=-1$, $\widehat{e}_k\cdot\widehat{e}_k=-3$, $\widehat{e}_k\cdot\widehat{\ell}_k=1$, $\widehat{e}_k\cdot\widehat{\ell}'_k=1$ on the surface $\widehat{F}_{3,i}$, and all other intersections are zero. This implies that the divisor $-K_Y\big|_{\widehat{F}_{3,i}}$ is nef in the third case, so that $-K_Y$ is also nef.

Therefore, we proved that the divisor $-K_Y$ is nef if and only if the curve $\widetilde{Z}$ does not intersect the curves $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$, and $\widetilde{\ell}_1+\widetilde{\ell}_2+\widetilde{\ell}_3+\widetilde{\ell}_4$. Observe that these curves intersects the ν-exceptional surface F by two (distinct) G-orbits of length 4, respectively. Denote these G-orbits by Θ and Θ', respectively. Hence, to complete the proof, it is enough to check that neither Θ nor Θ' is contained in the curve $\widetilde{Z}$.

We have $h^0(\mathcal{O}_V(-K_V))=9$ by the Riemann–Roch formula and the Kawamata–Viehweg vanishing, since $-K_V$ is big and nef by Remark 3.11. Moreover, we have

$-K_V|_F \sim s_F + 4f_F$ and $h^0(\mathcal{O}_F(s_F + 4f_F)) = 8$. Furthermore, we have $h^0(\mathcal{O}_V(-K_V - F)) = 1$, since the linear system $|-K_V - F|$ contains unique effective divisor: $\widetilde{F}_{3,i} + \widetilde{E}_i$. This gives the following exact sequence of G-representations:

$$0 \longrightarrow H^0\Big(\mathcal{O}_V\big(\widetilde{F}_{3,i} + \widetilde{E}_i\big)\Big) \longrightarrow H^0\Big(\mathcal{O}_V\big(-K_V\big)\Big) \longrightarrow H^0\Big(\mathcal{O}_F\big(s_F + 4f_F\big)\Big) \longrightarrow 0. \tag{3.2}$$

Here, the kernel of the third map is the one-dimensional G-representation generated by the section vanishing on the divisor $\widetilde{F}_{3,i} + \widetilde{E}_i + F$.

Note that $s_F \cong \mathbb{P}^1$ and $(s_F + 4f_F) \cdot s_F = 2$. Thus, the Riemann–Roch formula and the Kawamata–Viehweg vanishing give the following exact sequence of G-representations:

$$0 \longrightarrow H^0\big(\mathcal{O}_F(4f_F)\big) \longrightarrow H^0\big(\mathcal{O}_F(s_F + 4f_F)\big) \longrightarrow H^0\big(\mathcal{O}_{\mathbb{P}^1}(2)\big) \longrightarrow 0.$$

Since s_F does not have G-orbits of length 2, we have $H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \cong \mathbb{U}_3$, where $\mathbb{U}_3$ is the unique irreducible three-dimensional representation of the group G. Similarly, since Z has exactly two G-orbits of length 4, we have $H^0(\mathcal{O}_F(4f_F)) \cong \mathbb{U}_1 \oplus \mathbb{U}_1' \oplus \mathbb{U}_3$, where $\mathbb{U}_1$ and $\mathbb{U}_1'$ are different one-dimensional representations of the group G. Thus, one has

$$H^0\Big(\mathcal{O}_F\big(s_F + 4f_F\big)\Big) \cong \mathbb{U}_1 \oplus \mathbb{U}_1' \oplus \mathbb{U}_3 \oplus \mathbb{U}_3.$$

We may assume that $\mathbb{U}_1$ is generated by a section that vanishes at $s_F + e_1 + e_2 + e_3 + e_4$.

Let $\mathbb{V}$ and $\mathbb{V}'$ be sub-representations in $H^0(\mathcal{O}_F(s_F + 4f_F))$ that consist of all sections vanishing at the G-orbits Θ and Θ', respectively. Then $\dim(\mathbb{V}) = \dim(\mathbb{V}') = 4$, so that

$$\mathbb{V} \cong \mathbb{V}' \cong \mathbb{U}_1 \oplus \mathbb{U}_3,$$

since both G-orbits Θ and Θ' are contained in $s_F + e_1 + e_2 + e_3 + e_4$ by construction. Let $\widetilde{\mathbb{V}}$ and $\widetilde{\mathbb{V}}'$ be the the preimages in $H^0(\mathcal{O}_V(-K_V))$ via the restriction map in (3.2) of the sub-representations $\mathbb{V}$ and $\mathbb{V}'$, respectively. Then, as G-representations, we have

$$\widetilde{\mathbb{V}} \cong \widetilde{\mathbb{V}}' \cong \mathbb{U}_1 \oplus \mathbb{U}_1'' \oplus \mathbb{U}_3,$$

where $\mathbb{U}_1''$ is a one-dimensional representation of the group G. Since $\widetilde{\mathbb{V}}$ and $\widetilde{\mathbb{V}}'$ contain unique three-dimensional subrepresentation of the group G, these (two) three-dimensional subrepresentations define two G-invariant linear subsystems $\mathcal{M}_V$ and $\mathcal{M}_V'$ of the linear system $|-K_V|$, respectively. They can be characterized as (unique) three-dimensional G-invariant linear subsystems in $|-K_V|$ that contains G-orbits Θ and Θ', respectively. Then $\mathcal{M}_V|_F$ and $\mathcal{M}_V'|_F$ are (unique) three-dimensional G-invariant linear subsystems of the linear system $|s_F + 4f_F|$ that contain Θ and Θ', respectively. Thus, if $\Theta \subset \widetilde{Z}$, then

$$\mathcal{M}_V\big|_F = \widetilde{Z} + |2f_F|,$$

so that $\widetilde{Z} \subseteq \mathrm{Bs}(\mathcal{M}_V)$. Similarly, if $\Theta' \subset \widetilde{Z}$, then $\mathcal{M}'_V|_F = \widetilde{Z} + |2f_F|$, so that $\widetilde{Z} \subseteq \mathrm{Bs}(\mathcal{M}'_V)$.

Let $\mathcal{M}$ and $\mathcal{M}'$ be strict transforms on V_4 of the linear systems $\mathcal{M}_V$ and $\mathcal{M}'_V$, respectively. Then $\mathcal{M}$ and $\mathcal{M}'$ are linear subsystems in $|2H|$, so that they do not have fixed components, because $|H|$ does not have G-invariant divisors. Let M_1 and M_2 be two distinct surfaces in $\mathcal{M}$. If $\Theta \subset \widetilde{Z}$, then

$$\Big(M_1 \cdot M_2\Big)_{C_i} \geqslant 3, \tag{3.3}$$

where $(M_1 \cdot M_2)_{C_i}$ is the intersection multiplicity of the surfaces M_1 and M_2 at general point of the curve C_i. Similarly, if $\Theta' \subset \widetilde{Z}$, then

$$\Big(M'_1 \cdot M'_2\Big)_{C_i} \geqslant 3, \tag{3.4}$$

where M'_1 and M'_2 are two surfaces in $\mathcal{M}'$. Both conditions (3.3) and (3.4) are easy to check provided that we know generators of the linear system $\mathcal{M}$ and $\mathcal{M}'$.

Observe that the curve $\widetilde{\ell}_1 + \widetilde{\ell}_2 + \widetilde{\ell}_3 + \widetilde{\ell}_4$ is contained in the base locus of the linear system $\mathcal{M}_V$. Indeed, one has $\mathcal{M}_V \subset |-K_V|$ and $-K_V \cdot \widetilde{\ell}_i = 0$ for every $i \in \{1, 2, 3, 4\}$, while $\Theta \subseteq \mathrm{Bs}(\mathcal{M}_V)$ by construction, and Θ is contained in $\widetilde{\ell}_1 + \widetilde{\ell}_2 + \widetilde{\ell}_3 + \widetilde{\ell}_4$ by definition. Likewise, we see that $\widetilde{\ell}'_1 + \widetilde{\ell}'_2 + \widetilde{\ell}'_3 + \widetilde{\ell}'_4$ is contained in the base locus of the linear system $\mathcal{M}'_V$. Hence, the G-irreducible curves $\overline{\ell}_1 + \overline{\ell}_2 + \overline{\ell}_3 + \overline{\ell}_4$ and $\overline{\ell}'_1 + \overline{\ell}'_2 + \overline{\ell}'_3 + \overline{\ell}'_4$ are contained in the base loci of the linear systems $\mathcal{M}$ and $\mathcal{M}'$, respectively. Moreover, the base loci of these linear systems also contain the conic C_i. Using these linear conditions, we can find the generators of these linear systems, and check the conditions (3.3) and (3.4).

Since $X_1 \cong X_2$ and $X_3 \cong X_4$, it is enough to consider only the cases $i = 1$ and $i = 3$. First, we deal with the case $i = 1$. In this case, the curves $\overline{\ell}_1 + \overline{\ell}_2 + \overline{\ell}_3 + \overline{\ell}_4$ and $\overline{\ell}'_1 + \overline{\ell}'_2 + \overline{\ell}'_3 + \overline{\ell}'_4$ can be described as follows: up to a swap and a reshuffle, we may assume that

- $\overline{\ell}_1$ is the line $[\lambda : \omega\lambda : -(\omega+1)\lambda : \mu - (\omega+2)\lambda : \mu : \mu + (\omega-1)\lambda]$,
- $\overline{\ell}_2$ is the line $[\lambda : -\omega\lambda : -(\omega+1)\lambda : -\mu - (\omega+2)\lambda : \mu : -\mu + (\omega-1)\lambda]$,
- $\overline{\ell}_3$ is the line $[\lambda : \omega\lambda : (\omega+1)\lambda : \mu - (\omega+2)\lambda : \mu : -\mu + (-\omega+1)\lambda]$,
- $\overline{\ell}_4$ is the line $[\lambda : -\omega\lambda : (\omega+1)\lambda : -\mu - (\omega+2)\lambda : \mu : \mu + (-\omega+1)\lambda]$,

and

- $\overline{\ell}'_1$ is the line $[\lambda : \omega\lambda : -(\omega+1)\lambda : \mu + (2\omega+1)\lambda : \mu : \mu + (\omega+2)\lambda]$,
- $\overline{\ell}'_2$ is the line $[\lambda : -\omega\lambda : -(\omega+1)\lambda : -\mu + (2\omega+1)\lambda : \mu : -\mu + (\omega+2)\lambda]$,
- $\overline{\ell}'_3$ is the line $[\lambda : \omega\lambda : (\omega+1)\lambda : \mu + (2\omega+1)\lambda : \mu : -\mu - (\omega+2)\lambda]$,
- $\overline{\ell}'_4$ is the line $[\lambda : -\omega\lambda : (\omega+1)\lambda : -\mu + (2\omega+1)\lambda : \mu : \mu - (\omega+2)\lambda]$,

where $[\lambda : \mu] \in \mathbb{P}^1$. Therefore, the linear subsystem in $|2H|$ that consists of all surfaces containing the conic C_1 and the curve $\overline{\ell}_1 + \overline{\ell}_2 + \overline{\ell}_3 + \overline{\ell}_4$ is five-dimensional.

Moreover, it is generated by the G-invariant surfaces $\overline{F}_{1,1}$, $\overline{F}_{3,1}$, $\overline{F}_{2,3}$, and the G-invariant two-dimensional linear subsystem (net) that is cut out on V_4 by

$$\lambda\Big((1-\omega)x_0x_5-(2\omega+1)x_2x_3+3x_0x_2\Big)+ \\ +\mu\Big((\omega+1)x_1x_3+(2\omega+1)x_0x_1+x_4x_0\Big)+ \\ +\gamma\Big((\omega+2)x_1x_2-\omega x_1x_5+x_2x_4\Big)=0, \tag{3.5}$$

where $[\lambda:\mu:\gamma]\in\mathbb{P}^2$. Therefore, we conclude that (3.5) defines the linear system $\mathcal{M}$. It follows from (3.5) that the base locus of this linear system consists of the conic C_1, the curve $\overline{\ell}_1+\overline{\ell}_2+\overline{\ell}_3+\overline{\ell}_4$, and the conic C_3. Similarly, we see that $\mathcal{M}'$ is given by

$$\lambda\Big((2\omega+1)x_0x_5+(\omega+2)x_2x_3+3x_0x_2\Big)+ \\ +\mu\Big((\omega+1)x_1x_3+(1-\omega)x_0x_1+x_4x_0\Big)+ \\ +\gamma\Big((2\omega+1)x_1x_2+\omega x_1x_5-x_2x_4\Big)=0,$$

where $[\lambda:\mu:\gamma]\in\mathbb{P}^2$. We also see that the base locus of the linear system $\mathcal{M}'$ consists of the conic C_1, the curve $\overline{\ell}_1'+\overline{\ell}_2'+\overline{\ell}_3'+\overline{\ell}_4'$, and the conic C_4. Now one can check that neither (3.3) nor (3.4) holds. Thus, if $i=1$ or $i=2$, then $-K_Y$ is nef.

Finally, we consider the case $i=3$. Now, up to a swap, the linear system $\mathcal{M}$ is again given by (3.5), and the linear system $\mathcal{M}'$ is given by

$$\lambda\Big((\omega+1)x_1x_5+x_4x_2-(\omega-1)x_4x_5\Big)+ \\ +\mu\Big(\omega x_0x_5-x_3x_2+(2\omega+1)x_3x_5\Big)+ \\ +\gamma\Big(\omega x_0x_5-x_3x_2+(2\omega+1)x_3x_5\Big)=0,$$

where $[\lambda:\mu:\gamma]\in\mathbb{P}^2$. Note that the base locus of the net $\mathcal{M}'$ consists of the conic C_3, the curve $\overline{\ell}_1'+\overline{\ell}_2'+\overline{\ell}_3'+\overline{\ell}_4'$, and the conic C_2. As above, one can check that neither (3.3) nor (3.4) holds. Thus, the divisor $-K_Y$ is nef. $\square$

Let $\widehat{D}$ be the proper transform of the divisor D on the threefold Y. Then

$$\widehat{D}\sim_{\mathbb{Q}}(\pi_i\circ\nu\circ\rho)^*(2H)-(\nu\circ\rho)^*(E_i)-m\rho^*(F)-\widetilde{m}R.$$

Since $-K_Y$ is nef, we see that $-K_Y^2\cdot\widehat{D}\geqslant 0$. To compute $-K_Y^2\cdot\widehat{D}$, observe that

$$H^3 = 4,\ \pi_i^*(H)\cdot E^2 = -2,\ (\pi_i\circ\nu)^*(H)\cdot F^2 = -2,$$
$$(\pi_i\circ\nu\circ\rho)^*(H)\cdot R^2 = -2,\ E^3 = -2,\ F^3 = -2,\ R^3 = -2,$$

and other intersections involved in the computation $-K_Y^2\cdot\widehat{D}$ are all zero. This gives

$$0 \leqslant -K_Y^2\cdot\widehat{D} = \Big((\pi_i\circ\nu\circ\rho)^*(2H) - (\nu\circ\rho)^*(E_i) - \rho^*(F) - R\Big)^2\cdot\widehat{D} = 14 - 6(m+\widetilde{m}),$$

so that $m+\widetilde{m}\leqslant\frac{7}{3}$, which is impossible by (3.1). The obtained contradiction completes the proof of Lemma 3.7, which completes the proof of Proposition 3.8. Thus, we see that the threefolds X_1, X_2, X_3 and X_4 are K-polystable.

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References

1. Araujo, C., Castravet, A.-M., Cheltsov, I., Fujita, K., Kaloghiros, A.-S., Martinez-Garcia, J., Shramov, C., Süß, H., Viswanathan, N.: The Calabi problem for Fano threefolds. MPIM preprint 2021-31
2. Cheltsov, I., Shramov, C.: Log canonical thresholds of smooth Fano threefolds. Russ. Math. Surv. **63**, 71–178 (2008)
3. Fujita, K.: A valuative criterion for uniform K-stability of $\mathbb{Q}$- Fano varieties. J. Reine Angew. Math. **751**, 309–338 (2019)
4. Iskovskikh, V., Prokhorov, Yu.: Fano varieties. Encyclopaedia of Mathematical Sciences, vol. 47. Springer, Berlin (1999)
5. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties. Cambridge University Press, Cambridge (1998)
6. Li, C.: K-semistability is equivariant volume minimization. Duke Math. J. **166**, 3147–3218 (2017)
7. Mabuchi, T.: On the classification of essentially effective $\mathrm{SL_n}(\mathbb{C})$- actions on algebraic n-folds. Osaka J. Math. **16**, 745–758 (1979)
8. Tian, G.: On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M)>0$. Invent. Math. **89**, 225–246 (1987)
9. Zhuang, Z.: Optimal destabilizing centers and equivariant K-stability. Invent. Math. **226**, 195–223 (2021)

A Note on Families of K-Semistable Log-Fano Pairs

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Abstract In this short note, we give an alternative proof of the semipositivity of the Chow–Mumford line bundle for families of K-semistable log-Fano pairs, and of the nefness threeshold for the log-anti-canonical line bundle on families of K-stable log Fano pairs. We also prove a bound on the multiplicity of fibers for families of K-semistable log Fano varieties, which to the best of our knowledge is new.

Keywords K-stability · Log-Fano varieties

1 Introduction

K-polystability is an algebraic stability notion for log-Fano pairs, which, over the complex numbers, is equivalent to the existence of a Kähler–Einstein metric. Over an algebraically closed field of characteristic zero, K-polystable log-Fano pairs have a good projective moduli space. The Chow–Mumford (CM) line bundle is an ample line bundle on this moduli space. We refer to the introductions of [2, 11], to the survey [10] and to the recent groundbreaking paper [7] for an exhaustive discussion of these notions and a comprehensive bibliography.

We now recall the definition of the CM line bundle for families of log-Fano pairs over a curve, and in doing so we also establish some notations which will be used through all this article.

Notation 1.1 Let T be a smooth projective curve, and let (X, Δ) be an irreducible, normal pair of dimension $n + 1$, both defined over an algebraically closed field k

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of characteristic zero. Let $f\colon X \to T$ be a flat morphism such that $f_*\mathcal{O}_X = \mathcal{O}_T$. Assume that the relative log-canonical divisor $-K_{X/T} - \Delta$ is $\mathbb{Q}$-Cartier and f-ample; in other words, f is a family of log-Fano pairs. The Chow–Mumford line bundle is defined as

$$\lambda_{CM} = -f_*(-K_{X/T} - \Delta)^{n+1}$$

We refer to [2, Sect. 3] for the basic properties of λ_{CM} and its connection with the other definitions in the literature. Our first result is

Theorem 1.2 *In the situation of Notation 1.1, if there exists a t in T such that (X_t, Δ_t) is K-semistable, then λ_{CM} is nef.*

This result was already proved in [2, Theorem 1.8] and [11, Corollary 4.7]. That is, we give the third proof of Theorem 1.2. Our main contribution is that the present proof is particularly short and it uses very little of the theory of filtrations. In fact, all the above proofs use the Harder–Narasimhan filtration of $f_*\mathcal{O}_X(-qK_{X/T})$. However, our proof uses it in a quite minimalistic manner.

As shown in [2, Sect. 10], Theorem 1.2 *implies that the Chow–Mumford line bundle gives a nef line bundle on the moduli space of K-polystable log-Fano pairs.*

As explained in [8, Appendix], the anti-log-canonical bundle is not nef on X unless the family is locally isotrivial. With our methods we can also bound its nefness threeshold as follows:

Theorem 1.3 *In the situation of Notation 1.1, if there exists a t in T such that (X_t, Δ_t) is K-stable, then*

$$-K_{X/T} - \Delta + \frac{\delta f^*\lambda_{CM}}{(\delta - 1)\dim(X)v} F$$

is nef, where δ is the stability threshold of (X_t, Δ_t), see Sect. 2, F is the class of a fiber, and $v = \left(-K_{X_t} - \Delta_t\right)^n$.

The above result was proven in [2, Theorem 1.20], and in [11, Corollary 4.10], but here we give a different proof. The novelty of this new proof is similar as for the proof of Theorem 1.2, which was explained after the statement of Theorem 1.2. We also note that Theorem 1.3 is used to prove the positivity of the Chow–Mumford line bundle under convenient assumptions, see [2, 9, 11]. Theorems 1.2 and 1.3 have also been recently used in [3] to prove slope inequalities for families of K-stable Fano varieties.

Additionally, we prove the following result, which gives a bound on the number of non-reduced fibers of a family of K-semistable Fano varieties, under a semipositivity assumption on the top self-intersection of the anti-canonical bundle of the total space.

Proposition 1.4 *In the situation of Notation 1.1, assume that $(-K_X - \Delta)^{n+1} \geq 0$ and that $T \cong \mathbb{P}^1$ (both of these are satisfied if (X, Δ) is log-Fano). Additionally assume that there exists a t in T such that (X_t, Δ_t) is K-semistable. Then, denoting by d_i the multiplicity of the non-reduced irreducible fibers, we have*

$$\sum_i \left(1 - \frac{1}{d_i}\right) \leq 2\,.$$

In particular, there are at most 4 *non-reduced irreducible fibers.*

Example 6.1 shows that the above result is sharp. Proposition 1.4 should be compared with [2, Corollary 1.17], where, under similar assumptions, it is given an upper bound for the volume of (X, Δ).

2 Basis Type Divisors, Delta Invariants and K-Stability

For the whole article we work over a fixed algebraically closed field k of characteristic zero. Following [5], we introduce the notion of basis type divisor and *stability threshold* (formerly known as the delta invariant). Let (Z, Γ) be a normal projective pair, that is, Z is a normal projective variety and Γ is an effective $\mathbb{Q}$-divisor on X. Let H be a $\mathbb{Q}$-divisor on Z. Let q be a positive integer such that qH is Cartier. A q-basis type divisor for $(Z, \Gamma;\, H)$ is a $\mathbb{Q}$-Cartier divisor D on Z such that there exists a basis $s_1, \ldots, s_{N_q}$ of $H^0(Z, qH)$ with

$$D = \frac{1}{qN_q} \sum_{i=1}^{N_q} \{s_i = 0\}\,.$$

We define the qth stability threshold of the pair $(Z, \Gamma;\, H)$ as

$$\delta_q(Z, \Gamma;\, H) := \inf \left\{ \ \mathrm{lct}(Z, \Gamma;\, D) \ \middle|\ D \text{ is a } q\text{-basis type divisor} \right\},$$

We then define the stability threshold, also known as the delta invariant, as

$$\delta(Z, \Gamma;\, H) = \lim_{q \to \infty} \delta_q(Z, \Gamma;\, H)\,,$$

where the above limit does exist by [1]. If (Z, Γ) is a log-Fano pair, we let

$$\delta(Z, \Gamma) := \delta(Z, \Gamma;\, -K_Z - \Gamma)\,.$$

We can now give the key definitions

Definition 2.1 A log-Fano pair (Z, Γ) is K-semistable if $\delta(Z, \Gamma) \geq 1$, it is K-stable if $\delta(Z, \Gamma) > 1$.

Both K-semistability and K-stability are open conditions in families. For families of log-Fano pairs, the stability threshold of the fiber is a lower-semicontinuous function on the base. If the base field is uncountable, it attains its maximum on the very general geometric fiber. In particular, if the base field is uncountable, one can minimize the coefficient of F in Theorem 1.3 by taking t a very general point in T. If the base field is countable, one can make a field extension and then take a very general point defined over this bigger field.

3 Harder–Narasimhan Filtration and Lift of Basis Type Divisors

In the situation of Notation 1.1, for the values of q such that $-q(K_{X/T}+\Delta)$ is Cartier, we can look at the following vector bundles on T

$$\mathcal{E}_q := f_*\mathcal{O}_X\big(-q(K_{X/T}+\Delta)\big). \tag{3.1}$$

Let

$$0=\mathcal{F}_0^q \subsetneq \mathcal{F}_1^q \subsetneq \cdots \subsetneq \mathcal{F}_{c_q}^q = \mathcal{E}_q$$

be the Harder–Narashiman filtration of $\mathcal{E}_q$; denote its graded objects $\mathcal{F}_i^q/\mathcal{F}_{i-1}^q$ by $\mathcal{G}_i^q$ and their slopes by μ_i^q. Recall that $\mu_i^q > \mu_{i+1}^q$.

Lemma 3.1 *In the situation of Notation 1.1, fix a closed point t in T such that the fiber X_t over t is a normal variety. For every q divisible enough and for every rational number $\varepsilon > 0$ (the divisibility condition on q does not depend on ε), there exists an effective $\mathbb{Q}$-Cartier divisor $D_\varepsilon^{(q)}$ on X such that*

- *$D_\varepsilon^{(q)}$ is $\mathbb{Q}$-linearly equivalent to*

$$M_{q,\varepsilon} = -K_{X/T} - \Delta - \left(\frac{\deg \mathcal{E}_q}{qN_q} + \varepsilon\right) X_t$$

  where $N_q = h^0\big(X_t, -q(K_{X_t}+\Delta_t)\big)$.
- *the restriction of $D_\varepsilon^{(q)}$ to the fiber X_t is a q-basis type divisor.*

Proof Choose q divisible enough so that $f_*\mathcal{O}_X\big(-q(K_{X/T}+\Delta)\big)$ satisfies cohomology and base-change. This is possible by the relative ampleness assumption on $-K_{X/T}-\Delta$.

Fix an index i, and let a_i be a strictly positive integer such that $a_i\mu_i^q$ is an integer. Let g be the genus of T. All the slopes of the Harder–Narasimhan filtration of the vector bunde $\big(\mathcal{F}_i^q\big)^{\otimes a_i} \otimes \mathcal{O}_T\big(-(a_i\mu_i^q - 2g)t\big)$ are greater or equal to $2g$ (see the proof of [2, Proposition 5.9]), hence by [2, Proposition 5.7] the above vector bundle is globally generated.

Take an element s in the fiber

$$\mathcal{F}_i^q \otimes k(t) \hookrightarrow \mathcal{E}_q \otimes k(t) \cong H^0\big(T, -q(K_{X_t,}+\Delta_t)\big), \tag{3.2}$$

which by (3.2) corresponds to a divisor $\{s = 0\} \sim -q(K_{X_t,} + \Delta_t)$. By the above global generation statement, there exists a global section $\tilde{s}$ of $\mathcal{E}_q^{\otimes a_i} \otimes \mathcal{O}_T\big(-(a_i\mu_i^q - 2g)t\big)$ which over t equals $s^{\otimes a_i}$ (remark that $\mathcal{O}_T(t) \otimes k(t) \cong k(t)$ canonically, so this makes sense). Let $[\tilde{s}]$ be the image of $\tilde{s}$ in $\mathcal{E}_{qa_i} \otimes \mathcal{O}_T\big(-(a_i\mu_i^q - 2g)t\big)$, via the homomorphism induced by the multiplication map $\mathcal{E}_q^{\otimes a_i} \to \mathcal{E}_{qa_i}$.

Using (3.1) and the projection formula we obtain the isomorphism

$$H^0\Big(T, \mathcal{E}_{qa_i} \otimes \mathcal{O}_T\big(-(a_i\mu_i^q - 2g)t\big)\Big) \cong H^0\big(X, -qa_i(K_{X/T} + \Delta) - (a_i\mu_i^q - 2g)X_t\big).$$

Hence, we can consider the $\mathbb{Q}$-Cartier divisor $D_s = \frac{1}{a_i}\{[\tilde{s}] = 0\}$ on X; its restriction to X_t equals the Cartier divisor $\{s = 0\}$, and by (3.2) on X we have

$$D_s \sim_{\mathbb{Q}} -q(K_{X/T} + \Delta) - \left(\mu_i^q - \frac{2g}{a_i}\right) X_t. \tag{3.3}$$

For each integer $1 \le i \le c_q$, fix elements $s_{i,j}^q$ for $j = 1 \dots, \mathrm{rk}(\mathcal{G}_i^q)$ in $\mathcal{F}_i^q \otimes t$ whose image in $\mathcal{G}_i^q \otimes t$ give a basis. For each of them we perform the above construction, obtaining $\mathbb{Q}$-divisors $D_{i,j}^{(q)}$ on X. Let

$$D_{\text{pre}}^{(q)} := \frac{1}{qN_q} \sum_{i,j} D_{i,j}^{(q)}.$$

By construction, the above sum run over a set of N_q indexes (i, j); in other words, the number of divisors $D_{i,j}^{(q)}$ is equal to the rank of $f_*\mathcal{O}_X(-q(K_{X/T} + \Delta))$.

By (3.3) and by the fact that there are N_q appearances of the pairs of indices (i, j), we have

$$\begin{aligned} D_{\text{pre}}^{(q)} &\sim_{\mathbb{Q}} \frac{1}{qN_q} \sum_{i,j} \left(-q(K_{X/T} + \Delta) - \left(\mu_i^q - \frac{2g}{a_i}\right) X_t\right) \\ &= -K_{X/T} - \Delta - \sum_i \left(\frac{\mu_i^q \,\mathrm{rk}(\mathcal{G}_i^q)}{qN_q} - \frac{2g\,\mathrm{rk}(\mathcal{G}_i^q)}{qN_q a_i}\right) X_t \\ &= -K_{X/T} - \Delta - \left(\frac{\deg \mathcal{E}_q}{qN_q} - \sum_i \frac{2g\,\mathrm{rk}(\mathcal{G}_i^q)}{qN_q a_i}\right) X_t. \end{aligned}$$

Apart from the a_i's, everything in $\sum_i \frac{2g\,\mathrm{rk}(\mathcal{G}_i^q)}{qN_q a_i}$ is fixed, including the set of indices over which we do the sum. Hence, by choosing a_i big enough we may assume that $\sum_i \frac{2g\,\mathrm{rk}(\mathcal{G}_i^q)}{qN_q a_i} \le \varepsilon$. Then we may choose

$$D_{\varepsilon}^{(q)} = D_{\text{pre}}^{(q)} + \left(\varepsilon - \sum_i \frac{2g\,\mathrm{rk}(\mathcal{G}_i^q)}{qN_q a_i}\right) X_{t'}$$

where $t \neq t' \in T$ is another closed point. To show that this is a good choice, we have to show that $D_\varepsilon^{(q)}\big|_{X_t}$ is a basis type divisor. Indeed, the restriction of each $D_{i,j}^{(q)}$ to X_t gives an element of a basis of the linear system $|-q(K_{X_t}+\Delta_t)|$, hence the restriction of $D_\varepsilon^{(q)}$ gives a q-basis type divisor. □

4 Nefness Threshold

The following lemma is a consequence of [4, Theorem 1.13]

Lemma 4.1 *In the situation of Notation 1.1, let Γ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that the pair $(X_t, \Delta_t + \Gamma_t)$ is klt for some closed point $t \in T$ and $K_{X/T} + \Delta + \Gamma$ is f-ample, then $K_{X/T} + \Delta + \Gamma$ is nef.*

Proof By [4, Theorem 1.13], the vector bundle $f_*\mathcal{O}_X(q(K_{X/T}+\Delta+\Gamma))$ is nef for all q big and divisible enough. The evaluation map $f^*f_*\mathcal{O}_X(q(K_{X/T}+\Delta+\Gamma)) \to \mathcal{O}_X(q(K_{X/T}+\Delta+\Gamma))$ is surjective for all q divisible enough, as $K_{X/T}+\Delta+\Gamma$ is f-ample. We conclude that $q(K_{X/T}+\Delta+\Gamma)$ and hence $K_{X/T}+\Delta+\Gamma$ is nef. □

***Proof of Theorem* 1.3** We keep the notation of Lemma 3.1. Applying this lemma and using the definition of K-stability yields that for all rational number $\varepsilon' \in (0, \delta - 1)$, there exists an integer $q(\varepsilon')$ such that for all positive integers $q(\varepsilon')\big|q$ the pair $\left(X_t, (1+\varepsilon')D_t^{(q)} + \Delta_t\right)$ is klt. Fix such integer $q(\varepsilon')$. By Lemma 4.1, the $\mathbb{Q}$-Cartier divisor

$$N_{\varepsilon'} = K_{X/T} + \Delta + (1+\varepsilon')D_\varepsilon^{(q(\varepsilon'))} \sim_{\mathbb{Q}} -\varepsilon'(K_{X/T}+\Delta) - (1+\varepsilon')\left(\frac{\deg \mathcal{E}_{q(\varepsilon')}}{q(\varepsilon')N_{q(\varepsilon')}} + \varepsilon\right)X_t$$

is nef on X. Hence, also $\frac{N_{\varepsilon'}}{\varepsilon'}$ is nef. Since, this is true for every $\mathbb{Q} \ni \varepsilon > 0$, by limiting with ε to 0 we obtain that

$$-(K_{X/T}+\Delta) - \frac{(1+\varepsilon')\deg \mathcal{E}_{q(\varepsilon')}}{\varepsilon' q(\varepsilon')N_{q(\varepsilon')}}X_t \tag{4.1}$$

is nef. Next we note that if we limit with ε' to $\delta - 1$, then we may choose that at the same time $q(\varepsilon')$ limits to ∞. Indeed, this is possible, since $q(\varepsilon')$ can be replaced by each of its multiples. Additionally, $\lim_{q\to\infty} \frac{\deg \mathcal{E}_q}{qN_q} = -\frac{\deg(\lambda_{CM})}{\dim(X)v}$, see [2, Sect. 3]. So, by limiting ε' in (4.1) to $\delta - 1$ we obtain that

$$-(K_{X/T}+\Delta) - \frac{\delta \deg(\lambda_{CM})}{(\delta-1)\dim(X)v}X_t$$

is nef. □

5 Semipositivity

Proof of Theorem 1.2 Assume by contradiction that $\deg(\lambda_{CM}) < 0$. We keep the notation of Lemma 3.1.

Let a be a positive integer such that $E = -K_{X/T} - \Delta + aX_t$ is ample on X. Let $c, \varepsilon > 0$ be rational numbers such that for all q divisible enough we have

$$(1-c)\left(\frac{\deg(\lambda_{CM})}{\dim(X)v} + \varepsilon\right) + ca < 0\,.$$

The K-semistability assumption implies that $\delta(X_t, \Delta_t) \geq 1$, so for all q divisible enough we have $\delta_q(X_t, \Delta|_{X_t}) > 1 - c$. By the definition of δ_q in terms of log canonical threshold of q-basis type divisors, we have that $\left(X_t, (1-c)\left(D_\varepsilon^{(q)}\right)_t + \Delta|_{X_t}\right)$ is klt for all q divisible enough. This yields a contradiction with [2, Proposition 7.2] as we can write

$$(1-c)D_\varepsilon^{(q)} + cE \sim_{\mathbb{Q}} -K_{X/T} - \Delta + \left((1-c)\left(\frac{\deg(\lambda_{CM})}{\dim(X)v} + \varepsilon\right) + ca\right)X_t\,.$$

□

6 Bound on the Multiplicity of the Fibers

Proof of Proposition 1.4 Let $d_i F_i$ be the non-reduced fibers of f, and d a common multiple of the d_i. Let $\tau\colon S \to T$ be the degree d cover of T totally ramified at the points corresponding to the non-reduced fibers. Denote by Y the normalization of base change X_S, and by $\sigma\colon Y \to X$ and $g\colon Y \to S$ the induced maps. Let $\Delta_Y := \sigma^*\Delta$ (for the pull-back of a Weil divisor via a finite map between normal varieties see [6, Proof of Proposition 5.20]). We have

$$-K_{Y/S} - \Delta_Y - \sum_i \frac{d}{d_i}(d_i - 1)F_i = -\sigma^*(K_{X/T} + \Delta)\,.$$

As $d_i F_i \sim_f 0$, we have $F_i \sim_{f,\mathbb{Q}} 0$, and hence, $-K_{Y/S} - \Delta_Y$ is g-ample and g is a family of log-Fano varieties. As the generic fiber of g is isomorphic to the generic fiber of f, it is K-semistable, hence the Chow–Mumford line bundle of g is nef, in other words $(-K_{Y/S} - \Delta_Y)^{n+1} \leq 0$.

We now make the following direct computation.

$$\begin{aligned}-(n+1)\left(-K_{Y_t} - (\Delta_Y)_t\right)^n \cdot \sum_i \left(d - \frac{d}{d_i}\right) \\ \geq (-K_{Y/S} - \Delta_Y)^{n+1} - (n+1)\left(-K_{Y_t} - (\Delta_Y)_t\right)^n \cdot \sum_i \left(d - \frac{d}{d_i}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(-K_{Y/S} - \Delta_Y - \sum_i \frac{d}{d_i}(d_i - 1)F_i\right)^{n+1} \\
&= \left(-\sigma^* \left(K_{X/T} + \Delta\right)\right)^{n+1} \\
&= d(-K_{X/T} - \Delta)^{n+1} \\
&= d(-K_X - \Delta + f^* K_T)^{n+1} \\
&= d(-K_X - \Delta)^{n+1} - 2d(n+1)\left(-K_{X_t} - \Delta_t\right)^n \\
&\geq -2d(n+1)\left(-K_{X_t} - \Delta_t\right)^n .
\end{aligned}
$$

As (X_t, Δ_t) and $(Y_t, (\Delta_Y)_t)$ are isomorphic for generic t, we conclude that

$$
\sum_i \left(1 - \frac{1}{d_i}\right) \leq 2 .
$$

□

Example 6.1 This example shows that Proposition 1.4 is sharp, and the condition $(-K_X)^{n+1} \geq 0$ is necessary. Let C be a genus g hyperelliptic curve, and ι the hyperelliplict involution. Let X be the quotient of $\mathbb{P}^1 \times C$ by $G = \mathbb{Z}/2\mathbb{Z}$, which acts on C by ι and on $\mathbb{P}^1$ as the standard involution. Consider the morphism $f : X \to C/\iota \cong \mathbb{P}^1 =: T$. This map is a $\mathbb{P}^1$-bundle, so that the generic fiber is K-semistable, and it has $2g + 2$ non-reduced fibers of multiplicity 2. The condition $(-K_X)^2 \geq 0$ is fulfilled if and only if $g \leq 1$.

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References

1. Blum, H., Jonsson, M.: Thresholds, valuations, and K-stability (2017). http://arxiv.org/abs/1706.04548
2. Codogni, G., Patakfalvi, Zs.: Positivity of the CM line bundle for families of K-stable klt Fano varieties. Invent. Math. **223**(3), 811–894 (2021)
3. Codogni, G., Tasin, L., Viviani, F.: Slope inequalities for KSB-stable and K-stable families (2021). https://arxiv.org/abs/2107.09553
4. Fujino, O.: Semipositivity theorems for moduli problems. Ann. of Math. (2) **187**(3), 639–665 (2018)
5. Fujita, K., Odaka, Y.: On the K-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. (2) **70**(4), 511–521 (2018)

6. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties. Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998). With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original
7. Liu, Y., Xu, C., Zhuang, Z.: Finite generation for valuations computing stability thresholds and applications to K-stability (2021). https://arxiv.org/abs/2102.09405
8. Patakfalvi, Zs., Zdanowicz, M.: On the Beauville–Bogomolov decomposition in characteristic $p \geq 0$ (2019). arxiv:1912.12742
9. Posva, Q.: Positivity of the CM line bundle for K-stable log Fanos (2019). http://arxiv.org/abs/1910.12623
10. Xu, C.: K-stability of Fano varieties: an algebro-geometric approach (2020). https://arxiv.org/abs/2011.10477
11. Xu, C., Zhuang, Z.: On positivity of the CM line bundle on K-moduli spaces. Ann. of Math. (2) **192**(3), 1005–1068 (2020)

The Yau–Tian–Donaldson Conjecture for Cohomogeneity One Manifolds

Thibaut Delcroix

Abstract We prove the Yau–Tian–Donaldson conjecture for cohomogeneity one manifolds, that is, for projective manifolds equipped with a holomorphic action of a compact Lie group with at least one real hypersurface orbit. Contrary to what seems to be a popular belief, such manifolds do not admit extremal Kähler metrics in all Kähler classes in general. More generally, we prove that for rank one polarized spherical varieties, G-uniform K-stability is equivalent to K-stability with respect to special G-equivariant test configurations. This is furthermore encoded by a single combinatorial condition, checkable in practice. We illustrate on examples and answer along the way a question of Kanemitsu.

Keywords K-stability · Cohomogeneity one manifolds

1 Introduction

A compact complex manifold X equipped with a holomorphic action of a real compact Lie group K such that there is at least one real hypersurface orbit $K \cdot x$ in X is called a (compact) cohomogeneity one manifold. Such manifolds have played a key role in complex geometry, especially in Kähler geometry, for being the easiest non-homogeneous manifolds to study. Indeed, under the previous assumption, the generic orbit of K is a real hypersurface as well, so that any K-equivariant condition on the manifold must reduce to a one-variable condition. It is the underlying reason why Calabi's construction [7] of extremal Kähler metrics on Hirzebruch surfaces works, a construction which gave birth to the Calabi ansatz which applies in many more situations. It was also the method Koiso and Sakane [22] used to produce the first examples of non-homogeneous compact Kähler–Einstein manifolds with positive curvature.

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Though these initial examples are a bit old, very recent works highlight how these manifolds are still very useful in complex geometry. Let us simply illustrate this with one example, where cohomogeneity one manifolds appear under the guise of *two-orbits manifolds* (under a complex Lie group). A well studied and long-standing conjecture attributed to Iskovskikh stated that Picard rank one projective manifolds should have a semistable tangent bundle (in the sense of Mumford–Takemoto). Kanemitsu [19] disproved this conjecture by studying the Picard rank one, two-orbits manifolds whose classification was obtained by Pasquier [26]. It was actually not the first appearance of these manifolds in Kähler geometry to disprove a conjecture since the author proved in [10] that they provided infinitely many counterexamples to a shorter-lived conjecture of Odaka and Okada [25] stating that all Picard rank one Fano manifolds should be K-semistable. We must inform the reader here that the conjecture of Odaka and Okada was disproved as well by Fujita [14] with two counterexamples.

In the present note, we will not disprove any conjecture but confirm the Yau–Tian–Donaldson conjecture for projective cohomogeneity one manifolds.

Theorem 1.1 *On a projective cohomogeneity one manifold, a Kähler class admits a constant scalar curvature Kähler metric if and only if it is K-stable with respect to special equivariant test configurations. The latter amounts to a single combinatorial condition checkable in practice.*

The content of the note is as follows. In Sect. 2 we explain how projective cohomogeneity one manifolds coincide with (non-singular) rank one spherical varieties, briefly recall their classification, then recall some of the results in [12] for the special case of rank one spherical varieties. Section 3 is devoted to the proof of our main theorem, and of the corresponding K-stability statement which holds for singular varieties as well. In the remaining section, we illustrate the result on some examples. It appears that, due to various different hypotheses in papers dealing with cohomogeneity one manifolds, a common belief is that they admit extremal Kähler metrics in all Kähler classes (see e.g. [8, 18]). We thus exhibit an example of cohomogeneity one projective manifold which admits both Kähler classes with cscK metrics and Kähler classes with no extremal Kähler metrics. We then answer a question of Kanemitsu on the existence of Kähler–Einstein metrics on non-horospherical Picard rank one manifolds, then study two related Picard rank two cohomogeneity one manifolds and show that they are strong Calabi dream manifolds in the sense of [24].

2 Recollections

2.1 Cohomogeneity One Manifolds and Spherical Varieties

Let X be a projective complex manifold, equipped with a holomorphic action of a real connected compact Lie group K. The manifold X is of *cohomogeneity one* if there exists at least one orbit of K of real codimension one. It then follows from [17, Sect. 2] that the manifold is almost-homogeneous under the action of the complexification $G := K^{\mathbb{C}}$, that is, G acts with an open orbit $G \cdot x$ for some $x \in X$. We denote by H the stabilizer of such a point, and identify G/H with $G \cdot x \subset X$. Furthermore, the complement $X \setminus G/H$ consists of one or two orbits, on which K acts transitively.

If there are two orbits in the complement, they are disconnected and the manifold is G-equivariantly birational to a G-homogeneous $\mathbb{P}^1$-bundle over a generalized flag manifold for G [17, Proposition 3.1]. Such manifolds are, from a different point of view, called rank one horospherical varieties. They belong to the large class of spherical varieties, well-studied from the algebraic point of view. In fact, in the case where there is only one orbit in the complement, the manifold is a rank one spherical manifold as well [9, Corollary 2.4]. We now turn to rank one spherical varieties in general.

2.2 On Rank One Spherical Varieties

Let G denote a connected complex reductive group, and fix a Borel subgroup B of G and a maximal torus T in B. We let $\mathfrak{X}(T)$ denote the lattice of characters of T. We denote by R^+ the set of positive roots of G and by $2\varpi = \sum_{\alpha \in R^+} \alpha$ the sum of its positive roots.

A spherical subgroup of G is a subgroup H such that BH is open in G. The homogeneous space G/H is then called spherical. The rank of a spherical homogeneous space is the rank of its weight lattice M, defined as the set of weights of B-semi-invariant rational functions on G/H.

Spherical homogeneous spaces of rank one are completely classified up to parabolic induction, by the work of Akhiezer [1]. More precisely, there is a finite list of families of primitive cases (G, H) (we only list the groups G and H up to isogeny, there can be one or two corresponding homogeneous spaces G/H depending on the couple (G, H)):

- the reductive symmetric spaces of rank one
  - $(\mathrm{SO}_{m+1}, S(\mathrm{O}_1 \times \mathrm{O}_m))$ for $m \geq 1$,
  - $(\mathrm{SL}_{m+1}, S(\mathrm{GL}_1 \times \mathrm{GL}_m))$ for $m \geq 2$,
  - $(\mathrm{Sp}_{2m}, \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$ for $n \geq 3$,
  - (F_4, SO_9),
- four other affine homogeneous spaces corresponding to the couples (G_2, SL_3), (SO_7, G_2),
- and three non affine families described precisely in [1, p. 68, Examples].

An arbitrary spherical homogeneous space of rank one G/H is then obtained from the primitive cases as follows. There exists a parabolic subgroup P of G, and a reductive quotient $\tilde{G}$ of P such that G/H is the quotient of $G \times \tilde{G}/\tilde{H}$ where P acts diagonally and $\tilde{G}/\tilde{H}$ is in the list of primitive spherical homogeneous spaces of rank one.

As already highlighted in the discussion of cohomogeneity one manifolds, there are two very different types of rank one spherical homogeneous spaces, according to whether $\tilde{G} = \mathbb{C}^*$ or $\tilde{G}$ is semisimple. In the case when $\tilde{G} = \mathbb{C}^*$, the resulting homogeneous space G/H is called a rank one horospherical homogeneous space. Then the group of G-equivariant automorphisms of G/H is of dimension one. Otherwise, the group of G-equivariant automorphisms of G/H is finite.

Spherical varieties of rank one are the normal G-equivariant embeddings of rank one spherical homogeneous spaces G/H. They are classified by colored fans [21] in $N \otimes \mathbb{R}$, where $N = \mathrm{Hom}(M, \mathbb{Z})$. For primitive rank one spherical homogeneous spaces, there is a unique G-equivariant projective embedding, described in details as well (for most cases) in [1]. The corresponding fan is without colors and consists either of the toric fan of $\mathbb{P}^1$ (if $G = \mathbb{C}^*$), or consists of a single one-dimensional cone and its zero dimensional face $\{0\}$. If X is a projective horospherical rank one spherical variety, its colored fan consists again of the toric fan of $\mathbb{P}^1$, but now each of the two one-dimensional cones can carry colors (and if they do, the corresponding added G-orbits are not of codimension one). If X is a projective non-horospherical rank one spherical variety, the colored fan again consists of $\{0\}$ and a single one-dimensional cone, which can now be colored (again, in this case, the added G-orbit is not of codimension one). If X is a non-horospherical projective rank one spherical variety, the generator of M which evaluates negatively on the colored cone is called the *spherical root* of G/H.

2.3 On Uniform K-Stability

Our references for this section are [4, 15]. We recall the main notions for the reader's convenience. Let G be a complex reductive group, and let (X, L) be a G-polarized variety. A (normal, ample) *G-equivariant test configuration* for (X, L) consists of the data of a normal $(G \times \mathbb{C}^*)$-variety $\hat{X}$, a $(G \times \mathbb{C}^*)$-linearized ample line bundle $\hat{L}$ on $\hat{X}$, and a $\mathbb{C}^*$-equivariant flat morphism $\pi : \hat{X} \to \mathbb{C}$ whose fiber (X_1, L_1) over 1 is G-equivariantly isomorphic to (X, L^r) for some $r \in \mathbb{Z}_{>0}$.

For $k \in \mathbb{N}$, let d_k denote the dimension $\dim H^0(X_0, L_0^k)$, let $\lambda_{1,k}, \dots, \lambda_{d_k,k}$ denote the weights of the $\mathbb{C}^*$-action on $H^0(X_0, L_0^k)$ induced by the action of $\mathbb{C}^*$ on $\hat{X}$, which stabilizes the central fiber X_0. Let w_k denote the sum of the $\lambda_{i,k}$. The quotient $\frac{w_k}{kd_k}$ admits an expansion in powers of k at infinity:

$$\frac{w_k}{kd_k} = F_0 + F_1 k^{-1} + o(k^{-1}).$$

The *non-Archimedean J-functional* of a test configuration $(\hat{X}, \hat{L})$ is

$$J^{NA}(\hat{X}, \hat{L}) := \sup\{\lambda_{i,k}/k \mid k \in \mathbb{Z}_{>0}, 1 \le i \le d_k\} - F_0$$

and the *non-Archimedean Mabuchi functional* is defined by

$$M^{NA}(\hat{X}, \hat{L}) := -F_1$$

on test configurations with a reduced central fiber, and vary linearly with base changes of the form $z \mapsto z^m$.

The G-polarized variety (X, L) is *K-stable with respect to G-equivariant special test configurations* if $M^{NA}(\hat{X}, \hat{L}) \geq 0$ for all test configurations with a normal central fiber X_0, with equality if and only if $\hat{X}$ is G-equivariantly isomorphic to $X \times \mathbb{C}$.

A G-equivariant test configuration $(\hat{X}, \hat{L})$ may be twisted by a one-parameter subgroup $\mu : \mathbb{C}^* \to \mathrm{Aut}^G(X)$ of the group of automorphisms of X commuting with the action of G. This amounts to keeping the same total space, but changing the action of $\mathbb{C}^*$ using the one-parameter subgroup μ. More explicitly, over $\mathbb{C}^*$, $\hat{X}$ is $G \times \mathbb{C}^*$-equivariantly isomorphic to $X \times \mathbb{C}^*$, and one may consider the action of $\mathbb{C}^*$ given by $t \cdot (x, s) = (\mu(t) \cdot x, s)$. This action turns out to extend to $\hat{X}$, and the resulting data of $\hat{X}, \hat{L}$ equipped with this new action of $\mathbb{C}^*$ defines a new test configuration for (X, L). One can also make sense of twists by rational multiples of one-parameter subgroups *via* base change. The G-polarized variety $X, L)$ is *G-uniformly K-stable* if there exists an $\epsilon > 0$ such that $M^{NA}(\hat{X}, \hat{L}) \geq \epsilon \inf\{J^{NA}(\text{twist of } (\hat{X}, \hat{L}))\}$ for all test configurations.

2.4 On K-Stability of Spherical Varieties

Let X be a rank one G-spherical variety, with spherical lattice M. Let $N = \mathrm{Hom}(M, \mathbb{Z})$. If it is not horospherical, denote by σ its spherical root, and let $\sigma^* \in N$ be the dual element. If it is horospherical, choose any generator σ of M, and let again $\sigma^* \in N$ be the dual element.

Let L be an ample line bundle on X, with moment polytope Δ_+. Recall that the moment polytope is the closure of the set of all $\lambda/k \in \mathfrak{X}(T) \otimes \mathbb{R}$, where λ run over the weights of B-stable lines in $H^0(X, L^{\otimes k})$. It lies in an affine line in $\mathfrak{X}(T) \otimes \mathbb{R}$, with direction $M \otimes \mathbb{R}$. Choose an element χ of Δ_+, then there exists $s_- < s_+ \in \mathbb{Q}$ such that

$$\Delta_+ = \{\chi + t\sigma \mid t \in [s_-, s_+]\}.$$

Let R_X^+ denote the set of positive roots of G which do not vanish identically on Δ_+. Set for $t \in \mathbb{R}$,

$$P(t) := \prod_{\alpha \in R_X^+} \frac{\langle \alpha, \chi + t\sigma \rangle}{\langle \alpha, \varpi \rangle}, \qquad Q(t) := \left(\sum_{\alpha \in R_X^+} \frac{\langle \alpha, \varpi \rangle}{\langle \alpha, \chi + t\sigma \rangle} \right) P(t).$$

Finally, for any continuous function g on $[s_-, s_+]$, set

$$\mathcal{L}(g) = g(s_-)P(s_-) + g(s_+)P(s_+) - \int_{s_-}^{s_+} 2g(t)(aP(t) - Q(t))\, dt$$

where a is such that $\mathcal{L}(1) = 0$, and

$$\mathcal{J}(g) = \int_{s_-}^{s_+} (g(t) - \inf g) P(t)\, dt \, .$$

The constant a may be explicitly computed as

$$a = \frac{P(s_-) + P(s_+) + 2\int_{s_-}^{s_+} Q(t)dt}{2\int_{s_-}^{s_+} P(t)dt}$$

Note that the moment polytope lies in the positive Weyl chamber of G, that is, all positive roots evaluate non-negatively on elements of Δ_+. As a consequence, P and Q are positive on $]s_-, s_+[$, and $\mathcal{J}(g) = 0$ if and only if $g \equiv 0$.

The following criteria for K-stability of (X, L) follow from [12]. Note that we switch here from concave to convex functions to simplify notations.

Theorem 2.1

1. *A polarized rank one* horospherical *variety* (X, L) *is* G*-uniformly K-stable if and only if there exists* $\varepsilon > 0$ *such that*

$$\mathcal{L}(g) \geq \varepsilon \inf_{l \in \mathbb{R}^*} \mathcal{J}(g + l)$$

for all rational piecewise linear convex functions $g : [s_-, s_+] \to \mathbb{R}$.

2. *A polarized rank one spherical variety* (X, L) *which is* not horospherical *is* G*-uniformly K-stable if and only if there exists* $\varepsilon > 0$ *such that*

$$\mathcal{L}(g) \geq \varepsilon \mathcal{J}(g)$$

for all non-decreasing rational piecewise linear convex functions $g : [s_-, s_+] \to \mathbb{R}$.

3. *A polarized rank one* horospherical *variety* (X, L) *is K-stable with respect to* G*-equivariant special test configurations if*

$$\mathcal{L}(g) = 0$$

for all affine functions g *on* $[s_-, s_+]$.

4. *A polarized rank one spherical variety* (X, L) *which is* not horospherical *is K-stable with respect to* G*-equivariant special test configurations if*

$$\mathcal{L}(g) > 0$$

for all affine strictly increasing functions g *on* $[s_-, s_+]$.

Remark 2.2 The proof of the above theorem is detailed in several steps in [12]. It relies on translating all the data of test configurations and non-Archimedean functionals into combinatorial data, thanks to the general theory of spherical varieties.

More precisely, test configurations are classified by piecewise linear functions on the moment polytope of the spherical variety (X, L) (with some conditions on the slopes related to spherical roots), with affine functions corresponding to test configurations with a normal central fiber up to base change. This classification is obtained by seeing the moment polytope of the (compactified) test configuration $(\hat{X}, \hat{L})$ as cut out from the cylinder over the moment polytope of (X, L), an image originally used by Donaldson for toric varieties. The expression of the non-Archimedean functionals are then obtained through the relation between the representations given by spaces of sections of a line bundles over a spherical variety and the moment polytope.

Remark 2.3 Observe that $\mathcal{L}$ is linear and $\mathcal{L}(1) = 0$. As a consequence, since $[s_-, s_+]$ is one-dimensional, if $\mathcal{L}(\mathrm{id}) = 0$ for the identity function $\mathrm{id} : [s_-, s_+] \to \mathbb{R}, s \mapsto s$, then $\mathcal{L}(g) = 0$ for all affine functions on $[s_-, s_+]$. Similarly, if $\mathcal{L}(\mathrm{id}) > 0$, then $\mathcal{L}(g) > 0$ for all affine strictly increasing functions on $[s_-, s_+]$. Hence there is actually only one condition to check in order to check K-stability with respect to G-equivariant test configurations for a polarized rank one spherical variety. Furthermore, to avoid computing the contant a independently, one can note that

$$\left(2\int_{s_-}^{s_+} P(t)dt\right)\mathcal{L}(\mathrm{id}) = \left(2\int_{s_-}^{s_+} P(t)dt\right)\left(s_-P(s_-) + s_+P(s_+) + 2\int_{s_-}^{s_+} tQ(t)dt\right)$$
$$- \left(2\int_{s_-}^{s_+} 2tP(t)dt\right)\left(P(s_-) + P(s_+) + 2\int_{s_-}^{s_+} Q(t)dt\right)$$

where the multiplicative constant $2\int_{s_-}^{s_+} P(t)dt$ is positive, so it is enough to check vanishing or positivity of the above quantity in order to check K-stability with respect to G-equivariant special test configurations.

3 Uniform K-Stability of Rank One Spherical Varieties

In this section we will prove Theorem 1.1 as a consequence of the following K-stability result applying to singular varieties as well.

Theorem 3.1 *A polarized rank 1 G-spherical variety is G-uniformly K-stable if and only if it is K-stable with respect to G-equivariant special test configurations.*

Let us first show how it proves the Yau–Tian–Donaldson conjecture for cohomogeneity one manifolds.

Proof of Theorem 1.1 It suffices to work on rational Kähler classes since the extremal cone is open in the Kähler cone [23], and the Kähler cone coincides with the cone of ample real line bundles on spherical manifolds since these manifolds are Mori dream spaces [5, 16], hence rational Kähler classes are dense in the Kähler cone. One of the direction is known: existence of cscK metrics implies K-(poly)stability [3], hence in particular K-stability with respect to special equivariant test configurations. For the other direction, it suffices to apply Theorem 3.1 together with Odaka's appendix to [12], which shows that for spherical manifolds, G-uniform K-stability implies the existence of cscK metrics. □

The result is of course more precise in view of Theorem 2.1. It shows first that for rank one G-horospherical varieties, G-uniform K-stability is equivalent to the vanishing of the Futaki invariant on the (at most one-dimensional) center of the group of automorphism. Second, if the variety is not horospherical, it admits a unique G-equivariant special test configuration, and it suffices to check that its Donaldson–Futaki invariant is positive.

In the course of the proof, we will use the following remarkable properties for the sign of $aP - Q$. Recall first that by definition of a moment polytope, the polynomials P and Q are positive on $]s_-, s_+[$.

Lemma 3.2 *Assume that $P(s_\pm) = 0$, then $(aP - Q)(t)$ is negative for $t \in [s_-, s_+]$ close to $s_\pm$.*

Proof Let $V_\pm \subset R_X^+$ be the subset of roots $\alpha \in R_X^+$ such that $\langle \alpha, \chi + s_\pm \sigma \rangle = 0$. If $V_\pm$ is not empty, then the polynomial aP vanishes to the order exactly $\mathrm{Card}(V_\pm)$ at $s_\pm$, while the polynomial Q vanishes to the order exactly $\mathrm{Card}(V_\pm) - 1$ at $s_\pm$. It follows that in the same situation, since P and Q are positive on $]s_-, s_+[$, $(aP - Q)(t)$ is negative when $t \in]s_-, s_+[$ is close enough to $s_\pm$. □

Lemma 3.3 *The locus where $aP - Q$ is non-negative on Δ is $[t_-, t_+]$ for some $t_\pm \in [s_-, s_+]$.*

Proof Since P is positive on $[s_-, s_+]$, $aP - Q$ is of the same sign as

$$a - \sum_{\alpha \in R_X^+} \frac{\langle \alpha, \varpi \rangle}{\langle \alpha, \chi + t\sigma \rangle}$$

on $[s_-, s_+]$. Since the reciprocal of an affine function on $\mathbb{R}$ is convex on the locus where this affine function is positive, the above function is concave on $[s_-, s_+]$. It follows that its non-negative locus is a segment in $[s_-, s_+]$. □

***Proof of Theorem* 3.1** In order to show the main result by contradiction, we assume that (X, L) is a polarized rank one G-spherical variety which is K-stable with respect to G-equivariant special test configurations but not G-uniformly K-stable. For a convex function $g : [s_-, s_+] \to \mathbb{R}$, let us denote by $\|g\|$ the quantity $\inf_{l \in \mathbb{R}^*} \mathcal{I}(g + l)$ if X is horospherical, and $\mathcal{I}(g)$ if not. By Theorem 2.1, since (X, L) is not G-uniformly K-stable, one can find a sequence (f_n) of rational piecewise linear convex functions from $[s_-, s_+]$ to $\mathbb{R}$ such that for all n, $\|f_n\| = 1$ and the sequence $(\mathcal{L}(f_n))$ converges to a limit $l \leq 0$. Note that if the limit can be taken to be strictly negative, then the sequence can be assumed constant. Further note that, if X is not horospherical, the f_n can and are assumed to be non decreasing.

Let us first modify the sequence a bit. Since both $\mathcal{L}$ and $\|\cdot\|$ are invariant under addition of a constant, we may assume that all the functions in the sequence satisfy $\inf f_n = 0$. Fix some s_0 in $]t_-, t_+[$, where t_- and t_+ are provided by Lemma 3.3. If X is horospherical, we can further assume that $0 = \inf f_n = f_n(s_0)$, by adding to f_n one of its subdifferential at s_0. This does not change $\|f_n\|$ by definition, and it does not change $\mathcal{L}(f_n)$ by the assumption that (X, L) is K-stable with respect to G-equivariant special test configurations. If X is not horospherical, then since the f_n are non decreasing, the infimum is attained at s_-.

Under these modifications, the sequence $(\int f_n P)$ is bounded in $\mathbb{R}$. If X is not horospherical, this is immediate since $\int f_n P = \|f_n\| = 1$. If X is horospherical, we prove it by contradiction. Assume that there is a subsequence of (f_n) (still denoted by (f_n) for simplicity) such that $\int f_n P \to +\infty$. Consider the functions $g_n = \frac{f_n}{\int f_n P}$. Then $\|g_n\| \to 0$ while $\int g_n P = 1$. By the pre-compactness result in [12, Proposition 7.2], the sequence (g_n) converges (up to subsequence again) to a function g_∞ on $]s_-, s_+[$, and the convergence is uniform on all compact subsets. The latter ensures that $\|g_n\|$ converges to $\|g_\infty\|$, which is thus equal to zero. This is possible only if g_∞ is affine. Finally, since $0 = \inf g_n = g_n(s_0)$, this implies that g_∞ is the zero function. This is in contradiction with the convergence $\lim \int g_n P = 1$.[1]

Now we can apply the pre-compactness result [12, Proposition 7.2] to the sequence (f_n) itself. Replacing (f_n) by a subsequence, we can and do assume that (f_n) converges to a convex function f_∞ on $]s_-, s_+[$ and the convergence is uniform on compact subsets.

We want to show that, $\mathcal{L}(f_\infty)$ is well-defined and less than the limit l of $\mathcal{L}(f_n)$. Let us first isolate the negative contribution in $\mathcal{L}(f)$ for an arbitrary non-negative function $f : [s_-, s_+] \to \mathbb{R} \cup \{+\infty\}$ which takes finite values where P is positive and which is integrable with respect to $(aP(t) - Q(t))\,dt$. By Lemma 3.3, there exists $t_- < t_+$ in $[s_-, s_+]$ such that $aP - Q$ is non-negative exactly on $[t_-, t_+]$. It follows that $\int_{t_-}^{t_+} 2f(t)(aP(t) - Q(t))\,dt$ and $\mathcal{L}(f) + \int_{t_-}^{t_+} 2f(t)(aP(t) - Q(t))\,dt$ are both non-negative, for any non-negative function f. We claim that the first expression above is well defined and finite for $f = f_\infty$. If P is positive on $[t_-, t_+]$, then since $\int f_\infty P = 1$, the claim holds. Lemma 3.2 shows that, if $P(s_-) = 0$ then $s_- < t_-$ and if $P(s_+) = 0$, then $t_+ < s_+$. As a consequence, it is actually always the case that P is strictly positive on $[t_-, t_+]$ since P is positive on $]s_-, s_+[$.

Since we assumed $\lim \mathcal{L}(f_n) \leq 0$, and the negative contribution in the decomposition of $\mathcal{L}(f_n)$ above converges to $-\int_{t_-}^{t_+} 2f_\infty(t)(aP(t) - Q(t))\,dt$, the positive contribution of $\mathcal{L}(f_n)$ must converge as well. Since the positive contribution is the sum of positive terms

$$f_n(s_-)P(s_-) + f_n(s_+)P(s_+) - \int_{s_-}^{t_-} 2f_n(t)(aP(t) - Q(t))\,dt - \int_{t_+}^{s_+} 2f_n(t)(aP(t) - Q(t))\,dt$$

[1] This shortcut is actually incorrect, but may be completed into a full argument, see [12].

each of these terms must be bounded. This implies that f_∞ is integrable with respect to $(aP(t) - Q(t))\,dt$ and that $\lim_{t\to s_\pm} f_\infty(t)$ is finite when $P(s_\pm)$ is non-zero. By a slight abuse of notations, we let $f_\infty : [s_-, s_+] \to \mathbb{R} \cup \{+\infty\}$ be the unique lower semi-continuous extension of f_∞. The last part of the penultimate sentence shows that $f_\infty(s_\pm)$ is finite whenever $P(s_\pm)$ is, so $\mathcal{L}(f_\infty)$ is well defined and $\mathcal{L}(f_\infty) \leq l$.

Consider the affine function

$$h(t) = f_\infty(t_-) + \frac{t - t_-}{t_+ - t_-}(f_\infty(t_+) - f_\infty(t_-)).$$

Note that the values $f_\infty(t_\pm)$ are finite by the discussion above. Convexity of f_∞ implies $h \leq f_\infty$ on $[s_-, t_-] \cup [t_+, s_+]$ and $h \geq f_\infty$ on $[t_-, t_+]$. Thus both the positive and negative contribution in $\mathcal{L}(h)$ are lower than that in $\mathcal{L}(f_\infty)$, hence

$$0 \leq \mathcal{L}(h) \leq \mathcal{L}(f_\infty) \leq l \leq 0.$$

In particular, we have shown that K-stability with respect to G-equivariant special test configurations implies G-equivariant K-semistability.

To conclude the proof it remains to obtain a contradiction with the initial definition of the sequence (f_n). This final argument depends on the nature of X. If X is not horospherical then all the functions f_n are non decreasing, hence f_∞ and h as well. If the slope of h is strictly positive, then $\mathcal{L}(h) > 0$ by assumption which provides the contradiction. Else h is constant. Since f_∞ is non decreasing, f_∞ is constant on $[s_-, t_+]$. All f_n satisfy $\inf f_n = f_n(s_-) = 0$, so $f_\infty(s_-) = 0$. But then either $f_\infty \equiv 0$, which contradicts $\|f_\infty\| = 1$, or $\mathcal{L}(f_\infty) > 0$, which is another contradiction.

We now conclude the horospherical case. Assume first that f_∞ is affine. All the functions f_n satisfy $f_n(s_0) = \inf f_n = 0$, hence f_∞ as well. Since $s_- < s_0 < s_+$, this shows that f_∞ is the zero function, a contradiction with $\int f_\infty P = 1$. Assume now that f_∞ is not affine, hence that $f_\infty \neq h$. Since $aP - Q$ is a non-zero polynomial, by considering the positive and negative contribution in $\mathcal{L}$ as before, we see that $\mathcal{L}(f_\infty) > \mathcal{L}(h) = 0$. This is the final contradiction. □

4 Examples

4.1 An Example of Kähler Class with No Extremal Kähler Metrics

We will here consider an example initially encountered in [2]. There, we considered as an ingredient of the proof the existence of Kähler–Einstein metrics on some blow-down of the G_2-stable divisors in the wonderful compactification of G_2/SO_4. Such varieties are rank one spherical (horosymmetric) varieties, Fano with Picard rank one, and one of these does not admit (singular) Kähler–Einstein metrics. If we go back to the corresponding G_2-stable divisor in the wonderful compactification of G_2/SO_4, which is smooth, this should provide an example of cohomogeneity one manifold and Kähler classes on it with no extremal Kähler metrics. We verify this in the following paragraphs.

4.1.1 Recollection on the Group G_2

We consider the exceptional group G_2 with a fixed choice of Borel subgroup B and maximal torus T, and an ordering of simple roots as in Bourbaki's numbering, so that α_1 is the short root and α_2 is the long root.

Up to scaling, the Weyl group invariant scalar product on $\mathfrak{X}(T)$ satisfies $\langle\alpha_1, \alpha_1\rangle = 2$, $\langle\alpha_1, \alpha_2\rangle = -3$ and $\langle\alpha_2, \alpha_2\rangle = 6$. The fundamental weight for α_1 is $2\alpha_1 + \alpha_2$ and the fundamental weight for α_2 is $3\alpha_1 + 2\alpha_2$. The positive roots and their scalar product with an arbitrary element $x_1\alpha_1 + x_2\alpha_2$ read

$$\begin{aligned}
\langle\alpha_1, x_1\alpha_1 + x_2\alpha_2\rangle &= 2x_1 - 3x_2 \\
\langle\alpha_2, x_1\alpha_1 + x_2\alpha_2\rangle &= 3(-x_1 + 2x_2) \\
\langle\alpha_1 + \alpha_2, x_1\alpha_1 + x_2\alpha_2\rangle &= -x_1 + 3x_2 \\
\langle 2\alpha_1 + \alpha_2, x_1\alpha_1 + x_2\alpha_2\rangle &= x_1 \\
\langle 3\alpha_1 + \alpha_2, x_1\alpha_1 + x_2\alpha_2\rangle &= 3(x_1 - x_2) \\
\langle 3\alpha_1 + 2\alpha_2, x_1\alpha_1 + x_2\alpha_2\rangle &= 3x_2
\end{aligned}$$

The half-sum of positive roots is $\varpi = 5\alpha_1 + 3\alpha_2$.

4.1.2 The Facet of the Wonderful Compactification of G_2/SO_4 and Its Kähler Classes

Let P_1 be the parabolic subgroup of G_2 containing B such that $-\alpha_1$ is not a root of P_1. Its Levi factor has adjoint form PSL_2. From any P_1-variety Y, one may build a G_2 variety X which is a homogeneous bundle over G_2/P_1 with fiber Y, simply by considering the quotient of $Y \times G_2$ by the P_1-action $p \cdot (y, g) = (p \cdot y, gp^{-1})$. Such a construction is sometimes (and in the following) referred to as parabolic induction and it is particularly relevant for the geometry of horosymmetric varieties [11]. Let X be the (non-singular) horosymmetric variety obtained by parabolic induction from the P_1-variety $\mathbb{P}^2$, considered as the projectivization of the space of equations of quadrics in $\mathbb{P}^1$ on which P_1 acts via the natural action of PSL_2 on $\mathbb{P}^1$. Its open orbit G_2/H is the corresponding parabolic induction from $\mathrm{PSL}_2/\mathrm{PSO}_2$. Note that X is the wonderful compactification of G_2/H and that since it is a parabolic induction, $\mathrm{Aut}^0(X) = G_2$ [27, Proposition 3.4.1].

The variety X is a Picard rank two horosymmetric variety. Its spherical root is $\sigma = 2\alpha_2$, and its spherical lattice M is the lattice generated by σ. We can describe its

Kähler cone by using [6], recalled for the special case of horosymmetric varieties in [11, 13]. Any Kähler class on a projective spherical variety is the class of a real divisor. The vector space of classes of real divisor is generated by the classes of all prime B-stable divisors modulo the relations imposed by B-semi-invariant rational functions. Here the prime B-stable divisors are the closure E of the unique codimension one G_2-orbit (obtained by parabolic induction from the space of degenerate quadrics in $\mathbb{P}^1$, that is, double points), and the closures of the two colors D_1 and D_2 in G/H, where D_1 is the only codimension one B-orbit not stable under P_1 (the codimension one P_2-orbit obtained by moving the unique color in $\mathbb{P}^2$), and D_2 is the only codimension one B-orbit not stable under P_2. Note that D_1 is also the pull-back of the ample generator of the Picard group of G/P_1. Since the spherical rank of G/H is one, there is a single relation to consider, which amounts to $2D_2 - E - 6D_1 = 0$, since the image of D_1 by the color map is the restriction $-6\sigma^*$ of the coroot $\alpha_1^\vee$ to $M \otimes \mathbb{R}$, and the image by the color map of D_2 is the restriction $2\sigma^*$ of the positive restricted coroot $\frac{\alpha_2^\vee}{2}$ (the image of E is the primitive generator of the valuation cone $-\sigma^*$).

In view of the above presentation, we can write a real divisor as $sE + s_1 D_1$. Since K-stability is invariant under scaling of the Kähler class, we may as well assume $s_1 = 6$. Brion's ampleness criterion for the real line bundle $sE + 6D_1$ translates simply to the condition $0 < s < 1$, and the moment polytope is then

$$\Delta_+(s) = 6(2\alpha_1 + \alpha_2) + \{2t\alpha_2 \mid 0 \leq t \leq s\}.$$

4.1.3 K-Stability Condition

We have

$$P(t) = \frac{288}{5} t(1 - t^2)(9 - t^2)$$

and

$$\begin{aligned} Q(t) &= \left(\frac{1}{6(1-t)} + \frac{1}{4t} + \frac{2}{3(1+t)} + \frac{5}{12} + \frac{1}{3-t} + \frac{3}{2(3+t)} \right) P(t) \\ &= \frac{24}{5}(5t^5 + 15t^4 - 150t^3 - 90t^2 + 225t + 27). \end{aligned}$$

In view of Remark 2.3, we want to know when

$$\left(sP(s) + 2\int_0^s tQ(t)\,dt \right) \int_0^s P(t)\,dt - \left(P(s) + 2\int_0^s Q(t)\,dt \right) \int_0^s tP(t)\,dt > 0 \tag{1}$$

The left hand side above is the polynomial

$$R(s) := \frac{1152}{175} s^4 (11s^8 + 20s^7 - 348s^6 - 240s^5 + 3123s^4 + 1260s^3 - 9072s^2 + 5103).$$

One can plug in specific values to check that

$$R\left(\frac{1}{2}\right) = \frac{7315083}{5600} > 0$$

and

$$R\left(\frac{98}{100}\right) = -\frac{12097691278181901659043}{476837158203125000000} < 0.$$

In other words, there are Kähler classes on X with cscK metrics and Kähler classes with no cscK metrics. Since $\mathrm{Aut}^0(X) = G_2$ is semisimple, a Kähler class with no cscK metrics does not admit any extremal Kähler metric either. Using numerical approximation, one can be more precise: the Kähler class $sE + 6D_1$ contains a cscK metric if and only if $s < s_0$, where $s_0 \simeq 0.97202$.

4.2 Strong Calabi Dream Manifolds of Cohomogeneity One, and an Answer to a Question of Kanemitsu

We will now provide examples of cohomogeneity one manifolds which are not horospherical and are strong Calabi Dream manifolds in the sense of [24]. We take a small detour and choose slightly complicated manifolds to answer along the way a question of Kanemitsu [19, Remark 4.1]: when does there exist a Kähler–Einstein metric on cohomogeneity one manifolds with Picard rank one? The answer was already known for most of such manifolds, namely for homogeneous ones (which admit Kähler–Einstein metrics) and for horospherical, non-homogeneous ones (which have non-reductive automorphism group [26] hence no Kähler–Einstein metrics by Matsushima's obstruction).

By Pasquier's classification [26], there are two Picard rank one, non-horospherical cohomogeneity one manifolds, one acted upon by $\mathrm{PSL}_2 \times G_2$, that we will denote by $\mathbb{X}_1$, and one acted upon by F_4, that we will denote by $\mathbb{X}_2$. Both are two orbit varieties with semisimple automorphism group. More precisely, $\mathrm{Aut}^0(\mathbb{X}_1) = \mathrm{PSL}_2 \times G_2$ and $\mathrm{Aut}^0(\mathbb{X}_2) = F_4$. We will first prove:

Theorem 4.1 *There exist Kähler–Einstein metrics on $\mathbb{X}_1$ and $\mathbb{X}_2$.*

This provides the missing cases in Kanemitsu's question. Note that, after this preprint appeared, Kanemitsu also answered his own question in [20]. As a corollary, we also recover a result of [19].

Corollary 4.2 *The tangent bundles of $\mathbb{X}_1$ and $\mathbb{X}_2$ are Mumford–Takemoto stable.*

These manifolds $\mathbb{X}_i$ each admit a unique *discoloration* $\widetilde{\mathbb{X}_i}$ which is a smooth projective Picard rank two cohomogeneity one manifold which surjects equivariantly to $\mathbb{X}_i$ and where the complement of the open orbit is of codimension one. We will apply Theorem 1.1 to obtain:

Theorem 4.3 *The manifolds $\widetilde{\mathbb{X}_1}$ and $\widetilde{\mathbb{X}_2}$ admits a cscK metric in all Kähler classes. In other words, they are* Calabi dream manifolds *in the terminology of [8], and more precisely* strong Calabi dream manifolds *in the terminology of [24].*

We will in the paragraphs to follow provide the combinatorial data associated to the manifolds under study. It is rather easy since these are horosymmetric. For the discolorations, we will then determine the Kähler classes and compute the K-stability condition as in the previous example. For the Kähler–Einstein metrics, it is a bit faster to use directly the criterion in [10] since one needs only the polynomial P up to scalar, and the polynomial Q is not needed.

4.2.1 Kähler–Einstein Metrics on $\mathbb{X}_1$

Let G denote the group $\mathrm{PSL}_2 \times G_2$. We fix a choice of Borel subgroup B and of maximal torus $T \subset B$. Let α_0 denote the positive root of SL_2 and let α_1 and α_2 denote the simple roots of G_2, numbered so that α_1 is the short root (in accordance with Bourbaki's standard numbering and with the previous example). We can choose a Weyl group invariant scalar product on $\mathfrak{X}(T)$ satisfying $\langle \alpha_0, \alpha_0 \rangle = 1$ and the same scaling as in the previous example for the restriction to G_2. Of course, the root α_0 is orthogonal to α_1 and α_2.

It follows from the description of the variety $\mathbb{X}_1$ in [26] that its open orbit G/H is obtained by parabolic induction from the rank one symmetric space $\mathrm{PSL}_2 \times \mathrm{PSL}_2 \,/\, \mathrm{PSL}_2$, where the parabolic subgroup of G is the parabolic P_2 associated to the long root α_2, whose Levi factor has adjoint form $\mathrm{PSL}_2 \times \mathrm{PSL}_2$. The spherical lattice M for $\mathbb{X}_1$ is thus the lattice generated by $\alpha_0 + \alpha_1$.

Furthermore, the variety $\mathbb{X}_1$ is the unique fully colored compactification of G/H. It follows that the moment polytope Δ_+ corresponding to the anticanonical line bundle is the intersection with the positive Weyl chamber of the affine line with direction $\mathbb{R}(\alpha_0 + \alpha_1)$ passing through the sum of positive roots $\alpha_0 + 10\alpha_1 + 6\alpha_2$. If we write the moment polytope as

$$\Delta_+ = \{(1+t)\alpha_0 + (10+t)\alpha_1) + 6\alpha_2 \mid u \le t \le v\}$$

then we can determine u and v as the extreme values of t such that $\langle \alpha_i, (1+t)\alpha_0 + (10+t)\alpha_1) + 6\alpha_2 \rangle \ge 0$ for $i \in \{0, 1, 2\}$, that is, $u = -1$ and $v = 2$.

We may finally compute the K-stability condition, which is

$$\int_{-1}^{2} t(1+t)^2(2-t)(8-t)(10+t)(4+t)\,dt > 0.$$

By direct computation, the integral is equal to $\frac{120285}{56}$ hence the condition is satisfied.

4.2.2 CscK Metrics on the Discoloration $\widetilde{\mathbb{X}_1}$

The discoloration $\widetilde{\mathbb{X}_1}$ of $\mathbb{X}_1$ is obtained by the following parabolic induction procedure. Take the quotient of $\mathbb{P}^3 \times G$ by the diagonal action of the parabolic P_2, where the action on G is obvious and the action on $\mathbb{P}^3$ is induced by the action of the Levi factor of P_2 and the obvious structure of two-orbit $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-variety on $\mathbb{P}^3$ seen as the projectivization of 2 by 2 matrices. It is a homogeneous $\mathbb{P}^3$-bundle over the generalized flag manifold G/P_2.

It is a rank one horosymmetric variety with Picard rank two. We can describe its Kähler cone by using [6], recalled for the special case of horosymmetric varieties in [11, 13]. Any Kähler class on a projective spherical variety is the class of a real divisor. The vector space of classes of real divisor is generated by the classes of all prime B-stable divisors modulo the relations imposed by B-semi-invariant rational functions. Here the prime B-stable divisors are the (G-stable) exceptional divisor E, and the closures of the two colors D_{01} and D_2 in G/H, where D_{01} is the only codimension one B-orbit not stable under P_0 and P_1, and D_2 is the only codimension one B-orbit not stable under P_2. Note that D_2 is also the pull-back of the ample generator of the Picard group of G/P_2. Since the spherical rank of G/H is one, there is a single relation to consider, which amounts to $E + D_2 - 2D_1 = 0$, since the image of D_2 by the color map is the restriction of the coroot $\alpha_2^\vee$ to $M \otimes \mathbb{R}$ which coincides with the generator of the colorless ray corresponding to E, and the image by the color map of D_{01} is the only positive restricted coroot, the restriction of $\frac{1}{2}(\alpha_0^\vee + \alpha_1^\vee)$ to $M \otimes \mathbb{R}$.

The class of any real divisor is thus represented by a $s_E E + s_2 D_2$ for s_E and s_2 two real numbers. By Brion's ampleness criterion, it is a Kähler class if and only if $0 < s_E < s_2$, and the moment polytope is then

$$\Delta_+(s_E, s_2) := s_2(3\alpha_1 + 2\alpha_2) + \{t(\alpha_0 + \alpha_1) \mid 0 \leq t \leq s_E\},$$

where $3\alpha_1 + 2\alpha_2$ is to be though of as the fundamental weight of α_2 here.

We may now compute the unique condition for K-stability given by an equivariant special test configuration for these polarizations. Since a Kähler class is K-stable if and only if one of its positive multiple is, we can assume $s_2 = 1$ and write $s := s_E$ in the following, to simplify notations. The moment polytope is illustrated in Fig. 1. Let P and Q denote the polynomials

$$P(t) = \frac{t^2}{15}\left(t^4 - 10t^2 + 9\right)$$
$$Q(t) = \frac{t}{30}\left(3t^5 + 6t^4 - 90t^3 - 40t^2 + 135t + 18\right)$$

The Kähler class $sE + D_2$ is G-uniformly K-stable if and only if s satisfies the condition (1). By computing the polynomial on the left hand side, the condition is

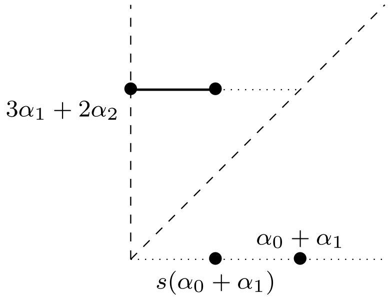

Fig. 1 Moment polytope $\Delta_+(s, 1)$

$$\frac{s^6}{132300}(9s^8 + 42s^7 - 266s^6 - 378s^5 + 2135s^4 + 1764s^3 - 5292s^2 + 2646) > 0,$$

and one can check that this condition is satisfied for all $s \in]0, 1[$.

4.2.3 Kähler–Einstein Metrics on the F_4-Variety $\mathbb{X}_2$

Let α_i denote the simple roots of F_4, ordered in accordance with Bourbaki's numbering. Up to scaling, the Weyl group invariant scalar product is such that the matrix of $\langle \alpha_i, \alpha_j \rangle$ is given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

It follows from the description of the variety $\mathbb{X}_2$ in [26] that its open orbit F_4/H is obtained by parabolic induction from the rank one symmetric space $\mathrm{Sp}_6 / \mathrm{Sp}_2 \times \mathrm{Sp}_4$, where the parabolic subgroup of F_4 is the parabolic P_1 associated to the root α_1. The spherical lattice M for $\mathbb{X}_2$ is the lattice generated by the restricted root of the symmetric space, $\beta := \alpha_2 + 2\alpha_3 + \alpha_4$.

Furthermore, the variety $\mathbb{X}_2$ is the unique fully colored compactification of F_4/H. It follows that the moment polytope corresponding to the anticanonical line bundle is the intersection with the positive Weyl chamber of the affine line with direction $\mathbb{R}\beta$ passing through $16\alpha_1 + 29\alpha_2 + 42\alpha_3 + 21\alpha_4$, the sum of positive roots minus the sum of positive roots of Sp_6 fixed by the involution defining the symmetric space. More explicitly, the moment polytope is

$$\Delta_+ = \{t\beta + 8\omega_1 \mid 0 \leq t \leq 8\}$$

where $\omega_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ is the fundamental weight for α_1.

The K-stability condition for the anticanonical line bundle thus reads

$$\int_0^8 (t-5)t^7(256-t^2)^2(64-t^2)^2\,dt > 0.$$

This condition is satisfied since the left hand side is equal to

$$\frac{3672386428957884416}{153153}.$$

4.2.4 CscK Metrics on the Discoloration $\widetilde{\mathbb{X}_2}$

The discoloration $\widetilde{\mathbb{X}_2}$ of $\mathbb{X}_2$ is obtained by the following parabolic induction procedure. Take the quotient of $\mathrm{Grass}(2,6) \times F_4$ by the diagonal action of the minimal parabolic P_1, where the action on F_4 is obvious and the action on $\mathrm{Grass}(2,6)$ is induced by the action of the Levi factor of P_1 and the structure of two-orbit Sp_6-variety on $\mathrm{Grass}(2,6)$ (this is the wonderful compactification of the symmetric space $\mathrm{Sp}_6 / \mathrm{Sp}_2 \times \mathrm{Sp}_4$).

It is a rank one horosymmetric variety with Picard rank two. Again, its Kähler cone is determined from combinatorial data using [6, 11, 13]. Here, the vector space of real divisors is the quotient of the three dimensional vector space generated by the exceptional divisor E and the closure of two colors D_1 and D_3, where D_i is the closure of the only codimension one B-orbit not stable under the minimal parabolic P_i where $i \in \{1,3\}$, by the relation $D_1 + E - 2D_3 = 0$. The relation follows from the fact that the image of D_1 under the color map is the restriction of the coroot $\alpha_1^\vee$ to $M \otimes \mathbb{R}$, which coincides with the primitive generator of the colorless ray corresponding to the F_4-orbit E, and the image by the color map of D_3 is the only positive restricted coroot, equal to (the restriction to $M \otimes \mathbb{R}$ of) $(\alpha_2 + 2\alpha_3 + \alpha_4)^\vee$, which coincides with the double of the opposite of the generator of E. Note that D_1 is the pull-back of the ample generator of the Picard group of F_4/P_1.

The class of any real divisor is thus represented, up to multiple, by some $sE + D_1$. By Brion's ampleness criterion, it is a Kähler class if and only if 0<s<1, and the moment polytope is then

$$\Delta_+ = \{t\beta + \omega_1 \mid 0 \le t \le s\}.$$

The Kähler class $sE + D_1$ is F_4-uniformly K-stable if and only if s satisfies the condition (1) where, here, the polynomials P and Q are given by

$$P(t) = \frac{1}{2^7} t^7 (4-t^2)^2 (1-t^2)^2$$
$$Q(t) = \frac{1}{2^8} t^6 (4-t^2)(1-t^2)(13t^5 + 22t^4 - 105t^3 - 110t^2 + 116t + 88).$$

It is a tedious but workable task to verify that the polynomial on the left hand side of condition (1) is positive for $s \in]0, 1[$.

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References

1. Ahiezer, D.: Equivariant completions of homogeneous algebraic varieties by homogeneous divisors. Ann. Global Anal. Geom. **1**(1), 49–78 (1983)
2. Biquard, O., Delcroix, T.: Ricci flat Kähler metrics on rank two complex symmetric spaces. J. Éc. Polytech. Math. **6**, 163–201 (2019)
3. Berman, R., Darvas, T., Lu, C.: Regularity of weak minimizers of the K-energy and applications to properness and K-stability. Ann. Sci. Éc. Norm. Supér. (4) **2**(53), 267–289 (2020)
4. Boucksom, S., Hisamoto, T., Jonsson, M.: Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs. Ann. Inst. Fourier (Grenoble) **67**(2), 743–841 (2017)
5. Brion, M., Knop, F.: Contractions and flips for varieties with group action of small complexity. J. Math. Sci. Univ. Tokyo **1**(3), 641–655 (1994)
6. Brion, M.: Groupe de Picard et nombres caractéristiques des variétés sphériques. Duke Math. J. **58**(2), 397–424 (1989)
7. Calabi, E.: Extremal Kähler metrics. In: Seminar on Differential Geometry. Annals of Mathematics Studies, vol. 102, pp. 259–290. Princeton University Press, Princeton (1982)
8. Chen, X., Cheng, J.: On the constant scalar curvature Kähler metrics, existence results (2018). arXiv:1801.00656
9. Cupit-Foutou, S.: Classification of two-orbit varieties. Comment. Math. Helv. **78**(2), 245–265 (2003)
10. Delcroix, T.: K-stability of Fano spherical varieties. Ann. Sci. Éc. Norm. Supér. (4) **3**(53), 615–662 (2020)
11. Delcroix, T.: Kähler geometry of horosymmetric varieties, and application to Mabuchi's K-energy functional. J. Reine Angew. Math. **763**, 129–199 (2020)
12. Delcroix, T.: Uniform K-stability of polarized spherical varieties (2020). arXiv:2009.06463
13. Delcroix, T., Hultgren, J.: Coupled complex Monge-Ampère equations on Fano horosymmetric manifolds (2018). arXiv:1812.07218. To appear in Journal de Mathématiques Pures et Appliquées
14. Fujita, K.: Examples of K-unstable Fano manifolds with the Picard number 1. Proc. Edinb. Math. Soc. (2) **60**(4), 881–891 (2017)
15. Hisamoto, T.: Stability and coercivity for toric polarizations (2020). arXiv:1610.07998v3
16. Hu, Y., Keel, S.: Mori dream spaces and GIT, vol. 48, pp. 331–348 (2000). Dedicated to William Fulton on the occasion of his 60th birthday
17. Huckleberry, A.T., Snow, D.M.: Almost-homogeneous Kähler manifolds with hypersurface orbits. Osaka Math. J. **19**(4), 763–786 (1982)
18. Hwang, A.D.: On existence of Kähler metrics with constant scalar curvature. Osaka J. Math. **31**(3), 561–595 (1994)
19. Kanemitsu, A.: Fano manifolds and stability of tangent bundles (2019). arXiv:1912.12617
20. Kanemitsu, A.: Kähler-einstein metrics on pasquier's two-orbit varieties. Manuscripta Math. (2021)
21. Knop, F.: The Luna-Vust theory of spherical embeddings. In: Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), pp. 225–249. Manoj Prakashan, Madras (1991)

22. Koiso, N., Sakane, Y.: Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds. In: Curvature and Topology of Riemannian Manifolds (Katata, 1985). Lecture Notes in Mathematics, vol. 1201, pp. 165–179. Springer, Berlin (1986)
23. LeBrun, C., Simanca, S.R.: On the Kähler classes of extremal metrics. In: Geometry and Global Analysis (Sendai, 1993), pp. 255–271. Tohoku University, Sendai (1993)
24. Martinez-Garcia, J.: Constant scalar curvature Kähler metrics on rational surfaces (2018). arXiv:1712.04857v2
25. Odaka, Y., Okada, T.: Birational superrigidity and slope stability of Fano manifolds. Math. Z. **275**(3–4), 1109–1119 (2013)
26. Pasquier, B.: On some smooth projective two-orbit varieties with Picard number 1. Math. Ann. **344**(4), 963–987 (2009)
27. Pezzini, G.: Automorphisms of wonderful varieties. Transform. Groups **14**(3), 677–694 (2009)

Fibrations by Affine Lines on Rational Affine Surfaces with Irreducible Boundaries

Adrien Dubouloz

Abstract We consider fibrations by affine lines on smooth affine surfaces obtained as complements of smooth rational curves B in smooth projective surfaces X defined over an algebraically closed field of characteristic zero. We observe that except for two exceptions, these surfaces $X \setminus B$ admit infinitely many families of $\mathbb{A}^1$-fibrations over the projective line with irreducible fibers and a unique singular fiber of arbitrarily large multiplicity. For $\mathbb{A}^1$-fibrations over the affine line, we give a new and essentially self-contained proof that the set of equivalence classes of such fibrations up to composition by automorphisms at the source and target is finite if and only if the self-intersection number B^2 of B in X is less than or equal to 6.

Keywords Affine surfaces · Cylinders

2000 Mathematics Subject Classification 14R25 · 14E05 · 14R05

1 Introduction

Affine surfaces whose automorphism groups act with a dense orbit with finite complement were studied by M. H. Gizatullin and V. I. Danilov in a series of seminal papers in the seventies [6, 7, 20, 21]. There, they established that except for finitely many special cases, these are the affine surfaces which admit projective completions whose boundaries are chains of smooth proper rational curves. Up to the exception $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1$, such surfaces are equivalently characterized by the property that they admit two fibrations over the affine line $\mathbb{A}^1$, whose general fibers are pairwise dis-

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tinct and isomorphic to $\mathbb{A}^1$, see [1, 10, 21]. Many of these surfaces actually admit infinitely many such $\mathbb{A}^1$-fibrations with pairwise different general fibers up to the equivalence relation given by composition by automorphisms on the source and the target. This richness of $\mathbb{A}^1$-fibrations contributes in a central way to the complexity of their automorphism groups, see e.g. [2, 3, 13, 19, 23, 24].

In this article, we consider $\mathbb{A}^1$-fibrations on the subclass consisting of affine surfaces S defined over an algebraically closed field k of characteristic zero and which admit smooth projective completions X with boundary $B = X \setminus S$ isomorphic to the projective line $\mathbb{P}^1$. A surface of this type is isomorphic either to the affine plane $\mathbb{A}^2$, or to the complement of a smooth conic in $\mathbb{P}^2$ or to the complement of an ample section of a $\mathbb{P}^1$-bundle $\pi_n : \mathbb{F}_n = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n)) \to \mathbb{P}^1$ for some $n \geq 0$. Furthermore, the famous Danilov–Gizatullin isomorphism theorem asserts that the isomorphism class of an affine surface of the form $\mathbb{F}_n \setminus B$ depends only on the self-intersection B^2 of the boundary B.

While $\mathbb{A}^2$ and the complements of smooth conics in $\mathbb{P}^2$ only admit $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$, a surface $\mathbb{F}_n \setminus B$ admits an $\mathbb{A}^1$-fibration $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ given by the restriction of the $\mathbb{P}^1$-bundle $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$. These fibrations are actually locally trivial $\mathbb{A}^1$-bundles, and one can check that their equivalence classes are in one-to-one correspondence with the orbits of the natural action of the group $\mathrm{PGL}_2(\mathrm{k})$ on the space $\mathbb{P}(H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-B^2))$ (see Remark 5). In particular, for $B^2 \geq 3$, these surfaces admit infinitely many equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{P}^1$. The geometry and equivalence classes of other families of $\mathbb{A}^1$-fibrations over $\mathbb{P}^1$ on surfaces $\mathbb{F}_n \setminus B$ has been much less studied than those of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$. Our first result, inspired by a construction due to Blanc-van Santen [4] of infinite families of pairwise non-equivalent closed embedding of the affine line in the complement of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, reads as follows:

Theorem 1 *Let $(\mathbb{F}_n, B)$ be a pair consisting of a Hirzebruch surface $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$ and an ample section B of π_n. Then for every $m \geq 4$, there exist infinite families of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique singular fiber, irreducible and of multiplicity m.*

The proof of Theorem 1 given in Sect. 3 actually provides a natural bijection between a set of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique singular fiber, irreducible and of multiplicity m, and the elements of the set-theoretic quotient of the set of closed points of a certain Zariski dense open subset of $\mathbb{P}^{m-1}$ by the algebraic action of a linear algebraic group whose general orbits are at most 2-dimensional. This construction strongly suggests that by replacing the consideration of set-theoretic quotients by, for instance, that of GIT quotients of suitable open subsets, one should be able to strengthen Theorem 1 in a form asserting the existence of algebraic families of $\mathbb{A}^1$-fibrations s $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique singular fiber, irreducible and of multiplicity m parametrized by the closed points of an algebraic variety of dimension $m - 3$. Tackling the necessary additional constructions which are needed to give a rigorous and accurate formulation of this coarse moduli viewpoint falls beyond the scope of the present article.

In a second step, we consider equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on affine surfaces $X \setminus B$. It is a well-known result of Danilov–Gizatullin [6, 7] that every such surface other than the complement of a smooth conic in $\mathbb{P}^2$ has a unique equivalence class of smooth $\mathbb{A}^1$-fibration over $\mathbb{A}^1$. As already observed by Danilov–Gizatullin again, for every $d = 1, \dots, 5$, the finiteness of the number of equivalences classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on surfaces $X \setminus B$ with $B^2 = d$ is then a consequence of the finiteness of isomorphism types of pairs (X, B) with $B^2 = d$. Over the field of complex numbers, equivalence classes of non-smooth $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$, equivalently $\mathbb{A}^1$-fibrations having non-reduced components in their degenerate fibers, have been extensively studied in [19] in a broader context, see especially Corollary 6.3.19 and Corollary 6.3.20 in *loc. cit.*. The techniques there consist in first constructing a finite-to-one correspondence between equivalence classes of such $\mathbb{A}^1$-fibrations and collections of points in a configuration space. The latter encodes the standard construction of a completion of an $\mathbb{A}^1$-fibered smooth affine surface $\pi : S \to \mathbb{A}^1$ into a $\mathbb{P}^1$-fibered smooth projective surface $\bar{\pi} : \bar{S} \to \mathbb{P}^1$ obtained by performing a suitable sequence of blow-ups of closed points starting from a Hirzerbruch surface $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$. The second step consists in describing the possible configurations and determining their respective numbers of moduli. A consequence of this extensive description is that for every $B^2 \geq 7$, the surface $X \setminus B$ admits families of pairwise non-equivalent $\mathbb{A}^1$-fibrations $X \setminus B \to \mathbb{A}^1$ parametrized by a space whose dimension is an increasing function of B^2 (see [19, Example 6.3.2]). Our second result consists of an alternative direct proof of the following theorem, based on the use of a different point of view.

Theorem 2 *Let (X, B) be a pair consisting of a smooth projective surface X and an ample smooth rational curve B on X. Then the following alternative holds:*

(a) If $B^2 \leq 6$ then $X \setminus B$ admits at most seven equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$,

(b) If $B^2 \geq 7$ then the set of equivalence classes of $\mathbb{A}^1$-fibrations $X \setminus B \to \mathbb{A}^1$ is infinite, of cardinality larger than or equal to that of the field k.

For $B^2 \leq 6$, the different equivalence classes are derived by an explicit case by case study. For $B^2 \geq 7$, our argument is based on the study of the equivalence classes of a subclass of $\mathbb{A}^1$-fibrations $\mathbb{F}_n \setminus B \to \mathbb{A}^1$ which have an irreducible component of multiplicity two inside their degenerate fiber. We show that for every $B^2 \geq 7$, the set of equivalence classes of $\mathbb{A}^1$-fibration of this type is infinite. More precisely, we actually construct explicit families of equivalences classes of $\mathbb{A}^1$-fibered smooth affine surfaces $S \to \mathbb{A}^1$ with a unique degenerate fiber, irreducible and of multiplicity two, depending algebraically on a parameter varying in an affine space of dimension $\left\lfloor \frac{B^2-5}{2} \right\rfloor$ (see Example 33) and which are all realized as restrictions of $\mathbb{A}^1$-fibrations $\mathbb{F}_n \setminus B \to \mathbb{A}^1$ on suitable Zariski open subsets. This indicates in an indirect fashion that the “number of moduli” of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on surfaces $X \setminus B$ with $B^2 = d$ is bounded from below by $\left\lfloor \frac{d-5}{2} \right\rfloor$.

The article is organized as follows. In section one, after setting some notations, we review basic properties of smooth affine surfaces completable by smooth rational curves. We then proceed in section two to the proof of Theorem 1. The third section is devoted to the proof of Theorem 2, which combines several known facts together with new results on equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ having an irreducible component of multiplicity two inside their unique degenerate fiber.

2 Preliminaries

All varieties and schemes considered are defined over a fixed algebraically closed field k of characteristic zero.

2.1 Notations and Basic Definitions

We briefly recall basic definitions on SNC divisors and standard properties of $\mathbb{A}^1$-fibrations and $\mathbb{P}^1$-fibrations which we use throughout the paper, see e.g [25, Chap. 3] for the details.

2.1.1 SNC Divisors and Rationals Trees on Smooth Surfaces

(i) Let X be a smooth projective surface. An *SNC divisor* on X is a curve $B \subset X$ with smooth irreducible components and ordinary double points only as singularities. We say that B is *SNC-minimal* if its image by any strictly birational proper morphism $\tau : X \to X'$ onto a smooth projective surface X' with exceptional locus contained in B is not an SNC divisor. A *rational tree* on X is an SNC divisor whose irreducible components are isomorphic to $\mathbb{P}^1$ and whose incidence graph is a tree. A *rational chain* is a rational tree whose incidence graph is a chain. We use the notation $B = B_0 \triangleleft B_1 \triangleleft \cdots \triangleleft B_r$ to indicate a rational chain whose irreducible components B_i are ordered in such a way that for $0 \leq i < j \leq r$, one has $B_i \cdot B_j = 1$ if $j = i + 1$ and 0 otherwise. The sequence of self-intersections $[B_0^2, \dots, B_r^2]$ is referred to as the *type* of the oriented rational chain B.

(ii) An *SNC completion* of a smooth quasi-projective surface V is a pair (X, B) consisting of a smooth projective surface X and an SNC divisor $B \subset V$ such that $X \setminus B \simeq V$. The completion is said to be SNC-minimal if B is SNC-minimal and to be smooth if B is smooth.

2.1.2 Recollection on $\mathbb{A}^1$-Fibrations and $\mathbb{P}^1$-Fibrations on Smooth Surfaces

(i) A $\mathbb{P}^1$*-fibration* on a smooth projective surface X is a morphism $\overline{\rho} : X \to C$ onto a smooth projective curve C whose generic fiber is isomorphic to the projective line over the function field of C. Every $\mathbb{P}^1$-fibration $\overline{\rho} : X \to C$ is obtained from a Zariski locally trivial $\mathbb{P}^1$-bundle over C by performing a finite sequence of blow-ups of points. In particular, every $\mathbb{P}^1$-fibration has a section and its singular fibers are supported by rational trees on X. If X is rational, then for every smooth proper rational curve F with self-intersection 0, the complete linear system $|F|$ of effective divisors on X linearly equivalent to F defines a $\mathbb{P}^1$-fibration $\overline{\rho}_{|F|} : X \to \mathbb{P}^1_k$ having F as a smooth fiber.

(ii) An $\mathbb{A}^1$*-fibration* on a smooth quasi-projective surface V is a surjective morphism $\rho : V \to A$ onto a smooth curve A whose generic fiber is isomorphic to the affine line over the function field of A. Every $\mathbb{A}^1$-fibration is the restriction of a $\mathbb{P}^1$-fibration $\overline{\rho} : X \to C$ over the smooth projective model C of A, defined on an SNC completion (X, B) of V with boundary $B = \bigcup_{c \in C \setminus A} F_c \cup H \cup \bigcup_{a \in A} G_a$ where, $F_c = \overline{\rho}^{-1}(c) \simeq \mathbb{P}^1$ for every $c \in C \setminus A$, H is a section of $\overline{\rho}$, and for every $a \in A$, G_a is a union of SNC-minimal rational subtrees of the rational tree $(\overline{\rho}^{-1}(a))_{\mathrm{red}}$. The pair (X, B) is called a relatively minimal $\mathbb{P}^1$-fibered completion of $\rho : V \to A$. If $\rho : V \to A$ is affine, every nonempty G_a is connected and has an irreducible component intersecting H, and the closure in X of every irreducible component of $\rho^{-1}(a)$ intersects G_a transversely in a unique point. In particular, every irreducible component of $\rho^{-1}(a)$ is isomorphic to $\mathbb{A}^1$ when equipped with its reduced structure. A scheme-theoretic closed fiber of $\rho : V \to A$ which is not isomorphic to $\mathbb{A}^1$ is called *degenerate*.

(iii) A smooth $\mathbb{A}^1$-fibered surface is a pair (V, π) consisting of a smooth quasi-projective surface V and an $\mathbb{A}^1$-fibration $\pi : V \to A$ onto a smooth curve A. The $\mathbb{A}^1$-fibration π is said to be of affine type if A is affine and of complete type otherwise. Two $\mathbb{A}^1$-fibered surfaces $(V, \pi : V \to A)$ and $(V', \pi' : V' \to A')$ are said to be *equivalent* if there exist an isomorphism $\Psi : V \to V'$ and an isomorphism $\psi : A \to A'$ such that $\pi' \circ \Psi = \psi \circ \pi$.

2.2 *Models of Smooth Affine Surfaces with Irreducible Rational Boundaries*

We review known basic properties of smooth affine surfaces admitting smooth completions (X, B) with boundaries $B \cong \mathbb{P}^1$. Recall [22, Theorem 2] that for such a pair (X, B), the affineness of $X \setminus B$ implies that B is the support of an ample effective divisor on X.

Lemma 3 ([20, Proposition 1]) *A pair (X, B) consisting of a smooth projective surface X and a divisor $B \cong \mathbb{P}^1$ such that $X \setminus B$ is affine is isomorphic to one of the following:*

(a) $(\mathbb{P}^2, B)$ where B is either a line L or a smooth conic Q,

(b) $(\mathbb{F}_n, B)$ where $\pi_n : \mathbb{F}_n = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n)) \to \mathbb{P}^1$, $n \geq 0$, is a $\mathbb{P}^1$-bundle and B is an ample section of π_n.

Proof The log-canonical divisor $K_X + B$ is not nef since $(K_X + B) \cdot B = -2$ by adjunction formula. Given a $K_X + B$ extremal smooth rational curve R on X, the conditions $B \cdot R > 0$ and $(K_X + B) \cdot R < 0$ imply that $R^2 \geq 0$. If $R^2 > 0$ then X is a smooth log del Pezzo surface of Picard rank 1, hence is isomorphic to $\mathbb{P}^2$, and B is either a line or a smooth conic. If $R^2 = 0$, then the associated extremal contraction is a Zariski locally trivial $\mathbb{P}^1$-bundle $h : X \to C$ over a smooth projective curve C and B is a section of h. Thus, $C \cong B \cong \mathbb{P}^1$ and $(X, h) \cong (\mathbb{F}_n, \pi_n)$ for some $n \geq 0$.

For a pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b), we denote by $C_0 \subset \mathbb{F}_n$ a section of π_n with self-intersection $C_0^2 = -n$ and by F a closed fiber of π_n. Recall [8, Corollary V.2.18] that for $m \geq 1$, the complete linear system $|C_0 + mF|$ contains prime divisors if and only if $m \geq n$. Since B is a section of π_n, we have $B \sim C_0 + \frac{1}{2}(B^2 + n)F$, where $B^2 \geq n + 2$ because B is ample. For a fixed $d \geq 2$, the Hirzebruch surfaces $\mathbb{F}_n$ containing an ample section B with $B^2 = d$ are those of the form $\mathbb{F}_{d-2i}$, $i = 1, \ldots, \left\lfloor \frac{d}{2} \right\rfloor$, with B belonging to the complete linear system $|C_0 + (d - i)F|$.

Since the divisor class group of $\mathbb{F}_n$ is freely generated by the classes of F and of a section of π_n, the divisor class group of $\mathbb{F}_n \setminus B$ is freely generated by the class of $F|_{\mathbb{F}_n \setminus B}$. A canonical divisor $K_{\mathbb{F}_n}$ of $\mathbb{F}_n$ being linearly equivalent to $-2C_0 - (n + 2)F$, we have $K_{\mathbb{F}_n} \sim -2B + (B^2 - 2)F$ and hence $K_{\mathbb{F}_n \setminus B} \sim (B^2 - 2)F|_{\mathbb{F}_n \setminus B}$. A result due to Danilov–Gizatullin asserts conversely that the integers B^2 are a complete invariant of the isomorphism classes of surfaces $\mathbb{F}_n \setminus B$, namely:

Theorem 4 ([7, Theorem 5.8.1] (see also [5, Corollary 4.8], [12, §3.1], [18] and [19, Corollary 6.2.4])) *The isomorphism class of the complement of an ample section B in a Hirzebruch surface $\mathbb{F}_n$ depends only on the self-intersection B^2 of B.*

Remark 5 For a pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b) with $B^2 = d$, the closed immersion $B \hookrightarrow \mathbb{F}_n$ is determined by a surjection $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n) \to \mathscr{L}$ onto an invertible sheaf $\mathscr{L}$ on $\mathbb{P}^1$, with kernel $\mathscr{K}$ isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-\frac{1}{2}(d + n))$. The locally trivial $\mathbb{A}^1$-bundle $\nu = \pi_n|_{\mathbb{F}_n \setminus B} : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ thus carries the additional structure of a non-trivial torsor under the line bundle associated to the invertible sheaf $\mathscr{L}^\vee \otimes \mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(-d)$. Isomorphism classes of such $\mathbb{A}^1$-bundles are in one-to-one correspondence with the elements of the projective space $\mathbb{P}(\mathrm{Ext}^1(\mathscr{L}, \mathscr{K})) \cong \mathbb{P}(H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-d))) \cong \mathbb{P}^{d-2}$. By [7, Remark 4.8.6] (see also [12, Sect. 3.1] and [11, Proposition 2]), every non-trivial $\mathscr{O}_{\mathbb{P}^1}(-d)$-torsor arises as a restriction $\pi_n|_{\mathbb{F}_n \setminus B} : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ for some pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b) with $B^2 = d$.

Example 6 The pairs (X, B) of Lemma 3 (a) are unique up to isomorphism. In particular every affine surface isomorphic to the complement of a smooth conic in $\mathbb{P}^2$ is isomorphic to the complement of the conic $Q_0 = \{xz + y^2 = 0\}$ in $\mathbb{P}^2_{[xy:z]}$. A model of an affine surface $\mathbb{F}_n \setminus B$ with $B^2 = d$ is given for $d = 2e \geq 2$ by the complement in $\mathbb{F}_0 = \mathbb{P}^1_{[u_0:u_1]} \times \mathbb{P}^1_{[v_0:v_1]}$ of the section $\Delta_e = \{u_1^e v_0 - u_0^e v_1 = 0\}$ of $\pi_0 = \mathrm{pr}_1$, and for $d = 2e + 1 \geq 3$ by the complement in $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ viewed as the blow-up of $\mathbb{P}^2_{[x:y:z]}$ at the point $p = [0 : 1 : 0]$ of the proper transform of the rational cuspidal curve $C_e = \{yz^e + x^{e+1} = 0\}$.

The next examples illustrate some other representatives of isomorphism classes of affine surfaces $\mathbb{F}_n \setminus B$.

Example 7 ([12]) For every $d \geq 2$ and every pair of integers $p, q \geq 1$ such that $p + q = d$, the geometric quotient S_d of the smooth affine threefold $X_{p,q} = \{x^p v - y^q u = 1\}$ in $\mathbb{A}^4$ by the free $\mathbb{G}_m$-action defined by $\lambda \cdot (x, y, u, v) = (\lambda x, \lambda y, \lambda^{-q} u, \lambda^{-p} v)$ is a representative of the isomorphism class of surfaces of the form $\mathbb{F}_n \setminus B$ such that $B^2 = d$. Indeed, S_d is isomorphic to the complement in the geometric quotient

$$\pi_{|p-q|} : \mathbb{F}_{|p-q|} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(q)) \to \mathbb{P}^1 = \mathrm{Proj}(k[x, y])$$

of $(\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$ by the $\mathbb{G}_m^2$-action $(\lambda, \mu) \cdot (x, y, u, v) = (\lambda x, \lambda y, \lambda^{-p} \mu u, \lambda^{-q} \mu v)$ of the ample section B of $\pi_{|p-q|}$ with self-intersection d determined by the vanishing of the polynomial $x^p v - y^q u$ of bi-homogeneous degree $(0, 1)$.

Example 8 ([11]) For every $d \geq 2$, the surface W_d in $\mathbb{A}^4 = \mathrm{Spec}(k[x_1, x_2, x_2, x_4])$ defined by the equations

$$x_1 x_3 - x_2(x_2 + 1) = 0, \;\; x_2^{d-2} x_4 - x_3^{d-1} = 0, \;\; x_1^{d-2} x_4 - (x_2 + 1)^{d-2} x_3 = 0$$

is a representative of the isomorphism class of surfaces of the form $\mathbb{F}_n \setminus B$ such that $B^2 = d$. Indeed, the morphism $\nu : W_d \to \mathbb{P}^1, (x_1, x_2, x_3, x_4) \mapsto [x_1 : x_2 + 1] = [x_2 : x_3]$ is a locally trivial $\mathbb{A}^1$-bundle with local trivializations

$$\nu^{-1}(\mathbb{P}^1 \setminus [0 : 1]) \cong \mathrm{Spec}(k[w][x_4]) \quad \text{and} \quad \nu^{-1}(\mathbb{P}^1 \setminus [1 : 0]) \cong \mathrm{Spec}(k[w'][x_1])$$

and gluing isomorphism $(w, x_4) \mapsto (w', x_1) = (w^{-1}, w^d x_4 - w^{d-1})$, hence is a torsor under the line bundle associated to $\mathcal{O}_{\mathbb{P}^1}(-d)$. By Remark 5, the surface W_d isomorphic to $\mathbb{F}_n \setminus B$ for some pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b) with $B^2 = d$.

The surface W_2 is isomorphic to the surface in $\mathbb{A}^3 = \mathrm{Spec}(k[x, y, z])$ given by the equation $xy = z(z + 1)$. For $d \geq 3$, the morphism $W_d \to \mathbb{A}^3, (x_1, x_2, x_3, x_4) \mapsto (x_1, x_4, x_2)$ has image equal to the non-normal surface V_d given by the equation $x^{d-1} y = z(z + 1)^{d-1}$ and the induced morphism $\nu_d : W_d \to V_d$ is finite and birational, expressing W_d as the normalization of V_d. This recovers the other description of representative of surfaces $\mathbb{F}_n \setminus B$ such that $B^2 = d$ as normalizations of surfaces of the form V_d given in [19, Sect. 1.0.8].

3 Families of $\mathbb{A}^1$-Fibrations of Complete Type

Since they have torsion class groups, the affine plane $\mathbb{A}^2 = \mathbb{P}^2 \setminus L$ and the complements of smooth conics in $\mathbb{P}^2$ do not admit $\mathbb{A}^1$-fibrations over complete curves. In contrast, a surface $\mathbb{F}_n \setminus B$ admits a smooth $\mathbb{A}^1$-fibration $\pi_n|_{\mathbb{F}_n \setminus B} : \mathbb{F}_n \setminus B \to \mathbb{P}^1$. In this section, we are interested in the properties of certain $\mathbb{A}^1$-fibrations $\mathbb{F}_n \setminus B \to \mathbb{P}^1$ with multiple fibers.

Lemma 9 *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b), let q be a point of B and let $m \geq 1$. Then the linear subsystem $\mathscr{Z}_q(m)$ of the complete linear system $|B + mF|$ consisting of divisors intersecting B with multiplicity $B^2 + m$ at q has dimension m. Furthermore, the open subset $\mathscr{U}_q(m)$ of $\mathscr{Z}_q(m)$ consisting of prime divisors is Zariski dense.*

Proof Put $d = B^2$ and $F_q = \pi_n^{-1}(\pi_n(q))$. Let $\mathscr{I}_B \cong \mathscr{O}_{\mathbb{F}_n}(-B)$ denote the ideal sheaf of B and consider the long exact sequence of cohomology associated to the short exact sequence of coherent sheaves

$$0 \to \mathscr{I}_B \otimes \mathscr{O}_{\mathbb{F}_n}(B + mF) \cong \mathscr{O}_{\mathbb{F}_n}(mF) \to \mathscr{O}_{\mathbb{F}_n}(B + mF) \to \mathscr{O}_{\mathbb{F}_n}(B + mF)|_B \cong \mathscr{O}_B(d + m) \to 0$$

Since $H^1(\mathbb{F}_n, \mathscr{O}_{\mathbb{F}_n}(mF)) = 0$, the map $H^0(\mathbb{F}_n, \mathscr{O}_{\mathbb{F}_n}(B + mF)) \to H^0(B, \mathscr{O}_B(d + m))$ is surjective, with kernel of dimension $\dim H^0(\mathbb{F}_n, \mathscr{O}_{\mathbb{F}_n}(mF)) = m + 1$. It follows that $\mathscr{Z}_q(m) \cong \mathbb{P}^m$. A singular element of $\mathscr{Z}_q(m)$ decomposes as the sum of a prime member $B_{m'}$ of the complete linear system $|B + m'F|$, $0 \leq m' < m$, intersecting B with multiplicity $d + m'$ at q and of $(m - m')F_q$. By the same computation as above, these elements form a closed linear subspace $\mathscr{Z}_q(m') \cong \mathbb{P}^{m'}$ of $\mathscr{Z}_q(m)$ and so, $\mathscr{U}_q(m) = \mathscr{Z}_q(m) \setminus \bigcup_{m'=0}^{m-1} \mathscr{Z}_q(m')$ is a dense open subset of $\mathscr{Z}_q(m)$.

Example 10 Let $\mathbb{F}_0 = \mathbb{P}^1_{[u_0:u_1]} \times \mathbb{P}^1_{[v_0:v_1]}$, $\Delta = \{u_1 v_0 - u_0 v_1 = 0\}$ and let $q = ([0 : 1], [0 : 1])$. For every $m \geq 1$, denote by $\mathscr{V}_m \subset k[t]$ the m-dimensional vector space of monic polynomials of degree m. Writing $P(u, v) = p(\frac{v}{u})u^m$ for the homogenization of a polynomial $p(t) \in k[t]$, the map which associates to $p \in \mathscr{V}_m$ the section

$$B_{m,p} = \{u_0 P(u_0, u_1) v_1 - (u_0^{m+1} + u_1 P(u_0, u_1)) v_0 = 0\}$$

of $\pi_0 = \mathrm{pr}_1$ defines an open immersion $\mathscr{V}_m \to \mathscr{U}_q(m) \subset \mathscr{Z}_q(m) \cong \mathbb{P}^m$. These curves $B_{m,p}$ were considered by Blanc-van Santen [4, Sect. 3] for the fact that their restrictions $B_{m,p} \cap (\mathbb{F}_0 \setminus \Delta) \cong \mathbb{A}^1$ provide examples of non-equivalent closed embeddings of the affine line into the smooth affine quadric surface $\mathbb{F}_0 \setminus \Delta$.

Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b), let $m \geq 2$ and let B_m be a section of π_n corresponding to a closed point of the scheme $\mathscr{U}_q(m)$ for some $q \in B$. Let $\mathscr{L}_{q,B_m} \subset |B + mF|$ be the pencil generated by the divisors B_m and $B + mF_q$.

Lemma 11 *Every member of $\mathscr{L}_{q,B_m}$ other than $B + mF_q$ is a smooth rational curve.*

Proof Every divisor in the complete linear system $|B + mF|$ has self-intersection $B^2 + 2m$. The minimal resolution $\tau : \tilde{\mathbb{F}}_n \to \mathbb{F}_n$ of the rational map $\gamma : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ defined by $\mathscr{L}_{q,B_m}$ is obtained by performing $B^2 + 2m$ successive blow-ups with center at q on the successive proper transforms of B_m, with exceptional divisors $E_1, \ldots, E_{B^2+2m}$. Letting $\tau_*^{-1} B_m$ be the proper transform of B_m, the composition $\tilde{\gamma} = \gamma \circ \tau : \tilde{\mathbb{F}}_n \to \mathbb{P}^1$ is the $\mathbb{P}^1$-fibration defined by the complete linear system $|\tau_*^{-1} B_m|$. The total transform of B_m is a rational chain $E_1 \triangleleft \cdots \triangleleft E_{B^2+2m} \triangleleft \tau_*^{-1} B_m$, where E_{B^2+2m} is a section of $\tilde{\gamma}$. Since every singular member of $\mathscr{L}_{q,B_m}$ is the sum of a prime member B' of the linear system $|B + m'F|$ for some $0 \leq m' < m$ and of $(m - m')F_q$, every fiber of $\tilde{\gamma}$ other than that containing the proper transform of $B \cup F_q$ is smooth. This implies that every member of $\mathscr{L}_{q,B_m}$ other than $B + mF_q$ is a smooth rational curve.

For every $q \in B$, the space of pencils $\mathscr{L}_{q,B_m}$ identifies with a dense open subset $\mathscr{S}_q(m)$ of the projective space $\mathbb{P}^{m-1}$ of lines passing through the point of $\mathscr{Z}_q(m) \setminus \mathscr{U}_q(m)$ corresponding to the reducible divisor $B + mF_q$ and a point of $\mathscr{U}_q(m)$. The linear action on $H^0(\mathbb{F}_n, \mathscr{O}_{\mathbb{F}_n}(B + mF))$ of the algebraic subgroup $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$ of automorphisms of $\mathbb{F}_n$ preserving $B \cup F_q$ induces an action of $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$ on $\mathscr{S}_q(m)$. On the other hand, the rational map $\gamma_{q,B_m} : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ associated to a pencil $\mathscr{L}_{q,B_m}$ restricts to a surjective $\mathbb{A}^1$-fibration $\delta_{q,B_m} : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ having $F_q \cap (\mathbb{F}_n \setminus B) \cong \mathbb{A}^1$ as a unique degenerate fiber, of multiplicity m. We have the following characterization:

Lemma 12 *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b), let $m \geq 3$ and let B_m and B'_m be sections of $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$ corresponding to points of the scheme $\mathscr{U}_q(m)$ for some $q \in B$. Then the $\mathbb{A}^1$-fibered surfaces $(\mathbb{F}_n \setminus B, \delta_{q,B_m})$ and $(\mathbb{F}_n \setminus B, \delta_{q,B'_m})$ are equivalent if and only if the pencils $\mathscr{L}_{q,B_m}$ and $\mathscr{L}_{q,B'_m}$ belong to the same* $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$*-orbit.*

Proof Let $\Psi : (\mathbb{F}_n \setminus B, \delta_{q,B_m}) \to (\mathbb{F}_n \setminus B, \delta_{q,B'_m})$ be an equivalence of $\mathbb{A}^1$-fibered surfaces, let $\bar{\Psi}$ be its extension to a birational automorphism of $\mathbb{F}_n$ and let $\mathbb{F}_n \xleftarrow{\sigma} Y \xrightarrow{\sigma'} \mathbb{F}_n$ be the minimal resolution of $\bar{\Psi}$. If σ' is not an isomorphism, then σ is not an isomorphism and Ψ contracts B onto a point. By the minimality assumption, the proper transform $\sigma_*^{-1}(B)$ of B is the only σ'-exceptional (-1)-curve contained in $\sigma^{-1}(B)$. It follows that σ has $B^2 + 1$ proper or infinitely near base points on B and hence, since $B \cdot B_m = B^2 + m$, that $\sigma_*^{-1} B_m \cdot \sigma_*^{-1}(B) \geq m - 1 \geq 2$. But then, $\sigma'(\sigma_*^{-1} B_m)$ is an irreducible singular member of $\mathscr{L}_{q,B'_m}$, which is impossible by Lemma 11. Thus, σ' is an isomorphism and $\bar{\Psi}$ is an automorphism of $\mathbb{F}_n$, which preserves B and the closure F_q of the unique common degenerate fiber $F_q \cap (\mathbb{F}_n \setminus B)$ of δ and δ'. Furthermore, $\bar{\Psi}$ maps B_m onto a certain smooth member of $\mathscr{L}_{q,B'_m}$, hence maps $\mathscr{L}_{q,B_m}$ onto $\mathscr{L}_{q,B'_m}$. The converse implication is clear.

Remark 13 For every $m \geq 2$ and every point $q \in B$, one can find distinct points B_m and B'_m in the scheme $\mathscr{U}_q(m)$ such that the pencils $\mathscr{L}_{q,B_m}$ and $\mathscr{L}_{q,B'_m}$ have distinct general members. The associated $\mathbb{A}^1$-fibrations δ_{q,B_m} and δ_{q,B'_m} have distinct general fibers but share $F_q \cap (\mathbb{F}_n \setminus B)$ as a degenerate fiber. This contrasts with $\mathbb{A}^1$-fibrations of affine type for which no curve can be contained simultaneously in fibers of two $\mathbb{A}^1$-fibrations with distinct general fibers, see [10, Corollary 2.22].

Proposition 14 *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b). Then for every $m \geq 4$, there exist infinitely many equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{P}^1$ with a unique degenerate fiber of multiplicity m.*

Proof By Theorem 4, it suffices to construct such families from the two pairs $(\mathbb{F}_n, B) = (\mathbb{F}_0, \Delta_e)$ and $(\mathbb{F}_1, \sigma_*^{-1}C_e)$ of Example 6. If $B^2 = 2e \geq 2$, let $q = ([1 : 0], [0 : 1]) \in \Delta_e = \{u_1^e v_0 - u_0^e v_1 = 0\} \subset \mathbb{F}_0$. The group $\mathrm{Aut}(\mathbb{F}_0, \Delta_e \cup F_q)$ is isomorphic to the affine group $\mathbb{G}_m \ltimes \mathbb{G}_a$ acting by

$$(\lambda, t) \cdot ([u_0 : u_1], [v_0 : v_1]) = ([\lambda u_0 + tu_1 : u_1], [v_0 + \lambda^{-1}tv_1 : \lambda^{-1}v_1])$$

if $e = 1$ and for every $e \geq 2$ to the group $\mathbb{G}_m$ acting by $\lambda \cdot ([u_0 : u_1], [v_0 : v_1]) = ([\lambda u_0 : u_1], [v_0 : \lambda^{-e}v_1])$. If $B^2 = 2e + 1 \geq 3$, viewing $\mathbb{F}_1$ as the blow-up $\sigma : \mathbb{F}_1 \to \mathbb{P}^2_{[x:y:z]}$ of the point $p = [0 : 1 : 0]$ with exceptional divisor C_0, let q be the intersection point of $\sigma_*^{-1}C_e$ with the proper transform of the tangent line $L = \{z = 0\}$ to $C_e = \{yz^e + x^{e+1} = 0\}$ at p. The group $\mathrm{Aut}(\mathbb{F}_0, \sigma_*^{-1}C_e \cup F_q)$ is then isomorphic to the group $\mathrm{Aut}(\mathbb{P}^2, C_e \cup L)$. The latter is isomorphic to $\mathbb{G}_m \ltimes \mathbb{G}_a$ acting by $(\lambda, t) \cdot [x : y : z] = [\lambda x + tz : \lambda^2 y - 2\lambda tx - t^2 z : z]$ if $e = 1$ and for every $e \geq 2$ to $\mathbb{G}_m$ acting by $\lambda \cdot [x : y : z] = [\lambda x : \lambda^{e+1} y : z]$.

In both cases, the $\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$-orbit of a point of the open subset $\mathscr{S}_q(m) \subset \mathbb{P}^{m-1}$ is at most 2-dimensional. Since $m - 1 \geq 3$, the set-theoretic orbit space $\mathscr{S}_q(m)/\mathrm{Aut}(\mathbb{F}_n, B \cup F_q)$ is infinite and the assertion follows from Lemma 12.

4 Equivalence Classes of $\mathbb{A}^1$-Fibrations of Affine Type

4.1 Special Pencils of Rational Curves and Associated $\mathbb{A}^1$-Fibrations of Affine Type

Let (X, B) be a pair as in Lemma 3. For every point $q \in B$, denote by $\mathscr{P}_q$ the linear subsystem of the complete linear system $|B|$ on X consisting of curves with local intersection number with B at q equal to B^2. If $(X, B) \cong (\mathbb{P}^2, L)$ where L is a line, then $\mathscr{P}_q$ is simply the pencil of lines through q. More generally, if $B^2 \geq 2$ then the same type of computation as in the proof of Lemma 11 implies that $\mathscr{P}_q$ is a pencil. The minimal resolution $\sigma : \tilde{X} \to X$ of the rational map $\rho_q : X \dashrightarrow \mathbb{P}^1$ defined by $\mathscr{P}_q$ is obtained by performing B^2 successive blow-ups with center at q on the successive proper transforms of B, with respective exceptional divisor $E_1, \ldots, E_{B^2}$. The total transform of B in $\tilde{X}$ is a rational chain $\sigma_*^{-1}B \triangleleft E_{B^2} \triangleleft E_{B^2-1} \triangleleft \cdots E_1$ of type $[0, -1, -2, \ldots, -2]$ and the morphism $\tilde{\rho}_q = \rho_q \circ \sigma : \tilde{X} \to \mathbb{P}^1$ is the $\mathbb{P}^1$-fibration defined by the complete linear system $|\sigma_*^{-1}B|$.

Example 15 For every point q on a smooth conic $Q \subset \mathbb{P}^2$, $\mathscr{P}_q$ is the pencil of conics intersecting Q with multiplicity 4 at q, generated by Q and twice its tangent line $T_q Q$ at q. The total transform of $Q \cup T_q Q$ by σ is the rational tree

$$\begin{array}{c} (\sigma_*^{-1}Q, 0) \triangleleft (E_4, -1) \triangleleft (E_3, -2) \triangleleft (E_2, -2) - (\sigma_*^{-1}T_q Q, -1) \\ \hspace{7em} | \\ \hspace{7em} (E_1, -2), \end{array}$$

where the displayed numbers in the parenthesis are the self-intersection numbers of the corresponding irreducible components. The $\mathbb{P}^1$-fibration $\tilde{\rho}_q : \tilde{\mathbb{P}}^2 \to \mathbb{P}^1$ has $\tilde{\rho}_q^{-1}(\rho_q(T_q Q)) = \bigcup_{i=1}^3 E_i \cup \sigma_*^{-1} T_q Q$ as a unique singular fiber.

For pairs $(\mathbb{F}_n, B)$ as in Lemma 3 (b), we have the following description (see also [13, Sect. 4]):

Lemma 16 *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b). Then for every point $q \in B$, the following hold:*

(a) The pencil $\mathscr{P}_q$ has a unique singular member consisting of a divisor of the form $C + m_q F_q$, where C is a section of π_n, $F_q = \pi_n^{-1}(\pi_n(q))$ and $m_q \in \{1, \ldots, B^2 - 1\}$.

(b) The $\mathbb{P}^1$-fibration $\tilde{\rho}_q : \tilde{\mathbb{F}}_n \to \mathbb{P}^1$ has $\tilde{\rho}_q^{-1}(\rho_q(C \cup F_q)) = \bigcup_{i=1}^{B^2-1} E_i \cup \sigma_*^{-1}(C \cup F_q)$ as a unique singular fiber.

(c) For a general point $q \in B$, the unique singular member of $\mathscr{P}_q$ is reduced.

Proof With the notation of Sect. 2.2, put $d = B^2 \geq 2$ and $B \sim C_0 + \ell F$ with $\ell = \frac{1}{2}(d + n) \leq d - 1$. Since $d \geq 2$ and $E_{d-1}^2 = -2$, $\tilde{\rho}_q^{-1}(\tilde{\rho}_q(E_{d-1}))$ is a singular fiber of $\tilde{\rho}_q$ and its image by σ is a singular member of $\mathscr{P}_q$. Since a singular member of $\mathscr{P}_q$ decomposes as the union of a section $C \sim C_0 + \ell' F$ of π_n for some $0 \leq \ell' < \ell \leq d - 1$ intersecting B with multiplicity $d - (\ell - \ell')$ at q and of $(\ell - \ell')F_q$, it follows that $\sigma(\tilde{\rho}_q^{-1}(\tilde{\rho}_q(E_{d-1}))) = C + (\ell - \ell')F_q$ is the unique singular member of $\mathscr{P}_q$. This proves a) and b). For assertion c), see [7, Proposition 4.8.11] or [12, Lemma 3.2].

For a pencil $\mathscr{P}_q$ on a pair (X, B) as in Lemma 3, the rational map $\rho_q : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ defined by $\mathscr{P}_q$ restricts to an $\mathbb{A}^1$-fibration $\pi_q : X \setminus B \to \mathbb{A}^1 = \mathbb{P}^1 \setminus \rho_q(B)$. In the case where $(X, B) = (\mathbb{P}^2, L)$ for some line L, π_q is a trivial $\mathbb{A}^1$-bundle on $\mathbb{A}^2 = \mathbb{P}^2 \setminus L$, and in the case where $(X, B) = (\mathbb{P}^2, Q)$, π_q has a unique degenerate fiber consisting of $T_q Q \cap (\mathbb{P}^2 \setminus Q) \cong \mathbb{A}^1$ occurring with multiplicity 2 (see Example 15). For pairs $(\mathbb{F}_n, B)$, it follows from Lemma 16 a) that $\pi_q : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ has unique degenerate fiber which is reducible, consisting of the disjoint union of $C \cap (\mathbb{F}_n \setminus B) \cong \mathbb{A}^1$ occurring with multiplicity 1 and of $F_q \cap (\mathbb{F}_n \setminus B) \cong \mathbb{A}^1$ occurring with a certain multiplicity $m_q \in \{1, \ldots, B^2 - 1\}$. The following lemma shows conversely that for a pair (X, B) as above, every $\mathbb{A}^1$-fibration $\pi : X \setminus B \to \mathbb{A}^1$ is induced by a pencil $\mathscr{P}_q$ on a suitable smooth completion (X', B') of $X \setminus B$.

Lemma 17 *Let (X, B) be a pair as in Lemma 3 and let $\pi : X \setminus B \to \mathbb{A}^1$ be an $\mathbb{A}^1$-fibration. Then there exists a smooth completion (X', B') of $X \setminus B$ by some pair* *as in Lemma 3, an isomorphism* $\Psi : X \setminus B \to X' \setminus B'$ *and a point* $q' \in B'$ *such that* $\pi = \pi_{q'} \circ \Psi$.

Proof Note that $X \setminus B$ admits an SNC completion by a rational chain of type $[0, -1]$ if $(X, B) \cong (\mathbb{P}^2, L)$ and of type $[0, -1, -2, \ldots, -2]$ with $B^2 - 1 \geq 1$ curves of self-intersection number -2 otherwise. Now let (Y, D) be a relatively minimal SNC completion of $\pi : X \setminus B \to \mathbb{A}^1$ into a $\mathbb{P}^1$-fibration $\bar{\pi} : Y \to \mathbb{P}^1$ as in Sect. 2.1.2 (ii). By [10, Proposition 2.15], D is a rational chain $F_\infty \triangleleft H \triangleleft E$, where $F_\infty \cong \mathbb{P}^1$ is the fiber of $\bar{\pi}$ over the point $\mathbb{P}^1 \setminus \mathbb{A}^1$, H is a section of $\bar{\pi}$ and E is either the empty divisor or a rational chain $E_1 \triangleleft \cdots \triangleleft E_{d-1}$ consisting of curves with self-intersection number ≤ -2 contained in a fiber of $\bar{\pi}$. By making elementary transformations consisting of the blow-up of a point of F_∞ followed by the contraction of the proper transform of F_∞, we can further assume from the beginning that $H^2 = -1$. By [6, Corollary 2] (see also [16, Corollary 3.32] or [2, Corollary 3.2.3]), the number of irreducible components of E and their self-intersection numbers are independent on (Y, D). Thus, D is a chain of one of the types listed above and so, letting $\tau : Y \to X'$ be the contraction of the subchain $H \triangleleft E$ onto a smooth point $q' \in B' = \tau(F_\infty) \cong \mathbb{P}^1$, we obtain a smooth completion (X', B') of $X \setminus B$ and an isomorphism $\Psi = \tau|_{Y \setminus D} : X \setminus B \cong Y \setminus D \to X' \setminus B'$ such that $\pi = \pi_{q'} \circ \Psi$.

Corollary 18 *The affine plane* $\mathbb{A}^2$*, the complement* $\mathbb{P}^2 \setminus Q$ *of a smooth conic* $Q \subset \mathbb{P}^2$ *and the affine quadric surface* $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ *all have a unique equivalence class of* $\mathbb{A}^1$*-fibrations over* $\mathbb{A}^1$.

Proof In each case, given an $\mathbb{A}^1$-fibration $\pi : X \setminus B \to \mathbb{A}^1$, Lemma 17 provides a smooth completion (X', B') of $X \setminus B$ such that $\pi = \pi_{q'} \circ \Psi$ for some isomorphism $\Psi : X \setminus B \to X' \setminus B'$ and some point $q' \in B'$. If $(X, B) = (\mathbb{P}^2, L)$ or $(\mathbb{P}^2, Q)$ then $X' = \mathbb{P}^2$ and B' is respectively a line L' or a smooth conic Q'. If $(X, B) = (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ then $X' = \mathbb{P}^1 \times \mathbb{P}^1$ and B' is a prime divisor of type $(1, 1)$. The assertion then follows from the fact that in each case, the automophism group of the surface $X' = X$ acts transitively on the set of pairs (B', q').

Corollary 19 *Every* $\mathbb{A}^1$*-fibration* $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ *on an affine surface* $\mathbb{F}_n \setminus B$ *has a unique degenerate fiber which consists of the disjoint union of a reduced irreducible component and an irreducible component of multiplicity* $m \in \{1, \ldots, B^2 - 1\}$.

In contrast to Corollary 18, the following lemma, whose proof reproduces that of [9, Theorem 16.2.1] (in french), implies that every affine surface $\mathbb{F}_n \setminus B$ with $B^2 \geq 3$ admits more than one equivalence class of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$.

Lemma 20 *Let* $(\mathbb{F}_n, B)$ *be a pair as in Lemma 3 (b). Then for every integer* $m \in \{1, \ldots, B^2 - 1\}$*, there exists an* $\mathbb{A}^1$*-fibration* $\pi_m : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ *whose degenerate fiber has a reduced component and a component of multiplicity* m.[1]

[1] In particular, there exist at least $B^2 - 1$ equivalence classes of $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ on $\mathbb{F}_n \setminus B$. The lower bound $\left\lfloor \frac{B^2-1}{2} \right\rfloor$ was discovered earlier by Peter Russell (unpublished) and was proven by Flenner–Kaliman–Zaidenberg [17, Corollary 5.16 a)] using a closely related construction.

Proof We use the notation introduced in Sect. 2.2 and put $d = B^2 \geq 2$. Let $n' = d - 2i$ for some $i = 1, \dots \lfloor \frac{d}{2} \rfloor$, let $C_{n'}$ be a prime member of the complete linear system $|C_0 + n'F|$ and let $q_0 \in C_0$ and $q_{n'} \in C_{n'}$ be a pair of closed points contained in two different fibers F_{q_0} and $F_{q_{n'}}$ of $\pi_{n'} : \mathbb{F}_{n'} \to \mathbb{P}^1$. Note that $B' \cdot C_0 = i$ and $B' \cdot C_{n'} = d - i$ for every member B' of $|C_0 + (d-i)F|$. Applying a sequence of i elementary transformations with center at q_0 followed by a sequence of $d - i$ elementary transformations with center at $q_{n'}$ yields a birational map $\beta : \mathbb{F}_{n'} \dashrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ such that $\pi_{n'} = \mathrm{pr}_1 \circ \beta$. The composition $\mathrm{pr}_2 \circ \beta : \mathbb{F}_{n'} \dashrightarrow \mathbb{P}^1$ is given by a pencil $\mathscr{L} \subset |C_0 + (d-i)F|$ whose general members are sections B' of $\pi_{n'}$ which satisfy $B' \cap C_0 = q_0$ and $B' \cap C_{n'} = q_{n'}$. Since $(B')^2 = 2(d-i) - (d-2i) = d$, $\mathbb{F}_{n'} \setminus B'$ is isomorphic to $\mathbb{F}_n \setminus B$ by Theorem 4. On other hand, the pencil $\mathscr{P}_{q_0}$ has a unique singular member equal to $C_0 + (d-i)F_{q_0}$ whereas the pencil $\mathscr{P}_{q_{n'}}$ has a unique singular member equal to $C_{n'} + iF_{q_{n'}}$. The degenerate fibers of the associated $\mathbb{A}^1$-fibrations $\pi_{q_0} : \mathbb{F}_{n'} \setminus B' \to \mathbb{A}^1$ and $\pi_{q_{n'}} : \mathbb{F}_{n'} \setminus B' \to \mathbb{A}^1$ have $F_{q_0} \cap (\mathbb{F}_{n'} \setminus B')$ and $F_{q_{n'}} \cap (\mathbb{F}_{n'} \setminus B')$ as irreducible components of multiplicity $d - i$ and i respectively. Since i ranges from 1 to $\lfloor \frac{d}{2} \rfloor$, the assertion follows.

4.2 *Some Classes of $\mathbb{A}^1$-Fibrations of Affine Type on Surfaces $\mathbb{F}_n \setminus B$*

By Corollary 19 and Lemma 20, the classification of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ is divided into that of each type according to the multiplicity $m \in \{1, \dots, B^2 - 1\}$ of the possibly non-reduced irreducible component of their unique degenerate fiber. Hereafter, we first recall known results on the two extremal cases: $\mathbb{A}^1$-fibrations with a component of maximal multiplicity $B^2 - 1$ on the one hand, and smooth $\mathbb{A}^1$-fibrations on the other hand. We then proceed to the study of equivalence classes of $\mathbb{A}^1$-fibrations with a component of multiplicity two in their unique degenerate fiber.

4.2.1 Equivalence Classes of $\mathbb{A}^1$-Fibrations with Maximal Multiplicity

Proposition 21 *For every pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b), the affine surface $\mathbb{F}_n \setminus B$ has a unique equivalence class of $\mathbb{A}^1$-fibration $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ with a degenerate fiber containing an irreducible component of multiplicity $B^2 - 1$.*

Proof A pair $(\mathbb{F}_n, B)$ with $B^2 = d + 2 \geq 2$ such that B contains a point q for which the singular member of the pencil $\mathscr{P}_q$ has the form $C + (d+1)F_q$ for some irreducible curve C is necessarily equal to $(\mathbb{F}_d, B)$ for some section $B \sim C_0 + (d+1)F$ of π_d intersecting C_0 transversely at the point q, the curve C being then equal to C_0. Then the assertion follows from the fact that the group $\mathrm{Aut}(\mathbb{F}_d)$ acts transitively on the set of sections $B \sim C_0 + (d+1)F$ of π_d. Let us recall the argument. Since the restriction homomorphism $\mathrm{Aut}(\mathbb{F}_d, C_0) \to \mathrm{Aut}(C_0)$ is surjective, it suffices to show that for some chosen fiber F_0 of π_d, the action of $\mathrm{Aut}(\mathbb{F}_d, C_0 \cup F_0)$ on the set of sections $B_0 \sim C_0 + (d+1)F$ intersecting C_0 at the point $F_0 \cap C_0$ is transitive. Identifying $\mathbb{F}_d \setminus (C_0 \cup F_0)$ to $\mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ in such a way that $\pi_d|_{\mathbb{A}^2} = \mathrm{pr}_x$ and that the closures in $\mathbb{F}_d$ of the level sets of y are sections of π_d linearly equivalent to $C_0 + dF$, these sections B_0 are the closures in $\mathbb{F}_d$ of curves $\Gamma_p \subset \mathbb{A}^2$ defined by equations of the form $y = p(x)$ where $p(x) \in k[x]$ is a polynomial of degree $d+1$. Since every automorphism of $\mathbb{A}^2$ of the form $(x, y) \mapsto (\lambda x + \mu, \nu y + r(x))$, where $r(x) \in k[x]$ is a polynomial of degree at most d, extends to an element of $\mathrm{Aut}(\mathbb{F}_d, C_0 \cup F_0)$, it follows that every section B_0 belongs to the $\mathrm{Aut}(\mathbb{F}_d, C_0 \cup F_0)$-orbit of the closure of the curve $\Gamma_{x^{d+1}} = \{y = x^{d+1}\}$.

4.2.2 Equivalence Classes of Smooth $\mathbb{A}^1$-Fibrations

The following lemma is a reformulation of [7, Lemma 5.5.5], which appeared, stated in a different language, in the Appendix of [12].

Lemma 22 *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b) and let $q \in B$ be a point such that the singular member of the pencil $\mathscr{P}_q$ is reduced. Then the isomorphism type of the $\mathbb{A}^1$-fibered surface $\pi_q : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ depends only on the integer B^2.*

Proof Put $d = B^2$ and $S = \mathbb{F}_n \setminus B$. The singular member of $\mathscr{P}_q$ has the form $C + F_q$ where C is a prime member of the complete linear system $|B - F_q|$. Without loss of generality, we can fix an isomorphism $A = \mathbb{P}^1 \setminus \rho_q(B) \cong \mathrm{Spec}(k[x])$ so that $\rho_q(C \cup F_q) = \{0\} \in A$. With the notation of Sect. 4.1, the total transform of $B \cup C \cup F_q$ in the minimal resolution $\sigma : \tilde{\mathbb{F}}_n \to \mathbb{F}_n$ of the rational map map $\rho_q : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ defined by $\mathscr{P}_q$ is a rational tree of the form

$$\begin{array}{ccccccccc} (\sigma_*^{-1}B, 0) \triangleleft & (E_d, -1) & \triangleleft\, (E_{d-1}, -2) & \triangleleft & (E_{d-2}, -2) \triangleleft & \cdots\cdots & \triangleleft\, (E_1, -2) & - & (\sigma_*^{-1}F_q, -1) \\ & & | & & & & & & \\ & & (\sigma_*^{-1}C, -1). & & & & & & \end{array}$$

Furthermore, the singular fiber $\tilde{\rho}_q^{-1}(\rho_q(C \cup F_q)) = \bigcup_{i=1}^{d-1} E_1 \cup \sigma_*^{-1}C \cup \sigma_*^{-1}F_q$ of the $\mathbb{P}^1$-fibration $\tilde{\rho}_q = \rho_q \circ \sigma : \tilde{\mathbb{F}}_n \to \mathbb{P}^1$ is reduced. By contracting successively $\sigma_*^{-1}F_q, E_1, \ldots, E_{d-2}$ and $\sigma_*^{-1}C$, we get a birational morphism $\overline{\tau} : \tilde{\mathbb{F}}_n \to \mathbb{F}_1$ of $\mathbb{P}^1$-fibered surfaces over $\mathbb{P}^1$. The later restricts to a morphism

$$\tau : S \cong \tilde{\mathbb{F}}_n \setminus \sigma^{-1}(B) \longrightarrow \mathbb{F}_1 \setminus \overline{\tau}(\sigma_*^{-1}(B) \cup E_d) \simeq A \times \mathbb{A}^1$$

of schemes over A, inducing an isomorphism $S \setminus (C \cup F_q) \xrightarrow{\sim} A \setminus \{0\} \times \mathbb{A}^1$ and contracting $C \cap S$ and $F_q \cap S$ to a pair of distinct points supported on $\{0\} \times \mathbb{A}^1$. One can choose a coordinate on the second factor of $A \times \mathbb{A}^1$ and a pair of isomorphisms of A-schemes $S \setminus F_q \simeq A \times \mathrm{Spec}(k[u])$ and $S \setminus C \simeq A \times \mathrm{Spec}([u'])$ so that the restrictions of τ to $S \setminus F_q$ and $S \setminus C$ coincide respectively with the morphisms

$$S \setminus F_q \to A \times \mathbb{A}^1,\ (x,u) \mapsto (x, xu+1) \quad \text{and} \quad S \setminus C \to A \times \mathbb{A}^1,\ \left(x,u'\right) \mapsto (x, x^{d-1}u').$$

Thus, $\pi_q : S \to A$ is A-isomorphic to the surface W_d obtained by gluing two copies $U_\pm = \mathrm{Spec}(k[x][v_\pm])$ of $A \times \mathbb{A}^1$ along the open subset $(A \setminus \{0\}) \times \mathbb{A}^1$ by the isomorphism $U_+ \ni (x, v_+) \mapsto (x, x^{2-d}v_+ + x^{1-d}) \in U_-$, endowed with the $\mathbb{A}^1$-fibration $\xi_d : W_d \to A$ induced by the first projections on each of the open subsets $U_\pm$.

Corollary 23 *For every pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b), the affine surface $\mathbb{F}_n \setminus B$ has a unique equivalence class of smooth $\mathbb{A}^1$-fibration $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$.*

Remark 24 The proofs of Lemma 16 and Lemma 22 do not depend on the Danilov–Gizatullin isomorphism theorem, and, when combined together, they actually provide a proof of Theorem 4. Indeed, Lemma 16 c) asserts in particular that for every pair $(\mathbb{F}_n, B)$ there exists a point $q \in B$ such that the $\mathbb{A}^1$-fibration $\pi_q : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ associated to the pencil $\mathscr{P}_q$ is a smooth morphism. On the other hand, Lemma 22 implies that the isomorphism type of $\pi_q : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ as an $\mathbb{A}^1$-fibered surface over $\mathbb{A}^1$, hence in particular as an abstract affine surface, depends only on the integer B^2.

4.2.3 Equivalence Classes $\mathbb{A}^1$-Fibrations with An Irreducible Component of Multiplicity Two

Given a pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b), denote by $\mathscr{A}_2(B^2)$ the set of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : \mathbb{F}_n \setminus B \to \mathbb{A}^1$ whose unique degenerate fiber has an irreducible component of multiplicity two. By Corollary 19 and Lemma 20, $\mathscr{A}_2(2) = \emptyset$ and $\mathscr{A}_2(d) \neq \emptyset$ for every $d \geq 3$. The aim of this subsection is to establish the following more precise description of the sets $\mathscr{A}_2(d)$ for $d \geq 3$.

Proposition 25 *With the notation above, the following hold:*

(a) The sets $\mathscr{A}_2(3)$ and $\mathscr{A}_2(4)$ both consist of a single element,

(b) The sets $\mathscr{A}_2(5)$ and $\mathscr{A}_2(6)$ both consist of two elements,

(c) For every $d \geq 7$, $\mathscr{A}_2(d)$ has cardinality larger than or equal to that of the field k.

The proof follows from a combination of several intermediate results established below. By Lemma 17, for a surface $S = \mathbb{F}_n \setminus B$ every element of $\mathscr{A}_2(B^2)$ is represented by an $\mathbb{A}^1$-fibration $\pi_q : S \to \mathbb{A}^1$ associated to a pencil $\mathscr{P}_q$ on some smooth completion $(\mathbb{F}_{n'}, B')$ of S whose unique singular member is a divisor of the form $C + 2F_q$ for some prime element C of the complete linear system $|B' - 2F|$. The unique degenerate fiber of π_q then consists of the disjoint union of $C \cap S$ with multiplicity one and of $F_q \cap S$ with multiplicity two.

Lemma 26 *Let $(\mathbb{F}_n, B)$ be pair as in Lemma 3 (b) with $d = B^2 \geq 3$ and such that there exists a point $q \in B$ for which the singular member of the pencil $\mathscr{P}_q$ is a divisor of the form $C + 2F_q$ for some prime element C of the complete linear system $|B - 2F|$. Then the total transform $\sigma_*^{-1}B \cup D_q \cup \sigma_*^{-1}(C) \cup \sigma_*^{-1}(F_q)$ of $B \cup C \cup F_q$*

in the minimal resolution $\sigma : \tilde{\mathbb{F}}_n \to \mathbb{F}_n$ *of the rational map* $\rho_q : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ *defined by* $\mathscr{P}_q$ *is a rational tree of the form*

$$\begin{array}{c}(\sigma_*^{-1}B, 0) \triangleleft (E_d, -1) \triangleleft (E_{d-1}, -2) \triangleleft (E_{d-2}, -2) \triangleleft \cdots\cdots \triangleleft (E_1, -2) - (\sigma_*^{-1}F_q, -1)\\ \qquad\qquad\qquad\qquad | \\ \qquad\qquad\qquad\qquad (\sigma_*^{-1}C, -2).\end{array}$$

Proof The assertion is straightforward to verify from the description of $\sigma : \tilde{\mathbb{F}}_n \to \mathbb{F}_n$ given in Sect. 4.1.

Notation 27 Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b) with $d = B^2 \geq 3$ and such that there exists a point $q \in B$ for which the singular member of $\mathscr{P}_q$ is a divisor of the form $C + 2F_q$. We denote by $\hat{S}_q$ the affine open subset $\mathbb{F}_n \setminus (B \cup C)$ of $S = \mathbb{F}_n \setminus B$ and we denote by $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ the $\mathbb{A}^1$-fibration with degenerate fiber $F_q \cap \hat{S}_q$ of multiplicity two induced by $\pi_q : S \to \mathbb{A}^1$. With the notation of Lemma 26, the $\mathbb{P}^1$-fibered surface $\tilde{\rho}_q = \rho_q \circ \sigma : \tilde{\mathbb{F}}_n \to \mathbb{P}^1$ is a relatively minimal SNC completion of $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ with boundary divisor $\hat{D}_q = \sigma_*^{-1}(B) \cup D_q \cup \sigma_*^{-1}C = \sigma_*^{-1}B \cup \bigcup_{i=1}^d E_i \cup \sigma_*^{-1}C$.

Let $(\mathbb{F}_n, B)$ and $(\mathbb{F}_{n'}, B')$ with $B^2 = (B')^2 = d \geq 3$ be smooth completions of an affine surface S such that for some points $q \in B$ and $q' \in B'$, the pencils $\mathscr{P}_q$ and $\mathscr{P}'_q$ have a singular member of the form $C + 2F_q$ and $C' + 2F'_{q'}$ respectively. Let $\pi_q : S \to \mathbb{A}^1$, $\hat{\pi}_q : \hat{S}_q = S \setminus C \to \mathbb{A}^1$, $\pi_{q'} : S \to \mathbb{A}^1$ and $\hat{\pi}_{q'} : \hat{S}_{q'} = S \setminus C' \to \mathbb{A}^1$ be the associated $\mathbb{A}^1$-fibrations. The next lemma reduces the study of equivalence classes of $\mathbb{A}^1$-fibered surfaces (S, π_q) to those of the simpler ones $(\hat{S}_q, \hat{\pi}_q)$.

Lemma 28 *The* $\mathbb{A}^1$*-fibered surfaces* (S, π_q) *and* $(S, \pi_{q'})$ *are equivalent if and only if* $(\hat{S}_q, \hat{\pi}_q)$ *and* $(\hat{S}_{q'}, \hat{\pi}_{q'})$ *are equivalent.*

Proof Since $C \cap S$ (resp. $C' \cap S$) is a reduced fiber of π_q (resp. $\pi_{q'}$) whereas $F_q \cap S$ (resp. $F_{q'} \cap S$) is a fiber of multiplicity two of it, every equivalence of $\mathbb{A}^1$-fibered surface $\Psi : (S, \pi_q) \to (S, \pi_{q'})$ maps $C \cap S$ onto $C' \cap S$ and $F_q \cap S$ onto $F_{q'} \cap S$, hence restricts to equivalence between $\hat{\Psi} : (\hat{S}_q, \hat{\pi}_q) \to (\hat{S}_{q'}, \hat{\pi}_{q'})$. Now assume conversely that there exists an equivalence of $\mathbb{A}^1$-fibered surfaces $\hat{\Psi} : (\hat{S}_q, \hat{\pi}_q) \to (\hat{S}_{q'}, \hat{\pi}_{q'})$. With the notation of Lemma 26, let

$$(\tilde{\mathbb{F}}_n, \hat{D}_q = \sigma_*^{-1}B \cup D_q \cup \sigma_*^{-1}C) \quad \text{and} \quad (\tilde{\mathbb{F}}_{n'}, \hat{D}'_{q'} = \sigma'^{-1}_* B' \cup D'_{q'} \cup \sigma'^{-1}_* C')$$

be the relatively minimal SNC completions of $\hat{\pi}_q$ and $\hat{\pi}_{q'}$ respectively. The isomorphism $\hat{\Psi}$ induces a birational map of $\mathbb{P}^1$-fibered surfaces $\hat{\Psi} : (\tilde{\mathbb{F}}_n, \tilde{\rho}_q) \dashrightarrow (\tilde{\mathbb{F}}_{n'}, \tilde{\rho}_{q'})$. In particular, $\hat{\Psi}$ maps the section E_d of $\tilde{\rho}_q$ isomorphically onto the section E'_d of $\tilde{\rho}_{q'}$, which implies that any proper base point of $\hat{\Psi}$ is supported either on $D_q \cup \sigma_*^{-1}C$ or on $\sigma_*^{-1}B$. Assume that $\hat{\Psi}$ has a proper base point supported on $D_q \cup \sigma_*^{-1}C$ and let $\tilde{\mathbb{F}}_n \stackrel{\eta}{\leftarrow} Z \stackrel{\eta'}{\rightarrow} \tilde{\mathbb{F}}_{n'}$ be the minimal resolution of $\hat{\Psi}$. Since $\hat{\Psi}$ maps E_d isomorphically onto E'_d and F_q onto $F_{q'}$, it follows that η' contracts $\eta^{-1}(D_q \cup \sigma_*^{-1}C)$

onto $D'_{q'} \cup \sigma'^{-1}_{*} C'$. Since $D_q \cup \sigma_*^{-1} C$ and $D'_{q'} \cup \sigma'^{-1}_{*} C'$ are SNC divisors with the same number of irreducible components, the minimality assumption implies that the proper transform in Z of $D_q \cup \sigma_*^{-1} C$ contains an η'-exceptional (-1)-curve. But this is impossible since all these curves have self-intersection ≤ -2 in Z. For the same reason, $\hat{\Psi}^{-1}$ has no proper base point on $D'_{q'} \cup \sigma'^{-1}_{*} C'$. So, $\hat{\Psi}$ is well-defined on an open neighborhood U of $\hat{D}_q \setminus \sigma_*^{-1} B$ in $\tilde{\mathbb{F}}_n$ and $\hat{\Psi}|_U$ is an isomorphism onto an open neighborhood U' of $\hat{D}_{q'} \setminus \sigma'^{-1}_{*} B'$ in $\tilde{\mathbb{F}}_{n'}$. The geometry of the divisors $\hat{D}_q \setminus \sigma_*^{-1} B$ and $\hat{D}_{q'} \setminus \sigma'^{-1}_{*} B'$ and the fact that $\hat{\Psi}$ maps $\sigma_*^{-1} F_q$ onto $\sigma'^{-1}_{*} F'_{q'}$ imply that $\hat{\Psi}(\sigma_*^{-1} C) = \sigma'^{-1}_{*} C'$ and hence, that $\hat{\Psi}$ induces an equivalence of $\mathbb{A}^1$-fibered surfaces $\Psi : (S, \pi_q) \to (S, \pi_{q'})$.

To study equivalence classes of $\mathbb{A}^1$-fibered surfaces $(\hat{S}_q, \hat{\pi}_q)$, we now introduce two auxiliary families of surfaces.

Notation 29 For every integer $\ell \geq 1$ and every polynomial $s \in k[x^2] \subset k[x]$ of degree $< \ell$ with $s(0) = 1$, denote by $\tilde{S}_{\ell,s}$ the surface in $\mathbb{A}^3 = \mathrm{Spec}(k[x, y, z])$ with equation $x^\ell z = y^2 - s^2(x)$. The morphism $\tilde{\pi}_{\ell,s} = \mathrm{pr}_x : \tilde{S}_{\ell,s} \to \mathbb{A}^1$ is a smooth $\mathbb{A}^1$-fibration with unique degenerate fiber $\tilde{\pi}_{\ell,s}^{-1}(0)$ consisting of two irreducible components $\{x = y \pm 1 = 0\}$. The morphism $\tilde{\pi}_{\ell,s}$ is equivariant for the actions of the group $\mu_2 = \{\pm 1\}$ given by $(-x, -y, (-1)^\ell z)$ on $\tilde{S}_{\ell,s}$ and by $x \mapsto -x$ on $\mathbb{A}^1$. As a scheme over $\mathbb{A}^1$, $\tilde{S}_{\ell,s}$ is μ_2-equivariantly isomorphic to the surface $W^{(-1)^{1-\ell}}_{2x^\ell s(x)}$ obtained by gluing two copies

$$U_\pm = \tilde{S}_{\ell,s} \setminus \{x = y \mp s(x) = 0\} = \mathrm{Spec}(k[x][u_\pm]), \quad \text{where } u_\pm = x^{-\ell}(y - s(x)) = (y + s(x))^{-1} z$$

of $\mathbb{A}^1 \times \mathbb{A}^1$ over $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$ by the isomorphism $U_+ \ni (x, u_+) \mapsto (x, u_+ + 2x^{-\ell} s(x)) \in U_-$, endowed with the μ_2-action $U_+ \ni (x, u_+) \mapsto (-x, (-1)^{1-\ell} u_+) \in U_-$ and with the $\mathbb{A}^1$-fibration $\theta_{2x^\ell s(x)} : W^{(-1)^{1-\ell}}_{2x^\ell s(x)} \to \mathbb{A}^1$ induced by the first projections on each of the open subsets $U_\pm$.

Notation 30 The categorical quotient $S_{\ell,s} = \tilde{S}_{\ell,s} /\!/ \mu_2$ in the category of affine schemes of the fixed point free μ_2-action on $\tilde{S}_{\ell,s}$ is a geometric quotient and the quotient morphism $\Phi_{\ell,s} : \tilde{S}_{\ell,s} \to S_{\ell,s}$ is a nontrivial μ_2-torsor, in particular, $S_{\ell,s}$ is a smooth affine surface. The $\mathbb{A}^1$-fibration $\tilde{\pi}_{\ell,s} : \tilde{S}_{\ell,s} \to \mathbb{A}^1$ descends to an $\mathbb{A}^1$-fibration $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1 /\!/ \mu_2 = \mathrm{Spec}(k[x^2])$ with $\pi_{\ell,s}^{-1}(0)$ as a unique degenerate fiber and we have a commutative diagram

$$\begin{tikzcd}
\tilde{S}_{\ell,s} \arrow[r, "\Phi_{\ell,s}"] \arrow[d, "\tilde{\pi}_{\ell,s}"'] & S_{\ell,s} = \tilde{S}_{\ell,s} /\!/ \mu_2 \arrow[d, "\pi_{\ell,s}"] \\
\mathbb{A}^1 = \mathrm{Spec}(k[x]) \arrow[r, "\phi"] & \mathbb{A}^1 = \mathrm{Spec}(k[x^2]),
\end{tikzcd}$$

where $\phi : \mathbb{A}^1 \to \mathbb{A}^1 /\!/ \mu_2$ is the quotient morphism induced by the inclusion $k[x^2] \subset k[x]$. Since $\Phi_{\ell,s}^{-1}(\pi_{\ell,s}^{-1}(0)) = \tilde{\pi}_{\ell,s}^{-1}(0)$ consists of two component which are exchanged by the μ_2-action and since Φ is étale whereas ϕ is totally ramified of ramification index 2 over 0, it follows that $\pi_{\ell,s}^{-1}(0)$ is irreducible, of multiplicity 2. The Picard group $\mathrm{Pic}(S_{\ell,s})$ is isomorphic to $\mathbb{Z}_2$, generated by the class of the ideal sheaf of the curve $F_{\ell,s} = (\pi_{\ell,s}^{-1}(0))_{\mathrm{red}}$ of $S_{\ell,s}$.

Lemma 31 *Every smooth affine surface S endowed with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ whose unique degenerate fiber is irreducible and of multiplicity two is equivalent to an $\mathbb{A}^1$-fibered surface $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1$ for some $\ell \geq 1$.*

Proof We can assume that $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ and that $\pi^{-1}(0)$ is the degenerate fiber of π. Let $\phi : A = \mathrm{Spec}(k[x]) \to \mathbb{A}^1$ be the ramified μ_2-cover $x \mapsto t = x^2$ and let $\nu : \tilde{S} \to S \times_{\pi,\mathbb{A}^1,\phi} A$ be the normalization of $S \times_{\pi,\mathbb{A}^1,\phi} A$, endowed with the μ_2-action lifting that on the second factor. By [15, Example 1.6 and Theorem 1.7], $\tilde{\pi} = \mathrm{pr}_1 \circ \nu : \tilde{S} \to A$ is μ_2-equivariantly isomorphic as an A-scheme to an affine surface W_f^{ε} obtained by gluing two copies $U_{\pm} = \mathrm{Spec}(k[x][u_{\pm}])$ of $A \times \mathbb{A}^1$ over $(A \setminus \{0\}) \times \mathbb{A}^1$ by an isomorphism of the form $U_+ \ni (x, u_+) \mapsto (x, u_+ + f(x)) \in U_-$ for some $f \in k[x^{-1}] \setminus k$, endowed with a μ_2-action of the form $U_+ \ni (x, u_+) \mapsto (-x, \varepsilon u_+) \in U_-$, where $\varepsilon = 1$ or -1, viewed as a scheme over A via the $\mathbb{A}^1$-fibration $\theta_f : W_f^{\varepsilon} \to A$ induced by the first projections on each of the open subsets $U_{\pm}$. If $\varepsilon = 1$, we have $f(-x) = -f(x)$, which implies that the pole order $\ell = -\mathrm{ord}_0 f$ of f at 0 is odd. The polynomial $\sigma = \frac{1}{2}x^{\ell} f(x) \in k[x^2] \setminus x^2 k[x^2]$ can be written in the form $\sigma = \lambda(s(x) + x^{\ell} r(x))$ for some $s \in k[x^2]$ of degree $< \ell$ with $s(0) = 1$ and $\lambda \in k^*$ and the local isomorphisms

$$U_{\pm} = \mathrm{Spec}(k[x][u_{\pm}]) \to U'_{\pm} = \mathrm{Spec}(k[x][u'_{\pm}]),\ (x, u_{\pm}) \to (x, \lambda^{-1}(u_{\pm} \pm r(x)))$$

glue to a μ_2-equivariant isomorphism $W_f^1 \to W^1_{2x^{-\ell}s(x)} \cong \tilde{S}_{\ell,s}$ of A-schemes. The latter induces in turn an isomorphism $S = \tilde{S}//\mu_2 = W_f^1 \cong \tilde{S}_{\ell,s}//\mu_2 = S_{\ell,s}$ of $\mathbb{A}^1$-fibered surfaces over $\mathbb{A}^1$. If $\varepsilon = -1$, then $f(-x) = f(x)$, $\ell = -\mathrm{ord}_0 f$ is even and the same argument shows that $\pi : S \to \mathbb{A}^1$ is isomorphic as a scheme over $\mathbb{A}^1$ to $\pi_{\ell,s} : S_{\ell,s} = W^{-1}_{2x^{-\ell}s(x)}//\mu_2 \to \mathbb{A}^1$ where s is the unique polynomial of degree $< \ell$ with $s(0) = 1$ such that $\frac{1}{2}x^{\ell} f(x) = \lambda(s(x) + x^{\ell} r(x)) \in k[x^2] \setminus x^2 k[x^2]$.

Lemma 32 *Let $(S_{\ell_i,s_i}, \pi_{\ell_i,s_i})$, $i = 1, 2$ be $\mathbb{A}^1$-fibered affine surfaces as in Notation 30. Then the following are equivalent:*

(a) The $\mathbb{A}^1$-fibered surfaces $(S_{\ell_1,s_1}, \pi_{\ell_1,s_1})$ and $(S_{\ell_2,s_2}, \pi_{\ell_2,s_2})$ are equivalent,
(b) The surfaces S_{ℓ_1,s_1} and S_{ℓ_2,s_2} are isomorphic,
(c) There exists $\lambda \in k^*$ such that $s_2(\lambda x) = s_1(x)$.

Proof The implication (a)$\Rightarrow$(b) is clear. Now put $S_1 = S_{\ell_1,s_1}$, $S_2 = S_{\ell_2,s_2}$ and assume that there exists an isomorphism $\Psi : S_1 \to S_2$. Since $\mathrm{Pic}(S_2) \cong H^1_{ét}(S_2, \mathscr{O}^*_{S_2}) \cong \mathbb{Z}_2$ and $H^0(S_2, \mathscr{O}^*_{S_2}) = k^*$, the long exact sequence of étale cohomology associated to the short exact sequence of sheaves of abelian groups

$$1 \to \mu_2 \to \mathscr{O}^*_{S_2} \xrightarrow{f \mapsto f^2} \mathscr{O}^*_{S_2} \to 1$$

implies that $H^1_{\text{ét}}(S_2, \mu_2) = \mathbb{Z}_2$, generated by the class of the μ_2-torsor $\Phi_{\ell,s} : \tilde{S}_{\ell_2,s_2} \to S_2$. Since $\Psi \circ \Phi_{\ell_1,s_1} : \tilde{S}_{\ell_1,s_1} \to S_2$ is a nontrivial μ_2-torsor, it follows that there exists a unique μ_2-equivariant isomorphism $\tilde{\Psi} : \tilde{S}_{\ell_1,s_1} \to \tilde{S}_{\ell_2,s_2}$ such that $\tilde{\Psi} \circ \Phi_{\ell_1,s_1} = \Phi_{\ell_2,s_2} \circ \Psi$. We deduce from [14, Proposition 3.6] that $\ell_1 = \ell_2$ and that there exists a pair $(\lambda, \mu) \in k^* \times k^*$ such that $s_2^2(\lambda x) = \mu^2 s_1^2(x)$. Since $s_1(0) = s_2(0) = 1$, the only possibility is that $\mu = \pm 1$ and the implication b)⇒c) follows. The last implication c)⇒a) follows from the observation that for $\ell_1 = \ell_2 = \ell$ and $s_2(\lambda x) = s_1(x)$, the morphism $\tilde{\Psi} : \tilde{S}_{\ell,s_1} \to \tilde{S}_{\ell,s_2}$ defined by $(x, y, z) \mapsto (\lambda x, y, \lambda^{-\ell} z)$ is a μ_2-equivariant equivalence between the $\mathbb{A}^1$-fibered surfaces $(\tilde{S}_{\ell,s_1}, \tilde{\pi}_{\ell,s_1})$ and $(\tilde{S}_{\ell,s_2}, \tilde{\pi}_{\ell,s_2})$ which descends to an equivalence between $(S_{\ell,s_1}, \pi_{\ell,s_1})$ and $(S_{\ell,s_2}, \pi_{\ell,s_2})$.

Example 33 For every $\ell \geq 5$, put $m = \left\lfloor \frac{\ell-3}{2} \right\rfloor$, $R = k[a_1, \ldots a_m]$ and $\mathfrak{s}(x) = 1 + x^2 + \sum_{i=2}^m a_i x^{2i} \in R[x^2]$. Let $V = \mathrm{Spec}(R) \cong \mathbb{A}^m$ and let $\mathfrak{S}_\ell$ be the quotient of the closed subscheme $\tilde{\mathfrak{S}}_\ell \subset V \times \mathbb{A}^3$ with equation $x^\ell z = y^2 - \mathfrak{s}^2(x)$ by the $\mu_{2,V}$-action $(x, y, z) \mapsto (-x, -y, (-1)^\ell z)$. By Lemma 32, the closed fibers of the smooth morphism $\Theta : \mathfrak{S}_\ell \to V$ induced by the μ_2-invariant projection $\mathrm{pr}_1 : \tilde{\mathfrak{S}}_\ell \to V$ are pairwise non-isomorphic surfaces of the form $S_{\ell,s}$.

We now relate the family of surfaces $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ of Notation 27 to those $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1$ of Notation 30.

Lemma 34 *An $\mathbb{A}^1$-fibered affine surface $\pi_{\ell,s} : S_{\ell,s} \to \mathbb{A}^1$ admits a relatively minimal SNC completion $(Y_{\ell,s}, D_{\ell,s})$ into a $\mathbb{P}^1$-fibered surface $\bar{\pi}_{\ell,s} : Y_{\ell,s} \to \mathbb{P}^1$ such that the union of $D_{\ell,s}$ and of the closure $\overline{F}_{\ell,s}$ of $F_{\ell,s}$ is a rational tree of the form*

$$\begin{array}{l} (F_\infty, 0) \triangleleft (H, -1) \triangleleft (G_0, -2) \triangleleft \underset{\displaystyle (G_1, -2),}{\overset{}{(G_2, -2)}} \triangleleft \cdots\cdots\cdots \triangleleft (G_{\ell+1}, -2) - (\bar{F}_{\ell,s}, -1) \end{array}$$

where F_∞ is the fiber of $\bar{\pi}_{\ell,s}$ over $\mathbb{P}^1 \setminus \mathbb{A}^1$, H is a section of $\bar{\pi}_{\ell,s}$ and $\bar{\pi}_{\ell,s}^{-1}(0) = \overline{F}_{\ell,s} \cup \bigcup_{i=0}^{\ell+1} G_i$.

Proof For every $\ell \geq 2$ and every polynomial $s_\ell \in k[x^2]$ of degree $< \ell$ with $s(0) = 1$, write $s_\ell = s_{\ell-1} + a x^{\ell-1}$ where $s_{\ell-1} \in k[x^2]$ is a polynomial of degree $< \ell - 1$ and $a \in k$. The endomorphism $(x, y, z) \mapsto (x, y, xz + 2a s_{\ell-1} + a^2 x^{\ell-1})$ of $\mathbb{A}^3$ induces a μ_2-equivariant birational morphism $\tilde{\sigma} : \tilde{S}_{\ell,s_\ell} \to \tilde{S}_{\ell-1,s_{\ell-1}}$ of $\mathbb{A}^1$-fibered surfaces. It descends to a birational morphism $\sigma : S_{\ell,s_\ell} \to S_{\ell-1,s_{\ell-1}}$ of $\mathbb{A}^1$-fibered surfaces restricting to an isomorphism over $\mathbb{A}^1 \setminus \{0\}$ and contracting F_{ℓ,s_ℓ} onto a point x_{ℓ,s_ℓ} of $F_{\ell-1,s_{\ell-1}}$. This morphism σ expresses S_{ℓ,s_ℓ} as the surface obtained from $S_{\ell-1,s_{\ell-1}}$ by blowing-up the point x_{ℓ,s_ℓ} and then removing the proper transform of F_{ℓ,s_ℓ}. Assume that $(Y_{\ell-1,s_{\ell-1}}, D_{\ell-1,s_{\ell-1}})$ is a relatively minimal SNC completion of $\pi_{\ell-1,s_\ell} : S_{\ell-1,s_{\ell-1}} \to \mathbb{A}^1$ into a $\mathbb{P}^1$-fibered surface $\overline{\pi}_{\ell-1,s_{\ell-1}} : Y_{\ell-1,s_{\ell-1}} \to \mathbb{P}^1$ which satisfies the claimed properties. Then the pair $(Y_{\ell,s_\ell}, D_{\ell,s_\ell})$, where $\tau : Y_{\ell,s_\ell} \to Y_{\ell-1,s_{\ell-1}}$ is the blow-up of the point $x_{\ell,s_\ell} \in \overline{F}_{\ell-1,s_{\ell-1}}$ and $D_{\ell,s}$ is the proper transform of $D_{\ell-1,s_{\ell-1}} \cup \overline{F}_{\ell-1,s_{\ell-1}}$, endowed the $\mathbb{P}^1$-fibration $\bar{\pi}_{\ell,s_\ell} = \overline{\pi}_{\ell-1,s_{\ell-1}} \circ \tau$ is a relatively minimal SNC completion of $\pi_{\ell,s_\ell} : S_{\ell,s_\ell} \to \mathbb{A}^1$ which also satisfies the claimed properties. Now the assertion follows by induction from Example 15 and the fact that $\pi_{1,1} : S_{1,1} = \{xz = y^2 - 1\}//\mu_2 \to \mathbb{A}^1$ is isomorphic to the complement of the smooth conic $Q = \{-xz + y^2 = 0\}$ in $\mathbb{P}^2_{[x:y,z]}$, endowed with the $\mathbb{A}^1$-fibration associated to the pencil $\mathscr{P}_{[0:0:1]}$.

By combining Lemmas 26, 31 and 34, we obtain that every $\mathbb{A}^1$-fibered surface $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ as in Notation 27 is equivalent to some surface of the form $\pi_{d-2,s} : S_{d-2,s} \to \mathbb{A}^1$ of Notation 30, and, conversely, that every equivalence class of $\mathbb{A}^1$-fibered surface $\pi_{d-2,s} : S_{d-2,s} \to \mathbb{A}^1$ is realized by an $\mathbb{A}^1$-fibration $\hat{\pi}_q : \hat{S}_q \to \mathbb{A}^1$ induced by restriction of an $\mathbb{A}^1$-fibration on an affine surface $S = \mathbb{F}_n \setminus B$ with $B^2 = d$. Combining in turn this result with Lemma 28, we obtain the following:

Corollary 35 *Let $(\mathbb{F}_n, B)$ be a pair as in Lemma 3 (b) with $d = B^2 \geq 3$. Then equivalence classes of $\mathbb{A}^1$-fibered surfaces $((\mathbb{F}_n \setminus B)_q, \pi_q)$ where q ranges through the set of closed points of the boundaries B' of smooth completions $(\mathbb{F}_{n'}, B')$ of $\mathbb{F}_n \setminus B$ such that $\mathscr{P}_q$ has a singular member of the form $C + 2F_q$ are in one-to-one correspondence with equivalence classes of $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ of Notation 30.*

Proposition 25 is now a straightforward consequence of Corollary 35 and of the description of equivalence classes of $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ given in Lemma 32. Namely, for $d = 3, 4$, the unique equivalences classes are those of $(S_{1,1}, \pi_{1,1})$ and $(S_{2,1}, \pi_{2,1})$ respectively. For $d = 5$, the two equivalence classes are those of the surfaces $(S_{3,1}, \pi_{3,1})$ and $(S_{3,x^2+1}, \pi_{3,x^2+1})$. The case $d = 6$ is similar. Finally, if $d \geq 7$, then Example 33 provides a family pairwise non-equivalent $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ parametrized by the elements of k^m, where $m = \lfloor \frac{d-5}{2} \rfloor \geq 1$, showing in particular that the cardinality of $\mathscr{A}_2(d)$ is at least equal to that of k.

Remark 36 The "number of moduli" $m = \lfloor \frac{d-5}{2} \rfloor \geq 1$ for equivalence classes of $\mathbb{A}^1$-fibered surfaces $(S_{d-2,s}, \pi_{d-2,s})$ with a unique singular fiber of multiplicity two deduced from the explicit family in Example 33 is the same as that computed by different techniques in [19], as can be seen by taking $k = 2$ in Corollary 6.3.20 of *loc. cit.*. The results in [19] apply more generally, in particular, to any smooth affine $\mathbb{A}^1$-fibered surface $S \to \mathbb{A}^1$ having a unique singular fiber, irreducible of arbitrary multiplicity $e \geq 2$. On the other hand, it follows from [15] that similarly as in the case $e = 2$ described above, every such surface can be realized as a quotient of smooth affine surface $\tilde{S}$ endowed with a smooth $\mathbb{A}^1$-fibration $\tilde{\pi} : \tilde{S} \to \mathbb{A}^1$ by a suitable free action of a cyclic group μ_e of eth roots of unity. This suggests the possibility to construct for every $e \geq 2$ explicit families as in Example 33 over a base scheme V whose dimension equals the number of moduli computed in [19, Corollary 6.3.20].

4.3 Proof of Theorem 2

In this subsection, we finish the proof of Theorem 2. Let $S = \mathbb{F}_n \setminus B$ for some pair $(\mathbb{F}_n, B)$ as in Lemma 3 (b) with $d = B^2 \geq 2$. If $d \geq 7$, then, by Proposition 25, S has infinitely many equivalence classes of $\mathbb{A}^1$-fibrations $\pi : S \to \mathbb{A}^1$. It remains to show that for every $d \leq 6$, the number of equivalence classes is finite. For every $d \geq 2$ and every $m \in 1, \dots, d-1$, denote by $\mathscr{A}_m(d)$ the set of equivalence classes of $\mathbb{A}^1$-fibrations $\pi : S \to \mathbb{A}^1$ whose unique degenerate fiber has an irreducible component of multiplicity m. The following table summarizes the properties of the sets $\mathscr{A}_m(d)$ (Table 1):

Table 1 Numbers of equivalence classes of $\mathbb{A}^1$-fibrations

	$\sharp\mathscr{A}_1(d)$	$\sharp\mathscr{A}_2(d)$	$\sharp\mathscr{A}_3(d)$	$\sharp\mathscr{A}_4(d)$	$\sharp\mathscr{A}_5(d)$	$\sum \sharp\mathscr{A}_i(d)$
$d = 2$	1	0	0	0	0	1
$d = 3$	1	1	0	0	0	2
$d = 4$	1	1	1	0	0	3
$d = 5$	1	2	1	1	0	5
$d = 6$	1	2	2	1	1	7

Indeed, we have $\mathscr{A}_m(d) = \emptyset$ if $m \geq d$ by Corollary 19. On the other hand, the cardinal $\sharp\mathscr{A}_m(d)$ of $\mathscr{A}_m(d)$ is larger than or equal to 1 for every $1 \leq m \leq d-1$ by Lemma 20. The sets $\mathscr{A}_{d-1}(d)$ and $\mathscr{A}_1(d)$ both consist of a single element by Proposition 21 and Corollary 23 respectively. By Proposition 25, we have $\sharp\mathscr{A}_2(d) = 1$ for $d = 3, 4$ and $\sharp\mathscr{A}_2(d) = 2$ for $d = 5, 6$. These observations settle the cases $d = 2$, 3 and 4. In the next paragraphs, we determine the remaining numbers of equivalence classes of $\mathbb{A}^1$-fibrations displayed in the table. We refer the reader to [13, Sect. 4] for the details of the reductions to the chosen particular models of pairs which are used in the argument.

4.3.1 The Case $d = 5$

The $\mathbb{A}^1$-fibrations on S representing elements of $\mathscr{A}_3(5)$ can only arise form pencils $\mathscr{P}_q$ on pairs $(\mathbb{F}_1, B)$ for which $B \sim C_0 + 3F$ intersects C_0 with multiplicity two at a single point q. Up to isomorphism, there is a unique such pair, which is given, under the identification of $\mathbb{F}_1$ with the blow-up $\sigma : \mathbb{F}_1 \to \mathbb{P}^2$ of $\mathbb{P}^2$ with homogeneous coordinates $[x : y : z]$ at the point $p = [0 : 1 : 0]$ with exceptional divisor C_0, by taking for B the proper transform of the cuspidal cubic $C = \{x^3 - z^2 y = 0\}$ in $\mathbb{P}^2$. The section $B = \sigma_*^{-1} C \sim C_0 + 3F$ intersects C_0 with multiplicity two at the intersection point q of C_0 with the proper transform of the tangent line $T_p C = \{z = 0\}$ to C at p and the pencil $\mathscr{P}_q$ is the proper transform of the pencil generated by C

and $3T_pC$. The associated $\mathbb{A}^1$-fibration $\pi_q : S \to \mathbb{A}^1$ has $F_q \cap S$ as a component of multiplicity three in its degenerate fiber. We conclude that $\sharp\mathscr{A}_3(5) = 1$.

4.3.2 The Case $d = 6$

The numbers to be computed are $\sharp\mathscr{A}_3(6)$ and $\sharp\mathscr{A}_4(6)$. The possible smooth completions $(\mathbb{F}_n, B)$ of S are either of the form $(\mathbb{F}_0, B)$ where $B \sim C_0 + 3F$ is a section of π_0, or of the form $(\mathbb{F}_2, B)$ where $B \sim C_0 + 4F$ is a section of π_2, or the form $(\mathbb{F}_4, B)$ where $B \sim C_0 + 5F$ is a section of π_4.

The $\mathbb{A}^1$-fibrations on S representing elements of $\mathscr{A}_4(6)$ can arise only from pairs $(\mathbb{F}_2, B)$ for which $B \sim C_0 + 4F$ intersects C_0 with multiplicity two in a single point. Up to isomorphism, there exists a unique such pair which is given, after fixing a fiber F_∞ of π_2 and an identification $\mathbb{F}_2 \setminus (C_0 \cup F_\infty) \cong \mathbb{A}^2 = \mathrm{Spec}(k[x, y])$ in such way that $\pi_2|_{\mathbb{A}^2} = \mathrm{pr}_x$ and that the closures in $\mathbb{F}_2$ of the level sets of y are sections of π_2 linearly equivalent to $C_0 + 2F$, by taking for B the closure of the curve $\Gamma_{x^4} = \{y = x^4\} \subset \mathbb{A}^2$. For the point $q = B \cap C_0 = B \cap F_\infty$, the singular member of the pencil $\mathscr{P}_q$ is equal to $C_0 + 4F_\infty$. The unique degenerate fiber of the corresponding $\mathbb{A}^1$-fibration $\pi_{q_4} : S \to \mathbb{A}^1$ has $F_\infty \cap S$ as a component of multiplicity four and we conclude that $\sharp\mathscr{A}_4(6) = 1$.

The $\mathbb{A}^1$-fibrations representing elements of $\mathscr{A}_3(6)$ can arise only from pairs $(\mathbb{F}_0, B)$ on $\pi_0 = \mathrm{pr}_1 : \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for which $B \sim C_0 + 3F$ intersects a fiber of the second projection with multiplicity three at some point q. Up to isomorphisms, there are exactly two such pairs $(\mathbb{F}_0, B_1)$ and $(\mathbb{F}_0, B_2)$ which, using bi-homogeneous coordinates $([u_0 : u_1], [v_0 : v_1])$ on $\mathbb{P}^1 \times \mathbb{P}^1$, are given by the curves $B_1 = \{u_1^3 v_0 + u_0^2(u_0 + u_1)v_1 = 0\}$ and $B_2 = \{u_1^3 v_0 + u_0^3 v_1 = 0\}$. The only fiber of pr_2 which intersects B_1 in a single point is the curve $C_{[1:0]} = \{v_1 = 0\}$ with $q = C_{[1,0]} \cap B_1 = ([1 : 0], [1 : 0])$. This yields an $\mathbb{A}^1$-fibration $\pi_q : S \to \mathbb{A}^1$ which has $F_{q_1} \cap S$ as component of multiplicity three in its degenerate fiber. In contrast, there are two fibers of pr_2 which intersects B_2 in a single point: the curve $C_{[0:1]} = \{v_0 = 0\}$ at the point $q_0 = ([0 : 1], [0 : 1])$ and the curve $C_{[1:0]}$ at the point $q_\infty = ([1 : 0], [1 : 0])$. The $\mathbb{A}^1$-fibrations $\pi_{q_0} : S \to \mathbb{A}^1$ and $\pi_{q_\infty} : S \to \mathbb{A}^1$ associated to the pencils $\mathscr{P}_{q_0}$ and $\mathscr{P}_{q_\infty}$ have $F_{q_0} \cap S$ and $F_{q_\infty} \cap S$ as components of multiplicity three of their respective degenerate fibers, and since the points q_0 and q_∞ belongs to the same orbit of the action of the group $\mathrm{Aut}(\mathbb{F}_0, B_2) \cong \mathbb{G}_m \times \mathbb{Z}_2$, these $\mathbb{A}^1$-fibrations represents a same element of $\mathscr{A}_3(6)$. The next lemma shows that $\mathscr{A}_3(6)$ consists of two elements and completes the proof.

Lemma 37 *The $\mathbb{A}^1$-fibration $\pi_q : S \to \mathbb{A}^1$ is not equivalent to $\pi_{q_0} : S \to \mathbb{A}^1$ (hence not equivalent to $\pi_{q_\infty} : S \to \mathbb{A}^1$).*

Proof The curve $C_{[1:0]} \cap S \cong \mathbb{A}^1$ is a 3-section of $\pi_{q_0} : S \to \mathbb{A}^1$ which intersects the multiple irreducible component $F_{q_0} \cap S$ of the degenerate fiber of π_{q_0} transversely in a single point. To verify that $\pi_q : S \to \mathbb{A}^1$ is not equivalent to $\pi_{q_0} : S \to \mathbb{A}^1$, it suffices to show that there is no 3-section of $\pi_q : S \to \mathbb{A}^1$ isomorphic to $\mathbb{A}^1$ and intersecting F_q transversely in a single point. Suppose on the contrary that such a

3-section D exists. Let $\sigma : \tilde{\mathbb{F}}_0 \to \mathbb{F}_0$ be the minimal resolution of the rational map $\rho_q : \mathbb{F}_0 \dashrightarrow \mathbb{P}^1$ defined by $\mathscr{P}_q$. The closure $\bar{D}$ of D in $\tilde{\mathbb{F}}_0$ is a rational 3-section of the $\mathbb{P}^1$-fibration $\tilde{\rho}_q = \rho_q \circ \sigma$ which intersects the proper transform $\sigma_*^{-1} B_1$ of B_1 with multiplicity 3 in a single point p. The total transform of $B_1 \cup F_q \cup C_{[1:0]}$ in $\tilde{\mathbb{F}}_0$ being rational tree of the form

$$\begin{array}{ccccccccccccccc} (\sigma_*^{-1}B, 0) & \triangleleft & (E_6, -1) & \triangleleft & (E_5, -2) & \triangleleft & (E_4, -2) & \triangleleft & (E_3, -2) & \triangleleft & (E_2, -2) & \triangleleft & (E_1, -2) & - & (\sigma_*^{-1}F_q, -1) \\ & & & & & & & & | & & & & & & \\ & & & & & & & & (\sigma_*^{-1}C_{[1:0]}, -3), & & & & & & \end{array}$$

there exists a unique birational morphism of $\mathbb{P}^1$-fibered surface $\tau : \tilde{\mathbb{F}}_0 \to \mathbb{F}_1$ which contracts $\sigma_*^{-1} F_q \cup \sigma_*^{-1} C_{[1:0]} \cup \bigcup_{i=1}^4 E_i$ onto a point $s \in \tau(E_5) \setminus \tau(E_6)$. The curve $\tau(\bar{D})$ is a 3-section of $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ which has a cusp of multiplicity 2 at s and intersects $\tau(E_5)$ with multiplicity 3 at s. Let C be the image of $\tau(\bar{D})$ by the contraction $\alpha : \mathbb{F}_1 \to \mathbb{P}^2$ of $\tau(E_6)$ to a point p'. Assume that $m = \tau(\bar{D}) \cdot \tau(E_6) \geq 1$. Then C is a curve of degree $m + 3$ which intersects the line $\alpha(\tau(\sigma_*^{-1} B_1))$ with multiplicity $m + 3$ at p' and the line $\alpha(\tau(E_5))$ with multiplicity m at p' and multiplicity 3 at $\alpha(s)$. Choosing homogeneous coordinates $[x : y : z]$ on $\mathbb{P}^2$ so that $\alpha(\tau(\sigma_*^{-1} B_1)) = \{z = 0\}$, $\alpha(\tau(E_5)) = \{x = 0\}$ and $\alpha(s) = [0 : 0 : 1]$, the curve C is thus given by an equation of the form $\lambda x^{m+3} - \mu y^3 z^m = 0$ for some $\lambda, \mu \in k^*$. But this is impossible since on the other hand $C = \alpha(\tau(\bar{D}))$ has multiplicity 2 at $\alpha(s)$. So $m = 0$ and hence, C is a cubic with a cusp at $\alpha(s)$ and intersecting $\tau(\sigma_*^{-1} B)$ with multiplicity 3 at a point other than p'. It follows that $\sigma(\bar{D})$ is a smooth rational curve which intersects F_q transversely at unique point of $F_q \setminus \{q\}$ and B_1 at a unique point of $B_1 \setminus \{q\}$, with multiplicity 3. Thus, $\sigma(\bar{D})$ is a fiber of $\mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ which intersects B_1 with multiplicity 3 at a point other than q, which is impossible.

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References

1. Bertin, J.: Pinceaux de droites et automorphismes des surfaces affines. J. Reine Angew. Math. **341**, 32–53 (1983)
2. Blanc, J., Dubouloz, A.: Automorphisms of $\mathbb{A}^1$-fibered affine surfaces. Trans. Amer. Math. Soc. **363**(11), 5887–5924 (2011)
3. Blanc, J., Dubouloz, A.: Affine surfaces with a huge group of automorphisms. Int. Math. Res. Not. IMRN (2), 422–459 (2015)
4. Blanc, J., van Santen, I.: Embeddings of affine spaces into quadrics. Trans. Amer. Math. Soc. **371**(12), 8429–8465 (2019)
5. Cassou-Noguès, P., Russell, P.: Birational morphisms $\mathbb{C}^2 \to \mathbb{C}^2$ and affine ruled surfaces, Affine algebraic geometry, 57–105. Osaka University Press, Osaka (2007)
6. Danilov, V.I., Gizatullin, M.H.: Automorphisms of affine surfaces, I. Izv. Akad. Nauk SSSR Ser. Mat. **39**(3), 523–565 (1975)

7. Danilov, V.I., Gizatullin, M.H.: Automorphisms of affine surfaces, II. Izv. Akad. Nauk SSSR Ser. Mat. **41**(1), 54–103 (1977)
8. Hartshorne, R.: Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer, New York (1977)
9. Dubouloz, A.: Sur une classe de schémas avec actions de fibrés en droites. Ph.D. Thesis, Grenoble (2004)
10. Dubouloz, A.: Completions of normal affine surfaces with a trivial Makar-Limanov invariant. Michigan Math. J. **52**(2), 289–308 (2004)
11. Dubouloz, A.: Complements of hyperplane sub-bundles in projective spaces bundles over $\mathbb{P}^1$. Math. Ann. **361**(1–2), 259–273 (2015)
12. Dubouloz, A., Finston, D.R.: On exotic affine 3-spheres. J. Algebraic Geom. **23**(3), 445–469 (2014)
13. Dubouloz, A., Lamy, S.: Automorphisms of open surfaces with irreducible boundary. Osaka J. Math. **52**(3), 747–791 (2015)
14. Dubouloz, A., Poloni, P.-M.: On a class of Danielewski surfaces in affine 3-space. J. Algebra **321**(7), 1797–1812 (2009)
15. Fieseler, K.-H.: On complex affine surfaces with $\mathbb{C}_+$ -action. Comment. Math. Helv. **69**(1), 5–27 (1994)
16. Flenner, H., Kaliman, S., Zaidenberg, M.: Birational transformations of weighted graphs, Affine algebraic geometry, 107–147. Osaka Univ. Press, Osaka (2007)
17. Flenner, H., Kaliman, S., Zaidenberg, M.: Completions of $\mathbb{C}^*$ -surfaces. Affine Algebraic Geometry, pp. 149–201. Osaka University Press, Osaka (2007)
18. Flenner, H., Kaliman, S., Zaidenberg, M.: On the Danilov-Gizatullin isomorphism theorem. Enseign. Math. (2) **55**(3–4), 275–283 (2009)
19. Flenner, H., Kaliman, S., Zaidenberg, M.: Smooth affine surfaces with non-unique $\mathbb{C}^*$ -actions. J. Algebraic Geom. **20**(2), 329–398 (2011)
20. Gizatullin, M.H.: On affine surfaces that can be completed by a nonsingular rational curve. Math. USSR Izv. **4**, 787–810 (1970)
21. Gizatullin, M.H.: Quasihomogeneous affine surfaces. Izv. Akad. Nauk SSSR Ser. Mat. **35**, 1047–1071 (1971)
22. Goodman, J.E.: Affine open subsets of algebraic varieties and ample divisors. Ann. Math. (2) **89**, 160–183 (1969)
23. Kovalenko, S.: Transitivity of automorphism groups of Gizatullin surfaces. Int. Math. Res. Not. IMRN **21**, 11433–11484 (2015)
24. Kovalenko, S., Perepechko, A., Zaidenberg, M.: On automorphism groups of affine surfaces. Algebraic Varieties and Automorphism Groups. Advanced Studies in Pure Mathematics, vol. 75, pp. 207–286. Mathematics Society, Japan, Tokyo (2017)
25. Miyanishi, M.: Open Algebraic Surfaces. CRM Monograph Series, vol. 12. American Mathematical Society, Providence, RI (2001)

On Fano Threefolds of Degree 22 After Cheltsov and Shramov

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Abstract It has been known that nonsingular Fano threefolds of Picard rank one with the anti-canonical degree 22 admitting faithful actions of the multiplicative group form a one-dimensional family. Cheltsov and Shramov showed that all but two of them admit Kähler–Einstein metrics. In this paper, we show that the remaining Fano threefolds also admit Kähler–Einstein metrics.

Keywords Fano varieties · K-stability · Kähler–Einstein metrics

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1 Introduction

Let X be a nonsingular Fano threefold over the complex number field with $\mathrm{Pic}(X) = \mathbb{Z}[-K_X]$. Such X with large automorphism groups have been studied by many authors. For example, Mukai and Umemura systematically studied in [24] the Fano threefold V^{MU} with the anti-canonical degree 22, so-called *the Mukai-Umemura threefold*, which is obtained by the unique $\mathrm{SL}(2, \mathbb{C})$-equivariant nonsingular projective compactification of $\mathrm{SL}(2, \mathbb{C})/I_h$ with the Picard rank one, where $I_h \subset \mathrm{SL}(2, \mathbb{C})$ is the icosahedral group. The automorphism group of V^{MU} is equal to $\mathbf{PGL}(2, \mathbb{C})$. On the other hand, Prokhorov showed in [25] that, if $\mathrm{Aut}(X)$ is not finite, then the anti-canonical degree of X must be equal to 22. Moreover, he determined all of such X. For example, there is a unique Fano threefold V^{a} such that $\mathrm{Aut}^0(V^{\mathrm{a}})$ is equal to the additive group $\mathbb{C}^+$.

Nowadays, the structures of such X are well-understood thanks to the works of Kuznetsov, Prokhorov, Shramov [19], and Kuznetzov, Prokhorov [18]. The family of nonsingular Fano threefolds X with $\mathrm{Pic}(X) = \mathbb{Z}[-K_X]$, $(-K_X)^{\cdot 3} = 22$, $X \not\simeq V^{\mathrm{a}}$

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and $\mathrm{Aut}(X)$ infinite are parametrized by $\mathbb{C} \setminus \{0, 1\}$. Let us denote the family by $\{V_u\}_{u\in\mathbb{C}\setminus\{0,1\}}$. Then $V_{-1/4}$ is isomorphic to V^{MU}, and the automorphism group of V_u is equal to $G := \mathbb{C}^* \rtimes (\mathbb{Z}/2\mathbb{Z})$ unless $u = -1/4$. Moreover, any V_u can be obtained by the two-ray game from the blowup of some 3-dimensional quadric hypersurface along a certain nonsingular sextic rational curve.

In [4], Cheltsov and Shramov considered the problem for the existence of Kähler–Einstein metrics for the above V_u. If $u = -1/4$, then the Fano threefold is the Mukai-Umemura threefold V^{MU}. In this case, Donaldson already showed in [7] the existence of Kähler–Einstein metrics on V^{MU} by showing that the $\mathbf{PGL}(2, \mathbb{C})$-invariant α-invariant $\alpha_{\mathbf{PGL}(2,\mathbb{C})}(V^{\mathrm{MU}})$ (see Sect. 2) of V^{MU} is equal to $5/6$. In fact, Tian showed in [27] that, for a Fano manifold X and a reductive subgroup $\Gamma \subset \mathrm{Aut}(X)$, if $\alpha_\Gamma(X) > \dim X/(\dim X + 1)$ holds, then X admits Kähler–Einstein metrics. Cheltsov and Shramov considered the remaining cases by evaluating the G-invariant α-invariant of V_u. More precisely, they showed the following:

Theorem 1.1 (Theorem 1.5, [4]) *We have*

$$\alpha_G(V_u) = \begin{cases} 4/5 & \text{if } u \neq 3/4 \text{ and } u \neq 2, \\ 3/4 & \text{if } u = 3/4, \\ 2/3 & \text{if } u = 2. \end{cases}$$

Thus, together with Tian's result, if $u \neq 3/4$ and $u \neq 2$, then V_u admits Kähler–Einstein metrics. However, if $u = 3/4$ or $u = 2$, then the existence of Kähler–Einstein metrics of V_u was not known at that time. The purpose of this paper is to show the existence of Kähler–Einstein metrics for $V_{3/4}$ and V_2. The main result is the following:

Theorem 1.2 *Both $V_{3/4}$ and V_2 admit Kähler–Einstein metrics.*

Thus, together with Matsushima's obstruction [23], we gave a complete answer for the existence of Kähler–Einstein metrics of nonsingular Fano threefolds X with $\mathrm{Pic}(X) = \mathbb{Z}[-K_X]$ and $\mathrm{Aut}(X)$ infinite; such X admits Kähler–Einstein metrics if and only if $X \not\simeq V^{\mathrm{a}}$.

It would be interesting to know what happen for $u = 0, 1, \infty$. When $u = 1$, then V_1 is a terminal Gorenstein Fano threefold with one ordinary double point, $\mathrm{Pic}(V_1) = \mathbb{Z}[-K_{V_1}]$ and $\mathrm{Cl}(V_1) \simeq \mathbb{Z}^2$ (see [18, Proposition 5.4] and [4, Remark 2.12]). The author does not know whether the V_1 admits (weak) Kähler–Einstein metrics or not.

The main technique to prove Theorem 1.2 is the "G-valuative criterion" (see Sect. 2, [1, 29]), which is a G-equivariant version of [22, Theorem 3.7] and [13, Theorem 1.6]. Moreover, we need deep analyses [4] of G-invariant curves on V_u which might be possible destabilizing centers of G-invariant prime divisors over V_u. Technically, the theory of quasi-log schemes [10] and the careful analysis of the volume functions (see Sect. 3) play important roles in order to show Theorem 1.2. Especially, we use a subadjunction-type result for projective qlc strata on quasi-log schemes [11] (see Theorem 4.2). We expect that these techniques will be applied for many other Fano varieties (cf. [1]).

2 Log Fano Pairs

In this section, we recall the definition of α-invariant for log Fano pairs and see the relationship between α-invariant and the existence of Kähler–Einstein metrics. For the minimal model program, we refer the readers to [17].

Definition 2.1 A *log Fano pair* (X, Δ) is a pair of a complex normal projective variety X and an effective $\mathbb{Q}$-Weil divisor Δ on X such that the pair (X, Δ) is klt and $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-divisor. If X is nonsingular and $\Delta = 0$, then X is said to be a *Fano manifold*.

Definition 2.2 (α*-invariant (see* [5] *for example)*) Let (X, Δ) be a log Fano pair and let $\Gamma \subset \mathrm{Aut}(X, \Delta)$ be an algebraic subgroup.

(1) The Γ-invariant α-invariant $\alpha_\Gamma(X, \Delta)$ of (X, Δ) is defined as the supremum of $\alpha \in \mathbb{Q}_{>0}$ such that the pair $\left(X, \Delta + \frac{\alpha}{m}\mathcal{D}\right)$ is lc for any $m \in \mathbb{Z}_{>0}$ with $-m(K_X + \Delta)$ Cartier and for any nonempty Γ-invariant sub-linear system $\mathcal{D} \subset |-m(K_X + \Delta)|$.
(2) For any scheme-theoretic point $\eta \in X$, we define $\alpha_{\Gamma,\eta}(X, \Delta)$ as the supremum of $\alpha \in \mathbb{Q}_{>0}$ such that the pair $\left(X, \Delta + \frac{\alpha}{m}\mathcal{D}\right)$ is lc at η for any $m \in \mathbb{Z}_{>0}$ with $-m(K_X + \Delta)$ Cartier and for any nonempty Γ-invariant sub-linear system $\mathcal{D} \subset |-m(K_X + \Delta)|$.
(3) If $\Delta = 0$, then we simply write $\alpha_\Gamma(X)$ and $\alpha_{\Gamma,\eta}(X)$ in place of $\alpha_\Gamma(X, 0)$ and $\alpha_{\Gamma,\eta}(X, 0)$, respectively.

Lemma 2.3 *Let (X, Δ) be a log Fano pair, $\Gamma \subset Aut(X, \Delta)$ be an algebraic subgroup, and $\eta \in X$ be a scheme-theoretic point. Assume that the identity component Γ_0 of Γ is solvable.*

(1) The value $\alpha_\Gamma(X, \Delta)$ is equal to the supremum of $\alpha \in \mathbb{Q}_{>0}$ such that the pair $(X, \Delta + \alpha D)$ is lc for any effective Γ-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$.
(2) The value $\alpha_{\Gamma,\eta}(X, \Delta)$ is equal to the supremum of $\alpha \in \mathbb{Q}_{>0}$ such that the pair $(X, \Delta + \alpha D)$ is lc at η for any effective Γ-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$.

Proof We only prove (2). By [3, Lemme 5.11], there is a finite algebraic subgroup $\Gamma_1 \subset \Gamma$ such that Γ_1 meets every connected component of Γ. Set $d := \#\Gamma_1$. Fix $\alpha \in \mathbb{Q}_{>0}$.

Assume that $\left(X, \Delta + \frac{\alpha}{m}\mathcal{D}\right)$ is lc at η for any m and $\mathcal{D} \subset |-m(K_X + \Delta)|$. Then, for any effective Γ-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$, since $\{mD\} \subset |-m(K_X + \Delta)|$ is a Γ-invariant sub-linear system for m sufficiently divisible, we know that the pair $\left(X, \Delta + \frac{\alpha}{m}\{mD\}\right)$ is lc at η, i.e., the pair $(X, \Delta + \alpha D)$ is lc at η.

Conversely, assume that $(X, \Delta + \alpha D)$ is lc at η for any effective Γ-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$. Take any Γ-invariant sub-linear system $\mathcal{D} \subset |-m(K_X + \Delta)|$. Since Γ_0 is connected and solvable, there exists a Γ_0-invariant divisor $D_0 \in \mathcal{D}$ by the Borel fixed point theorem [16, Sect. 21.2]. Let us set

$$\tilde{D} := \sum_{h \in \Gamma_1} h(D_0) \in d\mathcal{D}.$$

Since $h(D_0) \in \mathcal{D}$ is Γ_0-invariant for any $h \in \Gamma_1$, the divisor $\tilde{D} \in d\mathcal{D}$ is Γ-invariant. Set $D := \frac{1}{md}\tilde{D} \sim_{\mathbb{Q}} -(K_X + \Delta)$. Since $(X, \Delta + \alpha D)$ is lc at η, the pair $\left(X, \Delta + \frac{\alpha}{md}(d\mathcal{D})\right)$ is also lc at η. This is equivalent to the pair $\left(X, \Delta + \frac{\alpha}{m}\mathcal{D}\right)$ being lc at η. □

We recall the notion of *K-stability* for log Fano pairs. The original notion of K-stability was introduced by Tian [28] and Donaldson [6] by using the languages of *test configurations*. In this paper, we only treat its simplification due to Li [22] and the author [13].

Definition 2.4 Let (X, Δ) be an n-dimensional log Fano pair and let F be a prime divisor over X obtained by a log resolution $\pi : \tilde{X} \to X$ of (X, Δ) (that is, F is a prime divisor on $\tilde{X}$).

(1) Let $A_{X,\Delta}(F)$ be the log discrepancy of (X, Δ) along F, that is, 1 plus the coefficient of $K_{\tilde{X}} - \pi^*(K_X + \Delta)$ along F.
(2) For any effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D, let $\operatorname{ord}_F D \in \mathbb{Q}_{\geq 0}$ be the coefficient of $\pi^* D$ along F. For any $m \in \mathbb{Z}_{\geq 0}$ with $-m(K_X + \Delta)$ Cartier and for any $j \in \mathbb{R}_{\geq 0}$, let

$$H^0(X, -m(K_X + \Delta) - jF) \subset H^0(X, -m(K_X + \Delta))$$

be the sub-vector space corresponds to the sub-linear system $|-m(K_X + \Delta) - jF| \subset |-m(K_X + \Delta)|$ consisting all $D \in |-m(K_X + \Delta)|$ with $\operatorname{ord}_F D \geq j$.
(3) For any $x \in \mathbb{R}_{\geq 0}$, let us set

$$\operatorname{vol}(-(K_X + \Delta) - xF) := \lim_{m \to \infty} \frac{\dim H^0(X, -m(K_X + \Delta) - mxF)}{m^n / n!},$$

where m runs through all positive integers with $-m(K_X + \Delta)$ Cartier (the limit exists by [20, 21]). Obviously, $\operatorname{vol}(-(K_X + \Delta) - 0 \cdot F) = (-(K_X + \Delta))^{\cdot n}$ holds. It follows from the definition that

$$\operatorname{vol}(-(K_X + \Delta) - xF) = \operatorname{vol}_{\tilde{X}}(-\pi^*(K_X + \Delta) - xF).$$

In particular, by [20, 21], the function $\operatorname{vol}(-(K_X + \Delta) - xF)$ is a non-increasing and continuous function over $x \in [0, \infty)$. Moreover, if $x \gg 0$, then $\operatorname{vol}(-(K_X + \Delta) - xF) = 0$ holds. Let us set

$$\tau_{X,\Delta}(F) := \sup\{\tau \in \mathbb{R}_{>0} \mid \operatorname{vol}(-(K_X + \Delta) - \tau F) > 0\}.$$

(4) Let us set

$$S_{X,\Delta}(F) := \frac{1}{(-(K_X + \Delta))^{\cdot n}} \int_0^\infty \operatorname{vol}(-(K_X + \Delta) - xF) dx.$$

The above definitions do not depend on the choice of π. We often write $A(F)$, $\tau(F)$, $S(F)$ in place of $A_{X,\Delta}(F)$, $\tau_{X,\Delta}(F)$, $S_{X,\Delta}(F)$, just for simplicity.

The following lemma is well-known:

Lemma 2.5 *Let (X, Δ) be a log Fano pair, let $\Gamma \subset Aut(X, \Delta)$ be an algebraic subgroup, and let $\eta \in X$ be a scheme-theoretic point. For any Γ-invariant prime divisor F over X with $\eta \in c_X(F)$, we have*

$$\frac{A(F)}{\tau(F)} \geq \alpha_{\Gamma,\eta}(X, \Delta),$$

where $c_X(F)$ is the center of F on X.

Proof Take any $\tau = \frac{j}{m} \in (0, \tau(F)) \cap \mathbb{Q}$. Then the sub-linear system

$$|-m(K_X + \Delta) - jF| \subset |-m(K_X + \Delta)|$$

is nonempty, Γ-invariant, and vanishes along F at least j times. Thus we have

$$\begin{aligned}\alpha_{\Gamma,\eta}(X, \Delta) &\leq \operatorname{lct}_\eta \left(X, \Delta;\ \frac{1}{m}|-m(K_X + \Delta) - jF|\right)\\ &\leq \frac{A_{X,\Delta}(F)}{\operatorname{ord}_F\left(\frac{1}{m}|-m(K_X + \Delta) - jF|\right)} \leq \frac{A_{X,\Delta}(F)}{\tau},\end{aligned}$$

where lct_η is the log canonical threshold at η. □

We see a G-invariant version of [13, 15, 22]. See also [29] for more general frameworks.

Proposition 2.6 (see [1]) *Let X be an n-dimensional Fano manifold and let $\Gamma \subset Aut(X)$ be a reductive subgroup.*

(1) (see also [29, Corollary 4.13]) Assume that $A(F) > S(F)$ holds for any Γ-invariant prime divisor over X with the $\mathbb{C}$-algebra

$$\bigoplus_{m,j \in \mathbb{Z}_{\geq 0}} H^0(X, -mK_X - jF)$$

finitely generated. Then X admits Kähler–Einstein metrics.

(2) (see also [27] and [15, Theorem 1.3]) If $\alpha_\Gamma(X) \geq \frac{n}{n+1}$, then X admits Kähler–Einstein metrics.

For the complete proof, see [1]. We only give a sketch of the idea. For (1), for any Γ-equivariant special degeneration of X in the sense of [8], the corresponding prime divisor F over X in [13, Theorem 5.1] obviously satisfies the assumptions in (1). By [13, Theorem 5.1], the signs of the Donaldson–Futaki invariant of the special degeneration and $A(F) - S(F)$ are same. Thus we can apply [8, Theorem 1]. For (2), we may assume that there exists a Γ-invariant prime divisor F over X with $A(F) \leq S(F)$ such that the $\mathbb{C}$-algebra in (1) is finitely generated. By Lemma 2.5 and [15, Theorem 4.1], X is isomorphic to $\mathbb{P}^n$ and then we complete the proof.

3 On the Volume Functions

In this section, we generalize [14, Proposition 2.1] in order to show that V_2 admits Kähler–Einstein metrics.

Proposition 3.1 *Let (X, Δ) be an n-dimensional log Fano pair, let F be a prime divisor over X, and let $0 < a < b$ be positive real numbers. Assume that*

$$\mathrm{vol}(-(K_X + \Delta) - xF) = \left(\frac{b-x}{b-a}\right)^n \mathrm{vol}(-(K_X + \Delta) - aF)$$

for any $x \in [a, b]$. Then we have

$$S(F) \leq \frac{(n-1)a + b}{n+1}.$$

Proof The proof looks similar to the argument in the proof of [26, Theorem 1.2]. From the assumption, we have $\tau(F) = b$. Set $V := (-(K_X + \Delta))^{\cdot n}$. By [2, Theorem A], the function $\mathrm{vol}(-(K_X + \Delta) - xF)$ is $\mathcal{C}^1$ over $x \in [0, b)$. Let us set

$$f(x) := -\frac{1}{n}\frac{d}{dx}\mathrm{vol}(-(K_X + \Delta) - xF)$$

as in [14, Proof of Proposition 2.1]. (We note that $f(x)$ is a restricted volume function in the sense of [9].) Then, for any $x \in [a, b)$, we have

$$f(x) = \left(\frac{b-x}{b-a}\right)^{n-1} f(a).$$

As in [14, Proof of Proposition 2.1], we have

$$\begin{aligned} V &= n\int_0^b f(x)dx, \\ S(F) &= \frac{1}{V}\cdot n\int_0^b xf(x)dx, \\ f(x) &\geq \left(\frac{x}{a}\right)^{n-1} f(a) \quad \text{for any } x \in [0, a]. \end{aligned}$$

In particular, we get

$$\begin{aligned} V &\geq n\left(\int_0^a \left(\frac{x}{a}\right)^{n-1} f(a)dx + \int_a^b \left(\frac{b-x}{b-a}\right)^{n-1} f(a)dx\right) \\ &= b \cdot f(a). \end{aligned}$$

Set

$$g(x) := \begin{cases} \left(\frac{x}{a}\right)^{n-1} \cdot \frac{V}{b} & \text{for } x \in [0, a], \\ \left(\frac{b-x}{b-a}\right)^{n-1} \cdot \frac{V}{b} & \text{for } x \in [a, b]. \end{cases}$$

Then we have

$$\begin{aligned} V &= n \int_0^b g(x)dx, \\ g(x) &\geq f(x) \quad \text{for any } x \in [a, b]. \end{aligned}$$

Moreover, the function

$$f(x)^{\frac{1}{n-1}} - g(x)^{\frac{1}{n-1}} = f(x)^{\frac{1}{n-1}} - \frac{x}{a} \cdot \left(\frac{V}{b}\right)^{\frac{1}{n-1}}$$

is $\mathcal{C}^0$ and concave over $x \in [0, a]$ (by [9, Theorem A]). Moreover, we have $f(0)^{\frac{1}{n-1}} - g(0)^{\frac{1}{n-1}} \geq 0$ and $f(a)^{\frac{1}{n-1}} - g(a)^{\frac{1}{n-1}} \leq 0$. Thus there exists $c \in [0, a]$ such that

$$\begin{aligned} f(x) &\geq g(x) \quad \text{for any } x \in [0, c], \\ g(x) &\geq f(x) \quad \text{for any } x \in [c, a]. \end{aligned}$$

Thus we get

$$\begin{aligned} n\int_0^b xf(x)dx - cV &= n\int_0^b (x-c)f(x)dx \\ \leq n\int_0^b (x-c)g(x)dx &= n\int_0^b xg(x)dx - cV. \end{aligned}$$

Since

$$n\int_0^b xg(x)dx = \frac{(n-1)a+b}{n+1}V,$$

we get the assertion. □

Proposition 3.2 *Let (X, Δ) be an n-dimensional log Fano pair, let $\eta \in X$ be a scheme-theoretic point, and let $0 < t \le s$ be positive real numbers. Assume that there exists a prime divisor T on X with $T \sim_{\mathbb{Q}} -k(K_X + \Delta)$ for some $k \in \mathbb{Q}_{>0}$ such that*

- *the pair $(X, \Delta + \frac{t}{k}T)$ is lc at η, and*
- *for any effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} -(K_X + \Delta)$ with $T \not\subset \operatorname{Supp} D'$, the pair $(X, \Delta + sD')$ is lc at η.*

Then, for any prime divisor F over X with $\eta \in c_X(F)$, we have the following:

(1) If $s^{-1}A(F) \le \frac{1}{k} \operatorname{ord}_F T$, then we have

$$\begin{aligned}&\operatorname{vol}(-(K_X + \Delta) - xF)\\ &= \left(\frac{\frac{1}{k}\operatorname{ord}_F T - x}{\frac{1}{k}\operatorname{ord}_F T - s^{-1}A(F)}\right)^n \operatorname{vol}\left(-(K_X + \Delta) - s^{-1}A(F)F\right)\end{aligned}$$

for any $x \in [s^{-1}A(F), \frac{1}{k}\operatorname{ord}_F T]$.

(2) We have

$$S(F) \le \frac{A(F)}{n+1}\left((n-1)s^{-1} + t^{-1}\right).$$

Proof (1) Let us fix a log resolution $\pi : \tilde{X} \to X$ of (X, Δ) with $F \subset \tilde{X}$. Take any effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$. Then there uniquely exists $e \in [0, 1] \cap \mathbb{Q}$ such that we can write $D = \frac{e}{k}T + (1 - e)D'$ with $D' \sim_{\mathbb{Q}} -(K_X + \Delta)$ effective and $T \not\subset \operatorname{Supp} D'$. Since $(X, \Delta + sD')$ is lc at η, we have $\operatorname{ord}_F D' \le s^{-1}A(F)$. Assume that $x \in (s^{-1}A(F), \frac{1}{k}\operatorname{ord}_F T] \cap \mathbb{Q}$ satisfies that $\operatorname{ord}_F D \ge x$. (In other words, $mD \in |-m(K_X + \Delta) - mxF|$ holds for a sufficiently divisible $m \in \mathbb{Z}_{>0}$.) Then we have

$$e \ge \frac{x - s^{-1}A(F)}{\frac{1}{k}\operatorname{ord}_F T - s^{-1}A(F)},$$

since

$$x \le \operatorname{ord}_F D \le e \cdot \frac{1}{k}\operatorname{ord}_F T + (1 - e)s^{-1}A(F).$$

This implies that the linear system $|-m(K_X + \Delta) - mxF|$ has a fixed divisor

$$m \cdot \frac{x - s^{-1}A(F)}{\frac{1}{k}\operatorname{ord}_F T - s^{-1}A(F)} \cdot \frac{1}{k}T$$

for $m \in \mathbb{Z}_{>0}$ sufficiently divisible. Thus we get

$$
\begin{aligned}
&\operatorname{vol}(-(K_X+\Delta)-xF)\\
&=\operatorname{vol}_{\tilde{X}}\left(-\pi^*(K_X+\Delta)-xF-\frac{x-s^{-1}A(F)}{\frac{1}{k}\operatorname{ord}_F T-s^{-1}A(F)}\frac{1}{k}(\pi^*T-(\operatorname{ord}_F T)F)\right)\\
&=\left(\frac{\frac{1}{k}\operatorname{ord}_F T-x}{\frac{1}{k}\operatorname{ord}_F T-s^{-1}A(F)}\right)^n\operatorname{vol}\left(-(K_X+\Delta)-s^{-1}A(F)F\right).
\end{aligned}
$$

(2) Since the pair $(X, \Delta+\frac{t}{k}T)$ is lc at η, we have $\frac{1}{k}\operatorname{ord}_F T \le t^{-1}A(F)$. If $\frac{1}{k}\operatorname{ord}_F T \le s^{-1}A(F)$, then we have $\operatorname{ord}_F D \le s^{-1}A(F)$ for any effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X+\Delta)$. Thus we get the inequality $s \le \frac{A(F)}{\tau(F)}$. By [14, Proposition 2.1], we get

$$
\begin{aligned}
S(F) &\le \frac{n}{n+1}\tau(F) \le \frac{n}{n+1}s^{-1}A(F)\\
&\le \frac{A(F)}{n+1}\left((n-1)s^{-1}+t^{-1}\right).
\end{aligned}
$$

Thus we may assume that $\frac{1}{k}\operatorname{ord}_F T > s^{-1}A(F)$. In this case, we can apply (1). We have

$$
\begin{aligned}
S(F) &\le \frac{1}{n+1}\left((n-1)s^{-1}A(F)+\frac{1}{k}\operatorname{ord}_F T\right)\\
&\le \frac{A(F)}{n+1}\left((n-1)s^{-1}+t^{-1}\right)
\end{aligned}
$$

by Proposition 3.1. □

4 On Quasi-Log Schemes

In order to prove Theorem 1.2, we must consider log pairs such that their singularities are possibly worse than log canonical. The theory of *quasi-log schemes* is very powerful in order to overcome the difficulty. In this section, we see a kind of subadjunction theorem for projective qlc strata on quasi-log schemes, which is a direct consequence of the recent work [11]. We briefly recall the notion of quasi-log schemes. See [10] for detail.

Definition 4.1 ([10, Definition 6.1.1]) A *quasi-log scheme* consists of a scheme X which is separated and of finite type over $\mathbb{C}$, an $\mathbb{R}$-line bundle ω on X, a proper closed subscheme $X_{-\infty}$ called *the non-qlc locus*, and a finite set $\{C\}$ of closed subvarieties on X called the set of *qlc strata* such that there exists a globally embedded simple normal crossing pair (Y, B_Y) (see [10, Sect. 5.2]) and a proper morphism $f : Y \to X$ which satisfies the following:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural homomorphism $\mathcal{O}_X \to f_*\mathcal{O}_Y\left(\lceil -(B_Y^{<1})\rceil\right)$ gives an isomorphism $I_{X_{-\infty}} \simeq f_*\mathcal{O}_Y\left(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor\right)$, where $I_{X_{-\infty}} \subset \mathcal{O}_X$ is the ideal sheaf with respects to the closed subscheme $X_{-\infty} \subset X$.
(3) The set $\{C\}$ is equal to the set of the images of (Y, B_Y)-strata (see [10, Sect. 5.2]) whose image do not map into X_∞.

The above quasi-log scheme is often denoted by $[X, \omega]$ for simplicity.

Now we see a kind of subadjunction theorem for quasi-log schemes. An $\mathbb{R}$-Weil divisor D on a normal projective variety is said to be *pseudo-effective* in this paper if $D + A$ is big (i.e., there exists an effective $\mathbb{R}$-Weil divisor E such that $D + A - E$ is an ample $\mathbb{R}$-divisor) for any ample $\mathbb{R}$-divisor A.

Theorem 4.2 (see [11, Lemma 4.17, Theorems 1.9 and 7.1]) *Let $[X, \omega]$ be a quasi-log scheme, let $C \subset X$ be a qlc stratum of $[X, \omega]$, and let $\nu\colon \bar{C} \to C$ be the normalization of C. Assume that C is a projective variety. Then $\nu^*(\omega|_C) - K_{\bar{C}}$ is a pseudo-effective $\mathbb{R}$-Weil divisor on $\bar{C}$.*

Proof By [11, Lemma 4.19], we may assume that $X = C$. Moreover, by [11, Theorem 1.9], we may further assume that $X = C$ is normal. By [11, Theorem 7.1], there exists a projective birational morphism $p\colon X' \to X$ from a smooth projective variety X' such that we can write

$$K_{X'} + B_{X'} + M_{X'} = p^*\omega,$$

where $B_{X'}$ is an effective $\mathbb{R}$-Weil divisor on X' with $B_{X'}^{<0}$ p-exceptional, and $M_{X'}$ is a nef $\mathbb{R}$-divisor on X'. This immediately implies that

$$\omega - K_X = p_*\left(B_{X'} + M_{X'}\right)$$

is a pseudo-effective $\mathbb{R}$-Weil divisor on X. □

As a corollary, we get the following result, which is important for the proof of Theorem 1.2.

Corollary 4.3 *Let (X, Δ) be a log Fano pair, let $D \sim_{\mathbb{Q}} -(K_X + \Delta)$ be an effective $\mathbb{Q}$-divisor, and let $\alpha \in (0, 1) \cap \mathbb{Q}$. Assume that the pair $(X, \Delta + \alpha D)$ is not klt. Let* $\mathrm{Nklt}(X, \Delta + \alpha D)$ *be the locus of non-klt points of $(X, \Delta + \alpha D)$.*

(1) The locus $\mathrm{Nklt}(X, \Delta + \alpha D)$ *is connected.*
(2) Take any 1-dimensional irreducible component $B \subset \mathrm{Nklt}(X, \Delta + \alpha D)$ with its reduced scheme structure. Then B is a rational curve with

$$(-(K_X + \Delta) \cdot B) \le \frac{2}{1 - \alpha}.$$

Moreover, if any irreducible component of $\mathrm{Nklt}(X, \Delta + \alpha D)$ *is of dimension* ≤ 1*, then* $B \simeq \mathbb{P}^1$ *and the restriction homomorphism*

$$H^0(X, L) \to H^0(B, L|_B)$$

is surjective for any nef line bundle L *on* X*.*

Proof By [10, 6.4.1], the pair $(X, \Delta + \alpha D)$ admits a quasi-log structure $[X, \omega]$ with $\omega = K_X + \Delta + \alpha D$, and $N := \mathrm{Nklt}(X, \Delta + \alpha D)$ has a natural scheme structure with

$$N = \bigcup_C C \cup X_{-\infty},$$

where C are the lc centers of $(X, \Delta + \alpha D)$ and $X_{-\infty}$ is the non-qlc locus of $(X, \Delta + \alpha D)$.

For any nef line bundle L on X, since

$$L - (K_X + \Delta + \alpha D) \sim_{\mathbb{Q}} L + (1 - \alpha)(-K_X - \Delta)$$

is ample, we have

$$H^i(X, L \otimes I_N) = 0$$

for any $i > 0$ by [10, Theorem 6.3.5 (ii)], where $I_N \subset \mathcal{O}_X$ is the defining ideal sheaf of $N \subset X$.

(1) For $L := \mathcal{O}_X$, we get the surjection

$$H^0(X, \mathcal{O}_X) \twoheadrightarrow H^0(N, \mathcal{O}_N).$$

Thus N is connected.

(2) After replacing α with the log canonical threshold of $(X, \Delta; D)$ at the generic point of B, we may assume that B is an lc center of $(X, \Delta + \alpha D)$. Take the normalization $\nu \colon \bar{B} \to B$. By Theorem 4.2, the $\mathbb{Q}$-divisor

$$\nu^* \left((K_X + \Delta + \alpha D)|_B\right) - K_{\bar{B}}$$

is pseudo-effective. This implies that $\bar{B} \simeq \mathbb{P}^1$ and

$$\begin{aligned} 0 &\leq \deg_{\bar{B}} \left(\nu^* \left((K_X + \Delta + \alpha D)|_B\right) - K_{\bar{B}}\right) \\ &= -(1 - \alpha)\left(-(K_X + \Delta) \cdot B\right) + 2. \end{aligned}$$

Now we assume that $\dim N \leq 1$. Since $H^1(X, \mathcal{O}_X) = H^2(X, I_N) = 0$, we get $H^1(N, \mathcal{O}_N) = 0$. From the assumption $\dim N \leq 1$, we get $H^1(B, \mathcal{O}_B) = 0$, i.e., $B \simeq \mathbb{P}^1$. Moreover, by [17, Lemma 4.13 and the proof of Lemma 4.50], the restriction homomorphism

$$H^0(N, L|_N) \to H^0(B, L|_B)$$

is surjective for any nef line bundle L on X. Since $H^1(X, L \otimes I_N) = 0$,

$$H^0(X, L) \to H^0(B, L|_B)$$

is also surjective. Thus we get the assertions. $\square$

5 Proof of Theorem 1.2

By Theorem 1.1 and Proposition 2.6 (2), we may assume that $u = 2$. Set $X := V_2$. Take any G-invariant prime divisor F over X. (Recall that $G := \mathbb{C}^* \rtimes (\mathbb{Z}/2\mathbb{Z})$.) By Proposition 2.6 (1), it is enough to show the inequality $A(F) > S(F)$. Set $C := c_X(F)$ and let $\eta \in X$ be the generic point of C. By Lemma 2.5 and [14, Proposition 2.1], we may assume that $\alpha_{G,\eta}(X) \le \frac{3}{4}$. Note that C is a G-invariant subvariety on X. Thus C is not a closed point by [4, Lemma 2.23]. Moreover, if F is a prime divisor on X, then $A(F) > S(F)$ holds by [12, Corollary 9.3]. Thus we may assume that C is a G-invariant curve. By Lemma 2.3, there exists a G-invariant effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ and there exists $\alpha \in [\alpha_{G,\eta}(X), 4/5) \cap \mathbb{Q}$ such that the pair $(X, \alpha D)$ is lc but not klt at η. We note that the non-klt locus of the pair $(X, \alpha D)$ is of dimension ≤ 1 since $\operatorname{Pic}(X) = \mathbb{Z}[-K_X]$ and $\alpha < 1$.

By Corollary 4.3, C is a smooth rational curve with

$$(-K_X \cdot C) \le \frac{2}{1-\alpha} < 10.$$

Moreover, for any $m \in \mathbb{Z}_{\ge 0}$, the restriction homomorphism

$$H^0(X, \mathcal{O}_X(-mK_X)) \to H^0(C, \mathcal{O}_X(-mK_X)|_C)$$

is surjective. Thus $C \subset X \subset \mathbb{P}^{13} = \mathbb{P}^* H^0(X, -K_X)$ is a G-invariant rational normal curve in $\mathbb{P}^{13}$ with $\deg C < 10$.

By [4, Proposition 4.12, Lemma 7.7 and Corollary 7.10] and the assumption $\alpha_{G,\eta}(X) \le \frac{3}{4}$, we may assume that $C = \mathcal{C}_4$, where $\mathcal{C}_4 \subset X$ is the unique G-invariant rational normal curve of anti-canonical degree 4 in X (see [4] for the definition of the curve $\mathcal{C}_4$). By [4, Lemma 5.2 and the proof of Lemma 7.14], there exists a prime divisor $T'_{15} \sim -K_X$ on X such that

- the pair $(X, \frac{2}{3}T'_{15})$ is lc at η, and
- the pair (X, D') is lc at η for any effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} -K_X$ with $T'_{15} \not\subset \operatorname{Supp} D'$.

By Proposition 3.2 (2), we get

$$S(F) \le \frac{A(F)}{4}\left(2 \cdot 1^{-1} + \left(\frac{2}{3}\right)^{-1}\right) = \frac{7}{8}A(F).$$

Thus we get the desired inequality $A(F) > S(F)$ and we complete the proof of Theorem 1.2.

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References

1. Araujo, C., Castravet, A.-M., Cheltsov, I., Fujita, K., Kaloghiros, A.-S., Martinez-Garcia, J., Shramov, C., Süss, H., Viswanathan, N.: The Calabi problem for Fano threefolds. MPIM preprint 2021-31
2. Boucksom, S., Favre, C., Jonsson, M.: Differentiability of volumes of divisors and a problem of Teissier. J. Algebraic Geom. **18**(2), 279–308 (2009)
3. Borel, A., Serre, J.-P.: Théorèmes de finitude en cohomologie galoisienne. Comment. Math. Helv. **39**, 111–164 (1964)
4. Cheltsov, I., Shramov, C.: Kähler-Einstein Fano threefolds of degree 22. arXiv:1803.02774v4
5. Demailly, J.-P.: Appendix to I. Cheltsov and C. Shramov's article. "Log canonical thresholds of smooth Fano threefolds": On Tian's invariant and log canonical thresholds. Russ. Math. Surveys **63**(5), 945–950 (2008)
6. Donaldson, S.: Scalar curvature and stability of toric varieties. J. Diff. Geom. **62**(2), 289–349 (2002)
7. Donaldson, S.: A note on the α-invariant of the Mukai-Umemura 3-fold. arXiv:0711.4357v1
8. Datar, V., Székelyhidi, G.: Kähler-Einstein metrics along the smooth continuity method. Geom. Funct Anal. **26**(4), 975–1010
9. Ein, L., Lazarsfeld, R., Mustaţă, M., Nakamaye, M., Popa, M.: Restricted volumes and base loci of linear series. Amer. J. Math. **131**(3), 607–651 (2009)
10. Fujino, O.: Foundations of the minimal model program. MSJ Memoirs, 35. Mathematical Society of Japan, Tokyo (2017)
11. Fujino, O.: Cone theorem and Mori hyperbolicity. arXiv:2102.11986v1
12. Fujita, K.: On K-stability and the volume functions of $\mathbb{Q}$-Fano varieties. Proc. Lond. Math. Soc. **113**(5), 541–582 (2016)
13. Fujita, K.: A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. J. Reine Angew. Math. **751**, 309–338 (2019)
14. Fujita, K.: Uniform K-stability and plt blowups of log Fano pairs. Kyoto J. Math. **59**(2), 399–418 (2019)
15. Fujita, K.: K-stability of Fano manifolds with not small alpha invariants. J. Inst. Math. Jussieu **18**(3), 519–530 (2019)
16. Humphreys, J.: Linear algebraic groups. Graduate Texts in Mathematics, No. 21. Springer, New York (1975)
17. Kollár, J., Mori, S.: Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998)
18. Kuznetsov, A., Prokhorov, Y.: Prime Fano threefolds of genus 12 with a $\mathbb{G}_m$-action and their automorphisms. Épijournal Geom. Algébrique **2**, Art. 3, 14 (2018)
19. Kuznetsov, A., Prokhorov, Y., Shramov, C.: Hilbert schemes of lines and conics and automorphism groups of Fano threefolds. Jpn. J. Math. **13**(1), 109–185 (2018)

20. Lazarsfeld, R.: Positivity in algebraic geometry, I: Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete, (3), vol. 48. Springer, Berlin (2004)
21. Lazarsfeld, R.: Positivity in algebraic geometry, II: Positivity for Vector Bundles, and Multiplier Ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 49. Springer, Berlin (2004)
22. Li, C.: K-semistability is equivariant volume minimization. Duke Math. J. **166**(16), 3147–3218 (2017)
23. Matsushima, Y.: Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne. Nagoya Math. J. **11**, 145–150 (1957)
24. Mukai, S., Umemura, H.: Minimal rational threefolds. Algebraic geometry (Tokyo/Kyoto, 1982). Lecture Notes in Mathematics, vol. 1016, pp. 490–518. Springer, Berlin (1983)
25. Prokhorov, Y.: Automorphism groups of Fano 3-folds. Uspekhi Mat. Nauk **45**(3(273)), 195–196 (1990)
26. Stibitz, C., Zhuang, Z.: K-stability of birationally superrigid Fano varieties. Compos. Math. **155**, 1845–1852 (2019)
27. Tian, G.: On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M)>0$. Invent. Math. **89**(2), 225–246 (1987)
28. Tian, G.: Kähler-Einstein metrics with positive scalar curvature. Invent. Math. **130**(1), 1–37 (1997)
29. Zhuang, Z.: Optimal destabilizing centers and equivariant K-stability. Invent. Math. **226**(1), 195–223 (2021)

Lagrangian Skeleta, Collars and Duality

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Abstract We present a geometric realization of the duality between skeleta in $T^*\mathbb{P}^n$ and collars of local surfaces. Such duality is predicted by combining two auxiliary types of duality: on one side, symplectic duality between $T^*\mathbb{P}^n$ and a crepant resolution of the A_n singularity; on the other side, toric duality between two types of isolated quotient singularities. We give a correspondence between Lagrangian submanifolds of a cotangent bundle and vector bundles on a collar, and describe those birational transformations within the skeleton which are dual to deformations of vector bundles.

Keywords Crepant resolution · Vector bundles · Lagrangian submanifolds · Quotient singularities

1 Skeleton to Collar Duality

The simplest example of symplectic duality is the one between the cotangent bundle of projective space $T^*\mathbb{P}^{n-1}$ and the crepant resolution $\widetilde{Y}_n$ of the A_{n-1} singularity obtained as a quotient $Y_n = \mathbb{C}^2/\mathbb{Z}_n$ [4, 5]. There exists also a duality between $\widetilde{Y}_n$ and the surface $Z_n = \mathrm{Tot}\mathscr{O}_{\mathbb{P}^1}(-n)$, in the sense that they are both minimal resolutions of quotient singularities, but their respective singularities have dual toric fans. In fact,

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the singular surface $\mathscr{X}_n$ obtained from Z_n by contracting the zero section is also a quotient of $\mathbb{C}^2$ by the cyclic group of n elements, but the singularity of $\mathscr{X}_n$ is of type $\frac{1}{n}(1,1)$ whereas the singularity of Y_n is of type $\frac{1}{n}(1, n-1)$. Motivated by these two dualities we discuss some features of the resulting duality between $T^*\mathbb{P}^{n-1}$ and Z_n. On one side, we consider $T^*\mathbb{P}^{n-1}$ together with a complex potential, thus forming a Landau–Ginzburg model, and we study the Lagrangian skeleton of the corresponding Hamiltonian flow; on the other side, we describe the behaviour of vector bundles on the surfaces Z_n considered as algebraic varieties.

In both cases we will focus our attention on building blocks used for those types of gluing procedures which may be viewed as surgery operations. We will see that vector bundles on what we call the collar of Z_n (see Sect. 6) behave similarly to components of the Lagrangian skeleton of $T^*\mathbb{P}^{n-1}$.

Denoting by **bir** a birational transformation applied to a compactified Lagrangian and **def** a deformation of the complex structure of a vector bundle (without describing a categorial equivalence) we give a geometric description of a 1-1 correspondence between objects and some essential morphisms. Such a duality is described by the diagram in the following theorem.

Theorem 1.1 *The following diagram commutes:*

$$\begin{tikzcd}
L_j \subset T^*\mathbb{P}^{n-1} \arrow[r, Leftrightarrow, "\textbf{dual}"] \arrow[d, "\textbf{bir}"'] & \mathscr{O}_{Z_n^\circ}(j) \oplus \mathscr{O}_{Z_n^\circ}(-j) \\
L_{j+1} \subset T^*\mathbb{P}^{n-1} \arrow[r, Leftrightarrow, "\textbf{dual}"'] & \mathscr{O}_{Z_n^\circ}(j+1) \oplus \mathscr{O}_{Z_n^\circ}(-j-1). \arrow[u, "\textbf{def}"']
\end{tikzcd} \tag{1.2}$$

Duality between Lagrangians and vector bundles.

The surfaces Z_n have rich moduli spaces of vector bundles, but it is mainly the restriction of a vector bundle to the collar of Z_n° (see (6.4)) that plays a role in this duality. The cotangent bundle is taken with the canonical symplectic structure and Lagrangian skeleta are described in Sect. 2. Vector bundles on the local surfaces Z_n are building blocks for vector bundles on compact surfaces. In fact, a new gluing procedure called *grafting* introduced in [10] explores the local contribution of these building blocks to the top Chern class. This grafting procedure was successful in explaining the physics mechanism underlying the phenomenon of instanton decay around a complex line with negative self-intersection, showing that instantons may decay by inflicting curvature to the complex surface that holds them [10, Sect. 7]. For a line with self-intersection $-n$, grafting is done via cutting and gluing over a collar Z_n°. The set of isomorphisms classes of rank 2 vector vector bundles over such a collar Z_n° presents a behaviour similar to that of the Lagrangian skeleton of the cotangent bundle $T^*\mathbb{P}^{n-1}$. Therefore our construction here offers a geometric interpretation of this particular instance of duality by exploring building blocks of surgery operations on both sides. When considered in families, one Lagrangian in the skeleton is taken to the next via a birational transformation (Sect. 4) whereas a bundle on the collar is taken to another via deformation of the complex structure (Sect. 7). In this sense we may say that when considering objects of this duality, birational transformations on Lagrangian skeleta occur as dual to deformations of vector bundles.

2 Lagrangian Skeleton of $T^*\mathbb{P}^n$

In this section we will calculate skeleta of certain Landau–Ginzburg models. By a Landau–Ginzburg model we mean a complex manifold together with a complex valued function.

Let (M, ω) be a symplectic manifold together with a potential h. We assume that h is a Morse function. In the case when h is a real valued function, the stable manifold of a critical point p consists of all the points in M that are taken to p by the gradient flow of h. However, when h is a complex valued function, even though the stable manifold of a point p is still formed by points that flow to p, the natural choice is to use the Hamiltonian flow of h (which can be thought of as the symplectic gradient). Furthermore, in the cases considered here, the Hamiltonian flow is given by a torus action (as described in Sect. 3) and the critical points of h are the fixed points of such action.

Let L be the union of the stable manifolds of the Hamiltonian flow of h with respect to the Kähler metric. Then L is the isotropic skeleton of (M, ω). When L is of middle dimension, it is called the **Lagrangian skeleton** of (M, ω). In the case of exact symplectic manifolds, the Lagrangian skeleton of M is the complement of the locus escaping to infinity under the natural Liouville flow, see [11, 12].

To describe the Lagrangian skeleton of $T^*\mathbb{P}^n$, we will use the Hamiltonian torus action. We start out with $\mathbb{P}^n$ described by homogeneous coordinates $[x_0, x_1, \ldots, x_n]$, covered by the usual open charts $U_i = \{x_i \neq 0\}$. We then write trivializations of the cotangent bundle $T^*\mathbb{P}^n$ taking products $V_i = U_i \times \mathbb{C}^n$ and over the V_0 chart we write coordinates as $V_0 = \{[1, x_1, \ldots, x_n], (y_1, \ldots, y_n)\}$. In this chart, we write the Hamiltonian action of the torus $\mathbb{T}:=\mathbb{C} \setminus \{0\}$ on $T^*\mathbb{P}^n$ as

$$\mathbb{T} \cdot V_0 = \left\{[1, t^{-1}x_1, \ldots, t^{-n}x_n], (ty_1, \ldots, t^n y_n)\right\}. \tag{2.1}$$

Note that the same action can be written as

$$\mathbb{T} \cdot V_0 = \left\{[t^n, t^{n-1}x_1, \ldots, x_n], (ty_1, \ldots, t^n y_n)\right\}.$$

We will now describe the Lagrangian skeleton corresponding to this Hamiltonian action. We start by showing an example, i.e. the case of $T^*\mathbb{P}^3$ and then we present the general procedure.

Example: skeleton of $T^*\mathbb{P}^3$. We take $\mathbb{P}^3$ with homogeneous coordinates $[x_0, x_1, x_2, x_3]$, and cover it by open sets $U_i = \{x_i \neq 0\}$ and charts $\varphi_i : U_i \to \mathbb{C}^3$ given by $\varphi_i([x_0, x_1, x_2, x_3]) = \left(\frac{x_0}{x_i}, \ldots, \hat{x}_i, \ldots, \frac{x_3}{x_i}\right)$. The transition matrices for the cotangent bundle $T_{ij} : \varphi_i(U_i \cap U_j) \to \mathrm{Aut}(\mathbb{C}^3)$ are

$$T_{01} = \begin{pmatrix} -x_1^2 & -x_1x_2 & -x_1x_3 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix} \quad T_{02} = \begin{pmatrix} -x_1x_2 & -x_2^2 & -x_2x_3 \\ x_2 & 0 & 0 \\ 0 & 0 & x_2 \end{pmatrix} \quad T_{03} = \begin{pmatrix} -x_1x_3 & -x_2x_3 & -x_3^2 \\ x_3 & 0 & 0 \\ 0 & x_3 & 0 \end{pmatrix}.$$

Consequently, we can write down a cover for the cotangent bundle as $V_i = U_i \times \mathbb{C}^3$, and in coordinates

$$\begin{aligned}
V_0 &= \{[x_0, x_1, x_2, x_3], (y_1, y_2, y_3)\}, \\
V_1 &= \left\{[x_1^{-1}, 1, x_1^{-1}x_2, x_1^{-1}x_3], (-x_1^2y_1 - x_1x_2y_2 - x_1x_3y_3, x_1y_2, x_1y_3)\right\}, \\
V_2 &= \left\{[x_2^{-1}, x_2^{-1}x_1, 1, x_2^{-1}x_3], (-x_1x_2y_1 - x_2^2y_2 - x_2x_3y_3, x_2y_1, x_2y_3)\right\}, \\
V_3 &= \left\{[x_3^{-1}, x_3^{-1}x_1, x_3^{-1}x_2, 1], (-x_1x_3y_1 - x_2x_3y_2 - x_3^2y_3, x_3y_1, x_3y_2)\right\}.
\end{aligned}$$

Now we take the Hamiltonian action of the torus $\mathbb{T}$ on $T^*\mathbb{P}^3$ given by

$$\mathbb{T} \cdot V_0 = \left\{[1, t^{-1}x_1, t^{-2}x_2, t^{-3}x_3], (ty_1, t^2y_2, t^3y_3)\right\} = \left\{[t^3, t^2x_1, tx_2, x_3], (ty_1, t^2y_2, t^3y_3)\right\},$$

and compatibility on the intersections implies that

$$\begin{aligned}
\mathbb{T} \cdot V_1 &= \left\{[tx_1^{-1}, 1, t^{-1}x_1^{-1}x_2, t^{-2}x_1^{-1}x_3], (-t^{-1}(x_1^2y_1 + x_1x_2y_2 + x_1x_3y_3), tx_1y_2, t^2x_1y_3)\right\}, \\
\mathbb{T} \cdot V_2 &= \left\{[t^2x_2^{-1}, tx_2^{-1}x_1, 1, t^{-1}x_2^{-1}x_3], (-t^{-2}(x_1x_2y_1 + x_2^2y_2 + x_2x_3y_3), t^{-1}x_2y_1, tx_2y_3)\right\}, \\
\mathbb{T} \cdot V_3 &= \left\{[t^3x_3^{-1}, t^2x_3^{-1}x_1, tx_3^{-1}x_2, 1], (-t^{-3}(x_1x_3y_1 + x_2x_3y_2 + x_3^2y_3), t^{-2}x_3y_1, t^{-1}x_3y_2)\right\}.
\end{aligned}$$

Using these, we calculate the Lagrangians.

STABLE MANIFOLD OF e_0 - on V_0 we find the points satisfying

$$\begin{aligned}
&\lim_{t\to 0}[1, t^{-1}x_1, t^{-2}x_2, t^{-3}x_3], (ty_1, t^2y_2, t^3y_3) \\
&\quad = [1, 0, 0, 0], (0, 0, 0)
\end{aligned}$$

this requires $x_1 = x_2 = x_3 = 0$ and we obtain the fibre over the point $[1, 0, 0, 0]$, that is,

$$\boxed{L_0 = T^*_{[1,0,0,0]}\mathbb{P}^3 \sim \mathbb{C}^3.}$$

STABLE MANIFOLD OF e_1- on V_1 we look for the points satisfying

$$\lim_{t\to 0}[tx_1^{-1}, 1, t^{-1}x_1^{-1}x_2, t^{-2}x_1^{-1}x_3], (-t^{-1}(x_1^2y_1 + x_1x_2y_2 + x_1x_3y_3), tx_1y_2, t^2x_1y_3)$$
$$= [0, 1, 0, 0], (0, 0, 0).$$

This requires $x_1^{-1}x_2 = x_1^{-1}x_3 = 0 = x_1^2y_1 + x_1x_2y_2 + x_1x_3y_3$, but since $x_1 \neq 0$ in this chart, we get $x_2 = x_3 = 0 = y_1$. So, we are left with points having coordinates $[1, x_1, 0, 0], (0, y_2, y_3)$ on V_0 which on V_1 become $[x_1^{-1}, 1, 0, 0], (0, x_1y_2, x_1y_3)$. We obtain (after taking the closure, that is, by adding the point $[1, 0, 0, 0], (0, 0, 0)$) the set of points $\{[1, x_1, 0, 0], (0, y_2, y_3) \mapsto [x_1^{-1}, 1, 0, 0], (0, x_1y_2, x_1y_3)\}$ so that

$$\boxed{L_1 = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).}$$

STABLE MANIFOLD OF e_2 - on V_2 we look for the points satisfying

$$\lim_{t\to 0}[t^2x_2^{-1}, tx_2^{-1}x_1, 1, t^{-1}x_2^{-1}x_3], (-t^{-2}(x_1x_2y_1 + x_2^2y_2 + x_2x_3y_3), t^{-1}x_2y_1, tx_2y_3)$$
$$= [0, 0, 1, 0], (0, 0, 0).$$

This requires $x_2^{-1}x_3 = 0 = x_1x_2y_1 + x_2^2y_2 + x_2x_3y_3 = x_2y_1$ but since $x_2 \neq 0$ in this chart, we get $x_3 = 0$ and $x_1x_2y_1 + x_2^2y_2 = 0 = x_2y_1$ and since on this chart $x_2 \neq 0$ it follows that $y_1 = y_2 = 0$.

We obtain (after taking the closure) the set of points $\{[1, x_1, x_2, 0], (0, 0, y_3) \mapsto [x_2^{-1}, x_2^{-1}x_1, 1, 0], (0, 0, x_2y_3)\}$, so

$$\boxed{L_2 = \mathcal{O}_{\mathbb{P}^2}(-1).}$$

STABLE MANIFOLD OF e_3 - on V_3 we find the points satisfying

$$\lim_{t\to 0}[t^3x_3^{-1}, t^2x_3^{-1}x_1, tx_3^{-1}x_2, 1], (-t^{-3}(x_1x_3y_1 + x_2x_3y_2 + x_3^2y_3), t^{-2}x_3y_1, t^{-1}x_3y_2)$$
$$= [0, 0, 0, 1], (0, 0).$$

This requires $x_1x_3y_1 + x_2x_3y_2 + x_3^2y_3 = x_3y_1 = x_3y_2 = 0$ and since $x_3 \neq 0$ in this chart, we get that $y_1 = y_2 = y_3 = 0$. We obtain the set of points $\{[x_0, x_1, x_2, x_3], (0, 0, 0)\}$, so

$$\boxed{L_3 = \mathbb{P}^3.}$$

The generalization of this procedure to higher dimensions now becomes evident, giving:

General case: the skeleton of $T^*\mathbb{P}^n$. We take $\mathbb{P}^n$ with homogeneous coordinates $[x_0, x_1, x_2, \ldots, x_n]$, and cover it by standard open sets $U_i = \{x_i \neq 0\}$ and charts

$\varphi_i\colon U_i \to \mathbb{C}^n$ given by $\varphi_i([x_0, x_1, x_2, \ldots, x_n]) = \left(\frac{x_0}{x_i}, \ldots, \hat{x}_i, \ldots, \frac{x_n}{x_i}\right)$. The transition matrices for the cotangent bundle $T_{ij}\colon \varphi_i(U_i \cap U_j) \to \mathrm{Aut}(\mathbb{C}^n)$ are

$$T_{01} = \begin{pmatrix} -x_1^2 & -x_1x_2 & \cdots & -x_1x_n \\ 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_1 \end{pmatrix}, \qquad T_{0n} = \begin{pmatrix} -x_1x_n & -x_2x_n & \cdots & -x_2x_n & -x_n^2 \\ x_n & 0 & \cdots & 0 & 0 \\ 0 & x_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & x_n & 0 \end{pmatrix}.$$

$$T_{0j} = \begin{pmatrix} -x_jx_1 & -x_jx_2 & \cdots & -x_j^2 & \cdots & -x_jx_{n-1} & -x_jx_n \\ x_j & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & x_j & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & & \vdots \\ & & & 0 & & & \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & x_j & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & x_j \end{pmatrix}.$$

Consequently, we can write down a cover for the cotangent bundle as $V_i = U_i \times \mathbb{C}^n$, and in coordinates

$$\begin{aligned} V_0 &= \{[x_0, \ldots, x_n], (y_1, \ldots, y_n)\}, \\ V_1 &= \left\{[x_1^{-1}, 1, x_1^{-1}x_2, \ldots, x_1^{-1}x_{n-1}, x_1^{-1}x_n], (-x_1^2y_1 - x_1x_2y_2 - x_1x_3y_3, x_1y_2, \ldots, x_1y_n)\right\}, \\ &\ \ \vdots \\ V_j &= \left\{[x_j^{-1}, x_j^{-1}x_1, \ldots 1, \ldots, x_j^{-1}x_n], (-x_jx_1y_1 - \ldots - x_jx_ny_n, x_jy_2, \ldots, x_jy_n)\right\}, \\ &\ \ \vdots \\ V_n &= \left\{[x_n^{-1}, x_n^{-1}x_1, \ldots, x_n^{-1}x_{n-1}, 1], (-x_nx_1y_1 - \ldots - x_n^2y_n, x_ny_2, \ldots, x_ny_n)\right\}. \end{aligned}$$

Now we take the Hamiltonian action of the torus $\mathbb{T}$ on $T^*\mathbb{P}^n$ given by

$$\begin{aligned} \mathbb{T} \cdot V_0 &= \left\{[1, t^{-1}x_1, t^{-2}x_2, \ldots, t^{-n}x_n], (ty_1, t^2y_2, \ldots, t^ny_n)\right\} \\ &= \left\{[t^n, t^{n-1}x_1, t^{n-2}x_2, \ldots, x_n], (ty_1, t^2y_2, \ldots, t^ny_n)\right\}, \end{aligned}$$

and compatibility on the intersections implies that

$$\mathbb{T}\cdot V_1 = \left\{[tx_1^{-1}, 1, t^{-1}x_1^{-1}x_2, \dots, t^{n-1}x_1^{-1}x_n], (-t^{-1}(x_1^2y_1 + \dots + x_1x_ny_n), tx_1y_2, \dots, t^{n-1}x_1y_n)\right\},$$

$$\vdots$$

$$\mathbb{T}\cdot V_n = \left\{[t^nx_n^{-1}, t^{n-1}x_n^{-1}x_1, \dots, tx_n^{-1}x_{n-1}, 1], (-t^{-n}(x_1x_ny_1 + \dots, +x_n^2y_n), t^{-(n-1)}x_ny_1, \dots, t^{-1}x_ny_n)\right\}.$$

Using these, we calculate the Lagrangians.

STABLE MANIFOLD OF e_0 - on V_0 we find the points satisfying

$$\lim_{t\to 0}[1, t^{-1}x_1, \dots, t^{-n}x_n], (ty_1, t^2y_2, \dots, t^ny_n) = [1, 0, \dots, 0], (0, \dots, 0)$$

this requires $x_1 = x_2 = \cdots = x_n = 0$ and we obtain the fibre over the point $[1, 0, \dots, 0]$, that is,

$$\boxed{L_0 = T^*_{[1,0,\dots,0]}\mathbb{P}^n \sim \mathbb{C}^n.}$$

STABLE MANIFOLD OF e_1 - on V_1 we find the points satisfying

$$\lim_{t\to 0}[tx_1^{-1}, 1, t^{-1}x_1^{-1}x_2, \dots, t^{-(n-1)}x_1^{-1}x_n], (-t^{-1}(x_1^2y_1 + \dots + x_1x_ny_n), tx_1y_2, \quad \dots, t^{n-1}x_1y_n)$$
$$= [0, 1, 0 \dots, 0], (0, \dots, 0).$$

This requires $x_1^{-1}x_2 = \cdots = x_1^{-1}x_n = 0 = x_1^2y_1 + \dots + x_1x_ny_n$, but since $x_1 \neq 0$ in this chart, we get $x_2 = \cdots = x_3 = 0 = y_1$. So, we are left with points having coordinates $[1, x_1, 0, \dots, 0], (0, y_2, \dots, y_n)$ on V_0 which on V_1 become $[x_1^{-1}, 1, 0 \dots, 0], (0, x_1y_2, \dots, x_1y_n)$. We obtain (after taking the closure, that is adding the point $[1, 0, \dots, 0], (0, \dots, 0)$)

$$\boxed{\begin{aligned} L_1 = \{&[1, x_1, 0, \dots, 0], (0, y_2, \dots, y_n) \mapsto [x_1^{-1}, 1, 0, \dots, 0], (0, x_1y_2, \dots, x_1y_n)\} \\ &\sim \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-1).\end{aligned}}$$

($n - 1$ summands).

OTHER STABLE MANIFOLDS

Using similar computations, we have that the Lagrangian L_j corresponding to the fixed point e_j is

$$\boxed{L_j = \begin{cases} \mathbb{C}^n & \text{if} \quad j = 0, \\ \oplus_{i=1}^{n-j}\mathcal{O}_{\mathbb{P}^j}(-1) & \text{if} \quad 0 < j < n, \\ \mathbb{P}^n & \text{if} \quad j = n. \end{cases}} \tag{2.2}$$

3 Potentials on the Cotangent Bundle

In this section we consider the question: What choices of potential h for a Landau–Ginzburg model $(T^*\mathbb{P}^n, h)$ are compatible with the Hamiltonian action considered in the previous sections, and hence gives rise to the same skeleta? We obtain the following result.

Proposition 3.1 *Consider* $(T^*\mathbb{P}^n, h_c)$ *with coordinates* $[1, x_1, \dots, x_n]$, $(y_1, \dots, y_n)$. *Each potential*

$$h_c([1, x_1, \dots, x_n], (y_1, \dots, y_n)) = \sum_{i=1}^{n} -2i x_i y_i + c,$$

has a corresponding Hamiltonian flow that coincides with the flow obtained by the torus action given in (2.1), that is

$$\mathbb{T} \cdot V_0 = \left\{[1, t^{-1}x_1, \dots, t^{-n}x_n], (ty_1, \dots, t^n y_n)\right\}.$$

To prove this, first consider the vector field on $T^*\mathbb{P}^n$ corresponding to the Hamiltonian action given in coordinates by

$$\mathbb{T} \cdot V_0 = \left\{[1, t^{-1}x_1, \dots, t^{-n}x_n], (ty_1, \dots, t^n y_n)\right\}.$$

On the image of the V_0 chart, the right hand side becomes

$$\alpha(t) = (t^{-1}x_1, \dots, t^{-n}x_n, ty_1, \dots, t^n y_n)$$

so that the derivative gives

$$\alpha'(t) = (-t^{-2}x_1, \dots, -nt^{-n-1}x_n, y_1, \dots, nt^{n-1}y_n)$$

and evaluating at 1 we get

$$\alpha'(1) = (-x_1, \dots, -nx_n, y_1, \dots, ny_n).$$

From the action of this 1-parameter subgroup, we have obtained the flow $\alpha'(1)$. Now we wish to calculate a potential h corresponding to the vector field $X = \alpha'(1)$.

Let ω be the canonical symplectic form on $T^*\mathbb{P}^n$, then h must satisfy, for all vector fields $Z \in \mathfrak{X}(M)$

$$dh(Z) = \omega(X, Z).$$

In coordinates this gives

$$\begin{aligned}
&\left(\frac{\partial h}{\partial \mathbf{x}} \frac{\partial h}{\partial \mathbf{y}}\right)\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\\
&\quad = \sum_{i=1}^{n} dx_i \wedge dy_i\ ((-x_1, \ldots, -nx_n, y_1, \ldots, ny_n), (a_1, \ldots, a_n, b_1, \ldots, b_n))\\
&\quad = -2\sum i x_i b_i + i y_i a_i.
\end{aligned}$$

where $\mathbf{x}=(x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$. Comparing the terms multiplying a_k and b_k on each side of the equation, for $i = 1, \ldots, n$ we obtain the differential equations

$$\frac{\partial h}{\partial x_i} = -2iy_i, \qquad \frac{\partial h}{\partial y_i} = -2ix_i.$$

For $c \in \mathbb{C}$, the solutions are:

$$h_c = -2x_1y_1 - \ldots - 2nx_ny_n + c = \sum_{i=1}^{n} -2ix_iy_i + c. \tag{3.2}$$

We thus conclude that any Landau–Ginzburg model of the form $(T^*\mathbb{P}^n, h_c)$ will give rise to the same skeleta described above. This concludes the description of our Landau–Ginzburg models and their skeleta on $T^*\mathbb{P}^n$ and in the next section we discuss birational maps within each skeleton.

4 Birational Maps within the Skeleton

In this section we present the birational transformations between components of the skeleton that justify the vertical downarrow appearing on the left hand side of diagram (1.2). As we saw in (2.2), the component L_j of the skeleton of $T^*\mathbb{P}^n$ has the form

$$\mathscr{O}_{\mathbb{P}^j}(-1) \oplus \mathscr{O}_{\mathbb{P}^j}(-1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^j}(-1)$$

with $n - j$ factors. Projectivizing we obtain $\mathbb{P}^j \times \mathbb{P}^{n-j-1}$. Thus, the component L_{j+1} has the form

$$\mathscr{O}_{\mathbb{P}^{j+1}}(-1) \oplus \mathscr{O}_{\mathbb{P}^{j+1}}(-1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^{j+1}}(-1)$$

with $n - j - 1$ factors. Projectivizing we obtain $\mathbb{P}^{j+1} \times \mathbb{P}^{n-j-2}$. The projectivizations are birationally equivalent, as we describe next, and up to tensoring by $\mathscr{O}(+1)$, we may choose a birational map taking L_j to L_{j+1}.

The birational maps $\mathbb{P}^n \times \mathbb{P}^m \dashrightarrow \mathbb{P}^{n+m}$ we need here are well known, but we recall one construction for completeness. We take homogeneous coordinates $y_0, \ldots, y_n$ on $\mathbb{P}^n$ and $z_0, \ldots, z_m$ on $\mathbb{P}^m$. Set $r := (n + 1)(m + 1) - 1$ and take homogeneous coordinates $u_{ij}, 0 \leq i \leq n, 0 \leq j \leq m$, of $\mathbb{P}^r$. Let $\nu : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^r$ be the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^r$ given by the equations $u_{ij} = y_i z_j$. The birational map (not a morphism, since there is no birational morphism between these two varieties) is induced by a linear projection $\ell_M : \mathbb{P}^r \setminus M \to \mathbb{P}^{n+m}$, where M is an $(r - n - m - 1)$-dimensional linear subspace whose equations are coordinates $u_{ij} = 0$ for some i, j and the $n + m + 1$ homogeneous coordinates of $\mathbb{P}^{n+m+1}$ are the ones used to describe M. Recall that linear projections in suitable coordinates are just rational maps which forget some of the coordinates.

We start by considering the simplest example, that is, $n = m = 1$ and hence $r = 3$. Therefore M is a point, say $([0 : 1], [0 : 1])$. Take for $\mathbb{P}^3$ homogeneous coordinates

$$x_0 = y_0 z_0, \quad x_1 = y_1 z_0, \quad x_2 = y_0 z_1, \quad x_3 = y_1 z_1$$

with $M = [0 : 0 : 0 : 1]$ and use x_0, x_1, x_2 for coordinates of $\mathbb{P}^2$.

The next step is to consider $n = 2$, $m = 1$ and hence $r = 5$. Take $\mathbb{P}^5$ with homogeneous coordinates

$$x_0 = y_0 z_0, \quad x_1 = y_0 z_1, \quad x_2 = y_1 z_0, \quad x_3 = y_1 z_1, \quad x_4 = y_2 z_0, \quad x_5 = y_2 z_1.$$

Then M is a line contained in the first ruling of $\mathbb{P}^2 \times \mathbb{P}^1$ so it has the form $L \times \{p\}$ where $L \subset \mathbb{P}^2$ is a line, and $p \in \mathbb{P}^1$ is a point. If we take $L = \{y_0 = 0\}$ and $p = [0 : 1]$ we get the equations $x_0 = x_1 = x_2 = x_4 = 0$ and the coordinates of $\mathbb{P}^3$ should be x_0, x_1, x_2, x_4. We blow-up $L \times \{p\} \subset \mathbb{P}^2 \times \mathbb{P}^1$ and then we contract the strict transform of $\mathbb{P}^2 \times \{p\}$ and $L \times \mathbb{P}^1$. So, the birational map is clear.

Now, to take one Lagrangian to the next one, we argue in generality. Suppose we have 2 quasi-projective varieties X, X', with Zariski open subsets $U \subseteq X$, $V \subseteq X'$, $U \neq \emptyset$, such that there exists an isomorphism

$$s : U \to V.$$

If for a fixed quasi-projective variety Y, we need two proper birational morphisms

$$u_1 : Y \to X \qquad \text{and} \qquad u_2 : Y \to X'$$

compatible with s, then we have a single choice: take first the graph

$$W := \{(x, s(x)\}_{x \in U} \subset U \cup V,$$

then take the closure T of W in $X \times X'$. Then T has the two morphisms

$$v_1 : T \to X \qquad \text{and} \qquad v_2 : T \to X'$$

and any other (Y, u_1, u_2) must be obtained by composing (T, v_1, v_2) with a morphism $f : Y \to T$, in such a way that we obtain

$$u_1 := f \circ v_1 \qquad \text{and} \qquad u_2 := f \circ v_2.$$

The argument in this section shows that we have a birational transformation taking L_j to L_{j+1}, thus justifying the vertical downarrow **bir** appearing in Theorem 1.1 we now proceed to discuss the other side of the duality in focus here, namely singularities and vector bundles on their resolutions.

5 Duality for Multiplicity *n* Singularities

We describe a duality between vector bundles on 2 distinct minimal resolutions of toric singularities of multiplicity n, which are both quotients of $\mathbb{C}^2$ by the cyclic group of n elements $\mathbb{Z}/n\mathbb{Z}$, and whose toric cones are dual, they are:

$$\mathscr{X}_n := \frac{1}{n}(1, 1) \quad \text{and} \quad \mathscr{X}_n^\vee := \frac{1}{n}(1, n-1).$$

These singularities are obtained by the following actions:

$$\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \quad \text{for } \mathscr{X}_n \quad \text{and} \quad \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \quad \text{for } \mathscr{X}_n^\vee,$$

where ρ is a primitive n-th root of unity, that is, $\rho = e^{\frac{2\pi i}{n}}$. In general the singularity $\frac{1}{n}(1, a)$ is obtained from the action $(x, y) \mapsto (\rho x, \rho^a y)$.

A resolution of singularities $\widetilde{X} \to X$ is called **minimal** if $\widetilde{X} \to X' \to X$ with X' smooth imply $\widetilde{X} \simeq X'$. Let Z_n and $\widetilde{Y}_n$ denote the minimal toric resolutions of $\mathscr{X}_n$ and $\mathscr{X}_n^\vee$, respectively, depicted in Figs. 1 and 2. We observe that, in particular we have $Z_2 \simeq \widetilde{Y}_2$, but $Z_n \not\simeq \widetilde{Y}_n$ for $n \neq 2$.

The surface $Z_n = \mathrm{Tot}\mathscr{O}_{\mathbb{P}^1}(-n)$ contains a single rational curve with self-intersection $-n$, whereas $\mathscr{X}_n^\vee$ contains an isolated A_{n-1}-singularity and $\widetilde{Y}_n$ contains a chain of $n-1$ curves $E_i \simeq \mathbb{P}^1$ for $1 \leq i \leq n-1$ whose intersection matrix $(E_i \cdot E_j)$ coincides with the negative of the Cartan matrix of the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of type A_{n-1}.

Note that the Dynkin diagram of type A_{n-1} is precisely the graph dual to the system of curves E_i in the resolution of $\widetilde{Y}_n$.

The surfaces $\mathscr{X}_n$ and $\mathscr{X}_n^\vee$ are toric varieties having fans formed by a single cone, calling $\sigma_{\mathscr{X}_n}$ and $\sigma_{\mathscr{X}_n^\vee}$ their respective fans, we have that $\sigma_{\mathscr{X}_n}$ is dual to $\sigma_{\mathscr{X}_n^\vee}$. In particular, for the case of $n = 2$ we also have that $\sigma_{Z_2} \simeq \sigma_{\widetilde{Y}_2}$ is self-dual.

We now describe the coordinate rings of the singularities. We have $\mathscr{X}_n = \mathrm{Spec} A$, where

$$A = H^0(Z_n, \mathscr{O}) \simeq \mathbb{C}[x_0, \ldots, x_n]/(x_i x_{j+1} - x_{i+1} x_j)_{0 \leq i < j < n}. \tag{5.1}$$

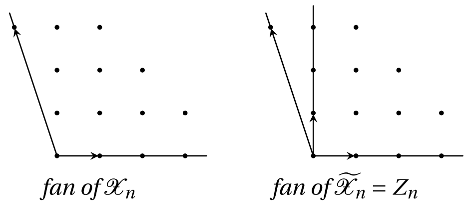

Fig. 1 Z_n as toric resolution of $\mathscr{X}_n$

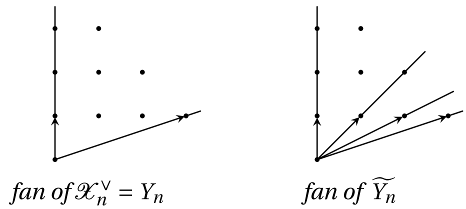

Fig. 2 $\widetilde{Y}_n$ as toric resolution of $\mathscr{X}_n^\vee = Y_n$

Given that $\mathscr{X}_n \simeq \mathbb{C}^2/\,\Gamma$, where Γ is the group generated by $\left(\begin{smallmatrix} \rho & 0 \\ 0 & \rho \end{smallmatrix}\right)$ for ρ a primitive n-th root of unity, we have $\Gamma \simeq \mathbb{Z}/n\mathbb{Z}$, with $j \in \mathbb{Z}/n\mathbb{Z}$ corresponding to $\left(\begin{smallmatrix} \rho^j & 0 \\ 0 & \rho^j \end{smallmatrix}\right)$.

Functions on the quotient $\mathbb{C}^2/\,\Gamma$ are given by those functions on $\mathbb{C}^2$ which are invariant under Γ.

The algebra of functions on $\mathbb{C}^2$ is $\mathbb{C}[a, b]$ and Γ acts by multiplication by ρ on both a and b. We thus have that $a^i b^j = (\rho a)^i (\rho b)^j = \rho^{i+j} a^i b^j$ if and only if $\rho^{i+j} = 1$, i.e. if and only if $i + j$ is a multiple of n. One sees that $\mathbb{C}[a, b]^\Gamma$ (functions on $\mathbb{C}^2$ invariant under Γ) are generated by

$$a^n, a^{n-1}b, \ldots, ab^{n-1}, b^n.$$

Now one can check that the invariants are

$$\mathbb{C}[a, b]^\Gamma = \mathbb{C}[a^n, a^{n-1}b, \ldots, ab^{n-1}, b^n] \simeq A$$

with the resolution mapping

$$a^i b^{n-i} \mapsto x_i \quad \text{for} \quad 0 \leq i \leq n,$$

so that $\mathbb{C}^2/\,\Gamma \simeq \mathscr{X}_n$. This map looks quite similar to the Veronese embedding. In fact, $\mathscr{X}_n$ is the so-called affine cone over the *Veronese curve* (or *rational normal curve*) of degree n, i.e. $\mathscr{X}_n \simeq \mathbb{C}^2/\,\Gamma$ is the affine cone over the image of the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^n$ given by $[a : b] \mapsto [a^n : a^{n-1}b : \cdots : ab^{n-1} : b^n]$.

The duality between $\mathscr{X}_n$ and $\mathscr{X}_n^\vee$ is made clear by their toric fans. Just observe that each fan consists of a single cone, and the vectors forming the fan of $\mathscr{X}_n$ are perpendicular to those of the fan of $\mathscr{X}_n^\vee$ as depicted in Figs. 1 and 2.

6 Vector Bundles on Local Surfaces

We now describe vector bundles on Z_n, the resolution of the isolated singularity $\mathscr{X}_n$. The surface Z_n is the local model of the neighborhood of a rational line ℓ with self-intersection $-n$ in a complex surface X. Thus, vector bundles on Z_n model vector bundles around such a line ℓ in X. The case $n = 1$ occurs when blowing-up a smooth point, and was explored in [6].

Recently, a new complex surgery operation on vector bundles over Z_n, named *grafting*, was introduced in the context of mathematical physics (see [10]). It provided an original explanation for the phenomenon of instanton decay in terms of curvature of the underlying space. Here we explore the geometric features of this grafting procedure. When considered from the point of view of grafting, bundles on Z_n occur as building blocks of vector bundles on surfaces, in a sense somewhat analogue (and dual) to the use of the Lagrangian skeleton for building a symplectic manifold.

Let E be a vector bundle on a compact complex surface X which contains a $-n$ line. Let $F = E|_N$ be the restriction of E to an open neighborhood N of ℓ in the analytic topology. Grafting is obtained by replacing F by another vector bundle F', which is then glued to $E|_{X \setminus N}$. Note that after grafting the top Chern class of E will in general change, but not the first one. Therefore, this surgery procedure is not obtained by an elementary transformation. The gluing itself is done over $N \setminus \ell$ which is identified with the complement of the zero section in Z_n called the collar defined below; such a gluing is possible because vector bundles on Z_n are completely determined by their restriction to a finite formal neighborhood of ℓ, see [2]. We now describe explicit local data on Z_n used to classify vector bundles on them. These vector bundles restricted to the collars will give rise to the dual objects to the components of the skeleta described above.

For each integer n, we have the surface $Z_n = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-n))$. The complex manifold structure can be described by gluing the open sets

$$U = \mathbb{C}[z, u] \quad \text{and} \quad V = \mathbb{C}[\xi, v]$$

by the relation

$$(\xi, v) = (z^{-1}, z^n u) \tag{6.1}$$

whenever z and ξ are not equal to 0. We call (6.1) the **canonical coordinates** for Z_n.

Using canonical coordinates, the contraction $Z_n := \mathrm{Tot}\mathscr{O}_{\mathbb{P}^1}(-n) \to \mathscr{X}_n$ sends $z^i u \mapsto x_i$, where x_i are the coordinates of $\mathscr{X}_n$ as described in (5.1).

Let E be a rank r holomorphic vector bundle on Z_n. The restriction of E to the zero section $\ell \simeq \mathbb{P}^1$ is a rank r bundle on $\mathbb{P}^1$, which by Grothendieck's lemma splits as a direct sum of line bundles. Thus, $E|_\ell \simeq \mathscr{O}_{\mathbb{P}^1}(j_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(j_r)$. Following [1], we call $(j_1, \ldots, j_r)$ the splitting type of E. When E is a rank 2 bundle with first Chern class 0, then the splitting type is $(j, -j)$ for some $j \geq 0$ and we say for short that E has **splitting type** j.

There are many rank 2 vector bundles on Z_n. For each fixed splitting type, they can be obtained as a quotient of $\mathrm{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))$. Considering isomorphism classes of vector bundles modulo holomorphic equivalence, moduli spaces were obtained as follows.

Proposition 6.2 ([2, Theorem 4.11]) *The moduli space of irreducible* SU(2) *instantons on Z_n with charge (and splitting type) j is a quasi-projective variety of dimension $2j - n - 2$.*

An equivalent formulation in terms of vector bundles is:

Corollary 6.3 *The moduli space of (stable) rank 2 bundles on Z_n with vanishing first Chern class and local second Chern class j is a quasi-projective variety of dimension $2j - n - 2$.*

Even though vector bundles on Z_n are many, their restrictions to the collars have very simple behaviour, as we now shall demonstrate.

We denote by ℓ the $\mathbb{P}^1$ contained in Z_n corresponding to the zero section of the corresponding vector bundles, and we set

$$Z_n^o := Z_n \setminus \ell. \tag{6.4}$$

We call Z_n^o the **collar** of ℓ in Z_n. Using the canonical coordinates for Z_n we obtain **canonical coordinates** for the collar by setting

$$Z_n^o = U^o \cup V^o,$$

with the complex manifold structure obtained by gluing the open sets

$$U^o = \mathbb{C} \times \mathbb{C} - \{0\} \simeq \mathbb{C}[z, u, u^{-1}] \quad \text{and} \quad V^o = \mathbb{C} \times \mathbb{C} - \{0\} \simeq \mathbb{C}[\xi, v, v^{-1}]$$

by the relation

$$(\xi, v) = (z^{-1}, z^n u).$$

Lemma 6.5 *The homotopy type of Z_n^o is that of an S^1-bundle over S^2, and $\pi_1(Z_n^o) = \mathbb{Z}/n\mathbb{Z}$.*

Proof Let $D = \{z, |z| \leq 1\}$ be the unit disc in $\mathbb{C}$, denoted D^+ when oriented positively, and D^- when oriented negatively. The homotopy type of Z_n^o is then that of

$$U^o \sim U^+ = D^+ \times S^1 = [z, u = e^{i\theta}] \quad \text{and} \quad V^o \sim U^- = D^- \times S^1 = [\xi, v = e^{i\phi}]$$

with identification in $U^+ \cap V^-$ given by

$$(\xi = e^{i\alpha}, v = e^{i\phi}) = (z^{-1} = e^{-i\alpha}, v = z^n u = e^{i(\theta + n\alpha)}).$$

The result of the identification is an S^1-bundle over $S^2 = D^+ \cup D^-$, with the S^1 fibers identified via the gluing map z^n which has degree $n \in \pi_1(SO(2)) = \mathbb{Z}$ since $SO(2) \simeq S^1$. $\square$

Let $\iota\colon Z_n^o \to Z_n$ denote the inclusion, and set

$$L_n(j) := \iota^* \mathcal{O}_{Z_n}(j).$$

Proposition 6.6 *For each n, the group of all isomorphism classes of line bundles $\{L_n(j),\ j \in \mathbb{Z}\}$ is cyclic of order n, hence $\mathbb{Z}/n\mathbb{Z}$.*

Proof Note that $\mathrm{Pic}(Z_n) = \mathbb{Z}$. Each line bundle over Z_n with first Chern class j is isomorphic to $\mathcal{O}_{Z_n}(j)$ and therefore can be represented by a transition matrix (z^{-j}). Since in canonical coordinates we have that $u^{-1} \neq 0$ and $v \neq 0$ on the collar Z_n^o, we may change coordinates as follows

$$(z^{-j}) \simeq (v)(z^{-j})(u^{-1}) = (z^n u \cdot z^{-j} \cdot u^{-1}) = (z^{-j+n}),$$

i.e., over Z_n^o, the bundles $L_n(j)$ and $L_n(j-n)$ (defined by (z^{-j}) and (z^{-j+n}) respectively) are isomorphic. Moreover, if $j_1 \equiv j_2 \mod n$, then $L_n(j_1)$ and $L_n(j_2)$ are isomorphic. The proof that the cases $1, 2, \ldots, n-1$ are not pairwise isomorphic is included in the proof of Proposition 6.7. $\square$

The following is a slightly rephrased version of [7, Proposition 4.1].

Proposition 6.7 *Let E_1 and E_2 be rank 2 bundles over Z_n with vanishing first Chern classes and splitting types j_1 and j_2, respectively. There exists an isomorphism $E_1|_{Z_n^o} \simeq E_2|_{Z_n^o}$ if and only if $j_1 \equiv j_2 \bmod n$. In particular, E_1 is trivial over Z_n if and only if $j_1 \equiv 0 \bmod n$.*

Proof We first claim that the bundle $\mathcal{O}_\ell(-n)$ is trivial on Z_n^o. In fact, if $u = 0$ is the equation of ℓ, then $s(z, u) = u$ determines a section of $\mathcal{O}_\ell(-n)$ that does not vanish on Z_n^o.

If a bundle E over Z_n has splitting type j, then by definition, $E|_\ell \cong \mathcal{O}_\ell(-j) \oplus \mathcal{O}_\ell(j)$. So there is a surjection $\rho\colon E|_\ell \to \mathcal{O}_\ell(j)$, and a corresponding elementary transformation, resulting in a vector bundle $E' = \mathrm{Elm}_{\mathcal{O}_\ell(j)}(E)$ which splits over ℓ as $\mathcal{O}_\ell(-n) \oplus \mathcal{O}_\ell(j+n)$, see [2, Sect. 3]. Therefore we can use the surjection $\rho\colon E'|_\ell \to \mathcal{O}_\ell(j+n)$ to perform a second elementary transformation, and we obtain the bundle $E'' = \mathrm{Elm}_{\mathcal{O}_\ell(j+n)}(E')$, which splits over ℓ as $\mathcal{O}_\ell(-j) \oplus \mathcal{O}_\ell(j+2n)$ and has first Chern class $2n$. Tensoring by $\mathcal{O}_\ell(-n)$ we get back to a $\mathfrak{sl}_2(\mathbb{C})$-bundle with splitting type $j+n$. Hence, the transformation

$$\Phi(E) = \otimes \mathcal{O}(-n) \circ \mathrm{Elm}_{\mathcal{O}_\ell(j+n)} \circ \mathrm{Elm}_{\mathcal{O}_\ell(j)}(E)$$

increases the splitting type by n while keeping the isomorphism type of E over Z_n^o. So we need only to analyze bundles with splitting type $j < n$.

If $j = 0$, the bundle is globally trivial on Z_n. If $j \neq 0$, then $E|_{Z_n^o}$ induces a non-zero element on the fundamental group $\pi_1(Z_n^o) = \mathbb{Z}/n\mathbb{Z}$.

By Lemma 6.5 the collar Z_n^o has the homotopy type of an S^1-bundle over S^2 and $\pi_1(Z_n^o) = \mathbb{Z}/n\mathbb{Z}$. Therefore $H_1(Z_n^o, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ and by Poincaré duality $H^2(Z_n^o, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. The exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathscr{O} \to \mathscr{O}^* \to 0$$

induces the first Chern class map

$$H^1(Z_n^o, \mathscr{O}^*) \to H^2(Z_n^o, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z},$$

and

$$L_n(j) \mapsto j \mod n.$$

□

In this section we have described vector bundles on Z_n, their moduli, and behaviour on collars. We will see next that each splitting type is connected to the lower ones by deformations.

7 Deformations

In this section we justify the vertical upwards arrow appearing in diagram (1.2). We start with a vector bundle E with splitting type $(j, -j)$ on Z_n, so that E may be written as an extension

$$0 \to \mathscr{O}(-j) \to E \to \mathscr{O}(j) \to 0. \tag{7.1}$$

Alternatively, we may also choose to write E as an extension of $\mathscr{O}(j+s)$ by $\mathscr{O}(-j-s)$ for any $s > 0$. To see this, just observe that there exist inclusions

$$H^1(\mathscr{O}(-2j)) = \mathrm{Ext}^1(\mathscr{O}(j), \mathscr{O}(-j)) \overset{\iota}{\hookrightarrow} \mathrm{Ext}^1(\mathscr{O}(j+s), \mathscr{O}(-j-s)) = H^1(\mathscr{O}(-2j-2s)).$$

Let p be the extension class corresponding to representing the bundle E by the exact sequence (7.1). Next, fixing an injection ι, consider the family $t \cdot \iota(p)$ of extensions of $\mathscr{O}(j+1)$ by $\mathscr{O}(-j-1)$. For such a family, when $t = 0$ we obtain $\mathscr{O}(j+1) \oplus \mathscr{O}(-j-1)$ but when $t = 1$ we obtain E.

Now, using induction on j, we conclude that every bundle on Z_n occurs as a deformation of another bundle with splitting type as high as desired. In particular, such behaviour of lowering the splitting type via deformations is also observed over the collars, justifying the vertical uparrow **def** appearing in Theorem 1.1. We now combine this vertical uparrow with the vertical downarrow **bir** described in Sect. 4. There is a 1-1 correspondence between elements of the skeleton and splitting types on the collar. Given that this correspondence is obtained via a combination of 2 dualities, we call it a duality transformation. We denote it by a horizontal double arrow:

$$L_j \Longleftrightarrow \mathcal{O}_{Z_n^\circ}(j) \oplus \mathcal{O}_{Z_n^\circ}(-j).$$

Collecting horizontal and vertical arrows together, we obtain the commutative diagram claimed in Theorem 1.1.

$$\begin{tikzcd}
L_j \subset T^*\mathbb{P}^{n-1} \arrow[r, Leftrightarrow, "\textbf{dual}"] \arrow[d, "\textbf{bir}"'] & \mathcal{O}_{Z_n^\circ}(j) \oplus \mathcal{O}_{Z_n^\circ}(-j) \\
L_{j+1} \subset T^*\mathbb{P}^{n-1} \arrow[r, Leftrightarrow, "\textbf{dual}"'] & \mathcal{O}_{Z_n^\circ}(j+1) \oplus \mathcal{O}_{Z_n^\circ}(-j-1). \arrow[u, "\textbf{def}"']
\end{tikzcd}$$

In conclusion, we have given an explicit geometric description of a duality between Lagrangians in the skeleta of cotangent bundles and vector bundles on collars. The symplectic side of the duality studies the components of the Lagrangian skeleta of cotangent bundles over n-dimensional projective spaces. The complex algebraic side considers only 2-dimensional complex varieties. These 2 are rather different types of objects. So, a priori this duality was not at all evident, but was abstractly predicted by a combination of 2 other types of duality.

In future work, we intend to pursue a generalization of this type of duality to the realm of Calabi–Yau threefolds, investigating what symplectic manifolds and Lagrangians are dual to vector bundles on local Calabi–Yau varieties and what operations occur as dual to deformations of vector bundles, see [8, 9]. The latter promises to be a challenging question, given the existence of infinite dimensional families of deformations in the case of 3-dimensional varieties, see [3].

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References

1. Ballico, E.: Rank 2 vector bundles in a neighborhood of an exceptional curve of a smooth surface. Rocky Mountain J. Math **29**(4), 1185–1193 (1999)
2. Ballico, E., Gasparim, E., Köppe, T.: Vector bundles near negative curves: moduli and local Euler characteristic. Comm. Algebra **37**, 2688–2713 (2009)

3. Ballico, E., Gasparim, E., Suzuki, B.: Infinite dimensional families of Calabi-Yau threefolds and moduli of vector bundles. J. Pure Appl. Algebra **225**(4), 106554 (2021)
4. Brion, M., Fu, B.: Symplectic resolutions for conical symplectic varieties. Int. Math. Res. Not. IMRN **2015**(12), 4335–4343 (2014)
5. Braden, T., Licata, A., Proudfoot, N., Webster, B.: Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality. arXiv:1407.0964
6. Gasparim, E.: The Atiyah-Jones conjecture for rational surfaces. Adv. Math. **218**, 1027–1050 (2008)
7. Gasparim, E., Koppe, T., Majumdar, P.: Local Holomorphic Euler characteristic and instanton decay. Pure Appl. Math. Q. **4**(2), 363–382 (2008)
8. Gasparim, E., Köppe, T., Rubilar, F., Suzuki, B.: Deformations of noncompact Calabi-Yau threefolds. Rev. Colombiana Mat. **52**(1), 41–57 (2018)
9. Gasparim, E., Rubilar, R.: Deformations of noncompact Calabi-Yau manifolds, families and diamonds. Contemp. Math. **766**, 117–132 (2021)
10. Gasparim, E., Suzuki, B.: Curvature grafted by instantons. Indian J. Phys. **95**(8), 1631–1638 (2021)
11. Ruan, W.-D.: The Fukaya category of symplectic neighborhood of a non-Hausdorff manifold. arXiv:math/0602119
12. Shende, V., Treumann, D., Williams, H.: On the combinatorics of exact Lagrangian surfaces. arXiv:1603.07449

Quot-Scheme Limit of Fubini–Study Metrics and Its Applications to Balanced Metrics

Yoshinori Hashimoto and Julien Keller

Abstract We present some results that complement our prequels [27, 28] on holomorphic vector bundles. We apply the method of the Quot-scheme limit of Fubini–Study metrics developed therein to provide a generalisation to the singular case of the result originally obtained by X. W. Wang for the smooth case, which states that the existence of balanced metrics is equivalent to the Gieseker stability of the vector bundle. We also prove that the Bergman 1-parameter subgroups form subgeodesics in the space of Hermitian metrics. This paper also contains a review of techniques developed in [27, 28] and how they correspond to their counterparts developed in the study of the Yau–Tian–Donaldson conjecture.

Keywords Fubini-Study metrics · Balanced metrics

1 Introduction

The theorem due to Donaldson [13–15] and Uhlenbeck–Yau [52] states that a holomorphic vector bundle over a smooth complex projective variety admits a Hermitian–Einstein metric if the bundle is slope stable. Together with the theorem by Kobayashi [32] and Lübke [36], it follows that the vector bundle admits a Hermitian–Einstein metric if and only if it is slope stable. This is an important theorem in complex geometry that provides an important link between differential and algebraic geometry, which was proved by using deep analytic results in [13–15, 52].

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In the papers [27, 28] we discussed attempts at establishing a more direct link between the Hermitian–Einstein metrics and slope stability, from the point of view of computing the asymptotic slope of the appropriate energy functional that arises in the variational formulation of the problem. Following these works, the purpose of this paper is twofold: one is to prove some results (Corollary 4.5 and Theorem 5.8) that complement the prequels [27, 28], and the other is to provide a summary (Sect. 2) of what is developed in [27, 28] and compare it with the analogous ideas in the study of Yau–Tian–Donaldson conjecture, i.e. the case of varieties (Sect. 3). More detailed content of this paper is summarised below.

Organisation of the paper. In Sect. 2, we survey the methods developed in [28] and the results proved in [27, 28]. We then present how these methods can be regarded as a vector bundle version of the ideas proposed in the study of constant scalar curvature Kähler and Kähler–Einstein metrics in Sect. 3; we also give brief comments on the Deligne pairing in Sect. 3.2, which plays an important role in the previous work by Boucksom–Hisamoto–Jonsson [8] and Phong–Ross–Sturm [42] for the case of varieties but not in our papers [27, 28], by discussing its relationship to the Bott–Chern class. After showing that the Bergman 1-parameter subgroups do indeed define subgeodesics in Corollary 4.5, Sect. 4, as expected from the case of varieties, we prove in Sect. 5 that the method of the Quot-scheme limit of Fubini–Study metrics provides a variational characterisation of the Gieseker stability (Theorem 5.8), generalising the results originally proved by X.W. Wang [54] and later by Phong–Sturm [43] to the case of torsion-free sheaves over a $\mathbb{Q}$-Gorenstein log terminal projective variety. Finally in Sect. 6, we provide some effective results and numerical aspects of the Donaldson functional.

Notation. Throughout in this paper (except for Remarks 2.5, 4.6, and Sect. 5), X stands for a smooth complex projective variety, and $\mathcal{E}$ stands for a holomorphic vector bundle of rank r over X, and we write $\mathcal{E}(k)$ for $\mathcal{E} \otimes \mathcal{O}_X(k)$ with a very ample line bundle $\mathcal{O}_X(1)$; we shall write $\mathcal{O}_X(k)$ for $\mathcal{O}_X(1)^{\otimes k}$ ($k \in \mathbb{Z}$). We assume, just in order to simplify the exposition, that $\mathcal{E}$ does not split into a direct sum of holomorphic subbundles (i.e. $\mathcal{E}$ is irreducible), noting that the reducible case can be treated similarly by considering each irreducible component.

We fix a Kähler metric ω on X once and for all, in the Kähler class $c_1(L)$ where $L = \mathcal{O}_X(1)$. We shall also write Vol_L for the volume $\int_X \omega^n/n!$ of X with respect to ω.

In Sect. 5, we treat the case when X is a singular variety, and its singular locus is denoted by $\mathrm{Sing}(X)$. Likewise, the singular locus of a coherent sheaf $\mathcal{E}$ is denoted by $\mathrm{Sing}(\mathcal{E})$. In this paper, the *regular locus* $X^{\mathrm{reg}} \subset X$ denotes a further subset of $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(\mathcal{E}))$ as given in Definition 2.2. We may also write $\mathrm{Sing}(X, \mathcal{E})$ for $\mathrm{Sing}(X) \cup \mathrm{Sing}(\mathcal{E})$.

2 Summary of the Methods Developed in [27, 28]

2.1 Fubini–Study Metrics

We start by recalling the classical Fubini–Study metrics for vector bundles. The reference is [54] or [38, Chap. 5].

Since $\mathcal{O}_X(1)$ is very ample, the vector bundle $\mathcal{E}(k)$ is globally generated for all large enough k, so that the sheaf map

$$\rho : H^0(X, \mathcal{E}(k)) \otimes \mathcal{O}_X(-k) \to \mathcal{E}, \tag{2.1}$$

defined by evaluation at each point, is surjective. In what follows, to streamline the exposition, we shall further assume $k > \mathrm{reg}(\mathcal{E})$, where $\mathrm{reg}(\mathcal{E})$ is an integer called the **Castelnuovo–Mumford regularity** defined by

$$\mathrm{reg}(\mathcal{E}) := \inf_{k \in \mathbb{Z}} \{\mathcal{E} \text{ is } k\text{-regular.}\}$$

where we recall that $\mathcal{E}$ is k-regular if $H^i(X, \mathcal{E}(k-i)) = 0$ for all $i > 0$ (the existence of the regularity number is justified by Serre vanishing theorem).

The sheaf surjection (2.1) implies that we have a holomorphic map

$$\Phi : X \to \mathrm{Gr}(r, H^0(X, \mathcal{E}(k))^\vee) \tag{2.2}$$

to the Grassmannian of r-planes in $H^0(X, \mathcal{E}(k))^\vee$, such that the universal bundle $\mathcal{U}$ over the Grassmannian (i.e. the dual of the tautological bundle) is pulled back by Φ to $\mathcal{E}(k)$.

A positive definite Hermitian form H on the vector space $H^0(X, \mathcal{E}(k))^\vee$ naturally defines a Hermitian metric $h_{FS,H}$ on the universal bundle $\mathcal{U}$ over the Grassmannian. Pulling this back by Φ we get a Hermitian metric on $\mathcal{E}(k)$; fixing a reference Hermitian metric on $\mathcal{O}_X(1)$, this means that we get a Hermitian metric that we shall still denote $\Phi^* h_{FS,H}$ on $\mathcal{E}$. The metric thus constructed is called the k**th Fubini–Study metric** on $\mathcal{E}$ defined by the Hermitian form H; note that this precisely agrees with the classical Fubini–Study metrics induced by the embedding to the projective space when $\mathcal{E}$ is a line bundle.

The construction above can also be described as follows. Note first that any positive definite Hermitian form $H^0(X, \mathcal{E}(k))^\vee$ can be written, up to an overall constant multiple which does not play an important role in this paper, as $\sigma^* \sigma$ for some $\sigma \in SL(H^0(X, \mathcal{E}(k))^\vee)$. Then, as pointed out by Wang [54, Remark 3.5] (see also [38, Theorem 5.1.16] and [28, Sect. 2.3]), there exists a C^∞-map

$$Q : \mathcal{E} \to H^0(X, \mathcal{E}(k)) \otimes C^\infty_X(-k), \tag{2.3}$$

where $C_X^{\infty}(-k)$ is the sheaf of smooth sections of $\mathcal{O}_X(-k)$, such that the kth Fubini–Study metric $h_\sigma = \Phi^* h_{FS,\sigma^*\sigma}$ defined by $\sigma^*\sigma$ can be written explicitly as

$$h_\sigma = Q^* \sigma^* \sigma Q, \tag{2.4}$$

where Q^* is the formal adjoint of Q with respect to some fixed reference Hermitian metrics.

We write $\mathcal{H}_k$ for the set of kth Fubini–Study metrics. Noting that the map $SL(H^0(X, \mathcal{E}(k))^\vee) \ni \sigma \mapsto h_\sigma = Q^*\sigma^*\sigma Q \in \mathcal{H}_k$ factors through $SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k)$ where $N_k = \dim H^0(X, \mathcal{E}(k))$, we find that $\mathcal{H}_k$ is parametrised by the homogeneous manifold $SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k)$.

Finally, we point out that the above construction works for the singular case that we treat in Sect. 5, by means of a pointwise argument over the regular locus. Note that this is not completely obvious from [54, Remark 3.5] as the construction in fact relies on fixing an L^2-inner product on $H^0(X, \mathcal{E}(k))$ which may be divergent in the singular case. But the pointwise equalities [38, (5.1.41), (5.1.44), (5.1.45) in the proof of Theorem 5.1.16] easily generalise to the case when the basis under consideration is not L^2-orthonormal, which is all that we need in this paper.

2.2 Quot-Scheme Limit of Fubini–Study Metrics

We recall some key concepts from [28] that we need in what follows, which the reader is referred to for more details on this section.

Recalling the description of the Fubini–Study metric h_σ as in (2.4) by using the map Q as defined in (2.3), we naturally get a family $\{h_{\sigma_t}\}_{t \geq 0}$ of Fubini–Study metrics defined by a 1-parameter subgroup (1-PS) $\{\sigma_t\}_{t \geq 0} \subset SL(H^0(X, \mathcal{E}(k))^\vee)$ as

$$h_{\sigma_t} := Q^* \sigma_t^* \sigma_t Q.$$

Assuming as we may that σ_t is generated by a Hermitian element $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ as $\sigma_t = e^{\zeta t}$, we call the above $\{h_{\sigma_t}\}_{t \geq 0}$ the **Bergman 1-PS** generated by $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$; further, when ζ has rational eigenvalues, it is called the **rational Bergman 1-PS**.

The main technique developed in [28] is to evaluate the limit of h_{σ_t} as $t \to +\infty$ for $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ with rational eigenvalues, in terms of the Quot-scheme limit. We give a quick summary of it below.

Suppose that $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ has eigenvalues $w_1, \dots, w_\nu \in \mathbb{Q}$, with the ordering

$$w_1 > \cdots > w_\nu. \tag{2.5}$$

We consider the action of ζ on $H^0(X, \mathcal{E}(k))$ which is not the natural dual action, but the one that is natural with respect to certain metric duals (see [28, (3.6)] and the discussion that follows). In any case, such an action gives us the weight decomposition

$$H^0(X, \mathcal{E}(k)) = \bigoplus_{i=1}^{\nu} V_{-w_i,k}$$

where ζ acts on $V_{-w_i,k}$ via the $\mathbb{C}^*$-action $T : \mathbb{C}^* \curvearrowright V_{-w_i,k}$ defined by $T \mapsto T^{-w_i}$ (cf. [28, Sect. 3.1]); here we introduced an auxiliary variable T by $T := e^{-t}$, so that the limit $t \to +\infty$ corresponds to $T \to 0$. The above decomposition naturally leads to the filtration

$$V_{\leq -w_i,k} := \bigoplus_{j=1}^{i} V_{-w_j,k}, \tag{2.6}$$

of $H^0(X, \mathcal{E}(k))$ by its vector subspaces.

Recalling the sheaf surjection (2.1), the filtration (2.6) also gives rise to the one

$$0 \neq \mathcal{E}_{\leq -w_1} \subset \cdots \subset \mathcal{E}_{\leq -w_\nu} = \mathcal{E} \tag{2.7}$$

of $\mathcal{E}$ by subsheaves, where $\mathcal{E}_{\leq -w_i}$ is a coherent subsheaf of $\mathcal{E}$ defined by the quotient map

$$\rho_{\leq -w_i} : V_{\leq -w_i,k} \otimes \mathcal{O}_X(-k) \to \mathcal{E}_{\leq -w_i}$$

induced from ρ as defined in (2.1). As in [28, Lemma 3.6], we can modify this filtration on a Zariski closed subset of X, to get a filtration

$$0 \neq \mathcal{E}'_{\leq -w_1} \subset \cdots \subset \mathcal{E}'_{\leq -w_\nu} = \mathcal{E} \tag{2.8}$$

of $\mathcal{E}$ by saturated subsheaves. We say that a filtration is trivial if it is equal to $0 \subsetneq \mathcal{E}$.

Remark 2.1 When the eigenvalues $w_1, \dots, w_\nu$ of ζ are only real, as opposed to rational, exactly the same argument applies so as to get the filtration (2.8), with the only difference being that the grading of the filtration is given by real numbers.

In describing the limit of the Bergman 1-PS $\{h_{\sigma_t}\}_{t \geq 0}$, only a certain subset of the subsheaves in (2.8) matters, in the sense that we only need to consider the subsheaves in (2.8) such that the associated graded sheaf has a nontrivial rank. More precisely, following [28, Definition 3.3], we can pick a certain subset

$$\{w_\alpha\}_{\alpha=\hat{1}}^{\hat{\nu}} \subset \{w_i\}_{i=1}^{\nu} \tag{2.9}$$

with $\{\hat{1}, \dots, \hat{\nu}\} \subset \{1, \dots, \nu\}$, by means of the Quot-scheme limit as explained below; what (2.9) precisely means is that the subscript α runs over a subset $\{\hat{1}, \dots, \hat{\nu}\}$ of $\{1, \dots, \nu\}$, with the ordering given by $\hat{1} < \hat{2} < \cdots < \hat{\nu}$.

We recall the quotient map (2.1)

$$\rho : H^0(X, \mathcal{E}(k)) \otimes \mathcal{O}_X(-k) \to \mathcal{E}$$

for $\mathcal{E}$, and note that its $\mathbb{C}^*$-orbit defined by $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ can be written as

$$\rho_T := \rho \circ T^\zeta : H^0(X, \mathcal{E}(k)) \otimes \mathcal{O}_X(-k) \to \mathcal{E}.$$

If we set

$$\mathcal{E}_{-w_i} := \mathcal{E}_{\leq -w_i} / \mathcal{E}_{\leq -w_{i-1}}, \tag{2.10}$$

we can consider the coherent sheaf $\bigoplus_{i=1}^{\nu} \mathcal{E}_{-w_i}$ that comes with the map

$$\hat{\rho} : H^0(X, \mathcal{E}(k)) \otimes \mathcal{O}_X(-k) = \bigoplus_{i=1}^{\nu} V_{-w_i,k} \otimes \mathcal{O}_X(-k) \to \bigoplus_{i=1}^{\nu} \mathcal{E}_{-w_i},$$

defined by the direct sum of sheaf maps $V_{-w_j,k} \otimes \mathcal{O}_X(-k) \to \mathcal{E}_{\leq -w_i} \to \mathcal{E}_{-w_i}$. It is well-known that $\hat{\rho}$ defined as above is equal to limit of ρ_T in the Quot-scheme under the $\mathbb{C}^*$-action T^ζ [29, Lemma 4.4.3]. The subset $\{\hat{1}, \dots, \hat{\nu}\}$ in (2.9) consists of the indices i such that $\mathrm{rk}(\mathcal{E}_{-w_i}) > 0$ (see also [28, Definition 3.3]). It turns out that $\hat{1} = 1$ (see [28, Remark 3.4]), and the reader is referred to [28, Sect. 3] for more details.

We shall often argue over a certain Zariski open subset of X as defined below.

Definition 2.2 We define X^{reg} be the Zariski open subset of X over which the sheaves $\mathcal{E}_{\leq -w_i}$ in (2.7) and $\mathcal{E}/\mathcal{E}_{\leq -w_i}$ are all locally free [28, Definition 3.5], and such that each $\mathcal{E}_{\leq -w_i}$ agrees with the saturated subsheaf $\mathcal{E}'_{\leq -w_i}$ in (2.8).

For each $\alpha \in \{\hat{1}, \dots, \hat{\nu}\}$ in (2.9), the quotient sheaf $\mathcal{E}_{-w_\alpha} = \mathcal{E}_{\leq -w_\alpha} / \mathcal{E}_{\leq -w_{\alpha-1}}$ is locally free over X^{reg} (and agrees as a C^∞-vector bundle with the quotient vector bundle of $\mathcal{E}_{\leq -w_\alpha}$ by $\mathcal{E}_{\leq -w_{\alpha-1}}$; see [28, discussion following Definition 3.5]).

The definition (2.10) of $\mathcal{E}_{-w_i}$, combined with the above definition of X^{reg}, we get a C^∞-isomorphism

$$\mathcal{E} \xrightarrow{\sim} \bigoplus_{\alpha=\hat{1}}^{\hat{\nu}} \mathcal{E}_{-w_\alpha} \tag{2.11}$$

of smooth complex vector bundles over X^{reg} [28, (3.8)]. Moreover, we have a gauge transformation on $\bigoplus_{\alpha=\hat{1}}^{\hat{\nu}} \mathcal{E}_{-w_\alpha}$ over X^{reg} by the constant endomorphism

$$e^{wt} := \mathrm{diag}(e^{w_{\hat{1}}t}, \cdots, e^{w_{\hat{\nu}}t}), \tag{2.12}$$

with $e^{w_\alpha t}$ acting on the factor $\mathcal{E}_{-w_\alpha}$.

An important observation is that the map Q^*, defined as the formal adjoint of Q (2.3) as

$$Q^* : \overline{H^0(X, \mathcal{E}(k))^\vee} \otimes \overline{C_X^\infty(-k)^\vee} \to \overline{\mathcal{E}^\vee},$$

can be regarded as a C^∞-version of the quotient map ρ (2.1), up to taking the metric dual in the domain and the range [28, Lemma 2.22]. This seems to suggest that the limit of the Bergman 1-PS $\{h_{\sigma_t}\}_{t \geq 0}$ as $t \to +\infty$ can be related to the Quot-scheme limit $\bigoplus_{\alpha=\hat{1}}^{\hat{\nu}} \mathcal{E}_{-w_\alpha}$, up to the metric duality isomorphism $\mathcal{E} \tilde{\to} \overline{\mathcal{E}^\vee}$ (as a C^∞-vector bundle), and indeed it is the main technical result that was established in [28]. More precisely, we define a Hermitian metric on $\mathcal{E}|_{X^{\text{reg}}}$ by

$$\hat{h}_{\sigma_t} := e^{-wt} h_{\sigma_t} e^{-wt}, \tag{2.13}$$

with (2.11) understood, which we call the **renormalised Bergman 1-PS** associated to σ_t [28, Definition 3.10]. An important fact is that this 1-PS is convergent in C^∞_{loc} over X^{reg} [28, Proposition 3.9], and we call the limit

$$\hat{h} := \lim_{t \to +\infty} e^{-wt} h_{\sigma_t} e^{-wt} \tag{2.14}$$

the **renormalised Quot-scheme limit** of h_{σ_t} [28, Definition 3.10], which is positive definite over X^{reg} [28, Lemma 3.12].

Proposition 2.3 (see [28, Proposition 3.9 and Lemma 3.12]) *The renormalised Bergman 1-PS converges in C^∞_{loc} over X^{reg} as $t \to +\infty$, and its limit defines a well-defined Hermitian metric on $\mathcal{E}$ via (2.11) over X^{reg}.*

The above limit (2.14) is only defined on a Zariski open subset X^{reg} of X, and may well be degenerate on $X \setminus X^{\text{reg}}$. In spite of this drawback, we can evaluate the degeneracy of $\hat{h}$ by comparing Q^* with ρ and by using the resolution of singularities. The argument is technical, but can be carried out by using the methods in Jacob [30] and Sibley [46], which occupies a large portion of the technical argument in [28, Sect. 4].

Remark 2.4 Throughout in what follows, we shall assume that the operator norm (i.e. the modulus of the maximum eigenvalue) of ζ is at most 1, as pointed out in [28, Remark 3.2].

Remark 2.5 Inspection of [28] reveals that almost all the arguments so far carry over word-by-word to the case when X is a (not necessarily smooth) projective variety and when $\mathcal{E}$ is a torsion-free sheaf, by considering Hermitian metrics on $X \setminus (\text{Sing}(X) \cup \text{Sing}(\mathcal{E}))$ instead of X; the only exception is that the map Φ in (2.2) needs to be replaced by a rational map that is well-defined on $X \setminus \text{Sing}(\mathcal{E})$. This means that X^{reg} as in Definition 2.2 should be replaced by the Zariski open subset in X excluding all the singular sets of sheaves $\mathcal{E}_{\leq -w_i}$ and $\mathcal{E}/\mathcal{E}_{\leq -w_i}$ and that of the background data, i.e. $\text{Sing}(X) \cup \text{Sing}(\mathcal{E})$. In the arguments later (except for Remark 4.6 and Sect. 5), however, it will be important that X is smooth.

2.3 The Non-Archimedean Donaldson Functional

We recall an important functional defined by Donaldson [14]. Let $\mathcal{H}_\infty$ be the set of all smooth Hermitian metrics on $\mathcal{E}$.

Definition 2.6 Given two Hermitian metrics h_0 and h_1 on $\mathcal{E}$, **the Donaldson functional** $\mathcal{M}^{Don} : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathbb{R}$ is defined as

$$\mathcal{M}^{Don}(h_1, h_0) := \mathcal{M}_1^{Don}(h_1, h_0) - \mu(\mathcal{E})\mathcal{M}_2^{Don}(h_1, h_0),$$

where

$$\mathcal{M}_1^{Don}(h_1, h_0) := \int_0^1 \int_X \mathrm{tr}\left(h_t^{-1}\partial_t h_t \cdot F_t\right) \frac{\omega^{n-1}}{(n-1)!} dt$$

and

$$\mathcal{M}_2^{Don}(h_1, h_0) := \frac{1}{\mathrm{Vol}_L} \int_X \log\det(h_0^{-1}h_1)\frac{\omega^n}{n!},$$

with

$$\mu(\mathcal{E}) := \frac{\int_X c_1(\mathcal{E}) \wedge \omega^{n-1}/(n-1)!}{\mathrm{rk}(\mathcal{E})}.$$

In the above, $\{h_t\}_{0 \le t \le 1} \subset \mathcal{H}_\infty$ is a smooth path of Hermitian metrics between h_0 to h_1 and F_t denotes $(\sqrt{-1}/2\pi)$ times the Chern curvature of h_t, with respect to the holomorphic structure of $\mathcal{E}$.

Remark 2.7 Throughout in what follows, we shall fix the second argument of the Donaldson functional as a reference metric. Thus $\mathcal{M}^{Don}(h, h_0)$ is regarded as a function of h with a fixed reference metric h_0.

The critical point of the Donaldson functional is the Hermitian–Einstein metric. An important property of the Donaldson functional is that it is convex along geodesics in $\mathcal{H}_\infty$, with an appropriate notion of geodesics in $\mathcal{H}_\infty$; see [14, 33] for more details. Thus, the existence of the critical point of the Donaldson functional can be, at least conceptually, characterised by the positivity of its asymptotic slope. The main result of [28] is the explicit description of the asymptotic slope of the Donaldson functional along the rational Bergman 1-PS by means of the algebro-geometric data. While the Bergman 1-PS is not a geodesic in $\mathcal{H}_\infty$, it turns out that it is a subgeodesic in $\mathcal{H}_\infty$ (Corollary 4.5).

We now recall the non-Archimedean Donaldson functional from [28], defined for a rational Bergman 1-PS generated by $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k)))$ which is Hermitian (with respect to a fixed Hermitian form as described in [28, Sect. 3.1]). Writing $w_1, \dots, w_\nu \in \mathbb{Q}$ for the weights of ζ as in (2.5), we choose $j(\zeta, k) \in \mathbb{N}$ to be the minimum integer so that

$$j(\zeta, k)w_i \in \mathbb{Z} \tag{2.15}$$

for all $i = 1, \dots, \nu$. Writing $\bar{w}_i := j(\zeta, k)w_i$, we may replace the filtration (2.8) by

$$0 \neq \mathcal{E}'_{\le -\bar{w}_1} \subset \cdots \subset \mathcal{E}'_{\le -\bar{w}_\nu} = \mathcal{E} \tag{2.16}$$

which is graded by integers. With this understood, the following was defined in [28, Definition 5.3].

Definition 2.8 **The non-Archimedean Donaldson functional** $\mathcal{M}^{\mathrm{NA}}(\zeta, k)$ is a rational number defined for $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k)))$ Hermitian with rational eigenvalues as

$$\mathcal{M}^{\mathrm{NA}}(\zeta, k) := \frac{2}{j(\zeta, k)} \sum_{q \in \mathbb{Z}} \mathrm{rk}(\mathcal{E}'_{\leq q}) \left(\mu(\mathcal{E}) - \mu(\mathcal{E}'_{\leq q}) \right).$$

We recall, in the above, that the **slope** of a (necessarily torsion-free) subsheaf $\mathcal{F} \subset \mathcal{E}$ on X is defined by

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})},$$

where, by noting that $\det \mathcal{F} := \left(\bigwedge^{\mathrm{rk}(\mathcal{F})} \mathcal{F} \right)^{\vee\vee}$ is a holomorphic line bundle on X [33, Proposition 5.6.10], the degree is given by $\deg(\mathcal{F}) := \int_X c_1(\det \mathcal{F}) \wedge c_1(\mathcal{O}_X(1))^{n-1} / (n-1)!$.

We can also define [27, Definition 4.1] a rational number $J^{\mathrm{NA}}(\zeta, k)$ for $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k)))$ with rational eigenvalues by

$$J^{\mathrm{NA}}(\zeta, k) := \max_{\alpha, \beta \in \{\hat{1}, \dots, \hat{\nu}\}} |w_\alpha - w_\beta| \tag{2.17}$$

where we recall (2.9) for the definition of $\hat{1}, \dots, \hat{\nu}$. We can easily show [27, Remark 4.2] that $J^{\mathrm{NA}}(\zeta, k) = 0$ if and only if the corresponding filtration (2.8) is trivial, i.e. equals $0 \subsetneq \mathcal{E}$.

An elementary yet important fact is that the positivity of $\mathcal{M}^{\mathrm{NA}}(\zeta, k)$ is equivalent to the slope stability of $\mathcal{E}$, as stated in the following.

Proposition 2.9 (see [28, Proposition 7.4]) *The non-Archimedean Donaldson functional $\mathcal{M}^{\mathrm{NA}}(\zeta, k)$ is positive (resp. nonnegative) for all $k \in \mathbb{N}$ such that $\mathcal{E}(k)$ is globally generated and all $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$, with rational eigenvalues, whose associated filtration (2.8) is nontrivial, if and only if $\mathcal{E}$ is slope stable (resp. semistable), i.e. for any subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ we have*

$$\mu(\mathcal{E}) > \mu(\mathcal{F}), \quad (\textit{resp. } \mu(\mathcal{E}) \geq \mu(\mathcal{F})).$$

This can be proved by an argument similar to the proof of Proposition 6.1 which is presented later (see also [28, Sect. 6]).

Remark 2.10 When $\mathcal{E}$ splits into a direct sum of holomorphic vector subbundles, the right notion to consider is the slope polystability: $\mathcal{E} = \bigoplus_{1 \leq l \leq m} \mathcal{E}_l$ is said to be slope polystable if each direct summand $\mathcal{E}_l$ is slope stable and $\mu(\mathcal{E}_{l_1}) = \mu(\mathcal{E}_{l_2})$ for all $1 \leq l_1, l_2 \leq m$.

2.4 Summary of Results in [27, 28]

We now recall the main results of [28] as follows.

Theorem 2.11 (see [28, Theorems 7.2, 7.3, and Corollary 7.5]) *There exists a constant $c_k > 0$ that depends only on the reference metric $h_{\mathrm{ref}} \in \mathcal{H}_\infty$ and $k \in \mathbb{N}$ such that*

$$\mathcal{M}^{Don}(h_{\sigma_t}, h_{\mathrm{ref}}) \geq \mathcal{M}^{\mathrm{NA}}(\zeta, k)t - c_k \tag{2.18}$$

holds for all $t \geq 0$ and all Hermitian $\zeta \in \mathfrak{sl}(H^0(\mathcal{E}(k))^\vee)$ with rational eigenvalues. We can further show that

$$\mathcal{M}^{Don}(h_{\sigma_t}, h_{\mathrm{ref}}) = \mathcal{M}^{\mathrm{NA}}(\zeta, k)t + O(1), \tag{2.19}$$

where $O(1)$ stands for the term that remains bounded as $t \to +\infty$, i.e $\mathcal{M}^{Don}$ has log norm singularities along any rational Bergman 1-PS. In particular, we have

$$\lim_{t \to +\infty} \frac{\mathcal{M}^{Don}(h_{\sigma_t}, h_{\mathrm{ref}})}{t} = \mathcal{M}^{\mathrm{NA}}(\zeta, k),$$

which shows that $\mathcal{M}^{\mathrm{NA}}(\zeta, k)$ is the term that controls the asymptotic behaviour of $\mathcal{M}^{Don}(h_{\sigma_t}, h_{\mathrm{ref}})$.

Another way of stating the property (2.18) is to say that $\mathcal{M}^{Don}$ is *coercive* (resp. bounded from below) along rational Bergman 1-PS if $\mathcal{E}$ is slope stable (resp. slope semistable), see [28, Sect. 7].

The corollary of the above result is that we can show that the existence of the Hermitian–Einstein metrics implies $\mathcal{E}$ is slope stable [28, Sect. 7]. It may also be worth noting that the analysis used to prove the above results is elementary, cf. [28, Sect. 4].

The reverse direction of the correspondence was discussed in [27]. Writing $\mathcal{H}_k$ for the set of kth Fubini–Study metrics, we have a natural inclusion $\mathcal{H}_k \subset \mathcal{H}_\infty$ for each k. While $\mathcal{H}_k$ is a "small" subset of $\mathcal{H}_\infty$ parametrised by a finite dimensional homogeneous manifold $SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k)$, it turns out that the union of $\mathcal{H}_k$'s is dense in $\mathcal{H}_\infty$ with respect to the C^p-topology (for any fixed $p \in \mathbb{N}$), i.e.

$$\mathcal{H}_\infty = \overline{\bigcup_{k \in \mathbb{N}} \mathcal{H}_k}. \tag{2.20}$$

This fact follows from a foundational result in Kähler geometry, called the **asymptotic expansion of the Bergman kernel**, which is also called the Tian–Yau–Zelditch expansion, and is essentially a theorem in analysis. We do not give a detailed account of this result, and refer the reader to [9, 55]; note that several proofs have been written especially when $\mathcal{E}$ has rank one (see e.g. the book [38] and references therein). An elementary proof can be found in [1].

The main result of [27] is that, *if* we assume that a uniform version of Theorem 2.11 holds, we can prove that the slope stability implies Hermitian–Einstein metrics by only using elementary analysis except for the asymptotic expansion of the Bergman kernel. The precise statement is as follows.

Theorem 2.12 (see [27, Theorem 1]) *Suppose that the estimate in the theorem above holds uniformly in k, i.e.*

$$\mathcal{M}^{Don}(h_{\sigma_t}, h_{\text{ref}}) \geq \mathcal{M}^{\text{NA}}(\zeta, k)t - c_{\text{ref}}$$

for a constant $c_{\text{ref}} > 0$ that depends only on the reference metric. Then we can prove that the stability implies the existence of the Hermitian–Einstein metric by using only elementary analytic methods except for $\mathcal{H}_\infty = \overline{\bigcup_{k\in\mathbb{N}} \mathcal{H}_k}$ in (2.20) which is a consequence of the asymptotic expansion of the Bergman kernel.

The hypothesis of the above theorem will be satisfied if the constant c_k in Theorem 2.11 can be bounded uniformly in k, but we do not discuss this point further as it seems to be a difficult problem; see also [27, Sect. 5.1].

3 Comparison to the Case of the Yau–Tian–Donaldson Conjecture

3.1 Dictionary Between Vector Bundles and Manifolds

The methods and results summarised above are motivated by the recent progress on the Yau–Tian–Donaldson conjecture, surveyed e.g. in [5, 11, 21, 49], and it seems reasonable to have a table of correspondence between the vector bundles case and the varieties case. Indeed, our approach in [28] can be regarded as a vector bundle version of the results concerning the Yau–Tian–Donaldson conjecture as established e.g. in [4, 5, 7, 8, 41, 42]. For example, one of our main results Theorem 2.11 (or rather its consequence (2.19)) can be regarded as a vector bundle version of a result by Boucksom–Hisamoto–Jonsson [8], Paul [41], and Phong–Ross–Sturm [42] (amongst many other related results).

It is well-known that the role played by the Mabuchi energy in the case of varieties is almost exactly the same as that of the Donaldson functional in the case of vector bundles; the critical point of these functionals are precisely the canonical metrics, and both of them are convex along geodesics in the space of metrics (although the convexity for the Mabuchi energy is a much more subtle issue due to the weaker regularity of the geodesics in the space of Kähler potentials [3, 10]). It is also well-known that the maximally destabilising subsheaf for vector bundles corresponds to the optimal destabilising test configuration. We list below (Table 1) how the objects reviewed in Sect. 2.2 correspond to the ones in the case of varieties, i.e. study of constant scalar curvature Kähler and Kähler–Einstein metrics.

Table 1 Dictionary

Vector bundles	Manifolds
Filtration $\mathcal{E}'_{\leq -w_1} \subset \cdots \subset \mathcal{E}'_{\leq -w_\nu}$ (2.8) of $\mathcal{E}$	Test configuration [17, 50]
The graded object $\bigoplus_{\alpha=\hat{1}}^{\hat{\nu}} \mathcal{E}_{-w_\alpha}$ of the filtration	Central fibre of a test configuration
$k \geq \mathrm{reg}(\mathcal{E})$ in $H^0(X, \mathcal{E}(k))$	Exponent of a test configuration
Non-Archimedean Donaldson functional (Definition 2.8)	Non-Archimedean Mabuchi functional ([41], [8, Sect. 5])
$J^{\mathrm{NA}}(\zeta, k)$ in (2.17)	Non-Archimedean J-functional or the minimum norm [7, 12]

It is well-known in the case of varieties that a test configuration defines a subgeodesic in the space of Kähler metrics (see e.g. [2, 8]). In Sect. 4 we provide a vector bundle version of this result in Corollary 4.5.

Another important topic in the study of constant scalar curvature Kähler and Kähler–Einstein metrics is what is known as Donaldson's quantisation, which can be regarded as a finite dimensional approximation of the canonical metric by a sequence of balanced metrics [16]. The vector bundle version of this result was established by Wang [55]. He also proved that the existence of the balanced metrics is equivalent to the Gieseker stability of the vector bundle [54], where we note that the analogous result for the varieties case is due to Luo [37] and Zhang [57].

In Sect. 5, we apply the method of the Quot-scheme limit of Fubini–Study metrics that we reviewed in Sect. 2 to give a generalisation of this result. The dictionary also extends to the balancing flow for manifolds defined in [16, 23] and for the bundle version in [31]. The first one provides a quantisation of the Calabi flow while the second one provides a quantisation of the Yang–Mills flow.

On a Fano manifold (without nontrivial holomorphic vector field), the behaviour of Mabuchi energy restricted to Fubini–Study metrics of level k_0 (for a certain k_0 sufficiently large) is sufficient to test K-stability. Actually, building on the partial C^0 estimate from Székelyhidi [47] solving a conjecture of Tian and the work Paul [41] about CM-stability, Boucksom–Hisamoto–Jonsson [8] get that coercivity of the Mabuchi energy on the space of positive metrics implies uniform K-stability and thus the existence of a Kähler–Einstein metric by Chen–Donaldson–Sun. Moreover, as explained in the discussion after [40, Theorem 2.9] or [35, page 3], in order to test K-stability it is sufficient to work with 1-PS degenerations in a fixed projective space (induced by the space of holomorphic section of a fixed power of the anticanonical bundle). This is actually a consequence of Chen–Donaldson–Sun too. Then the fact that it is sufficient to consider a fixed k_0 is a consequence of [8, Theorem C] (more precisely $(i) \Rightarrow (ii)$ which can be obtained from Theorem A and the equivalence $(ii) \Leftrightarrow (iii)$). Eventually, Sect. 6 is addressing the counterpart of this result for the Mabuchi energy to the bundle case for the Donaldson functional.

Remark 3.1 We have not yet found an appropriate analogue of the J-functional for vector bundles, while the quantity $J^{\mathrm{NA}}(\zeta, k)$ defined in (2.17) does seem to play a role analogous to the non-Archimedean J-functional.

3.2 Comments on the Deligne Pairing

While our results can be regarded as a vector bundle version of the results by Boucksom–Hisamoto–Jonsson [8], Paul [41], or Phong–Ross–Sturm [42], the proof is not a naive transplantation of the methods used therein. Our method relies on the materials reviewed in Sect. 2.2, whereas the Deligne pairing (resp. the Bott–Chern class) plays a crucially important role in [8, 42] (resp. [41]).

The method of the Deligne pairing was not used extensively in establishing the results in [27, 28], unlike in [8, 42]. We look at the Donaldson functional from the point of view of the Deligne pairing in this section, but the result we get is not as clear-cut as [8, Sect. 1.5]; the difficulty seems to arise from the fact that there is no explicit formula for the second Bott–Chern class yet (which seems to indicate the difficulty in naively transplanting Paul's argument [41] to vector bundles).

In this paper we do not give detailed definitions concerning the Deligne pairing, since we only focus on the very special case; the reader is referred for its proper treatment e.g. to [8, Sect. 1.2], [20, 39], or [42, Sect. 2] and the references cited therein. The only Deligne pairing that we use in this paper is for the (trivial) flat projective morphism $\pi : X \to \mathrm{pt}$ which maps X to a point. For holomorphic line bundles $L_1, \dots, L_{n+1}$ we can define the Deligne pairing line bundle $\langle L_1, \dots, L_{n+1} \rangle_X$ over the point (i.e. a $\mathbb{C}$-vector space). Given Hermitian metrics $\phi_1, \dots, \phi_{n+1}$ on $L_1, \dots, L_{n+1}$ we can furthermore define a continuous metric $\langle \phi_1, \dots, \phi_{n+1} \rangle_X$ on $\langle L_1, \dots, L_{n+1} \rangle_X$. Moreover, the construction is "functorial" in the sense as explained in the references cited above. When we give another Hermitian metric ϕ_1 on L_1, we have the change of metric formula (see e.g. [8, (1.5)] or [42, (2.5)])

$$\langle \phi_1 - \phi_1', \dots, \phi_{n+1} \rangle_X = \int_X (\phi_1 - \phi_1') \eta_{\phi_2} \wedge \cdots \wedge \eta_{\phi_{n+1}}$$

where η_{ϕ_i} is the curvature form of ϕ_i ($i = 2, \dots, n+1$), and the additive notation is used to denote the tensor product of Hermitian metrics. In what follows, we shall consider the Deligne pairing for the case $L_2 = \cdots = L_{n+1} = L$. It is well-known that many important functionals that appear in the study of constant scalar curvature Kähler or Kähler–Einstein metrics can be written as a change of metric formula of an appropriate Deligne pairing line bundle (cf. [8, Sect. 1.5]). An analogous result holds for the vector bundle case, as stated below.

Proposition 3.2 *There exists a $\mathbb{Q}$-line bundle $\mathcal{L}$ on X such that $c_1(\mathcal{L}) = \Lambda ch_2(\mathcal{E})$, where Λ is the adjoint Lefschetz operator on $H^*(X, \mathbb{C})$, such that the Donaldson functional $\mathcal{M}^{Don}$ can be written as a change of metric formula*

$$\langle \psi_1 - \psi_0, \phi, \dots, \phi \rangle_X - \frac{\mu(\mathcal{E})}{\mathrm{Vol}_L} \langle \det h_1 - \det h_0, \phi, \dots, \phi \rangle_X$$

for the Deligne pairing

$$\langle \mathcal{L}, L, \dots, L \rangle_X - \frac{\mu(\mathcal{E})}{\mathrm{Vol}_L} \langle \det \mathcal{E}, L, \dots, L \rangle_X,$$

which is a $\mathbb{Q}$-line bundle over a point, where we wrote ϕ for the Hermitian metric on L whose associated Kähler form is ω, and ψ_1 (resp. ψ_0) is a certain Hermitian metric on $\mathcal{L}$ which depends on h_1 (resp. h_0).

In terms of the Fubini–Study metrics that we discussed earlier, the right family to look at should be $Y := X \times SL(H^0(X, \mathcal{E}(k))^\vee)$ with the flat projective morphism $\pi : Y \to SL(H^0(X, \mathcal{E}(k))^\vee)$ defined by the second projection, as opposed to the trivial $\pi : X \to \mathrm{pt}$, but we get an analogous result for this case by the functoriality of the Deligne pairing.

Proof It is well-known [14, Sect. 1.2] that the Donaldson functional

$$\mathcal{M}^{Don}(h_1, h_0) = \mathcal{M}_1^{Don}(h_1, h_0) - \mu(\mathcal{E}) \mathcal{M}_2^{Don}(h_1, h_0)$$

can be written in terms of the Bott–Chern characteristic forms, with

$$\mathcal{M}_1^{Don}(h_1, h_0) = \int_X \mathbf{BC}_2(\mathcal{E}, h_1, h_0) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

and

$$\mathcal{M}_2^{Don}(h_1, h_0) = \int_X \mathbf{BC}_1(\mathcal{E}, h_1, h_0) \wedge \frac{\omega^n}{n!},$$

where the Bott–Chern characteristic forms $\mathbf{BC}_i(\mathcal{E}, h_1, h_0)$ ($i = 1, \dots, n$) are a collection of certain secondary characteristic forms, defined modulo ∂- and $\bar{\partial}$-exact forms, such that

$$-\sqrt{-1}\partial\bar{\partial}\mathbf{BC}_i(\mathcal{E}, h_1, h_0) = ch_i(\mathcal{E}, h_1) - ch_i(\mathcal{E}, h_0),$$

where $ch_i(\mathcal{E}, h)$ stands for the ith term of the Chern character form

$$ch(\mathcal{E}, h) = \sum_{i=1}^{n} ch_i(\mathcal{E}, h) = \mathrm{tr}\,(\exp(F_h))$$

where F_h is $\sqrt{-1}/2\pi$ times the curvature form of h. While it is easy to see $\mathbf{BC}_1(\mathcal{E}, h_1, h_0) = \log\det h_1 h_0^{-1}$, an explicit formula for $\mathbf{BC}_2(\mathcal{E}, h_1, h_0)$ does not seem to be known yet. See [14, 48, 51] for more details on the above.

We now write Λ for the adjoint Lefschetz operator $*^{-1}(\omega \wedge \cdot)*$ on differential forms on X, defined with respect to ω, where $*$ is the Hodge star operator with respect to ω. Note also that the adjoint Lefschetz operator on 2-forms equals the metric contraction by ω. We find

$$\mathcal{M}_1^{Don}(h_1, h_0) = \int_X \Lambda \mathbf{BC}_2(\mathcal{E}, h_1, h_0) \frac{\omega^n}{n!}.$$

Recalling the well-known Kähler identities [53, Proposition 6.5]

(1) $[\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*$,
(2) $[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*$,

we find, modulo ∂- and $\bar{\partial}$-exact forms, that

$$\begin{aligned}
&\sqrt{-1}\partial\bar{\partial}\Lambda \mathbf{BC}_2(\mathcal{E}, h_1, h_0) \\
&= \sqrt{-1}\left(\Lambda\partial\bar{\partial} - \sqrt{-1}\bar{\partial}^*\bar{\partial} + \sqrt{-1}\partial\partial^*\right)\mathbf{BC}_2(\mathcal{E}, h_1, h_0) \\
&= \Lambda(\mathrm{tr}(F_{h_1}^2) - \mathrm{tr}(F_{h_0}^2)) + \Delta_{\bar{\partial}}\mathbf{BC}_2(\mathcal{E}, h_1, h_0) \quad \mathrm{mod}\ \mathrm{im}\partial + \mathrm{im}\bar{\partial}.
\end{aligned}$$

Note that the second term involving the Laplacian, as well as the ∂- and $\bar{\partial}$-exact forms, vanish under the integration.

Now, as in [8], we would like to regard $\mathcal{M}^{Don}(h_1, h_0)$ as a change-of-metric formula for the Deligne pairing. It is easier to deal with $\mathcal{M}_2^{Don}$, since it can be written manifestly as a change of metric formula

$$\frac{1}{\mathrm{Vol}_L}\langle \det h_1 - \det h_0, \phi, \dots, \phi\rangle_X$$

on the line bundle $\frac{1}{\mathrm{Vol}_L}\langle \det \mathcal{E}, L, \dots, L\rangle_X$ over the point.

We consider $\mathcal{M}_1^{Don}(h_1, h_0)$. Let $\mathcal{L}$ be a holomorphic line bundle on X defined as follows. Recalling that Λ induces an operator on the cohomology ring $H^*(X, \mathbb{Z})/\mathrm{torsion}$, since $[\omega] = c_1(L)$ is an integral cohomology class, we find that $\Lambda ch_2(\mathcal{E})$ defines a closed real rational $(1,1)$-form on X, which in turn can be realised as the first Chern class of a holomorphic $\mathbb{Q}$-line bundle $\mathcal{L}$ by the Lefschetz $(1,1)$-theorem (see e.g. [53, Theorem 7.2]). This is the line bundle that we are after, which is well-defined up to an element of the Picard variety of X.

Let $h_{\mathcal{L}}$ be a reference Hermitian metric on $\mathcal{L}$. We then define a Hermitian metric $e^{-\psi_1}h_{\mathcal{L}}$ so that its curvature form is equal to $\Lambda ch_2(\mathcal{E}, h_1) = \Lambda\mathrm{tr}(F_{h_1}^2)$, modulo $\mathrm{im}(\partial) + \mathrm{im}(\bar{\partial}) + \mathrm{im}(\Delta_{\bar{\partial}})$. We can always find such ψ_1 by solving Laplace's equation, which depends on h_1, since $\Lambda\mathrm{tr}(F_{h_1}^2)$ is a de Rham representative of $c_1(\mathcal{L}) = \Lambda ch_2(\mathcal{E})$. Likewise we define ψ_0 so that the curvature form of $e^{-\psi_0}h_{\mathcal{L}}$ is equal to $\Lambda ch_2(\mathcal{E}, h_0) = \Lambda\mathrm{tr}(F_{h_0}^2)$, modulo $\mathrm{im}(\partial) + \mathrm{im}(\bar{\partial}) + \mathrm{im}(\Delta_{\bar{\partial}})$. $\square$

As we can see in the proof above, the dependence of ψ_i on h_i $(i = 1, 2)$ is not straightforward; it seems this is partially because no explicit formula is known for the second Bott–Chern class $\mathbf{BC}_2(\mathcal{E}, h_1, h_0)$.

Remark 3.3 While it is difficult to explicitly write down $\mathcal{L}$ in terms of $\mathcal{E}$ as pointed out in the above, we can be slightly more specific about it by recalling the Lefschetz decomposition of $H^*(X, \mathbb{C})$ [53, Theorem 6.4]. This implies that $ch_2(\mathcal{E})$ can be written uniquely as

$$ch_2(\mathcal{E}) = aL_\omega^2 \cdot 1 + L_\omega \cdot \eta_1 + \eta_2$$

where $a \in \mathbb{Q}$, L_ω is the operator defined by $[\omega] \wedge \cdot$, and η_1, η_2 are primitive forms, i.e. $\Lambda\eta_i = 0$ for $i = 1, 2$. A well-known result in Kähler geometry [53, Lemma 6.19] says $[L_\omega, \Lambda] = (k - n)\mathrm{id}$ on real k-forms. Applying this, we have

$$\begin{aligned} \Lambda L_\omega^2 \cdot 1 &= (L_\omega(L_\omega\Lambda - n) - (2 - n)L_\omega) \cdot 1 = -2L_\omega \cdot 1 \\ \Lambda L_\omega \cdot \eta_1 &= -(2 - n)\eta_1 \end{aligned}$$

and hence

$$\Lambda ch_2(\mathcal{E}) = -2a[\omega] - (2 - n)\eta_1,$$

which implies

$$c_1(\mathcal{L}) = -2a[\omega] - (2 - n)\eta_1.$$

4 Bergman 1-Parameter Subgroups as Subgeodesics

In the above dictionary (Table 1) we saw that a 1-PS in the Quot-scheme can be regarded as a test configuration for vector bundles. In the case of varieties, it is well-known that test configurations define subgeodesics (see e.g. [2, Sect. 2.4] or [8, Sect. 3.1]). In this section we prove that an analogous result holds for the vector bundle case. We start by recalling [33, Sect. 6.2] that the geodesic (with respect to the natural L^2-metric on $\mathcal{H}_\infty$) is a piecewise C^1-family $\{h_t\}_{t\geq 0}$ of smooth Hermitian metrics satisfying

$$\partial_t(h_t^{-1}\partial_t h_t) = 0.$$

Thus, we aim to prove that we have

$$\partial_t(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}) \geq 0$$

for a Bergman 1-PS $\{\sigma_t\}_{t\geq 0}$. A more precise statement can be found in Proposition 4.3. The proof is given by re-writing $\partial_t(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t})$ in an appropriate manner, which occupies most of what follows.

Let $\sigma_t = e^{\zeta t}$ for a trace-free Hermitian matrix ζ, and write $u := \zeta^* + \zeta = 2\zeta$. Recalling the notation (2.3) and (2.4), we compute $\partial_t (h_{\sigma_t}^{-1} \partial_t h_{\sigma_t})$ as

$$\begin{aligned} &\partial_t((Q^*\sigma_t^*\sigma_t Q)^{-1}(\partial_t Q^*\sigma_t^*\sigma_t Q)) = \partial_t((Q^*\sigma_t^*\sigma_t Q)^{-1} Q^*\sigma_t^* u\sigma_t Q) \qquad (4.1)\\ &= (Q^*\sigma_t^*\sigma_t Q)^{-1} Q^*\sigma_t^* u^2\sigma_t Q - (Q^*\sigma_t^*\sigma_t Q)^{-1} Q^*\sigma_t^* u\sigma_t Q (Q^*\sigma_t^*\sigma_t Q)^{-1} Q^*\sigma_t^* u\sigma_t Q. \end{aligned}$$

Our aim is to simplify this expression. We start with the following lemma.

Lemma 4.1 *Any two of $Q^*\sigma_t^*\sigma_t Q$, $Q^*\sigma_t^* u\sigma_t Q$, $Q^*\sigma_t^* u^2\sigma_t Q$ pairwise commute.*

In the above statement, $Q^*\sigma_t^*\sigma_t Q$, $Q^*\sigma_t^* u\sigma_t Q$, $Q^*\sigma_t^* u^2\sigma_t Q$ are regarded as Hermitian endomorphisms on $\mathcal{E}$ by fixing a Q^*Q-orthonormal frame for $\mathcal{E}$. In what follows, we shall also write h_{ref} for Q^*Q.

Proof We may fix a point $x \in X$ once and for all, and work on the fibre $\mathcal{E}_x$ over x. Choosing an orthonormal frame of $\mathcal{E}_x$ with respect to the reference Hermitian metric $h_{\mathrm{ref}} = Q^*Q$, we may assume $Q^*Q = I_r$, where I_r is the $r \times r$ identity matrix. Further, by choosing an appropriate basis for $H^0(\mathcal{E}(k))$, we may assume that we can write $Q^* = \begin{pmatrix} I_r & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$ with respect to a certain decomposition $H^0(\mathcal{E}(k)) = V_r \oplus V_{N-r}$.

Suppose that we have two Hermitian matrices P_1 and P_2, which can be written as $P_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ with respect to the above block decomposition. Then

$$\begin{aligned} Q^* \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} Q Q^* \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} Q &= Q^* \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} Q \\ &= \begin{pmatrix} I_r & 0 \end{pmatrix} \begin{pmatrix} A_1A_2 & A_1B_2 \\ C_1A_2 & C_1B_2 \end{pmatrix} \begin{pmatrix} I_r \\ 0 \end{pmatrix} \\ &= A_1A_2. \end{aligned}$$

Thus, to prove commutativity of Q^*P_1Q and Q^*P_2Q, it suffices to show $A_1A_2 = A_2A_1$, which is in turn equivalent to showing that $U_r^*A_1U_r$ commutes with $U_r^*A_2U_r$ for some $r \times r$ unitary matrix U_r.

Recalling that $\zeta = \zeta^*$ and $\sigma_t = e^{\zeta t}$, we find that $\sigma_t^*\sigma_t, \sigma_t^* u\sigma_t, \sigma_t^* u^2\sigma_t$ are all simultaneously diagonalisable, and hence they commute. For the choice of the subspace $V_r \leq H^0(\mathcal{E}(k))$ as above, we have Hermitian forms on V_r defined by restriction of $\sigma_t^*\sigma_t, \sigma_t^* u\sigma_t$, and $\sigma_t^* u^2\sigma_t$; these are the matrices denoted by A_t, A_t', A_t'' in the formulae below:

$$\sigma_t^*\sigma_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}, \quad \sigma_t^* u\sigma_t = \begin{pmatrix} A_t' & B_t' \\ C_t' & D_t' \end{pmatrix}, \quad \sigma_t^* u^2\sigma_t = \begin{pmatrix} A_t'' & B_t'' \\ C_t'' & D_t'' \end{pmatrix},$$

where the block decomposition is in terms of $H^0(\mathcal{E}(k)) = V_r \oplus V_{N-r}$. By writing down a basis for V_r in terms of the diagonalising basis for ζ, we find that all the three

Hermitian forms (or, more precisely, the associated Hermitian endomorphisms) A_t, A'_t, A''_t on V_r thus defined pairwise commute; note that such a basis for V_r can be given by a one that is unitarily equivalent to the one we started with. Thus, combined with the general argument above, for each fixed $x \in X$, we find that all of $Q^* \sigma_t^* \sigma_t Q$, $Q^* \sigma_t^* u \sigma_t Q$, $Q^* \sigma_t^* u^2 \sigma_t Q$ pairwise commute at $\mathcal{E}_x$. Since this holds for all $x \in X$ and $Q^* \sigma_t^* \sigma_t Q$, $Q^* \sigma_t^* u \sigma_t Q$, $Q^* \sigma_t^* u^2 \sigma_t Q$ are tensorial, we conclude the required commutativity among them. $\square$

The following definition, artificial as it may seem, plays an important role.

Definition 4.2 For a kth Fubini–Study metric h_{σ_t} we define $\mathcal{F}(h_{\sigma_t}) \in \mathrm{Hom}_{C^\infty_X}(\mathcal{E}, H^0(\mathcal{E}(k)) \otimes C^\infty_X)$ as

$$\mathcal{F}(h_{\sigma_t}) = \left(\frac{d\sigma_t}{dt} Q - \sigma Q \left(h_{\sigma_t}^{-1} \partial_t h_{\sigma_t} \right) \right) h_{\sigma_t}^{-1/2},$$

where $h_{\sigma_t}^{-1/2}$ (regarded as a Hermitian endomorphism on $\mathcal{E}$) is defined fibrewise with respect to the h_{ref}-orthonormal frame.

Note that $\mathcal{F}(h_{\sigma_t})$, a fibrewise $N \times r$ matrix varying smoothly in x, is a tensorial quantity since $h_{\sigma_t}^{-1/2}$ is tensorial.

Proposition 4.3 *For $\mathcal{F}(h_{\sigma_t})$ defined as above, we have*

$$\partial_t (h_{\sigma_t}^{-1} \partial_t h_{\sigma_t}) = \mathcal{F}(h_{\sigma_t})^* \mathcal{F}(h_{\sigma_t}) \geq 0,$$

where $\mathcal{F}(h_{\sigma_t})^*$ is the (fibrewise) conjugate transpose of $\mathcal{F}(h_{\sigma_t})$ with respect to h_{ref}, and the inequality is that of the fibrewise Hermitian form.

Remark 4.4 The weak form of the above proposition $\mathrm{tr}(\partial_t (h_{\sigma_t}^{-1} \partial_t h_{\sigma_t})) \geq 0$ is due to Phong–Sturm [43, Lemma 2.2] and this was used to prove the convexity of the balancing energy (see Wang [54, Lemma 3.5] and also Lemma 5.7).

The proposition above immediately implies the following result.

Corollary 4.5 *The Bergman 1-PS define a subgeodesic in the space of Hermitian metrics.*

Proof of Proposition 4.3 Fixing an h_{ref}-orthonormal frame to identify Hermitian forms and endomorphisms, we compute

$$\begin{aligned}
&\left(Q^* \sigma_t^* u - (Q^* \sigma_t^* \sigma_t Q)^{-1} (Q^* \sigma_t^* u \sigma_t Q Q^* \sigma_t^*) \right) \\
&\quad \times \left(u \sigma_t Q - (\sigma_t Q Q^* \sigma_t^* u \sigma_t Q)(Q^* \sigma_t^* \sigma_t Q)^{-1} \right) \\
&= Q^* \sigma_t^* u^2 \sigma_t Q \\
&\quad - (Q^* \sigma_t^* u \sigma_t Q)^2 (Q^* \sigma_t^* \sigma_t Q)^{-1} - (Q^* \sigma_t^* \sigma_t Q)^{-1} (Q^* \sigma_t^* u \sigma_t Q)^2 \\
&\quad + (Q^* \sigma_t^* \sigma_t Q)^{-1} (Q^* \sigma_t^* u \sigma_t Q)(Q^* \sigma_t^* \sigma_t Q)(Q^* \sigma_t^* u \sigma_t Q)(Q^* \sigma_t^* \sigma_t Q)^{-1} \\
&= Q^* \sigma_t^* u^2 \sigma_t Q - (Q^* \sigma_t^* u \sigma_t Q)(Q^* \sigma_t^* \sigma_t Q)^{-1} (Q^* \sigma_t^* u \sigma_t Q),
\end{aligned}$$

where we used Lemma 4.1 in the last equality. We recall the Eq. (4.1) and apply $(Q^*\sigma_t^*\sigma_t Q)^{-1}$ from the left to get the claimed result

$$
\begin{aligned}
&\partial_t(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t})\\
&= (Q^*\sigma_t^*\sigma_t Q)^{-1}Q^*\sigma_t^*u^2\sigma_t Q - (Q^*\sigma_t^*\sigma_t Q)^{-1}Q^*\sigma_t^*u\sigma_t Q(Q^*\sigma_t^*\sigma_t Q)^{-1}Q^*\sigma_t^*u\sigma_t Q\\
&= (Q^*\sigma_t^*\sigma_t Q)^{-1/2}\left(Q^*\sigma_t^*u - (Q^*\sigma_t^*\sigma_t Q)^{-1}(Q^*\sigma_t^*u\sigma_t Q Q^*\sigma_t^*)\right)\\
&\quad\times\left(u\sigma_t Q - (\sigma_t Q Q^*\sigma_t^*u\sigma_t Q)(Q^*\sigma_t^*\sigma_t Q)^{-1}\right)(Q^*\sigma_t^*\sigma_t Q)^{-1/2},
\end{aligned}
$$

where in the last line we used the fact that $(Q^*\sigma_t^*\sigma_t Q)^{-1}$ is positive definite Hermitian, and hence $(Q^*\sigma_t^*\sigma_t Q)^{-1/2}$, defined with respect to the fixed h_{ref}-orthonormal frame, commutes with $Q^*\sigma_t^*\sigma_t Q$, $Q^*\sigma_t^*u\sigma_t Q$, and $Q^*\sigma_t^*u^2\sigma_t Q$, again by Lemma 4.1. $\square$

An interesting question is to consider when we have $\mathcal{F}(h_{\sigma_t}) = 0$, for which $\partial_t(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}) = 0$. Note that $\mathcal{F}(h_{\sigma_t}) = 0$ is equivalent to

$$\zeta Q = Q\left(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}\right)$$

since σ_t commutes with ζ. By taking the fibrewise Hermitian conjugate, this is further equivalent to

$$Q^*\zeta = \left(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}\right)Q^*$$

by noting that ζ and h_{σ_t} are both Hermitian. This means that the operator $\mathcal{F}$ captures the failure of commutativity of the following diagrams:

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}"] \arrow[d, "Q"'] & \mathcal{E} \arrow[d, "Q"] \\
H^0(\mathcal{E}(k)) \arrow[r, "\zeta"'] & H^0(\mathcal{E}(k)),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E} \arrow[r, "h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}"] & \mathcal{E} \\
H^0(\mathcal{E}(k)) \arrow[r, "\zeta"'] \arrow[u, "Q^*"] & H^0(\mathcal{E}(k)). \arrow[u, "Q^*"']
\end{tikzcd}
$$

An example of when it happens is when $\mathcal{E}$ splits in a direct sum of holomorphic vector bundles $\mathcal{E} = \bigoplus_i \mathcal{E}_i$ and ζ acts as a constant scalar multiplication on each $H^0(\mathcal{E}_i(k))$. Whether this is the only case when the above diagrams commute may be an interesting problem, but we do not touch on it in this paper.

Remark 4.6 The argument above being local, the same result holds over the non-singular locus $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(\mathcal{E}))$ of X even when X is a singular variety and $\mathcal{E}$ is a torsion-free sheaf.

5 Gieseker Stability and Balanced Metrics

In this section, we apply the method of the Quot-scheme limit as surveyed in Sect. 2.2 to provide a variational characterisation of the Gieseker stability of a torsion-free sheaf on a $\mathbb{Q}$-Gorenstein log terminal projective variety (Theorem 5.8); this generalises the result first proved by Wang [54] for holomorphic vector bundles over a smooth projective variety (with an alternative proof given by Phong–Sturm [43]). While this can be seen as an application of materials in Sect. 2.2 to the singular case, as pointed out in Remark 2.5, the method of our proof is new even for the regular case considered by [43, 54], in that it does not use the Chow-type norm of the Gieseker point, which was an essential ingredient of the proofs in [43, 54]. Instead, in our proof presented below, the inequality for the Gieseker stability appears explicitly as the positivity of the asymptotic slope of the appropriate energy functional (see Proposition 5.9).

Remark 5.1 It is perhaps worth pointing out that a related argument was carried out in the papers by García-Fernandez–Keller–Ross [24] and García-Fernandez–Ross [25], for the special case when the filtration is a two-step filtration defined by a saturated subsheaf of $\mathcal{E}$.

5.1 *Variational Formulation of the Problem*

We start by recalling the Gieseker stability.

Definition 5.2 (*Gieseker stability*) A torsion-free sheaf $\mathcal{E}$ is said to be **Gieseker stable** if the following inequality

$$\frac{P_{\mathcal{E}}(k)}{\mathrm{rk}(\mathcal{E})} > \frac{P_{\mathcal{F}}(k)}{\mathrm{rk}(\mathcal{F})} \quad \text{for } k \gg 0,$$

holds for all coherent subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, where the Hilbert polynomial $P_{\mathcal{G}}(k)$ for a coherent sheaf $\mathcal{G}$ on X is defined by $P_{\mathcal{G}}(k) := \sum_{i=0}^{n} h^i(X, \mathcal{G}(k))$. Gieseker semistability (resp. polystability) can be defined analogously to the slope semistable (resp. polystable) case as mentioned in Proposition 2.9.

The energy functional that we need to consider in this section is the following, which appeared in [43, 54]; we first present the version for a holomorphic vector bundle on a smooth variety.

Definition 5.3 Let X be a smooth projective variety and $\mathcal{E}$ be a holomorphic vector bundle on X. The functional

$$\mathcal{M}_2^{Don} : SL(H^0(X, \mathcal{E}(k))^{\vee})/SU(N_k) \to \mathbb{R}$$

is defined by

$$\mathcal{M}_2^{Don}(\sigma) = \frac{1}{\mathrm{Vol}_L} \int_X \log\det(h_\sigma h_{\mathrm{ref}}^{-1}) \frac{\omega^n}{n!},$$

where $h_\sigma = Q^* \sigma^* \sigma Q$ is the Fubini–Study metric defined (as in (2.4)) by the Hermitian form $\sigma^*\sigma$ on $H^0(X, \mathcal{E}(k))$ by $\sigma \in SL(H^0(X, \mathcal{E}(k))^\vee)$, and we take the reference metric h_{ref} to be Q^*Q.

In the above definition and in what follows, we identify an element $\sigma \in SL(H^0(X, \mathcal{E}(k))^\vee)$ with its coset class in $SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k)$, noting that $\mathcal{M}_2^{Don}(\sigma)$ depends only on $\sigma^*\sigma$.

Remark 5.4 Note that $\mathcal{M}_2^{Don}(\sigma_t)$ as defined above is clearly equal to the functional $\mathcal{M}_2^{Don}(h_\sigma, h_{\mathrm{ref}})$ in Definition 2.6. We use the above notation in what follows since we only consider the Fubini–Study metrics defined by an element of the coset space $SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k)$.

Remark 5.5 The method of Quot-scheme limits as discussed in Sect. 2.2 involves an implicit identification between $\mathcal{E}$ and its dual $\mathcal{E}^\vee$ (see [28, Remark 4.5]), which is assumed in what follows, but this does not affect the formulae that appear below as they do not contain the curvature term.

Suppose now that X is singular and that $\mathcal{E}$ is a torsion-free sheaf on X. Instead of the embedding (2.2), we have a rational map

$$\Phi : X \dashrightarrow \mathrm{Gr}(\mathrm{rk}(\mathcal{E}), H^0(X, \mathcal{E}(k))^\vee), \tag{5.1}$$

birational onto its image, by taking $k > \mathrm{reg}(\mathcal{E})$, which is defined on a Zariski open set $X \setminus \mathrm{Sing}(X, \mathcal{E})$, where we defined

$$\mathrm{Sing}(X, \mathcal{E}) := \mathrm{Sing}(X) \cup \mathrm{Sing}(\mathcal{E}).$$

With this understood and writing h_{σ_t} for the pullback of the Fubini–Study metric by $\Phi|_{X \setminus \mathrm{Sing}(X,\mathcal{E})}$, we can define the functional $\mathcal{M}_2^{Don}$ for the singular case as stated more precisely below.

Definition 5.6 Let X be a normal $\mathbb{Q}$-Gorenstein projective variety with log terminal singularities and that $\mathcal{E}$ is a torsion-free sheaf on X. We define the functional $\mathcal{M}_2^{Don}$: $SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k) \to \mathbb{R}$ by

$$\mathcal{M}_2^{Don}(\sigma) = \frac{1}{\mathrm{Vol}} \int_{X \setminus \mathrm{Sing}(X,\mathcal{E})} \log\det(h_\sigma h_{\mathrm{ref}}^{-1}) dV_X,$$

where dV_X is a volume form on $X \setminus \mathrm{Sing}(X)$ defined by the sections of the pluricanonical bundle $K_X^{\otimes m}$, $m \in \mathbb{N}$ (see e.g. [6, (4.35)] or [22, Sect. 6.2]), which has locally finite mass near $\mathrm{Sing}(X)$ since X is log terminal; this in turn means that

$$\mathrm{Vol} := \int_{X \setminus \mathrm{Sing}(X,\mathcal{E})} dV_X = \int_{X \setminus \mathrm{Sing}(X)} dV_X < \infty$$

is well-defined.

We note that the above is indeed a well-defined integral since X is log terminal and Φ in (5.1) is a rational map. More precisely, first note that by taking a resolution $\pi : \tilde{X} \to X$ we may write

$$\mathcal{M}_2^{Don}(\sigma) = \frac{1}{\mathrm{Vol}} \int_{\tilde{X} \setminus \pi^{-1}(\mathrm{Sing}(X,\mathcal{E}))} \log \pi^* \det(h_\sigma h_{\mathrm{ref}}^{-1}) \pi^*(dV_X).$$

We may assume that $\pi^{-1}(\mathrm{Sing}(X)) = \sum_j E_j$ is a simple normal crossing divisor and that locally in a neighbourhood $U \subset \tilde{X}$, we have $E_j = \{z_j = 0\}$ and

$$\pi^*(dV_X)\big|_{U \setminus \pi^{-1}(\mathrm{Sing}(X))} = \prod_j |z_j|^{2a_j} dV_U$$

with $a_j > -1$ for all j (as X is log terminal) and some smooth volume form dV_U on U [6, Lemma 4.6.5]. By composing π with further blowups, we may assume that $\pi^{-1}(\mathrm{Sing}(\mathcal{E})) = \sum_l F_l$ is also a simple normal crossing divisor. Writing $F_l = \{y_l = 0\}$ locally in a neighbourhood $U \subset \tilde{X}$ and noting that $\pi^* \det(h_\sigma h_{\mathrm{ref}}^{-1})$ has at most poles and zeros of finite order as Φ is rational, we find that

$$\begin{aligned} &\log \pi^* \det(h_\sigma h_{\mathrm{ref}}^{-1}) \pi^*(dV_X)\big|_{U \setminus \pi^{-1}(\mathrm{Sing}(X,\mathcal{E}))} \\ &= \left(\sum_l m_l \log |y_l| + O(1) \right) \prod_j |z_j|^{2a_j} dV_U \end{aligned}$$

with some integers m_l and terms denoted by $O(1)$ that stay bounded over U. This is integrable, since $a_j > -1$ for all j.

Note also that Definition 5.6 is consistent with the one for the smooth varieties (Definition 5.3), since by Yau's theorem [56] we can always find a smooth Kähler metric $\omega_\phi \in c_1(L)$ such that $\omega_\phi^n = dV_X$, up to rescaling dV_X by a constant. Thus, from now on, without ambiguity we shall adopt Definition 5.6 for the definition of $\mathcal{M}_2^{Don}$.

A straightforward computation yields that, given a (smooth) path $\{\sigma_t\}_{t\geq 0} \subset SL(H^0(X, \mathcal{E}(k))^\vee)$, we have

$$\frac{d^2}{dt^2}\mathcal{M}_2^{Don}(\sigma_t) = \frac{1}{\mathrm{Vol}}\int_{X\setminus \mathrm{Sing}(X,\mathcal{E})} \mathrm{tr}(\partial_t(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}))dV_X.$$

Note that this integral makes sense. As the integrand is invariant under the unitary change of frames of $\mathcal{E}$ (over $X \setminus \mathrm{Sing}(X, \mathcal{E})$), we may simultaneously diagonalise h_{σ_t} and $\partial_t h_{\sigma_t}$ by Lemma 4.1 (regarded as Hermitian endomorphisms on $\mathcal{E}$). Since ∂_t does not introduce further poles and does not decrease the order of zeros, we find that $h_{\sigma_t}^{-1}\partial_t h_{\sigma_t}$ (and hence $\partial_t(h_{\sigma_t}^{-1}\partial_t h_{\sigma_t})$) is bounded on $X \setminus \mathrm{Sing}(X, \mathcal{E})$ as poles and zeros cancel each other.

The following lemma was first proved for smooth X and locally free $\mathcal{E}$ by Wang [54, Lemma 3.5], and also by Phong–Sturm [43, Lemma 2.2], but we observe that it can be obtained as an immediate consequence of Proposition 4.3 and the above formula.

Lemma 5.7 *$\mathcal{M}_2^{Don} : SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k) \to \mathbb{R}$ is convex along Bergman 1-PS's.*

The above lemma implies that any critical point of $\mathcal{M}_2^{Don}$ is necessarily the global minimum.

The critical point $\sigma \in SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k)$, or the associated Fubini–Study metric h_σ, of $\mathcal{M}_2^{Don}$ is called the **balanced metric**. We can characterise the balanced metric as the one whose Bergman kernel is a constant multiple of the identity, or the one whose associated centre of mass is a constant multiple of the identity, and they can be regarded as providing a finite dimensional approximation of the Hermitian–Einstein metric; we will not discuss these topics here and the reader is referred to [43, 54, 55] for the details. Note however that these results make sense, at least naively, only when X is smooth and $\mathcal{E}$ is locally free.

5.2 *Main Result and Proof*

The main theorem of this section, stated below, is a generalisation of the result of Wang [54] (and also Phong–Sturm [43]).

Theorem 5.8 *Let X be a normal $\mathbb{Q}$-Gorenstein projective variety with log terminal singularities and that $\mathcal{E}$ is a torsion-free sheaf on X. $\mathcal{M}_2^{Don} : SL(H^0(X, \mathcal{E}(k))^\vee)/SU(N_k) \to \mathbb{R}$ admits a critical point for all large enough k if and only if $\mathcal{E}$ is Gieseker stable.*

A novel point of our proof is the following formula for the asymptotic slope of $\mathcal{M}_2^{Don}$ in terms of the invariant that defines the Gieseker stability, which relies on the Quot-scheme limit of Fubini–Study metrics as reviewed in Sect. 2.

Proposition 5.9 *Let $\{h_{\sigma_t}\}_{t\geq 0}$ be the Bergman 1-PS generated by $\zeta \in \mathfrak{sl}(H^0(\mathcal{E}(k))^{\vee})$ that has rational eigenvalues. Then we have*

$$\lim_{t\to+\infty} \frac{\mathcal{M}_2^{Don}(\sigma_t)}{t} = \frac{2}{j(\zeta,k)} \frac{\mathrm{rk}(\mathcal{E})}{h^0(X,\mathcal{E}(k))} \sum_{q\in\mathbb{Z}} \mathrm{rk}(\mathcal{E}_{\leq q}) \left(\frac{h^0(X,\mathcal{E}(k))}{\mathrm{rk}(\mathcal{E})} - \frac{\dim V_{\leq q}}{\mathrm{rk}(\mathcal{E}_{\leq q})} \right).$$

The rest of this section is devoted to the proof of Theorem 5.8. We first prove Proposition 5.9, which is a key ingredient in the proof of Theorem 5.8, which in turn follows from Lemmas 5.10 and 5.11 presented below: Lemma 5.10 is where we critically make use of the renormalised Quot-scheme limit surveyed in Sect. 2.2, but Lemma 5.11 is mostly a repetition of what is well-known to the experts [29]. The proof of Theorem 5.8 is presented after proving these lemmas. It comes down to proving that the asymptotic slope of $\mathcal{M}_2^{Don}$ is positive if and only if $\mathcal{E}$ is Gieseker stable, but the hypothesis on rational eigenvalues in Proposition 5.9 presents subtleties that need to be taken care of. This issue is addressed by a slight modification of the argument in [26].

Lemma 5.10 *Let $\{h_{\sigma_t}\}_{t\geq 0}$ be the Bergman 1-PS generated by a Hermitian matrix $\zeta \in \mathfrak{sl}(H^0(X,\mathcal{E}(k))^{\vee})$, which need not have rational eigenvalues. Then*

$$\lim_{t\to+\infty} \frac{\mathcal{M}_2^{Don}(\sigma_t)}{t} = 2\sum_{\alpha=\hat{1}}^{\hat{\nu}} w_\alpha \mathrm{rk}(\mathcal{E}'_{-w_\alpha}),$$

where the sheaves $\{\mathcal{E}'_{-w_\alpha}\}_{\alpha=\hat{1}}^{\hat{\nu}}$ are defined by (2.8); see also Remark 2.1.

Proof By using the renormalised metric (2.13), we write

$$h_{\sigma_t} = e^{wt}\hat{h}_{\sigma_t}e^{wt}.$$

Then, we write

$$\begin{aligned} &\int_{X\setminus \mathrm{Sing}(X,\mathcal{E})} \log\det(h_{\sigma_t}h_{\mathrm{ref}}^{-1})dV_X \\ &= \int_{X^{\mathrm{reg}}} \log\det(e^{2wt})\det(\hat{h}_{\sigma_t}h_{\mathrm{ref}}^{-1})dV_X \\ &= 2t\cdot \mathrm{Vol}\sum_{\alpha=1}^{\hat{\nu}} w_\alpha \mathrm{rk}(\mathcal{E}'_{-w_\alpha}) + \int_{X^{\mathrm{reg}}} \log\det(\hat{h}_{\sigma_t}h_{\mathrm{ref}}^{-1})dV_X. \end{aligned}$$

Note first that the integral $\int_{X^{\mathrm{reg}}} \log\det(\hat{h}_{\sigma_t}h_{\mathrm{ref}}^{-1})dV_X$ is well-defined, by recalling the comments after Definition 5.6 and the definition (2.13) for $\hat{h}_{\sigma_t}$.

Thus it suffices to prove that the integral $\int_{X^{\mathrm{reg}}} \log\det(\hat{h}_{\sigma_t}h_{\mathrm{ref}}^{-1})dV_X$ remains bounded as $t\to+\infty$. The argument is similar to the proof of [28, Lemma 4.13], except for that the case under consideration here is much easier. The key ingredient is that $\hat{h}$ degenerates only on a Zariski closed subset $X \setminus X^{\mathrm{reg}}$, and that $\hat{h}$ has at worst zeros and poles of finite order by [28, Lemma 2.22] which follows from the fact that the quotient map ρ in [28, (2.1)] is algebraic and that Φ in (5.1) is rational. Since $\hat{h}$ is well-defined as a Hermitian metric over X^{reg} [28, Lemmas 3.12 and 3.13], we find that the integral on the right hand side remains bounded as $t \to +\infty$, which gives the claimed result. □

Lemma 5.11 (cf. [29, Sect. 4.A]) *Suppose that we have $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ that gives rise to the filtrations (2.6), (2.7), and (2.8) by taking the saturation. Then*

$$\sum_{\alpha=1}^{\hat{\nu}} w_\alpha \mathrm{rk}(\mathcal{E}'_{-w_\alpha}) = \frac{2}{j(\zeta, k)} \frac{\mathrm{rk}(\mathcal{E})}{h^0(X, \mathcal{E}(k))} \sum_{q \in \mathbb{Z}} \mathrm{rk}(\mathcal{E}_{\le q}) \left(\frac{h^0(X, \mathcal{E}(k))}{\mathrm{rk}(\mathcal{E})} - \frac{\dim V_{\le q}}{\mathrm{rk}(\mathcal{E}_{\le q})} \right).$$

Proof Recall the definition (2.15) and that we have an integrally graded filtration (2.16) of $\mathcal{E}$ by subsheaves. We then observe

$$\sum_{\alpha=\hat{1}}^{\hat{\nu}} w_\alpha \mathrm{rk}(\mathcal{E}'_{-w_\alpha}) = \frac{1}{j(\zeta, k)} \sum_{\alpha=\hat{1}}^{\hat{\nu}} \bar{w}_\alpha \mathrm{rk}(\mathcal{E}'_{-\bar{w}_\alpha}) = \frac{1}{j(\zeta, k)} \sum_{i=1}^{\nu} \bar{w}_i \mathrm{rk}(\mathcal{E}'_{-\bar{w}_i}).$$

by recalling the definition (2.9) of $\hat{1}, \dots, \hat{\nu}$, which implies $\mathrm{rk}(\mathcal{E}'_{-w_i}) = 0$ if and only if $i \notin \{\hat{1}, \dots, \hat{\nu}\}$. Note further that

$$\sum_{i=1}^{\nu} \bar{w}_i \mathrm{rk}(\mathcal{E}'_{-\bar{w}_i}) = -\sum_{q \in \mathbb{Z}} q \cdot \mathrm{rk}(\mathcal{E}'_q).$$

We now perform the calculation that is identical to the one carried out in [29, Sect. 4.A]: since $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$, we have

$$\begin{aligned} \dim V \sum_{q \in \mathbb{Z}} q \cdot \mathrm{rk}(\mathcal{E}'_q) &= \sum_{q \in \mathbb{Z}} q \cdot \left(\mathrm{rk}(\mathcal{E}'_q) \cdot \dim V - \mathrm{rk}(\mathcal{E}) \cdot \dim V_q \right) \\ &= -\sum_{q \in \mathbb{Z}} \left(\mathrm{rk}(\mathcal{E}'_{\le q}) \cdot \dim V - \mathrm{rk}(\mathcal{E}) \cdot \dim V_{\le q} \right). \end{aligned}$$

Combining these equalities we get

$$\sum_{\alpha=1}^{\hat{\nu}} w_\alpha \mathrm{rk}(\mathcal{E}'_{-w_\alpha}) = \frac{1}{j(\zeta, k)} \frac{1}{\dim V} \sum_{q \in \mathbb{Z}} \left(\mathrm{rk}(\mathcal{E}'_{\le q}) \cdot \dim V - \mathrm{rk}(\mathcal{E}) \cdot \dim V_{\le q} \right).$$

By substituting in $\dim V = h^0(X, \mathcal{E}(k))$ and tidying up the terms, we get the desired result. □

Proof of Proposition 5.8 We first recall [29, Theorem 4.4.1] which is credited to Le Potier in [29], where we note that the multiplicity of a sheaf is just the rank since X is a projective variety [29, page 11]. It states that for all sufficiently large integers k, $\mathcal{E}$ being Gieseker stable is equivalent to

$$\frac{h^0(X, \mathcal{F}(k))}{\mathrm{rk}(\mathcal{F})} < \frac{P_{\mathcal{E}}(k)}{\mathrm{rk}(\mathcal{E})} = \frac{h^0(X, \mathcal{E}(k))}{\mathrm{rk}(\mathcal{E})}$$

for all subsheaves $\mathcal{F} \subset \mathcal{E}$ of rank $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$. This proves that $\mathcal{E}$ is Gieseker stable if and only if for all sufficiently large k and all $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ with rational eigenvalues we have, for the 1-PS $\{\sigma_t\}_{t\geq 0}$ defined by $\sigma_t = e^{\zeta t}$,

$$\lim_{t\to+\infty} \frac{\mathcal{M}_2^{Don}(\sigma_t)}{t} > 0.$$

On the other hand, the convexity of $\mathcal{M}_2^{Don}$ along Bergman 1-PS (Lemma 5.7) implies that $\mathcal{M}_2^{Don}$ has a critical point if and only if the above inequality holds for all $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ that is Hermitian but *not* necessarily having rational eigenvalues. We follow the argument in [26, Lemmas 3.15 and 3.17] to prove that the Gieseker stability in fact implies this seemingly stronger condition. First note that by continuity and Lemma 5.10 we have

$$\lim_{t\to+\infty} \frac{\mathcal{M}_2^{Don}(\sigma_t)}{t} = 2\sum_{\alpha=1}^{\hat{\nu}} w_\alpha \mathrm{rk}(\mathcal{E}'_{-w_\alpha}) \geq 0 \tag{5.2}$$

for all $\zeta \in \mathfrak{sl}(H^0(\mathcal{E}(k))^\vee)$, not necessarily having rational eigenvalues, by recalling that $\mathcal{E}_{-w_i}$ is well-defined even when w_i is not rational (Remark 2.1); in the above we wrote $w_1, \cdots, w_\nu \in \mathbb{R}$ for the eigenvalues of ζ. Now, we choose a Hermitian matrix $\tilde{\zeta}$ with rational eigenvalues, say $\tilde{w}_1, \dots, \tilde{w}_\nu \in \mathbb{Q}$, so that

1. $V_{-w_i} = V_{-\tilde{w}_i}$ for all $i = 1, \dots, \nu$,
2. $\tilde{w}_1 > \tilde{w}_2 > \cdots > \tilde{w}_\nu$,
3. $w_1 - \tilde{w}_1 > w_2 - \tilde{w}_2 > \cdots > w_\nu - \tilde{w}_\nu$,

which is possible since $\mathbb{Q}$ is dense in $\mathbb{R}$. Note also that the first item above implies that we have $\mathcal{E}_{-w_i} = \mathcal{E}_{-\tilde{w}_i}$ for all $i = 1, \dots, \nu$. We can then re-write (5.2) as

$$\lim_{t\to+\infty} \frac{\mathcal{M}_2^{Don}(\sigma_t)}{t} = 2\sum_{\alpha=1}^{\hat{\nu}} \tilde{w}_\alpha \mathrm{rk}(\mathcal{E}'_{-\tilde{w}_\alpha}) + 2\sum_{\alpha=1}^{\hat{\nu}} (w_\alpha - \tilde{w}_\alpha) \mathrm{rk}(\mathcal{E}'_{-w_\alpha}).$$

The first term on the right hand side is strictly positive since $\tilde{\zeta}$ has rational eigenvalues, and we claim that the second term is nonnegative. This is a consequence of the inequality (5.2) and the following observation: the second term is equal to the asymptotic slope $\lim_{t\to+\infty} \mathcal{M}_2^{Don}(\eta_t)/t$, where $\{\eta_t\}_{t\geq 0}$ is the 1-PS defined by $\eta_t = \exp((\zeta - \tilde{\zeta})t)$ in $SL(H^0(X, \mathcal{E}(k))^\vee)$, by observing $V_{-(w_i - \tilde{w}_i)} = V_{-\tilde{w}_i}$ and $\mathcal{E}_{-(w_i - \tilde{w}_i)} = \mathcal{E}_{-\tilde{w}_i}$ for all $i = 1, \dots, \nu$. Thus, the sum of these two terms is strictly positive, finally implying

$$\lim_{t \to +\infty} \frac{\mathcal{M}_2^{Don}(\sigma_t)}{t} > 0$$

for all Hermitian $\zeta \in \mathfrak{sl}(H^0(\mathcal{E}(k))^\vee)$, as required.

Thus, identifying $t > 0$ with the radial direction in $\mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee)$ by recalling $\|\zeta\|_{op} \leq 1$ (Remark 2.4), and also noting that we have

$$\mathfrak{sl}(H^0(X, \mathcal{E}(k))^\vee) = \mathfrak{su}(N_k) \oplus \sqrt{-1}\mathfrak{su}(N_k)$$

and that $\sqrt{-1}\mathfrak{su}(N_k)$ is the set of Hermitian matrices, the geodesic convexity of $\mathcal{M}_2^{Don}(\sigma_t)$ (Lemma 5.7) implies that the map

$$SL(H^0(X, \mathcal{E}(k))^\vee)/U(N_k) \xrightarrow{\sim} \sqrt{-1}\mathfrak{su}(N_k) \ni \zeta t \mapsto \mathcal{M}_2^{Don}(\sigma_t) \in \mathbb{R}$$

is bounded below and proper, where the first arrow is the diffeomorphism given by the global Cartan decomposition. Hence we finally conclude that there exists $\tilde{\sigma} \in SL(H^0(X, \mathcal{E}(k))^\vee)/U(N_k)$ that attains the global minimum of $\mathcal{M}_2^{Don}$. $\square$

6 Towards Effective Results and an Algorithm for Computing Hermitian–Einstein Metrics

One can notice that in Proposition 2.9, in order to prove slope stability of the bundle, it is sufficient to check $\mathcal{M}^{\mathrm{NA}}(\zeta, k_0) > 0$ for $\zeta \in \mathfrak{sl}(H^0(X, \mathcal{E}(k_0))^\vee)$ with *some* $k_0 \in \mathbb{N}$ satisfying

$$k_0 \geq \max\{\mathrm{reg}(\mathcal{E}), \mathrm{reg}(\mathcal{F}_{\max})\},$$

where $\mathcal{F}_{\max} \subset \mathcal{E}$ is the maximally destabilising subsheaf; recall that given a holomorphic vector bundle $\mathcal{E}$, there exists a unique maximal destabilising saturated subsheaf $\mathcal{F}_{\max}$ for $\mathcal{E}$ which satisfies

- if $\mathcal{F} \subset \mathcal{E}$ is a proper subsheaf of $\mathcal{E}$, then $\mu(\mathcal{F}) \leq \mu(\mathcal{F}_{\max})$;
- if $\mu(\mathcal{F}) = \mu(\mathcal{F}_{\max})$, then $\mathrm{rk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{F}_{\max})$.

For the existence and uniqueness of the maximal destabilising subsheaf the reader is referred to [29, Lemma 1.3.5] and [33, Sect. V.7, Lemma 7.17]. Note that $\mathcal{F}_{max}$ is slope semistable. With above notations, we have the following proposition.

Proposition 6.1 *Let $\mathcal{E}$ a holomorphic vector bundle over a polarized manifold. Set*

$$k_0 = \max(\mathrm{reg}(\mathcal{E}), \mathrm{reg}(\mathcal{F}_{max})).$$

The following assertions are equivalent:

(1) $\mathcal{E}$ *is slope stable,*
(2) *The Donaldson functional* $\mathcal{M}^{Don}$ *is coercive along rational Bergman 1-PS at level* k_0*,*
(3) *The Donaldson functional* $\mathcal{M}^{Don}$ *is coercive along rational Bergman 1-PS at level* k *for any* $k \geq k_0$*.*

The following assertions are equivalent:

(1') $\mathcal{E}$ *is slope semistable,*
(2') *The Donaldson functional* $\mathcal{M}^{Don}$ *along rational Bergman 1-PS at level* k_0 *is bounded from below,*
(3') *The Donaldson functional* $\mathcal{M}^{Don}$ *along rational Bergman 1-PS at level* k *is bounded from below for any* $k \geq k_0$*.*

Proof If $\mathcal{E}$ is slope stable (resp. slope semistable), then one can apply Theorem 2.11 since $\mathcal{E}$ is k-regular for any $k \geq \mathrm{reg}(\mathcal{E})$, see [29, Lemma 1.7.2]. This gives (1) $\Rightarrow$ (3) and obviously (3) $\Rightarrow$ (2) (resp. $(1') \Rightarrow (3') \Rightarrow (2')$).

A special case of our study is given when one is considering a 2-step filtration associated to a regular saturated subsheaf $\mathcal{F} \subset \mathcal{E}$ as described in [28, Proposition 6.2]. For $\zeta_{\mathcal{F}} \in \mathfrak{sl}(H^0(\mathcal{E}(k))^\vee)$ the element defining this filtration, with weights well chosen (cf. [28, Sect. 6]), one can consider the Bergman 1-PS $\{h_{\sigma_t}\}_{t\geq 0}$ emanating from h_k and induced by $\zeta_{\mathcal{F}}$ as above. Then we proved the existence of a constant $c_k = c(h_k, k) > 0$ such that

$$\mathcal{M}^{Don}(h_{\sigma_t}, h_k) \geq \mathrm{rk}(\mathcal{F})(\mu(\mathcal{E}) - \mu(\mathcal{F})) \cdot 2t - c_k \tag{6.1}$$

for all $t \geq 0$, and a constant $c'_k = c(h_k, \zeta_{\mathcal{F}}, k) > 0$ such that

$$\mathcal{M}^{Don}(h_{\sigma_t}, h_k) \leq \mathrm{rk}(\mathcal{F})(\mu(\mathcal{E}) - \mu(\mathcal{F})) \cdot 2t + c'_k \tag{6.2}$$

holds for all sufficiently large $t > 0$. If we have coercivity at level k_0, we apply (6.2) to $\mathcal{F}_{max}$. Since $\mathcal{M}^{Don}$ along $\{h_{\sigma_t}\}$ grows to $+\infty$ when $t \to +\infty$, this provides $\mu(\mathcal{E}) > \mu(\mathcal{F}_{max})$ and by definition of $\mathcal{F}_{max}$, $\mathcal{E}$ is actually slope stable. Thus (2) $\Rightarrow$ (1). If we have boundedness from below, we apply both (6.2) and (6.1) when $t \to +\infty$ to conclude that $\mu(\mathcal{E}) = \mu(\mathcal{F}_{max})$. This shows $(2') \Rightarrow (1')$. $\square$

Remark 6.2 The proof shows that the slope stability actually implies coercivity of the Donaldson function on the Bergman space at the minimum level k for which $\mathcal{E}(k)$ is globally generated and of course $k \leq k_0$. We don't expect the converse to be true.

The regularity of coherent sheaves has been studied since decades and bounds on the regularity have been made more or less explicit. Let's provide some details. If $\iota : X \to \mathbb{CP}^N$ is a holomorphic embedding and $\mathcal{F}$ a coherent sheaf on X then by the projection formula $\mathrm{reg}(\mathcal{F}) = \mathrm{reg}(\iota_* \mathcal{F})$. Using this argument, one can obtain information on the regularity of sheaves by restricting to the projective case. From the fundamental work of Mumford, it is known that for any coherent sheaf $\mathcal{F}$ on $\mathbb{CP}^N$, which is isomorphic to a subsheaf of $\bigoplus_{j=1}^{N_0} \mathcal{O}_{\mathbb{CP}^N}$, with Hilbert polynomial

$$\chi(\mathcal{F}(k)) = \sum_{i=0}^{N} a_i \binom{k}{i}$$

($a_i \in \mathbb{Z}$), one has the k_0-regularity of $\mathcal{F}$ for $k_0 = F(a_0, .., a_N)$ where F is a universal polynomial in $N+1$ variables that depends on (N, N_0) that can be made explicit. For instance, the case of semistable bundles over $\mathbb{CP}^2$ is studied in [34, Corollary 5.5], see also [19]. The interest of making effective the Castelnuovo–Mumford regularity k_0 in Proposition 6.1 becomes clear when one is considering numerical applications. In [45] it is presented an algorithm based on ideas of S. Donaldson [18] to compute balanced metrics (cf. Sect. 5) in the set of kth Fubini–Study metrics $\mathcal{B}_k$ on a stable bundle $\mathcal{E}$ that approximate the Hermitian–Einstein metric living on the bundle when $k \to +\infty$. Nevertheless, with the notion of balanced metrics, it remains unclear which minimal k can be chosen to run the algorithm, see for instance [44]. If $\mathcal{E}$ is a stable bundle, Proposition 6.1 ensures that the Donaldson functional is coercive on the Bergman space $\mathcal{B}_{k_0}$ which has finite dimension, and thus it attains a minimum, say at the metric $h_k^{\min}$. If one denotes h_{HE} the Hermitian–Einstein metric on $\mathcal{E}$, $h \mapsto \mathcal{M}^{Don}(h, h_{HE})$ reaches its minimum at $h = h_{HE}$ where it vanishes. Technically, in order to find $h_k^{\min}$, one can apply Levenberg–Marquardt algorithm to $h \mapsto |\mathcal{M}^{Don}(h, h_{HE})|$ restricted to $\mathcal{B}_{k_0}$. By density of the Bergman spaces (cf. [27, Corollary 1.9]),

$$\mathcal{M}^{Don}(h_k^{\min}, h_{HE}) \leq \frac{C_{HE}}{k},$$

where C_{HE} is a constant that depends only on the Hermitian–Einstein metric and its covariant derivatives. Moreover, once the minimum is achieved, it is possible to estimate how far is h_k^{min} from h_{HE} using [27, Theorem B.5]. Actually, in this view, one can introduce

$$\delta := \inf_{x \in X} \frac{\lambda_{\min}}{\lambda_{\max}}$$

where $\lambda_{\max}, \lambda_{\min}$ are the maximum and minimum eigenvalues of $h_k^{\min} h_{HE}^{-1} := e^v$. The problems turns out to measure $1 - \delta$ when this quantity is small. But the proof of [27, Theorem B.5] shows that we have the inequality

$$\mathcal{M}^{Don}(h_k^{min}, h_{HE}) \geq \frac{\delta - 1 - \log(\delta)}{\log(\delta)^2} C_{\nabla^* \bar{\partial}}^{-1} \|v - \bar{v}\|_{L^2}^2,$$

where $\bar{v} = \frac{1}{r \mathrm{Vol}_L} \int_X \mathrm{tr}(v) \frac{\omega^n}{n!} \cdot \mathrm{Id}_{\mathcal{E}}$ is the average of v, and $C_{\nabla^* \bar{\partial}}$ can be interpreted as the first non zero eigenvalue of the operator $\sqrt{-1}\Lambda \bar{\partial}\partial$ acting on endomorphisms of E

and that depends on the metric h_{HE}. Denoting r the rank of $\mathcal{E}$, one has $\|v-\bar{v}\|^2_{L^2} = \|v\|^2_{L^2} - \|\bar{v}\|^2_{L^2} \geq \frac{1}{r}(\log\delta)^2$ and consequently,

$$\mathcal{M}^{Don}(h_k^{min}, h_{HE}) \geq \frac{1}{r}(\delta - 1 - \log(\delta))C^{-1}_{\nabla^*\bar{\partial}} \sim C^{-1}_{\nabla^*\bar{\partial}} \frac{(1-\delta)^2}{2r}.$$

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References

1. Berman, R., Berndtsson, B., Sjöstrand, J.: A direct approach to Bergman kernel asymptotics for positive line bundles. Ark. Mat. **46**(2), 197–217, MR2430724 (2008)
2. Berman, R.J.: K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics. Invent. Math. **203**(3), 973–1025, MR3461370 (2016)
3. Berman, R.J., Berndtsson, B.: Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics. J. Amer. Math. Soc. **30**(4), 1165–1196, MR3671939 (2017)
4. Berman, R.J., Boucksom, S., Jonsson, M.: A variational approach to the Yau-Tian-Donaldson conjecture. J. Amer. Math. Soc. **34**(3), 605–652, MR4334189 (2021)
5. Boucksom, S.: Variational and non-archimedean aspects of the Yau-Tian-Donaldson conjecture. In: Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018, vol. II. Invited Lectures, pp. 591–617, MR3966781 (2018)
6. Boucksom, S., Guedj, V.: Regularizing properties of the Kähler-Ricci flow. An Introduction to the Kähler-Ricci Flow, vol. 2086, pp. 189–237, MR3185334 (2013)
7. Boucksom, S., Hisamoto, T., Jonsson, M.: Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs. Ann. Inst. Fourier (Grenoble) **67**(2), 743–841, MR3669511 (2017)
8. Boucksom, S., Hisamoto, T., Jonsson, M.: Uniform K-stability and asymptotics of energy functionals in Kähler geometry. J. Eur. Math. Soc. (JEMS) **21**(9), 2905–2944, MR3985614 (2019)
9. Catlin, D.: The Bergman kernel and a theorem of Tian. Analysis and Geometry in Several Complex Variables (Katata, 1997), pp. 1–23, MR1699887 (1999)
10. Chen, X.: The space of Kähler metrics. J. Diff. Geom. **56**(2), 189–234, MR1863016 (2000)
11. Demailly, J.-P.: Variational approach for complex Monge-Ampère equations and geometric applications. Astérisque **390**, Exp. No. 1112, 245–275 (2017). Séminaire Bourbaki, vol. 2015/2016. Exposés, pp. 1104–1119, MR3666028
12. Dervan, R.: Uniform stability of twisted constant scalar curvature Kähler metrics. Int. Math. Res. Not. IMRN (15), 4728–4783, MR3564626 (2016)
13. Donaldson, S.K.: A new proof of a theorem of Narasimhan and Seshadri. J. Diff. Geom. **18**(2), 269–277, MR710055 (1983)
14. Donaldson, S.K.: Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. Proc. London Math. Soc. (3) **50**(1), 1–26, MR765366 (1985)
15. Donaldson, S.K.: Infinite determinants, stable bundles and curvature. Duke Math. J. **54**(1), 231–247, MR885784 (1987)
16. Donaldson, S.K.: Scalar curvature and projective embeddings. I. J. Diff. Geom. **59**(3), 479–522, MR1916953 (2001)
17. Donaldson, S.K.: Scalar curvature and stability of toric varieties. J. Diff. Geom. **62**(2), 289–349, MR1988506 (2002)

18. Donaldson, S.K.: Some numerical results in complex differential geometry. Pure Appl. Math. Q. **5**(2), Special Issue: In honor of Friedrich Hirzebruch. Part 1, 571–618, MR2508897 (2009)
19. Elencwajg, G., Forster, O.: Bounding cohomology groups of vector bundles on $\mathbb{P}_n$. Math. Ann. **246**, 251–270 (English) (1980)
20. Elkik, R.: Métriques sur les fibrés d'intersection. Duke Math. J. **61**(1), 303–328, MR1068389 (1990)
21. Eyssidieux, P.: Métriques de Kähler-Einstein sur les variétés de Fano [d'après Chen-Donaldson-Sun et Tian]. Astérisque (380), Exp. No. 1095, 207–229 (2016), Séminaire Bourbaki, vol. 2014/2015, MR3522175
22. Eyssidieux, P., Guedj, V., Zeriahi, A.: Singular Kähler-Einstein metrics. J. Amer. Math. Soc. **22**(3), 607–639, MR2505296 (2009)
23. Fine, J.: Calabi flow and projective embeddings. J. Diff. Geom. **84**(3), 489–523, With an appendix by Kefeng Liu and Xiaonan Ma, MR2669363 (2010)
24. Garcia-Fernandez, M., Keller, J., Ross, J.: Quantization of Hitchin's equations for Higgs Bundles I (2016). arXiv:1601.04960
25. Garcia-Fernandez, M., Ross, J.: Balanced metrics on twisted Higgs bundles. Math. Ann. **367**(3–4), 1429–1471, MR3623230 (2017)
26. Hashimoto, Y., Keller, J.: About J-flow, J-balanced metrics, uniform J-stability and K-stability. Asian J. Math. **22**(3), 391–411, MR3845084 (2018)
27. Hashimoto, Y., Keller, J.: A variational approach to the Hermitian-Einstein metrics and the Quot-scheme limit of Fubini-Study metrics (2019). arXiv:1907.05770. To appear in Geom. Dedicata
28. Hashimoto, Y., Keller, J.: Quot-scheme limit of Fubini-Study metrics and Donaldson's functional for vector bundles. Épijournal de Géométrie Algébrique **5** (2022)
29. Huybrechts, D., Lehn, M.: The geometry of moduli spaces of sheaves, 2nd edn. Cambridge Mathematical Library, MR2665168. Cambridge University Press, Cambridge (2010)
30. Jacob, A.: Existence of approximate Hermitian-Einstein structures on semi-stable bundles. Asian J. Math. **18**(5), 859–883, MR3287006 (2014)
31. Keller, J., Seyyedali, R.: Quantization of Donaldson's heat flow over projective manifolds. Math. Z. **282**(3–4), 839–866, MR3473646 (2016)
32. Kobayashi, S.: Differential geometry of holomorphic vector bundles. Math. Seminar Notes (by I. Enoki) **41** (1982). University of Tokyo, in Japanese
33. Kobayashi, S.: Differential geometry of complex vector bundles. Publications of the Mathematical Society of Japan, Kanô Memorial Lectures, vol. 15, p. 5, MR909698. Princeton University Press, Princeton, NJ; Princeton University Press, Princeton, NJ (1987)
34. Langer, A.: Moduli spaces and Castelnuovo-Mumford regularity of sheaves on surfaces. Am. J. Math. **128**(2), 373–417 (English) (2006)
35. Li, C., Wang, X., Xu, C.: On the proper moduli spaces of smoothable Kähler-Einstein Fano varieties. Duke Math. J. **168**(8), 1387–1459, MR3959862 (2019)
36. Lübke, M.: Stability of Einstein-Hermitian vector bundles. Manuscripta Math. **42**(2–3), 245–257, MR701206 (1983)
37. Luo, H.: Geometric criterion for Gieseker-Mumford stability of polarized manifolds. J. Diff. Geom. **49**(3), 577–599, MR1669716 (1998)
38. Ma, X., Marinescu, G.: Holomorphic Morse inequalities and Bergman kernels. Progress in Mathematics, vol. 254, MRMR2339952 (2008g:32030). Birkhäuser Verlag, Basel (2007)
39. Moriwaki, A.: The continuity of Deligne's pairing. Int. Math. Res. Notices (19), 1057–1066, MR1725483 (1999)
40. Odaka, Y.: On the moduli of Kähler-Einstein Fano manifolds. In: Proceeding of Kinosaki Algebraic Geometry Symposium 2013 (2012). arXiv:1211.4833
41. Paul, S.T.: Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics. Ann. Math. (2) **175**(1), 255–296, MR2874643 (2012)
42. Phong, D.H., Ross, J., Sturm, J.: Deligne pairings and the Knudsen-Mumford expansion. J. Diff. Geom. **78**(3), 475–496, MR2396251 (2008)

43. Phong, D.H., Sturm, J.: Stability, energy functionals, and Kähler-Einstein metrics. Comm. Anal. Geom. **11**(3), 565–597, MR2015757 (2003)
44. Douglas, M.R., Karp, R.L., Lukic, S., Reinbacher, R.: Numerical solution to the hermitian Yang-Mills equation on the Fermat quintic. J. High Energy Phys. **2007**(12), 24, Id/No 083 (2007)
45. Seyyedali, R.: Numerical algorithm for finding balanced metrics on vector bundles. Asian J. Math. **13**(3), 311–321, MR2570441 (2009)
46. Sibley, B.: Asymptotics of the Yang-Mills flow for holomorphic vector bundles over Kähler manifolds: the canonical structure of the limit. J. Reine Angew. Math. **706**, 123–191, MR3393366 (2015)
47. Székelyhidi, G.: The partial C^0-estimate along the continuity method. J. Amer. Math. Soc. **29**(2), 537–560, MR3454382 (2016)
48. Takhtajan, L.A.: Explicit computation of the Chern character forms. Geom. Dedicata **181**, 223–237, MR3475747 (2016)
49. Thomas, R.P.: Notes on GIT and symplectic reduction for bundles and varieties. Surveys in differential geometry, vol. X. Surveys in Differential Geometry, vol. 10, pp. 221–273, MR2408226. International Press, Somerville, MA (2006)
50. Tian, G.: Kähler-Einstein metrics with positive scalar curvature. Invent. Math. **130**(1), 1–37, MR1471884 (1997)
51. Tian, G.: Bott-Chern forms and geometric stability. Discr. Contin. Dyn. Syst. **6**(1), 211–220, MR1739924 (2000)
52. Uhlenbeck, K., Yau, S.-T.: On the existence of Hermitian-Yang-Mills connections in stable vector bundles. Comm. Pure Appl. Math. **39**(S, suppl.), S257–S293 Frontiers of the mathematical sciences: 1985 (New York, 1985), MR861491 (1986)
53. Voisin, C.: Hodge theory and complex algebraic geometry. I. English, Cambridge Studies in Advanced Mathematics, vol. 76, MR2451566. Cambridge University Press, Cambridge (2007). Translated from the French by Leila Schneps
54. Wang, X.: Balance point and stability of vector bundles over a projective manifold. Math. Res. Lett. **9**(2–3), 393–411, MR1909652 (2002)
55. Wang, X.: Canonical metrics on stable vector bundles. Comm. Anal. Geom. **13**(2), 253–285, MR2154820 (2005)
56. Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. **31**(3), 339–411, MR480350 (1978)
57. Zhang, S.: Heights and reductions of semi-stable varieties. Compositio Math. **104**(1), 77–105, MR1420712 (1996)

Existence of Canonical Models for Kawamata Log Terminal Pairs

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Abstract We prove that a Kawamata log terminal pair has the canonical model.

Keywords Kawamata log terminal pair · Klt pair · Canonical model

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1 Introduction

We work over an algebraically closed field of characteristic zero.

Our main result is the existence of canonical models for Kawamata log terminal pairs.

Theorem 1.1 *Let $(X/Z, B)$ be a Kawamata log terminal pair with the Kodaira dimension $\kappa_\iota(X/Z, K_X + B) \geq 0$. Then, $(X/Z, B)$ has the canonical model.*

If B is a $\mathbb{Q}$-divisor, then Theorem 1.1 is [5, Corollary 1.1.2]. In this paper, we prove it for the general case. The idea of proof is to reduce Theorem 1.1 to [5, Theorem 1.2], by a canonical bundle formula of Fujino-Mori type for $\mathbb{R}$-divisors (cf. [9]).

Theorem 1.2 *Let $f : X \to Y$ be a contraction of normal varieties and (X, B) be a klt pair such that $\kappa_\iota(X/Y, K_X + B) = 0$. Then, there exists a commutative diagram*

$$\begin{tikzcd}
(X', B') \arrow[r, dash, "\pi"] \arrow[d, "f'"'] & (X, B) \arrow[d, "f"] \\
Y' \arrow[r, dash, "\phi"] & Y
\end{tikzcd}$$

which consists of birational models $\pi : X' \to X$ and $\phi : Y' \to Y$, such that:

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(1) $K_{X'} + B' = \pi^*(K_X + B) + E$ *where* E *is exceptional*$/X$ *and* $B', E \geq 0$ *have no common components.*
(2) $K_{X'} + B' \sim_{\mathbb{R}} f'^*(K_{Y'} + B_{Y'} + M_{Y'}) + R$ *where* $R \geq 0$ *and* $(Y', B_{Y'} + M_{Y'})$ *is a g-klt generalised pair with the moduli b-divisor* $\mathbf{M}$.
(3) $\kappa(X'/Y', R^h) = 0$ *and* R^v *is very exceptional*$/Y'$, *where* R^h *(resp.* R^v*) denotes the horizontal (resp. vertical) part over* Y'.

One can easily generalise the above theorem to log canonical pairs. See Remark 3.3.

2 Preliminaries

In this section we collect definitions and some important results. Throughout this paper all varieties are quasi-projective over a fixed algebraically closed field of characteristic zero and a divisor refers to an $\mathbb{R}$-Weil divisor unless stated otherwise.

2.1 Notations and Definitions

We collect some notations and definitions. We use standard definitions of Kawamata log terminal (klt, for short) pair and sub-klt pair (for example, see [11, Sect. 2.1]).

Contractions. In this paper a *contraction* refers to a proper morphism $f: X \to Y$ of varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. In particular, f has connected fibres. Moreover, if X is normal, then Y is also normal. A birational map $\pi : X \dashrightarrow Y$ is a *birational contraction* if the inverse of π does not contract divisors. Note that π is not necessarily a morphism unless stated otherwise.

Very exceptional divisors. Let $f : X \to Y$ be a dominant morphism from a normal variety to a variety, D a divisor on X, and $Z \subset X$ a closed subset. We say Z is *horizontal* over Y if $f(Z)$ dominates Y, and we say Z is *vertical* over Y if $f(Z)$ is a proper subset of Y.

Suppose f is a contraction of normal varieties. Recall that a divisor D is *very exceptional*$/Y$ if D is vertical$/Y$ and for any prime divisor P on Y there is a prime divisor Q on X which is not a component of D but $f(Q) = P$, i.e. over the generic point of P we have $\operatorname{Supp} f^*P \nsubseteq \operatorname{Supp} D$.

If $\operatorname{codim} f(D) \geq 2$, then D is very exceptional. In this case we say D is f-*exceptional*.

Generalised pairs. For the basic theory of generalised polarised pairs (generalised pairs for short) we refer to [6, Sect. 4]. Below we recall some of the main notions and discuss some basic properties.

A *generalised sub-pair* consists of

- a normal variety X equipped with a proper morphism $X \to Z$,
- an $\mathbb{R}$-divisor B on X, and
- a b-$\mathbb{R}$-Cartier b-divisor over X represented by some projective birational morphism $\overline{X} \xrightarrow{\phi} X$ and an $\mathbb{R}$-Cartier divisor $\overline{M}$ on $\overline{X}$ such that $\overline{M}$ is nef$/Z$ and $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := \phi_* \overline{M}$.

A generalised sub-pair is a *generalised pair* if B is effective. We usually refer to the sub-pair by saying $(X/Z, B + M)$ is a generalised sub-pair *with data* $\overline{M}$ or *with the moduli b-divisor* $\mathbf{M}$, where $\mathbf{M}$ is represented by $\overline{M}$. We will use standard definitions of b-divisors, generalised singularities and log minimal models (for example, see [11, Sect. 2.1]).

2.2 Iitaka Dimension and Iitaka Fibration

In this subsection we introduce the notion of invariant Iitaka dimension and invariant Iitaka fibration.

Recall the following definitions of Iitaka dimension, which is a birational invariant integer given by the growth of the quantity of sections.

Definition 2.1 (*Invariant Iitaka dimension*) Let X be a normal projective variety, and D be an $\mathbb{R}$-Cartier divisor D on X. We define the *invariant Iitaka dimension* of D, denoted by $\kappa_\iota(X, D)$, as follows (see also [8, Definition 2.5.5]): If there is an $\mathbb{R}$-divisor $E \geq 0$ such that $D \sim_{\mathbb{R}} E$, set $\kappa_\iota(X, D) = \kappa(X, E)$. Here, the right hand side is the usual Iitaka dimension of E. Otherwise, we set $\kappa_\iota(X, D) = -\infty$. We can check that $\kappa_\iota(X, D)$ is well-defined, i.e., when there is $E \geq 0$ such that $D \sim_{\mathbb{R}} E$, the invariant Iitaka dimension $\kappa_\iota(X, D)$ does not depend on the choice of E. By definition, we have $\kappa_\iota(X, D) \geq 0$ if and only if D is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor.

Let $X \to Z$ be a projective morphism from a normal variety to a variety, and let D be an $\mathbb{R}$-Cartier divisor on X. Then the *relative invariant Iitaka dimension* of D, denoted by $\kappa_\iota(X/Z, D)$, is defined by $\kappa_\iota(X/Z, D) = \kappa_\iota(X, D|_F)$, where F is a very general fibre (i.e. the fibre over a very general point) of the Stein factorisation of $X \to Z$. Note that the value $\kappa_\iota(X, D|_F)$ does not depend on the choice of F (see [10, Lemma 2.10]).

For basic properties of the invariant Iitaka dimension, we refer to [10, Remark 2.8].

Definition 2.2 (*Invariant Iitaka fibration*) Let X be a normal variety, $f : X \to Z$ be a proper morphism, and D be an $\mathbb{R}$-Cartier divisor on X with $\kappa_\iota(X/Z, D) \geq 0$. Pick an $\mathbb{R}$-Cartier divisor $E \geq 0$ such that $D \sim_{\mathbb{R}} E/Z$. Then there exists a contraction $\phi : X' \to Y$ of smooth varieties such that for all sufficiently large integers $m > 0$, the rational maps $\phi_m : X \dashrightarrow Y_m$ given by $f^* f_* \mathcal{O}_X(\lfloor mE \rfloor)$ are birationally equivalent to ϕ, that is, there exists a commutative diagram

$$\begin{tikzcd} X \arrow[d, dashed, "\phi_m"'] & X' \arrow[l, "\pi"'] \arrow[d, "\phi"] \\ Y_m & Y \arrow[l, dashed, "\varphi_m"] \end{tikzcd}$$

of rational maps ϕ_m, φ_m and a contraction π, where the horizontal maps are birational, $\dim Y = \kappa_\iota(X/Z, D) + \dim Z$, and $\kappa(X'/Y, \pi^* E) = 0$. Such a fibration is called an *Iitaka fibration* of D. It is unique up to birational equivalence.

Lemma 2.3 *The definition above is well-defined and independent of the choice of E.*

Proof By compactification, we may assume Z is projective, and hence X, Y projective. The definition is well-defined by [12, II,3.14]. Let $\phi' : X' \to Y'$ be a relative Iitaka fibration over Z associated to an $\mathbb{R}$-Cartier divisor $E' \geq 0$ such that $D \sim_{\mathbb{R}} E'/Z$. Pick a very general closed point $y' \in Y'$. By [8, Proof of Lemma 2.5.6], for any sufficiently large positive integer m, there is an injection

$$H^0(\phi'^{-1}(y'), \mathcal{O}_X(\lfloor m\pi^* E|_{\phi'^{-1}(y')}\rfloor)) \hookrightarrow H^0(\phi'^{-1}(y'), \mathcal{O}_X(\lfloor (m+1)\pi^* E'|_{\phi'^{-1}(y')}\rfloor)) \simeq k$$

where k is the ground field. We infer that the image of $\phi'^{-1}(y')$ under ϕ_m is a point. Therefore, by the rigidity lemma [12, II,1.12], ϕ' induces a birational map $\psi_m : Y' \dashrightarrow Y_m$ such that $\phi_m \circ \pi = \psi_m \circ \phi'$, which completes the proof. $\square$

Canonical models. Recall that, given a proper morphism $h\colon X \to Z$ from a normal variety to a variety, an $\mathbb{R}$-Cartier divisor D is *semi-ample* over Z if there exist a proper surjective morphism $g : X \to Y$ over Z and an ample$/Z$ divisor D_Y of Y such that $D \sim_{\mathbb{R}} g^* D_Y$.

Remark 2.4 ([11]) Notation as above, let D be an $\mathbb{R}$-Cartier divisor.

(1) D is semi-ample if and only if D is a convex combination of semi-ample $\mathbb{Q}$-divisors.
(2) Let D' be another $\mathbb{R}$-Cartier divisor. If D, D' are semi-ample, then so is $D + D'$.
(3) Let $f : W \to X$ be a proper surjective morphism. Then, D is semi-ample if and only if $f^* D$ is semi-ample.

Given an $\mathbb{R}$-linear system $|D/Z|_{\mathbb{R}}$, we say a divisor $E \geq 0$ is *contained in the fixed part of* $|D/Z|_{\mathbb{R}}$ if, for every $B \in |p^* D/Z|_{\mathbb{R}}$, then $B \geq E$.

Definition 2.5 ([5, Definitions 3.6.5 and 3.6.7]) Let $h : X \to Z$ be a projective morphism of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X.

(1) We say that a birational contraction $f : X \dashrightarrow X'$ over Z is a *semi-ample model* of D over Z, if f is D-non-positive, X' is normal and projective over Z and $D' = f_* D$ is semi-ample over Z.

(2) We say that $g : X \dashrightarrow Y$ is the *ample model* of D over Z, if g is a rational map over Z, Y is normal and projective over Z and there is an ample divisor H over Z on Y such that if $p : W \to X$ and $q : W \to Y$ resolve g then q is a contraction morphism and we may write $p^*D \sim_{\mathbb{R}} q^*H + E/Z$, where $E \geq 0$ is contained in the fixed part of $|p^*D/Z|_{\mathbb{R}}$. By [5, Lemma 3.6.6], the ample model is unique up to isomorphism.
(3) (Canonical model.) If (X, B) is a klt pair and $D = K_X + B$, then we say Y in (2) is the *canonical model* of (X, B) over Z.

2.3 Klt-trivial Fibrations

Recall that the discrepancy b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ of a pair (X, B) is the b-divisor of X with the trace $\mathbf{A}_Y$ defined by the formula

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. By the definition, we have $\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = \mathcal{O}_X$ when (X, B) is klt (see [7, Lemma 3.19]).

Definition 2.6 ([11, Definition 2.21]) Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. A $\mathbb{K}$-*klt-trivial fibration* $f : (X, B) \to Y$ consists of a contraction $f : X \to Y$ of normal varieties and a subpair (X, B) satisfying the following properties:

(1) (X, B) is sub-klt over the generic point of Y;
(2) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = 1$;
(3) There exists an $\mathbb{R}$-Cartier divisor D on Y such that

$$K_X + B \sim_{\mathbb{K}} f^*D.$$

Notation as above, we set

$$b_P = \max \left\{ t \in \mathbb{R} | (X, B + tf^*P) \text{ is sub-lc over the generic point of } P \right\}$$

and set

$$B_Y = \sum_P (1 - b_P)P,$$

where P runs over prime divisors on Y. Then it is easy to see that B_Y is well defined since $b_P = 1$ for all but a finite number of prime divisors and it is called the *discriminant divisor*. Furthermore, we set

$$M_Y = D - K_Y - B_Y$$

and call M_Y the *moduli divisor*.

Let $\phi : Y' \to Y$ be a birational contraction from a normal variety Y'. Let X' be a resolution of the main component of $X \times_Y Y'$ which dominates Y'. The induced morphism $\pi : X' \to X$ is birational, and $K_{X'} + B' = \pi^*(K_X + B)$. Let $B_{Y'}$ be the discriminant of $K_{X'} + B'$ on Y'. Since the definition of the discriminant is divisorial and ϕ is an isomorphism over codimension one points of Y, we have $B_Y = \phi_*(B_{Y'})$. This means that there exists a unique b-divisor $\mathbf{B}$ of Y such that $\mathbf{B}_{Y'}$ is the discriminant on Y' of the induced fibre space $f' : (X', B') \to Y'$, for every birational model Y' of Y . We call $\mathbf{B}$ the *discriminant b-divisor*. We define the *moduli b-divisor* $\mathbf{M}$ in a similar way.

Note that if $\mathbb{K} = \mathbb{Q}$, thanks to the important result [3][Theorem 2.5] obtained by the theory of variations of Hodge structure, the moduli b-divisor $\mathbf{M}$ of a $\mathbb{Q}$-klt-trivial fibration is $\mathbb{Q}$-b-Cartier and b-nef. Hence $\mathbf{K} + \mathbf{B}$ is $\mathbb{R}$-b-Cartier.

The arguments for next lemma are taken from [11].

Lemma 2.7 ([11, Lemma 2.22]) *Let $f : (X, B) \to Y$ be an $\mathbb{R}$-klt-trivial fibration. Then, B is a convex combination of $\mathbb{Q}$-divisors B_i such that $f : (X, B_i) \to Y$ is $\mathbb{Q}$-klt-trivial. Moreover, if (X, B) is sub-klt, then we can choose B_i so that (X, B_i) is sub-klt for each i.*

Proof Replacing X we may assume it is smooth. Let $f : (X, B) \to Y$ be an $\mathbb{R}$-klt-trivial fibration, $\varphi = \prod_{i=1}^{k} \varphi_i^{\alpha_i}$ be an $\mathbb{R}$-rational function so that $K_X + B + (\varphi) = f^*D$. Let $\mathcal{V} \subset \mathrm{CDiv}_{\mathbb{R}}(Y)$ be a finite dimensional rational linear subspace containing D, $\mathcal{L} \subset \mathrm{CDiv}_{\mathbb{R}}(X)$ be a rational polytope containing B such that, for every $\Delta \in \mathcal{L}$, we have (X, Δ) is a sub-pair which is sub-klt over the generic point of Y. Now we consider the rational polytope

$$\mathcal{P} := \left\{ \Delta \in \mathcal{L} | \Delta + \sum_{i=1}^{k} \mathbb{R}(\varphi_i) \text{ intersects } f^*\mathcal{V} \right\}.$$

For every $\Delta \in \mathcal{P}$, we have further $K_X + \Delta \sim_{\mathbb{R}} 0/Y$. It is obvious that $B \in \mathcal{P}$.

It suffices to show that, there exists a convex combination $B = \sum_j r_j B_j$ of $\mathbb{Q}$-divisors $B_j \in \mathcal{P}$ with rank $f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B_j) \rceil) = 1$. To this end, pick a log resolution $\pi : \overline{X} \to X$ of $(X, \sum_j \Gamma_j)$ where every element of $\mathcal{P}$ is supported by $\sum_j \Gamma_j$. Note that the proofs of [7, Lemmas 3.19 and 3.20] are still valid for $\mathbb{R}$-sub-boundaries. Hence, by shrinking Y, we may assume (X, Δ) is sub-klt for every $\Delta \in \mathcal{P}$, and we have

$$f_*\mathcal{O}_X(\lceil \mathbf{A}(X, \Delta) \rceil) = f_*\pi_*\mathcal{O}_{\overline{X}}(\sum \lceil a_i \rceil A_i)$$

where $K_{\overline{X}} = \pi^*(K_X + \Delta) + \sum a_i A_i$. Consider the rational sub-polytope

$$\mathcal{Q} = \{\Delta \in \mathcal{P} | \lceil \mathbf{A}(X, \Delta)_{\overline{X}} \rceil \leq \lceil \mathbf{A}(X, B)_{\overline{X}} \rceil \}.$$

Then, for any $B_j \in \mathcal{Q}$, we have rank $f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B_j) \rceil) = 1$ which completes the first assertion. The last statement is obvious. □

Lemma 2.8 *Let $f : (X, B) \to Y$ be an $\mathbb{R}$-klt-trivial fibration from a sub-klt pair, B_Y be the discriminant divisor and M_Y be the moduli divisor. Then, there exists a b-divisor* $\mathbf{M}$ *satisfying:*

(1) The trace $\mathbf{M}_Y = M_Y$.
(2) $(Y, B_Y + M_Y)$ is a g-sub-klt generalised pair with the moduli b-divisor $\mathbf{M}$.

Proof Replacing X, we may assume it is smooth. By Lemma 2.7, there exists a convex combination of $B = \sum_i r_i B_i$ of $\mathbb{Q}$-divisors such that $f : (X, B_i) \to Y$ is $\mathbb{Q}$-klt-trivial. Let $\mathcal{P} \subset \mathrm{CDiv}_{\mathbb{R}}(X)$ be the polytope defined by B_i's. For any prime divisor P on Y, we set the function b_P on $\mathcal{P}$:

$$b_P(\Delta) = \max\{t \in \mathbb{R} | (X, \Delta + tf^*P) \text{ is sub-lc over the generic point of } P\}.$$

We note that the b_P is piecewisely affine and gives a rational polyhedral decomposition of $\mathcal{P}$. Also note that there are only finitely many P such that b_P is not identically one on $\mathcal{P}$. Therefore, there exists a rational sub-polytope $\mathcal{Q}$ containing B such that b_P is affine on $\mathcal{Q}$, for any prime divisor P. In particular, replacing B_i's and r_i's, we have $B_Y = \sum_i r_i B_{Y,i}$ and $M_Y = \sum_i r_i M_{Y,i}$, where B_Y, $B_{Y,i}$ are discriminant divisors and M_Y, $M_{Y,i}$ are moduli divisors of $f : (X, B) \to Y$, $f : (X, B_i) \to Y$ respectively. Letting $\mathbf{M} = \sum_i r_i \mathbf{M}_i$, where $\mathbf{M}_i$ is the moduli b-divisor of $f : (X, B_i) \to Y$ for each i, we conclude the lemma by [3, Theorem 2.5]. □

Remark 2.9 If the rational polytope $\mathcal{Q}$ in the above argument is sufficiently small, then $\mathbf{M}$ is precisely the moduli b-divisor of the $\mathbb{R}$-klt-trivial fibration $f : (X, B) \to Y$. For a detailed proof, see [11, Theorem 1.1].

Lemma 2.10 *Let $f : (X, B) \to Y$ be a contraction of normal varieties from a klt pair (X, B). Suppose $K_X + B \sim_{\mathbb{R}} R/Y$ where $R \geq 0$, and $\kappa(X/Y, R) = 0$, then*

$$\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B - R) \rceil) = 1.$$

Proof Let $\pi : X' \to X$ be a log resolution of (X, B) and write $\Delta = B - R$ and $K_{X'} = \pi^*(K_X + \Delta) + \sum_i a_i A_i$. By [7, Proof of Lemmas 3.19 and 3.20], we have $f_*\mathcal{O}_X(\lceil \mathbf{A}(X, \Delta) \rceil) = (f \circ \pi)_*\mathcal{O}_{X''}(\sum_i \lceil a_i \rceil A_i)$. Because we have

$$\operatorname{Supp} \sum_i \lceil a_i \rceil A_i \subseteq \operatorname{Supp} \pi^* R \bigcup \operatorname{Ex}(\pi),$$

we deduce $\kappa(X'/Y, \sum_i \lceil a_i \rceil A_i) = 0$ and hence the lemma. □

3 Existence of Canonical Models

Lemma 3.1 *Let $f : X \to Y$ be a contraction of normal varieties over Z, D be an $\mathbb{R}$-Cartier divisor on X and D_Y be an $\mathbb{R}$-Cartier divisor on Y. We suppose that:*

- *$D \sim_{\mathbb{R}} f^*D_Y + E/Z$ for some divisor $E \geq 0$, such that $\kappa(X/Y, E^h) = 0$ and E^v is very exceptional/Y, where E^h (resp. E^v) denotes the horizontal (resp. vertical) part over Y.*
- *There is a semi-ample model of D_Y/Z.*

Then, there exists the ample model of D/Z.

Proof We first reduce the lemma to the case D_Y is semi-ample/Z. Let $\varphi : Y \dashrightarrow Y'$ be the birational contraction to a semi-ample model of D_Y/Z, and $p : \overline{Y} \to Y$ and $q : \overline{Y} \to Y'$ which resolve φ. We write $D_{Y'}$ for the birational transform of D_Y and $p^*D_Y = q^*D_{Y'} + F$ where $F \geq 0$ is exceptional/Y'. Pick a resolution $\pi : \overline{X} \to X$ such that the induced map $\overline{f} : \overline{X} \dashrightarrow \overline{Y}$ is a morphism. We write $\overline{D} = \pi^*D$, $\overline{E} = \pi^*E + \overline{f}^*F$, and $\overline{D} \sim_{\mathbb{R}} (q \circ \overline{f})^*D_{Y'} + \overline{E}$. If we denote by $\overline{E}^h$ and $\overline{E}^v$ the horizontal and vertical part over Y', then one can easily verify that $\kappa(\overline{X}/Y', \overline{E}^h) = 0$, and $\overline{E}^v = \pi^*E^v + \overline{f}^*F$ is very exceptional/Y'. Replacing X, Y with $\overline{X}, Y'$ and the other data accordingly, we may assume D_Y is semi-ample/Z.

It remains to check that E is contained in the fixed part of $|D/Z|_{\mathbb{R}}$. To this end, pick any $D' \sim_{\mathbb{R}} D/Z$. Since $D'|_F \sim_{\mathbb{R}} D|_F$ where F is a general fbre of f, we have E^h is contained in the fixed part of $|D/Z|_{\mathbb{R}}$. Replacing D with $D - E^h$, we may assume E is vertical and very exceptional/Y. Hence, the lemma follows from the Negativity lemma [4, Lemma 3.3]. □

Remark 3.2 The lemma above also holds when f is a proper surjective morphism instead of a contraction.

Proof (*Proof of Theorem* 1.2) Since $\kappa_\iota(X/Y, K_X + B) = 0$, by [10, Lemma 2.10], there exists an $\mathbb{R}$-Cartier divisor $D \geq 0$ such that $K_X + B \sim_{\mathbb{R}} D/Y$. Applying [2, Theorem 2.1, Proposition 4.4], there exist birational models $\pi : (X', \Delta') \to X$, $\phi : (Y', \Delta_{Y'}) \to Y$ such that the induced morphism $f' : (X', \Delta') \to (Y', \Delta_{Y'})$ is toroidal and equidimensional to a log smooth pair. Moreover, writing $K_{X'} + B' = \pi^*(K_X + B) + E$ as in (1), by [1, Theorem 1.1], we have $B' \leq \Delta'$ and $\mathrm{Supp} D' \subseteq \Delta'$ where $D' = \pi^*D + E$. Hence, there exists an $\mathbb{R}$-Cartier divisor $G \geq 0$, supported by $\Delta_{Y'}$, such that $D'^v - f'^*G$ is very exceptional/Y', where D'^v denotes the vertical/Y' part. Set $R = D' - f'^*G$. We see R satisfies (3).

Finally, by Lemma 2.10, $f' : (X', \Theta) \to Y'$ is an $\mathbb{R}$-klt-trivial fibration, where $\Theta := B' - R$. Hence, by Lemma 2.8, we apply a canonical bundle formula to obtain $K_{X'} + \Theta \sim_{\mathbb{R}} f'^*(K_{Y'} + B_{Y'} + M_{Y'})$, such that $(Y', B_{Y'} + M_{Y'})$ is a g-sub-klt generalised pair with the moduli b-divisor $\mathbf{M}$. It remains to check that $(Y', B_{Y'} + M_{Y'})$ is g-klt. Indeed, the effectiveness of $B_{Y'}$ follows from the construction of discriminant divisor. □

Remark 3.3 Since the arguments for Lemmas 2.7, 2.8 and 2.10 are still valid for lc-trivial fibrations and lc pairs, one can easily generalise Theorem 1.2 to lc pairs with the above argument. Note that, in this case, with notation from Theorem 1.2, $(Y', B_{Y'} + M_{Y'})$ is a g-lc generalised pair, and it is g-klt if all lc centres of (X, B) are horizontal$/Y$.

Proof (*Proof of Theorem* 1.1) Take a relative Iitaka fibration $f : \overline{X} \to Y$ over Z. Replacing (X, B), we may assume $X = \overline{X}$. By definition, we have $\kappa_\iota(X/Y, K_X + B) = 0$. So, by a canonical bundle formula, there exists a commutative diagram

$$\begin{tikzcd} (X', B') \arrow[r, dash, "\pi"] \arrow[d, "f'"'] & (X, B) \arrow[d, "f"] \\ Y' \arrow[r, dash, "\phi"] & Y \end{tikzcd}$$

which consists of birational models $\pi : X' \to X, \phi : Y' \to Y$, satisfying the conditions listed in Theorem 1.2. Replacing (X, B), Y with (X', B'), Y', we have $K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y + M_Y) + R$. Since $(Y, B_Y + M_Y)$ is g-klt and $K_Y + B_Y + M_Y$ is big$/Z$, $K_Y + B_Y + M_Y$ has a semi-ample model$/Z$ by [6, Lemma 4.4(2)]. Because $R \geq 0, \kappa(X/Y, R^h) = 0$ and R^v is very exceptional$/Y$, where R^h (resp. R^v) denotes the horizontal (resp. vertical) part over Y, by Lemma 3.1, we deduce that $(X/Z, B)$ has the canonical model. ☐

References

1. Abramovich, D., Denef, J., Karu, K.: Weak toroidalization over non-closed fields. Manuscripta Math. **142**(1–2), 257–271 (2013)
2. Abramovich, D., Karu, K.: Weak semistable reduction in characteristic 0. Invent. Math. **139**(2), 241–273 (2000)
3. Ambro, F.: Shokurov's boundary property. J. Differ. Geom. **67**(2), 229–255 (2004)
4. Birkar, C.: Existence of log canonical flips and a special LMMP. Publ. Math. Inst. Hautes Études Sci. **115**(1), 325–368 (2012)
5. Birkar, C., Cascini, P., Hacon, C.D., McKernan, J.: Existence of minimal models for varieties of log general type. J. Am. Math. Soc. **23**(2), 405–468 (2010)
6. Birkar, C., Zhang, D.Q.: Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs. Publ. Math. Inst. Hautes Études Sci. **123**(1), 283–331 (2016)
7. Fujino, O.: Basepoint-free theorems: saturation, *b*-divisors, and canonical bundle formula. Algebra Number Theory **6**(4), 797–823 (2012)
8. Fujino, O.: Foundations of the minimal model program. MSJ Mem. **35**, Mathematical Society in Japan, Tokyo, (2017)
9. Fujino, O., Mori, S.: A canonical bundle formula. J. Differ. Geom. **56**(1), 167–188 (2000)
10. Hashizume, K., Hu, Z.: On minimal model theory for log abundant lc pairs. J. Reine Angew. Math. **767**, 109–159 (2020)

11. Hu, Z.: Log abundance of the moduli b-divisors of lc-trivial fibrations, preprint (2020). arXiv:2003.14379
12. Nakayama, N.: Zariski-decomposition and abundance, MSJ Mem., **14**, Mathematical Society of Japan, Tokyo, (2004)

Generalized Thomas–Yau Uniqueness Theorems

Yohsuke Imagi

Abstract We generalize Thomas–Yau's uniqueness theorem [17,Theorem 4.3] in two ways. We prove a stronger statement for special Lagranigans and include minimal Lagrangians in Kähler–Einstein manifolds or more generally J-minimal Lagrangians introduced by Lotay and Pacini [13,14]. In every case the heart of the proof is to make certain Hamiltonian perturbations. For this we use the method by Imagi, Joyce and Oliveira dos Santos [8,Theorem 4.7].

Keywords Fukaya category · Special Lagrangians · Kähler–Einstein manifolds

1 Introduction

In this paper we improve and generalize Thomas–Yau's theorem [17, Theorem 4.3]. Our first main result is the following. For the more complete statement see Corollary 6.3 (ii).

Theorem 1.1 *Let X be a Kähler manifold equipped with a holomorphic volume form. Let the cohomology Fukaya category $H\mathcal{F}(X)$ have two isomorphic objects supported near two closed irreducibly-immersed special Lagrangians L_1, L_2 respectively. Then $L_1 = L_2 \subseteq X$.*

Remark 1.2 **(i)** Thomas and Yau prove their uniqueness theorem for closed special Lagrangians L_1, L_2 in the same Hamiltonian isotopy class. But their proof works under the weaker hypothesis as above; that is, we need only that L_1, L_2 have the same isomorphism class in $H\mathcal{F}(X)$. This $H\mathcal{F}(X)$ is a (usual) associative category obtained from the A_∞ category $\mathcal{F}(X)$ by taking the $(\mathfrak{m}^1)$ cohomology groups of the

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hom spaces in it. The key fact is that if L_1, L_2 are Hamiltonian isotopic then L_1, L_2 will define isomorphic objects in $H\mathcal{F}(X)$ (after given the additional data including bounding cochains).

It is clear that we need only the weaker hypothesis because Thomas and Yau use in their proof only the Floer cohomology group $HF^*(L_1, L_2)$, the hom space in $H\mathcal{F}(X)$. But when Thomas and Yan wrote their paper, the theory of Fukaya categories was much less developed at that time, so the statement was presumably better-sounding with Hamiltonian isotopies. In this paper we improve Thomas–Yau's theorem by using the more developed theory of Fukaya categories as follows.

(ii) Theorem 1.1 is stronger than the original theorem in the respect that L_1, L_2 need not have (even after shift) the same phase. This would also follow from the following statement: two Lagrangians underlying the same isomorphism class of objects in $H\mathcal{F}(X)$ should have the same homology class in X; although we shall not discuss this in the present paper. The fact that L_1, L_2 have the same shift up to shift will be proved in Corollary 4.5 (before giving the full proof of the theorem), for which we shall use some simple fact about Maslov indices and some nonvanishing results for Floer cohomology groups (which we recall in Sect. 4).

Also we do not suppose that either L_1 or L_2 itself underlies an object of $H\mathcal{F}(X)$ nor do we suppose even that it is cleanly immersed as Fukaya [3] does. But we do suppose that we can perturb L_1, L_2 both to generically immersed Lagrangians which underlie objects of $H\mathcal{F}(X)$. This is how we deal with badly immersed Lagrangians in the Floer-theory context in this paper.

(iii) The idea of the proof of Theorem 1.1 is the same as that of Thomas and Yau. But we include the modification by Imagi, Joyce and Oliveira dos Santos [8][Theorem 4.7] because we can unfortunately not justify the original Morse-theory argument [17, Theorem 4.3].

Another natural generalization is to minimal Lagrangians in Kähler–Einstein manifolds or more generally to J-minimal Lagrangians introduced by Lotay and Pacini [13, 14]. At the moment we state the result only in the Kähler–Einstein case. For the more complete statement see Corollary 6.3 (i).

Theorem 1.3 *Let X be a $\mathbb{Z}_k$-graded Kähler–Einstein manifold of complex dimension $n \leq k$; and $L_1, L_2 \subseteq X$ two closed irreducibly-immersed $\mathbb{Z}_k$-graded minimal Lagrangians with the same grading on $L_1 \cap L_2$. Let $\mathbf{b}_1, \mathbf{b}_2 \in H\mathcal{F}(X)$ be two objects supported respectively near L_1, L_2 and such that:*

$$\text{either } \mathbf{b}_1 \cong \mathbf{b}_2 \textit{ or } k \geq 2n-1 \textit{ and } \mathbf{b}_1 \cong \mathbf{b}_2 \textit{ up to shift; and} \tag{1}$$

$$HF^i(\mathbf{b}_1, \mathbf{b}_1) \cong HF^i(\mathbf{b}_2, \mathbf{b}_2) \neq 0 \textit{ for } i = 0, n. \tag{2}$$

Then $L_1 = L_2 \subseteq X$.

Remark 1.4 The more precise meaning of L_1, L_2 having the same grading on $L_1 \cap L_2$ is given in Definition 2.1 below. We make a more geometric definition of gradings than Seidel [16] does.

In the circumstances of Theorem 1.3, if $c_1(X) > 0$ there may be many automorphisms of X and many zero objects of $H\mathcal{F}(X)$ especially in the toric case as Fukaya, Oh, Ohta and Ono [5] study. So we can hardly expect a general uniqueness statement to hold. But for instance, to the *Clifford torus*

$$T^n := \{[z_0, \dots, z_n] \in \mathbb{C}P^n : |z_0| = \cdots = |z_n|\} \tag{3}$$

we can apply Theorem 1.3 with the latter alternative of (1). We can in fact take $k = 2n + 2$ as we show in Example 1 below. Also according to Fukaya, Oh, Ohta and Ono [5] there are $(n + 1)$ objects $\mathbf{b} \in H\mathcal{F}(X)$ supported on T^n with $HF^*(\mathbf{b}, \mathbf{b}) \cong H^*(T^n)$. So (2) holds and we have:

Corollary 1.5 *Let $L \subseteq \mathbb{C}P^n$ be a closed irreducibly-immersed $\mathbb{Z}_{2n+2}$-graded minimal Lagrangian such that L, T^n have the same grading on $L \cap T^n$. Let $H\mathcal{F}(X)$ have an object which is supported near L and isomorphic to one of the $(n + 1)$ objects* $\mathbf{b}$*'s above (all supported on T^n). Then $L = T^n$.*

On the other hand, if $c_1(X) \leq 0$ the nonzero condition (2) holds automatically in certain circumstances. For instance, if either L_1 or L_2 is embedded then (2) holds automatically by Fukaya, Oh, Ohta and Ono's theorem [4][Theorem E]. If $c_1(X) = 0$ there is a stronger version of it which we use for Theorem 1.1.

If $c_1(X) < 0$ Theorem 1.2 extends to J-minimal Lagrangians. They are locally unique, so the result is interesting only in the global context.

For the proofs we follow in outline that by Thomas and Yau [17, Theorem 4.3]. In every case the heart of the proof is to make certain Hamiltonian perturbations of the two Lagrangians. The Floer-theory condition implies that the two perturbed Lagrangians have at least one intersection point of index 0 or n at which we do by analysis a sort of unique continuation.

But for the Hamiltonian perturbation process we use the method by Imagi, Joyce and Oliveira dos Santos [8][Theorem 4.7] as mentioned above. The modified method works only in the real analytic category. This causes no problem in the Kähler–Einstein case because in that case everything is automatically analytic by elliptic regularity. On the other hand, in Theorem 1.1 and in the J-minimal version of Theorem 1.3 we work in the C^∞ category and the real analyticity condition is achieved by another Hamiltonian-perturbation process.

We begin in Sect. 2 with our geometric treatment of graded Lagrangians. In Sect. 3 we recall the relevant facts about Maslov forms in Lotay–Pacini's sense. In Sect. 4 we give a minimum account of Floer theory which we use in this paper. In Sect. 5 we carry out the key step of making Hamiltonian perturbations. In Sect. 6 we state and prove all our results. They are essentially at the chain level because Sect. 5 is independent of Floer theory.

Finally we remark that Thomas–Yau's uniqueness theorem is just a bit of their whole proposal [17] which is now improved by Joyce [10].

2 Geometric Gradings

We make a geometric definition of graded Lagrangians:

Definition 2.1 Let (X, ω) a symplectic manifold, J an ω-compatible almost-complex structure on X and K_X the canonical bundle over (X, J). Let k be either a positive integer or infinity, write $\mathbb{Z}_k := \mathbb{Z}/k\mathbb{Z}$ for k finite and write $\mathbb{Z}_\infty := \mathbb{Z}$. By a $\mathbb{Z}_k$-*grading* of (X, ω, J) we mean for $k < \infty$ the pair of a complex line bundle $K_X^{2/k}$ and a bundle isomorphism $(K_X^{2/k})^k \cong K_X^2$, and for $k = \infty$ a nowhere vanishing section Ψ of K_X^2. Suppose given such a $\mathbb{Z}_k$-grading of X and a Hermitian metric g on (X, J). Then for every $x \in X$ and for every Lagrangian plane $\ell \subseteq T_x X$ there is a g-orthogonal decomposition $T_x X = \ell \oplus J\ell$ and accordingly a canonical g-unit section of K_X^2 which we denote by Ω_ℓ^2; that is, if $e_1, \dots, e_n \in \ell$ are a g-orthonormal basis then

$$\Omega_\ell^2 := \pm e^1 \wedge \cdots \wedge e^n \wedge Je^1 \wedge \cdots \wedge Je^n.$$

There is also a canonical k-fold cover $\operatorname{Lag}^k TX$ of the Lagrangian Grassmannian $\operatorname{Lag} TX$ defined for $k < \infty$ by

$$\operatorname{Lag}^k TX := \{(\ell, \alpha) \in \operatorname{Lag} TX \times_X K_X^{2/k} : \Omega_\ell^2 = \alpha^k\}$$

and for $k = \infty$ by $\operatorname{Lag}^\infty TX := \{(\ell, \phi) \in \operatorname{Lag} TX \times_X \mathbb{R} : \Omega_\ell^2 = e^{i\phi}\Psi\}$. By a $\mathbb{Z}_k$-grading of an immersed Lagrangian $L \subseteq (X, \omega)$ we mean a lift to $\operatorname{Lag}^k TX$ of the tangent-space map $L \to \operatorname{Lag} TX$, that is, for $k < \infty$ a section $\alpha : L \to K_X^{2/k}$ with $\alpha^k = \Omega_L^2$ and for $k = \infty$ a section $\phi : L \to S^1$ with $e^{i\phi}\Psi = \Omega_L^2$. It is unique up to $\mathbb{Z}_k$-shifts, where the shift [1] is defined for $k < \infty$ by $(L, \alpha)[1] := (L, e^{2\pi i/k}\alpha)$ and for $k = \infty$ by $(L, \phi)[1] := (L, \phi + 2\pi)$.

Remark 2.1 The advantage of this definition is that given two Lagrangians L_i $(i = 1, 2)$ graded by α_i or ϕ_i as above, we can compare the two gradings α_i or ϕ_i. In particular, it makes sense to say that they are equal or not.

We make another definition in the circumstances of Definition 2.1:

Definition 2.2 For $i = 1, 2$ let $L_i \subseteq (X, \omega)$ be a Lagrangian with a $\mathbb{Z}_k$-grading $\alpha_i : L \to K_X^{2/k}$, $k < \infty$. Suppose L_1, L_2 intersect transversely at a point $x \in L_1 \cap L_2$. We define the Maslov index $\mu_{L_1, L_2}(x) \in \mathbb{Z}_k$. Define $e^{i\tau} \in S^1$ by $\alpha_2 = e^{i\tau}\alpha_1 \in K_X^{2/k}$ over x. Take an isomorphism $(T_x X, J|_x, g|_x) \cong \mathbb{C}^n$ which maps $T_x L_1$ to $\mathbb{R}^n \subseteq \mathbb{C}^n$ and $T_x L_2$ to $\{(e^{i\theta_1}x_1, \dots, e^{i\theta_n}x_n) \in \mathbb{C}^n : x_1, \dots, x_n \in \mathbb{R}\}$. for some $\theta_1, \dots, \theta_n \in (0, \pi)$. These $\theta_1, \dots, \theta_n$ are unique up to order and we have $\Omega_{L_2}^2 = \pm\, e^{-2i(\theta_1 + \cdots + \theta_n)}\Omega_{L_1}^2$. So we can define

$$\mu_{L_1, L_2}(x) := \frac{1}{2\pi}(2\theta_1 + \cdots + 2\theta_n + k\tau) \in \mathbb{Z} \text{ modulo } k\mathbb{Z}. \tag{4}$$

If $k = \infty$ and if for $i = 1, 2$ the L_i is $\mathbb{Z}$-graded by $\phi_i : L_i \to \mathbb{R}$ then

$$\mu_{L_1,L_2}(x) := \frac{1}{2\pi}(2\theta_1 + \cdots + 2\theta_n - \phi_1(x) + \phi_2(x)) \in \mathbb{Z}. \tag{5}$$

We recall now the notion of *special* Lagrangians:

Definition 2.3 Let (X, ω) a symplectic manifold, J an ω-compatible almost-complex structure on X and Ψ a $\mathbb{Z}$-grading of (X, ω, J). We say that an immersed Lagrangian in (X, ω, J, Ψ) is *special* if it is $\mathbb{Z}$-graded by a constant.

Remark 2.2 In the symplectic context including this definition and Sect. 4 we do not need J to be integrable. But in the more geometric context we suppose J integrable and Harvey–Lawson's theorem applies as in Theorem 3.1.

We prove a lemma which we use in Corollary 4.5:

Lemma 1 *Let (X, ω, J, Ψ) be such as in Definition 2.3 and $L_1, L_2 \subseteq X$ two mutually-transverse special Lagrangian submanifolds which have not even after shifts the same grading. Then there exists $i \in \mathbb{Z}$ such that the Maslov indices of L_1, L_2 all belong to $[i, i+n-1]$. This will still hold under Lagrangian perturbations of L_1, L_2.*

Proof Let $\phi_1, \phi_2 \in \mathbb{R}$ be gradings of L_1, L_2. Then $\phi - \phi' \notin \mathbb{Z}$ and this condition is preserved under Lagrangian perturbations of L_1, L_2. Let $x \in L_1 \cap L_2$ and let $\theta_1, \ldots, \theta_n \in (0, \pi)$ be as in (4). Then by (5) we have $\mu_{L_1,L_2}(x) \in [i, i+n-1]$ for some $i \in \mathbb{Z}$. This i is independent of x which completes the proof. □

We prove another lemma which we use in Sect. 6:

Lemma 2 *Let (X, ω) be a symplectic manifold of real dimension $2n$, $\mathbb{Z}_k$-graded with $k \geq n$, equipped with a compatible almost complex-structure J and equipped with a Hermitian metric g. Let $L_1, L_2 \subseteq X$ be two closed immersed $\mathbb{Z}_k$-graded Lagrangians with the same grading on $L_1 \cap L_2$. Denote by $S \subseteq L_1 \cap L_2$ the set of points at which L_1, L_2 have at least one common tangent space. Then for every open neighbourhood $U \subseteq X$ of S there exists $\epsilon > 0$ such that every Hamiltonian ϵ-perturbation of L_2 that intersects L_1 generically (that is, only at transverse double points) has no intersection point with $L_1 \cap U$ of index 0 or n modulo $k\mathbb{Z}$.*

Proof If this fails there are an open neighbourhood $U \subseteq X$ of S, a sequence of Hamiltonian perturbations L_2^i of L_2 all transverse to L_1 and tending smoothly to L_2, and a sequence of intersection points $x^i \in L_1 \cap L_2^i - U$ of index 0 or n. By hypothesis $L_1 \cap L_2^i\ U$ is compact, so there is a subsequence of x^i tending to some point $x \in L_1 \cap L_2\ U$. Suppose now that $k < \infty$; the other case $k = \infty$ may be treated in the same manner. Give L_2^i a grading α^i by the obvious homotopy, and write $\alpha^i = \exp(\sqrt{-1}\tau^i)\alpha$ over x^i. By hypothesis L_1, L_2 have the same grading at x^i so we may suppose that for each i we have $|\tau^i| < 2\pi/k$ and the sequence τ^i tends to 0 as $i \to \infty$. Denote by $\theta_1^i, \ldots, \theta_n^i \in (0, \pi)$ the n angles between the two tangent spaces at x^i. Then $2\theta_1^i + \cdots + 2\theta_n^i + k\tau^i = 0$ or $n\pi$ modulo $2k\pi\mathbb{Z}$. In fact, since $k \geq n$ it follows that this holds without modulo $2k\pi\mathbb{Z}$. So the limits of the n angles sum up to 0 or $n\pi$; that is, at the limit point x there is a common tangent space to L_1, L_2. But this contradicts the definition of S. □

3 Maslov Forms

Following Lotay and Pacini [14] we make:

Definition 3.1 Let (X, ω, J) be a Kähler manifold of complex dimension n and K_X the canonical bundle over (X, J). Let $L \subseteq (X, \omega)$ be a Lagrangian submanifold, and g a Hermitian metric on (X, J) which need not be the Kähler metric of (X, ω, J) nor even Kähler. As in Definition 2.1 take the canonical g-unit section Ω^2_L of K^2_X over L. Denote by ∇ the Chern connection on (X, J, g) and by A the real 1-form on K^2_X over L defined by $\nabla\Omega^2_L = 2iA \otimes \Omega^2_L$. We call A the *Maslov form* on L relative to g and say that L is *g-Maslov-zero* if $A \equiv 0$.

Here it is not essential that J is integrable. Many results by Lotay and Pacini [13, 14] hold with J an ω-compatible almost-complex structure and with ∇ a connection such that $\nabla J = \nabla g = 0$. But it is convenient for us to suppose J integrable. We can then take the Chern connection; in Sect. 5 we can use the underlying real analytic structure of (X, J) which will be convenient for us to state our results; and there are as in Theorem 3.3 nice deformation theory results for special Lagrangians and J-minimal Lagrangians which we use in Corollary 6.2.

As Lotay and Pacini [13, Theorem 2.4] prove, if g is Kähler the g-Maslov-zero condition and the J-minimal condition are equivalent. If also L is g-Lagrangian—that is, $g(Jv, v') = 0$ for any $v, v' \in TL$—then L is J-minimal if and only if L is g-minimal in the ordinary sense.

We work with g-Maslov-zero Lagrangians rather than with J-minimal Lagrangians because for special Lagrangians we can take g to be merely *conformally* Kähler as we recall now. Let (X, ω, J) a Kähler manifold and Ψ a J-holomorphic volume form on X. Joyce [9] and other authors call (X, ω, J, Ψ) an almost Calabi–Yau manifold but this does not mean that J is merely an almost complex structure; J is integrable. It means that ω is Andnot necessarily Ricci-flat. Denote by g the conformally Kähler metric on (X, J) associated with $\psi^2\omega$ where $\psi : X \to \mathbb{R}^+$ is defined by

$$\psi^{2n}\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n\Psi \wedge \overline{\Psi}. \tag{6}$$

Then for every Lagrangian submanifold $L \subseteq X$ the canonical g-unit section ω^2_L is of the form $\alpha\Psi^2|_L$ where $\alpha : L \to S^1$ is a smooth function. This L has zero Maslov 1-form if and only if α is constant, that is, if and only if L is special. In these circumstances Harvey–Lawson's theorem [7] may be stated as follows:

Theorem 3.1 *If L is special then L is orientable and calibrated by $\sqrt{\alpha}\Psi$ in Harvey–Lawson's sense where the choice of $\sqrt{\alpha}$ corresponds to the orientation of L. Furthermore, L is area-minimizing with respect to g and J-minimal in (X, J, g).*

Lotay and Pacini [14, Proposition 4.5] prove a formula for the Maslov 1-form A and the J-mean-curvature vector H_J:

$$-JA = (H_J + T_J) \lrcorner g \tag{7}$$

where T_J comes from the torsion of the Chern connection ∇. If L is special then by definition we have $A = 0$ and H_J is the ordinary mean curvature which is also zero. So by (7) we have $T_J = 0$ too. Conversely, if L is merely J-minimal then T_J may be nonzero. This justifies us working with g-Maslov-zero Lagrangians rather than J-minimal Lagrangians.

We compute now the *relative first Chern class* $2c_1(X, L) \in H^2(X, L; \mathbb{Z})$ for g-Maslov-zero Lagrangians $L \subseteq X$. We recall therefore:

Lemma 3 *Let (X, ω) be a symplectic manifold, J an ω-compatible complex structure, K_X the canonical bundle over (X, J) and $L \subseteq (X, \omega)$ a Lagrangian submanifold. Then the relative de-Rham class $2c_1(X, L) \in H^2(X, L; \mathbb{R})$, which is integral, may be represented by the curvature 2-form of any connection on K_X^2 that is flat over L.*

So by Definition 3.1 we have:

Corollary 3.2 *Let (X, ω, J) be a Kähler manifold, g a Hermitian metric on (X, J), and $L \subseteq (X, \omega)$ a g-Maslov-zero Lagrangian submanifold. Then $2c_1(X, L)$ may be represented by the curvature $(1, 1)$-form of K_X^2 relative to the Chern connection on (X, J, g).*

Example 1 Let $X = \mathbb{C}P^n$, ω the Fubini–Study form and g the Fubini–Study metric. This is Kähler–Einstein and the curvature $(1, 1)$-form of K_X relative to g is $-(n + 1)\omega$. The Clifford torus $T^n \subseteq \mathbb{C}P^n$ defined by (3) is a minimal Lagrangian. It is J-minimal and g-Maslov-zero so we can apply to it Corollary 3.2.

We prove that $\mathbb{C}P^n$ and T^n may both be $\mathbb{Z}_{2n+2}$-graded. Since $H_1(\mathbb{C}P^n, \mathbb{Z}) = 0$ it follows according to Seidel [16, Lemma 2.6] that $\mathbb{C}P^n$ may be $\mathbb{Z}_{2n+2}$-graded because $2c_1(\mathbb{C}P^n) = -2(n + 1)[\omega]$. Also T^n may be $\mathbb{Z}_k$-graded if and only if $2c_1(\mathbb{C}P^n, T^n)$ is divisible by k. By Corollary 3.2 we have $c_1(\mathbb{C}P^n, T^n) = -(n + 1)[\omega] \in H^2(\mathbb{C}P^n, T^n; \mathbb{R})$. Computation shows that $H_2(\mathbb{C}P^n, T^n; \mathbb{Z})$ is generated by the image of $H_2(\mathbb{C}P^n, \mathbb{Z})$ and the n discs $D_a \subseteq \mathbb{C}P^n$, $a \in \{1, \ldots, n\}$, defined by $|z_a| \leq 1$ and $z_b = 1$ for every $b \neq a$. We have

$$\omega|_{D_a} = \frac{ni}{2\pi} \frac{dz_a \wedge d\bar{z}_a}{(n + |z_a|^2)^2} \text{ and } \int_{D_a} \omega = 1.$$

So $[\omega] \in H^2(\mathbb{C}P^n, T^n; \mathbb{R})$ is integral, q.e.d.

We turn now to the deformation theory for g-Maslov-zero Lagrangians. We recall Lotay–Pacini's lemma [13, Lemma 4.1] which we use in Sect. 5:

Lemma 4 *Let (X, ω, J) be a Kähler manifold and g a Hermitian metric on (X, J). Let $L \subseteq (X, \omega, J)$ be a closed immersed g-Maslov-zero Lagrangian and NL the g-normal bundle to $L \subseteq X$. For every sufficiently small $v \in C^\infty(NL)$ denote by Av the g-Maslov 1-form on the graph of v embedded in X by the g-exponential map, and denote by A' the linearization of A at $0 \in C^\infty(NL)$. Then for every $v \in C^\infty(NL)$ we have*

$$A'v = dd^*\hat{v} + \sum_{i=1}^{n} g(T_\nabla(v, e_i), e_i) + v \lrcorner F_\nabla \tag{8}$$

where $\hat{v} := -Jv \lrcorner g$, $\{e_1, \dots, e_n\}$ *is a* g*-orthonormal frame on* TL, ∇ *the Chern connection of* (X, J, g), T_∇ *its torsion tensor and* F_∇ *its curvature* $(1, 1)$*-form for* K_X.

Proof This is a version of Lotay–Pacini's lemma [13, Lemma 4.1]. They suppose that (X, J, g) is almost Kähler but again this is not essential; the result applies to every almost Hermitian manifold (X, J, g) with connection ∇ such that $\nabla J = \nabla g = 0$. Our formula (8) is simpler than Lotay–Pacini's in two respects. One is that their J-volume function $\rho_J : L \to \mathbb{R}^+$ is identically equal to 1 and the other is that their projection operator π_L is g-orthogonal. These both hold because L is g-Lagrangian, which completes the proof. □

We remark that we can compute the torsion term in the conformally Kähler case. If g is a conformally Kähler metric on (X, J) associated with $\psi^2\omega$ then $\nabla - \partial \log \psi^2 \otimes \mathrm{id}$ is the Levi-Civita connection for ω. Hence by computation we see that for any vector fields u, v on X we have

$$T_\nabla(u, v) = d \log \psi \llcorner (u \otimes v + Ju \otimes Jv - v \otimes u - Jv \otimes Ju).$$

So the torsion term on (8) is equal to $(n-1)d \log \psi$ and

$$A'v = d(\psi^{n-1} d^* \psi^{1-n} \hat{v}) + v \lrcorner F_\nabla. \tag{9}$$

This has application to deformation theory; that is, by (9) we can apply Lotay–Pacini's result [13, Proposition 4.5] with ψ^{n-1} in place of ρ_J. So Lotay–Pacini's uniqueness and persistence theorem [13, Theorem 5.2] extends to g-Maslov-zero Lagrangians with g conformally Kähler. But the corresponding perturbations are just Lagrangian and not necessarily Hamiltonian as we want in the Floer theory context.

On the other hand, we can make (domain) Hamiltonian perturbations of special Lagrangians and J-minimal Lagangians. The result for special Lagrangians goes back to McLean's theorem [15] and that for J-minimal Lagrangians is proved by Lotay–Pacini [13, Theorem 5.6].

Theorem 3.3 **(i)** *Let* (X, ω, J) *be a Kähler manifold,* Ψ *a* J*-holomorphic volume form on* X *and* $L \subseteq X$ *a closed immersed special Lagrangian relative to* (J, Ψ). *Then for every sufficiently small perturbation* (J', Ψ') *of* (J, Ψ) *as* ω*-compatible complex structures and holomorphic volume forms relative to them, there exists a domain Hamiltonian perturbation of* L *which is special with respect to* (J', Ψ').
(ii) *Let* (X, ω, J) *be a Kähler manifold such that* $-\omega$ *is the Ricci* $(1, 1)$*-form of another Kähler metric* g *on* (X, J), *and* $L \subseteq X$ *a closed immersed* J*-minimal Lagrangian. Then for every sufficiently small perturbation* (J', g') *of* (J, g) *as* ω*-*

compatible complex structures and Kähler metrics relative to them whose Ricci $(1,1)$*-forms are all equal to* $-\omega$*, there exists a domain Hamiltonian perturbation of* L *which is* J*-minimal with respect to* (J', g')*.*

4 HF^* Nonvanishing Theorems

Following Fukaya, Oh, Ohta and Ono [4] given a unital commutative ring R we define the Novikov ring

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in R[e, e^{-1}], \lambda_i \in \mathbb{R} \text{ for each } i \text{ and } \lim_{i\to\infty} \lambda_i = +\infty \right\} \quad (10)$$

where T, e are two formal variables. They are related respectively to the areas and Maslov numbers of pseudoholomorphic curves. We denote by $\Lambda^0 \subseteq \Lambda$ the subring defined by the same formula (10) as Λ but with $\lambda_i \geq 0$ in place of $\lambda_i \in \mathbb{R}$. We denote by $\Lambda^+ \subseteq \Lambda^0$ the ideal defined by (10) with $\lambda_i > 0$ in place of $\lambda_i \in \mathbb{R}$. We write $\approx$ for $=$ modulo Λ^+.

Fukaya categories are defined over Λ. If the ambient symplectic manifold is spherically nonnegative we can take $R = \mathbb{Z}$ as Fukaya, Oh, Ohta and Ono [6] do. But in general we have to do the pseudoholomorphic-curve counts over $\mathbb{Q}$ and we shall therefore suppose that R contains $\mathbb{Q}$. Otherwise, in our applications we need no condition on R; that is, we work under the following:

Hypothesis 1 Let (X, ω) be a $\mathbb{Z}_k$-graded symplectic manifold of real dimension $2n$, either compact or ω-convex at infinity. Let R be a unital commutative ring such that if X is not spherically nonnegative then $\mathbb{Q} \subseteq R$. If $2 \neq 0$ in R, we shall suppose that X is given a background bundle for relative spin structures on Lagrangians in X, and that the Lagrangians concerned are all given relative spin structures.

Under this hypothesis there is a Fukaya category $\mathcal{F}(X)$ and its cohomology category $H\mathcal{F}(X)$. In this paper, following Akaho and Joyce [1] we suppose that every object of $\mathcal{F}(X)$ consists of a closed generically-immersed Lagrangian $L \subseteq X$ and its additional structures such as bounding cochains. More precisely, we should write the immersion map to L, say $\widehat{L} \to L$, and 'closed' means $\widehat{L}$ being a closed manifold. We can include also a suitable class of local systems on L and the results in this paper will hold for them but we shall omit this in our formal treatment. We recall now the notion of bounding cochains now.

We recall first the homologically perturbed version of Floer complexes. Given a closed manifold $\widehat{L}$ and a generic Lagrangian immersion $\widehat{L} \to L \subseteq X$ we write $L \cap L := \widehat{L} \times_X \widehat{L}$. This is the disjoint union of the diagonal $\widehat{L}$ and self-intersection pairs. It depends upon the choice of $\widehat{L} \to X$ but is unique up to automorphisms of $\widehat{L}$. We write $C_L^* := H^*(L \cap L, R)[-\mu_{L,L}]$ where the diagonal has no degree shift and every self-intersection pair has degree equal to its Maslov index. If L is embedded, this C_L^* is $\mathbb{Z}$-graded and supported in degrees $0, \ldots, n$. If L is $\mathbb{Z}_k$-graded, so is C_L^*. If also L has on its every self-intersection pair two gradings nearly equal, then C_L^* is supported in degrees $0, \ldots, n$ modulo $k\mathbb{Z}$.

We use the following version of A_∞ algebra structure constructions. There are on C_L^0 a unit and the intersection pairing so we can speak of unitality and cyclic symmetry as Fukaya [2] does.

Theorem 4.1 *There is on $C_L^* \otimes \Lambda^0$ a cohomologically-unital cyclic curved A_∞ algebra structure $(\mathfrak{m}^i)_{i=0}^\infty$ with $\mathfrak{m}^1 \approx 0$ and $\mathfrak{m}^2 \approx \pm\wedge$ where $\wedge$ is the cup-product map. If also L is embedded, we can make this strictly unital.*

Remark 4.2 The embedded case is proved by Fukaya [2]. This will extend to the immersed case if we give up having the strict unit and satisfy ourselves with a cohomological unit. To have the strict unit we ought to perturb in a certain manner the relevant pseudoholomorphic-curve moduli spaces. But in the immersed case there would be moduli spaces of constant maps to self-intersection points. If we perturbed these, the cyclic symmetry would fail.

The cyclic symmetry is used in Lemma 5 below. But also we give it another proof in which we use the open-closed map.

By a *bounding cochain* on $(C_L^* \otimes \Lambda^0; \mathfrak{m}^0, \mathfrak{m}^1, \mathfrak{m}^2, \cdots)$ we mean an element $\mathbf{b} \in C_L^1 \otimes \Lambda^+$ with $\sum_{i=0}^\infty \mathfrak{m}^i(\mathbf{b}, \ldots, \mathbf{b}) = 0$. Given such a $\mathbf{b}$ there is a natural way of giving $C_L^* \otimes \Lambda^0$ a cohomologically-unital cyclic *ordinary* A_∞ algebra structure $(\mathfrak{m}_\mathbf{b}^i)_{i=0}^\infty$ with $\mathfrak{m}_\mathbf{b}^1 \approx 0$ and $\mathfrak{m}_\mathbf{b}^2 \approx \pm\wedge$.

We denote by $1 \in C_L^0$ the unit, which is the cohomological unit in Theorem (4.1). We denote by $*1 \in C_L^n$ the volume form supported on the diagonal, which is if $2 \neq 0$ in R the Poincaré dual to a point. We denote also symbolically by $(x, y) := \pm\int_{L\cap L} x \wedge y$ the intersection pairing on C_L^*. The sign does not matter to us. We prove now:

Lemma 5 *For every bounding cochain $\mathbf{b} \in C_L^1 \otimes \Lambda^+$ we have:*

(i) *if L is embedded, we have $*1 \notin \operatorname{im} \mathfrak{m}_\mathbf{b}^1$; and*
(ii) *if $\mathfrak{m}_\mathbf{b}^1 * 1 = 0$ then $[*1] \neq 0$ in $H^n(C_L^* \otimes \Lambda^0, \mathfrak{m}_\mathbf{b}^1)$.*

Proof Suppose contrary to (i) that $*1 = \mathfrak{m}_\mathbf{b}^1 x$ for some $x \in C_L^* \otimes \Lambda$. Then using the strict unit we find

$$0 = (\mathfrak{m}_\mathbf{b}^1 1, x) = \pm(1, \mathfrak{m}_\mathbf{b}^1 x) = \pm(1, *1) \approx \pm\int_{L\cap L} \mathfrak{m}_\mathbf{b}^2(1, *1) \approx \pm\int_{L\cap L} *1 = 1. \tag{11}$$

This is impossible which proves (i).

We give (ii) two proofs. Firstly, if $*1 = \mathfrak{m}_\mathbf{b}^1 x$ for some $x \in C_L^* \otimes \Lambda$ then as in (11) we have

$$0 \approx \pm\int_{L\cap L} \mathfrak{m}_\mathbf{b}^2(1, *1) \tag{12}$$

but $\mathfrak{m}_\mathbf{b}^2(1, *1)$ need not be $*1$. However 1 is a cohomological unit and $*1$ an $\mathfrak{m}_\mathbf{b}^1$-cocycle, so

$$\mathfrak{m}^2_{\mathbf{b}}(1, *1) = *1 + \mathfrak{m}^1_{\mathbf{b}} y \approx *1 \approx *1. \tag{13}$$

Now by (12) and (13) we have $0 \approx \pm \int_{L \cap L} *1 \approx \pm 1 \in R\ \{0\}$. This is impossible and so completes the first proof.

In the other proof of (ii) we use neither the cyclic symmetry nor the unitality but the open-closed map constructed by Fukaya, Oh, Ohta and Ono [4, Sect. 3.8]. This extends to generically immersed Lagrangians. We use in particular the map $\mathfrak{p}_{\mathbf{b}} : H^n(C^*_L \otimes \Lambda^0, \mathfrak{m}^1_{\mathbf{b}}) \to H^{2n}(X, R) \otimes \Lambda^0$. Denote by $*_X 1$ the volume form on X and to distinguish it from that on L write $*_L 1 := *1 \in C^n_L$. Then by the extended version of Fukaya, Oh, Ohta and Ono's lemma [4, Lemma 6.4.2] we have $\mathfrak{p}_{\mathbf{b}}[*_L 1] \approx [*_X 1]$. So $[*_L 1] \neq 0$. □

We prove now our version of Fukaya, Oh, Ohta and Ono's theorem [4, Theorem E]. We recall therefore that the A_∞ structure of Theorem 4.1 is gapped; that is, if we take on X an ω-compatible almost-complex structure J we can write $\mathfrak{m}^1_{\mathbf{b}}$ as the sum of $\mathfrak{m}^1_{\mathbf{b},\beta}$ of degree $1 - [\beta] \cdot 2c_1(X, L)$ where $\beta \in H_2(X, L; \mathbb{Z})$ is a genus-zero stable J-holomorphic curve class. Also, given a bounding cochain $\mathbf{b} \in C^1_L \otimes \Lambda^+$ we define $HF^*(\mathbf{b}, \mathbf{b}) := H^*(C^*_L \otimes \Lambda, \mathfrak{m}^1_{\mathbf{b}}) \otimes \Lambda$.

Theorem 4.3 *Let Hypothesis 1 hold and let $L \subseteq X$ be a closed generically-immersed Lagrangian. Suppose also that:*

- **(i)** *L is embedded and there is on X an ω-compatible almost-complex structure J such that for every J-holomorphic curve class $\beta \in H_2(X, L; \mathbb{Z})$ we have $\beta \cdot 2c_1(X, L) \leq 0$; or*
- **(ii)** *$k \geq n + 2$ and L has on its every self-intersection pair the two gradings nearly equal.*

Then for $i = 0, n$ and for every object $\mathbf{b} \in H\mathcal{F}(X)$ supported on L we have $HF^i(\mathbf{b}, \mathbf{b}) \neq 0$.

Proof The case (i) is due to Fukaya, Oh, Ohta and Ono [4, Theorem E] but we recall the proof. Since L is embedded it follows that C^*_L is $\mathbb{Z}$-graded and supported in degrees $0, \ldots, n$. Also, since $\beta \cdot 2c_1(X, L) \leq 0$ it follows that $\mathfrak{m}^1_{\mathbf{b}}$ has degree ≥ 1. So under $\mathfrak{m}^1_{\mathbf{b}}$ nothing goes to 1 and $*1$ goes to zero. Since the cohomological unit is by definition an $\mathfrak{m}^1_{\mathbf{b}}$-cycle it follows now that $[1] \neq 0$ in $H^0(C^*_L \otimes \Lambda^0, \mathfrak{m}^1_{\mathbf{b}})$ and accordingly in $HF^0(\mathbf{b}, \mathbf{b})$. On the other hand, we have proved that $*1$ is also an $\mathfrak{m}^1_{\mathbf{b}}$-cycle. So by Lemma 5 (i) we have $[*1] \neq 0$ in $H^n(C^*_L \otimes \Lambda^0, \mathfrak{m}^1_{\mathbf{b}})$ and accordingly in $HF^n(\mathbf{b}, \mathbf{b})$.

In the case (ii) we have C^*_L merely $\mathbb{Z}_k$-graded. But $k \geq n + 2$ so $C^{-1}_L = C^{n+1}_L = 0$. Now $d_{\mathbf{b}}$ has degree 1 modulo $k\mathbb{Z}$ so again under $\mathfrak{m}^1_{\mathbf{b}}$ nothing goes to 1 and $*1$ goes to zero. So $[1] \neq 0$ in $HF^0(\mathbf{b}, \mathbf{b})$ and now by Lemma 5 (ii) we have $[*1] \neq 0$ in $HF^n(\mathbf{b}, \mathbf{b})$. This completes the proof. □

We apply Theorem 4.3 to g-Maslov-zero Lagrangians. We say that a closed immersed Lagrangian $L \subseteq X$ *nearly* underlies $\mathcal{F}(X)$ if there is an arbitrarily-small Hamiltonian deformation of L which underlies an object of $H\mathcal{F}(X)$. Such an object is said to be *supported near* L.

Corollary 4.4 *Let Hypothesis 1 hold with X given an ω-compatible almost-complex structure J. Let g be another Hermitian metric on (X, J) and $L \subseteq (X, \omega, J, g)$ a close $\mathbb{Z}_k$-graded immersed g-Maslov-zero Lagrangian. Suppose that either:* **(i)** *L is embedded; or* **(ii)** *$k \geq n + 2$ and L has on its every self-intersection pair the two gradings nearly equal. Then for $i = 0, n$ and for every object $\mathbf{b} \in H\mathcal{F}(X)$ supported near L we have $HF^i(\mathbf{b}, \mathbf{b}) \neq 0$.*

Proof In the case (i) it follows from Corollary 3.2 that the condition (i) of Theorem 4.3 holds. This concerns only integral homology and cohomology classes and so is preserved under perturbations of L. Taking one of them which underlies $\mathbf{b}$ we see from Theorem 4.3 that $HF^i(\mathbf{b}, \mathbf{b}) \neq 0$ as we want. Also in the case (ii) the condition (ii) of Theorem 4.3 is preserved under perturbations of L so the conclusion follows in the same way. □

For special Lagrangians we can say more:

Corollary 4.5 *Let Hypothesis 1 hold with X given an ω-compatible almost-complex structure J and a J-holomorphic volume form. Then we have:*

- **(i)** *For $i = 0, n$ and for every object $\mathbf{b} \in H\mathcal{F}(X)$ supported near a closed immersed special Lagrangian, we have $HF^i(\mathbf{b}, \mathbf{b}) \neq 0$.*
- **(ii)** *Let $L_1, L_2 \subseteq X$ be two closed immersed special Lagrangians and let $H\mathcal{F}(X)$ have two isomorphic objects $\mathbf{b}_1, \mathbf{b}_2$ supported respectively near L_1, L_2. Then L_1, L_2 have up to shift the same grading.*

Proof The part (i) follows immediately from Corollary 4.4 with $k = \infty$. If (ii) fails then by Lemma 1 we have $HF^*(\mathbf{b}_1, \mathbf{b}_2)$ supported in degrees $i, \ldots, i + n - 1$ for some $i \in \mathbb{Z}$. So either $HF^0(\mathbf{b}_1, \mathbf{b}_2) = 0$ or $HF^n(\mathbf{b}_1, \mathbf{b}_2) = 0$. But $\mathbf{b}_1 \cong \mathbf{b}_2$ so either $HF^0(\mathbf{b}_1, \mathbf{b}_1) = 0$ or $HF^n(\mathbf{b}_1, \mathbf{b}_1) = 0$. This however contradicts (i). □

5 Hamiltonian Perturbation Theorem

In this section we prove the following theorem. We work in the real analytic category and for our applications in Sect. 6 we can take the underlying real analytic structure of a complex manifold (X, J). In fact any other real analytic structure will do if it makes everything analytic but the statement will then be too long.

Theorem 5.1 **(i)** *Let (X, ω, J) be a real analytic Kähler manifold of complex dimension n and g a real analytic Hermitian metric on (X, J). Let $L_1, L_2 \subseteq X$ be two distinct closed irreducibly-immersed g-Maslov-zero Lagrangians and let S be as in Lemma 2. Then there exists a neighbourhood U of $S \subseteq X$ and arbitrarily-small Hamiltonian deformations L_1', L_2' of L_1, L_2 respectively which intersect generically each other with no intersection point in U of index 0 or n. This will still hold if L_1, L_2 are g-Maslov-zero only near $L_1 \cap L_2$. If also Hypothesis 1 holds and L_1, L_2 nearly underlie $H\mathcal{F}(X)$ we can make L_1', L_2' underlie $H\mathcal{F}(X)$.*

(ii) *The assertion will hold even if ω, g are not analytic but if there are ω-compatible complex structures J_t, J_t-Hermitian metrics g_t and g_t-Maslov-zero Hamiltonian perturbations L_{1t}, L_{2t} of L_1, L_2 all parametrized smoothly by $t \in \mathbb{R}$ with $(J_0, g_0, L_{10}, L_{20}) = (J, g, L_1, L_2)$ and such that for $t \neq 0$ the ω, g_t are J_t-analytic.*

Remark 5.2 **(i)** The theorem is of local nature so in fact we need the g-Maslov-zero condition and the real analyticity condition only near $L_1 \cap L_2$.
(ii) The index in U may be defined without grading L_1', L_2'; that is, near S we write L_2' over L_1' as the graph of an exact 1-form df and we take the Morse indices of f. This definition agrees with that in Sect. 2 if L_1, L_2 are graded with the same grading on $L_1 \cap L_2$ and if L_1', L_2' are graded by the obvious homotopies.

There are a few key estimates which we use for the proof of Theorem 5.1. Firstly, according to Łojasiewicz [12] for every function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(0) = 0$ and *analytic* near $0 \in \mathbb{R}^n$ there exist two constants $c > 0$ and $p \in (0, 1)$ such that near $0 \in \mathbb{R}^n$ we have $|f|^p \leq c|df|$. This implies readily that we have:

Lemma 6 *In the circumstances of Theorem 5.1* (i) *take a Weinstein neighbourhood of $L_2 \subseteq (X, \omega)$ and write L_1 near S as the graph over L_2 of a closed 1-form u. Then near S we have $|u| \ll |\nabla u| \ll 1$; that is, for every $\epsilon > 0$ there exists a neighbourhood of $S \subseteq L_2$ on which $|u| \leq \epsilon|\nabla u| \leq \epsilon^2$. Here $| \bullet |, \nabla$ are both computed with respect to the induced metric on L_2.*

We show next that the Maslov 1-form operator may be well approximated by dd^*:

Lemma 7 *Let (X, ω, J) be a Kähler manifold and let g be a Hermitian metric on (X, J). Let $L \subseteq X$ be a closed immersed g-Maslov-zero Lagrangian and let NL be the g-normal bundle to $L \subseteq X$. Then there exists a constant $c > 0$ such that for every sufficiently C^1-small $v \in C^\infty(NL)$ whose graph (embedded in X by the g-exponential map) is g-Maslov-zero, we have at every point of L*

$$|dd^* \hat{v}| \leq c|v|_{C^1}(1 + |v|_{C^2}) \tag{14}$$

where $\hat{v}$ is the same as in Lemma 4, $|v|_{C^1} := |v| + |\nabla v|$ and $|v|_{C^2} := |v| + |\nabla v| + |\nabla^2 v|$; the latter two are computed pointwise on L with respect to g.

Proof Let $v \in C^\infty(NL)$ be sufficiently C^1-small and $Av \in C^\infty(T^*L)$ the Maslov 1-form of the graph of v. The tangent space to the graph of v involves the first derivative ∇v so the canonical section of K_X^2 over the graph of v is of the form $B(v, \nabla v)$ where $B : NL \oplus (T^*L \otimes NL) \to NL$ is a smooth function. Then Av involves the differentiation of $B(v, \nabla v)$ so

$$Av = A_1(v, \nabla v) + A_2(v, \nabla v) \cdot \nabla^2 v \tag{15}$$

where $A_1 : NL \oplus (T^*L \otimes NL) \to NL$ and $A_2 : NL \oplus (T^*L \otimes NL) \to NL \otimes TL \otimes TL$ are some smooth functions. Denote by A' the linearization of A at $0 \in C^\infty(NL)$. Then by (15) there is a v-independent constant $c_1 > 0$ such that

$$|(A - A')v| \leq c_1[|v|^2 + |\nabla v|^2 + |\nabla^2 v|(|v| + |\nabla v|)] \leq c_1|v|_{C^1}|v|_{C^2} \tag{16}$$

where the same constant c_1 will do for both the inequalities. Now by Lemma 4 A' is, up to lower-order terms, equal to $dd^*\hat{v}$; that is, there is a v-independent constant $c_2 > 0$ such that

$$|A'v - dd^*\hat{v}| \leq c_2|v|_{C^1}.$$

So by (16) we have $|(A - A')v| \leq \max\{c_1, c_2\}|v|_{C^1}(1 + |v|_{C^2})$. This implies indeed that if $Av = 0$ then (14) holds. □

We give a corollary to the lemma above:

Corollary 5.3 *In the circumstances of Lemma 7 take a Weinstein neighbourhood of $L \subseteq X$. Then there exists a constant $c > 0$ such that for every sufficiently C^1-small 1-form $u \in C^\infty(T^*L)$ whose graph is also minimal, we have at every point of L*

$$|dd^*u| \leq c|u|_{C^1}(1 + |u|_{C^2}). \tag{17}$$

Proof The two embeddings, one given by the exponential map and the other given by the Weinstein neighbourhood, are mutually related by a diffeomorphism F between neighbourhoods of the zero sections of T^*L, NL respectively which induces the identity on the zero section L, and over every point of L, a diffeomorphism of the two fibres. So there is a smooth function $F_1 : T^*L \to NL \otimes TL$ such that for every sufficiently C^1-small $u \in C^\infty(T^*L)$ we have $Fu = F_1 u \cdot u$. In particular there is a u-independent constant $c > 0$ such that

$$|Fu|_{C^1} \leq |u|_{C^1} \text{ and } |Fu|_{C^2} \leq c|u|_{C^2}. \tag{18}$$

The function $Gu := -JFu \lrcorner g$ is also of the same form; that is, there is a smooth function $G_1 : T^*L \to T^*L \otimes TL$ such that $Gu = G_1 u \cdot u$. This implies in turn that we have, making c large enough,

$$|dd^*Gu| \leq c|dd^*u| + c|u|_{C^1}(1 + |u|_{C^2}) \tag{19}$$

Now by (14) we have, making c large enough,

$$|dd^*Gu| \leq c|Fu|_{C^1}(1 + |Fu|_{C^2}).$$

So by (19) we have (17) with c large enough. □

Now we prove Theorem 5.1 in three steps:

Proof (Step 1: proof of the first part of (i)) In this case we take $L_1' = L_1$ and perturb only L_2. We take a Weinstein neighbourhood of $L_2 \subseteq (X, \omega)$ and define L_2' as the graph of a certain exact 1-form df on L_2 with $f : L_2 \to \mathbb{R}$ taken as follows. Since L_1, L_2 are both connected and analytic with $L_1 \neq L_2$ as subsets of X it follows that L_2 L_1 is dense in L_2. So there is a Morse function $f : L_2 \to \mathbb{R}$ with $df \neq 0$ on $L_1 \cap L_2$. Since $S \subseteq L_1 \cap L_2$ it follows then that there is a constant $c > 0$ such that we have near S

$$|f|_{C^2} < c|df|. \tag{20}$$

This condition is preserved under rescalings and smooth perturbations of f so we can make L_2' arbitrarily C^∞ close to L_2 and transverse to L_1.

Near S take a local component of L_1 if need be (in the strictly immersed case) and write it as the graph over L_2 of some closed 1-form u. This is possible by the definition of S and in fact we have also $u = \nabla u = 0$ on S. So by Lemma 6 we have near S

$$|u| \ll |\nabla u| \ll 1. \tag{21}$$

On the other hand, by Corollary 5.3 with L_2 in place of L, there is a constant $c' > 0$, which depends upon $|u|_{C^2}$, such that we have near S

$$|dd^*u| \le c'(|u| + |\nabla u|) \le 2c'|\nabla u| \ll 1 \tag{22}$$

where the second inequality follows from (21). Applying to d^*u the Łojasiewicz estimate, we find a constant $\epsilon > 0$ such that we have near S

$$|d^*u| \le |dd^*u|^{1+\epsilon} \le (2c'|\nabla u|)^{1+\epsilon} \ll |\nabla u| \tag{23}$$

where the second inequality follows from (22).

We show now that we have on $L_1 \cap L_2'$ near S

$$|d^*(u - df)| \le n^{-1/2}|\nabla(u - df)|. \tag{24}$$

On $L_1 \cap L_2'$ near S we have $u = df$ and so by (20)

$$|\nabla(u - df)| \ge |\nabla u| - |\nabla df| \ge |\nabla u| - c|df| = |\nabla u| - c|u| \ge \frac{1}{2}|\nabla u| \tag{25}$$

where the last inequality follows from (21). On the other hand by (23) we have on $L_1 \cap L_2'$ near S

$$|d^*(u - df)| \le |d^*u| + c|df| \le \frac{1}{4}n^{-1/2}|\nabla u| + c|u| \le \frac{1}{2}n^{-1/2}|\nabla u| \tag{26}$$

where the last inequality follows again from (21). By (25) and (26) we have (24) as we want.

Finally we take a point $x \in L_1 \cap L_2'$ near S and compute its index, which is the index of the nondegenerate symmetric bilinear form $\nabla(u - df)$ on $T_x L_2$. Diagonalize it by an orthonormal basis of $T_x L_2$ and denote by $\lambda_1, \dots, \lambda_n$ the diagonal entries. Then by (24) we have

$$|\lambda_1 + \cdots + \lambda_n| \le n^{-1/2}(\lambda_1^2 + \cdots + \lambda_n^2)^{1/2} \le \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

But $\lambda_1, \dots, \lambda_n$ are all nonzero so they have not all the same sign; that is, the index is neither 0 nor n as we want. □

Proof (Step 2: proof of the latter part of (i)) We make first a generic Hamiltonian perturbation of L_2 which underlies an object of $H\mathcal{F}(X)$ and we define L_2' as a further Hamiltonian perturbation of it. As in step 1 we take a Weinstein neighbourhood of L_2 and define L_2' as the graph of some exact 1-form dh_2 over L_2 with $h_2 \neq 0$ on S. Also as in step 1 write L_1 near S as the graph of some real analytic closed 1-form u on L_2 near S. Take a generic Hamiltonian perturbation L_1' of L_1 which underlies $H\mathcal{F}(X)$ and is close enough to L_1 to be written near S as the graph of $u + dh_1$ over L_2 with

$$|h_1|_{C^2} \ll |h_2|_{C^2}. \tag{27}$$

Since $dh_2 \neq 0$ on S it follows that there is a constant $c > 0$ such that near S we have $|h_2|_{C^2} < \frac{c}{4}|dh_2|$. And then putting $f := h_2 - h_1$ we have

$$|h_2|_{C^2} < \frac{c}{4}(|df| + |dh_1|) \le \frac{c}{4}|df| + \frac{1}{2}|h_2|_{C^2} \tag{28}$$

where the last inequality follows from (27). The whole estimate (28) implies $|h_2|_{C^2} < \frac{c}{2}|df|$. Hence using again (27) we find

$$|f|_{C^2} \le |h_2|_{C^2} + |h_1|_{C^2} < \frac{c}{2}|df| + |h_2|_{C^2} < c|df|;$$

that is, the estimate (20) holds as in step 1. Following the subsequent estimates we see also that (24) holds now on $L_1' \cap L_2'$ near S. This implies again that L_1', L_2' have near S no intersection point of index 0 or n. □

Proof (Step 3: proof of Theorem 5.1 (ii)) As in step 2 we make Hamiltonian perturbations L_1', L_2' of L_1, L_2 both underlying $H\mathcal{F}(X)$. The L_2' is the graph of dh_2 over L_2 and the L_1' near S is the graph of $u + dh_1$ over L_2 near S. The difference $f := h_2 - h_1$ satisfies (20) as in step 1. We extend these to smooth families. By Theorem 3.3 there are g_t-Maslov-zero Hamiltonian perturbations L_{1t}, L_{2t} of L_1, L_2. Extend the Weinstein neighbourhood of L_2 to those of L_{2t} and write L_2' as the graph of some dh_{2t} over L_{2t}. Denote by $S_t \subseteq L_{1t} \cap L_{2t}$ the set of intersection points at which L_{1t}, L_{2t} have at least one common tangent space. Write L_{1t} near S_t as the graph of a real analytic closed 1-form u_t over L_{2t} near S_t and write L_1' near S_t as the graph of $u_t + dh_{1t}$ over L_{2t} near S_t. Then with t small enough the estimate (20) holds for $f_t := h_{2t} - h_{1t}$ in place of f and we can follow the subsequent estimates in step 1. We can do this in a t-independent neighbourhood U of $S \subseteq X$ because S_t tends to S. So L_1', L_2' have in U no intersection point of index 0 or n. □

6 Conclusions

From Lemma 2 and Theorem 5.1 we deduce:

Theorem 6.1 *Let (X, ω, J) be a real analytic Kähler manifold of complex dimension n, $\mathbb{Z}_k$-graded with $k \geq n$ and given another real analytic Hermitian metric g. Let $L_1, L_2 \subseteq X$ two distinct closed irreducibly-immersed g-Maslov-zero Lagrangians with the same grading on $L_1 \cap L_2$. Then there exist arbitrarily-small Hamiltonian deformations L_1', L_2' of L_1, L_2 which intersect generically each other with no intersection point of index 0 or n. If also Hypothesis 1 holds and L_1, L_2 nearly underlie objects of $H\mathcal{F}(X)$ then we can make L_1', L_2' underly objects of $H\mathcal{F}(X)$.*

Proof Let U be as in Theorem 5.1 and corresponding to this U let ϵ be as in Lemma 2. Then corresponding to this ϵ let L_i' $(i = 1, 2)$ be such ϵ-perturbations of L_i as in Theorem 5.1. Applying Lemma 2 to the intersection points outside U and Theorem 5.1 to those in U we see that L_1', L_2' are such as we want. □

We make now the C^∞ version of Theorem 6.1:

Theorem 6.2 *Let (X, ω, J) be a Kähler manifold of complex dimension n which is either: $\mathbb{Z}$-graded by a J-holomorphic volume form Ψ and given the conformally Kähler metric g as in §2 by using (6); or $\mathbb{Z}_k$-graded with $k \geq n$ and given another Kähler metric g whose the Ricci $(1, 1)$-form is $-\omega$. Let $L_1, L_2 \subseteq (X, \omega, J, g)$ be two distinct closed irreducibly-immersed g-Maslov-zero Lagrangians with the same grading on $L_1 \cap L_2$. Suppose that one of the following three conditions holds:* **(i)** *X is compact;* **(ii)** *X is ω-convex, ω is exact and J is Stein; or* **(iii)** *either L_1 or L_2 is generically immersed. Then the same conclusion holds as in Theorem 6.1 (both the two sentences in it).*

Proof We reduce the problem to the real analytic case by a further perturbation process. In the case (i) we use the fact that for every compact Kähler manifold (X, ω, J) there exists on (X, J) a smooth family $(\omega_t)_{t \in \mathbb{R}}$ of Kähler forms with $\omega_0 = \omega$ and ω_t, $t \neq 0$, analytic. This seems well known but for clarity we give it a proof. Denote by g_ω the Kähler metric of (X, ω, J) and take on (X, J) a smooth family of *Riemannian* metrics g_t' with $g_0' = g_\omega$ and g_t', $t \neq 0$, analytic. Then $g_t := \frac{1}{2}(g_t' + J^* g_t')$ defines on (X, J) a smooth family of Hermitian metrics with $g_0 = g_\omega$ and g_t, $t \neq 0$, analytic. Denote by ∂_t^*, $\bar\partial_t^*$ the respective formal g_t-adjoints of ∂, $\bar\partial$ and following Kodaira and Spencer [11, Sect. 6] introduce the smooth family of elliptic operators

$$E_t := \partial\bar\partial\bar\partial_t^*\partial_t^* + \bar\partial_t^*\partial_t^*\partial\bar\partial + \bar\partial_t^*\partial\partial_t^*\bar\partial + \partial_t^*\bar\partial\bar\partial_t^*\partial + \partial_t^*\partial + \bar\partial_t^*\bar\partial.$$

Since g_0 is Kähler it follows readily that E_0 is apart from the last two terms just the squared Laplacian. The last two terms are so added that the kernel of E_t consists of closed forms. Denote by ω_t' the projection of ω onto the kernel of E_t in the space of $(1, 1)$-forms. Then by Kodaira–Spencer's result [11, Proposition 8] the kernel of E_t has dimension independent of t. So by another result of them [11, Theorem 5] ω_t' is smooth with respect to t. Putting $\omega_t := \frac{1}{2}(\omega_t' + \overline{\omega}_t')$ we see that ω_t is such as we want.

In the case (ii) we write $\omega = dJdp$ for some $p : X \to \mathbb{R}$ and perturb p to a real analytic function. In the case (iii) we take a Stein open set in (X, J) on which ω is exact and which contains the generically immersed Lagrangian, say L_1. This is possible because every self-intersection point of L_1 is a transverse double point; the Stein structure near it is given by taking the product of squared distances from the two components of the generically immersed Lagrangian, which is a plurisubharmonic function.

In every case there are Kähler perturbations ω_t of ω and by Moser's theorem diffeomorphisms $\Phi_t : X \to X$ with $\omega = \Phi_t^* \omega_t$. Then ω is analytic with respect to $J_t := \Phi_t^* J$ and for $i = 1, 2$ the Lagrangian $\Phi_t^* L_i \subseteq (X, \omega)$ is a nearly-zero Maslov-form relative to $g_t := \Phi_t^* g$. So by Theorem 3.3 there is a g_t-Maslov-zero Hamiltonian perturbation L_{it} of $\Phi_t^* L_i$ to which we can apply Theorem 5.1 (ii). Combining it with Lemma 2 as in the proof of Theorem 6.1 we see that Theorem 6.2 holds. □

Finally using the results of Sect. 4 we get:

Corollary 6.3 **(i)** *Let $(X, \omega, J, g, L_1, L_2)$ be such as in Theorem 6.1 or Theorem 6.2 except that $L_1, L_2 \subseteq X$ may be equal. Let Hypothesis 1 hold and let $\mathbf{b}_1, \mathbf{b}_2 \in H\mathcal{F}(X)$ be two objects supported respectively near L_1, L_2 such that:*

$$\textit{either} \mathbf{b}_1 \cong \mathbf{b}_2 \textit{or} k \geq 2n - 1 \textit{and} \mathbf{b}_1 \cong \mathbf{b}_2 \textit{up to shift}. \tag{29}$$

Suppose also that

$$HF^i(\mathbf{b}_1, \mathbf{b}_1) \cong HF^i(\mathbf{b}_2, \mathbf{b}_2) \neq 0 \textit{ for } i = 0, n. \tag{30}$$

More generally, suppose as in Corollary 4.4 that either: $c_1(X) \leq 0$ and one of L_1, L_2 is embedded; or $k \geq n + 2$ and one of L_1, L_2 has on its every self-intersection pair the two gradings nearly equal. Then $L_1 = L_2 \subseteq X$.
(ii) *Let (X, ω, J) be a Kähler manifold equipped with a holomorphic volume form; and $L_1, L_2 \subseteq X$ two closed irreducibly-immersed special Lagrangians. Suppose that either ω is J-analytic or one of* (i)–(iii) *of Theorem 6.2 holds. Let Hypothesis 1 hold and let $H\mathcal{F}(X)$ have two isomorphic objects supported respectively near L_1, L_2. Then $L_1 = L_2 \subseteq X$.*

Proof Suppose contrary to the assertion that $L_1 \neq L_2$. Then perturbing L_1, L_2 as in Theorem 6.1 or Theorem 6.2 we see that $HF^*(\mathbf{b}_1, \mathbf{b}_2)$ is supported in degrees $1, \ldots, n-1$ modulo $k\mathbb{Z}$. But by hypothesis we have $k \geq n$ so $HF^0(\mathbf{b}_1, \mathbf{b}_2) = HF^n(\mathbf{b}_1, \mathbf{b}_2) = 0$. If also $\mathbf{b}_1 \cong \mathbf{b}_2$ then $HF^0(\mathbf{b}_1, \mathbf{b}_1) = HF^n(\mathbf{b}_1, \mathbf{b}_1) = 0$ which however contradicts (30) or Corollary 4.4. This is the first case of (29).

In the second case we have $k \geq 2n-1$ but $\mathbf{b}_1 \cong \mathbf{b}_2$ only up to shift. Then from Theorem 6.1 we see only that $HF^*(\mathbf{b}_1, \mathbf{b}_1)$ is supported in degrees $i, \ldots, i+n-2$ for some $i \in \mathbb{Z}$. But again we have $HF^0(\mathbf{b}_1, \mathbf{b}_1) \neq 0$ and $HF^n(\mathbf{b}_1, \mathbf{b}_1) \neq 0$ so after translating $[i, i+n-2]$ by $k\mathbb{Z}$ we may suppose $0 \in [i, i+n-2]$ and $n \in [i+j, i+j+n-2]$ for some $j \in k\mathbb{Z}$. Then $i \in [2-n, 0] \cap [2-j, n-j]$ so $0 \geq 2-j$ and $2-n \leq n-j$; that is, $j \in [2, 2n-2]$. So $0 < j < 2n-1 \leq k$ which however contradicts $j \in k\mathbb{Z}$. This proves Corollary 6.3 (i).

For Corollary 6.3 (ii) we use Corollary 4.5 (ii) to see that L_1, L_2 have up to shifts the same grading. We can then apply Corollary 6.3 (i) with $k = \infty$ which completes the proof. □

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References

1. Akaho, M., Joyce, D.: Immersed Lagrangian Floer theory. J. Differ. Geom. **86**, 381–500 (2010)
2. Fukaya, K.: Cyclic symmetry and adic convergence in Lagrangian Floer theory. Kyoto J. Math. **50**, 521–590 (2010)
3. Fukaya, K.: Unobstructed immersed Lagrangian correspondence and filtered A_∞ functor. arXiv:1706.02131
4. Fukaya, K., Oh, Y-G., Ohta, H., Ono, K.: Lagrangian intersection Floer theory—anomaly and obstruction. Parts I and II AMS/IP Studies in Advanced Mathematics 46.1 and 46.2 (2009)
5. Fukaya, K., Oh, Y.-G.: H Ohta and K Ono 'Lagrangian Floer theory on compact toric manifolds. I' Duke Math. J. **151**, 23–174 (2010)
6. Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K.: Lagrangian Floer theory over integers: spherically positive symplectic manifolds. Pure Appl. Math. Q. **9**, 189–289 (2013)
7. Harvey, R., Lawson Jr, H.B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)
8. Imagi, Y., Joyce, D., Oliveira dos Santos, J.: Uniqueness results for special Lagrangians and Lagrangian mean curvature flow in $\mathbb{C}^m$. Duke Math. J **165**, 847–933 (2016)
9. Joyce, D., Riemannian holonomy groups and calibrated geometry. Oxford Graduate Texts in Mathematics, vol. 12 (2007)
10. Joyce, D.: Conjectures on Bridgeland stability for Fukaya categories of Calabi–Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow. EMS Surv. Math. Sci. **2**, 1–62 (2015)
11. Kodaira, K., Spencer, D.C.: On deformations of complex analytic structures, III. Stability theorems for complex structures. Ann. Math. **71**, 43–76 (1960)
12. Łojasiewicz, S.: Ensembles semi-analytiques. Inst. Hautes Études Sci. (1965)
13. Lotay, J.D., Pacini, T.: From minimal Lagrangian to J-minimal submanifolds: persistence and uniqueness. Boll. Unione Mat. Ital. **12**, 63–82 (2019)
14. Lotay, J.D., Pacini, T.: From Lagrangian to totally real geometry: coupled flows and calibrations. Comm. Anal. Geom. 607–675 (2020)

15. McLean, R.C.: Deformations of calibrated submanifolds. Commun. Anal. Geom. **6**, 705–747 (1998)
16. Seidel, P.: Graded Lagrangian submanifolds. Bull. Soc. Math. Fr. **128**, 103–149 (2000)
17. Thomas, R.P., Yau, S.-T.: Special Lagrangians, stable bundles and mean curvature flow. Commun. Anal. Geom. **10**, 1075–1113 (2002)

Birationally Rigid Complete Intersections of Codimension Three

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Abstract We prove that the complement to the set of birationally superrigid Fano complete intersections of index 1 and codimension 3 in $\mathbb{P}^{M+3}$ is at least

$$\frac{1}{2}(M-10)(M-11)-2$$

for $M \geq 30$.

Keywords Rationality · Fano varieties · Birational rigidity

1 Statement of the Main Result

Let $M \geq 30$ be an integer and $\mathbb{P} = \mathbb{P}^{M+3}$ the complex projective space. For any integral triple $\underline{d} = (d_1, d_2, d_3)$, such that

$$2 \leq d_1 \leq d_2 \leq d_3,$$

and

$$|d| = d_1 + d_2 + d_3 = M + 3,$$

let

$$\mathcal{P}(\underline{d}) = \prod_{i=1}^{3} \mathcal{P}_{d_i, M+4}$$

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be the space of triples (f_1, f_2, f_3) of homogenous polynomials of degree d_1, d_2, d_3, respectively, in the homogeneous coordinates $(x_0 : x_1 : \cdots : x_{M+3})$ on the projective space $\mathbb{P}$.

For a triple $\underline{f} = (f_1, f_2, f_3)$, the scheme of common zeros, defined by the polynomials f_1, f_2, f_3, is denoted by $V(\underline{f})$. The following claim is the main result of this paper.

Theorem 1 *There exists a Zariski open subset $\mathcal{P}_{\text{reg}}(\underline{d}) \subset \mathcal{P}(\underline{d})$ such that:*

(i) for every triple $\underline{f} \in \mathcal{P}_{\text{reg}}(\underline{d})$, the scheme $V(\underline{f})$ is an irreducible reduced factorial complete intersection of codimension 3 with at most terminal singularities, and a Fano variety of dimension M and index 1,
(ii) the Fano variety $V(\underline{f})$ is birationally superrigid, for $\underline{f} \in \mathcal{P}_{\text{reg}}(\underline{d})$,
(iii) the estimate

$$\operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\text{reg}}(\underline{d})\right) \subset \mathcal{P}(\underline{d})\right) \geq \frac{(M-10)(M-11)}{2} - 2$$

holds.

We call the claim above the *effective birational superrigidity*, because it includes an effective estimate of the codimension of the complement to the set of birationally superrigid varieties. Effective birational superrigidity was shown for hypersurfaces in [1], for complete intersections of codimension 2 in [2], and for complete intersections of codimension ≥20 in [3]. Thus far, there have been no effective results (in the sense above) for complete intersections of codimension 3, 4, . . . , 19. This paper treats the case of codimension 3.

Birational superrigidity of a Zariski general complete intersection of index one without an estimate for the codimension of the set of non-rigid varieties was shown for the overwhelming majority of families in [4–6]. In [15], birational supperigidity and K-stability were shown for singular Fano complete intersections of index one. Later on in [16], it was shown that every smooth Fano complete intersection of index one and codimension r in $\mathbb{P}^{n+r}$ is birationally supperigid and K-stable for $n \geq 10r$.

Obtaining a strong explicit estimate requires a study of singularities of complete intersections. We use this approach in the present work.

2 Multi-quadratic Singularities

Take an arbitrary point $o \in \mathbb{P}$ and assume that f_1, f_2, and f_3 vanish there. We fix a system of affine coordinates $z_* = (z_1, \ldots, z_{M+3})$ on an affine chart $\mathbb{C}^{M+3} \subset \mathbb{P}$ with the origin at o, and write the corresponding dehomogenized polynomials f_1, f_2, f_3 (for simplicity of notation, we use the same symbols we used for the original polynomials to denote their dehomogenizations):

$$
\begin{aligned}
f_1 &= q_{1,1} + q_{1,2} + \cdots + q_{1,d_1} \\
f_2 &= q_{2,1} + q_{2,2} + \cdots + q_{1,d_2} \\
f_3 &= q_{3,1} + q_{3,2} + \cdots + q_{3,d_3},
\end{aligned} \tag{1}
$$

where $q_{i,j}$ is a homogeneous polynomial of degree j in z_*. If the linear forms

$$q_{i,1}, \quad i = 1, 2, 3,$$

are linearly independent, then the scheme of zeros $V(\underline{f})$ near the point o is a non-singular (irreducible and reduced) complete intersection of codimension 3.

Assume that

$$\dim\langle q_{1,1}, \ldots, q_{3,1}\rangle = 3 - l,$$

where $l \in \{1, 2, 3\}$. As described in [3], we say that $o \in V(\underline{f})$ is a *correct multi-quadratic singularity* of type 2^l, if one (and only one) of the conditions below is satisfied:

(1) $l = 1$ (quadratic singularities):

$$\dim\langle q_{1,1}, q_{2,1}, q_{3,1}\rangle = 2,$$

say, for certainty, $q_{1,1}$ and $q_{2,1}$ are linearly independent and

$$q_{3,1} = \lambda_{3,1} q_{1,1} + \lambda_{3,2} q_{2,1};$$

the rank of the quadratic form

$$\left(q_{3,2} - \lambda_{3,1} q_{1,2} - \lambda_{3,2} q_{2,2}\right)\Big|_{\{q_{1,1} = q_{2,1} = 0\}}$$

is at least 7.

(2) $l = 2$ (bi-quadratic singularities):

$$\dim\langle q_{1,1}, q_{2,1}, q_{3,1}\rangle = 1,$$

say, for certainty, $q_{1,1} \neq 0$,

$$
\begin{aligned}
q_{2,1} &= \lambda_{2,1} q_{1,1} \\
q_{3,1} &= \lambda_{3,1} q_{1,1}
\end{aligned}
$$

for some $\lambda_{2,1}, \lambda_{3,1} \in \mathbb{C}$; the system of quadratic equations

$$(q_{2,2} - \lambda_{2,1} q_{1,2})\Big|_{\{q_{1,1}=0\}} = (q_{3,2} - \lambda_{3,1} q_{1,2})\Big|_{\{q_{1,1}=0\}} = 0$$

defines in $\mathbb{P}(\{q_{1,1} = 0\}) = \mathbb{P}^{M+1}$, an irreducible reduced complete intersection of two quadrics, the singular locus of which is of codimension at least 12 in $\mathbb{P}^{M+1}$.

(3) $l = 3$ (multi-quadratic singularities): all forms $q_{i,1} = 0$, $i = 1, 2, 3$, and the three quadratic equations

$$q_{1,2} = q_{2,2} = q_{3,2} = 0$$

define in $\mathbb{P}^{M+2}$ an irreducible reduced complete intersection of three quadrics, the singular locus of which is of codimension at least 14 in $\mathbb{P}^{M+2}$.

We treat the case when the three linear forms are linearly independent, that is the non-singular case, as the multi-quadratic singularity of type 2^0. Also, by $\mathcal{P}_{\text{mq}}(\underline{d})$, we denote the set of all triples $\underline{f} \in \mathcal{P}(\underline{d})$ such that every point o where f_1, f_2 and f_3 vanish is a multi-quadratic singularity of type 2^l. It is obvious that for $\underline{f} \in \mathcal{P}_{\text{mq}}(\underline{d})$, the scheme of zeros $V(\underline{f})$ is an irreducible reduced complete intersection of codimension 3 in $\mathbb{P}$. By [7], the variety $V(\underline{f})$ is factorial, and by [3], its singularities are terminal.

The following fact is a particular case of Theorem 0.2 in [3].

Theorem 2 *The following estimate holds*

$$\operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\text{mq}}(\underline{d})\right) \subset \mathcal{P}(\underline{d})\right) \geq \frac{(M-10)(M-11)}{2} - 2.$$

Now for a triple $\underline{f} \in \mathcal{P}_{\text{mq}}(\underline{d})$, the claim (i) of Theorem 1 is satisfied. We will construct inside $\mathcal{P}_{\text{mq}}(\underline{d})$, a smaller open subset, by removing subsets, the codimension of which is at least

$$\frac{(M-10)(M-11)}{2} - 2.$$

For triples in that smaller subset, the claim (ii) of Theorem 1 will be satisfied, which will complete the proof of our main result.

3 The Regularity Conditions

To prove birational superrigidity, we impose some conditions of general position, called the regularity conditions, on the triples in $\mathcal{P}(\underline{d})$.

Consider a triple $\underline{f} = (f_1, f_2, f_3) \in \mathcal{P}_{\text{mq}}(\underline{d})$ and let $o \in V(\underline{f})$ be a point. Set

$$\Pi = \{q_{1,1} = q_{2,1} = q_{3,1} = 0\} \subset \mathbb{C}^{M+3}$$

in terms of the presentation (1). For $i \in \{1, 2, 3\}$, and $j > 2$, we arrange the forms $q_{i,j}$ in the lexicographic order for the pairs (i, j). That is, that (i_1, j_1) precedes (i_2, j_2), if

$$j_1 < j_2 \text{ or}$$

$$j_1 = j_2 \text{ and } i_1 < i_2.$$

What results is the sequence

$$q_{1,2}|_\Pi, q_{2,2}|_\Pi, q_{3,2}|_\Pi, \ldots, q_{3,d_3}|_\Pi. \qquad (2)$$

We say that the triple $\underline{f}$ is *regular* at the point o, if the sequence above, with the last 3 polynomials removed, is regular in $\mathcal{O}_{\Pi,o}$. Furthermore, we say that the triple $\underline{f}$ is *regular* if it is regular at every point in $V(\underline{f})$. Denote the set of regular triples $\underline{f} \in \mathcal{P}_{\mathrm{mq}}(\underline{d})$ by the symbol $\mathcal{P}_{\mathrm{reg}}(\underline{d})$. By construction, $\mathcal{P}_{\mathrm{reg}}(\underline{d}) \subset \mathcal{P}_{\mathrm{mq}}(\underline{d})$ is a Zariski open subset.

Theorem 3 *For every triple $\underline{f} \in \mathcal{P}_{\mathrm{reg}}(\underline{d})$, the Fano variety $V(\underline{f})$ is birationally superrigid.*

What remains for the completion of the proof of Theorem 1, is to prove Theorem 3, and the inequality in part (iii) of Theorem 1.

4 Birational Superrigidity

For a regular triple $\underline{f}$, we prove the birational superrigidity of the variety $V = V(\underline{f})$ in almost exactly the same way as in Sect. 1 of [3]. That is, we first assume that V is not birationally superrigid. Then there is a mobile linear system $\Sigma \subset |nH|$ (where H is the class of a hyperplane section of $V \subset \mathbb{P}$), and an exceptional divisor E over V, satisfying the Noether-Fano inequality

$$\mathrm{ord}_E\,\Sigma > na(E).$$

Furthermore, let $B \subset V$ be the centre of E on V. Then we note that by [3, Sect. 1, Lemma 1.1],

$$\mathrm{codim}(B \subset V) \geq 3.$$

Assume first that $B \not\subset \mathrm{Sing}\, V$, and take a general point $o \in B$, which is non-singular on V. Arguing in word for word the same way as in [3, Sect. 1.3], we construct, using the technique of hypertangent divisors, a sequence of irreducible subvarieties

$$Y_2, Y_3, \ldots, Y_{M-3}$$

of V, where Y_2 is a component of the self-intersection $Z = (D_1 \circ D_2)$ of the mobile system Σ (where D_1 and D_2 are general divisors in Σ) with the maximal value of the ratio $\mathrm{mult}_o / \deg$, and where $\mathrm{codim}(Y_i \subset V) = i$. For the last subvariety Y_{M-3} in this sequence, we get the estimate

$$\frac{\mathrm{mult}_o\, Y_{M-3}}{\deg_o Y_{M-3}} > \frac{4}{3\beta},$$

where β is the product of the slopes of the last 3 omitted hypertangent divisors (see proof of Proposition 1.3 in [3]). The only part of the proof given in [3, Sect. 1.3] that needs to be modified is Lemma 1.3, and this takes the form of Lemma 4 below.

In the case considered in the present paper, we observe that β has the highest value when one of the following three options takes place:

(0) $d_1 = d_2 = d_3$ and $M \equiv 0 \bmod 3$,
(1) $d_1 = d_2 = d_3 - 1$ and $M \equiv 1 \bmod 3$,
(2) $d_1 + 1 = d_2 = d_3$ and $M \equiv 2 \bmod 3$,

as in these cases the slopes $\beta_{M-2}, \beta_{M-1}, \beta_M$ of the 3 omitted hypertangent divisors, (see the proof of Proposition 1.3 in [3]) take the highest values.

Lemma 4 *The inequality* $4 \geq 3\beta$ *holds.*

Proof Supposing the scenario (0) takes place, we have

$$\beta_{M-2} = \beta_{M-1} = \beta_M = \frac{\frac{M}{3} + 1}{\frac{M}{3}},$$

so that

$$\beta = \frac{(M+3)^3}{M^3},$$

and the inequality $4 \geq 3\beta$ takes the form

$$M^3 - 27M^2 - 81M - 81 \geq 0.$$

The highest root of the polynomial on the left hand side is 29.80850, so (because $M \equiv 0 \bmod 3$) the inequality is satisfied for $M \geq 30$.

If scenario (1) occurs, then $4 \geq 3\beta$ for $M \geq 31$, and in the case of the option (2), the lemma holds for $M \geq 32$. Since $M \geq 30$ by our general assumption, the proof of Lemma 4 is complete. Therefore

$$\mathrm{mult}_o\, Y_{M-3} > \deg Y_{M-3},$$

which is impossible. This contradiction excludes the case $B \not\subset \mathrm{Sing}\, V$.

If $B \subset \mathrm{Sing}\, V$, then a general point $o \in B$ is either a quadratic, bi-quadratic, or multi-quadratic singularity of V. Then we argue as in [3, Sect. 1.4], while using the $4n^2$-inequality for complete intersection singularities shown in [8]. In each of these singular settings, the computations involving the slopes of hypertangent divisors are similar to those in the non-singular case and lead to a contradiction in exactly the same way. Therefore, we have shown that the mobile linear system Σ cannot have a maximal singularity. This completes the proof of Theorem 3. □

5 Codimension of the Complement

The final major task is to prove the claim (iii) of Theorem 1. We will show that for M large enough, the codimension of the complement $\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\mathrm{reg}}(\underline{d})$ is given by the same quadratic polynomial in M. We use the projection method explained, for instance in Chap. 3 of [9] to estimate the codimension of triples (f_1, f_2, f_3) such that $V(\underline{f})$ does not satisfy the regularity condition. This is the most difficult part of the work, because while the arguments of [3, Sect. 3] produce a similar estimate, they use a somewhat different approach from the one we use here.

First fix a point $o \in V$. We start in (word for word) the same way as in [3, Sect. 3.1]: the sequence (2) (see p. 4) with the last 3 polynomials removed, consists of

$$(d_1 - 1) + (d_2 - 1) + (d_3 - 1) - 3 = M - 3$$

homogeneous polynomials on $\Pi = \mathbb{C}^{M+l}$. Let us consider these as polynomials on $\mathbb{P}(\Pi) \cong \mathbb{P}^{M+l-1}$, and denote them by

$$g_1, \dots, g_{M-3}.$$

Set $m_i = \deg g_i$, so that

$$m_1 = m_2 = m_3 = 2,$$

and the degrees m_i are non-decreasing. Following [3, Sect. 3.1], we define the space of all such sequences

$$\mathcal{G}(\underline{d}, l) = \prod_{i=1}^{M-3} \mathcal{P}_{m_i, M+l}$$

and the closed set

$$\mathcal{Y} = \mathcal{Y}(\underline{d}, l) \subset \mathcal{G}(\underline{d}, l)$$

of non-regular sequences.

Theorem 5 *For $M \geq 30$ the inequality*

$$\operatorname{codim}\left(\mathcal{Y} \subset \mathcal{G}(\underline{d}, l)\right) \geq \frac{(M-10)(M-11)}{2} - 2 + M,$$

holds.

Proof Let

$$P = \frac{(M-10)(M-11)}{2} - 2 + M.$$

Now, analogously to what was described in the paragraph after the statement of Theorem 3.1 in [3, Sect. 3.1], Theorem 5 implies (iii) of Theorem 1. Now, arguing as in that subsection, we see that in order to show Theorem 5, it is sufficient to demonstrate that each of the following $M+3$ integers

$$\binom{M+l-e+m_e}{M+l-e}$$

is not smaller than the right hand side of Theorem 5 (see also [9, Chap. 3] for the details of the projection method). As in [3, Sect. 3.2], we use a number of reductions to simplify the task: first, [3, Sect. 3.1] allows us to consider only the options (0),(1), and (2), introduced above. Then [3, Proposition 3.3] tells us that we can only consider the non-singular case $l=0$. The final reduction, explained at the end of [3, Sect. 3.2], shows that the minimum of the integers

$$\binom{M-e+m_e}{M-e}$$

is attained for e either divisible by 3, or the very last one $e=M-3$. It is from here that our arguments cease to be identical to those of [3, Sect. 3].

We always write the binomial coefficients in the form $\binom{A}{B}$ with $B \leq [A/2]$ (replacing, if necessary, B by $A-B$), so we get a sequence of integers, the first part of which is

$$\binom{M-1}{2}, \binom{M-3}{3}, \binom{M-5}{4}, \binom{M-7}{5}, \ldots.$$

Each upper number is decreasing at every step, and each lower one increasing by 1 (recall that we are in one of the options (0),(1) or (2), so the degrees d_1, d_2, d_3 are equal or "almost equal"), until the bottom number gets to the half of the top one. More precisely, for

$$j \leq \frac{M-5}{4}, \quad j = 0, 1, 2, \ldots,$$

we have the integers

$$\binom{M-2j-1}{j+2}.$$

The second part of our sequence of binomial coefficients takes the form

$$\binom{M-2j-1}{M-3j-3}$$

for $j > \frac{M-5}{4}$ until the last term, which in the case of the option (0), is equal to

$$\binom{\frac{M}{3}+3}{3}, \tag{2}$$

in the case of option (1) is equal to

$$\binom{\frac{M+2}{3}+3}{3}, \tag{3}$$

and in the case of option (2) is equal to

$$\binom{\frac{M+1}{3}+2}{3}, \tag{4}$$

when we remove 3 hypertangent divisors.

Now we use the obvious property of a binomial coefficient $\binom{A}{B}$, that it is increasing when A is increasing, and when B is increasing while $B \leq [A/2]$, to easily check that the minimum of our sequence of integers is attained at one of the endpoints. Since $\binom{M-1}{2}$ is obviously higher than the right hand side of the inequality of Theorem 5, we only need to check that the integers (2)–(4) are not smaller than P for certain values of M.

We go on to identify, for $M \equiv 0, 1$, and 2 modulo 3, the final elements of the sequences of binomial coefficients which occur when we remove 3, 4 and 5 hypertangent divisors, and we compare the polynomials which we obtain from each sequence against P. This is done by directly observing where the graph of each polynomial lies with with respect to the others over various ranges of values of M. For instance, when the degrees are equal and we remove 4 hypertangent divisors, the polynomials

$$\binom{M-1}{2}, \binom{M-3}{3}, \binom{\frac{M}{3}+4}{4}$$

are greater than P for $M \geq 9$.

In the end we find that when M is congruent to 0, 1, and 2:

(i) if we remove the last 3 hypertangent divisors,

$$\operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\mathrm{reg}}(\underline{d})\right) \subset \mathcal{P}(\underline{d})\right) \geq P$$

for M not less than 9, 10, and 8 respectively.

(ii) when we remove 4 divisors, then

$$\operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\mathrm{reg}}(\underline{d})\right) \subset \mathcal{P}(\underline{d})\right) \geq P$$

for M not less than 9, 13 and 11 respectively.

(iii) when $a = 5$,

$$\operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\mathrm{reg}}(\underline{d})\right) \subset \mathcal{P}(\underline{d})\right) \geq P$$

for M not less than 18, 19 and 17 respectively.

Consequently, the proof of Theorem 5, and thus of Theorem 1, is complete. □

6 Concluding Remarks

Besides the papers cited above, the problem of birational superrigidity for singular Fano varieties was considered in many papers, see (for instance [10–12], or [13]). However, no estimates for the codimension of the set of non-rigid varieties were given there. While such estimates are interesting by themselves, they are especially important in birational geometry because they help to prove birational rigidity of Fano-Mori fibre spaces (see for instance [14]), since they show the existence of fibre spaces over a higher-dimensional base, every fibre of which is birationally superrigid.

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References

1. Eckl, T., Pukhlikov, A.V.: On the locus of nonrigid hypersurfaces, Automorphisms in birational and affine geometry. In: Springer Proceedings in Mathematics & Statistics, vol. 79, pp. 121–139. Springer, Cham (2014)
2. Evans, D., Pukhlikov. A.V.: Birationally Rigid Fano Complete Intersections of Codimension Two (2016). arXiv:1604.00512
3. Evans. D., Pukhlikov. A.V.: Birationally rigid Fano complete intersections of high codimension. Izvestiya: Math. **83**(4), 743 (2019)
4. Pukhlikov. A.V.: Birationally rigid Fano complete intersections. Crelle Journal für die reine und angewandte Mathematik **Issue 541**, 55–79 (2001)
5. Pukhlikov. A.V.: Birationally rigid Fano complete intersections. II. Crelle Journal für die reine und angewandte Mathematik **Issue 688**, 209–218 (2014)
6. Pukhlikov. A.V.: Birationally rigid complete intersections of quadrics and cubics. Izvestiya: Math. **77**(4), 795–845 (2013)
7. Call, F., Lyubeznik, G.: A simple proof of Grothendieck's theorem on the parafactoriality of local rings. Contemp. Math. **159**, 15–18 (1994)
8. Pukhlikov, A.V.: The $4n^2$-inequality for complete intersection singularities. Arnold Math. J. **3**(2), 187–196 (2017)
9. Pukhlikov, A.V.: Birationally Rigid Varieties. Mathematical Surveys and Monographs, p. 190. American Mathematical Society, Providence (2013)
10. Pukhlikov. A.V.: Birationally rigid Fano hypersurfaces with isolated singularities. Sbornik: Math. **193**(3), 445–471 (2002)

11. Cheltsov. I.A.: Birationally superrigid cyclic triple spaces. Izvestiya: Math. **68**(6), 1229–1275 (2004)
12. Cheltsov, I.A.: Double cubics and double quartics. Math. Z. **253**, 75–86 (2006)
13. Cheltsov. I.A.: Local inequalities and birational superrigidity of Fano varieties. Izvestiya: Math. **70**(3), 605 (2006)
14. Pukhlikov, A.V.: Birational automorphisms of Fano hypersurfaces. Invent. Math. **134**(2), 401–426 (1998)
15. Zhuang, Z., Liu, Y.: Birational superrigidity and K-stability of singular Fano complete intersections. Int. Math. Res. Not. **1**, 382–401 (2021)
16. Zhuang, Z., Liu, Y.: Birational superrigidity and K-stability of Fano complete intersections of Index One. Duke Math. J. **169**(1), 2205–2229 (2020)

Characterizing Terminal Fano Threefolds with the Smallest Anti-canonical Volume

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Abstract It was proved by Chen and Chen that a terminal Fano 3-fold X satisfies $(-K_X)^3 \geq \frac{1}{330}$. We show that a non-rational $\mathbb{Q}$-factorial terminal Fano 3-fold X with $\rho(X) = 1$ and $(-K_X)^3 = \frac{1}{330}$ is a weighted hypersurface of degree 66 in $\mathbb{P}(1, 5, 6, 22, 33)$.

Keywords Fano threefolds · Anti-canonical volumes

2000 Mathematics Subject Classification 14J45 · 14J30 · 14J17

1 Introduction

Throughout this paper, we work over the field of complex numbers $\mathbb{C}$.

A normal projective variety X is called a *Fano variety* (or $\mathbb{Q}$-*Fano variety* in some literature) if the anti-canonical divisor $-K_X$ is ample. A normal projective variety X is called a *weak Fano variety* if the anti-canonical divisor $-K_X$ is nef and big. A *terminal* (weak) Fano variety is a (weak) Fano variety with at worst terminal singularities.

According to the minimal model program, Fano varieties form a fundamental class among research objects of birational geometry. Motivated by the classification theory of 3-dimensional algebraic varieties, we are interested in the study of explicit geometry of terminal Fano 3-folds.

Given a terminal weak Fano 3-fold X, it was proved in [4, Theorem 1.1] that $(-K_X)^3 \geq \frac{1}{330}$. This lower bound is optimal, as it is attained when $X = X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$ is a general weighted hypersurface of degree 66. Moreover, [4, Theorem 1.1] showed that when $(-K_X)^3 = \frac{1}{330}$, then X has exactly the same virtual

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orbifold singularities and the same Hilbert series as X_{66} (see Proposition 2.1). So it is interesting to ask the following question:

Question 1.1 *Let X be a terminal (weak) Fano 3-fold with $(-K_X)^3 = \frac{1}{330}$. Is X (a $\mathbb{Q}$-Gorenstein deformation of) a quasi-smooth weighted hypersurface of degree 66 in $\mathbb{P}(1, 5, 6, 22, 33)$?*

Every quasi-smooth weighted hypersurface of degree 66 in $\mathbb{P}(1, 5, 6, 22, 33)$ is a $\mathbb{Q}$-factorial terminal Fano 3-fold with $\rho = 1$ by [8, 9] and is birationally rigid (and in particular, non-rational) by [3, 6]. So the main goal of this note is to give a partial answer to Question 1.1 in the category of non-rational $\mathbb{Q}$-factorial terminal Fano 3-folds with $\rho = 1$.

Theorem 1.2 *Let X be a terminal Fano 3-fold and $(-K_X)^3 = \frac{1}{330}$. Assume further that*

(1) X is $\mathbb{Q}$-factorial with $\rho(X) = 1$; and
(2) X is **non-rational***, that is, X is not birational to $\mathbb{P}^3$.*

Then X is a weighted hypersurface of degree 66 in $\mathbb{P}(1, 5, 6, 22, 33)$ defined by a weighted homogeneous polynomial F of degree 66, where

$$F(x, y, z, w, t) = t^2 + F_0(x, y, z, w)$$

in suitable homogeneous coordinates $[x : y : z : w : t]$ of $\mathbb{P}(1, 5, 6, 22, 33)$.

Remark 1.3 Theorem 1.2 shows that X is a $\mathbb{Q}$-Gorenstein deformation of general weighted hypersurfaces of degree 66 in $\mathbb{P}(1, 5, 6, 22, 33)$ (here the $\mathbb{Q}$-Gorensteinness follows from [7, Theorem B.1]). It is natural to ask whether X itself is quasi-smooth or not. In fact, by [2, P. 463], X is quasi-smooth if and only if X has only cyclic quotient singularities. From the basket B_X in Proposition 2.1, we know that all non-Gorenstein singularities of X are cyclic quotient singularities, but in general X might have Gorenstein terminal singularities (which are not cyclic quotient singularities).

The idea of the proof is as the following: as $X \simeq \operatorname{Proj} R(X, -K_X)$ where $R(X, -K_X)$ is the section ring of $-K_X$, it suffices to show that $R(X, -K_X) \simeq R(Y, \mathcal{O}_Y(1))$ for Y a weighted hypersurface of degree 66 in $\mathbb{P}(1, 5, 6, 22, 33)$. By [4, Theorem 1.1] (see Proposition 2.1), these two graded $\mathbb{C}$-algebras have the same dimension on homogeneous parts of each degree, but this is not sufficient to conclude that they are isomorphic as $\mathbb{C}$-algebras. The goal is to determine generators of $R(X, -K_X)$ and their relations.

The key ingredient is the special geometry of anti-pluri-canonical systems of $\mathbb{Q}$-factorial terminal Fano 3-fold with $\rho = 1$ proved in [5] (see Lemma 3.1) which was first observed by Alexeev [1] for anti-canonical systems, where the assumption that X is $\mathbb{Q}$-factorial with $\rho(X) = 1$ is essentially used. Also we note that the "non-rational" assumption in Theorem 1.2 is essential in our proof. In order to drop these assumptions, it is natural to ask the following question.

Question 1.4 *Let X be a terminal weak Fano 3-fold with $(-K_X)^3 = \frac{1}{330}$.*

(1) Is $|-6K_X|$ not composed with a pencil of surfaces?
(2) Is X non-rational?

If we have an affirmative answer to Question 1.4(1), then we can remove assumption (1) in Theorem 1.2, see Remark 3.2 for more details. If we have an affirmative answer to Question 1.4(2), then we can remove assumption (2) in Theorem 1.2.

Remark 1.5 The method in this note can be used to characterize other terminal Fano 3-folds. In fact, by the same method, it can be shown that, if X is a non-rational $\mathbb{Q}$-factorial terminal Fano 3-fold with $\rho(X) = 1$ such that there exists a general weighted hypersurface

$$X_{6d} \subset \mathbb{P}(1, a, b, 2d, 3d)$$

of degree $6d$ as in [9, List 16.6, Table 5, No. 14, No. 34, No. 53, No. 70, No. 72, No. 82, No. 88–90, No. 92, No. 94] with $(-K_X)^3 = (-K_{X_{6d}})^3$ and $B_X = B_{X_{6d}}$ (see Sect. 2), then X is a weighted hypersurface of degree $6d$ in $\mathbb{P}(1, a, b, 2d, 3d)$. We hope that similar characterization could also be done for some other examples in [9, List 16.6, Table 5] or possibly even for weighted complete intersections of lower codimensions, but more details should be checked.

Lastly we remark that the situation for upper bound of $(-K)^3$ of terminal weak Fano 3-folds is more complicated, see [11] for recent progress. It is worth to mention that by [12, 13], for a $\mathbb{Q}$-factorial terminal Fano 3-fold with $\rho(X) = 1$, it is known that $(-K_X)^3 \leq 64$ and the equality holds if and only if $X \simeq \mathbb{P}^3$; if moreover X is non-Gorenstein, then $(-K_X)^3 \leq 125/2$ and the equality holds if and only if $X \simeq \mathbb{P}(1, 1, 1, 2)$.

2 Reid's Riemann–Roch Formula

A *basket* B is a collection of pairs of integers (permitting weights), say $\{(b_i, r_i) \mid i = 1, \ldots, s;\ b_i \text{ is coprime to } r_i\}$.

Let X be a terminal weak Fano 3-fold. According to Reid [14], there is a basket of orbifold points (called *Reid basket*)

$$B_X = \left\{(b_i, r_i) \mid i = 1, \cdots, s;\ 0 < b_i \leq \frac{r_i}{2};\ b_i \text{ is coprime to } r_i\right\}$$

associated to X, where a pair (b_i, r_i) corresponds to a (virtual) orbifold point Q_i of type $\frac{1}{r_i}(1, -1, b_i)$.

Recall that for a Weil divisor D on X,

$$H^0(X, D) = \{f \in \mathbb{C}(X)^\times \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

By Reid's Riemann–Roch formula and the Kawamata–Viehweg vanishing theorem, for any positive integer m,

$$\begin{aligned}h^0(X, -mK_X) &= \chi(X, \mathcal{O}_X(-mK_X))\\ &= \frac{1}{12}m(m+1)(2m+1)(-K_X)^3 + (2m+1) - l(m+1)\end{aligned}$$

where $l(m+1) = \sum_i \sum_{j=1}^{m} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}$ and the first sum runs over all orbifold points in Reid basket ([5, 2.2]). Here $\overline{jb_i}$ means the smallest non-negative residue of jb_i mod r_i.

Proposition 2.1 *Let X be a terminal weak Fano 3-fold with $(-K_X)^3 = \frac{1}{330}$. Then $B_X = \{(1, 2), (2, 5), (1, 3), (2, 11)\}$. Moreover,*

$$\sum_{m \geq 0} h^0(X, -mK_X)q^m = \tilde{H}_{66}(q).$$

Here

$$\tilde{H}_{66}(q) = \frac{1 - q^{66}}{(1-q)(1-q^5)(1-q^6)(1-q^{22})(1-q^{33})}.$$

Proof The characterization of B_X is given in [4, Theorem 1.1(iii)]. For a general weighted hypersurface

$$X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$$

of degree 66, $(-K_X)^3 = (-K_{X_{66}})^3 = \frac{1}{330}$ and $B_X = B_{X_{66}}$ ([9, List 16.6, Table 5, No. 95]). By Reid's Riemann–Roch formula, $h^0(X, -mK_X)$ depends only on $(-K_X)^3$ and B_X. Note that $\mathcal{O}_{X_{66}}(-K_{X_{66}}) = \mathcal{O}(1)|_{X_{66}}$. So

$$\sum_{m \geq 0} h^0(X, -mK_X)q^m = \sum_{m \geq 0} h^0(X_{66}, -mK_{X_{66}})q^m = \tilde{H}_{66}(q)$$

by [8, Theorem 3.4.4]. □

3 Proofs

We recall the following lemma as a special case proved in [5, Theorem 1.4]. We refer to [5] for basic definitions. Here we should remind that in [5], a $\mathbb{Q}$-factorial terminal Fano 3-fold with $\rho = 1$ is called a $\mathbb{Q}$-Fano 3-fold.

Lemma 3.1 *Let X be a $\mathbb{Q}$-factorial terminal Fano 3-fold with $\rho(X) = 1$ and $(-K_X)^3 = \frac{1}{330}$. Then for*

(1) $h^0(X, -K_X) = 1$ and the unique divisor in $|-K_X|$ is prime;

(2) $|-5K_X|$ *is composed with an irreducible pencil of surfaces;*
(3) $|-6K_X|$ *is not composed with a pencil of surfaces.*

Proof By Proposition 2.1,

$$h^0(X, -mK_X) = \begin{cases} 1 & \text{if } 1 \leq m \leq 4; \\ 2 & \text{if } m = 5; \\ 3 & \text{if } m = 6. \end{cases}$$

(1) is a direct consequence of [5, Theorem 3.2] for $m = 1$ (or [1, Theorem 2.18]), (2) is a direct consequence of the fact that $h^0(X, -5K_X) = 2$, and (3) is a direct consequence of [5, Theorem 3.4]. □

Remark 3.2 We do not know whether Lemma 3.1(3) remains true or not if we only assume that X is a terminal weak Fano 3-fold. The current proof uses the fact that Lemma 3.1(3) is implied by Lemma 3.1(1) (by [5, Theorem 3.4]), meanwhile Lemma 3.1(1) essentially relies on the assumption that X is $\mathbb{Q}$-factorial with $\rho(X) = 1$ as in [5, Theorem 3.2] or [1, Theorem 2.18]. If one can drop these conditions in Lemma 3.1(3), then one can drop these conditions in Theorem 1.2 as suggested by Theorem 3.3.

The following theorem is a slightly more general version of Theorem 1.2.

Theorem 3.3 *Let X be a terminal Fano* 3*-fold and* $(-K_X)^3 = \frac{1}{330}$. *Assume further that*

(1) $|-6K_X|$ *is not composed with a pencil of surfaces; and*
(2) X *is non-rational.*

Then X is a weighted hypersurface of degree 66 *in* $\mathbb{P}(1, 5, 6, 22, 33)$ *defined by a weighted homogeneous polynomial F of degree* 66*, where*

$$F(x, y, z, w, t) = t^2 + F_0(x, y, z, w)$$

in suitable homogeneous coordinates $[x : y : z : w : t]$ *of* $\mathbb{P}(1, 5, 6, 22, 33)$.

Proof Recall that for a Weil divisor D on X, $H^0(X, D)$ can be viewed as a $\mathbb{C}$-linear subspace of the function field $\mathbb{C}(X)$. For $m \in \{1, 5, 6, 22, 33\}$, take $f_m \in H^0(X, -mK_X) \setminus \{0\}$ to be a general element. We can define 3 rational maps by these functions:

$$\begin{aligned} \Phi_6 : X &\dashrightarrow \mathbb{P}(1, 5, 6); \\ P &\mapsto [f_1(P) : f_5(P) : f_6(P)]; \\ \Phi_{22} : X &\dashrightarrow \mathbb{P}(1, 5, 6, 22); \\ P &\mapsto [f_1(P) : f_5(P) : f_6(P) : f_{22}(P)]; \\ \Phi_{33} : X &\dashrightarrow \mathbb{P}(1, 5, 6, 22, 33); \\ P &\mapsto [f_1(P) : f_5(P) : f_6(P) : f_{22}(P) : f_{33}(P)]. \end{aligned}$$

We claim that they have the following geometric properties. □

Proposition 3.4 *Keep the above settings.*

(1) Φ_6 *is dominant;*
(2) Φ_{22} *is dominant and generically finite of degree* 2*;*
(3) Φ_{33} *is birational onto its image;*
(4) *let* Y *be the closure of* $\Phi_{33}(X)$ *in* $\mathbb{P}(1,5,6,22,33)$*, then* Y *is defined by a weighted homogeneous polynomial* F *of degree* 66*, where*

$$F(x, y, z, w, t) = t^2 + F_0(x, y, z, w)$$

in suitable homogeneous coordinates $[x : y : z : w : t]$ *of* $\mathbb{P}(1,5,6,22,33)$.

Proof (1) As $h^0(X, -5K_X) = 2$, $|-5K_X|$ is composed with an irreducible pencil of surfaces. By assumption, $|-6K_X|$ is not composed with a pencil of surfaces. Recall that $h^0(X, -6K_X) = 3$, so $H^0(X, -6K_X)$ is spanned by $\{f_1^6, f_1 f_5, f_6\}$. Hence Φ_6 is birational to the rational map $X \dashrightarrow \mathbb{P}^2$ defined by $|-6K_X|$, which is obviously dominant.

(2) By [10, Theorem 4.4.11] (taking $m_0 = \mu_0 = 5$ and $m_1 = 6$), we conclude that $|-22K_X|$ defines a generically finite map onto its image. Hence a general f_{22} is not constant along general fibers of Φ_6. Therefore Φ_{22} is generically finite onto its image. In particular, Φ_{22} is dominant by dimension reason. To compute the degree of Φ_{22}, take a resolution $\pi : W \to X$ such that for $m \in \{5, 6, 22\}$, $\pi^*(-mK_X) = M_m + F_m$ where M_m is free and F_m is the fixed part. Then

$$\deg \Phi_{22} = (M_5 \cdot M_6 \cdot M_{22}) \leq (\pi^*(-5K_X) \cdot \pi^*(-6K_X) \cdot \pi^*(-22K_X)) = 2.$$

As X is non-rational, we conclude that $\deg \Phi_{22} = 2$.

(3) By [5, Theorem 5.11] (taking $m_0 = \mu_0 = 5$ and $m_1 = 6$), we conclude that $|-33K_X|$ defines a birational map onto its image. As f_{33} is general, it can separate two points in general fibers of Φ_{22}, so Φ_{33} is birational onto its image.

(4) Note that $h^0(X, -66K_X) = 172$ by Proposition 2.1. On the other hand, the equation

$$n_1 + 5n_2 + 6n_3 + 22n_4 + 33n_5 = 66$$

has exactly 173 solutions in $\mathbb{Z}^5_{\geq 0}$. So there exists a weighted homogeneous polynomial $F(x, y, z, w, t)$ of degree 66 with $\mathrm{wt}(x, y, z, w, t) = (1, 5, 6, 22, 33)$ such that

$$F(f_1, f_5, f_6, f_{22}, f_{33}) = 0.$$

So Y is contained in $(F = 0) \subset \mathbb{P}(1,5,6,22,33)$.

We claim that $Y = (F = 0)$ and t^2 has non-zero coefficient in F. Otherwise, Y is defined by a weighted homogeneous polynomial $\tilde{F}$ of degree ≤ 66 of the form

$$\tilde{F}(x, y, z, w, t) = t\tilde{F}_1(x, y, z, w) + \tilde{F}_2(x, y, z, w).$$

Here note that $\tilde{F}_1 \neq 0$, otherwise the image of Φ_{22} is contained in $(\tilde{F}_2 = 0) \subset \mathbb{P}(1, 5, 6, 22)$, which contradicts the fact that Φ_{22} is dominant. Then Y is birational to $\mathbb{P}(1, 5, 6, 22)$ under the rational projection map

$$\mathbb{P}(1, 5, 6, 22, 33) \dashrightarrow \mathbb{P}(1, 5, 6, 22);$$
$$[x : y : z : w : t] \mapsto [x : y : z : w].$$

This contradicts the assumption that X is non-rational. So $Y = (F = 0)$ and t^2 has non-zero coefficient in F. After a suitable coordinate change we may assume that $F = t^2 + F_0(x, y, z, w)$.

This finishes the proof of the proposition. □

Proof Now go back to the proof of Theorem 3.3. By the above proposition, F is the only relation on $f_1, f_5, f_6, f_{22}, f_{33}$. Denote $\mathcal{R}$ to be the graded sub-$\mathbb{C}$-algebra of

$$R(X, -K_X) = \bigoplus_{m\geq 0} H^0(X, -mK_X)$$

generated by $\{f_1, f_5, f_6, f_{22}, f_{33}\}$. Then we have a natural isomorphism between graded $\mathbb{C}$-algebras

$$\mathcal{R} \simeq \mathbb{C}[x, y, z, w, t]/(t^2 + F_0)$$

by sending $f_1 \mapsto x$, $f_5 \mapsto y$, $f_6 \mapsto z$, $f_{22} \mapsto w$, $f_{33} \mapsto t$ and the right hand side is exactly the weighted homogeneous coordinate ring of Y. Write $\mathcal{R} = \bigoplus_{m\geq 0} \mathcal{R}_m$ where $\mathcal{R}_m$ is the homogeneous part of degree m. Then by [8, 3.4.2],

$$\sum_{m\geq 0} \dim_{\mathbb{C}} \mathcal{R}_m \cdot q^m = \tilde{H}_{66}(q).$$

So by Proposition 2.1, $\mathcal{R}_m = H^0(X, -mK_X)$ for any $m \in \mathbb{Z}_{\geq 0}$, and hence the inclusion $\mathcal{R} \subset R(X, -K_X)$ is an isomorphism. This implies that

$$X \simeq \operatorname{Proj} R(X, -K_X) \simeq \operatorname{Proj} \mathcal{R} \simeq Y.$$

This finishes the proof. □

Proof (*Proof of Theorem* 1.2) It follows directly from Lemma 3.1 and Theorem 3.3. □

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References

1. Alexeev, V.: General elephants of $\mathbb{Q}$-Fano 3-folds. Compositio Math. **91**(1), 91–116 (1994)
2. Buckley, A., Reid, M., Zhou, S.: Ice cream and orbifold Riemann-Roch. Izv. Math. **77**(3), 461–486 (2013)
3. Cheltsov, I., Park, J.: Birationally rigid Fano threefold hypersurfaces. Mem. Amer. Math. Soc. **246**(1167), v+117 (2017)
4. Chen, J.A., Chen, M.: An optimal boundedness on weak $\mathbb{Q}$-Fano 3-folds. Adv. Math. **219**(6), 2086–2104 (2008)
5. Chen, M., Jiang, C.: On the anti-canonical geometry of $\mathbb{Q}$-Fano threefolds. J. Differ. Geom. **104**(1), 59–109 (2016)
6. Corti, A., Pukhlikov, A., Reid, M.: Fano 3-fold hypersurfaces, Explicit birational geometry of 3-folds, London Mathematical Society. Lecture Note Series, vol. 281, pp. 175–258. Cambridge University Press, Cambridge (2000)
7. de Fernex, T., Ein, L., Mustaţă, M.: Log canonical thresholds on varieties with bounded singularities. Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., pp. 221–257 Zürich (2011)
8. Dolgachev, I.: Weighted projective varieties. Group Actions and Vector Fields (Vancouver, B.C., 1981). Lecture Notes in Mathematics, vol. 956, pp. 34–71. Springer, Berlin (1982)
9. Iano-Fletcher, A.R.: Working with weighted complete intersections, Explicit birational geometry of 3-folds. London Mathematical Society. Lecture Note Series, vol. 281, pp. 101–173. Cambridge University Press, Cambridge (2000)
10. Jiang, C.: On boundedness of volumes and birationality in birational geometry, Ph.D. thesis, University of Tokyo, vi+115 pp (2015). https://chenjiangfudan.github.io/home/thesis.pdf
11. Jiang, C., Zou, Y.: An effective upper bound for anti-canonical volumes of canonical $\mathbb{Q}$-Fano threefolds, to appear in Int. Math. Res. Not. arXiv:2107.01037
12. Namikawa, Y.: Smoothing Fano 3-folds. J. Algebraic Geom. **6**(2), 307–324 (1997)
13. Prokhorov, Y.G.: The degree of $\mathbb{Q}$-Fano threefolds. Mat. Sb. **198** (11), 153–174 (2007). Translation in Sb. Math. **198**(11–12), 1683–1702 (2007)
14. Reid, M.: Young person's guide to canonical singularities. Algebraic Geometry, Bowdoin (1985) (Brunswick, Maine, 1985). Proceedings of Symposium Pure Mathematics, vol. 46, Part 1, pp. 345–414. American Mathematical Society, Providence, RI (1987)

Rationality of Quotients by Finite Heisenberg Groups

Stanislav Grishin, Ilya Karzhemanov, and Ming-chang Kang

Abstract We prove rationality of the quotient $\mathbb{C}^n/H_n$ for the finite Heisenberg group H_n, any $n \geq 1$, acting on $\mathbb{C}^n$ via its irreducible representation.

Keywords Heisenberg group · Quotient · Log pair · Toric variety

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1 Introduction

1.1. In the present paper, we study rationality of the quotient $\mathbb{C}^n/G$ (*Noether's problem*) for the affine space $\mathbb{C}^n$, $n \geq 1$, equipped with a linear action of an algebraic group G. Recall that for *finite* G variety $\mathbb{C}^n/G$ can be non-rational (e.g. this is the case for certain p-groups in [24]). At the same time, for *connected* G the quotient $\mathbb{C}^n/G$ is typically *stably rational*, that is the product $\mathbb{C}^k \times (\mathbb{C}^n/G)$ is rational for some k (see [3, Theorem 2.1]).

Note that variety $\mathbb{C}^n/G$ is rational when G is *Abelian* (see [7]). Some rationality constructions for $\mathbb{C}^n/G$ with non-Abelian G can be found in [21] (see also [14]). In the present paper, we consider a particular case of the *Heisenberg group* $G := H_n$ generated by two elements ξ, η, which act on $\mathbb{C}^n$ as follows (*Schrödinger representation*):

$$\xi : x_i \mapsto \omega^{-i} x_i, \qquad \eta : x_i \mapsto x_{i+1} \quad (i \in \mathbb{Z}/n,\ \omega := e^{\frac{2\pi\sqrt{-1}}{n}}),$$

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where $x_1, \ldots, x_n$ form a basis in $\mathbb{C}^n$ (up to a choice of ω this is the only irreducible linear representation of H_n).

When studying rationality problem for $\mathbb{C}^n/H_n$ it is reasonable to pass to the *projectivization* and consider the quotient $X := \mathbb{P}^{n-1}/H_n$ (cf. [21, Proposition 1.2]). Here is our main result:

Theorem 1.2 *Variety X is* rational *for every n.*

The group H_n is a *central extension* of $\mathbb{Z}/n \oplus \mathbb{Z}/n$ by $\mathbb{Z}/n \ni [\xi, \eta]$ and so the action of H_n on $\mathbb{P}^{n-1}$ factors through that of $\mathbb{Z}/n \oplus \mathbb{Z}/n$. Thus Theorem 1.2 is a natural generalization of *linear* Abelian case mentioned above. Let us also point out that the case of central extensions of *cyclic* groups has been treated in [25].

Our result confirms in addition (a stronger version of) Conjecture 15 in [5]. Actually, stable rationality of X can be proved via a direct argument by considering diagonal action of H_n on $V \times V$, with linear action of $\mathbb{Z}/n \oplus \mathbb{Z}/n$ on the second factor. Note also that Theorem 1.2 is evident when $n \le 3$ and the case $n = 4$ has been treated in [21, Theorem 5.2] (compare with [3, Lemma 3.1]).

1.3 Let us outline our approach towards the proof of Theorem 1.2. One may observe that the quotient $\mathbb{C}^n/G$ is a *toric* variety for Abelian group G. In our case of $G = H_n$, its action on $\mathbb{P}^{n-1}$ is also Abelian, and so it is reasonable to expect that X is toric as well. This turns out to be (almost) so.

Namely, one employs an instance of the *toric conjecture* after Shokurov, characterizing toric varieties in terms of the log pairs (see e.g. [22]): we construct a $\mathbb{Q}$-divisor D on X satisfying the assumptions of Proposition 2.1 below and reduce rationality problem for X to that for a *cyclic* quotient of $\mathbb{P}^{n-1}$ (the latter is rational by the discussion in **1.1**). In turn, the explicit action of H_n on $\mathbb{P}^{n-1}$ allows one to find appropriate invariant divisors descending to the components of D, which is done in **2.3**.

Our point was, more generally, to develop a *geometric* approach to the Noether's problem for central extensions of Abelian groups (cf. **3.1** below). Thus the case of $\mathbb{C}^n/H_n$ is a special corollary of this approach. On the other hand, after our paper appeared online, Professor Ming-chang Kang has kindly communicated to us an *algebraic* proof of Theorem 1.2 (see Appendix after Sect. 3).

Remark 1.3 We show in Proposition 2.1 that X is actually a cyclic quotient of $\mathbb{P}^{n-1}/\widetilde{G}$ for a *linearized Abelian* group $\widetilde{G}$. Thus X resembles the so-called *fake weighted projective space* (see [15]). Note however that X *need not* be toric. Let us consider the first non-trivial case $n = 3$. Here the group H_3 acts on $\mathbb{P}^2$ preserving the *Hesse pencil* $\left\{E_t : x^3 + y^3 + z^3 + txyz = 0 \mid t \in \mathbb{P}^1\right\}$ and on the smooth cubic E_t the H_3-action coincides with the one of the group of 3-torsion points $E_t[3]$ (see [1]). The quotient surface $X = \mathbb{P}^2/H_3$ has 4 singular points of type A_2 and so can not be toric because X has Picard number 1 (cf. [8, Sect. 3.4]). One may also observe that the algebra of invariants of H_3 in $\mathbb{C}[x, y, z]$ is generated by polynomials xyz, $x^3 + y^3 + z^3$, $x^3y^3 + y^3z^3 + z^3x^3$ and $x^3y^6 + y^3z^6 + z^3x^6$ (cf. [1, Sect. 6]).

2 Proof of Theorem 1.2

We will be using freely standard notions and facts about the singularities of pairs (see e.g. [17, Chap. 5]). All varieties are assumed to be normal, projective, over $\mathbb{C}$, and all divisors are $\mathbb{Q}$-Cartier with rational coefficients.

Our proof of Theorem 1.2 is based on the following:

Proposition 2.1 (cf. [12, 16, 22]) Let V be a d-dimensional variety with a boundary divisor $D = \sum_{i=1}^{d+1} d_i D_i$, where D_i are prime Weil divisors, such that the following holds:

- the Picard number of V is 1,
- the log pair (V, D) is log canonical,
- $K_V + D \sim_{\mathbb{Q}} 0$,
- $d_i D_i \sim_{\mathbb{Q}} d_j D_j$ for all $1 \le i, j \le d+1$,
- there exists a *finite, étale in codimension* 1 *cyclic* cover $p : V' \longrightarrow V$ such that $p^*(d_i D_i) \sim_{\mathbb{Q}} W_i$, $1 \le i \le d+1$, where W_i are *distinct* Weil divisors on V'.

Then V' is a *toric* quotient $\mathbb{P}^d/\widetilde{G}$ for a finite Abelian group $\widetilde{G}$ with *linearized* action on $\mathbb{P}^d$. In particular, if $\Gamma \simeq \mathbb{Z}/m\mathbb{Z}$ is the Galois group of p, then $V = V'/\Gamma$ is birational to $\mathbb{P}^d/\Gamma$ (hence V is rational).

Proof We follow the proof of Lemma 3.1 in [22]. Namely, after repeated finite, étale in codimension 1 cyclic covers $V \xleftarrow{p} V' \longleftarrow \ldots \longleftarrow \widetilde{V}$ we obtain a new log pair $(\widetilde{V}, \widetilde{D} = \varphi^*(D))$, where $\varphi : \widetilde{V} \longrightarrow V'$ is the resulting morphism, such that all $\varphi^* p^*(W_i)$ are *Cartier*. Furthermore, we have

$$K_{\widetilde{V}} + \widetilde{D} \sim_{\mathbb{Q}} \varphi^* p^*(K_V + D) \sim_{\mathbb{Q}} 0$$

and $(\widetilde{V}, \widetilde{D})$ is log canonical, i.e. $\widetilde{V}$ is a log Fano (note that $\varphi^* p^*(D)$ is ample).

The Fano index of $\widetilde{V}$ is $\ge d+1$, since $-K_{\widetilde{V}} \sim_{\mathbb{Q}} \widetilde{D}$ and $\varphi^* p^*(d_i D_i) \sim_{\mathbb{Q}} \varphi^* p^* (d_j D_j)$ for all $1 \le i, j \le d+1$. This implies that $\widetilde{V} = \mathbb{P}^d$ (see e.g. [11, Theorem 3.1.14]) and φ coincides with the quotient morphism by some finite group $\widetilde{G}$ (the Galois group of the field extension $\mathbb{C}(\widetilde{V})/\varphi^*\mathbb{C}(V')$). Also, by construction $\widetilde{G}$ leaves invariant $d+1$ hyperplanes $\varphi^* p^*(W_i)$ in $\mathbb{P}^d$, whence it is Abelian.

Further, variety V' is toric by construction, so let us a fix an open torus $T := (\mathbb{C}^*)^d \subset V'$ with coordinates $z_1, \ldots, z_d$. Let also $\widetilde{W}_i \subset V'$ be the Zariski closure of the zero-locus $(z_i = 0)$, $1 \le i \le d$, and $\widetilde{W}_{d+1} \subset V'$ be the closure of $(z_{d+1} := (z_1 \ldots z_d)^{-1} = 0)$. Then, since each $p^*(d_i D_i)$ generates the $\mathbb{Q}$-Weil group of V', we get that $V' \setminus T = \bigcup_{i=1}^{d+1} \widetilde{W}_i$, with $\widetilde{W}_i$ corresponding to the rays of the fan of V' (cf. [8, Sect. 3.4]). The induced action of Γ on the fan either preserves all these rays or permutes them cyclicly. In particular, the same applies to the divisors $\widetilde{W}_i$,

$1 \le i \le d+1$, and to their defining functions z_i. Hence, compactifying the torus T by $\mathbb{P}^d$, we obtain that T/Γ is birational to the rational variety $\mathbb{P}^d/\Gamma$. $\square$

2.3 We now turn to the variety $X = \mathbb{P}^{n-1}/H_n$ from Theorem 1.2. Let $\pi : \mathbb{P}^{n-1} \longrightarrow X$ be the quotient morphism.

Lemma 2.2 *π is étale in codimension 1 and $K_{\mathbb{P}^{n-1}} \sim_{\mathbb{Q}} \pi^*(K_X)$.*

Proof The first assertion follows from the fact that every $\ne 1$ element in H_n has non-multiple spectrum (see **1.1**). Then the equivalence $K_{\mathbb{P}^{n-1}} \sim_{\mathbb{Q}} \pi^*(K_X)$ is the usual Hurwitz formula. $\square$

Identify $x_0, \dots, x_{n-1}$ from **1.1** with projective coordinates on $\mathbb{P}^{n-1}$. Put $f_k := \sum_{i \in \mathbb{Z}/n} x_i^k x_{i+1}^{n-k}$ for $1 \le k \le n$. We have $\xi^* f_k = \omega^k f_k$ and $\eta^* f_k = f_k$. Hence polynomials f_k^n are H_n-invariant.

Lemma 2.3 *The linear system $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^{n-1}}(n^2)|$ spanned by $f_1^n, \dots, f_n^n$ and $(x_0 \dots x_{n-1})^n$ is basepoint-free.*

Proof It suffices to show that $f_1, \dots, f_n$ and $x_0 \dots x_{n-1}$ span a basepoint-free linear system. Fix an arbitrary $m \ge n$ and consider the polynomials $f_k^{(m)} := \sum_{i \in \mathbb{Z}/n} x_i^k x_{i+1}^{m-k}$ for various $1 \le k \le m$. Let $\mathcal{L}^{(m)}$ be the linear system spanned by $f_1^{(m)}, \dots, f_m^{(m)}$ and $x_0 \dots x_{n-1}^{m-n+1}$. Then we claim that $\mathcal{L}^{(m)}$ is basepoint-free (note that $m = n$ corresponds to our case). Indeed, for $n = 2$ this is trivially true, whereas for $n > 2$ we restrict to the hyperplanes $(x_i = 0)$ and argue by induction. $\square$

Let $B_1, \dots, B_n$ be generic elements in the linear system $\mathcal{L}$ from Lemma 2.3. We may assume the pair $(\mathbb{P}^{n-1}, \sum_{i=1}^{n} B_i)$ is *log canonical.*

Further, put $D_i := \pi(B_i)$, $1 \le i \le n$, so that $B_i = \pi^*(D_i)$,

$$K_{\mathbb{P}^{n-1}} + \sum_{i=1}^{n} B_i \sim_{\mathbb{Q}} \pi^* \left(K_X + \sum_{i=1}^{n} D_i \right) \tag{1}$$

(cf. Lemma 2.2) and the pair $(X, \sum_{i=1}^{n} D_i)$ is also log canonical.

Lemma 2.4 *π factorizes as $\mathbb{P}^{n-1} \xrightarrow{q} X' \xrightarrow{p} X$, where q, p are both* degree n, étale in codimension 1 cyclic *covers, $p^*(d_i D_i) \sim_{\mathbb{Q}} W_i$, $1 \le i \le n$, for $d_i := 1/n^2$ and some* distinct *Weil divisors W_i.*

Proof Note that the field extension $\mathbb{C}(\mathbb{P}^{n-1})/\pi^*\mathbb{C}(X)$ is Galois with the group $\mathfrak{S} := \mathbb{Z}/n \oplus \mathbb{Z}/n$. Restricting to the field of ξ-invariants yields an intermediate field $\pi^*\mathbb{C}(X) \subset F \subset \mathbb{C}(\mathbb{P}^{n-1})$. Note that extension $\mathbb{C}(\mathbb{P}^{n-1})/F$ corresponds to the quotient morphism $q : \mathbb{P}^{n-1} \longrightarrow X'$ for $X' = \mathbb{P}^{n-1}/\langle\xi\rangle >$ and the cyclic subgroup $\langle\xi\rangle \subset \mathfrak{S}$. Finally, $F/\mathbb{C}(X)$ is also Galois, corresponding to the quotient morphism $p : X' \longrightarrow X = X'/\langle\eta\rangle$.

Further, consider the divisors $B_0 := ((x_0 \dots x_{n-1})^n = 0)$ and $H_i := (x_i = 0), 0 \leq i \leq n-1$, so that $B_0 = n \sum_{i=0}^{n-1} H_i$. We have $\frac{1}{n^2} B_0 \sim_{\mathbb{Q}} H_i$ for all i and hence

$$q_*\left(\frac{1}{n^2} B_0\right) \sim_{\mathbb{Q}} q_*(H_i) \sim_{\mathbb{Q}} nq(H_i)$$

because $q_*(H_i) = nq(H_i)$ for H_i being ξ-invariant hyperplanes. This implies that

$$p^*\left(\frac{1}{n^2} D_i\right) = \frac{1}{n^2} q(B_i) = \frac{1}{n^3} q_*(B_i) \sim_{\mathbb{Q}} q_*\left(\frac{1}{n^3} B_0\right) \sim_{\mathbb{Q}} q(H_{i-1}) =: W_i$$

for all $1 \leq i \leq n$. □

Put $D := \frac{1}{n^2} \sum_{i=1}^{n} D_i$. Then it follows immediately from (1) and Lemma 2.4 that the log pair (X, D) satisfies all the assumptions in Proposition 2.1 (for $V := X$). Thus X is rational and the proof of Theorem 1.2 is complete.

3 Miscellany

3.1 It would be interesting to extend the technique presented in Sect. 2 to the case of quotients $\mathbb{P}^{n-1}/G$ by other finite central extensions of Abelian groups. This requires, however, analogs of technical lemmas from **2.3**, where we have crucially used that $G = H_n$. So we plan to consider this matter elsewhere.

More generally, it would be interesting to give a characterization of those finite groups G, for which $\mathbb{P}^{n-1}/G$ is a cyclic quotient of toric variety (cf. Remark 1.3). Observe at this point that the singularities of $X = \mathbb{P}^{n-1}/H_n$ are *non-exceptional* (cf. [20, Proposition 3.4]), and one might try to look for a similar property of singularities, distinguishing (cyclic quotients of) toric $\mathbb{P}^{n-1}/G$.

3.2 Initially, our interest was in constructing a *mirror dual* Y^+ for Calabi–Yau threefolds Y, studied in [9]. Recall that Y is a small resolution of a nodal Calabi–Yau $V \subset \mathbb{P}^{n-1}$, *invariant under* H_n, such that there is a pencil of $(1, n)$-*polarized Abelian surfaces* $A \subset V$. The action of H_n extends to a *free* one on Y and it is expected that $Y^+ = Y/H_n$. Indeed, when $n = 8$ the *derived equivalence* between Y and Y/H_n was established in [23], which on the level of Abelian surfaces is the Mukai equivalence between A and $\mathrm{Pic}^0(A) = A/H_n$ (note that H_n acts on A via shifts by n-torsion points).

In particular, when $n = 5$ and V is the *Horrocks–Mumford quintic* (see [9, Sect. 3]), V/H_5 is a Calabi–Yau hypersurface in (almost) toric variety $\mathbb{P}^4/H_5$. This brings in a possibility for applying Batyrev's construction of mirror pairs (see [2]) as well as other explicit methods: matrix factorizations, period integrals, etc. (see e.g. [10, 19]). We plan to return to this subject elsewhere.

3.3 As a complement to **3.1**, one may try to attack (stable) rationality problem for various quotients $\mathbb{P}^{n-1}/G$ by considering their classes $[\mathbb{P}^{n-1}/G]$ in $K_0(\mathrm{Var})$, the *Grothendieck ring* of complex algebraic varieties, and applying [18, Corollary 2.6] to them. It is thus important to compute $[\mathbb{P}^{n-1}/G]$ explicitly (compare with [13]).

For instance, we have $[\mathbb{P}^{n-1}/H_n] = [\mathbb{P}^{n-1}]$ modulo $\mathbb{L} := [\mathbb{A}^1]$ by [18, Proposition 2.7] (cf. Theorem 1.2), and it would be interesting to obtain a similar mod $\mathbb{L}$-relation in general. Perhaps the fact that any such variety is *stably birationally infinitely transitive* (see [4, Corollary 3.2]) might be of some use here.

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Appendix by Ming-chang Kang

The algebraic proof of Theorem 1.2 follows the same lines as in [6]. Namely, put $\lambda := [\xi, \eta]$ and $y_0 := x_0^n$, $y_i := x_i/x_{i-1}$, $1 \le i \le n-1$ (see **1.1**). Then we have $\mathbb{C}(x_0, \dots, x_{n-1})^{\langle\lambda\rangle} = \mathbb{C}(y_0, \dots, y_{n-1})$. By [6, Theorem 4.1] it suffices to prove rationality of $\mathbb{C}(y_1, \dots, y_{n-1})^{\langle\xi,\eta\rangle}$. Note that the action of ξ on y_i, $1 \le i \le n-1$, is given by $\xi : y_i \mapsto \omega y_i$.

Define $z_1 := y_1^n$, $z_i := y_i/y_{i-1}$, $2 \le i \le n-1$. Then we have $\mathbb{C}(y_1, \dots, y_{n-1})^{\langle\xi\rangle} = \mathbb{C}(z_1, \dots, z_{n-1})$. Note that the action of η on z_i is the same as the action of τ in [6, p. 686] (by replacing p with n everywhere).

Now define $w_1 := z_2$, $w_i := \eta^{i-1}(z_2)$, $2 \le i \le n-1$. Then we have $\mathbb{C}(z_1, \dots, z_{n-1}) = \mathbb{C}(w_1, \dots, w_{n-1})$ and the action of η is as follows:

$$\eta : w_1 \mapsto w_2 \mapsto w_3 \mapsto \dots \mapsto w_{n-1} \mapsto \frac{1}{w_1, \dots w_{n-1}}.$$

The latter action can be linearized exactly as in the middle of [6, p. 687]. Hence we can apply Fischer's Theorem (see [7]).

References

1. Artebani, M., Dolgachev, I.: The Hesse pencil of plane cubic curves. Enseign. Math. (2) **55**(3–4), 235–273 (2009)
2. Batyrev, V.V.: Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. J. Algebraic Geom. **3**(3), 493–535 (1994)
3. Bogomolov, F.A.: The stable rationality of quotient spaces for simply connected groups, Math. USSR - Sb. **58**(1), 1–14 (1987). Translated from Mat. Sb. (N.S.) **130(172)**(1), 3–17, 128 (1986)
4. Bogomolov, F., Karzhemanov, I., Kuyumzhiyan, K.: Unirationality and existence of infinitely transitive models. In: Birational Geometry, Rational Curves, and Arithmetic, pp. 77–92. Springer, New York (2013)
5. Bogomolov, F., Tschinkel, Y.: Noether's problem and descent. In: Proceedings of the Seventh International Congress of Chinese Mathematicians, Vol. I. Advanced Lectures in Mathematics (ALM), vol. 43, pp. 3–15. International Press, Somerville, MA
6. Chu, H., Kang, M.: Rationality of p - group actions. J. Algebra **237**(2), 673–690 (2001)
7. Fischer, E.: Zur Theorie der endlichen Abelschen Gruppen. Math. Ann. **77**(1), 81–88 (1915)
8. Fulton, W.: Introduction to Toric Varieties, Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton, NJ (1993) 131 131. Princeton University Press, Princeton, NJ (1993)
9. Gross, M., Popescu, S.: Calabi–Yau threefolds and moduli of abelian surfaces I. Compositio Math. **127**(2), 169–228 (2001)
10. He, W., Polishchuk, A., Shen, Y., Vaintrob, A.: A Landau – Ginzburg mirror theorem via matrix factorizations. arXiv:2001.00536
11. Iskovskikh, V.A. (RS-AOS), Prokhorov, Yu.G. (RS-MOSC): Fano Varieties. Algebraic Geometry, V. Encyclopaedia of Mathematical Sciences, vol. 47, pp. 1 – 247. Springer, Berlin (1999)
12. Karzhemanov, I.: On characterization of toric varieties. arXiv:1306.4131
13. Karzhemanov, I.: On the cut-and-paste property of algebraic varieties. Bull. Sci. Math. **166**, 8 pp (2021), Paper No. 102920
14. Karzhemanov, I.: On the quotient of $\mathbb{C}^4$ by a finite primitive group of type (I). Math. Res. Lett. **21**(1), 133–139 (2014)
15. Kasprzyk, A.M.: Bounds on fake weighted projective space. Kodai Math. J. **32**(2), 197–208 (2009)
16. Kollár, J. et al.: Flips and abundance for algebraic threefolds. A summer seminar at the University of Utah, Salt Lake City, 1991. Ast'erisque. **211** (1992)
17. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties, Translated from the 1998 Japanese Original. Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998)
18. Larsen, M., Lunts, V.A.: Motivic measures and stable birational geometry. Mosc. Math. J. **3**(1), 85–95, 259 (2003)
19. Lee, T.-J., Lian, B.H., Yau, S.-T.: On Calabi–Yau fractional complete intersections. arXiv:2008.04039
20. Markushevich, D., Prokhorov, Yu.G.: Exceptional quotient singularities. Am. J. Math. **121**(6), 1179–1189 (1999)
21. Prokhorov, Y.G.: Fields of invariants of finite linear groups. In: Cohomological and Geometric Approaches to Rationality Problems. Progress in Mathematics, vol. 282, pp. 245–273. Birkhäuser Boston, Boston, MA
22. Prokhorov, Yu.G.: On a conjecture of Shokurov: characterization of toric varieties. Tohoku Math. J. (2) **53**(4), 581–592 (2001)
23. Schnell, C.: The fundamental group is not a derived invariant. In: Derived Categories in Algebraic Geometry, EMS Ser. Congr. Rep, Eur. Math. Soc., pp. 279–285. Zürich
24. Shafarevich, I.R.: The Lüroth problem. Trudy Mat. Inst. Steklov. **183**, 199–204, 229 (1990)
25. Swan, R.G.: Noether's problem in Galois theory. In: Emmy Noether in Bryn Mawr (Bryn Mawr, Pa., pp. 21–40. Springer, New York (1982)

Interpretations of Spectra

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Abstract The studies of homological mirror symmetry as correspondence of Lefshetz pencils was initiated as part of the general theory of categorical linear systems. In this paper, we look at the monodromy of these linear systems via a new notion of noncommutative spectrum.

Keywords Mirror symmetry · Landau-Ginzburg models · Spectra

1 Introduction

The studies of homological mirror symmetry (HMS) as correspondence of Lefshetz pencils was initiated in [31] as part of the general theory of categorical linear systems. In this paper, we look at the monodromy of these linear systems. We utilise these monodromies by introducing a new notion of noncommutative spectrum. We will use the setup and the notations from [31]. We start with a pencil where the fibers are CY varieties and the global pencils constitute mirrors of Fano manifolds. We have the following category diagram:

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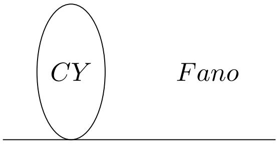


$$F(CY) \to F(Fano)$$

Here $F(CY)$, $F(Fano)$ are the corresponding Fukaya–Seidel categories. Im $\Phi(F(CY)) = \mathcal{A}$ is a localization category $F(CY)/\sim$. (Using HMS we can use $D^b(X)$—the category of coherent sheaves on algebraic varieties X.)

This localization category has a filtration:

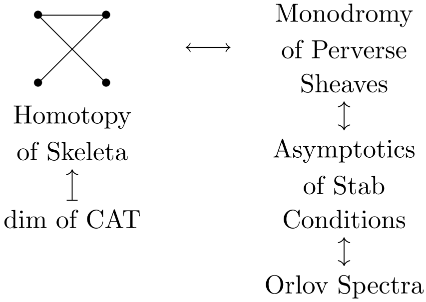
$$\mathcal{A} \supset \mathcal{F}_{\lambda_1} \supset \cdots \supset \mathcal{F}_{\lambda_n}$$

where:

- λ_i are the asymptotics of limiting stability conditions.
- $Z = z^{\lambda_i}(\cdots)$
- $\mathcal{F}_{\lambda_i} = \{F \text{ s.t. } Z(F) = z^{\lambda_i}(\cdots)\}$
- λ_i are also the asymptotics of the PDE

$$\left(\frac{\partial}{\partial u} + u^{-2}K + u^{-1}G\right)$$

The above filtration can also be seen as the monodromy of the perverse sheaf of categories over the skeleton. Following [31] we think of the category as a perverse sheaf of categories over lagrangian skeleton. In the diagram bellow we describe our findings in [31].



The main idea in current paper is to give an interpretation of the above λ_i filtration as a noncommutative spectrum and a spectrum of Landau-Ginzburg (LG) models. We use the theory of LG models as generalized theory of singularity.

The above considerations lead to birational invariants, which will appear in more details in [29, 34]. (For definitions and general theory of LG models and HMS we refer to [30].)

We will base our birational considerations on the following major notions and ideas:

(1) **Quantum spectrum**. The quantum spectrum is defined in [29]. Let $K\cdot$ be the quantum multiplication by canonical class. It defines the following splitting of cohomology:

$$\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}.$$

Here λ_i are the eigenvalues of $K\cdot$. We call these eigenvalues *quantum spectrum*. The main theorem proven in [29] is:

MAIN THEOREM: **The splitting $\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}$ is a birational invariant.**

(2) **Noncommutative spectrum**. The noncommutative spectrum is defined in [29]. In the current paper we extend these ideas and give some examples.

(A) We build analogues with low dimensional topology and give several new directions for research.

(B) We extend the definition of a noncommutative spectrum to multispectra. Possible applications are discussed.
Our considerations are only the tip of the iceberg. We propose a correspondence between nonrationality over algebraically nonclosed fields and complexity of the discriminant loci of the moduli space of LG models. We will consider some arithmetics applications in Sect. 3. In fact one can define several different spectra.
In addition to the **quantum spectrum** mentioned above, one can define several other spectra:

- **Noncommutative spectrum;**
  defined by the asymptotics of the quantum equation.
- **Givental spectrum;**
  defined by the solutions of the Givental's equation.
- **Spectrum of LG model—multiplier ideal sheaf;**
  defined as the Steenbrink spectrum of a new singularity theory of the LG model.
- **Asymptotics of stability conditions—stability spectrum;**
  defined as asymptotics of limiting stability conditions.
- **Serre dimension of the Kuznetsov's component;**
  defined as a categorical dimension.
- **Arnold-Varchenko-Steenbrink spectrum of the affine cone.**
  defined as the classical spectrum of the affine cone singularity over X.
- **R-charges**—the assymptotics of RG flow—the same as asymptotics of Kähler-Ricci flow—see Sect. 6.

We will discuss relations among some of them. Understanding the complete scope of relations is an intriguing problem. We initiate the study of these connections in this paper. We will develop this connections in upcoming papers [27, 32].

(C) We also propose a parallel between the existence of Kähler-Einstein metrics and the top number of the noncommutative spectra. Recall that

$$lct(X, G) = \sup\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ l.c.s. } \forall G \text{ inv. } D\}$$

We note the following parallel:

nonrationality of (X, G) orbifold	$\exists$of K.E. metric on (X, G)
$\delta > \dim X - 2$ X is not rational δ is lct for sing	$lct(X, G) > \dfrac{\dim X}{\dim X + 1}$

In the above table lct is the log canonical threshold.
We take this parallel further:

(D) We connect the noncommutative spectra with elliptic genus and conformal field theory. We connect orbifoldization of elliptic genus with spectra of singular varieties. This leads to a categorical interpretation of Birkar's boundness theorem. We propose the idea of categorical resolution and "boundness" of conformal field theories—the central charges correspond to the noncommutative spectra.

As a consequence we propose a parallel between Zamolodchikov's c-theorem and uppersemicontinuity condition of noncommutative spectra.

We will call the monotonicity of the highest number of the spectrum uppersemicontinuity. In other words, the highest number of the spectrum is decreasing monotonically when moving from the boundary of Frobenius manifold to its general point.

The paper is organized as follows. We explain the general theory in Sect. 2. The Fano applications are considered in Sect. 2. The arithmetics applications are considered in Sect. 3. The parallel with 3-dimensional topology are discussed in Sect. 4. The extension to multispectra is discussed in Sect. 5. In Sect. 6, we consider the connection of spectra with elliptic genus. We make a connection between Birkar's theory and the conformal field theories.

2 Noncommutative Spectra

In this section we introduce the idea of noncommutative spectra—an idea which belongs to M. Kontsevich. We describe new birational invariants and describe some easy applications.

2.1 Definitions of Quantum and Nc Spectra

Let X be a projective algebraic variety over $\mathbb{C}$, with a given ample line bundle. The Gromov-Witten invariants in genus zero define a potential $\mathcal{F}_0$: formal series on $H^\bullet(X)$ with coefficients in $\mathbb{Q}[[T]]$—see e.g. [30]. We briefly recall two conjectures (see e.g. [29]).

1. First we have:

Conjecture 2.1 $\mathcal{F}_0$ is convergent for a point $\gamma \in H^\bullet(X)$ and for $T \in \mathbb{C}$, both close to 0.

2. Assuming Γ-conjecture (see e.g. [30]) we get that **nc** Hodge structures are parametrized by a domain

$$M \subset H^\bullet(X, \mathbb{C})/H^2(X, 2\pi i\mathbb{Z}),$$

which is a meromorphic connection on the trivial bundle over u-plane $\mathbb{C}_u$ with fiber $H^\bullet(X)$:

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G$$

(Recall that the Γ-conjecture gives a lattice, hypothetically compatible with Stokes filtrations along rays at $u \to 0$. For more details see [30].)

We define the operator $K = K(\gamma)$ as the quantum product with $c_1(T_X) + \sum_{i\neq 2}(2-i)\gamma_i$. It depends on the point $\gamma = (\gamma_i \in H^i(X))_{i=0,\dots,2\dim_{\mathbb{C}} X}$ in Frobenius manifold $\mathcal{M}$. We also define the operator G as a constant operator given by $G_{|H^i(X)} = \frac{i-\dim_{\mathbb{C}} X}{2} \cdot id_{H^i(X)}$.

We use the example bellow to introduce and demonstrate two important definitions. Let X be a smooth 3-dimensional cubic in $\mathbb{P}^4$. Operators K, G on 4-dimensional space $H^{\text{even}}(X) = \oplus_{i=0}^3 H^{2i}(X)$ with the basis being powers of the hyperplane section, at point $\gamma = 0 \in \mathcal{M}$, are:

$$K = 2 \cdot \begin{pmatrix} 0 & 6 & 0 & 36 \\ 1 & 0 & 15 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

Solutions of the quantum equation

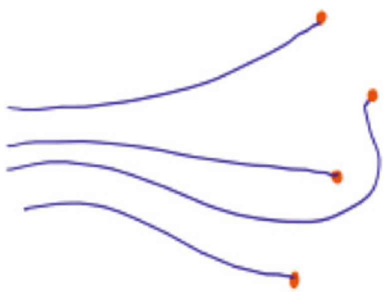


Fig. 1 Gabrielov paths *(Red dots correspond to eigenvalues of quantum multiplication)*

$$\left(\frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G\right)\psi(u) = 0 \tag{1}$$

grow at $u \to 0$ as

$$\sim u^{-\frac{5}{6}}, \sim u^{-\frac{1}{6}}.$$

Definition 2.2 Quantum spectrum is the spectrum of K, a finite subset $\{z_a\} = Spec_X \subset \mathbb{C}$ (depends on the point γ in $\mathcal{M}$).

Definition 2.3 Noncommutative spectrum: The asymptotics of the sub-exponential growth solutions of the Eq. 1 above form the **noncommutative spectrum** or **nc spectrum**.

In what follows we will denote by δ minus two times the lowest number of **noncommutative spectrum**. In the above example

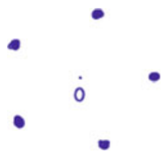
$$\delta = \frac{5}{3}.$$

Consider a purely even affine submanifold $\mathcal{M}^{\text{alg}} \subset \mathcal{M}$, given by deformations of quantum product by linear combinations of algebraic classes $H^{\text{alg}}_{\mathbb{Q}}(X) \subset H^{\text{even}}(X, \mathbb{Q})$.

Conjecture 2.4 For any point in $\mathcal{M}^{\text{alg}}$ and a choice of disjoint paths from ∞ to points of the corresponding quantum spectrum (see Fig. 1), we obtain a semi-orthogonal decomposition $\mathrm{D}^{\mathrm{b}}(Coh(X)) = \langle \mathcal{C}_1, \ldots, \mathcal{C}_r \rangle$ where r is the number of elements of the spectrum.

All categories $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are saturated (i.e. smooth and proper), equal to local Fukaya-Seidel categories for the mirror LG dual $(Y, W : Y \to \mathbb{C})$, if it exists.

Example 1 (1) $X = \mathbb{P}^n$, the **quantum spectrum** is $\mu_{n+1} = \{z \in \mathbb{C} \mid z^{n+1} = 1\}$ (for some point in $\mathcal{M}$)



This gives $\mathrm{D}^{\mathrm{b}}(Coh X) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$.

(2) Conjectural blow-up formula: If $\widetilde{X} = Bl_Y(X)$ where $Y \subset X$ is a smooth closed subvariety of codimension $m \geq 2$, then the **quantum spectrum** $Spec_{\widetilde{X}}$ looks like with $(m-1)$ shifted copies of $Spec_Y$ around one copy of $Spec_X$. (Here the blue dots correspond to eigenvalues of quantum multiplication added after blow ups.)

(3) If X is a Calabi-Yau manifold or a manifold of general type the **quantum spectrum** is just a point.

(4) The above considerations lead to the following theorem proven in [29]: **MAIN THEOREM**: The splitting $\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}$ is a birational invariant.

2.2 Dimension Theory

In this section, we introduce Serre dimension which (with some exceptions) is equal to the number δ from the noncommutative spectrum. We see that sometimes elementary pieces $\mathcal{C}_a = \mathcal{C}_{z_a}$, $z_a \in Spec_X$ (could be combined as some points of the spectrum collide), are themselves equivalent to derived categories of coherent sheaves on some varieties, of certain dimensions $\leq \dim X$.

In general, for a saturated category $\mathcal{C}$ one can define its **Serre dimension** [49]

$$\dim_{\text{Serre}} \mathcal{C} := \lim_{|k| \to +\infty} \left\{ \frac{i}{k} \mid Ext^i(Id_{\mathcal{C}}, S_{\mathcal{C}}^k) \neq 0 \right\} \subset \mathbb{R}.$$

Here $S_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the Serre functor [48]:

$$Hom_{\mathcal{C}}(E, F)^{\star} = Hom_{\mathcal{C}}(F, S_{\mathcal{C}} E), \quad \forall E, F \in Ob(\mathcal{C}).$$

In general, Serre dimension could be an empty set, or an interval.

For categories $\mathrm{D}^{\mathrm{b}}(Coh(X))$, it is exactly the dimension $\dim X \in \mathbb{Z}_{\geq 0}$. For a fractional Calabi-Yau category $S_{\mathcal{C}}^k \sim [n]$, the Serre dimension is equal to Calabi-Yau dimension $\frac{n}{k}$, hence fractional.

Example 2 Fukaya-Seidel category of $Y = \mathbb{C}_x$, $W = x^d$, $d \geq 2$: $\dim_{\text{Serre}} = 1 - \frac{2}{d}$.

Let us assume that (H, ∇) is a connection with second order pole and regular singularity (i.e. all solutions have polynomial growth). Then the order of growth defines a filtration by subbundles, preserved by connection ∇, the indices form the **subexponential growth spectrum = nc spectrum**.

Essential Example
Consider the hypersurface $X \subset \mathbb{P}^n$ of Calabi-Yau/general type. The connection on the image of $H^\bullet(\mathbb{P}^n)$ in $H^\bullet(X)$ under restriction map, i.e. the span of powers of $c_1(\mathcal{O}(1)) \in H^2(X)$:

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G,\ K = \text{ classical product with } c_1(T_X)$$

The **nc spectrum** is

$$(-\dim X/2, -\dim X/2, \ldots)$$

for X a manifold of general type and so

$$\delta = \dim X.$$

For X a Calabi-Yau manifold **nc spectrum** is

$$(-\dim X/2, 1 - \dim X/2, \ldots, +\dim X/2)$$

and $\delta = \dim X$. Similar behavior happens for Calabi-Yau when we replace the multiplication by $c_1(T_X) = 0$, by the multiplication by an inhomogeneous class $c_1(T_X) + \sum_{i \neq 2}(2 - i)\gamma_i$, $\gamma_i \in H^i(X)$, $i \in 2\mathbb{Z}$.

2.2.1 More General Example

Let us consider a weighted projective space $\mathbb{P}^{\omega_0,\ldots,\omega_n}$ and generic complete intersection X of hypersurfaces of degrees $d_1, \ldots, d_m$. In what follows we investigate the connection between **nc spectrum**, **Givental spectrum** and **Steenbrink spectrum** in this example.

Recall that such a complete intersection is called **well-formed** iff (here unions are understood **with multiplicities**)

$$\bigcup_i \left\{\frac{1}{\omega_i}, \ldots, \frac{\omega_i - 1}{\omega_i}\right\} \subset \bigcup_j \left\{\frac{1}{d_j}, \ldots, \frac{d_j - 1}{d_j}\right\} \qquad \star$$

We call the numbers from $\star$ **Givental spectrum**.

Well formed X is smooth, and does not meet singularities of $\mathbb{P}^{\omega_0,\dots,\omega_n}$. Let us assume that X is a Fano variety, i.e. $\sum_i \omega_i > \sum_j d_j$.

We define the Givental's hypergeometric operator:

$$\prod_i \omega_i^{\omega_i} \cdot \partial^{\dim X} - \prod_j d_j^{d_j} \cdot q \cdot \frac{\prod_j (\partial + \frac{1}{d_j}) \cdots (\partial + \frac{d_j - 1}{d_j})}{\prod_i (\partial + \frac{1}{\omega_i}) \cdots (\partial + \frac{\omega_i - 1}{\omega_i})}, \quad \partial := q\frac{d}{dq}, u = c \cdot q^{-\frac{1}{\sum_i \omega_i - \sum_j d_j}}$$

The **nc spectrum** of the Laplace operator of the Givental's hypergeometric operator is:

$-\frac{\dim X}{2} + \{\text{complement in } (\star)\} \cdot (\sum_i \omega_i - \sum_j d_j) \to$ numbers $s_0 \leq s_1 \leq \cdots$.

The **adjusted Steenbrink spectrum** is:

$(s_0, s_1 + 1, s_2 + 2, \dots)$.

The adjusted Steenbrink spectrum is symmetric with center at 0.

Example 3 Let use consider complete intersection of two hypersurfaces of degree $d_1 = 2, d_2 = 4$ in $\mathbb{P}^6 = \mathbb{P}^6(1, 1, 1, 1, 1, 1, 1)$.

The **growth spectrum** is

$$\left(-\frac{7}{4}, -\frac{6}{4}, -\frac{6}{4}, -\frac{5}{4}\right)$$

In other words the solutions of the quantum equation grow as

$$u^{-\frac{7}{4}}, \log(u)u^{-\frac{6}{4}}, u^{-\frac{6}{4}}, u^{-\frac{5}{4}}$$

.

Adding (0, 1, 2, 3) to **nc spectrum** we obtain **adjusted Steenbrink spectrum**:

$$\left(-\frac{7}{4}, -\frac{1}{2}, +\frac{1}{2}, +\frac{7}{4}\right)$$

.

2.3 Some Computational Tools

We briefly discuss some methods for calculations. We start with:

Theorem 2.5 (Saito's Theorem) ([46]) $P_f(t) = Sp_f(t)$.

Here $P_f(t) = \sum_\alpha (\dim J_\alpha) t^\alpha$ is the Poincare series and $Sp_f(t) = \sum_i (n_i . t^i)$—is the spectrum polynomial and n_i—are the multiplicity of spectral number.

Recall that for $f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda f(x_1, \dots, x_n)$ we define weight $wt.(x_1^{a_1}, \dots, x_n^{a_n}) = \sum_{i=1}^n (1 + a_i) w_i$.

Example 4 Let us look at the example of three dimensional cubic from a new point of view:
$f(x_1, \ldots, x_5) = x_1^3 + \cdots + x_5^3$
$P_f(t) = t^{\frac{5}{3}} + 5t^2 + 10t^{\frac{7}{3}} + 5t^3 + t^{\frac{10}{3}}$
$\delta = \frac{10}{3} - \frac{5}{3} = \frac{5}{3}$.

Let us denote by $Cone(X)$ the cone over a hypersurface X and C is the Fukaya-Seidel category associated with the most singular fiber of the LG model of X. By Orlov's theorem we have $D^b(Cone(X/G)) = C$.

Denote by S_l the lowest number of the Steenbrink spectrum and by S_h the highest number of the Steenbrink spectrum for $Cone(X/G)$. An A-side conjectural version of Orlov's theorem suggests:

Conjecture 2.6 The Steenbrink spectra of $Cone(X)$ determines noncommutative spectrum associated with X. The following identity holds

$$\delta = S_h - S_l.$$

Let $\mathcal{C}$ be a Calabi-Yau category s.t. Serre functor satisfies $S^a = [b]$.
$HH_\bullet(\mathcal{C}) = \oplus HH^i(\mathcal{C})[\delta]$

Definition 2.7 The homomorphism

$$\epsilon : (Q \times \mathbb{Z}_2) \to Aut(\mathcal{C})$$

defines a categorical covering. The covering structure is recorded by multiplication in the A_∞.

In the example 2.8 we get $t^{\frac{10}{3}}$, $t^{\frac{5}{3}}$ define $\frac{10}{3} - \frac{5}{3}$, which produces degree of a covering.

Example 5 $x_1^4 + \cdots + x_5^4$. We consider this hypersurface as an affine cone. We compute the Poincare polynomial and obtain:
$P_f = t^{\frac{5}{4}} + \cdots + t^{\frac{15}{4}} \Rightarrow \delta = \frac{15}{4} - \frac{5}{4}$.

Example 6 $x_1^3 + \cdots + x_5^3$. We consider this hypersurface as an affine cone. Here we can compute the Bernstein polynomial
$b_f(t) = (t+1)(t+2)(t+3)(t+\frac{5}{3})(t+\frac{7}{3})(t+\frac{8}{3})(t+\frac{10}{3})$
and obtain:
$\delta = \frac{10}{3} - \frac{5}{3}$.

2.4 New Nonrationality Results

In this section we record the results of our method and compare them with already known results. We use the simplest of invariants—δ. We hope that other numbers of the noncommutative spectrum can be used as well. In fact it seems that these numbers mirror classical theory of multiplier ideal sheaves and characterize the stratification of the base loci of the anticanonical system for Fano's.

We have defined

$$\delta = dim(X) - 2(N-d)/d$$

As an immediate consequence we get in [29].

Theorem 2.8 *(1) Let X be a Fano smooth hypersurface of degree d in $\mathbb{P}^{5-1}$ such that*

$$d > 5/2.$$

Then X is not rational.

(2) Let X be a Fano smooth hypersurface of degree d in $\mathbb{P}^{6-1}$ such that

$$d \geq 6/2$$

and $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$. Then X is not rational.

(3) Let us assume uppersemicontinuity condition. Let X be a Fano smooth hypersurface of odd dimension and of degree d in $\mathbb{P}^{N-1}$ such that

$$d > N/2$$

Then X is not rational.

(4) Let X be a Fano smooth hypersurface of even dimension $k = (N-2)/2$ and of degree d such that

$$d > N/2$$

and $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$. Then X is not rational.

We briefly describe the idea of the proof.

Proof The above formulae is equivalent to $\delta > dim(X) - 2$.

(1) $dim(X) = 3$ Assume that X is rational so it is obtained via sequence of blow ups and blow downs with centers curves.
According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to 1.
Our **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2 = 1.$$

- a contradiction.

(2) $dim(X) = 4$. Assume $\delta > 2$. The fact that $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ ensures that $\delta > 2$ stays unchanged under deformations. Assume that X is rational so it is obtained via sequence of blow ups and blow downs with centers points, surfaces, curves. According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to 2.
The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2 = 2.$$

- a contradiction.
The case $d = 3$, $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ will be treated in [29]. Let us briefly mention the idea. We have a splitting

$$\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}.$$

With the exception of one all of these H_{λ_i} are one dimensional. The high dimensional one has a symmetric noncommutative Hodge structure. With 20 dimensional space of deformation this noncommutative Hodge structure cannot come from a commutative surface.

(3) $dim(X) = N - 2$, $N - 2$ is odd. In this case $\delta > dim(X) - 2$.
Assume that X is rational so it is obtained via sequence of blow ups and blow downs.
According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to $dim(X) - 2$. According to uppersemicontinuity these asymptotics can only go down. The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2.$$

- a contradiction.

(4) $dim(X) = N - 2 = 2k$, $N - 2$ is even $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$. In this case $\delta > dim(X) - 2$. The fact that $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$ ensures that $\delta > dim(X) - 2$ does not go down.
Assume that X is rational so it is obtained via sequence of blow ups and blow downs.
According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to $dim(X) - 2$. According to uppersemicontinuity these asymptotics can go only down. The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2.$$

- a contradiction.

Similarly we have [29].

Theorem 2.9 *Let X be a smooth Fano complete intersection of hypersurfaces of degrees $d_1, \dots, d_m$ in $\mathbb{P}^N$. Denote by d_t the sum $d_1 + \dots + d_n$ and by d_m the minimal degree.*

In this case the Arnold number (the largest number of the noncommutative spectrum) is equal to:

$$\delta = dim(X) - 2((d_t - d_m)/d_m)$$

.

1. *Let X be 3 dimensional and $\delta > 1$. Then X is not rational.*
2. *Let X be 4 dimensional, $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ and $\delta > 2$. Then X is not rational.*

Let us assume uppersemicontinuity condition.

3. *Let X be of odd dimension and $\delta > dim(X) - 2$. Then X is not rational.*
4. *Let X be of even dimension $2k$, $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$ and $\delta > dim(X) - 2$. Then X is not rational.*

The same result works for well formed complete intersection in weighted projective spaces. The formulae for δ is similar:

$$\delta = \dim X - 2\frac{\omega_{\text{sum}} - d_{\text{sum}}}{d_{\max}}, \quad \omega_{\text{sum}} := \sum_j \omega_j \text{ for } \mathbb{P}^{\omega_0, \dots, \omega_n}$$

3 Application to Arithmetics

The GW invariants can be defined over algebraically nonclosed fields L. Therefore the techniques of noncommutative spectrum can be used to investigate nonrationality over algebraically nonclosed fields L. Of course changing the fields does not change the GW invariants but it changes algebraic cycles. Changing algebraic cycles affects deformations of LG models and as a result the spectrum of quantum multiplication by the canonical class. In this case we do not need an uppersemicontinuity—the restriction on deformation comes from algebraic cycles.

Recall the example from the introduction—the two dimensional cubic: $X : X_0^3 + \dots + X_3^3 = 0$. Consider X over algebraically nonclosed field L s.t. Pic $X_L = 1$. After analyzing the Sarkisov links we conclude that X is not rational.

We will look at this example from the point of view of the spectrum. We begin with:

Theorem 3.1 *Let X be a Fano stack of dimension at most* 4 *over a field L such that image of $CH(X)$ in $\sum_i H^i(X,\mathbb{Z})$ is generated by powers of anticanonical class. Assume that Arnold constant (the highest number in the spectrum) is bigger than $dim(X) - 2$. Then X is not rational.*

The same theorem works in the case when dimension of X is greater than four but with the assumption of uppersemicontinuity condition.

Proof We give a proof under assumption of an isomorphism between the quantum cohomologies and Jacobian ring proven in many cases. The quantum multiplication by the canonical class K corresponds to multiplication of the class of W.

$$\begin{array}{ccc} QH(H^r) & \cong & \mathrm{Jac}\,(W) \\ \text{multi } K & & \text{mult by } W \end{array}$$

$$\begin{array}{ccc} QH & \cong & \mathrm{Jac}\,(W) \\ \cup & & \cup \\ \text{subring} & & \text{subring} \\ \text{generated by } K & & \text{generated by } W \end{array}$$

$$\begin{array}{c} 2 \text{ def of } K \to P \text{ polynomial of } W \\ W + P(W) \\ \text{all deformations} \\ \text{have the same critical values} \\ \text{as } W \end{array}$$

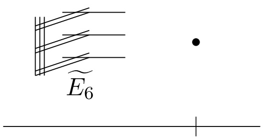
It follows that the spectrum of the most singular fiber of W does not go down since this most singular fiber does not split further under deformations. So we have $\delta > \dim X_L - 2 = 2$.

From another point the main assumption and the fact that we blow up points, curves and surfaces implies that $\delta = 2$—a contradiction. In the case of dimension higher than 4 the proof is the same.

We return to the case of cubic surface. We assume existence of a point in X_L over L. Its Landau–Ginzburg models is:

$$w = \frac{(x+y+1)^3}{xy} \text{ for cubic}$$

$$w = \frac{(x+y+1)^3}{xy} \text{ for cubic}$$


$\widetilde{E}_6$

the deformation
does not change

If the $PicX_L = \mathbb{Z}$ then W have only two singular fibers.
We compute:

$$\delta = 2 - 2\frac{4-3}{3} = \frac{4}{3}$$

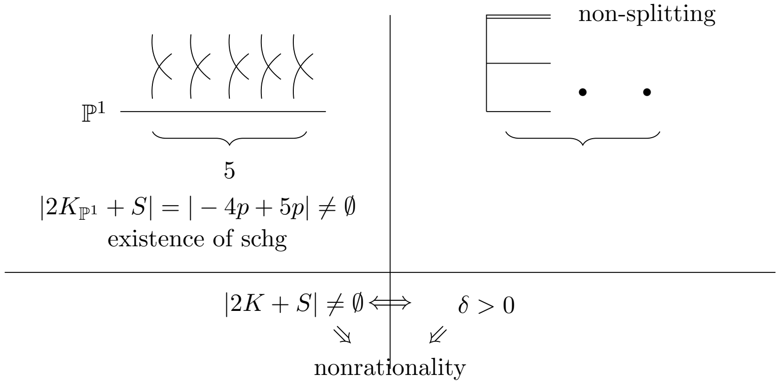
$$\Rightarrow X \text{ is not rational}$$

Since the Pic $X_L = \mathbb{Z}$ the deformation of W is restricted so we cannot morsify and δ does not go down to 0. So X_L is not rational. We move to considering a cubic with Pic $X_L = \mathbb{Z} + \mathbb{Z}$:

(1) In the case Pic $X_L = \mathbb{Z} + \mathbb{Z} \Rightarrow$ we get a conic bundle with 5 singular fibers. By Noether formulae:

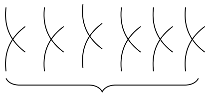
$$8 - S = k^2 = 3,$$

so we have 5 singular fibers. (The classical Iskovskikh criteria $|2K_{\mathbb{P}^1} + S| = |-4p + 5p| \neq \emptyset$ gives nonrationality.)
We will use spectrum in order to compute nonrationality. We compute the Bernstein polynomial for a cubic as an affine cone with a singularity at zero.
We have $8 - C = 3$. $C = 5$ pts.
$|2K + C| = |-4 + 5| = \mathcal{O}_{\mathbb{P}^1}(1) \neq \emptyset$
$f = a^5x^2 + b^5y^2 + c^5z^2$
$f = (s+1)^2(s+2)^2(s+\frac{3}{2})^2 \cdots (s+\frac{3}{10})$
So $\delta = \frac{3}{2} - \frac{3}{10} \neq 0$ and X_L is nonrational.



(2) We consider del Pezzo surface $X_L =$ of degree 4 in $\mathbb{P}^3(1,1,1,2)$ with Pic $X_L = \mathbb{Z} + \mathbb{Z}$ It is a conic bundle with 6 singular fibers. (The classical Iskovskikh criteria $|2K_{\mathbb{P}^1} + S| = |-4p + 6p| \neq \emptyset$ gives nonrationality.)
As before we use the Bernstein polynomial to show that $\delta > 0$ and X_L are not rational.

$$8 - S = K^2 = 2$$

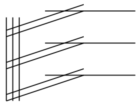


$$6$$

$$|2K + S| \neq \emptyset$$

2 points

$$\delta = 2 - 2\tfrac{5-4}{4}$$

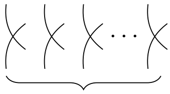


$\longrightarrow$ no splitting

(3) Consider del Pezzo surface X_L of degree 6 in $\mathbb{P}(1, 1, 2, 3)$.

$$8 - S = K^2 = 1$$

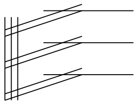
$$S = 7$$



$$7$$

$$|2K + 7| \neq \emptyset$$

$$\delta = 2 - 2\tfrac{7-6}{6}$$



$\longrightarrow$ no splitting

As before we use the Bernstein polynomial to show that $\delta > 0$ and X_L are not rational.

The above observations suggest the following conjecture.

Conjecture 3.2 Let X_L be a conic bundle over $\mathbb{P}^2$ (or another rational surface). Assume that the following holds:

$$\begin{tikzcd} |2K+S| \neq \emptyset \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & nonsplitting \arrow[dl, leftrightarrow] \\ & \delta > dim(X/L) - 2 & \end{tikzcd}$$

Then X_L is not rational.

Let us consider a stack X/G. In this case the GW invariant of X are different from the ones of X/G. From another point the new contributions to cohomologies do form as twisted sectors which do not interact with the quantum span of the anticanonical divisor.

We denote the cohomologies associated to twisted sectors by $H_{\gamma_1}, ... + \cdots , H_{\gamma_k}$. We have the following splitting of quantum cohomologies.

$$QH(X)^G = H + H_{\gamma_1} + \cdots H_{\gamma_k}$$

It leads to the following conjecture.

Conjecture 3.3 Let X/G be a stack defined over a field L such that the image of $CH(X)$ in $\sum_i H^i(X, \mathbb{Z})$ is generated by powers of anticanonical class.
Assume that $\delta > dim(X/G) - 2$. Then X/G is not rational.

The proof is very similar to the proof of the previous theorem. As before we have:

$$\begin{array}{ccc} QH = H + H_{\gamma_1} + \cdots H_{\gamma_k} & \longrightarrow & \text{Jac}\,(W_m) + J_{\gamma_1} + \cdots J_{\gamma_k} \\ < 1, K(1)_1 > \text{ deformed} & \cong & < W_m > + P(W_m) \\ & & = \text{ no new eigenvalues} \end{array}$$

Here we denote by W_m the potential modified by the contributions of the age factors. As before we do not have further splitting of the cohomology and the inequality $\delta > dim(X/G) - 2$ implies nonrationality.

We will look at some examples of del Pezzo stacks.

Using this theorem we consider several examples of del Pezzo stacks—all hypersurfaces in weighted projective $\mathbb{P}^3$. Consider the case of weights: 3, 3, 5, 5 and a hypersurface of degree 15. In this case $\delta = 2 - 2(16 - 15)/15 = 28/15 > 0$ so we have nonrationality. We can compute the spectrum applying theorem 5.5. Using Singular we compute the Steenbrink spectrum of $Cone(X)$-(0, 1), ..., (28/15, 1). So $\delta = 48/15$. We obtain nonrationality.

Remark 3.4 Observe that choice of the field L and the condition $Im(CH \to H) = \langle 1, K(1), K^2(1), \cdots \rangle$ are essential. Without these assumptions the most singular fiber of W_m splits to singularities A_4, A_2, A_2 and further which makes $\delta = 0$.

Similarly consider the weights: 3, 5, 7, 11 and a hypersurface of degree 25. The Steenbrink spectrum of $Cone(X)$ is (0, 1), ..., (48/25, 1). So $\delta = 48/25$. We obtain nonrationality.

This methods work in all Johnson-Kollár examples as well as in higher dimension—for more see [35].

4 Low Dimensional Topology Invariants

We explain a parallel between quantum spectrum and classical 3-dim, 4-dim invariants. First we recall the classical theory. We start with theory of knots and Alexander polynomials. Consider the singular curve:

$$f(z, w) = z^p + w^q, (z, w) \in \mathbb{C}^2$$

$$S_\epsilon = \{|z|^2 + |w|^2 = \epsilon^2\} \subset \mathbb{C}^2, 0 < \epsilon << 1$$

$$K_{p,q} = f^{-1}(0) \cap S_\epsilon \text{ a knot}$$

Alexander polynomial of this torus knot is:

$$\Delta_{p,q} = t^{-\frac{(p-1)(q-1)}{2}} \cdot \frac{(t-1)(t^{pq}-1)}{(t^p-1)(t^q-1)}$$

We define $Sp(f) := \sum_{\alpha\in\mathbb{Q}} n_{f,\alpha} t^\alpha$ the *Steenbrink spectrum*

$$Steen = \{\alpha_1, \alpha_2, \ldots, \alpha_\mu\}, \mu = (p-1)(q-1)$$

Fact $\Delta_{K_{p,q}} = t^{-\frac{\mu}{2}} \prod_{i=1}^{\mu} \Phi_{\alpha_i}(t)$, $\Phi_{\alpha_i}(t) = (t - e^{2\pi i\alpha_i})$

Example 7 *((p, q) = (2, 3))*

$$\Delta_{K_{2,3}} = t^{-\frac{\mu}{2}} \frac{(t^6-1)(t-1)}{(t^2-1)(t^3-1)} = t^{-\frac{\mu}{2}}(t - e^{2\pi i\frac{5}{6}})(t - e^{2\pi i\frac{7}{6}})$$

$Steen = \{\frac{5}{6}, \frac{7}{6}\}$. Also using Thom-Sebastiani theorem we get:

$$Steen = \{Steen(z^2)\} + \{Steen(w^3)\} = \left\{\frac{1}{2}\right\} + \left\{\frac{1}{3}, \frac{2}{3}\right\} = \left\{\frac{5}{6}, \frac{7}{6}\right\}$$

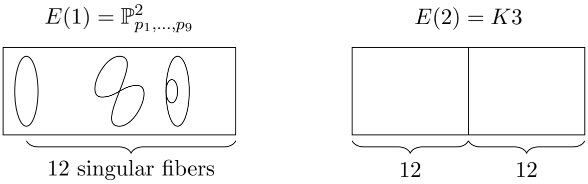
Example 8 *((p, q) = (2, 5))*

$$\Delta_{K_{2,5}} = t^{-\frac{\mu}{2}} \frac{(t^{10}-1)(t-1)}{(t^2-1)(t^5-1)}$$

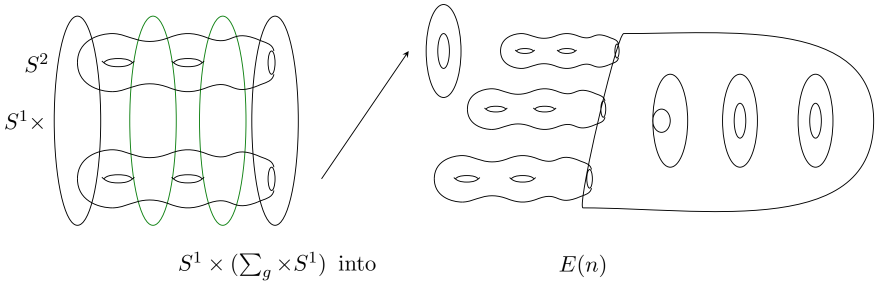
$Steen = \{\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\}$. Using Thom-Sebastiani we get:

$$Steen(z^2 + w^5) = \{Steen(z^2)\} + \{Steen(w^5)\} = \left\{\frac{1}{2}\right\} + \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}$$

We move 1 dimension higher. Consider an elliptic surface $E(n)$: an elliptic fibration.



We describe fibered knot surgery and its connections with Seiberg Witten invariants SW.



Under surgery:

$$SW_{E_K(n)} = \sum_{K \in \mathbb{Z}} SW(K[F])t^K = SW_{E(n)}(t)\Delta_K(t), SW_{E(n)} = (t - t^{-1})^{n-2}$$

where F is the fiber of $E_K(n)$.

Theorem 4.1 (Gr = SW) *Coefficients of Δ_K count holomorphic curves $g = 1$ in the class $K[F]$ in $E_K(n)$.*

We explore the connection with spectra. Recall that:

$$\begin{array}{ccc} \sum_g & \to & S^3 - K \\ & & \downarrow \\ & & S_1 \end{array}$$

Φ the monodromy of the surgery (char polynomial of $\Delta_k(t)$) produces an endofunctor on $Fuk(\sum_g)$ and $Fuk(Sym^k \sum_g)$ (or $FS(\sum_g)$?).

Conjecture 4.2 Φ defines filtration on $HH(Fuk(\sum_g))$ which corresponds to Steen.

Conjecture 4.3 $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f)$ has a filtration

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f) \supset \mathcal{F}_{\alpha_1} \supset \mathcal{F}_{\alpha_2} \cdots$$

given by the spectra.

Let $\mathcal{F}$ be mirror of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f)$. Consider the quantum differential Eq. 1

$$\{\text{asymptotics of } 2.1\} \leftrightarrow \{\text{Spectrum of } f\}$$

Conjecture 4.4 Entropy of Φ: $\eta(\Phi)$ is the first coefficient of $\Delta_K(t)$.

These simple observations suggest the following questions:

Question 4.5 Does the spectrum define canonical filtration on Floer homology?

Question 4.6 What is the symplectic meaning of this filtration? We expect it is connected with the structures of the Lagrangian skeleta.

We discuss further applications. We define modular spectrum of a link M - link of singularity $X_f \leftarrow Y_{1,q}$ as the Steenbrink $Steen(Y_{1,q})$. We give a brief example to fix notations.

Example 9 $M = \Sigma(2, 3, 5)$

$$Y_{1,q} - E_8$$

$$WRT(M) \leftrightarrow (1, 7, 11, 13, 17, 19, 23, 29)$$

Here $WRT(M)$ is the Witten-Reshetikhin-Turaev (WRT) invariant of the 3-manifold M.

We pose the following:

Question 4.7 Is there a categorical meaning of WRT?

We will discuss some of these questions in the next section.

4.1 Spectra and WRT

Let M be a smooth 3-manifold which is a link of an isolated normal surface singularity in $\mathbb{C}^3$. In the following sections, we study topological invariants of M and their relation to spectra. GPPV invariants[1] $\hat{Z}_b(q)$ [37, 38] are q-series that refine the WRT invariants.

Series $Z_b(q)$ can be expressed as a linear combination of false theta functions in the case of Seifert manifolds with 3 singular fibres. Corresponding theta functions can be conjecturally written as components of a vector-valued modular form, which is know for some examples, including links of ADE singularities [37]. Induced representation of $SL(2, \mathbb{Z})$ is a subrepresentation of $2m$-dimensional Weil representation for some integer m and θ functions are labelled by residue classes modulo $2m$. We are interested in these residue classes for all components of the modular form, not just those that correspond to $\hat{Z}_b$. We call this set *Modular spectrum* for convenience. A precise definition depends on the conjectural existence of a natural vector-valued modular form. It was posed as a question in [37] what is a deeper meaning of these residue classes.

[1] also called BPS q-series or homological blocks.

Example 10 The relation with the spectrum started with an observation about E_8 singularity, defined by the equation $x^2 + y^3 + z^5 = 0$. Its link is a PoincarÃƒÆ’Ã‚Â© homology sphere, Seifert manifold $M(-2, 1/2, 2/3, 4/5)$. WRT invariants of this manifold have been studied in [40]. Lawrence and Zagier defined two functions holomorphic inside the unit circle:

$$\theta_+(\tau) = q^{1/120}(1 + 11q + 19q^3 + 29q^7 - 31q^8 - 41q^{14} - \dots$$
$$\theta_-(\tau) = q^{49/120}(7 + 13q + 17q^2 + 23q^4 - 37q^{11} - 43q^{15} - \dots$$

The first function gives WRT as the radial limits at the roots of unity. Both functions together form a vector-valued modular form for $SL(2, \mathbb{Z})$.

Those functions can be written as a linear combination of theta functions assigned to residue classes modulo 60 (see Sect. 2):

$$\theta_+(\tau) = \theta^1_{30,1}(\tau) + \theta^1_{30,11}(\tau) + \theta^1_{30,19}(\tau) + \theta^1_{30,29}(\tau) + \dots$$
$$\theta_-(\tau) = \theta^1_{30,7}(\tau) + \theta^1_{30,13}(\tau) + \theta^1_{30,17}(\tau) + \theta^1_{30,23}(\tau) + \dots$$

The spectrum of E_8 singularity is

$$\{1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30\}$$

and we can see that the numerators of the elements of spectrum correspond to residue classes of the theta functions while the denominator corresponds to the modulus.

This example can be generalized in two ways. One is the class of Brieskorn homology spheres $x^{p_1} + y^{p_2} + z^{p_3} = 0$ for a_0, a_1, a_2 pairwise coprime. An analogical relation of theta functions and spectrum is true for them as described in Sect. 3. It is remarkable since the spectrum contains negative numbers and this is reflected in topology.

Theorem 4.8 *Let M be a Brieskorn homology sphere, i.e. the link of the singularity X given by the equation* $x^{p_1} + y^{p_2} + z^{p_3} = 0$ *Then*

Modular spectrum of M = Steenbrink spectrum of X.

Another generalization is the class of ADE singularities. Here we need to take a spectrum of a different but related singularity—universal Abelian cover.

Theorem 4.9 *Let M be a link of ADE singularity X and Y be the corresponding maximal Abelian cover. Then*

Modular spectrum of M = Steenbrink spectrum of Y.

This phenomenon can be certainly generalized to Seifert manifolds, where $\hat{Z}_b$ have been explicitly computed recently. For more general plumbed 3-manifolds, the singularities to consider are splice-quotients and their universal covers, where the spectrum is difficult to compute, however much can be said about the topology itself using ideas from singularity theory and simpler invariants than spectrum. For these generalizations, see [32]. On the topology side, since the description of $\hat{Z}_b$ using false theta functions is limited to 3 singular fibres of Seifert fibration on M, we need to replace theta function labels by something more general. The poles of Borel plane [43] seem to be a good candidate.

4.1.1 Theta Functions

We will follow the notation in [37]. In particular we denote $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$.

Definition 4.10 Let m be a positive integer and r a residue class $\mod 2m$. We define weight 1/2 theta function and weight 3/2 unary theta function as (respectively)

$$\theta_{m,r}(\tau, z) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} q^{\ell^2/4m} y^{\ell}; \; \theta^1_{m,r}(\tau) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} \ell\, q^{\ell^2/4m}, \tag{2}$$

Unary theta functions form a (rank $2m$) vector-valued modular form of weight 3/2. Its matrices S and T define *Weil representation* of $\widetilde{SL}(2, \mathbb{Z})$, the double cover of $SL(2, \mathbb{Z})$.

Definition 4.11 *False theta function (or Eichler integral)* of $\theta_{m,r}$ is

$$\Psi_{m,r}(\tau) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} \operatorname{sgn}(\ell)\, q^{\ell^2/4m}. \tag{3}$$

False theta functions keep a weaker modular property—quantum modularity [41]. Note also the obvious relations:

$$\Psi_{m,r}(\tau) = \Psi_{m,-r}(\tau) \tag{4}$$

$$\Psi_{m,r+2m}(\tau) = \Psi_{m,r}(\tau) \tag{5}$$

The basic idea is the correspondence $\frac{r}{m}$ as an element of the spectrum of certain singularity related to the 3-manifold and $\Psi_{m,r}(\tau)$ as an Eichler integral of a certain theta function assigned to a 3-manifold.

4.1.2 GPPV Invariants

A plumbed 3-manifold M admits GPPV invariants [38], which are q-series $\hat{Z}_b(q)$ defined using plumbing graph of M and labeled by elements of $H_1(M)$ or $spin^c$ structures. These invariants can be computed by an explicit integral formula [37]. It is an intriguing question whether the series Z_b can be written as components of (quantum) modular forms.

The vector-valued modular forms described in [37] have usually more components than is the number of $Z_b(q)$ (as in the example E_8 in the introduction). It is not clear what is the meaning of these components for the 3-manifold and how to get an intrinsic definition of them.

4.1.3 Example of Brieskorn Homology Sphere $\boldsymbol{\Sigma(3,4,5)}$

Here we give an example of theorem 4.8. Homology sphere $\Sigma(3,4,5)$ is the link of $x^3+y^4+z^5=0$. This case has been studied in [37], p. 67. They describe a representation of $\widetilde{SL}(2,\mathbb{Z})$ given by theta functions $\theta^1_{m,r}$ and corresponding false theta functions $\Psi_{m,r}$. The number m is $3\cdot 4\cdot 5=60$.

False theta functions:

$$\begin{gathered}
\Psi_{60,1}-\Psi_{60,31}-\Psi_{60,41}-\Psi_{60,49}\\
\Psi_{60,2}+\Psi_{60,22}+\Psi_{60,38}+\Psi_{60,58}\\
\Psi_{60,7}+\Psi_{60,17}+\Psi_{60,23}-\Psi_{60,47}\\
\Psi_{60,11}+\Psi_{60,19}+\Psi_{60,29}-\Psi_{60,59}\\
\Psi_{60,13}-\Psi_{60,37}-\Psi_{60,43}-\Psi_{60,53}\\
\Psi_{60,14}+\Psi_{60,26}+\Psi_{60,34}-\Psi_{60,46}
\end{gathered}$$

If we use the relation $\Psi_{m,2m+r}=\Psi_{m,r}$ and multiply first and fifth row by -1 (change of the basis of the representation) we obtain

$$\begin{gathered}
\Psi_{60,-1}+\Psi_{60,31}+\Psi_{60,41}+\Psi_{60,49}\\
\Psi_{60,2}+\Psi_{60,22}+\Psi_{60,38}+\Psi_{60,58}\\
\Psi_{60,7}+\Psi_{60,17}+\Psi_{60,23}+\Psi_{60,73}\\
\Psi_{60,11}+\Psi_{60,19}+\Psi_{60,29}+\Psi_{60,61}\\
\Psi_{60,-13}+\Psi_{60,37}+\Psi_{60,43}+\Psi_{60,53}\\
\Psi_{60,14}+\Psi_{60,26}+\Psi_{60,34}+\Psi_{60,46}
\end{gathered}$$

Now the labels r of $\Psi_{m,r}$ are exactly the numerators of the elements of Steenbrink spectrum of $x^3+y^4+z^5=0$. The terms in each sum correspond to the orbits of a natural action of $\mathbb{Z}_2^2$ on the spectrum. Note that since the theta functions only depend on $r \pmod{2m}$ the relevant spectrum is spectrum modulo 2 (we cannot hope to recover the full Hodge-theoretic information from topology).

The series $Z_0(q)$ is at the fifth row. It contains the term labelled by the smallest number in the spectrum: $-13/60$.

Remark 4.12 As conjectured in [37], components of the representation should correspond to non-abelian $SL(2,\mathbb{C})$ connections (it is true for Brieskorn spheres). If

Table 1 Labels of false theta functions for M, the link of singularity X, correspond to the spectrum of the universal Ab. cover Y of X

Manifold M	X	Y	False thetas of M	Spectrum of Y
Lens space	A_n	$\mathbb{C}^2$	No thetas	Empty
$M(-2; \frac{1}{2}, \frac{1}{2}, \frac{n-3}{n-2})$	D_n	A_{n-3}	$\Psi_{1,n-2}, \Psi_{2,n-2}, \dots, \Psi_{n-3,n-2}$	$(1, 2, \dots, n-3)/(n-2)$
$M(-2; \frac{1}{2}, \frac{2}{3}, \frac{2}{3})$	E_6	D_4	$\Psi_{6,1} + \Psi_{6,5}, 2\Psi_{6,3}$	$(1, 3, 3, 5)/6$
$M(-1; \frac{1}{2}, \frac{2}{3}, \frac{3}{4})$	E_7	E_6	$\Psi_{12,1} + \Psi_{12,7}, \Psi_{12,4}+$ $\Psi_{12,8}, \Psi_{12,5} + \Psi_{12,11}$	$(1, 4, 5, 7, 8, 11)/12$
$\Sigma(2, 3, 5)$	E_8	E_8	10, [40]	$(1, 7, 11, 13, 17, 19, 23, 29)/30$

we use this identification and restrict it to real connections, we recover the classical relation of the signature of Milnor fiber of the Brieskorn singularity and Casson invariant of M [44].

4.1.4 ADE Singularities

Before we get to the relation of GPPV and the spectrum, we need to recall the notion of universal Abelian cover of an isolated singularity (see, for example, [42]). Recall that a closed oriented 3-manifold M is a $\mathbb{Q}$-homology sphere if $H_*(M, \mathbb{Q}) = H_*(S^3, \mathbb{Q})$.

Definition 4.13 Let X be a germ of an isolated normal surface singularity whose link M is a $\mathbb{Q}$-homology sphere. The universal Abelian cover Y of X is a maximal Abelian cover of the germ ramified at the singular point.[2]

$\hat{Z}_b$ and modular forms of the links of ADE singularities were computed in [37], see also [39]. Using their results, we obtain Theorem 4.9. All ADE singularities, their Abelian covers and invariants are summarized in Table 1.

4.2 Topological Invariants of Plane Curve Singularity

We give some ideas of the categorical origin of these topological invariants. Let $C = \{f(x, y) = 0\}$ be a germ of a plane curve having an isolated singularity at the origin p and $L_{C,p}$ be an algebraic link of the plane curve singularity. There have been lots of works studying relations between algebraic geometry of C and topology of $L_{C,p}$. For example, the Alexander polynomial of $L_{C,p}$ can be computed via the ring of functions $\mathcal{O}_C$ thanks to the works of Campillo-Delgado-Gusein-Zade (cf. [5]) and the HOMFLY-PT polynomial of $L_{C,p}$ can be expressed in terms of Hilbert schemes of the plane curve singularity thanks to the works of Oblomkov-Shende (cf. [45]) and Maulik (cf. [21]). On the other hand, there have been lots of interests in mirror

[2] The covering group is then $H_1(M, \mathbb{Z})$.

symmetry of hypersurface singularities these days (see [15] and references therein for more details) and plane curve singularities again have provided natural testing grounds for mirror symmetry conjecture. Takahashi conjectured that for an invertible polynomial f, the category of graded matrix factorization $\mathrm{HMS}^{L_f}(f)$ will be equivalent to the Fukaya-Seidel category $\mathrm{Fuk}^{\to}(f^T)$ of the Berglund-Hübsch mirror polynomial f^T and recently there have been lots of works in this direction and both categories have been intensively studied. For example, it turns out that $\mathrm{HMS}^{L_f}(f)$ has a full exceptional collection and admits a Gepner type stability condition when f is of ADE type. Here, we will discuss the relation between Hilbert schemes of plane curve singularities, certain topological data of some algebraic links, and matrix factorizations.

To be more precise, we will consider the images of ideals which belong to certain Hilbert scheme $C_p^{[*]}$ in the category $\mathrm{HMF}^{L_f}(f)$ when $f = x^2 + y^3$. Then we can check that the images have interesting properties. For example, a natural stratification on (some parts of) the Hilbert scheme $C_p^{[*]}$ corresponds to an indecomposable object in $\mathrm{HMS}^{L_f}(f)$. We can also verify that the difference between the Alexander polynomial and the HOMFLY-PT polynomial of $L_{C,p}$ can be expressed in terms of $\mathrm{HMF}^{L_f}(f)$.

4.2.1 Hilbert Schemes

Let $C = \{f(x, y) = 0\}$ be the germ of a plane curve with an isolated singularity at the origin at $p = (0, 0)$.

Definition 4.14 Let $C_p^{[l]}$ be the Hilbert scheme of length l zero dimensional subschemes of C which are set-theoretically supported at p. And let $C_p^{[*]} := \bigcup_l C_p^{[l]}$.

The normalization induces an embedding $\mathcal{O}_C \to \mathbb{C}[[t]]$. And the natural valuation induces a valuation $\mathcal{O}_C \to \mathbb{N}$. Let $\Gamma = \nu(\mathcal{O})$ be the semigroup. Let $I \subset \mathcal{O}_C$ be a L_f-graded ideal. Then $\mathcal{O}_C/I$ gives an element in $D_{\mathrm{sg}}^{L_f}(R_f)$.

Proposition 4.15 *Let f be a weighted homogeneous polynomial. Then there is a $\mathbb{C}^*$-action on $C_p^{[*]}$. A $\mathbb{C}^*$-invariant ideal gives an $\mathbb{Z}$-graded ideal.*

Proof The obvious $\mathbb{C}^*$-action on f induces an action on $C_p^{[*]}$ and having a $\mathbb{C}^*$-action is equivalent to having a $\mathbb{Z}$-grading.

The following remark tells us that not all ideals of $\mathcal{O}_C$ give nontrivial elements in $\mathrm{HMF}^{L_f}(f)$.

Remark 4.16 Let g be a nonzero divisor in $\mathcal{O}_C$. Then $\mathcal{O}/(g)$ is a perfect complex.

Proof We have the following short exact sequence.

$$0 \to \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_C/(g) \to 0$$

Therefore $\mathcal{O}/(g)$ is a perfect complex.

4.2.2 Example $f = x^2 + y^3$

We can compute L_f as follows.

$$L_f = \mathbb{Z}\overrightarrow{x} \oplus \mathbb{Z}\overrightarrow{y} \oplus \mathbb{Z}\overrightarrow{f}/(\overrightarrow{f} - 2\overrightarrow{x} - 3\overrightarrow{y}) \cong \mathbb{Z}$$

$$R_f = \mathcal{O}_C = \mathbb{C}[[x, y]]/(x^2 + y^3) = \mathbb{C}[[t^2, t^3]]$$

There is a stratification on the Hilbert scheme as follows.

$$(1)$$

$$(t^i + ut^{i+1}), \quad i \geq 2, u \in \mathbb{C}$$

$$(t^i, t^{i+1}), \quad i \geq 2$$

The $\mathbb{C}^*$-invariant parts of the Hilbert scheme are as follows.

$$(1)$$

$$(t^i), \quad i \geq 2$$

$$(t^i, t^{i+1}), \quad i \geq 2$$

The semigroup Γ is $\{0, 2, 3, 4, 5, 6, 7, \cdots\}$.

The Koszul resolution of $\mathbb{C}[[x, y]]/(x, y)$ induces an L_f-graded matrix factorization $F = (F_0, F_1, f_0, f_1)$ of f where $P(\overrightarrow{f}) := S(-\overrightarrow{x}) \oplus S(-\overrightarrow{y})$ and

$$F_0 := S \oplus \wedge^2 P(\overrightarrow{f}), \quad F_1 := P(\overrightarrow{f}).$$

Proposition 4.17 *The matrix factorizations correspond to the ideal* (t^i, t^{i+1}) *is the image of the above matrix factorization under the autoequivalence* $(\overrightarrow{l})$ *for some* $\overrightarrow{l} \in L_f$.

Proof Let $M = \mathbb{C}[[x, y]]/(x, y)$. Let M^{stab} be the above matrix factorization. Note that (t^i, t^{i+1}) is isomorphic to (t^2, t^3) as an R_f-modules. The only difference between them is grading and hence we obtain the desired conclusion.

Proposition 4.18 *The ideal* (t^i, t^{i+1}) *is an exceptional object in* $\mathrm{HMF}^{L_f}(f)$.

Proof Because $\mathbb{C}[[x, y]]/(x, y)$ is an exceptional object (cf. [16]), we see that (t^i, t^{i+1}) is also exceptional.

Then we have the following.

Corollary 4.19 *The ideal* (t^i, t^{i+1}) *is an indecomposable object in* $\mathrm{HMF}^{L_f}(f)$.

It is well-known that there are only finitely many indecomposable objects in $\mathrm{HMF}^{L_f}(f)$ up to autoequivalences.

Theorem 4.20 *The difference between the Alexander polynomial and the HOMFLY-PT polynomial is a categorical invariant.*

Proof The difference between the Alexander polynomial and the HOMFLY-PT polynomial of $L_{C,p}$ is the integration over ideals of type (t^i, t^{i+1}). And every element of the form (t^i, t^{i+1}) can be obtained from (t^2, t^3) by applying translations. From the above discussion, we see that these ideals give nontrivial elements in $\mathrm{HMF}^{L_f}(f)$. Therefore, one can see that the difference can be written in terms of $\mathrm{HMF}^{L_f}(f)$.

5 Generalization of Spectra

We extend the connection of spectra with Alexander polynomial initiated in the previous section. We extend the correspondence:

$$\boxed{\text{Multivariable Alexander Polynomials}} \longleftrightarrow \boxed{\text{multispectra}}$$

Theorem of Libgober [26] says that we can associate to spectrum of $f_1, f_2, \ldots \leftrightarrow$ faces of quasiadjunction. We will give a categorical version of this process:

5.1 Splitting of a Potential

Consider a Landau–Ginzburg model with a potential $W = W_1 + W_2$ We consider the associated Fukaya-Seidel categories $FS(W_1)$, $FS(W_2)$, $FS(W)$.

We start with the tower:

$$\begin{tikzcd} FS(W_1+W_2) \arrow[r] \arrow[d] & FS(W_1) \arrow[d] \\ FS(W_1) \arrow[r] & FS(W_1 \cap W_2) \end{tikzcd}$$

Example 11 *($X_3^5 \subset \mathbb{P}^6$ 5-dim cubic)*

$$\mathrm{D^b}(X_3^5) \cong FS(W_1 + W_2)$$
$$\mathrm{D^b}(X_6^4) \cong FS(W_1)$$
$$\mathrm{D^b}(X_6^4) \cong FS(W_2)$$

Conjecture 5.1 The NC spectra of X_3^5 is a superposition of X_6^4 and X_6^4.

We have the P.D.E.

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G$$

Conjecture 5.2 The P.D.E. of X_6^4 and P.D.E. of X_6^4 produce the P.D.E. of X_3^5 via convolution.

$$PDE(X_6^4) *_A PDE(X_6^4) \cong PDE(X_3^5)$$

We see that asymptotics are superposition of asymptotics.

Corollary 5.3 *Let $\widetilde{\mathbb{P}_X^N}$ is a blow-up of $\mathbb{P}^N$ along X. Then the faces of quasiadjuction contain*

$$(-(\dim X)/2, \ldots, -(\dim X)/2)$$

In general, we have

$$Spec(\{A_i\}) \twoheadrightarrow Spec(\{K\})$$

Here the algebra $\{K\}$ is the algebra generated by canonical bundle. $\{A_i\}$ is the algebra generated by algebraic cycles. The above epimorphism defines a deeper filtration.

Question 5.4 Is this new filtration a birational invariant?

Question 5.5 Does the algebra defined by splitting produce birational invariants?

We consider the example of 5-dim cubics.

$$\begin{tikzcd}
 & \mathrm{D^b}(X_3^5) & \\
\mathrm{D^b}(X_{3,2}^4) \arrow[ur, hook] & & \mathrm{D^b}(X_{3,2}^4) \arrow[ul, hook']
\end{tikzcd}$$

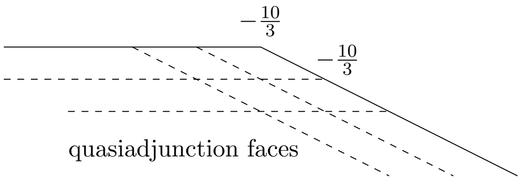
$$\delta_1(X_3^5) = \frac{7}{3}$$

$$\delta_1(X_{3,2}^5) = 4 - 2\frac{6-3-2}{3} = \frac{10}{3}$$

$$\delta_1(X_3^5) = \frac{7}{3}$$

$$\delta_1(X_{3,2}^5) = 4 - 2\frac{6-3-2}{3} = \frac{10}{3}$$

We compute the quasiadjunction of the above splitting.



Observation
We notice that in the above spliting $-(\dim X)/2, \dots, (\dim X)/2$ do not belong to quasiadjunction faces of the polygon. This suggest a different proof of the nonrationality 3-dimensional cubic.

5.2 *Category Filtrations*

For a category $\mathcal{C}$ and A, B and a noncommutative Hodge structure $\mathcal{H}$, ∇, $Herm > 0$, we define a sequence of stability conditions $\mathcal{J}_1, \dots, \mathcal{J}_k$ corresponding to asymptotics of stability spectrum.

We consider the asymptotics of integral $\int_{\Gamma'(0)} \alpha_{(0)} \sim$ Asymptotics at $z = 0$. These asymptotics define

stability spectrum.

Example 12 Consider the category A_n—1 dimensional Fukaya-Seidel categories. So we have $x^j e^{\frac{p}{u}} dx$ is a stability condition. Here p is a polynomial of degree $<$ $(n-1)$.

Step 1 We have $\alpha = dx$.
Step 2 We move to define Kähler metric on moduli space of stability conditions. We begin with $K_{ij}(u, \bar{u}) = \iint_{\mathbb{C}} x^i x^j e^{\frac{p}{u} - \frac{\bar{p}}{\bar{u}}} dx d\bar{x}$

$$\Phi : |u| \leq 1 \to GL(n+1, \mathbb{C})$$

$$\forall |u| = 1,\ \Phi(u)\Phi^t(u) = K_{ij}$$

We define Hermitian form

$$H(u) = \Phi(u)\Phi^t(u)$$

$$\text{Asymptotics} \int x^i e^{\frac{p}{u}} dx$$

define asymptotics and the noncommutative spectrum.

As we saw the asymptotics of the integral $\lim_{n \to 0} Z_n = \sum u^{\alpha_i}$ define stability and nc spectra. We move in to investigate the connection with analysis.

We have the following:

Theorem 5.6 *The stability conditions $\mathcal{J}_1, \ldots, \mathcal{J}_k$ define a filtration on $\mathcal{C}$:*

$$\mathcal{F}_{\leq i}(\mathcal{C}) = \textit{semistable } Obj(\mathcal{E})$$

such that

$$Z_{\mathcal{J}_i}(\mathcal{E}) \leq \mathcal{O}(|\mathcal{J}|^j)$$

This theorem will be discussed in detail in [29]. We will make some use of this filtration in what follows. We consider a Fano X and a splitting of a canonical divisor $K_X = D_1 + D_2$.

$$X - \text{Fano}$$
$$K_X = D_1 + D_2$$

On the mirror side we have spitting of the potential $W = W_1 + W_2$.

$$\begin{tikzcd} FS(W_1) \arrow[r] & FS(W) \\ Fuk(CY) \arrow[u] \arrow[r] & FS(W_2) \arrow[u] \end{tikzcd}$$

Monodromy of W_1 gives a filtration:

$$FS(W_1) \supset \mathcal{F}_{\lambda_1} \supset \cdots \supset \mathcal{F}_{\lambda_n}$$

Monodromy of W_2 gives a filtration:

$$FS(W_2) \supset \mathcal{F}_{\mu_1} \supset \cdots \supset \mathcal{F}_{\mu_n}$$

giving a double filtration

$$FS(W) \supset \mathcal{F}_{\mu_1, \lambda_1} \supset \cdots$$

$$FS(W) \supset \mathcal{F}_{\nu_1} \supset \cdots$$

The behavior of λ_i, μ_j is of Thom Sebastiani type generalized

$$\nu_i \overset{Thom Sebastiani}{=} (\lambda_i, \mu_i)$$

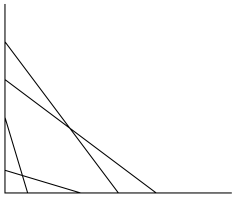
In fact, we have a correspondence:

$$\left\{\begin{matrix} Choices \\ of \\ W_1, W_2, \ldots \end{matrix}\right\} \to \left\{\begin{matrix} generalized \\ Thom Sebastiani \\ \lambda_i\ \mu_i\ \nu_i \\ \vdots\ \vdots\ \vdots \end{matrix}\right\}$$

Question 5.7 Can one produce out of λ_i, μ_i, ν_i new birational invariants?

We discuss briefly a couple of examples.

Example 13 (*Polytope of quasiadjunction* $(x^2+y^3)(x^3+y^2)$)



$(x^2+y^3)(x^3+y^2)$

$$2u+3v = \tfrac{1}{2}\,\tfrac{3}{2}\,\tfrac{5}{2}$$

$$3u+2v = \tfrac{1}{2}\,\tfrac{3}{2}\,\tfrac{5}{2}$$

The Alexander polynomial is:

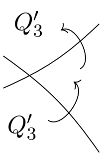
$$(t_1^2t_2^3+1)(t_1^3t_2^2+1)$$

Example 14 (*3-dim cubic*)

$$-K_X = 2H$$

$$f = Q_3'Q_3' \qquad \text{two cubics}$$

$$\begin{matrix} \lambda_1 = \dfrac{5}{3} & & \\ & \to & \delta = \dfrac{5}{3} \\ \lambda_2 = \dfrac{5}{3} & & \end{matrix}$$



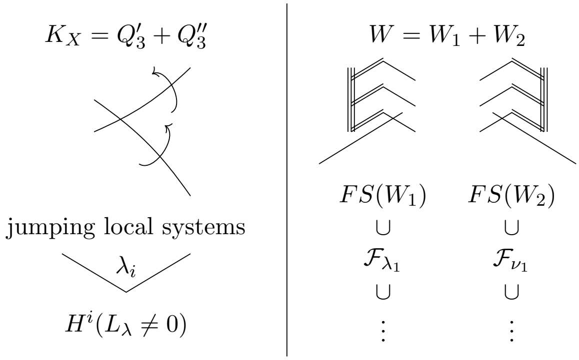
local Alexander polynomials

$$\Downarrow$$

$$\frac{5}{3}$$

Mirror

$$\begin{array}{cccl} W = & W_1 & + W_2 & \\ & | & | & \\ & Q_3'' & Q_3'' & \quad \text{no deformations} \\ & \frac{5}{3} & \frac{5}{3} & \end{array}$$



6 Spectrum, Orbifoldization and Conformal Field Theory

In this section we propose a new point of view of noncommutative spectra. Details will appear elsewhere see e.g. [27, 32].

Our approach is based on the parallel between:

- Birkar's proof [1] of boundness of Fano's.
- Zamolodchikov's [7] c-theorem.

We combine these two directions with categorical resolution of singularities. The final outcome is creating theory of noncommutative spectra similar to Arnold-Varchenko-Steenbrink spectrum.

We will describe a procedure of computing noncommutative spectrum as equivariant part of Steenbrink spectrum of the corresponding affine cone.

Steenbrink Spectrum $\xrightarrow[Equivariant]{Elliptic}$ Noncommutative Spectrum.

We consider the following examples.

1. Let X be a hypersurface (Fermat) of degree d in $\mathbb{P}^N$

$$x_0^d + \cdots x_N^d$$

by Steenbrink $(y^{\frac{1}{d}} + \cdots + y^{\frac{d-1}{d}})^{N+1}$.

This is the fixed part of the Elliptic genus when applied to 5-dim. cubic.

Recall that

$$x_0^3 + \cdots x_6^3 = 0$$

has Steenbrink Spectrum

$$(y^{\frac{1}{3}} + y^{\frac{2}{3}})^7$$

We orbitalize using action of $\mathbb{Z}_3$

$$\frac{1}{3} y^{-\frac{7}{2}} \left(\sum_{0 \le a \le 3} \left(\frac{y^{\frac{1}{3}} - y\omega^{-a}}{y^{\frac{1}{3}} - \omega^{-a}} \right)^7 + \sum \left(y^{\frac{6}{3}} \right)^7 \right)$$

So after that, we get

$$-21(y^{-\frac{7}{2}} + y^{\frac{1}{2}}) + y^{-\frac{7}{6}} + y^{\frac{7}{6}}$$

$$\Rightarrow \left(-\frac{7}{6}, \frac{7}{6} \right) \text{ - noncommutative spectrum}$$

2. Similarly for 2-dim. cubic $y^{-\frac{2}{3}} + 2 + y^{\frac{2}{3}}$.
For K3 $(x_0^4 + \cdots + x_3^4 = 0)$, we have $2y^{-1} + 20 + 2y$.

Proposition 6.1 *For CY, the procedure gives* $-\frac{\dim X}{2}, \ldots, \frac{\dim X}{2}$.

Proposition 6.2 *For general type, the procedure gives* $-\frac{\dim X}{2}, \ldots, \frac{\dim X}{2}$.

Proposition 6.3 *The uppersemicontinuity for Steenbrink spectrum brings uppersemicontinuity for noncommutative spectrum.*

We consider the Berglund-Hübsch Mirror Symmetry.

$$X^\vee = \mathbb{C}^{n+1}/\Gamma \xrightarrow{f} \mathbb{C}$$

where $X^\vee$ is the mirror of $X \subset \mathbb{P}^N$. So we have:

Conjecture 6.4 $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(X^\vee, f)^{\mathrm{eq}} = Fuk^0(X)$.

Now we present a program which not only explains Conjecture 6.1 but suggests a far going program of categorical resolutions. We begin by:

Conjecture 6.5 Let $r : X \to X_{\mathrm{sing}}$ be a resolution of singularity. There exists a category $\mathcal{C}_0$ which does not depend on r.

In the case of orbifold we can be more precise:

Conjecture 6.6 There exists a piece $\mathcal{H}_0 \subset H^i(X)$ which does not depend on r. Then $\mathcal{H}_0 \cong IH(X_{\mathrm{sing}})$.

We have:

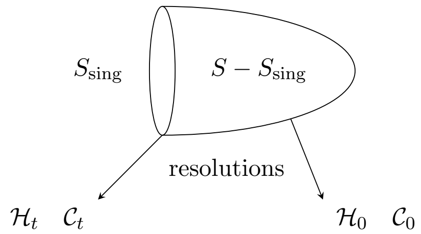
$$H^{\text{String}}(X_{\text{sing}}) = IH(X_{\text{sing}}) + T_{S_1} + \cdots T_{S_w}$$

Here IH are the intersection cohomologies of X. The noncommutative spectrum is defined over $IH(X_{\text{sing}})$. We can combine above conjecture with our orbifoldization procedure. We observe that the twisted sectors we need to take are precisely the ones on which the group acts with determinant equal to one. The above considerations can be lifted to categorical level.

Conjecture 6.7 Consider a resolution $S' \xleftarrow{\text{res}} S$ of terminal singularities. Assume $S - S_{\text{sing}}$ has a volume form. Then

(1) $\mathcal{H}_0$ is independent of r;
(2) $\mathcal{C}_0$ is a CY-category, subcategory of $Perf(X)$ is independent of r.

We would like to make a parallel between Birkar's theory and category theory.



In the above setting $S - S_{\text{sing}}$ determines $\mathcal{H}_0$ and S_{sing} the rest of semi-orthogonal decomposition.

We have a correspondence between classical and categorical notions:

$$K_X, B \longleftrightarrow S_{\text{sing}}$$

$$B'_{\text{complement}} \longleftrightarrow S/S_{\text{sing}}$$

$$\text{volumes} \longleftrightarrow \text{Categorical Entropy } h$$

Let $\mathcal{C}^d_{\mathcal{E}}$ be a log Calabi-Yau category. (We fix the biggest number in the spectra and d is the categorical dimension.)

Question 6.8 Φ is a functor of $\mathcal{C}^d_{\mathcal{E}}$. Are $h(\Phi)$ bounded?

Question 6.9 Is Aut($\mathcal{C}^d_{\mathcal{E}}$) of Jordan type? (Here Aut($\mathcal{C}^d_{\mathcal{E}}$) is the group of autoequivalences).

Question 6.10 Is F($\mathcal{C}^d_{\mathcal{E}}$) a bounding family? (Here F($\mathcal{C}^d_{\mathcal{E}}$) is the family parametrizing the categories with dimension d and bounded the biggest number of the spectra from below. Proper definition will take effort.)

Question 6.11 Consider the splitting

$$\mathcal{C} = \bigcup_{i \geq 0}^{\lambda(\mathcal{E},d)} \mathcal{C}_i$$

$$\mathcal{H} = \bigcup_{i \geq 0}^{\lambda(\mathcal{E},d)} \mathcal{H}_i$$

Show that $\lambda(\mathcal{E}, d)$ is finite.

Question 6.12 Are categorical dimensions of $\mathcal{C}^{\lambda_i}_{\mathcal{E},d}$ bounded?

The above considerations suggest the following parallels.

Fano	Category	CFT
Birkar's Theory $\mathcal{E}, d$ Boundness	σ, d Boundness of log CY theory	Behavior of σ, d theory
Jordan Property of Birational Aut	Jordan Property of Aut D^{b}	
	uppersemicontinuity of Spectra	Zamolodchikov Theorem

The Zamolodchikov's c theorem suggests semicontinuity of the noncommutative spectra—see [6, 8]. This correspondence will be discussed elsewhere.

Our findings in the previous sections suggest that in the case of X, an algebraic surface, we have the following correspondence.

$$\left\{ \begin{array}{c} \text{Additional basic} \\ \text{for } H^2(X) \text{ classes} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{Phantoms} \\ \text{of } \mathrm{D}^{\mathrm{b}}(X) \text{ classes} \end{array} \right\} \qquad \left\{ \ \delta > 0 \ \right\}$$

The above findings suggest that new (A, B) structures can be used to define new invariants, A side invariants for the B side.

We have the following parallel:

Resolution of singularity	Surgery
Creation of Spectra	Creation of Spectra

Conjecture 6.13 Log transform (rational blow down) creates nontrivial $\delta > 0$.

This suggests the following questions.

Question 6.14 Can we have symplectic 4-fold with the same basic classes but different spectra?

We have a connection with k-spectra of CFT. This observations lead to: symplectic Poincare conjectures.

- Find a 4-dim symplectic manifold s.t. $X \overset{\text{homeo}}{\cong} \mathbb{P}^2$ and $\delta(X) > 0$.
- Find a 4-dim symplectic manifold s.t. $X \overset{\text{homeo}}{\cong} \mathbb{P}^1 \times \mathbb{P}^1$ and $\delta(X) > 0$.
- Find a $2n$-dim symplectic manifold s.t. $X \cong \mathbb{P}^n$ and $\delta(X) > 0$.

The parallel between RG flow and Kaehler Ricci flow suggests that the other R-charges can also lead to birational invariants.

Renormalisation group flow and defects lines in the LG model could lead to higher invariants. We investigate these phenomena further in [33].

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References

1. Birkar, C.: Anti-pluricanonical Systems on Fano Varieties, vol. 190, Issue 2, pp. 345–463 (2019)
2. Hassett, B., Pirutka, A., Tschinkel, Yu.: A very general quartic double fourfold is not stably rational. Algebr. Geom. **6**(1), 64–75 (2019)
3. Iskovskikh, V., Pukhlikov, A.: Birational automorphisms of multidimensional algebraic manifolds. Algebraic Geometry, 1. J. Math. Sci. **82**(4), 3528–3613 (1996)
4. Colliot-Thelene, J.-L., Pirutka, A.: Hypersurfaces quartiques de dimension 3: non-rationalite stable. Ann. Sci. Ee. Norm. Super. (4) **49**(2), 371–397 (2016)
5. Campillo, A., Delgado, F., Gusein-Zade, S.M.: The Alexander polynomial of a plane curve singularity via the ring of functions on it. Duke Math. J. **117**(1), 125–156 (2003)
6. Cecotti, S., Vafa, C.: Nucl. Phys. B **367**, 359–461 (1991)
7. Graham Shor, M.: The c and a theorems and the local renormalization group. hep -th. arXiv:1601.06662
8. Gukov, S.: Counting RG flows, hepth arXiv:1503.01474
9. Iskovskih, V., Manin, Yu.: Three-dimensional quartics and counterexamples to the Luroth problem. Mat. Sb. (N.S.) **86**(128), 140–166 (1971)
10. Beauville, A.: Varietes de Prym et jacobiennes intermediaires. Ann. Sci. Ecole Norm. Sup. (4), **10**(3), 309–391 (1977)
11. Grinenko, M.: Mori structures on a Fano threefold of index 2 and degree l. Proc. Steklov Inst. Math. **246**(3), 103–128 (2004)
12. Voisin, C.: Unirational threefolds with no universal codimension 2 cycle. Invent. Math. **201**(1), 207–237 (2015)
13. Voisin, C.: Sur la jacobienne intermediaire du double solide d'indice deux. Duke Math. J. **57**(2), 629–646 (1988)
14. Clemens, H., Griffiths, Ph.: The intermediate Jacobian of the cubic threefold. Ann. Math. **2**(95), 281–356 (1972)

15. Ebeling, W.: Homological mirror symmetry for singularities . arXiv:1601.06027
16. Futaki, M., Ueda, K.: Homological mirror symmetry for Brieskorn-Pham singularities. Sel. Math. **17**(2), 435–452 (2011)
17. Okada, T.: Stable rationality of cyclic covers of projective spaces. Proc. Edinb. Math. Soc. (2) **62**(3), 667–682 (2019)
18. Totaro, B.: Hypersurfaces that are not stably rational. J. Am. Math. Soc. **29**(3), 883–891 (2016)
19. Schreieder, S.: Stably irrational hypersurfaces of small slopes (2018). arXiv:1801.05397
20. Kollar, J.: The rigidity theorem of Fano-Segre-Iskovskikh-Manin-PukhlikovCorti- Cheltsov-de Fernex- Ein- Mustata- Zhuang (2018). arXiv:1807.00863
21. Maulik, D.: Stable pairs and the HOMFLY polynomial. Invent. math. **204**, 787–831 (2016)
22. Nicaise, J., Ottem, J.: Tropical degenerations and stable rationality (2019). arXiv:1911.06138
23. Hassett, B., Pirutka, A., Tschinkel, Yu.: Intersections of three quadrics in $\mathbb{P}^7$ (2017). arXiv:1706.01371
24. Hassett, B., Pirutka, A., Tschinkel, Yu.: A very general quartic double fourfold is not stably rational. Algebr. Geom. **6**(1), 64–75 (2019)
25. Russo, F., Stagliano, G.: Explicit rationality of some cubic fourfolds (2018). arXiv:1811.03502
26. Libgober, A.: Elliptic genus of singular algebraic varieties and quotients. arXiv:1702.03580
27. Katzarkov, L., Lee, K.-S., Libgober, A.: In preparation
28. Katzarkov, L., Kontsevich, M., Pantev, T., Yu, T.: Singularity theory, for LG theory. In preparation
29. Katzarkov, L., Kontsevich, M., Pantev, T., Yu, T.: Blow up formulae. In preparation
30. Katzarkov, L., Kontsevich, M., Pantev, T.: Hodge theoretic aspects of mirror symmetry. From Hodge Theory to Integrability and TQFT: tt*-geometry. In: Proceedings of Symposia in Pure Mathematics, vol. 78 (2008)
31. Katzarkov, L., Liu, Y.: Categorical base loci and spectral gaps, via okounkov bodies and nevanlinna theory L Katzarkov, Y Liu - String-Math (2013)
32. Gukov, S., Katzarkov, L., Svoboda, J.: GPPV invariants and splicing. In preparation
33. Gukov, S., Katzarkov, L., Svoboda, J.: RG and birational invariants. In preparation
34. Katzarkov, L., Prizhalkovsky, V.: tLandau-Ginzburg models old new. In: Giokova Proceedings (2012)
35. Johnson, J., Kollar, J.: Kahler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces
36. Pukhlikov, A.: Birationally rigid Fano cyclic covers (2004). arXiv:0411198
37. Cheng, M.C., Chun, S., Ferrari, F., Gukov, S., Harrison, S.: 3d modularity. JHEP (10)(2018)
38. Gukov, S., Pei, D., Putrov, P., Vafa, C.: BPS spectra and 3-manifold invariants. J. Knot Theor. Ramificat. **2** (2020)
39. Hikami, K.: On the quantum invariants for the spherical seifert manifolds. Commun. Math. Phys. **2**, 285–319 (2006)
40. Lawrence, R., Zagier, D.: Modular forms and quantum invariants of 3-manifolds. Asian J. Math. **1**, 93–108 (1999)
41. Zagier, D.: Quantum Modular Forms. Clay Mathematics Proceedings Volume 12 (2010)
42. Neumann, W.D., Wahl, J.: Universal Abelian Covers of Surface Singularities. Birkhäuser Basel, pp. 181–190 (2002)
43. Gukov, S., Marino, M., Putrov, P.: Resurgence in complex Chern-Simons theory
44. Fintushel, R., Stern, R.J.: Instanton homology of Seifert fibred homology three spheres. Proc. Lond. Math. Soc. **3**, 61 (1990)
45. Oblomkov, A., Shende, V.: The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link. Duke Math. J. **161**(7), 1277–1303 (2012)
46. Singular - handbook
47. Tyurin, A.: On intersections of quadrics. Russ. Math. Surv. **30**(6), 51–105 (1975)
48. Bondal, A.I., Kapranov, M.M.: Representable functors, Serre functors, and mutations. Izv. Akad. Nauk SSSR Ser. Mat. **53**(6), 1183–1205 (1989)
49. Elagin, A., Lunts, V.A.: Three notions of dimension for triangulated categories. J. Algebra **569**, 334–376 (2021). Accessed from 1 March 2021

On Singular Del Pezzo Hypersurfaces of Index 3

In-Kyun Kim, Nivedita Viswanathan, and Joonyeong Won

Abstract We determine the existence of Kähler-Einstein metrics on singular del Pezzo surfaces with quotient singularities which are hypersurfaces of index 3 in weighted projective spaces.

Keywords K-stability · Fano varieties · del Pezzo surfaces

1 Introduction

Let S be a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d, where $a_0 \leq a_1 \leq a_2 \leq a_3$. Then S is given by a quasihomogeneous polynomial

$$f(x, y, z, t) = 0 \subset \mathbb{P}(a_0, a_1, a_2, a_3)$$

of degree d with weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$ and $\mathrm{wt}(t) = a_3$. Since S is quasismooth and well-formed it satisfies the adjunction formula

$$-K_S \sim_{\mathbb{Q}} \mathcal{O}_S(a_0 + a_1 + a_2 + a_3 - d)$$

(see [14, Sect. 6.14]). We define the index of S to be $I = a_0 + a_1 + a_2 + a_3 - d$ and in this paper we only consider the case when I is positive, that is, when S is a singular del Pezzo hypersurface with quotient singularities.

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The problem of determining the existence of Kähler-Einstein metrics on these surfaces has been studied using Sasakian geometry in [3, 4, 25].

For $I = 1$, the existence of Kähler-Einstein metrics on S is completely solved. It has been proven that S admits an orbifold Kähler-Einstein metric if $I = 1$. (See [1, 7–9, 15]).

In the case when $I = 2$ or $I = 3$, the existence of Kähler-Einstein metrics on the surface S has been studied in [4, 8, 9] using α-invariant (See Definition 2.1) under the assumption that $I < 3a_0/2$. This is because $I \geq 3a_0/2$ implies that $\alpha(S) \leq 2/3$ and hence Tian's criterion (see [11, 23, 28]) cannot be used to determine if the surface S admits Kähler-Einstein metrics.

For instance, when

$$(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12) \text{ and equation of } S \text{ does not contain } yzt$$

the δ-invariant has been used to prove that S is Kähler-Einstein [7].

When $I = 2$, it was conjectured in [7] that all such surfaces S admit orbifold Kähler-Einstein metrics.

But this conjecture [7, Conjecture 1.10], was disproven in [16]. That is, if the quintuple (a_0, a_1, a_2, a_3, d) is any one of the following

$$\begin{aligned}
&(1, 6, 9, 13, 27), (1, 9, 15, 22, 45),\\
&(1, 3, 3n + 3, 3n + 4, 6n + 9),\\
&(1, 1, n + 1, m + 1, n + m + 2),\\
&(1, 3, 3n + 4, 3n + 5, 6n + 11)
\end{aligned}$$

where n and m are non-negative integers with $n < m$, then S does not have an orbifold Kähler-Einstein metric.

Later it was verified that these are all the cases when the index $I = 2$ and the surface does not admit a Kähler-Einstein metric (see [17, 22]).

In this paper, we consider hypersurfaces S of index $I = 3$. That is, S is a quasismooth member of family No.i where $\mathsf{i} \in \{1, 1^{\dagger}, 2, \ldots, 17\}$ of Table 1 in Sect. 6. When $\mathsf{i} \in \{11, \ldots, 17\}$ then the existence of Kähler-Einstein metric is proven in [8]. In this paper, we determine the existence of Kähler-Einstein metrics for all the remaining del Pezzo hypersurfaces of index 3. That is, we only consider the case when $\mathsf{i} \in \{1, 1^{\dagger}, 2, \ldots, 10\}$.

Surprisingly the problem of existence of Kähler-Einstein metrics on these surfaces has a very simple answer.

Theorem 1.1 *Suppose that S is quasismooth and has index $I = 3$. Then S admits Kähler-Einstein metrics if and only if $I < 3a_0/2$ or the quintuple* (a_0, a_1, a_2, a_3, d) *is* $(1, 2, 2n + 3, 2n + 3, 4n + 6)$ *where n is a nonnegative integer.*

We therefore expect that there is no Kähler-Einstein del Pezzo hypersurface of very high index if $I \geq 3a_0/2$.

Conjecture 1.2 Let the index I of a singular del Pezzo hypersurface S be large enough. If S has Kähler-Einstein metric then $I < 3a_0/2$.

2 Preliminaries

The Kähler-Einstein K-stability correspondence for Fano varieties has been intensively studied and was primarily motivated by the Chen-Tian-Donaldson conjecture. This conjecture was first proven for Fano varieties with anti-canonical polarisation in [10, 27] and for del Pezzo surfaces with quotient singularities in [20, 21].

In this section, we will introduce the various invariants that are used in this paper to prove Theorem 1.1.

Notations: Throughout the paper we use the following notations:

- $\mathbb{P}(a_0, a_1, a_2, a_3)$ is the weighted projective space where a_0, a_1, a_2 and a_3 are positive integers such that $a_0 \leq a_1 \leq a_2 \leq a_3$.
- x, y, z and t are the weighted homogeneous coordinates of $\mathbb{P}(a_0, a_1, a_2, a_3)$ with weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$ and $\mathrm{wt}(t) = a_3$.
- $S \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ denotes a quasismooth hypersurface given by a quasihomogeneous polynomial of degree d.
- H_* is the hyperplane that is cut out by the equation $* = 0$ in S.
- p_x denotes the point on S given by $y = z = t = 0$. The points p_y, p_z and p_t are given by $x = z = t = 0$, $x = y = t = 0$, $x = y = z = 0$.
- $-K_S$ denotes the anti-canonical divisor of S.

Let X be a $\mathbb{Q}$-Fano variety, i.e. a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities such that $-K_X$ is ample.

2.1 α-Invariant of Tian

Definition 2.1 Let (X, D) be a pair, that is, D is an effective $\mathbb{Q}$-divisor, and let $\mathsf{p} \in X$ be a point. We define the *log canonical threshold* (LCT, for short) of (X, D) and the *log canonical threshold* of (X, D) *at* p to be the numbers

$$\mathrm{lct}(X, D) = \sup\{ c \mid (X, cD) \text{is log canonical} \},$$
$$\mathrm{lct}_{\mathsf{p}}(X, D) = \sup\{ c \mid (X, cD) \text{is log canonical at} \, \mathsf{p} \},$$

respectively. We define

$$\mathrm{lct}_{\mathsf{p}}(X) = \inf\{ \mathrm{lct}_{\mathsf{p}}(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor, } D \equiv -K_X \},$$

and for a subset $\Sigma \subset X$, we define

$$\mathrm{lct}_{\Sigma}(X) = \inf\{\, \mathrm{lct}_{\mathsf{p}}(X) \mid \mathsf{p} \in \Sigma \,\}.$$

The number $\alpha(X) := \mathrm{lct}_X(X)$ is called the *global log canonical threshold* (GLCT, for short) or the *α-invariant* of X.

2.2 δ-Invariant

The δ-invariant of a variety X (see [13] for the definition of the δ-invariant) is called the *stability threshold* because of the following.

Theorem 2.2 ([5, Theorem B]) *Let X be a $\mathbb{Q}$-Fano variety.*

- *X is K-semistable if and only if $\delta(X) \geq 1$;*
- *X is uniformly K-stable if and only if $\delta(X) > 1$.*

By [5, Theorem A], we have the following:

$$\left(\frac{\dim(X)+1}{\dim(X)}\right)\alpha(X) \leq \delta(X) \leq (\dim(X)+1)\alpha(X). \tag{1}$$

We will use this inequality extensively to prove that the surface S is not K-polystable, when S is a quasismooth, well-formed hypersurface of Index 3 that belongs to Family No.i, when $\mathsf{i} \in \{2, \ldots, 10\}$.

2.3 β-Invariant

Let X be a n-dimensional $\mathbb{Q}$-Fano variety. Let E be an arbitrary prime divisor *over* X, that is, there exists a birational morphism $\sigma\colon Y \to X$ with a normal variety Y and $E \subset Y$ is a prime divisor in Y (not necessarily f-exceptional). Let

$$A_X(E) = 1 + \mathrm{ord}_E\left(K_Y - f^*(K_X)\right), \tag{2}$$

and we let

$$S_X(E) = \frac{1}{(-K_X)^n}\int_0^{\tau} \mathrm{vol}(f^*(-K_X) - uE)\,\mathrm{d}u, \tag{3}$$

where $\tau = \tau(E)$ is the pseudo-effective threshold of E with respect to $-K_X$, i.e. we have

$$\tau(E) = \sup\left\{u \in \mathbb{Q}_{>0} \mid f^*(-K_X) - uE \text{ is big}\right\}.$$

Then,

$$\beta(E) = A_X(E) - S_X(E).$$

Then, we have the following.

Theorem 2.3 ([6, 12, 19]) *The following assertions hold:*

- *X is K-stable $\iff$ $\beta(E) > 0$ for every prime divisor E over S;*
- *X is K-semistable $\iff$ $\beta(E) \geqslant 0$ for every prime divisor E over S.*

Using the following criterion, we will establish the existence of Kähler-Einstein metrics on a surface S belonging to Family No $1^\dagger$.

Theorem 2.4 ([29, Corollary 4.14]) *Let G be a reductive subgroup in* $\mathrm{Aut}(X)$. *Suppose that $\beta(E) > 0$ for every G-invariant dreamy prime divisor E (see [12, Definition 1.3] for the definition) over X. Then X is K-polystable.*

In order to compute $\beta(E)$ for divisors E *over* the surface S, we will first describe the properties of the volume function.

Let S be a surface with quotient singularities. And let D be an effective $\mathbb{Q}$-divisor. If D is Cartier then its volume is the number

$$\mathrm{vol}_S(D) = \limsup_{k\in\mathbb{N}} \frac{h^0(\mathcal{O}_S(kD))}{k^2/2!}. \tag{4}$$

When D is a $\mathbb{Q}$-divisor one can define the volume $\mathrm{vol}_S(D)$ of D as in (4) taking the lim sup over those k for which kD is integral. Moreover we can define its volume using the identity

$$\mathrm{vol}_S(D) = \frac{1}{\lambda^2}\,\mathrm{vol}_S(\lambda D)$$

for an appropriate positive rational number λ. The volume of D depends only upon its numerical equivalence class (see [18, Proposition 2.2.41] for details).

If D is not pseudoeffective, then $\mathrm{vol}_S(D) = 0$. If D is pseudoeffective, its volume can be computed using its Zariski decomposition [2, 26]. Namely, if D is pseudoeffective, then there exists a nef $\mathbb{R}$-divisor P on the surface S such that

$$D \sim_{\mathbb{R}} P + \sum_{i=1}^{n} a_i C_i$$

where each C_i is an irreducible curve on S with $P \cdot C_i = 0$, each a_i is a non-negative real number, and the intersection form of the curves $C_1, \ldots, C_n$ is negative definite. Such decomposition is unique, and it follows from [2, Corollary 3.2] that

$$\mathrm{vol}_S(D) = \mathrm{vol}_S(P) = P^2.$$

Next, to find a prime divisor E over S satisfying $\beta(E) < 0$ we consider the following birational morphism. This is used in proving that the surface S which is a quasismooth, well-formed hypersurface of index 3 belonging to Family No. 1, is not K-semistable.

From our assumption, S has a cyclic quotient singularity of type $\frac{1}{m}(a_1, a_2)$ at the point p. Let the weighted coordinates around this point p be x and y. Let $\phi\colon \widetilde{S} \to S$ be the weighted blow-up at p of S with weights $\mathrm{wt}(x) = a_1$ and $\mathrm{wt}(y) = a_2$ and let E be the exceptional divisor. Then we have the following:

$$K_{\widetilde{S}} \sim_{\mathbb{Q}} \phi^*(K_S) + \left(-1 + \frac{a_1}{m} + \frac{a_2}{m}\right) E$$

where E is the exceptional divisor of ϕ and

$$E^2 = -\frac{m}{a_1 a_2}.$$

Let H_x be a curve on S that is locally given by $x = 0$ near p. Then we have

$$\widetilde{H_x} \sim_{\mathbb{Q}} \phi^*(H_x) - \frac{a_1}{m} E$$

where $\widetilde{H_x}$ is the strict transform of H_x.

3 Small α-Invariants

Throughout this section every family is contained in Table 1 of Sect. 6. Denote $\mathsf{I}_\alpha = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. In this section we estimate the α-invariants of quasismooth members of families No. i with $\mathsf{i} \in \mathsf{I}_\alpha$.

Proposition 3.1 *Let S be a quasismooth member of family No.* i *with* $\mathsf{i} \in \mathsf{I}_\alpha$. *Then S is not K-semistable.*

Let S be a quasismooth member of family No. 2. By a suitable coordinate change S is given by a quasihomogeneous polynomial

$$tx + \sum_{i=0}^{n+2} \xi_i z^i y^{2n+4-2i} = 0$$

in $\mathbb{P}(1, 1, 2, 2n+3)$ where n is a nonnegative integer and ξ_i are constants. Let H_x denote the curve in S cut out by $x = 0$. Then H_x is isomorphic to the curve given by a quasihomogeneous polynomial

$$\sum_{i=0}^{n+2} \xi_i z^i y^{2n+4-2i} = 0$$

in $\mathbb{P}(1, 2, 2n+3) = \mathrm{Proj}(\mathbb{C}[y, z, t])$. From this equation we have $\mathrm{lct}(S, H_x) = 3/(2n+4)$. Similarly, we can obtain the following.

Let S be a quasismooth member of family No. i with $\mathsf{i} \in \mathsf{I}_\alpha$. Let H_x denote the curve in S cut out by $x = 0$. Then H_x is isomorphic to the following curve:

- No. 2 : $z^{n+2} + z^n y^2 + \cdots + y^{2n+4} = 0$ in $\mathbb{P}(1, 2, 2n+3)$;
- No. 3 : $z^2 y + zy^{2n+2} + y^{4n+3} = 0$ in $\mathbb{P}(5, 10n+5, 10n+7)$;
- No. 4 : $ty^{2n+2} + z^2 y = 0$ in $\mathbb{P}(5, 10n+7, 10n+9)$;
- No. 5 : $ty^2 + z^3 = 0$ in $\mathbb{P}(7, 9, 13)$;
- No. 6 : $t^2 + ty^2 + y^4 = 0$ in $\mathbb{P}(7, 9, 14)$;
- No. 7 : $t^2 + zy^3 = 0$ in $\mathbb{P}(9, 13, 20)$;
- No. 8 : $t^2 + z^3 = 0$ in $\mathbb{P}(13, 22, 33)$;
- No. 9 : $t^2 + y^5 = 0$ in $\mathbb{P}(14, 23, 35)$;
- No. 10 : $z^3 + y^5 = 0$ in $\mathbb{P}(15, 25, 37)$

where n is a nonnegative integer. Then we have the following:

- No. 2: $\mathrm{lct}(S, H_x) = 3/(2n+4)$;
- No. 3: $\mathrm{lct}(S, H_x) = (2n+2)/(4n+3)$;
- No. 4: $\mathrm{lct}(S, H_x) = (2n+3)/(4n+4)$;
- No. 5: $\mathrm{lct}(S, H_x) = 5/6$;
- No. 6: $\mathrm{lct}(S, H_x) = 3/4$;
- No. 7: $\mathrm{lct}(S, H_x) = 5/6$;
- No. 8: $\mathrm{lct}(S, H_x) = 5/6$;
- No. 9: $\mathrm{lct}(S, H_x) = 7/10$;
- No. 10: $\mathrm{lct}(S, H_x) = 8/15$

where n is a nonnegative integer. Therefore we obtain the following proof.

Proof of Proposition 3.1 Using the above computations and the inequalities in (1) we have,

$$\delta(S) \leq 3\alpha(S) \leq 3\,\mathrm{lct}(S, 3H_x) = \mathrm{lct}(S, H_x) < 1.$$

By Theorem 2.2 S is not K-semistable.

4 *K*-stable Singular del Pezzo Hypersurfaces

Let S be a quasismooth member of family No. $1^\dagger$. By a suitable coordinate change $S \subset \mathbb{P}(1, 2, 2n+3, 2n+3)$ is given by a quasihomogeneous polynomial

$$tz + \prod_{i=1}^{2n+3}(a_i x^2 + b_i y) = 0$$

where n is a nonnegative integer and $[a_i : b_i]$ with $i \in \{1, \dots, 2n+3\}$ are distinct points in $\mathbb{P}^1$. The points p_z and p_t are singular points of type $\frac{1}{2n+3}(1, 2)$.

Proposition 4.1 *S is K-polystable.*

Proof Let G be the subgroup of the $\mathrm{Aut}(S)$ that is generated by ψ and ϕ_ξ, where $\xi \in \mathbb{C}^*$, whose actions on S are as described below,

$$\psi : [x : y : z : t] \longmapsto [x : y : t : z]$$
$$\phi_\xi : [x : y : z : t] \longmapsto [x : y : \xi z : \xi^{-1} t].$$

Let $\mathcal{P} = \mathcal{O}_S(2)$ be the pencil of G-invariant curves on S. That is, any G-invariant curve on S is given by $ax^2 + by = 0$ for some $[a : b] \in \mathbb{P}^1$.

In order to prove Proposition 4.1, we will compute $\beta(E)$ for any G-invariant prime divisor E in the surface S and all the G-invariant prime divisors over the surface S.

Lemma 4.2 *Let E be a G-invariant prime divisor in S. Then $\beta(E) > 0$.*

Proof Consider the divisor $-K_S - uE \equiv (\frac{3}{2} - u)E$. Then, we have that $\tau(E) = \frac{3}{2}$. Therefore,

$$\beta(E) = A_S(E) - \frac{1}{-K_S^2} \int_0^{\frac{3}{2}} \mathrm{vol}_S(-K_S - uE)\, \mathrm{d}u$$
$$= 1 - \int_0^{\frac{3}{2}} \left(1 - \frac{2}{3}u\right)^2 \mathrm{d}u = \frac{1}{2} > 0.$$

Next, we consider a G-invariant prime divisor E over S. This divisor is mapped to G-invariant points on S and are given by $z = t = 0$ in S. Thus the points p_i, given by $a_i x^2 + b_i y = z = t = 0$ in $\mathbb{P}(1, 2, 2n+3, 2n+3)$, are all the points on S fixed by G. Without loss of generality, let P be one such point. Therefore using Theorem 2.4 we consider the blow-ups at P.

Lemma 4.3 *Let $\pi : \widetilde{S} \to S$ be the blow-up at P with exceptional divisor E. Then* $\beta(E) > 0$.

Proof We consider the curve $C_i \in \mathcal{P}$ given by $a_i x^2 + b_i y = 0$ in S, containing P. It can be written as

$$C_i = C_z + C_t \tag{5}$$

where C_z and C_t are given by $a_i x^2 + b_i y = z = 0$ and $a_i x^2 + b_i y = t = 0$ in $\mathbb{P}(1, 2, 2n+3, 2n+3)$, respectively. Then $C_z \cap C_t = P$.

We have the following intersection numbers:

$$\widetilde{C}_z \cdot \widetilde{C}_t = 0, \qquad \widetilde{C}_z^2 = \widetilde{C}_t^2 = -\frac{4n+4}{2n+3}, \qquad \widetilde{C}_z \cdot E = \widetilde{C}_t \cdot E = 1$$

where $\widetilde{C}_z$ and $\widetilde{C}_t$ are the strict transforms of C_z and C_t, respectively. To find $\beta(E)$ we have to calculate $\mathrm{vol}_S(-K_S - uE)$. Consider the divisor

$$\pi^*(-K_S) - uE \equiv \frac{3}{2}\left(\widetilde{C}_z + \widetilde{C}_t\right) + (3-u)E.$$

Since the intersection form of $\widetilde{C}_z$ and $\widetilde{C}_t$ are negative definite we have $\tau(E) = 3$, that is, $\mathrm{vol}_S(-K_S - uE) = 0$ for $u > 3$.

Since

$$\left(\frac{3}{2}\left(\widetilde{C}_z + \widetilde{C}_t\right) + (3-u)E\right) \cdot \widetilde{C}_z = \frac{3}{2n+3} - u,$$
$$\left(\frac{3}{2}\left(\widetilde{C}_z + \widetilde{C}_t\right) + (3-u)E\right) \cdot \widetilde{C}_t = \frac{3}{2n+3} - u,$$

the divisor $\pi^*(-K_S) - uE$ is nef for $0 \leq u \leq \frac{3}{2n+3}$. When $u \in [\frac{3}{2n+3}, 3]$, the Zariski decomposition of the divisor $\pi^*(-K_S) - uE$ is given by

$$\pi^*(-K_S) - uE = (3-u)\left(\frac{2n+3}{4n+4}\left(\widetilde{C}_z + \widetilde{C}_t\right) + E\right)$$
$$+ \left(\left(\frac{6(n+1) + (2n+3)(u-3)}{4(n+1)}\right)(\widetilde{C}_z + \widetilde{C}_t)\right).$$

This implies that

$$\mathrm{vol}_S(-K_S - uE) = \begin{cases} \frac{9}{2n+3} - u^2 & \text{for } 0 \leq u \leq \frac{3}{2n+3}, \\ \frac{1}{2n+2}(3-u)^2 & \text{for } \frac{3}{2n+3} \leq u \leq 3. \end{cases}$$

and therefore,

$$\begin{aligned} S_S(E) =& \frac{1}{(-K_S)^2}\int_0^3 \mathrm{vol}_S(-K_S - uE)\,\mathrm{d}u \\ =& \frac{2n+3}{9}\left(\int_0^{\frac{3}{2n+3}} \frac{9}{2n+3} - u^2\,\mathrm{d}u + \int_{\frac{3}{2n+3}}^3 \frac{1}{2n+2}(3-u)^2\,\mathrm{d}u\right) \\ =& \frac{2n+4}{2n+3}. \end{aligned}$$

Then

$$\beta(E) = A_S(E) - S_S(E) = 2 - \frac{2n+4}{2n+3} = \frac{2n+2}{2n+3} > 0.$$

Therefore by Theorem 2.4, the surface S is K-polystable, thus completing the proof of Proposition 4.1.

5 Non K-semistable Singular del Pezzo Hypersurfaces

In this section we prove that there are prime divisors E over the following singular del Pezzo hypersurfaces such that $\beta(E) < 0$, that is, they are not K-semistable.

Lemma 5.1 *Let S be a quasismooth member of family No. 1. Then S is not K-semistable.*

Proof By a suitable coordinate change we can assume that $S \subset \mathbb{P}(1, 2, 2n+3, 2m+3)$, where n and m are nonnegative integers, is given by a quasihomogeneous polynomial

$$tz + \prod_{i=1}^{n+m+3} (a_i x^2 + b_i y) = 0$$

where $[a_i : b_i]$ are distinct points in $\mathbb{P}^1$.

S is singular at the point p_z and p_t of type $\frac{1}{2n+3}(1, 2)$ and $\frac{1}{2m+3}(1, 2)$. In a neighborhood of p_t, we may regard that x and y are local coordinates.

Let $\pi : \widetilde{S} \to S$ be the weighted blow-up at p_t with weights $\mathrm{wt}(x) = 1$ and $\mathrm{wt}(y) = 2$. Then we have

$$K_{\widetilde{S}} \sim_{\mathbb{Q}} \pi^*(K_S) - \frac{2m}{2m+3} E,$$

where E is the exceptional divisor of π.

Consider the hyperplane H_z given by $z = 0$ in S. Then

$$H_z = \sum_{i=1}^{n+m+3} L_i$$

where L_i with $i \in \{1, 2, \dots, n+m+3\}$ is the curve given by $z = a_i x^2 + b_i y = 0$ in $\mathbb{P}(1, 2, 2n+3, 2m+3)$. Since

$$\widetilde{L}_i \sim_{\mathbb{Q}} \pi^*(L_i) - \frac{2}{2m+3} E,$$

where $\widetilde{L}_i$ is the strict transform of L_i on $\tilde{S}$, we have

$$\pi^*(H_z) \sim_{\mathbb{Q}} \sum_{i=1}^{n+m+3} \widetilde{L}_i + \frac{2(n+m+3)}{2m+3} E.$$

Also the various intersection numbers are

$$\widetilde{L}_i^2 = -1, \qquad \widetilde{L}_i \cdot \widetilde{L}_j = 0, \qquad \widetilde{L}_i \cdot E = 1$$

where $i \neq j$.

Since $-K_S \equiv \frac{3}{2n+3}H_z$, we consider the divisor

$$\pi^*(-K_S) - uE \equiv \frac{3}{2n+3}\sum_{i=1}^{n+m+3}\widetilde{L}_i + \left(\frac{6(n+m+3)}{(2n+3)(2m+3)} - u\right)E.$$

Since the intersection form of the curves $L_1, L_2, \ldots, L_{n+m+3}$ is negative definite we have $\tau(E) = \frac{6(n+m+3)}{(2n+3)(2m+3)}$, that is, $\mathrm{vol}_S(\pi^*(-K_S) - uE) = 0$ when $u > \tau(E)$.

Meanwhile for every $i \in \{1, 2, \ldots, n+m+3\}$, we determine the nefness of the divisor $\pi^*(-K_S) - uE$, by intersecting the divisor with the curves L_i as below.

$$(\pi^*(-K_S) - uE)\cdot L_i = -\frac{3}{2n+3} + \left(\frac{6(n+m+3)}{(2n+3)(2m+3)} - u\right) = \frac{3}{2m+3} - u.$$

Therefore, if $u \in [0, \frac{3}{2m+3}]$, $\pi^*(-K_S) - uE$ is nef. When $u \in [\frac{3}{2m+3}, \frac{6(n+m+3)}{(2n+3)(2m+3)}]$, the Zariski decomposition of the divisor is as below.

$$\begin{aligned}\pi^*(-K_S) - uE = &\left(\frac{6(n+m+3)}{(2n+3)(2m+3)} - u\right)\left(\sum_{i=1}^{n+m+3}\widetilde{L}_i + E\right)\\ &+ \left(u - \frac{3}{2n+3}\right)\sum_{i=1}^{n+m+3}\widetilde{L}_i.\end{aligned}$$

Thus

$$\mathrm{vol}_S(-K_S - uE) = \begin{cases} \frac{9(n+m+3)}{(2n+3)(2m+3)} - \frac{2m+3}{2}u^2 & \text{for } u \in [0, \frac{3}{2m+3}],\\ \left(\frac{6(n+m+3)}{(2n+3)(2m+3)} - u\right)^2\left(\frac{2n+3}{2}\right) & \text{for } u \in [\frac{3}{2m+3}, \frac{6(n+m+3)}{(2n+3)(2m+3)}].\end{cases}$$

From these we have

$$\begin{aligned}S_S(E) =& \frac{1}{-K_S^2}\int_0^{\tau(E)} \mathrm{vol}_S(-K_S - uE)\,\mathrm{d}u\\ =& \frac{(2n+3)(2m+3)}{9(n+m+3)}\left[\int_0^{\frac{3}{2m+3}} \frac{9(n+m+3)}{(2n+3)(2m+3)} - \frac{2m+3}{2}u^2\,\mathrm{d}u\right.\\ &\left.+ \int_{\frac{3}{2m+3}}^{\tau(E)}\left(\frac{6(n+m+3)}{(2n+3)(2m+3)} - u\right)^2\left(\frac{2n+3}{2}\right)\mathrm{d}u\right]\\ =& \frac{3}{2m+3} - \frac{2n+3}{2(2m+3)(n+m+3)} + \frac{2m+3}{2(2n+3)(n+m+3)}.\end{aligned}$$

Since $A_S(E) = \frac{3}{2m+3}$, we have that,

$$\beta(E) = A_S(E) - S_S(E) = \frac{2n+3}{2(2m+3)(n+m+3)} - \frac{2m+3}{2(2n+3)(n+m+3)} < 0.$$

By Theorem 2.3 S is not K-semistable.

6 Table

In [24] there is the list of the quasismooth del Pezzo hypersurfaces with index 3. For the quintuple $(1, 2, 2n+3, 2m+3, 2(n+m)+6)$, where n and m are nonnegative integers, we split it into the two families No. 1 and No. $1^\dagger$.

Each row in Table 1 is the family of quasismooth hypersurfaces in the weighted projective space. Each quadruple of the weights column is the weights of the weighted projective space. And each number of the degree column is the degree of defining equations of quasismooth del Pezzo hypersurfaces. Finally, KE means the existence of orbifold Kähler-Einstein metrics.
where n and m are nonnegative integers with $n < m$.

Table 1 Index 3

No.	Weights	Degree	KE
1	$(1, 2, 2n+3, 2m+3)$	$2(n+m)+6$	No
$1^\dagger$	$(1, 2, 2n+3, 2n+3)$	$4n+6$	Yes
2	$(1, 1, 2, 2n+3)$	$2n+4$	No
3	$(1, 5, 10n+5, 10n+7)$	$20n+15$	No
4	$(1, 5, 10n+7, 10n+9)$	$20n+19$	No
5	$(1, 7, 9, 13)$	27	No
6	$(1, 7, 9, 14)$	28	No
7	$(1, 9, 13, 20)$	40	No
8	$(1, 13, 22, 33)$	66	No
9	$(1, 14, 23, 35)$	70	No
10	$(1, 15, 25, 37)$	75	No
11	$(5, 7, 11, 13)$	33	Yes
12	$(5, 7, 11, 20)$	40	Yes
13	$(11, 21, 29, 37)$	95	Yes
14	$(11, 37, 53, 98)$	196	Yes
15	$(13, 17, 27, 41)$	95	Yes
16	$(13, 27, 61, 98)$	196	Yes
17	$(15, 19, 43, 74)$	148	Yes

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References

1. Araujo, C.: Kähler-Einstein metrics for some quasi-smooth log del Pezzo surfaces. Trans. Am. Math. Soc. **354**, 4303–4312 (2002)
2. Bauer, Th., Küronya, A., Szemberg, T.: Zariski chambers, volumes, and stable base loci. J. Reine Angew. Math. **576**, 209–233 (2004)
3. Boyer, C., Galicki, K.: Sasakian Geometry. Oxford University Press (2008)
4. Boyer, C., Galicki, K., Nakamaye, M.: On the geometry of Sasakian-Einstein 5-manifolds. Math. Ann. **325**, 485–524 (2003)
5. Blum, H., Jonsson, M.: Thresholds, valuations, and K-stability. Adv. Math. **365**, 107062, 57 (2020)
6. Blum, H., Xu, C.: Uniqueness of K-polystable degenerations of Fano varieties. Ann. Math. **190**(2), 609–656 (2019)
7. Cheltsov, I., Park, J., Shramov, C.: Delta invariants of singular del Pezzo surfaces. J. Geom. Anal. **31**(3), 2354–2382 (2021)
8. Cheltsov, I., Park, J., Shramov, C.: Exceptional del Pezzo hypersurfaces. J. Geom. Anal. **20**(4), 787–816 (2010)
9. Cheltsov, I., Shramov, C.: Del Pezzo zoo. Exp. Math. **22**(3), 313–326 (2013)
10. Chen, X.-X., Donaldson, S., Sun, S.: Kähler–Einstein metrics on Fano manifolds. I, II, III. J. Am. Math. Soc. **28**(1), 183–197, 199–234, 235–278 (2015)
11. Demailly, J.-P., Kollár, J.: Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. Ann. Sci. École Norm. Sup. **34**(4), 525–556 (2001)
12. Fujita, K.: A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. J. Reine Angew. Math. **751**, 309–338 (2019)
13. Fujita, K., Odaka, Y.: On the K-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. (2) **70**(4), 511–521 (2018)
14. Iano-Fletcher, A.R.: Working with weighted complete intersections, explicit birational geometry of 3-folds, 101–173. London Mathematical Society Lecture Note Series, vol. 281. Cambridge University Press, Cambridge (2000)
15. Johnson, J., Kollár, J.: Kähler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces. Ann. Inst. Fourier (Grenoble) **51**(1), 69–79 (2001)
16. Kim, I., Won, J.: Unstable singular del Pezzo hypersurfaces with lower index. Commun. Algebra **49**(6), 2679–2688 (2021)
17. Kim, I., Viswanathan, N., Won, J.: K-stability of log del Pezzo hypersurfaces with index 2 (2022). arXiv:2202.03473
18. Lazarsfeld, R.: Positivity in Algebraic Geometry, I. Springer (2004)
19. Li, C.: K-semistability is equivariant volume minimization. Duke Math. J. **166**(16), 3147–3218 (2017)
20. Li, C., Tian, G., Wang, F.: On the Yau-Tian-Donaldson conjecture for singular Fano varieties. Commun. Pure Appl. Math. **74**(8), 1748–1800 (2021)
21. Li, C., Tian, G., Wang, F.: The uniform version of Yau–Tian–Donaldson conjecture for singular Fano varieties. Peking Math. J. to appear (2021). https://doi.org/10.1007/s42543-021-00039-5
22. Liu, Y., Petracci, A.: On K-stability of some del Pezzo surfaces of Fano index 2. Bull. London Math. Soc. **54**, 517–525 (2022)

23. Nadel, A.: Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. Ann. Math. **132**, 549–596 (1990)
24. Paemurru, E.: Del Pezzo surfaces in weighted projective spaces. Proc. Edinb. Math. Soc. **61**(2), 545–572 (2018)
25. Park, J., Won, J.: Simply connected Sasaki-Einstein rational homology 5-spheres. Duke Math. J. **170**(6), 1085–1112 (2021)
26. Prokhorov, Yu.: On the Zariski Decomposition Problem. Proc. Steklov Inst. Math. **240**, 37–65 (2003)
27. Tian, G.: K-stability and Kähler-Einstein metrics. Commun. Pure Appl. Math. **68**, 1085–1156 (2015)
28. Tian, G.: On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M)>0$. Invent. Math. **89**, 225–246 (1987)
29. Zhuang, Z.: Optimal destabilizing centers and equivariant K-stability. Invent. Math. **226**, 195–223 (2021)

Blow-ups of Three-dimensional Toric Singularities

S. A. Kudryavtsev

Abstract The purely log terminal blow-ups of three-dimensional terminal toric singularities are described. The three-dimensional divisorial contractions $f: (Y, E) \to (X \ni P)$ are described provided that $\operatorname{Exc} f = E$ is an irreducible divisor, $(X \ni P)$ is a toric terminal singularity, $f(E)$ is a toric subvariety and Y has canonical singularities.

Keywords Toric singularities · Threefolds · Extremal contractions

Introduction

Let $(X \ni P)$ be a log canonical singularity and let $f: Y \to X$ be its blow-up. Suppose that the exceptional locus of f consists of only one irreducible divisor: $\operatorname{Exc} f = E$. Then $f: (Y, E) \to (X \ni P)$ is called a *purely log terminal blow-up*, *canonical blow-up* or *terminal blow-up*, if (1), (2) or (3) are satisfied respectively: (1) $K_Y + E$ is plt and $-E$ is f-ample; (2) $-K_Y$ is f-ample and Y has canonical singularities; (3) $-K_Y$ is f-ample and Y has terminal singularities.

The definition of plt blow-up implicitly requires that the divisor E be $\mathbb{Q}$-Cartier. Hence Y is a $\mathbb{Q}$-gorenstein variety. By the inversion of adjunction (see [11, Theorem 17.6]) $K_E + \operatorname{Diff}_E(0) = (K_Y + E)|_E$ is klt.

The importance of study of purely log terminal blow-ups is that: some very important questions of birational geometry for n-dimensional varieties, contractions can be reduced to the smaller dimension $n-1$, using purely log terminal blow-ups (for instance, see the papers [20–22, 26]). In dimension two, purely log terminal

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blow-ups are completely classified and the classification of two-dimensional non-divisorial log terminal extremal contractions of local type is obtained using them [22]. For three-dimensional varieties the first similar problem is to get the same explicit geometric classification of three-dimensional Mori contraction of local type as in two-dimensional case. The next problem is the first difficulty to realize this approach.

Problem. Describe the class of all log del Pezzo surfaces, generic $\mathbb{P}^1$-fibrations which can be the exceptional divisors of some purely log terminal blow-ups of three-dimensional terminal singularities.

Suppose that $f(E) = P$ is a point. Then we solve this problem in the case of terminal toric singularities (Theorem 6.2). Moreover we obtain the description of plt blow-ups of $\mathbb{Q}$-factorial three-dimensional toric singularities (Theorem 6.4). Purely log terminal and canonical blow-ups are divided into toric and non-toric blow-ups up to analytic isomorphism. The study of non-toric plt blow-ups is reduced to the description of plt triples (S, D, Γ) in dimension two (Definition 4.9).

Also we obtain the description of canonical blow-ups of three-dimensional terminal toric singularities (Theorem 6.5). The study of non-toric canonical blow-ups is reduced to the description of the following two interrelated objects: (a) toric canonical blow-ups of $(X \ni P)$ and (b) some triples (S, D, Γ) in dimension two.

Immediate corollary of Theorem 6.5 is that the terminal blow-ups of three-dimensional terminal toric singularities are toric up to analytic isomorphism. This corollary was proved in the papers [2, 6, 8] by another methods.

Suppose that $f(E)$ is a one-dimensional toric subvariety (curve) of the toric singularity $(X \ni P)$. Then the description of plt and canonical blow-ups is given in Theorems 3.7, 3.8, 3.9 and in Corollary 3.10.

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1 Preliminary Results and Facts

All varieties are algebraic and are assumed to be defined over $\mathbb{C}$, the complex number field. The main definitions, notations and notions used in the paper are given in [9, 11, 22]. See [1, Sect. 3.10] on minimal model program with scaling. The definition of Diff and its main properties are given in the papers [25, Sect. 3], [11, Chap. 16]. By $(X \ni P)$ denote the algebraic germ of the variety X at the point P.

A smooth point is a special case of *singularity* by our definition. For example, Du Val singularity of type $\mathbb{A}_0$ is a smooth point.

Let $f: Y \dashrightarrow X$ be a birational map and let D be a divisor on the variety X. By D_Y denote the proper transform of D on the variety Y. If $Y = \widetilde{X}$, $Y = X'$ or $Y = \overline{X}$, then for notational convenience we use the notation $\widetilde{D} = D_{\widetilde{X}}$, $D' = D_{X'}$ or $\overline{D} = D_{\overline{X}}$ respectively. The similar notation is used for subvarieties of X.

The contraction $f: Y \to X$ is a projective morphism of the normal variety such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. *A blow-up* is a birational divisorial contraction. A $\mathbb{Q}$-*factoriality* means analytical $\mathbb{Q}$-factoriality in this paper.

The proper irreducible subvariety Γ of X is said to be *a center of canonical singularities* of (X, D), if there exist the birational morphism $f: Y \to X$ and the exceptional divisor $E \subset Y$ such that $\Gamma = f(E)$ and $a(E, D) \leq 0$. The set of canonical singularity centers of (X, D) and X is denoted by $\mathrm{CS}(X, D)$ and $\mathrm{CS}(X)$ respectively.

By our definition the *toric varieties, toric morphisms* are considered up to analytic isomorphism (analytical identification), if they are not explicitly defined by fans. Shokurov's (hypothetical) criterion on the characterization of toric varieties is formulated in [26, Chap. 6]. By definition of *weighted blow-up*, its center is a point always, that is, its every weight is positive.

We write all singularities of surface in brackets. For example, the notation $S(\mathbb{A}_1 + \frac{1}{5}(1, 2))$ means that the surface S has two singular points of types $\mathbb{A}_1$ and $\frac{1}{5}(1, 2)$ exactly.

We actively use a structure of the local toric conic bundle $f: S \to (C \ni P)$, where $\dim S = 2$ and $\rho(S/C) = 1$. By [22, Lemma 7.1.11] the surface S has two singularities of types $\frac{1}{r}(1, q)$ and $\frac{1}{r}(1, -q)$ over the point P only, where $r \geq 1$.

Proposition 1.1 ([11, Lemma 6.2]) *Let* $f_i: Y_i \to X$ *be two divisorial contractions of normal varieties, where* $\mathrm{Exc}\, f_i = E_i$ *are irreducible divisors and* $-E_i$ *are* f_i*-ample divisors. If* E_1 *and* E_2 *define the same discrete valuation of the function field* $\mathcal{K}(X)$*, then the contractions* f_1 *and* f_2 *are isomorphic.*

Proposition 1.2 *Let* $f_i: Y_i \to (X \ni P)$ *be two divisorial contractions to a point* P*, where* $\mathrm{Exc}\, f_i = E_i$ *are irreducible divisors. Suppose that the varieties* Y_i*,* X *have log terminal singularities,* E_1 *and* E_2 *define the same discrete valuation of the function field* $\mathcal{K}(X)$*, the divisor* $-E_1$ *is* f_1*-ample, the divisor* $-E_2$ *is not* f_2*-ample. Then there exists the small flopping contraction* (*with respect to* K_{Y_2}) $g: Y_2 \to Y_1$ *such that* f_2 *and* $f_1 \circ g$ *are isomorphic.*

Proof Let $K_{Y_2} = f_2^* K_X + aE_2$. If $a > 0$, then we put $L = -K_{Y_2}$. If $a \leq 0$, then we put $L = -(K_{Y_2} + (-a + \varepsilon)E_2)$, where ε is a sufficiently small positive rational number. Since $-E_2$ is a f_2-nef divisor, then the linear system $|nL|$ is free over X for $n \gg 0$ and gives a contraction $g: Y_2 \to Y_2'$ over X by the base point free theorem [9, Remark 3.1.2]. A curve C is exceptional for g if and only if $L \cdot C = E_2 \cdot C = K_{Y_2} \cdot C = 0$. Therefore g is a flopping contraction and $Y_2' \cong Y_1$ by Proposition 1.1. □

The next example shows the idea of Proposition 1.2.

Example 1.3 Let $(X \ni P) \cong (\{x_1x_2 + x_3^2 + x_4^4{=}0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$. Consider the divisorial contraction $f_1: Y_1 \to (X \ni P)$ induced by the blow-up of the maximal ideal of the point $(\mathbb{C}^4 \ni 0)$. Then $\mathrm{Exc}\, f_1 \cong \mathbb{P}(1, 1, 2)$, the variety Y_1 has only one singular point denoted by Q, and $(Y_1 \ni Q) \cong (\{y_1y_2 + y_3^2 + y_4^2 = 0\} \subset (\mathbb{C}^4_{y_1y_2y_3y_4}, 0))$.

This singularity is not $\mathbb{Q}$-factorial and let $g\colon Y_2 \to (Y_1 \ni P)$ be its $\mathbb{Q}$-factorialization. We obtain the divisorial contraction $f_2\colon Y_2 \to (X \ni P)$, where Y_2 is a smooth 3-fold, $\operatorname{Exc} f_2 \cong \mathbb{F}_2$, and $-K_{Y_2}$ is not a f_2-ample divisor.

Definition 1.4 Let $(X \ni P)$ be a log canonical singularity and let $f\colon Y \to X$ be its blow-up. Suppose that the exceptional locus of f consists of only one irreducible divisor: $\operatorname{Exc} f = E$. Then $f\colon (Y, E) \to (X \ni P)$ is called a *canonical blow-up* if $-K_Y$ is f-ample and Y has canonical singularities. Note that the definition of canonical blow-up implies that $(X \ni P)$ is a canonical singularity. The canonical blow-up is said to be *a terminal blow-up* if Y has terminal singularities.

Remark 1.5 Using the notation of Definition 1.4, we have the following properties of canonical blow-ups.

(1) The definition of canonical (resp. terminal) blow-up implies easily that $(X \ni P)$ is a canonical (resp. terminal) singularity.
(2) The divisor $-E$ is f-ample and $a(E, 0) > 0$.
(3) Let $f_i\colon (Y_i, E_i) \to (X \ni P)$ be two canonical blow-ups. If E_1 and E_2 define the same discrete valuation of the function field $\mathcal{K}(X)$ then the blow-ups f_1 and f_2 are isomorphic by Proposition 1.1.
(4) Let $(X \ni P)$ be a $\mathbb{Q}$-factorial singularity. Then Y is a $\mathbb{Q}$-factorial variety also, $\rho(Y/X) = 1$ and $\rho(E) = 1$ [4, Sect. 5].

Theorem 1.6 *Let $(X \ni P)$ be a canonical singularity and $(X \ni P, D)$ be a pair with canonical singularities, where D is a boundary. Assume that $a(E, D) = 0$ and $a(E, 0) > 0$ for some irreducible exceptional divisor E. Then there exists a canonical blow-up such that its exceptional divisor and E define the same discrete valuation of the function field $\mathcal{K}(X)$. Moreover, if E is a unique exceptional divisor with $a(E, D) = 0$ then its canonical blow-up is a terminal blow-up.*

Proof By Proposition 21.6.1 of the paper [11] we consider the birational contraction $\widetilde{f}\colon (\widetilde{Y}, \widetilde{E}) \to (X \ni P)$ with the following three properties:

(1) $\widetilde{E}$ is a unique irreducible exceptional divisor of $\operatorname{Exc} \widetilde{f}$;
(2) $\widetilde{E}$ and E define the same discrete valuation of the function field $\mathcal{K}(X)$;
(3) if $(X \ni P)$ is $\mathbb{Q}$-factorial then $\rho(\widetilde{Y}/X) = 1$ and $\operatorname{Exc} \widetilde{f} = \widetilde{E}$.

The proof of Proposition 21.6.1 of [11] holds in any dimension since we can apply MMP with scaling to prove it. Let $\widetilde{f}$ be not the required canonical blow-up. If $\operatorname{Exc} \widetilde{f} = \widetilde{E}$ then by Proposition 1.2 we have $\widetilde{f} \cong f \circ g$, where f is the required blow-up. Consider the remaining case when $\operatorname{Exc} \widetilde{f} = \widetilde{E} \cup \Delta$, where $\Delta \neq \emptyset$ and $\operatorname{codim}_{\widetilde{Y}} \Delta \geq 2$. Let H be a general Cartier divisor containing the set $\widetilde{f}(\operatorname{Exc} \widetilde{f})$. Then $K_{\widetilde{Y}} + D_{\widetilde{Y}} + \varepsilon H_{\widetilde{Y}} \equiv -\varepsilon a \widetilde{E}$ over X, where $a > 0$. For $0 < \varepsilon \ll 1$ we apply $K_{\widetilde{Y}} + D_{\widetilde{Y}}$—MMP with scaling of $H_{\widetilde{Y}}$. We obtain a birational map $\varphi\colon \widetilde{Y} \dashrightarrow Y'$, which is a composition of log flips, and we also obtain a divisorial contraction $f'\colon Y' \to X$ such that $\operatorname{Exc} f' = E'$, where E' is an irreducible divisor. Therefore, by Proposition 1.2 we have the required canonical blow-up. $\square$

Definition 1.7 Let $(X \ni P)$ be a log canonical singularity and let $f: Y \to X$ be its blow-up. Suppose that the exceptional locus of f consists of only one irreducible divisor: $\operatorname{Exc} f = E$. Then $f: (Y, E) \to (X \ni P)$ is called a *purely log terminal blow-up* if the divisor $K_Y + E$ is purely log terminal and $-E$ is f-ample.

Remark 1.8 Definition 1.7 implicitly requires that the divisor E be $\mathbb{Q}$-Cartier. Hence Y is a $\mathbb{Q}$-gorenstein variety. By the inversion of adjunction $K_E + \operatorname{Diff}_E(0) = (K_Y + E)|_E$ is klt.

Remark 1.9 Using the notation of Definition 1.7 we have the following properties of purely log terminal blow-ups.

(1) The variety $f(E)$ is normal [19, Corollary 2.11].
(2) If $(X \ni P)$ is a log terminal singularity then $-(K_Y + E)$ is a f-ample divisor. A purely log terminal blow-up of log terminal singularity always exists by Theorem 1.5 of [13] since we can apply MMP with scaling to prove it (see also Theorem 1.10).
(3) If $(X \ni P)$ is a strictly log canonical singularity then $a(E, 0) = -1$. A purely log terminal blow-up of strictly log canonical singularity exists if and only if there is only one exceptional divisor with discrepancy -1 [13, Theorem 1.9], since we can apply MMP with scaling to prove Theorem 1.9 of [13].
(4) If $(X \ni P)$ is a $\mathbb{Q}$-factorial singularity then Y is a $\mathbb{Q}$-factorial variety also, $\rho(Y/X) = 1$ and $\rho(E) = 1$ [19, Remark 2.2], [4, Sect. 5]. Hence, for $\mathbb{Q}$-factorial singularity we can omit the requirement that $-E$ be f-ample in Definition 1.7 because it holds automatically.
(5) Let $f_i: (Y_i, E_i) \to (X \ni P)$ be two purely log terminal blow-ups. If E_1 and E_2 define the same discrete valuation of the function field $\mathcal{K}(X)$ then the blow-ups f_1 and f_2 are isomorphic by Proposition 1.1.
(6) Let $-E$ be not a f-ample divisor in Definition 1.7. Then such blow-up can differ from some plt blow-up by a small flopping contraction only (with respect to the canonical divisor K_Y) [13, Corollary 1.13]. This statement is similar to Proposition 1.2.
(7) Let $f: (Y, E) \to (X \ni P)$ be a toric blow-up of a toric $\mathbb{Q}$-gorenstein singularity. Assume that Y is a $\mathbb{Q}$-gorenstein variety and $\operatorname{Exc} f = E$ is an irreducible divisor. It is obvious that $K_Y + E$ is a plt divisor. Therefore, if $(X \ni P)$ is $\mathbb{Q}$-factorial singularity then f is a plt blow-up.

Theorem 1.10 ([13, Theorem 1.5], [19, Proposition 2.9]) *Let X be a kawamata log terminal variety and let $D \neq 0$ be a boundary on X such that (X, D) is log canonical, but not purely log terminal. Then there exists an inductive blow-up $f: Y \to X$ such that:*

(1) the exceptional locus of f contains only one irreducible divisor E $(\operatorname{Exc}(f) = E)$*;*
(2) $K_Y + E + D_Y = f^*(K_X + D)$ is log canonical;

(3) $K_Y + E + (1-\varepsilon)D_Y$ *is purely log terminal and anti-ample over* X *for any* $\varepsilon > 0$*;*
(4) if X *is* $\mathbb{Q}$*-factorial then* Y *is also* $\mathbb{Q}$*-factorial and* $\rho(Y/X) = 1$.

Proof The proofs of [13, Theorem 1.5], [19, Proposition 2.9] hold in any dimension since we can apply MMP with scaling to prove them. □

Remark 1.11 Inductive blow-up is a plt blow-up. Conversely, for any plt blow-up $f\colon (Y, E) \to (X \ni P)$ there exists a pair (X, D) such that f is its inductive blow-up. Indeed, put $D = f(\frac{1}{n}D_Y)$, where $D_Y \in |-n(K_Y + E)|$ is a general element for $n \gg 0$.

Definition 1.12 Let $(X/Z, D)$ be a contraction of varieties, where D is a subboundary. Then a $\mathbb{Q}$-*complement* of $K_X + D$ is an effective $\mathbb{Q}$-divisor D' such that $D' \geq D$, $K_X + D'$ is log canonical and $K_X + D' \sim_{\mathbb{Q}} 0/Z$ for some $n \in \mathbb{N}$.

Definition 1.13 Let $(X/Z, D)$ be a contraction of varieties. Let $D = S + B$ be a subboundary on X such that B and S have no common components, S is an effective integral divisor and $\llcorner B \lrcorner \leq 0$. Then we say that $K_X + D$ is n*-complementary* if there is a $\mathbb{Q}$-divisor D^+ (called an n*-complement*) such that

(1) $n(K_X + D^+) \sim 0/Z$ (in particular, nD^+ is an integral divisor);
(2) the divisor $K_X + D^+$ is log canonical;
(3) $nD^+ \geq nS + \llcorner (n+1)B \lrcorner$.

The divisor $K_X + D^+$ is also called an n*-complement*.

Definition 1.14 For $n \in \mathbb{N}$ put

$$\mathcal{P}_n = \{a \mid 0 \leq a \leq 1,\ \llcorner (n+1)a \lrcorner \geq na\}.$$

Proposition 1.15 ([25, Lemma 5.4]) *Let* $f\colon X \to Y$ *be a birational contraction and let* D *be a subboundary on* X*. Assume that* $K_X + D$ *is* n*-complementary for some* $n \in \mathbb{N}$*. Then* $K_Y + f(D)$ *is also* n*-complementary.*

Proposition 1.16 ([26, Lemma 4.4]) *Let* $f\colon X \to Z$ *be a birational contraction of varieties and let* D *be a subboundary on* X*. Assume that*

(1) the divisor $K_X + D$ *is* f*-nef;*
(2) the coefficient of every non-exceptional component of D *meeting* Exc f *belongs to* $\mathcal{P}_n$*;*
(3) the divisor $K_Z + f(D)$ *is* n*-complementary.*

Then the divisor $K_X + D$ *is also* n*-complementary.*

Proposition 1.17 ([22, Proposition 4.4.1]) *Let* $f\colon X \to (Z \ni P)$ *be a contraction and* D *be a boundary on* X*. Put* $S = \llcorner D \lrcorner$ *and* $B = \{D\}$*. Assume that*

(1) the divisor $K_X + D$ *is purely log terminal;*
(2) the divisor $-(K_X + D)$ *is* f*-nef and* f*-big;*
(3) $S \neq 0$ *near* $f^{-1}(P)$*;*
(4) every coefficient of D *belongs to* $\mathcal{P}_n$.

Further, assume that near $f^{-1}(P) \cap S$ *there exists an* n*-complement* $K_S + \mathrm{Diff}_S(B)^+$ *of* $K_S + \mathrm{Diff}_S(B)$*. Then near* $f^{-1}(P)$ *there exists an* n*-complement* $K_X + S + B^+$ *of* $K_X + S + B$ *such that* $\mathrm{Diff}_S(B)^+ = \mathrm{Diff}_S(B^+)$.

2 Toric Blow-ups

We refer the reader to [18] for the basics of toric geometry.

Definition 2.1 Let N be the lattice $\mathbb{Z}^n$ in the vector linear space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and M be its dual lattice $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ in the vector linear space $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. We have a canonical pairing $\langle\ ,\ \rangle\colon N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$.

For a fan Δ in N the corresponding toric variety is denoted by $T_N(\Delta)$. For a k-dimensional cone $\sigma \in \Delta$ the closure of corresponding orbit is denoted by $V(\sigma)$. This is a closed subvariety of codimension k in $T_N(\Delta)$.

Example 2.2 (1) Let the vectors $e_1, \ldots, e_n$ be a $\mathbb{Z}$-basis of N, where $n \geq 2$. Consider the cone

$$\sigma = \mathbb{R}_{\geq 0}e_1 + \ldots + \mathbb{R}_{\geq 0}e_{n-1} + \mathbb{R}_{\geq 0}(a_1e_1 + \ldots + a_{n-1}e_{n-1} + re_n).$$

Let the fan Δ consists of the cone σ and its faces. Then the affine toric variety $T_N(\Delta)$ is the quotient space $(\mathbb{C}^n \ni 0)/\mathbb{Z}_r$ with the action $\frac{1}{r}(-a_1, \ldots, -a_{n-1}, 1)$.

(2) Let

$$\sigma = \langle e_1, e_2, e_3, e_4\rangle = \langle (1,0,0), (0,1,0), (0,0,1), (1,1,-1)\rangle$$

for the lattice $N \cong \mathbb{Z}^3$. Let the fan Δ consists of the cone σ and its faces. The affine toric variety $(X \ni P) = T_N(\Delta)$ is a three-dimensional non-degenerate quadratic cone in $\mathbb{C}^4$. Let

$$\Delta^1 = \{\langle e_1, e_2, e_3\rangle, \langle e_1, e_2, e_4\rangle, \text{their faces}\}$$

and

$$\Delta^2 = \{\langle e_1, e_3, e_4\rangle, \langle e_2, e_3, e_4\rangle, \text{their faces}\}.$$

Then the birational contractions $\psi_i\colon T_N(\Delta^i) \to T_N(\Delta)$ are small resolutions for $i = 1, 2$, and $\mathrm{Exc}\,\psi_1 = V(\langle e_1, e_2\rangle)$, $\mathrm{Exc}\,\psi_2 = V(\langle e_3, e_4\rangle)$. The birational map $T_N(\Delta^1) \dashrightarrow T_N(\Delta^2)$ is a flop.

Let $f\colon (Y, E) \to (X \ni P)$ be a toric blow-up, where Y is $\mathbb{Q}$-gorenstein, $\mathrm{Exc}\, f = E$ is an irreducible divisor. Then f is a plt blow-up. Let us prove it. The divisor

$K_Y + E$ is plt. Let $a = (a_1, a_2, a_3)$ be a primitive vector defining f. Consider any three-dimensional cone σ' giving non-$\mathbb{Q}$-factorial singularity of subdivision of the cone σ by a. Then the cone σ' gives non-$\mathbb{Q}$-gorenstein singularity by Proposition 4.3 (i) [24], since there is no any vector $m \in M_{\mathbb{Q}}$ such that $\langle m, e_i\rangle = 1$ for every i and $\langle m, a\rangle = 1$. Hence $-E$ is a f-ample divisor. This completes the proof.

Let $f(E) = P$. Then $Y = T_N(\widetilde{\Delta})$ and

$$\widetilde{\Delta} = \{\langle e_1, e_3, a\rangle, \langle e_1, e_4, a\rangle, \langle e_2, e_3, a\rangle, \langle e_2, e_4, a\rangle, \text{their faces}\},$$

where $a = (a_1, a_2, a_3)$, $\gcd(a_1, a_2, a_3) = 1$, $a_1 > 0$, $a_2 > 0$, $a_1 + a_3 > 0$ and $a_2 + a_3 > 0$.

Obviously, the converse is also true. Any such vector a defines a plt blow-up.

Let $f(E) = C$ and $\dim C = 1$. Then, up to a permutation of the faces of the cone σ we have $C = \langle e_2, e_3\rangle$, $Y = T_N(\widehat{\Delta})$ and

$$\widehat{\Delta} = \{\langle e_2, e_4, a\rangle, \langle e_1, e_3, a\rangle, \langle e_1, e_4, a\rangle, \text{their faces}\},$$

where $a = (0, a_2, a_3)$, $\gcd(a_2, a_3) = 1$, $a_2 > 0$, $a_3 > 0$.

Obviously, the converse is also true. Any such vector a defines a plt blow-up.

The variety Y has the singularities $\frac{1}{a_3}(0, -a_2, 1)$, $\frac{1}{a_2}(0, 1, -a_3)$, $\frac{1}{a_2+a_3}(-a_3, -a_2, 1)$. The surface E is a toric conic bundle, $\rho(E/C) = 2$, the single singular point of E (with a center of the third singularity of Y) has type $\mathbb{A}_{a_2+a_3-1}$ and $\mathrm{Diff}_E(0) = \frac{a_2-1}{a_2}E_1 + \frac{a_3-1}{a_3}E_2$, where E_1, E_2 are corresponding sections.

We will calculate a structure of f by the following way (for convenience). Let us consider $(X \ni P) \subset (\mathbb{C}^4, 0)$ as the embedding $\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0)$. The weighted blow-up of $(\mathbb{C}^4, 0)$ with weights $w = (w_1, w_2, w_3, w_4)$ provided that $w_1 + w_2 = w_3 + w_4$ induces a toric blow-up $f' \colon (Y', E') \to (X \ni P)$, where

$$\mathrm{Exc}\, f' = E' \cong \{x_1x_2 + x_3x_4 \subset \mathbb{P}_{x_1x_2x_3x_4}(w_1, w_2, w_3, w_4)\} -$$

is an irreducible divisor. If put $w_1 = a_1 + a_3$, $w_2 = a_2$, $w_3 = a_2 + a_3$ and $w_4 = a_1$, then we can easily compare the natural affine covers of Y and Y' and prove that f and f' are isomorphic blow-ups. Note that $C = \{x_1 = x_2 = x_3 = 0\}$ in the case $C = f'(E')$.

Proposition 2.3 ([18, pages 36–37]) *The following statements are satisfied:*

(1) $(X \ni P)$ *is a three-dimensional* $\mathbb{Q}$*–factorial toric terminal singularity if and only if* $(X \ni P) \cong (\mathbb{C}^3 \ni 0)/\mathbb{Z}_r(q, -1, 1)$, *where* $\gcd(r, q) = 1$;

(2) $(X \ni P)$ *is a three-dimensional non-*$\mathbb{Q}$*–factorial toric terminal singularity if and only if* $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$.

Theorem 2.4 ([17]) *Let* $(X \ni P)$ *be a three-dimensional cyclic singularity of type* $\frac{1}{r}(a_1, a_2, a_3)$. *Then* $(X \ni P)$ *is a canonical singularity if and only if one of the following holds:*

(1) $a_1 + a_2 + a_3 \equiv 0 (\bmod r)$;
(2) $a_i + a_j \equiv 0 (\bmod r)$ *for some* $i \neq j$;
(3) $(X \ni P)$ *has type* $\frac{1}{9}(1, 4, 7)$ *or type* $\frac{1}{14}(1, 9, 11)$.

Proposition 2.5 *Let* $f : (Y, E) \to (X \ni P)$ *be a toric canonical blow-up of three-dimensional toric terminal singularity,* $f(E) = C$ *and* $\dim C = 1$. *Then we have the following statements.*

(1) *Let* $(X \ni P)$ *be a* $\mathbb{Q}$*-factorial singularity, that is, it is* $(\mathbb{C}^3_{x_1x_2x_3} \ni 0)/\mathbb{Z}_r(-1, -q, 1)$, *where* $\gcd(r, q) = 1$, $0 < q \leq r - 1$ *and* $r \geq 1$. *Determine the numbers* u, v *by the equality* $uq + vr = 1$, *where* $0 \leq u \leq r - 1$ *and* $u, v \in \mathbb{Z}$. *Consider the cone* σ *defining* $(X \ni P)$ *(see example 2.2 (1)). Let* (w_1, w_2, w_3) *be a primitive vector defining* f.

Then we have one of the two following cases up to permutation of coordinates: either 2*A)* $C = \{x_1 = x_2 = 0\}/\mathbb{Z}_r$, $(w_1, w_2, w_3) = (1, w_2, 0)$, *or* 2*B)* $C = \{x_2 = x_3 = 0\}/\mathbb{Z}_r$, $(w_1, w_2, w_3) = (0, w_2, 1)$. *The variety* Y *has the singularities* $\frac{1}{r}(-1, w_2 - q, 1)$, $\frac{1}{rw_2}(-1 + uw_2, -uw_2, 1)$ *in Case* 2*A) and* $\frac{1}{r}(-1, -w_2 - q, 1)$, $\frac{1}{rw_2}(uw_2, -uw_2 - 1, 1)$ *in Case* 2*B).*

Converse is also true: every such numbers (w_1, w_2, w_3) *define a canonical blow-up.*

A general element of the linear system $|-K_Y|$ *has Du Val singularities.*

Let Q *be a central point of second singularity in each of the two cases. Then* $Q \in \mathrm{CS}(Y)$ *if and only if* $r \geq 2$. *Therefore* f *is a terminal blow-up if and only if it is the blow-up of the ideal of the curve* C *[8].*

(2) *Let* $(X \ni P)$ *be a non-*$\mathbb{Q}$*-factorial singularity, that is,* $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$. *Then* $C = \{x_1 = x_2 = x_3 = 0\}$ *up to permutation of coordinates,* f *is induced by the blow-up of* $(\mathbb{C}^4, 0)$ *with weights* $(w_1, w_2, w_1 + w_2, 0)$, *where* $w_1 = 1$, $w_2 > 0$ *or* $w_1 > 0$, $w_2 = 1$. *Converse is also true: every such numbers induce a canonical blow-up. A general element of the linear system* $|-K_Y|$ *has Du Val singularities.*

The morphism f *is a terminal blow-up if and only if* $(w_1, w_2, w_3, w_4) = (1, 1, 2, 0)$.

Proof Let us prove (1). Put $e'_1 = e_1$, $e'_2 = e_2$ and $e'_3 = e_1 + qe_2 + re_3$ (see Example 2.2 (1)). Then $w = w_ie'_i + w_je'_j$ for some $i < j$ and $w_i, w_j \in \mathbb{Z}_{\geq 1}$. We have $Y = T_N(\Delta)$ and

$$\Delta = \{\langle e'_k, e'_i, w\rangle, \langle e'_k, e'_j, w\rangle, \text{their faces}\},$$

where k is a third index other than the indices i and j. Consider an induced blow-up of general hyperplane section passing through the general point of C. Then $w_1 = 1$ or $w_2 = 1$. Now the statement is proved by a simple enumeration of the indices i and j. As an example, consider $i = 1$, $j = 2$. There are the two possibilities of weights: $(w_1, 1, 0)$ and $(1, w_2, 0)$. Let $(w_1, 1, 0)$. The variety Y is covered by two affine charts with singularities of types $\frac{1}{r}(-q, qw_1 - 1, 1)$ and $\frac{1}{rw_1}(-w_1, qw_1 - 1, 1)$. By Theorem 2.4 applied to the second singularity it follows that either $q = 1$, or $w_1 = 1$, or $r = 1$. All these variants are realized, it is Case (2A). The possibility $(1, w_2, 0)$ is considered similarly.

The proper transform of $\{x_2 = 0\}/\mathbb{Z}_r(-1, -q, 1)$ is Du Val element of $|-K_Y|$.

The statement $Q \in \mathrm{CS}(Y)$ is obvious if we consider a blow-up with the weights $(-1 + uw_2, (r-u)w_2, 1)$ in Case $(2A)$ and $(uw_2, (r-u)w_2 - 1, 1)$ in Case $(2B)$ provided that $r \geq 2$.

Statement (2) obviously follows from Example 2.2 (2). The proper transform of $\{x_1^{w_2} + x_2 = 0\}|_X$ ($\{x_1 + x_2^{w_1} = 0\}|_X$) is Du Val element of $|-K_Y|$ for the first (second) possibility. □

Proposition 2.6 *Let $f\colon (Y, E) \to (X \ni P)$ be a toric canonical blow-up of three-dimensional toric terminal point, where $f(E) = P$. Then we have the following statements.*

(1) *Let $(X \ni P)$ be a smooth point. Then f is a weighted blow-up with weights $(w_1, w_2, 1)$, $(l, l-1, 2)$, $(15, 10, 6)$, $(12, 8, 5)$, $(10, 7, 4)$, $(9, 6, 4)$, $(8, 5, 3)$, $(7, 5, 3)$, $(6, 4, 3)$, $(5, 3, 2)$ or $(9, 5, 2)$ in some coordinate system, where $l \geq 3$. Converse is also true: every such weights define a canonical blow-up. In all cases, except case $(9, 5, 2)$, a general element of the linear system $|-K_Y|$ has Du Val singularities. In case $(9, 5, 2)$ we have*

$$\min\{m | \exists D \in |-mK_Y| \text{ such that } (Y, (1/m)D) \text{has canonical singularities}\} = 3.$$

The morphism f is a terminal blow-up if and only if it is a weighted blow-up with weights $(w_1, w_2, 1)$ in some coordinate system, where $\gcd(w_1, w_2) = 1$.

(2) *Let $(X \ni P)$ be a $\mathbb{Q}$-factorial singularity of an index ≥ 2, that is, it is of type $\frac{1}{r}(-1, -q, 1)$, where $\gcd(r, q) = 1$, $0 < q \leq r - 1$ and $r \geq 2$. Let us consider the cone σ defining the singularity $(X \ni P)$ (see Example 2.2 (1)). Determine the numbers u, v by the equality $uq + vr = 1$, where $0 \leq u \leq r - 1$ and $u, v \in \mathbb{Z}$. Let (w_1, w_2, w_3) be a primitive vector defining f.*

Then we have one of the two following cases: either $2A$) $(w_1, w_2, w_3) = (1, w_2, w_3)$, $w_3 \leq \min(r-1, \frac{rw_2 - 1}{q})$ up to permutation of the numbers w_1 and w_2 provided that $q = 1$, or $2B$) $(w_1, w_2, w_3) = (w_1, w_2, w_1 + w_2 - 1)$, $w_1 \geq 2$, $w_2 \geq 2$, $0 \leq w_1(r-1) - w_2 \leq r - 2$, $q = r - 1$. Converse is also true: every such numbers (w_1, w_2, w_3) define a canonical blow-up. A general element of the linear system $|-K_Y|$ has Du Val singularities.

The morphism f is a terminal blow-up if and only if it is a weighted blow-up with weights $(u, 1, r-u)$ [8].

(3) *Let $(X \ni P)$ be a non-$\mathbb{Q}$-factorial singularity, that is, $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$. Then f is induced by the weighted blow-up of $(\mathbb{C}^4, 0)$ with weights (w_1, w_2, w_3, w_4) up to analytical isomorphism of $(\mathbb{C}^4, 0)$, where $1 + w_2 = w_3 + w_4$, $w_1 = 1$. Converse is also true: every such weights induce a canonical blow-up. A general element of the linear system $|-K_Y|$ has Du Val singularities.*

The morphism f is a terminal blow-up if and only if $(w_1, w_2, w_3, w_4) = (1, 1, 1, 1)$ [2].

Proof Let us prove (1). Now we classify canonical blow-ups. To be definite, assume that $w_1 \geq w_2 \geq w_3$, where (w_1, w_2, w_3) are primitive weights of f. By P_1, P_2

and P_3 denote the zero-dimensional orbits (points) of Y. These points have types $\frac{1}{w_1}(w_2, w_3, w_1 - 1)$, $\frac{1}{w_2}(w_1, w_3, w_2 - 1)$ and $\frac{1}{w_3}(w_1, w_2, w_3 - 1)$ respectively.

Assume that Cases (1) and (1) of Theorem 2.4 are satisfied at the points P_1 and P_2 respectively. Then $w_1 = w_2 + w_3 - 1$ and $w_2 | (2w_3 - 2)$. Thus we obtain the weights $(l, l, 1)$, where $l \geq 1$ and $(3w_3 - 3, 2w_3 - 2, w_3)$, where $w_3 \geq 2$. For the second possibility, the singularity is of type $\frac{1}{w_3}(3, 2, 1)$ at the point P_3, therefore $w_3 \leq 6$, and it is easy to prove that every value $w_3 = 2, \ldots, 6$ is realized.

Assume that Cases (1) and (2) of Theorem 2.4 are satisfied at the points P_1 and P_2 respectively. As above we obtain $w_1 = w_2 + w_3 - 1$ and have one of the following possibilities: (i1) $w_3 = 1$, $w_3 = 2$ or (i2) $2w_3 - 1 = w_2$, $w_2 = 1, \ldots, 4$. These possibilities are realized.

Assume that Cases (1) and (3) of Theorem 2.4 are satisfied at the points P_1 and P_2 respectively. Then $w_1 = w_2 + w_3 - 1$. Let the singularity be of type $\frac{1}{9}(1, 4, 7) = \frac{1}{9}(5, 2, 8)$ at the point P_2, in particular, $w_2 = 9$. Hence $w_3 = 2$ or $w_3 = 5$. It follows easily that these possibilities are not realized. Let the singularity be of type $\frac{1}{14}(1, 9, 11) = \frac{1}{14}(5, 3, 13)$ at the point P_2, in particular, $w_2 = 14$. Hence $w_3 = 3$ or $w_3 = 5$. It follows easily that these possibilities are not realized.

Assume that Cases (2) and (1) of Theorem 2.4 are satisfied at the points P_1 and P_2 respectively. Then we obtain the two possibilities: (i) $w_1 = w_2 + w_3$, $w_2 = 2w_3 - 1$, $w_3 = 2, 3$ or (ii) $w_3 = 1$. These possibilities are realized.

Assume that Cases (2) and (2) of Theorem 2.4 are satisfied at the points P_1 and P_2 respectively. As above it is easy to prove that new weights do not appear.

Assume that Cases (2) and (3) of Theorem 2.4 are satisfied at the points P_1 and P_2 respectively. As above it is easy to prove that this case is not realized.

Assume that Cases (3) of Theorem 2.4 are satisfied at the point P_1. Then $(w_1, w_2, w_3) = (9, 5, 2)$ or $(14, 5, 3)$. It is obvious that only the first possibility is realized.

For any weights obtained, except case $(9, 5, 2)$, we can easily find a surface $S \subset X$ with Du Val singularity at the point P such that $a(S, E) = 0$. For example, the surface S is given (locally at the point P) by the equations $x_1 x_2 + x_3^{w_1 + w_2} = 0$ and $x_1^2 + x_2^3 + x_2 x_3^3 = 0$ for cases $(w_1, w_2, 1)$ and $(5, 3, 2)$ respectively. Therefore $S_Y \in |-K_Y|$ has Du Val singularities.

In case $(9, 5, 2)$ the variety Y has the two non-terminal isolated singularities at the points P_1 and P_2 ($\mathrm{CS}(Y) = \{P_1, P_2\}$). Let $C \subset E = \mathbb{P}(9, 5, 2)$ be a curve not passing through the points P_1 and P_2. Then a (quasihomogeneous) degree of C is at least 45 since it must be divided by 9 and 5. Hence $m \geq 3$, and the required element D is the proper transform of $x_1^5 + x_2^9 + x_3^{23} = 0$. The other statements of (1) are obvious.

Let us prove (2). Now we classify canonical blow-ups. The variety Y is covered by three affine charts with singularities of types $\frac{1}{w_3}(-w_1, -w_2, 1)$, $\frac{1}{rw_2 - qw_3}(-w_1 + uw_2 + vw_3, -uw_2 - vw_3, 1)$ and $\frac{1}{rw_1 - w_3}(-w_1, qw_1 - w_2, 1)$ respectively. The corresponding zero-dimensional orbits of Y are denoted by P_1, P_2 and P_3. Note that $rw_1 - w_3, rw_2 - qw_3 \in \mathbb{Z}_{\geq 1}$. Obviously, $a(S, 0) = \frac{1}{r}(w_3 + rw_2 - qw_3 + rw_1 - w_3) - 1$. The minimal discrepancy of $(X \ni P)$ is equal to $\frac{1}{r}$. If $a(S, 0) = \frac{1}{r}$, that it is easy to calculate that f is a terminal blow-up, that is, a weighted blow-up with weights $(u, 1, r-u)$ [8]. Therefore we suppose that $a(S, 0) > \frac{1}{r}$.

Since Y has canonical singularities, then for some $j \in \{1, 2, 3\}$ we have the inequality $\frac{1}{r} \geq a(S, 0)/N_j$ and one of the two following requirements: either $P_j \in \mathrm{CS}(Z)$, or the singularity at the point P_j is of type $\frac{1}{N_j}(1, -1, 0)$, where $N_j \geq 2$, $N_1 = w_3$, $N_2 = rw_1 - w_3$, $N_3 = rw_2 - qw_3$. This is called *Property* R_j. Note if $j = 3$ then $w_1 = 1$. Therefore we suppose that $j \leq 2$.

Let $w_1 = \max\{w_1, w_2, w_3\}$. Assume that Case (1) of Theorem 2.4 is satisfied at the point P_2. Then $q = 1$ and $w_2 = 1$. Assume that Case (2) of Theorem 2.4 is satisfied at the point P_2. Then, either $w_1 = w_2 = w_3 = 1$, or $q = 2$, $w_1 = w_2$, $w_1 \geq 2$, $r \geq 3$. Since the inequality of Property R_2 holds then the second possibility is not realized. It is not hard to prove that Case (3) of Theorem 2.4 is not realized at the point P_2.

Let $w_2 = \max\{w_1, w_2, w_3\}$. Property R_1 is not realized. Therefore Property R_2 holds. Then $w_2 = w_3$, and we have $w_1 = 1$ by Theorem 2.4 for the point P_1.

Let us consider the last case $w_3 > \max\{w_1, w_2\}$. The possibility $w_1 = 1$ holds. Therefore we suppose that $w_1 \geq 2$. If $w_2 = 1$ then Theorem 2.4 for the point P_2 implies $q = 1$. Therefore we suppose that $w_2 \geq 2$.

Assume that Case (1) of Theorem 2.4 is satisfied at the point P_1. Then $w_1 + w_2 - 1 = w_3$. If the inequality of Property R_1 holds then $q = r - 1$. Therefore we suppose that Property R_2 holds and $N_2 > w_3$. It is not hard to prove that Case (3) of Theorem 2.4 is not realized at the point P_2. If Case (1) of Theorem 2.4 is satisfied at the point P_2 then the inequality of Property R_2 implies that $(q-1)w_1 - w_2 + 1 = 0$, but this equality contradicts the same inequality. Therefore the singularity is of type $\frac{1}{N_2}(1, -1, 0)$ at the point P_2. Therefore $w_1 = 1$. We obtain the contradiction.

Assume that Case (2) of Theorem 2.4 is satisfied at the point P_1. Then $w_1 + w_2 = w_3$ and Property R_2 holds. Let Case (3) of Theorem 2.4 be satisfied at the point P_2. Then it is not hard to prove that $(w_1, w_2, w_3, r) = (2, 2q+5, 2q+7, q+8)$. We obtain a contradiction with Theorem 2.4 for the point P_3 since $0 < uw_2 + vw_3 \leq N_3$. Let Case (1) of Theorem 2.4 be satisfied at the point P_2. The inequality of Property R_2 implies that $(q-1)w_1 - w_2 + 1 = 0$, but this equality contradicts the same inequality. Therefore the singularity is of type $\frac{1}{N_2}(1, -1, 0)$ at the point P_2. Considering two possibilities: $N_2 \leq w_1$ and $N_2 > w_1$, it is easy to obtain a contradiction.

Now, applying the blow-up classification obtained, we can prove that the proper transform of the divisor

$$S_k = \{x_k = 0\}/\mathbb{Z}_r \subset (\mathbb{C}^3_{x_1x_2x_3}, 0)/\mathbb{Z}_r(-1, -q, 1)$$

is Du Val element of $|-K_Y|$ for some k. The other statements of 2) are obvious.

Let us prove (3). Consider Example 2.2 (2). Now we classify canonical blow-ups. Obviously, $a(S, 0) = w_1 + w_2 - 1 = w_3 + w_4 - 1$. The variety Y is covered by three affine charts with singularities of types $\frac{1}{w_1}(w_3, w_4, -1)$, $\frac{1}{w_2}(w_3, w_4, -1)$, $\frac{1}{w_3}(w_1, w_2, -1)$ and $\frac{1}{w_4}(w_1, w_2, -1)$ respectively. The minimal discrepancy of $(X \ni P)$ is equal to 1. If $a(S, 0) = 1$ then it is easy to calculate that f is a terminal blow-up induced by the weighted blow-up with weights $(1, 1, 1, 1)$ [2]. Therefore we suppose that $a(S, 0) > 1$. Since Y has canonical singularities then $1 \geq a(S, 0)/w_j$ for some j. Hence $w_i = 1$ for some $i \neq j$ such that $w_i + w_j - 1 = a(S, 0)$. The proper transform of $\{x_i^{w_j} + x_j = 0\}|_X$ is Du Val element of $|-K_Y|$. The other statements of (3) are obvious. $\square$

Definition 2.7 Let $(X \ni P)$ be an n-dimensional $\mathbb{Q}$-factorial toric singularity. Then $(X \ni P) \cong (\mathbb{C}^n \ni 0)/G$, where G is an abelian group acting freely in codimension 1. The singularity $(\mathbb{C}^n \ni 0)/G$ is given by the simplicial cone σ_G in the lattice $N = \mathbb{Z}^n$.

Let a power series (polynomial) $\varphi = \sum_m a_m x^m \in \mathbb{C}[[x_1, x_2, \dots, x_n]]$ be G-semiinvariant.

The *Newton polyhedron* $\Gamma_+(\varphi)$ in $\mathbb{R}^n$ is the convex hull of the set

$$\bigcup_{x^m \in \varphi} (m + \sigma_G^\vee), \text{ where } \sigma_G^\vee \text{ is a dual cone in } M_{\mathbb{R}}.$$

For any face γ of $\Gamma_+(\varphi)$ we define

$$\varphi_\gamma = \sum_{m \in \gamma} a_m x^m.$$

The function φ is said to be *non-degenerate* if, for any compact face γ of the Newton polyhedron, the polynomial equation $\varphi_\gamma = 0$ defines a smooth hypersurface in the complement of the set $x_1 x_2 \dots x_n = 0$. The effective Weil divisor D on X is said to be *non-degenerate* if the G-semiinvariant polynomial φ defining D in $\mathbb{C}^n$ is non-degenerate.

For any effective Weil divisor D there exists the fan Δ depending on Newton polyhedron $\Gamma_+(\varphi)$ such that $T_N(\Delta)$ is a smooth variety and a toric birational morphism $\psi \colon T_N(\Delta) \to \mathbb{C}^n$ is a resolution of non-degenerate singularities of D. So, ψ is said a *partial resolution of* (X, D). In particular, if D is a non-degenerate boundary then ψ is a toric log resolution of the pair (X, D). If $(X \ni P)$ is a smooth variety then this statement was proved in the paper [27]. Note that the proof from the paper [27] is rewritten immediately in our case if we will use our Newton polyhedron instead of standard Newton polyhedron.

The next Theorems 2.8 and 2.9 are criteria of the characterization of toric plt and canonical blow-up respectively. They explicitly show a nature of non-toric contractions.

Theorem 2.8 *Let $f \colon (Y, E) \to (X \ni P)$ be a plt blow-up of $\mathbb{Q}$-factorial toric singularity, and let $f(E)$ be a toric subvariety. Then f is a toric morphism (under a suitable identification) if and only if there exists an effective non-degenerate Weil divisor D on $(X \ni P)$ and a number $d > 0$ with the following properties:*

(1) $a(E, dD) = -1$;
(2) *E is a unique exceptional divisor of (X, dD) with discrepancy ≤ -1 and* $\lfloor dD \rfloor = 0$.

Proof First let us prove the necessary condition. Let $D_Y \in |-n(K_Y + E)|$ be a general element for $n \gg 0$. Put $D = f(D_Y)$ and $d = \frac{1}{n}$. Then $K_Y + E + dD_Y = f^*(K_X + dD)$ is a plt divisor. Since D_Y is a general divisor by construction, then D is an irreducible reduced non-degenerate divisor.

Finally let us prove the sufficient condition. Consider the toric log resolution $\psi\colon Z \to X$ of (X, dD). Write

$$K_Z + dD_Z + \sum a_i E_i = \psi^*\Big(K_X + dD\Big).$$

By theorem assertion $(Z, dD_Z + \sum a_i E_i)$ is a plt pair. Therefore $E \subset \operatorname{Exc} \psi$.

Considering corresponding fans (see [24]) we have the composition of toric log flips $Z \dashrightarrow Z'$ over $(X \ni P)$ such that the (induced) toric divisorial contraction $\psi'\colon Z' \to (X \ni P)$ is isomorphic to $\psi'_1 \circ \psi'_2$, where ψ'_1, ψ'_2 are toric divisorial contractions and $E = \operatorname{Exc} \psi'_1$. Therefore f and ψ'_1 are isomorphic by Remark 1.9 (5). □

Theorem 2.9 *Let $f\colon (Y, E) \to (X \ni P)$ be a canonical blow-up of $\mathbb{Q}$-factorial toric singularity, and let $f(E)$ be a toric subvariety. Then f is a toric morphism (under a suitable identification) if and only if there exists an effective non-degenerate Weil divisor D on $(X \ni P)$ and a number $d > 0$ with the following properties:*

(1) $a(E, dD) = 0$;
(2) (X, dD) *has canonical singularities and* $\llcorner 2dD \lrcorner = 0$.

Proof First let us prove the necessary condition. Let $D_Y \in |-nK_Y|$ be a general element for $n \gg 0$. Put $D = f(D_Y)$ and $d = \frac{1}{n}$. Then the divisor $K_Y + dD_Y = f^*(K_X + dD)$ has canonical singularities. Since D_Y is a general divisor by construction, then D is an irreducible reduced non-degenerate divisor.

Finally let us prove the sufficient condition. Consider the toric log resolution $\psi\colon Z \to X$ of (X, dD). Write

$$K_Z + dD_Z + \sum a_i E_i = \psi^*\Big(K_X + dD\Big).$$

By theorem assertion $(Z, dD_Z + \sum a_i E_i)$ is a terminal pair. Therefore $E \subset \operatorname{Exc} \psi$. Considering corresponding fans (see [24]) we have the composition of toric log flips $Z \dashrightarrow Z'$ over $(X \ni P)$ such that the (induced) toric divisorial contraction $\psi'\colon Z' \to (X \ni P)$ is isomorphic to $\psi'_1 \circ \psi'_2$, where ψ'_1, ψ'_2 are toric divisorial contractions and $E = \operatorname{Exc} \psi'_1$. Therefore f and ψ'_1 are isomorphic by Proposition 1.1. □

Definition 2.10 The subvariety Y is said to be a *non-toric subvariety* of the toric pair (X, D), if there is not any toric structure of X such that (X, D) is a toric pair and Y is a toric subvariety.

Example 2.11 Consider the toric variety $X = \mathbb{P}_{x_1x_2x_3}(1, 2, 3)$.

(1) Let $D = 0$. The point P is a non-toric subvariety of (X, D) if and only if $P = (0 : 1 : a)$, where $a \neq 0$. The irreducible curve C is a non-toric subvariety of (X, D) if and only if $C \neq \{x_1 = 0\}$, $C \neq \{x_2 + ax_1^2 = 0\}$ and $C \neq \{x_3 + ax_2x_1 + bx_1^3 = 0\}$.

(2) Let $D = \{x_1 = 0\} + \{x_2 = 0\}$. The point P is a non-toric subvariety of (X, D) if and only if $P = (0 : 1 : a)$, where $a \neq 0$. The irreducible curve C is a non-toric subvariety of (X, D) if and only if $C \neq \{x_1 = 0\}$, $C \neq \{x_2 = 0\}$ and $C \neq \{x_3 + ax_2x_1 + bx_1^3 = 0\}$.

(3) Let $D = \{x_1 = 0\} + \{x_2 = 0\} + \{x_3 = 0\}$. The point P is a non-toric subvariety of (X, D) if and only if $P \neq (1 : 0 : 0)$, $P \neq (0 : 1 : 0)$ and $P \neq (0 : 0 : 1)$. The irreducible curve C is a non-toric subvariety of (X, D) if and only if $C \neq \{x_1 = 0\}$, $C \neq \{x_2 = 0\}$ and $C \neq \{x_3 = 0\}$.

Next Theorems 2.12 and 2.13 are two-dimensional analogs of main theorems. Their proofs clearly describe the main method used in this paper.

Theorem 2.12 ([22]) *Let $f\colon (Y, E) \to (X \ni P)$ be a plt blow-up of two-dimensional toric singularity. Then f is a toric morphism* (*under a suitable identification*).

Proof A two-dimensional toric singularity is always $\mathbb{Q}$-factorial. Let f be a non-toric morphism (up to identification). Let $D_Y \in |-n(K_Y + E)|$ is a general element of $n \gg 0$. Put $D_X = f(D_Y)$ and $d = \frac{1}{n}$. Then (X, dD_X) is a log canonical pair, $a(E, dD_X) = -1$ and E is a unique exceptional divisor with discrepancy -1.

By Criterion 2.8 there exists a toric divisorial contraction $g\colon Z \to X$ with the following properties.

(A) The exceptional set $\operatorname{Exc} g = S$ is an irreducible divisor ($S \cong \mathbb{P}^1$), the divisors S and E define the different discrete valuations of the function field $\mathcal{K}(X)$.

(B) By Γ denote the center of E on S. Then the point Γ is a non-toric subvariety of Z for any toric structure of $(X \ni P)$. In the other words, Γ is a non-toric subvariety of the toric pair $(S, \operatorname{Diff}_S(0))$.

Condition (B) implies that the surface Z has the two singular points P_1 and P_2, which lie on the curve S. Also Γ is a non-toric point of $(S, \operatorname{Diff}_S(0)) \cong (\mathbb{P}^1, \frac{n_1-1}{n_1}P_1 + \frac{n_2-1}{n_2}P_2)$, where $n_1 \geq 2$, $n_2 \geq 2$. Write

$$K_Z + dD_Z + aS = g^*\big(K_X + dD_X\big),$$

where $a < 1$. Hence

$$a\big(E, S + dD_Z\big) < a\big(E, aS + dD_Z\big) = -1.$$

Therefore $K_Z + S + dD_Z$ is not a log canonical divisor at the point Γ and is an anti-ample over X divisor. Hence, by the inversion of adjunction, $K_S + \operatorname{Diff}_S(dD_Z)$ is not a log canonical divisor at the point Γ and is an anti-ample divisor. We obtain the contradiction

$$0 > \deg\big(K_S + \operatorname{Diff}_S(dD_Z)\big) > -2 + \frac{n_1 - 1}{n_1} + \frac{n_2 - 1}{n_2} + 1 \geq 0.$$

$\square$

Theorem 2.13 *[16] Let $f\colon (Y, E) \to (X \ni P)$ be a canonical blow-up of two-dimensional toric singularity. Then $(X \ni P)$ is a smooth point, and f is a weighted blow-up with weights $(1, \alpha)$ (under a suitable identification).*

Proof Theorem assertion implies that $(X \ni P)$ is a terminal point, therefore it is smooth.

Assume that f is a toric morphism then f is a weighted blow-up of the smooth point with weights (β, α). Since Y is Du Val surface then $\alpha = 1$ or $\beta = 1$.

Let f be a non-toric morphism (up to identification). Let $D_Y \in |-nK_Y|$ be a general element for $n \gg 0$. Put $D_X = f(D_Y)$ and $d = \frac{1}{n}$. The pair (X, dD_X) has canonical singularities and $a(E, dD_X) = 0$.

By Criterion 2.9 there exists a toric divisorial contraction $g\colon Z \to X$ with the following properties.

(A) The exceptional set $\operatorname{Exc} g = S$ is an irreducible divisor ($S \cong \mathbb{P}^1$), the divisors S and E define the different discrete valuations of the function field $\mathcal{K}(X)$.
(B) By Γ denote the center of E on S. Then the point Γ is a non-toric subvariety of Z for any toric structure of $(X \ni P)$. In the other words, Γ is a non-toric subvariety of the toric pair $(S, \operatorname{Diff}_S(0))$.

Condition (B) implies that the surface Z has the two singular points P_1 and P_2, which lie on the curve S. Also Γ is a non-toric point of $(S, \operatorname{Diff}_S(0)) \cong (\mathbb{P}^1, \frac{n_1-1}{n_1}P_1 + \frac{n_2-1}{n_2}P_2)$, where $n_1 \geq 2, n_2 \geq 2$. Write

$$K_Z + dD_Z + S = g^*\big(K_X + dD_X\big) + (a(S, dD_X) + 1)S,$$

where $a(S, dD_X) \geq 0$. Since S is (locally) Cartier divisor at the point Γ, then

$$a\big(E, S + dD_Z\big) \leq a\big(E, dD_X\big) - 1 = -1.$$

Therefore $K_Z + S + dD_Z$ is not a plt divisor at the point Γ and is an anti-ample divisor over X. Hence, by the inversion of adjunction $K_S + \operatorname{Diff}_S(dD_Z)$ is not a klt divisor at the point Γ and is an anti-ample divisor. We obtain the contradiction

$$0 > \deg\big(K_S + \operatorname{Diff}_S(dD_Z)\big) \geq -2 + \frac{n_1 - 1}{n_1} + \frac{n_2 - 1}{n_2} + 1 \geq 0.$$

□

Example 2.14 Theorems 2.12 and 2.13 cannot be generalized in dimension at least three for divisorial contraction to a point. Consider the blow-up $g\colon Z \to (X \ni P)$ with the weights $(1, \dots, 1)$, where $(X \ni P) \cong (\mathbb{C}^n_{x_1 \dots x_n} \ni 0)$ and consider the divisors $D = \{x_1^2 + \cdots + x_n^2 = 0\}$, $T^i = \{x_i = 0\}$, where $i = 1, \dots, n$ and $n \geq 3$. The exceptional set $\operatorname{Exc} g = S$ is isomorphic to $\mathbb{P}^{n-1}$, $Q = S \cap D_Z$ is a smooth quadric. Let $\widetilde{g}\colon \widetilde{Z} \to Z$ be the standard blow-up of the ideal I_Q. By the base point free theorem [9] the linear system $|mD_{\widetilde{Z}}|$ gives a divisorial contraction $\varphi\colon \widetilde{Z} \to Y$, which contracts the divisor $S_{\widetilde{Z}} \cong \mathbb{P}^{n-1}$ for $m \gg 0$. Since the divisor $K_{\widetilde{Z}} + S_{\widetilde{Z}} + \sum_{i=1}^n T^i_{\widetilde{Z}} \sim 0/Y$ has log canonical singularities, then by Shokurov's criterion on the characterization of toric varieties for divisorial contractions to a $\mathbb{Q}$-factorial singularity [11, Theorem 18.22], the morphism φ is toric. Hence Y has only one singularity and its type is $\frac{1}{r}(1, \ldots, 1)$. Let l be a straight line in a general position in $S_{\widetilde{Z}}$. Considering φ we have $S_{\widetilde{Z}} \cdot l = -r$, and considering $g \circ \widetilde{g}$ we have $S_{\widetilde{Z}} \cdot l = -3$, hence $r = 3$.

We obtain a non-toric divisorial contraction $f : Y \to (X \ni P)$. The variety Y has only one singularity and its type is $\frac{1}{3}(1, \ldots, 1)$. Thus, if $n \geq 4$, then Y is a terminal variety, and if $n = 3$, then Y is a canonical non-terminal variety (cf. [6]). The blow-up f is plt since the exceptional set $\operatorname{Exc} f$ is a cone over a smooth $(n-2)$-dimensional quadric.

We will apply the following special case of Shokurov's criterion on the characterization of toric varieties.

Proposition 2.15 *Let $f : (X, D) \to (Z \ni P)$ be a small contraction of the $\mathbb{Q}$-factorial threefold X. Assume that $D = \sum_{i=1}^{r} D_i$, where D_i is a prime divisor for each i. Assume that $K_X + D$ is a log canonical divisor, $-(K_X + D)$ is a f-nef divisor and* $\operatorname{Exc} f = C$ *is an irreducible curve* $(\rho(X/Z) = 1)$. *Then* $r \leq 4$. *Moreover, the equality holds if and only if the pair $(X/Z \ni P, D)$ is analytically isomorphic to a toric pair, in particular, $K_X + D \sim 0/Z$.*

Proof If the pair $(X/Z \ni P, D)$ is analytically isomorphic to a toric pair then all statements immediately follow from the description of toric log flips [24]. Let $r \geq 4$. Let the divisor $K_X + D'$ be a $\mathbb{Q}$-complement of $K_X + D$. It exists, since we can add to the divisor D the necessary number of general hyperplane sections of X. So, by abundance theorem [11, Theorem 8.4] the $\mathbb{Q}$-complement D' required is constructed for our contraction $(X/Z \ni P, D)$.

Put $D' = \sum d_i D_i'$. We will prove that $D' = D$. For any $\mathbb{Q}$-Weil divisor $B = \sum b_i B_i$ we define $||B|| = \sum b_i$. Put

$$D^{\text{hor}} = \sum_{i:\ D_i' \cdot C > 0} d_i D_i' \quad \text{and} \quad D^{\text{vert}} = \sum_{i:\ D_i' \cdot C \leq 0} d_i D_i'.$$

Let $f^+ : X^+ \to Z$ be a log flip of f and $\operatorname{Exc} f^+ = C^+$. □

Lemma 2.16 ([23, Lemma 2.10]) *We have* $||D^{\text{hor}}|| = ||D^{\text{vert}}|| = 2$. *Hence,* $D = D'$. *Moreover,* $C \not\subset \operatorname{Supp} D^{\text{hor}}$, $C^+ \not\subset \operatorname{Supp}(D^{\text{vert}})^+$ *and* $D_i' \cdot C \neq 0$ *for all* i.

Proof Since $K_X + D$ is a log canonical divisor then $||D^{\text{vert}}|| \leq 2$. Since $K_{X^+} + D^+$ is a log canonical divisor then $||D^{\text{hor}}|| \leq 2$. The statements remained are obvious. □

Let S be an irreducible component of the divisor D^{vert} and let $F = D - S$. The divisorial log contraction $(S, \operatorname{Diff}_S(F)) \to (f(S) \ni P)$ is toric by the two-dimensional Shokurov's criterion on the characterization of toric varieties [26, Theorem 6.4]. In particular, it is a toric blow-up of cyclic singularity. Thus, the singularities of X are toric by three-dimensional Shokurov's criterion on the characterization of toric varieties for $\mathbb{Q}$-factorial singularities [11, Theorem 18.22]. Replacing X by X^+ it can be assumed that $-(K_X + S)$ is a f-ample divisor and $S \cdot C < 0$.

In order to prove the proposition we will apply some modification, which is a toric one by its nature. After it we will get some small contraction, which is analytically isomorphic to a small toric contraction of Example 2.2 (2). Therefore the initial contraction is a toric up to analytical isomorphism.

Now, taking toric blow-ups of X (every time we take an one blow-up with a unique exceptional divisor that has a minimal discrepancy of a singularity considered and consider two extremal rays on a variety obtained), it can be assumed that S is a smooth surface, and X is a smooth variety outside the curves C. The condition that $-(K_X + S)$ is f-ample holds is preserved, since the discrepancies of exceptional divisors of (X, S) are less than and equal to 0. In some analytical neighborhood of every point of C the variety X is analytically isomorphic to $\frac{1}{k}(q, 1) \times \mathbb{C}^1$, where $(k, q) = 1$.

Assume that $k \geq 2$. Consider a natural cyclic cover $\psi\colon \overline{X} \to X$ of degree k. Put $\overline{C} = \psi^{-1}(C)$ and let $\overline{Z}$ be the normalization of Z in the function field of $\overline{X}$. Let $\overline{f}\colon \overline{X} \to (\overline{Z} \ni \overline{P})$ be the induced small contraction of the curve $\overline{C}$. Thus we can assume that $k = 1$, that is, X is a smooth variety.

Since $-K_S$ is a f-ample divisor then $f\colon S \to f(S)$ is the contraction of the (-1) curve C and $(K_X + S) \cdot C = -1$. We have $S \cdot C = -m$, $K_X \cdot C = m - 1$ for some $m \in \mathbb{Z}_{\geq 1}$.

Let $m \geq 2$. Using the natural section of $\mathcal{O}_X(S)$ we can construct a degree m-cyclic cover $\varphi\colon \widetilde{X} \to X$ ramified along S (cf. [11, Theorem 5.4]). Let $\widetilde{C} = \varphi^{-1}(C)$ and let $\widetilde{Z}$ be the normalization of Z in the function field of $\widetilde{X}$. Let $\widetilde{f}\colon \widetilde{X} \to (\widetilde{Z} \ni \widetilde{P})$ be the induced small contraction of the curve $\widetilde{C}$. By the ramification formula

$$K_{\widetilde{X}} \cdot \widetilde{C} = \varphi^*\Big(K_X + \frac{m-1}{m} S\Big) \cdot \widetilde{C} = K_X \cdot C + \frac{m-1}{m} S \cdot C = 0.$$

Thus we can assume that f is a small flopping contraction with respect to K_X $(K_X \cdot C = 0)$, that is, we can assume that $m = 1$.

Since the minimal discrepancy of three-dimensional terminal non-cDV singularity is strict less than 1 then $(Z \ni P) \cong (g = 0 \subset (\mathbb{C}^4, 0))$ is an isolated cDV (terminal) singularity. Note that $(D_1 + D_2) \cdot C = (D_3 + D_4) \cdot C = 0$ up to permutation of components of D. Hence L_1 and L_2 are Cartier divisors, where $L_1 = f(D_1) + f(D_2)$ and $L_2 = f(D_3) + f(D_4)$. By Bertini theorem [12, Theorem 4.8] the pair $(Z \ni P, H + L_i)$ is log canonical for any $i = 1, 2$, where H is a general hyperplane section passing through the point P. By the inversion of adjunction $(H \ni P, L_i|_H)$ is a log canonical pair. Thus, the classification of two-dimensional log canonical pairs [11] implies that $(H \ni P)$ is a cyclic singularity at the point P, that is, it has type $\mathbb{A}_k$. By the paper [5] or the paper [7] the singularity $(H \ni P)$ is of type $\mathbb{A}_1$. Thus

$$(Z \ni P) \cong (xy + z^2 + t^{2l} = 0 \subset (\mathbb{C}^4, 0))$$

and $f(D) = \{x = 0\}|_Z + \{y = 0\}|_Z$. Since $(Z \ni P, f(D))$ is a log canonical pair then we can take the weighted blow-up of $(\mathbb{C}^4, 0)$ with the weights $(l, l, l, 1)$ and obtain $l = 1$. This completes the proof.

Remark 2.17 Let $\rho(P)$ be a rank of local analytic group of Weil divisors at the point P. Then the Proposition 2.15 implies easily Shokurov's criterion on the characterization of toric varieties for three-dimensional singularities $(Z \ni P)$ if $\rho(P) = 1$, and hence the same criterion for three-dimensional divisorial contractions $f\colon X \to (Z \ni P)$ if $\rho(P) = 1$.

3 Three-dimensional Blow-ups. Case of Curve

Example 3.1 Now we construct the examples of three-dimensional non-toric plt blow-ups $f\colon (Y, E) \to (X \supset C \ni P)$ provided that $(X \ni P)$ is a $\mathbb{Q}$-gorenstein toric singularity, $\dim f(E) = 1$ and the curve $C = f(E)$ is a toric (smooth) subvariety. Depending on a type of $(X \ni P)$ we consider two Cases **A1)** and **A2)**.

(A1) Let $(X \ni P)$ be a $\mathbb{Q}$-factorial toric singularity, that is, $(X \ni P) \cong (\mathbb{C}^3 \ni 0)/G$, where G is an abelian group acting freely in codimension 1.

All plt blow-ups are constructed by the procedure illustrated on the next diagram (Fig. 1) and defined below.

First step. Let $g_0\colon (Z_0, S_0) \to (X \supset C \ni P)$ be a toric blow-up, where $\operatorname{Exc} g_0 = S_0$ is an irreducible divisor and $g_0(S_0) = C$. Recall that g_0 is a plt blow-up, the surface S_0 is a toric conic bundle, $\rho(S_0/C) = 1$ and $\operatorname{Diff}_{S_0}(0) = \frac{w_0^1-1}{w_0^1}E_0^1 + \frac{w_0^2-1}{w_0^2}E_0^2 + \frac{d_0-1}{d_0}F_0$, where E_0^1, E_0^2 are some sections of conic bundle, F_0 is a fiber over P and $w_0^1, w_0^2, d_0 \in \mathbb{Z}_{\geq 1}$. Let us remark that the numbers w_0^1, w_0^2 determine g_0. Moreover, $d_0 = 1$ if $(X \ni P)$ is a smooth point.

Assume that there exists a curve $\Gamma_0 \subset S_0$ with the following two properties: (1) $K_{S_0} + \operatorname{Diff}_{S_0}(0) + \Gamma_0$ is a plt and g_0-anti-ample divisor; (2) Γ_0 is a non-toric subvariety in any analytical neighborhood of the fiber F_0 on the toric variety Z_0 for any toric structure of $(X \ni P)$, that is, the curve Γ_0 is a non-toric subvariety of $(S_0, \operatorname{Diff}_{S_0}(0))$ in any analytical neighborhood of F_0 on S_0.

By considering the general fiber over a general point of C we obtain $w_0^i = 1$ for some $i = 1, 2$. To be definite, put $w_0^1 = 1$ and let $Q_0 = E_0^2 \cap F_0$. Applying the adjunction formula it is easy to prove that $\Gamma_0 \cap F_0 = Q_0$, $w_0^2 \geq 2$, $d_0 = 1$, $(S_0 \ni Q_0)$ is of type $\frac{1}{r_0}(1, 1)$ $(r_0 \geq 1)$ and $\Gamma_0 \cdot F_0 = \Gamma \cdot E_0^2 = \frac{1}{r_0}$.

$$\begin{tikzcd}
 & Y_0 \arrow[dl, "h_0"'] \arrow[d, "h_0'"] & Y_1 \arrow[l, "h_1"'] \arrow[d, dashed, "h_1'"] & Y_2 \arrow[l, "h_2"'] \arrow[d, dashed, "h_2'"] \\
Z_0 \arrow[d, "g_0"'] & Z_1 \arrow[d, "g_1"'] & Z_2 \arrow[d, "g_2"'] & Z_3 \arrow[d, "g_3"'] \\
X & X & X & X
\end{tikzcd}$$

Fig. 1 Case of Curve. Construction in $\mathbb{Q}$-factorial case

Remark 3.2 Let $(X \ni P)$ be a terminal singularity, that is, $(X \ni P) \cong (\mathbb{C}^3_{x_1,x_2,x_3} \ni 0)/\mathbb{Z}_r(-1,-q,1)$. Then $r = r_0$ and one of the following cases holds by simple calculation.

(1) $C = \{x_1 = x_2 = 0\}$, g_0 is a blow-up with weights $(w_0^2, 1, 0)$, $r_0 | w_0^2$ or $(1, w_0^2, 0)$, $r_0 | (w_0^2 - q + 1)$.

(2) $C = \{x_1 = x_3 = 0\}$, g_0 is a blow-up with weights $(w_0^2, 0, 1)$, $r_0 | (w_0^2 + 1 + q)$ or $(1, 0, w_0^2)$, $r_0 | (w_0^2 - q + 1)$.

(3) $C = \{x_2 = x_3 = 0\}$, g_0 is a blow-up with weights $(0, w_0^2, 1)$, $r_0 | (w_0^2 + 1 + q)$ or $(0, 1, w_0^2)$, $r_0 | w_0^2$.

Consider an arbitrary toric structure of Z_0 in any neighborhood of the point Q_0 such that Γ_0 is also a toric subvariety of Z_0. Let $h_0 \colon (Y_0, (S_1)_{Y_0}) \to (Z_0 \supset \Gamma_0 \ni Q_0)$ be an arbitrary toric blow-up of the curve Γ_0 with an unique exceptional divisor $(\operatorname{Exc} h_0 = (S_1)_{Y_0})$. The structures of h_0 and g_0 are similar, in particular, h_0 is determined by some numbers w_1^1 and w_1^2, $(S_0)_{Y_0} \cong S_0$.

The set of all possible blow-ups h_0 for any toric structure of $(Z_0 \ni Q_0, \Gamma_0)$ is denoted by $\mathcal{H}_0$.

Let $(D_0)_{Z_0}$ be a toric Weil divisor of $(Z_0 \ni Q_0)$ such that $(D_0)_{Z_0}|_{S_0} = \Gamma_0$ and $a((S_1)_{Y_0}, (D_0)_{Z_0} + S_0) = -1$. Let T be a toric Weil divisor of $(X \ni P)$ such that $T_{Z_0} \cap S_0 = E_0^2$. Then $K_{Y_0} + (S_1)_{Y_0} + (S_0)_{Y_0} + (D_0)_{Y_0} + T_{Y_0} \sim 0$ is lc by Inversion of Adjunction. The ray $\mathbb{R}_+[(F_0)_{Y_0}]$ gives the divisorial contraction of $(S_0)_{Y_0}$ onto a curve, denoted by h_0' in our diagram. We obtain a non-toric blow-up $g_1 \colon (Z_1, S_1) \to (X \supset C \ni P)$, where $S_1 = \operatorname{Exc} g_1$, $g_1(S_1) = C$ and $(S_1)_{Y_0} \cong S_1$. Since g_1 be a toric blow-up (under identification) in some neighborhood of any point other than P, then $\operatorname{Diff}_{S_1}(0) = \frac{w_1^3-1}{w_1^3} E_1^2 + \frac{w_1^j-1}{w_1^j} E_1^1 + \frac{d_1-1}{d_1}(F_1)_{Z_0}$, $j \in \{1, 2\}$, $E_1^2 = h_0'((S_0)_{Y_0})$ and E_1^1 are some sections, F_1 is a fiber over P, $w_1^3 \in \mathbb{Z}_{\geq 3}$ and $d_1 \in \mathbb{Z}_{\geq 1}$. Hence g_1 is a plt blow-up.

Second step. Assume that there exists a curve $\Gamma_1 \subset (S_1)_{Y_0}$ with the following two properties: (1) $K_{(S_1)_{Y_0}} + \operatorname{Diff}_{(S_1)_{Y_0}}(0) + \Gamma_1$ is a plt and h_0-anti-ample divisor, $h_0 \colon \Gamma_1 \to \Gamma_0$ is a surjective morphism and (2) Γ_1 is not a center of any blow-up of $\mathcal{H}_0$, that is, Γ_1 is a non-toric subvariety of $((S_1)_{Y_0}, \operatorname{Diff}_{(S_1)_{Y_0}}(0))$ in any analytical neighborhood of the fiber $(F_1)_{Y_0}$ over P.

The triples $((S_1)_{Y_0}, \operatorname{Diff}_{(S_1)_{Y_0}}(0), \Gamma_1)$ and $(S_0, \operatorname{Diff}_{S_0}(0), \Gamma_0)$ have the same structures and (with similar notation) $w_1^1 = 1$, $Q_1 = (E_1^2)_{Y_0} \cap (F_1)_{Y_0}$, $\Gamma_1 \cap (F_1)_{Y_0} = Q_1$, $w_1^2 \geq 1$, $d_1 = 1$, $((S_1)_{Y_0} \ni Q_1)$ is of type $\frac{1}{r_1}(1,1)$ $(r_1 \geq 1)$ and $\Gamma_1 \cdot (F_1)_{Y_0} = \Gamma_1 \cdot (E_1^2)_{Y_0} = \frac{1}{r_1}$.

Consider an arbitrary toric structure of Y_0 in any neighborhood of the point Q_1 such that Γ_1 is also a toric subvariety of Y_0. Let $h_1 \colon (Y_1, (S_2)_{Y_1}) \to (Y_0 \supset \Gamma_1 \ni Q_1)$ be an arbitrary toric blow-up of the curve Γ_1 with an unique exceptional divisor $(\operatorname{Exc} h_1 = (S_2)_{Y_1})$, $(S_1)_{Y_1} \cong (S_1)_{Y_0}$.

The set of all possible blow-ups h_1 for any toric structure of $(Y_0 \ni Q_1, \Gamma_1)$ is denoted by $\mathcal{H}_1$.

Let $(D_1)_{Y_0}$ be a toric Weil divisor of $(Y_0 \ni Q_1)$ such that $(D_1)_{Y_0}|_{S_1} = \Gamma_1$ and $a((S_2)_{Y_1}, (D_1)_{Y_0} + (S_0)_{Y_0} + (S_1)_{Y_0}) = -1$. We have 1-complement $K_{Y_1} + (S_2)_{Y_1} +$

$(S_1)_{Y_1} + (S_0)_{Y_1} + (D_1)_{Y_1} \sim 0/X$ by Inversion of Adjunction applied to the surfaces $(S_i)_{Y_1}$. By the cone theorem we have:

(1) there exists a divisorial contraction $h'_{1,1}: Y_1 \to Y_{1,1}$ of $(S_1)_{Y_1}$ onto a curve, $(S_2)_{Y_1} \cong (S_2)_{Y_{1,1}}$;

(2) there exists a small contraction $\varphi_{1,1}$ of an extremal ray generated by $(F_0)_{Y_{1,1}}$. Let $\varphi^+_{1,1}$ be a log flip of $\varphi_{1,1}$, $\operatorname{Exc}\varphi^+_{1,1} = (F_0^+)_{Y_{1,2}}$, $h'_{1,2}: Y_{1,1} \dashrightarrow Y_{1,2}$ be a corresponding birational map;

(3) there exists a divisorial contraction $h'_{1,3}: Y_{1,2} \to Z_2$ of $(S_0)_{Y_{1,2}}$ onto a curve.

Thus we obtain a birational map $h'_1 = h'_{1,3} \circ h'_{1,2} \circ h'_{1,1}: Y_1 \dashrightarrow Z_2$. Put $S_2 = (S_2)_{Z_2}$. Since $(E_0^2)_{Y_{1,1}} \cap (F_0)_{Y_{1,1}} = (Q_0)_{Y_{1,1}}$ then $(D_1)_{Y_{1,1}} \cdot (F_0)_{Y_{1,1}} > 0$ and the divisor $(D_1)_{Z_2}$ contains the fiber $(F_0^+)_{Z_2}$ and two sections of the local conic bundle $S_2 \to C$, $\rho(S_2/C) = 1$, $K_{Z_2} + S_2 + (D_1)_{Z_2} \sim 0/X$ is lc. By Shokurov's criterion on the characterization of toric varieties $(S_2, \operatorname{Diff}_{S_2}(0)) \to C$ is a toric conic bundle [26]. We obtain a non-toric plt blow-up $g_2: (Z_2, S_2) \to (X \supset C \ni P)$.

We prove the following proposition.

Proposition 3.3 *The pair $(S_i, \operatorname{Diff}_{S_i}(0))$ is klt and local toric conic bundle* (1-*complementary), $\rho(S_i/C) = 1$, g_i is a non-toric plt blow-up for $i = 1, 2$.*

Third step. Assume that there exists a curve $\Gamma_2 \subset (S_2)_{Y_1}$ with the following two properties: (1) $K_{(S_2)_{Y_1}} + \operatorname{Diff}_{(S_2)_{Y_1}}(0) + \Gamma_2$ is a plt and h_1-anti-ample divisor, $h_1: \Gamma_2 \to \Gamma_1$ is a surjective morphism and (2) Γ_2 is not a center of any blow-up of $\mathcal{H}_1$, that is, Γ_2 is a non-toric subvariety of $((S_2)_{Y_1}, \operatorname{Diff}_{(S_2)_{Y_1}}(0))$ in any analytical neighborhood of the central fiber F_2 of $(S_2)_{Y_1}$ over P.

The triple $((S_2)_{Y_1}, \operatorname{Diff}_{(S_2)_{Y_1}}(0), \Gamma_2)$ has the same structures as the previous ones. In particular (with similar notation), $w_2^1 = 1$ and $w_2^2 \geq 1$.

Proposition 3.4 *There is no any blow-up $h_2: (Y_2, (S_3)_{Y_2}) \to (Y_1 \supset \Gamma_2)$ of the curve Γ_2 with unique exceptional divisor such that $(S_3)_{Y_2}$ is realized by some plt blow-up $g_3: (Z_3, (S_3)_{Z_3}) \to (X \supset C \ni P)$.*

Proof Assume the converse. Consider a general point of C. Let F_3 be a fiber of $(S_3)_{Y_2}$ over P. Put $\Theta = \operatorname{Diff}_{(S_3)_{Z_3}}(0)$ for simplicity. Since $w_0^2 + w_1^2 + w_2^2 + 1 \geq 5$ then Θ has some component (a section of conic bundle) with a coefficient $\geq 4/5$.

We claim that $K_{(S_3)_{Z_3}} + \Theta$ is 1 or 2-complementary. Assume that $K_{(S_3)_{Z_3}} + \Theta$ is not 1-complementary. Then the divisor $K_{(S_3)_{Z_3}} + \alpha F_3 + \Theta$ is lc, but not plt for some $\alpha \leq 1$, and consider its inductive blow-up $\sigma: \widetilde{X} \to (S_3)_{Z_3}$ with exceptional divisor $\widetilde{E}$. The curve $(F_3)_{\widetilde{X}}$ can be contracted in the appropriate MMP over C. Denote this contraction by $\widetilde{X} \to \overline{X}$. The divisor $K_{\overline{X}} + \overline{E} + \Theta_{\overline{X}}$ is plt.

Let $K_{\widetilde{X}} + \widetilde{E} + \Theta_{\widetilde{X}}$ be nonnegative on $(F_3)_{\widetilde{X}}$. We can extend complement of $K_{\overline{E}} + \operatorname{Diff}_{\overline{E}}(\Theta_{\overline{X}})$ on $\overline{X}$, pull back on $\widetilde{X}$ and push-down them on $(S_3)_{Z_3}$. There are only two cases: (1) $\operatorname{Diff}_{\overline{E}}(\Theta_{\overline{X}}) = 1/2P_1 + 1/2P_2 + (1 - 1/m)P_3$ and 2) $\operatorname{Diff}_{\overline{E}}(\Theta_{\overline{X}}) = 1/2P_1 + 2/3P_2 + 4/5P_3$, where $\{P_i\}$ are some points, $m \geq 5$. We obtain 2- or 6-complement.

Let $K_{\widetilde{X}} + \widetilde{E} + \Theta_{\widetilde{X}}$ be negative on $(F_3)_{\widetilde{X}}$. The divisor $-(K_{\widetilde{X}} + \widetilde{E} + \Theta_{\widetilde{X}})$ is ample over C. Similarly 2- or 6-complement of $K_{\widetilde{E}} + \operatorname{Diff}_{\widetilde{E}}(\Theta_{\widetilde{X}})$ can be extended on $\widetilde{X}$ and we have 2- or 6-complement D^+ of K_X with $a((S_3)_{Y_2}, D^+) = -1$.

Fig. 2 Case of Curve. Construction in non-$\mathbb{Q}$-factorial case

$$\begin{tikzcd} & Y_0 \arrow[dl, "h_0"'] \arrow[d, dashed, "h_0'"] & Y_1 \arrow[l, "h_1"'] \arrow[d, dashed, "h_1'"] \\ Z_0 \arrow[d, "g_0"] & Z_1 \arrow[d, "g_1"] & Z_2 \arrow[d, "g_2"] \\ X & X & X \end{tikzcd}$$

Consider the case of 6-complement. Since $a((S_3)_{Y_2}, D^+) = -1$ then there is one possibility $a((S_0)_{Y_2}, D^+) = -1/2$, $D^+|_{S_0} = (7/6)\Gamma_0 + \ldots$ and $a((S_1)_{Y_2}, D^+) \le -2/3$. Since $F_3 \subset (S_i)_{Y_2}$ for $i = 0, 1$ then $K_{Y_2} + a((S_0)_{Y_2}, D^+)(S_0)_{Y_2} + a((S_1)_{Y_2}, D^+)(S_1)_{Y_2} + (S_3)_{Y_2}$ is not lc, the contradiction.

Thus we have 1- or 2-complement. Therefore the coefficients of D^+ are equal 1 or 1/2 and $a((S_0)_{Y_2}, D^+) \le -1/2$. We have the same contradiction as above. $\square$

(A2) Let $(X \ni P)$ be a non-$\mathbb{Q}$-factorial terminal toric three-dimensional singularity, that is, $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$ by Proposition 2.3.

Let $f\colon (Y, E) \to (X \supset C \ni P)$ be some plt blow-up. Let $\varphi_i\colon X_i \to (X \ni P)$ be any of two $\mathbb{Q}$-factorializations, $\operatorname{Exc}\varphi_i = C_i$ $(i = 1, 2)$. Let $\psi_i\colon (Y_i, E_i) \to (X_i \supset C_{X_i} \ni P_{X_i})$ be a plt blow-up of C_{X_i} such that E_i and E define the same discrete valuation of the function field $\mathcal{K}(X)$, $\rho(E_i/C) = 1$. The blow-up ψ_i was constructed in the previous case of $\mathbb{Q}$-factorial singularities. Let $Y_i \dashrightarrow Y$ be a log flip for the curve $(C_i)_{Y_i}$. Thus f has constructed and $\rho(E/C) = 2$.

We give another construction and prove that $(E, \operatorname{Diff}_E(0)) \to C$ is a toric conic bundle by the procedure illustrated on the next diagram (Fig. 2) and defined below.

First step. Let $g_0\colon (Z_0, S_0) \to (X \supset C \ni P)$ be any toric plt blow-up, where $g_0(S_0) = C$. Its description is given in example 2.2 2), whose notation is used. Let $F_0 = F_0^1 + F_0^2$ be a fiber over the point P. Put $Q_0 = F_0^1 \cap F_0^2$.

Second step. Assume that there exists a curve $\Gamma_0 \subset S_0$ with the following two properties: (1) $K_{S_0} + \operatorname{Diff}_{S_0}(0) + \Gamma_0$ is a plt and g_0-anti-ample divisor; 2) Γ_0 is a non-toric subvariety in any analytical neighborhood of the fiber F_0 on the toric variety Z_0 for any toric structure of $(X \ni P)$, that is, the curve Γ_0 is a non-toric subvariety of $(S_0, \operatorname{Diff}_{S_0}(0))$ in any analytical neighborhood of F_0 on S_0.

Considering a fiber over a general point of C we have $a_2 = 1$ or $a_3 = 1$. To be definite, put $a_2 = 1$ and $F_0^2 \cap E_2 \ne \emptyset$. By simple calculations $\Gamma_0 \cap (F_0^1 \cup F_0^2) = Q_0$, $F_0^1 \cdot \Gamma_0 = \frac{a_3}{a_3+1}$ and $F_0^2 \cdot \Gamma_0 = \frac{1}{a_3+1}$.

Consider an arbitrary toric structure of Z_0 in any neighborhood of the point Q_0 such that Γ_0 is a toric subvariety of Z_0 also. Let $h_0\colon (Y_0, (S_1)_{Y_0}) \to (Z_0 \supset \Gamma_0 \ni Q_0)$ be an arbitrary toric blow-up of the curve Γ_0 with an unique exceptional divisor $(\operatorname{Exc} h_0 = (S_1)_{Y_0})$, $(S_0)_{Y_0} \cong S_0$.

The set of all possible blow-ups h_0 for any toric structure of $(Z_0 \ni Q_0, \Gamma_0)$ is denoted by $\mathcal{H}_0$.

Let $(D_0)_{Z_0}$ be a toric Weil divisor of $(Z_0 \ni Q_0)$ such that $(D_0)_{Z_0}|_{S_0} = \Gamma_0$ and $a((S_1)_{Y_0}, (D_0)_{Z_0} + S_0) = -1$. Let T_1 and T_2 be toric Weil divisors of $(X \ni P)$ such that

$$K_{S_0} + \operatorname{Diff}_{S_0}((T_1 + T_2)_{Z_0} + (D_0)_{Z_0}) = K_{S_0} + F_0^2 + E_2 + \Gamma_0 \sim 0.$$

The pairs $(X \ni P, T_1 + T_2 + (D_0)_X)$ and $((S_1)_{Y_0}, \operatorname{Diff}_{(S_1)_{Y_0}}((T_1 + T_2)_{Y_0} + (D_0)_{Y_0} + (S_0)_{Y_0}))$ are lc. Since $T_1 + T_2$ is Cartier divisor then $(D_0)_X$ is Cartier divisor. The curves $(F_0^1)_{Y_0}$ and $(F_0^2)_{Y_0}$ generate extremal rays of $\overline{\mathrm{NE}}(Y_0/X)$ that give small contractions. Let $h'_{0,1}\colon Y_0 \dashrightarrow Y_{0,1}$ be any of two log flips. Since our pairs are lc then $\rho((S_0)_{Y_{0,1}}/C) = 1$. Let $h'_{0,2}\colon Y_{0,1} \to Z_1$ be a divisorial contraction of $(S_0)_{Y_{0,1}}$ onto a curve.

Thus we obtain a birational map $h'_0 = h'_{0,2} \circ h'_{0,1}\colon Y_0 \dashrightarrow Z_1$ and a non-toric blow-up $g_1\colon (Z_1, S_1) \to (X \supset C \ni P)$, $\rho(S_1/C) = 2$. It can be proved by direct computation that $-S_1$ is g_1-ample divisor, but if we consider the construction of g_1 through two $\mathbb{Q}$-factorializations of $(X \ni P)$ as done above, then it is obvious that the divisor $-S_1$ is g_1-ample. The divisor $\operatorname{Diff}_{S_1}((T_1 + T_2)_{Z_1} + (D_0)_{Z_1})$ consists of four curves and gives 1-complement of $K_{S_1} + \operatorname{Diff}_{S_1}(0)$. By Shokurov's criterion on the characterization of toric varieties $(S_1, \operatorname{Diff}_{S_1}((T_1 + T_2)_{Z_1} + (D_0)_{Z_1}) \to C$ is a toric conic bundle [26]. Thus g_1 is a plt blow-up.

Third step. Assume that there exists a curve $\Gamma_1 \subset (S_1)_{Y_0}$ with the following two properties: (1) $K_{(S_1)_{Y_0}} + \operatorname{Diff}_{(S_1)_{Y_0}}(0) + \Gamma_1$ is plt and h_0-anti-ample divisor, $h_0\colon \Gamma_1 \to (\Gamma_0)_{Z_0}$ is a surjective morphism and (2) Γ_1 is not a center of any blow-up of $\mathcal{H}_0$, that is, Γ_1 is a non-toric subvariety of $((S_1)_{Y_0}, \operatorname{Diff}_{(S_1)_{Y_0}}(0))$ in any analytical neighborhood of the central fiber F_1 of $(S_1)_{Y_0}$ over P.

The triple $((S_1)_{Y_0}, \operatorname{Diff}_{(S_1)_{Y_0}}(0), \Gamma_1)$ has the same structures as in the previous case of $\mathbb{Q}$-factorial singularities, and we use its notation.

Consider an arbitrary toric structure of Y_0 in any neighborhood of the point Q_1 such that Γ_1 is also a toric subvariety of Y_0. Let $h_1\colon (Y_1, (S_2)_{Y_1}) \to (Y_0 \supset \Gamma_1 \ni Q_1)$ be an arbitrary toric blow-up of the curve Γ_1 with an unique exceptional divisor $(\operatorname{Exc} h_1 = (S_2)_{Y_1})$, $(S_1)_{Y_1} \cong (S_1)_{Y_0}$.

Let $(D_1)_{Y_0}$ be a toric Weil divisor of $(Y_0 \ni Q_1)$ such that $(D_1)_{Y_0}|_{S_1} = \Gamma_1$ and $a((S_2)_{Y_1}, (D_1)_{Y_0} + (S_0)_{Y_0} + (S_1)_{Y_0}) = -1$. Considering the case of $\mathbb{Q}$-factorial singularities and construction of $g_0 \circ h_0$ through $\mathbb{Q}$-factorializations of $(X \ni P)$ we have $(E_2)_{Y_0} \subset (D_1)_{Y_0}$ and hence $F_0^2 \subset (D_1)_{Y_0}$. Thus we have 1-complement $K_{Y_1} + (S_2)_{Y_1} + (S_1)_{Y_1} + (S_0)_{Y_1} + (D_1)_{Y_1} \sim 0/X$ by Inversion of Adjunction applied to the surfaces $(S_i)_{Y_1}$. By the cone theorem we have:

(1) there exists a divisorial contraction $h'_{1,1}\colon Y_1 \to Y_{1,1}$ of $(S_1)_{Y_1}$ onto a curve, $(S_2)_{Y_1} \cong (S_2)_{Y_{1,1}}$;

(2) there exists a small contraction of $(F_0^1)_{Y_{1,1}}$, $h'_{1,2}\colon Y_{1,1} \dashrightarrow Y_{1,2}$ is a corresponding log flip;

(3) there exists a small contraction of $(F_0^2)_{Y_{1,2}}$, $h'_{1,3}\colon Y_{1,2} \dashrightarrow Y_{1,3}$ is a corresponding log flip;

(4) there exists an divisorial contraction $h'_{1,4}\colon Y_{1,3} \to Z_2$ of $(S_0)_{Y_{1,3}}$ onto a curve.

Thus we obtain a birational map $h'_1 = h'_{1,4} \circ h'_{1,3} \circ h'_{1,2} \circ h'_{1,1}\colon Y_1 \dashrightarrow Z_2$, the local conic bundle $(S_2)_{Z_2} \to C$, $\rho((S_2)_{Z_2}/C) = 2$ and $K_{Z_2} + (S_2)_{Z_2} + (D_1)_{Z_2} \sim$

$0/X$ is lc. Let $F_2 = F_2^1 + F_2^2$ be a fiber over P and the curves F_2^1, F_2^2 appear due to log flips $h'_{1,2}, h'_{1,3}$ respectively. By the construction the divisor $(D_1)_{Z_2}$ contains two sections of $(S_2)_{Z_2}$ and F_2^1.

If we consider this construction through two $\mathbb{Q}$-factorializations of $(X \ni P)$ then $(S_2)_{Z_2}$ is anti-ample over C and $(F_0^1)_{Y_{1,1}} \cap (F_0^2)_{Y_{1,1}} = (Q_0)_{Y_{1,1}}$. Since $(F_0^2)_{Y_{1,2}} \cdot (F_0^2)_{Y_{1,2}} = 0$, $K_{Y_{1,2}} + (S_0)_{Y_{1,2}} + (S_2)_{Y_{1,2}} + (D_1)_{Y_{1,2}} \sim 0$ then for some $e > 0$ we have $(D_1)_{Y_{1,2}} \cdot (F_0^2)_{Y_{1,2}} = e(E_2)_{Y_{1,2}} \cdot (F_0^2)_{Y_{1,2}} > 0$ and $(D_1)_{Z_2}$ contains F_2^2.

By Shokurov's criterion on the characterization of toric varieties $((S_2)_{Z_2}, \operatorname{Diff}_{(S_2)_{Z_2}}(0)) \to C$ is a toric conic bundle [26]. We obtain a non-toric plt blow-up $g_2\colon (Z_2, S_2) \to (X \supset C \ni P)$, where $S_2 = (S_2)_{Z_2}$.

We prove the following proposition.

Proposition 3.5 *The pair* $(S_i, \operatorname{Diff}_{S_i}(0))$ *is klt and local toric conic bundle* (1-*complementary*), $\rho(S_i/C) = 1$, g_i *is a non-toric plt blow-up for* $i = 1, 2$.

Example 3.6 Let us describe the non-toric canonical blow-ups (they will be non-terminal blow-ups always) $g\colon (Y, E) \to (X \supset C \ni P)$ provided that $(X \ni P)$ is a toric terminal singularity, $C = g(E)$ is a toric (smooth) subvariety and $\dim C = 1$. Depending on a type of $(X \ni P)$ we consider two Cases **(B1)** and **(B2)**.

(B1) Let $(X \ni P)$ be a $\mathbb{Q}$-factorial terminal singularity. Let $g\colon (Z, S) \to (X \supset C \ni P)$ be any toric canonical blow-up (see Proposition 2.5).

Assume that there exists a curve $\Gamma \subset S$ with the following two properties: (1) $K_S + \operatorname{Diff}_S(0) + \Gamma$ is g-anti-ample divisor, and Γ does not contain any center of canonical singularities of Z; (2) Γ is a non-toric subvariety in any analytical neighborhood of the fiber F (over P) on the toric variety Z for any toric structure of $(X \ni P)$, that is, the curve Γ is a non-toric subvariety of $(S, \operatorname{Diff}_S(0))$ in any analytical neighborhood of F on S.

Thus $(X \ni P)$ is a smooth point, S is a smooth surface, $\operatorname{Diff}_S(0) = \frac{k-1}{k}E$, where $k \geq 2$ and E is some section by Proposition 2.5. By adjunction formula Γ is smooth, $Q = \Gamma \cap F \cap E$, $\Gamma \cdot F = 1$.

Let $(X \ni P, D)$ be any pair with canonical singularities such that D is a boundary, $\Gamma \in \operatorname{CS}(Z, D_Z - a(S, D)S)$. Obviously, $D_Z|_S = \Gamma + aF$ and $a(S, D) = 0$, where $a \geq 0$.

Considering the blow-up $(\mathbb{C}^3_{x_1x_2x_3} \ni 0) \cong (X \supset C \ni P)$ with weights $(k, 1, 0)$, $C = \{x_1 = x_2 = 0\}$ and the divisor given by the equation $x_1^2 + x_1x_2 + x_1x_3^m + bx_2^k = 0$, then clearly, there is a divisor D for any such curve Γ.

By Theorem 1.6 there exists a divisorial contraction $h\colon (\widetilde{Y}, \widetilde{E}) \to (Z \supset \Gamma)$ such that $a(\widetilde{E}, D) = 0$, $\operatorname{Exc} h = \widetilde{E}$ is an irreducible divisor and $h(\widetilde{E}) = \Gamma$. Applly $K_{\widetilde{Y}} + D_{\widetilde{Y}} + \varepsilon\widetilde{S}$–MMP. Since $\rho(\widetilde{Y}/X) = 2$ and $K_{\widetilde{Y}} + D_{\widetilde{Y}} + \varepsilon\widetilde{S} \equiv \varepsilon\widetilde{S}$ over X, then after log flips $\widetilde{Y} \dashrightarrow \overline{Y}$ (perhaps their lack) we obtain a divisorial contraction $h'\colon \overline{Y} \to Y$, which contracts $\overline{S}$ onto a curve C_Y.

Thus we obtain a non-toric canonical blow-up f. Since $C_Y \in \operatorname{CS}(Y)$ by the construction then f is not a terminal blow-up.

(B2) Let $(X \ni P)$ be a non-$\mathbb{Q}$-factorial terminal toric three-dimensional singularity, that is, $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$. Consider a $\mathbb{Q}$-factorialization $g\colon \widetilde{X} \to X$, $\widetilde{T} = \operatorname{Exc} g$ and $\widetilde{P} = \widetilde{T} \cap \widetilde{C}$. We apply the construction from **(B1)** for the curve $\widetilde{C} \subset (\widetilde{X} \ni \widetilde{P})$ such that the divisor D from the construction has the form g^*D_X, where D_X is a $\mathbb{Q}$-Cartier divisor. We obtain a non-toric canonical blow-up $f\colon Y^+ \to \widetilde{X}$. Let $Y^+ \dashrightarrow Y$ be a log flip for the curve T_{Y^+}. Thus we obtain a required non-toric canonical blow-up f (anti-amplness of E is proved as in case **(A2)**).

Let us describe the curves Γ. Let $g\colon (Z, S) \to (\widetilde{X} \ni \widetilde{P})$ be a toric canonical blow-up obtained in the first step of the construction. Let $\psi\colon Z \dashrightarrow Z^+$ be a toric log flip for the curve T_Z. So $g^+\colon (Z^+, S^+) \to (X \ni P)$ is a toric canonical blow-up. The structure of the curve Γ_{S^+} is completely identical to the structure of the curve Γ considered in case **(A2)**. To prove that any such curve Γ_{S^+} is realizable, it suffices to consider a divisor of the form $x_{i_1} + bx_{i_2}^k = 0$ on $(X \ni P)$ for some b, k, $\{i_1, i_2\} = \{1, 2\}$ or $\{3, 4\}$.

Theorem 3.7 *Let $f\colon (Y, E) \to (X \supset C \ni P)$ be a plt blow-up of three-dimensional toric terminal singularity, where* $\dim f(E) = 1$. *Assume that the curve $C = f(E)$ is a toric subvariety of $(X \ni P)$. Then, either f is a toric morphism (see Example 2.2), or f is a non-toric morphism described in Example 3.1.*

Proof By Example 3.1 we must only consider the case when $(X \ni P)$ is a $\mathbb{Q}$–factorial singularity. Let f be a non-toric morphism (up to analytic isomorphism). Let $D_Y \in |-n(K_Y + E)|$ be a general element for $n \gg 0$. Put $D_X = f(D_Y)$ and $d = \frac{1}{n}$. The pair (X, dD_X) is log canonical, $a(E, dD_X) = -1$, and E is a unique exceptional divisor with discrepancy -1.

By the construction of partial resolution of (X, dD_X) (see Definition 2.7 and the paper [27]) and by Criterion 2.8, there exists a toric divisorial contraction $g\colon Z \to X$ dominated by partial resolution of (X, dD_X) (up to toric log flips) and the following properties are fulfilled.

(A) The exceptional set $\operatorname{Exc} g = S$ is an irreducible divisor, the divisors S and E define the different discrete valuations of the function field $\mathcal{K}(X)$, and $g(S) = C$.
(B) By Γ denote the center of E on the surface S. Then the curve Γ is a non-toric subvariety of Z. In the other words, Γ is a non-toric subvariety of $(S, \operatorname{Diff}_S(0))$.

Obviously, $a(S_0, dD_X) < 0$. By Example 3.1 (in its notation) we must prove only that the anti-ample over X divisor $K_{S_0} + \operatorname{Diff}_{S_0}(0) + \Gamma_0$ is plt in some analytical neighborhood of the fiber $F_0 \subset S_0$. We can choose the divisor dD_X such that $\operatorname{Supp}(dD_X|_{S_0}) \subset \Gamma_0 \cup F \cup \Gamma_0' \cup E_0^2$, where Γ_0' is a general divisor on S_0.

Assume that $K_{S_0} + \operatorname{Diff}_{S_0}(0) + \Gamma_0$ is not a plt divisor. By the adjunction formula the curve Γ_0 is smooth. By connectedness lemma $K_{S_0} + \operatorname{Diff}_{S_0}(0) + \Gamma_0$ is not a plt divisor at unique point, and denote this point by G_0. The point G_0 is a non-toric subvariety of $(S_0, \operatorname{Diff}_{S_0}(0))$. Moreover, the curve Γ_0 is locally a non-toric subvariety at the point G_0 only. By the construction of partial resolution [27] (in a small analytical neighborhood of the point G_0) there exists a divisorial toric contraction $\widehat{g}_0\colon \widehat{Z}_0 \to Z_0$ such that $\operatorname{Exc} \widehat{g}_0 = S_0''$ is an irreducible divisor, $\widehat{g}(S_0'') = G_0$ and the two following conditions are satisfied.

(1) Put $S_0' = (S_0)_{\widehat{Z}_0}$ and $C_0 = S_0' \cap S_0''$. Let $c(\Gamma_0)$ be the log canonical threshold of Γ_0 for the pair $(S_0, \operatorname{Diff}_{S_0}(0))$. Then $\widehat{g}_0|_{S_0'}\colon S_0' \to S_0$ is the toric inductive blow-up

of $K_{S_0} + \operatorname{Diff}_{S_0}(0) + c(\Gamma_0)\Gamma_0$ (see Theorems 1.10 and 2.12), and the point $\widehat{G}_0 = C_0 \cap (\Gamma_0)_{S'_0}$ is a non-toric subvariety of $(S''_0, \operatorname{Diff}_{S''_0}(0))$.

(2) The divisor $\operatorname{Diff}_{S''_0}(dD_{\widehat{Z}_0} + a(S_0, dD_X)S'_0)$ is a boundary in some small analytical neighborhood of the point $\widehat{G}_0$.

Let H be a general hyperplane section of sufficiently large degree passing through the point P such that it does not contain the curve C. Then there exists a number $h > 0$ such that $a(S''_0, dD_X + hH) > -1$, and the point $\widehat{G}_0$ is a center of $(S''_0, \operatorname{Diff}_{S''_0}(dD_{\widehat{Z}_0} + a(S_0, dD_X)S'_0 + hH_{\widehat{Z}_0}))$. Therefore we obtain a contradiction for the pair $(S''_0, \operatorname{Diff}_{S''_0}(dD_{\widehat{Z}_0} + a(S_0, dD_X)S'_0 + hH_{\widehat{Z}_0}))$ and the point $\widehat{G}_0$ by Theorem 4.2. $\square$

We have proved the next theorem too.

Theorem 3.8 *Let $f : (Y, E) \to (X \supset C \ni P)$ be a plt blow-up of three-dimensional toric $\mathbb{Q}$-factorial singularity, where* $\dim f(E) = 1$. *Assume that the curve $C = f(E)$ is a toric subvariety of $(X \ni P)$. Then, either f is a toric morphism (see Example 2.2), or f is a non-toric morphism described in Example 3.1.*

Theorem 3.9 *Let $f : (Y, E) \to (X \supset C \ni P)$ be a canonical blow-up of three-dimensional toric terminal singularity, where* $\dim f(E) = 1$. *Assume that the curve $C = f(E)$ is a toric subvariety of $(X \ni P)$. Then, either f is a toric morphism (see Proposition 2.5), or f is a non-toric morphism and described in Example 3.6.*

Proof Let f be a non-toric morphism (up to analytic isomorphism). Let $D_Y \in |-nK_Y|$ be a general element for $n \gg 0$. Put $D_X = f(D_Y)$ and $d = \frac{1}{n}$. The pair (X, dD_X) has canonical singularities and $a(E, dD_X) = 0$. Now the arguments of the proof of Theorem 3.7 can be obviously applied, and we have $a(S, dD_X) = 0$, this completes the proof. $\square$

Corollary 3.10 *Under the same assumption as in Theorem 3.9 the two following statements are satisfied:*

(1) *[8] if f is a terminal blow-up then the (toric) morphism f is isomorphic to the blow-up of the ideal of the curve C and an index of $(X \ni P)$ is equal to* 1, *that is, either $(X \ni P)$ is a smooth point or $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$;*

(2) *if f is a non-toric morphism then an index of $(X \ni P)$ is equal to* 1.

4 Toric Log Surfaces

Definition 4.1 Let $\mathbb{P}(\mathbf{w}) = \mathbb{P}_{x_1x_2x_3x_4}(w_1, w_2, w_3, w_4)$, where $w_1 + w_2 = w_3 + w_4$ and $\gcd(w_1, w_2, w_3, w_4)$=1. Put $(w_1, w_2, w_3, w_4) = (a_1d_{23}d_{24}, a_2d_{13}d_{14}, a_3d_{14}d_{24}, a_4d_{13}d_{23})$, where $d_{ij} = \gcd(w_k, w_l)$ and i, j, k, l are mutually distinct indices from 1 to 4. The toric pair

$$\big(S, D\big) = \big(x_1x_2 + x_3x_4 \subset \mathbb{P}(\mathbf{w}), \operatorname{Diff}_{S/\mathbb{P}(\mathbf{w})}(0)\big)$$

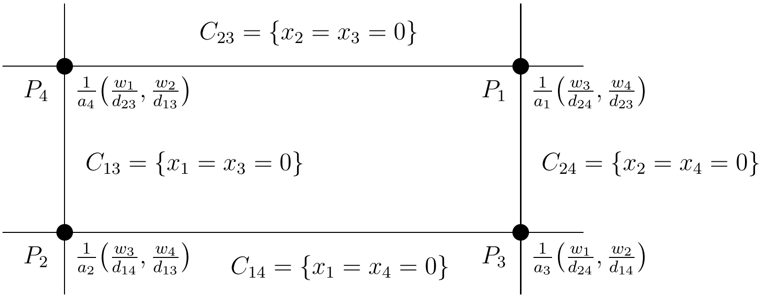

Fig. 3 ODP surface

is called *an ODP pair*, $\rho(S) = 2$. By Proposition 1.6 of [14] we have $D = \sum_{i<j, 1\leq i\leq 2} \frac{d_{ij}-1}{d_{ij}} C_{ij}$, where $C_{ij} = \{x_i = x_j = 0\} \cap S$.

Let $f\colon (Y, E) \to (X \ni P)$ be a toric plt blow-up of three-dimensional ordinary double point. Then $(E, \mathrm{Diff}_E(0))$ is an ODP pair by Example 2.2. Converse is also true: every ODP pair is realized by some toric plt blow-up of three-dimensional ordinary double point.

To be definite, assume that $w_1 \leq w_2, w_3 \leq w_4, w_2 \leq w_4, P_1 = (1:0:0:0), \ldots, P_4 = (0:0:0:1)$. The surface S has a cyclic singularity at the point P_i for every $i = 1, 2, 3, 4$ (see Fig. 3).

Since $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(w_i)|_S = \{x_i = 0\}|_S = \frac{1}{d_{ik}} C_{ik} + \frac{1}{d_{il}} C_{il}$ for the corresponding different indices k and l, then it is easy to calculate that $C_{13}^2 = d_{13}^2(w_3 - w_2)/(w_2 w_4) \leq 0$, $C_{23}^2 = d_{23}^2(w_2 - w_4)/(w_1 w_4) \leq 0$, $C_{14}^2 = d_{14}^2(w_4 - w_2)/(w_2 w_3) \geq 0$ and $C_{24}^2 = d_{24}^2(w_2 - w_3)/(w_1 w_3) \geq 0$. In particular, Mori cone $\overline{\mathrm{NE}}(S)$ is generated by the two rays $\mathbb{R}_+[C_{13}], \mathbb{R}_+[C_{23}]$.

Now we prove a two-dimensional *non-toric point theorem*. An one-dimensional analog ($\dim S = 1$) of Theorem 4.2 (1) is obvious (see the proofs of Theorems 2.12 and 2.13 also).

Theorem 4.2 *Let (S, D) be a toric pair, where S is a normal projective surface. Assume that $D = \sum_{i=1}^r d_i D_i$, where D_i is a prime divisor and $\frac{1}{2} \leq d_i \leq 1$ for each i. Assume that there exists the boundary T such that $T \geq D$ and $-(K_S + T)$ is an ample divisor. Assume that some point Γ is a center of* $\mathrm{LCS}(S, T)$*, and there exists the analytical neighborhood U of Γ such that $K_S + T$ is a log canonical divisor in the punctured neighborhood $U \backslash \Gamma$. Then the point Γ is a toric subvariety of (S, D) if one of the two following conditions is satisfied:*

(1) $\rho(S) = 1$*;*

(2) $\rho(S) = 2$*, two different extremal rays of* $\overline{\mathrm{NE}}(S)$ *give two toric conic bundles;*

(3) (S, D) *is ODP pair.*

Proof Let the point Γ be a non-toric subvariety of (S, D). We will obtain a contradiction.

Consider Condition (1). It is clear that this theorem is sufficient to prove in the case $d_i = \frac{1}{2}$ for all i.

Since $-(K_S + T)$ is an ample divisor, then replacing T by some divisor we can assume that $\mathrm{LCS}(S, T) \cap U = \Gamma$. Hence, connectedness lemma implies that $\mathrm{LCS}(S, T) = \Gamma$.

The toric projective surface S (with Picard number $\rho(S) = 1$) is determined by the fan Δ in the lattice $N \cong \mathbb{Z}^2$, where

$$\Delta = \big\{\langle n_1, n_2\rangle, \langle n_2, n_3\rangle, \langle n_1, n_3\rangle, \text{their faces}\big\}.$$

Thus surface S has at most three singular points. If the number of singularities is less than or equal to two, then there exists an isomorphism of the lattice N such that $n_1 = (1, 0)$, $n_2 = (0, 1)$, and therefore $S \cong \mathbb{P}_{x_1x_2x_3}(a_1, a_2, 1)$.

Suppose that the point Γ is a non-toric subvariety of (S, D'), where $D' = D - \frac{1}{2}D_j = \sum_{i\neq j} \frac{1}{2}D_i$. Then the divisor D can be replaced by the other divisor $D' < D$. Therefore we have the four possibilities for the pair (S, D) and the point Γ.

(A) S has three singular points and $D = 0$. In this possibility $\Gamma \notin \mathrm{Supp}(\mathrm{Sing}\, S)$.

(B) $\Gamma \notin D_{i_1} \cup D_{i_2}$, where $i_1 \neq i_2$. To be definite, let $D_{i_1} - D_{i_2}$ be a nef divisor.

(C) S has two singular points, that is, $S \cong \mathbb{P}(a_1, a_2, 1)$, where $a_1 \geq 3$, $a_2 \geq 2$ and $\Gamma = (b : 1 : 0)$, where $b \neq 0$.

(D) $S \cong \mathbb{P}(a_1, a_2, 1)$, $D = \frac{1}{2}\{x_1 = 0\} + \frac{1}{2}\{x_2 = 0\}$, $a_1 \geq 2$, $a_2 \geq 1$ and $\Gamma = (1 : 0 : b)$, where $b \neq 0$.

Possibility **(B)** is impossible since $\mathrm{LCS}(S, T - \frac{1}{2}D_{i_1} + \frac{1}{2}D_{i_2}) = \Gamma \cup D_{i_2}$, that is, we have the contradiction with connectedness lemma. Possibility **(D)** is impossible since $\mathrm{LCS}(S, T - \frac{1}{2}\{x_1 = 0\} + \{x_3 = 0\}) = \Gamma \cup \{x_3 = 0\}$, that is, we have the contradiction with connectedness lemma. Consider possibility **(C)**. Write $T = a\{x_3 = 0\} + T'$, where $\{x_3 = 0\} \not\subset \mathrm{Supp}(T')$ and $0 \leq a < 1$. The divisor $K_S + \{x_3 = 0\} + T'$ is not log canonical at the point Γ, therefore by the inversion of adjunction we have $\big(\{x_3 = 0\} \cdot T'\big)_\Gamma > 1$. We obtain the contradiction

$$1 < \big(\{x_3 = 0\} \cdot T'\big)_\Gamma < \{x_3 = 0\} \cdot (-K_S) = \frac{a_1 + a_2 + 1}{a_1a_2} \leq 1.$$

Consider possibility **(A)**. Let $f\colon (Y, E) \to (S \ni \Gamma)$ be an inductive blow-up of (S, T) (see Theorem 1.10). By Theorem 2.12 the morphism f is a weighted blow-up of smooth point with weights (α_1, α_2). Write $K_Y + E + T_Y = f^*(K_S + T)$. □

Lemma 4.3 *The divisor K_S has a 1-complement B^+ such that Γ is a center of* $\mathrm{LCS}(S, B^+)$.

Proof The divisor $K_Y + E + (1 - \delta)T_Y$ is plt and anti-ample for $0 < \delta \ll 1$. Since $\rho(Y) = 2$ then the cone $\overline{\mathrm{NE}}(Y)$ is degenerated by two extremal rays. By R_1 and R_2 denote these two rays. To be definite, let R_1 gives the contraction f. If $-(K_Y + E)$ is a nef divisor then a 1–complement of $K_E + \mathrm{Diff}_E(0) = K_E + \frac{\alpha_1-1}{\alpha_1}P_1 + \frac{\alpha_2-1}{\alpha_2}P_2$ is extended to a 1–complement of $K_Y + E$ by Proposition 1.17, therefore we obtain the required 1–complement of K_S by Proposition 1.15.

Consider the last possibility: $(K_Y + E) \cdot R_2 > 0$, $T_Y \cdot R_2 < 0$. Let $L(\delta) \in |-n(K_Y + E + (1-\delta)T_Y)|$ be a general element for $n \gg 0$ and let $M = (1-\delta)T_Y + \frac{1}{n}L(\delta)$, where $\delta > 0$ is a sufficiently small fixed rational number. By construction, $K_Y + E + (1+\varepsilon)M \equiv \varepsilon M$, $K_Y + E + (1+\varepsilon)M$ is a plt divisor. Therefore, applying $(K_Y + E + (1+\varepsilon)M)$–MMP is a contraction of the ray R_2 for $0 < \varepsilon \ll 1$. The corresponding divisorial contraction is denoted by $h\colon Y \to \overline{S}$, and the image of E on the surface $\overline{S}$ is denoted by $\overline{E}$, put $\operatorname{Exc} h = C_Y$ and $C_S = f(C_Y)$. The divisor $K_{\overline{S}} + \overline{E}$ is plt and anti-ample. Therefore, if 1–complement of $K_{\overline{E}} + \operatorname{Diff}_{\overline{E}}(0)$ exists then we consistently apply Theorems 1.17, 1.16 and 1.15 and obtain the required 1–complement of K_S.

Suppose that there does not exist any 1–complement of $K_{\overline{E}} + \operatorname{Diff}_{\overline{E}}(0)$. It is possible if and only if there are three singular points of $\overline{S}$ lying on the curve $\overline{E}$. It implies that $\alpha_1 \geq 2$, $\alpha_2 \geq 2$, the curve C_Y is contracted to a cyclic singularity, and the curve C_S passes through at most one singularity of S (see [11, Chap. 3]). Let us apply Corollary 9.2 of the paper [10] for $K_{\overline{S}} + \overline{E}$. We obtain that S has the two singularities of type $\mathbb{A}_1$, which do not lie on the curve C_S. Let $V(\langle n_1 \rangle)$ be the closure of one-dimensional orbit passing through the two singular points of type $\mathbb{A}_1$. Then there exists an isomorphism of the lattice N such that $n_1 = (1,0)$, $n_2 = (1,2)$, and therefore $n_3 = (-2n+1, -2)$, where $n \geq 2$. By considering the cone $\langle n_2, n_3 \rangle$ we obtain that the third singularity of S is of type $\frac{1}{4n-4}(2n-1, 1)$, its minimal resolution graph consists of three exceptional curve chain with the self-intersection indices -2, $-n$ and -2 respectively. The following two cases are possible: (i) $\Gamma \in V(\langle n_2 \rangle) \cup V(\langle n_3 \rangle)$ and (ii) $\Gamma \notin V(\langle n_2 \rangle) \cup V(\langle n_3 \rangle)$.

Consider former Case (i). To be definite, let $\Gamma \in V(\langle n_2 \rangle)$, then $V(\langle n_2 \rangle) \cdot (-K_S) = \frac{n}{2n-2} \leq 1$, and therefore we obtain a contradiction for the same reason as in Case **(C)**.

Consider latter Case (ii). Let $g\colon S^{\min} \to S$ be a minimal resolution. Let us contract all curves of $\operatorname{Exc} g$, except the exceptional curve of the singularity $\frac{1}{4n-4}(2n-1, 1)$ with the self-intersection index $-n$. We obtain the divisorial contractions $S^{\min} \to \widetilde{S}$ and $\widetilde{S} \to S$. Note that $\rho(\widetilde{S}) = 2$ and $\widetilde{S} = T_N(\widetilde{\Delta})$, where the fan $\widetilde{\Delta}$ is given by Δ with the help of subdivision of the cone $\langle n_2, n_3 \rangle$ into the two cones $\langle n_2, n_4 \rangle$, $\langle n_4, n_3 \rangle$, where $n_4 = (-1, 0)$. The surface $\widetilde{S}$ is a conic bundle with irreducible fibers, and its two fibers are non-reduced. These two fibers are the curves $V(\langle n_2 \rangle)$, $V(\langle n_3 \rangle)$, and every such curve contains the two singularities of type $\mathbb{A}_1$. By $\widetilde{\Gamma}$ denote the transform of Γ on the surface $\widetilde{S}$. We have $K_{\widetilde{S}} + \widetilde{B}_1^+ + \widetilde{B}_2^+ + V(\langle n_4 \rangle) \sim 0$, where $\widetilde{B}_1^+ \sim V(\langle n_2 \rangle) + V(\langle n_3 \rangle)$ is the fiber passing through the point $\widetilde{\Gamma}$, and $\widetilde{B}_2^+ \sim V(\langle n_1 \rangle)$ is the section passing through the point $\widetilde{\Gamma}$. By Proposition 1.15 we obtain the required 1–complement of K_S. $\square$

Assume that $B^+ = B_1^+ + B^{+'}$, where the irreducible curve B_1^+ has an ordinary double point singularity at the point Γ. By the inversion of adjunction we have $B^{+'} = 0$, $B_1^+ \cap \operatorname{Supp}(\operatorname{Sing} S) = \emptyset$ and $K_S + B_1^+ \sim 0$, therefore K_S is Cartier divisor. Classification of Del Pezzo surfaces with Du Val singularities (in our case Du Val singularities are cyclic), with Picard number 1 and with three singular points implies $K_S^2 \leq 4$ [3]. Write $T = aB_1^+ + T'$, where $B_1^+ \not\subset \operatorname{Supp}(T')$ and $0 \leq a < 1$. Since $0 \sim K_Y + E + \widetilde{B_1^+} = f^*(K_S + B_1^+)$ then we obtain the contradiction

$$0 > (K_Y + E + T_Y) \cdot \widetilde{B_1^+} \geq (-1 + a)\Big(\widetilde{B_1^+}\Big)^2 =$$
$$= (-1 + a)\Big(K_S^2 - \frac{(\alpha_1 + \alpha_2)^2}{\alpha_1 \alpha_2}\Big) \geq 0.$$

Consider the last case $B^+ = B_1^+ + B_2^+ + B^{+'}$, where the irreducible curves B_1^+ and B_2^+ have a simple normal crossing at the point Γ. We have $(B_1^+ \cup B_2^+) \supset \mathrm{Supp}(\mathrm{Sing}\ S)$ according to Corollary 9.2 of the paper [10] applied for $K_S + B_1^+ + B_2^+$. To be definite, let the curve B_1^+ contains two singular points of S. By the inversion of adjunction, $\deg \mathrm{Diff}_{B_1^+}(0) \leq 1$, and therefore the curve B_1^+ passes through two singular points only, and they are of type $\mathbb{A}_1$. Such surfaces were classified in the proof of Lemma 4.3, and therefore it can be assumed that the third singularity of S is of type $\frac{1}{4n-4}(2n-1, 1)$, $B^{+'} = 0$, $B_1^+ \cap B_2^+ = \Gamma$, $(B_1^+)^2 = n - 1$ and $(B_2^+)^2 = \frac{1}{n-1}$, where $n \geq 2$. To be definite, assume that $f^*(B_1^+) = \widetilde{B_1^+} + \alpha_1 E$ and $f^*(B_2^+) = \widetilde{B_2^+} + \alpha_2 E$. Thus $(\widetilde{B_1^+})^2 = n - 1 - \alpha_1/\alpha_2$, $(\widetilde{B_2^+})^2 = \frac{1}{n-1} - \alpha_2/\alpha_1$, and therefore $(\widetilde{B_k^+})^2 \leq 0$ for either $k = 1$ or $k = 2$. Write $T = a_1 B_1^+ + a_2 B_2^+ + T'$, where $B_1^+, B_2^+ \not\subset \mathrm{Supp}(T')$, $0 \leq a_1 < 1$, $0 \leq a_2 < 1$. Since $0 \sim K_Y + E + \widetilde{B_1^+} + \widetilde{B_2^+} = f^*(K_S + B_1^+ + B_2^+)$, then we obtain the contradiction

$$0 > (K_Y + E + T_Y) \cdot \widetilde{B_k^+} = (-1 + a_k)\Big(\widetilde{B_k^+}\Big)^2 + T_Y' \cdot \widetilde{B_k^+} \geq$$
$$\geq (-1 + a_k)\Big(\widetilde{B_k^+}\Big)^2 \geq 0.$$

Consider Condition (2). Such toric surface is determined by the fan Δ in the lattice $N \cong \mathbb{Z}^2$, where

$$\Delta = \Big\{\langle m_1, m_2\rangle, \langle m_2, m_3\rangle, \langle m_3, m_4, \rangle, \langle m_4, m_1\rangle, \text{their faces}\Big\},$$

$m_1 = (1, 0), m_2 = (q, r), m_3 = (-1, 0), m_4 = (-q, -r), q \geq 1, r \geq 1$ and $\gcd(q, r) = 1$. Therefore S has four singularities of types $\frac{1}{r}(1, -q)$, $\frac{1}{r}(1, q)$, $\frac{1}{r}(1, -q)$ and $\frac{1}{r}(1, q)$ respectively.

Two different fibers passing through the point Γ are denoted by F_1 and F_2. Since $T \cdot F_i \geq 1$ by Lemma 4.4 for $i = 1, 2$, then $T - F_1 - F_2$ is nef.

Lemma 4.4 *Let O be a smooth point of the surface M. Assume (M, N) is not a log canonical pair at the point O, where $N = dI + \Sigma \geq 0$, $I \not\subset \mathrm{Supp}\,\Sigma$, $d \leq 1$, I is an irreducible curve which is a smooth at the point O. Then $(\Sigma \cdot I)_O > 1$.*

Proof The proof follows by the inversion of adjunction, see, for example, [12, Theorem 7.5]. □

Consider the index j such that F_j is non-toric subvariety of (S, D). Let F' and F'' be the closures of one-dimensional toric orbits provided that $F' \sim_{\mathbb{Q}} F'' \sim_{\mathbb{Q}} \frac{1}{r} F_j$. We obtain the contradiction $(K_S + T) \cdot F_i \geq (-F' - F'' + D + F_j) \cdot F_i \geq 0$, where the index $i \in \{1, 2\}$ satisfies the condition $i \neq j$.

Consider Condition (3). Assume that either $\Gamma \in C_{13}$, or $\Gamma \in C_{23}$. Let us consider the first possibility. The second possibility is considered similarly. If Γ is a non-toric point of $(C_{13}, \mathrm{Diff}_{C_{13}}(D))$ then we have a contradiction with one-dimensional analog of this theorem since $C_{13}^2 < 0$. Therefore, $a_4 = 1$ and $d_{23} = 1$. The case $C_{23}^2 = 0$ is impossible also (in this case the surface S is a toric conic bundle, and we use its structure). Thus $C_{23}^2 < 0$ and consider the contraction $\psi\colon S \to S'$ of C_{23}. If $\psi(\Gamma)$ is a non-toric point of $(S', \psi(D))$ then we have a contradiction with this theorem under Condition (1). Therefore the curve C_{23} is contracted to a smooth point and $d_{24} = 1$. We obtain the contradiction $a_2 d_{13} d_{14} = w_2 > w_4 = d_{13}$.

Assume that $\Gamma \in C_{13} \cup C_{23}$. Let $C_{23}^2 = 0$. Then $(w_3, w_4) = (w_1, w_2)$, $(S, D) \cong \left(\mathbb{F}_{w_1 - w_2}, \frac{w_2 - 1}{w_2} C_{13} + \frac{w_1 - 1}{w_1} C_{24}\right)$ and $2 \le w_1 < w_2$. By F_j denote a fiber of S passing through Γ. Then $T' \cdot F_j \ge 1$ by Lemma 4.4, where $T = T' + \alpha F_j$, $F_j \not\subset \mathrm{Supp}(T')$, and we have the contradiction $0 > (K_S + T) \cdot F_j \ge (K_S + F_j + T') \cdot F_j \ge 0$.

Therefore $C_{23}^2 < 0$, $C_{13}^2 < 0$. Considering case by case the contractions of the curves C_{13} and C_{23}, we obtain that these curves are contracted to smooth points and $d_{13} = d_{23} = d_{24} = d_{14} = 1$. Since $C_{13}^2 = -\frac{1}{a_2 a_4}$, $C_{23}^2 = -\frac{1}{a_1 a_4}$ then $(w_1, w_2, w_3, w_4) = (a_2, a_2, a_2 - 1, a_2 + 1)$, $a_2 \ge 3$. It is easy to find a birational map

$$S \dashrightarrow S'\left(\frac{1}{a_2 - 1}(1, -1) + \frac{1}{a_2 - 1}(1, 1) + \frac{1}{a_2 - 1}(1, -1) + \frac{1}{a_2 - 1}(1, 1)\right),$$

where $\rho(S') = 2$, and in result of this map we obtain a contradiction with this theorem under Condition (2). To find this map it is enough to consider two (required) toric blow-ups at the points P_2, P_4 and a contraction of proper transforms of C_{13} and C_{23}. □

Remark 4.5 Theorem 4.2 (1) can not be generalized to the case $\rho(S) \ge 2$. Consider the toric pair$(S, D) = (\mathbb{F}_1, \frac{1}{2} E_0)$ and the divisor $T = \frac{1}{2} E_0 + E_0' + F + \delta E_\infty$ provided that $F \cap E_0' \notin E_0 \cup E_\infty$, where E_0, E_0' are two different zero sections, E_∞ is the infinity section, F is a fiber and $0 < \delta < \frac{1}{2}$. Put $\Gamma = F \cap E_0'$. Then Γ is a non-toric point of (S, D), $T \ge D$, $K_S + T$ is anti-ample log canonical divisor and $\Gamma \in \mathrm{LCS}(S, T)$.

Nevertheless, it is expected that Theorem 4.2 can be generalized to every dimension and every Picard number $\rho(S)$, if we require the following condition, instead of Conditions (1), (2) and (3): $(S, D) = (E, \mathrm{Diff}_E(0))$, where $f\colon (Y, E) \to (X \ni P)$ is a toric plt blow-up of some toric singularity.

Definition 4.6 Let $(\Gamma, D_\Gamma) \cong (\mathbb{P}^1, \sum_{i=1}^r \frac{m_i - 1}{m_i} P_i)$. Assume that $-(K_\Gamma + D_\Gamma)$ is an ample divisor. Then, for set $(m_1, \dots, m_r)$ we have one of the following cases up to permutations: (m_1, m_2), it is of type A; $(2, 2, m)$, $m \ge 2$, it is of type D_{m+2}; $(2, 3, 3)$, it is of type E_6; $(2, 3, 4)$, it is of type E_7; $(2, 3, 5)$, it is of type E_8. In Propositions 4.7 and 4.8 the classification according to types corresponds to the types of $(\Gamma, D_\Gamma) = (\Gamma, \mathrm{Diff}_\Gamma(D))$.

Proposition 4.7 *Let (S, D) be a toric pair, where S is a normal projective surface with $\rho(S) = 1$, and let D be a divisor with standard coefficients. Assume that there* *exists a curve* Γ *such that* $-(K_S + D + \Gamma)$ *is an ample divisor and* $(S, D + \Gamma)$ *is a plt non-toric pair. Let us denote a hypersurface of degree* d *in a weighted projective space by* X_d*. Then one of the following cases is satisfied.*

(1) $(S, D, \Gamma) \cong (\mathbb{P}^2_{x_1x_2x_3}, \frac{d_1-1}{d_1}\{x_1 = 0\}, X_2)$ *and* $d_1 \geq 1$*. It is of type* A*.*

(2) $(S, D, \Gamma) \cong (\mathbb{P}^2_{x_1x_2x_3}, \sum_{i=1}^{3} \frac{d_i-1}{d_i}\{x_i = 0\}, X_1)$*, the integer number triple* (d_1, d_2, d_3) *is either* $(2, 2, k)$*,* $(2, 3, 3)$*,* $(2, 3, 4)$ *or* $(2, 3, 5)$*, where* $k \geq 2$*. They are of types* D_{k+2}*,* E_6*,* E_7 *and* E_8 *respectively.*

(3) $(S, D, \Gamma) \cong (\mathbb{P}_{x_1x_2x_3}(a_1, 1, 1), \sum_{i=1}^{2} \frac{d_i-1}{d_i}\{x_i = 0\}, X_{a_1})$*, the integer number triple* (a_1, d_1, d_2) *is either* $(2, 2, k_1)$*,* $(2, 3, k_2)$*,* $(2, k_3, 1)$ *or* $(3, 2, 1)$*, where* $k_1 \geq 1$*,* $1 \leq k_2 \leq 2$*,* $k_3 \geq 4$*. In the first possibility, if* $k_1 \geq 2$ *then it is of type* D_{k_1+2}*. In the second possibility, if* $k_2 = 2$ *then it is of type* E_6*. The other possibilities are of type* A *always.*

(4) $(S, D, \Gamma) \cong (\mathbb{P}_{x_1x_2x_3}(a_1, 1, 1), \frac{d_1-1}{d_1}\{x_2 = 0\}, X_{a_1+1})$*,* $a_1 \geq 2$ *and* $d_1 \geq 1$*. It is of type* A*.*

(5) $(S, D, \Gamma) \cong (\mathbb{P}_{x_1x_2x_3}(a_2 + 1, a_2, 1), \sum_{i=1}^{2} \frac{d_i-1}{d_i}\{x_i = 0\}, X_{a_2+1})$*, the integer number triple* (a_2, d_1, d_2) *is either* $(2, 2, k_1)$*,* $(k_2, 2, k_3)$ *or* $(k_4, k_5, 1)$*, where* $k_1 \leq 3$*,* $k_2 \geq 3$*,* $k_3 \leq 2$*,* $k_4 \geq 2$ *and* $k_5 \geq 3$*. In the first possibility, if* $k_1 = 2$ *then it is of type* D_6*, and, if* $k_1 = 3$ *then it is of type* E_7*. In the second possibility, if* $k_3 = 2$ *then it is of type* D_{2k_2+2}*. The other possibilities are of type* A *always.*

(6) $(S, D, \Gamma) \cong (\mathbb{P}_{x_1x_2x_3}(2a_2 + 1, a_2, 1), \frac{1}{2}\{x_1 = 0\}, X_{2a_2+1})$*,* $a_2 \geq 2$*. It is of type* D_{2a_2+2}*.*

(7) $(S, D, \Gamma) \cong (\mathbb{P}_{x_1x_2x_3}(la_2 - 1, a_2, 1), \sum_{i=1}^{2} \frac{d_i-1}{d_i}\{x_i = 0\}, X_{la_2})$*,* $a_2 \geq 2$*, the integer number triple* (l, d_1, d_2) *is either* $(2, 2, 1)$ *or* $(k_1, 1, k_2)$*, where* $k_1 \geq 2$ *and* $k_2 \geq 1$*. They are of types* D_{2a_2+1} *and* A *respectively.*

(8) $(S, D, \Gamma) \cong (\mathbb{P}_{x_1x_2x_3}(a_1, a_2, 1), \frac{d_1-1}{d_1}\{x_3 = 0\}, X_{a_1+a_2})$*,* $a_1 > a_2 \geq 2$ *and* $d_1 \geq 1$*. It is of type* A*.*

(9) $(S, D) \cong (S(\frac{1}{r_1}(1, 1) + \frac{1}{r_2}(1, 1) + \mathbb{A}_{r_1+r_2-1}), \frac{d_1-1}{d_1} D_3)$*,* $\Gamma \sim_{\mathbb{Q}} D_3$ *is an irreducible curve being different from* D_3*, where* D_3 *is the closure of one-dimensional orbit passing through the first and second singular points,* $d_1 \geq 2$ *and* $r_1, r_2 \geq 2$*. It is of type* A*.*

(10) $(S, D) \cong (S(\frac{1}{r_1}(l, 1) + \frac{1}{r_2}(l, 1) + \mathbb{A}_{(r_1+r_2)/l-1}), \frac{d_1-1}{d_1} D_3)$*, the surface* S *has three singular points,* $\Gamma \sim D_1 + D_2$*, where* D_i *is the closure of one-dimensional orbit not passing through the* i*-th singular point of* S*,* $d_1 \geq 1$*,* $l \geq 2$ *and* $l|(r_1 + r_2)$*. It is of type* A*.*

Proof By the adjunction formula the curve Γ is smooth and irreducible. It follows easily that, if $P \in \operatorname{Supp} D \cap \Gamma$ then $(S, D + \Gamma)$ is a toric pair in a sufficiently small analytical neighborhood of P. If S is a smooth surface then $S \cong \mathbb{P}^2$ and we have two Cases (1) and (2).

Assume that S is a non-smooth surface having at most two singular points. Then we have $S \cong \mathbb{P}_{x_1x_2x_3}(a_1, a_2, 1)$ as before in the proof of Theorem 4.2. At first let us consider the case of one singular point, that is, $a_1 \geq 2$ and $a_2 = 1$. Then either $\Gamma \sim \mathcal{O}_S(1), \mathcal{O}_S(a_1)$ or $\mathcal{O}_S(a_1 + 1)$. The variant $\Gamma \sim \mathcal{O}_S(1)$ is impossible since $K_S + D + \Gamma$ is not a plt divisor at the point $(1 : 0 : 0)$. The other variants lead us to Cases (3) and (4) respectively. At second let us consider the case of two singular points, that is, $a_1 > a_2 \geq 2$. Put $\Gamma = \{\psi(x_1, x_2, x_3) = 0\}$. Suppose that $\Gamma \nsim \mathcal{O}_S(a_1 + a_2)$, $\mathcal{O}_S(a_1)$, $\mathcal{O}_S(a_2)$, $\mathcal{O}_S(1)$ then $\psi(x_1, x_2, x_3) = bx_1x_3^l + \varphi(x_2, x_3)$, and by considering the point $(1 : 0 : 0)$ we obtain $b \neq 0$, $l = 1$, $\Gamma \sim \mathcal{O}_S(a_1 + 1)$ and $x_2^m \in \varphi(x_2, x_3)$. It leads us to Case (7). If $\Gamma \sim \mathcal{O}_S(a_1)$ then by considering the point $(0 : 1 : 0)$ we obtain $x_1, x_2^l x_3 \in \psi(x_1, x_2, x_3)$. It leads us to Cases (5) and 6). It is easy to prove that cases $\Gamma \sim \mathcal{O}_S(a_2)$ and $\Gamma \sim \mathcal{O}_S(1)$ are not realized. If $\Gamma \sim \mathcal{O}_S(a_1 + a_2)$ then $x_1x_2, x_3^{a_1+a_2} \in \psi(x_1, x_2, x_3)$, and we have Case (8).

Assume that S is a surface having three singular points (it is the last possibility for S). According to Corollary 9.2 of the paper [10] for the divisor $K_S + \Gamma$, we obtain that the curve Γ contains a singular point of S.

Suppose that the curve Γ contains only one singular point of S, then arguing as above in the proof of Theorem 4.2, we obtain $S = S(2\mathbb{A}_1 + \frac{1}{4n-4}(2n-1, 1))$, where $n \geq 2$, and Γ is locally a toric subvariety of $(S \ni P)$, where $(S \ni P)$ is of type $\frac{1}{4n-4}(2n-1, 1)$. By T_1 and T_2 denote the closures of one-dimensional orbits passing through the singular point P. Since $T_1 \sim T_2$ and $(\Gamma \cdot T_1)_P \neq (\Gamma \cdot T_2)_P$ then $\Gamma \cdot T_i > 1$. Therefore $\Gamma - (4n-4)T_1$ is an ample divisor, and we obtain the contradiction with ampleness of $-(K_S + \Gamma) \sim 2nT_1 - \Gamma$. Thus this possibility is not realized.

Suppose that the curve Γ passes through the two singular points P_1 and P_2 of S only. There exists a 1-complement of $K_\Gamma + \mathrm{Diff}_\Gamma(0)$, and we obtain the 1-complement $K_S + \Gamma + T \sim 0$ of $K_S + \Gamma$ by Proposition 1.17. There are two Cases **(A)** and **(B)**.

(A) Let T is a reducible divisor. By the two-dimensional criterion on the characterization of toric varieties [26, Theorem 6.4] we have $T = T_1 + T_2$, $\Gamma \sim T_3$, $D = \frac{d_1-1}{d_1}T_3$, the singularities at the points P_j are of type $\frac{1}{r_j}(1, 1)$, where $d_1 \geq 2$, $r_j \geq 2$ and T_i are the closures of one-dimensional orbits, and $P_1 \in T_1$. Let $f\colon \widetilde{S} \to S$ be a minimal resolution at the points P_1 and P_2 only. By E_1 denote the curve such that $f(E_1) = P_1$. By the inversion of adjunction $\Gamma \cdot T_3 = \frac{1}{r_1} + \frac{1}{r_2}$, hence $(\Gamma_{\widetilde{S}})^2 = \Gamma_{\widetilde{S}} \cdot (T_3)_{\widetilde{S}} = 0$, and the linear system $|E_1 + m\Gamma_{\widetilde{S}}|$ gives the birational morphism $g\colon \widetilde{S} \to \mathbb{F}_{r_1}$ for $m \gg 0$ [15, Proposition 1.10] such that the curve $(T_2)_{\widetilde{S}}$ is contracted to a smooth point. The morphism g is toric and the third singularity of S is of type $\mathbb{A}_{r_1+r_2-1}$. We obtain Case (9).

(B) Let T is an irreducible divisor. To be definite, let D_i be the closures of one-dimensional orbits not passing through the i-th singular point of $S = S(\frac{1}{r_1}(a_1, 1) + \frac{1}{r_2}(a_2, 1) + \frac{1}{r_3}(a_3, 1))$. We have $\frac{1}{r_1}D_1 \equiv \frac{1}{r_2}D_2 \equiv \frac{1}{r_3}D_3$. To be definite, the curve Γ passes through the first and second singular point of S. By the definition of 1–complement we obtain $\Gamma \cdot T = \frac{1}{r_1} + \frac{1}{r_2}, \Gamma + T \sim \sum_{i=1}^{3} D_i$. Hence, either $\Gamma \sim D_1 + D_2$, $T \sim D_3$ or $\Gamma \sim D_3$, $T \sim D_1 + D_2$. Since 1–complement not passing through the third singular point of S then it is of type $\mathbb{A}_{r_3-1}$. The case $\Gamma \sim D_3$ was considered in Case (A). Since the curve Γ does not pass through the third singular point then we have to consider the possibility remained: $\Gamma \sim D_1 + D_2 \sim lD_3$, where $l \geq 2, l \in \mathbb{Z}$. We obtain Case (10).

Suppose that the curve Γ passes through three singular points of S with the indices r_1, r_2 and r_3 respectively. By the inversion of adjunction the triple (r_1, r_2, r_3) is either $(2, 2, k)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$, where $k \geq 2$. For the second and third variants there does not exist any surface S. For the first and fourth variants we have $S = S(2\mathbb{A}_1 + \frac{1}{4n-4}(2n-1, 1))$ and $S \cong \mathbb{P}(2, 3, 5)$ respectively, where $n \geq 2$. These variants are considered as above mentioned case, when the curve Γ contains only one singular point of S. $\square$

Proposition 4.8 *Let (S, D) be ODP pair. Assume that there exist a curve Γ and an effective $\mathbb{Q}$-divisor Γ' such that $K_S + D + \Gamma + \Gamma'$ is an anti-ample and plt divisor, and $(S, D + \Gamma)$ is a non-toric pair. Then $d_{23} = d_{24} = 1$, $a_1 | a_2$ and $\Gamma \sim \mathcal{O}_{\mathbb{P}(\mathbf{w})}(w_2)|_S$ up to permutation of the coordinates. In particular, $-(K_S + D + \Gamma)$ is an ample divisor and $w_1 | w_2$. It is of type A.*

Proof The sets $\Gamma \cap C_{13}$, $\Gamma \cap C_{23}$ consist of at most one point by the adjunction formula. Moreover, we may assume that $\Gamma' = \gamma_1 C_{13} + \gamma_2 C_{23}$, where $\gamma_1 < 1$ and $\gamma_2 < 1$. If $C_{i3}^2 = 0$ then $\gamma_i = 0$, where $i = 1, 2$.

Let us prove that $\Gamma \cdot C_{13} > 0$ and $\Gamma \cdot C_{23} > 0$. Assuming the converse: $\Gamma \cdot C_{13} = 0$, that is, $\Gamma \sim dC_{24}$. The possibility $\Gamma \cdot C_{23} = 0$ is considered similarly. Since $C_{23} \cdot C_{24} = \frac{1}{a_1}$, $a_1(C_{23} \cdot \Gamma) \in \mathbb{Z}_{>0}$ then $d \in \mathbb{Z}_{>0}$. The divisor $C_{24} - \gamma C_{13}$ is nef for $0 \leq \gamma \leq \frac{1}{d_{13}}$, hence it is semiample by the base point free theorem [9]. Therefore, if $d \geq 2$ then we have a contradiction with connectedness lemma, since there exists a $\mathbb{Q}$–divisor Γ'' such that $\llcorner \Gamma'' \lrcorner = 0$ and $D + \Gamma + \Gamma' \sim_{\mathbb{Q}} C_{24} + C_{13} + \Gamma''$. Thus, $d = 1$. Since the curve Γ is a non-toric subvariety of (S, D) then $d_{24} \geq 2$, and we have $d_{13} = 1$ by connectedness lemma again. We obtain the contradiction

$$0 > (K_S + D + \Gamma + \Gamma') \cdot C_{23} \geq$$
$$\geq \Big(\frac{d_{24} - 1}{d_{24}} C_{24} - C_{13} - C_{23} - C_{14} + \Gamma'\Big) \cdot C_{23} \geq$$
$$\geq \frac{d_{24} - 1}{d_{24}} C_{24} \cdot C_{23} - C_{13} \cdot C_{23} = d_{23}\Big(\frac{d_{24} - 1}{w_1} - \frac{1}{w_4}\Big) \geq 0.$$

Thus, we proved that the sets $\Gamma \cap C_{13}$ and $\Gamma \cap C_{23}$ consist of one point only.

Suppose that $P_4 \notin \Gamma$. Then $\Gamma \sim_{\mathbb{Q}} \alpha_1 C_{14} + \alpha_2 C_{24}$, $\alpha_1 = a_2(\Gamma \cdot C_{13}) \in \mathbb{Z}_{>0}$ and $\alpha_2 = a_1(\Gamma \cdot C_{23}) \in \mathbb{Z}_{>0}$. By applying connectedness lemma we have $\alpha_1 = \alpha_2 = 1$. Let us prove that $d_{14} = d_{24} = 1$. Assuming the converse: $d_{14} \geq 2$. The possibility $d_{24} \geq 2$ is considered similarly. In order to apply connectedness lemma and obtain a contradiction (for the disjoint curves C_{14}, C_{23}) we must only prove that $D_1 = \frac{d_{14}-1}{d_{14}} C_{14} + C_{24} + \frac{d_{24}-1}{d_{24}} C_{24} - \frac{1}{d_{23}} C_{23}$ is a semiample divisor. Since $D_1 \cdot C_{23} > 0$ and $D_1 \cdot C_{13} = d_{13}(\frac{d_{14}-1}{w_2} - \frac{1}{w_4}) \geq 0$ then D_1 is a nef divisor and it is semiample by the base point free theorem [9]. Finally, since $K_S + \Gamma + C_{13} + C_{23} \sim 0$ then K_S is Cartier divisor at the point P_3, and the singularity at the point P_3 is Du Val of type $\frac{1}{w_3}(w_1, w_2)$. Therefore $w_3 + w_4 = w_1 + w_2 \equiv 0 (\mathrm{mod}\, w_3)$, $w_3 | w_4$ and $a_3 | a_4$.

Suppose that $P_4 \in \Gamma$. Since the curve Γ is a (locally) toric orbit in some analytical neighborhood of P_4 then either $\Gamma \cdot C_{13} = \frac{1}{a_4}$ or $\Gamma \cdot C_{23} = \frac{1}{a_4}$. Let us consider the former case. The latter case is considered similarly. Write $\Gamma \sim_{\mathbb{Q}} \alpha_1 C_{23} + \alpha_2 C_{24}$, $\alpha_1 = a_4(\Gamma \cdot C_{13}) = 1$ and $\alpha_2 = a_3(\Gamma \cdot C_{14}) \in \mathbb{Z}_{>0}$. Arguing as above, we see that $\alpha_2 = 1$, $d_{24} = 1$. If $d_{23} = 1$ then this proposition is proved. Let $d_{23} \geq 2$. By the plt assumption of this proposition $\Gamma \cdot C_{23} = \frac{1}{a_4}$ and $d_{13} = 1$. Considering $\Gamma \sim_{\mathbb{Q}} C_{13} + \alpha_2' C_{14}$ we obtain $\alpha_2' = 1$, $d_{14} = 1$. This completes the proof. $\square$

Definition 4.9 The triple (S, D, Γ) determined by the assertions of Propositions 4.7 or 4.8 is said to be a *purely log terminal triple*.

The following problem is important for the classification of plt blow-ups of three-dimensional toric non-$\mathbb{Q}$-factorial singularity (if we follow the method described in this paper).

Problem. Let $(S, D) = (E, \operatorname{Diff}_E(0))$, where $f: (Y, E) \to (X \ni P)$ is a toric plt blow-up of some toric three-dimensional (non-$\mathbb{Q}$-factorial) singularity. Assume that there exist a curve Γ and an effective $\mathbb{Q}$-divisor Γ' such that $K_S + D + \Gamma + \Gamma'$ is an anti-ample plt divisor, and Γ is a non-toric subvariety of (S, D). Classify the triples (S, D, Γ).

5 Non-toric Three-dimensional Blow-ups. Case of Point

Example 5.1 Now we construct the examples of three-dimensional non-toric plt blow-ups $f: (Y, E) \to (X \ni P)$ provided that $(X \ni P)$ is a $\mathbb{Q}$-gorenstein toric singularity and $P = f(E)$. Depending on a type of $(X \ni P)$ we consider two Cases **(A1)** and **(A2)**.

(A1). Let $(X \ni P)$ be a $\mathbb{Q}$-factorial toric singularity, that is, $(X \ni P) \cong (\mathbb{C}^3 \ni 0)/G$, where G is an abelian group acting freely in codimension 1. All plt blow-ups are constructed by the procedure illustrated on the next diagram (Fig. 4) and defined below.

First step. Let $g_0: (Z_0, S_0) \to (X \ni P)$ be a toric blow-up, where $\operatorname{Exc} g_0 = S_0$ is an irreducible divisor and $g_0(S_0) = P$. Assume that there exists a curve $\Gamma_0 \subset S_0$ such that $(S_0, \operatorname{Diff}_{S_0}(0), \Gamma_0)$ is a plt triple (see Definition 4.9). Such triples are classified in Proposition 4.7 and are divided into the five types: A, D_l, E_6, E_7 and E_8.

Remark 5.2 There exists an irreducible reduced Weil divisor Ω on X such that $\Omega_{Z_0}|_{S_0} = \Gamma_0$. The surface Ω has a log terminal singularity at the point P. A singularity

Fig. 4 Case of Point. Construction in $\mathbb{Q}$-factorial case

$$\begin{tikzcd} & Y_0 \arrow[dl, "h_0"'] \arrow[d, "h_0'"] & Y_1 \arrow[l, "h_1"'] \arrow[d, dashed, "h_1'"] \\ Z_0 \arrow[d, "g_0"'] & Z_1 \arrow[d, "g_1"'] & Z_2 \arrow[d, "g_2"'] \\ X & X & X \end{tikzcd}$$

type coincides with a type of the triple $(S_0, \mathrm{Diff}_{S_0}(0), \Gamma_0)$. In particular, if ψ is a G–semi-invariant polynomial in $\mathbb{C}^3$ determining Ω then Du Val singularity $\{\psi = 0\} \subset (\mathbb{C}^3 \ni 0)$ is of the same type.

The following lemma gives a restriction on the triple $(S_0, \mathrm{Diff}_{S_0}(0), \Gamma_0)$ in the case of terminal singularities.

Lemma 5.3 *Let $(X \ni P)$ be a terminal singularity, that is, it is of type $\frac{1}{r}(-1, -q, 1)$, where* $\gcd(r, q) = 1$ *and* $1 \le q \le r$. *Write* $\mathrm{Diff}_{S_0}(0) = \sum_{i=1}^{3} \frac{d_i - 1}{d_i} D_i$, *where D_i are the closures of corresponding one-dimensional orbits of the toric surface S_0. Then* $\gcd(d_i, d_j) = 1$ *for $i \neq j$.*

Proof It is sufficient to prove that the singularities of Z_0 are cyclic. Consider the cone σ determining the singularity $(X \ni P)$ (see Example 2.2 (1)). By (w_1, w_2, w_3) denote the primitive vector defining the blow-up g_0. Then Z_0 is covered by three affine charts with the singularities of types $\frac{1}{w_3}(-w_1, -w_2, 1)$, $\frac{1}{rw_2 - qw_3}(-w_1 + uw_2 + vw_3, -uw_2 - vw_3, 1)$ and $\frac{1}{rw_1 - w_3}(-w_1, qw_1 - w_2, 1)$, where $uq + vr = 1$ and u, $v \in \mathbb{Z}$. □

According to Proposition 4.7 the curve Γ_0 is locally a toric subvariety of Z_0 in every sufficiently small analytic neighborhood of each point of Γ_0. Note also that Z_0 is a smooth variety at a general point of Γ_0.

Let $h_0\colon (Y_0, \widetilde{S}_1) \to (Z_0 \supset \Gamma_0)$ be an arbitrary blow-up of the curve Γ_0 with an unique exceptional divisor ($\mathrm{Exc}\, h_0 = \widetilde{S}_1$) for which the following three conditions are satisfied.

(1) The morphism h_0 is locally toric at every point of Γ_0. In particular, $\widetilde{S}_0 \cong S_0$, $\rho(\widetilde{S}_0) = 1$.

(2) Let H_0 be a general hyperplane section of Z_0 passing through the general point $Q_0 \in \Gamma_0$. Then the morphism h_0 induces a weighted blow-up of the smooth point $(H_0 \ni Q_0)$ with weights (β_0^1, β_0^2).

(3) $h_0^* S_0 = \widetilde{S}_0 + \beta_0^2 \widetilde{S}_1$.

The set of all possible blow-ups h_0 is denoted by $\mathcal{H}_0$. The morphism h_0' gives the divisorial contraction $h_0'\colon Y_0 \to Z_1$ which contracts the divisor $\widetilde{S}_0$ to a point. We obtain a non-toric blow-up $g_1\colon (Z_1, S_1) \to (X \ni P)$, where $\mathrm{Exc}\, g_1 = S_1$ is an irreducible divisor and $g_1(S_1) = P$.

Lemma 5.4 *Let $\widetilde{\Gamma}_0 = \widetilde{S}_0 \cap \widetilde{S}_1$. Then*

$$(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1} = \beta_0^1 \frac{\big(K_{S_0} + \mathrm{Diff}_{S_0}(0)\big) \cdot \Gamma_0}{a(S_0, 0) + 1} - \beta_0^2 (\Gamma_0^2)_{S_0}.$$

Proof This formula follows from the following equalities

$$\begin{aligned}(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1} &= \beta_0^1 \widetilde{S}_0 \cdot \widetilde{\Gamma}_0 = \beta_0^1 (S_0 \cdot \Gamma_0 - \beta_0^2 \widetilde{S}_1 \cdot \widetilde{\Gamma}_0) = \beta_0^1 S_0 \cdot \Gamma_0 - \\ &- \beta_0^2 (\widetilde{\Gamma}_0^2)_{\widetilde{S}_0} = \beta_0^1 S_0 \cdot \Gamma_0 - \beta_0^2 (\Gamma_0^2)_{S_0} = \\ &= \beta_0^1 ((K_{Z_0} + S_0) \cdot \Gamma_0)/(a(S_0, 0) + 1) - \beta_0^2 (\Gamma_0^2)_{S_0}.\end{aligned}$$

□

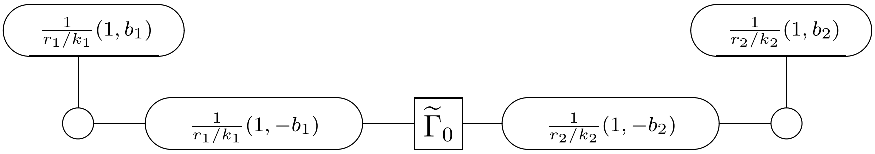

Fig. 5 Type A

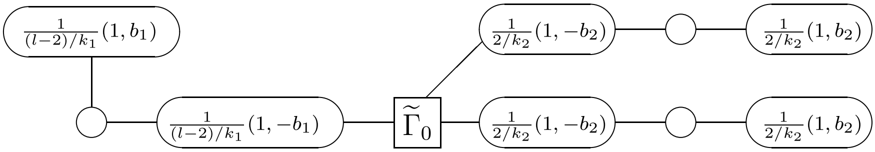

Fig. 6 Type D_l

In next Proposition 5.5 we will describe the pair $(S_1, \mathrm{Diff}_{S_1}(0))$. The surface $\widetilde{S}_1$ is a conic bundle with $\rho(\widetilde{S}_1) = 2$, in particular, every geometric fiber is irreducible. If we contract the section $\widetilde{\Gamma}_0 = \widetilde{S}_0 \cap \widetilde{S}_1$ of $\widetilde{S}_1$ then we obtain the surface S_1. The curve Γ_0 passes through a finite number of the singular points $Q_1, \ldots, Q_r$ of Z_0 ($r \le 3$), and by $\widetilde{F}_1, \ldots, \widetilde{F}_r$ denote the fibers of $\widetilde{S}_1$ over these points. In small analytic neighborhoods of a general point of $\widetilde{\Gamma}_0$ and a general point of some section $\widetilde{E}_0$ the variety Y_0 has the singularities of types $\mathbb{C}^1 \times \frac{1}{\beta_0^1}(-\beta_0^2, 1)$ and $\mathbb{C}^1 \times \frac{1}{\beta_0^2}(-\beta_0^1, 1)$ respectively. By $F_1, \ldots, F_r$, E_0 denote the transforms of $\widetilde{F}_1, \ldots, \widetilde{F}_r$, $\widetilde{E}_0$ on the surface S_1 respectively. The empty circles are $\widetilde{F}_1, \ldots, \widetilde{F}_r$ in the figures of Proposition 5.5. The singularities of $\widetilde{S}_1$ are into ovals. Note that the self-intersection index $(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1}$ was calculated in Lemma 5.4.

Proposition 5.5 *Depending on a type of the triple* $(S_0, \mathrm{Diff}_{S_0}(0), \Gamma_0)$ *we have the following structure of* $(S_1, \mathrm{Diff}_{S_1}(0))$.

(1) *Type A (Fig. 5),*

and

$$\mathrm{Diff}_{S_1}(0) = \frac{k_1 - 1}{k_1} F_1 + \frac{k_2 - 1}{k_2} F_2 + \frac{\beta_0^2 - 1}{\beta_0^2} E_0.$$

The pair $(S_1, \mathrm{Diff}_{S_1}(0))$ *is toric.*

(2) *Type* D_l $(l \ge 4)$ *(Fig. 6),*

and

$$\mathrm{Diff}_{S_1}(0) = \frac{k_1 - 1}{k_1} F_1 + \frac{k_2 - 1}{k_2} F_2 + \frac{k_2 - 1}{k_2} F_3 + \frac{\beta_0^2 - 1}{\beta_0^2} E_0.$$

(3) *Type* E_6 *(Fig. 7),*

and

$$\mathrm{Diff}_{S_1}(0) = \frac{k_1 - 1}{k_1} F_1 + \frac{k_2 - 1}{k_2} F_2 + \frac{k_3 - 1}{k_3} F_3 + \frac{\beta_0^2 - 1}{\beta_0^2} E_0.$$

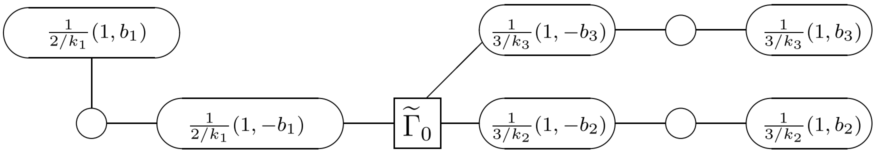

Fig. 7 Type E_6

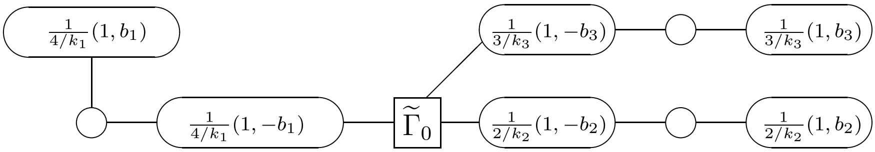

Fig. 8 Type E_7

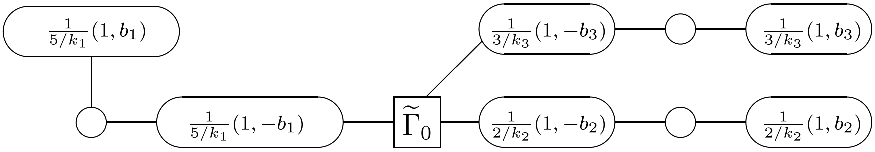

Fig. 9 Type E_8

(4) *Type E_7, (Fig. 8)*
and

$$\mathrm{Diff}_{S_1}(0) = \frac{k_1 - 1}{k_1} F_1 + \frac{k_2 - 1}{k_2} F_2 + \frac{k_3 - 1}{k_3} F_3 + \frac{\beta_0^2 - 1}{\beta_0^2} E_0.$$

(5) *Type E_8, (Fig. 9)*
and

$$\mathrm{Diff}_{S_1}(0) = \frac{k_1 - 1}{k_1} F_1 + \frac{k_2 - 1}{k_2} F_2 + \frac{k_3 - 1}{k_3} F_3 + \frac{\beta_0^2 - 1}{\beta_0^2} E_0.$$

The pair $(S_1, \mathrm{Diff}_{S_1}(0))$ is klt, therefore $g_1\colon (Z_1, S_1) \to (X \ni P)$ is a non-toric plt blow-up.

In cases A, D_l, E_6, E_7 and E_8 we have a non-plt 1-, 2-, 3-, 4- *and* 6-*complement of* $(S_1, \mathrm{Diff}_{S_1}(0))$ *respectively.*

Proof By the construction, the morphism $h_0|_{\widetilde{S}_1}\colon \widetilde{S}_1 \to \Gamma_0$ is locally toric. Therefore, the surface $\widetilde{S}_1$ has either no singularities in a fiber or only two singularities of types $\frac{1}{r_1}(1, b_1)$ and $\frac{1}{r_1}(1, -b_1)$. Let us show the local calculations. Consider the singularity at the point Q_1 of Z_0 such that the curve Γ_0 contains it. Let the cone $\langle e_1, e_2, e_3\rangle$ determines locally the variety Z_0 in some analytical neighborhood of Q_1, $\Gamma_0 = V(\langle e_2, e_3\rangle)$ and $S_0 = V(\langle e_3\rangle)$. According to Proposition 4.7 we may assume $e_1 = (1, 0, 0)$. We locally have $Y_0 = T_N(\Delta')$, where

$$\Delta' = \{\langle \beta, e_1, e_2 \rangle, \langle \beta, e_1, e_3 \rangle, \text{ their faces}\},$$

$\beta = \beta_0^1 e_2 + \beta_0^2 e_3$ and $N \cong \mathbb{Z}^3$. Note that $V(\langle \beta \rangle) = \widetilde{S}_1$ and $\widetilde{F}_1 = V(\langle \beta, e_1 \rangle)$ is the fiber of $\widetilde{S}_1$ over the point Q_1. Write $(Z_0 \ni Q_1) \cong (\mathbb{C}^3 \ni 0)/G$, $(Y_0 \ni Q_1') \cong (\mathbb{C}^3 \ni 0)/G_1$, $(Y_0 \ni Q_1'') \cong (\mathbb{C}^3 \ni 0)/G_2$, where $Q_1' = \widetilde{F}_1 \cap \widetilde{E}_0$, $Q_1'' = \widetilde{F}_1 \cap \widetilde{S}_0$, and G, G_1, G_2 are the abelian groups acting freely in codimension 1. Hence, $\beta_0^2|G| = |G_1|$ and $\beta_0^1|G| = |G_2|$.

Finally, a corresponding complement of the pair $(E_0, \mathrm{Diff}_{E_0}(\mathrm{Diff}_{S_1}(0)))$ is extended to a required complement of $(S_1, \mathrm{Diff}_{S_1}(0))$ by Proposition 1.17. □

Second step. Assume that there exists a curve $\Gamma_1 \subset S_1$ with the following two properties: (1) $K_{S_1} + \mathrm{Diff}_{S_1}(0) + \Gamma_1$ is an anti-ample divisor, $h_0\colon (\Gamma_1)_{\widetilde{S}_1} \to \Gamma_0$ is a surjective morphism and (2) Γ_1 is not a center of any blow-up of $\mathcal{H}_0$, in particular, if $(S_1, \mathrm{Diff}_{S_1}(0))$ is a toric pair then Γ_1 is its non-toric subvariety. For convenience, we put $\widetilde{\Gamma}_1 = (\Gamma_1)_{\widetilde{S}_1}$.

Lemma 5.6 *The triples* $(S_0, \mathrm{Diff}_{S_0}(0), \Gamma_0)$ *and* $(S_1, \mathrm{Diff}_{S_1}(0), \Gamma_1)$ *are of type A. Moreover,* $\Gamma_1 \sim E_0 + F_j$ *for some index j and* $\beta_0^2 = 1$ (*that is,* $E_0 \not\subset \mathrm{Supp}(\mathrm{Diff}_{S_1}(0))$).

Proof Let us remember that the pairs $(S_1, \mathrm{Diff}_{S_1}(0))$ were classified in Proposition 5.5, and we will use the same notation.

Put $M = (K_{\widetilde{S}_1} + \mathrm{Diff}_{\widetilde{S}_1}(0) + \widetilde{\Gamma}_1) \cdot \widetilde{E}_0$. Note that $M < 0$. There are two possibilities:

(1) $\widetilde{\Gamma}_1 \sim \widetilde{E}_0$, $\widetilde{E}_0 \subset \mathrm{Supp}(\mathrm{Diff}_{\widetilde{S}_1}(0))$ and $\widetilde{\Gamma}_1 \neq \widetilde{E}_0$;

(2) $\widetilde{\Gamma}_1 \not\sim \widetilde{E}_0$, $\widetilde{\Gamma}_1 \sim a_0 \widetilde{E}_0 + \sum_{i=1}^r a_i \widetilde{F}_i$, where $a_i \in \mathbb{Z}_{\geq 0}$ and $a_0 \geq 1$.

Suppose that the triple $(S_0, \mathrm{Diff}_{S_0}(0), \Gamma_0)$ does not have type A. We will prove that it is impossible. Proposition 4.7 and Lemma 5.4 imply that $(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1} < -\beta_0^2 (\Gamma_0^2)_{S_0} \leq -\beta_0^2 \leq -1$. Hence the proper transform of $\widetilde{\Gamma}_0$ has the self-intersection index ≤ -2 on the minimal resolution of $\widetilde{S}_1$. Consider possibility (1). Then $M = -2 + \deg(\mathrm{Diff}_{\widetilde{E}_0}(0)) + \frac{1}{2}\widetilde{E}_0^2 = 1 - \sum_{i=1}^3 \frac{1}{n_i} + \frac{1}{2}\widetilde{E}_0^2$, where $n_i \geq 2$ for all i. Since the linear system $|\widetilde{E}_0|$ is movable then $\widetilde{E}_0^2 = \widetilde{E}_0 \cdot \widetilde{\Gamma}_1 \geq \frac{1}{n_{i_1}} + \frac{1}{n_{i_2}}$ (it is possible that $i_1 = i_2$), and hence $M \geq 0$. Consider possibility (2). If $a_i \geq 1$ for some $i \geq 1$ then it is obvious that $M \geq 0$. Therefore we have to consider the last case $\widetilde{\Gamma}_1 \sim a_0 \widetilde{E}_0$, where $a_0 \geq 2$. Arguing as in possibility (1) and in its notation we have $\widetilde{E}_0^2 = \frac{1}{a_0}\widetilde{E}_0 \cdot \widetilde{\Gamma}_1 \geq \frac{2}{a_0} \sum_{k=1}^{a_0} \frac{1}{n_{i_k}}$, where $i_k \in \{1, 2, 3\}$, and hence $M \geq 0$.

Suppose that the triple $(S_0, \mathrm{Diff}_{S_0}(0), \Gamma_0)$ is of type A. We will prove that possibility (1) is not realized, and $a_0 = 1$, $r = 1$, $a_1 = 1$ in possibility (2).

Let $m_i = r_i/k_i$ be an index of the singularity at the point $\widetilde{F}_i \cap \widetilde{E}_0 \in \widetilde{S}_1$, where $i = 1, 2$. Lemma 5.4 implies that

$$(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1} < -\beta_0^2 (\Gamma_0^2)_{S_0} \leq -\beta_0^2 \Big(\frac{1}{m_1 k_1} + \frac{1}{m_2 k_2} \Big). \tag{1}$$

The morphism $h_0'|_{\widetilde{S}_1}\colon \widetilde{S}_1 \to S_1$ contracts $\widetilde{\Gamma}_0$ to a point of type $\frac{1}{m_3}(m_1, m_2)$ and $h_0'|_{\widetilde{S}_1}$ is a toric blow-up corresponding to the weights (m_1, m_2). Hence

$$(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1} = -\frac{m_3}{m_1 m_2}. \tag{2}$$

Therefore $m_3 > \beta_0^2(m_1/k_2 + m_2/k_1)$. The toric surface S_1 is completely determined by the triple (m_1, m_2, m_3). For possibility (1) (recall that $\beta_0^2 \geq 2$) we obtain the contradiction

$$M \geq -2 + \deg\left(\operatorname{Diff}_{\widetilde{E}_0}\left(\frac{k_1-1}{k_1}\widetilde{F}_1 + \frac{k_2-1}{k_2}\widetilde{F}_2\right)\right) + \frac{1}{2}\widetilde{E}_0^2 =$$
$$= -\frac{1}{m_1 k_1} - \frac{1}{m_2 k_2} + \frac{m_3}{2m_1 m_2} > 0.$$

The same calculations for possibility (2) imply $a_0 = 1$, and since $\widetilde{\Gamma}_1$ is an irreducible curve that the same calculations imply $r = 1$ and $a_1 = 1$.

In order to prove the lemma we must prove only that the plt triple $(S_1, \operatorname{Diff}_{S_1}(0), \Gamma_1)$ is of type A. Assuming the converse: its type differs from type A. For instance, let us consider Case 6) of Proposition 4.7, the other cases are considered similarly. Thus $(S_1, \operatorname{Diff}_{S_1}(0), \Gamma_1) = (\mathbb{P}_{x_1x_2x_3}(2b_2+1, b_2, 1), \frac{1}{2}\{x_1 = 0\}, \mathcal{O}_{S_1}(2b_2+1))$, where $b_2 \geq 2$. Since $\widetilde{S}_1 \to \Gamma_0$ is a toric conic bundle then there are one possibility only: $\widetilde{S}_1 \to S_1$ is the weighted blow-up of singularity of type $\frac{1}{b_2}(1,1)$ at the point $(0:1:0)$ with the weights $(2b_2+1, 1)$. Now $(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1} = -\frac{b_2}{2b_2+1}$ by equality (2) and $(\widetilde{\Gamma}_0^2)_{\widetilde{S}_1} \leq -(\frac{1}{2} + \frac{1}{2b_2+1})$ by inequality (1). This contradiction concludes the proof. $\square$

Remark 5.7 A klt singularity is called *weakly exceptional* if there exists its unique plt blow-up (see [13, 19]). A two-dimensional klt singularity is weakly exceptional if and only if it is of type $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. Lemma 5.6 shows the interesting correspondence of the types.

Let $h_1 \colon (Y_1, (S_2)_{Y_1}) \to (Y_0 \supset \widetilde{\Gamma}_1)$ be a blow-up of the curve $\widetilde{\Gamma}_1$ with an unique exceptional divisor ($\operatorname{Exc} h_1 = (S_2)_{Y_1}$), $(S_1)_{Y_1} \cong (S_1)_{Y_0}$ and the same structure as h_0. The set of all possible blow-ups h_1 is denoted by $\mathcal{H}_1$.

By Proposition 4.7 there is 1-complement of $K_{S_1} + \operatorname{Diff}_{S_1}(0) + \widetilde{\Gamma}_1$ that extends to 1-complement of $K_{Z_1} + S_1$. Therefore we have 1-complement $K_{Y_0} + \widetilde{S}_1 + \widetilde{S}_0 + (D_1)_{Y_0} \sim 0$. Since $(D_1)_X = (\psi = 0 \subset (\mathbb{C}^3 \ni 0))/G$ we can slightly change the function ψ keeping all properties. Therefore there is at least a pencil of $(D_1)_{Y_1}$ by proof of Proposition 4.4.1 [22], and we can assume that $a((S_2)_{Y_1}, (D_1)_X) = -1$.

If $a(S_0, (D_1)_X) \geq 0$ then $S_0 \cdot (D_1)_{Z_0} \geq 2\Gamma_0$, hence $K_{S_0} + \operatorname{Diff}_{S_0}((D_1)_{Z_0})$ is nef by Proposition 4.7 and $a(S_0, (D_1)_X) \leq -1$.

So we have 1-complement $K_{Y_1} + (S_2)_{Y_1} + (S_1)_{Y_1} + (S_0)_{Y_1} + (D_1)_{Y_1} \sim 0$. By the cone theorem we have:

(1) there exists an divisorial contraction $h'_{1,1} \colon Y_1 \to Y_{1,1}$ of $(S_1)_{Y_1}$ onto a curve, $(S_2)_{Y_1} \cong (S_2)_{Y_{1,1}}$;

(2) apply $K_{Y_{1,1}} + (S_0)_{Y_{1,1}} + (S_2)_{Y_{1,1}}$-MMP to contract small extremal ray by a small contraction $\varphi_{1,1}$. Put $\operatorname{Exc} \varphi_{1,1} = (F_0)_{Y_{1,1}}$. Let $\varphi^+_{1,1}$ be a log flip of $\varphi_{1,1}$, $\operatorname{Exc} \varphi^+_{1,1} = (F^+_0)_{Y_{1,2}}$, $h'_{1,2} \colon Y_{1,1} \dashrightarrow Y_{1,2}$ be a corresponding birational map;

(3) there exists a divisorial contraction $h'_{1,3} \colon Y_{1,2} \to Z_2$ of $(S_0)_{Y_{1,2}}$ to a point.

Thus we obtain a birational map $h'_1 = h'_{1,3} \circ h'_{1,2} \circ h'_{1,1} \colon Y_1 \dashrightarrow Z_2$. Since $(D_1)_{Y_{1,1}} \cdot (F_0)_{Y_{1,1}} = -(K_{Y_{1,1}} + (S_0)_{Y_{1,1}} + (S_2)_{Y_{1,1}}) \cdot (F_0)_{Y_{1,1}} > 0$, $(D_1)_{Y_{1,1}}$ contains a some fiber of $(S_2)_{Y_{1,1}}$ and $(D_1)_{Y_{1,1}} \not\supset (F_0)_{Y_{1,1}}$ by Proposition 4.7, then the divisor $(D_1)_{Z_2}$ contains the fiber $(F^+_0)_{Z_2}$ and $((S_2)_{Z_2}, \operatorname{Diff}_{(S_2)_{Z_2}}(0))$ is a toric pair by Shokurov's criterion on the characterization of toric varieties [26]. We obtain a non-toric blow-up $g_2 \colon (Z_2, S_2) \to (X \ni P)$.

We prove the following proposition.

Proposition 5.8 *The pair* $(S_2, \operatorname{Diff}_{S_2}(0))$ *is toric* (1*-complementary*) *with the structure described in Proposition 5.5* (*Type A*)*,* g_2 *is a non-toric plt blow-up.*

Third step. Assume that there exists a curve $\Gamma_2 \subset S_2$ with the following two properties: (1) $K_{S_2} + \operatorname{Diff}_{S_2}(0) + \Gamma_2$ is an anti-ample divisor, $h_0 \circ h_1 \colon (\Gamma_2)_{Y_1} \to \Gamma_0$ is a surjective morphism and (2) Γ_2 is not a center of any blow-up of $\mathcal{H}_1$, in particular, Γ_2 is a non-toric subvariety of $(S_2, \operatorname{Diff}_{S_2}(0))$.

Proposition 5.9 *There is no any blow-up* $h_2 \colon (Y_2, (S_3)_{Y_2}) \to (Y_1 \supset (\Gamma_2)_{Y_1})$ *of the curve* $(\Gamma_2)_{Y_1}$ *with unique exceptional divisor such that* $(S_3)_{Y_2}$ *is realized by some plt blow-up* $g_3 \colon Z_3 \to (X \ni P)$*.*

Proof Assume the converse. Repeat the procedure described in Diagram 4, but with one change, replace the blow-up $g_0 \colon Z_0 \to X$ by the blow-up $g_1 \colon Z_1 \to X$. Therefore, returning to the main procedure, we can assume that there is 1-complement $K_{Y_2} + (S_3)_{Y_2} + (S_2)_{Y_2} + (S_1)_{Y_2} + (S_0)_{Y_2} + (D_2)_{Y_2} \sim 0$. Apply MMP to contract S_1 and S_2. Let $Y_2 \dashrightarrow Y_{2,2}$ be a corresponding birational map. If $(S_0)_{Y_{2,2}}$ contains one fiber of $(S_3)_{Y_{2,2}}$ then $(S_1)_{Y_2}$ and $(S_0)_{Y_2}$ contain a fiber of $(S_3)_{Y_2}$, a contradiction with log canonicity. Therefore $(S_0)_{Y_{2,2}}$ contains two fibers of $(S_3)_{Y_{2,2}}$. Then we obtain the contradiction $(K_{(S_3)_{Y_{2,2}}} + \operatorname{Diff}_{(S_3)_{Y_{2,2}}}((S_0)_{Y_{2,2}} + (D_2)_{Y_{2,2}})) \cdot C > 0$, where C is any section of the conic bundle $(S_3)_{Y_{2,2}}$. $\square$

(A2). Let $(X \ni P)$ be a non-$\mathbb{Q}$-factorial terminal toric three-dimensional singularity, that is, $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$.

Let $f \colon (Y, E) \to (X \ni P)$ be some non-toric plt blow-up. Let $\varphi_i \colon X_i \to (X \ni P)$ be two $\mathbb{Q}$-factorializations, $\operatorname{Exc} \varphi_i = C_i$ $(i = 1, 2)$. Let $\psi_i \colon (Y_i, E_i) \to (X_i \ni Q_i)$ be a plt blow-up for some i such that E_i and E define the same discrete valuation of the function field $\mathcal{K}(X)$, Q_i is a point. The blow-up ψ_i was constructed in the previous case of $\mathbb{Q}$-factorial singularities, $\rho(E_i) = 1$.

Let $Y_i \dashrightarrow \overline{Y}_i$ be a log flip for the curve $(C_i)_{Y_i}$. Considering another value of i we see that $-(E_i)_{\overline{Y}_i}$ is ample. Therefore $\overline{Y}_i = Y$ and $\rho(E) = 2$.

We give another construction and prove that $(E, \operatorname{Diff}_E(0))$ is a toric pair by the procedure illustrated on the next diagram (Fig. 10) and defined below.

First step. Let $g_0 \colon (Z_0, S_0) \to (X \ni P)$ be a toric plt blow-up, where $\operatorname{Exc} g_0 = S_0$ and $g_0(S_0) = P$ (see Definition 4.1 and its notation). Assume that there exists a curve

$$\begin{tikzcd} & Y_0 \arrow[dl, "h_0"'] \arrow[d, dashed, "h'_0"] & Y_1 \arrow[l, "h_1"'] \arrow[d, dashed, "h'_1"] \\ Z_0 \arrow[d, "g_0"'] & Z_1 \arrow[d, "g_1"'] & Z_2 \arrow[d, "g_2"'] \\ X & X & X \end{tikzcd}$$

Fig. 10 Case of Point. Construction in non-$\mathbb{Q}$-factorial case

$\Gamma_0 \subset S_0$ such that $(S_0, \mathrm{Diff}_{S_0}(0), \Gamma_0)$ is a plt triple (see Definition 4.9). Such triples are classified in Proposition 4.8.

Remark 5.10 Note that there exists the divisor $\Omega = \{x_2 + \gamma x_1^{w_2/w_1} + \ldots = 0\}|_X$ such that $\Omega_Z|_S = \Gamma_0$, and it has Du Val singularity of type $\mathbb{A}_{w_2/w_1}$, where $\gamma \neq 0$.

Let $h_0\colon (Y_0, \widetilde{S}_1) \to (Z_0 \supset \Gamma_0)$ be an arbitrary blow-up of the curve Γ_0 with an unique exceptional divisor $(\mathrm{Exc}\, h_0 = \widetilde{S}_1)$ as in case **(A1)**. The set of all possible blow-ups h_0 is denoted by $\mathcal{H}_0$.

There are two possibilities. The first possibility is as follows. There is a divisorial contraction of $\widetilde{S}_0$ to a curve: $h'_0\colon Y_0 \to Z_1$, and we obtain a non-toric plt blow-up $g_1\colon (Z_1, S_1) \to (X \ni P)$, where $\mathrm{Exc}\, g_1 = S_1$ and $g_1(S_1) = P$. The pair $(S_1, \mathrm{Diff}_{S_1}(0))$ is toric as in Proposition 5.5 (1).

The second possibility is when the first possibility is not realized. The curves $(C_{13})_{Y_0}$ and $(C_{23})_{Y_0}$ (see Definition 4.1) generate extremal rays of $\overline{\mathrm{NE}}(Y_0/X)$ that give small contractions. Let us contract the second one and $h'_{0,1}\colon Y_0 \dashrightarrow Y_{0,1}$ be a log flip. Let $h'_{0,2}\colon Y_{0,1} \to Z_1$ be a divisorial contraction of $(S_0)_{Y_{0,1}}$ to a point. Thus we obtain a birational map $h'_0 = h'_{0,2} \circ h'_{0,1}\colon Y_0 \dashrightarrow Z_1$. As in case **(A1)** 1-complement $K_{S_0} + C_{13} + C_{23} + \Gamma_0$ of $K_{S_0} + \mathrm{Diff}_{S_0}(0)$ extends to 1-complement $K_{Z_0} + S_0 + (D_0)_{Z_0}$ such that $a((S_1)_{Y_0}, (D_0)_{Z_0} + S_0) = -1$. Therefore the divisor $\mathrm{Diff}_{S_1}((D_0)_{Z_1})$ consists of four curves and is 1-complement of $K_{S_1} + \mathrm{Diff}_{S_1}(0)$. By Shokurov's criterion on the characterization of toric varieties $(S_1, \mathrm{Diff}_{S_1}(0))$ is a toric pair. Thus $g_1\colon Z_1 \to (X \ni P)$ is a non-toric plt blow-up.

Second step. Assume that there exists a curve $\Gamma_1 \subset S_1$ with the following two properties: (1) $K_{S_1} + \mathrm{Diff}_{S_1}(0) + \Gamma_1$ is an anti-ample divisor, $h_0\colon (\Gamma_1)_{\widetilde{S}_1} \to \Gamma_0$ is a surjective morphism and (2) Γ_1 is not a center of any blow-up of $\mathcal{H}_0$, Γ_1 is a non-toric subvariety of $(S_1, \mathrm{Diff}_{S_1}(0))$.

The self-intersection index Γ_0^2 is calculated by Proposition 4.8. Lemmas 5.4 and 5.6 are also true in this case. So we have 1-complement $K_{Y_1} + (S_2)_{Y_1} + (S_1)_{Y_1} + (S_0)_{Y_1} + (D_1)_{Y_1} \sim 0$. By the cone theorem we have:

(1) there exists an divisorial contraction $h'_{1,1}\colon Y_1 \to Y_{1,1}$ of $(S_1)_{Y_1}$ onto a curve, $(S_2)_{Y_1} \cong (S_2)_{Y_{1,1}}$;

(2) apply $K_{Y_{1,1}} + (S_0)_{Y_{1,1}} + (S_2)_{Y_{1,1}}$-MMP to contract small extremal ray, let $h'_{1,2}\colon Y_{1,1} \dashrightarrow Y_{1,2}$ be a corresponding log flip;

(3) apply $K_{Y_{1,2}} + (S_0)_{Y_{1,2}} + (S_2)_{Y_{1,2}}$-MMP to contract either small extremal ray or the divisor $(S_0)_{Y_{1,2}}$ onto a curve; we obtain a birational map $h'_{1,3}\colon Y_{1,2} \dashrightarrow Y_{1,3}$ or a morphism $h'_{1,4}\colon Y_{1,3} \to Z_2$ respectively;

(4) in the first case of (3) there exists a divisorial contraction $h'_{1,3}\colon Y_{1,3} \to Z_2$ of $(S_0)_{Y_{1,2}}$ to a point.

Thus we obtain a birational map $h'_1\colon Y_1 \dashrightarrow Z_2$ and a non-toric blow-up $g_2\colon (Z_2, S_2) \to (X \ni P)$. The pair $(S_2, \mathrm{Diff}_{S_2}(0))$ is toric by the same arguments as in case **(A1)**.

We prove the following proposition.

Proposition 5.11 *The pair* $(S_i, \mathrm{Diff}_{S_i}(0))$ *is klt and toric* (1-*complementary*), $\rho(S_i) = 2$, g_i *is a non-toric plt blow-up for* $i = 1, 2$.

Example 5.12 In this case we will construct examples of non-toric canonical blow-ups and prove that they are not terminal blow-ups. Depending on a type of $(X \ni P)$ there are two Cases **(B1)** and **(B2)**.

(B1). Let $(X \ni P) \cong (\mathbb{C}^3_{x_1x_2x_3} \ni 0)$. Let us consider a weighted blow-up $g\colon (Z, S) \to (X \ni P)$ with weights (w_1, w_2, w_3) such that $g(S) = P$ (that is, $w_i > 0$ for all $i = 1, 2, 3$), where $\gcd(w_1, w_2, w_3) = 1$. Write $(w_1, w_2, w_3) = (a_1q_2q_3, a_2q_1q_3, a_3q_1q_2)$, where $q_i = \gcd(w_k, w_l)$ and i, k, l are mutually distinct indices from 1 to 3. Then

$$\Big(S, \mathrm{Diff}_S(0)\Big) \cong \Big(\mathbb{P}_{x_1x_2x_3}\big(a_1, a_2, a_3\big), \sum_{i=1}^{3} \frac{q_i - 1}{q_i}\{x_i = 0\}\Big).$$

Assume that g is a canonical blow-up.

Proposition 5.13 *Let the curve* Γ *be a non-toric subvariety of* $(S, \mathrm{Diff}_S(0))$. *Assume that* Γ *does not contain any center of canonical singularities of* Z *and* $-(K_S + \mathrm{Diff}_S(0) + \Gamma)$ *is an ample divisor. Then we have one of the following possibilities for weights* (w_1, w_2, w_3) *up to permutation of coordinates.*

Type $(\mathbb{A})$. $(w_1, w_2, w_3) = (a_1q_3, a_2q_3, 1)$, $\Gamma \sim \mathcal{O}_S(a_1 + a_2)$.

Type $(\mathbb{D})$. $(w_1, w_2, w_3) = (l, l-1, 2)$, $(l+1, l, 1)$, $(l, l, 1)$ *and* $\Gamma \sim \mathcal{O}_S(l)$, $\mathcal{O}_S(2l)$, $\mathcal{O}_S(2)$ *respectively, where* $l \geq 2$.

Type $(\mathbb{E}_6)$. $(w_1, w_2, w_3) = (3, 2, 2)$, $(6, 4, 3)$, $(5, 3, 2)$, $(4, 2, 1)$ *and* $\Gamma \sim \mathcal{O}_S(3)$, $\mathcal{O}_S(2)$, $\mathcal{O}_S(9)$, $\mathcal{O}_S(3)$ *respectively.*

Type $(\mathbb{E}_7)$. $(w_1, w_2, w_3) = (3, 2, 2)$, $(6, 4, 3)$, $(9, 6, 4)$, $(3, 3, 1)$, $(5, 4, 2)$, $(7, 5, 3)$, $(5, 3, 2)$ *and* $\Gamma \sim \mathcal{O}_S(3)$, $\mathcal{O}_S(2)$, $\mathcal{O}_S(3)$, $\mathcal{O}_S(2)$, $\mathcal{O}_S(5)$, $\mathcal{O}_S(14)$, $\mathcal{O}_S(6)$ *respectively.*

Type $(\mathbb{E}_8)$. $(w_1, w_2, w_3) = (3, 2, 2)$, $(6, 4, 3)$, $(9, 6, 4)$, $(12, 8, 5)$, $(15, 10, 6)$, $(5, 4, 2)$, $(10, 7, 4)$, $(8, 5, 3)$ *and* $\Gamma \sim \mathcal{O}_S(3)$, $\mathcal{O}_S(2)$, $\mathcal{O}_S(3)$, $\mathcal{O}_S(6)$, $\mathcal{O}_S(1)$, $\mathcal{O}_S(5)$, $\mathcal{O}_S(10)$, $\mathcal{O}_S(15)$ *respectively.*

In all possibilities there is Du Val element $\Omega_Z \in |-K_Z|$ *such that* $\Omega_Z|_S = \Gamma + \sum_{i=1}^{r} \gamma_i \Gamma_i$. *Moreover,* $\Omega_Z|_S = \Gamma$, *except the two possibilities:* $(l+1, l, 1)$, $\Gamma \sim \mathcal{O}_S(2l)$(*type* $\mathbb{D}$) *and* $(5, 3, 2)$, $\Gamma \sim \mathcal{O}_S(6)$(*type* $\mathbb{E}_7$). *In these two possibilities we have* $\Omega_Z|_S = \Gamma + \Gamma_1$, *where* $\Gamma_1 \sim \mathcal{O}_S(1)$ *and* $\mathcal{O}_S(3)$ *respectively.*

Proof The proof follows from Proposition 2.6 by enumeration of cases. $\square$

Remark 5.14 Proposition 5.13 is similar to Proposition 4.7. Note that there is one-to-one correspondence between the sets (w_1, w_2, w_3, Γ) and the exceptional curves of minimal resolution of Du Val singularity $(\Omega \ni P)$, where $\Omega = g(\Omega_Z)$. Types in Proposition 5.13 correspond to Du Val types of the singularity $(\Omega \ni P)$.

By Theorem 1.6 there exists a divisorial contraction $h\colon (\widetilde{Y}, \widetilde{E}) \to (Z \supset \Gamma)$ for any weights $(\beta_1, 1)$ such that

(1) $\mathrm{Exc}\, h = \widetilde{E}$ is an irreducible divisor and $h(\widetilde{E}) = \Gamma$;

(2) the morphism h is locally toric for a general point of Γ;

(3) if H is a general hyperplane section passing through the general point $Q \in \Gamma$, then h induces the weighted blow-up of the smooth point $(H \ni Q)$ with weights $(\beta_1, 1)$;

(4) $h^*S = \widetilde{S} + \widetilde{E}$ and $h^*\Omega_Z = \Omega_{\widetilde{Y}} + \beta_1\widetilde{E}$.

Apply $K_{\widetilde{Y}} + \Omega_{\widetilde{Y}} + \varepsilon\widetilde{S}$–MMP. Since $\rho(\widetilde{Y}/X) = 2$ and $K_{\widetilde{Y}} + \Omega_{\widetilde{Y}} + \varepsilon\widetilde{S} \equiv \varepsilon\widetilde{S}$ over X, then we obtain a sequence of log flips $\widetilde{Y} \dashrightarrow \overline{Y}$, and after it we obtain the divisorial contraction $h'\colon \overline{Y} \to Y$ which contracts the proper transform $\overline{S}$ of $\widetilde{S}$.

Thus we obtain a required non-toric blow-up $f\colon (Y, E) \to (X \ni P)$, where $\mathrm{Exc}\, f = E$ is an irreducible divisor and $f(E) = P$. Since $K_Y + \Omega_Y = f^*(K_X + \Omega)$ then f is a canonical blow-up.

Finally let us prove that f is a non-terminal blow-up, that is, the singularities of Y are non-terminal. We must prove only that the center of $\overline{S}$ on Y does not lie in Ω_Y, since $0 = a(S, \Omega)$. Let $\widetilde{Y} = \overline{Y}_1 \dashrightarrow \overline{Y}_2 \dashrightarrow \ldots \dashrightarrow \overline{Y}_n = \overline{Y}$ be a decomposition of log flip sequence into elementary steps. If $\Omega_{\overline{Y}_i}$ is a nef divisor then by the base point free theorem [9] the linear system $|m\Omega_{\overline{Y}_i}|$ gives the birational contraction h' for $m \gg 0$. It contracts the proper transform of $\widetilde{S}$ to a point, $i = n$, and this completes the proof. Suppose that $\Omega_{\overline{Y}_i}$ is not a nef divisor. The cone $\overline{\mathrm{NE}}(\overline{Y}_i/X)$ is generated by two extremal rays. By Q_i, R_i denote them, and to be definite, assume that the ray R_i determines the next step of MMP. By construction, we have $\Omega_{\overline{Y}_i} \cdot Q_i > 0$, and hence $-K_{\overline{Y}_i} \cdot R_i = \Omega_{\overline{Y}_i} \cdot R_i < 0$. Since $K_{\overline{Y}_i} \cdot R_i > 0$ and the singularities of MMP are canonical, then the ray R_i gives a log flip (that is, $i < n$), and after it we have $\Omega_{\overline{Y}_{i+1}} \cdot Q_{i+1} > 0$. At the end we obtain that $\Omega_{\overline{Y}_j}$ is a nef divisor for some j. This completes the proof.

(B2). Let $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$. Let us consider a toric canonical blow-up $g\colon (Z, S) \to (X \ni P)$ (see Proposition 2.6 (3)).

Proposition 5.15 *Let a curve Γ be a non-toric subvariety of $(S, \mathrm{Diff}_S(0))$. Assume that Γ does not contain any center of canonical singularities of Z and $-(K_S + \mathrm{Diff}_S(0) + \Gamma + \Gamma')$ is an ample divisor, where Γ' is some effective $\mathbb{Q}$-divisor. Then $w_1 = 1$ and $\Gamma \sim \mathcal{O}_{\mathbb{P}(w_1,w_2,w_3,w_4)}(w_2)|_S$ up to permutation of coordinates. There exists Du Val element $\Omega_Z \in |-K_Z|$ such that $\Omega_Z|_S = \Gamma$. In particular, $-(K_S + \mathrm{Diff}_S(0) + \Gamma)$ is an ample divisor and $(\Omega \ni P)$ is Du Val singularity of type $\mathbb{A}_{w_2}$, where $\Omega = g(\Omega_Z)$.*

Proof The proof follows from Proposition 2.6 (3). $\square$

Now we can apply the construction of Case (**B1**).

Another construction of same non-toric canonical blow-ups is the following one. Consider a $\mathbb{Q}$-factorialization $g\colon \widetilde{X} \to X$ and $\widetilde{T} = \operatorname{Exc} g$. By G denote the center of E on $\widetilde{X}$. Applying (if necessary) a flop $\widetilde{X} \dashrightarrow \widetilde{X}^+$ we may assume that G is a point. Let us apply the above mentioned construction in Case (**B1**) for singularity $(\widetilde{X} \ni G)$. We obtain a non-toric canonical blow-up $f\colon Y \to \widetilde{X}$. Let $Y \dashrightarrow Y^+$ be a log flip for the curve T_Y. Thus we obtain a non-toric canonical blow-up $f^+\colon (Y^+, E^+) \to (X \ni P)$, where $E^+ = \operatorname{Exc} f^+$ and $f^+(E^+) = P$.

6 Main Theorems. Case of Point

Example 6.1 Let $(X \ni P) \cong (\mathbb{C}^3_{x_1x_2x_3} \ni 0)$. Let us consider the weighted blow-up $g\colon (Z, S) \to (X \ni P)$ with the weights $(15, 10, 6)$. Then

$$\Big(S, \operatorname{Diff}_S(0)\Big) \cong \Big(\mathbb{P}^2, \frac{1}{2}L_1 + \frac{2}{3}L_2 + \frac{4}{5}L_3\Big),$$

where L_i are the straight lines, and the divisor $\sum L_i$ is a complement to open toric orbit of S.

Let $\Omega = \{x_1^2 + x_2^3 + x_3^5 = 0\} \subset (X \ni P)$ be a divisor with Du Val singularity of type $\mathbb{E}_8$. Then $L = \Omega_Z|_S$ is a straight line. Put $P_i = L_i \cap L$. Then the points P_i are non-toric subvarieties of $(S, \operatorname{Diff}_S(0))$.

The main difference of structure of non-toric canonical blow-ups from the structure of non-toric plt blow-ups is shown in the following statements.

(1) We have $P_i \in \operatorname{CS}(Z, \Omega_Z)$ for every i. Thus P_i are the centers of some non-toric canonical blow-ups of $(X \ni P)$, that is, there exists the canonical blow-up $(Y, E_i) \to (X \ni P)$ such that the center of E_i on Z is the point P_i for every i.

(2) The points P_i are not the centers of any non-toric plt blow-ups of $(X \ni P)$. The proof of this fact is given in Theorem 6.2.

The origin of this difference is that S is not (locally) Cartier divisor at the points P_i (cf. Theorem 2.13).

The straight line $L \in \operatorname{CS}(Z, \Omega_Z)$ is a center of some non-toric canonical and plt blow-ups of $(X \ni P)$. As might appear at first sight the class of non-toric canonical blow-ups is much wider than the class of non-toric plt blow-ups, but it is not true. To construct the non-toric canonical blow-ups, some necessary conditions used implicitly in this example must be satisfied. Namely, g is a canonical blow-up, $a(S, \Omega) = 0$, the straight line L does not contain any center of canonical singularities of Z.

Theorem 6.2 *Let $f\colon (Y, E) \to (X \ni P)$ be a plt blow-up of three-dimensional toric terminal singularity, where $f(E) = P$. Then, either f is a toric morphism, or f is a non-toric morphism described in Sect. 5.*

Proof Let f be a non-toric morphism (up to analytical isomorphism). Let $D_Y \in |-n(K_Y + E)|$ be a general element for $n \gg 0$. Put $D_X = f(D_Y)$ and $d = \frac{1}{n}$. The pair (X, dD_X) is log canonical, $a(E, dD_X) = -1$ and E is a unique exceptional divisor with discrepancy -1.

Let $(X \ni P)$ be a $\mathbb{Q}$–factorial singularity. According to the construction of partial resolution of (X, dD_X) (see Definition 2.7) and Criterion 2.8 there exists a toric divisorial contraction $g \colon Z \to X$ such that it is dominated by partial resolution of (X, dD_X) (up to toric log flips), and one of the following Cases I and II occurs.

Case I. The exceptional set $\operatorname{Exc} g = S$ is an irreducible divisor, the divisors S and E define the different discrete valuations of the function field $\mathcal{K}(X)$, and $g(S) = P$. By Γ denote the center of E on the surface S. Then the center Γ is a non-toric subvariety of Z. In the other words Γ is a non-toric subvariety of $(S, \operatorname{Diff}_S(0))$. If Γ is a point then we assume that it does not lie on any one-dimensional orbit of the surface S (up to analytical isomorphism $(X \ni P)$ of course).

Case II. The variety Z is $\mathbb{Q}$-gorenstein, hence it is $\mathbb{Q}$-factorial. The exceptional set $\operatorname{Exc} g = S_1 \cup S_2$ is the union of two exceptional irreducible divisors, S_1, S_2 and E define mutually distinct discrete valuations of the function field $\mathcal{K}(X)$ and $g(S_1) = g(S_2) = P$. To be definite, let $\rho(S_1) = 1$, $\rho(S_2) = 2$, and $C = S_1 \cap S_2$ is a closure of one-dimensional orbit of Z. By Γ denote the center of E on Z. In this case Γ is a point and a non-toric subvariety of $(S_1, \operatorname{Diff}_{S_1}(0))$, $\Gamma \in C$, and the curve C has the coefficient 1 in the divisor $\operatorname{Diff}_{S_1}(S_2 + dD_Z)$. Mori cone $\overline{\mathrm{NE}}(Z/X)$ is generated by two extremal rays, denote them by R_1 and R_2. To be definite, let R_1 gives the divisorial contraction which contracts the divisor S_1 to some point P_1. Considering toric blow-ups of P_1 we may assume that $\operatorname{Diff}_{S_1}(S_2 + dD_Z)$ is a boundary in some analytical neighborhood of the point Γ.

If R_2 gives the divisorial contraction which contracts the divisor S_2 (onto curve) then it is *Case* IIa. If R_2 gives a small flipping contraction then it is *Case* IIb.

Let us consider *Case* IIb in more detail. Let $Z \dashrightarrow Z^+$ be a toric log flip induced by R_2. The corresponding objects on Z^+ are denoted by the index $^+$. For the toric divisorial contraction $g^+ \colon Z^+ \to X$ we have $\rho(S_1^+) = 2$, $\rho(S_2^+) = 1$. Note that the point $\Gamma^+ \in C^+ = S_1^+ \cap S_2^+$ of E on Z^+ can be a toric subvariety of $(S_2^+, \operatorname{Diff}_{S_2^+}(0))$. The morphism g^+ is dominated by partial resolution of (X, dD_X) (up to toric log flips), and the curve C^+ has the coefficient 1 in the divisor $\operatorname{Diff}_{S_2^+}(S_1^+ + dD_{Z^+})$.

Note that the equality $g(\operatorname{Exc} g) = P$ is proved similarly to Theorem 2.12 in both *Cases* I and II.

Now, according to Sect. 5 the following lemma implies the proof of theorem (for $\mathbb{Q}$–factorial singularities). □

Lemma 6.3 *It is possible Case* I *only. Moreover,* Γ *is a curve and* $K_S + \operatorname{Diff}_S(0) + \Gamma$ *is a plt divisor.*

Proof Let us consider Case I. Write

$$K_Z + dD_Z + aS = g^*\big(K_X + dD_X\big),$$

where $a < 1$. Hence

$$a\big(E, S + dD_Z\big) < a\big(E, aS + dD_Z\big) = -1.$$

Therefore $\Gamma \subset \mathrm{LCS}(S, \mathrm{Diff}_S(dD_Z))$ and $-(K_S + \mathrm{Diff}_S(dD_Z))$ is an ample divisor.

Assume that Γ is a (irreducible) curve. We must prove that $K_S + \mathrm{Diff}_S(0) + \Gamma$ is a plt divisor. Assume the converse. By the adjunction formula, Γ is a smooth curve, and by connectedness lemma the divisor $K_S + \mathrm{Diff}_S(0) + \Gamma$ is not a plt one at unique point denoted by G. The point G is a toric subvariety of $(S, \mathrm{Diff}_S(0))$ by Theorem 4.2. Moreover, the curve Γ is locally a non-toric subvariety at the point G only. According to the construction of partial resolution [27] there exists the divisorial toric contraction $\widehat{g}\colon \widehat{Z} \to Z$ such that $\mathrm{Exc}\,\widehat{g} = S_2$ is an irreducible divisor, $\widehat{g}(S_2) = G$ and the following two conditions are satisfied.

(1). Put $S_1 = S_{\widehat{Z}}$ and $C = S_1 \cap S_2$. Let $c(\Gamma)$ be the log canonical threshold of Γ for the pair $(S, \mathrm{Diff}_S(0))$. Then $\widehat{g}|_{S_1}\colon S_1 \to S$ is the inductive toric blow-up of $K_S + \mathrm{Diff}_S(0) + c(\Gamma)\Gamma$ (see Theorems 1.10 and 2.12), and the point $\widehat{G} = C \cap \Gamma_{S_1}$ is a non-toric subvariety of $(S_2, \mathrm{Diff}_{S_2}(0))$.

(2). The divisor $\mathrm{Diff}_{S_2}(dD_{\widehat{Z}} + S_1)$ is a boundary at the point $\widehat{G}$.

Let H be a general hyperplane section of large degree passing through the point P. Then we have $a(S_i, dD_X + hH) = -1$ and $a(S_j, dD_X + hH) > -1$ for some $h > 0$, $i \neq j$. If $i = 1$ and $j = 2$ then we have the contradiction with Theorem 4.2 for the pair $(S_2, \mathrm{Diff}_{S_2}(dD_{\widehat{Z}} + S_1))$. Hence, we may assume that $i = 2$ and $j = 1$. Mori cone $\overline{\mathrm{NE}}(\widehat{Z}/X)$ is generated by two rays, denote them by $\widehat{R}_1$ and $\widehat{R}_2$. To be definite, let $\widehat{R}_2$ gives the contraction $\widehat{g}$.

At first assume that $\widehat{R}_1$ gives the contraction $g_1\colon \widehat{Z} \to Z_1$ which contracts S_1 (onto a curve). The contraction g_1 is an isomorphism for the surface S_2, therefore we denote $g_1(S_2)$ by S_2 again for convenience. If $\mathrm{Diff}_{S_2}(dD_{Z_1})$ is a boundary then we have the contradiction with Theorem 4.2 applied for the pair $(S_2, \mathrm{Diff}_{S_2}(dD_{Z_1}))$. If it is not a boundary then we have the following contradiction

$$
\begin{aligned}
0 > {}&(1 + a(S_1, dD_X + hH))S_1 \cdot C_0 = \\
&= (K_{S_1} + \mathrm{Diff}_{S_1}(dD_{\widehat{Z}} + S_2 + hH_{\widehat{Z}})) \cdot C_0 \geq \\
\geq{}&(K_{S_1} + \mathrm{Diff}_{S_1}(0)' + \Gamma_{S_1} + C + C_0) \cdot C_0 \geq (-F_1 - F_2 + \Gamma_{S_1}) \cdot C_0 \geq 0,
\end{aligned}
$$

where C_0 is the closure of one-dimensional orbit of S_1, having zero-intersection with C, and F_1, F_2 are the two toric fibers (the closures of corresponding one-dimensional toric orbits) of the toric conic bundle $S_1 \to g_1(S_1)$, and the divisor $\mathrm{Diff}_{S_1}(0)'$ is a part of $\mathrm{Diff}_{S_1}(0)$ provided that we equate to zero the coefficients of C and C_0 in $\mathrm{Diff}_{S_1}(0)$.

At last assume that $\widehat{R}_1$ gives a flipping contraction. Let $\widehat{Z} \dashrightarrow \widehat{Z}^+$ be a corresponding toric log flip. The corresponding objects on $\widehat{Z}^+$ are denoted by the index $^+$. If the point $\widehat{G}^+$ is a non-toric subvariety of $(S_1^+, \mathrm{Diff}_{S_1^+}(0))$ then we have the contradiction with Theorem 4.2 applied for the pair $(S_1^+, \mathrm{Diff}_{S_1^+}(S_2^+) + \widehat{\Gamma}^+)$. Therefore we can assume that the point G^+ is a toric subvariety. If the curve $\widehat{\Gamma}^+$ is a non-toric subvariety of $(S_1^+, \mathrm{Diff}_{S_1^+}(0))$, then by the inversion of adjunction the pair $(S_1^+, \mathrm{Diff}_{S_1^+}(S_2^+) + \widehat{\Gamma}^+)$ is plt outside $\widehat{G}^+$, and we have the contradiction with Proposition 4.7. Thus we have proved that $\widehat{\Gamma}^+$ and G^+ are the toric subvarieties of $(S_1^+, \mathrm{Diff}_{S_1^+}(0))$. In particular, $S_1^+ \cong \mathbb{P}(1, r_1, r_2)$, where $\gcd(r_1, r_2) = 1$ and $(\widehat{\Gamma}^+)^2 = r_1/r_2$. Considering the divisor $D(\delta) = (d - \delta)D + h(\delta)H$ for some $\delta \geq 0$ and $h(\delta) > 0$ ($h(0) = 1$) instead of the divisor $D(0) = dD$, we may assume that the whole construction is satisfied and $a(E, D(\delta)) = -1$.

Let $\mathrm{Diff}_{S_2}(D(\delta) - a(S_1, D(\delta))S_1) \geq 0$ (for example, it holds if $a(S_1, D(\delta)) < 0$). Replacing the divisor H by other general divisor with $\widehat{G} \in \mathrm{Supp}(H_{\widehat{Z}})$, we may assume that the three following conditions are satisfied: (1) $\mathrm{Diff}_{S_2}(D(\delta) - a(S_1, D(\delta))S_1) \geq 0$; (2) $\widehat{G}$ is a center of $\mathrm{LCS}(\widehat{Z}, D(\delta)_{\widehat{Z}} - a(S_1, D(\delta))S_1 - a(S_2, D(\delta))S_2)$; (3) $a(S_2, D(\delta)) > -1$. We obtain the contradiction with Theorem 4.2 for the pair $(S_2, \mathrm{Diff}_{S_2}(D(\delta) - a(S_1, D(\delta))S_1))$.

Let $\mathrm{Diff}_{S_2}(D(\delta) - a(S_1, D(\delta))S_1)$ is not an effective divisor. The curve $\widehat{\Gamma}^+$ is locally a toric subvariety in some analytical neighborhood of every point of $\widehat{Z}^+$, therefore there exists a blow-up $\overline{g}\colon (\overline{Z} \supset \overline{S}_3) \to (\widehat{Z}^+ \supset \widehat{\Gamma}^+)$, where $\mathrm{Exc}\,\overline{g} = \overline{S}_3$ is an irreducible divisor such that $\overline{g}(\overline{S}_3) = \widehat{\Gamma}^+$ and the following three conditions are satisfied.

(A) The morphism $\overline{g}$ is locally a toric one at every point of $\widehat{\Gamma}^+$, in particular, $\overline{S}_1 \cong S_1$.
(B) Let H be a general hyperplane section of $\widehat{Z}^+$ passing through the general point $\widehat{Q} \in \widehat{\Gamma}^+$. Then $\overline{g}$ induces a weighted blow-up of $(H \ni \widehat{Q})$ with weights (β_1, β_2), and $\overline{g}^* S_1^+ = \overline{S}_1 + \beta_2 \overline{S}_3$.
(C) Either the divisors $\overline{S}_3$ and E define the same discrete valuation of the function field $\mathcal{K}(X)$ (*Case* C1), or the curve $\overline{\Gamma} \subset \overline{S}_3$ being the center of E on $\overline{Z}$ is a non-toric subvariety of $(\overline{S}_3, \mathrm{Diff}_{\overline{S}_3}(0))$ (*Case* C2).

By $\overline{C}_0$ and $\overline{F}$ denote zero-section and a general fiber of $\overline{S}_3$ respectively.

Let us consider *Case* C1. Then $\overline{D}(\delta)|_{\overline{S}_3} \sim_{\mathbb{Q}} a\overline{C}_0 + b\overline{F}$ by the generality of D, where $b \geq 0$ and $a = 2 + a(S_1, D(\delta))/\beta_1 - \frac{\beta_2 - 1}{\beta_2} - \frac{\beta_1 - 1}{\beta_1} \geq 1 + \frac{1}{\beta_2}$. We obtain the contradiction (the calculations are similar to Lemma 5.4 and Proposition 5.5)

$$
\begin{aligned}
0 =& (K_{\overline{S}_3} + \mathrm{Diff}_{\overline{S}_3}(\overline{D}(\delta) + \overline{S}_2^+ - a(S_1, D(\delta))\overline{S}_1^+)) \cdot \overline{C}_0 \geq \\
&\geq -2 + 1 + \frac{r_2 - 1}{r_2} + \overline{C}_0^2 > (r_1 - 1)/r_2 \geq 0.
\end{aligned}
$$

Let us consider *Case* C2. If $a(\overline{S}_3, D(\delta)) \leq -1$ then we require the condition $a(\overline{S}_3, D(\delta)) = -1$ to be satisfied instead of the condition $a(E, D(\delta)) = -1$ in the construction of $D(\delta)$, and we obtain similar contradiction as in *Case* C1. Therefore we may assume that $a(\overline{S}_3, D(\delta)) > -1$. Then $\overline{\Gamma} \sim a\overline{C}_0 + b\overline{F}$, where either $a \geq 1$, $b \geq 1$, or $a \geq 2$, $b \geq 0$, or $a = 1$, $b = 0$, $\overline{\Gamma} \neq \overline{C}_0$, $\beta_2 \geq 2$. Continuing this line of reasoning, we have the same contradictions for any possibility of $\overline{\Gamma}$.

Now assume that Γ is a point. Theorem 4.2 implies that $\mathrm{Diff}_S(dD_Z)$ is not a boundary in any analytical neighborhood of Γ. Moreover, there is unique curve passing through Γ with the coefficient ≥ 1 in the divisor $\mathrm{Diff}_S(dD_Z)$. It is clear that it is smooth at the point Γ, it is a non-toric subvariety of $(S, \mathrm{Diff}_S(0))$ and denote it by T.

Let us prove that $(S, \mathrm{Diff}_S(0) + T)$ is a plt pair. Let H be a general hyperplane section of large degree passing through the point P such that $\Gamma \in H_Z$.

As above by Theorem 4.2, there exist some rational numbers $0 < \delta < d$, $h > 0$ and the divisor $D' = (d - \delta)D_X + hH$ such that (X, D') is a log canonical pair, $\mathrm{LCS}(Z, D'_Z - a(S, D')S) = T$ and Γ is a center of $(Z, D'_Z - a(S, D')S)$. Moreover, we may assume that there are not another centers differing from Γ and T by connectedness lemma. Now, according to the standard Kawamata's perturbation trick, there exists an effective $\mathbb{Q}$-divisor D'' on X such that the curve T is unique minimal center of $(Z, D''_Z - a(S, D'')S)$. So, by the previous statement proved (when Γ is a curve) $(S, \mathrm{Diff}_S(0) + T)$ is a plt pair.

Let us consider the blow-up $\overline{g}\colon (\overline{Z} \supset \overline{S}_3) \to (Z \supset T)$ for the pair (X, D') which is similar to the blow-up $\overline{g}\colon (\overline{Z} \supset \overline{S}_3) \to (\widehat{Z}^+ \supset \widehat{\Gamma}^+)$, where $\mathrm{Exc}\,\overline{g} = \overline{S}_3$. Let $\overline{\Gamma} \subset \overline{Z}$ be a center of E. There are two cases $\overline{\Gamma} = \overline{F}$, $\overline{\Gamma}$ is a point, where $\overline{F}$ is a fiber over the point Γ. Applying Lemma 4.4 if Γ is a point, we obtain the contradiction in same way as above

$$0 = (K_{\overline{S}_3} + \mathrm{Diff}_{\overline{S}_3}(\overline{D'} - a(S, D')\overline{S})) \cdot \overline{C}_0 > 0.$$

Let us prove that *Case* II is impossible. Let H be a general hyperplane section of large degree passing through the point P. Then we have $a(S_i, dD_X + hH) = -1$ and $a(S_j, dD_X + hH) > -1$ for some $h > 0$.

Let us introduce the following notation: let $M = \sum m_i M_i$ be the divisor decomposition on irreducible components, then we put $M^b = \sum_{i\colon m_i > 1} M_i + \sum_{i\colon m_i \le 1} m_i M_i$.

If $i = 2$ and $j = 1$ then we obtain the contradiction with Theorem 4.2 for the pair $(S_1, \mathrm{Diff}_{S_1}(dD_Z + S_2)^b)$. Therefore $i = 1$ and $j = 2$.

Let us consider *Case* IIb. If Γ^+ is a non-toric subvariety of $(S_2^+, \mathrm{Diff}_{S_2^+}(0))$ then we obtain the contradiction with Theorem 4.2 for the pair $(S_2^+, \mathrm{Diff}_{S_2^+}(dD_{Z^+} + S_1^+)^b)$. Therefore we assume that Γ^+ is a toric subvariety of $(S_2^+, \mathrm{Diff}_{S_2^+}(0))$. The similar (related) case have been considered, when Γ was a curve, therefore we do not repeat its complete description. By construction, the curve $C^+ \subset S_1^+$ is exceptional and contains at most one singularity of S_1^+. Since the pair $(S_1^+, \mathrm{Diff}_{S_1^+}(dD_{Z^+} + hH_{Z^+})$ is not log canonical at the point Γ^+, then $(dD_{Z^+} + hH_{Z^+}) \cdot C^+ = 1 + \sigma$, where $\sigma > 0$. Since the divisor $-K_{S_1^+}$ is a sum of four one-dimensional orbit closures, then

$$\begin{aligned} &a(S_2^+, dD_{Z^+} + hH_{Z^+})S_2^+ \cdot C^+ = \\ &= (K_{S_1^+} + \mathrm{Diff}_{S_1^+}(dD_{Z^+} + hH_{Z^+})) \cdot C^+ \ge \\ &\ge -(C^+)^2_{S_1^+} - 1 - \frac{1}{r_1} + 1 + \sigma \ge \sigma > 0. \end{aligned}$$

Since $S_2^+ \cdot C^+ < 0$ then $a(S_2^+, dD_{Z^+} + hH_{Z^+}) < 0$. Now, to obtain the contradiction with Theorem 4.2 for the pair $(S_1, \mathrm{Diff}_{S_1}(dD_Z + hH_Z - a(S_2, dD + hH)S_2)^b)$, it is sufficient to decrease the coefficient h slightly (then $a(S_1, dD + hH) > -1$).

Let us consider *Case* IIa. Let $g_1\colon Z \to Z_1$ be a contraction of R_2. The contraction g_1 is an isomorphism for the surface S_1, therefore we denote $g_1(S_1)$ by S_1 again for convenience. If the divisor $\mathrm{Diff}_{S_1}(dD_{Z_1})$ is a boundary then we have the contradiction with Theorem 4.2 for the pair $(S_1, \mathrm{Diff}_{S_1}(dD_{Z_1}))$, and if it is not a boundary then we have the following contradiction

$$\begin{aligned}&0 > (1 + a(S_2, dD_X + hH))S_2 \cdot C_0 =\\&= (K_{S_2} + \operatorname{Diff}_{S_2}(dD_Z + S_1 + hH_Z)) \cdot C_0 \geq\\&\geq (K_{S_2} + \operatorname{Diff}_{S_2}(0)' + F + C + C_0) \cdot C_0 \geq 0,\end{aligned}$$

where C_0 is the closure of one-dimensional orbit of S_2 having zero-intersection with C, and F is a general fiber of the conic bundle $S_2 \to g_1(S_2)$, and the divisor $\operatorname{Diff}_{S_2}(0)'$ is a part of $\operatorname{Diff}_{S_2}(0)$ provided that we equate to zero the coefficients of C and C_0 in $\operatorname{Diff}_{S_2}(0)$. Note that the equality $(D_Z|_{S_2} \cdot C)_\Gamma \geq 1$ have been applied here (see Lemma 4.4); it is true since $(S_2, C + D_Z|_{S_2})$ is not a plt pair at the point Γ by the construction. □

Let $(X \ni P)$ be a non-$\mathbb{Q}$–factorial singularity, that is, $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$. We repeat the arguments given in Sect. 5. Let $g\colon \widetilde{X} \to X$ be a $\mathbb{Q}$-factorialization and let $C = \operatorname{Exc} g \cong \mathbb{P}^1$. Note that $\widetilde{X}$ is a smooth variety. By G denote the center of E on $\widetilde{X}$. If G is a point then it is a toric subvariety, and hence the main theorem is reduced to the case of $\mathbb{Q}$-factorial singularities. If $G = C$ then we consider the flop $\widetilde{X} \dashrightarrow \widetilde{X}^+$, and we may assume that G is a point by replacing $\widetilde{X}$ by $\widetilde{X}^+$.

Theorem 6.4 *Let $f\colon (Y, E) \to (X \ni P)$ be a plt blow-up of three-dimensional toric $\mathbb{Q}$–factorial singularity, where $f(E) = P$. Then, either f is a toric morphism, or f is a non-toric morphism described in Sect. 5.*

Proof We can repeat the proof of Theorem 6.2 without any changes in our case. Lemma 5.3 gives some restrictions, when $(X \ni P)$ is a terminal singularity, but it is not used in what follows. □

Theorem 6.5 *Let $f\colon (Y, E) \to (X \ni P)$ be a canonical blow-up of three-dimensional toric terminal singularity, where $f(E) = P$. Then, either f is a toric morphism (see Proposition 2.6), or f is a non-toric morphism described in Sect. 5.*

Proof Let f be a non-toric morphism (up to analytical isomorphism). Let $D_Y \in |-nK_Y|$ be a general element for $n \gg 0$. Put $D_X = f(D_Y)$ and $d = \frac{1}{n}$. The pair (X, dD_X) has canonical singularities and $a(E, dD_X) = 0$.

Let $(X \ni P)$ be a $\mathbb{Q}$–factorial singularity. There is one of two *Cases* I and II described in the proof of Theorem 6.2. We will use the notation from the proof of Theorem 6.2. According to Sect. 5 the following proposition implies the proof of theorem for $\mathbb{Q}$–factorial singularities. □

Proposition 6.6 *There exists a toric blow-up g such that we have Case I always, the center Γ is a curve, $a(S, dD_X) = 0$ and $(X \ni P)$ is a smooth point, in particular, g is a canonical blow-up.*

Proof Let us consider *Case* II. We may assume that $C \not\subset \operatorname{Supp}(\operatorname{Sing} Z)$. Actually, by taking toric blow-ups with the center C we obtain either the requirement, or *Case* I (that is, there is some blow-up g such that the center of E is a curve and a non-toric subvariety of corresponding exceptional divisor). Therefore S_1 and S_2 are Cartier divisors at the point Γ. Therefore we have

$$a(E, S_i + dD_Z) \leq a(E, -a(S_i, dD_X)S_i + dD_Z) - 1 \leq -1$$

for $i = 1, 2$

Let H be a general hyperplane section of large degree passing through the point P and let $\Gamma \in H_Z$. For any $\delta > 0$ there exists a number $h(\delta) > 0$ such that $(X, D(\delta) = (d - \delta)D_X + h(\delta)H)$ is a log canonical and not plt pair. Let $D_Z|_S = \sum d_i D_i^S$ be a decomposition on the irreducible components ($S = S_1 + S_2$). If it is necessary we replace the divisor D_X by D'_X in order to $D'_Z|_S = \sum_{i\colon \Gamma \in D_i^S} d_i D_i^S$. By the generality of H and connectedness lemma, there exists $\delta > 0$ with the following two properties.

(1) The pair $(X, D(\delta))$ defines a plt blow-up $(Y(\delta), E(\delta)) \to (X \ni P)$.

(2) By T denote the center of $E(\delta)$ on Z. Then, either $T = \Gamma$, or T is a curve provided that $T \subset S_2$ and $\Gamma \in T$ (note that case $T \subset S_1$ is impossible, since it was proved in *Case* I of Theorem 6.2).

Let $T = \Gamma$. Then we have *Case* II of Theorem 6.2, but it was proved that this case is impossible.

Let T be a curve and let $\psi\colon Z \to Z'$ be a contraction of R_1. The morphism ψ contracts the divisor S_1 to the point P_1. By construction, $K_{S'_2} + \mathrm{Diff}_{S'_2}(0) + T_{S'_2}$ is not a plt divisor at the point P_1, and it was proved in *Case* I of Theorem 6.2 that this case is impossible.

Let us consider *Case* I. Write $K_Z + dD_Z = g^*(K_X + dD_X) + a(S, dD_X)S$, where $a(S, dD_X) \geq 0$. Since S is Cartier divisor at a general point of Γ then

$$a(E, S + dD_Z) \leq a(E, -a(S, dD_X)S + dD_Z) - 1 = -1.$$

Hence $\Gamma \subset \mathrm{LCS}(S, \mathrm{Diff}_S(dD_Z))$.

Let $a(S, dD_X) = 0$. Then Z has canonical singularities.

Assume that Γ is a curve. Then $(X \ni P)$ is a smooth point by Lemma 6.7, which is of independent interest. □

Lemma 6.7 *Let $g\colon (Z, S) \to (X \ni P)$ be a toric canonical blow-up of three-dimensional $\mathbb{Q}$-factorial terminal toric singularity. Assume that there exists a curve $\Gamma \subset S$ such that it is a non-toric subvariety of $(S, \mathrm{Diff}_S(0))$, and it does not contain any center of canonical singularities of Z. Let $-(K_S + \mathrm{Diff}_S(0) + \Gamma)$ be an ample divisor. Assume that there exists a divisor $D'_Z \in |-mK_Z|$ for some $m \in \mathbb{Z}_{>0}$ such that $\left(Z, \frac{1}{m}D'_Z\right)$ is a canonical pair and $\left(\frac{1}{m}D'_Z\right)|_S = \Gamma + \sum \gamma_i \Gamma_i$, where $\gamma_i \geq 0$ for all i. Then $(X \ni P)$ is a smooth point.*

Proof Assume the converse. We suppose that the reader knows the proof of Proposition 2.6 (2), and we use its terminology. We have $a(S, 0) = \frac{1}{r}(w_3 + rw_2 - qw_3 + rw_1 - w_3) - 1$. If $a(S, 0) = \frac{1}{r}$ then we have a contradiction obviously. Therefore we suppose that $a(S, 0) > \frac{1}{r}$. For some $j \in \{1, 2, 3\}$ we have the inequality $\frac{1}{r} \geq a(S, 0)/N_j$ and one of the two following requirements: either $P_j \in \mathrm{CS}(Z)$, or the singularity at the point P_j is of type $\frac{1}{N_j}(1,-1,0)$, where $N_j \geq 2$, $N_1 = w_3$, $N_2 = rw_1 - w_3$, $N_3 = rw_2 - qw_3$.

The non-toric curve Γ is conveniently represented as $\Gamma = D_Z \cap S$, where $D = (\psi(x_1, x_2, x_3) = 0)/\mathbb{Z}_r \subset (\mathbb{C}^3 \ni 0)/\mathbb{Z}_r(-1, -q, 1)$ and ψ is a quasihomogeneous polynomial with respect to (N_1, N_2, N_3).

Then $P_j \in \Gamma$, the singularity is of type $\frac{1}{N_j}(1,-1,0)$ at the point P_j and $N_j/r \geq 1$. Let us prove it. Let $D' = g(D'_Z)$. If $P_j \notin \Gamma$ then we have the contradiction $a(S, \frac{1}{m}D') < a(S, 0) - N_j/r \leq 0$, since Γ is a non-toric subvariety. Let $P_j \in \Gamma$. Then $P_j \notin \mathrm{CS}(Z)$, and if $N_j/r < 1$, then we have the contradiction $a(S, \frac{1}{m}D') \leq N_j/r - 1 < 0$ since Γ is a non-toric subvariety.

Assume that *Case* (2A) of Proposition 2.6 takes place. Then $j = 3$. Since $N_3 > \max\{N_1, N_2\}$ then the singularity must be isolated at the point P_3. We obtain the contradiction. It is not hard to prove that *Case* (2B) of Proposition 2.6 is impossible. □

Assume that Γ is a point. Then $\mathrm{Diff}_S(dD_X)$ is a boundary, and hence we obtain the contradiction with Theorem 4.2 for the pair $(S, \mathrm{Diff}_S(dD_X))$ and the point Γ.

Let $a(S, dD_X) > 0$. We will obtain a contradiction. Note that the number of exceptional divisors with discrepancy 0 is finite for the pair (X, dD_X). Now we will carry out the procedure consisting of the two steps: (i1) replacing dD_X by $D(\delta)$ and (i2) replacing (X, dD_X) by other pair with canonical singularities (the variety X is replaced by other variety also). As the result of finite number of steps of this procedure we will obtain a contradiction. Let H_1 be a general hyperplane section of large degree containing the center of S on X (at this first step the point P is this center, and note that this center can be a curve after replacing X as a result of step (i2)). Also we require that $(H_1)_Z|_S \subset S$ is an irreducible reduced subvariety (curve) not containing any zero-dimensional orbit of S. This last condition is necessary to our procedure terminates obviously after a finite number of steps.

Let us consider the numbers $\delta \geq 0$, $h(\delta) \geq 0$ and the divisor $D(\delta) = (d - \delta)D_X + h(\delta)H_1$ such that $(X, D(\delta))$ has canonical singularities, Γ is a center of canonical singularities of $(Z, D(\delta)_Z - a(S, D(\delta))S)$, and one of the two following conditions are satisfied: either (a1) $a(S, D(\delta)) = 0$ or (a2) $a(S, D(\delta)) > 0$ and there exists a center of canonical singularities different from Γ for the pair $(Z, D(\delta)_Z - a(S, D(\delta))S)$. Take the maximal number δ with such properties. By E again (for convenience) we denote some exceptional divisor with discrepancy 0 for $(X, D(\delta))$ such that its center is Γ on Z. It is step (i1).

Let $a(S, D(\delta)) = 0$ and Γ be a curve. By the above statement $(X \ni P)$ is a smooth point. We claim that $h(\delta) = 0$, and thus we have the contradiction. Let us prove it. Consider the general point Q of Γ and the general (smooth) hyperplane section H passing through this point. Then $(H \ni Q, (D(\delta)_Z)|_H)$ has canonical non-terminal singularities. This is equivalent to $\mathrm{mult}_Q(D(\delta)_Z)|_H = 1$. Let us apply the construction of non-toric canonical blow-ups from Sect. 5 to the curve Γ provided that $\beta_1 = 1$. As the result we obtain the non-toric canonical non-terminal blow-up $(Y'', E'') \to (X \ni P)$. By the above $a(E'', D(\delta)) = 0$. Since $\Gamma \not\subset (H_1)_Z$ then the divisor $(H_1)_{Y''}$ contains the center of canonical singularities of Y'' (see Sect. 5) always. Therefore $h(\delta) = 0$.

Let $a(S, D(\delta)) = 0$ and Γ be a point. Then $\mathrm{Diff}_S(D(\delta))$ is a boundary and we have the contradiction with Theorem 4.2.

Let $a(S, D(\delta)) > 0$. Let $\widehat{X} \to X$ be a log resolution of $(X, D(\delta))$. Let us consider the set $\mathcal{E}$ consisting of all exceptional divisors E' on $\widehat{X}$ with the two conditions: (1) E' can be realized by some toric blow-up of $(X \ni P)$ and (2) $a(E', D(\delta)) = 0$.

Let $\mathcal{E} = \emptyset$. Hence, if $T \in \mathrm{CS}(Z, D(\delta)_Z - a(S, D(\delta))S)$ and T is a curve, then T is a non-toric subvariety of $(S, \mathrm{Diff}_S(0))$. Let us consider the variety $T \in \mathrm{CS}(Z, D(\delta)_Z - a(S, D(\delta))S)$ which is the maximal obstruction to increase a coefficient δ, that is, if put $\Gamma = T$ then we can more increase the coefficient δ as the result of step (i1). If T is a curve then we consider T instead of Γ and repeat the first step (i1) to increase the coefficient δ (for the sake to be definite, we denote the curve T by Γ). If T is a non-toric point lying on some toric orbit, then we are in *Case* II. We have proved that *Case* II is reduced to *Case* I, besides we can assume that we consider the pair $(X, D(\delta))$ for some $\delta > 0$. If T is a point not lying on any toric orbit then we can consider the point T instead of Γ and increase δ as the result of step (i1). If T is a toric point then we can consider the point T instead of Γ and increase δ and repeat the procedure from the beginning with the same notation.

Let $\mathcal{E} \neq \emptyset$. Let us consider the toric divisorial contraction $g_1 \colon Z_1 \to (X \ni P)$ which realizes the set $\mathcal{E}$ exactly. In particular, $K_{Z_1} + D(\delta)_{Z_1} = g_1^*(K_X + D(\delta))$. Let P_1 be a center of E on Z_1. Let us consider locally the pair $(Z_1 \supset P_1, D_1 = D(\delta)_{Z_1})$ instead of $(X \ni P, D(\delta))$. It is step i2). Let us repeat the whole procedure. We obtain a new divisor $D_1(\delta)$ on Z_1. Let $a(S, D_1(\delta)) = 0$. If the center of S on Z_1 is a point then we have the contradiction as above. If the center of S on Z_1 is a closure of one-dimensional toric orbit then we have the similar contradiction, but we must use the results of Sect. 3 (Example 3.6 and Theorem 3.9) to prove $h(\delta) = 0$. Let $a(S, D_1(\delta)) > 0$. The case $\mathcal{E} = \emptyset$ is considered as above (the set $\mathcal{E}$ will be another one). In the case $\mathcal{E} \neq \emptyset$ we obtain a toric divisorial contraction $g_2 \colon Z_2 \to (Z_1 \supset P_1)$, which is constructed similarly to the construction of g_1. After it let us repeat the whole procedure. By construction of partial resolution of (X, dD_X) we obtain some pair $(Z_k, D_k(\delta))$ in a finite numbers of steps such that $a(S, D_k(\delta)) = 0$, and hence we have the contradiction.

Let $(X \ni P)$ be a non-$\mathbb{Q}$–factorial singularity, that is, $(X \ni P) \cong (\{x_1x_2 + x_3x_4 = 0\} \subset (\mathbb{C}^4_{x_1x_2x_3x_4}, 0))$. According to Sect. 5 it is sufficient to prove that the analog of Proposition 6.6 is satisfied for this singularity. Arguing as above in Theorem 6.2, the required statement is reduced to the case of $\mathbb{Q}$-factorial singularities, this concludes the proof. □

Corollary 6.8 *Under the same assumption as in Theorem 6.5 the two following statements are satisfied:*

(1) *[2, 6, 8] if f is a terminal blow-up then f is a toric morphism (see Proposition 2.6);*

(2) *if f is a non-toric morphism then an index of $(X \ni P)$ is equal to* 1. □

References

1. Birkar, C., Cascini, P., Hacon, C., McKernan, J.: Existence of minimal models for varieties of log general type. J. Am. Math. Soc. **23**(2), 405–468 (2010)
2. Corti, A.: Singularities of linear systems and 3-fold birational geometry Explicit birational geometry of 3-folds, Cambridge LMS, vol. 281. pp. 259–312 (2000)
3. Furushima, M.: Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $\mathbb{C}^3$. Nagoya Math. J. **104**, 1–28 (1986)
4. Ishii, I., Prokhorov, Yu.G.: Hypersurface exceptional singularities. Int. J. Math. **12**(6), 661–687 (2001)
5. Katz, S., Morrison, D.R.: Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups. J. Alg. Geom. **1**(3), 449–530
6. Kawakita, M.: Divisorial contractions in dimension three which contract divisors to smooth points. Invent. Math. **145**, 105–119 (2001)
7. Kawamata, Y.: General hyperplane sections of nonsingular flops in dimension 3. Math. Res. Lett. **1**(1), 49–52 (1994)
8. Kawamata, Y.: Divisorial contractions to 3-dimensional terminal quotient singularities Higher-dimensional complex varieties (Trento 1994), de Gruyter, pp. 241–246 1996
9. Kawamata, Y.,Matsuda, K., Matsuki, K.: Introduction to the minimal model program Algebraic Geometry, Sendai. Adv. Stud. Pure Math. **10**, 283–360 (1987)
10. Keel, S., McKernan, J.: Rational curves on quasi-projective surfaces. Mem. AMS **140**(669) (1999)
11. Kollar, J. et al.: Flips and abundance for algebraic threefolds Astérisque, vol. 211 (1992)
12. *Kollár J.* Singularities of pairs. Proc. Symp. Pure Math. **62** Part 1, 221–287 (1997)
13. Kudryavtsev, S.A.: On plt blow-UPS. Math. Notes **69**(6), 814–819 (2001)
14. Kudryavtsev, S.A.: Classification of three-dimensional exceptional log canonical hypersurface singularities II Izvestiya. Mathematics **68**(2), 355–364 (2004)
15. Kudryavtsev, S.A.: Classfication of exceptional log del Pezzo surfaces with $\delta = 1$ Izvestiya. Mathematics **67**(3), 461–497
16. Morrison, D.: The birational geometry of surfaces with rational double points. Math. Ann. **271**, 415–438 (1985)
17. Morrison, D.: Canonical quotient singularities in dimension three. Proc. Am. Math. Soc. Mar. **93**(3), 393–396 (1985)
18. Oda, T.: Convex bodies and algebraic geometry. An Introduction to the Theory of Toric Varieties. Springer (1988)
19. Prokhorov, Y.G.: Blow-ups of canonical singularities Algebra. In: Bahturin, Y. (ed.), Proceedings of the International Conference on the Occasion of the 90th Birthday of A.G. Kurosh, Moscow, Russia, 25–30 May 1998, Walter de Gruyter, Berlin, pp. 301–317 (2000)
20. Prokhorov, Yu.G., Shokurov, V.V.: The first main theorem on complements: from global to local Izvestiya. Mathematics **65**(6), 1169–1196 (2001)
21. Prokhorov, Yu.G.: Classification of Mori contractions: the case of an elliptic curve Izvestiya. Mathematics **65**(1), 75–84 (2001)
22. Prokhorov, Y.G.: Lectures on Complements on Log Surfaces MSJ Memoirs, vol. 10 (2001)
23. Prokhorov, Yu.G.: An application of the canonical bundle formula. Proc. Steklov Inst. Math. **241**, 210–217 (2003)
24. Reid, M.: Decomposition of toric morphisms Arithmetic and Geometry II. In: Artin, M., Tate, J. (eds.), Birkhauser. Progress in Mathematics, vol. 36, pp. 395–418 (1983)
25. Shokurov, V.V.: 3-fold log flips Izvestiya: Mathematics **40**(1), 95–202 (1993)
26. Shokurov, V.V.: Complements on surfaces. J. Math. Sci. **102**(2), 3876–3932 (2000)
27. Varchenko, A.N.: Zeta-function of monodromy and Newton's diagram Invent. Mathematics **37**, 253–262 (1976)

Automorphisms of Hyperkähler Manifolds and Groups Acting on CAT(0) Spaces

Nikon Kurnosov and Egor Yasinsky

Abstract We study groups of biholomorphic and bimeromorphic automorphisms of projective hyperkähler manifolds. Using a geometric action of these groups on some non-positively curved space, we immediately deduce many of their properties, including finite presentation, strong form of Tits' alternative, and some structural results about groups consisting of transformations with infinite order. We also consider some obstructions to being an automorphism group of a hyperkähler manifold.

Keywords Hyperkahler manifolds · CAT(0) spaces

1 Introduction

The purpose of this note is to explain some boundedness properties on biholomorphic and bimeromorphic automorphism groups of hyperkähler manifolds via geometric group theory. Most of these facts should be known to experts, but their proofs (sometimes quite recent) have different nature. Our goal is to put them into the context of groups acting geometrically on CAT(0) spaces, and to explain some new and old statements from that point of view. The advantage of this approach is that a group Γ

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acting on a CAT(0) space <<nicely>> (properly and cocompactly by isometries) automatically has a lot of good properties which are classically known to people doing metric geometry. For example, a consequence of the Švarc-Milnor lemma states that Γ is finitely presented in this case. On the other hand, we would like to restrict the class of *infinite* groups which can appear as automorphism groups of hyperkähler manifolds and, in particular, of K3 surfaces. To the best of our knowledge, there is no general understanding of how complicated such groups can be, at the moment.

Throughout this note, we work over the complex number field $\mathbb{C}$. By a *hyperkähler manifold* we mean a compact simply-connected complex Kähler manifold M having everywhere non-degenerate holomorphic 2-form ω_M such that $H^0(M, \Omega_M^2) = \mathbb{C}\omega_M$. These manifolds play a very important role in classification of compact Kähler manifolds with vanishing first Chern class. The known examples include K3 surfaces, the Hilbert schemes $\mathrm{Hilb}^n(S)$ of 0-dimensional closed subschemes of length n of a K3 surface S, generalized Kummer varieties, i.e. the kernels of the composition $\mathrm{Hilb}^n(T) \to \mathrm{Sym}^n T \xrightarrow{s} T$, where T is a complex torus and s is the sum morphism, and O'Grady's two sporadic examples of dimension 6 and 10.

Let M be a hyperkähler manifold. In present note we are interested in groups of its biholomorphic and bimeromorphic automorphisms, $\mathrm{Aut}(M)$ and $\mathrm{Bir}(M)$ respectively. In [33] K. Oguiso also asked (Question 1.5) if the groups $\mathrm{Bir}(M)$ and $\mathrm{Aut}(M)$ are finitely generated for *projective* hyperkähler manifolds.[1] Using Global Torelli Theorem, S. Boissière and A. Sarti proved that $\mathrm{Bir}(M)$ is finitely generated. This does not imply that $\mathrm{Aut}(M)$ is finitely generated since $\mathrm{Aut}(M)$ is not necessarily of finite index in $\mathrm{Bir}(M)$. The question of finite generation of $\mathrm{Aut}(M)$ remained open until the recent paper of Cattaneo and Fu [18], where the authors were able to give an affirmative answer to Oguiso's question.

Theorem 1.1 ([18, Theorem 1.5]) *Let M be a projective hyperkähler manifold. Then the group* $\mathrm{Aut}(M)$ *is finitely presented.*

Another curious group-theoretic property that was recently investigated in different geometric contexts is a *Tits' alternative*; see e.g. the works of Cantat [16], Oguiso [32] or Arzhantsev-Zaidenberg [6, 7]. Recall that a classical Tits' alternative states that any finitely generated linear algebraic group over a field is either virtually solvable (i.e. has a solvable subgroup of finite index), or contains a non-abelian free group. Following [32] let us say that a group G is *almost abelian of finite rank r* if there are a normal subgroup $G' \lhd G$ of finite index and a finite group K which fit in the exact sequence

$$\mathrm{id} \to K \to G' \to \mathbb{Z}^r \to 0.$$

Then one has the following analogue of Tits' alternative for hyperkähler manifolds:

Theorem 1.2 *([32, Theorem 1.1]) Let M be a projective hyperkähler manifold and G be a subgroup of* $\mathrm{Bir}(M)$. *Then G satisfies either:*

[1] For a non-projective hyperkähler manifold M, the group $\mathrm{Bir}(M)$ (and hence $\mathrm{Aut}(M)$) is almost abelian of rank at most $\max(\rho(M) - 1, 1)$. In particular, these groups are finitely generated [33, Theorem 1.5].

(1) G is an almost abelian group of finite rank, or
(2) G contains a non-abelian free group.

The goal of this note is to show that many finiteness properties of automorphism groups of projective hyperkähler manifolds (including Theorems 1.1 and 1.2) follow from the fact that these groups act geometrically on some metric space of non-positive curvature. These metric spaces are the so-called CAT(0) spaces. Roughly speaking, these are spaces which are at least as non-positively curved as the Euclidean plane. An advantage of this point of view is that it provides a quick and easy way of translating results from pure metric geometry to hyperkähler world.

Our method is not new: its rough sketch can be found already in [44] (in the context of K3 surfaces), and then it was applied in [9] for proving that rational algebraic surfaces with a structure of so-called klt Calabi-Yau pair have finitely many real forms. However, to apply the same strategy to hyperkähler manifolds one needs to have some tools which were developed only recently (e.g. that the Kawamata-Morrison cone conjecture holds for hyperkähler manifolds). So, in this note we tried to give a reasonably self-contained account of the corresponding construction. Our first main result is the following:

Theorem A *Let M be a projective hyperkähler manifold. Then the groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Bir}(M)$ *are CAT(0) groups. In particular, they are finitely presented.*

To the best of our knowledge, nowadays there is no general understanding of how complicated automorphism groups of hyperkähler manifolds can be, even for K3 surfaces (although lots of interesting examples are known). For instance, in [44] Totaro provides an example of K3 surface whose automorphism group is not even commensurable with an arithmetic group. On the other hand, in recent years there has been numerous results which *prevent* some groups from acting geometrically on CAT(0) spaces. Thus, Theorems A can be used to obtain some restrictions on possible automorphism groups of hyperkähler manifolds. We give some explicit examples in Sect. 4.5.

Further, from Theorem A we easily deduce the following reinforcement of Tits' alternative for hyperkähler manifolds:

Theorem B *Let M be a projective hyperkähler manifold, and $G \subseteq \mathrm{Bir}(M)$ be a subgroup. Then*

(1) either G contains a finite index subgroup isomorphic to $\mathbb{Z}^n$;
(2) or G contains a non-commutative free group.

Another immediate application of Theorem A is

Theorem C (cf. Theorem 7.1 in *[18]*) *Let M be a projective hyperkähler manifold. Then the groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Bir}(M)$ *have finitely many conjugacy classes of finite subgroups. In particular, there exists a constant* $\mathrm{B} = \mathrm{B}(M)$ *such that* $|G| \leqslant \mathrm{B}$ *for any finite subgroup* $G \subset \mathrm{Bir}(M)$.

Remark 1.3 A reader familiar with some basic properties of hyperkähler manifolds might have an impression that Theorem C easily follows from the fact that both $\mathrm{Aut}(M)$ and $\mathrm{Bir}(M)$ admit a natural representation in $\mathrm{GL}(\mathrm{NS}(M))$ with a finite kernel, and the groups $\mathrm{GL}_n(\mathbb{Z})$ are known to have only finitely many conjugacy classes of finite subgroups. However, finitely generated subgroups of $\mathrm{GL}_n(\mathbb{Z})$ may violate the latter property. Indeed, in [22] Grunewald and Platonov give an example of a finitely generated subgroup of $\mathrm{SL}_4(\mathbb{Z})$ that contains infinitely many conjugacy classes of elements of order 4.

Remark 1.4 Let $\mathscr{G}$ be a family of groups. Following the terminology introduced in [35, 36] we say that $\mathscr{G}$ is *uniformly Jordan* (resp. has *uniformly bounded finite subgroups*) if there is a constant $\mathrm{J} = \mathrm{J}(\mathscr{G})$ (resp. $\mathrm{B} = \mathrm{B}(\mathscr{G})$) such that for any group $\Gamma \in \mathscr{G}$ and any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J (resp. $|G| \leqslant \mathrm{B}$). We say that Γ is *Jordan* (resp. has *bounded finite subgroups*) if the family $\{\Gamma\}$ is uniformly Jordan (resp. bounded). In view of Theorem C and Remark 4.6 it is natural then to ask if the following group-theoretic analog of Beauville's finiteness conjecture is true:

Question 1.5 *Consider the family*

$$\mathscr{G}_n = \left\{ \mathrm{Bir}(M) : \ M \textit{ is a projective hyperkähler manifold of dimension } 2n \right\}.$$

Does the familly $\mathscr{G}_n$ have uniformly bounded finite subgroups (with a constant $\mathrm{B} = \mathrm{B}(n)$ depending only on n)? Is it at least uniformly Jordan with $\mathrm{J} = \mathrm{J}(n)$? Same questions for $\mathrm{Aut}(M)$.

In some particular cases one can hope to obtain such bounds using results of [23, 27, 39].

2 Preliminaries

2.1 Hyperkähler Manifolds

By a *hyperkähler* (or *irreducible holomorphic symplectic*) manifold we mean a compact simply-connected complex Kähler manifold M having everywhere non-degenerate holomorphic 2-form ω_M such that $H^0(M, \Omega_M^2) = \mathbb{C}\omega_M$. These manifolds are even dimensional Calabi-Yau manifolds and play a very important role in classification of Kähler manifolds with trivial Chern class. Namely, Beauville–Bogomolov decomposition theorem [11] states that for any compact Kähler manifold with trivial Chern class there exists a finite étale cover $\widetilde{M} \to M$ such that

$$\widetilde{M} \cong T \times \prod_{i=1}^{n} Y_i \times \prod_{j=1}^{m} Z_j,$$

where T is a complex torus, Y_i are strict Calabi–Yau manifolds (with $\pi_1(Y_i) = 0$, $K_{Y_i} = \mathscr{O}_{Y_i}$ and $h^{0,p} = 0$ for $0 < p < \dim Y_i$), and Z_j are hyperkähler manifolds.

One of the most important properties of a hyperkähler manifold is the existence of Beauville–Bogomolov–Fujiki form (BBF-form for short) [8, 19]. This is an integral symmetric bilinear form q_{BBF} on $H^2(M, \mathbb{Z})$ of signature $(3, 0, b_2(M) - 3)$. By means of BBF-form, the signature of the Neron-Severi group $\mathrm{NS}(M)$ is one of the following:

$$(1, 0, \rho(M) - 1), \quad (0, 1, \rho(M) - 1), \quad (0, 0, \rho(M)).$$

We call these three cases hyperbolic, parabolic, and elliptic respectively. Due to a deep result of Huybrechts, M is projective if and only if $\mathrm{NS}(M)$ is hyperbolic [25], [21, Proposition 26.13].

Let $f : M \dashrightarrow M'$ be a bimeromorphic map between Calabi–Yau manifolds. By [21, III.25.14] this map f is an isomorphism in codimension 1 and induces a linear isomorphism

$$f^* : H^2(M', \mathbb{Z}) \xrightarrow{\sim} H^2(M, \mathbb{Z})$$

which in the case of hyperkähler manifolds preserves the BBF-form. In particular, there is a group homomorphism

$$\Phi_{\mathrm{NS}} : \mathrm{Bir}(M) \to \mathrm{O}(\mathrm{NS}(M) \otimes \mathbb{R}, q_{BBF}).$$

Put

$$\mathrm{Bir}^*(M) = \Phi_{\mathrm{NS}}(\mathrm{Bir}(M)), \quad \mathrm{Aut}^*(M) = \Phi_{\mathrm{NS}}(\mathrm{Aut}(M)).$$

We have the following important fact.

Proposition 2.1 ([34, Proposition 2.4]) *Let M be a projective Calabi–Yau manifold. Then the kernel of a homomorphism*

$$\Phi_{\mathrm{NS}} : \mathrm{Bir}(M) \to \mathrm{GL}(\mathrm{NS}(M))$$

is a finite group.

2.2 The Kawamata–Morrison Conjecture

In this note we shall consider various cones (i.e. subsets stable under multiplication by $\mathbb{R}_{>0}$) in the finite-dimensional vector space $\mathrm{NS}(M)_{\mathbb{R}} = \mathrm{NS}(M) \otimes \mathbb{R}$ equipped with $\mathbb{Z}$-structure given by $\mathrm{NS}(M) = H^{1,1}(M, \mathbb{R}) \cap H^2(M, \mathbb{Z})$.

Let M be a compact, Kähler manifold. In what follows $\mathrm{Kah}(M) \subset H^{1,1}(M, \mathbb{R})$ will denote the open convex *Kähler cone* of M. Its closure $\mathrm{Nef}(M) = \overline{\mathrm{Kah}(M)}$ in $H^{1,1}(M, \mathbb{R})$ is called the *nef cone*. Further, $\mathrm{Amp}(M)$ will denote the *ample cone* of M. For some varieties these cones have a nice structure, e.g. for Fano varieties they are rational polyhedral. However in general they can be quite mysterious: they can have infinitely many isolated extremal rays or <<round>> parts. Both phenomena occur already for K3 surfaces. The Kawamata–Morrison cone conjecture predicts that for Calabi–Yau varieties the structure of these cones (or rather some closely related cones) is nice "up to the action of the automorphism group". Before stating a suitable version of this conjecture, we need some definitions.

Let V be a finite-dimensional real vector space equipped with a fixed $\mathbb{Q}$-structure. A *rational polyhedral cone* in V is a cone, which is an intersection of finitely many half spaces defined over $\mathbb{Q}$. In particular, such a cone is convex and has finitely many faces. For an open convex cone $\mathscr{C} \subset V$ we denote by $\mathscr{C}^+$ the convex hull of $\overline{\mathscr{C}} \cap V(\mathbb{Q})$.

Let Γ be a group acting on a topological space X. A *fundamental domain* for the action of Γ is a connected open subset $D \subset X$ such that

$$\bigcup_{\gamma \in \Gamma} \gamma \cdot \overline{D} = X,$$

and the sets $\gamma \cdot D$ are pairwise disjoint. Let X be a subset of a metric space Y (typically Y will be either Euclidean or hyperbolic n-space). A *side* of a convex subset $C \subset Y$ is a maximal nonempty convex subset of ∂C. A *polyhedron* in Y is a nonempty closed convex subset whose collection of sides is locally finite. A *fundamental polyhedron* for the action of a discrete isometry group Γ on X is a convex polyhedron D whose interior is a locally finite fundamental domain for Γ. Local finiteness means that for each point $x \in X$ there is an open neighborhood U of x such that U meets only finitely many sets $\gamma \overline{D}$, $\gamma \in \Gamma$. Obviously, this also implies that every compact subset $K \subset X$ intersects only finitely many sets $\gamma \overline{D}$.

One of the versions of the Kawamata–Morrison cone conjecture says that the action of the automorphism group of a Calabi–Yau variety on the cone $\mathrm{Amp}(M)^+$ has a rational polyhedral fundamental domain. There is also a birational version for $\mathrm{Bir}(M)$ and $\mathrm{Mov}(M)^+$ respectively, where $\mathrm{Mov}(M)$ denotes the *movable cone*, i.e. the convex hull in $\mathrm{NS}(M)_{\mathbb{R}}$ of all classes of movable line bundles on M. The conjecture has been proved for K3 surfaces by Sterk and Namikawa [31, 41] using the Torelli theorem of Piatetski–Shapiro and Shafarevich, and generalized later on 2-dimensional Calabi–Yau pairs by Totaro [42]. For projective hyperkähler manifolds the following versions of the Kawamata–Morrison conjectures were recently proved by E. Markman, E. Amerik and M. Verbitsky:

Theorem 2.2 *Let M be a projective simple hyperkähler manifold. Then*

(1) [4, Theorem 5.6] The group $\mathrm{Aut}(M)$ *has a finite polyhedral fundamental domain on* $\mathrm{Amp}(M)^+$.
(2) [30, Theorem 6.25] The group $\mathrm{Bir}(M)$ *has a rational polyhedral fundamental domain on* $\mathrm{Mov}(M)^+$.

Remark 2.3 For the reader who would like to follow Ratcliffe's [37] exposition of geometrically finite groups, while reading Sect. 3.3, it may be useful to keep in mind a somewhat more explicit construction of fundamental polyhedrons in Theorem 2.2. This is due to E. Looijenga [28, Proposition 4.1 and Application 4.14]. Let C be a non-degenerate open convex cone in a finite dimensional real vector space V equipped with a fixed $\mathbb{Q}$-structure. Let Γ be a subgroup of $\mathrm{GL}(V)$ which stabilizes C and some lattice in $V(\mathbb{Q})$. Assume that there exists a polyhedral cone Π in C^+ such that $\Gamma \cdot \Pi \supseteq C$ and there is an element $\xi \in C^\circ \cap V^*(\mathbb{Q})$ whose stabilizer Γ_ξ is trivial[2] (here C° denotes the open dual cone of C). Then Γ admits a rational polyhedral fundamental domain Σ on C^+. Moreover, as was noticed before [42, Theorem 3.1] (and proved in [43, Lemma 2.2]) Looijenga's fundamental domain coincides with a *Dirichlet domain* of Γ when the representation preserves a bilinear form of signature $(1, *)$. Recall that for a discontinuous group Γ of isometries of a metric space (X, d) and a point $\xi \in X$ with a trivial stabilizer Γ_ξ one defines the Dirichlet domain for Γ as the set

$$D_\xi(\Gamma) = \left\{x \in X :\ d(x, \xi) \leqslant d(x, g\xi) \text{ for all } g \in \Gamma\right\}.$$

Dirichlet polyhedrons are known to have many good properties, in particular they are locally finite in the interior of the positive cone [43, Corollary 2.3]) and exact [37, Theorem 6.6.2] (meaning that for each side S of $D = D_\xi(\Gamma)$ there is an element $\gamma \in \Gamma$ such that $S = D \cap \gamma D$).

Theorem 2.2 (1) has been initially proved in an assumption $b_2 \neq 5$. Below we sketch a proof for the case $b_2 = 5$, which follows from the results of Amerik and Verbitsky (see also [5, Remark 1.5]).

Proposition 2.4 *Let M be a projective hyperkähler manifold with $b_2 = 5$. The automorphism group has a rational polyhedral fundamental domain on the ample cone of M.*

Recall that the *mapping class group* is the group $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$, where $\mathrm{Diff}_0(M)$ is a connected component of diffeomorphism group of M (the group of isotopies). Consider the subgroup of the mapping class group which fixes the connected component of our chosen complex structure. The *monodromy group* is the image of this subgroup in $\mathrm{O}(H^2(M, \mathbb{Z}))$.

Denote by Hyp an infinite-dimensional space of all quaternionic triples I, J, K on M which are induced by some hyperkähler structure, with the same C^∞-topology of convergence with all derivatives. Identify $\mathrm{Hyp}_m = \mathrm{Hyp}/\mathrm{SU}(2)$ with the space of all hyperkähler metrics of fixed volume. Define the *Teichmüller space* Teich_h of hyperähler structures as the quotient $\mathrm{Hyp}_m/\mathrm{Diff}_0$. Define the period space of hyperkähler structures by the space $\mathrm{Per}_h = \mathrm{Gr}_{+++}(H^2(M, \mathbb{R}))$ of all positive oriented 3-dimensional subspaces in $H^2(M, \mathbb{R})$.

Remark 2.5 The period space Per_h is naturally diffeomorphic to $\mathrm{SO}(b_2 - 3, 3)/\mathrm{SO}(3) \times \mathrm{SO}(b_2 - 3)$. The map $\mathscr{P}er_h : \mathrm{Teich}_h \to \mathrm{Per}_h$ is the period

[2] In our situation ξ exists automatically even without assuming that there is a fundamental domain for Γ, see e.g. [18, Proposition 6.6].

map associating the 3-dimensional space generated by the three Kähler forms $\omega_I, \omega_J, \omega_K$ to a hyperkähler structure (M, I, J, K, g). This map by [3, Theorem 4.9] is an open embedding for each connected component. Moreover, its image is the set of all spaces $W \in \mathrm{Per}_h$ such that the orthogonal complement $W^\perp$ contains no MBM classes (see below).

A non-zero negative rational homology class (1, 1)-class z is called *monodromy birationally minimal (MBM)* if for some isometry $\gamma \in O(H^2(M, \mathbb{Z}))$ belonging to the monodromy group, $\gamma(z)^\perp \subset H^{1,1}(M)$ contains a face of the pull-back of the Kähler cone of one of birational models M' of M.

Proof of Proposition 2.4 For each primitive MBM class r, denote by S_r the set of all 3-planes $W \in \mathrm{Gr}_{+++}$ orthogonal to r. Consider the union $\cup_r S_r$ of this sets. Its complement in Gr_{+++} is identified to a connected component of the Teichmüller space by [3, Theorem 4.9]. So it is open. From the [5, Theorem 1.7] for $X = G/K$, where $G = \mathrm{SO}(3, 2)$ and $K = \mathrm{SO}(3) \times \mathrm{SO}(2)$[3] it follows that monodromy group acts on the set of MBM classes with finite number of orbits. Recall that the monodromy acts by isometries, thus the square of a primitive MBM class in respect with the Beauvile-Bogomolov-Fujiki form on M is bounded in a absolute value. This is key assumption in Amerik-Verbitsky's proof of Kawamata-Morrison cone conjecture. Indeed, the [2, Theorem 6.6] implies the finitness of orbits for the Kähler cone.

Consider the quotient $S = (\mathrm{Pos}(M) \cap \mathrm{NS}(M) \otimes \mathbb{R})/\Gamma$, where $\mathrm{Pos}(M)$ is positive cone and Γ is the Hodge monodromy group. Then by Borel and Harish-Chandra theorem S is a complete hyperbolic manifold of finite volume. Since $\mathrm{Aut}(M)$ acts with finite number of orbits on $\mathrm{Kah}(M)$, then the image of $\mathrm{Amp}(M)$ in S is a hyperbolic manifold T with finite boundary. One can prove that T admits decomposition by finitely many cells with finite piecewise geodesic boundary. Finite polyhedral fundamental domain on the ample cone of M is obtained by suitable liftings of this cells. We refer the reader to [4, Theorem 5.6] for the further details. □

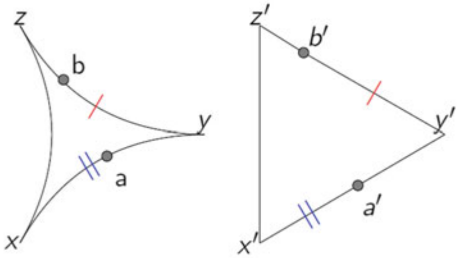
3 The CAT(0) Space

Let (X, dist) be a metric space. Recall that a geodesic segment $[x, y]$ joining two points $x, y \in X$ is the image of a path of length $\mathrm{dist}(x, y)$ joining x and y. A metric space is said to be geodesic if every two poins in X can be joined by a geodesic. A *geodesic triangle* in X consists of three points $x, y, z \in X$ and a choice of geodesic segments $[x, y]$, $[y, z]$ and $[x, z]$.

A geodesic metric space (X, dist) is said to be a *CAT(0) space* if for every geodesic triangle $\Delta \subset X$ there exists a triangle $\Delta' \subset \mathbb{E}^n$ (here and throughout the paper $\mathbb{E}^n$ denotes the Euclidean n-space with a standard metric) with sides of the same length as the sides of Δ, such that distances between points on Δ are less or equal to the

[3] In the general case $G = \mathrm{SO}(3, b_2 - 3)$ and $K = \mathrm{SO}(3) \times \mathrm{SO}(b_2 - 3)$.

distances between corresponding points on Δ'. Informally speaking, this means that geodesic triangles in X are <<not thicker>> than Euclidean ones.



Definition 3.1 (*CAT(0) groups*) Let Γ be a group acting by isometries on a metric space X. This action is *proper* or *properly discontinous* if for each $x \in X$ there exists $r > 0$ such that the set of $\gamma \in \Gamma$ with

$$\gamma \cdot B(x, r) \cap B(x, r) \neq \varnothing$$

is finite (here and throughout the paper $B(x, r)$ denotes an open ball with center x and radius r). The action is *cocompact* is there exists a compact set $K \subset X$ such that $X = \Gamma \cdot K$. The action is called *geometric* if it is proper and cocompact. Finally, we say that Γ is a *CAT(0) group* if it acts geometrically on a CAT(0) space.

To prove Theorem A we shall construct a CAT(0) space where the groups $\mathrm{Bir}(M)$ and $\mathrm{Aut}(M)$ act properly and cocompactly by isometries. As was mentioned in Introduction, here we basically summarize the ideas sketched in [9, 44], but try to make our exposition accessible for a non-expert in metric geometry. The main reference where the reader can find most technical facts used here is [1, 37]. Our construction will involve a hyperbolic space, so we first recall some basic definitions.

3.1 Hyperbolic Space and its Isometries

A hyperbolic n-dimensional space is an n-dimensional Riemannian simply connected space of constant negative curvature. Throughout this note we use several models of hyperbolic space, which we briefly describe below to establish notation.

3.1.1 Standard Models

Let $\mathscr{V}$ be a Minkowski vector space of dimension $n + 1$ with the quadratic form $\mathfrak{q} : \mathscr{V} \to \mathbb{R}$ of signature $(1, n)$ and inner product of two vectors v_1, v_2 denoted by $\langle v_1, v_2\rangle$. We will choose the coordinates $x_0, \dots, x_n$ in $\mathscr{V}$ such that $\mathfrak{q} = x_0^2 - x_1^2 - \cdots - x_n^2$. The vectors $v \in V$ with $\mathfrak{q}(v) = 1$ form an n-dimensional hyperboloid consisting of two connected components: $H^+ = \{x_0 > 0\}$, and $H^- = \{x_0 < 0\}$. The points of the *hyperboloid model* $\mathbb{H}^n$ are the points on H^+. The distance function is given by $\mathrm{dist}(u, v) = \operatorname{argcosh}\langle u, v\rangle$. The m-planes are represented by the intersections of the $(m+1)$-planes in $\mathscr{V}$ with H^+. The *Poincaré model* of $\mathbb{H}^n$ has its points lying inside the unit open disk

$$\mathbb{B}^n = \{(0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_n^2 < 1\}$$

and is obtained from the hyperboloid model by means of the stereographic projection ζ from the south pole of the unit sphere in $\mathscr{V}$ (i.e. the point $(-1, 0, \dots, 0)$) on the hyperplane $V_0 = \{x_0 = 0\}$. A subset $P \subset \mathbb{B}^n$ is called a hyperbolic m-plane if and only if $\zeta(P)$ is a hyperbolic m-plane of $\mathbb{H}^n$.

Denote by $\mathbb{E}^n$ the Euclidean n-space with standard Euclidean metric. By $\widehat{\mathbb{E}}^n = \mathbb{E}^n \cup \{\infty\}$ we denote its one-point compactification (e.g. $\widehat{\mathbb{C}}$ is the Riemann sphere). If a is a unit vector in $\mathbb{E}^n$ and $r \in \mathbb{R}$, then $P(a, r)$ is the hyperplane with unit normal vector a passing through the point ra. Further, $S(a, r)$ denotes the sphere of radius r centered at a. We shall also consider extended planes $\widehat{P}(a, r) = P(a, r) \cup \{\infty\}$. By a *sphere* in $\widehat{\mathbb{E}}^n$ we mean either a Euclidean sphere or an extended plane (so, topologically a sphere too). A p-sphere and a q-sphere of $\widehat{\mathbb{E}}^n$ are said to be orthogonal if they intersect and at each finite point of intersection their tangent planes are orthogonal. One can show that $P \subset \mathbb{B}^n$ is a hyperbolic m-plane of $\mathbb{B}^n$ if and only if P is the intersection of $\mathbb{B}^n$ either with an m-dimensional vector subspace of $V_0 = \mathbb{E}^n$, or an m-sphere of V_0 orthogonal to $\partial\overline{\mathbb{B}}^n$.

In the Poincare ball model $\mathbb{B}^n$, a *horoball* based at $a \in \partial\mathbb{B}^n$ is an Euclidean ball contained in $\overline{\mathbb{B}}^n$ which is tangent to $\partial\mathbb{B}^n$ at the point a. Assume Γ contains a parabolic element having $a \in \partial\mathbb{B}^n$ as its fixed point. A *horocusp region* is an open horoball B based at a point $a \in \partial\mathbb{B}^n$ such that for all $\gamma \in \Gamma \setminus \mathrm{Stab}_\Gamma(a)$ one has $\gamma(B) \cap B = \varnothing$.

Finally, we mention the *upper half-space model*

$$\mathbb{U}^n = \{(x_1, \dots, x_n) \in \mathbb{E}^n : x_n > 0\}$$

with a metric induced from $\mathbb{B}^n$ in the following way. Let σ be the reflection of $\widehat{\mathbb{E}}^n$ in the sphere $S(e_n, \sqrt{2})$ and ρ be the reflection of $\widehat{\mathbb{E}}^n$ in $\widehat{\mathbb{E}}^{n-1}$. Then $\eta = \sigma \circ \rho$ maps homeomorphically $\mathbb{U}^n$ to $\mathbb{B}^n$. Put $\mathrm{dist}_{\mathbb{U}^n}(u, v) = \mathrm{dist}_{\mathbb{B}^n}(\eta(u), \eta(v))$. A subset $P \subset \mathbb{U}^n$ is called a hyperbolic m-plane if and only if $\eta(P)$ is a hyperbolic m-plane of $\mathbb{B}^n$. One can show that $P \subset \mathbb{U}^n$ is a hyperbolic m-plane of $\mathbb{U}^n$ if and only if P is the intersection of $\mathbb{U}^n$ either with an m-plane of $\mathbb{E}^n$ orthogonal to $\mathbb{E}^{n-1}$, or an m-sphere of $\mathbb{E}^n$ orthogonal to $\mathbb{E}^{n-1}$.

Recall that a *geodesic line* (or just *geodesic*) in a Riemannian manifold M is a continuous map $\gamma : \mathbb{R} \to M$ such that $\mathrm{dist}_M(\gamma(x), \gamma(y)) = |x - y|$. We also refer to the image of γ as a geodesic line. For any two distinct points $x, y \in M$ there exists a closed interval $[a; b] \subset \mathbb{R}$ and a geodesic γ with $\gamma(a) = x$, $\gamma(b) = y$, which is called the *geodesic segment*. In all described models of hyperbolic space, its geodesics are just hyperbolic lines, i.e. 1-planes.

3.1.2 Isometries

A *Möbius transformation* of $\widehat{\mathbb{E}}^n$ is a finite composition of reflections of $\widehat{\mathbb{E}}^n$ in spheres. Consider $\mathbb{E}^{n-1} \equiv \mathbb{E}^{n-1} \times \{0\} \subset \mathbb{E}^n$. Any $f \in \text{Möb}(\widehat{\mathbb{E}}^{n-1})$ can be extended to an element of $\text{Möb}(\widehat{\mathbb{E}}^n)$ as follows. If f is a reflection of $\widehat{\mathbb{E}}^{n-1}$ in $\widehat{P}(a,r)$ then $\widetilde{f}$ is the reflection of $\widehat{\mathbb{E}}^n$ in $\widehat{P}(\widetilde{a},r)$ where $\widetilde{a} = (a,0)$. If f is a reflection of $\widehat{\mathbb{E}}^{n-1}$ in $S(a,r)$ then $\widetilde{f}$ is the reflection of $\widehat{\mathbb{E}}^n$ in $S(\widetilde{a},r)$. The *Poincaré extension* of an arbitrary $f = f_1 \circ \ldots \circ f_m \in \text{Möb}(\widehat{\mathbb{E}}^{n-1})$ is then defined as $\widetilde{f} = \widetilde{f_1} \circ \ldots \circ \widetilde{f_m}$.

If $Y = \mathbb{U}^n$ or $\mathbb{B}^n$, a *Möbius transformation* $f \in \text{Möb}(Y)$ is a Möbius transformation of $\widehat{\mathbb{E}}^n$ that leaves Y invariant. The element $f \in \text{Möb}(\mathbb{U}^n)$ is a Möbius transformation if and only if it is the Poincaré extension of an element of $\text{Möb}(\widehat{\mathbb{E}}^{n-1})$, so $\text{Möb}(\mathbb{U}^n) \cong \text{Möb}(\widehat{\mathbb{E}}^{n-1})$ [37, Sect. 4.4]. Similar statement holds for $\text{Möb}(\mathbb{B}^n)$. Every Möbius transformation of $\mathbb{B}^n$ restricts to an isometry of the conformal ball model $\mathbb{B}^n$, and every isometry of $\mathbb{B}^n$ extends to a unique Möbius transformation of $\mathbb{B}^n$ [37, Theorem 4.5.2]. In particular, $\text{Isom}(\mathbb{B}^n) \cong \text{Möb}(\mathbb{B}^n)$.

Let $f \in \text{Möb}(\mathbb{B}^n)$ be a Möbius transformation (an isometry of the hyperbolic n-space). Then f maps $\overline{\mathbb{B}}^n$ into itself and by the Brouwer fixed point theorem, f has a fixed point in $\overline{\mathbb{B}}^n$. Recall that f is said to be *elliptic* if f fixes a point of $\mathbb{B}^n$; *parabolic* if f fixes no point of $\mathbb{B}^n$ and fixes a unique point of $\partial\overline{\mathbb{B}}^n = \mathbb{S}^{n-1}$; *loxodromic* if f fixes no point of $\mathbb{B}^n$ and fixes two points of $\mathbb{S}^{n-1}$, say a and b. A hyperbolic line L joining a and b is called the *axis* of f, and f acts as a translation along L. If f translates L in the direction of a, then for any $x \in \overline{\mathbb{B}}^n$, $x \neq b$, one has $f^m(x) \to a$ as $m \to \infty$, i.e. a is an *attractive fixed point* (and b is *repulsive*).

3.2 *Construction of a CAT(0) Space*

We are ready to explain the main technical result.

Theorem 3.2 *Let $\mathscr{V}$ be a vector space of dimension $n+1$, $n \geqslant 2$, with a fixed $\mathbb{Z}$-structure $\Lambda \cong \mathbb{Z}^{n+1}$, $\Lambda \otimes \mathbb{R} = \mathscr{V}$. Assume there is a quadratic form $\mathfrak{q} : \mathscr{V} \to \mathbb{R}$ of signature $(1,n)$, a convex cone $\mathscr{C}$ in $\mathscr{V}$ and a group action $\Phi : \Gamma \to \mathrm{GL}(\Lambda) \subset \mathrm{GL}(\mathscr{V})$ with discrete image $\Phi(\Gamma)$ and finite kernel, such that Γ preserves $\mathscr{C}$, $\mathfrak{q}$, and has a rational polyhedral (locally finite) fundamental domain Ω on $\mathscr{C}$. Then Γ is a CAT(0) group.*

Proof Let $\mathbb{H}^n$ be the hyperbolic space associated with $(V, \mathfrak{q})$ and set $\mathrm{D} = \mathrm{pr}\left(\mathscr{C} \cap \mathbb{H}^n\right)$, where $\mathrm{pr} : \mathscr{V} \to \mathscr{V}$ is the projection from the origin (inducing an isometry $\mathbb{H}^n \to \mathbb{B}^n$). Then D is a convex subset of $\mathbb{B}^n$. The group Γ acts on D with a fundamental domain Π_{D}, which moreover has finitely many sides. We are going to show that D can be "improved" so that Γ acts properly and cocompactly on the resulting CAT(0) space. Recall that a point $a \in \partial\overline{\mathbb{B}}^n = \mathbb{S}^{n-1}$ is a *limit point* of a subgroup $\Gamma \subset \text{Möb}(\mathbb{B}^n)$ if there is a point $b \in \mathbb{B}^n$ and a sequence $\{\gamma_i \in \Gamma\}_{i=1}^{\infty}$ such that $\{\gamma_i b\}_{i=1}^{\infty}$ converges to a. Let $C(\Gamma)$ denote the convex hull of the set of limit points of Γ on $\overline{\mathbb{B}}^n$. Note that this is a closed subset of $\overline{\mathbb{B}}^n$ [37, Sect. 12.1]. Put

$$X = \mathrm{D} \cap C(\Gamma), \quad \Pi = \Pi_{\mathrm{D}} \cap C(\Gamma).$$

Step 1: There exists a finite family U of horocusp regions with disjoint closures such that $\Pi \backslash U$ is compact. See paragraph Sect. 3.3 for details.

Step 2: Put $U' = \bigcup_{\gamma \in \Gamma} \gamma(U)$. Step 1 shows that Γ acts cocompactly on $X \backslash U'$.

Step 3: Besides, Γ acts properly discontinuously on $X \backslash U'$. Since we assume that the kernel of the induced homomorphism $\Phi : \Gamma \to \text{Isom}(\mathbb{B}^n)$ is finite, it suffices to show that $\Phi(\Gamma)$ acts properly on $\mathbb{B}^n$. By [47, Lemma 3.1.1] if H and K are subgroups of a group G with K compact and G locally compact, then H is properly discountinous on G/K if and only if H is discrete in G. Now take $G = \text{Isom}(\mathbb{H}^n) \cong \mathrm{O}^+(1, n)$, $H = \Phi(\Gamma)$, and $K = \text{Stab}(x) \cong \mathrm{O}_n(\mathbb{R})$, where $x \in \mathbb{H}^n$. Notice that $\mathrm{O}^+(1, n)$ is transitive on $\mathbb{H}^n$ and $\mathbb{H}^n \cong \mathrm{O}^+(1, n)/\mathrm{O}_n(\mathbb{R})$, see [12, I.2.24].

Step 4: The radii of the horoballs of U' can be decreased such that we obtain a new collection W of open horoballs with disjoint closures and $X \backslash W$ is a CAT(0) space. This is explained in paragraph Sect. 3.4.

Step 5: The action of Γ on $X \backslash W$ clearly remains properly discountinuous. It also remains cocompact by Remark 3.4. This completes the proof.

□

3.3 Explanation of Step 1

A group $G \subset \text{Möb}(\mathbb{U}^n)$ is called *elementary* if G has a finite orbit in $\widehat{\mathbb{E}}^n$. An elementary group G is said to be of *parabolic type* if G fixes a point on $\widehat{\mathbb{E}}^n$ and has no other finite orbits. Let $\Gamma \subset \text{Möb}(\mathbb{U}^n)$ be a discrete subgroup such that ∞ is fixed by a parabolic element of Γ. Then $\Gamma_\infty = \text{Stab}_\Gamma(\infty)$ is an elementary group of parabolic type. Thus Γ_∞ corresponds under Poincarè extension (see above) to a discrete subgroup of $\text{Isom}(\mathbb{E}^{n-1})$ [37, Theorem 5.5.5].

By the Bieberbach theorem [37, Theorem 5.4.6, 7.4.2] there is a Γ_∞-invariant m-plane Q of $\mathbb{E}^{n-1}$ with Q/Γ_∞ compact. Denote by $N(Q, \varepsilon)$ the ε-neighborhood of Q in $\mathbb{E}^n$. Then $N(Q, \varepsilon)$ is Γ_∞-invariant. Set

$$U(Q, \varepsilon) = \overline{\mathbb{U}}^n \backslash \overline{N}(Q, \varepsilon).$$

This is an open Γ_∞-invariant subset of $\overline{\mathbb{U}}^n$. It is called a *cusped region* for Γ based at ∞ if for all $\gamma \in \Gamma \backslash \Gamma_\infty$ we have

$$U(Q, \varepsilon) \cap \gamma U(Q, \varepsilon) = \varnothing. \tag{1}$$

Viewed in $\mathbb{B}^n$ model when $m = n-1$, the sets $U(Q, \varepsilon)$ are just horocusp regions based at ∞, as defined in Sect. 3.1. Let $c \in \widehat{\mathbb{E}}^{n-1}$ be a point fixed by a parabolic element of a discrete subgroup $\Gamma \subset \text{Möb}(\mathbb{U}^n)$. A subset $U \subset \overline{\mathbb{U}}^n$ is a *cusped region* for Γ based at c if upon conjugating Γ so that $c = \infty$, the set U transforms to a cusped region for Γ based at ∞. A *cusped limit point* of Γ is a fixed point c of a parabolic element of Γ such that there is a cusped region U for Γ based at c. Recall that by $L(\Gamma)$ we denoted the set of limit points of Γ in $\overline{\mathbb{B}}^n$, and by $C(\Gamma)$ the convex hull of $L(\Gamma)$ in $\overline{\mathbb{B}}^n$. Now Step 1 is the content of the following claim.[4]

Proposition 3.3 ([37, Theorem 12.4.5]) *Let* $\Gamma \subset \text{Isom}(\mathbb{B}^n)$ *be a discrete subgroup, and* Z *be a closed* Γ*-invariant convex subset of* $\mathbb{B}^n$*. Assume that the action of* Γ *on* Z *has a finitely sided (locally finite) polyhedral fundamental domain* Π*. Then there exists a finite union* V *of horocusp regions with disjoint closures such that* $(\Pi \cap C(\Gamma))\backslash V$ *is compact in* Z*.*

Proof This is essentially the content of [37, Theorem 12.4.5]. Since there is minor difference in the setting, we outline the proof for the reader's convenience. Let $\overline{\Pi}$ denote the closure of fundamental polyhedron Π in $\overline{\mathbb{B}}^n$.

By [1, Lemma 4.10] or [37, Sect. 12, Corollary 3] the set $P = \overline{\Pi} \cap L(\Gamma)$ is a finite set of cusped limit points of Γ. Let $\overline{\Pi} \cap L(\Gamma)$ consist of cusped limit points $c_1, \dots, c_m$. Choose a proper (i.e. non-maximal) cusped region U_i for Γ based at c_i for each i such that $\overline{U}_1, \dots, \overline{U}_m$ are disjoint and $\overline{U}_i$ meets just the sides of Π incident with c_i. Further, let B_i be a horoball based at c_i and contained in U_i such that if $gc_i = c_j$ then $gB_i = B_j$. Then B_i is a proper horocusped region for Γ based at c_i. Set

$$K = (\Pi \cap C(\Gamma)) \setminus \bigcup_i B_i.$$

As $C(\Gamma)$ is closed in $\overline{\mathbb{B}}^n$ [37, §12.1] (and Π is closed in $\mathbb{B}^n$ by definition), the set K is closed in $\mathbb{B}^n$. Now exactly the same argument as in [37, Theorem 12.4.5] shows that K is also bounded. □

Remark 3.4 Note that after finding horoballs B_i corresponding to cusped regions U_i, we can further shrink them if needed, and the proof of the compactness of $(\Pi \cap C(\Gamma)\backslash V)$ still remains valid.

3.4 *Explanation of Step 4 (cf. [9, Lemma 2.10])*

Let $U = B_1 \sqcup \ldots \sqcup B_N$, where B_i are horocusp regions with disjoint closures, constructed in Step 1. By the definition of a horocusp region, for each i the set $\cup_{\gamma \in \Gamma} \gamma(B_i)$ consists of pairwise disjoint balls. One can view the set

[4] It is clear that at this step we may assume that Γ acts effectively.

$$U' = \bigcup_{\gamma \in \Gamma} \gamma(U) = \bigcup_{i=1}^{N} \bigcup_{\gamma \in \Gamma} \gamma(B_i)$$

constructed in Step 2 as a finite collection of disconnected sets in a metric space. Clearly one can decrease the radii of B_i so that U' is a family of disjoint open horoballs. Denote this resulting family by U''. We are now going to show that U'' can be shrunk further so that $X \backslash U''$ is a CAT(0) space. First, note that $\mathbb{H}^n \backslash U''$ is a complete CAT(0) space by the following general fact:

Theorem 3.5 ([12, II.11.27]) *Let $Y \subset \mathbb{B}^n$ be a subspace obtained by deleting a family of disjoint open horoballs. When endowed with the induced length metric, Y is a complete CAT(0) space.*

Obviously, a convex subset of a CAT(0) space is itself a CAT(0) space when endowed with the induced metric. In view of Theorem 3.5, to conclude that $X \backslash U''$ is a CAT(0) space we need to check its convexity in $\mathbb{B}^n \backslash U''$ (for the induced length metric), possibly after decreasing radii of U''.

First let us mention the description of geodesics in the truncated hyperbolic space $\mathbb{B}^n \backslash U''$. Let Y be as in Theorem 3.5. By [12, Corollary 11.34], a path $c : [a, b] \to Y$ parametrized by arc length is a geodesic in Y if and only if it can be expressed as a concatenation of non-trivial paths $c_1, \dots, c_n$ parametrized by arc length, such that:

(1) each of the paths c_i is either a hyperbolic geodesic or else its image is contained in one of the horospheres bounding Y and in that horosphere it is a Euclidean geodesic;
(2) if c_i is a hyperbolic geodesic then the image c_{i+1} is contained in a horosphere and vice versa.

Now pick two different points $x, y \in X \backslash U''$. Let ℓ be the geodesic of $\mathbb{B}^n$ joining them. Note that $\ell \subset X$, as X is convex in $\mathbb{B}^n$. If $\ell \cap U'' = \varnothing$, then $\ell \subset X \backslash U''$ and we are done. So, let us assume that ℓ intersects U''. The geodesic of $X \backslash U''$ is a concatenation of hyperbolic geodesics α_i and Euclidean geodesics β_j lying on horospheres. Note that both endpoints of each α_i belong to X, hence all α_i lie in $X \backslash U''$ (as X is convex in $\mathbb{B}^n$ and α_i are hyperbolic geodesics of $\mathbb{B}^n$). To make sure that β_j lie in X we can decrease the radius of each B_i so that the antipodal point of its base point belongs to X. This will hold for all $\gamma(B_i)$ in fact, as X is Γ-invariant. Denote by W the resulting disjoint family of open horoballs. Now all β_j are contained in X, so the whole geodesic between x and y is contained in $X \backslash W$. So, $X \backslash W$ is a CAT(0) space.

4 Applications

We are ready apply the results of the previous section in the hyperkähler setting. Let us start with the following trivial

Lemma 4.1 *Let G be a group with a finite normal subgroup H such that G/H is cyclic. Then G is virtually cyclic, i.e. there is a finite index normal cyclic subgroup in G.*

Proof We may assume that $G/H \simeq \mathbb{Z}$, otherwise the statement is trivial. Consider the canonical projection $\pi : G \to G/H$ and suppose that $\pi(g)$ generates G/H, where $g \in G$. Then g is of infinite order and $G = H \rtimes \langle g \rangle$. Hence some power g^m must centralize H and $\langle g^m \rangle$ is a normal cyclic subgroup of index $m|H|$. □

Proof of Theorem A We simply apply Theorem 3.2. For $\rho(M) \geqslant 2$ it follows with $\Lambda = \mathrm{NS}(M)$, $\mathfrak{q} = q_{BBF}$, $\mathscr{C} = \mathrm{Amp}(M)^+$ for $\Gamma = \mathrm{Aut}(M)$ and $\mathscr{C} = \mathrm{Mov}(M)^+$ for $\Gamma = \mathrm{Bir}(M)$, and Ω given by Theorem 2.2 and Remark 2.3. Note that the kernel of $\Phi = \Phi_{\mathrm{NS}}$ is finite by Proposition 2.1.

Now let us treat the case $\rho(M) = 2$ separately (as will be clear from below, case $\rho(M) = 1$ is easier). Let Γ be either $\mathrm{Aut}(M)$ or $\mathrm{Bir}(M)$. By Proposition 2.1 there is a short exact sequence

$$1 \to K \to \Gamma \xrightarrow{\Phi_{\mathrm{NS}}} \Gamma^* \to 1$$

with $\Gamma^* = \Phi_{\mathrm{NS}}(\Gamma)$ and K a finite group. For any $g \in \Gamma^*$ one has $\det g = \pm 1$. Put $\Gamma^+ = \{g \in \Gamma^* : \det g = 1\}$. By [29, Theorem 3.9] one has either $\Gamma^+ = 1$ or $\Gamma^+ \cong \mathbb{Z}$. It suffices to consider only the last case. Here we have a short exact sequence

$$1 \to K \to \Gamma' = \Phi_{\mathrm{NS}}^{-1}(\Gamma^+) \to \Gamma^+ \to 1.$$

By Lemma 4.1, a finite-by-cyclic group is always virtually cyclic. This means that Γ', and hence Γ, is either finite, or $\mathbb{Z}$ up to finite index. But all such groups are CAT(0) groups (this follows e.g. from the Bieberbach Theorem, see [12, II.7, Remark 7.3]). □

The following properties of CAT(0) groups will be crucial for us.

Theorem 4.2 *([12, III.Γ, Theorem 1.1]) Every CAT(0) group Γ satisfies the following properties:*

(1) Γ finitely presented;
(2) Γ has finitely many conjugacy classes of finite subgroups;
(3) Every solvable subgroup of Γ has an abelian subgroup of finite index;
(4) Every abelian subgroup of Γ is finitely generated.

Corollary 4.3 (Theorem A) *Let M be a hyperkähler manifold. Then the groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Bir}(M)$ *are finitely presented.*

Proof If M is non-projective, then the groups $\mathrm{Aut}(M)$ and $\mathrm{Bir}(M)$ are almost abelian by [32, Theorem 1.5], hence finitely presented. For projective hyperkähler manifolds the statement follows from Theorem 4.2 (1). □

Corollary 4.4 (Theorem C) *Let M be a projective hyperkähler manifold. Then the groups $\mathrm{Aut}(M)$ and $\mathrm{Bir}(M)$ have finitely many conjugacy classes of finite subgroups. In particular, there exists a constant $B = B(M)$ such that for every finite subgroup $G \subset \mathrm{Bir}(M)$ one has $|G| \leqslant B$ (i.e. $\mathrm{Bir}(M)$ has bounded finite subgroups).*

Proof Follows from Theorem 4.2 (2). □

Of course, the second part of this statement (as well as Corollary 4.7 below) can be obtained using Minkowski's Theorem, which states that $\mathrm{GL}_n(\mathbb{Q})$ has bounded finite subgroups (see e.g. [40]). However, as we mentioned in Remark 1.3 the finiteness of conjugacy classes of finite subgroups is a much more subtle issue.

Remark 4.5 One can compare this result with [35, Theorem 1.8] which states that $\mathrm{Bir}(X)$ has bounded finite subgroups provided that X is an irreducible algebraic variety which is non-uniruled and has $h^1(X, \mathscr{O}_X) = 0$. The latter condition is clearly true for any projective hyperkähler manifold X, and the former one holds since complex uniruled varieties have Kodaira dimension $-\infty$, while Calabi-Yau manifolds have Kodaira dimension zero.

Remark 4.6 In dimension two, i.e. for projective K3 surfaces, it is known that the orders of their finite automorphism groups are bounded by 3840 (and this bound is sharp) [26].

Recall that a torsion group is a group in which each element has finite order. In general it is an open question whether torsion subgroups of any CAT(0) group are always finite. However in our case the answer to this question is positive.

Corollary 4.7 (Burnside property) *Let M be a projective hyperkähler manifold. Then every torsion subgroup $G \subseteq \mathrm{Bir}(M)$ is finite.*

Proof Put $G^* = \Phi_{\mathrm{NS}}(G)$ and $G_0 = G \cap \ker \Phi_{\mathrm{NS}}$. One has a short exact sequence

$$1 \to G_0 \to G \to G^* \to 1$$

with G_0 finite, and G^* a torsion group. By Theorem 4.2 (2) (or Corollary 4.4) the group G^* has bounded exponent, i.e. there exists $d \in \mathbb{Z}_{>0}$ such that the order of any $g \in G^*$ is $\leqslant d$. Since G^* is linear, it must be finite by Burnside's theorem. Therefore G is finite too. □

4.1 Tits' Alternative

In this subsection we show how our method implies a strong form of Tits' alternative (Theorem B) for projective hyperkähler manifolds. In general this is a well-known open question whether CAT(0) groups always satisfy Tits' alternative, but in our case the usual Tits' alternative for $\mathrm{GL}_n(\mathbb{Q})$ and some properties of CAT(0) groups give even stronger restrictions than in the classical settings. The heart of the proof of Oguiso's Theorem 1.2 was the fact that a virtually solvable subgroup of $\mathrm{O}(L)$, where L is a hyperbolic lattice of finite rank, must be almost abelian of finite rank. The proof of the latter involves Lie-Kolchin Theorem and various properties of Salem polynomials. In our case the key ingredient of Oguiso's proof follows from the fact that $\mathrm{Bir}(M)$ is a CAT(0) group. But in fact we are able to prove something stronger, namely, that in the first case of Tits alternative our group is just $\mathbb{Z}^n$ up to finite index.

Theorem 4.8 (Theorem B) *Let M be a projective hyperkähler manifold, and $G \subseteq \mathrm{Bir}(M)$ be a subgroup. Then*

(1) either G contains a finite index subgroup isomorphic to $\mathbb{Z}^n$;
(2) or G contains a non-commutative free group.

Proof Put $G^* = \Phi_{\mathrm{NS}}(G)$. Then one has a short exact sequence of groups

$$1 \to N \to G \to G^* \to 1,$$

with N being a finite group by Proposition 2.1. Assume that G^* does not contain a non-abelian free subgroup. Then by usual Tits' alternative for $\mathrm{GL}(\mathrm{NS}(M) \otimes \mathbb{R})$ the group G^* has a solvable subgroup S^* of finite index. Put $S = \Phi_{\mathrm{NS}}^{-1}(S^*)$. We have a short exact sequence

$$1 \to N \to S \to S^* \to 1$$

with $[G : S] < \infty$, N finite, and S^* solvable. The centralizer $C = C_S(N)$ of N in S has finite index in S (indeed, S acts on N by conjugation, which gives a homomorphism $S \to \mathrm{Aut}(N)$ with kernel $C_S(N)$ and $\mathrm{Aut}(N)$ a finite group). Thus we have an extension

$$1 \to A \to C \to C^* \to 1$$

with $A = N \cap C$ abelian and C^* solvable group. Clearly $[G : C] < \infty$. Since both A and C^* are solvable, the group $C \subset \mathrm{Bir}(M)$ is solvable. By Theorem 4.2 (3) and (4) it then contains $F \cong \mathbb{Z}^n$ with $[C : F] < \infty$. Hence G contains a finite index subgroup isomorphic to $\mathbb{Z}^n$. □

4.2 Some Applications to Dynamics

Let (X, dist) be a metric space and $f \in \mathrm{Isom}(X)$ be its isometry. Then one can consider the *displacement function* of f

$$d_f : X \to \mathbb{R}_{\geqslant 0}, \quad d_f(x) = \mathrm{dist}(f(x), x).$$

The *translation length* of f is the number $\|f\| = \inf\{d_f(x) : x \in X\}$. The set of points where d_f attains the infimum is denoted by $\mathrm{Min}(f)$. If d_f attains a strictly positive minimum, then f is called *loxodromic*; if this minimum is 0 (i.e. f has a fixed point), then f is called *elliptic*; if d_f does not attain the minimum (i.e. $\mathrm{Min}(f) = \varnothing$), then f is called *parabolic*. Elliptic and loxodromic isometries are also called *semi-simple*. In the case $X = \mathbb{H}^n$ these definitions agree with the old ones.

Now let M be a projective hyperkähler manifold, and $f \in \mathrm{Bir}(M)$ be birational automorphism. According to the action of f^* on the corresponding hyperbolic space $(\mathrm{NS}(M)_{\mathbb{R}}, q_{BBF})$ one can classify f as elliptic, parabolic or loxodromic. Denote by $\mathscr{X}_M$ the CAT(0) space constructed in Sect. 3, i.e. the space on which $\mathrm{Bir}(M)$ acts properly and cocompactly by isometries of $\mathscr{X}_M$. Then one has a group homomorphism

$$\Theta_M : \mathrm{Bir}(M) \to \mathrm{Isom}(\mathscr{X}_M).$$

We should warn the reader that in general Θ_M does not preserve[5] the type of an isometry. In fact, $\Theta(\mathrm{Bir}(M))$ does not contain parabolic isometries by [12, II.6.10 (2)].

Lemma 4.9 *The images of M-loxodromic and M-parabolic birational automorphisms under Θ_M are $\mathscr{X}_M$-loxodromic. The images of M-elliptic elements are $\mathscr{X}_M$-elliptic. In particular, Θ_M maps semi-simple isometries to semi-simple ones.*

Proof First note that if a group Γ acts geometrically on a proper CAT(0) space then $\gamma \in \Gamma$ has finite order if and only if γ is elliptic.

Let $f \in \mathrm{Bir}(M)$ be of infinite order, i.e. either M-loxodromic or M-parabolic. Then, as was noticed above, $\Theta_M(f)$ is either $\mathscr{X}_M$-loxodromic, or $\mathscr{X}_M$-elliptic. In the latter case $\Theta_M(f)^n = \mathrm{id}$ for some $n > 0$. Thus $f^n \in \ker \Theta_M$, i.e. f^n acts as identity on $\mathscr{X}_M$. But this also means that f has a fixed point locus on the underlying hyperbolic space $(\mathrm{NS}(M)_{\mathbb{R}}, q_{BBF})$, i.e. f^n is M-elliptic. By [12, II.6.7] we have that f must be M-elliptic too, contradiction. Finally, the image of an element of finite order is of finite order, hence M-elliptic elements map to $\mathscr{X}_M$-elliptic elements. □

4.3 Structure of Centralizers

Let X be an algebraic variety. Given an element of infinite order $f \in \mathrm{Bir}(X)$, it is often useful to understand the structure of its centralizer $C(f)$, see e.g. [14] or [48]. To clarify this structure in our case, we shall use the following important Flat Torus Theorem.

Theorem ([12, II.7.1]) *Let A be a free abelian group of rank n acting properly by semi-simple isometries on a CAT(0) space X. Then:*

(1) $\mathrm{Min}(A) = \cap_{\alpha \in A} \mathrm{Min}(\alpha)$ *is non-empty and splits as a product* $Y \times \mathbb{E}^n$*;*

[5] It is not surprising since the metric on $\mathscr{X}_M$ is not the same as on the initial hyperbolic space.

(2) Every element $\alpha \in A$ leaves $\mathrm{Min}(A)$ *invariant and respects the product decomposition; α acts as the identity on Y and as a translation on $\mathbb{E}^n$;*
(3) If a finitely generated subgroup $\Gamma \subset \mathrm{Isom}(X)$ normalizes A, then Γ has a subgroup of finite index that contains A as a direct factor.

Proposition 4.10 *Let M be a projective hyperkähler manifold, and $f \in \mathrm{Bir}(M)$ be either parabolic, or loxodromic. Denote by $C(f)$ the centralizer of f in* $\mathrm{Bir}(M)$. *Then $C(f)$ has a finite index subgroup H which splits as a direct product: $H = N \times \langle f \rangle$.*

Proof Suppose that a group Γ acts geometrically on a CAT(0) space X, and γ is an element of infinite order. Then its centralizer $C(\gamma)$ acts geometrically on the CAT(0) subset $\mathrm{Min}(\gamma)$ of X [38, Theorem 3.2]. Thus we see that $C(f)$ is a CAT(0) group, hence finitely generated by Theorem 4.2 (1). It remains to apply the Flat Torus Theorem (3) to $A = \langle f \rangle$. □

Given a finitely generated group and its arbitrary element, it is natural to ask how the iterates of this element behave with respect of generators. Namely, let Γ be a finitely generated group with finite symmetric generating set $\Sigma = \Sigma^{-1}$. Recall that the word metric on Γ is defined as

$$w_\Sigma(\gamma_1, \gamma_2) = \min\{n : \ \gamma_1^{-1}\gamma_2 = \sigma_1\sigma_2 \ldots \sigma_n, \ \sigma_i \in \Sigma\},$$

and the length of $\gamma \in \Gamma$ is $|\gamma|_\Sigma = w_\Sigma(\mathrm{id}, \gamma)$. An element $\gamma \in \Gamma$ is called *distorted* if

$$\lim_{n\to\infty} \frac{|\gamma^n|_\Sigma}{n} = 0$$

and *undistorted* otherwise. The property of being undistorted is well known to be independent of choice of Σ.

Proposition 4.11 *Let M be a projective hyperkähler manifold. Then its loxodromic and parabolic birational automorphisms are undistorted.*

Proof Let $\gamma \in \mathrm{Bir}(M)$ be a M-loxodromic or M-parabolic automorphism. Then $\Theta(\gamma) = \Theta_M(\gamma)$ is of infinite order by Lemma 4.9. By [12, I.8.18] for any choice of basepoint $x_0 \in \mathscr{X}_M$ there exists a constant $\mu > 0$ such that

$$\mathrm{dist}_{\mathscr{X}_M}(\gamma_1 x_0, \gamma_2 x_0) \leq \mu w_\Sigma(\gamma_1, \gamma_2).$$

Then one has

$$\lim_{n\to\infty} \frac{|\gamma^n|_\Sigma}{n} \geqslant \lim_{n\to\infty} \frac{|\Theta(\gamma)^n|_{\Theta(\Sigma)}}{n} = \lim_{n\to\infty} \frac{w_{\Theta(\Sigma)}(\mathrm{id}, \Theta(\gamma)^n)}{n} \geqslant \lim_{n\to\infty} \frac{\mu^{-1}\,\mathrm{dist}_{\mathscr{X}_M}(x_0, \Theta(\gamma)^n x_0)}{n}, \tag{2}$$

where $x_0 \in \mathscr{X}_M$ is an arbitrary point. Now let X be a CAT(0) space, δ be a semi-simple isometry, and $x \in X$ be any point. Then it is easy to check that

$$\|\delta\| = \frac{\mathrm{dist}_X(x, \delta^n x)}{n}. \tag{3}$$

By the Flat Torus Theorem, the set $\mathrm{Min}(\langle\delta\rangle) \equiv \cap_k \mathrm{Min}(\gamma^k)$ is γ-invariant ans splits as a product $Y \times \mathbb{E}^1$ such that δ acts identically on Y and by translations on $\mathbb{E}^1$. It then easily follows that $\|\delta^n\| = n \cdot \|\delta\|$. Now taking $X = \mathscr{X}_M$ and $\delta = \Theta(\gamma)$ we get from (2) and (3) that

$$\lim_{n\to\infty} \frac{|\gamma^n|_\Sigma}{n} \geqslant \mu^{-1}\|\Theta(\gamma)^n\| = \mu^{-1}n\|\Theta(\gamma)\| > 0,$$

since $\|\Theta(\gamma)\| > 0$. □

4.4 Cohomological Properties

Finally we would like to show that cohomological properties of $\mathrm{Bir}(M)$ and $\mathrm{Aut}(M)$ mentioned in [18] can be also obtained using our approach. Recall that a group Γ is called *of type FL* if the trivial $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$ has a finite resolution by free $\mathbb{Z}[\Gamma]$-modules of finite rank:

$$0 \to \mathbb{Z}[\Gamma]^{n_k} \to \ldots \to \mathbb{Z}[\Gamma]^{n_1} \to \mathbb{Z} \to 0.$$

We say that Γ is *of type VFL*[6] if it is virtually FL, i.e. admits a finite-index subgroup satisfying property FL.

Proposition 4.12 *Let M be a projective hyperkähler manifold. Then the groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Bir}(M)$ *are of type VFL.*

Proof Let Γ denote either $\mathrm{Aut}(M)$ or $\mathrm{Bir}(M)$. By Selberg's lemma, the group $\Phi_{\mathrm{NS}}(\Gamma)$ is virtually torsion-free. Since $\ker \Phi_{\mathrm{NS}}$ is finite, Γ aslo contains a finite-index torsion-free subgroup, say Γ_0. Consider the action of Γ on the associated CAT(0) space $\mathscr{X}_M$. Note that cocompactness is inherited under restriction of the action of Γ to any finite-index subgroup, and properness holds for any subgroup of Γ. So, the action of Γ_0 on $\mathscr{X}_M$ is proper and cocompact (and free). By [12, III.Γ.1.1, II.5.13], Γ_0 has a finite CW complex as classifying space. By [15, VIII.6.3] Γ_0 is of type FL then. □

We refer to [15] for further finiteness properties of groups.

4.5 Infinite Automorphism Groups of Hyperkähler Manifolds

Let M be a projective hyperkähler manifold. As was mentioned in the Introduction, at the moment there is no general understanding of how complicated the automorphism

[6] Virtuellement une résolution Libre de type Finie.

groups of hyperkähler manifolds can be, even in the case of K3 surfaces. Here are just some particular examples in this direction:

- Let X be a K3 surface with Picard number 2. Then $\mathrm{Aut}(X)$ is finite precisely when the Picard lattice contains divisors L with $L^2 = 0$ or with $L^2 = -2$. Otherwise, $\mathrm{Aut}(X)$ is either infinite cyclic, or the infinite dihedral $\mathbb{Z}/2 * \mathbb{Z}/2$ [20].
- For any K3 surface with $\mathrm{NS}(X) \simeq \mathbb{Z}(2nd) \oplus \mathbb{Z}(-2n)$ with $n \geqslant 2$ and d not a square one has $\mathrm{Aut}(X) \simeq \mathbb{Z}$ [10].
- A *Wehler surface* is a K3 surface X given as intersection of two divisors of bidegrees (1, 1) and (2, 2). In the generic case $\mathrm{NS}(X)$ is of rank two with intersection matrix $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ and $\mathrm{Aut}(X) \simeq \mathbb{Z}/2 * \mathbb{Z}/2$. The two generators correspond to the covering involutions $X \to \mathbb{P}^2$ of the projections to the two factors (they are not symplectic, but their product is, and of infinite order), see [17] for more details and generalizations.
- In 1977 Shioda and Inose classified K3 surfaces with maximal Picard rank in terms of their transcendental lattices. In particular, they discussed two K3 surfaces with maximal Picard rank which are the simplest in the sense that their transcendental lattices have the smallest possible discriminants equal to 3 and 4. Then Vinberg [46] called these surfaces *the most algebraic K3 surfaces*, X_3 and X_4. The surface X_m is birational to $(Y_m \times Y_m)/\Gamma_m$, where $Y_m = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega_m)$, $\omega_m = \exp(2\pi i/m)$ is an elliptic curve, Γ_m is the cyclic group of order m generated by $(z_1, z_2) \mapsto (\omega_m z_1, \omega_m^{-1} z_2)$. Moreover, Vinberg showed that $\mathrm{Aut}(X_4)$ is a non-trivial central extension of
$$\underbrace{\mathbb{Z}/2 * \ldots * \mathbb{Z}/2}_{5\text{ times}} \rtimes \mathfrak{S}_5$$
by $\mathbb{Z}/2$ and $\mathrm{Aut}(X_3)$ is the trivial central extension of
$$\underbrace{\mathbb{Z}/2 * \ldots * \mathbb{Z}/2}_{12\text{ times}} \rtimes ((\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}/2)$$
by $\mathbb{Z}/3$.

Of course, none of our main results provide sufficient condition for being an automorphism group of a projective hyperkähler manifold. For example, it is known that the (finitely presented) Thompson group F does not satisfy Tits alternative. Below we would like to collect some examples of groups which *cannot* be the automorphism groups of any hyperkähler manifolds.

Example 4.13 For each integer m and n consider the Baumslag-Solitar group
$$\mathrm{BS}(m, n) = \langle a, b \mid a^{-1} b^m a = b^n \rangle.$$

Let M be a projective hyperkähler manifold. By Lemma 4.9, $\mathrm{Aut}(M)$ acts on $\mathscr{X}_M$ by semi-simple isometries. Since $\mathrm{Aut}(M)$ is finitely generated, [12, Theorem 1.1 (iii), III.Γ] implies that $\mathrm{Aut}(M)$ cannot contain a copy of $\mathrm{BS}(m, n)$ with $|m| \neq |n|$.

Example 4.14 Our next example of impossible automorphism group has a completely different flavor. By [12, III.Γ, Theorem 1.4], every CAT(0) group has solvable word and conjugacy problems. It is due to Boone and Novikov that there exist finitely presented groups with an unsolvable word problem (and some explicit presentations are also known after D. J. Collins). Note that the Baumslag-Solitar group of Example 4.13 has a solvable word problem (e.g. because the classical result of Magnus states that every one-relator group does).

Finally, there are examples of very simple group extensions which cannot be automorphism groups of any hyperkähler manifolds.

Example 4.15 Assume that $\mathrm{Aut}(M)$ contains a semi-direct product $G = \mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$. As G is solvable, Theorem 4.2 (3) implies that G has an abelian subgroup of finite index. It is easy to see that φ must have a finite order then. For example, when $n = 2$ this shows that the integer Heisenberg group

$$\mathscr{H}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & t & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid t, x, y \in \mathbb{Z} \right\} \cong \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$$

cannot embed into $\mathrm{Aut}(M)$. Here we identify the matrix in the brackets with the pair $((x, y), t)$ and φ is given by $\varphi(t) \cdot (x, y) = (x + ty, y)$.

Example 4.16 Now consider the free group $F = F(a, b, c)$ on three letters and the automorphism

$$\varphi : \quad a \mapsto a, \quad b \mapsto ba, \quad c \mapsto ca^2.$$

Then the group

$$F \rtimes_\varphi \mathbb{Z} \cong \langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ba, tct^{-1} = ca^2 \rangle$$

is not a CAT(0) group [24, Proposition 2.1]. The proof relies on the study of translation lengths introduced in paragraph Sect. 4.2.

We believe that a better understanding of CAT(0) groups will provide some insights about the structure of automorphism groups of hyperkähler manifolds.

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References

1. Apanasov, B.N.: Conformal Geometry of Discrete Groups and Manifolds, Degruyter Expositions in Mathematics (Book 32) (2000)
2. Amerik, E., Verbitsky, M.: Rational curves on Hyperkähler manifolds. Int. Math. Res. Not. **23**, 13009–13045 (2015)
3. Amerik, E., Verbitsky, M.: Teichmuller space for hyperkahler and symplectic structures. J. Geom. Phys. **97**, 44–50 (2015)
4. Amerik, E., Verbitsky, M.: Morrison-Kawamata cone conjecture for hyperkähler manifolds. Ann. Scientifiques de l'École normale supérieure **50**(4), 973–993 (2017)
5. Amerik, E., Verbitsky, M.: Collections of parabolic orbits in homogeneous spaces, homogeneous dynamics and hyperkähler geometry. Int. Math. Res. Not. **2020**(1), 25–38 (2020)
6. Arzhantsev, I., Zaidenberg, M.: Tits-type alternative for groups acting on toric affine varieties. Int. Math. Res. Not. rnaa342
7. Arzhantsev, I., Zaidenberg, M.: Tits-type alternative for certain groups acting on algebraic surfaces. arxiv:2111.06659
8. Beauville, A.: Variete Kähleriennes dont la premiere classe de chern est nulle. J. Diff. Deom. **18**, 755–782 (1983)
9. Benzerga, M.: Finiteness of real structures on KLT Calabi-Yau regular smooth pairs of dimension 2. arxiv:1702.08808
10. Bini, G.: On the automorphism group of some K3 surfaces with Picard number two. MCFA Ann. **4**, 1–3
11. Bogomolov, F.: On the decomposition of Kähler manifolds with trivial canonical class. Math. USSR-Sb **22**, 580–583 (1974)
12. Bridson, M.R., Häfliger, A.: Metric Spaces of Non-Positive Curvature. Springer (1999)
13. Boissière, S., Sarti, A.: A note on automorphisms and birational transformations of holomorphic symplectic manifolds. Proc. Am. Math. Soc. **140**(12), 4053–4062 (2012)
14. Blanc, J., Cantat, S.: Dynamical degrees of birational transformations of projective surfaces. J. Am. Math. Soc. **29**(2), 415–471 (2016)
15. Brown, K.: Cohomology of Groups. Springer (1982)
16. Cantat, S.: Sur les groupes de transformations birationnelles des surfaces. Ann. Math. **174**, 299–340 (2011)
17. Cantat, S., Oguiso, K.: Birational automorphism groups and the movable cone theorem for Calabi-Yau manifolds of Wehler type via universal Coxeter groups. Am. J. Math. **137**, 1013–1044 (2015)
18. Cattaneo, A., Fu, L.: Finiteness of Klein actions and real structures on compact hyperhahler manifolds. Math. Ann. **375**, 1783–1822 (2019)
19. Fujiki, A.: On the de Rham cohomology group of a compact Kähler symplectic manifold. In: Algebraic geometry, Sendai. Advanced Studies in Pure Mathematic, vol. 10, pp. 105–165 (1985)
20. Galluzzi, F., Lombardo, G., Peters, C.: Automorphs of indefinite binary quadratic forms and K3-surfaces with Picard number 2. Rend. Sem. Mat. Univ. Politec. Torino **68**(1), 57–77 (2010)
21. Gross, M., Huybrechts, D., Joyce, D.: Calabi-Yau Manifolds and Related Geometries Universitext. Springer, Berlin (2003)
22. Grunewald, F., Platonov, V.: Rigidity results for groups with radical cohomology of finite groups and arithmeticity problems. Duke Math. J. **100**(2), 321–358 (1999)
23. Guan, D.: On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four. Math. Res. Lett. **8**(5–6), 663–669 (2001)
24. Gersten, S.M.: The automorphism group of a free group is not a CAT(0) group. Proc. Am. Math. Soc. **121**(4), 999–1002 (1994)
25. Huybrechts, D.: Compact hyperkähler manifolds: basic results. Invent. Math. **135** 63–113 (1999). Erratum: Compact hyperkähler manifolds: basic results. Invent. Math. **152**, 209–212 (2003)

26. Kondo, S.: The maximum order of finite groups of automorphisms of K3 surfaces. Am. J. Math. **121**(6), 1245–1252 (1999)
27. Kurnosov, N.: Boundness of b_2 for hyperkähler manifolds with vanishing odd-Betti numbers. arxiv:1511.02838
28. Looijenga, E.: Discrete automorphism groups of convex cones of finite type. Compos. Math. **150**(11), 1939–1962 (2014)
29. Lazić, V., Peternell, Th.: On the cone conjecture for Calabi-Yau manifolds with Picard number two. Math. Res. Lett. **20**(6), 1103–1113 (2013)
30. Markman, E.: A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In: Proceedings of the Conference Complex and Differential Geometry, Springer Proceedings in Mathematics, vol 8, pp. 257–322 (2011)
31. Namikawa, Y.: Periods of Enriques surfaces. Math. Ann. **270**(2), 201–222 (1985)
32. Oguiso, K.: Tits alternative in hyperkähler manifolds. Math. Res. Lett. **13**(2), 307–316 (2006)
33. Oguiso, K.: Bimeromorphic automorphism groups of non-projective hyperkähler manifolds—a note inspired by C. T. McMullen. J. Diff. Geom. **78**(1), 163–191 (2008)
34. Oguiso, K.: Automorphism groups of Calabi-Yau manifolds of Picard number 2. J. Algeb. Geom. **23**, 775–795 (2014)
35. Prokhorov, Yu., Shramov, C.: Jordan property for groups of birational selfmaps. Compos. Mathematica **150**(12), 2054–2072 (2014)
36. Prokhorov, Y., Shramov, C.: Automorphism groups of compact complex surfaces. arxiv:1708.03566
37. Ratcliffe, J.G.: Foundations of Hyperbolic Manifolds, volume 149 de Graduate Texts in Mathematics (2006)
38. Ruane, K.: Dynamics of the action of a CAT(0) group on the boundary. Geom. Dedicata **84**(1–3), 81–99 (2001)
39. Sawon, J.: A bound on the second Betti number of hyperkähler manifolds of complex dimension six. arxiv:1511.09105
40. Serre, J.-P.: Bounds for the Orders of the Finite Subgroups of G(k) in Group Representation Theory, pp. 405–450. EPFL Press, Lausanne (2007)
41. Sterk, H.: Finiteness results for algebraic K3 surfaces. Math. Z. **189**, 507–513 (1985)
42. Totaro, B.: The cone conjecture for Calabi-Yau pairs in dimension 2. Duke Math. J. **154**(2), 241–263 (2010)
43. Totaro, B.: The cone conjecture for Calabi-Yau pairs in dimension 2. arXiv:0901.3361
44. Totaro, B.: Algebraic surfaces and hyperbolic geometry. In: Current Developments in Algebraic Geometry, volume 59 de Mathematical Sciences Research Institute Publication, pp. 405–426. Cambridge University Press, Cambridge (2012)
45. Verbitsky, M.: Mapping class group and a global Torelli theorem for hyperkähler manifolds. Duke Math. J. **162**(15), 2929–2986 (2013)
46. Vinberg, E.: The two most algebraic K3 surfaces. Mathematische Annalen **265**, 1–21 (1983)
47. Wolf, J.A.: Spaces of Constant Curvature. AMS (2011)
48. Zhao, S.: Centralizers of elements of infinite order in plane Cremona groups. Ann. Scuola Normale Sup. Pisa, to appear

On Generalized Büchi Surfaces

Antonio Laface and Rodrigo Quezada

Abstract The aim of this note is to introduce a class of smooth projective surfaces, named Generalized Büchi surfaces. These are complete intersections in $\mathbb{P}^n$ which generalize the surfaces arising from the Büchi problem in number theory [9]. We show that a Generalized Büchi surface of $\mathbb{P}^n$ determines, up to projectivities, a subset of cardinality $n+1$ of $\mathbb{P}^1$ and viceversa it is determined, up to projectivities, by such a subset.

Keywords Algebraic surfaces · Hyperelliptic curves · Büchi surfaces

2010 Mathematics Subject Classification Primary 14J25. Secondary 14J29, 14H55

1 Introduction

In what follows all the varieties that we consider are defined over the complex numbers. Let n be an integer with $n \geq 3$ and let $\{x_i\}_{i=1}^n$ be a sequence of n integers satisfying the system of second order difference equations

$$(x_{i+2}^2 - x_{i+1}^2) - (x_{i+1}^2 - x_i^2) = 2 \qquad \text{for } i \in \{1, \ldots, n-2\}.$$

For any integer x, the sequence of consecutive integers $\{x_i = x + i\}_{i=1}^n$ is a solution. An integral solution is *trivial* if it is obtained from a sequence of consecutive integers

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by arbitrary sign changes, like e.g. $(5, -6, -7, 8, \dots)$, and it is *non-trivial* otherwise. The Büchi problem is the well known question asking whether there exists a positive integer n such that all integral solutions of the above equations are trivial. A positive answer to Büchi's problem would imply, using the negative answer to Hilbert's Tenth Problem by Yu. Matiyasevich, that there is no algorithm to decide whether a system of diagonal quadratic forms with integer coefficients admits an integer solution. For a recent survey on the problem, we refer the reader to [9].

The above equations define an affine surface whose closure in the complex projective space is the *Büchi surface* $S_n \subseteq \mathbb{P}^n$. The Büchi problem, in its geometric formulation, asks if there exists a positive integer n such that all the rational points of S_n lie on the *trivial lines* of S_n, see Definition 2.2. In [1] the authors prove that S_5 is determined by its trivial lines: each such line intersects exactly six other trivial lines and the set of intersection points are all projectively equivalent. In fact S_5 is the minimal resolution of the Kummer surface of the genus two curve $y^2 = (x-2)(x-1)x(x+1)(x+2)$ defined by the six intersection points. Our aim is to generalize the geometric description of S_5 given in [1] to the following class of surfaces.

Definition. Given an $(n+1)$-tuple $\alpha = (\alpha_0, \dots, \alpha_n)$ of distinct points of $\mathbb{P}^1$, the *Generalized Büchi surface* $S_n(\alpha) \subseteq \mathbb{P}^n$ is the complete intersection of the following $n-2$ diagonal quadrics of $\mathbb{P}^n$:

$$x_i^2 - \frac{(\alpha_0-\alpha_i)(\alpha_1-\alpha_i)}{(\alpha_0-\alpha_2)(\alpha_1-\alpha_2)}x_2^2 + \frac{(\alpha_0-\alpha_i)(\alpha_2-\alpha_i)}{(\alpha_0-\alpha_1)(\alpha_1-\alpha_2)}x_1^2 - \frac{(\alpha_1-\alpha_i)(\alpha_2-\alpha_i)}{(\alpha_0-\alpha_1)(\alpha_0-\alpha_2)}x_0^2 = 0$$

where $i \in \{3, \dots, n\}$.

The name "Generalized Büchi surfaces" is because $S_n := S_n(\infty, 1, 2, \dots, n)$, as shown at the beginning of Sect. 2. Before stating our first result recall that a graph is *bipartite of type* (a, b) if it set of vertices is the union of two disjoint sets of cardinality a and b respectively.

Theorem 1 *Let $n \geq 4$ be an integer. Then the following hold for $S_n(\alpha)$.*

(1) The 2^n trivial lines are the only lines on the surface.
(2) Each line meets exactly other $n+1$ lines along a subset of points which is projectively equivalent to α.
(3) If n is odd then the intersection graph of the lines of the surface is bipartite of type $(2^{n-1}, 2^{n-1})$.
(4) The defining ideal of the surface is generated by the quadrics which vanish along the lines.

An immediate consequence of Theorem 1 is that $S_n(\alpha)$ is uniquely determined by its trivial lines. Before stating our next result let us recall that $M_{0,n}$ denotes the variety which parametrizes n-tuples of points of $\mathbb{P}^1$. This variety admits a natural action by the symmetric group $\mathfrak{S}_n$. We denote by $\texttt{GBS}_n$ the set of generalized Büchi surfaces up to projectivities. Our next result is the following.

Theorem 2 *Let $n \geq 4$ be an integer. The assignment $\alpha \mapsto S_n(\alpha)$ induces a bijection $\Phi_n : M_{0,n+1}/\mathfrak{S}_{n+1} \to \mathtt{GBS}_n$.*

In particular if n is odd the above map establishes a bijection between the moduli space of hyperelliptic curves of genus $\frac{1}{2}(n-1)$ and $\mathtt{GBS}_n$. In this case we show that the quotient $Y_n(\alpha)$ of $S_n(\alpha)$ by the subgroup of even changes of signs is dominated by the second symmetric power of the corresponding hyperelliptic curve C. More precisely there is a degree two map $\mathrm{Sym}^2(C) \to Y_n(\alpha)$ which generalizes the classical construction of the Kummer surface when C has genus 2.

The paper is organized as follows. In Sect. 2 we prove the main geometric properties of Generalized Büchi surfaces and show how to reconstruct such a surface from its lines. Sect. 3 is devoted to describe the geometry of lines of a Generalized Büchi surface. The main result here is Theorem 1. In Sect. 4 we prove Theorem 2. Sect. 5 introduces the surface $Y_n(\alpha)$ as a quotient of a Generalized Büchi surface $S_n(\alpha)$ by the subgroup of even sign changes. Finally in Sect. 6 we show how to associate a Generalized Büchi surface to any hyperelliptic curve C and prove that, if α is the $2g+2$-tuple of branch points in the hyperelliptic involution $C \to \mathbb{P}^1$, then the quotient $\mathrm{Sym}^2(C)$ by the natural action of the hyperelliptic involution is isomorphic to the surface $Y_{2g+1}(\alpha)$. This is Theorem 6.1.

2 Generalized Büchi Surfaces

In this section we discuss the basic properties of Generalized Büchi surfaces. In what follows we will denote by $\mathcal{Q}_i$ the i-th defining polynomial for the Generalized Büchi surface and by $\beta_0^i, \beta_1^i, \beta_2^i$ the opposite of the coefficient of x_0^2, x_1^2, x_2^2 in $\mathcal{Q}_i$, so that $\mathcal{Q}_i = x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \beta_0^i x_0^2 = 0$. Our first result is the following.

Proposition 2.1 *Generalized Büchi surfaces are irreducible smooth projective surfaces. These surfaces are rational for $n = 3$ and 4, $K3$ for $n = 5$ and of general type for $n \geq 6$.*

Proof To prove the smoothness of $S_n(\alpha)$ observe that, the $(n-2) \times (n+1)$ Jacobian matrix of the equations that define $S_n(\alpha)$ is

$$2\begin{pmatrix} -\beta_0^3 x_0 & -\beta_1^3 x_1 & -\beta_2^3 x_2 & x_3 & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ -\beta_0^i x_0 & -\beta_1^i x_1 & -\beta_2^i x_2 & & & x_i & \\ \vdots & \vdots & \vdots & & & \ddots & \\ -\beta_0^n x_0 & -\beta_1^n x_1 & -\beta_2^n x_2 & & & & x_n \end{pmatrix}. \tag{2.1}$$

We thus have to show that the above Jacobian matrix has maximal rank. First of all we claim that a point of a General Büchi surface has at most two coordinates equal to zero. Indeed assume that three coordinates are zero. From the equations of a Generalized Büchi surface it immediately follows that these coordinates cannot be x_0, x_1, x_2, otherwise all the coordinates would vanish. Similarly it cannot be that two of the three vanishing coordinates are among the first three. Suppose now that only one of the vanishing coordinates is among the first three, let us say x_0, and the remaining two are x_i, x_j. Then

$$\beta_1^i x_1^2 + \beta_2^i x_2^2 = 0 \qquad \text{and} \qquad \beta_1^j x_1^2 + \beta_2^j x_2^2 = 0.$$

Since the determinant of the 2×2 matrix is

$$\begin{vmatrix} \beta_1^i & \beta_2^i \\ \beta_1^j & \beta_2^j \end{vmatrix} = \frac{(\alpha_i - \alpha_j)(\alpha_0 - \alpha_j)(\alpha_0 - \alpha_i)}{(\alpha_1 - \alpha_2)(\alpha_0 - \alpha_2)(\alpha_0 - \alpha_1)} \neq 0,$$

the only solution of the above equations is $x_1 = x_2 = 0$, so that again the first three variables would vanish giving a contradiction. Finally if none of the three vanishing variables x_i, x_j, x_k is among the first three then

$$\beta_0^i x_0^2 + \beta_1^i x_1^2 + \beta_2^i x_2^2 = 0, \quad \beta_0^j x_0^2 + \beta_1^j x_1^2 + \beta_2^j x_2^2 = 0 \quad \text{and} \quad \beta_0^k x_0^2 + \beta_1^k x_1^2 + \beta_2^k x_2^2 = 0.$$

Since the determinant of the 3×3 matrix is

$$\begin{vmatrix} \beta_0^i & \beta_1^i & \beta_2^i \\ \beta_0^j & \beta_1^j & \beta_2^j \\ \beta_0^k & \beta_1^k & \beta_2^k \end{vmatrix} = \frac{(\alpha_j - \alpha_k)(\alpha_i - \alpha_k)(\alpha_i - \alpha_j)}{(\alpha_1 - \alpha_2)(\alpha_0 - \alpha_2)(\alpha_0 - \alpha_1)} \neq 0,$$

the only solution of the above equations is again $x_0 = x_1 = x_2 = 0$, a contradiction. The claim is proved. As a consequence of the claim one can always find $n - 2$ columns of the above Jacobian matrix whose determinant is non-zero, so that the matrix has maximal rank.

Now we will prove, by induction on n, the irreducibility of $S_n(\alpha)$. For $n = 3$ the surface is a smooth quadric of $\mathbb{P}^3$. For $n \geq 4$ observe that the map $S_n(\alpha) \to S_{n-1}(\alpha)$ induced by the projection on the first n coordinates is a double covering branched along the curve B of equations

$$\beta_2^n x_2^2 + \beta_1^n x_1^2 + \beta_0^n x_0^2 = Q_{n-1} = \cdots = Q_3 = 0.$$

Since $S_{n-1}(\alpha)$ is irreducible by induction, if $S_n(\alpha)$ were reducible then it would be union of two surfaces intersecting at B, so that B would be contained in the singular locus of the surface. We now show that $S_n(\alpha)$ is smooth at a point of B, proving in this way that the surface is irreducible. Let $p \in B$ be a point such that $x_3 \cdots x_{n-1} \neq 0$ (just take a point $[x_0 : x_1 : x_2]$ on the conic $\beta_2^n x_2^2 + \beta_1^n x_1^2 + \beta_0^n x_0^2 = 0$ with $x_0 x_1 x_2 \neq 0$ which lies outside the union of the conics $\beta_2^i x_2^2 + \beta_1^i x_1^2 + \beta_0^i x_0^2 = 0$ for any $i < n$). The jacobian criterion implies that B is smooth at p if the following matrix has maximal rank at p.

$$2\begin{pmatrix} -\beta_0^3 x_0 & -\beta_1^3 x_1 & -\beta_2^3 x_2 & x_3 & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ -\beta_0^i x_0 & -\beta_1^i x_1 & -\beta_2^i x_2 & & & x_i & \\ \vdots & \vdots & \vdots & & & & \ddots \\ -\beta_0^{n-1} x_0 & -\beta_1^{n-1} x_1 & -\beta_2^{n-1} x_2 & & & & x_{n-1} \\ -\beta_0^n x_0 & -\beta_1^n x_1 & -\beta_2^n x_2 & 0 & & \cdots & 0 \end{pmatrix}. \tag{2.2}$$

This is the case because the first $n-3$ rows are linearly independent and the last row is not in the row space of the first $n-3$ due to the condition $x_3 \cdots x_{n-1} \neq 0$.

Finally, if we denote by H a hyperplane section of $S_n(\alpha)$ then, by the adjunction formula a canonical divisor of $S_n(\alpha)$ is $K_{S_n(\alpha)} = (n-5)H$ and the last part of the statement follows, being $S_n(\alpha)$ a smooth complete intersection. □

Observe that Büchi surfaces are an example of Generalized Büchi surfaces. Indeed, the projectivity $[x_0 : x_1 : \cdots : x_n] \mapsto [\alpha_0^{-1} x_0 : x_1 : \cdots : x_n]$ maps Q_i to $x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \alpha_0^2 \beta_0^i x_0^2$. Sending α_0 to ∞ and putting $\alpha_k = k$ for the other values of k, the polynomial Q_i is mapped to

$$x_i^2 - (i-1)x_2^2 + (i-2)x_1^2 - (1-i)(2-i)x_0^2.$$

According to the proof of [1, Corollary 2.2] the zero locus of the above polynomials is the n-th Büchi surface. Before stating the second result recall the following definition.

Definition 2.2 The *trivial lines* of the Generalized Büchi surface $S_n(\alpha)$ are the 2^n lines of $\mathbb{P}^n$ defined parametrically, in an affine chart, by

$$t \mapsto [\pm(t-\alpha_0) : \cdots : \pm(t-\alpha_n)].$$

We show that $S_n(\alpha)$ is uniquely determined by its trivial lines. More precisely we have the following.

Proposition 2.3 *Each Generalized Büchi surface is the zero locus of the ideal generated by the quadrics which vanish along its trivial lines.*

Proof Let $\mathcal{L}$ be the set of trivial lines of $S_n(\alpha)$. The statement is equivalent to show that the quadratic part $I(\mathcal{L})_2$ of the ideal $I(\mathcal{L})$ is generated by the homogeneous polynomials Q_i given in the definition of a Generalized Büchi surface. Let $\langle Q_3, ..., Q_n\rangle$ be the linear span of the quadratic polynomials. The inclusion $\langle Q_3, ..., Q_n\rangle \subseteq I(\mathcal{L})_2$ is obvious. To prove the opposite inclusion, let $\sigma_k : \mathbb{P}^n \to \mathbb{P}^n$ be the involution which exchanges the sign of the k-th coordinate. Let $p \in \mathcal{L}$ be a point with all coordinates non-zero and observe that $\sigma_k(p) \in \mathcal{L}$. Given $Q \in I(\mathcal{L})_2$ we have

$$Q = x_k^2 + x_k l_k + g_k, \qquad \text{where } l_k, g_k \in \mathbb{C}[x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n].$$

From $Q(p) = Q(\sigma_k(p)) = 0$, we deduce that $l_k(p) = 0$. This argument shows that the linear polynomial l_k must vanish at a general point of $\mathcal{L}$ and thus it vanishes along $\mathcal{L}$. Since the linear span of $\mathcal{L}$ is the whole space, we conclude that l_k vanishes identically. Repeating the argument for each k proves that Q is a degree two diagonal homogeneous polynomial, $Q = \sum_{i=0}^{n} C_i x_i^2$. Then the polynomial

$$Q' := Q - \sum_{i=3}^{n-2} C_i \mathcal{Q}_i \in I(\mathcal{L})_2$$

has the form $\gamma_0 x_0^2 + \gamma_1 x_1^2 + \gamma_2 x_2^2$, for some complex numbers $\gamma_0, \gamma_1, \gamma_2$. By evaluating Q' at a line of $\mathcal{L}$ one gets a linear combination of the following three polynomials $(t-\alpha_0)^2$, $(t-\alpha_1)^2$, $(t-\alpha_2)^2$ of $\mathbb{C}[t]$. This linear combination must be identically zero because Q' vanishes on $\mathcal{L}$. Since the above three polynomials are linearly independent we conclude that $\gamma_0 = \gamma_1 = \gamma_2 = 0$. Therefore $Q \in \langle \mathcal{Q}_3, ..., \mathcal{Q}_n \rangle$. □

3 Proof of Theorem 1

The aim of this section is to describe the geometry of lines of a Generalized Büchi surface. All the results are summarized in the proof of Theorem 1.

Proof of Theorem 1 We prove (1). If $n = 4$ the surface S is a del Pezzo surface of degree four which thus contains exactly $2^4 = 16$ lines [7, Example 8.6.6.], the trivial ones. The double cover map $S_{n+1}(\alpha) \to S_n(\alpha)$, defined by $[x_0 : \cdots : x_{n+1}] \mapsto [x_0 : \cdots : x_n]$, sends lines to lines. Thus, by induction on n, the surface $S_{n+1}(\alpha)$ contains at most 2^{n+1} lines. On the other hand $S_{n+1}(\alpha)$ contains 2^{n+1} trivial lines, the statement follows.

We prove (2). Since the lines of $S_n(\alpha)$ form one orbit with respect to the group of sign changes, it suffices to prove the statement for the line L parametrized by $t \mapsto [t-\alpha_0 : \cdots : t-\alpha_n]$. Let L' be another trivial line parametrized by $u \mapsto [\varepsilon_0(u-\alpha_0) : \cdots : \varepsilon_n(u-\alpha_n)]$, where $\varepsilon_i \in \{-1, 1\}$ for any i. If $L \cap L'$ is not empty then there exist $t, u \in \mathbb{P}^1$ such that the following matrix has rank one

$$\begin{pmatrix} t-\alpha_0 & \ldots & t-\alpha_n \\ \varepsilon_0(u-\alpha_0) & \ldots & \varepsilon_n(u-\alpha_n) \end{pmatrix}.$$

Let i, j be two indices such that $\varepsilon_i = \varepsilon_j$. The corresponding 2×2 minor is $\varepsilon_i(t-\alpha_i)(u-\alpha_j) - \varepsilon_i(t-\alpha_j)(u-\alpha_i) = \varepsilon_i(t-u)(\alpha_i - \alpha_j)$, so that $u = t$. Applying this substitution, all the 2×2 minors with $\varepsilon_i = \varepsilon_j$ vanish, while each minor with $\varepsilon_i = -\varepsilon_j$ is equal to $\varepsilon_j(t-\alpha_i)(t-\alpha_j) - \varepsilon_i(t-\alpha_j)(t-\alpha_i) = 2\varepsilon_j(t-\alpha_i)(t-\alpha_j)$. Thus in this last case we conclude $t \in \{\alpha_i, \alpha_j\}$. Assume $t = \alpha_i$, then the i-th column of the above matrix is the zero vector and, from the above discussion, we conclude that $\varepsilon_j = \varepsilon_k$ for any j, k different from i. The intersection point is $[\alpha_i - \alpha_0 : \cdots :$

$\alpha_i - \alpha_n]$. This proves that L intersects exactly $n+1$ trivial lines. By applying the morphism $L \to \mathbb{P}^1$, defined by $[z_0 : \cdots : z_n] \mapsto [z_0 - z_1 : \alpha_1 z_0 - \alpha_0 z_1]$, the above intersection point is mapped to $[1 : \alpha_i]$.

We prove (3). Denote by $[n+1]$ the set $\{0, \dots, n\}$. For each subset $\mathcal{P} \subset [n+1]$, denote by $\sigma_{\mathcal{P}}$ the automorphism of $S_n(\alpha)$ that changes the sign of all the variables with indices in $\mathcal{P}$, we observe that $\sigma_{\mathcal{P}} = \sigma_{[n+1]\setminus\mathcal{P}}$. Then each line of $S_n(\alpha)$ can be identified with a subset $\mathcal{P}$ of $[n+1]$, also we note that the complement of $\mathcal{P}$ is identified with the same line. Let us denote by $\mathscr{P}$ the power set of $[n+1]$ and let

$$\mathscr{P}_i := \{\mathcal{P} \in \mathscr{P} : |\mathcal{P}| \equiv i \pmod 2\}.$$

The partition $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_1$ induces a partition $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1$ of the set of lines of $S_n(\alpha)$ because $\mathcal{P} \in \mathscr{P}_i$ if and only if $[n+1] \setminus \mathcal{P} \in \mathscr{P}_i$, being n odd. A consequence of (4) is that given $\mathcal{P}, \mathcal{P}' \in \mathscr{P}$, the corresponding lines, say $L_{\mathcal{P}}$ and $L_{\mathcal{P}'}$ intersect if and only if the automorphism $\sigma_{\mathcal{P}} \circ \sigma_{\mathcal{P}'}$ is the sign change of a single variable, equivalently if, up to relabelling and taking complements, $\mathcal{P}' \subset \mathcal{P}$ and $|\mathcal{P} \setminus \mathcal{P}'| = 1$. From this it follows that $L_{\mathcal{P}}$ and $L_{\mathcal{P}'}$ can not belong to $\mathcal{L}_0$ or $\mathcal{L}_1$ at the same time, since in both cases the lines $L_{\mathcal{P}}$ and $L_{\mathcal{P}'}$ have opposite signs in at least two coordinates, thus the graph is bipartite. Finally, since $|\mathscr{P}_0| = |\mathscr{P}_1|$, the graph is of type $(2^{n-1}, 2^{n-1})$.

We prove (4). This is the content of Proposition 2.3. □

4 Proof of Theorem 2

Let $\Delta \subseteq (\mathbb{P}^1)^{n+1}$ be the Zariski closed subset defined by the equality of two or more factors. Consider the function

$$(\mathbb{P}^1)^{n+1} \setminus \Delta \to \mathtt{GBS}_n \qquad \alpha \mapsto S_n(\alpha).$$

In this section we show that if α' is a $(n+1)$-tuple projectively equivalent to α or it is a permutation of α then $S_n(\alpha)$ is projectively equivalent to $S_n(\alpha')$. As a consequence the above function descends to a function

$$\Phi_n : \mathrm{M}_{0,n+1}/\mathfrak{S}_{n+1} \to \mathtt{GBS}_n,$$

where $\mathrm{M}_{0,n+1}$ is the moduli space of $(n+1)$-tuples of points in $\mathbb{P}^1$.

Lemma 4.1 *Let α and α' be two projectively equivalent $(n+1)$-tuples of distinct points of $\mathbb{P}^1$. Then $S_n(\alpha)$ is projectively equivalent to $S_n(\alpha')$.*

Proof By Proposition 2.3 it is enough to prove that the corresponding unions of trivial lines $\mathcal{L}, \mathcal{L}'$ are projectively equivalent. For simplicity we work with Möbius transformations instead of projectivities of $\mathbb{P}^1$. The group of Möbius transformations is generated by the maps $t \mapsto at + b$, and $t \mapsto t^{-1}$, where t is the complex coordinate and $a, b \in \mathbb{C}$ with $a \neq 0$. The first transformation maps each line of $\mathcal{L}$ to itself. Indeed the line $t \mapsto [\pm(t-\alpha_0) : \cdots : \pm(t-\alpha_n)]$ is mapped to $t \mapsto [\pm(t-(a\alpha_0 + b)) : \cdots : \pm(t-(a\alpha_n + b))]$ and the latter is the same as the original line after reparametrizing t with $at + b$. The second transformation maps $\mathcal{L}$ to the union of lines parametrized by

$$t \mapsto \left[\pm(t - \frac{1}{\alpha_0}) : \cdots : \pm(t - \frac{1}{\alpha_n})\right].$$

The projectivity $\mathbb{P}^n \to \mathbb{P}^n$, defined by $[x_0 : \cdots : x_n] \mapsto [\alpha_0 x_0 : \cdots : \alpha_n x_n]$, maps this set of lines back to $\mathcal{L}$, as one can see after reparameterizing t with t^{-1}. □

Lemma 4.2 *Let α be an $(n+1)$-tuple of distinct points in $\mathbb{P}^1$ and let α' be a permutation of α. Then $S_n(\alpha')$ is projectively equivalent to $S_n(\alpha)$.*

Proof Let $\sigma \in \mathfrak{S}_{n+1}$ be the permutation such that $\alpha'_i = \alpha_{\sigma(i)}$. Let $\phi \in \mathrm{PGL}(n+1, \mathbb{C})$ be the projectivity defined by $\phi([x_0 : \cdots : x_n]) = [x_{\sigma(0)} : \cdots : x_{\sigma(n)}]$. Then $\phi(S_n(\alpha')) = S_n(\alpha)$. □

Proof of Theorem 2 Φ_n is well-defined by Lemma 4.1 and Lemma 4.2. Moreover it is surjective by definition. We now show that Φ_n is injective. Let α, α' be two $(n+1)$-tuples such that $\Phi_n(\alpha) = \Phi_n(\alpha')$, that is $S_n(\alpha)$ is projectively equivalent to $S_n(\alpha')$. In particular the union of lines of the two surfaces are projectively equivalent and thus α and α' are projectively equivalent by Theorem 1. When n is odd the moduli space $\mathrm{M}_{0,n+1}/\mathfrak{S}_{n+1}$ of subsets of cardinality $n+1$ of $\mathbb{P}^1$ is isomorphic to the moduli space of hyperelliptic curves of genus $\frac{1}{2}(n-1)$, the isomorphism being given by taking the double cover of $\mathbb{P}^1$ branched along the $n+1$ points. □

Remark 4.3 Observe that if $Z \subseteq \mathbb{P}^n$, with $n > 5$, is a surface isomorphic to a Generalized Büchi surface $S_n(\alpha) \subseteq \mathbb{P}^n$, then Z is projectively equivalent to $S_n(\alpha)$. Indeed, by adjunction formula a canonical divisor of $S_n(\alpha)$ is $K = (n-5)H$, where H is a hyperplane section. Since $S_n(\alpha)$ is a complete intersection, its Picard group is torsion-free by [6, Theorem 1.8, pag. 49] or [2, Theorem B] and the same holds for Z. In particular there is a unique divisor class $[D]$ on Z such that $K_Z := (n-5)D$ is a canonical divisor. An isomorphism $f\colon S_n(\alpha) \to Z$ maps $[K]$ to $[K_Z]$ and thus, by the above unicity, must map $[D]$ to $[H]$. It follows that $f^*(D)$ is linearly equivalent to H, so that there exists a rational function h on $S_n(\alpha)$ with $\mathrm{div}(h) = f^*(D) - H$. The following map

$$H^0(Z, D) \to H^0(S_n(\alpha), H) \qquad \gamma \mapsto h \cdot (\gamma \circ f)$$

is linear so it induces a projectivity between Z and $S_n(\alpha)$.

Remark 4.4 Summarizing we have that from a Generalized Büchi surface $S \subseteq \mathbb{P}^n$ we can recover a subset $\{\alpha_0, \ldots, \alpha_n\}$ of $\mathbb{P}^1$, up to projective equivalence, which consists of the intersection points of a given line of S with the other lines of S. When

$n = 2g + 1$ is odd one associates to S the genus g hyperelliptic curve C, of affine equation

$$y^2 = (x - \alpha_0) \cdots (x - \alpha_{2g+1}). \tag{4.1}$$

This is the double cover of $\mathbb{P}^1$ branched along the $2g + 2$ points. On the other hand, if we start with a genus $g \geq 2$ hyperelliptic curve, its set of $2g + 2$ Weierstraß points, $\alpha_0, \ldots, \alpha_{2g+1}$, can be made to correspond to a Generalized Büchi surface $S_{2g+1}(\alpha)$ in $\mathbb{P}^{2g+1}$, which is uniquely defined up to projectivities by Lemma 4.1.

5 The Quotient by the Group of Even Sign Changes

Let $g \geq 2$ be an integer and let $S := S_{2g+1}(\alpha) \subseteq \mathbb{P}^{2g+1}$ be a Generalized Büchi surface.

Definition 5.1 Let G be the subgroup of $\mathrm{GL}(2g + 2, \mathbb{C})$ generated by the change of sign of one of the coordinates. The *group of even sign changes* is the index two subgroup

$$G_0 := G \cap \mathrm{SL}(2g + 2, \mathbb{C}).$$

Proposition 5.2 *The surface $Y := S/G_0$ is isomorphic to the following hypersurface of degree $2g + 2$ of $\mathbb{P}(1, 1, 1, g + 1)$:*

$$w^2 = z_0 z_1 z_2 \prod_{i=3}^{2g+1} (\beta_2^i z_2 + \beta_1^i z_1 + \beta_0^i z_0),$$

where the coefficients are the ones appearing in the equations of a Generalized Büchi surface. In particular Y has $\binom{2g+2}{2}$ ordinary double points, coming from the intersection points of the $2g + 2$ lines defined by the vanishing of the right hand side of the above equation. The surface is K3 if $g = 2$ and of general type if $g \geq 3$.

Proof Let $R := \mathbb{C}[x_0, \ldots, x_{2g+1}]$. Since the action is diagonal, the invariant ring R^{G_0} is generated by monomials. Given a monomial $m := \prod_i x_i^{a_i}$, the element of G_0 which changes exactly the signs of the i-th and j-th coordinates maps m to $(-1)^{a_i + a_j} m$. Thus, if m is invariant, $a_i + a_j$ must be even for any pair of distinct indices i, j. This implies that all the a_i have the same parity. It follows that R^{G_0} is generated by $x_0^2, \ldots, x_{2g+1}^2, x_0 \cdots x_{2g+1}$. The corresponding quotient morphism, at the level of (weighted) projective spaces is

$$\mathbb{P}^{2g+1} \to \mathbb{P}(1, \ldots, 1, g + 1) \qquad (x_0, \ldots, x_{2g+1}) \mapsto (x_0^2, \ldots, x_{2g+1}^2, x_0 \cdots x_{2g+1}).$$

The image is the hypersurface of equation $w^2 = z_0 \cdots z_{2g+1}$. The quadratic polynomial $x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \beta_0^i x_0^2$, appearing among the defining equations of S, becomes $z_i - \beta_2^i z_2 - \beta_1^i z_1 - \beta_0^i z_0$ in the new variables. This gives the claimed equation for Y in $\mathbb{P}(1, 1, 1, g+1)$.

From the equation one deduces that Y is a double covering of $\mathbb{P}^2$ branched along the union of $2g+2$ lines. In particular Y is singular at each intersection point of two such lines and the singularity is an ordinary double point. To compute a canonical divisor K_Y for Y we apply the adjunction formula together with the observation that, being Y a normal surface, its canonical divisor is the closure of a canonical divisor of the smooth locus. First of all we recall that a canonical divisor of the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is $K_{\mathbb{P}} = -(a_0 + \cdots + a_n)H$, where H is degree one Weil divisor [5, Theorem 8.2.3]. In particular a canonical divisor of $\mathbb{P}(1, 1, 1, g+1)$ is $(-g-4)H$. Thus we get

$$K_Y = K_{\mathbb{P}} + Y|_Y \sim (-g-4)H + (2g+2)H|_Y = (g-2)H|_Y.$$

The last part of the statement follows. □

Proposition 5.3 *The surface $Y := S/G_0$ is isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of $2g+2$ lines tangent to the conic Γ parametrized by*

$$t \mapsto [(\alpha_0 - t)^2 : (\alpha_1 - t)^2 : (\alpha_2 - t)^2].$$

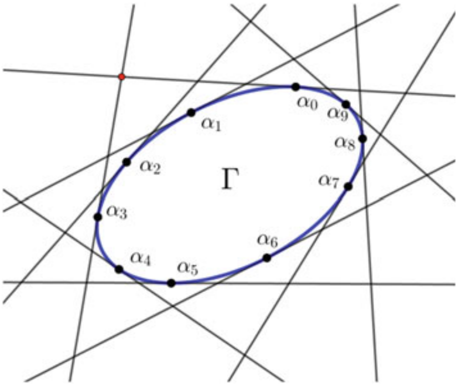
The set of tangency points is projectively equivalent to $\{\alpha_0, \dots, \alpha_{2g+1}\} \subseteq \mathbb{P}^1$.

Proof To show that the $2g+2$ lines are tangent to a conic Γ is equivalent to prove that the points corresponding to these lines, in the dual projective plane, lies on a conic Γ^*. The first three points, corresponding to z_0, z_1 and z_2, are the fundamental points of the dual projective plane, while the remaining ones are of the form $[\beta_0^i : \beta_1^i : \beta_2^i]$, where the β_k^i are the ones appearing in the definition of the Generalized Büchi surfaces. A parametrization of the conic Γ^* is the following:

$$t \mapsto \left[\frac{(\alpha_1 - t)(\alpha_2 - t)}{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)} : -\frac{(\alpha_0 - t)(\alpha_2 - t)}{(\alpha_0 - \alpha_1)(\alpha_1 - \alpha_2)} : \frac{(\alpha_0 - t)(\alpha_1 - t)}{(\alpha_0 - \alpha_2)(\alpha_1 - \alpha_2)} \right].$$

Indeed α_0, α_1 and α_2 are mapped to the fundamental points of the projective plane, while α_i is mapped to $[\beta_0^i : \beta_1^i : \beta_2^i]$. By taking the derivative of the above parametrization one can compute the parametrization for the dual conic Γ, which turns out to be the stated one. This proves both claims in the statement. □

Remark 5.4 We summarize the content of Proposition 5.3 in the following picture.



Each trivial line of S is mapped to Γ by the quotient map $S \to S/G \simeq \mathbb{P}^2$. Indeed the map is defined by $[x_0 : \cdots : x_{2g+1}] \mapsto [x_0^2 : x_1^2 : x_2^2]$, because one can use the equations of S to express all the x_i^2, with $i \geq 3$, as functions of the first three squares. Thus the parametrized line $t \mapsto [\pm(t-\alpha_0) : \cdots : \pm(t-\alpha_{2g+1})]$ is sent to the parametrized conic $t \mapsto [(t-\alpha_0)^2 : (t-\alpha_1)^2 : (t-\alpha_2)^2]$, which is Γ. By replacing this parametrization into the equation of Y we deduce that the double cover $Y = S/G_0 \to S/G$ is trivial over Γ since the curve has the following two preimages of parametric equation

$$t \mapsto \left[(t-\alpha_0)^2 : (t-\alpha_1)^2 : (t-\alpha_2)^2 : \pm \prod_{i=0}^{2g+1} (t-\alpha_i) \right].$$

In particular if we denote by $\Gamma_0, \Gamma_1 \subseteq Y$ the above two curves, corresponding respectively to the sign $+$ and $-$, then the trivial lines of S mapped to Γ_0 are exactly those with an even number of negative signs.

Proposition 5.5 *The surface S is birational to $Y := S/G_0$ if and only if $g = 2$.*

Proof If $g = 2$ then both S and Y are birational to the Kummer surface of the jacobian variety of the genus two curve of equation $y^2 = (x-\alpha_1)\cdots(x-\alpha_5)$. This is classical known, see e.g. [7, Theorem 10.3.16].

Assume now $g > 2$. Denote by K_S and $e(S)$ a canonical divisor and the Euler characteristic of the surface S, respectively. Since $S \subseteq \mathbb{P}^{2g+1}$ is a complete intersection of $2g-1$ quadrics then by [10, Example 2.3] we have

$$K_S^2 = 4(g-2)^2 2^{2g-1} \qquad e(S) = (2g^2 - 5g + 5)2^{2g-1}.$$

Replacing these values in the Noether's formula [3, I.14], and using $h^1(S, \mathcal{O}_S) = 0$ [6, Theorem 1.5 $(iii)_a$] one deduces

$$h^0(S, K_S) = h^2(S, \mathcal{O}_S) = \chi(\mathcal{O}_S) - 1 = (2g^2 - 7g + 7)2^{2g-3} - 1,$$

where the first equality is by Serre's duality. On the other side, consider the surface $Y \subseteq \mathbb{P}(1, 1, 1, g+1)$. The fundamental sequence of Y is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-Y) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Taking tensor product with $\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + Y)$, passing to the long exact sequence in cohomology and recalling that $K_Y \sim K_{\mathbb{P}} + Y|_Y$, we get

$$H^0(\mathbb{P}, K_{\mathbb{P}}) \longrightarrow H^0(\mathbb{P}, K_{\mathbb{P}} + Y) \longrightarrow H^0(Y, K_Y) \longrightarrow H^1(\mathbb{P}, K_{\mathbb{P}}).$$

Since $K_{\mathbb{P}}$ has degree $-g - 4 < 0$, we have $H^0(\mathbb{P}, K_{\mathbb{P}}) = 0$. Moreover by Batyrev-Borisov vanishing [5, Theorem 9.2.7] we have $H^1(\mathbb{P}, K_{\mathbb{P}}) = 0$. It follows that $h^0(Y, K_Y) = h^0(\mathbb{P}, K_{\mathbb{P}} + Y)$, where the last number equals the number of monomials of degree $2g + 2 - (g + 4) = g - 2$ in $\mathbb{P}(1, 1, 1, g + 1)$. Thus we conclude

$$h^0(Y, K_Y) = \binom{g}{2} = \frac{g(g-1)}{2}.$$

Then $h^0(Y, K_Y) < h^0(S, K_S)$ for any $g > 2$. By Proposition 5.2 the surface Y has only ordinary double points, thus its minimal resolution of singularities $\pi\colon \tilde{Y} \to Y$ is crepant (see [8, Theorem 7.5.1]), that is $K_{\tilde{Y}} = \pi^* K_Y$. As a consequence $p_g(\tilde{Y}) = h^0(\tilde{Y}, K_{\tilde{Y}}) = h^0(Y, K_Y)$. Since two smooth projective birational surfaces have the same p_g and $p_g(S) > p_g(\tilde{Y})$ we conclude that S cannot be birational to Y. $\square$

6 Hyperelliptic Curves and Generalized Büchi Surfaces

Let C be hyperelliptic curve of genus $g \geq 2$ and let ι be the hyperelliptic involution. Let $N \simeq (\mathbb{Z}/2\mathbb{Z})^2$ be the subgroup of $\mathrm{Aut}(C \times C)$ generated by the involutions $(p, q) \mapsto (\iota(p), q)$ and $(p, q) \mapsto (p, \iota(q))$. Denote by H the subgroup of $\mathrm{Aut}(C \times C)$ generated by the involution $(p, q) \mapsto (q, p)$. A direct calculation shows that H is in the normalizer of N so that NH is a group which contains N as a normal subgroup. Let

$$\mathrm{Sym}^2(C) := (C \times C)/H$$

be the second symmetric power of C with itself, let $C \times C \to \mathrm{Sym}^2(C)$ be the quotient map and denote by $p + q$ the image of (p, q). Observe that H conmutes with the subgroup N_0 of N generated by the involution $(p, q) \mapsto (\iota(p), \iota(q))$. As a consequence this involution descends to $\mathrm{Sym}^2(C)$ acting as $p + q \mapsto \iota(p) + \iota(q)$. We denote by $\mathrm{Sym}^2(C)/\langle\iota\rangle$ the corresponding quotient surface. Recalling that the symmetric product of $\mathbb{P}^1$ with itself is $\mathbb{P}^2$ we summarize the above construction in the following commutative diagram

$$\begin{tikzcd} C \times C \arrow[r, "/H"] \arrow[d, "/N"', "\pi"] & \mathrm{Sym}^2(C) \arrow[r, "/\langle\iota\rangle"] \arrow[d, "\pi^{(2)}"] & \mathrm{Sym}^2(C)/\langle\iota\rangle \arrow[d, "\pi_\iota^{(2)}"] \\ \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "/H"] & \mathbb{P}^2 \arrow[r, equal] & \mathbb{P}^2 \end{tikzcd} \tag{6.1}$$

where $\pi^{(2)}$ is the degree 4 morphism induced by π. Observe that $\pi^{(2)}$ is *not* a quotient morphism by a group action. In the next theorem we will use the following linear polynomials.

$$\begin{aligned} f_0 &:= (\alpha_1 - \alpha_2)z_0 - (\alpha_0 - \alpha_2)z_1 + (\alpha_0 - \alpha_1)z_2 \\ f_1 &:= (\alpha_1^2 - \alpha_2^2)z_0 - (\alpha_0^2 - \alpha_2^2)z_1 + (\alpha_0^2 - \alpha_1^2)z_2 \\ f_2 &:= \alpha_1\alpha_2(\alpha_1 - \alpha_2)z_0 - \alpha_0\alpha_2(\alpha_0 - \alpha_2)z_1 + \alpha_0\alpha_1(\alpha_0 - \alpha_1)z_2, \end{aligned} \tag{6.2}$$

Theorem 6.1 *Let C be a hyperelliptic curve of genus $g \geq 2$ and let $\alpha_0, \ldots, \alpha_{2g+1} \in \mathbb{P}^1$ be the images of its Weierstraß points. Then $Sym^2(C)/\langle\iota\rangle$ is isomorphic to the hypersurface of $\mathbb{P}(1, 1, 1, g+1)$ of equation*

$$w^2 = \prod_{i=0}^{2g+1} (z_2 - \alpha_i z_1 + \alpha_i^2 z_0).$$

The automorphism of the ambient weighted projective space defined by $[z_0, z_1, z_2, w] \mapsto [f_0, f_1, f_2, \gamma^{g+1}w]$, where $\gamma = (\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)(\alpha_1 - \alpha_2)$, maps Y to $Sym^2 (C)/\langle\iota\rangle$.

Proof We are going to show that $\pi_\iota^{(2)}$ is a double cover branched along the union of the $2g+2$ lines of equations $z_2 - \alpha_i z_1 + \alpha_i^2 z_0 = 0$. These are images of the curves $\{p\} \times C$, where p varies along the Weierstraß points of C. We describe the morphisms of (6.1) in an invariant affine chart of $C \times C$. An affine equation of the curve C is $y^2 = f(x)$, where

$$f(x) = \prod_{i=1}^{2g+1} (x - \alpha_i).$$

Thus in an affine chart, with coordinates x, y, u, v, the equations of $C \times C$ are $y^2 = f(x)$, $v^2 = f(u)$, the generator of H is $(x, u, y, v) \mapsto (u, x, v, y)$ and the generator of N_0 is $(x, y, u, v) \mapsto (x, -y, u, -v)$. Using elementary invariant theory, see e.g. [12], we see that the ring of invariants of H is generated by $x + u$, $y + v$, xu, $xy + uv$, $y^2 + v^2$. The latter can be omitted when restricting to $C \times C$ because it is expressed as a symmetric function of x and u. Similarly one sees that the ring of invariants of N_0 is generated by $x + u$, xu, yv, $y^2 + v^2$, $xy^2 + uv^2$. The latter two can be omitted when restricting to $C \times C$ because they are expressed as symmetric functions of x and u. Thus, in these coordinates, the morphisms in (6.1) are given by

$$\begin{tikzcd}
(x,u,y,v) \arrow[r] \arrow[d, "\pi"] & (x+u, xu, y+v, xy+uv) \arrow[r] \arrow[d, "\pi^{(2)}"] & (x+u, xu, yv) \arrow[d, "\pi_\iota^{(2)}"] \\
(x,u) \arrow[r] & (x+u, xu) \arrow[r, equal] & (x+u, xu).
\end{tikzcd} \tag{6.3}$$

From the above description of $\pi_\iota^{(2)}$ we see that it ramifies along the curves with $yv = 0$. By the equation of $C \times C$ these are the images of the curves $\{p\} \times C$, where p is a Weierstraß point. These are the lines of parametric equation $t \mapsto (\alpha_i + t, \alpha_i t)$, whose homogeneous cartesian equation is $z_2 - \alpha_i z_1 + \alpha_i^2 z_0 = 0$.

To prove the last part of the statement it suffice to evaluate the latter polynomial at $z_i = f_i$, for $i = 0, 1, 2$. A direct calculation shows that the following holds:

$$f_2 - \alpha_i f_1 + \alpha_i^2 f_0 = \gamma(\beta_2^i z_2 + \beta_1^i z_1 + \beta_0^i z_0),$$

proving the statement. $\square$

6.1 Rationals Points

The isomorphism $Y \simeq \mathrm{Sym}^2(C)/\langle\iota\rangle$ given in Theorem 6.1 is defined over the field $\mathbb{Q}(\alpha_0, \alpha_1, \alpha_2)$. On the other hand, by exchanging the labels of the Weierstraß points one can construct a similar isomorphism defined over the field $\mathbb{Q}(\alpha_i, \alpha_j, \alpha_k)$, for any i, j, k. In particular if the vector α has at least three rational entries then the isomorphism is defined over the rationals. Let us denote by $F : C \times C \to \mathrm{Sym}^2(C)/\langle\iota\rangle$ the degree four quotient map which in affine coordinates is given by

$$(x, u, y, v) \mapsto (xu, x + u, yv).$$

Given a point $(p, q) \in C \times C$ we denote by $\mathbb{Q}(p, q)$ the extension of the rationals obtained by adding the affine coordinates of the points p and q. Thus if $p = (x, y)$ and $q = (u, v)$ then $\mathbb{Q}(p, q) := \mathbb{Q}(x, u, y, v)$. We say that a point of $\mathrm{Sym}^2(C)/\langle\iota\rangle$ is *rational* if all of its coordinates are rational numbers.

Proposition 6.2 *Let $(p, q) \in C \times C$ be the preimage, via F, of a rational point of $Sym^2(C)/\langle\iota\rangle$. Then the field extension $\mathbb{Q}(p, q)/\mathbb{Q}$ is Galois with group G_{pq} isomorphic to a subgroup of the Klein group.*

Proof By assumption xu, $x + u$, yv are rational numbers so that $\mathbb{Q}(p, q) = \mathbb{Q}(x, u, y, v) = \mathbb{Q}(x, y)$. By the tower law it follows that $d := [\mathbb{Q}(x, y) : \mathbb{Q}]$ is a divisor of 4. To prove that the extension is Galois observe that an element of the absolute Galois group preserves the set $\{x, u\}$ and the set $\{-y, y, -v, v\}$, so that it preserves $\mathbb{Q}(x, y)$.

Since the degree of the extension is a divisor of 4, the only possibilities for the Galois group are: a subgroup of the Klein group or the order four cyclic group. The last possibility cannot occur since an element $\sigma \in G_{pq}$ of order four would act, up to exchange the roles of y and v, as $\sigma(y) = v$ and $\sigma(v) = -y$, so that yv would not be invariant, a contradiction. □

The above proposition shows that the search for rational points on the surface $\mathrm{Sym}^2(C)/\langle\iota\rangle$ leads to look for points of C which live in an abelian extension of the rationals whose Galois group is a subgroup of the Klein group. In case this group is trivial, then both p and q would be rational points of C. We show in the following proposition that when $g = 4$ we know all such points.

Proposition 6.3 *The only rational points of the hyperelliptic curve of equation* $y^2 = \prod_{n=-4}^{4}(x-n)$ *are the Weierstraß points.*

Proof Let C be the hyperelliptic curve. By [11, pp. 15] if p is a prime of good reduction of C then

$$\#C(\mathbb{Q}) \le \#\overline{C}(\mathbb{F}_p) + 2r + \left\lfloor \frac{2r}{p-2} \right\rfloor,$$

where $\overline{C}$ is the reduction of C modulo p and r is the rank of the group $JC(\mathbb{Q})$ of rational points on the Jacobian variety. The discriminant of C is $2^{62}3^{18}5^{8}7^{4}$, so $p = 13$ is a prime of good reduction for C. Moreover $\#\overline{C}(\mathbb{F}_{13}) = 10$ and $r = 0$. All these calculations can be checked by means of the following Magma [4] code.

```
> R<x> := PolynomialRing(Rationals());
> C := HyperellipticCurve(&*[x-n : n in [-4..4]]);
> Factorization(Numerator(Discriminant(C)));
[ <2, 62>, <3, 18>, <5, 8>, <7, 4> ]
> #Points(ChangeRing(C,GF(13)));
10
> RankBounds(Jacobian(C));
0 0
```

Since $C(\mathbb{Q})$ contains the nine Weierstraß points together with the point at infinity, the statement follows. □

References

1. Artebani, M., Laface, A., Testa, D.: On Büchi's K3 surface. Math. Z. **278**(3–4), 1113–1131 (2014)
2. Bădescu, L.: A remark on the Grothendieck-Lefschetz theorem about the Picard group. Nagoya Math. J. **71**, 169–179 (1978)

3. Beauville, A.: Complex algebraic surfaces. London Mathematical Society Student Texts, 2nd ed., vol. 34. Cambridge University Press, Cambridge (1996). Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid
4. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. J. Symbol. Comput. **24** (3–4), 235–265 (1997). Computational algebra and number theory (London, 1993)
5. Cox, D.A., Little, J.B., Schenck, H.K.: Toric varieties. Graduate Studies in Mathematics, vol. 124. American Mathematical Society, Providence, RI (2011)
6. Groupes de monodromie en géométrie algébrique. II. Lecture Notes in Mathematics, Vol. 340. Springer-Verlag, Berlin-New York: Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). Dirigé par P. Deligne et N, Katz (1973)
7. Dolgachev, I.V.: Classical Algebraic Geometry. Cambridge University Press, Cambridge (2012). A modern view
8. Ishii, S.: Introduction to Singularities, 2nd ed. Springer, Tokyo (2018)
9. Pasten, H., Pheidas, T., Vidaux, X.: A survey on Büchi's problem: new presentations and open problems. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 377 (Issledovaniya po Teorii Chisel. 10), 111–140, 243 (2010); English transl.,. J. Math. Sci. (N.Y.) **171**(6), 765–781, 2010
10. Peters, C., Sterk, H.: Complete intersections of quadrics and complete intersections on Segre varieties with common specializations. Online notes (2017)
11. Stoll, M.: Rational Points on curves, Universitat Bayreuth. Online notes, Germany (2015)
12. Sturmfels, B.: Algorithms in Invariant Theory, 2nd edn. Springer Wien- NewYork, Vienna (2008)

Fano Visitors, Fano Dimension and Fano Orbifolds

Young-Hoon Kiem and Kyoung-Seog Lee

Abstract Bondal asked whether the derived category of any smooth projective variety can be embedded into the derived category of a Fano variety. In [31], the authors proved that every complete intersection smooth projective variety Y is a Fano visitor, i.e. its derived category $D^b(Y)$ is equivalent to a full triangulated subcategory of the derived category $D^b(X)$ of a smooth Fano variety X, called a Fano host of Y. They also introduced the notion of Fano dimension of Y as the smallest dimension of a Fano host X and obtained an upper bound for the Fano dimension of each complete intersection variety. In this paper, we generalize Bondal's question and study triangulated subcategories of derived categories of Fano orbifolds. We proved that there are many interesting triangulated categories which can be embedded into derived categories of Fano orbifolds. We also find more examples of Fano visitors and determine the Fano dimensions precisely for many interesting examples.

Keywords Fano visitors · Fano dimension · Fano orbifolds

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1 Introduction

If one were to write up a list of keywords that describe recent developments in algebraic geometry, it would be hard to miss the words like "derived category" or "categorification". The derived category $D^b(X)$ of bounded complexes of coherent sheaves of a projective variety X was found to be a sophisticated invariant which categorifies geometric invariants such as Hochschild homology, Hochschild cohomology and Grothendieck groups of algebraic varieties (cf. [37]). One of the basic problems in algebraic geometry is to study how invariants of a given variety can be encoded in invariants of the other varieties. In 2011, Bondal raised the following question (cf. [7]).

Question 1.1 (Fano visitor problem)
Let Y be a smooth projective variety. Is there a Fano variety X equipped with a fully faithful embedding $D^b(Y) \to D^b(X)$?

If the answer is yes, we call Y a *Fano visitor* and X a *Fano host* of Y. From the categorical point of view, Fano varieties are of particular interest because they admit natural semiorthogonal decompositions and many examples have been explicitly calculated (cf. [7, 10, 35, 36, 54]). They are also one of the main objects in birational geometry and mirror symmetry. If the answer to Question 1.1 is yes for all smooth projective varieties, some problems about derived categories may be effectively reduced to those of Fano varieties. Moreover the geometry and invariants of X are closely related to those of Y. Especially, it turns out that moduli spaces of rational curves or vector bundles on X are closely related to the geometry of Y. See [14, 39, 40, 44] for such examples.

Bondal and Orlov in [10] proved that the derived category of a hyperelliptic curve Y of genus g is embedded into the derived category of the intersection of two quadrics in $\mathbb{P}^{2g+1}$. Kuznetsov in [35] proved that the derived categories of some K3 surfaces are embedded into special cubic 4-folds. He also found that some Fano 3-folds contain the derived categories of certain smooth projective curves (cf [36]). Bernardara, Bolognesi and Faenzi in [7] proved that every smooth plane curve is a Fano visitor. Segal and Thomas in [54] proved that a general quintic 3-fold is a Fano visitor by finding an 11-dimensional Fano host. In [31], the authors proved the following.

Theorem 1.2 ([31, Theorem 4.1]) *All smooth projective complete intersections are Fano visitors.*

Moreover, they defined the *Fano dimension* of a smooth projective variety Y as the minimum dimension of Fano hosts X of Y. The Fano dimension is defined to be infinite if no Fano hosts exist. It was also proved that an arbitrary complete intersection Calabi-Yau variety Y of codimension ≤ 2 or a general complete intersection Calabi-Yau variety of codimension ≥ 3 has Fano dimension at most $\dim Y + 2$.

We generalize the construction and technique of [31] for complete intersections in more general varieties such as Grassmannians (cf. Theorem 3.1) or other homogeneous varieties. Using this, we prove that smooth curves of genus at most 4 are all Fano visitors and general curves of genus at most 9 are Fano visitors. For surfaces and higher dimensional varieties, we find more examples of Fano visitors and raise natural questions. We also provide a Hodge-theoretic criterion for the existence of a Fano host and decide Fano dimensions of several interesting examples. For instance, an arbitrary complete intersection Calabi-Yau variety Y of codimension ≤ 2 or a general complete intersection Calabi-Yau variety Y of codimension ≥ 3, the Fano dimension is precisely $\dim Y + 2$.

From the perspective of recent developments of the theory of Fano varieties, it seems to be natural to consider Fano varieties having singularities. By works of Kawamata (cf. [28–30]), it turns out that considering derived categories of smooth Deligne-Mumford stacks instead of considering derived categories of their coarse moduli spaces has many advantages. Therefore we investigate derived categories of Fano orbifolds instead of derived categories of Fano varieties having only quotient singularities. Here a Fano orbifold means a smooth Deligne-Mumford stack whose coarse moduli space is Fano. Moreover Fano orbifolds naturally appear in many context, e.g. mirror symmetry, orbifold Kahler-Einstein metric, etc. It will be nice if one can find a way to relate every Fano variety a smooth Artin stack whose coarse moduli space is the Fano variety, but it seems that we do not have such a method yet. Therefore a natural generalization of Bondal's original Fano visitor problem will be as follows.

Question 1.3 *(1) Which triangulated categories can be embedded into derived categories of smooth Deligne-Mumford stacks or smooth Artin stacks whose coarse moduli spaces are Fano?*
(2) For a smooth projective variety Y, is there a Fano orbifold $\mathcal{X}$ such that $D^b(\mathcal{X})$ contains $D^b(Y)$ as a full triangulated subcategory?

In this paper, we restrict ourselves to consider only triangulated subcategories of Fano orbifolds and found many examples of varieties whose derived categories are contained in derived categories of Fano orbifolds.

Definition 1.4 Let $\mathcal{Y}$ be an algebraic stack. If there is a Fano orbifold $\mathcal{X}$ such that $D^b(\mathcal{X})$ contains $D^b(\mathcal{Y})$ then we say $\mathcal{Y}$ has an orbifold Fano host $\mathcal{X}$.

Then we can find many examples of varieties which have orbifold Fano hosts. For example, we have the following result. See Theorem 3.5 for more precise statement and details.

Theorem 1.5 *Every quasi-smooth weighted complete intersection orbifold in a weighted projective space has an orbifold Fano host.*

Then it immediately follows that hyperelliptic curves, 95 families of (orbifold) K3 surfaces of Reid and many other well-known examples of quasi-smooth weighted complete intersection orbifolds (cf. [1]) have orbifold Fano hosts. Moreover we can find Fano orbifolds whose derived categories contain derived categories of many interesting varieties, e.g. Jacobians of curves, generic Enriques surfaces, some families of Kummer surfaces, bielliptic surfaces, certain surfaces with $\kappa = 1$, classical Godeaux surfaces, product-quotient surfaces, holomorphic symplectic varieties, etc. However we do not know whether there are smooth projective Fano varieties whose derived categories contain derived categories of these varieties yet.

An interesting recent discovery in the theory of derived categories is the existence of quasi-phantom subcategories in derived categories of some surfaces of general type with $p_g = q = 0$ [8, 9, 18, 42, 43, 45]. But no examples of Fano with quasi-phantom subcategories have been found. From the above constructions, we found Fano orbifolds whose derived categories contain quasi-phantom categories or phantom categories (cf. Example 6.18). As far as we know, this is the first discovery of a quasi-phantom category in the realm of Fano (orbifolds). However we do not know whether there is a smooth Fano variety with a (quasi-)phantom category.

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Notation. In this paper, all schemes and stacks are defined over the complex number field $\mathbb{C}$. For a vector bundle E on S, the projectivization $\mathbb{P}E := \operatorname{Proj}\left(\operatorname{Sym}^{\cdot} E^\vee\right)$ of E parameterizes one dimensional subspaces in fibers of E. For an algebraic stack $\mathcal{X}$, $D^b(\mathcal{X})$ denotes the bounded derived category of coherent sheaves on $\mathcal{X}$. The zero locus $s^{-1}(0)$ of a section $s : \mathcal{O}_X \to E$ of a vector bundle E over a scheme X is the closed subscheme of X whose ideal is the image of $s^\vee : E^\vee \to \mathcal{O}_X$.

2 Preliminaries

In this section we recall several definitions and facts which we will use later.

2.1 Fano Visitor Problem

Let us recall several definitions on Fano varieties.

Definition 2.1 ([27]) A normal projective variety X is Fano if $-K_X$ is $\mathbb{Q}$-Cartier and ample.

We learned the definition of Fano visitor from [7].

Definition 2.2 An algebraic stack $\mathcal{Y}$ is called a *Fano visitor* if there is a smooth projective Fano variety X together with a fully faithful (exact) embedding $D^b(\mathcal{Y}) \to D^b(X)$. We call such a Fano X a *Fano host* of $\mathcal{Y}$. If there is a smooth Deligne-Mumford stack $\mathcal{X}$ whose coarse moduli space is Fano and $D^b(\mathcal{X})$ contains $D^b(\mathcal{Y})$ as a full triangulated subcategory, then $\mathcal{X}$ is called an orbifold Fano host of $\mathcal{Y}$.

Remark 2.3 Let Y be a singular variety. Then there are objects e_1 and e_2 such that $Hom(e_1, e_2[i])$ is nonzero for infinitely many i. Therefore there is no Fano orbifold $\mathcal{X}$ such that $D^b(\mathcal{X})$ contains $D^b(Y)$ as a full triangulated subcategory.

Bondal's question (Question 1.1) asks whether a smooth projective variety is a Fano visitor. It is easy to see that a Fano host X of a smooth projective variety Y is not unique because for instance the product X and any smooth Fano variety is also a Fano host of Y. So we may ask for a Fano host of minimal dimension.

Definition 2.4 ([31]) The *Fano dimension* of a smooth projective variety Y is the minimum among the dimensions $\dim X$ of Fano hosts X of Y.

See [31] for more discussions and questions related to Fano visitors.

2.2 Coarse Moduli Spaces of Deligne-Mumford Stacks

We refer the reader to [19, 28, 29, 50, 55] for basic definition and properties about algebraic stacks and coherent sheaves on them. From the famous Keel-Mori theorem, we know that every algebraic stack locally of finite presentation with finite diagonal has a coarse moduli space (cf. [50, Theorem 11.1.2]). Sometimes we need to compare derived categories of Deligne-Mumford stacks and their coarse moduli spaces.

Lemma 2.5 *Let $\mathcal{X}$ be a Deligne-Mumford stack locally of finite type with finite diagonal and X be its coarse moduli space. Suppose that X is a smooth projective variety. Then we have a fully faithful functor $L\pi^* : D^b(X) \to D^b(\mathcal{X})$.*

Proof From [50, Theorem 11.1.2] and [50, Proposition 11.3.4], we have an isomorphism $\mathcal{O}_X \to R\pi_*\mathcal{O}_{\mathcal{X}}$. Because X is a smooth projective variety we know that every object in $D^b(X)$ is a perfect complex. Then we have a canonical isomorphisms $Hom^k(L\pi^*a, L\pi^*b) \cong Hom^k(a, R\pi_*L\pi^*b) \cong Hom^k(a, b)$ for any k and

a, b which are objects in $D^b(X)$ by the adjunction formula and the projection formula (cf. [24, Corollary 4.12], [50, Proposition 9.3.6]). Therefore we see that $L\pi^*$ is a fully faithful functor. □

We will use that quotient stacks of Fano varieties by finite groups form a natural class of Fano orbifolds.

Corollary 2.6 *Let X be a smooth Fano variety and G be a finite group acting on X. Suppose that the locus with nontrivial stabilizer on X has codimension at least 2. Then $[X/G]$ is a Fano orbifold.*

Proof It is easy to see that $[X/G]$ is a smooth Deligne-Mumford stack. From [55, Proposition 2.11], we see that X/G is the coarse moduli space of $[X/G]$. From the assumption on the G-action, there is an open subset of X where G acts freely and the complement of it has codimension at least 2. Over this open subset, the canonical bundle is the pullback of the canonical bundle of the image of the open subset. Because the canonical divisors can be extended into the whole spaces by taking closures, we see that the canonical divisor of X is the pullback of the canonical divisor of X/G. Therefore the anticanonical divisor of X/G is ample since its pullback to X is ample (cf. [41, Corollary 1.2.28]). □

2.3 Semiorthogonal Decomposition

We recall the definition and examples of semiorthogonal decompositions of derived categories of coherent sheaves.

Definition 2.7 Let $\mathcal{T}$ be a triangulated category. A *semiorthogonal decomposition* of $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, \cdots, \mathcal{A}_n$ satisfying the following properties:
(1) $Hom_{\mathcal{T}}(a_i, a_j) = 0$ for any $a_i \in \mathcal{A}_i, a_j \in \mathcal{A}_j$ with $i > j$;
(2) the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{A}_1, \cdots, \mathcal{A}_n$ is $\mathcal{T}$.
We will write $\mathcal{T} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_n \rangle$ to denote the semiorthogonal decomposition.

Let E be a vector bundle of rank $r \geq 2$ over a smooth variety S and let $Y = s^{-1}(0) \subset S$ denote the zero locus of a regular section $s \in H^0(S, E)$ such that $\dim Y = \dim S - \text{rank } E$. Let $X = w^{-1}(0) \subset \mathbb{P}E^\vee$ be the zero locus of the section $w \in H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1))$ determined by s under the natural isomorphisms

$$H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1)) \cong H^0(S, q_*\mathcal{O}_{\mathbb{P}E^\vee}(1)) \cong H^0(S, E)$$

where $q : \mathbb{P}E^\vee \to S$ is the projection map of the projective bundle.

Orlov proved in [52] that $D^b(X)$ has the following semiorthogonal decomposition which was subsequently generalized to higher degree hypersurface fibrations by Ballard, Deliu, Favero, Isik and Katzarkov in [3].

Theorem 2.8 ([52, Proposition 2.10], [3]) *There is a natural semiorthogonal decomposition*

$$D^b(X) = \langle q^*D^b(S), \cdots, q^*D^b(S) \otimes_{\mathcal{O}_X} \mathcal{O}_X(r-2), D^b(Y)\rangle.$$

Remark 2.9 Orlov proved in particular that there is a fully faithful exact functor from $D^b(Y)$ to $D^b(X)$ (cf. [52, Proposition 2.2]). When an algebraic group G acts on S and E compatibly and s is a G-invariant section, there is an induced action of G on X and Y. His proof also works for this equivariant setting to give us a fully faithful exact functor from $D^b([Y/G])$ to $D^b([X/G])$. See [52, Remark 2.9].

We can provide many examples of orbifold Fano hosts of interesting algebraic varieties using the following result of Ploog in [53] which was generalized by Krug and Sosna in [34].

Theorem 2.10 ([34, 53]) *Let X, Y be smooth projective varieties with G-action where G is a finite group. Suppose that $\Phi_K : D^b(Y) \to D^b(X)$ is a fully faithful functor and K has a G-linearization with respect to the diagonal G-action on $Y \times X$. Then K induces a functor $\Phi_K^G : D^b([Y/G]) \to D^b([X/G])$ which is also fully faithful.*

3 Cayley's Trick and Weighted Complete Intersections

In this section, we recall and generalize the main construction and result in [31].

3.1 Cayley's Trick

Let S be a smooth variety and $s \in H^0(S, E)$ be a regular section of a vector bundle of rank $r \geq 2$ such that $Y = s^{-1}(0)$ is smooth of dimension $\dim S - r$. Let $\mathbb{P}E^\vee = \mathrm{Proj}\,(\mathrm{Sym}^{\cdot} E)$ denote the projectivization of $E^\vee$. Then we have an isomorphism

$$H^0(S, E) \cong H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(1))$$

which gives us a section w of $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ corresponding to s. Let $X = w^{-1}(0)$. Since Y is smooth, X is also smooth by a local computation. We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}N^\vee \arrow[r, dash, "i"] \arrow[d, "p"'] & X \arrow[r, dash] & \mathbb{P}E^\vee \arrow[d] \\
Y \arrow[rr, dash] & & S
\end{tikzcd}$$

where N is the normal bundle of Y in S. By Orlov's theorem (cf. Theorem 2.8), there is a fully faithful embedding $Ri_*Lp^* : D^b(Y) \to D^b(X)$. Therefore if X is Fano, then X is a Fano host of Y and Y is a Fano visitor.

Note that there is an embedding $\mathbb{P}N^\vee \to Y \times X$ induced from the above diagram and the functor $Ri_*Lp^* : D^b(Y) \to D^b(X)$ is a Fourier-Mukai transform Φ_K whose kernel K is $\mathcal{O}_{\mathbb{P}N^\vee}$. Suppose that there is an algebraic group G acting on Y and the action extends to S and E and Y is given by an invariant section s. Then G acts on $\mathbb{P}N^\vee$ and $\mathcal{O}_{\mathbb{P}N^\vee}$ has a canonical G-linearization induced by the group action. When G is a finite group, we can recover the Remark 2.9 of Orlov from the Theorem 2.10. Moreover it holds when G is a reductive algebraic group (cf. [52, Remark 2.9]).

3.2 Complete Intersections in Projective Space

When $Y \subset \mathbb{P}^m$ is a smooth complete intersection defined by a section s' of $\oplus_{i=1}^l \mathcal{O}_{\mathbb{P}^m}(a_i)$ with $a_i > 0$ and $l \geq 0$, we enlarge the ambient space $\mathbb{P}^m$ to $\mathbb{P}^{m+c} = S$ and extend the vector bundle $\bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^m}(a_i)$ to

$$\bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^{m+c}}(a_i) \oplus \mathcal{O}_{\mathbb{P}^{m+c}}(1)^{\oplus c} = E$$

for $c \geq 0$. The section s' together with a choice of defining linear equations for $\mathbb{P}^m \subset \mathbb{P}^{m+c}$ gives us a section s of E with $s^{-1}(0) = s'^{-1}(0) = Y$. Applying Cayley's trick above, we obtain a hypersurface $X = w^{-1}(0)$ of $\mathbb{P}E^\vee$ whose dimension is $m + 2c + l - 2 = \dim Y + 2c + 2l - 2$.

The authors proved in [31, Sect. 4.2] that if c is greater than $\sum_{i=1}^l a_i - m - l$ and $1 - l$, then X is Fano. This proves the main result (Theorem 1.2) of [31] because X is a Fano host of Y by the discussion in Sect. 3.1.

3.3 A Generalization

We can capture the essence of the proof of Theorem 1.2 in [31] as follows.

Theorem 3.1 *Let S be a smooth projective variety and s be a section of a vector bundle E of rank $r \geq 2$ over S whose zero locus is a smooth subvariety $Y = s^{-1}(0)$ of codimension r. Let X and w be as in Cayley's trick in Sect. 3.1. Suppose that*

(1) E is ample and $K_S^\vee \otimes \det E^\vee$ is nef, or
(2) there is a nef line bundle H such that $F := E \otimes H^\vee$ is a nef vector bundle and that $K_S^\vee \otimes \det E^\vee \otimes H^{r-1}$ is ample.

Then $X = w^{-1}(0)$ is a Fano host of $Y = s^{-1}(0)$.

Proof By Theorem 2.8, it suffices to show that X is Fano. For (1), see [31, Lemma 3.1]. For (2), let $q : \mathbb{P}E^\vee \to S$ denote the canonical projection. Let us compute K_X. From the relative Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}E^\vee} \longrightarrow q^*E^\vee \otimes \mathcal{O}_{\mathbb{P}E^\vee}(1) \longrightarrow T_{\mathbb{P}E^\vee/S} \longrightarrow 0,$$

we have $K^\vee_{\mathbb{P}E^\vee/S} = (q^* \det E^\vee) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(r)$. From $K_{\mathbb{P}E^\vee} = q^*K_S \otimes K_{\mathbb{P}E^\vee/S}$ we have

$$K^\vee_{\mathbb{P}E^\vee} \cong q^*(K_S^\vee \otimes \det E^\vee) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(r).$$

Therefore we get

$$\begin{aligned} K_X^\vee = K^\vee_{\mathbb{P}E^\vee} \otimes \mathcal{O}(-1)|_X &\cong q^*(K_S^\vee \otimes \det E^\vee) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(r-1)|_X \\ &\cong q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(r-1)|_X. \end{aligned}$$

By assumption, both $q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1})$ and $\mathcal{O}_{\mathbb{P}F^\vee}(r-1)$ are nef line bundles, hence so is $K_X^\vee$. To see that $K_X^\vee$ is big, let us compute the intersection number $(K_X^\vee)^{\dim X}$ as follows:

$$\begin{aligned} (K_X^\vee)^{\dim X} &= (q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(r-1)|_X)^{\dim X} \\ &= (q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(r-1))^{\dim X} \cdot \mathcal{O}_{\mathbb{P}E^\vee}(1) \\ &= (q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(r-1))^{\dim X} \cdot (q^*H \otimes \mathcal{O}_{\mathbb{P}F^\vee}(1)). \end{aligned}$$

By the binomial expansion formula, we see that $(K_X^\vee)^{\dim X}$ is positive since every term is a multiple of a nef line bundle and $q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1})^{\dim S} \cdot \mathcal{O}_{\mathbb{P}F^\vee}(1)^{r-1}$ is strictly positive by our assumption. Therefore $K_X^\vee$ is nef and big, i.e. X is a weak Fano variety. Then the Mori cone of X is rational polyhedral and the extremal rays are generated by rational curves by [56, Theorem 1.4].

Finally we claim that $K_X^\vee$ intersects positively with all irreducible curves. Let C be an irreducible curve in $\mathbb{P}E^\vee = \mathbb{P}F^\vee$. If $q(C)$ is a point, then the degree of $\mathcal{O}_{\mathbb{P}F^\vee}(r-1)|_C$ is positive because $\mathcal{O}_{\mathbb{P}F^\vee}(1)$ is ample on each fiber of $q : \mathbb{P}E^\vee \to S$. If $q(C)$ is a curve, then the degree of $q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1})|_C$ is positive since $K_S^\vee \otimes \det E^\vee \otimes H^{r-1}$ is ample. From our assumptions, we find that the degree of the line bundle $q^*(K_S^\vee \otimes \det E^\vee \otimes H^{r-1}) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(r-1)|_C$ is always positive. Since the Mori cone is polyhedral, this implies that $K_X^\vee$ is ample and X is a Fano variety. □

Remark 3.2 In the proof of Theorem 1.2 in [31, Sect. 4.2], we used $H = \mathcal{O}_{\mathbb{P}^{m+c}}(1)$ and chose sufficient large c as written in Sect. 3.2. However when the degrees of defining equations of Y are large enough, then the above theorem tells us that we can choose larger H and smaller c. This often gives a Fano host of smaller dimension as in the following example.

Example 3.3 Let C be a non-hyperelliptic curve of genus 4. Then C is the complete intersection of a quadric and a cubic in $\mathbb{P}^3$, i.e. C is the zero locus of a regular section s of $E = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(3)$ over $S = \mathbb{P}^3$. Let $F = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ with $H = \mathcal{O}_{\mathbb{P}^3}(2)$. From the above theorem, we find that $X = w^{-1}(0)$ in Cayley's trick (cf. Sect. 3.1) is a 3-dimensional Fano host of C because F is nef and $K_S^\vee \otimes \det E^\vee \otimes H^{r-1} = \mathcal{O}_{\mathbb{P}^3}(1)$ is ample. Note that if we insist on using $H = \mathcal{O}_{\mathbb{P}^3}(1)$ instead, we have to enlarge $\mathbb{P}^3$ to $\mathbb{P}^4$ and extend $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(3)$ to $\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3) \oplus \mathcal{O}_{\mathbb{P}^4}(1)$, so that the Fano host is 5 dimensional.

By Example 3.3, we find that a non-hyperelliptic curve C of genus 4 has Fano dimension at most 3. We will see below that indeed 3 is the Fano dimension of C.

3.4 *Weighted Complete Intersections*

We can also extend the main theorem of [31] for the following situation. Let $\bar{a} = (a_0, a_1, \cdots, a_n)$ be a sequence of positive integers. Then consider the graded polynomial ring $\mathbb{C}[z_0, \cdots, z_n]$ with degree of $z_i = a_i$. Then we can define the weighted projective space $\mathbb{P}(\bar{a})$ as the projective variety $Proj(\mathbb{C}[z_0, \cdots, z_n])$. Note that giving a grading on $\mathbb{C}[z_0, \cdots, z_n]$ corresponds to giving a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$. We also define $\mathcal{P}(\bar{a})$ to be the smooth Deligne-Mumford stack $[\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*]$ whose coarse moduli space is $\mathbb{P}(\bar{a})$.

When we consider derived categories then it is more natural to consider a weighted projective space as a smooth Deligne-Mumford stack $\mathcal{P}(\bar{a})$. However when we do geometry it is easier to consider a weighted projective space as a projective variety $\mathbb{P}(\bar{a})$. See [15] for more details. Let us recall some of relevant definitions.

Definition 3.4 ([15]) (1) For a closed subscheme $Y \subset \mathbb{P}(\bar{a})$, we can associate a quasi-cone C_Y which is the scheme closure of the inverse of Y in the $\mathbb{C}^{n+1}$. Let us also denote C_Y^* to be $C_Y - \{0\}$.
(2) A closed subscheme $Y \subset \mathbb{P}(\bar{a})$ is called quasi-smooth if C_Y is smooth outside of its vertex.
(3) Y is a weighted complete intersection of multidegree $\bar{d} = (d_1, \cdots, d_c)$ if I_Y is generated by a regular sequence of homogeneous elements $f_1, \cdots, f_c$ where the degree of f_i is d_i.

For every quasi-smooth weighted complete intersection Y we consider its associated stack $\mathcal{Y} = [C_Y^*/\mathbb{C}^*]$. It is a smooth Deligne-Mumford stack with coarse moduli space Y. We call $\mathcal{Y}$ a quasi-smooth weighted complete intersection orbifold in a weighted projective space.

Theorem 3.5 *Every quasi-smooth weighted complete intersection orbifold $\mathcal{Y}$ in a weighted projective space has an orbifold Fano host.*

Proof Let Y be a quasi-smooth weighted complete intersection is a weighted projective space $\mathbb{P}(\bar{a})$. We can embed $\mathbb{P}(\bar{a})$ into $\mathbb{P}(\bar{a}, 1, \ldots, 1)$ and Y is again a quasi-smooth weighted complete intersection in a weighted projective space $\mathbb{P}(\bar{a}, 1, \cdots, 1)$. Therefore we may assume that the dimension n of $\mathbb{P}(\bar{a})$ is large enough. Then C_Y^* is a complete intersection in $\mathbb{C}^{n+1} - \{0\}$. We can regard C_Y^* as a zero set of a section of the rank c trivial vector bundle on $\mathbb{C}^{n+1} - \{0\}$. We can use Cayley's trick to construct C_X^* as follows.

$$\begin{tikzcd} & C_X^* \arrow[r, dash] & \mathbb{C}^{n+1} - \{0\} \times \mathbb{P}^{c-1} \arrow[d] \\ C_Y^* \arrow[rr, dash] & & \mathbb{C}^{n+1} - \{0\} \end{tikzcd}$$

Because C_Y^* is defined by a $\mathbb{C}^*$-invariant section, we can naturally extend the $\mathbb{C}^*$-action to $\mathbb{C}^{n+1} - \{0\} \times \mathbb{P}^{c-1}$ and C_X^*. Let $\mathcal{X}$ denote the quotient stack $[C_X^*/\mathbb{C}^*]$ and $\mathcal{Y}$ denotes the quotient stack $[C_Y^*/\mathbb{C}^*]$. From the definition we see that C_Y^* is smooth and $[C_Y^*/\mathbb{C}^*]$ is a smooth Deligne-Mumford stack whose coarse moduli space is Y. From Theorem 2.8 and Remark 2.9, we see that $D^b([C_Y^*/\mathbb{C}^*])$ can be embedded into $D^b([C_X^*/\mathbb{C}^*])$. Therefore we get the desired result from the following Lemma. $\square$

Lemma 3.6 *$\mathcal{X}$ is a smooth Deligne-Mumford stack whose coarse moduli space is X and X is a Fano variety when n is large enough.*

Proof Because C_Y^* is smooth we see that C_X^* is also smooth by a local calculation. For every point of C_X^* the stabilizer of the induced $\mathbb{C}^*$-action is a finite abelian group. Therefore $\mathcal{X}$ is a smooth Deligne-Mumford stack whose coarse moduli space is $X = C_X^*/\mathbb{C}^*$. Therefore X has only quotient singularities and K_X is a $\mathbb{Q}$-Cartier divisor.

Recall that $\mathbb{P}(\bar{a})$ is a quotient of $\mathbb{P}^n$ by $\mu_{\bar{a}}$-action where $\mu_{\bar{a}}$ acts on $\mathbb{P}^n$ via

$$(\mu_0, \cdots, \mu_n) \cdot [z_0 : \cdots : z_n] = [\epsilon_0^{\mu_0} \cdot z_0 : \cdots : \epsilon_n^{\mu_n} \cdot z_n]$$

where ϵ_i is a primitive a_i-th root of unity (cf. [15]).

Then we can construct $C_{\tilde{Y}}^*$

$$\begin{tikzcd} C_{\tilde{Y}}^* \arrow[r, dash] \arrow[d] & \mathbb{C}^{n+1} - \{0\} \arrow[d] \\ C_Y^* \arrow[r, dash] & \mathbb{C}^{n+1} - \{0\} \end{tikzcd}$$

by lifting of the equations defining C_Y. Note that $C_{\tilde{Y}}^*$ is also a complete intersection in $\mathbb{C}^{n+1} - \{0\}$ and we can also apply Cayley's trick to $C_{\tilde{Y}}^*$. By taking $\mathbb{C}^*$-quotient we have the following diagram.

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[r, dash] \arrow[d] & \mathbb{P}E^{\vee} \arrow[d] \arrow[dll] \\
\widetilde{Y} \arrow[r, dash] \arrow[d] & \mathbb{P}^n \arrow[d] & X = \widetilde{X}/\mu_{\bar{a}} \arrow[r, dash] & \mathbb{P}E^{\vee}/\mu_{\bar{a}} \arrow[dll] \\
Y \arrow[r, dash] & \mathbb{P}(\bar{a}) & &
\end{tikzcd}$$

In other words we can construct a ramified covering $\widetilde{X}$ of X by applying Cayley's trick to $\mathbb{P}^n$ which is also a ramified covering of $\mathbb{P}(\bar{a})$. When n is large enough, we can see that $\widetilde{X}$ is a Fano variety from adjuction (cf. [33, Proposition 5.73]) and one can compute the canonical divisor of $\widetilde{X}$ as in Theorem 3.1. One can also directly check that the ramification divisors are bounded since the group $\mu_{\bar{a}}$ is fixed even though n is very large. Therefore we see that X is also a Fano variety. □

Remark 3.7 Note that if Y is singular quasi-smooth weighted complete intersection then Y itself cannot have an orbifold Fano host. Therefore we should consider the derived category of smooth stack $\mathcal{Y}$ whose coarse moduli space is Y instead of the derived category of Y itself.

However if Y is smooth then we see that Y itself has an orbifold Fano host.

Corollary 3.8 *Every smooth weighted complete intersection Y in a weighted projective space has an orbifold Fano host.*

Proof We see that $D^b(Y)$ is contained in $D^b(\mathcal{Y})$ from Lemma 2.5 and $\mathcal{Y}$ has an orbifold Fano host from Theorem 3.5. Therefore we see that Y has an orbifold Fano host. □

4 Fourier-Mukai Transforms and an Embeddability Criterion

In this section, we use the Fourier-Mukai transform to give a Hodge-theoretic criterion for the existence of a fully faithful functor $D^b(Y) \to D^b(X)$ for smooth projective varieties X and Y. Let us recall the following proposition which is well-known to experts. We give a proof for convenience of readers.

Proposition 4.1 *If a Fourier-Mukai transform $\Phi_K : D^b(Y) \to D^b(X)$ is fully faithful, then the induced cohomological Fourier-Mukai transform $\Phi_K^H : H^*(Y, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ yields an injective homomorphism*

$$\bigoplus_{p-q=i} H^{p,q}(Y) \subset \bigoplus_{p-q=i} H^{p,q}(X).$$

Hence, we have the inequality

$$\sum_{p-q=i} h^{p,q}(Y) \leq \sum_{p-q=i} h^{p,q}(X) \quad \textit{for all } i.$$

Proof We will follow the arguments in [25]. There exists a right adjoint Φ_{K_R} of Φ_K and $\Phi_{K_R} \circ \Phi_K \cong id \cong \Phi_{\mathcal{O}_\Delta}$ from the uniqueness of the Fourier-Mukai kernel. Then we get $\Phi^H_{K_R} \circ \Phi^H_K \cong \Phi^H_{\mathcal{O}_\Delta} \cong id$ (cf. [25, Proposition 5.33]). Therefore Φ^H_K induces an inclusion $\Phi^H_K : H^*(Y, \mathbb{C}) \to H^*(X, \mathbb{C})$ which satisfies

$$\Phi^H_K(H^{p,q}(Y)) \subset \bigoplus_{r-s=p-q} H^{r,s}(X)$$

by the arguments in [25, Proposition 5.39]. □

A first consequence of the above inclusion is the following lower bound.

Corollary 4.2 *Let Y be an n-dimensional smooth projective variety with $h^{n,0}(Y) > 0$ for $n > 0$. Then its Fano dimension is at least $n+2$.*

Proof Suppose that there is a smooth Fano variety X of dimension at most $n+1$ and a fully faithful exact functor $F : D^b(Y) \to D^b(X)$. From the inclusion $\bigoplus_{p-q=i} H^{p,q}(Y) \subset \bigoplus_{p-q=i} H^{p,q}(X)$, we have the inequality

$$0 < h^{n,0}(Y) \leq \bigoplus_{p-q=n} h^{p,q}(X).$$

Obviously the right hand side is zero unless $\dim X$ is n or $n+1$. By Kodaira vanishing theorem, we have $h^{n,0}(X) = 0$ and $\dim X \geq n+1$. When $\dim X = n+1$, we have $h^{n+1,1}(X) = \dim H^1(X, K_X) = \dim H^n(X, \mathcal{O}_X) = h^{0,n}(X) = h^{n,0}(X) = 0$ from Serre duality and complex conjugation. Hence the right hand side is always zero if $\dim X \leq n+1$. This proves the proposition. □

When $\dim Y = 1$ and Y is not rational, $h^{1,0}(Y) > 0$ and so we obtain the following.

Corollary 4.3 *The Fano dimension of a smooth projective curve which is not rational is at least 3.*

We will see below that the Fano dimension of a curve Y is exactly 3 when the genus is 1 or 2 or when Y is a general curve of genus 4.

Combining Corollary 4.2 with the construction of Fano hosts in [31], we can determine the Fano dimension of a general complete intersection Calabi-Yau variety.

Proposition 4.4 *Let $Y \subset \mathbb{P}^{n+c}$ be a smooth projective complete intersection Calabi-Yau variety of dimension n defined by the vanishing of homogeneous polynomials $f_1, \cdots, f_c$. Suppose $c \le 2$ or Y is general in the sense that we can choose the defining polynomials such that the projective variety S defined by the vanishing of $f_3, \cdots, f_c$ is smooth. Then the Fano dimension of Y is precisely $n+2$.*

Proof By [31, Proposition 3.6], the Fano dimension of Y is at most $n+2$. By Corollary 4.2, the Fano dimension is at least $n+2$. This proves the proposition. □

For instance, the Fano dimension of an arbitrary quintic 3-fold is 5 and the Fano host constructed in [31] is of minimal dimension.

5 Curves and their Jacobians

In this section we search for Fano visitors among smooth projective curves. Curves in this section mean smooth projective curves.

5.1 Hyperelliptic Curves

Bondal and Orlov proved that every hyperelliptic curve is a Fano visitor.

Theorem 5.1 ([10]) *Let C be a hyperelliptic curve of genus g. Then there are two quadric hypersurfaces in $\mathbb{P}^{2g+1}$ whose intersection is a Fano host of C.*

From Bondal and Orlov's work we see that a hyperelliptic curve C of genus g is a Fano visitor whose Fano dimension is at most $2g-1$. And this observation indicates that the Fano dimension of a curve of genus g might increase as g increases. Indeed the Fano dimension of a curve of genus g may grow arbitrarily large as g increases.

Proposition 5.2 *Let $\mathrm{fd}(g)$ be the minimum among the Fano dimensions of curves of genus g. Then $\lim_{g\to\infty} \mathrm{fd}(g) = \infty$.*

Proof For any natural number n, there are only finitely many deformation equivalence classes of Fano varieties of dimension n. Therefore there are only finitely many possible values of $\sum_{i-j=1} h^{i,j}(X)$ for n-dimensional Fano varieties X. When the genus $g = h^{1,0}(C)$ of a curve C is greater than all these possible values, there can be no n-dimensional Fano host of C. Therefore for any integer $n > 0$ there is an integer g_0 such that any curve of genus $g \ge g_0$ has Fano dimension greater than n. This proves the proposition. □

Remark 5.3 We can also prove that every hyperelliptic curve has an orbifold Fano host since it is a complete intersection in a weighted projective space $\mathbb{P}(1, 1, 1, g+1)$ of dimension 3.

5.2 Low Genus Curves

In this subsection we prove that all curves C of genus $g \leq 4$ and generic curves of genus $g \leq 9$ are Fano visitors. We will use several classical results about low genus curves and refer [2, 16] fore more details. If $g = 0$, $C = \mathbb{P}^1$ itself is a Fano variety. If $g = 1$, $C \subset \mathbb{P}^2$ is a complete intersection Calabi-Yau variety of codimension 1 and hence its Fano dimension is 3 by Proposition 4.4. If $g = 2$, C is a hyperelliptic curve and hence the Fano dimension is at most 3 by Theorem 5.1. By Corollary 4.3, the Fano dimension of C is at least 3. Therefore we see that every curve of genus 2 is a Fano visitor with Fano dimension 3.

If $g = 3$, it is well known that C is either a plane quartic or a hyperelliptic curve. In the former case, we use the construction in Sect. 3.2 with $l = 1$, $m = 2$, $a_1 = 4$, $c = 2$ to obtain a Fano host X of dimension 5. In the latter case, Bondal-Orlov's semi-orthogonal decomposition gives a Fano host of dimension 5.

If $g = 4$, it is well known that C is either the complete intersection of a quadric and a cubic in $\mathbb{P}^3$ or a hyperelliptic curve. In the former case, the Fano dimension is exactly 3 by Example 3.3. In the latter case, Bondal-Orlov's semi-orthogonal decomposition gives a Fano host of dimension 7.

A general curve C of genus 5 has canonical embedding into $\mathbb{P}^4$ whose image is the intersection of three general quadrics. Let S be one of the quadric hypersurfaces and let s be the section of $E = \mathcal{O}_{\mathbb{P}^4}(2)^{\oplus 2}|_S$ defined by the remaining two quadrics, so that $C = s^{-1}(0)$. Then $\mathbb{P}E^\vee \cong S \times \mathbb{P}^1$ is a Fano variety and hence the Mori cone of $\mathbb{P}E^\vee$ is rational polyhedral. Let $H = \mathcal{O}_{\mathbb{P}^4}(2)|_S$. Then $F = E \otimes H^{-1} = \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$ is nef and $K_S^\vee \otimes \det E^\vee \otimes H = \mathcal{O}_{\mathbb{P}^4}(1)|_S$ is ample. Therefore we see that a general curve of genus 5 is a Fano visitor and its Fano dimension is 3 from Theorem 3.1.

For curves of genus $g \leq 9$, Mukai proved that a general curve of genus $g \leq 9$ can be written as a complete intersection of a linear section of a homogenous space (cf. [47]). Therefore we obtain the following conclusion.

Theorem 5.4 *Generic curves of genus $g \leq 9$ are Fano visitors.*

Proof We already proved that general curves of genus ≤ 5 are Fano visitors by using their canonical embeddings. Let $C \subset Z \subset \mathbb{P}^N$ be a curve of genus $6 \leq g \leq 9$ which is a complete intersection in a homogeneous variety Z embedded in $\mathbb{P}^N$ via the Plücker embedding. From the adjunction formula we see that $K_C \cong \mathcal{O}_{\mathbb{P}^N}(1)|_C$. In each case, we can find varieties $C \subset S \subset Z \subset \mathbb{P}^N$ where S is a 4-dimensional complete intersection in Z and C is the zero locus of a section of a rank 3 vector bundle on S. We then find that the variety S and the rank 3 vector bundle satisfy the assumptions of Theorem 3.1. Therefore C is a Fano visitor. Moreover we see that the Fano dimensions of general curves of genus $6 \leq g \leq 9$ are at most 5. $\square$

Theorem 3.1 enables us to provide many more examples of curves of genus ≥ 10 which are Fano visitors.

Remark 5.5 After we finished writing this paper, we received a manuscript from Narasimhan [48] in which he proves that all curves of genus at least 6 are Fano visitors. He also proves that all non-hyperelliptic curves of genus 3, 4 or 5 are Fano visitors in [49]. Combined with our previous discussions, we see that all curves are Fano visitors.

It is well known that the moduli space of rank 2 stable vector bundles over a curve with fixed odd determinant is Fano. Narasimhan proves that the Fourier-Mukai transform defined by the universal bundle is fully faithful. Recently, Fonarev and Kuznetsov obtained similar results for generic curves, especially for all hyperelliptic curves via different method (cf. [17]). It follows that the Fano dimension of an arbitrary curve of genus $g \geq 2$ is at most $3g - 3$. But our discussion above for curves of low genus indicates that this upper bound is far from being optimal.

5.3 *Jacobians of Curves*

Let C be a curve and $J(C)$ be the Jacobian of C. It is a classical topic in algebraic geomety to study interactions between C and $J(C)$. Because every curve is a Fano visitor we can prove that every Jacobian of a curve has orbifold Fano hosts.

Proposition 5.6 *Let C be a curve and $J(C)$ be its Jacobian. Then $J(C)$ has orbifold Fano hosts.*

Proof Let F be a smooth projective Fano host of C. It is well-known that there is a surjection $\phi^{(n)} : C^{(n)} \to J(C)$ such that $R\phi_*^{(n)}\mathcal{O}_{C^{(n)}} = \mathcal{O}_{J(C)}$ for $n > 2g - 2$ and we see that this surjection induces a fully faithful functor $D^b(J(C)) \to D^b(C^{(n)})$. From the Lemma 2.5 we see that there is a fully faithful functor $D^b(C^{(n)}) \to D^b([C^n/S_n])$. And from the Theorem 2.10 we see that there is a fully faithful functor $D^b([C^n/S_n]) \to D^b([F^n/S_n])$. Therefore we see that for every Jacobian of curve $J(C)$ there is an orbifold Fano host $[F^n/S_n]$. $\square$

6 Surfaces

In this section we discuss the Fano visitor problem for surfaces. A surface in this section always means a normal projective surface. Unfortunately, we do not know whether every smooth projective surface has a Fano host or an orbifold Fano host. Therefore we raise many questions and give some partial results. Let Y be a surface and κ denote its Kodaira dimension. First, one can ask whether it is enough to consider the Fano visitor problem for minimal surfaces only.

Question 6.1 *Let Y be a smooth projective surface and $\tilde{Y}$ denote the blowup of Y at a point. Is $\tilde{Y}$ a Fano visitor if Y is a Fano visitor? More generally, is a variety birational to a Fano visitor a Fano visitor?*

From now on we study Fano visitor problem for minimal surfaces.

6.1 $\kappa = -\infty$ Case

If the answer to Question 6.1 is yes, then we may assume Y is either $\mathbb{P}^2$, a Hirzebruch surface or a ruled surface. Now let us provide several examples of ruled surfaces having Fano hosts.

Proposition 6.2 *Let $C = s^{-1}(0)$ be a smooth projective variety which is defined by a regular section of a rank $r \geq 2$ vector bundle E on S and let F be a rank 2 vector bundle on S. Suppose that there are line bundles H_1 and H_2 such that $q^*(E \otimes H_1^\vee)$, $F \otimes H_2^\vee$, $K_S^\vee \otimes \det E^\vee \otimes det F^\vee \otimes H_1^{r-1} \otimes H_2^2$ are nef vector bundles and at least one of them is ample. Then $\mathbb{P}(F^\vee|_C)$ is a Fano visitor.*

Proof It is obvious that $\mathbb{P}(F^\vee|_C)$ is a complete intersection of a regular section of q^*E in $\mathbb{P}(F^\vee)$. We can use Cayley's trick to construct a Fano host X of $\mathbb{P}(F^\vee|_C)$ because it is a complete intersection of a regular section q^*E in $\mathbb{P}(F^\vee)$. Then we have the following diagram

$$\begin{tikzcd}
 & X \arrow[r, dash] & \mathbb{P}(q^*E^\vee) \arrow[d, "p"] \\
\mathbb{P}(F^\vee|_C) \arrow[rr, dash] \arrow[d] & & \mathbb{P}F^\vee \arrow[d, "q"] \\
C \arrow[rr, dash] & & S
\end{tikzcd}$$

and

$$K_{\mathbb{P}F^\vee}^\vee \cong q^*(K_S^\vee \otimes det F^\vee) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(2)$$

and

$$K_{\mathbb{P}(q^*E^\vee)}^\vee \cong p^*(K_{\mathbb{P}F^\vee}^\vee \otimes det(q^*E^\vee)) \otimes \mathcal{O}_{\mathbb{P}(q^*E^\vee)}(r)$$

$$\cong p^*(q^*(K_S^\vee \otimes det F^\vee \otimes det E^\vee) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(2)) \otimes \mathcal{O}_{\mathbb{P}(q^*E^\vee)}(r).$$

From the construction we see that $D^b(\mathbb{P}(F^\vee|_C))$ can be embedded into $D^b(X)$ and

$$K_X^\vee \cong p^*(q^*(K_S^\vee \otimes det F^\vee \otimes det E^\vee) \otimes \mathcal{O}_{\mathbb{P}F^\vee}(2)) \otimes \mathcal{O}_{\mathbb{P}(q^*E^\vee)}(r-1).$$

Using the same argument of proof of Theorem 3.1, we can see that X is Fano. $\square$

Therefore we get the following results.

Corollary 6.3 *All Hirzebruch surfaces are Fano visitors.*

Proof Let $Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ be a Hirzebruch surface. We can embed $\mathbb{P}^1$ into $\mathbb{P}^{r+1}$. Let $S = \mathbb{P}^{r+1}$, $F = \mathcal{O}_{\mathbb{P}^{r+1}} \oplus \mathcal{O}_{\mathbb{P}^{r+1}}(a)$, $E = \mathcal{O}(1)^{\oplus r}$, and $E' = \mathcal{O}^{\oplus r}$. From the same construction and notation as the above Proposition 6.2 we have

$$K^{\vee}_{\mathbb{P}(q^*E^{\vee})} \cong p^*(K^{\vee}_{\mathbb{P}F^{\vee}} \otimes det(q^*E^{\vee})) \otimes \mathcal{O}_{\mathbb{P}(q^*E^{\vee})}(r)$$

$$\cong p^*(q^*(\mathcal{O}(r+2) \otimes \mathcal{O}(-a) \otimes \mathcal{O}(-r)) \otimes \mathcal{O}_{\mathbb{P}F^{\vee}}(2)) \otimes \mathcal{O}_{\mathbb{P}(q^*E^{\vee})}(r)$$

$$\cong p^*(q^*(\mathcal{O}(r+2-a)) \otimes \mathcal{O}_{\mathbb{P}F^{\vee}}(2)) \otimes \mathcal{O}_{\mathbb{P}(q^*E'^{\vee})}(r).$$

When r is sufficiently large, the above construction gives a Fano host of Y. □

By the same proof we obtain the following.

Corollary 6.4 *Let C be a curve which is a complete intersection in a projective space. Let F be a vector bundle of rank 2 on the projective space which is a direct sum of two line bundles. Then $\mathbb{P}(F^{\vee}|_C)$ is a Fano visitor.*

We thank the referee for letting us know the following result.

Remark 6.5 Ballard, Favero and Katzarkov proved that the derived category of a projetive toric variety can be embedded into the derived category of a weak Fano toric Deligne-Mumford stack (cf. [4, Proposition 5.2.5]) using variation of GIT.

6.2 $\kappa = 0$ *Case*

6.2.1 Abelian Surfaces

An Abelian surface which is the product of two elliptic curves is a Fano visitor (cf. [25, Corollary 7.4]). We also saw that the Jacobian of any genus 2 curve always has an orbifold Fano host from Proposition 5.6.

6.2.2 K3 Surfaces

The following is a consequence of Theorem 3.1 for K3 surfaces.

Corollary 6.6 *Let Y be a $K3$ surface which is the zero locus of a section of an ample vector bundle E of rank r on a Fano variety S of dimension of $r+2$ where $r \geq 2$. Then Y is a Fano visitor. The Fano dimension of Y is at most $2r$.*

Example 6.7 Let V be a Fano 3-fold and let Y be a smooth divisor in $|K_V^\vee|$ which is a $K3$ surface by adjunction. When V is the zero locus of a regular section of an ample vector bundle on another Fano manifold W and the line bundle K_V is the restriction of an ample line bundle on W, we find that Y is a Fano visitor by Theorem 3.1. For example, general $K3$ surfaces of genus $6 \leq g \leq 10$ satisfy these conditions (cf. [47]). Therefore general $K3$ surfaces of genus $6 \leq g \leq 10$ are Fano visitors and their Fano dimensions are 4.

The above result can be used to find orbifold Fano hosts of holomorphic symplectic varieties. Recall that the Hilbert schemes of points on K3 surfaces are holomorphic symplectic varieties.

Corollary 6.8 *Let Y be a K3 surface and X be a Fano host of Y. Then $[X^n/S_n]$ is an orbifold Fano host of $Y^{[n]}$.*

Proof From the Bridgeland-King-Reid-Haiman correspondence (cf. [11, 23]) we see that $D^b(Y^{[n]}) \simeq D^b([Y^n/S_n])$. Then from Theorem 2.10 we see that $D^b([Y^n/S_n])$ can be embedded into $D^b([X^n/S_n])$. Therefore we get the desired result. □

Now let us consider Kummer surfaces. Consider an Abelian surface A having Fano host and consider an involution σ on A which send $x \mapsto -x$ with respect to the group structure on A. Then σ has 16 fixed points and the minimal resolution of A/σ is a K3 surface S. We call S a Kummer surface. One can prove that if A has a Fano host F such that σ extends to F and the Fourier-Mukai kernel of the embedding is σ-invariant, then S has an orbifold Fano host. Let us give such examples as follows.

Proposition 6.9 *Let E_1, E_2 be elliptic curves, $A = E_1 \times E_2$ be an Abelian surface and let S be the associated Kummer surface. Then S has an orbifold Fano host.*

Proof In this case σ is induced by two involutions σ_1, σ_2 on E_1, E_2 respectively. For each i, the 2-torsion points of E_i form a σ_i-invariant divisor of degree 4 on E_i and gives an embedding $E_i \to \mathbb{P}^3$. Then σ_i-action extends to $\mathbb{P}^3$ and it also extends to $F_i = Bl_{E_i}\mathbb{P}^3$. Note that there exists an embedding $\Phi_{K_i} : D^b(E_i) \to D^b(F_i)$ for $i = 1, 2$. Then we have a fully faithful functor $\Phi_{K_1 \boxtimes K_2} : D^b(E_1 \times E_2) \to D^b(F_1 \times F_2)$. Let us consider the diagonal $\mathbb{Z}_2$-actions on $E_1 \times E_2$, $F_1 \times F_2$ and $E_1 \times E_2 \times F_1 \times F_2$. It is easy to see that $K_1 \boxtimes K_2$ is $\mathbb{Z}_2$-invariant hence $\mathbb{Z}_2$-linearized. By the McKay correspondence (cf. [11]) we see that $D^b(S) \cong D^b([(E_1 \times E_2)/\mathbb{Z}_2])$ can be embedded into $D^b([(F_1 \times F_2)/\mathbb{Z}_2])$. Therefore S has an orbifold Fano host. □

Reid constructed 95 families of orbifold K3 surfaces as complete intersections in weighted projective spaces (cf. [1]). We can apply our method to prove that they have orbifold Fano hosts.

6.2.3 Enriques Surfaces

We will review a construction of an Enriques surface described in [6, Example 8.18]. Let $Q_1(z_0, z_1, z_2) + Q_1'(z_3, z_4, z_5)$, $Q_2(z_0, z_1, z_2) + Q_2'(z_3, z_4, z_5)$,

$Q_3(z_0, z_1, z_2) + Q'_3(z_3, z_4, z_5)$ be three quadric forms with variables $z_0, \cdots, z_5$ and let Y be a K3 which is an intersection of three quadrics hypersurfaces defined by these three quadric forms in $\mathbb{P}^5$. Then Y is a smooth K3 surface if we choose Q_i, Q'_i generically. Let σ be an involution on $\mathbb{P}^5$ defined as follows.

$$\sigma \cdot [z_0 : z_1 : z_2 : z_3 : z_4 : z_5] = [z_0 : z_1 : z_2 : -z_3 : -z_4 : -z_5].$$

Then σ induces a fixed point free involution σ on a K3 surface Y when we choose Q_i, Q'_i generically. It is known that the generic Enriques surface can be obtained in the above construction. See [6, Example 8.18] for more details.

Proposition 6.10 *A generic Enriques surface has an orbifold Fano host.*

Proof Let S be an Enriques surface obtained as the quotient of a K3 surface Y which is constructed as above and let X be the Fano host of Y constructed by Cayley's trick. Note that σ induces an involution σ_X on X because Y is defined by σ-invariant sections. Let S be the Enriques surface whose double cover is Y and we see that $D^b(Y/\langle\sigma\rangle) = D^b([Y/\langle\sigma\rangle]) \hookrightarrow D^b([X/\langle\sigma_X\rangle])$. Therefore the Enriques surface $S = Y/\langle\sigma\rangle$ has an orbifold Fano host. □

Recently, Kuznetsov proved that general Enriques surfaces can be embedded into dervied categories of certain Fano varieties in [38]. His constructions also provide examples of Fano varieties whose Grothendieck groups contain torsion groups as direct summands.

6.2.4 Bielliptic Surfaces

Recall that a bielliptic surface is the quotient of product of two elliptic curves by a finite abelian group. They were classified by Bagnera and de Franchis (cf. [6, 12]).

Lemma 6.11 *Let E be an elliptic curve and let G be a finite group acting on E. (Here, we do not assume that the G-action preserves the identity of E so G can includes translations.) Then there is a G-invariant ample divisor which induces an embedding $E \to \mathbb{P}^n$ and the G-action on E extends to $\mathbb{P}^n$. Moreover if $2 \leq |G| \leq 4$ then E is a zero locus of G-invariant regular section of vector bundle in $\mathbb{P}^n$*

Proof Let us consider the quotient map $E \to E/G$. By considering the pullback of a general point of E/G, we see that there is a G-equivariant line bundle $\mathcal{O}(D) = L$ of degree $|G|$ on E. If $|G| = 3, 4$, then L induces an embedding of E to $\mathbb{P}^{|G|-1}$ (cf. [22, Example 3.3.3]). Since L is a G-equivariant line bundle, $H^0(E, L)$ has a G-module structure. Therefore the G-action on E extends to $\mathbb{P}^{|G|-1}$. Because G is an abelian group, $H^0(E, L)$ is a direct sum of 1-dimensional representations of G. If $|G| = 2$, then we can use $L^{\otimes 2}$ and obtain similar result. Therefore we get the desired result from the information about the syzygies of elliptic curves in projective spaces of low dimensions (cf. [2, 16]). □

Then we have the following result.

Proposition 6.12 *A bielliptic surface $S = (E_1 \times E_2)/G$ where $|G| \leq 4$ has an orbifold Fano host.*

Proof Let $S = (E_1 \times E_2)/G$ be a bielliptic surface. From the above Lemma 6.11, we can construct two Fano hosts F_i of E_i with embedding $\Phi_{K_i} : D^b(E_i) \to D^b(F_i)$ for $i = 1, 2$. For example, let F_i be the blowup of E_i in $\mathbb{P}^3$. Then we have a fully faithful functor $\Phi_{K_1 \boxtimes K_2} : D^b(E_1 \times E_2) \to D^b(F_1 \times F_2)$. It is easy to check that $K_1 \boxtimes K_2$ can be G-linearized. Therefore we have the desired embedding $\Phi_{K_1 \boxtimes K_2} : D^b([E_1 \times E_2/G]) \to D^b([F_1 \times F_2/G])$ and $(F_1 \times F_2)/G$ is a Fano variety. □

6.3 $\kappa = 1$ *Case*

Let us discuss some examples of surfaces with $\kappa = 1$.

Example 6.13 Let E be an elliptic curve and C be a curve of genus $g \geq 2$. Let G be a finite group of translations of E and suppose that G is acting on C. Consider $E \times C$ and the diagonal G-action on it. Let $Y = (E \times C)/G$. Because the diagonal action on $E \times C$ is free, Y is a surface with $\kappa = 1$.

In order to construct an orbifold Fano host of Y, we will use the moduli space of rank 2 stable vector bundles whose determinants are isomorphic to a fixed odd degree line bundle on C. Note that the moduli space turns out to be a Fano host of C (cf. [17, 48]). Let us fix a G-invariant fixed odd degree line bundle ξ where G is a finite group acting on the curve C.

Lemma 6.14 *Let C be a curve with a G-action where G is a finite group. Suppose that C has a G-invariant line bundle ξ of odd degree. Then there is a natural action on the moduli space M of rank 2 stable vector bundles on C whose determinants are isomorphic to ξ and the universal bundle is a G-invariant vector bundle with respect to the diagonal action.*

Proof Let $E \in M$ and $g \in G$. We can define the G-action on M by $g \cdot E = (g^{-1})^* E$. Therefore we have a diagonal action on $C \times M$. Let U be the universal vector bundle on $C \times M$. Note that $g^* U$ is a flat family of rank 2 vector bundles on $C \times M$. Therefore $g^* U$ induces an isomorphism $\phi_g : M \to M$ such that $g^* U \cong (Id_C \times \phi_g)^* U$. From the definition of the action one can check that ϕ_g is an identity morphism from M to M. Therefore U is a G-invariant vector bundle on $C \times M$. □

We proved that U is a G-invariant vector bundle if ξ is G-invariant line bundle of odd degree. However it does not mean that U is a G-equivariant vector bundle (or U is G-linearizable). Indeed if ξ is not G-equivariant line bundle then U is not G-equivariant vector bundle. We have the following numerical condition when U being G-invariant imply U being G-equivariant.

Lemma 6.15 *Let G be a finite group acting on a variety X and let U be a G-invariant rank 2 simple vector bundle whose determinant ξ is a G-equivariant line bundle. Suppose that $gcd(2, |H^2(G, \mathbb{C}^*)|) = 1$ then U is a G-equivariant vector bundle.*

Proof Because U is a G-invariant vector bundle we have an isomorphism $\theta_g : g^*U \to U$ for each $g \in G$. Because U is simple, we have an element $(\theta_{gh})^{-1} \cdot h^*(\theta_g) \cdot \theta_h \in \mathbb{C}^*$ for any pair $g, h \in G$ and this assignment gives an element in $H^2(C, \mathbb{C}^*)$. When we take determinant of each θ_g we have $((\theta_{gh})^{-1} \cdot h^*(\theta_g) \cdot \theta_h)^2$ which gives the trivial element of $H^2(G, \mathbb{C}^*)$ since ξ is a G-equivariant line bundle. Because $gcd(2, |H^2(G, \mathbb{C}^*)|) = 1$ we see that θ_g gives a trivial element in $H^2(G, \mathbb{C}^*)$. Therefore U is a G-equivariant vector bundle on X. □

Then we can construct orbifold Fano hosts of elliptic surfaces with $\kappa = 1$ constructed above.

Proposition 6.16 *An elliptic surface $Y = (E \times C)/G$ constructed above where $|G| \leq 3$ and there is a G-equivariant odd degree line bundle on C. Then Y has an orbifold Fano host.*

Proof From Lemma 6.11 we see that E has a Fano host F_1 with G-action and and the Fourier-Mukai kernel K_1 is a G-linearized object with respect to the diagonal action. From the assumption there is a G-equivariant odd degree line bundle on C. Again from the above two Lemmas 6.14, 6.15, we see that C has a Fano host F_2 with G-action and and the Fourier-Mukai kernel K_2 is also a G-linearized object with respect to the diagonal action. From the Theorem 2.10, we have a fully faithful functor $D^b(Y) \simeq D^b([(E \times C)/G]) \to D^b([(F_1 \times F_2)/G])$. Therefore we obtain an orbifold Fano host of Y. □

We expect to obtain many more examples of orbifold Fano hosts of surfaces with $\kappa = 1$ via the above method.

6.4 $\kappa = 2$ Case

Surfaces of general type are still mysterious objects. A very simple way to construct surfaces of general type is to consider complete intersection in projective spaces or product of two curves. From Remark 5.5, it is very easy to see that they are Fano visitors. By Theorem 3.1, we can provide many examples of surfaces of general type which are Fano visitors. However we do not know whether all surfaces of general type are Fano visitors or not, since many of them, e.g. surfaces of general type with $p_g = q = 0$, cannot be embedded in projective spaces as complete intersections.

Recently, interesting new categories in the derived categories of surfaces of general type with $p_g = q = 0$ were discovered (cf. [8, 9, 18, 42, 43, 45]). Their Grothendieck groups are finite torsion and their Hochschild homology groups vanish. We call them *quasi-phantom categories*. If the Grothendieck group of a quasi-phantom category also vanishes, then we call it a *phantom category*. On the other hand, no smooth projective Fano variety is known to have a quasi-phantom subcategory. Therefore the following question seems interesting.

Question 6.17 *Is there a Fano variety X whose derived category contains a quasi-phantom category?*

Obviously this question is closely related to the Fano visitor problem.

Question 6.18 *Let Y be a surface of general type with $p_g = q = 0$. Is there a Fano host of Y?*

For example, a Fano host of the determinantal Barlow surface will give us a Fano variety containing a phantom category. Although we do not know the answer to Question 6.17, we can construct a Fano orbifold whose derived category contains a quasi-phantom category following [9]. Then we will improve this by constructing an Fano orbifold containing phantom subcategory, following [21].

6.4.1 Classical Godeaux Surfaces

Let $Y \subset \mathbb{P}^3$ be the variety defined by Fermat quintic $f = z_0^5 + z_1^5 + z_2^5 + z_3^5 = 0$ and let $G = \mathbb{Z}_5 = \langle \epsilon \rangle$ act on Y by $\epsilon \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : \epsilon z_1 : \epsilon^2 z_2 : \epsilon^3 z_3]$ where $\epsilon = e^{\frac{2\pi\sqrt{-1}}{5}}$ is a primitive fifth root of unity. The G-action on Y is free and Y/G is the classical Godeaux surface.

Proposition 6.19 *The classical Godeaux surface has an orbifold Fano host.*

Proof Let $X = w^{-1}(0) \subset \mathbb{P}E^\vee$ be a Fano host of $Y = s^{-1}(0) \subset \mathbb{P}^5$ obtained by Cayley's trick, where s is the section of $E = \mathcal{O}_{\mathbb{P}^5}(5) \oplus \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2}$ defined by the Fermat quintic f and two linear polynomials z_4, z_5 that cut out $\mathbb{P}^3$ in $\mathbb{P}^5$. Let G act on z_4 and z_5 trivially. Then G acts on $\mathbb{P}^5$ and E compatibly. Moreover the section $s = (f, z_4, z_5)$ is G-invariant. By Orlov's theorem (Remark 2.9), we see that there is a fully faithful embedding $D^b(Y/G) \to D^b([X/G])$ of the derived category of the classical Godeaux surface into the derived category of the Fano orbifold $[X/G]$. Since the derived category of the classical Godeaux surface contains a quasi-phantom category (cf. [9]), $D^b([X/G])$ also contains a quasi-phantom category. □

6.4.2 Product-Quotient Surfaces

Let us briefly recall the definition of product-quotient surfaces.

Definition 6.20 An algebraic surface S is called a product-quotient surface if there exist a fiinite group G and two algebraic curves C, D with G-action such that S is isomorphic to the minimal resolution of $(C \times D)/G$ where G acts on $C \times D$ diagonally.

Product-quotient surfaces provide surprisingly many new examples of surfaces of general type and play an important role in the theory of algebraic surfaces (cf. [5]). Recently derived categories of some product-quotient surfaces were studied and it turns out that some of them have quasi-phantom categories in their derived categories(cf. [18, 32, 42, 43, 45]). We can construct orbifold Fano hosts of some of product-quotient surfaces as follows.

Proposition 6.21 *Let S be a product-quotient surface which is the minimal resolution of $(C \times D)/G$. Suppose that C, D have G-equivariant odd degree line bundles and $gcd(2, |H^2(G, \mathbb{C}^*)|) = 1$. Then S has an orbifold Fano host.*

Proof Let C, D be algebraic curve with G-action such that S is a minimal resolution of $(C \times D)/G$. Then $D^b(S)$ is embedded into $D^b([(C \times D)/G])$ by the McKay correspondence (cf. [26]). From Lemma 6.15 we see that C (resp. D) has a Fano host F_1 (resp. F_2) with G-action and and the Fourier-Mukai kernel K_1 (resp. K_2) is a G-linearized object with respect to the diagonal action. From the Theorem 2.10, we have a fully faithful functor $D^b([(C \times D)/G]) \to D^b([(F_1 \times F_2)/G])$. Finally $[(F_1 \times F_2)/G]$ is a smooth Deligne-Mumford stack whose coarse moduli space $(F_1 \times F_2)/G$ is a Fano variety. Therefore we get the desired result. □

Example 6.22 Let S be a product-quotient surface where the order of G is odd. Then S satisfies the conditions of the above theorem. See [5, 18, 42] for examples of these surfaces.

Corollary 6.23 *There are Fano orbifolds whose derived categories contain phantom categories.*

Proof Let S_1 be the classical Godeaux surface and S_2 be the project-quotient surface obtained by the quotient of product two genus 4 curves with free $\mathbb{Z}_3^2$-action. Let $\mathcal{X}_1$ be an orbifold Fano host of S_1 and $\mathcal{X}_2$ be an orbifold Fano host of S_2 where we know the existence from the above discussion. Then $\mathcal{X}_1 \times \mathcal{X}_2$ is an orbifold Fano host of $S_1 \times S_2$.

It was proved that $D^b(S_1)$ contains a quasi-phantom category in [9] and $D^b(S_2)$ contains a quasi-phantom category in [42]. Then $D^b(S_1 \times S_2)$ contains a phantom category by the result of [21]. Therefore we have an example of Fano orbifold whose derived category contains a phantom category. Indeed, we can find more examples of such Fano orbifolds from the results of [18, 42]. □

7 Discussions

7.1 Phantom Categories

From the theorem of [21] we see that there are Fano orbifolds contain phantom categories. However we do not know any single example of smooth projective Fano variety whose derived category contains a (quasi-)phantom category. Recently several examples of surfaces whose derived categories containing (quasi-)phantom categories were constructed. Fano hosts of these surfaces will give us examples of smooth projective Fano varieties whose derived categories contain (quasi-)phantom categories.

Question 7.1 *(1) Is there a smooth projective Fano variety whose derived category contains a (quasi-)phantom category?*
(2) Is there a smooth projective Fano variety (or a Fano orbifold) whose derived category contains the derived category of a determinantal Barlow surface (cf. [8])?
(3) Is there a smooth projective Fano variety (or a Fano orbifold) whose derived category contains the derived category of an elliptic surface construced by Cho and Lee (cf. [13])?

It will be very interesting if one can see these phantom categories in the Landau-Ginzburg mirror of (orbifold) Fano hosts.

7.2 Noncommutative Varieties

There are many examples of noncommutative varieties in derived categories of Fano varieties. For example, Kuznetsov proved there are K3 categories not equivalent to derived cateogies of K3 surfaces inside derived categories of cubic 4-folds. These K3 categories provide a natural explanation why many holomorphic symplectic varieties arise from cubic 4-folds. Noncommutative varieties also appear in derived categories of cubic 3-folds and interesting applications of these noncommutative varieties were found (cf. [40]). It will be an interesting question which noncommutative varieties can be embedded into derived categories of Fano orbifolds. It is also an interesting problem to find another geometric description of these noncommutative varieties (cf. [40]) via different Fano hosts.

7.3 Applications and Perspectives

It will be very interesting to find applications of Fano visitor problem to arithmetic geometry, birational geometry and (homological) mirror symmetry. Indeed, understanding derived categories of Fano varieties is very important for all these areas. For example, it was conjectured by Orlov that semiorthogonal decomposition of the derived category of a variety will be closely related to motivic decomposition of the variety (cf. [51]). This idea leads to find new motivic decompositions of moduli spaces of stable vector bundles on curves (cf [20, 46]). From this perspective, we raise the following question.

Question 7.2 *Let Y be a smooth projective variety. Is there a Fano variety (or orbifold) whose motive (in a suitable category of motives) contains the motive of Y as a direct summand?*

References

1. Altinok, S., Brown, G., Reid, M.: Fano 3-folds, K3 surfaces and graded rings. Topology and Geometry: Commemorating SISTAG, vol. 314, pp. 25–53, Contemporary Mathematics. American Mathematical Society, Providence, RI (2002)
2. Arbarello, E., Cornalba, M., Griffiths, P.A, Harris, J.: Geometry of algebraic curves. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. I, 267, xvi+386 pp. Springer, New York (1985)
3. Ballard, M., Deliu, D., Favero, D., Isik, M.U., Katzarkov, L.: On the derived categories of degree d hypersurface fibrations. Math. Ann. **371**(1–2), 337–370 (2018)
4. Ballard, M., Favero, D., Katzarkov, L.: Variation of geometric invariant theory quotients and derived categories. J. Reine Angew. Math. **746**, 235–303 (2019)
5. Bauer, I., Catanese, F., Grunewald, F., Pignatelli, R.: Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups. Am. J. Math. **134**(4), 993–1049 (2012)
6. Beauville, A.: Complex algebraic surfaces. Translated from the 1978 French original by R. Barlow, with assistance from N.I. Shepherd-Barron and M. Reid. 2nd ed. London Mathematical Society Student Texts, vol. 34, x+132 pp. Cambridge University Press, Cambridge (1996)
7. Bernardara, M., Bolognesi, M., Faenzi, D.: Homological projective duality for determinantal varieties. Adv. Math. **296**, 181–209 (2016)
8. Böhning, C., von Bothmer, H.-C.G., Katzarkov, L., Sosna, P.: Determinantal Barlow surfaces and phantom categories. J. Eur. Math. Soc. (JEMS) **17**(7), 1569–1592 (2015)
9. Böhning, C., von Bothmer, H.-C.G., Sosna, P.: On the derived category of the classical Godeaux surface. Adv. Math. **243**, 203–231 (2013)
10. Bondal, A., Orlov, D.: Semiorthogonal decomposition for algebraic varieties. arXiv:alg-geom/9506012
11. Bridgeland, T., King, A., Reid, M.: The McKay correspondence as an equivalence of derived categories. J. Am. Math. Soc. **14**(3), 535–554 (2001)
12. Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A.: Compact complex surfaces. 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, xii+436 pp. Springer, Berlin (2004)
13. Cho, Y., Lee, Y.: Exceptional collections on Dolgachev surfaces associated with degenerations. Adv. Math. **324**, 394–436 (2018)
14. Cho, Y., Kim, Y., Lee, K.-S.: Ulrich bundles on intersections of two 4-dimensional quadrics. Int. Math. Res. Notices rnz320. https://doi.org/10.1093/imrn/rnz320
15. Dolgachev, I.: Weighted projective varieties. Group actions and vector fields (Vancouver, B.C., 1981). Lecture Notes in Mathematics, vol. 956, pp. 34–71. Springer, Berlin (1982)
16. Eisenbud, D.: The geometry of syzygies. A second course in commutative algebra and algebraic geometry. Graduate Texts in Mathematics, vol. 229. Springer, New York (2005)
17. Fonarev, A., Kuznetsov, A.: Derived categories of curves as components of Fano manifolds. J. Lond. Math. Soc. **97**(2), no. 1, 24–46 (2018)
18. Galkin, S., Shinder, E.: Exceptional collections of line bundles on the Beauville surface. Adv. Math. **244**, 1033–1050 (2013)
19. Gomez, T.L.: Algebraic stacks. Proc. Indian Acad. Sci. Math. Sci. **111**(1), 1–31 (2001)
20. Gomez, T.L., Lee, K.-S.: Motivic decompositions of moduli spaces of vector bundles on curves. arXiv:2007.06067

21. Gorchinskiy, S., Orlov, D.: Geometric phantom categories. Publ. Math. Inst. Hautes Etudes Sci. **117**, 329–349 (2013)
22. Hartshorne, R.: Algebraic geometry. Graduate Texts in Mathematics, No. 52, xvi+496 pp. Springer, New York-Heidelberg (1977)
23. Haiman, M.: Hilbert schemes, polygraphs and the Macdonald positivity conjecture. J. Am. Math. Soc. **14**(4), 941–1006 (2001)
24. Hall, J., Rydh, D.: Perfect complexes on algebraic stacks. Compos. Math. **153**(11), 2318–2367 (2017)
25. Huybrechts, D.: Fourier-Mukai transforms in algebraic geometry. Oxford Mathematical Monographs, viii+307 pp. The Clarendon Press, Oxford University Press, Oxford (2006)
26. Ishii, A., Ueda, K.: The special McKay correspondence and exceptional collections. Tohoku Math. J. **67**(2), no. 4, 585–609 (2015)
27. Iskovskih, V.A., Prokhorov, Y.G.: Algebraic Geometry V: Fano varieties. Encyclopaedia of Mathematics Sciences, vol. 47, Springer, Berlin (1999)
28. Kawamata, Y.: Francia's flip and derived categories. Algebraic Geometry, de Gruyter, Berlin, pp. 197–215 (2002)
29. Kawamata, Y.: Equivalences of derived categories of sheaves on smooth stacks. Am. J. Math. **126**(5), 1057–1083 (2004)
30. Kawamata, Y.: Log crepant birational maps and derived categories. J. Math. Sci. Univ. Tokyo **12**(2), 211–231 (2005)
31. Kiem, Y.-H., Kim, I.-K., Lee, H., Lee, K.-S.: All complete intersection varieties are Fano visitors. Adv. Math. **311**, 649–661 (2017)
32. Kim, H.K., Kim, Y.-H., Lee, K.-S.: Quasiphantom categories on a family of surfaces isogenous to a higher product. J. Algebra **473**, 591–606 (2017)
33. J. Kollar and S. Mori. *Birational geometry of algebraic varieties* With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998. viii+254 pp
34. Krug, A., Sosna, P.: Equivalences of equivariant derived categories. J. Lond. Math. Soc. **92**(2), no. 1, 19–40 (2015)
35. Kuznetsov, A.: Derived categories of cubic fourfolds. Cohomological and geometric approaches to rationality problems. Progress in Mathematics, vol. 282, pp. 219-243. Birkhauser Boston, Inc., Boston, MA (2010)
36. Kuznetsov, A.: Derived categories of Fano threefolds. (Russian) Tr. Mat. Inst. Steklova, vol. 264, Mnogomernaya Algebraicheskaya Geometriya (2009); translation in Proc. Steklov Inst. Math. 264 (2009), no. 1, 110–122
37. Kuznetsov, A.: Hochschild homology and semiorthogonal decompositions (2009). arXiv:0904.4330
38. Kuznetsov, A.: Embedding derived category of an Enriques surface into derived category of a Fano variety. Izvestiya: Math. **83**(3), 534–539 (2019)
39. Lahoz, M., Macri, E., Stellari, P.: Arithmetically Cohen-Macaulay bundles on cubic fourfolds containing a plane. Brauer Groups and Obstruction Problems. Progress in Mathematics, vol. 320, pp. 155–175. Birkhauser/Springer, Cham (2017)
40. Lahoz, M., Macri, E., Stellari, P.: Arithmetically Cohen-Macaulay bundles on cubic threefolds. Algebr. Geom. **2**(2), 231–269 (2015)
41. Lazarsfeld, R.: Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, xviii+387 pp. Springer, Berlin (2004)
42. Lee, K.-S.: Derived categories of surfaces isogenous to a higher product. J. Algebra **441**, 180–195 (2015)
43. Lee, K.-S.: Exceptional sequences of maximal length on some surfaces isogenous to a higher product. J. Algebra **454**, 308–333 (2016)
44. Lee, K.-S., Park, K.-D.: Moduli spaces of Ulrich bundles on the Fano 3-fold V_5. arXiv:1711.08305

45. Lee, K.-S., Shabalin, T.: Exceptional collections on some fake quadrics. Proc. Am. Math. Soc. **146**(6), 2299–2313 (2018)
46. Lee, K.-S.: Remarks on motives of moduli spaces of rank 2 vector bundles on curves. arXiv:1806.11101
47. Mukai, S.: Curves, K3 surfaces and Fano 3-folds of genus ≤ 10. Algebraic geometry and commutative algebra, vol. I, pp. 357–377. Kinokuniya, Tokyo (1988)
48. Narasimhan, M.S.: Derived categories of moduli spaces of vector bundles on curves. J. Geom. Phys. **122**, 53–58 (2017)
49. Narasimhan, M.S.: Derived categories of moduli Spaces of vector bundles on curves II. In: Akbary, A., Gun, S. (eds.), Geometry, Algebra, Number Theory, and Their Information Technology Applications. GANITA 2016. Springer Proceedings in Mathematics and Statistics, vol. 251. Springer (2018)
50. Olsson, M.: Algebraic spaces and stacks. American Mathematical Society Colloquium Publications, vol. 62, xi+298 pp. American Mathematical Society, Providence, RI (2016)
51. Orlov, D.O.: Derived categories of coherent sheaves and motives. Russian Math. Surv. **60**, 1242–1244 (2005)
52. Orlov, D.O.: Triangulated categories of singularities, and equivalences between Landau-Ginzburg models. (Russian) Mat. Sb. **197**(12), 117–132 (2006); translation in Sb. Math. 197 (2006), no. 11–12, 1827–1840
53. Ploog, D.: Equivariant autoequivalences for finite group actions. Adv. Math. **216**(1), 62–74 (2007)
54. Segal, E., Thomas, R.: Quintic threefolds and Fano elevenfolds. arXiv:1410.6829
55. Vistoli, A.: Intersection theory on algebraic stacks and on their moduli spaces. Invent. Math. **97**(3), 613–670 (1989)
56. Yasutake, K.: On the classification of rank 2 almost Fano bundles on projective space. Adv. Geom. **12**(2), 353–363 (2012)

K-stability and Fujita Approximation

Chi Li

Abstract This note is a continuation to the paper [26]. We derive a formula for non-Archimedean Monge-Ampère measures of big models. As applications, we derive a positive intersection formula for non-Archimedean Mabuchi functional, and further reduces the $(\mathrm{Aut}(X, L)_0)$-uniform Yau-Tian-Donaldson conjecture for polarized manifolds to a conjecture on the existence of approximate Zariski decompositions that satisfy some asymptotic vanishing condition. In an appendix, we also verify this conjecture for some of Nakayama's examples that do not admit birational Zariski decompositions.

Keywords K-stability · Fujita approximation

1 Introduction

Let (X, L) be a polarized projective manifold. The Yau-Tian-Donaldson (YTD) conjecture predicts that the existence of constant scalar curvature Kähler (cscK) metrics in the Kähler class $c_1(L)$ is equivalent to a K-stability condition for the pair (X, L). The K-stability condition is usually expressed as a positivity condition on the Futaki invariants of test configurations. In a recent work [26], it was proved that the existence of cscK metrics is equivalent to the uniform positivity of Mabuchi slopes along all maximal geodesic rays. Here the maximal geodesic rays, as introduced by Berman-Boucksom-Jonsson [1], are essentially the geodesic rays in the space of (mildly singular) positive metrics in $c_1(L)$ that can be algebraically approximated by the data of test configurations. It is known that for test configurations, the Mabuchi slopes (of geodesic rays associated to test configurations) are the Futaki invariants. So our result is of a Yau-Tian-Donaldson type. However the approximability of Mabuchi slopes of (maximal) geodesic rays, is not well-understood yet. In [26], we did a partial com-

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parison between the Mabuchi slope with non-Archimedean Mabuchi functional and reduced the (G-)uniform version of YTD conjecture to a non-Archimedean version of entropy regularization conjecture of Boucksom-Jonsson [12].

Furthermore we carried out a partial regularization process (based on Boucksom-Favre-Jonsson's work on Non-Archimedean Calabi-Yau theorems) and proved that uniform K-stability for models (or for filtrations associated to models) is a sufficient (and conjecturally also a necessary) condition for the existence of cscK metrics. By a model filtration, we mean a filtration of the section ring $R(X, L) = \bigoplus_{m=0}^{+\infty} H^0(X, mL)$ induced by a model $(\mathcal{X}, \mathcal{L})$ of (X, L). See Definition 2.1 for the definition of a model, for which the $\mathbb{Q}$-line bundle $\mathcal{L}$ is not assumed to be semiample compared to a test configuration in the usual definition of K-stability (see [19, 31]). Moreover, if $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ denotes the canonical compactification of $(\mathcal{X}, \mathcal{L})$ over $\mathbb{P}^1$ (see Definition 2.1), then, for the study of K-stability, one can assume that $(\mathcal{X}, \mathcal{L})$ is a big model which means that $\overline{\mathcal{L}}$ is big over $\overline{\mathcal{X}}$ (see Remark 3.2). The main goal of this paper is to further reduce Boucksom-Jonsson's non-Archimedean regularization conjecture and hence the YTD conjecture to some purely algebro-geometric conjecture about big line bundles (see Conjecture 4.4, or more generally Conjecture 4.7, for the conjectural statements), which could be studied even without the background on K-stability or non-Archimedean geometry.

More specifically, we will first derive a formula for the non-Archimedean Monge-Ampère measure of big models, which implies a positive intersection formula for the non-Archimedean Mabuchi functional of model filtrations. We refer to Sect. 2 for definitions of terms in the following statement of our main results.

Theorem 1.1 *For any normal and big model $(\mathcal{X}, \mathcal{L})$ of (X, L), if $\phi_{(\mathcal{X},\mathcal{L})}$ denotes the associated non-Archimedean psh metric, then the following statements hold true.*

(i) If the central fibre is given by $\mathcal{X}_0 = \sum_{i=1}^{I} b_i E_i$, and $x_i = r(b_i^{-1}\mathrm{ord}_{E_i})$ is the Shilov point associated to E_i, then the non-Archimedean Monge-Ampère measure of $\phi_{(\mathcal{X},\mathcal{L})}$ is given by the formula:

$$\mathrm{MA}^{\mathrm{NA}}(\phi_{(\mathcal{X},\mathcal{L})}) = \sum_{i=1}^{I} b_i \left(\langle \bar{\mathcal{L}}^n \rangle \cdot E_i\right) \delta_{x_i}, \tag{1}$$

where $\langle \bar{\mathcal{L}}^n \rangle \in H^{n,n}(\bar{\mathcal{X}})$ is the positive intersection product of big line bundles (see Sect. 2.2).

(ii) The non-Archimedean Mabuchi functional of any big model $(\mathcal{X}, \mathcal{L})$ is given by:

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \langle \bar{\mathcal{L}}^n \rangle \cdot \left(K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} + \frac{\underline{S}}{n+1}\bar{\mathcal{L}}\right). \tag{2}$$

The non-Archimedean Monge-Ampère measure on Berkovich spaces were introduced by A. Chambert-Loir [13] and the formula (1) is a generalization of the formula of non-Archimedean Monge-Ampère measures for smooth semipositive non-Archimedean metrics. We refer to Sect. 2.2 for the definition of positive intersection numbers that arise in the study of restricted volumes of big line bundles.

The formula (2), which was announced in [26], generalizes the intersection formula for non-Archimedean Mabuchi functional of a test configuration [10, 32] which coincides with the CM weight when the central fibre of the test configuration is reduced (see [27, 29, 34]). As mentioned above, it together with the work in [26] further reduce the proof of YTD conjecture to some algebraic conjecture (Conjecture 4.4). Here for the convenience of the reader we recall the main result from [26], which is the recent progress in the variational approach to the YTD conjecture (as proposed in [1, 4]) and incorporates the analytic existence result of Chen-Cheng [14].

Definition 1.2 (X, L) is uniformly K-stable for models if there exists $\gamma > 0$ such that for any model $(\mathcal{X}, \mathcal{L})$, we have:

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \cdot \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \tag{3}$$

where $\mathbf{M}^{\mathrm{NA}}$ and $\mathbf{J}^{\mathrm{NA}}$ are given in (23)-(24).

Theorem 1.3 ([26]) *If a polarized manifold (X, L) is uniformly K-stable for models, then (X, L) admits a cscK metric.*

We will see that the positive intersection formula (2) implies that it suffices to test the uniform K-stability for the models with reduced central fibres in which case $K^{\log}_{\tilde{\mathcal{X}}/\mathbb{P}^1} = K_{\mathcal{X}/\mathbb{P}^1}$ (see Proposition 3.5).

The converse direction of Theorem 1.3 is expected to be true if $\mathrm{Aut}(X, L)_0$ is discrete. Indeed, it is implied by Conjecture 4.4. Moreover there is a version in the case when $\mathrm{Aut}(X, L)_0$ is not discrete (see [26] for details). As observed by Y. Odaka, such results can be applied to get immediately the G-uniform version of Yau-Tian-Donaldson conjecture for polarized spherical manifolds (see some beautiful refinement by Delcroix [16, 17] in this case and Remark 4.14).

We end this introduction with the organization of this paper. In Sect. 2.1, we recall the construction of non-Archimedean psh metrics from models. In Sect. 2.2 we recall the concepts related to restricted volumes of big line bundles and positive intersection products, and important results from [6, 20] about the relation between them. In Sect. 3, we prove Theorem 1.1. In the Sect. 4, we propose a general conjecture which strengthens the usual Fujita approximation theorem and (in the $\mathbb{C}^*$-equivariant case) would imply the uniform YTD conjecture for cscK metrics. In the appendix, we verify this algebraic conjecture for some of Nakayama's examples that do not admits birational Zariski decompositions.

2 Preliminaries

2.1 Non-archimedean Metrics Associated to Models

This paper is a following-up work of [26] and we will mostly follow the notations from that work.

Definition 2.1 • A model of (X, L) is a flat family of projective varieties $\pi : \mathcal{X} \to \mathbb{C}$ together with a $\mathbb{Q}$-line bundle $\mathcal{L}$ satisfying:

(i) There is a $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ such that π is $\mathbb{C}^*$-equivariant;
(ii) There is a $\mathbb{C}^*$-equivariant isomorphism $(\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}} \mathbb{C}^* \cong (X, L) \times \mathbb{C}^*$.

- The trivial model of (X, L) is given by $(X \times \mathbb{C}, L \times \mathbb{C}) =: (X_{\mathbb{C}}, L_{\mathbb{C}})$.
  Two models $(\mathcal{X}_i, \mathcal{L}_i)$, $i = 1, 2$ are called equivalent if there exists a model $(\mathcal{X}_3, \mathcal{L}_3)$ and two $\mathbb{C}^*$-equivariant birational morphisms $\mu_i : \mathcal{X}_3 \to \mathcal{X}_i$ such that $\mu_1^*\mathcal{L}_1 = \mu_2^*\mathcal{L}_2$.
- If we forget about the data L and $\mathcal{L}$, then we say that $\mathcal{X}$ is a model of X.
  If there is a $\mathbb{C}^*$-equivariant birational morphism $r_{\mathcal{X}_1,\mathcal{X}_2} : \mathcal{X}_1 \to \mathcal{X}_2$ for two models $\mathcal{X}_i$, $i = 1, 2$, then we say that $\mathcal{X}_1$ dominates $\mathcal{X}_2$ and write $\mathcal{X}_1 \geq \mathcal{X}_2$. If $\mathcal{X} \geq X_{\mathbb{C}}$, then we say that $\mathcal{X}$ is dominating.
  If $\mathcal{X}$ is normal, we say that $\mathcal{X}$ is a normal model. We say a model $\mathcal{X}$ is a SNC (i.e. simple normal crossing) if $(\mathcal{X}, \mathcal{X}_0^{\mathrm{red}})$ is a simple normal crossing pair.
- Let $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ be the canonical $\mathbb{C}^*$-equivariant compactification of $(\mathcal{X}, \mathcal{L})$ over $\mathbb{P}^1$ by adding the trivial (X, L) at $\infty \in \mathbb{P}^1$.
  We say that $(\mathcal{X}, \mathcal{L})$ is a big model if $\bar{\mathcal{L}}$ is a big $\mathbb{Q}$-line bundle over $\bar{\mathcal{X}}$ and the stable base ideal of $m\bar{\mathcal{L}}$ is the same as the π-base ideal of $m\mathcal{L}$ for $m \gg 1$. In particular, the stable base locus satisfies $\mathbf{B}(\mathcal{L}) \subseteq \mathcal{X}_0 = \pi^{-1}(\{0\})$. (This definition is motivated by [12, Lemma A.6].)
  In the following for simplicity of notations, if there is no confusion, we also just write $(\mathcal{X}, \mathcal{L})$ for $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$.
- If $\mathcal{L}$ is semiample over $\mathbb{C}$, then we call the model $(\mathcal{X}, \mathcal{L})$ to be a test configuration of (X, L).

Remark 2.2 Rigorously speaking, the model of (X, L) should be called the model of $(X \times \mathbb{C}, L \times \mathbb{C})$. In other words, with the language of [11], we used the base change from the trivially valued case to the discrete valued case.

In the original literature of K-stability, which we adopt in this paper, the line bundle $\mathcal{L}$ is assumed to be semi-ample. For us this is the only difference between the definition of test configurations and models.

We refer to [7, 11] for the definition of Berkovich analytification $(X^{\mathrm{NA}}, L^{\mathrm{NA}})$ of (X, L) with respect to the trivially valued field $\mathbb{C}$ and the definition of non-Archimedean psh metrics L^{NA} which are represented by ϕ_{triv}-psh functions on X^{NA} (where ϕ_{triv} is the metric associated to the trivial test configuration).

For each model $(\mathcal{X}, \mathcal{L})$ of (X, L), we can associate a non-Archimedean psh metric $\phi_{(\mathcal{X},\mathcal{L})}$ in the following way. If $\mathfrak{b}_m$ denotes the π-relative base ideal of $m\mathcal{L}$ and $\mu_m : \mathcal{X}_m \to \mathcal{X}$ is the normalized blowup of $\mathfrak{b}_m$ with the exceptional divisor denoted by $\tilde{E}_m$, then $(\mathcal{X}_m, \mathcal{L}_m = \mu_m^*\mathcal{L} - \frac{1}{m}\tilde{E}_m)$ is a semiample test configuration. $(\mathcal{X}_m, \mathcal{L}_m)$ defines a smooth non-Archimedean metric $\phi_{(\mathcal{X}_m,\mathcal{L}_m)} \in \mathcal{H}^{\mathrm{NA}}(L)$ and we set

$$\phi_{(\mathcal{X},\mathcal{L})} = \lim_{m\to+\infty} \phi_{(\mathcal{X}_m,\mathcal{L}_m)}. \tag{4}$$

If the base variety $\mathcal{X}$ is clear, we just write $\phi_{(\mathcal{X},\mathcal{L})}$ as $\phi_{\mathcal{L}}$. It is easy to see that equivalent models define the same non-Archimedean psh metrics. Moreover, if $\mathcal{L}$ is semiample, then $\phi_{(\mathcal{X},\mathcal{L})} = \phi_{(\mathcal{X}_m,\mathcal{L}_m)}$ for m sufficiently divisible.

By resolution of singularities, we can assume that $\mathcal{X}$ is dominating via a $\mathbb{C}^*$-equivariant birational morphism $\rho : \mathcal{X} \to X_{\mathbb{C}}$. Write $\mathcal{L} = \rho^*L + D$ with D supported on $\mathcal{X}_0$. Then $\mathcal{L}$ defines a model function $f_{\mathcal{L}}$ on $X_{\mathbb{Q}}^{\mathrm{div}}$ (the set of divisorial valuations on X) given by:

$$f_{\mathcal{L}}(v) = G(v)(D), \quad \forall v \in X_{\mathbb{Q}}^{\mathrm{div}} \tag{5}$$

where $G(v) : X_{\mathbb{Q}}^{\mathrm{div}} \to (X \times \mathbb{C})_{\mathbb{Q}}^{\mathrm{div}}$ is the Gauss extension, i.e. $G(v)$ is a $\mathbb{C}^*$-invariant valuation on $X \times \mathbb{C}$ that extends v and satisfies $G(v)(t) = 1$. Set $\tilde{\phi}_{\mathcal{L}} = \phi_{\mathrm{triv}} + f_{\mathcal{L}}$.

The ϕ_{triv}-psh upper envelope of $f_{\mathcal{L}}$ is defined as:

$$P(f_{\mathcal{L}})(v) = \sup\left\{(\phi - \phi_{\mathrm{triv}})(v); \phi \in \mathrm{PSH}^{\mathrm{NA}}(L), \phi - \phi_{\mathrm{triv}} \le f_{\mathcal{L}}\right\}. \tag{6}$$

By [8, Theorem 8.5] we have the identity $\phi_{(\mathcal{X},\mathcal{L})} = \phi_{\mathrm{triv}} + P(f_{\mathcal{L}}) =: P(\tilde{\phi}_{\mathcal{L}})$. Moreover, by [8, Theorem 8.3], $P(f_{\mathcal{L}})$ is a continuous ϕ_{triv}-psh function.

Because $\bar{\mathcal{L}}$ is $\bar{\pi}$-big over the compactification $\bar{\mathcal{X}} \xrightarrow{\bar{\pi}} \mathbb{P}^1$, when $c \gg 1$, the $\mathbb{Q}$-line bundle $\bar{\mathcal{L}}_c := \bar{\mathcal{L}} + c\mathcal{X}_0$ is big over $\bar{\mathcal{X}}$. Moreover, by [7, Lemma A.8], when $c \gg 1$, the π-relative base ideal of $m\bar{\mathcal{L}}$ is the same as the absolute base ideal of $m\bar{\mathcal{L}}$ for all m sufficiently divisible. In other words we know that $(\mathcal{X}, \mathcal{L}_c)$ is a big model in the sense in Definition 2.1. Note that we have $P(\tilde{\phi}_{\mathcal{L}}) + c = P(\tilde{\phi}_{\mathcal{L}_c}) = \phi_{\mathrm{triv}} + P(f_{\mathcal{L}_c})$. As to be explained in Remark 3.2, it suffices to consider big models in the study of K-stability for models.

2.2 *Restricted Volumes and Positive Intersection Products*

In this section, we (change the notation and) assume that $\mathcal{X}$ is a compact projective manifold and $\mathcal{L}$ is a big line bundle over $\mathcal{X}$ of dimension $n + 1$. Recall that the volume of $\mathcal{L}$ is defined as:

$$\mathrm{vol}_{\mathcal{X}}(\mathcal{L}) = \limsup_{m\to+\infty} \frac{h^0(\mathcal{X}, m\mathcal{L})}{m^{n+1}/(n+1)!}. \tag{7}$$

Denote by $N^1(\mathcal{X}) = \mathrm{Div}(\mathcal{X})/\equiv$ the Néron-Severi group. Then the volume functional extends to be a continuous function on $N^1(\mathcal{X})_{\mathbb{R}} = N^1(\mathcal{X}) \otimes_{\mathbb{Q}} \mathbb{R}$. By Fujita's approximation theorem, this invariant can be calculated as the movable intersection number of $\mathcal{L}$ (see [18, 25]). In other words, if we let $\mu_m : \mathcal{X}_m \to \mathcal{X}$ be the normalized blowup of $\mathfrak{b}(|mL|)$ (or its resolution) with exceptional divisor $\tilde{E}_m$ and set $\mathcal{L}_m = \mu_m^*\mathcal{L} - \frac{1}{m}\tilde{E}_m$, then

$$\mathrm{vol}_{\mathcal{X}}(\mathcal{L}) = \lim_{m\to+\infty} \mathcal{L}_m^{n+1}. \tag{8}$$

As a consequence, the limsup in (7) is indeed a limit.

Next we recall the notion of restricted volume [6, 20, 33] and the asymptotic intersection number that calculates the restricted volume.

Definition 2.3 ([20]) For any irreducible (d-dimensional) subvariety $Z \subset \mathcal{X}$

- The restricted volume of $\mathcal{L}$ along Z is defined as

$$\mathrm{vol}_{\mathcal{X}|Z}(\mathcal{L}) = \limsup_{m\to+\infty} \frac{\dim_{\mathbb{C}} \mathrm{Im}\left(H^0(\mathcal{X}, m\mathcal{L}) \to H^0(Z, m\mathcal{L}|_Z)\right)}{m^d/d!}. \tag{9}$$

- For any $Z \not\subseteq \mathbf{B}(\mathcal{L})$ (the stable base locus of $\mathcal{L}$), the asymptotic intersection number of $\mathcal{L}$ and Z is defined as:

$$\|\mathcal{L}^d \cdot Z\| := \limsup_{m\to+\infty} \mathcal{L}_m^d \cdot \tilde{Z}_m, \tag{10}$$

where $\tilde{Z}_m$ is the strict transform of Z under the normalized blowup $\mu_m : \mathcal{X}_m \to \mathcal{X}$ of base ideal of $|m\mathcal{L}|$.

Remark 2.4 It is shown in [20] that the limsup in the formula (9) and (10) are actually limits.

Boucksom-Favre-Jonsson [6] proved that the restricted volume is equal to a positive intersection product.

Definition 2.5 [[6, Definition 2.5]] Let $\mathcal{L}$ be a big $\mathbb{Q}$-line bundle. For any effective divisor D, define:

$$\langle \mathcal{L}^n \rangle \cdot D = \sup_{\mu, E} (\mu^*L - E)^n \cdot \mu^* D, \tag{11}$$

where supremum is taken over all birational morphism $\mu : \tilde{\mathcal{X}} \to \mathcal{X}$ and an effective divisor E such that $\mu^*L - E$ is nef. If $D = \sum_i b_i D_i$ with $b_i \in \mathbb{R}$ with D_i effective, then we extend the definition (11) linearly:

$$\langle \mathcal{L}^n \rangle \cdot D = \sum_i b_i \langle \mathcal{L}^n \rangle \cdot D_i.$$

Remark 2.6 In [6], Boucksom-Favre-Jonsson defined positive intersection product $\langle \xi^p \rangle$ for any big class $\xi \in N^1(\mathcal{X})_{\mathbb{R}}$ and $1 \le p \le n+1$, by developing an intersection theory on the Riemann-Zariski space. For example, when $p = n+1$, $\langle \xi^{n+1} \rangle = \mathrm{vol}(\xi)$; when $p = 1$, $\langle \xi \rangle$ is the collection of positive parts of divisorial Zariski decomposition of $\pi^*\xi$ for all smooth blowups $\pi : \mathcal{X}_\pi \to \mathcal{X}$. We refer to [6] for details on these more general definitions.

Moreover an analytic definition of the positive intersection product was defined even earlier in [5, Theorem 3.5] (called movable intersection product there). For each semipositive class $\alpha \in H^{1,1}(\mathcal{X}, \mathbb{R})$, define:

$$\langle \mathcal{L}^n \rangle \cdot \alpha = \sup_{T,\mu} \{\beta^n \cdot \mu^*\alpha\} \tag{12}$$

where T ranges over all Kähler currents in $c_1(\mathcal{L})$ that have logarithmic poles and $\mu : \tilde{\mathcal{X}} \to \mathcal{X}$ ranges over the set of those log resolutions satisfying $\mu^*T = \{E\} + \beta$ (with $\{E\}$ an effective divisor and β smooth and semipositive). By Poincaré duality the class $\langle \mathcal{L}^n \rangle$ is uniquely defined as a semipositive class in $H^{n,n}(\mathcal{X}, \mathbb{R})$.

In the above definitions, we see that the left-hand-side of (11) depends only on the numerical class of $\mathcal{L}$ and D.

Recall that the augmented base locus of $\mathcal{L}$ is defined as (see [20]):

$$\mathbf{B}_+(\mathcal{L}) = \bigcap_{\mathcal{L}=A+E} \mathrm{Supp}(E), \tag{13}$$

where the intersection is over all decompositions of $\mathcal{L} = A + E$ into $\mathbb{Q}$-divisors with A ample and E effective. It is know that the augmented base locus depends only on the numerical class of $\mathcal{L}$ (see [20] and reference therein). We will use the following important results:

Theorem 2.7 *If $\mathcal{L} \to \mathcal{X}$ is a big line bundle, and $Z \subset \mathcal{X}$ is a prime divisor, then the following statements are true:*

1. *([20, Theorem 2.13, Theorem C]) If $Z \nsubseteq \mathbf{B}_+(\mathcal{L})$ then $\mathrm{vol}_{\mathcal{X}|Z}(\mathcal{L}) = \|\mathcal{L}^n \cdot Z\|$. If $Z \subseteq \mathbf{B}_+(\mathcal{L})$, then $\mathrm{vol}_{\mathcal{X}|Z}(\mathcal{L}) = 0$.*
2. *([6, Theorem B]) There is an identity $\mathrm{vol}_{\mathcal{X}|Z}(\mathcal{L}) = \langle \mathcal{L}^n \rangle \cdot Z$. As a consequence, $\mathrm{vol}_{\mathcal{X}|Z}(\mathcal{L})$ depends only on the numerical class of $\mathcal{L}$ and Z.*

As a consequence of these results, we know that in the definition of positive intersection number in (11), it suffices to take the supremum along the sequence $\mu_m : \mathcal{X}_m \to \mathcal{X}$ which is the normalized blowup of $\mathfrak{b}(|mL|)$ (or its resolution) with exceptional divisor $\tilde{E}_m$. In other words, if we set $\mathcal{L}_m = \mu_m^*\mathcal{L} - \frac{1}{m}\tilde{E}_m$, then for any divisor D, we have the identity:

$$\langle \mathcal{L}^n \rangle \cdot D = \lim_{m \to +\infty} \mathcal{L}_m^n \cdot \mu^* D. \tag{14}$$

3 Positive Intersection Formula

Let $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ be a *big* model of (X, L). By resolution of singularities, we can assume that $(\mathcal{X}, \mathcal{X}_0^{\mathrm{red}})$ is a dominating and SNC model of (X, L). From now on, for simplicity of notation, we still denote by $(\mathcal{X}, \mathcal{L})$ its natural compactification over $\mathbb{P}^1$.

Because $\mathcal{L}$ is big over $\mathcal{X}(= \bar{\mathcal{X}})$ (by the definition of big model), $\mathbf{B}_+(\mathcal{L}) \neq \mathcal{X}$, there exists a fiber $\mathcal{X}_t = \pi^{-1}(\{t\})$ for some $t \in \mathbb{P}^1 \setminus \{0\}$ such that $\mathcal{X}_t \not\subseteq \mathbf{B}_+(\mathcal{L})$. In particular, $\mathcal{X}_t \not\subseteq \mathbf{B}(\mathcal{L})$. We then apply Theorem 2.7 to get

$$\langle \mathcal{L}^n \rangle \cdot \mathcal{X}_t = \mathrm{vol}_{\mathcal{X}|X}(\mathcal{L}) = \|\mathcal{L}^n \cdot X\| = V. \tag{15}$$

Because $\langle \mathcal{L}^n \rangle \cdot \mathcal{X}_t$ depends only on numerical classes of $\mathcal{L}$ and $\mathcal{X}_t$ (see Definition 2.5), we can use $\mathcal{X}_t \equiv \mathcal{X}_0 = \sum_{i=1}^{I} b_i E_i$ to get:

$$V = \langle \mathcal{L}^n \rangle \cdot \mathcal{X}_t = \sum_{i=1}^{I} b_i \left(\langle \mathcal{L}^n \rangle \cdot E_i \right). \tag{16}$$

Now we can prove the formula (1) for the non-Archimedean Monge-Ampère measure of non-Archimedean metrics associated to model filtrations. This result refines and generalizes [7, Lemma 8.5].

Proof of Theorem 1.1 *(i)* We will use the notations in Sect. 2.1. Via the resolution of singularity, we can first replace $\mathcal{X}$ by any SNC model $\mathcal{X}'$ that dominates $\mathcal{X}$ via $\pi' : \mathcal{X}' \to \mathcal{X}$ and replace $\mathcal{L}$ by $\mathcal{L}' = \pi'^* \mathcal{L}$. For simplicity of notations, we will still use the notation $(\mathcal{X}, \mathcal{L})$ instead of $(\mathcal{X}', \mathcal{L}')$.

Because the sequence of continuous metrics $\phi_m := \phi_{(\mathcal{X}_m, \mathcal{L}_m)} \in \mathcal{H}^{\mathrm{NA}}$ increases to the continuous metric $\phi_{\mathcal{L}} = \phi_{\mathrm{triv}} + P(f_{\mathcal{L}})$, by Dini's theorem we know that ϕ_m converges to $\phi_{\mathcal{L}}$ uniformly. In particular, ϕ_m converges to $\phi_{\mathcal{L}}$ in the strong topology and $\mathrm{MA}^{\mathrm{NA}}(\phi_m)$ converges strongly, and hence also weakly, to $\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}})$.

Set $\nu_{\mathcal{X},m} = (r_{\mathcal{X}})_*(\mathrm{MA}^{\mathrm{NA}}(\phi_m))$ and $\nu_{\mathcal{X}} = (r_{\mathcal{X}})_* \mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}})$ where $r_{\mathcal{X}} : X^{\mathrm{NA}} \to \Delta_{\mathcal{X}}$ is the natural retraction to the dual complex of $\mathcal{X}$ (see [11]). Then they are supported on $\Delta_{\mathcal{X}}$ and it is easy to see that $\nu_{\mathcal{X},m}$ converges to $\nu_{\mathcal{X}}$ weakly. By Portmanteau's theorem for weak convergence of measures (see [2, Theorem 2.1]), we have:

$$\limsup_{m \to +\infty} \nu_{\mathcal{X},m}(\{x_i\}) \leq \nu_{\mathcal{X}}(\{x_i\}). \tag{17}$$

On the other hand, we clearly have

$$\begin{aligned} \nu_{\mathcal{X},m}(\{x_i\}) &= (r_{\mathcal{X}})_* \mathrm{MA}^{\mathrm{NA}}(\phi_m)(\{x_i\}) \\ &= \mathrm{MA}^{\mathrm{NA}}(\phi_m)((r_{\mathcal{X}})^{-1}\{x_i\}) \geq \mathrm{MA}^{\mathrm{NA}}(\phi_m)(\{x_i\}). \end{aligned}$$

So we combine the above two inequalities to get:

$$\limsup_{m\to+\infty} \mathrm{MA}^{\mathrm{NA}}(\phi_m)(\{x_i\}) \leq \nu_{\mathcal{X}}(\{x_i\}) =: V_i. \tag{18}$$

We consider two cases:

1. If $E_i \not\subseteq \mathbf{B}_+(\mathcal{L})$, then E_i is not contained in $\mathbf{B}(\mathcal{L})$. By the formula of non-Archimedean Monge-Ampère measures of test configurations (see [11, Sect. 3.4]) we get that:
$$b_i\mathcal{L}_m^n \cdot \tilde{E}_i = \mathrm{MA}^{\mathrm{NA}}(\phi_m)(\{x_i\}), \tag{19}$$
where $\tilde{E}_i$ is the strict transform of E_i under μ_m. So by (19) and (10) we get
$$\limsup_{m\to+\infty} \mathrm{MA}^{\mathrm{NA}}(\phi_m)(\{x_i\}) = \limsup_{m\to+\infty} b_i\mathcal{L}_m^n \cdot \tilde{E}_i = b_i\|\mathcal{L}^n \cdot E_i\|. \tag{20}$$
So by Theorem 2.7 and the inequality (18) we have
$$b_i\langle\mathcal{L}^n\rangle \cdot E_i = b_i \cdot \mathrm{vol}_{\mathcal{X}|E_i}(\mathcal{L}) = b_i\|\mathcal{L}^n \cdot E_i\| \leq V_i. \tag{21}$$
2. If $E_i \subseteq \mathbf{B}_+(\mathcal{L})$, then $b_i\langle\mathcal{L}^n\rangle \cdot E_i = 0 \leq V_i$.

Combining these with (16), we have:

$$V = \sum_i b_i\langle\mathcal{L}^n\rangle \cdot E_i \leq \sum_i V_i = \sum_i \nu_{\mathcal{X}}(\{x_i\}) \leq V.$$

So the inequalities in the above chain are actually equalities. So $b_i\langle\mathcal{L}^n\rangle \cdot E_i = V_i = \nu_{\mathcal{X}}(\{x_i\})$ for $i = 1, \ldots, I$ and $\nu_{\mathcal{X}} = (r_{\mathcal{X}})_*\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}})$ is supported on the finite set $\{x_i; i = 1, \ldots, I\}$. In other words, we have

$$(r_{\mathcal{X}})_*\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \sum_{i=1}^{N} b_i\left(\langle\mathcal{L}^n\rangle \cdot E_i\right)\delta_{x_i}. \tag{22}$$

But we have said that $(\mathcal{X}, \mathcal{L})$ can be replaced by any SNC model that dominates $\mathcal{X}$. Moreover, the pairs $\{(x_i, V_i); V_i \neq 0\}$ do not depend on the choice of such SNC models. By using the homemorphism $X^{\mathrm{NA}} = \varprojlim \Delta_{\mathcal{X}}$, it is then easy to conclude that the Radon measure $\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}})$ is indeed only supported on the finite set $\{x_i; i = 1, \ldots, I\}$ and the identity (1) holds true. □

Remark 3.1 Although our work on K-stability is the through the study of non-Archimedean geometry in the trivially valued case (which is base-changed to the discretely valued case, following [11, 12]), the proof of formula for non-Archimedean Monge-Ampère measure also holds true for more general discrete valued case.

To be more precise, let $C^\circ = C \setminus \{p\}$ be a punctured algebraic curve and $\pi : (\mathcal{X}^\circ, \mathcal{L}^\circ) \to C^\circ$ be a flat family of smooth polarized projective manifolds. Any compactification $(\mathcal{X}, \mathcal{L}) \to C$ of $(\mathcal{X}^\circ, \mathcal{L}^\circ)$ defines a non-Archimedean metric $\phi_{(\mathcal{X},\mathcal{L})}$ on

$(\mathcal{X}^\circ, \mathcal{L}^\circ)^{\mathrm{NA}}$ which is the Berkovich analytification with respect to the discrete valuation ord_t on $\mathbb{C}(t)$ (where $p = \{t = 0\}$). See [7] for details of the terminology. Without the loss of generality, we can also assume that $(\mathcal{X}, \mathcal{X}_p = \pi^{-1}(\{p\}))$ is simple normal crossing by using resolution of singularities. Then the same proof as above proves a formula similar to (1) for $\mathrm{MA}^{\mathrm{NA}}(\phi_{(\mathcal{X},\mathcal{L})})$ (see also [7, Lemma 8.5]).

We recall that the formula for non-Archimedean functionals following the works in [9, 11, 12] (see also [26]). For any continuous psh metric ϕ on L^{NA}, the non-Archimedean Mabuchi functional is given by:

$$\mathbf{M}^{\mathrm{NA}}(\phi) = \mathbf{H}^{\mathrm{NA}}(\phi) + (\mathbf{E}^{K_X})^{\mathrm{NA}}(\phi) + \underline{S}\, \mathbf{E}^{\mathrm{NA}}(\phi) \tag{23}$$

where the terms on the right-hand-side are given by the following non-Archimedean integrals:

$$\begin{aligned}
\mathbf{H}^{\mathrm{NA}}(\phi) &= \int_{X^{\mathrm{NA}}} A_X(x)\mathrm{MA}^{\mathrm{NA}}(\phi),\\
(\mathbf{E}^{K_X})^{\mathrm{NA}}(\phi) &= \sum_{i=0}^{n-1} \int_{X^{\mathrm{NA}}} (\phi - \phi_{\mathrm{triv}})\mathrm{dd^c}\psi \wedge \mathrm{MA}^{\mathrm{NA}}(\phi_{\mathrm{triv}}^{[i]}, \phi^{[n-1-i]})\\
\mathbf{E}^{\mathrm{NA}}(\phi) &= \frac{1}{n+1}\sum_{i=0}^{n+1} \int_{X^{\mathrm{NA}}} (\phi - \phi_{\mathrm{triv}})\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathrm{triv}}^{[i]}, \phi^{[n-i]}),
\end{aligned}$$

where in the second identity ψ is a Hermitian metric on K_X^{NA}. We also recall the $\mathbf{J}^{\mathrm{NA}}$-functional:

$$\mathbf{J}^{\mathrm{NA}}(\phi) = L^n \cdot \sup(\phi - \phi_{\mathrm{triv}}) - \mathbf{E}^{\mathrm{NA}}(\phi). \tag{24}$$

Remark 3.2 Note that $\mathbf{F}^{\mathrm{NA}} \in \{\mathbf{M}^{\mathrm{NA}}, \mathbf{J}^{\mathrm{NA}}\}$ satisfies $\mathbf{F}^{\mathrm{NA}}(\phi + c) = \mathbf{F}^{\mathrm{NA}}(\phi)$ for any $c \in \mathbb{R}$. In particular, we have $\mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) = \mathbf{F}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$. By the last paragraph in Sect. 2.1, we can assume that the models $(\mathcal{X}, \mathcal{L})$ in the definition of uniform K-stability in Definition 1.2 are always big.

Proposition 3.3 *With the above notation, we have:*

$$\mathbf{H}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \langle \mathcal{L}^n \rangle \cdot K^{\log}_{\mathcal{X}/X_{\mathbb{P}^1}}. \tag{25}$$

Proof Note that we have the identity:

$$\begin{aligned}
K^{\log}_{\mathcal{X}/X_{\mathbb{P}^1}} &= K_{\mathcal{X}} + \mathcal{X}_0^{\mathrm{red}} - (K_{X_{\mathbb{P}^1}} + \mathcal{X}_0) = \sum_i (A_{X_{\mathbb{P}^1}}(E_i) - b_i)E_i\\
&= \sum_i b_i(A_{X_{\mathbb{P}^1}}(b_i^{-1}\mathrm{ord}_{E_i}) - 1)E_i = \sum_i b_i A_X(x_i)E_i.
\end{aligned}$$

So we can use (1) to get the identity:

$$\mathbf{H}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \int_{X^{\mathrm{NA}}} A_X(x)\mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \sum_i A_X(x_i) b_i \langle \mathcal{L}^n \rangle \cdot E_i$$
$$= \langle \mathcal{L}^n \rangle \cdot K^{\log}_{\mathcal{X}/X_{\mathbb{P}^1}}.$$

□

Proposition 3.4 *With the above notation, we have the following identities:*

$$(\mathbf{E}^{K_X})^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \langle \mathcal{L}^n \rangle \cdot \rho^* K_X, \tag{26}$$

$$\mathbf{E}^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \frac{1}{n+1} \langle \mathcal{L}^{n+1} \rangle = \frac{1}{n+1} \langle \mathcal{L}^n \rangle \cdot \mathcal{L}. \tag{27}$$

Proof Because ϕ_m converges to $\phi_{\mathcal{L}}$ strongly, by [11] we have:

$$(\mathbf{E}^{K_X})^{\mathrm{NA}}(\phi_{\mathcal{L}}) = \lim_{m\to+\infty} (\mathbf{E}^{K_X})^{\mathrm{NA}}(\phi_m) = \lim_{m\to+\infty} \mathcal{L}_m^n \cdot \mu_m^* \rho^* K_X. \tag{28}$$

Write $\rho^* K_X = A_1 - A_2$ with A_1, A_2 very ample. Moreover we can choose $A_i, i = 1, 2$ to be sufficient general such that $A_i, i = 1, 2$ do not contain the centers of Rees valuations of $\mathfrak{b}_m$ for all m. Then the strict transforms of $A_i, i = 1, 2$ under $\mu_m : \mathcal{X}_m \to \mathcal{X}$ are the same as the total transform of $A_i, i = 1, 2$. By using Theorem 2.7 we see that the right-hand-side of (28) is equal to

$$\|\mathcal{L}^n \cdot A_1\| - \|\mathcal{L}^n \cdot A_2\| = \langle \mathcal{L}^n \rangle \cdot (A_1 - A_2) = \langle \mathcal{L}^n \rangle \cdot \rho^* K_X. \tag{29}$$

For the first equality in (27), we can again use $\phi_m = \phi_{(\mathcal{X}_m, \mathcal{L}_m)}$ (for which (27) is known to be true) to approximate and directly apply the Fujita approximation result in [25, Theorem 11.4.11]. The last equality in (27) follows from the orthogonality property proved in [5, Corollary 4.5] or [6, Corollary 3.6]. □

We can complete the proof the formula for the non-Archimedean Mabuchi functional.

Proof of Theorem 1.1 *(ii)* The formula (2) follows immediately from the decomposition $\mathbf{M}^{\mathrm{NA}} = \mathbf{H}^{\mathrm{NA}} + (\mathbf{E}^{K_X})^{\mathrm{NA}} + \underline{S}\mathbf{E}^{\mathrm{NA}}$ in (23) and the formula for each part in (25), (26) and (27). □

As an application of the positive intersection formula, we get:

Proposition 3.5 *To check the (G-) uniform K-stability for models (see Definition 1.2 and [26]), it suffices to consider models with reduced central fibres.*

Proof Let $(\mathcal{X}, \mathcal{L})$ be any big model. We can take a base change $(\mathcal{X}^{(d)}, \mathcal{L}^{(d)}) = (\mathcal{X}, \mathcal{L}) \times_{\mathbb{C}, t\mapsto t^d} \mathbb{C}$ such that its normalization $\tilde{\mathcal{X}}$ has reduced central fibers. Let $f : \tilde{\mathcal{X}} \to \mathcal{X}$ be the natural finite morphism and set $\tilde{\mathcal{L}} = f^*\mathcal{L}$. Then we have the identity

$$K^{\log}_{(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)} := K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0 = f^*(K_{\mathcal{X}} + \mathcal{X}_0^{\mathrm{red}}) = f^* K^{\log}_{(\mathcal{X}, \mathcal{X}_0)}.$$

It is known that volumes of big line bundles are multiplicative under generically finite morphisms (see [21, Lemma 4.3]). So we get the identity

$$\langle(\tilde{\mathcal{L}}+\epsilon K^{\log}_{(\tilde{\mathcal{X}},\tilde{\mathcal{X}}_0)})^{n+1}\rangle = d\cdot\langle(\mathcal{L}+\epsilon K^{\log}_{(\mathcal{X},\mathcal{X}_0)})^{n+1}\rangle. \tag{30}$$

Taking derivative with respect to ϵ at $\epsilon=0$, we also get:

$$\langle\tilde{\mathcal{L}}^n\rangle\cdot K^{\log}_{(\tilde{\mathcal{X}},\tilde{\mathcal{X}}_0)} = d\cdot\langle\mathcal{L}^n\rangle\cdot K^{\log}_{(\mathcal{X},\mathcal{X}_0)}. \tag{31}$$

Moreover (30) for $\epsilon=0$ gives $\mathbf{E}^{\mathrm{NA}}(\phi_{\tilde{\mathcal{L}}}) = d\cdot\mathbf{E}^{\mathrm{NA}}(\phi_{\mathcal{L}})$. On the other hand, it is known we have the formula (see [11])

$$(\phi_{\tilde{\mathcal{L}}}-\phi_{\mathrm{triv}})(x) = d\cdot(\phi_{\mathcal{L}}-\phi_{\mathrm{triv}})(d^{-1}x), \quad \text{for all } x\in X^{\mathrm{NA}}. \tag{32}$$

So we get the identity $\mathbf{J}^{\mathrm{NA}}(\phi_{\tilde{\mathcal{L}}}) = d\cdot\mathbf{J}^{\mathrm{NA}}(\phi_{\mathcal{L}})$ by (24). Combining these identities with the positive intersection formula (2), the statement now follows easily. □

4 First Riemann-Roch Coefficients of Big Line Bundles and Fujita Approximations

In view of the above intersection formula, it seems natural to consider the following invariant for big line bundles.

Definition 4.1 Let $\mathcal{L}$ be a big line bundle over a projective manifold $\mathcal{X}$ of dimension $n+1$. The first Riemann-Roch coefficient (1st-RR coefficient) of $\mathcal{L}$ is defined to be:

$$\mathfrak{r}_1(\mathcal{X},\mathcal{L}) = \langle\mathcal{L}^n\rangle\cdot K_{\mathcal{X}}. \tag{33}$$

If the base manifold $\mathcal{X}$ is clear, we just write $\mathfrak{r}_1(\mathcal{X},\mathcal{L})$ as $\mathfrak{r}_1(\mathcal{L})$.

The zero-th Riemann-Roch coefficient is of course the volume of $\mathcal{L}$:

$$\mathfrak{r}_0(\mathcal{X},\mathcal{L}) := \mathrm{vol}_{\mathcal{X}}(\mathcal{L}) = \langle\mathcal{L}^{n+1}\rangle. \tag{34}$$

One would hope that $\mathfrak{r}_1(\mathcal{X},\mathcal{L})$ is the second order coefficients in the expansion of $h^0(\mathcal{X},m\mathcal{L})$. This is true if $\mathcal{L}$ is big and nef by Fujita's vanishing theorem. But due to the example in [15], this does not seem to be true for general big line bundles.

Lemma 4.2 *If $\mu:\mathcal{Y}\to\mathcal{X}$ is a birational morphism between smooth projective manifold, which is a composition of blowups along smooth subvarieties. Then we have:*

$$\mathfrak{r}_1(\mathcal{L}) = \mathfrak{r}_1(\mu^*\mathcal{L}). \tag{35}$$

Proof Write $K_{\mathcal{X}}$ as the difference of very ample divisors $A_1 - A_2$ and arguing as in the proof of Proposition 3.4, we see that:

$$\langle \mathcal{L}^n \rangle \cdot K_{\mathcal{X}} = \langle \mu^* \mathcal{L}^n \rangle \cdot \mu^* K_{\mathcal{X}}. \tag{36}$$

Let E_i be the exceptional divisor of μ. We just need to show that $\langle \mu^* \mathcal{L}^n \rangle \cdot E_i = \mathrm{vol}_{\mathcal{Y}|E_i}(\mu^* \mathcal{L}^n) = 0$. This can be seen by the inclusion:

$$\mathrm{Im}\left(H^0(\mathcal{Y}, m\mu^*\mathcal{L}) \to H^0(E_i, m\mu^*\mathcal{L}|_{E_i})\right) \subseteq H^0(E_i, m\mu^*\mathcal{L}|_{E_i}) = H^0(\mu_*(E_i), m\mathcal{L}|_{\mu_*(E_i)}) \tag{37}$$

and using the fact that the right-hand-side is equal to $o(m^n)$ because $\dim(\mu_*(E_i)) < n$. □

If we consider $\mathcal{L}$ as a Cartier b-divisor in the sense of Shokurov, then because of identity (35), $\mathfrak{r}_1(\mathcal{L})$ is an invariant of the Cartier b-divisor $\mathcal{L}$.

The following lemma follows immediately from the results in [28, Sect. 3.1].

Lemma 4.3 *Let $\mathcal{L} = \mathcal{P} + \mathcal{N}$ be the divisorial Zariski decomposition of $\mathcal{L}$. Then we have:*

$$\mathfrak{r}_1(\mathcal{L}) = \mathfrak{r}_1(\mathcal{P}). \tag{38}$$

Moreover if $\mathcal{L}$ admits a Zariski decomposition, i.e. if $\mathcal{P}$ is nef, then we have:

$$\mathfrak{r}_1(\mathcal{L}) = \mathcal{P}^n \cdot K_{\mathcal{X}}. \tag{39}$$

We propose the following main conjecture.

Conjecture 4.4 Let $(\mathcal{X}, \mathcal{L})$ be a *big* model of (X, L). Then there exists a sequence of blowups $\mu_m : \mathcal{X}_m \to \mathcal{X}$ along $\mathbb{C}^*$-equivariant ideal sheaves cosupported on $\mathcal{X}_0$ and decompositions into $\mathbb{Q}$-divisors $\mu_m^* \mathcal{L} = \mathcal{L}_m + E_m$ with $\mathcal{L}_m$ semiample and E_m effective supported on the exceptional divisor of μ_m such that:

$$\lim_{m \to +\infty} \mathrm{vol}_{\mathcal{X}_m}(\bar{\mathcal{L}}_m) = \mathrm{vol}_{\mathcal{X}}(\bar{\mathcal{L}}) \quad \text{and} \quad \lim_{m \to +\infty} \mathfrak{r}_1(\bar{\mathcal{L}}_m) = \mathfrak{r}_1(\bar{\mathcal{L}}). \tag{40}$$

Because of the positive intersection formula in (2) and the reduction in [26], this indeed implies Boucksom-Jonsson's regularization conjecture. Moreover by the following lemma and the work in [26], it would complete the solution of Yau-Tian-Donaldson conjecture for cscK metrics.

Lemma 4.5 *For any big model $(\mathcal{X}, \mathcal{L})$, Conjecture 4.4 implies that there exists $\phi_m \in \mathcal{H}^{\mathrm{NA}}$ such that ϕ_m converges to $\phi_{(\mathcal{X},\mathcal{L})}$ in the strong topology and $\mathbf{M}^{\mathrm{NA}}(\phi_m) \to \mathbf{M}^{\mathrm{NA}}(\phi_{(\mathcal{X},\mathcal{L})})$.*

Proof By the same base change construction as in the proof of Proposition 3.5, we can assume that $\mathcal{X}$ has a reduced central fibre.

For simplicity of notations, we denote by $\phi = \phi_{(\mathcal{X},\mathcal{L})}$ (resp. ϕ_m) the non-Archimedean metrics associated to $\mathcal{L}$ (resp. $\mathcal{L}_m$). Then because E_m is effective, we have $\phi \geq \phi_m$. We claim that $\phi_m \to \phi$ strongly. Indeed, by [12, Proposition 6.26], it suffices to show the following non-negative quantity converges to 0 as $m \to +\infty$:

$$\mathbf{J}_\phi^{\mathrm{NA}}(\phi_m) = \int_{X^{\mathrm{NA}}} (\phi_m - \phi)\mathrm{MA}^{\mathrm{NA}}(\phi_m) - \mathbf{E}^{\mathrm{NA}}(\phi_m) + \mathbf{E}^{\mathrm{NA}}(\phi). \tag{41}$$

This follows immediately from $\phi_m \leq \phi$ and (27):

$$0 \leq \mathbf{J}_\phi^{\mathrm{NA}}(\phi_m) \leq -\mathbf{E}^{\mathrm{NA}}(\phi_m) + \mathbf{E}^{\mathrm{NA}}(\phi) = \frac{\mathrm{vol}_{\mathcal{X}}(\mathcal{L})}{n+1} - \frac{\mathrm{vol}_{\mathcal{X}_m}(\mathcal{L}_m)}{n+1}.$$

By the positive intersection formula (2) the second identity in (40) implies $\mathbf{M}^{\mathrm{NA}}(\phi_m) \to \mathbf{M}^{\mathrm{NA}}(\phi)$. □

We hope the Conjecture 4.4 can be studied by using the geometric tools introduced in the study of Fujita's approximation theorem. We recall the following definition

Definition 4.6 (*see* [25, Definition 11.4.3]) Let $\mathcal{L}$ be a big line bundle. A Fujita approximation of $\mathcal{L}$ consists of a projective birational morphism $\mu : \mathcal{X}' \to \mathcal{X}$ with $\mathcal{X}'$ irreducible together with a decomposition $\mu^*\mathcal{L} = A + E$ in $N^1(\mathcal{X})_{\mathbb{Q}}$ such that A is big and semiample and E is effective.

By using the above definition, we generalize the Conjecture 4.4 for all big line bundles.

Conjecture 4.7 Let $\mathcal{L}$ be a big line bundle over a smooth projective manifold $\mathcal{X}$. Then there exists a sequence of birational morphisms $\mu_m : \mathcal{X}_m \to \mathcal{X}$ with Fujita approximations $\mu_m^*\mathcal{L} = \mathcal{L}_m + E_m$ as in Definition 4.6 such that:

$$\lim_{m\to+\infty} \mathrm{vol}_{\mathcal{X}_m}(\mathcal{L}_m) = \mathrm{vol}_{\mathcal{X}}(\mathcal{L}) \quad \text{and} \quad \lim_{m\to+\infty} \mathfrak{r}_1(\mathcal{L}_m) = \mathfrak{r}_1(\mathcal{L}). \tag{42}$$

Remark 4.8 Sébastien Boucksom pointed out to me that this conjecture could be formulated using the language of b-divisors. Such a formulation has some consequences and (hopefully) might be useful for studying this problem.

Let's recall an orthogonality estimate by Boucksom-Demailly-Păun-Peternell (see also [25, Theorem 11.4.21]):

Theorem 4.9 *([5, Theorem 4.1]) Fix any ample line bundle H on $\mathcal{X}$. There exists a constant $C = C(\mathcal{X}, H) > 0$ such that any Fujita decomposition $(\mu : \mathcal{X}' \to \mathcal{X}, \mu^*\mathcal{L} = A + E)$ satisfies the estimate:*

$$(A^n \cdot E)^2 \leq C \cdot (\mathrm{vol}_{\mathcal{X}}(\mathcal{L}) - \mathrm{vol}_{\mathcal{X}'}(A)). \tag{43}$$

We observe an immediate consequence of this estimate.

Lemma 4.10 *Let $\mu_m : \mathcal{X}_m \to \mathcal{X}$ be a sequence of birational morphisms such $\mu_m^*\mathcal{L} = \mathcal{L}_m + E_m$ where $\mathcal{L}_m$ is ample and E_m is effective. Assume that the following conditions are satisfied:*

1. $\lim_{m\to+\infty} \mathrm{vol}_{\mathcal{X}_m}(\mathcal{L}_m) = \mathrm{vol}_{\mathcal{X}}(\mathcal{L})$.
2. $\lim_{m\to+\infty} \mathcal{L}_m^n \cdot \mu_m^* K_{\mathcal{X}} = \langle \mathcal{L}^n \rangle \cdot K_{\mathcal{X}}$.

Then $\lim_{m\to+\infty} \mathfrak{r}_1(\mathcal{L}_m) = \mathfrak{r}_1(\mathcal{L})$ *if and only if*

$$\lim_{m\to+\infty} \mathcal{L}_m^n \cdot K_{\mathcal{X}_m/\mathcal{X}} = 0. \tag{44}$$

In particular, if there exists a constant $C > 0$ *independent of* m *such that for any irreducible component* F *of* E_m *we have* $\mathrm{ord}_F(K_{\mathcal{X}_m/\mathcal{X}}) \leq C \cdot \mathrm{ord}_F(E_m)$*, then we have the convergence:* $\lim_{m\to+\infty} \mathfrak{r}_1(\mathcal{L}_m) = \mathfrak{r}_1(\mathcal{L})$.

Remark 4.11 The above lemma suggests that the techniques from birational algebraic geometry might be useful for achieving (44). Indeed, our hope is that the MMP techniques (based on the work of Birkar-Casini-Hacon-McKernan) could be used to extract suitable exceptional divisors satisfying the conditions in the above lemma. Note that such type of techniques has prove to be very powerful in the study of K-stability for Fano varieties (see for example [3]).

By the works in [18, 25] and [6, 20], the sequence $\{\mathcal{L}_m\}$ that satisfy the first two conditions can be obtained by blowing up base ideals. Moreover one can also get $\mathcal{L}_m$ by blowing up appropriate asymptotic multiplier ideals, which satisfy the important Nadel-vanishing and global generation properties. We review the construction in [25, 11.4.B, Proof of Theorem 11.4] for the reader's convenience. Fix a very ample bundle H on $\mathcal{X}$ such that $G := K_{\mathcal{X}} + (n+2)H$ is very ample. For $m \geq 0$, set $M_m = m\mathcal{L} - G$. Given $\epsilon > 0$ there exists $m \gg 1$ such that $\mathrm{vol}(M_m) \geq m^{n+1}(\mathrm{vol}(\mathcal{L}) - \epsilon)$. Set $\mathcal{J} = \mathcal{J}(\mathcal{X}, \|M_m\|)$, let $\mu_m : \mathcal{X}_m \to \mathcal{X}$ be a common resolution of $\mathcal{J}$ such that $\mu_m^* \mathcal{J} = \mathcal{O}(-\tilde{E}_m)$. Then $\mathcal{L}_m := \mathcal{L} - \frac{1}{m}\tilde{E}_m$ is semiample and $\mathcal{L}_m^{n+1} \geq \mathrm{vol}(\mathcal{L}) - \epsilon$ (see [25, 11.4.B] for more details). By letting $\epsilon \to 0$, we see that the first condition in Lemma 4.10 is thus satisfied.

Now we claim that in this construction, the second condition in Lemma 4.10 can also be satisfied. This fact will be used in the calculations of appendix Sect. 5. To see this, we use some similar argument as in [20, Proof of Theorem 2.13]. Choose $m_0 \gg 1$ such that $m_0\mathcal{L} - G = N'$ is effective. Fix a very ample divisor H such that $N := N' + H$ is ample. Then we have the inclusion

$$\begin{aligned}\mathfrak{b}(|(m-m_0)\mathcal{L}|)\mathcal{O}_{\mathcal{X}}(-N) &\subseteq \mathfrak{b}(|(m-m_0)\mathcal{L})\mathcal{O}(-N') \\ &\subseteq \mathfrak{b}(|m\mathcal{L} - G|) = \mathfrak{b}(|M_m|) \subseteq \mathcal{J}(\|M_m\|).\end{aligned}$$

We can also assume that μ_m is both resolutions of $\mathfrak{b}(|M_m|)$ and $\mathfrak{b}(|(m-m_0)\mathcal{L}|)$ satisfying the identities $\mu_m^*\mathfrak{b}(|m\mathcal{L} - G|) = \mathcal{O}_{\mathcal{X}_m}(-\tilde{F}_m)$ and $\mu_m^*\mathfrak{b}(|(m-m_0)\mathcal{L}|) = \mathcal{O}_{\mathcal{X}_m}(-\tilde{Q}_m)$. Set $\mathcal{L}'_m = \mu_m^*(\mathcal{L} - \frac{G}{m}) - \frac{1}{m}\tilde{F}_m$ and $\mathcal{L}''_m = \mu_m^*\mathcal{L} - \frac{1}{m-m_0}\tilde{Q}_m$.

Fix any effective divisor D on $\mathcal{X}$. Let $\tilde{D}_m$ be the strict transform of D under μ_m. Then the above inclusion implies:

$$\mathrm{vol}((\mathcal{L}_m + \frac{N}{m})|_{\tilde{D}_m}) \geq \mathrm{vol}((\mathcal{L}'_m + \frac{N}{m})|_{\tilde{D}_m}) \geq \mathrm{vol}(\mathcal{L}''_m|_{\tilde{D}_m}).$$

Because $\mathcal{L}_m$ and $\mathcal{L}''_m$ are both semiample, this implies

$$(\mathcal{L}_m + \frac{N}{m})^n \cdot \tilde{D}_m \geq \mathcal{L}''^n_m \cdot \tilde{D}_m. \tag{45}$$

Now we can fix a very ample line bundle H such that $\mu_m^*(mH) - \mathcal{L}_m = m\mu^*(H - \mathcal{L}) + \tilde{E}_m$ is effective. Then for any $1 \leq i \leq n$, we have:

$$\limsup_{m\to+\infty} \frac{1}{m^n}(m\mathcal{L}_m^{n-i}) \cdot N^i \cdot \tilde{D}_m \leq \limsup_{m\to+\infty} \frac{1}{m^n}(mH)^{n-i} \cdot N^i \cdot \tilde{D}_m = 0.$$

By expanding the left-hand-side of (45), this implies

$$\langle \mathcal{L}^n \rangle \cdot D \geq \limsup_{m\to+\infty} \mathcal{L}_m^n \cdot \tilde{D}_m \geq \limsup_{m\to+\infty} \mathcal{L}''^n_m \cdot \tilde{D}_m = \|\mathcal{L}^n \cdot D\|$$

which, by using Theorem 2.7, implies the equality

$$\lim_{m\to+\infty} \mathcal{L}_m^n \cdot \mu_m^* D = \|\mathcal{L}^n \cdot D\| = \langle \mathcal{L}^n \rangle \cdot D.$$

Writing $-K_{\mathcal{X}} = D_1 - D_2$ with D_1, D_2 effective, we then see that the second condition of Lemma 4.10 is satisfied too.

Finally we point out that Conjecture 4.7 holds true any for any big line bundle that admits a birational Zariski decomposition (in the sense of Cutkosky-Kawamata-Moriwaki). Unfortunately not all big line bundles admit such birational Zariski decomposition by the counterexamples of Nakayama [30]. On the other hand, we verify in the appendix that Conjecture 4.7 indeed holds for some of Nakayama's examples. Indeed, we will show that in these examples the bound of discrepancies in the above lemma is indeed satisfied. So it seems to be very interesting to know whether (44) can be achieved in general.

Definition 4.12 We say a big line bundle $\mathcal{L}$ admits a birational Zariski decomposition if there is a modification $\mu : \tilde{\mathcal{X}} \to \mathcal{X}$, a nef $\mathbb{R}$-divisor $\mathcal{P}$ and an $\mathbb{R}$-effective divisor $\mathcal{N}$ on $\tilde{\mathcal{X}}$ with the following properties:

- $\mu^*\mathcal{L} = \mathcal{P} + \mathcal{N}$.
- For any positive integer $m > 0$, the map

$$H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(\lfloor mP \rfloor)) \to H^0(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}(m\mathcal{L})) \tag{46}$$

  induced by the section e_m is an isomorphism, where e_m is the canonical section of $\lceil m\mathcal{N} \rceil$.

Lemma 4.13 *If a big line bundle $\mathcal{L}$ admits a birational Zariski decomposition, then the Conjecture 4.7 for $\mathcal{L}$ is true.*

Proof By Lemma 4.2 and (39), we have $\mathfrak{r}_1(\mathcal{L}) = \mathfrak{r}_1(\mu^*\mathcal{L}) = \mathfrak{r}_1(\mathcal{P}) = \mathcal{P}^n \cdot K_{\tilde{\mathcal{X}}}$. Choose any ample divisor A on $\tilde{\mathcal{X}}$. Because $\mathcal{P}$ is big and nef, we know that for $k \gg 1$, $k\mathcal{P} - A = \Delta_k$ is effective. So we get:

$$(m + k)\mathcal{P} = m\mathcal{P} + A + E_k, \tag{47}$$

which implies the decomposition over $\tilde{\mathcal{X}}$:

$$\mu^*\mathcal{L} = \mathcal{P} + \mathcal{N} = \frac{1}{m+k}(m\mathcal{P} + A) + \frac{1}{m+k}\Delta_k + \mathcal{N}. \tag{48}$$

By perturbing the coefficients of A, we can assume that $m\mathcal{P} + A$ is a $\mathbb{Q}$-divisor. Set $\mathcal{L}_m = \frac{1}{m+k}(m\mathcal{P} + A)$. Then it is easy to see that (42) holds true. $\square$

Remark 4.14 If (X, L) is a polarized spherical manifold, it is known that its models in the sense of Definition 2.1 is a Mori dream space (see the appendix A by Y. Odaka to [16]). Since Zariski decomposition of big lines bundles always exist on Mori dream spaces, the above lemma in the $\mathbb{C}^*$-equivariant setting gives an explanation why the Yau-Tian-Donaldson conjecture holds for polarized spherical manifolds. See [16, Appendix A] for a slightly different proof of this fact (again based on Theorem 1.3).

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5 Conjecture 4.7 for Nakayama's Examples Without Birational Zariski Decomposition

In this appendix, we will use Lemma 4.10 to show that Conjecture 4.7 is indeed true for some examples of big line bundles that do not have a birational Zariski decomposition. Such examples were first discovered by Nakayama [30]. Here we will do a case study based on the construction of Fujita approximation in [25, Theorem 11.4.4] and the calculation of asymptotic multiplier ideals for Nakayama's examples in the work of Koike [23].

We first write down some notations. Set $S = E \times E$ for an elliptic curve E without complex multiplication. Then the pseudoeffective cone $\mathrm{PE}(S)$ coincides with the nef cone $\mathrm{Nef}(S)$. Fix a point $p \in E$ and consider in $N^1(S)_{\mathbb{R}}$ three classes:

$$f_1 = [\{P\} \times E] =: [F_1], \quad f_2 = [E \times \{P\}] =: [F_2], \quad \delta = [\Delta]$$

where $\Delta \subset E \times E$ is the diagonal. Then $N^1(S)_{\mathbb{R}}$ is spanned by $\{f_1, f_2, \delta\}$ and the description of the nef cone is known (see [24, Lemma 1.5.4]): $\alpha = x \cdot f_1 + y \cdot f_2 + z \cdot \delta \in N^1(S)_{\mathbb{R}}$ is nef if and only if

$$xy + xz + yz \geq 0, \quad x + y + z \geq 0. \tag{49}$$

By standard linear algebra, we can use the following linear transformation to diagonalize the above relation:

$$l_1 = \frac{1}{6}(f_1 + f_2 - 2\delta), \quad l_2 = \frac{1}{6}(-\sqrt{3} f_1 + \sqrt{3} f_2), \quad \frac{1}{6}(f_1 + f_2 + \delta)$$
$$a = x + y - 2z, \quad b = -\sqrt{3}x + \sqrt{3}y, \quad c = 2(x + y + z).$$

such that $\alpha = al_1 + bl_2 + cl_3 \in N^1(S)_{\mathbb{R}}$ is nef if and only if

$$c^2 \geq a^2 + b^2, \quad c \geq 0. \tag{50}$$

Let L_i, $i = 0, 1, 2$ be three line bundles over S. Set

$$X = \mathbb{P}(\mathcal{O}_S \oplus (L_1 - L_0) \oplus (L_2 - L_0)) \cong \mathbb{P}(L_0 \oplus L_1 \oplus L_2). \tag{51}$$

Denote by $H = \mathcal{O}_X(1)$ the tautological line bundle for the first projectivization in (51).

We use the description of X as a toric bundle over S as in [30]. Let Σ denote the standard fan of $\mathbb{P}^2$, i.e. the fan generated by three cones:

$$\sigma_1 = \mathrm{Cone}\{e_1, e_2\}, \quad \sigma_2 = \mathrm{Cone}\{e_2, -(e_1 + e_2)\}, \quad \sigma_3 = \mathrm{Cone}\{-(e_1 + e_2), e_1\}.$$

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the piecewise linear function on Σ satisfying $h(e_1) = h(e_2) = 0$ and $h(-(e_1 + e_2)) = -1$. Then X is the toric bundle associated to Σ and h determines the line bundle H. Set $\mathbb{L} = \pi^* L_0 + D_h$. By a result of Cutkosky (see [23, Lemma 6.1]), $\mathbb{L}$ is a big line bundle if and only if there exists $(k_0, k_1, k_2) \in \mathbb{N}^3$ such that $L_0^{k_0} \otimes L_1^{k_1} \otimes L_2^{k_3}$ is an ample line bundle over S. Moreover it is well-known that the canonical line bundle of the projective bundle X is given by:

$$K_X = \pi^*(K_S + (L_1 - L_0) + (L_2 - L_0)) - 3H = L_1 + L_2 - 2L_0 - 3H. \tag{52}$$

The last identity uses the triviality of K_S.

We will consider the example in [23, Example 6.5]. Set $L_0 = 4F_1 + 4F_2 + \Delta$, $L_1 = \mathcal{O}_V$, $L_2 = \mathcal{O}_V(-F_1 + 9F_2 + \Delta)$. Then

$$c_1(L_0) = 6(l_1 + 3l_3), \quad c_1(L_1) = 0, \quad c_1(L_2) = 6l_1 + 10\sqrt{3}l_2 + 18l_3. \tag{53}$$

Because $c_1(L_0)$ is in the interior of the nef cone, L_0 is ample. Note that H is relatively ample. So it is easy to see that there exist $a, b \in \mathbb{Z}_{>0}$ such that $\frac{aL_0+bH-K_X}{n+1}$ is a very ample line bundle. Set $G = aL_0 + bH$ and

$$\begin{aligned} M_p = p\mathbb{L} - G &= p(L_0 + H) - (aL_0 + bH) = (p-a)L_0 + (p-b)H \\ &= (p-b)\left(\frac{p-a}{p-b}L_0 + H\right) =: (p-b)(Q_p + H). \end{aligned}$$

Set $\mathcal{J}_p := \mathcal{J}(X, \|M_p\|)$. Let $\mu_p : Y_p \to X$ be the normalized blowup of $\mathcal{J}_p$ with $E_p := \mu_p^*\mathcal{J}_p = \mathcal{O}_{Y_p}(-\sum_i c_{p,i}E_{p,i})$. Set $A_p = \mu_p^*(\mathbb{L}) - \frac{1}{p}E_p$.

By the discussion after Lemma 4.10, we known that (Y_p, A_p) satisfies the first two conditions of Lemma 4.10. So, by Lemma 4.10, it suffices to show that there exists $C > 0$ such that $A_X(E_{i,p}) \le Cp^{-1}c_{p,i}$ for any i, p.

For any $\mathbb{Q}$-line bundle L on S and with h as above, define a compact convex set following [30, Sect. 2.b] (we identify line bundles with their Chern classes):

$$\square(L, h) = \{(x, y) \in \mathbb{R}^2_{\ge 0};\, x + y \le 1 \text{ and } L + x(L_1 - L_0) + y(L_2 - L_0) \in \mathrm{PE}(S)\}. \tag{54}$$

Then it is straight-forward to use (50) and (53) to get:

$$\square(Q_p, h) = \left\{(x, y) \in \mathbb{R}^2_{\ge 0};\quad x + \frac{5}{2\sqrt{6}}\frac{p-b}{p-a}y \le 1\right\}.$$

Let $\varphi_{p,\min}$ be the metric of minimal singularity on M_p. Then it is known that $\mathcal{J}(\varphi_{p,\min}) = \mathcal{J}(\|M_p\|) = \mathcal{J}_p$. For each fan σ_i, $i = 0, 1, 2$, there exists an open set $U_i \cong S \times \mathbb{C}^2$ which is an affine toric bundle over S. Applying the result in [23, 5.2], $\mathcal{J}_p$ is trivial on U_0, U_1, and over U_2 we can choose the canonical affine coordinate (z_1, z_2) on $\mathbb{C}^2$ such that the multiplier ideal is generated by monomials:

$$\mathcal{J}_p(U_2) = \left\langle z_1^{m_1} z_2^{m_2};\quad (m_1 + 1, m_2 + 1) \in \mathrm{Int}((p-b)S_p) \cap \mathbb{Z}^2\right\rangle \tag{55}$$

where

$$S_p = \left\{(x, y) \in \mathbb{R}^2_{\ge 0};\quad x + \frac{y}{1 - \frac{2\sqrt{6}(p-a)}{5(p-b)}} \ge 1\right\}, \tag{56}$$

is generated by the exponents $(0, 1)$, $(1, 0)$, $(0, 1 - \frac{2\sqrt{6}(p-a)}{5(p-b)})$, which are the images of $(0, 0)$, $(1, 0)$, $(0, \frac{2\sqrt{6}(p-a)}{5(p-b)})$ under the linear map $(a, b) \mapsto (\langle (a, b) - m_{\sigma_3}, v_i\rangle)_{i=1,2}$ $= (a, 1 - a - b)$ with $m_{\sigma_3} = (0, 1)$, $v_1 = (1, 0)$, $v_2 = (-1, -1)$ (see [23, Definition 4.1]). In particular, the multiplier ideal sheaf is co-supported on $\mathbb{P}(L_2) \subset \mathbb{P}(L_0 \oplus L_1 \oplus L_2)$. Moreover, as pointed out in [23], by using [30, Theorem 2.10], we know that the line bundle $\mathbb{L}$ does not admit birational Zariski decomposition.

Equivalently, we have:

$$\mathcal{J}_p(U_2) = \left\langle z_1^{m_1} z_2^{m_2};\quad (m_1, m_2) \in \mathbb{Z}_{\geq 0}, \frac{m_1}{\alpha_p} + \frac{m_2}{\beta_p} > 1 \right\rangle. \tag{57}$$

where $\alpha_p = \frac{p-b}{d_p}$, $\beta_p = \frac{(1-\frac{2\sqrt{6}}{5})p+\frac{2\sqrt{6}}{5}a-b}{d_p}$, $d_p = 1 - \frac{1}{p-b} - \frac{1}{(1-\frac{2\sqrt{6}}{5})p+\frac{2\sqrt{6}}{5}a-b}$. We only need to know that there exists $C = \frac{4.9-2\sqrt{6}}{5} > 0$ such that $\alpha_p \geq Cp$, $\beta_p \geq Cp$ for $p \gg 1$ since $d_p = 1 + O(p^{-1})$.

Because the multiplier ideal is monomial, we can use the result about Rees valuations of monomial ideals to see that the blow-up of $\mathcal{J}_p$ corresponds to the sides of the Newton-polygons of $\mathcal{J}_p$ (see [22, 15.4]). Indeed, such blowup also corresponds to a subdivision of the cone σ_2.

Now let the sides of the Newton polygon be given by $\overline{P_{i-1}P_i}$, $1 \leq i \leq r$ with $P_i = (x_i, y_i)$. Then it is easy to see that $P_0 = (0, \lfloor\beta\rfloor + 1)$ and $P_r = (\lfloor\alpha\rfloor + 1, 0)$. Note that $(a_i, b_i) = (y_{i-1} - y_i, x_i - x_{i-1}) \in \mathbb{R}^2_{>0}$ is a normal vector of $\overline{P_{i-1}P_i}$. The monomial valuation ord_{E_i} that corresponds to the side $\overline{P_iP_{i+1}}$ can be chosen to be given by the weighted blowup with weights (a_i, b_i). Set $\tau_i = b_i/a_i > 0$. It is easy to see that:

$$\begin{aligned} w(E_i) := \frac{A(E_i)}{\mathrm{ord}_{E_i}(\mathcal{J}_p)} &= \frac{a_i + b_i}{a_i x_{i-1} + b_i y_{i-1}} = \frac{1+\tau_i}{x_{i-1} + y_{i-1}\tau_i} \\ &= \frac{a_i + b_i}{a_i x_i + b_i y_i} = \frac{1+\tau_i}{x_i + y_i\tau_i}. \end{aligned}$$

As a consequence, we have:

$$w(E_i) - w(E_{i+1}) = \frac{1+\tau_i}{x_i + y_i\tau_i} - \frac{1+\tau_{i+1}}{x_i + y_i\tau_{i+1}} = \frac{(y_i - x_i)(\tau_{i+1} - \tau_i)}{(x_i + y_i\tau_i)(x_i + y_i\tau_{i+1})}. \tag{58}$$

From this identity, we easily see that $\max\{w(E_i); 1 \leq i \leq r\} = \max\{w(E_1), w(E_r)\}$. Now note that τ_1^{-1} is at most the absolute value of the slope of the line $\overline{P_0P'}$ where P' is the point $(1, -\frac{\beta}{\alpha} + \beta)$ one the line connecting $(\alpha, 0)$ and $(0, \beta)$, which gives the inequality:

$$\begin{aligned} w(E_1) &= \frac{1+\tau_1}{(\lfloor\beta\rfloor + 1)\tau_1} = \frac{1}{\lfloor\beta\rfloor + 1} + \frac{1}{(\lfloor\beta\rfloor + 1)\tau_1} \leq \frac{1}{\beta} + \frac{1}{\lfloor\beta\rfloor + 1}(\lfloor\beta\rfloor + 1 - \beta + \frac{\beta}{\alpha}) \\ &= \frac{1}{\beta} + \frac{\lfloor\beta\rfloor + 1 - \beta}{\lfloor\beta\rfloor + 1} + \frac{\beta}{\lfloor\beta\rfloor + 1}\frac{1}{\alpha} = O(p^{-1}). \end{aligned}$$

By the same argument (or just by symmetry), we also get $w(E_r) = O(p^{-1})$. According to the previous discussion, the verification of 3rd condition in Lemma 4.10 is complete.

Remark 5.1 It is easy to see that the above arguments, which reduce the problem to the estimates for Rees valuations of monomial ideals, works for many more examples of Nakayama type.

References

1. Berman, R., Boucksom, S., Jonsson, M.: A variational approach to the Yau-Tian-Donaldson conjecture. arXiv:1509.04561v2
2. Billingsley, P.: Convergence of Probability Measures, 2nd edn. Wiley, New York (1999)
3. Blum, H., Liu, Y., Xu, C.: Openness of K-semistability for Fano varieties. arXiv:1907.02408
4. Boucksom, S.: Variational and non-Archimedean aspects of Yau-Tian-Donaldson conjecture. arXiv:1805.03289
5. Boucksom, S., Demailly, J.-P., Păun, M., Peternell, T.: The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. J. Algebraic Geom. **22**(2), 201–248 (2013)
6. Boucksom, S., Favre, C., Jonsson, M.: Differentiability of volumes of divisors and a problem of Teissier. J. Algebraic Geom. **18**(2), 279–308 (2009)
7. Boucksom, S., Favre, C., Jonsson, M.: Solution to a non-Archimedean Monge-Ampère equation. J. Am. Math. Soc. **28**, 617–667 (2015)
8. Boucksom, S., Favre, C., Jonsson, M.: Singular semipositive metrics in non-Archimedean geometry. J. Algebraic Geom. **25**(1), 77–139 (2016)
9. Boucksom, S., Hisamoto, T., Jonsson, M.: Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs. Ann. Inst. Fourier (Grenoble) **67**, 87–139 (2017)
10. Boucksom, S., Hisamoto, T., Jonsson, M.: Uniform K-stability and asymptotics of energy functionals in Kähler geometry. J. Eur. Math. Soc. (JEMS) **21**(9), 2905–2944 (2019)
11. Boucksom, S., Jonsson, M.: Singular semipositive metrics on line bundles on varieties over trivially valued fields. arXiv:1801.08229
12. Boucksom, S., Jonsson, M.: A non-Archimedean approach to K-stability. arXiv:1805.11160v1
13. Chambert-Loir, A.: Mesures et équidistribution sur des espaces de Berkovich. J. Reine Angew. Math. **595**, 215–235 (2006)
14. Chen, X., Cheng, J.: On the constant scalar curvature Kähler metrics, general automorphism group. arXiv:1801.05907
15. Cutkosky, S.D., Srinivas, V.: On a problem of Zariski on dimensions of linear systems. Ann. Math. **137**, 531–559 (1993)
16. Delcroix, T.: Uniform K-stability of polarized spherical varieties. arXiv:2009.06463
17. Delcroix, T.: The Yau-Tian-Donaldson conjecture for cohomogeneity one manifolds. arXiv:2011.07135
18. Demailly, J.-P., Ein, L., Lazarsfeld, R.: A subadditivity property of multiplier ideals. Michigan Math. J. **48**, 137–156 (2000)
19. Donaldson, S.: Scalar curvature and stability of toric varieties. J. Diff. Geom. **62**(2), 289–349 (2002)
20. Ein, L., Lazarsfeld, R., Mustaţă, M., Nakamaye, M., Popa, M.: Restricted volumes and base loci of linear series. Am. J. Math. **131**(3), 607–651 (2009)
21. Holschbach, A.: A Chebotarev-type density theorem for divisors on algebraic varieties. arXiv:1006.2340
22. Huneke, C., Swanson, I.: Integral closure of ideals, rings, and modules. London Mathematical Society Lecture Note Series, vol. 336. Cambridge University Press, Cambridge (2006)
23. Koike, T.: Minimal singular metrics of a line bundle admitting no Zariski decomposition. Tohoku Math. J. **67**, 297–321 (2015)
24. Lazarsfeld, R.: Positivity in algebraic geometry I, classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48. Springer, Berlin (2004)
25. Lazarsfeld, R.: Positivity in algebraic geometry II, positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49. Springer, Berlin (2004)
26. Li, C.: Geodesic rays and stability in the cscK problem, accepted by Annales Scientifiques de l'ENS. arXiv:2001.01366
27. Li, C., Xu, C.: Special test configurations and K-stability of Fano varieties. Ann. Math. **180**(2), no.1, 197–232 (2014)

28. Matsumura, S.: Restricted volumes and divisorial Zariski decomposition. Am. J. Math. **135**, 637–662 (2013)
29. Odaka, Y.: A generalization of the Ross-Thomas slope theory. Osaka J. Math. **50**, 171–185 (2013)
30. Nakayama, N.: Zariski-decomposition and abundance. MSJ Memoirs, vol. 14. Mathematical Society of Japan, Tokyo (2004)
31. Tian, G.: Kähler-Einstein metrics with positive scalar curvature. Invent. Math. **130**, 239–265 (1997)
32. Tian, G.: K-stability implies CM-stability, Geometry, analysis and probability, 245-261, Progress in Mathematics, vol. 310. Birkhäuser, Springer, Cham (2017)
33. Tsuji, H.: Pluricanonical systems of projective varieties of general type. I. Osaka J. Math. **43**(4), 967–995 (2006)
34. Wang, X.: Heights and GIT weight. Math. Res. Lett. **19**, 909–926 (2012)

On a Conjecture of Fulton on Isotropic Grassmannians

Yan Li and Zhenye Li

Abstract In this note, we confirm a positivity conjecture of Fulton (Conjecture 1 below) for isotropic Grassmannians $\mathrm{IGr}(2, 2n)$ with $n \geq 3$. Namely, the quantum deformation of the basis, formed by the Schubert cycles, is trivial. To the author's knowledge, this is the first time that the conjecture holds for a Grassmannian of type C_n.

Keywords Quantum cohomology · Isotropic Grassmannian

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1 Introduction and Preliminaries

Let V be a $2n$-dimensional complex linear space and $\langle\cdot,\cdot\rangle$ be a non-degenerated skew-symmetric bilinear form on it. Let $m \in [0, n]$ be a fixed integer. Then we define $\mathrm{IGr}(m, 2n)$ to be the variety of all m-dimensional isotropic subspace of V. This is an algebraic variety of dimensional $2m(n-m) + \frac{1}{2}m(m+1)$. The isotropic Grassmanian can also be realized as a flag variety G/P, where $G = \mathrm{Sp}_{2n}(\mathbb{C})$ and P is a maximal parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$, which just consists of the stabilizer of a chosen m-dimensional isotropic subspace in G. The isotropic Grassmannians form a family called Grassmannians of type C_n (cf. [6, Sect. 2]).

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To describe the classical cohomology ring $\mathrm{H}^*(\mathrm{IGr}(m, 2n); \mathbb{Q})$ of $\mathrm{IGr}(m, 2n)$ we first give the Schubert cells of $\mathrm{IGr}(m, 2n)$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0)$ be a partition. We identify it with its Young diagram of boxes (cf. [7, 9]). For $k \in \mathbb{Q}_{\geq 0}$, we call λ a *k-strict partition* if $\lambda_{j+1} < \lambda_j$ whenever $\lambda_j > k$. Then the Schubert cells of $\mathrm{IGr}(m, 2n)$ are in one-to-one correspondence to the $(n-m)$-strict partitions contained in an $m \times (2n-m)$-rectangle [2, Sect. 1]. In the following we will denote $k = n - m$.

Let us recall some useful results there: Denote by $\mathcal{P}(n-m, n)$ all such partitions. Fix an isotropic flag F:

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_{2n} = V.$$

For each $\lambda \in \mathcal{P}(n-m, n)$, the corresponding Schubert cell is defined by

$$X_\lambda = \{\Pi \in \mathrm{IGr}(m, 2n) | \dim(\Pi \cap F_{p(j)}) \leq j, \ \forall 1 \leq j \leq \mathrm{length}(\lambda)\},$$

where

$$p(j) = 2n - m + 1 - \lambda_j + \#\{i | i < j, \ \lambda_i + \lambda_j \leq 2n - 2m + j - i\}$$

and

$$\mathrm{length}(\lambda) = \#\{i | \lambda_i \neq 0\}.$$

Then the collection of all

$$\tau_\lambda := [X_\lambda], \ \lambda \in \mathcal{P}(n-m, n)$$

forms a basis of $\mathrm{H}^*(\mathrm{IGr}(m, 2n); \mathbb{Q})$. For this result, we refer the reader to [1, 2, 6], or [8] for a comprehensive study.

Multiplication in this classical cohomology ring is calculated by the Pieri rule (cf. [2, Theorem 1.1], or [8, Sects. 4 and 5] for origins). In [2] the Pieri rule was generalized for the small quantum cohomology ring $\mathrm{QH}^*(\mathrm{IGr}(m, 2n); \mathbb{Q})$, which is defined by

$$\mathrm{QH}^*(\mathrm{IGr}(m, 2n); \mathbb{Q}) := \mathrm{H}^*(\mathrm{IGr}(m, 2n); \mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}[q],$$

where q is the formal variable. Denote

$$|\lambda| := \lambda_1 + \cdots + \lambda_m$$

for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0)$. The multiplication law in this ring is defined by

$$\tau_\lambda \cdot \tau_\mu := \sum_{\substack{d \geq 0,\, \mu \text{ is } (n-m)\text{-strict}, \\ |\nu| = |\lambda| + |\mu| - (2n-m+1)d}} \langle \tau_\lambda, \tau_\mu, \tau_{\nu^\vee} \rangle_d \tau_\mu q^d,$$

where $\langle \cdot, \cdot, \cdot \rangle_d$ denotes the Gromov-Witten invariant (cf. [2, Sect. 1.4]) and $\nu^\vee$ is the dual partition of ν. There is an explicit formula of this multiplication, called the quantum Pieri rule, proved in [2, Theorem 1.4], which we will frequently use in the following. To state the formula, we first recall some notations from [2, Sect. 1]:

Definition 1 ([2, Definition 1.2]) Let λ be a k-strict partition, which is identified with a Young diagram. Two boxes $B(r_i, c_i)$ on the r_i-row, c_i-column ($i = 1, 2$) are called k-related to each other if $|c_1 - k - 1| + r_1 = |c_2 - k - 1| + r_2$;

Definition 2 ([2, Definition 1.3]) We say that two k-strict partitions λ and μ satisfies $\lambda \to \mu$ if μ can be obtained by removing a vertical strip from the first k columns of λ and then adding a horizontal strip so that:

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \setminus \lambda$;
and
(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be k-related to exactly one box of $\mu \setminus \lambda$, and these boxes of $\mu \setminus \lambda$ must all lie in the same row.

Suppose that $\lambda \to \mu$. Denote by $\mathcal{B}_1 = \{B_1, ..., B_m\}$ the bottom boxes and their associated k-related boxes in Definition 2 (1), and $\mathcal{B}_2$ the remaining bottom boxes in μ and their associated k-related ones in Definition 2 (2). Consider the $(k+1)$-th to $(k+n)$-th columns $(\mu \setminus \lambda)_{n+1}^{n+k}$ in $\mu \setminus \lambda$. Set $N(\lambda, \mu)$ to be the number of connected components of $(\mu \setminus \lambda)_{n+1}^{n+k} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ which do not have a box in the $(k+1)$-th column. Note that two boxes are connected if they have at least one common vertex.

Now we state the quantum Pieri rule for $\mathrm{IGr}(m, 2n)$:[1]

Theorem 1 ([2, Theorm 1.4]) *For any $(n-m)$-strict $\lambda \in \mathcal{P}(n-m, n)$ and $p \in \mathbb{Q} \cap [1, 2n-m]$,*

$$\tau_p \cdot \tau_\lambda = \sum_{\substack{\mu \in \mathcal{P}(n-m, n),\, \lambda \to \mu, \\ |\mu| = |\lambda| + p}} 2^{N(\lambda,\mu)} \tau_\mu + \sum_{\substack{\mu \in \mathcal{P}(n-m, n+1),\, \lambda \to \mu, \\ |\mu| = |\lambda| + p \\ \mu_1 = 2n-m+1}} 2^{N(\lambda,\mu)-1} \tau_{\mu^*} q,$$

where $\mu^* = (\mu_2 \geq \mu_3 \geq ...)$ for $\mu = (\mu_1 \geq \mu_2 \geq ...)$.[2]

[1] Unless otherwise stated, all multiplications below are considered in the quantum sense.

[2] We write τ_p in short of $\tau_{(p)}$ if the partition p has only one row.

We refer to the reader [2, Example 1.1] for an example of these terminologies and application of Theorem 1.

It is obvious that the Schubert classes $\{\tau_\lambda | \lambda \in \mathcal{P}(k,n)\}$ forms a basis of the small quantum cohomology ring of $\mathrm{IGr}(m, 2n)$ over $\mathbb{Q}[q]$. Then we can consider the quantum deformation of this canonical basis, which is another basis $\{\sigma_\lambda | \lambda \in \mathcal{P}(k,n)\}$ of the small quantum cohomology ring over $\mathbb{Q}[q]$. More precisely, for any given collection of constants $\{a_\mu \in \mathbb{Q} | \mu \in \mathcal{P}(k,n)\}$, the corresponding basis $\{\sigma_\lambda\}$ is defined as the solution of the following system:

$$\tau_\lambda = \sigma_\lambda + \sum_{j\geq 1}\left(\sum_{|\mu|+(n+k-1)j=|\lambda|} a_\mu \sigma_\mu q^j\right), \ \lambda \in \mathcal{P}(k,n). \tag{1}$$

Note that (1) is an upper-triangular system of the σ_λ's. It always admits a solution.

Fulton first raised the following conjecture (cf. [4, Conjecture 1]):

Conjecture 1 Suppose that $\{\sigma_\lambda | \lambda \in \mathcal{P}(k,n)\}$ is a quantum deformation of the basis (1) formed by the Schubert cycles such that:

($\star$) The coefficients of the the quantum multiplication of any σ_λ and σ_μ in the basis $\{\sigma_\lambda | \lambda \in \mathcal{P}(k,n)\}$ are polynomials in q with nonnegative coefficients.

Then the deformation is trivial. That is, in (1) $\tau_\lambda = \sigma_\lambda$ for all $\lambda \in \mathcal{P}(k,n)$.

In this note, we will confirm this conjecture for $\mathrm{IGr}(2, 2n)$ when $n \geq 3$. Our method is to use the quantum Pieri formula Theorem 1 developed in [2]. In particular we will frequently use the multiplication by a special cycle $\tau_{(1,1)}$ and its higher power, which exists only when $n \geq 3$. To our knowledge, this is the first time that Conjecture 1 holds for a Grassmannian of type C_n.

2 Proof of the Positive Conjecture for IGr(2, 2n) with $n \geq 3$

We will prove Conjecture 1 for $\mathrm{IGr}(2, 2n)$, $n \geq 3$ in this section. Also we denote $k = n - m = n - 2$. For the proof, we need the following result concerning the multiplication with a special Schubert class $\tau_{(1,1)}$, corresponding to the partition $(1, 1)$, of $\mathrm{IGr}(2, 2n)$:

Lemma 1 *Suppose that $n \geq 3$. Let t be a positive integer and $t \leq k$. Let $\mu = (\mu_1, \mu_2) \in \mathcal{P}(k,n)$. Then we have the following:*

(1) If $2t + |\mu| \leq 2k + 1$, then

$$\tau_{(t,t)}\tau_\mu = \tau_{(\mu_1+t,\mu_2+t)};$$

(2) If $2t + |\mu| = 2k + 2$ *and* $\mu_1 \neq \mu_2$ *or* $2t + |\mu| = 2k + 3$*, then*

$$\tau_{(t,t)}\tau_\mu = \tau_{(\mu_1+t,\mu_2+t)} + \tau_{(\mu_1+t+1,\mu_2+t-1)};$$

(3) If $2t + |\mu| = 2k + 2$ *and* $\mu_1 = \mu_2$*, then*

$$\tau_{(t,t)}\tau_\mu = \tau_{(\mu_1+t+1,\mu_2+t-1)};$$

(4) If $|\mu| \geq 2k + 2$ *and* $\mu_1 + t \leq 2k + 2$*, then*

$$\tau_{(t,t)}\tau_\mu = \tau_{(\mu_1+t,\mu_2+t)};$$

(5) If $|\mu| \geq 2k + 2$ *and* $\mu_1 + t = 2k + 3$*, then*

$$\tau_{(t,t)}\tau_\mu = q\tau_{\mu_2+t}.$$

Proof We prove the Lemma by an induction on t. First consider the case $t = 1$, this is a tedious calculation using quantum Pieri's rule.

By Theorem 1 we can calculate the multiplication of τ_1 with an arbitrary Schubert class as the following:

$$\tau_{(a,b)}\tau_1 = \begin{cases} \tau_{(a+2,b-1)} + \tau_{(a,b+1)} + 2\tau_{(a+1,b)}, & \text{if } a \geq b+2, b > 0 \text{ and } a+b = 2k+1; \\ \tau_{(a+2,b-1)} + 2\tau_{(a+1,b)}, & \text{if } (a,b) = (k+1,k); \\ \tau_{(a,b+1)} + 2\tau_{(a+1,b)}, & \text{if } (a,b) = (2k+1,0); \\ \tau_{(a+1,b)}, & \text{if } a = b+1, a \geq k+2 \text{ or } a = b; \\ \tau_{(a,b+1)} + q\tau_b, & \text{if } a = 2k+2, b \leq 2k; \\ q\tau_b, & \text{if } a = 2k+2, b = 2k+1; \\ \tau_{(a+1,b)} + \tau_{(a,b+1)}, & \text{otherwise.} \end{cases} \quad (2)$$

The expression of $\tau_2\tau_\mu$ is much more complicated. We have the following cases: When $\mu_2 \leq 2k$, there are no quantum terms except the case $\mu = (2k, 1)$. Thus

$$\tau_2\tau_\mu = \tau_{(2k,3)} + 2\tau_{(2k+1,2)} + 2\tau_{(2k+2,1)} + q.$$

When $\mu = (2k + 1, a)$, we have:

- if $a = 0$, then $\tau_2\tau_\mu = \tau_{(2k+1,2)} + 2\tau_{(2k+2,1)} + q$;
- if $1 \leq a \leq 2k - 2$, then $\tau_2\tau_\mu = \tau_{(2k+1,a+2)} + \tau_{(2k+2,a+11)} + q\tau_a$;
- if $a = 2k - 1$ or $2k$, then

$$\tau_2\tau_\mu = \tau_{(2k+2,2k)} + q\tau_a \text{ or } q\tau_a, \text{ respectively.}$$

When $\mu = (2k+2, a)$, $a \in \{1, 2, ..., 2k\}$, $a \in \{1, 2, ..., 2k\}$, we have:

- if $a = 0$, then $\tau_2\tau_\mu = \tau_{(2k+2,2)} + q\tau_1$;
- if $1 \leq a \leq 2k-1$, then $\tau_2\tau_\mu = \tau_{(2k+2,a+2)} + q(\tau_{(a,1)} + \tau_{a+1})$;
- if $a = 2k$ or $2k+1$, then $\tau_2\tau_\mu = q(\tau_{(a,1)} + \tau_{a+1})$.

Then we assume that $\mu_1 \leq 2k$. When $|\mu| \neq 2k, 2k+1$, we have:

- if $\mu = (a+2, a)$ and $a+2 \geq k+1$, the result is $\tau_2\tau_\mu = \tau_{(a+4,2)} + \tau_{(a+3,1)}$;
- if $\mu = (a+1, 1)$, the result is

$$\tau_2\tau_\mu = \begin{cases} \tau_{(a+3,1)} + \tau_{(a+2,2)}, & \text{when } a \leq k-1, \\ (a+3, 1), & \text{when } a \geq k+1; \end{cases}$$

- if $\mu = (a, a)$ with $a \leq k-1$, the result is $\tau_2\tau_\mu = \tau_{(a+2,a)}$;
- in other remaining cases the result is

$$\tau_2\tau_\mu = \tau_{(\mu_1+2,\mu_2)} + \tau_{(\mu_1+1,\mu_2+1)} + \tau_{(\mu_1,\mu_2+2)}.$$

When $|\mu| = 2k$ and $\mu = (a, b)$, where $a + b = 2k$ and $a \geq b \geq 0$, we have:

- if $b = 0$, the result is $\tau_2\tau_\mu = \tau_2\tau_\mu = \tau_{(2k,2)} + 2\tau_{(2k+1,1)} + 2\tau_{2k+2}$;
- if $a - b = 2$, the result is $\tau_2\tau_\mu = \tau_2\tau_\mu = \tau_{(k+4,k-2)} + 2\tau_{(k+2,k)} + 2\tau_{(k+3,k-1)}$;
- if $a = b$, the result is $\tau_2\tau_\mu = \tau_2\tau_\mu = \tau_{(k+3,k-1)} + \tau_{(k+2,k)}$;
- in other cases,

$$\tau_2\tau_\mu = \tau_2\tau_\mu = \tau_{(a+3,b-1)} + \tau_{(a,b+2)} + 2\tau_{(a+1,b+1)} + 2\tau_{(a+2,b)}.$$

When $|\mu| = 2k+1$ and $\mu = (a, b)$ with $b > 1$, we have:

- if $a - b = 1$, the result is

$$\tau_2\tau_\mu = \tau_{(k+2,k+1)} + \tau_{(k+4,k-1)} + 2\tau_{(k+3,k)};$$

- for other cases the result is

$$\tau_2\tau_\mu = \tau_{(a+3,b-1)} + \tau_{(a,b+2)} + 2\tau_{(a+1,b+1)} + 2\tau_{(a+2,b)}.$$

Since

$$\tau_{(1,1)} = \tau_1\tau_1 - \tau_2,$$

again by Theorem 1, we can calculate the multiplication with $\tau_{(1,1)}$ by plugging in the above relations. This gives the results in items (1)–(5) for $t = 1$. We give item (1) as an example.

For item (1):

- if $\mu_1 = \mu_2$, we get

$$\tau_1 \tau_{(\mu_1+1,\mu_2)} - \tau_{(\mu_1+2,\mu_2)} = \tau_{(\mu_1+1,\mu_2+1)};$$

- if $\mu_1 \neq \mu_2 + 2$, we get

$$\tau_1(\tau_{(\mu_1+1,\mu_2)} + \tau_{(\mu_1,\mu_2+1)}) - (\tau_{(\mu_1+2,\mu_2)} + \tau_{(\mu_1+1,\mu_2+1)} + \tau_{(\mu_1,\mu_2+2)}) = \tau_{(\mu_1+1,\mu_2+1)};$$

- if $\mu_1 = \mu_2 + 1$, we get

$$\tau_1(\tau_{(\mu_1+1,\mu_2)} + \tau_{(\mu_1,\mu_2+1)}) - (\tau_{(\mu_1+2,\mu_2)} + \tau_{(\mu_1+1,\mu_2+1)}) = \tau_{(\mu_1+1,\mu_2+1)}.$$

For general t in item (1), since $t \leq k$, then by item (1) with $t = 1$, we know that

$$\tau_{(t,t)} = \tau_{(1,1)} \tau_{(t-1,t-1)},$$

Thus

$$\tau_{(t,t)} \tau_\mu = \tau_{(1,1)} \tau_{(t-1,t-1)} \tau_\mu = \tau_{(1,1)} \tau_{(\mu_1+t-1,\mu_2+t-1)}$$

where μ satisfies the conditions in each item (1) to (5). The last equality follows from the inductive condition with respect to item (1) and (4). Then the final results follow from the case $t = 1$. □

Remark 1 It is crucial to use the cycle $\tau_{(1,1)}$ in the proof of Lemma 1. We should notice that the above calculation is not valid when $n = 2$ since the cycle $\tau_{(1,1)}$ does not exist when $n = 2$. Indeed, Conjecture 1 fails for $\mathrm{IGr}(2, 4)$. Recall [5, Sect. 3.1]. The Schubert classes of dimensional 0, 2, 4 and 6 (denoted by $\tau_0 = 1$, τ_1, τ_2 and τ_3, respectively) forms a basis of $\mathrm{H}^*(\mathrm{IGR}(2, 4); \mathbb{Q})$ with classical multiplication law:

$$\tau_1^2 = 2\tau_2, \ \ \tau_1 \tau_2 = \tau_3.$$

The quantum multiplication law of the τ_i's is then given by (cf. [5, Sections 3.2–3.3])

$$\begin{cases} \tau_1^2 = 2\tau_2, \\ \tau_1 \cdot \tau_2 = \tau_3 + q, \\ \tau_2^2 = q\tau_1, \\ \tau_i \cdot \tau_3 = q\tau_i, \ i = 1, 2, \\ \tau_3^2 = q^2. \end{cases}$$

Consider a quantum deformation

$$\begin{cases} \tau_0 = \sigma_0, \\ \tau_1 = \sigma_1, \\ \tau_2 = \sigma_2, \\ \tau_3 = \sigma_3 + a\sigma_2 q, \end{cases} \qquad a \in \mathbb{Q}. \tag{3}$$

Using a programme "`Quantum Calculator`" from A. S. Buch [3], it can be checked that the quantum deformation gives new relations:

$$\begin{cases} \sigma_1^2 = 2\sigma_2, \\ \sigma_1 \cdot \sigma_2 = \sigma_3 + (1+a)q, \\ \sigma_2^2 = q\sigma_1, \\ \sigma_i \cdot \sigma_3 = (1-a)q\sigma_i, \ i = 1, 2, \\ \sigma_3^2 = -2aq\sigma_3 + (1-a^2)q^2, \end{cases} \qquad a \in \mathbb{Q}. \tag{4}$$

Obviously the system (4) satisfies the condition ($\star$) in Conjecture 1 for all $a \in [-1, 0]$ and for $a \neq 0$ the deformation (3) is non-trivial.

Now we prove the main result:

Theorem 2 *Conjecture 1 holds for* $\mathrm{IGr}(2, 2n)$ *for* $n \geq 3$.

Proof We follow the assumptions in Conjecture 1. Since

$$|\lambda| \leq 2(2n-2) - 1 = 4n - 5 < 2(2n-1)$$

for any $\lambda \in \mathcal{P}(n-2, n)$, wee see that the monomials with q^j, $j \geq 2$ vanish in (1). This gives

$$\tau_\lambda = \sigma_\lambda + \sum_{|\mu|+2n-1=|\lambda|} a_\mu \sigma_\mu q. \tag{5}$$

For our purpose, in the following, we denote $k = n - 2$.

Note that when $|\lambda| < 2k + 3$, the monomial concerning q vanishes in (5), whence

$$\tau_\lambda = \sigma_\lambda.$$

Thus it is sufficient to deal with the case $|\lambda| \geq 2n - 1 = 2k + 3$,

First, we consider the case $|\lambda| = 2k + 3$. Let $\lambda = (\lambda_1, \lambda_2)$. On one hand, we can show that $a_\mu \leq 0$, according to the following:

Case-1.1: $\lambda_2 \leq \lambda_1 - 2$. Since in this case the only μ appeared in the monomial concerning q is $\emptyset$, we may assume $\tau_\lambda = \sigma_\lambda + aq$. Since $\lambda_1 + \lambda_2 = |\lambda| = 2k + 3$, we have

$$\lambda_1 \geq k + 3.$$

Hence the number

$$t := 2k + 2 - \lambda_1 + 1 \leq k.$$

Thus $\tau_{(t,t)}$ is a Schubert class of IGr$(2, 2n)$. Using item (5) of Lemma 1, we have

$$\tau_{(t,t)}\tau_\lambda = q\tau_{(|\lambda|+2t-2n+1)}.$$

Since $2t \leq 2k < 2n - 1$, we see from (1) that

$$\tau_{(t,t)} = \sigma_{(t,t)}.$$

Thus we get

$$\sigma_{(t,t)}\sigma_\lambda = q\tau_{(|\lambda|+2t-2n+1)} - a\sigma_{(t,t)}q. \tag{6}$$

On the other hand, since

$$2t + |\lambda| - 2n + 1 = 2k + 2 - \lambda_1 + 1 + \lambda_2 \leq 2k + 1,$$

by (1) we have

$$\tau_{(|\lambda|+2t-2n+1)} = \sigma_{(|\lambda|+2t-2n+1)}.$$

Plugging this relation into (6) and combining with the assumption $(\star)$ in Conjecture 1, since the multiplications of the basis $\{\sigma_\lambda\}$ has nonnegative coefficients we get $a \leq 0$ in (6).

Case-1.2: $\lambda = (k + 2, k + 1)$. Then we consider the remaining case when $\lambda = (k + 2, k + 1)$. There is only one choice $j = 1$ in (1), we have

$$\tau_\lambda = \sigma_\lambda + aq.$$

In this case, by Theorem 1, we conclude that

$$\tau_{(2k+2)}\tau_\lambda = q\tau_{(k+2,k)}.$$

By (1),

$$\sigma_{(2k+2)} = \tau_{(2k+2)} \text{ and } \sigma_{(k+2,k)} = \tau_{(k+2,k)}.$$

It follows that

$$\sigma_{(2k+2)}\sigma_\lambda = q\sigma_{(k+2,k)} - aq\sigma_{2k+2}.$$

Again by the assumption ($\star$) in Conjecture 1, $a \le 0$ in the above equation.

Combining the above two cases, we see that $a_\mu \le 0$ always holds.

Next we show that $a_\mu \ge 0$. We see that

$$\lambda = (k+2+i, k+1-i), \text{ for } i = 0, 1, \ldots, k$$

and we assume

$$\tau_{(k+2+i,k+1-i)} = \sigma_{(k+2+i,k+1-i)} + a_i q.$$

For $0 \le j \le k-1$, by item (1) of Lemma 1, we have

$$\tau_{(1,1)}\tau_{(k+1+j,k-j)} = \tau_{(k+2+j,k+1-j)} + \tau_{(k+3+j,k-j)}.$$

Since $\sigma_{(k+1+j,k-j)} = \tau_{(k+1+j,k-j)}$, we get

$$\begin{aligned}\sigma_{(1,1)}\sigma_{(k+1+j,k-j)} &= \tau_{(1,1)}\tau_{(k+1+j,k-j)}\\ &= \tau_{(k+2+j,k+1-j)} + \tau_{(k+3+j,k-j)}\\ &= \sigma_{(k+2+j,k+1-j)} + \sigma_{(k+3+j,k-j)} + (a_j + a_{j+1})q.\end{aligned}$$

By ($\star$) in Conjecture 1, the coefficients

$$a_j + a_{j+1} \ge 0, \; j = 0, \ldots, k-1.$$

However, we have already concluded that

$$a_i \le 0, \; i = 0, \ldots, k.$$

Thus it must hold $a_i = 0$ for all $i = 0, \ldots, k$ when $|\lambda| = 2k+3$.

Now we turn to the general cases. We will adopt the induction argument. Assume that we have proved that

$$\tau_\lambda = \sigma_\lambda \text{ for all } |\lambda| \le s, \tag{7}$$

where $s \ge 2k+3$. Then we consider the case $|\lambda| = s+1$.

As before, we first show that $a_\mu \ge 0$. Since $|\lambda| \ge 2k+4$ for $|\lambda| = s+1$, thus by item (4) of Lemma 1 we have

$$\tau_{(1,1)}\tau_{(\lambda_1-1,\lambda_2-1)} = \tau_\lambda.$$

Plugging (7) into (1), we get

$$\tau_{(\lambda_1-1,\lambda_2-1)} = \sigma_{(\lambda_1-1,\lambda_2-1)}, \tau_{(1,1)} = \sigma_{(1,1)}.$$

Thus we have that

$$\sigma_{(1,1)}\sigma_{(\lambda_1-1,\lambda_2-1)} = \sigma_\lambda + \sum_{|\mu|+2k+3=|\lambda|} a_\mu \sigma_\mu.$$

So we conclude that the coefficients $a_\mu \geq 0$.

Then we can show that $a_\mu \leq 0$. We have the following two cases:

Case-2.1: $\lambda_2 \leq \lambda_1 - 2$. In this case, we see that $\lambda_1 \geq k+3$. Hence

$$t = 2k + 2 - \lambda_1 + 1 \leq k,$$

and $\tau_{(t,t)}$ is a Schubert class of IGr(2, 2n). Using item (5) of Lemma 1, we have

$$\tau_{(t,t)}\tau_\lambda = q\tau_{(|\lambda|+2t-2n+1)}.$$

Since $2t \leq 2k < 2n - 1$, we see from (1) that

$$\tau_{(t,t)} = \sigma_{(t,t)}.$$

Thus we get

$$\sigma_{(t,t)}\sigma_\lambda = q\tau_{(|\lambda|+2t-2n+1)} - \sigma_{(t,t)}(\sum_{|\mu|+2n-1=|\lambda|} a_\mu \sigma_\mu q). \tag{8}$$

Since

$$\begin{aligned} 2t + |\mu| &= 2t + |\lambda| - 2n + 1 \\ &= 2k + 2 - \lambda_1 + 1 + \lambda_2 \\ &\leq 2k + 1, \end{aligned}$$

we have

$$\tau_{(|\lambda|+2t-2n+1)} = \sigma_{(|\lambda|+2t-2n+1)}$$

and by item (1) of Lemma 1

$$\begin{aligned} \sigma_{(t,t)}\sigma_\mu &= \tau_{(t,t)}\tau_\mu \\ &= \tau_{(\mu_1+t,\mu_2+t)} \\ &= \sigma_{(\mu_1+t,\mu_2+t)}. \end{aligned}$$

Plugging the above two relations into (8), we get $a_\mu \leq 0$. On the other hand, we have already proved that $a_\mu \geq 0$. Hence we get $a_\mu = 0$ in this case.

Case-2.2. $\lambda = (l+1, l)$, *where* $k+1 \leq l \leq 2k+1$.

In this case, the degree of each μ in (5) is $2l+1-(2k+3)$, thus μ has the form

$$((l-k-1)+i, (l-k-1)-i), \text{ where } i = 0, 1, \ldots, l-k-1.$$

Thus, we may assume that

$$\tau_\lambda = \sigma_\lambda + \sum_{i=0}^{l-k-1} a_i \sigma_{((l-k-1)+i,(l-k-1)-i)} q.$$

Since $|\lambda| = 2l+1 \geq 2k+4$, so in fact it holds $l \geq k+2$. Taking

$$t = 2k+2-l,$$

we see that $t \leq k$ and $\tau_{(t,t)}$ is a Schubert class of $\mathrm{IGr}(2, 2n)$. By item (5) of Lemma 1, we get

$$\tau_{(t,t)}\tau_\lambda = q\tau_{(2k+2)}.$$

As $2t \leq 2k$, we have $\tau_{(t,t)} = \sigma_{(t,t)}$. Thus we have that,

$$\sigma_{(t,t)}\sigma_\lambda = q\sigma_{(2k+2)} - \sigma_{(t,t)}\Big(\sum_{i=0}^{l-k-1} a_i \sigma_{((l-k-1)+i,(l-k-1)-i)} q\Big).$$

Also, by item (3) of Lemma 1, we have

$$\begin{aligned}\sigma_{(t,t)}\sigma_{(l-k-1,l-k-1)} &= \tau_{(t,t)}\tau_{(l-k-1,l-k-1)}\\ &= \tau_{(t+l-k,t+l-k-2)}\\ &= \sigma_{(t+l-k,t+l-k-2)}\end{aligned}$$

where in last equality holds because the degree of the class is $2k+2$.

Similarly, by item (2) of Lemma 1, we have

$$\begin{aligned}\sigma_{(t,t)}\sigma_{(l-k-1+i,l-k-1-i)} &= \tau_{(t,t)}\tau_{(l-k-1+i,l-k-1-i)}\\ &= \tau_{(t+l-k-1+i,t+l-k-1-i)} + \tau_{(t+l-k+i,t+l-k-2-i)}\\ &= \sigma_{(t+l-k-1+i,t+l-k-1-i)} + \sigma_{(t+l-k+i,t+l-k-2-i)}, \quad i > 0.\end{aligned}$$

Thus we get

$$\begin{aligned} a_0 &\leq 0, \\ a_0 + a_1 &\leq 0, \\ a_1 + a_2 &\leq 0, \\ &\ldots \end{aligned}$$

and

$$a_{l-k-1} + a_{l-k-2} \leq 0.$$

Recall that $a_i \geq 0$. We conclude that $a_i = 0$ for any $i \in \mathbb{N}$.

Remark 2 For a general $\mathrm{IGr}(m, 2n)$, we may first consider the multiplication with the special Schubert class $\tau_{(1,\ldots,1)}$ of degree m. After such multiplications are computed out as in Lemma 1, we may find a way to run the calculation and prove the the conjecture for general m using such multiplications as we did in the special case $m = 2$.

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References

1. Brion, M.: Lectures on the geometry of flag varieties. Notes de l'ècole d'ètè "Schubert Varieties", Varsovie (2003)
2. Buch, A.S., Kresch, A., Tamvakis, H.: Quantum Pieri rules for isotropic Grassmannians. Invent. Math. **178**, 345–405 (2009)
3. Buch, A.S.: Quantum Calculator (Maple package). Homepage of Anders Skovsted Buch at Rutgers University. https://sites.math.rutgers.edu/~asbuch/
4. Buch, A.S. et al.: Positivity determines the quantum cohomology of Grassmannians, preprint. Homepage of Anders Skovsted Buch at Rutgers University. https://sites.math.rutgers.edu/~asbuch/papers/qhunique.pdf
5. Kresch, A., Tamvakis, H.: Quantum cohomology of the Lagrangian Grassmannian. J. Algeb. Geom. **12**, 777–810 (2003)
6. Li, C., Ravikumar, V.: Equivariant Pieri rules for isotropic Grassmannians. Math. Ann. **365**, 881–909 (2016)
7. Stanley, R.: Enumerative combinatorics. Cambridge Studies in Advanced Mathematics, vol. 1, 49. Cambridge University Press, Cambridge (1997)

8. И. Н. Бернштейн, И. М. Гельфанд и С. И. Гельфанд, Клетки Шуберта и кгомологий пространств *G/B*, Успехи Математических Наук, **28**, 3–26 (1973)
9. Е. Ю. Смирнов, Диаграммы Юнга, плоские разбиения и знакочередуюшиеся матрицы, Издательство МЦНМО 2014

On Locally Nilpotent Derivations of Danielewski Domains

L. Makar-Limanov

Abstract Let $p(Z) \in \mathbb{C}[Z]$ be a polynomial of degree d. In this note I'll show that if positive natural numbers n, m, and d are relatively prime then up to an automorphism there is at most one nonzero irreducible locally nilpotent derivation on the domain $\mathbb{C}[X, Y, Z]/(X^n Y^m - p(Z))$.

Keywords Locally nilpotent derivations · Danielewski surfaces

1 Introduction

In this note we take the field $\mathbb{C}$ of complex numbers as the ground field. In fact it is essential only that the ground field has characteristic zero. Also all appearing rings are domains.

Let $R = \mathbb{C}[x, y]$. It is well known (see [7]) that the kernel of a nonzero locally nilpotent derivation of R is $\mathbb{C}[u]$ where u is an image of x under an automorphism. More recently a similar result was proved for domains $\mathbb{C}[X, Y, Z]/(X^n Y - p(Z))$, $n > 1$, $\deg(p(Z)) > 1$ (see [3, 4]) and $\mathbb{C}[X, Y, Z]/(XY - p(Z))$ where $\deg(p(Z)) > 0$ (see [1, 5]). Here we will look from this point of view on the domains R given by $\mathbb{C}[X, Y, Z]/(X^n Y^m - p(Z))$ where $p(Z)$ is a monic polynomial and n, m, and $d = \deg(p(Z))$ are relatively prime positive natural numbers. If $d = 1$ then the corresponding domains are actually isomorphic to $\mathbb{C}[x, y]$. It turns out that a nonzero locally nilpotent derivation (lnd for short) exists on R only if

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$(d-1)(n-1)(m-1)=0$. So as far as the description of lnds is concerned we have only previously described cases. On the other hand we have here more direct proofs, which are substantially shorter.

2 Definitions, Notations and Technical Lemmas

Here we recall briefly some necessary notions and facts.

Let A be a $\mathbb{C}$-algebra. A $\mathbb{C}$-homomorphism ∂ of A is called a *derivation* of A if it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + a\partial(b)$.

A derivation is *irreducible* if $\partial(A)$ does not belong to a proper principal ideal. (So, since (0) is not a proper ideal, according to this definition zero derivation is irreducible!)

We will be using in the next section so called Jacobian derivations on $\mathbb{C}[X, Y, Z]$. Let us take any two $p, q \in C[X, Y, Z]$. Then $\partial(r) = \mathrm{J}(p, q, r)$ where $\mathrm{J}(p, q, r)$ denotes the Jacobian, i. e. the determinant of the corresponding Jacobi matrix, is a derivation. Let us recall that $\mathrm{J}(p, q, r)$ is also skew symmetric.

Any derivation ∂ determines two subalgebras of A. One is the kernel of ∂ which is usually denoted by A^{∂} and is called the *ring of* ∂*-constants*.

The other is $\mathrm{Nil}_A(\partial)$, the *ring of nilpotency of* ∂ : $\mathrm{Nil}_A(\partial) = \{a \in A | \partial^n(a) = 0, n >> 1\}$.
In other words $a \in \mathrm{Nil}_A(\partial)$ if for a sufficiently large natural number n we have $\partial^n(a) = 0$.

Both A^{∂} and $\mathrm{Nil}_A(\partial)$ are subalgebras of A because of the Leibniz rule.

We will call a derivation *locally nilpotent* if $\mathrm{Nil}_A(\partial) = A$.

The best examples of lnds (locally nilpotent derivations) are the partial derivatives on the rings of polynomials $\mathbb{C}[x_1, ..., x_n]$.

With the help of a locally nilpotent derivation acting on A, we can define a function $\deg_\partial$ by $\deg_\partial(a) = \max(n|\partial^n(a) \neq 0)$ if $a \in A^* = A\backslash 0$ and $\deg_\partial(0) = -\infty$.

Then the function $\deg_\partial$ is a degree function, i.e.,
$\deg_\partial(a + b) \leq \max(\deg_\partial(a), \deg_\partial(b))$ and
$\deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b)$.

Two locally nilpotent derivations are *equivalent* if the corresponding degree function are the same.

By definition $\deg_\partial$ has only nonnegative values on A^* and $a \in A^{\partial}\backslash 0$ if $\deg_\partial(a) = 0$. So it is clear that the ring A^{∂} is "factorially closed"; i. e., if $a, b \in A^*$ and $ab \in A^{\partial}$, then $a, b \in A^{\partial}$.

Let F be the field of fractions of A. Any derivation ∂ can be extended to a derivation on F by the "calculus" formula $\partial(ab^{-1}) = (\partial(a)b - a\partial(b))b^{-2}$. We will denote this extended derivation also by ∂.

Lemma 1 *Let ∂ be a locally nilpotent nonzero derivation of A. Then there exists an element $t \in F$ for which $\partial(t) = 1$ and $Nil_F(\partial) = F^{\partial}[t]$.*

Proof ∂ is a nonzero derivation so $A \neq A^\partial$ and there exists an $a \in A \backslash A^\partial$. Put $r = \partial^n(a)$ and $s = \partial(r)$ where $n = \deg_\partial(a) - 1$. Then $r \notin A^\partial$, $s \in A^\partial$ and $\partial(t) = 1$ for $t = rs^{-1}$. Observe that $s \in A^\partial$, we will use this fact later.

It is clear that $F^\partial[t] \subset \mathrm{Nil}_F(\partial)$. Let $a \in \mathrm{Nil}_F(\partial)$. We will use induction on $\deg_\partial(a) = n$ to show the opposite inclusion. If $a \in F$ and $\deg_\partial(a) = 0$ then $a \in F^\partial$ by definition. Let us make the step from $\deg_\partial(a) = n - 1$ to $\deg_\partial(a) = n$. If $\deg_\partial(a) = n$ then $\deg_\partial(\partial(a)) = n - 1$ and by induction $\partial(a) = \sum_{i=0}^{n-1} a_i t^{n-1-i}$ for some $a_i \in F^\partial$. Let $f = \sum_{i=0}^{n-1} (n-i)^{-1} a_i t^{n-i}$. Then $\partial(f) = \partial(a)$. So $\partial(a - f) = 0$ which means that $a = f + a_n$ where $a_n \in F^\partial$. □

Remark 1 It is clear that $\deg_\partial$ and $\deg_t$ are the same functions. This, of course, gives a proof of the properties of $\deg_\partial$ mentioned above. See also [2].

Remark 2 A^∂ is algebraically closed in A. Indeed, if $a \notin A^\partial$ then it is represented by a polynomial of positive degree and $p(a)$ also has a positive degree for any nonzero polynomial p.

Lemma 2 *Let ∂ be a nonzero lnd of A. If $\partial = a\epsilon$ where $a \in A$ and ϵ is a derivation of A then $\partial(a) = 0$ and ϵ is an lnd.*

Proof We want to show that $\deg_\partial(a) = 0$. It is clear that $A^\epsilon = A^\partial$. If $\deg_\partial(a) > 0$ then $\deg_\partial(\partial(b)) = \deg_\partial(a\epsilon(b)) = \deg_\partial(a) + \deg_\partial(\epsilon(b)) > 0$ for any $b \notin A^\partial$. So if $b \notin A^\partial$ then $\partial(b) \notin A^\partial$ which means that ∂ is not an lnd if $A \neq A^\partial$ i. e. $\partial \neq 0$. So $\partial(a) = 0$. Therefore $\deg_\partial(\epsilon(b)) = \deg_\partial(b) - 1$ for any $b \notin A^\partial$. Hence ϵ is an lnd. Even more, $\deg_\partial = \deg_\epsilon$. □

Remark 3 We see that any nonzero lnd is equivalent to an irreducible lnd.

Lemma 3 *F^∂ is the field of fractions of A^∂.*

Proof This proof was suggested by Ofer Hadas. Let $a, b \in A$ and $r = ab^{-1} \in F^\partial$. Assume also that $\deg_\partial(a)$ is minimal possible for all presentations of r as a fraction. Now, $\partial(r) = (\partial(a)b - a\partial(b))b^{-2} = 0$. So $ab^{-1} = \partial(a)\partial(b)^{-1}$ and $\deg_\partial(\partial(a)) < \deg_\partial(a)$. To avoid a contradiction we have to assume that $\deg_\partial(a) = 0$, so a and b are in A^∂. □

Remark 4 Since $F = F^\partial(t)$ the transcendence degree $\mathrm{trdeg}(F^\partial) = \mathrm{trdeg}(F) - 1$. Furthermore, $\mathrm{trdeg}(F) = \mathrm{trdeg}(A)$, $\mathrm{trdeg}(F^\partial) = \mathrm{trdeg}(A^\partial)$ and $\mathrm{trdeg}(A^\partial) = \mathrm{trdeg}(A) - 1$.

Lemma 4 *Let $Q \in \mathbb{C}[X, Y, Z]$ be an irreducible polynomial, $S = \mathbb{C}[X, Y, Z]/(Q)$ be the corresponding factor ring, and π the projection of $\mathbb{C}[X, Y, Z]$ on S. Assume that there is a nonzero lnd ∂ on S. Let $H \in \mathbb{C}[X, Y, Z]$ be such that $\pi(H) \in S^\partial \backslash \mathbb{C}$. Then $\epsilon(\pi(G)) = \pi(J(Q, H, G))$ defines an lnd on S which is equivalent to ∂.*

Proof Expression $\pi(\mathrm{J}(Q, H, G))$ defines a derivation on S. To check this we should first verify that if $\pi(G_1) = \pi(G_2)$ then $\epsilon(\pi(G_1)) = \epsilon(\pi(G_2))$. In this case $G_2 = G_1 + PQ$ and

$J(Q, H, G_2) = J(Q, H, G_1 + PQ) = J(Q, H, G_1) + J(Q, H, PQ) =$
$J(Q, H, G_1) + J(Q, H, P)Q$.
Since $J(Q, H, P)Q \in (Q)$ we see that $\pi(J(Q, H, G_2)) = \pi(J(Q, H, G_1))$.
The linear homomorphism ϵ is a derivation because

$$J(Q, H, G_1G_2) = J(Q, H, G_1)G_2 + G_1J(Q, H, G_2)$$

and π is a linear homomorphism.

Lnd ∂ defines a degree function on S and we can lift $\deg_\partial$ on $\mathbb{C}[X, Y, Z]$ to obtain a function deg on $\mathbb{C}[X, Y, Z] :\ \deg(G) = \deg_\partial(\pi(G))$. This function is nearly an ordinary degree function with the only difference being that there are many polynomials in $\mathbb{C}[X, Y, Z]$ with $\deg = -\infty$: if $G \in (Q)$ then (and only then) $\deg(G) = -\infty$.

Consider the subring of the field of fractions of S consisting of fractions with denominators in $S^\partial \backslash 0$ and denote the result by $\mathcal{B}$. This is a subring since S^∂ is closed under multiplication. As we know ∂ can be extended on $\mathcal{B}$ and by the proof of Lemma 1 $\mathcal{B}$ contains an element t for which $\partial(t) = 1$. (The derivation ∂ is an lnd on $\mathcal{B}$.)

Denote by K the set of all polynomials in $\mathbb{C}[X, Y, Z]$ with degree zero, i.e. the preimage of $S^\partial \backslash 0$. Let $\mathcal{A} = \mathbb{C}[X, Y, Z]_K$ be the subring of the field of rational functions $\mathbb{C}(X, Y, Z)$ consisting of fractions with denominators in K. Since K is closed under multiplication $\mathcal{A}$ is a ring. The projection π can be extended to $\mathcal{A}$ with image $\mathcal{B}$. Take any preimage T of $t :\ \pi(T) = t$.

By Lemma 1 any element $b \in \mathcal{B}$ can be written as $b = \sum_{i=0}^n b_i t^{n-i}$ where $b_i \in \mathcal{B}^\partial$. Hence any element a of $\mathcal{A}$ can be written as $a = \sum_{i=0}^n a_i T^{n-i}$ where $\pi(a_i) \in \mathcal{B}^\partial$, i.e. $a_i \in L$, the field of fractions of K. So

$$1 = J(X, Y, Z) = \sum J(X_i T^i, Y_j T^j, Z_k T^k)$$

where $\pi(X_i),\ \pi(Y_j),\ \pi(Z_k) \in \mathcal{B}^\partial$.

Using that the Jacobian is skew-symmetric and is a derivation in every argument we can rewrite each of these summands as a linear combination with coefficients in $\mathcal{A}$ of the Jacobians of the following two types: $J(U_1, U_2, U_3)$ and $J(U_1, U_2, T)$ where $\pi(U_i) \in \mathcal{B}^\partial$.

We are going to show that $J(U_1, U_2, U_3) \in (Q)$ and that $J(U_1, U_2, T)$ is congruent modulo (Q) to $J(Q, H, T)$ multiplied by an element of $\mathcal{A}$.

Since $\pi(U_i) \in \mathcal{B}^\partial$ and $\mathrm{trdeg}(\mathcal{B}^\partial) = 1$ (Remark 4) elements $\pi(U_i)$ and $\pi(H)$ are algebraically dependent. Therefore for any pair $U_i,\ H$ there is a polynomial f_i such that $f_i(H, U_i) = P_i Q$. We can assume that all f_i are irreducible.

Now, some boring computations.
$J(f_1(H, U_1), U_2, U_3) = J(H, U_2, U_3)\frac{\partial f_1}{\partial H} + J(U_1, U_2, U_3)\frac{\partial f_1}{\partial U_1} = J(P_1Q, U_2, U_3) \equiv$
$P_1J(Q, U_2, U_3) \pmod{(Q)}$.
Since f_1 is irreducible and $H,\ U_1 \in K$ both $\frac{\partial f_1}{\partial H}$ and $\frac{\partial f_1}{\partial U_1}$ are in $K\backslash(Q)$ and it remains to show that $J(H, U_2, U_3) \in (Q)$ and $J(Q, U_2, U_3) \in (Q)$.

Next, $J(H, f_2, U_3) = J(H, H, U_3)\frac{\partial f_2}{\partial H} + J(H, U_2, U_3)\frac{\partial f_2}{\partial U_2} = J(H, P_2Q, U_3) \equiv$
$P_2J(H, Q, U_3) \pmod{(Q)}$ and $J(H, U_2, U_3)\frac{\partial f_2}{\partial U_2} \equiv P_2J(H, Q, U_3) \pmod{(Q)}$;

$$J(Q, f_2, U_3) = J(Q, H, U_3)\frac{\partial f_2}{\partial H} + J(Q, U_2, U_3)\frac{\partial f_2}{\partial U_2} = J(Q, P_2 Q, U_3) \equiv 0 \pmod{(Q)}.$$

It remains to show that $\mathrm{J}(Q, H, U_3) \equiv 0 \pmod{(Q)}$.

$\mathrm{J}(Q, H, f_3) = \mathrm{J}(Q, H, H)\frac{\partial f_3}{\partial H} + \mathrm{J}(Q, H, U_3)\frac{\partial f_3}{\partial U_3} = \mathrm{J}(Q, H, P_3 Q) \equiv 0 \pmod{(Q)}$. Hence $\mathrm{J}(Q, H, U_3) \equiv 0 \pmod{(Q)}$ and $\mathrm{J}(U_1, U_2, U_3) \equiv 0 \pmod{(Q)}$.

Finally we will check that Jacobians $\mathrm{J}(U_1, U_2, T)$ are congruent modulo (Q) to $\mathrm{J}(Q, H, T)$ multiplied by an element of $\mathcal{A}$.

$\mathrm{J}(f_1(H, U_1), U_2, T) = \mathrm{J}(H, U_2, T)\frac{\partial f_1}{\partial H} + \mathrm{J}(U_1, U_2, T)\frac{\partial f_1}{\partial U_1} = \mathrm{J}(P_1 Q, U_2, T) \equiv P_1 \mathrm{J}(Q, U_2, T) \pmod{(Q)}$.

$\mathrm{J}(H, f_2, T) = \mathrm{J}(H, H, T)\frac{\partial f_2}{\partial H} + \mathrm{J}(H, U_2, T)\frac{\partial f_2}{\partial U_2} = \mathrm{J}(H, P_2 Q, T) \equiv P_2 \mathrm{J}(H, Q, T) \pmod{(Q)}$; $\mathrm{J}(H, U_2, T)\frac{\partial f_2}{\partial U_2} \equiv P_2 \mathrm{J}(H, Q, T) \pmod{(Q)}$.

$\mathrm{J}(Q, f_2, T) = \mathrm{J}(Q, H, T)\frac{\partial f_2}{\partial H} + \mathrm{J}(Q, U_2, T)\frac{\partial f_2}{\partial U_2} = \mathrm{J}(Q, P_2 Q, T) \equiv 0 \pmod{(Q)}$.

The derivative $\frac{\partial f_2}{\partial U_2}$ is a polynomial in H and U_2 which are preimages of elements from $\mathcal{B}^\partial$. The projection $\pi(\frac{\partial f_2}{\partial U_2}) \in S^\partial \setminus 0$ because we assumed that f_2 is an irreducible polynomial. Hence $\mathrm{J}(H, U_2, T)$ and $\mathrm{J}(Q, U_2, T)$ are proportional to $\mathrm{J}(Q, H, T)$ with coefficients from $\mathcal{A}$ and thus $\mathrm{J}(U_1, U_2, T)$ is congruent modulo (Q) to $\mathrm{J}(Q, H, T)$ multiplied by an element of $\mathcal{A}$.

Therefore $1 = \mathrm{J}(X, Y, Z) \equiv a\mathrm{J}(Q, H, T) \pmod{(Q)}$ for some $a \in \mathcal{A}$, i. e. $1 = \pi(a\mathrm{J}(Q, H, T)) = \pi(a)\pi(\mathrm{J}(Q, H, T))$. Since $\pi(a) \in \mathcal{B}$ its ∂-degree is nonnegative. Hence $\deg_\partial(a) = \deg_\partial(\mathrm{J}(Q, H, T)) = 0$.

To finish the proof observe that we showed that
(a) $\mathrm{J}(Q, H, U) \in (Q)$ if $\deg_\partial(U) = 0$, so $\epsilon(u) = 0$ if $u \in S^\partial$;
(b) $\deg_\partial(\mathrm{J}(Q, H, T)) = 0$, so $\epsilon(t) \in S^\partial \setminus \mathbb{C}$.
So ϵ is an lnd on S and $\ker(\epsilon) = \ker(\partial)$ since $\ker(\epsilon) \supset \ker(\partial)$ and $\ker(\epsilon)$ and $\ker(\partial)$ are algebraically closed in S (see Remark 2). Then (b) shows that ∂ and ϵ give the same degree function and therefore are equivalent. □

Remark 5 We will be using the following description of $\epsilon(g)$:
$\epsilon(g) \equiv \mathrm{J}(Q, H, G)$ where G is a preimage of g. To make it a derivation on S we will consider the right side modulo the ideal (Q).

Remark 6 It turns out that a similar description of lnds is possible for any finitely generated domain (see [6]).

Let us also recall the following construction for $\mathbb{C}[X, Y, Z]$. We can take some real valued weights $w(X)$, $w(Y)$, and $w(Z)$, define $w(X^i Y^j Z^k) = iw(X) + jw(Y) + kw(Z)$, and extend w to polynomials by defining $w(p)$ be the maximal weight among the weights of all monomials which are present in p with nonzero coefficients. Then any $p \in \mathbb{C}[X, Y, Z]$ can be written as $p = \sum_{i=u}^{v} p_i$ where each p_i is homogeneous, i. e. consists only of monomials with the same weight, and $w(p_i) < w(p_{i+1})$. We will call $\bar{p} = p_v$ the leading form of p.

3 Results and Proofs

Theorem 1 *If $R = \mathbb{C}[X, Y, Z]/(Q)$, where $Q = X^n Y^m - p(Z)$ and p is a polynomial of degree d, has a nonzero lnd and m, n, and d are relatively prime then $(d-1)(n-1)(m-1) = 0$.*

Proof Let ∂ be a nonzero lnd on R and let $h \in \ker(\partial)\backslash\mathbb{C}$. We may assume that $\deg_z(h) < d$ since $z^d = x^n y^m + (z^d - p(z))$ in R. Let us replace this derivation by ϵ described in Remark 5: $\epsilon(g) \equiv \mathrm{J}(Q, H, G)$ where H is a preimage of h such that $\deg_Z(H) < d$.

Let us take weights $w(X) = m + dN$, $w(Y) = -n$, and $w(Z) = nN$ where N is a natural number. The leading form $\bar{Q}$ of Q is $X^n Y^m - Z^d$ for any N. The leading form $\bar{H}$ of H may depend on N. Let us check that by taking N sufficiently large we can make $\bar{H} = X^i Y^j Z^k$. Indeed, if monomials $X^{i_1} Y^{j_1} Z^{k_1}$ and $X^{i_2} Y^{j_2} Z^{k_2}$ are in $\bar{H}$ then $N(di_1 + nk_1) + mi_1 - nj_1 = N(di_2 + nk_2) + mi_2 - nj_2$. If $N > m \deg_X(H) + n \deg_Y(H)$ then $di_1 + nk_1 = di_2 + nk_2$ and therefore $mi_1 - nj_1 = mi_2 - nj_2$. Hence

$$d(i_1 - i_2) + n(k_1 - k_2) = 0$$

and

$$m(i_1 - i_2) - n(j_1 - j_2) = 0$$

We assumed that $(n, m, d) = 1$. Therefore $i_1 - i_2 = ns$, $k_1 - k_2 = -ds$, and $j_1 - j_2 = ms$ where s is an integer. But then $s = 0$ since $|k_1 - k_2| < d$.

Let us fix such a sufficiently large N for which $\bar{H}$ is a monomial $X^i Y^j Z^k$.

Consider now a derivation $\bar{\epsilon}(G) = \mathrm{J}(\bar{Q}, \bar{H}, G)$. We can observe that the projection of this derivation on $\bar{R} = \mathbb{C}[X, Y, Z]/(\bar{Q})$ is locally nilpotent on $\bar{R}$. Indeed, it is easy to see that $\mathrm{J}(\bar{Q}, \bar{H}, \bar{G})$ is either $\overline{\mathrm{J}(Q, H, G)}$ or zero. Since ϵ is lnd on R we know that after several applications of a derivation $D(-) = \mathrm{J}(Q, H, -)$ to G we obtain a polynomial which is divisible by Q. It implies, of course, that the leading form of this polynomial is divisible by the leading form of Q. So if we apply at most the same number of times $\bar{\epsilon}$ to $\bar{G}$ we get a polynomial which is divisible by $\bar{Q}$. It may happen that we'll get zero or a polynomial which is divisible by $\bar{Q}$ on one of the previous steps.

Condition $(n, m, d) = 1$ makes $\bar{Q} = X^n Y^m - Z^d$ irreducible. Hence $\bar{R}$ is a domain. As we saw, in this setting the product of two nonzero elements is an $\bar{\epsilon}$-constant only if both factors are constants. Since $\bar{\epsilon}(\bar{H}) = \bar{\epsilon}(X^i Y^j Z^k) = 0$ we can conclude that either x, or y, or z is a constant of $\pi(\bar{\epsilon})$. (Here x, y, and z are the images of X, Y, and Z in $\bar{R}$.) So according to Lemma 4 one of the derivations $\epsilon_x(-) = J(X^n Y^m - Z^d, X, -)$, $\epsilon_y(-) = J(X^n Y^m - Z^d, Y, -)$, $\epsilon_z(-) = J(X^n Y^m - Z^d, Z, -)$ induces a locally nilpotent derivation on $\bar{R}$.

Now, $\epsilon_x(X) = 0$, $\epsilon_x(Y) = -dZ^{d-1}$, $\epsilon_x(Z) = -mX^n Y^{m-1}$. To see when the induced derivation is an lnd let us use the degree function defined by this derivation on $\bar{R}$. Denote by d_x, d_y, and d_z the degrees of x, y, and z correspondingly. Then $d_x = 0$,

$d_y - 1 = (d-1)d_z$, and $d_z - 1 = (m-1)d_y$. Thus $-2 = (m-2)d_y + (d-2)d_z$. Since d_y and d_z are natural numbers this equality is possible only if either $m = 1$ or $d = 1$. In both these cases $\pi(\epsilon_x)$ is an lnd.

For ϵ_y we have $\epsilon_y(X) = dZ^{d-1}$, $\epsilon_y(Y) = 0$, $\epsilon_y(Z) = nX^{n-1}Y^m$. This case is similar to the previous one and $\pi(\epsilon_y)$ is an lnd if and only if either $n = 1$ or $d = 1$.

Finally $\epsilon_z(X) = mX^nY^{m-1}$, $\epsilon_z(Y) = -nX^{n-1}Y^m$, $\epsilon_z(Z) = 0$. Using the degree function which would be defined by $\pi(\epsilon_z)$ we can see that $\pi(\epsilon_z)$ is never an lnd.

This finishes the proof of Theorem 1. □

We have now the following cases in which there is a nonzero lnd on R:
$d = 1$ which corresponds to the polynomial ring in two variables independently of values of n and m; $n = 1$; $m = 1$.

If $d > 1$ and R has a nonzero lnd then either $n = 1$ of $m = 1$ and we may assume without loss of generality that $m = 1$: if $m \neq 1$, $n = 1$ we will switch x and y.

From now on $d > 1$ and R is given by a relation $x^n y = p(z)$.

Theorem 2 *Let ∂ be a nonzero lnd of $R = \mathbb{C}[X, Y, Z]/(Q)$ where $Q = X^nY - p(Z)$ and let $h \in \ker(\partial)\backslash\mathbb{C}$. Then there exists an automorphism α of R such that $\alpha(h) = q(x)$.*

Proof We will be choosing different weights for X, Y, and Z in the course of the proof of this Theorem. Since for all these choices the weight of Z will be positive and $nw(X) + w(Y) = dw(Z)$, the leading form of Q for these weights will be $X^nY - Z^d$.

As above, we can take a preimage H of h for which $\deg_Z(H) < d$. Let us use again the weights $w_1(X) = 1 + dN$, $w_1(Y) = -n$, and $w_1(Z) = nN$. As we saw in the proof of Theorem 1 we can conclude that if N is very large then the leading form $\bar{H}$ of H is either X^i or Y^j. (It cannot be a product X^iY^j since then $\ker(\pi(\bar{\epsilon})) \ni x, y$ which is possible only if $\pi(\bar{\epsilon}) = 0$.) We can also observe that if $\bar{H} = Y^j$ with our choice of N then $h \in \mathbb{C}[y]$ since then $w_1(H) < 0$ while the weight of any monomial which contains X or Z is positive if N is large enough. (This, of course, imply that $\ker(\partial) = \mathbb{C}[y]$ and $(n-1)(d-1) = 0$; so $n = 1$ and there exists an automorphism of R sending y to x.)

Let us use now different weights: $w_2(X) = -1$, $w_2(Y) = n + dN$, and $w_2(Z) = N$. Again, if N is sufficiently large the leading form of H is a monomial. We already know that this monomial is either X^i or Y^j.

If it is X^i then $h \in \mathbb{C}[x]$ and $\pi(\epsilon_x(g)) \equiv \mathrm{J}(X^nY - p(Z), X, G)$ is indeed an lnd, and if it is Y^j then $n = 1$.

So we see that if $(n-1)(d-1) \neq 0$ then $h \in \mathbb{C}[x]$. It remains to consider the case $n = 1$ with an additional assumption that $h \notin (\mathbb{C}[x] \cup \mathbb{C}[y])$. Then the leading form of H relative to w_1 is and X^a and the leading form of H relative to w_2 is Y^b.

Since $x \to y$, $y \to x$, $z \to z$ is an automorphism of R when $n = 1$ we may also assume that $a \geq b$.

Let us now chose natural positive weights $w_3(X) = \rho$, $w_3(Y) = \sigma$, $w_3(Z) = \tau$ so that $a\rho = b\sigma$, $\rho + \sigma = d\tau$, and ρ and τ are relatively prime. (If k divides ρ and τ then k divides σ and we can cancel it.)

Denote by $\bar{H}_3$ the leading form of H relative to w_3. Then $\bar{H}_3$ contains both X^a and Y^b. Indeed if $w_3(H) = a\rho = b\sigma$, then both X^a and Y^b are in $\bar{H}_3$. Otherwise, since $\rho > 0$, $w_3(H) > a\rho$ and $\bar{H}_3$ contains a monomial $X^iY^jZ^k$ for which $i\rho + j\sigma + k\tau > a\rho$.

To bring this to a contradiction let us consider the weights
$w_4(X) = \rho - d\delta_1$, $w_4(Y) = \sigma - d\delta_2$, $w_4(Z) = \tau - \delta_1 - \delta_2$ where δ_1 and δ_2 satisfy the following conditions:
(1) $da\delta_1 + db\delta_2 + \deg_Z(\bar{H}_3)(\delta_1 + \delta_2) < w_3(\bar{H}_3) - a\rho$.
(2) δ_1 and δ_2 are positive irrational numbers which are linearly independent over the field of rational numbers.
(3) $w_4(X) > 0, \quad w_4(Y) > 0, \quad w_4(Z) > 0.$

Then $\bar{H}_4$ for w_4 is a monomial in force of condition (2) and this monomial cannot be neither X^a nor Y^b since in force of condition 1) $w_4(X^iY^jZ^k) = w_3(H) - id\delta_1 - jd\delta_2 - k(\delta_1 + \delta_2) > w_3(X^a) = w_3(Y^b)$ while $w_4(X^a) = a\rho - ad\delta_1 < w_3(X^a)$ and $w_4(Y^b) = b\sigma - bd\delta_2 < w_3(Y^b)$. As we already know it is impossible and hence $\bar{H}_3 = \mu X^a + \cdots + \nu Y^b$.

Consider now $X^b\bar{H}_3$. This polynomial can be rewritten as a polynomial $\psi \in \mathbb{C}[X, Z]$ since $XY = Z^d$ in $\bar{R}$.

The polynomial ψ is ρ, τ homogeneous, so $\psi = c\prod_i(X^\tau - c_iZ^\rho)$ and $\bar{H}_3 = c\prod_i(X^\tau - c_iZ^\rho)X^{-b}$. Let us replace $\bar{H}_3$ by $\bar{H}_3^d$.

Lemma 5 $(x^\tau - c_iz^\rho)^dx^{-\rho} \in \bar{R}$.

Proof It is sufficient to show that any monomial $x^{i\tau-\rho}z^{(d-i)\rho} \in \bar{R}$. Of course, any monomial of this kind with $i\tau - \rho \geq 0$ is in $\bar{R}$. If $i\tau - \rho < 0$ then $(d - i)\rho > d(\rho - i\tau)$ since $d\tau = \rho + \sigma$ and the corresponding monomial is equal to $y^{\rho-i\tau}z^{(d-i)\rho-d(\rho-i\tau)} \in \bar{R}$. $\square$

The form $\bar{H}_3^d$ can be written as $c\prod_i[(X^\tau - c_iZ^\rho)^dX^{-\rho}]$ and each of the factors $(x^\tau - c_iz^\rho)^dx^{-\rho}$ belongs to $\bar{R}$.

As we know the derivation which is induced on $\bar{R}$ by $\bar{\epsilon}(-) = \mathrm{J}(XY - Z^d, \bar{H}_3, -)$ is an lnd. Since $\bar{\epsilon}(\bar{H}_3^d) = 0$ each of these factors is in the kernel of the $\pi(\bar{\epsilon})$ and if there are two different factors then $\ker(\pi(\bar{\epsilon}))$ has the transcendence degree 2 and $\pi(\bar{\epsilon}) = 0$. Since it is not the case, there is just one factor. Furthermore, since $x \to \lambda x$, $y \to \lambda^{-1}y$, $z \to z$ is an automorphism of $\bar{R}$ it remains to find out for which ρ, τ, and d the derivation of $\bar{R}$ given by $\pi(\bar{\epsilon})(g) \equiv \mathrm{J}(XY - Z^d, (X^\tau - Z^\rho)^dX^{-\rho}, G)$ is an lnd.

Let us compute $\pi(\bar{\epsilon})(z)$:
$\pi(\bar{\epsilon})(z) \equiv \mathrm{J}(XY - Z^d, (X^\tau - Z^\rho)^dX^{-\rho}, Z) = \mathrm{J}_{X,Y}(XY, (X^\tau - Z^\rho)^dX^{-\rho}) = -X[d(X^\tau - Z^\rho)^{d-1}\tau X^{\tau-1-\rho} - \rho(X^\tau - Z^\rho)^dX^{-\rho-1}] \equiv [\rho - \tau dx^\tau(x^\tau - z^\rho)^{-1}](x^\tau - z^\rho)^dx^{-\rho}$.

Now, $\pi(\bar{\epsilon})((x^\tau - z^\rho)^dx^{-\rho}) = 0$, so $\pi(\bar{\epsilon})(x^\tau(x^\tau - z^\rho)^{-1}) \neq 0$ since $\rho \neq d\tau$.

Let us denote by deg the degree induced by $\bar{\epsilon}$. Then $\deg((x^\tau - z^\rho)^d x^{-\rho}) = 0$ and $\deg(z) - 1 = \deg(\rho - \tau d x^\tau (x^\tau - z^\rho)^{-1}) \neq 0$.

We see that $\deg(x^\tau (x^\tau - z^\rho)^{-1}) = \deg(z) - 1 > 0$. This is possible only if $\deg(x^\tau) = \deg(z^\rho) > \deg(x^\tau - z^\rho)$. So
$\tau \deg(x) - \rho \deg(z) = 0$,
$\tau \deg(x) - \deg(z) - \deg(x^\tau - z^\rho) = -1$ and
$\rho \deg(x) - d \deg(x^\tau - z^\rho) = 0$.

Solving this system we obtain $\deg(x^\tau - z^\rho) = \rho^2[\rho^2 - \tau d(\rho - 1)]^{-1}$. Now, $\rho^2 - \tau d(\rho - 1) = \rho^2 - (\rho + \sigma)(\rho - 1) = \rho + \sigma - \rho\sigma = 1 - (\rho - 1)(\sigma - 1)$ since $\tau d = \rho + \sigma$. Since $\deg(x^\tau - z^\rho) > 0$ we should have $1 - (\rho - 1)(\sigma - 1) > 0$ which is possible only if $(\rho - 1)(\sigma - 1) = 0$. Since $\rho a = \sigma b$ and $a \geq b$ we have $\sigma \geq \rho$ and so $\rho = 1$ if $\bar{\partial} = \pi(\bar{\epsilon})$ is an lnd on $\bar{R}$.

If $\rho = 1$ then $\deg(x^\tau - z) = 1$, $\deg(x) = d$, $\deg(z) = d\tau$. Hence if $\bar{\partial}$ is an lnd then $\bar{\partial}(x^\tau - z) = \lambda_1 \in \bar{R}^{\bar{\partial}}$.

Since $\bar{\partial}(x^\tau - z) \equiv \mathrm{J}(XY - Z^d, (X^\tau - Z)^d X^{-1}, X^\tau - Z) = -(X^\tau - Z)^d X^{-1}$ $\equiv -(x^\tau - z)^d x^{-1} = \lambda_1 \in \bar{R}^{\bar{\partial}}$ we can put $x^\tau - z = \lambda_1 t$ where $\bar{\partial}(t) = 1$. Then $x = -\lambda_1^{d-1} t^d \in \mathrm{Nil}_{\bar{R}}(\bar{\partial})$, $z = x^\tau - \lambda_1 t \in \mathrm{Nil}_{\bar{R}}(\bar{\partial})$ and $y = z^d x^{-1} = -\lambda_1^{1-d}((-\lambda_1)^{(d-1)\tau} t^{d\tau - 1} - \lambda_1)^d \in \mathrm{Nil}_{\bar{R}}(\bar{\partial})$, i.e. $\bar{\partial}$ is an lnd on $\bar{R}$.

We checked that if $a \geq b$ then $\bar{H}_3 = c(X^\tau - c_1 Z)^k X^{-b}$. Therefore the leading form of h relative to the weight given by $w_3(x) = \rho$, $w_3(y) = \sigma$, $w_3(z) = \tau$ is $c(x^\tau - c_1 z)^k x^{-b}$.

Observe that a homomorphism β given by $x \to x$, $y \to (p(z + c_1^{-1} x^\tau)) x^{-1}$, $z \to z + c_1^{-1} x^\tau$ is an automorphism of R. If we apply this automorphism to h then the leading form of h, as an element of $\bar{R}$ becomes $c[x^\tau - c_1(z + c_1^{-1} x^\tau)]^k x^{-b} = c(-c_1 z)^k x^{-b} = \nu y^b$. (Hence $k = bd$.)

Therefore $\deg_y(\beta(h)) = \deg_y(h)$ while $\deg_x(\beta(h)) < \deg_x(h)$. If $\beta(h) \in \mathbb{C}[y]$ we can finish the proof since $x \to y$, $y \to x$, $z \to z$ is an automorphism of R. If $\beta(h) \notin \mathbb{C}[y]$ we can find an automorphism which will decrease either $\deg_x$ or $\deg_y$ of $\beta(h)$. Since these degrees cannot decrease indefinitely, a composition of several automorphisms of this type and, possibly, an automorphism exchanging x and y gives an automorphism α such that $\alpha(h) = q(x)$. □

4 Conclusion

We proved that there is only the zero lnd on $R = \mathbb{C}[X, Y, Z]/(X^n Y^m - p(z))$, $\deg(p) = d$ when $(d-1)(m-1)(n-1) \neq 0$ and $(d, m, n) = 1$; when $(d-1)(m-1) \neq 0$ and $n = 1$ or when $(d-1)(n-1) \neq 0$ and $m = 1$ all nonzero lnds have the same kernel; when $d = 1$ or when $n = m = 1$ there are lnds with different kernels but each kernel can be mapped on a "standard" one by an automorphism.

Lemma 6 *Locally nilpotent derivations of a domain A with the same kernel are equivalent to each other.*

Proof Assume that nonzero lnds ∂_1 and ∂_2 of A have the same kernel K. We know that $\mathrm{Nil}_F(\partial_1) = F^{\partial_1}[t_1]$ and $\mathrm{Nil}_F(\partial_2) = F^{\partial_2}[t_2]$ where F is the field of fractions of A (Lemma 1) and that $F^{\partial_1} = F^{\partial_2} = L = \mathrm{Frac}(K)$ (Lemma 3). We may assume that $at_1 \in A$ for some $a \in K\backslash 0$ (see the proof of Lemma 1). Then $\partial_2^i(at_1) = a\partial_2^i(t_1)$ for any i. Hence $t_1 \in \mathrm{Nil}_F(\partial_2)$ and $t_1 = \sum_i f_i t_2^i$ where $f_i \in L$. Similarly, $t_2 = \sum_j f_j t_1^j$ where $f_j \in L$. Hence $\deg_{t_2}(t_1) = \deg_{t_1}(t_2) = 1$ and Lemma is proved.

Remark 7 All these derivations are proportional to each other over F^∂ and any linear combination of these derivations with coefficients in K is again an lnd with the kernel K. By Lemma 2 at least one of these derivations is irreducible. If A is not a unique factorization domain then there may be several irreducible derivations among these derivations (It would be interesting to find an example).

Theorem 3 *If R is a ring satisfying conditions of Theorem 1 then, up to an automorphism (and multiplication by $c \in \mathbb{C}$), there is just one nonzero irreducible lnd of R. It is defined by $\partial(x) = 0,\ \partial(y) = p'(z),\ \partial(z) = x^n$.*

Proof If ϵ is an lnd of R with $R^\epsilon = \mathbb{C}[x]$ then $\epsilon = \frac{q_1(x)}{q_2(x)}\partial$ and we can assume that polynomials $q_1,\ q_2$ are relatively prime. We can find two polynomials $p_1,\ p_2 \in \mathbb{C}[x]$ such that the lnd $\epsilon_1 = p_1\epsilon + p_2\partial = \frac{1}{q_2(x)}\partial$. Therefore $\epsilon_1(y) = \frac{p'(z)}{q_2(x)} \in R$ and $\epsilon_1(z) = \frac{x^n}{q_2(x)} \in R$. If $q_2(x) \notin \mathbb{C}$ then $\frac{p'(z)}{x} \in R = \mathbb{C}[x, \frac{p(z)}{x^n}, z]$. Assume that $\frac{p'(z)}{x} = r(x, \frac{p(z)}{x^n}, z)$ where $r(x, y, z) \in \mathbb{C}[x, y, z]$. Let us take $w(x) = 1,\ w(z) = \lambda$ where λ is a positive irrational number, such that all monomials of $r(x, y, z)$ have different weights. Then $w(\frac{p'(z)}{x}) = i + j(d\lambda - n) + k\lambda$ for some nonnegative integers $i,\ j,\ k$, i.e. $(d-1)\lambda - 1 = i + j(d\lambda - n) + k\lambda$. Since λ is irrational, $i - jn + 1 = 0$ and $jd + k - d + 1 = 0$. Hence $j = 0$. But then $i = -1$, which is impossible. Hence $q_2 \in \mathbb{C}$. $\square$

References

1. Daigle, D.: On locally nilpotent derivations of $k[X_1, X_2, Y]/(\phi(Y) - X_1X_2)$. J. Pure Appl. Algebr. **181**(2–3), 181–208 (2003)
2. Ferrero, M., Lequain, Y., Nowicki, A.: A note on locally nilpotent derivations. J. Pure Appl. Algebr. **79**, 45–50 (1992)
3. Freudenburg, G., Moser-Jauslin, L.: Embeddings of danielewski surfaces. Math. Z. **245**(4), 823–834 (2003)
4. Makar-Limanov, L.: On the group of automorphisms of a surface $x^n y = P(z)$. Isr. J. Math. **121**, 113–123 (2001)
5. Makar-Limanov, L.: Locally nilpotent derivations of the surface $xy = p(z)$. In: Proceedings of the Third International Algebra Conference. Chang Jung Christian University, Tainan, Taiwan, 16 June–1 July 1, 2002
6. Makar-Limanov, L.: Locally nilpotent derivations of affine domains, affine algebraic geometry. CRM Proceedings Lecture Notes, vol. 54, pp. 221–229. American Mathematical Society, Providence, RI (2011)
7. Rentschler, R.: Operations du groupe additif sur le plane affine. C.R. Acad. Sci. Paris. **267**, 384–387 (1968)

Action of the Automorphism Group on the Jacobian of Klein's Quartic Curve

Dimitri Markushevich and Anne Moreau

Abstract Klein's simple group H of order 168 is the automorphism group of the plane quartic curve $C4$, called Klein quartic. By Torelli Theorem, the full automorphism group G of the Jacobian $J = J(C)$ is the group of order 336, obtained by adding minus identity to H. The quotient variety J/G can be alternatively represented as the quotient $\mathbb{C}^3/\widetilde{G}$ of the complex 3-space by the complex crystallographic group $\widetilde{G}$, the extension of G by the period lattice of the Klein quartic. Moreover, it turns out that $\widetilde{G}$ is generated by affine complex reflections. According to a conjecture of Bernstein-Schwarzman, a quotient of $\mathbb{C}^n$ by an irreducible complex crystallographic group generated by reflections is a weighted projective space. The conjecture is known in dimension two and for complexifications of the real crystallographic groups generated by reflections. The case of $\widetilde{G}$ is the first, and in a sense the smallest of the unknown cases. We compute the orbits and the stabilizers of the action of G on J and deduce that $J/G = \mathbb{C}^3/\widetilde{G}$ is a strongly simply connected variety with the same singularities as the weighted projective space $\mathbb{P}(1, 2, 4, 7)$.

Keywords Klein's quartic curve · Automorphism group · Jacobian variety

1 Introduction

Klein's simple group H_{168} of order 168 can be defined by $H_{168} \simeq \mathbf{PSL}(2, 7) \simeq \mathbf{GL}(3, 2)$, where $\mathbf{GL}(n, q)$, resp. $\mathbf{PSL}(n, q)$ stands for the linear (resp. projective special linear) group of automorphisms of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the finite field with q elements. Klein introduced this group in 1879 [12] and described its irreducible 3-dimensional complex representation by automorphisms of the plane

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quartic curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ with equation $x^3y + y^3z + z^3x = 0$, called Klein's quartic curve. See, for example [1, 8] for a modern exposition, some applications and interesting ramifications.

Klein's simple group also appears in the context of groups generated by complex reflections. Consider it as a complex linear group acting on the 3-dimensional complex vector space $V \simeq \mathbb{C}^3$, whose projectivization is the projective plane containing the Klein quartic: $\mathbb{P}(V) = \mathbb{P}^2_{\mathbb{C}}$. This representation embeds H_{168} into $\mathbf{SL}(3, \mathbb{C})$. If we extend this copy of H_{168} by adding $-\,\mathrm{id}_{\mathbb{C}^3}$, we will obtain a subgroup of $\mathbf{GL}(3, \mathbb{C})$ of order 336, which we will denote G_{336}. This extension of H_{168} is not just split, it is simply a direct product: $G_{336} = \{\pm\,\mathrm{id}\} \times H_{168}$. In spite of the apparent triviality of this step, it brings in a new very important property: G_{336} is one of the finite complex reflection groups classified by Shephard–Todd [21]; see also [5] for a simplified approach to the classification.

On the other hand, the action of G_{336} on $\mathbb{C}^3$ is of arithmetic nature, as it preserves a rank-6 lattice in $\mathbb{C}^3$. One can easily see the existence of such a lattice Λ. Indeed, as H_{168} acts on Klein's curve C, it also acts on its Jacobian $\mathcal{J} = \mathcal{J}(C)$, a 3-dimensional abelian variety. So we can represent $\mathcal{J}(C)$ as the complex torus $\mathbb{C}^3/\Lambda$, where Λ is the period lattice of C, and then the action of H_{168} lifts to a linear action on $\mathbb{C}^3$ leaving invariant Λ. The fact that the action on $\mathbb{C}^3$ is the same as that on V can be verified by using the canonical identification $\mathcal{J} = H^0(C, \Omega^1_C)^*/\Lambda$, and the action on $H^0(C, \Omega^1_C)$ can be deduced from the representation of the 1-forms on C via the Poincaré residue. In this way we recover a lattice Λ, invariant under G_{336}, as the period lattice of C in $H^0(C, \Omega^1_C)^*$.

As G_{336} leaves invariant the lattice Λ, one can construct the extension $\widetilde{G}_{336}$ of G_{336} by adding the translations by vectors from Λ:

$$0 \longrightarrow \Lambda \longrightarrow \widetilde{G}_{336} \longrightarrow G_{336} \longrightarrow 0. \tag{1}$$

The thus obtained group $\widetilde{G}_{336}$ of affine transformations of $\mathbb{C}^3$ is a complex crystallographic group generated by reflections, or a CCR group for short. Moreover, G_{336} is a maximal finite group of linear transformations leaving invariant Λ. By Torelli theorem, the order of the automorphism group of $\mathcal{J}(C)$ as a principally polarized abelian variety is twice the order of the automorphism group of C, and the latter is 168, which is the maximal order of the automorphism group of a curve of genus $g = 3$ by the Hurwitz inequality $|\,\mathrm{Aut}(C)| \leq 84(g-1)$.

The main object of interest of the present study is the quotient variety $X = \mathcal{J}/G_{336}$, which can also be viewed as the quotient $\mathbb{C}^3/\widetilde{G}_{336}$ by the CCR group. This quotient can be thought of as the projective spectrum of the algebra of G_{336}-invariant theta functions for $\mathcal{J}$. For finite reflection groups acting on $\mathbb{C}^n$, we have the Chevalley–Shephard–Todd Theorem, which states that the algebra of polynomial invariants of the action is also polynomial, that is freely generated by n basic generators. It is a natural conjecture that the analogue of the Chevalley–Shephard–Todd Theorem also holds for irreducible affine CCR groups. The conjecture can be stated in other words by saying that for such a group Γ, the quotient variety $\mathbb{C}^n/\,\Gamma$ is a weighted projective space. This conjecture, taken in full generality, persists for more than 40 years, since Looijenga [15] established the result for the CCR groups

Γ obtained as the extensions of the Weyl group of a real irreducible root system in $\mathbb{R}^n$ by a complexification of its root lattice. Such complexified real crystallographic reflection groups depend on one complex parameter τ. Several papers generalized and improved this result in several ways, and at present it is known to be true for all CCR groups of Coxeter type [2–4, 9, 11, 23].

The conjecture was also claimed to be proven in dimension two, see [14, 19, 22], but the proofs were based on an incomplete classification of rank-2 CCR groups. For example, as we know from [7, 13], the weighted projective plane $\mathbb{P}(1, 3, 8)$ is a CCR quotient, but it is missing in the above references; see also [18], [10, Sect. 5]. In dimension > 2, not a single result of this type is known for any one of the genuinely complex crystallographic reflection groups, i. e. those which are not of Coxeter type. A classification of such groups can be found in [18]. By contrast with the CCR groups of Coxeter type, genuinely complex CCR groups are all rigid: there is no continuous parameter τ. According to Popov's classification, there exists a unique complex crystallographic reflection group with point group G_{336}: it is listed as $[K_{24}]$ in Table 2 in loc. cit. (24 being the number of G_{336} in the classification table of [21]). From Popov's table, one also reads off the generators of the invariant lattice Λ and the extension cocycle, which can only be zero in this case. Thus an extension of G_{336} by Λ is always split, so that $\tilde{G}_{336} = \Lambda \rtimes G_{336}$ is a unique such extension, and the G_{336}-invariant lattice Λ is unique modulo equivalence. We will use a slightly different, more symmetric representation of Λ from [17].

Our results on the singularities of X make it plausible that X is the weighted projective space $\mathbb{P}(1, 2, 4, 7)$. We look into the combinatorics of the action of G_{336} and list the stabilizers and the orbits in $\mathcal{J}$. As follows from Theorem 3.1, X and $\mathbb{P}(1, 2, 4, 7)$ have the same singularities.

2 Klein's Group H_{168}, Its Double G_{336} and the Invariant Lattice Λ

We introduce the group $G = G_{336}$ directly in its embedding in $\mathbf{U}(3)$ as the group generated by reflections in the roots of the complex root system, usually denoted $J_3(4)$, but we will fix the notation $\mathcal{R}$ for it. We describe it following [17, pp. 235–236]. The root system $\mathcal{R}$ is the set of vectors of $\mathbb{C}^3$, obtained from $(2, 0, 0)$, $(0, \alpha, \alpha)$ and $(1, 1, \bar{\alpha})$, where $\alpha = \frac{1+i\sqrt{7}}{2}$, by sign changes and permutations of coordinates. The root lattice $\Lambda = Q(\mathcal{R})$ generated by $\mathcal{R}$ can be given by

$$\Lambda = \{(z_1, z_2, z_3) \in \mathcal{O}^3 \mid z_1 \equiv z_2 \equiv z_3 \bmod \alpha,\ z_1 + z_2 + z_3 \equiv 0 \bmod \bar{\alpha}\},$$

where $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\alpha = \mathbb{Z}[\alpha]$ is the ring of integers of the quadratic field $K = \mathbb{Q}(\alpha)$. The group G is the subgroup of $\mathbf{U}(3)$ leaving invariant Λ. The translations by Λ extend G to an affine crystallographic reflection group $\tilde{G}$.

The standard Hermitian scalar product of $\mathbb{C}^3$ is not primitive when restricted to Λ, so we will endow $\mathbb{C}^3$ with the Hermitian scalar product which is half the standard one:

$$\forall x,\ y \in \mathbb{C}^3,\ \ (x, y) := \frac{1}{2}\sum_{i=1}^{3} \bar{x}_i y_i .$$

With these definitions, $\mathcal{R}$ contains 42 roots e, all of them being of square 2: $(e, e) = 2$. They are divided in 21 pairs of opposite roots $\pm e$. Choosing one representative from each pair in an arbitrary way, we obtain the subset $\mathcal{R}_0$ of 21 roots which will be called positive roots. The group G is generated by the 21 reflections in the positive roots $e \in \mathcal{R}_0$,

$$r_e : \mathbb{C}^3 \to \mathbb{C}^3,\ \ x \mapsto x - (e, x)e,$$

and Klein's simple group is the unimodular part of G:

$$H_{168} = \{h \in G \mid \det(h) = 1\}.$$

It can be thought of as the group generated by the 21 antireflections $\rho_e := -r_e$, or by products $r_e r_{e'}$ of pairs of reflections $(e, e' \in \mathcal{R}_0)$. These generating sets are redundant; to generate G, it suffices to use three reflections. We choose the three "basic" roots as $e_1 = (0, \alpha, \alpha)$, $e_2 = (0, 0, 2)$ and $e_3 = (1, 1, \bar{\alpha})$ in such a way that the corresponding generators of G are the same as chosen in [21, (10.1)]:

$$r_1 = r_{e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ r_2 = r_{e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ r_3 = r_{e_3} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\alpha \\ -1 & 1 & -\alpha \\ -\bar{\alpha} & -\bar{\alpha} & 0 \end{pmatrix}.$$

These generators satisfy the following relations:

$$r_1^2 = r_2^2 = r_3^2 = (r_1 r_2)^4 = (r_2 r_3)^4 = (r_3 r_1)^3 = (r_1 r_2 r_1 r_3)^3 = 1.$$

By loc. cit., this is a presentation of G by generators and relations.

Obviously, $\rho_i = -r_i$ $(i = 1, 2, 3)$ generate H_{168}. As a minimal set of generators of H_{168} one can choose

$$r_3 r_1 = \frac{1}{2}\begin{pmatrix} 1 & -\alpha & -1 \\ -1 & -\alpha & 1 \\ -\bar{\alpha} & 0 & -\bar{\alpha} \end{pmatrix} \text{ and } r_1 r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix};\ \ (r_3 r_1)^3 = (r_1 r_2)^4 = 1.$$

The orders of elements of G are 1, 2, 3, 4, 6, 7, 14. An example of an element of maximal order in G (an analogue of a Coxeter element) is

$$r_1 r_2 r_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\alpha \\ \bar{\alpha} & \bar{\alpha} & 0 \\ -1 & 1 & -\alpha \end{pmatrix},\ \ (r_1 r_2 r_3)^7 = -1. \tag{2}$$

Remark that Λ is a free $\mathcal{O}$-module of rank 3, generated by the basic roots e_1, e_2, e_3 introduced above:

$$\Lambda = \mathcal{O}e_1 + \mathcal{O}e_2 + \mathcal{O}e_3.$$

This representation of Λ implies that the elements of H and G can be given by matrices from $M_3(\mathcal{O})$ in the basis (e_1, e_2, e_3). The disadvantage of this representation is that it is not unitary. So we stick to the representation of G by unitary matrices in the standard basis of $\mathbb{C}^3$ from which we started, though the elements of these unitary matrices are half-integers from $\mathcal{O}$. The columns of each matrix in G are roots from $\mathcal{R}$ divided by 2, so making a complete list of elements of G amounts to the enumeration of all the triples of mutually orthogonal roots in $\mathcal{R}$. Over $\mathbb{Z}$, we will fix

$$(\epsilon_1, \dots, \epsilon_6) = (\alpha e_1, \alpha e_2, \alpha e_3, \bar{\alpha} e_1, \bar{\alpha} e_2, \bar{\alpha} e_3)$$

as the "standard" $\mathbb{Z}$-basis of Λ.

The famous equation of Klein's quartic $x^3y + y^3z + z^3x = 0$ is referred to coordinates in which an order-7 element of H_{168} is diagonalized with eigenvalues $\zeta,\ \zeta^4,\ \zeta^2$, where $\zeta = \exp\frac{2\pi i}{7}$, but in the coordinates used in our representation it becomes

$$x^4 + y^4 + z^4 - 3\bar{\alpha}(x^2y^2 + x^2z^2 + y^2z^2) = 0.$$

The next table from [6] provides a list of the 15 conjugacy classes of subgroups of H_{168} with their minimal overgroups and maximal subgroups; these data determine a structure of a lattice (partially ordered set) on the set of subgroups of H_{168}. The notation for groups used in the column "Structure" is standard for papers in the theory of finite groups; we explain some of them that are unusual in other fields of mathematics: n is a cyclic group of order n; m^n is the direct product of n copies of a cyclic group of order m; $N : L$ is a semi-direct product of N and L with N a normal subgroup; $L_n(q)$ is what we denote $PSL(n, q)$, so that $L_2(7) \simeq H_{168}$. The repetition of a type of a subgroup means that there are two orbits under conjugation, their lengths are given in the column "Length". The last two columns refer to subgroups by their numbers from the first column, the integers between parentheses indicating the number of distinct subgroups of given type that are minimal overgroups or maximal subgroups for the subgroup from the current line.

Nr.	Structure	Order	Length	Maximal subgroups	Minimal overgroups
1	$L_2(7)$	168	1	2 (7), 3 (7), 4 (8)	
2	$2^2 : S_3$	24	7	5, 7 (3), 9 (4)	1
3	$2^2 : S_3$	24	7	6, 7 (3), 9 (4)	1
4	$7 : 3$	21	8	8, 13 (7)	1
5	A_4	12	7	10, 13 (4)	2
6	A_4	12	7	11, 13 (4)	3
7	D_8	8	21	10, 12, 11	2, 3
8	7	7	8	15	4
9	S_3	6	28	13, 14 (3)	2, 3
10	2^2	4	7	14 (3)	5, 7 (3)
11	2^2	4	7	14 (3)	6, 7 (3)
12	4	4	21	14	7
13	3	3	28	15	4 (2), 5, 6, 9
14	2	2	21	15	9 (4), 10, 11, 12
15	1	1	1		8 (8), 13 (28), 14 (21)

We will not list all the subgroups of G, but just note that each subgroup K of H_{168} has a degree-two extension in G, denoted $\pm K$:

$$\pm K = \langle -1, K\rangle = \{\pm k \mid k \in K\} \simeq \{\pm 1\} \times K.$$

Of course, G also has other types of subgroups.

For future reference, we provide some explicit examples of subgroups of H_{168} from the table:

$$D_8 = \langle s, t \mid s^4 = t^2 = 1,\ tst = s^{-1}\rangle = \\ \{1, s = h_4, h_4^2, h_4^3, t = \rho_1, \rho_2, \rho_2 h_4, h_4\rho_1\},\ h_4 = \rho_1\rho_2; \quad (3)$$

$$7 \simeq G_7 = \{1, g_7, \dots, g_7^6\},\ g_7 = \rho_1\rho_2\rho_3 = -r_1r_2r_3 = \frac{1}{2}\begin{pmatrix} -1 & 1 & \alpha \\ -\bar{\alpha} & -\bar{\alpha} & 0 \\ 1 & -1 & \alpha \end{pmatrix}; \quad (4)$$

$$7:3 \simeq G_{21} = \langle g_7, h_3 \mid g_7^7 = h_3^3 = 1,\ h_3 g_7 h_3^{-1} = g_7^2\rangle,\quad h_3 = \rho_1\rho_3\rho_1\rho_2; \quad (5)$$

$$2^2 : S_3 \simeq S_4 \simeq G_{24} = \left\{\gamma = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \end{pmatrix} \ \middle|\ \det\gamma = 1 \right\}. \quad (6)$$

In the next table we list the conjugacy classes of H:

$\mathrm{ord}\,(\gamma)$	1	2	3	4	7	7
$\lvert \mathrm{Cl}_H(\gamma)\rvert$	1	21	56	42	24	24
γ	1	ρ_1	h_3	h_4	g_7	g_7^{-1}

The representatives h_3, h_4, g_7 are defined in (3)–(5). The conjugacy classes of G are deduced from these in an obvious way: to every conjugacy class $\mathrm{Cl}_H(\gamma)$ in H correspond two conjugacy classes in G of the same length: $\mathrm{Cl}_G(\gamma) = \mathrm{Cl}_H(\gamma)$ and $\mathrm{Cl}_G(-\gamma) = -\mathrm{Cl}_H(\gamma)$.

3 Fixed Loci of Elements of G_{336} Acting on $\mathcal{J} = \mathbb{C}^3/\Lambda$

We divide the elements of G in two classes, elliptic and parabolic; the parabolic ones are defined as those having 1 among their eigenvalues, and all the remaining elements are called elliptic. The elliptic elements are -1, the 42 elements of order 4 with determinant -1, the 56 elements of order 6, and those of order 7 and 14. There are also 42 elements of order 4 with determinant 1, but they are parabolic. For both orders 7 and 14, there are two conjugacy classes of length 24, but what we need for enumerating the fixed points is the number of cyclic subgroups generated by them, and there are fewer classes of elliptically generated cyclic subgroups.

Proposition 3.1 *G has the following cyclic subgroups generated by elliptic elements:*

(i) One group of order 2, $C_2 = \{\pm 1\}$, *with* 64 *fixed points in* $\mathcal{J}$ *that are images of the half-periods of* Λ*:*

$$\{\xi_0, \dots, \xi_{63}\} = \left\{ \sum_{i=1}^{6} x_i \epsilon_i,\ x_i \in \left\{0, \tfrac{1}{2}\right\} \right\}.$$

(ii) One conjugacy class of 21 *cyclic subgroups of order* 4, $C_4^{(1)}, \dots, C_4^{(21)}$, *having each* 16 *fixed points in* $\mathcal{J}$. *Choosing* $C_4^{(1)} = \langle h_4' \rangle$, $h_4' = -r_1 r_2 : (z_1, z_2, z_3) \mapsto (-z_1, z_3, -z_2)$, *we find the representatives of the* 16 *fixed points of* h_4' *in the form*

$$\{\beta_0, \dots, \beta_{15}\} = \iota_0(1,0,0) + \iota_1(\alpha, 0, 0) + \iota_2(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}, -\tfrac{\alpha}{2}) + \iota_3(\tfrac{\bar{\alpha}}{2}, 1, 0),\quad \iota_k \in \{0, 1\}.$$

(iii) One conjugacy class of 28 *cyclic subgroups of order* 6, $C_6^{(1)}, \dots, C_6^{(28)}$, *having each* 4 *fixed points in* $\mathcal{J}$. *Choosing* $C_6^{(1)} = \langle c \rangle$ *with* $c = (z_1, z_2, z_3) \mapsto (-z_3, -z_1, -z_2)$ *we identify the representatives of the* 4 *fixed points in* Π *as:*

$$\omega_{ij} = \frac{i}{2}(\bar{\alpha}, \bar{\alpha}, \bar{\alpha}) + j(1, 1, 1),\ (i, j) \in \{0, 1\}^2,$$

so that $\omega_{00} = 0$ *and the remaining* 3 *points* ω_{ij} *belong to the set of* 64 *fixed points of* C_2 *from item (i).*

(iv) One conjugacy class of 8 *subgroups* $C_7^{(1)}, \dots, C_7^{(8)}$ *of order* 7, *having each* 7 *fixed points on* $\mathcal{J}$. *Choosing* $C_7^{(1)} = \langle g_7 \rangle$, *where* g_7 *is defined in* (5), *we find the following representatives of the* 7 *fixed points of* $C_7^{(1)}$*:*

$$\eta_0 = 0,\ \eta_i = \frac{1}{7}\Big(-i\epsilon_1 - i\epsilon_2 + i\epsilon_3 + i\epsilon_4 + i\epsilon_5 - i\epsilon_6 \Big),\ i = 1, \dots, 6.$$

(v) One conjugacy class of 8 *cyclic subgroups* $C_{14}^{(1)}, \dots, C_{14}^{(8)}$ *of order* 14, *having each a unique fixed point, the zero of* $\mathcal{J}$.

Proof Let γ be an elliptic element and $z \in \mathbb{C}^3$ a fixed point of γ modulo Λ. This means that $\gamma z - z \in \Lambda$, or else $z \in (\gamma - \mathrm{id}_{\mathbb{C}^3})^{-1}(\Lambda)$. Thus the number of fixed points modulo Λ is equal to $[(\gamma - \mathrm{id}_{\mathbb{C}^3})^{-1}(\Lambda) : \Lambda]$. Hence to evaluate the number of fixed points on $\mathcal{J}$, it suffices to calculate the determinant of $\gamma - \mathrm{id}_\Lambda$, where γ is viewed as an automorphism of the rank-6 $\mathbb{Z}$-module Λ. When working with 3×3 complex matrices, this determinant becomes $|\det(\gamma - \mathrm{id}_{\mathbb{C}^3})|^2$. The calculation of $\det(\gamma - \mathrm{id}_{\mathbb{C}^3})$ for $\gamma = -1,\ h_4',\ c,\ g_7,\ -g_7$ gives, respectively, the values

$-8,\ -4,\ -2,\ i\sqrt{7},\ -1$. This implies the assertion on the numbers of fixed points. The explicit representatives produced in the statement are obtained by a direct calculation. It is quite easy for the orders < 7, and for order 7, we wrote down g_7 by an integer matrix in the $\mathbb{Z}$-basis (ϵ_i) of Λ and searched for the fixed points in the unit cube of $\mathbb{R}^6$. We omit further details. $\square$

The non-elliptic elements different from 1 have fixed loci of positive dimension in $\mathcal{J}$, which are translates of elliptic curves or abelian surfaces. We denote the eigenspace of γ corresponding to an eigenvalue λ by $V_\lambda^{(\gamma)}$, or simply by V_λ. We also denote $\Lambda_\lambda^{(\gamma)}$ or just Λ_λ the intersection $\Lambda \cap V_\lambda^{(\gamma)}$. When this is a full-rank lattice in $V_\lambda^{(\gamma)}$, the quotient $\mathcal{J}_\lambda^{(\gamma)} = \mathcal{J}_\lambda := V_\lambda^{(\gamma)}/\Lambda_\lambda^{(\gamma)}$ is an abelian variety of dimension $\dim V_\lambda^{(\gamma)}$.

When $\lambda = 1$ is among the eigenvalues of γ, $\mathcal{J}_1^{(\gamma)}$ is the connected component of 0 in the fixed locus $\mathcal{J}^\gamma = \mathrm{Fix}_{\mathcal{J}}(\gamma)$, but the latter fixed locus can contain several connected components, which are translates of $\mathcal{J}_1^{(\gamma)}$. The number of components can be determined as follows. Let $\Lambda_{\mathrm{a}}^{(\gamma)}$, $V_{\mathrm{a}}^{(\gamma)}$ be the anti-invariant parts of γ in Λ, respectively V, that is the orthogonal complements of $\Lambda_1^{(\gamma)}$, $V_1^{(\gamma)}$, and $\mathcal{J}_{\mathrm{a}} = V_{\mathrm{a}}^{(\gamma)}/\Lambda_{\mathrm{a}}^{(\gamma)}$ (the superscript (γ) can be omitted when there is no risk of confusion). Then $\mathcal{J}_1$ and $\mathcal{J}_{\mathrm{a}}$ are complementary in the sense that $\mathcal{J}_1 + \mathcal{J}_{\mathrm{a}} = \mathcal{J}$ and $\mathcal{J}_1 \cap \mathcal{J}_{\mathrm{a}}$ is finite. As the action of γ restricted to $\mathcal{J}_{\mathrm{a}}$ is elliptic, we can determine the number of fixed points $\#\mathcal{J}_{\mathrm{a}}^\gamma$ for this action as we did before, for example by computing the determinant of $(\gamma - \mathrm{id}_{\mathbb{C}^3})|_{V_{\mathrm{a}}}$. Then we have for the group of connected components of $\mathcal{J}^\gamma$:

$$\mathcal{J}^\gamma/\mathcal{J}_1 \simeq \mathcal{J}_{\mathrm{a}}^\gamma/(\mathcal{J}_{\mathrm{a}}^\gamma \cap \mathcal{J}_1).$$

Hence to know the number of components, we have to determine $\#(\mathcal{J}_{\mathrm{a}}^\gamma \cap \mathcal{J}_1)$, that is, the number of points of $\mathcal{J}_{\mathrm{a}}^\gamma$ whose representatives in $\mathbb{C}^3$ are zero modulo $V_1 + \Lambda$.

For reflections the eigenspace V_1 is a plane, in which case we call it the mirror. For the remaining non-trivial parabolic elements γ, V_1 is 1-dimensional, and we call it the axis of γ. For both reflections and antireflections, we have $V_{\mathrm{a}} = V_{-1}$.

Proposition 3.2 *G has the following cyclic subgroups of order* > 1 *generated by parabolic elements:*

(i) One conjugacy class of 21 *subgroups of order* 2 *generated by reflections; the fixed locus in* $\mathcal{J}$ *of each of them is the abelian surface, the image of the mirror of the reflection.*

(ii) One conjugacy class of 21 *subgroups of order* 2 *generated by antireflections; the fixed locus in* $\mathcal{J}$ *of each of them is the union of* 4 *translates of the elliptic curve* $\mathcal{J}_1$ *in* $\mathcal{J}$*, the image of the axis of the antireflection.*

(iii) One conjugacy class of 28 *subgroups of order* 3*; the fixed locus in* $\mathcal{J}$ *of each of them is the elliptic curve* $\mathcal{J}_1$*, the image of the axis of the generator.*

(iv) One conjugacy class of 21 *cyclic subgroups of order* 4*; the fixed locus in* $\mathcal{J}$ *of each of them is the elliptic curve* $\mathcal{J}_1$*, the image of the axis of the generator.*

Proof (i) As the reflections form one conjugacy class, it suffices to compute the fixed locus just for one of them; choose $r_2 : (z_1, z_2, z_3) \mapsto (z_1, z_2, -z_3)$. For $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, the point $z + \Lambda \in \mathcal{J} = \mathbb{C}^3/\Lambda$ is fixed under r_2 if and only if $r_2(z) - z = -z_3(0, 0, 2) \in \Lambda$, which is equivalent to $z_3 \in \mathcal{O}$. Then there exists $v \in V_1 = \ker(r_2 - \mathrm{id}_{\mathbb{C}^3})$ such that $z = v + z_3(\bar{\alpha}, 1, 1)$, hence $z \equiv v \mod \Lambda$ and thus z represents the point $v + \Lambda$ of the abelian surface $\mathcal{J}_1 = V_1/(V_1 \cap \Lambda)$, the image of the mirror V_1 in $\mathcal{J}$. We see that the restricted action on $\mathcal{J}_{\mathrm{a}} = \mathcal{J}_{-1}$ is by multiplication by -1, so the fixed locus $\mathcal{J}_{\mathrm{a}}^{r_2}$ consists of 4 points, images of the half-periods of Λ_{-1}, but all the 4 fixed points are contained in $\mathcal{J}_1$, so $\mathcal{J}_{\mathrm{a}}^{r_2}/(\mathcal{J}_{\mathrm{a}}^{r_2} \cap \mathcal{J}_1)$ is trivial.

(ii) Compute the fixed locus of $\rho_2 = -r_2$. Here V_1 is the z_3-axis. For $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, the point $z + \Lambda \in \mathcal{J} = \mathbb{C}^3/\Lambda$ is fixed under ρ_2 if and only if $\rho_2(z) - z = (-2z_1, -2z_2, 0) \in \Lambda$, which is equivalent to

$$(z_1, z_2, 0) \in \tfrac{1}{2}\Lambda_{\mathrm{a}}, \text{ where } \Lambda_{\mathrm{a}} := \Lambda \cap \{z_3 = 0\} = \mathcal{O}(2, 0, 0) + \mathcal{O}(\alpha, \alpha, 0).$$

The latter condition means that $(z_1, z_2, 0)$, modulo Λ_{a}, is one of the 16 linear combinations of the vectors

$$(1, 0, 0),\ (\bar{\alpha}, 0, 0), (\tfrac{\alpha}{2}, \tfrac{\alpha}{2}, 0), (1, 1, 0)$$

with coefficients from $\{0, 1\}$. As $(1, 1, 0) \equiv (1, 1, \bar{\alpha}) \mod V_1$ and $(1, 1, 0) + (1, 0, 0) + (\bar{\alpha}, 0, 0) \equiv (\bar{\alpha}, 1, 1) + (2, 0, 0) \mod V_1$, we see that only four of the 16 linear combinations are distinct modulo $V_1 + \Lambda$, which implies the conclusion.

(iii) We will determine the fixed locus of the order-3 element $c^4 = -c$, where c is the order-6 element from Proposition 3.1 (iii). For $z \in \mathbb{C}^3$ the property of being a fixed point of the order-3 element $-c$ modulo Λ can be given the following characterization:

$$(z_3 - z_1, z_1 - z_2, z_2 - z_3) \in \Lambda_{\mathrm{a}} = \Lambda \cap \{z_1 + z_2 + z_3 = 0\} = \mathcal{O}(\alpha, 0, -\alpha) + \mathcal{O}(0, \alpha, -\alpha).$$

Looking at the induced action on the abelian surface $\mathcal{J}_{\mathrm{a}}$, we easily find 9 fixed points, whose representatives modulo Λ_{a} can be given by

$$\theta_{ij} = \tfrac{i}{3}(-\alpha, -\alpha, 2\alpha) + \tfrac{j}{3}(-2, -2, 4),\ i, j = 0, 1, 2$$

The existence of exactly nine fixed points for the induced action on $\mathcal{J}_{\mathrm{a}}$ can be confirmed by the calculation of the determinant of $(-c - \mathrm{id})|_{V_{\mathrm{a}}}$. Now we easily see that the θ_{ij} are 0 modulo $V_1 + \Lambda$, for example, $\theta_{1,0} = \frac{1}{3}(-\alpha, -\alpha, 2\alpha) = -(\alpha, \alpha, 0) + \frac{2}{3}(\alpha, \alpha, \alpha)$, where $-(\alpha, \alpha, 0) \in \Lambda$ and $\frac{2}{3}(\alpha, \alpha, \alpha) \in V_1$. Hence the images of $\theta_{ij} + V_1$ in $\mathcal{J}$ are one and the same elliptic curve passing through zero.

(iv) We will determine the fixed locus of the order-4 parabolic element $h_4 = -h_4'$, where h_4' was defined in Proposition 3.1 (ii): $h_4 : (z_1, z_2, z_3) \mapsto (z_1, -z_3, z_2)$. Here V_1 is the z_1-axis. A point $z \in \mathbb{C}^3$ is fixed under h_4 modulo Λ if and only if

$$(0, z_2 + z_3, z_3 - z_2) \in \Lambda_{\mathrm{a}} = \Lambda \cap \{z_1 = 0\} = \mathcal{O}(0, 2, 0) + \mathcal{O}(0, \alpha, \alpha).$$

There are 4 solutions modulo Λ_{a}: 0, $(0, 1, 1)$, $(0, 0, \alpha)$, and $(0, 1, 1 + \alpha)$. All of them are in $\Lambda + V_1$, for example, $(0, 1, 1) = (\bar{\alpha}, 1, 1) + (-\bar{\alpha}, 0, 0)$ with $(\bar{\alpha}, 1, 1) \in \Lambda$, $(-\bar{\alpha}, 0, 0) \in V_1$. Hence the fixed locus of h_4 is connected. □

4 Orbits with Elliptic Stabilizers

We want to enumerate all the possible stabilizers $G_u = \mathrm{Stab}_G(u)$ and $H_u = \mathrm{Stab}_H(u)$ of points $u \in \mathcal{J}$. In this section we will consider the points u fixed by at least one elliptic element of G. Such points and their stabilizers will be called *elliptic*. In the case when the stabilizer G_u is non-trivial but contains no elliptic elements, we will call u and its stabilizer *parabolic*. The parabolic points will be studied in the next section.

The knowledge of the stabilizer provides the length of the orbit of u, which is the index of the stabilizer, and determines the singularities of the quotient varieties at $G \cdot u$ and $H \cdot u$, the orbits of u viewed as points of the respective quotients that are the images of u. The image of u is a nonsingular point of the quotient if and only if the stabilizer is generated by reflections, otherwise it is a singularity, locally analytically equivalent to the linear quotient $\mathbb{C}^3/G_u$, resp. $\mathbb{C}^3/H_u$.

The points of a Zariski open set of $\mathcal{J}$ have trivial stabilizer in G or H; we call this Zariski open set the *free locus* of G, resp. H.

The non-free locus of G is the union of two-dimensional images of mirrors of reflections, of a number of curves and of a number of isolated points. The union of images of mirrors will be called the *discriminant arrangement* in $\mathcal{J}$. By Proposition 3.2 (i), the discriminant arrangement is the union of 21 abelian surfaces passing through zero, which we will also call, by abuse of language, *mirrors* or *mirror abelian surfaces*. A generic point of a mirror abelian surface which is the image of the mirror of a reflection r has minimal stabilizer, equal to $\langle r \rangle$. The stabilizer can jump along some curves, called *special curves*. The special curves that belong to the discriminant arrangement are the intersection curves of two or more mirrors. Such curves are called *special discriminant curves*.

The points of a special curve with stabilizer bigger than that of the generic point of the curve will be called *dissident points* of the special curve. The curve components of the non-free locus will be called *off-discriminant special curves*, and the points of the zero-dimensional irreducible components of the non-free locus will be called *isolated special points*.

We will also distinguish the points u of the non-free locus in $\mathcal{J}$ according to the property whether their stabilizer G_u in G is cyclic or not; we will say that u is a *cyclic point* if G_u is a cyclic subgroup of G. The most special point is $0 \in \mathcal{J}$; it is stabilized by the whole of G and is a smooth point of $X = \mathcal{J}/G$, as G is generated by reflections.

Now we are turning to the locus of elliptic points. It turns out that the only *isolated* special points are the elliptic cyclic points fixed by elements of order 7. They are treated in the next Proposition; we determined six of them in Proposition 3.1 (iv), and all of them belong to the orbits of these six.

We denote by C_d a cyclic group of order d, and by $\frac{1}{d}(\nu_1, \nu_2, \nu_3)$ the (analytic equivalence class of the) cyclic quotient singularity $\mathbb{C}^3/C_d$, where the generator c_d of C_d acts by $c_d : (z_1, z_2, z_3) \mapsto (\epsilon^{\nu_1} z_1, \epsilon^{\nu_2} z_2, \epsilon^{\nu_3} z_3)$, $\epsilon = \exp\left(\frac{2\pi i}{d}\right)$.

Proposition 4.1 *Let T_7 denote the set of 48 non-zero points of $\mathcal{J}$ fixed by elements of order 7.*

(i) Suppose $\eta \in T_7$ is fixed under the action of an element $\sigma \in H_{168}$ of order 7. Then $\mathrm{Stab}_{H_{168}}(\eta) = \langle\sigma\rangle$ is of order 7, so T_7 is the union of two H_{168}-orbits of length 24.

(ii) In the notation of i), the normalizer $N_{H_{168}}(\langle\sigma\rangle) \simeq G_{21}$, where G_{21} is the group of order 21 introduced in (5), and there exists an element τ of order 3 in $N_{H_{168}}(\langle\sigma\rangle)$ such that $\tau(\eta) = 2\eta$, hence η, 2η, 4η belong to one of the two H_{168}-orbits in T_7, while 3η, 5η, 6η belong to the other. As representatives of the two orbits, one can choose η_1 and η_3, where η_i ($i = 1, \dots, 6$) were introduces in Proposition 3.1 (iv), and we denote by the same symbol η_i the fixed points of g_7 on $\mathcal{J}$ represented by the vectors $\eta_i \in \mathbb{C}^3$.

(iii) The images in $Y = \mathcal{J}/H_{168}$ of the 2 H_{168}-orbits in T_7 are 2 isolated cyclic quotient singularities of Y of local analytic type $\frac{1}{7}(1, 2, 4)$.

(iv) The action of -1 permutes the two H_{168}-orbits, hence T_7 is just one G-orbit, whose image in the quotient $X = \mathcal{J}/G$ is an isolated singularity of local analytic type $\frac{1}{7}(1, 2, 4)$.

Proof For any element σ of order 7, $\mathrm{Fix}_{\mathcal{J}}(\sigma)$ is the same as $\mathrm{Fix}_{\mathcal{J}}(\langle\sigma\rangle)$. As there is only one conjugacy class of subgroups of order 7 in H_{168}, we can restrict ourselves to one particular element of order 7, say the element g_7 introduced in (5). So we assume $\sigma = g_7$. As we know that H_{168} has eight 7-Sylow subgroups, the order of the normalizer of $\langle g_7\rangle$ is 21. The formula (5) presents a subgroup of order 21 normalizing $\langle g_7\rangle$, which is G_{21}, hence $N_{H_{168}}(\langle g_7\rangle) = G_{21}$. The order-3 element h_3 from (5) normalizes $\langle g_7\rangle$, hence leaves invariant $\mathrm{Fix}_{\mathcal{J}}(\langle g_7\rangle)$. By a direct calculation we check that h_3 doubles each fixed point of $\langle g_7\rangle$. Indeed, expressing η_1 in coordinates of $\mathbb{C}^3$, we obtain

$$\eta_1 = \begin{pmatrix} \frac{i\sqrt{7}}{7} \\ \frac{7+i\sqrt{7}}{14} \\ 1 - \frac{2i\sqrt{7}}{7} \end{pmatrix}, \quad h_3 = \frac{1}{2}\begin{pmatrix} 1 & -\alpha & 1 \\ -\bar{\alpha} & 0 & -\bar{\alpha} \\ -1 & -\alpha & -1 \end{pmatrix}, \quad h_3(\eta_1) - 2\eta_1 = \begin{pmatrix} 1-\alpha \\ -1-\alpha \\ -3+\alpha \end{pmatrix} \in \Lambda.$$

As $\eta_i = i\eta_1$ for all $i = 1, \dots, 6$, we deduce that the $\langle h_3\rangle$-orbit of η_1 consists of the three points η_1, η_2, η_4. This implies (i) and (ii), and the remaining assertions easily follow. □

We will now compute the stabilizers of the remaining points from fixed loci of elliptic elements. The next proposition uses the notation of Proposition 3.1.

Proposition 4.2 *(i) The locus T_6 of non-zero fixed points of elements of order 6 in G is the union of orbits of the 3 fixed points ω_{ij} $((i,j)\neq(0,0))$ of the order-6 element $c=(z_1,z_2,z_3)\mapsto(-z_3,-z_1,-z_2)$. We have:* $\mathrm{Stab}_{H_{168}}(\omega_{10})\simeq\mathrm{Stab}_{H_{168}}(\omega_{11})\simeq S_4$, $\mathrm{Stab}_{H_{168}}(\omega_{01})=\mathrm{Stab}_{H_{168}}(\omega_{10})\cap\mathrm{Stab}_{H_{168}}(\omega_{11})\simeq S_3$. *The three points are contained in $\mathcal{J}^{\langle -1\rangle}$ and are special on the off-discriminant special curve $\mathcal{J}_1^{(-c)}$, the image of the axis $z_1=z_2=z_3$ of $-c:(z_1,z_2,z_3)\mapsto(z_3,z_1,z_2)$. Moreover, ω_{10} and ω_{11} are quadruple points of the configuration of special curves, as they each belong to and are dissident on three special discriminant curves which are fixed by the order-4 elements in their stabilizers. Say, for ω_{10}, the stabilizer is nothing else but the monomial subgroup (6), the three axes of its 6 order-4 elements are just the coordinate axes of $\mathbb{C}^3$, and the three extra special curves passing through ω_{10} are the images of the coordinate axes. The stabilizers in G are twice bigger,* $\mathrm{Stab}_G(\omega_{ij})=\pm\,\mathrm{Stab}_{H_{168}}(\omega_{ij}):=\{\pm1\}\times\mathrm{Stab}_{H_{168}}(\omega_{ij})$, *and they are generated by reflections, so that the images of ω_{ij} in X are smooth points.*

(ii) The locus T_4' of non-zero fixed points of elements of order 4 with determinant -1 is the union of the orbits of the 15 fixed points β_i $(i\neq0)$ of the order-4 element h_4'. In the notation of β_i we will understand i as a binary multiindex $\iota_0\iota_1\iota_2\iota_3$ varying from 0000 to 1111. The next table lists the stabilizers of β_i (except for $\beta_{0000}=0$), up to isomorphism, and the singularities at the images of the corresponding points β_i in X. We mark by the plus sign the β_i that are fixed by -1; the numbers between brackets in the last line indicate the number of images in X of the points β_i from the current column; D_8, D_8' denote dihedral groups of order 8, the first of which is a subgroup of H_{168}, the second is not; similarly for the pair S_4, S_4'.

$\mathrm{Stab}_{H_{168}}(\beta_i)$	S_4	A_4	D_8	$C_2\times C_2$	C_2
$\mathrm{Stab}_G(\beta_i)$	$\pm S_4$	S_4'	$\pm D_8$	D_8'	C_4
		β_{0001}		β_{1010}	β_{0011}
		β_{0010}		β_{0101}	β_{1011}
	$\beta_{0100}(+)$	β_{0110}		β_{1001}	β_{0111}
β_i	$\beta_{1100}(+)$	β_{1101}	$\beta_{1000}(+)$	β_{1110}	β_{1111}
Image in X	Smooth [2]	Smooth [2]	Smooth [1]	Smooth [2]	$\frac{1}{4}(1,2,3)[1]$

All the G-stabilizers except for C_4 are generated by reflections and the corresponding points β_i are mapped to smooth points of $X=\mathcal{J}/G$. The image in X of the points β_i with stabilizer C_4 is a non-isolated cyclic quotient singularity of analytic type $\mathbb{C}^3/C_4$, where C_4 acts with weights 1, 2, 3.

(iii) The locus T_2 of 63 points fixed by the action of -1 on $\mathcal{J}\setminus\{0\}$ decomposes into the following G-orbits:

- *the two orbits of the points β_{0100}, β_{1100} from (ii) (or of ω_{10}, ω_{11} from (i)) with G-stabilizers $\pm S_4$, of length 7 each;*
- *the orbit of the point β_{1000} with G-stabilizer $\pm D_8$ of length 21;*
- *the orbit of the point ω_{01} from (i) with G-stabiliser $\pm S_3$ of length 28.*

These stabilizers are generated by reflections, so the image of T_2 in X consists of 4 *smooth points.*

Proof (i) The first assertion follows from the fact that the elements of order 6 form one orbit under conjugation by H_{168} (and by G). All the remaining assertions but the last one are proved by a routine verification, which we performed using the computer algebra system Macaulay2 [16]. For the last assertion, remark that the groups S_3, S_4 are generated by their elements of order 2, and all the elements of order 2 in H_{168} are antireflections. Hence the stabilizers of ω_{ij} in H_{168} are generated by antireflections. Passing to the stabilizers of ω_{ij} in G, we extend the stabilizers in H_{168} by adding -1, and this obviously provides groups generated by reflections.

(ii) As in (i), the proof is obtained by a computer-assisted enumeration of the elements of the stabilizers, followed by the inspection of the elements of order 2.

(iii) All the points of T_2 belong to orbits already enumerated in (i), (ii), so (iii) is an obvious consequence of (i), (ii). □

5 Parabolic Orbits and Singularities of $\mathcal{J}/G_{336}$

In the previous section, we enumerated all the *elliptic* special points in $\mathcal{J}$. All of them, except for those belonging to the orbit in the last column of the table in Proposition 4.2 (ii), turn out to be non-cyclic, that is have non-cyclic stabilizer in G. Now we will enumerate the parabolic points.

An obvious way to obtain a curve whose generic point is non-cyclic is to take the intersection of two mirror abelian surfaces fixed by reflections. Recall what happens in the case when the two reflections, say r, r', commute: they generate a subgroup $(\mathbb{Z}/2\mathbb{Z})^2$, their product $\rho = rr'$ is an anti-reflection, and there is a unique cyclic subgroup of order 4 in H containing ρ. This follows from the description of the lattice of subgroups of H in Sect. 1. Thus the curve which is the intersection of the mirrors of two commuting reflections r, r' can be also characterized as the image $\mathcal{J}_1^{(\rho)}$ in $\mathcal{J}$ of the axis of the antireflection $\rho = rr'$, and the full fixed locus $\mathcal{J}^\rho$ of ρ is the union of four translates of the elliptic curve $\mathcal{J}_1^{(\rho)}$ (Proposition 3.2, (ii)).

We will start by enumerating the parabolic points u with cyclic H_u.

Proposition 5.1 *Let $u \in \mathcal{J}$ be a parabolic point, and assume that H_u is cyclic. Then one of the following three cases is realized:*

(a) $H_u = \langle\rho\rangle$ is of order 2. In this case ρ is an anti-reflection and its fixed locus $\mathcal{J}^\rho$ is the disjoint union of 4 translates $\kappa_i + \mathcal{J}_1^{(\rho)}$ ($i = 0, 1, 2, 3$) of the elliptic curve $\mathcal{J}_1^{(\rho)}$. The points κ_i can be choosen in such a way that the following is true: $\kappa_0 = 0$, κ_1, κ_2, $\kappa_3 = \kappa_1 + \kappa_2$ are points of order 2, and u belongs to one of three curves $\kappa_i + \mathcal{J}_1^{(\rho)}$, $i = 1, 2, 3$. For generic $u_i \in \kappa_i + \mathcal{J}_1^{(\rho)}$, $i = 1, 2$, the H-stabilizer $H_{u_i} = \langle\rho\rangle$ is of order 2, while the G-stabilizer $G_{u_i} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is generated by two reflections r_i, r_i' such that $\rho = r_i r_i'$. For generic $u_3 \in \kappa_3 + \mathcal{J}_1^{(\rho)}$,

the H- and G-stabilizers coincide: $G_{u_3} = H_{u_3} = \langle\rho\rangle = G_{u_1} \cap G_{u_2}$. For all the three curves $\kappa_i + \mathcal{J}_1^{(\rho)}$, $i = 1, 2, 3$, the subgroup of H leaving invariant each of them is isomorphic to D_8.

(b) $u \in \mathcal{J}^{c_3}$ for some element $c_3 \in H$ of order 3, $H_u = \langle c_3\rangle$, and G_u is of type S_3' (a subgroup, isomorphic to S_3 and not contained in H). The subgroup of H (resp. G) leaving invariant $\mathcal{J}^{c_3}$ is of type S_3 (resp. $\pm S_3$).

(c) $u \in \mathcal{J}^{c_4}$ for some element $c_4 \in H$ of order 4, $H_u = \langle c_4\rangle$, and G_u is of type D_8'. The subgroup of H (resp. G) leaving invariant $\mathcal{J}^{c_4}$ is D_8 (resp. $\pm D_8$), where we denote, as before, by D_8 (resp. D_8') a dihedral subgroup of order 8 embedded in H (resp. in G in such a way, that the image contains four reflections).

In the cases (b), (c), G_u is generated by reflections and the image of u in X is nonsingular. In the case (a), the subgroups G_{u_1}, G_{u_2} are generated by reflections and G_{u_3} is not, where u_i denotes a generic point of the curve $\kappa_i + \mathcal{J}_1^{(\rho)}$, so the images of u_1, u_2 in X are nonsingular and the image of u_3 is a non-isolated singularity of type $\frac{1}{2}(1, 1, 0)$.

Proof The cyclic subgroups of H are all conjugate to those generated by ρ_1, h_3, h_4 or g_7. Only ρ_1, h_3, h_4 are parabolic. We have $|G_u| = 2|H_u|$ or $G_u = H_u$. In the case $|H_u| = 2$, we have $H_u = \langle\rho\rangle$ for an element ρ of order 2; all the 21 elements of order 2 in H are anti-reflections, conjugate to each other, so we may assume $\rho = \rho_2$. It is impossible that $u \in \mathcal{J}_1^{(\rho)}$, because every element of order 2 in H is the square of an element of order 4 fixing the same axis, and hence u would then be fixed by a subgroup of order 4 in H at least. Hence u belongs to $\mathcal{J}^{\rho}\setminus\mathcal{J}_1^{(\rho)}$, which is the union of the three translates of $\mathcal{J}_1^{(\rho)}$ according to Proposition 3.2 (ii):

$$[(1, 0, 0)] + \mathcal{J}_1^{(\rho)},\ [(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}, 0)] + \mathcal{J}_1^{(\rho)},\ [(1 + \tfrac{\alpha}{2}, \tfrac{\alpha}{2}, 0)] + \mathcal{J}_1^{(\rho)}.$$

We can set $\kappa_1 = [(1, 0, 0)]$, $\kappa_2 = [(\frac{\alpha}{2}, \frac{\alpha}{2}, 0)]$; the assertions about the stabilizers are verified by a direct calculation. This provides the case (a).

If $|H_u| = 3$, then $H_u = \langle c_3\rangle$ for some element c_3 of order 3. Each element of order 3 is a product of two reflections, so $G_u \supset \langle r, c_3\rangle \simeq S_3$, where r is one of those reflections. Let $K = \langle -r, c_3\rangle$. Obviously, $K \simeq \langle r, c_3\rangle \simeq S_3$. From the table of Sect. 1 describing the lattice of subgroups of H, we see that each 3 is a subgroup of index 2 in a unique S_3, its normalizer. The subgroups S_3 form one orbit in H, so we may choose $K = \langle -r_1, c_3\rangle$, where r_1 is one of our basic reflections and $c_3 = c^4 = -c$ is the same order-3 element as the one used in the proof of Proposition 3.2 (iii). We saw there that the fixed locus $\mathcal{J}^{c_3}$ is the elliptic curve obtained as the image of the diagonal locus of points $(x, x, x) \in \mathbb{C}^3$ in $\mathbb{C}^3/\Lambda$. Now $z = (z_1, z_2, z_3) + \Lambda$ is fixed under $r_1 : (z_1, z_2, z_3) \mapsto (z_1, z_3, z_2)$ if and only if $r_1(z) - z \in \Lambda$, or $(0, z_3 - z_2, z_2 - z_3) \in \Lambda$. Obviously, this condition is automatically satisfied for any z of the form (x, x, x), which implies that $G_u \supset \langle r, c_3\rangle \simeq S_3$. This provides the case (b).

By a similar argument, assuming $|H_u| = 4$, we reduce the proof to the case when $H_u = \langle h_4\rangle$, where h_4 is the element of order 4 from the proof of Proposition 3.2 (iv). The axis of h_4 is the first coordinate axis of $\mathbb{C}^3$, and one easily verifies that $G_u = D_8'$

for generic point u of the form $(z_1, 0, 0) + \Lambda$. For non-generic points of this form the stabilizer may be bigger, but then H_u is non-cyclic, and as we will see in the next proposition, this implies that u is non-parabolic, so all such cases have been treated in the previous section. $\square$

Now we consider the case when H_u is non-cyclic.

Proposition 5.2 *Let $u \in \mathcal{J}$, $u \neq 0$ and assume H_u non-cyclic. Then one of the following cases is realized.*

(d) H_u contains S_3. Then $u \in T_6$, where T_6 is the locus of nonzero points fixed by elements of order 6. This locus, described in Proposition 4.2 (i), is the union of orbits of the three points $\omega_{01}, \omega_{10}, \omega_{11}$ with G-stabilizers $\pm S_4$ or $\pm S_3$.
(e) H_u contains $(\mathbb{Z}/2\mathbb{Z})^2$. Then u belongs to the orbit of one of the 16 fixed points of the elliptic order-4 element h_4' from Proposition 3.1 (ii), and the possible G-stabilizers of u are D_8', $\pm D_8$, S_4' and $\pm S_4$.

In particular, none of these points u is parabolic. Their G-stabilizers are generated by reflections, so their images in X are smooth points.

Proof As H_u is non-cyclic, it contains at least two distinct cyclic subgroups generated by elements from the orbits of ρ_1, h_3, h_4, g_7 or g_7^{-1}. We can disregard the elements of order 7, because for a nonzero fixed point of such an element, its stabilizer is of order 7 and hence is cyclic. So, we have to consider only the cases when H_u contains two cyclic subgroups of orders 2, 3 or 4.

The first case we will consider is when H_u contains subgroups of orders 2 and 3. From the table in Sect. 1 describing the lattice of subgroups of H we see that then H_u is one of the subgroups S_3, A_4, S_4.

Assume that $H_u \supset S_3$. As the subgroups S_3 form one orbit, we can choose $S_3 = \langle -r_1, c_3 \rangle$ as in the proof of the previous proposition. As before, $\mathcal{J}^{c_3}$ is the elliptic curve obtained as the image of the diagonal of $\mathbb{C}^3$, that is the locus of points of the form (x, x, x) modulo Λ, and $z = (z_1, z_2, z_3) + \Lambda$ is fixed under $-r_1 : (z_1, z_2, z_3) \mapsto (-z_1, -z_3, -z_2)$ if and only if $r_1(z) + z \in \Lambda$, or $(2z_1, z_2 + z_3, z_3 + z_2) \in \Lambda$. For a point z of the form (x, x, x) the latter condition is equivalent to $x(2, 2, 2) \in \Lambda$, which gives four points stabilized by S_3:

$$\mathcal{J}^{S_3} = \left\{ \iota_1 \tfrac{(\bar{\alpha}, \bar{\alpha}, \bar{\alpha})}{2} + \iota_2(1, 1, 1) \right\}_{\iota_1, \iota_2 = 0, 1} \pmod{\Lambda}.$$

We now see that the three of these points different from 0 belong to the locus T_6 from Proposition 4.2 (i), which ends the proof for the case when $H_u \supset S_3$.

We will not consider separately the cases $H_u \supset A_4$ or S_4, because in these cases H_u contains a subgroup $\simeq (\mathbb{Z}/2\mathbb{Z})^2$. So we will just consider one case when $H_u \supset (\mathbb{Z}/2\mathbb{Z})^2$.

There are two orbits of subgroups 2^2 in H, and respectively two orbits of their normalizers S_4. For each “positive” root $e \in \mathcal{R}_0$, there are two pairs (e', e''), (f', f'') of orthogonal roots in $\mathcal{R}_0$, such that $e' \perp e''$, $f' \perp f''$, roots from different pairs being non-orthogonal. Say, if $e = (2, 0, 0)$, then, for an appropriate choice of $\mathcal{R}_0$, the two orthogonal pairs are $(e', e'') = ((0, 2, 0), (0, 0, 2))$ and $(f', f'') = ((0, \alpha, \alpha), (0, \alpha, -\alpha))$. We can choose for representatives of the two orbits of 2^2 in H the subgroups $H_{2^2} = \langle \rho_{e'}, \rho_{e''} \rangle$ and $H'_{2^2} = \langle \rho_{f'}, \rho_{f''} \rangle$, and $H_{2^2} \cap H'_{2^2} = \langle \rho_e \rangle$. We have:

$$(z_1, z_2, z_3) + \Lambda \in \mathcal{J}^{H_{2^2}} \iff (2z_1, 2z_2, 0) \equiv (0, 2z_2, 2z_3) \equiv 0 \mod \Lambda$$

$$\iff z = (z_1, z_2, z_3) \in \bar{\Lambda} = \mathbb{Z}\frac{(\alpha, \alpha, \alpha)}{2} + \mathcal{O}^3.$$

As $[\bar{\Lambda} : \Lambda] = 16$, $\#\mathcal{J}^{H_{2^2}} = 16$. Similarly one verifies that $\#\mathcal{J}^{H'_{2^2}} = 16$. Moreover, by inspecting the G-stabilizers of the 16 fixed points, we observe that each of them contains at least one elliptic element of order 4. All such elements are conjugate to h'_4, thus the possible stabilizers G_u in (d) are those appearing in Proposition 3.1 (ii), except for $D_u \simeq C_4$, for which $H_u \simeq C_2$ is too small.

It remains to consider the case when H_u contains two cyclic subgroups, one of which has order 4. Denoting by c_4 a generator of the latter subgroup of order 4, we see that H_u contains two distinct cyclic subgroups, one of which is of order 2, generated by c_4^2, and this brings us to one of the cases treated above. □

Now we are ready to enumerate the singularities of the quotient variety X. We say that a variety (as always in this paper, over $\mathbb{C}$) is strongly simply connected if its smooth locus is connected and simply connected.

Theorem 3.1 *The quotient $X = \mathcal{J}/G$ is a normal strongly simply connected variety whose singular locus is the union of two irreducible components, $\mathbb{P}^1 = \ell$ and an isolated point p. Denoting $\pi : \mathcal{J} \to X$ the natural map, we have $p = \pi(T_7)$ and $\ell = \pi(\kappa_3 + \mathcal{J}_1^{(\rho)})$, where T_7 is the orbit of fixed points of elements of order 7, described in Proposition 4.1, ρ is an anti-reflection and $\kappa_3 + \mathcal{J}_1^{(\rho)}$ is the elliptic curve in the fixed locus of ρ defined in Proposition 5.1 (a).*

The singularity at p is of analytic type $\frac{1}{7}(1, 2, 4)$. At all but one points of ℓ, the singularity of X is of type $\frac{1}{2}(1, 0, 1)$, that is $\mathbb{C} \times A_1$, the Cartesian product of $\mathbb{C}$ with a surface du Val singularity of type A_1. The unique point q of ℓ where the type of singularity changes is the image of the orbit of one of the points $\beta_{\iota_0\iota_1\iota_2\iota_3}$ from the last column of the table in Proposition 4.2 (ii), say β_{0011}. The type of singularity at q is $\frac{1}{4}(1, 2, 3)$.

Proof The strong simply-connectedness follows from [22, Theorem 3.2.1]; see also [20] or [4, Proposition 0.1]. In fact, for the quotients of $\mathbb{C}^n$ by complex crystallographic groups, the property of the group to be generated by affine complex reflections is *equivalent* to the strong simply-connectedness of the quotient.

Singularities of X may only occur in the image of the points of $\mathcal{J}$ whose G-stabilizers are not generated by reflections. We made a complete inventory of possible G-stabilizers. The orbits of points whose G-stabilizers are not generated by reflections are those mentioned in the statement of the theorem. □

We note that the weighted projective space $\mathbb{P}(1, 2, 4, 7)$ is also strongly simply connected and has the same singularities as X, which provides some evidence towards the conjecture stated in the introduction.

References

1. Bauer, M., Itzykson, C.: A case study in finite groups: $\mathbf{PSL}_2(\mathbb{F}_7)$. Int. J. Mod. Phys. **A5**, 3125–3153 (1990)
2. Bernšteĭn, I.N., Švarcman, O.V.: Chevalley's theorem for complex crystallographic Coxeter groups. (Russian) Funktsional. Anal. I Prilozhen. **12**, 79–80 (1978)
3. Bernstein, J., Schwarzman, O.: Complex crystallographic coxeter groups and affine root systems. J. Nonlinear Math. Phys. **13**, 163–182 (2006)
4. Bernstein, J., Schwarzman, O.: Chevalley's theorem for the complex crystallographic groups. J. Nonlinear Math. Phys. **13**, 323–351 (2006)
5. Cohen, A.M.: Finite complex reflection groups. Ann. Sci. Ecole Norm. Sup. **9**(4), 379–436 (1976). Erratum, **11**, 613 (1978)
6. Connor, T., Leemans, D.: An atlas of subgroup lattices of finite almost simple groups. Ars Math. Contemp. **8**, 259–266 (2015)
7. Deraux, M.: Non-arithmetic lattices from a configuration of elliptic curves in an Abelian surface. Comment. Math. Helv. **93**, 533–554 (2018)
8. Levy, S.: The eightfold way. The beauty of Klein's quartic curve. Mathematical Sciences Research Institute Publications, vol. 35, x+331 pp. Cambridge University Press, Cambridge (1999)
9. Friedman, R., Morgan, J.W., Witten, E.: Principal G-bundles over elliptic curves. Math. Res. Lett. **5**, 97–118 (1998)
10. Goryunov, V., Man, S. H.: The complex crystallographic groups and symmetries of J_{10}, Singularity theory and its applications. In: Advanced Studies in Pure Mathematics, vol. 43, pp. 55–72. Mathematics Society, Japan, Tokyo (2006)
11. Kac, V.G., Peterson, D.H.: Infinite-dimensional Lie algebras, theta functions and modular forms. Adv. Math. D **53**, 125–264 (1984)
12. Klein, F.: Über die transformationen siebenter ordnung der elliptischen Funktionen. Math. Ann. **14**, 428–471: Œuvres. Tome **III**, 90–136 D (1879)
13. Koziarz, V., Rito, C., Roulleau, X.: The bolza curve and some orbifold ball quotient surfaces. Mich. Math. J. **70**, 423–448 (2021)
14. Kaneko, J., Tokunaga, S., Yoshida, M.: Complex crystallographic groups II. J. Math. Soc. Jpn. **34**(1982), 595–605 (1982)
15. Looijenga, E.: Root systems and elliptic curves. Invent. Math. **38**, 17–32 (1976/77)
16. Grayson, D., Stillman, M.: Macaulay2, a software system for research in algebraic geometry. www.math.uiuc.edu/Macaulay2/
17. Mazur, B.: Arithmetic on curves. Bull. Am. Math. Soc. (N.S.) **14**, 207–259 (1986)
18. Popov, V.L.: Discrete complex reflection groups. Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, 15. Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 89 pp (1982)
19. Švarcman, O.V.: A Chevalley theorem for complex crystallographic groups generated by reflections in the affine space $\mathbb{C}^2$. Uspekhi Mat. Nauk **34**, 249–250 (1979)
20. Shvartsman, O.V.: Cocycles of complex reflection groups and the strong simple-connectedness of quotient spaces. In: Problems in Group Theory and Homological Algebra, Yaroslav, pp. 32–39. Gos. University, Yaroslavl (1991)
21. Shephard, G.C., Todd, J.A.: Finite unitary reflection groups. Canad. J. Math. **6**, 274–304 (1954)
22. Tokunaga, S., Yoshida, M.: Complex crystallographic groups I. J. Math. Soc. Jpn. **34**(1982), 581–593 (1982)
23. Wirthmüller, K.: Root systems and Jacobi forms. Compos. Math. **82**, 293–354 (1992)

Some Observations on the Dimension of Fano K-Moduli

Jesus Martinez-Garcia and Cristiano Spotti

Abstract In this short note we show the unboundedness of the dimension of the K-moduli space of n-dimensional Fano varieties, and that the *dimension of the stack* can also be unbounded while, simultaneously, the dimension of the corresponding coarse space remains bounded.

Keywords K-stability · K-moduli · Fano varieties

1 Main Statement

Moduli spaces of K-stable Fano varieties have been intensively investigated in the last decade, both from a general theory point of view as well as via the study of explicit examples. There are two objects of interest, the moduli stack of K-semistable Fano varieties $\mathcal{M}_K$ and its good moduli space, in the sense of Alper, M_K which parametrises K-polystable varieties. We refer the reader to [16] for a survey in the construction of these objects in the case of smoothable varieties (cf. [17]) and to [11] for the most recent construction in the general case. In this note, we observe the following:

Theorem 1.1 *For each $n > 1$ the dimension (as a variety) of the K-moduli spaces M_K of n-dimensional Fano varieties is unbounded. Moreover, the* dimension of the K-moduli stack $\mathcal{M}_K$ *can be arbitrarily big, while the dimension of its coarse variety M_K remains bounded.*

Here, for *dimension of the K-moduli stack $\mathcal{M}_K$* at a given K-polystable point $[X]$ we mean the difference between the dimension of the versal space of (K-semistable)

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$\mathbb{Q}$-Gorenstein deformations of the variety X minus the dimension of its reductive automorphism group, cf. [15, Sect. 0AFL].

It is well-known that smooth Fano manifolds, and more generally ε-log terminal Fano varieties (where $\varepsilon > 0$ is fixed), form a bounded family in a fixed dimension [4]. Thus to construct such examples we need to consider non-smoothable varieties whose Kawamata log terminal (klt) singularities get worse and worse.

Our main theorem is a quick consequence of these two dimensional easy examples.

Proposition 1.2 *Consider the following two families of K-polystable normal surfaces:*

1. $X_l := (\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_l$, *for* $l \geq 2$, *where the generator* 1 *of the cyclic group* $\mathbb{Z}_l$ *acts by*
$$([z_0 : z_1], [w_0 : w_1]) \mapsto ([\zeta z_0 : z_1], [\zeta^{-1} w_0 : w_1]),$$
where ζ *is a primitive* l*-root of unity.*
2. $Y_l := \mathbb{P}^2/\mathbb{Z}_l$ *for* $l \geq 3$, l *odd, where the generator* 1 *of the cyclic group* $\mathbb{Z}_l$ *acts by*
$$[z_0 : z_1 : z_2] \mapsto [\zeta z_0 : \zeta^{-1} z_1 : z_2],$$
where ζ *is a primitive* l*-root of unity.*

Then

1. *the dimension as a variety of the K-moduli space* M_K *at* $[X_l]$ *is equal to* $2l - 3$ *if* $l \neq 2, 4$, *and equal to* 2 *(resp.* 6*) for* $l = 2$ *(resp.* $l = 4$*).*
2. *the dimension of the K-moduli stack* $\mathcal{M}_K$ *at* $[Y_l]$ *is equal to* $l - 3$ *for* $l \neq 3, 9$ *and equal to* 4 *(resp.* 8*) for* $l = 3$ *(resp.* 9*). However,* $[Y_l]$ *is an* isolated *K-polystable point for* $l \neq 3, 9$.

The surfaces X_l with $l = 2, 4$ and Y_l with $l = 3, 9$ are actually $\mathbb{Q}$-Gorenstein smoothable and they appear in the boundary of K-moduli of smooth del Pezzo surfaces of degree 4, 2, 3 and 1 respectively [13, (Example 5.7, (4.3), Example 3.10)].

Note that if we had considered Y_l for l even, we would have a $\mathbb{Z}_2$ subgroup fixing the line $z_2 = 0$ (thus the more natural way to think about the quotient is as a pair (X, D), considering the line at infinity with weight $\frac{2}{l}$). This pair will be only *log*-K-polystable, not X.

Proof (*Proof of Theorem* 1.1) It simply follows by taking $\tilde{X}_l = X_l \times \mathbb{P}^{n-2}$. Of course, being the product of two K-polystable varieties, such n-dimensional varieties are still K-polystable [18] and hence the dimension of the K-moduli spaces at $[\tilde{X}_l]$ tends to infinity with l. Similarly, one may take $\tilde{Y}_l = Y_l \times \mathbb{P}^{n-2}$ and have that the moduli stack has arbitrary dimension while $[\tilde{Y}_l]$ is still an isolated K-polystable point. Note, moreover, that Y_l (or $\tilde{Y}_l$) for $l \neq 3, 9$ actually give examples where the K-moduli reduces to a point while there are many non isomorphic strictly K-semistable Fano varieties around Y_l (we are unaware if a similar phenomenon can occur for smooth Fano manifolds too; note that Y_l has klt singularities).

These are toric examples, thus they suggest the following problem:

Problem 1.3 Study in detail the local theory of K-moduli of toric del Pezzo surfaces.

We expect that such investigations are interesting and important when studying moduli spaces of non-necessarily $\mathbb{Q}$-Gorenstein smoothable del Pezzo surfaces.

The proof of the above Proposition is based on the local study of K-stability for $\mathbb{Q}$-Gorenstein deformations of the surfaces, which is possible even in this non-smoothable setting thanks to the recent works [5, 6]. This type of computations have been performed for the $\mathbb{Q}$-Gorenstein smoothable cases of the above examples in [13]. A similar strategy to show interesting behaviour of K-moduli spaces near toric varieties has also been considered in [8] to show that the moduli can be reducible and non-reduced.

After writing up a first draft of this problem in late December 2020, we found out that the first example in Proposition 1.2 was considered a few weeks before in [10], when studying the K-stability of hypersurfaces in $\mathbb{P}(1, 1, a, a)$. We would like to thank A. Petracci for having a look at an early draft of our manuscript and giving us very useful comments which improved our manuscript.

2 Proof of Proposition 1.2

Proposition 1.2 is a consequence of the next few lemmas.

Lemma 2.1 *For the surface X_l above we have that $Sing(X_l) = \{2A_{l-1}, 2\frac{1}{l}(1, 1)\}$, and the connected component to the identity is $Aut_0(X_l) = (\mathbb{C}^*)^2$. Similarly for Y_l we have that $Sing(Y_l) = \{A_{l-1}, 2\frac{1}{l}(1, 2)\}$ and $Aut_0(Y_l) = (\mathbb{C}^*)^2$.*

Proof Let's consider the Y_l case (X_l is completely analogous and we omit it). The singularities of Y_l correspond to points on $\mathbb{P}^2$ where $\mathbb{Z}_l$ acts with non-trivial stabilizer. Near $[0 : 0 : 1]$ the action has weight $(1, -1)$ resulting in a A_{l-1} canonical singularity. Similarly near the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ the action as weight $(1, 2)$ resulting in $\frac{1}{l}(1, 2)$ quotient singularities The statement about the automorphism follows by noting that $\mathrm{Aut}_0(Y_l) \cong \mathrm{Aut}_0(\mathbb{P}^2; S) \cong (\mathbb{C}^*)^2$, where $S = \{[0 : 0 : 1], [1 : 0 : 0], [0 : 1 : 0]\}$ and $\mathrm{Aut}_0(\mathbb{P}^2; S)$ is the connected component of the automorphism group containing the identity and fixing the subset S. For a similar computation see [12, Lemma 3.1].

Remark 2.2 Note that there is no $\epsilon > 0$ such that the non-Du Val singularities of the set of varieties $\{X_l\}_{l\geqslant 2}$ and $\{Y_l\}_{l\geqslant 3,\, l\,\mathrm{odd}}$, are ϵ-log terminal. Indeed, each of the two singular points $\frac{1}{l}(1, 1)$ in X_l is locally analytically isomorphic to the affine cone over the rational normal curve $C_l \subset \mathbb{P}^l$ and its resolution has exceptional locus $E \cong \mathbb{P}^1$ with $E^2 = -l$. It follows that their log discrepancies equal to $\frac{2}{l} - 1 \to -1$ as $l \to \infty$, and moreover $-K_{X_l}$ is $\mathbb{Q}$-Cartier (with Cartier index going to infinity) with $(-K_{X_l})^2 = \frac{8}{l} \to 0$ as $l \to \infty$, and similarly for Y_l.

Lemma 2.3 *X_l and Y_l are K-polystable Fano variety whose space of $\mathbb{Q}$-Gorenstein deformations are given by*

1. $\mathrm{qDef}(X_l) \cong \mathrm{qDef}(A_{l-1}) \oplus \mathrm{qDef}(A_{l-1}) \cong \mathbb{C}^{2(l-1)}$ *for* $l \neq 2, 4$.
2. $\mathrm{qDef}(Y_l) \cong \mathrm{qDef}(A_{l-1}) \cong \mathbb{C}^{(l-1)}$ *for* $l \neq 3, 9$.

Proof (*Proof of Lemma* 2.3) Y_l (and X_l) is a K-polystable del Pezzo surface as $\mathbb{Z}_l$ acts by isometries with respect to the Fubini-Study metric in $\mathbb{P}^2$ (with respect to the product in $\mathbb{P}^1 \times \mathbb{P}^1$ of the product of the Fubini-Study metrics in $\mathbb{P}^1$, respectively). Hence, both X_l and Y_l inherit an (orbifold) Kähler-Einstein metric and consequently they are K-polystable by [3].

By [1, Lemma 6], it follows that there are no local-to-global obstructions to $\mathbb{Q}$-Gorenstein deformations on del Pezzo surfaces. Since X_l is toric it does not admit *equisingular deformations* (i.e. non-trivial deformations to a non-isomorphic projective variety with the same singularities), e.g., [14, Lemma 4.4]. Hence all $\mathbb{Q}$-Gorenstein deformations must come from local $\mathbb{Q}$-Gorenstein deformations of the singularities. Thus

$$\mathrm{qDef}(Y_l) = \prod_{p \in \mathrm{Sing}(Y_l)} \mathrm{qDef}(p) \tag{1}$$

and similar for X_l.

Note that any deformation of A_{l-1} is $\mathbb{Q}$-Gorenstein and given by the versal family $xy = z^l + a_{l-2}z^{l-2} + \cdots + a_0$. Hence the vector $(a_0, a_1, \ldots, a_{l-2})$ defines a point in $\mathrm{qDef}(A_{l-1})$ and $\mathrm{qDef}(A_{l-1}) \cong \mathbb{C}^{l-1}$. The proof follows from Lemma 2.1, once we show that $\mathrm{qDef}(p) = \{0\}$ for p non-Du Val. We will do this for Y_l, since the case of X_l is very similar.

We claim the two $\frac{1}{l}(1, 2)$ singularities of Y_l are $\mathbb{Q}$-Gorenstein rigid (i.e. they do not admit $\mathbb{Q}$-Gorenstein deformations) if $l \neq 3, 9$, and $\mathbb{Q}$-Gorenstein smoothable otherwise. Let $w = \mathrm{hcf}(l, 3)$, $r > 0$ such that $l = wr$, $m \geqslant 0$ and $0 \leqslant w_0 < r$ such that $w = mr + w_0$. It is well known (see e.g. [1]) that a quotient singularity $\frac{1}{l}(1, 2)$ is $\mathbb{Q}$-Gorenstein rigid if and only if $m = 0$, or equivalently if $w = w_0$. Moreover, $\frac{1}{l}(1, 2)$ is $\mathbb{Q}$-Gorenstein smoothable (often known as a *T-singularity*) if and only if $w_0 = 0$ and a *primitive T-singularity* if in addition $m = 1$.

The number $w = \mathrm{hcf}(l, 3)$ can only be 1 or 3. If $w = 1$, then $l = wr = r$ and $1 = w = mr + w_0$ implies that $m = 0$ so $\frac{1}{l}(1, 2)$ is $\mathbb{Q}$-Gorenstein rigid. If $w = 3$ then $l = 3k$ for some $k \in \mathbb{N}$ but in fact, that means that $l = 3k = wr = 3r$, so $l = 3r$. If $r = 1$ then $m = 1$ and $w_0 = 0$ so $\frac{1}{3}(1, 2)$ is a primitive T-singularity. The case $r = 2$ is excluded, otherwise l would be even. If $r = 3$, then $m = 1$ and $w_0 = 0$ and $\frac{1}{9}(1, 2)$ is $\mathbb{Q}$-Gorenstein smoothable. If $r \geqslant 4$, (r odd) then $m = 0$ and $\frac{1}{3r}(1, 2)$ is $\mathbb{Q}$-Gorenstein rigid. Hence, whenever $l \neq 3, 9$ we have $\mathrm{qDef}(\frac{1}{l}(1, 2)) = \{0\}$.

For X_l similar computations show that the singularities $\frac{1}{l}(1, 1)$ are $\mathbb{Q}$-Gorenstein rigid for $l \neq 2, 4$.

Remark 2.4 For $l = 2$, X_l has four A_1 singularities giving a four dimensional versal space of deformation. For $l = 4$, the deformation space has (beside the deformations coming from the two A_3 singularities) $\mathbb{Q}$-Gorenstein deformations coming from the one dimensional family of $\mathbb{Q}$-Gorenstein smoothings of the $\frac{1}{4}(1,1)$ singularities. For $l=3$, Y_l is just the unique cubic surface with $3A_2$-singularities, given by $xyz=t^3$ (and the only strictly K-polystable surface in the K-moduli of del Pezzo surfaces of degree 3). The case $l=9$ was studied in [13, Example 3.10] and it appears in the boundary of the K-moduli compactification of smooth del Pezzo surfaces of degree 1.

Lemma 2.5 *The natural action of* $G = Aut_0(X_l) \cong (\mathbb{C}^*)^2$ *(*$G'{=}Aut_0(Y_l) \cong (\mathbb{C}^*)^2$*) on* $\mathrm{qDef}(X_l)$ *(respectively* $\mathrm{qDef}(Y_l)$*) for* $l \neq 2,4$ *(resp.* $l \neq 3,9$*) is not effective. Moreover:*

1. *The action on* $\mathrm{qDef}(X_l) \cong \mathbb{C}^{2(l-1)}$ *of* $G/\cap_x (G_x) \cong \mathbb{C}^*$ *with* $t=\lambda_1\lambda_2 \in G/\cap_x (G_x)$*, is given by*

$$(a_0, a_1, \ldots, a_{l-2}, a_0', \ldots, a_{l-2}') \mapsto (t^l a_0, t^{l-1}a_1, \ldots, t^2 a_{l-2}, t^{-l}a_0', \ldots, t^{-2}a_{l-2}');$$

2. *The action on* $\mathrm{qDef}(Y_l) \cong \mathbb{C}^{l-1}$ *of* $G'/\cap_x (G_x') \cong \mathbb{C}^*$ *with* $t=\lambda_1\lambda_2 \in G'/\cap_x (G_x')$*, is given by*

$$(a_0, a_1, \ldots, a_{l-2}) \mapsto (t^l a_0, t^{l-1}a_1, \ldots, t^2 a_{l-2}).$$

Proof Let us start with Y_l. In local coordinates near the A_{l-1}-point $[0:0:1]$ we can take coordinates on $Aut_0(Y_l) \cong (\mathbb{C}^*)^2$-action such that the action is just given by

$$(u, v) \mapsto (\lambda_1^{-1}u, \lambda_2^{-1}v).$$

Taking invariants for the $\mathbb{Z}_l$-action $x=u^l$, $y=v^l$ and $z=uv$, we get the induced action on the A_{l-1}-quotient singularity $xy=z^l$ given by $(\lambda_1^{-l}x, \lambda_2^{-l}y, (\lambda_1\lambda_2)^{-1}z)$. Considering then the natural action induced on the versal deformation family of the singularity $xy=z^l+a_{l-2}z^{l-2}+\cdots+a_0$, we get that

$$(a_0, a_1, \ldots, a_{l-2}) \mapsto ((\lambda_1\lambda_2)^l a_0, (\lambda_1\lambda_2)^{l-1}a_1, \ldots, (\lambda_1\lambda_2)^2 a_{l-2}).$$

In particular note that the action is non effective since the action of the subtorus $(s, s^{-1}) \subseteq (\mathbb{C}^*)^2$ is clearly trivial. Finally, putting $t=\lambda_1\lambda_2$ we obtain our statement for Y_l.

The statement for X_l is completely analogous, but (crucially) noticing that if we take coordinates on $Aut_0(X_l)$ to be such that near the point $([0:1],[0:1])$ the action is again by given by $(u, v) \mapsto (\lambda_1^{-1}u, \lambda_2^{-1}v)$, then near the point $([1:0],[1:0])$ one get an action with *opposite* weights. From there the statements follows immediately.

Descriptions of the local actions for the smoothable cases of X_l and Y_l can be found in [13]. Also note that since the above action is not effective (with a $\mathbb{C}^*$ as stabilizer) all the small deformations will have a residual $\mathbb{C}^*$-action on them.

Lemma 2.6 *When* $l \neq 2, 4$ *the K-moduli space near* $[X_l]$ *is (étale locally) described by the affine GIT quotient* $\mathbb{C}^{2(l-1)}//\mathbb{C}^*$*, where the* $\mathbb{C}^*$*-action is given as in Lemma 2.5. Similarly for* Y_l *when* $l \geqslant 4$ $l \neq 9$*, the K-moduli space near* $[Y_l]$ *is (étale locally) described by the affine GIT quotient* $\mathbb{C}^{l-1}//\mathbb{C}^*$*,*

Proof Any $\mathbb{Q}$-Gorenstein deformation of X_l and Y_l is still a Fano variety since the canonical $K_{\mathcal{X}}$ of the total space of a deformation $\mathcal{X}$ is $\mathbb{Q}$-Cartier and ampleness is an open condition. Moreover, the deformation is singular, since it is flat and $K_X^2 \notin \mathbb{Z}$ (alternatively, as pointed out by a referee, it is singular because there are rigid singularities). Then the characterization of those varieties in the deformation which are K-polystable follows by the local GIT description of non-necessarily smoothable Fano varieties in [5, Proof of Theorem 4.5], cf. [2, Remark 2.11], where it is shown that K-semistability is an open condition and that K-polystability can be checked locally by considering the action of the automorphisms.

We are now ready to conclude the proof of our Proposition 1.2:

Proof (*Proof of Proposition* 1.2). For Y_l it is clear that all points near zero in $\mathrm{qDef}(Y_l)$ are K-semistable by openness. However, note that all such points are destabilized to zero since

$$\lim_{t \to 0}(t^l a_0, \ldots, t^2 a_{l-2}) = 0.$$

Hence only 0 is GIT polystable, and Y_l is an isolated K-polystable variety. However, by [15, Lemma 98.12.1], the dimension of the stack at the point Y_l is equal to

$$\dim_{Y_l}(\mathcal{M}_K) = \dim \mathrm{qDef}(Y_l) - \dim \mathrm{Aut}(Y_l) = (l-1) - 2 = l - 3.$$

For X_l it is now sufficient to compute the dimension (as a variety) of the GIT quotient $\mathbb{C}^{2(l-1)}//\mathbb{C}^*$ above. But it is clear that the generic orbit is closed (with no further stabilizer). Indeed, if coordinates a_j and a_j' in Lemma 2.5 are all non-zero, then the orbits are given by the closed set $a_j a_j' = c_j \neq 0$, with $j = 0, \ldots, l-2$. Hence $\dim_{\mathbb{C}} M_K$ near $[X_l]$ is simply given by $2(l-1) - 1 = 2l - 3$ as claimed.

Observe that if we consider a deformation of X_l which smooths only one of the two A_{l-1} singularities, the resulting variety is strictly K-semistable and never K-polystable, since in order to obtain K-polystable varieties we need to deform the two A_{l-1} singularities *simultaneously* by the same computation as for the Y_l case in the last paragraph of the proof of Proposition 1.2.

Note also that for X_l, $l \neq 2, 4$, since the action is not effective, we also have a discrepancy between the dimension of the stack and the dimension of the coarse space (which is then one dimension *bigger* than expected).

3 Some Final Comments

The general small deformation X_t of X_l is then a K-polystable variety which is also Kähler-Einstein by [9]. Moreover the second Betti number gets bigger and bigger as l goes to infinity: indeed, smoothing out an A_{l-1}-singularity introduces a chain of S^2 of length $l - 1$, giving distinct homological classes. Hence:

Corollary 3.1 *There are K-polystable/Kähler-Einstein del Pezzo surfaces with arbitrarily big second Betti number.*

We should also observe that this moduli space corresponds to the moduli of Kähler-Einstein orbifolds with positive cosmological constant, hence giving also examples of moduli spaces of positive Einstein orbifolds of unbounded dimension. Thus, from a more differential geometric perspective, it would be interesting to know if a bound on the second Betti number would instead force the dimension of the moduli spaces of such metrics to stay bounded.

Finally, note that the unboundedness of the dimension can be avoided by bounding below either the volume or the singularities. Indeed, that is what [7] proves, where the measure of boundedness used for the singularities is the alpha-invariant. This does not contradict our example, as we had that $K_{X_l}^2 \to 0$ as l grows and the log discrepancies were monotonously decreasing with l towards -1. What is remarkable of this example is not that a bound below on the volume or the singularities are required to achieve boundedness of families, there were plenty of examples of this behaviour in [7]. What is remarkable is that removing such bounds not only gives an infinite number of families (whose dimension, one may think could, in principle, be uniformly bounded), but it also gives infinite dimension of the moduli.

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References

1. Akhtar, M., et al.: Mirror symmetry and the classification of orbifold del Pezzo surfaces. Proc. Amer. Math. Soc. **144**(2), 513–527 (2016)
2. Alper, J., et al.: Reductivity of the automorphism group of K-polystable Fano varieties. Invent. Math. **222**(3), 995–1032 (2020)
3. Berman, R.J.: K-polystability of Q-Fano varieties admitting Käahler-Einstein metrics. Invent. Math. **203**(3), 973–1025 (2016)
4. Birkar, C.: Singularities of linear systems and boundedness of Fano varieties. Ann. Math. (2), **193**(2), 347–405 (2021)
5. Blum, H., Liu, Y., Xu, C.: Openness of K-semistability for Fano varieties (2020). arXiv: 1907.02408 [math.AG]

6. Blum, H., et al.: On properness of K-moduli spaces and optimal degenerations of Fano varieties. Selecta Math. (N.S.), **27**(4), Paper No. 73 (2021)
7. Jiang, C.: Boundedness of Q-Fano varieties with degrees and alpha-invariants bounded from below. Ann. Sci. Éc. Norm. Supéer. (4), **53**(5), 1235–1250 (2020)
8. Kaloghiros, A.-S., Petracci, A.: On toric geometry and K-stability of Fano varieties. Trans. Am. Math. Soc. Ser. B **8**, 548–577 (2021)
9. Li, C., Tian, G., Wang, F.: On the Yau-Tian-Donaldson conjecture for singular Fano varieties. Comm. Pure Appl. Math. **74**(8), 1748–1800 (2021)
10. Liu, Y., Petracci, A.: On K-stability of some del Pezzo surfaces of Fano index 2 (2020). arXiv: 2011.05094 [math.AG]
11. Liu, Y., Xu, C., Zhuang, Z.: Finite generation for valuations computing stability thresholds and applications to k-stability (2021). arXiv: 2102.09405 [math.AG]
12. Martinez-Garcia, J.: Constant scalar curvature Käahler metrics on rational surfaces. Math. Nachr. In press
13. Odaka, Y., Spotti, C., Sun, S.: Compact moduli spaces of del Pezzo surfaces and Käahler-Einstein metrics. J. Differ. Geom. **102**, 127–172 (2016)
14. Petracci, A.: On deformations of toric Fano varieties (2019). arXiv: 1912.01538
15. Stacks project authors, T.: The stacks project (2020). https://stacks.math.columbia.edu
16. Wang, X.: GIT stability, K-stability and the moduli space of Fano varieties. In: Moduli of K-stable Varieties, vol. 31, Springer INdAM Series, pp. 153–181. Springer, Cham (2019)
17. Xu, C.: K-stability of fano varieties: an algebro-geometric approach (2020). arXiv: 2011.10477 [math.AG]
18. Zhuang, Z.: Product theorem for K-stability. Adv. Math. **371**(18), 107250 (2020)

Okounkov Bodies and the Kähler Geometry of Projective Manifolds

David Witt Nyström

Abstract Given a projective manifold X equipped with an ample line bundle L, we show how to embed certain torus-invariant domains $D \subseteq \mathbb{C}^n$ into X so that the Euclidean Kähler form on D extends to a Kähler form on X lying in the first Chern class of L. This is done using Okounkov bodies $\Delta(L)$, and the image of D under the standard moment map will approximate $\Delta(L)$. This means that the volume of D can be made to approximate the Kähler volume of X arbitrarily well. As a special case we can let D be an ellipsoid. We also have similar results when L is just big.

Keywords Okounkov bodies · Kahler currents

1 Introduction

In toric geometry there is a beautiful correspondence between Delzant polytopes Δ and toric manifolds X_Δ equipped with an ample torus-invariant line bundles L_Δ. This is important since many properties of L_Δ can be read directly from the polytope Δ. Okounkov found in [16, 17] a generalization of sorts, namely a way to associate a convex body $\Delta(L)$ to an ample line bundle L on a projective manifold X, depending on the choice of a flag of smooth irreducible subvarieties in X. In the toric case, if one uses a torus-invariant flag, one essentially gets back the polytope Δ. The convex bodies $\Delta(L)$ are now called Okounkov bodies. They were popularized by the work of Kaveh-Khovanskii [7, 8] and Lazarsfeld-Mustaţă [13], where it was shown that the construction works in far greater generality, e.g. big line bundles (for more references see the exposition [2]).

Recall that the volume of a line bundle measures the asymptotic growth of $h^0(X, kL) := \dim_{\mathbb{C}} H^0(X, kL)$:

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$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n} h^0(X, kL).$$

L is then said to be big if $\mathrm{vol}(L) > 0$. When L is ample or nef, asymptotic Riemann-Roch together with Kodaira vanishing shows that $\mathrm{vol}(L) = (L^n)$. This is not true in general, since (L^n) can be negative while the volume always is nonnegative.

The key fact about Okounkov bodies is that they capture this volume:

$$\mathrm{vol}(L) = n!\mathrm{vol}(\Delta(L)). \tag{1}$$

Here the volume of the Okounkov body is calculated using the Lebesgue measure. This means that results from convex analysis, e.g. the Brunn-Minkowski inequality, can be applied to study the volume of line bundles.

In the toric setting, a fruitful way of thinking of Δ is to treat it as the image of a moment map. There is a holomorphic $(\mathbb{C}^*)^n$-action on X_Δ which lifts to L_Δ and choosing an $(S^1)^n$-invariant Kähler form $\omega_\Delta \in c_1(L_\Delta)$ gives rise to a symplectic moment map μ_{ω_Δ} whose image can be identified with Δ.

Building on joint work with Harada [4], Kaveh shows in the recent work [6] how Okounkov body data can be used to gain insight into the symplectic geometry of (X, ω), where ω is some Kähler form in $c_1(L)$ (it does not matter which Kähler form $\omega \in c_1(L)$ one uses since by Moser's trick all such Kähler manifolds are symplectomorphic).

In short, Kaveh constructs symplectic embeddings $f_k : ((\mathbb{C}^*)^n, \eta_k) \hookrightarrow (X, \omega)$ where η_k are $(S^1)^n$-invariant Kähler forms that depend on data related to a certain non-standard Okounkov body $\Delta(L)$ (i.e. the order on $\mathbb{N}^n$ used is not the lexicographic one). As k tends to infinity the image of the corresponding moment map will fill up more and more of $\Delta(L)$, showing that the symplectic volume of $((\mathbb{C}^*)^n, \eta_k)$ approaches that of (X, ω). Just as in [4] the construction uses the gradient-Hamiltonian flow introduced by Ruan [18], and is thus fundamentally symplectic in nature.

1.1 Main Results

We first introduce the following notion:

Definition 1.1 We say that a Kähler manifold (Y, η) fits into a polarized projective manifold (X, L) if for every relatively compact open set $U \subseteq Y$ one can choose a holomorphic embedding f of U into X so that $f_*\eta$ extends to some Kähler form ω on X lying in $c_1(L)$. If $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X = n$ and

$$\int_Y \eta^n = \int_X c_1(L)^n$$

we say that (Y, η) fits perfectly into (X, L).

The special case of $(\mathbb{C}^n, \eta)$ (η being some nonstandard S^1-invariant Kähler form) fitting into (X, L) was considered in [20].

Let

$$\mu(z) := (|z_1|^2, \dots, |z_n|^2),$$

which we note is a moment map of $(\mathbb{C}^n, \omega_{st})$ with respect to the standard torus-action (here $\omega_{st} := dd^c|z|^2$ denotes the standard Euclidean Kähler form on $\mathbb{C}^n$).

Pick a complete flag $X_\bullet := X = X_0 \supset X_1 \supset X_{n-1} \supset X_n = \{p\}$ of smooth irreducible subvarieties. One can then define the associated Okounkov body $\Delta(L)$. We introduce the notion of the Okounkov domain $D(L) \subseteq \mathbb{C}^n$ which is a torus-invariant domain with the property that

$$\Delta(L)^\circ \subseteq \mu(D(L)) \subseteq \Delta(L),$$

(where in general both inclusions are strict). We note that by (1)

$$\int_{D(L)} \omega_{st}^n = \int_X c_1(L)^n. \tag{2}$$

Theorem A. We have that $(D(L), \omega_{st})$ fits perfectly into (X, L). For any relatively compact subset $U \subset D(L)$ we can furthermore choose the embedding $f : U \to X$ so that

$$f^{-1}(X_i) = \{z_1 = ... = z_i = 0\} \cap U.$$

So on $(f(U), f_*\omega_{st}) \subseteq (X, \omega)$ there is a torus-action with moment map $\mu \circ f^{-1}$ whose image approximates $\Delta(L)$ and for any $\epsilon > 0$ we can choose U so that

$$\int_{f(U)} \omega^n > (1 - \epsilon) \int_X \omega^n.$$

These results are still true if we use some nonstandard additive order on $\mathbb{N}^n$ to define the Okounkov body $\Delta(L)$. Of particular interest here is the deglex order, which gives rise to the infinitesimal Okounkov bodies that appear in [13] and in the recent work of Küronya-Lozovanu [10, 11].

When L is very ample there is a particular choice of flag $X_\bullet$ which makes $D(L)$ an ellipsiod, namely the ellipsiod $E(1, \dots, 1, (L^n))$ defined by the inequality

$$\sum_{i=1}^{n-1} |z_i|^2 + (L^n)^{-1}|z_n|^2 < 1.$$

This leads to the following theorem.

Theorem B. If L is very ample, then we have that $(E(1, \dots, 1, (L^n)), \omega_{st})$ fits perfectly into (X, L), and the associated embeddings can be chosen to be centered at any point $p \in X$.

There is an interesting connection between this result and the notion of Seshadri constants.

Recall the definition of the Seshadri constant $\epsilon(X, L, p)$, introduced by Demailly [3].

Definition 1.2 The Seshadri constant of an ample line bundle L at a point p is given by

$$\epsilon(X, L, p) := \inf_C \frac{L \cdot C}{\text{mult}_p C},$$

where the infimum is taken over all curves C in X.

One can show that the Seshadri constant $\epsilon(X, L, p)$ also measures the maximal size of embedded balls centered at p such that the restricted Kähler structure is standard.

Theorem 1.3 *We have that $\epsilon(X, L, p)$ is equal to the supremum of r such that $(B_r, 0, \omega_{st})$ fits into (X, L) with the embeddings centered at p.*

This result can be extracted from Lazarsfeld [12] (see Theorem 5.1.22 and Proposition 5.3.17); the main argument is due to McDuff-Polterovic [14].

From Theorem 1.3 follows the inequality

$$\epsilon(X, L, p) \leq (L^n)^{\frac{1}{n}}.$$

When this inequality is strict for (X, L, p) (which is the general case) it means that no ball $(B_r, 0, \omega_{st})$ can fit perfectly into (X, ω_L) centered at p. Nevertheless Theorem B. says that one always can find an ellipsoid which fits perfectly into (X, ω_L) centered at p.

Let $\Delta(L)$ be an infinitesimal Okounkov body at p and $D(L)$ the corresponding Okounkov domain. One can easily show that

$$\epsilon(X, L, p) = \sup\{r : B_r \subseteq D(L)\}$$

so thus Theorem A. can be thought of as strengthening of Theorem 1.3.

The proof of Theorem A. relies on finding suitable toric degenerations. Here we follow [1], but as in [5, 6] we do not degenerate the whole section ring $R(L)$ but rather $H^0(X, kL)$ for fixed k. We couple the degeneration with a max construction to find a suitable positive hermitian metric of L, whose curvature form will provide the appropriate Kähler form ω in the theorem. We recently used this technique to construct Kähler embeddings related to canonical growth conditions [20, Theorem C].

1.2 The Big Case

We have similar results when L is just big. Then there are no longer any Kähler forms in $c_1(L)$ so instead we use Kähler currents in $c_1(L)$ with analytic singularities.

Definition 1.4 If L is big we say that a Kähler manifold (Y, η) fits into (X, L) if for every relatively compact open set $U \subseteq Y$ there is a holomorphic embedding f of U into X such that $f_*\eta$ extends to a Kähler current with analytic singularities on X lying in $c_1(L)$. If $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X = n$ and

$$\int_Y \eta^n = \int_X c_1(L)^n,$$

then we say that (Y, η) fits perfectly into (X, L).

Theorem C. We have that $(D(L), \omega_{st})$ fits perfectly into (X, L). We can furthermore choose each embedding $f : U \to X$ $(U \subset D(L))$ so that

$$f^{-1}(X_i) = \{z_1 = \cdots = z_i = 0\} \cap U.$$

1.3 Related Work

The work of Kaveh [6] which inspired this paper has already been mentioned. This built on joint work with Harada [4], which in turn used the work of Anderson [1] on toric degenerations.

Anderson showed in [1] how, given some assumptions, the data generating the Okounkov body also gives rise to a degeneration of (X, L) into a possibly singular toric variety (X_Δ, L_Δ), where $\Delta = \Delta(L)$ (the assumptions force $\Delta(L)$ to be a polytope, which is not the case in general). In their important work [4] Harada-Kaveh used this to, under the same assumptions, to construct a completely integrable system $\{H_i\}$ on (X, ω), with ω a Kähler form in $c_1(L)$, such that $\Delta(L)$ precisely is the image of the moment map $\mu := (H_1, \ldots, H_n)$. More precisely, they find an open dense subset U and a Hamiltonian $(S^1)^n$-action on (U, ω) such that the corresponding moment map $\mu := (H_1, \ldots, H_n)$ extends continuosly to the whole of X. Their construction uses the gradient-Hamiltonian flow introduced by Ruan [18].

In the recent work [20], given an ample line bundle L and a point $p \in X$, we show how to construct an (S^1)-invariant plurisubharmonic function $\phi_{L,p}$ on T_pX, such that the corresponding growth condition $\phi_{L,p} + O(1)$ is canonically defined. We then prove that the growth condition provides a sufficient condition for certain Kähler balls (B_1, η) to be embeddable into some (X, ω) with $\omega \in c_1(L)$ and Kähler [20, Theorem D].

The very general Seshadri constant $\epsilon(X, L; 1)$ is defined as the supremum of $\epsilon(X, L; p)$ over all points p of X, which is the same as the Seshadri constant at a very general point. In [5] Ito proved that if Δ is an integer polytope such that $\frac{1}{k}\Delta \subset \Delta(L)$ then

$$\epsilon(X, L; 1) \geq \frac{1}{k}\epsilon(X_\Delta, L_\Delta; 1).$$

He did this using the same kind of toric degeneration as was later used by Kaveh in [6] and that we use here. One can easily show that this also follows from our results. This illustrates the difference between our results and those of Kaveh in [6]. Since Kaveh's construction is symplectic that only implies the weaker symplectic version of Ito's theorem, namely the corresponding lower bound on the Gromov width [6, Corollary 8.4].

2 Okounkov Bodies and Domains

Let L be a big line bundle on a projective manifold X. Choose a complete flag $X = X_0 \supset X_1 \supset X_{n-1} \supset X_n = \{p\}$ of smooth irreducible subvarieties X_i such that $\text{codim} X_i = i$. We can then choose local holomorphic coordinates z_i centered at p such that in some neighbourhood U of p,

$$X_i \cap U = \{z_1 = \cdots = z_i = 0\} \cap U.$$

Also pick a local trivialization of L near p. Locally near p we can then write any section $s \in H^0(X, kL)$ as a Taylor series

$$s = \sum_\alpha a_\alpha z^\alpha.$$

When s is nonzero we let

$$v(s) := \min\{\alpha : a_\alpha \neq 0\},$$

where the mininum is taken with respect to the lexicographic order (or some other additive order of choice). The Okounkov body $\Delta(L)$ of L (for ease of notation the dependence of the flag is usually not written out) is then defined as

$$\Delta(L) := Conv\left(\left\{\frac{v(s)}{k} : s \in H^0(X, kL) \setminus \{0\}, k \geq 1\right\}\right).$$

Here $Conv$ means the closed convex hull.

Remark 2.1 Another natural choice of order on $\mathbb{N}^n$ to use is the deglex order. This means that $\alpha < \beta$ if $|\alpha| < |\beta|$ ($|\alpha| := \sum_i \alpha_i$) or else if $|\alpha| = |\beta|$ and α is less than β lexicographically. If one uses this order to define the Okounkov body, this will only depend on the flag of subspaces of T_pX given by T_pX_i, and it will be equivalent to the infinitesimal Okounkov body considered in [13] and in the recent work of Küronya-Lozovanu [10, 11] (see [20]).

Let us define

$$\mathcal{A}(kL) := \{v(s) : s \in H^0(X, kL) \setminus \{0\}\}.$$

By elimination we can find sections $s_\alpha \in H^0(X, kL)$, $\alpha \in \mathcal{A}(kL)$, such that

$$s_\alpha = z^\alpha + \sum_{\beta > \alpha, \beta \notin \mathcal{A}(kL)} a_\beta z^\beta.$$

If

$$s = \sum_{\alpha \in \mathcal{A}(kL)} a_\alpha z^\alpha + \sum_{\beta \notin \mathcal{A}(kL)} a_\beta z^\beta$$

then we must have that

$$s = \sum_{\alpha \in \mathcal{A}(kL)} a_\alpha s_\alpha,$$

because otherwise we would have that $v(s - \sum a_\alpha s_\alpha) \notin \mathcal{A}(kL)$. It follows that s_α is a basis for $H^0(X, kL)$ so

$$|\mathcal{A}(kL)| = h^0(X, kL), \tag{3}$$

where $|\mathcal{A}(kL)|$ denotes the number of points in $\mathcal{A}(kL)$.

If $s = z^{\alpha_1} + \sum_{\beta > \alpha_1} a_\beta z^\beta$ and $t = z^{\alpha_2} + \sum_{\beta > \alpha_2} b_\beta z^\beta$, then

$$st = z^{\alpha_1 + \alpha_2} + \sum_{\beta > \alpha_1 + \alpha_2} c_\beta z^\beta$$

and hence $v(st) = v(s) + v(t)$. This implies that for $k, m \in \mathbb{N}$:

$$\mathcal{A}(kL) + \mathcal{A}(mL) \subseteq \mathcal{A}((k+m)L) \tag{4}$$

and thus

$$\Gamma(L) := \bigcup_{k \geq 1} \mathcal{A}(kL) \times \{k\} \subseteq \mathbb{N}^{n+1}$$

is a semigroup.

Combined with a result by Khovanskii [9, Proposition 2] it leads to the proof of the key result (see e.g. [7, 8] or [13]).

Theorem 2.2 *We have that*

$$vol(L) = n! vol(\Delta(L)),$$

where the volume of $\Delta(L)$ is calculated using the Lebesgue measure.

From this we see that when X has dimension one, $\Delta(L)$ is an interval of length $deg(L)$. When L is ample one gets that $0 \in \Delta(L)$ and thus

$$\Delta(L) = [0, deg(L)]. \tag{5}$$

Let

$$\Delta_k(L) := \frac{1}{k} Conv(\mathcal{A}(kL)).$$

From (4) we see that for $k, m \in \mathbb{N}$:

$$\Delta_k(L) \subseteq \Delta_{km}(L). \tag{6}$$

The following lemma is also an immediate consequence of the result of Khovanskii (see e.g. [19, Lemma 2.3]).

Lemma 2.3 *Let K be a compact subset of $\Delta(L)^\circ$. Then for $k > 0$ divisible enough we have that*

$$K \subset \Delta_k(L).$$

From this it follows that

$$\Delta(L)^\circ = \bigcup_{k \geq 1} \Delta_k(L)^\circ.$$

Let $\Delta_k(L)^{ess}$ denote the interior of $\Delta_k(L)$ as a subset of $\mathbb{R}^n_{\geq 0}$ with its induced topology.

Definition 2.4 We define the essential Okounkov body $\Delta(L)^{ess}$ as

$$\Delta(L)^{ess} := \bigcup_{k \geq 1} \Delta_k(L)^{ess}.$$

By (6) we get that for any $k, m \in \mathbb{N}$, $\Delta_k(L)^{ess} \subseteq \Delta_{km}(L)^{ess}$ and thus

$$\Delta(L)^{ess} = \bigcup_{k \geq 1} \Delta_{k!}(L)^{ess}.$$

We also see that $\Delta_{k!}(L)^{ess}$ is increasing in k which then implies that $\Delta(L)^{ess}$ is an open convex subset of $\mathbb{R}^n_{\geq 0}$.

Lemma 2.5 *Let K be a compact subset of $\Delta(L)^{ess}$. Then for $k > 0$ divisible enough we have that*

$$K \subset \Delta_k(L)^{ess}.$$

This is proved in the same way as Lemma 2.3.

It is easy to see that

$$\Delta(L) \cap \{x_1 = 0\} \subseteq \Delta(L_{|X_1}),$$

where $\Delta(L_{|X_1})$ is defined using the induced flag $X_1 \supset X_2 \supset ... \supset X_n$. When L is ample the restriction map from $H^0(X, L^k)$ to $H^0(X_1, L_{|X_1})$ is surjective for k large enough (see e.g. [12]), hence we have an equality

$$\Delta(L) \cap \{x_1 = 0\} = \Delta(L_{|Y_1}). \tag{7}$$

Let L_1 denote the holomorphic line bundle associated with the divisor X_1. An important fact, proved by Lazarsfeld-Mustaţă in [13] is that

$$\Delta(L) \cap \{x_1 \geq r\} = \Delta(L - rL_1) + re_1. \tag{8}$$

For $a \in \mathbb{R}^n$ we let Σ_a denote the convex hull of $\{0, a_1e_1, a_2e_2, \ldots, a_ne_n\}$ and Σ_a^{ess} the interior of Σ_a as a subset of $\mathbb{R}^n_{\geq 0}$.

Proposition 2.6 *If L is very ample, then there is a flag $X = X_0 \supset X_1 \supset \cdots \supset X_n = \{p\}$ of smooth irreducible subvarieties of X such that*

$$\Delta(L) = \Sigma_{(1,\ldots,1,(L^n))}$$

and

$$\Delta(L)^{ess} = \Sigma^{ess}_{(1,\ldots,1,(L^n))}.$$

Proof Since L is very ample, we can find a flag $X = X_0 \supset X_1 \supset \cdots \supset Y_n = \{p\}$ of smooth irreducible subvarieties of X such that for each $i \in \{1, \ldots, n\}$ the line bundle $L_{|X_{i-1}}$ is associated with the divisor X_i in X_{i-1}.

From repeated use of (7) and (8) we get that

$$\begin{aligned}\Delta(L) \cap \{x_1 = r_1, \ldots, x_{n-1} = r_{n-1}\} =& \Delta\left(\left(1 - \sum_i r_i\right) L_{|X_{n-1}}\right) \\ =& \left[0, \left(\left(1 - \sum_i r_i\right)\right)(L^n)\right],\end{aligned}$$

using (5) and the fact that $deg(L_{X_{n-1}}) = (L^n)$. In other words,

$$\Delta(L) = \Sigma_{(1,\ldots,1,(L^n))}.$$

Since

$$\Delta(L_{|Y_{n-1}})^{ess} = [0, (L^n)),$$

we similarly get that

$$\Delta(L)^{ess} = \Sigma^{ess}_{(1,\dots,1,(L^n))}.$$

$\square$

Recall that

$$\mu(z) := (|z_1|^2, \dots, |z_n|^2).$$

Definition 2.7 We define the Okounkov domain $D(L)$ to be

$$D(L) := \mu^{-1}(\Delta(L)^{ess}).$$

We note that $D(L)$ is a bounded domain in $\mathbb{C}^n$. We also note that when $\Delta(L)^{ess} = \Sigma^{ess}_{(1,\dots,1,(L^n))}$ we get that $D(L) = E(1,\dots,1,(L^n))$, i.e. the ellipsoid defined by the inequality

$$\sum_{i=1}^{n-1} |z_i|^2 + (L^n)^{-1}|z_n|^2 < 1.$$

3 Torus-Invariant Kähler Forms and Moment Maps

Let (M, ω) be a symplectic manifold. Assume that there is an S^1-action on M which preserves ω and let V be the generating vector field. We must have that $\mathcal{L}_V\omega = 0$. By Cartan's formula we have that

$$d(\omega(V, \cdot)) = \mathcal{L}_V\omega - d\omega(V, \cdot) = 0,$$

so the one-form $\omega(V, \cdot)$ is closed. A function H is called a Hamiltonian for the S^1-action if

$$dH = \omega(V, \cdot).$$

If H is a Hamiltonian, then clearly so is $H + c$ for any constant c. If M has an $(S^1)^n$-action which preserves ω, and each individual S^1-action has a Hamiltonian H_i, we call the map $\mu := (H_1, \dots, H_n)$ a moment map for the $(S^1)^n$-action. There is a more invariant way of defining the moment map so that it takes values in the dual of the Lie algebra of the acting group, but we will not go into that here.

Let $\mathcal{A} \subseteq \mathbb{N}^n$ be a finite set and assume that $Conv(\mathcal{A})^{ess}$ is nonempty. Let

$$D_{\mathcal{A}} := \mu^{-1}(Conv(\mathcal{A})^{ess}) = \mu^{-1}(Conv(\mathcal{A}))^{\circ}$$

and let $X_{\mathcal{A}}$ denote the manifold we get by removing from $\mathbb{C}^n$ all the submanifolds of the form $\{z_{i_1} = \cdots = z_{i_k} = 0\}$ which do not intersect $D_{\mathcal{A}}$. Then

$$\phi_{\mathcal{A}} := \ln\left(\sum_{\alpha\in\mathcal{A}} |z^\alpha|^2\right)$$

is a smooth strictly psh function on $X_{\mathcal{A}}$ and we denote by $\omega_{\mathcal{A}} := dd^c\phi_{\mathcal{A}}$ the corresponding Kähler form.

Note that we can write

$$\phi_{\mathcal{A}}(z) = u_{\mathcal{A}}(x) := \ln\left(\sum_{\alpha\in\mathcal{A}} e^{x\cdot\alpha}\right),$$

where $x_i := \ln|z_i|^2$ and $u_{\mathcal{A}}$ is a convex function on $\mathbb{R}^n$.

Let us think of $(X_{\mathcal{A}}, \omega_{\mathcal{A}})$ as a symplectic manifold. The symplectic form ω_A is clearly invariant under the standard $(S^1)^n$-action on $X_{\mathcal{A}}$ and it is a classical fact that $\mu_{\mathcal{A}} : z \mapsto \nabla u(x)$ is a moment map for this action. To see this we define $u_{\mathcal{A}}(w) := u_{\mathcal{A}}(\mathrm{Re} w)$ for $w \in X_{\mathcal{A}}$ and note that $u_{\mathcal{A}}$ is the pullback of $\phi_{\mathcal{A}}$ by the holomorphic map $f : w \to e^{w/2}$. We then have that $f^*\omega_\alpha = dd^c u_\alpha$. The pullback of the vector field generating the i:th S^1-action is $(2\pi)\partial/\partial x_i$, so to show that $\partial/\partial x_i u_\alpha$ is a Hamiltonian we need to establish that

$$d\frac{\partial}{\partial x_i} u_{\mathcal{A}} = dd^c u_{\mathcal{A}}((2\pi)\partial/\partial x_i, \cdot).$$

This is easily checked using that

$$dd^c u_{\mathcal{A}} = \frac{1}{2\pi i}\sum_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} dw_i \wedge d\bar{w}_j.$$

Clearly

$$\mu_{\mathcal{A}}(\mathbb{C}^*)^n = Conv(\mathcal{A})^\circ$$

while

$$\mu_{\mathcal{A}}(X_{\mathcal{A}}) = Conv(\mathcal{A})^{ess}.$$

Another classical fact is that for any open $(S^1)^n$-invariant set $U \subseteq X_{\mathcal{A}}$ we have that

$$\int_U \omega_{\mathcal{A}}^n = \mathrm{vol}(\mu_{\mathcal{A}}(U)).$$

To see this, write $f^{-1}(U) = V \times (i\mathbb{R})^n$ and thus

$$\int_U \omega_{\mathcal{A}}^n = \int_{V\times(i[0,2\pi])^n} (dd^c u_{\mathcal{A}})^n = \int_V \det(\mathrm{Hess}(u)) = \int_{\nabla u(V)} dx,$$

where $\mathrm{Hess}(u)$ denotes the Hessian of u, and in the last step we used that this is equal to the Jacobian of ∇u.

Lemma 3.1 *Let U be a relatively compact open subset of $D_{\mathcal{A}}$. Then there exists a smooth function $g : X_{\mathcal{A}} \to \mathbb{R}$ with compact support such that $\omega := \omega_{\mathcal{A}} + dd^c g$ is Kähler and on U we have that $\omega = \omega_{st}$.*

Proof Using Legendre transforms one can find a smooth $(S^1)^n$-invariant strictly psh function ϕ on $X_{\mathcal{A}}$ which is equal to $|z|^2$ on U and such that the image of the gradient of $u(x) := \phi(e^{x_1/2}, \dots, e^{x_n/2})$ is compactly supported in $Conv(\mathcal{A})^{ess}$. One sees then that $\phi_{\mathcal{A}} - \phi$ is proper on $X_{\mathcal{A}}$. Let C be a constant such that $\phi + C > \phi_{\mathcal{A}}$ on D. Pick some $\delta > 0$ and let $\max_{reg}(x, y)$ be a smooth convex function such that $\max_{reg}(x, y) = \max(x, y)$ whenever $|x - y| > \delta$. Then $\phi' := \max_{reg}(\phi + C + \delta, \phi_{\mathcal{A}})$ is a smooth strictly psh function on $X_{\mathcal{A}}$ which is equal to $\phi + C + \delta$ on U while being equal to $\phi_{\mathcal{A}}$ outside of some compact set. It follows that $g := \phi' - \phi_{\mathcal{A}}$ has the desired properties. $\square$

4 Kähler Embeddings of Domains

In the introduction we had the following definition.

Definition 4.1 We say that a Kähler manifold (Y, η) fits into (X, L) if for every relatively compact open set $U \Subset Y$ there is a holomorphic embedding f of U into X such that $f_*\eta$ extends to a Kähler form on X lying in $c_1(L)$. If in addition

$$\int_Y \eta^n = \int_X c_1(L)^n$$

then we say that (Y, η) fits perfectly into (X, L).

Recall that $\mathcal{A}(kL) := \{v(s) : s \in H^0(X, kL)\}$.

Theorem 4.2 *Assume that L is ample. Then for k large enough, $(X_{\mathcal{A}(kL)}, \omega_{\mathcal{A}(kL)})$ fits into (X, kL), and each associated Kähler embedding $f : U \to X$ can be chosen so that*

$$f^{-1}(X_i) = \{z_1 = \cdots = z_i = 0\} \cap U.$$

Before proving Theorem 4.2 we need a simple lemma.

Lemma 4.3 *For any finite set $\mathcal{A} \subseteq \mathbb{N}^n$ there exists a $\gamma \in (\mathbb{N}_{>0})^n$ such that for all $\alpha \in \mathcal{A}$:*

$$\alpha < \beta \in \mathbb{N}^n \Longrightarrow \alpha \cdot \gamma < \beta \cdot \gamma. \tag{9}$$

This is a standard fact which is true for any additive order, see e.g. [1, Lemma 8]. It plays a key role in constructing toric degenerations.

Proof Pick a number $C \in \mathbb{N}$ such that $C > |\alpha|$ for all $\alpha \in \mathcal{A}$. We claim that

$$\gamma := \sum_i (2C)^{n-i} e_i$$

has the desired property (9). Assume that $\alpha < \beta$. By definition there is an index j such that $\alpha_i = \beta_i$ for $i < j$ while $\beta_j > \alpha_j$. It follows that

$$(\beta - \alpha) \cdot \gamma = \sum_i (2C)^{n-i} (\beta_i - \alpha_i) = (2C)^{n-j} (\beta_j - \alpha_j) + \sum_{i>j} (2C)^{n-i} (\beta_i - \alpha_i) \geq$$
$$\geq (2C)^{n-j} - |\alpha| \sum_{i>j} (2C)^{n-i} \geq C^{n-j} > 0.$$

□

We can now prove Theorem 4.2. As in [6] the proof relies on a toric deformation, given by a suitable choice of γ. However, instead of coupling it with a gradiant-Hamiltonian flow, we finish the proof using a max construction. This is similar to the proof of Theorem D in [20].

Proof Recall that we have local holomorphic coordinates z_i centered at p. We assume that the unit ball $B_1 \subset \mathbb{C}^n$ lies in the image of the coordinate chart $z : V \to \mathbb{C}^n$.

Let k be large enough so that $Conv(\mathcal{A}(kL))$ has nonempty interior and let U be a relatively compact open set in $X_{\mathcal{A}(kL)}$.

Pick a basis s_α for $H^0(X, kL)$ indexed by $\mathcal{A}(kL)$ such that locally

$$s_\alpha = z^\alpha + \sum_{\beta > \alpha} a_\beta z^\beta.$$

Pick a γ as in Lemma 4.3 with $\mathcal{A} := \mathcal{A}(kL)$ and let $\tau^\gamma z := (\tau^{\gamma_1} z_1, \ldots, \tau^{\gamma_n} z_n)$. It follows that

$$s_\alpha(\tau^\gamma z) = \tau^{\alpha \cdot \gamma} (z^\alpha + o(|\tau|)) \tag{10}$$

for $\tau^\gamma z \in B_1$.

Let $f : X_{\mathcal{A}(kL)} \to [0, 1]$ be a smooth function such that $f \equiv 0$ on U and $f \equiv 1$ on the complement of some smoothly bounded compact set $K \subseteq X_{\mathcal{A}(kL)}$. Pick $0 < \delta \ll 1$ such that

$$\phi := \phi_{\mathcal{A}(kL)} - 4\delta f$$

is still strictly psh. It follows from (10) that we can pick $0 < \tau \ll 1$ such that $\tau^\gamma z \in B_1$ whenever $z \in K$ and so that

$$\phi > \ln \left(\sum_{\alpha \in \mathcal{A}(kL)} \left| \frac{s_\alpha(\tau^\gamma z)}{\tau^{\alpha \cdot \gamma}} \right|^2 \right) - \delta$$

on U while

$$\phi < \ln\left(\sum_{\alpha\in\mathcal{A}(kL)}\left|\frac{s_\alpha(\tau^\gamma z)}{\tau^{\alpha\cdot\gamma}}\right|^2\right) - 3\delta$$

near ∂K.

Let $\max_{reg}(x, y)$ be a smooth convex function such that $\max_{reg}(x, y) = \max(x, y)$ whenever $|x - y| > \delta$. Then the regularized maximum

$$\phi' := \max_{reg}\left(\phi, \ln\left(\sum_{\alpha\in\mathcal{A}(kL)}|\frac{s_\alpha(\tau^\gamma z)}{\tau^{\alpha\cdot\gamma}}|^2\right) - 2\delta\right)$$

is smooth and strictly plurisubharmonic on $X_{\mathcal{A}(kL)}$, identically equal to ϕ on U while identically equal to $\ln(\sum_{\alpha\in\mathcal{A}(kL)}|\frac{s_\alpha(\tau^\gamma z)}{\tau^{\alpha\cdot\gamma}}|^2) - 2\delta$ near the boundary of K. We get that

$$\omega := dd^c\phi'$$

is equal to $\omega_{\mathcal{A}(kL)}$ on U.

If we assume that k is large enough so that kL is very ample then $\ln(\sum_{\alpha\in\mathcal{A}(kL)}|\frac{s_\alpha(\tau^\gamma z)}{\tau^{\alpha\cdot\gamma}}|^2)$ extends as a positive metric of kL and thus ω extends to a Kähler form in $c_1(kL)$.

Since U was arbitrary this shows that $(X_{\mathcal{A}(kL)}, \omega_{\mathcal{A}(kL)})$ fits into (X, kL). We also note that the embedding f of U into X was given by $z \mapsto \tau^\gamma z$, and thus we have that

$$f_R^{-1}(X_i) = \{z_1 = \cdots = z_i = 0\} \cap U.$$

□

We can now combine this result with Lemma 3.1 to obtain Theorem A..

Theorem A. We have that $(D(L), \omega_{st})$ fits perfectly into (X, L) and each associated Kähler embedding $f : U \to X$ can be chosen so that

$$f^{-1}(X_i) = \{z_1 = \cdots = z_i = 0\} \cap U.$$

Proof If U is a relatively compact open set in $D(L)$ then by Lemma 2.5 for $k > 0$ divisible enough the closure of U is contained in $\mu^{-1}(\Delta_k(L)^{ess})$, or in other words, $\sqrt{k}U$ is relatively compact in $D_{\mathcal{A}(kL)}$, which is in turn relatively comapact in $X_{\mathcal{A}(kL)}$. Thus by Lemma 3.1 there exists a smooth function $g : X_{\mathcal{A}(kL)} \to \mathbb{R}$ with support on a relatively compact set U' such that $\omega := \omega_{\mathcal{A}(kL)} + dd^c g$ is Kähler and on $\sqrt{k}U$ we have that $\omega = \omega_{st}$. By Theorem 4.2, if k is large enough, $(X_{\mathcal{A}(kL)}, \omega_{\mathcal{A}(kL)})$ fits into (X, kL). Thus we can find a holomorphic embedding $f' : U' \to X$ such that $f'_*\omega_{\mathcal{A}(kL)}$ extends to a Kähler form $\omega \in c_1(kL)$. Then letting $f : U \to X$ be defined as $f(z) := f'(\sqrt{k}z)$ we get that $f_*\omega_{st} = \frac{1}{k}f'_*\omega_{st|\sqrt{k}U}$ extends to a Kähler form $\omega \in c_1(L)$.

That
$$\int_{D(L)} \omega_{st}^n = \int_X c_1(L)^n$$
followed from Theorem 2.2 and it is clear that the $f : U \to X$ we found had the property that
$$f^{-1}(X_i) = \{z_1 = \cdots = z_i = 0\} \cap U.$$
□

Theorem B. If L is very ample then we have that $(E(1, \ldots, 1, (L^n)), 0, \omega_{st})$ fits perfectly into (X, L), and the associated embeddings can be chosen to be centered at any point $p \in X$.

Proof This follows directly from combining Theorem A. with Proposition 2.6. □

5 Big Line Bundles

If L is big but not ample there are no Kähler forms in $c_1(L)$. Instead one can consider Kähler currents with analytic singularities that lies in $c_1(L)$. We can use these to define what it should mean for a Kähler manifold (Y, η) to fit into (X, L) when L is just big.

Definition 5.1 If L is big we say that a Kähler manifold (Y, η) fits into (X, L) if for every relatively compact open set $U \subseteq Y$ there is a holomorphic embedding f of U into X such that $f_*\eta$ extends to a Kähler current with analytic singularities on X lying in $c_1(L)$. If $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X = n$ and
$$\int_Y \eta^n = \int_X c_1(L)^n$$
we say that (Y, η) fits perfectly into (X, L)

Theorem C. We have that $(D(L), \omega_{st})$ fits perfectly into (X, L). We can furthermore choose each embedding $f : U \to X$ $(U \subset D(L))$ so that
$$f^{-1}(X_i) = \{z_1 = \cdots = z_i = 0\} \cap U.$$

For a big line bundle L and sufficiently large k, if s_m is a basis for $H^0(kL)$ we get that $dd^c \ln(\sum_m |s_m|^2)$ is a Kähler current with analytical singularities which lies in $c_1(kL)$. Thus one proves Theorem C. exactly as in the ample case.

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References

1. Anderson, D.: Okounkov bodies and toric degenerations. Math. Ann. **356**(3), 1183–1202 (2013)
2. Boucksom, S.: Corps d'Okounkov (d'après Okounkov, Lazarsfeld-Mustaţă et Kaveh-Khovanskii) Astérisque No. 361, Exp. No. 1059, vii, 1–41 (2014)
3. Demailly, J.-P.: Singular Hermitian metrics on positive line bundles. In: Complex Algebraic Varieties, pp. 87–104. Bayreuth (1990). Lecture Notes in Mathematics, vol. 1507, Springer, Berlin (1992)
4. Harada, M., Kaveh, K.: Integrable systems, toric degenerations and Okounkov bodies. Invent. Math. (2015). https://doi.org/10.1007/s00222-014-0574-4
5. Ito, A.: Okounkov bodies and Seshadri constants. Adv. Math. **241**, 246–262 (2013)
6. Kaveh, K.: Toric degenerations and symplectic geometry of smooth projective varieties (2015). arXiv:1508.00316
7. Kaveh, K., Khovanskii, A.: Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. Ann. of Math. **176**(2), 925–978 (2012)
8. Kaveh, K., Khovanskii, A.: Algebraic equations and convex bodies. In: Perspectives in Analysis, Geometry, and Topology, vol. 296, Progress in Mathematics, pp. 263–282. Birkhäuser/Springer, New York (2012)
9. Khovanskii, A.: Newton polyhedron, Hilbert polynomial and sums of finite sets. Funct. Anal. Appl. **26**, 331–348 (1993)
10. Küronya, A., Lozovanu, V.: Positivity of line bundles and Newton-Okounkov bodies (2015). arXiv:1506.06525
11. Küronya, A., Lozovanu, V.: Infinitesimal Newton-Okounkov bodies and jet separation (2015). arXiv:1507.04339
12. R. Lazarsfeld, R.: Positivity in Algebraic Geometry, I & II. In: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48 & 49. Springer, Berlin (2004)
13. Lazarsfeld, R., Mustaţă, M.: Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Supér. (4) **42**(5), 783–835 (2009)
14. McDuff, D., Polterovich, L.: Symplectic packings and algebraic geometry Invent. Math. **115**(3), 405–434 (1994)
15. Nakamaye, M.: Base loci of linear series are numerically determined. Trans. Am. Math. Soc. **355**, 551–566 (2002)
16. Okounkov, A.: Brunn-Minkowski inequality for multiplicities. Invent. Math. **125**, 405–411 (1996)
17. Okounkov, A.: Why would multiplicities be log-concave? In: The Orbit Method in Geometry and Physics. Progress in Mathematics, vol. 213, pp. 329–347 (2003)
18. Ruan, W.-D.: Lagrangian torus fibration of quintic hypersurfaces. I. Fermat quintic case. In: Winter School on Mirror Symmetry. Vector Bundles and Lagrangian Submanifolds. Cambridge, MA (1999). AMS/IP Studies in Advanced Mathematics, vol. 23, pp. 297–332. American Mathematical Society, Providence (2001)
19. Witt Nyström, D.: Transforming metrics on a line bundle to the Okounkov body. Ann. Sci. Éc. Norm. Supér. (4) **47**(6), 1111–1161 (2014)
20. Witt Nyström, D.: Canonical growth conditions associated to ample line bundles (2015). arXiv:1509.05528

Fano Shimura Varieties with Mostly Branched Cusps

Yota Maeda and Yuji Odaka

Abstract We prove that the Satake-Baily-Borel compactification of certain Shimura varieties are Fano varieties, Calabi-Yau varieties or have ample canonical divisors with mild singularities. We also prove some variants statements, give applications and discuss various examples including new ones, for instance, the moduli spaces of unpolarized (log) Enriques surfaces.

Keywords Satake-Baily-Borel compactification · Shimura varieties · Fano varieties · Calabi-Yau varieties

1 Introduction

We prove that the Satake-Baily-Borel compactification of certain Shimura varieties are Fano varieties or with ample canonical divisor by means of special modular forms (see Theorem 2.4). Their unbranched open subsets are always quasi-affine, and in Fano Shimura varieties case, we observe that most of cusps are covered by the closure of branch divisors. In Sect. 3, we give various concrete examples, which include the moduli of (log) Enriques surfaces, those corresponding to $II_{2,26}$, and those associated to various Hermitian lattices which we construct.

The study of birational types of Shimura varieties is a semi-classical topic; Tai [60], Freitag [19] and Mumford [52] (resp. Kondō [40, 42], Gritsenko-Hulek-Sankaran [30] and Ma [48]) showed some Siegel (resp. orthogonal) modular varieties are of general type. Recently, the first author studied a similar problem for unitary modular varieties [50].

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On the other hand, in order to prove that Shimura varieties have negative Kodaira dimension, one of the powerful tools for it is the use of certain reflective modular forms [25, 27, 28, 47, 49].

For this recurring theme, our main idea in this paper is to focus on the Satake-Baily-Borel compactification, study it through modern birational geometry adapted to singular varieties and give applications. In this paper, we define "special" reflective modular forms, motivated by the work of Gritsenko-Hulek [28], and show a criterion for proving the Satake-Baily-Borel compactification of Shimura varieties are Fano varieties.

Then, we discuss examples in Sect. 3, including new ones, to which we apply our criterion. For instance, it follows that the Satake-Baily-Borel compactification of the moduli spaces of unpolarized (log) Enriques surfaces are Fano varieties; see Example 3.13, 3.17.

We also give some applications to the understanding of cusps and rationality problems. That is, for these Fano-like Shimura varieties, all but one compact cusps are shown to be contained in the closure of branch divisors. In the same setup, we also show that if there are no such compact cusps, two general points are connected by a rational curve i.e., rationally connected by [68]. See Corollary 2.8 for details. The former uses [4, 22], and in particular it logically relies on a vanishing theorem proven in *loc.cit*. We do not know of another proof which does not use a vanishing theorem (Problem 2.14). See Corollaries 2.8, 2.10, 2.12 for the details and more assertions proven. For instance, the moduli space of (unpolarized) Enriques surface is shown to be rationally connected, which is a weaker version of a famous result of Kondō [41].

2 Main Results and Proofs

In this section, we prove general theorems which are mentioned in the introduction. In the later Sect. 3, we apply them to various concrete examples. First, we introduce some notations.

2.1 Convention and Notation

Below, we discuss the linear equivalence class of a Cartier divisor and the corresponding holomorphic line bundle interchangeably. Similarly, we do not distinguish the $\mathbb{Q}$-linear equivalence class of a $\mathbb{Q}$-Cartier divisor and the corresponding $\mathbb{Q}$-line bundle. We use the following notations throughout.

- $\mathbb{G}$ is a simple algebraic group over $\mathbb{Q}$, not isogenous to $\mathrm{SL}(2)$.
- G is the identity component of $\mathbb{G}(\mathbb{R})$, which we assume to be a simple Lie group.
- K is a maximal compact subgroup of G,

- The corresponding Hermitian symmetric domain is G/K.
- Take an arithmetic subgroup $\Gamma \subset \mathbb{G}(\mathbb{Q})$ i.e., commensurable to $\mathbb{G}(\mathbb{Z})$.
- $X := \Gamma\backslash G/K$ and its Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ [6, 58].
- $\mathbb{H}$ denotes the upper half plane (which is an example of X).
- $\partial\overline{X}^{\mathrm{SBB}}$ denotes the boundary of the Satake-Baily-Borel compactification, i.e., $\overline{X}^{\mathrm{SBB}} \setminus X$.
- Denote a toroidal compactification of X in the sense of [5], with an arbitrary fixed cone decompositions, simply as $\overline{X}$. (The choice of cone decompositions do not affect the following discussions.)
- Denote the boundary divisor $\overline{X} \setminus X$ as Δ (with coefficients 1).
- Denote the branch divisor of $G/K \to \Gamma\backslash G/K$ to be $\cup_i B_i (\subset X)$ with prime divisors B_i and branch (or ramification) degree d_i. We denote the closure of B_i in $\overline{X}$ (resp., $\overline{X}^{\mathrm{SBB}}$) as $\overline{B_i}$ (resp., $\overline{B_i}^{\mathrm{SBB}}$).
- $X^o := X \setminus \cup_i B_i$.
- $L := K_{\overline{X}} + \Delta + \sum_i \frac{d_i-1}{d_i}\overline{B_i} \in \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}$ and its descended (automorphic) $\mathbb{Q}$-line bundle on $\overline{X}^{\mathrm{SBB}}$, i.e., $K_{\overline{X}^{\mathrm{SBB}}} + \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}}$.
- Recall from [6] and [51, 3.4, 4.2 (also see 1.3)] that L is ample (resp., semiample) on $\overline{X}^{\mathrm{SBB}}$ (resp., $\overline{X}$) and a meromorphic section of $L^{\otimes t}$ for $t \in \mathbb{Z}_{>0}$ corresponds to meromorphic automorphic form of arithmetic weight ct for some $c \in \mathbb{Z}$. In this paper, weight always simply refers to the arithmetic weight (in the sense of e.g., [30]) and call c the *canonical weight*, following e.g., [30]. See also Lemma 2.3 for the calculation of c.

2.2 Special Reflective Modular Forms

Recall that reflective modular form is the concept originally formulated in [25] for orthogonal case, which means that the divisor is defined by reflections. In this paper, we consider the following stronger properties, or proper subclass of reflective modular forms. The upshot of our general observation is that the existence of such special reflective modular forms give strong implications on the birational properties of modular varieties (see Theorem 2.4). These modular forms are rare, but luckily still various interesting examples are known (cf., [27], our Sect. 3). We also construct new examples in the Sect. 3.

Assumption 2.1 (*Special reflective modular forms—General case*) Consider the following subclasses of reflective modular forms.

(i) A non-vanishing holomorphic section f of

$$\mathcal{O}_X(N(s(X)L - \sum_i \frac{d_i-1}{d_i}B_i))\Big(:= L^{\otimes aN}\Big(-\sum_i \frac{N(d_i-1)}{d_i}B_i\Big)\Big)$$

for some $N \in \mathbb{Z}_{>0}$, $s(X) \in \mathbb{Q}_{>0}$ with $s(X)N, \frac{N}{d_i} \in \mathbb{Z}_{>0}$.

(ii) A non-vanishing holomorphic section f of $\mathcal{O}_X(N(s(X)L - \sum_i c_i B_i))$ for some $N \in \mathbb{Z}_{>0}$, $s(X) \in \mathbb{Q}_{>0}$, and $c_i \in \mathbb{Q}$ with $0 \leq c_i \leq \frac{d_i-1}{d_i}$ for all i, such that $s(X)N, Nc_i \in \mathbb{Z}$.

We follow the same convention below.

For a specific choice of $\mathbb{G}$ and Γ that we are about to specify, Assumption 2.1(i) specializes to the following simpler condition.

Assumption 2.2 (*Special reflective modular forms—orthogonal case*) For $n > 2$, assume that there is a quadratic lattice Λ of signature $(2, n)$ such that $\mathbb{G} = O(\Lambda \otimes \mathbb{Q})$ with $\Gamma \subset O(\Lambda)$. In this situation, we consider the following subclasses of reflective modular forms.

(i) A non-vanishing holomorphic section f of $\mathcal{O}_X(N(s(X)L - \frac{1}{2}\sum_i B_i))$ for some $N \in \mathbb{Z}_{>0}$, $s(X) \in \mathbb{Q}_{>0}$ with $s(X)N, \frac{N}{2} \in \mathbb{Z}_{>0}$.

Indeed, for the above $\mathbb{G}$ and Γ, Gritsenko-Hulek-Sankaran showed that every branch divisor arises from reflections (of order 2) [30, 2.12, 2.13], i.e., the ramification degrees d_i are all 2.

Note that N is unessential as it gets multiplied when replacing f by its power, while the quantity $s(X)$ is more essential and sometimes called a *slope* in the literature. When we work on the cases $G = O(2, n)$ or $G = U(1, n)$ and regard f as a modular form, we call its arithmetic weight, in the sense of [30] for instance, simply as a weight from now on.

We also review the following well-known fact for the convenience.

Lemma 2.3 (cf., [19, Hilfsatz 2.1], [30, Sect. 6.1]) *In the orthogonal case $G = O(2, n)$ (resp., in the unitary case $G = U(1, n)$), the canonical weight c in the sense of Sect. 2.1 is n (resp., $n + 1$).*

Proof Recall that the compact dual D^c of D in the orthogonal case $G = O(2, n)$ is the n-dimensional quadratic hypersurface (resp., $D^c = \mathbb{P}^n$ in the unitary case $G = U(1, n)$), its canonical divisor is $K_{D^c} = \mathcal{O}_{\mathbb{P}^{n+1}}(-n)|_{Q^n}$ (resp., $K_{D^c} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$) so that the canonical weight c is n (resp., $n + 1$). □

Note that the quantity $s(X)$ in Theorem 2.4 is the (arithmetic) weight of the modular form s divided by such canonical weight c and some constant; see Remarks 3.8 and 3.27.

Below, we discuss various Shimura varieties X which can be roughly divided into two types, i.e., those with modular forms satisfying Assumption 2.1(i), and those with modular forms satisfying Assumption 2.1(ii).

The former is discussed in the next Sect. 2.3, with examples given in Sect. 3, and the latter is discussed in the Sect. 2.4 while some examples are given in [28, 49].

2.3 Main General Results and Proofs

Here is our first general theorem.

Theorem 2.4 (Birational properties) *We follow the notation as above. If there is a reflective modular form which satisfies Assumption 2.1(i) with some $s(X) \in \mathbb{Q}_{>0}$, then the Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of $X = \Gamma\backslash D$ only has log canonical singularities and X^o is quasi-affine. In addition,*

(i) if $s(X) > 1$, then $\overline{X}^{\mathrm{SBB}}$ is a Fano variety i.e., $-K_{\overline{X}^{\mathrm{SBB}}}$ is ample ($\mathbb{Q}$-Cartier),
(ii) if $s(X) = 1$, then $\overline{X}^{\mathrm{SBB}}$ is a Calabi-Yau variety i.e., $K_{\overline{X}^{\mathrm{SBB}}} \sim_{\mathbb{Q}} 0$, or
(iii) if $s(X) < 1$, then $K_{\overline{X}^{\mathrm{SBB}}}$ is ample.

Terminology. In this paper, we often say a normal variety is a *log canonical model* (resp., *canonical model*) in the sense that it only has log canonical singularities (resp., canonical singularities) and the canonical class is ample. Hence, in the case (iii) above, $\overline{X}^{\mathrm{SBB}}$ is a log canonical model. For the basics of birational geometry, we refer to e.g., [39].

Proof Note that the codimension of the boundary of the Satake-Baily-Borel compactification $\partial\overline{X}^{\mathrm{SBB}} := \overline{X}^{\mathrm{SBB}} \setminus X$ is at least 2, following from our assumption that $\mathbb{G}$ is not isogenous to SL(2). Indeed, for such G, any maximal real parabolic subgroup P has unipotent radical of dimension at least 2 so that Levi part of P has real codimension at least 3. The existence of the special reflective modular form implies

$$\sum_i \frac{d_i - 1}{d_i} B_i \sim_{\mathbb{Q}} s(X)L. \tag{1}$$

If we regard the holomorphic section satisfying Assumption 2.1(i) as a section of the ample line bundle $L^{\otimes s(X)N}$, it follows that the complement of the vanishing locus is affine but that is nothing but $\overline{X}^{\mathrm{SBB}} \setminus \cup_i \overline{B_i}^{\mathrm{SBB}}$ which includes X^o. This proof reflects the idea of [10].

From (1) and the definition of L it follows that

$$-K_{\overline{X}^{\mathrm{SBB}}} \sim_{\mathbb{Q}} (s(X) - 1)L \tag{2}$$

in $\mathrm{Pic}(\overline{X}^{\mathrm{SBB}}) \otimes \mathbb{Q}$. Hence, $-K_{\overline{X}^{\mathrm{SBB}}}$ is ample $\mathbb{Q}$-Cartier if $s(X) > 1$. Similarly, $K_{\overline{X}^{\mathrm{SBB}}}$ is ample $\mathbb{Q}$-Cartier (resp., $K_{\overline{X}^{\mathrm{SBB}}} = 0$) if $s(X) < 1$ (resp., if $s(X) = 1$). On the other hand, from [51, 3.4, 4.2 (also see 1.3)], $\overline{X}^{\mathrm{SBB}}$ is obtained as a projective spectrum of a certain log canonical ring, hence the pair $(\overline{X}^{\mathrm{SBB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}})$ has only log canonical singularity (as a pair) and $K_{\overline{X}^{\mathrm{SBB}}} + \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}}$ is ample (see also [1, 3.4, 3.5]). Thus $\sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}}$ is also $\mathbb{Q}$-Cartier so that X itself is also log canonical.

On the other hand, recall that the construction of the Baily-Borel compactification [6] is a projective spectrum of the graded ring of automorphic forms and L is the c

multiple tensors of its tautological line bundle $\mathcal{O}(1)$ in the construction. Hence, it is ample so that our latter statements of the above theorem all follow from (2). This fact is more clarified in [51, Sects. 3, 4]. We complete the proof. $\square$

Remark 2.5 The above results are analogous to the Fanoness results in [18], (resp., [36, Sect. 2] also [45, Sect. 4]) in the context of moduli of (semi)stable bundles over curves (resp., surfaces). For the case over surfaces, the determinant line bundle which descends to the Donaldson-Uhlenbeck compactification is used in the place of automorphic line bundle L.

Remark 2.6 Case (iii) is a variant of the so-called "low weight cusp form trick" (cf., e.g., [30]). See also [25], [27, Sect. 5.5] and references therein.

We introduce the following notion.

Definition 2.7 We call a cusp F of $\overline{X}^{\mathrm{SBB}}$ *naked* if it is not contained in $\mathrm{Supp}(\overline{B_i}^{\mathrm{SBB}}) \cap \partial\overline{X}^{\mathrm{SBB}}$ for any i. Further, we call it *minimal naked* if it is minimal with respect to the closure relation among naked cusps, i.e., $\overline{F} \setminus F$ is contained in $(\cup_i \mathrm{Supp}(\overline{B_i}^{\mathrm{SBB}})) \cap \partial\overline{X}^{\mathrm{SBB}}$. Also, we call $\partial\overline{X}^{\mathrm{SBB}} \setminus \bigcup_i \overline{B_i}^{\mathrm{SBB}}$ *the naked locus*.

Below, we observe a certain weakening of connected-ness of cusps closure in the case of $s(X) > 1$, i.e., Fano case. This follows from [4, 4.4, 6.6 (ii)], [22, 8.1], [21, Sect. 3], [24, 1.2] as the proof below, which is essentially just a review to make our logic more self-contained. Compare with our examples of the modular varieties given in the next section.

Corollary 2.8 (Boundary structure for Fano Shimura varieties) *Let us assume the same assumption of Theorem 2.4 and further that* $s(X) > 1$. *Then, the naked locus*

$$\partial\overline{X}^{\mathrm{SBB}} \setminus \bigcup_i \overline{B_i}^{\mathrm{SBB}}$$

is connected and its closure is nothing but the non-log-terminal locus of $\overline{X}^{\mathrm{SBB}}$. *More strongly, there is at most one minimal naked cusp with respect to the closure relation.*

Furthermore, if we suppose such a minimal naked cusp F exists, there is an effective $\mathbb{Q}$-divisor D_F such that $(\overline{F}, D_F)$ has only klt singularities and is a log Fano pair, i.e., $-K_F - D_F$ is ample and $\mathbb{Q}$-Cartier. For instance, if F is a modular curve, it is rational i.e., $\overline{F} \simeq \mathbb{P}^1$ (with "Hauptmodul").

Proof Firstly, we prepare the following general lemma (compare with e.g., [1, Sect. 3]).

Lemma 2.9 (Log canonical centers)

(i) Under the notation of Sect. 2.1 for general Shimura varieties, without the above assumptions in Corollary 2.8, the log canonical centers of $(\overline{X}^{\mathrm{SBB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}})$ *are nothing but cusps of the Satake-Baily-Borel compactification* $\overline{X}^{\mathrm{SBB}}$.

(ii) *Under the above assumptions in Corollary 2.8, the log canonical centers of $\overline{X}^{\mathrm{SBB}}$ are nothing but cusps of the Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ which are not contained in $\cup_i \mathrm{Supp}(\overline{B_i}^{\mathrm{SBB}})$.*

proof of Lemma 2.9 As in [5, Chap. III, Sect. 7], we replace the (implicit dividing) discrete group Γ in the notation Sect. 2.1 by its neat subgroup (cf., [5]) of finite index. In that way, we replace X (and $\overline{X}^{\mathrm{SBB}}$) by its finite cover so that the first desired claim (i) for the log canonical centers of $(\overline{X}^{\mathrm{SBB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}})$ is reduced to the case when there is no B_i.

Then, there is a log resolution of $(\overline{X}^{\mathrm{SBB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}})$ as a toroidal compactificaftion [5, chapter III], see especially *loc.cit* 6.2. By its construction in *op.cit* of toroidal nature (see again e.g., [1, Sect. 3]), all the exceptional prime divisors have the discrepancy -1 and hence the claim (i) for the log canonical centers of $(\overline{X}^{\mathrm{SBB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}})$ follows.

For the proof of latter claim (ii), note that the existence of special reflective modular form implies $\sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}}$ is a $\mathbb{Q}$-Cartier divisor by (1) of the proof of Theorem 2.4. Hence, the note that log canonical centers of $\overline{X}^{\mathrm{SBB}}$ form a subset of the lc centers of (i) which are not contained in the support of the effective $\mathbb{Q}$-Cartier divisor $\sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}}$. Hence, the claim of Lemma 2.9 (ii).

Now we start the proof of Corollary 2.8. We take the union of the minimal naked cusps of $\overline{X}^{\mathrm{SBB}}$ as W and put the reduced scheme structure on it. We denote the corresponding coherent ideal sheaf of $\mathcal{O}_{\overline{X}^{\mathrm{SBB}}}$ as I_W.

From a vanishing theorem of [4, 4.4], [22, 8.1], whose absolute non-log version is enough for our particular purpose here, we have $H^1(\overline{X}^{\mathrm{SBB}}, I_W) = 0$. On the other hand, $H^0(\overline{X}^{\mathrm{SBB}}, I_W) = 0$ also holds since it is a linear subspace of $H^0(\overline{X}^{\mathrm{SBB}}, \mathcal{O})$ which is identified with $\mathbb{C}$ because of the properness of $\overline{X}^{\mathrm{SBB}}$, combined with the fact that $W \neq \emptyset$. Hence, combined with standard cohomology exact sequence arguments, $H^0(\mathcal{O}_W) \simeq \mathbb{C}$ follows. Hence, it implies the connectivity of W, so that there is at most 1 minimal naked cusp F.

For such F, the existence of D_F on the closure $\overline{F}$ follows from applying the log canonical subadjunction [24, 1.2] to $F \subset (\overline{X}^{\mathrm{SBB}}, 0)$. □

We make a caution that the above Corollary 2.8 does not claim the naked cusp always has log terminal singularity. Nevertheless, in the $\mathbb{Q}$-rank 1 case, we have the following.

Corollary 2.10 ($\mathbb{Q}$-rank 1 case) *Under the same assumptions of Theorem 2.4 with > 1, if further $\mathbb{Q}$-rank of $\mathbb{G}$ is 1 (e.g., when $G \simeq U(1, n)$ for some n so that G/K is an n-dimensional complex unit ball), only either one of the followings hold.*

(i) *There is exactly one naked cusp F of $\overline{X}^{\mathrm{SBB}}$ which is an isolated non-log-terminal locus but at worst log canonical. Furthermore, there is an effective $\mathbb{Q}$-divisor*

D_F such that (F, D_F) is a klt log Fano pair hence in particular, the modular branch divisor in F is nonzero effective.

(ii) No naked cusp exists and X is rationally connected, i.e., two general points are connected by a rational curve and has at worst log terminal singularities. Furthermore, $X \setminus \mathrm{Supp} \cup_i B_i$ is affine (not only quasi-affine).

Proof Note that the condition that $\mathbb{Q}$-rank of $\mathbb{G}$ is 1 implies that the boundary strata of the Satake-Baily-Borel compactification of X are all compact and do not have closure relations. Thus, among the above statements, the only assertion which does not follow trivially from Corollary 2.8 is the rationally connected assertion for the latter case (ii). We confirm it as follows: the non-existence of naked cusp means $\overline{X}^{\mathrm{SBB}} \setminus X$ is included in $\cup_i \mathrm{Supp}(\overline{B_i}^{\mathrm{SBB}})$ which implies the log terminality of X. Hence, it is rationally connected by a theorem of Zhang [68]. Finally, $X \setminus \mathrm{Supp} \cup_i B_i$ is affine by the proof of Theorem 2.4 and the assumption that there are no naked cusps. □

Here is a version of the converse direction of Theorem 2.4.

Theorem 2.11 (Abstract existence of special modular forms) *We follow the notation of Theorem 2.4. If $\overline{X}^{\mathrm{SBB}}$ satisfies either*

- *$K_{\overline{X}^{\mathrm{SBB}}} \equiv 0$ or*
- *either $K_{\overline{X}^{\mathrm{SBB}}}$ or $-K_{\overline{X}^{\mathrm{SBB}}}$ is ample with Picard number 1,*

then there are special reflective modular forms satisfying Assumption 2.1(i) for some $s(X) \in \mathbb{Q}_{>0}$ and sufficiently divisible $N \in \mathbb{Z}_{>0}$. Furthermore, if it is of a certain orthogonal type, i.e., $\mathbb{G}$ is isogenous to $SO(\Lambda)$ for $\Lambda = U \oplus U(l) \oplus N$ with some negative definite lattice N and $l \in \mathbb{Z}_{>0}$, the modular forms are necessarily Borcherds lift of some nearly holomorphic elliptic $Mp_2(\mathbb{Z})$-modular forms of a specific principal part of the Fourier expansion in the sense of [11], [13, Sects. 1.3, 3.4].

Proof Given the proof of Theorem 2.4, we can almost trace back the arguments as follows. In either cases, the automorphic line bundle L is proportional to $K_{\overline{X}^{\mathrm{SBB}}}$ in $\mathrm{Pic}(\overline{X}^{\mathrm{SBB}})$, hence so is it to $\sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}}$. Therefore, $\mathcal{O}(N(s(X)L - \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}}))$ is trivial for some $s(X), N$. The last assertion follows from [13, 5.12], [14, 1.2]. □

2.4 Modular Varieties with Big Anti-Canonical Classes

Recall that Gritsenko-Hulek [28] (resp., Maeda [49]) discuss the classes of reflective orthogonal modular forms (resp., unitary modular forms) satisfying Assumption 2.1(ii) with $s(X) > 1$ and proved uniruledness of X and constructs some examples.

This subsection proves the following a slight refinement of their results, which applies to the examples constructed in *loc.cit.*

Theorem 2.12 (cf., [28, 2.1], [49, 4.1]) *We follow the notation of Sect. 2.1, and discuss Shimura varieties $X = \Gamma\backslash D$ for a priori general G. If there is a reflective modular form Φ which satisfies Assumption 2.1(ii) with some $s(X) \in \mathbb{Q}_{>1}$, , we define $V_\Phi := \cup_F \overline{F} \subset \partial\overline{X}^{\mathrm{SBB}}$ where F runs through all cusps along which Φ does not vanish (as a function, or a section of $L^{\otimes s(X)N}$). Then, the following holds.*

(i) The Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of $X = \Gamma\backslash D$ only has log canonical singularities, X^o is quasi-affine and $-K_{\overline{X}^{\mathrm{SBB}}}$ is big.

(ii) For any two closed points $x, y \in \overline{X}^{\mathrm{SBB}}$, there are union of rational curves C such that $C \cup V_\Phi$ is connected (i.e., rationally chain connected modulo V_Φ cf., [33, 1.1]). In particular, X is uniruled. If $G = U(1, n)$ for some n, then $\overline{X}^{\mathrm{SBB}}$ is even rationally chain connected.

(iii) If we consider the set of cusps outside V_Φ, there is at most 1 minimal element (cusp) with respect to the closure relation.

Proof We first consider (i) of the above theorem. From the existence of Φ, it follows in the same way that

$$-K_{\overline{X}^{\mathrm{SBB}}} \sim_{\mathbb{Q}} (s(X) - 1)L + \sum_i \left(\frac{d_i - 1}{d_i} - c_i\right)\overline{B_i}^{\mathrm{SBB}},$$

hence it is big. The proofs of the other assertions in (i) are the same as those of Theorem 2.4. For (ii), note that the non-klt locus of $(\overline{X}^{\mathrm{SBB}}, \sum_i(\frac{d_i-1}{d_i} - c_i)\overline{B_i}^{\mathrm{SBB}})$ is the union of log canonical centers of $(\overline{X}^{\mathrm{SBB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}})$ which are not inside $\mathrm{Supp}(\mathrm{div}(\Phi))$. Hence, the assertion (ii) directly follows from [33, 1.2] for $(\overline{X}^{\mathrm{SBB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{SBB}})$. The assertion for the unitary case holds since the cusps are all 0-dimensional (cf., e.g., [8, Sect. 4]). Indeed, it follows since the Levi part of real parabolic subgroup of G corresponding to the cusps are $U(0, n-1)$, which is trivial. For (iii), the same arguments as Corollary 2.8, similarly applying [4, 4.4, 6.6(ii)] or [22, 8.1] to the log canonical Fano pair $\left(\overline{X}^{\mathrm{SBB}}, \sum_i \left(\frac{d_i-1}{d_i} - c_i\right)\overline{B_i}^{\mathrm{SBB}}\right)$, give a proof. □

Remark 2.13 We can also show a variant of Corollary 2.8, Theorem 2.12(iii) under general *meromorphic* modular forms if we replace the use of [4, 6.6(ii)] by [4, 4.4] or [23, 6.1.2]. However, because the obtained statement is rather complicated and no interesting applications have been found (yet at least), we omit it in this paper.

We conclude this section by posing a natural problem.

Problem 2.14 *In specific situations, e.g., when $\mathbb{G} = SO(\Lambda \otimes \mathbb{Q})$ for a quadratic lattice Λ, or in the unitary modular case corresponding to a Hermitian lattice as later Sect. 3.4, the assertions of Corollaries 2.8, 2.10, Theorem 2.12(iii) can be phrased in a purely lattice theoretic manner. Is there a more lattice theoretic or number theoretic proof without the use of a vanishing theorem in algebraic geometry?*

3 Examples of Fano and K-ample Cases

We provide examples of which Theorems 2.4, Corollarys 2.8, 2.10, Theorem 2.11 in Sect. 2.3 apply. In the examples, the compactified modular varieties are either Fano varieties or with ample canonical classes. There are also some examples with $s(X) = 1$, for instance [20] (cf., also earlier [7] with a weaker statement) but we do not focus such cases in this paper.

3.1 Siegel Modular Cases

We start by discussing the Satake-Baily-Borel compactifications of some semi-classical modular varieties, which we show to fit our picture. The examples in this subsection and the next Sect. 3.2 do not use explicit modular forms but they are Fano varieties so that the converse Theorem 2.11 applies to imply the (abstract) existence of special reflective modular forms.

The examples with explicit special reflective modular forms, to which we can apply Theorem 2.4 will be discussed from the next Sect. 3.3. Here are two examples of Siegel modular varieties whose Satake-Baily-Borel compactifications are Fano varieties.

Example 3.1 ([38]) The Satake-Baily-Borel compactification of the moduli of principally polarized abelian surfaces $\overline{A_2}^{\mathrm{SBB}}$ is known to be a weighted projective hypersurface in $\mathbb{P}(4, 6, 10, 12, 35)$ of degree 70 with the coarse moduli isomorphic to $\mathbb{P}(2, 3, 5, 6)$ by relating to the invariants of genus 2 curves, hence binary sextics. Note that the adjunction does not work due to non-well-formedness, as indeed one has non-trivial isotropy (μ_2) along a divisor in the moduli stack. The reduction of the natural Faltings-Chai model over $\mathbb{F}_p$ are also determined (cf., [37, 62]).

Example 3.2 (cf., [61, 5.2] (also [38])) The Satake-Baily-Borel compcatification of the moduli of principally polarized abelian surfaces with level 2 structure $\overline{\Gamma(2)\backslash\mathfrak{H}}^{\mathrm{SBB}}$ is known to be a quartic 3-fold

$$\sum_{i=0}^{5} x_i = \left(\sum_{i=0}^{5} x_i^2\right)^2 - 4\left(\sum_{i=0}^{5} x_i^4\right) = 0, \tag{3}$$

with non-isolated singularities along 15 lines. Since this is a hypersurface, it is clearly Gorenstein and has ample anticanonical class. It also follows from [51, Sects. 3, 4] (cf., also [1, 3.5]) again that it is at least log canonical.

3.2 *Orthogonal Modular Cases, Part I*

Below, we consider the cases where $\mathbb{G} = SO(\Lambda \otimes \mathbb{Q})$ for a quadratic lattice $(\Lambda, (\ ,\ ))$ of signature $(2, n)$ with $n \in \mathbb{Z}_{>0}$. We realize the Hermitian symmetric domain $X = G/K$ as $G/K \simeq \mathcal{D}_\Lambda$ which is defined as one of (the isomorphic two) connected components of

$$\{v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (v, v) = 0,\ (v, \overline{v}) > 0\}.$$

We keep this notation throughout in the discussion of orthogonal modular varieties. Our first two examples in this Part I are understood via moduli-theoretic methods and GIT as follows.

Example 3.3 (*Hilbert*) The GIT compactification of the moduli of cubic surfaces ([55, Sect. 4.2]) is known to be isomorphic to the Satake-Baily-Borel compactification of the stable locus which admits uniformization of complex ball (cf., [3]). Hilbert's invariant calculation in his thesis tells this is $\mathbb{P}(1, 2, 3, 4, 5)$, hence the only cusp is not naked because of the log terminality. Obviously, it is also a ($\mathbb{Q}$-)Fano variety. This is also one of the simplest examples of the K-moduli variety of Fano varieties ([55, Sect. 4.2]).

Given [48], it is reasonable to ask the following problem in general.

Problem 3.4 *Classify the lattices Λ of signature $(2, n)$ such that the Satake-Baily-Borel compactification $\Gamma\backslash\mathcal{D}_\Lambda$ are Fano varieties, especially when $\Gamma = O^+(\Lambda)$ or $\widetilde{O}^+(\Lambda)$.*

From what follows, our arithmetic subgroup satisfies Γ is either $O^+(\Lambda)$ or the stable orthogonal group $\widetilde{O}^+(\Lambda)$.

Example 3.5 (*Moduli of elliptic K3 surfaces*) We consider the moduli M_W of Weierstrass elliptic K3 surfaces, which is an open subset of $O^+(\Lambda)\backslash\mathcal{D}_\Lambda$ for $\Lambda := U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$. We consider its Satake-Baily-Borel compactification ([56, Theorem 7.9]), which we denote $\overline{M_W}^{\mathrm{SBB}}$ here. Recall from *loc.cit* Sect. 7.1 that there are exactly two 1-cusps intersecting at the only 0-cusp. Two 1-cusps are M_W^{nn} with canonical Gorenstein singularity and M_W^{seg} with toroidal singularity (including the 0-cusp $\overline{M_W^{\mathrm{nn}}} \cap \overline{M_W^{\mathrm{seg}}}$) hence $\overline{M_W}^{\mathrm{SBB}}$ also only has log terminal singularity ([57, Part I, Sect. 2]). The notation of our superscripts "nn" and "seg" follow that of [56, Chapter 7] where some collapsing of hyperKähler metrics to *segment* i.e., $[0, 1]$ is partially observed along M_W^{seg}, and also that *non-normal* degenerations are parametrized by M_W^{nn}.

We recall that $\overline{M_W}^{\mathrm{SBB}}$ coincides with a certain GIT quotient of a weighted projective space ([56, Theorem 7.9]). Using the fact as well as some analysis of singularities along the 1-cusps in [57, Part I], we prove the following.

Theorem 3.6 *$\overline{M_W}^{\mathrm{SBB}}$ is a 18-dimensional log terminal rational Fano variety of Picard rank 1, although not isomorphic to any weighted projective space. Its two 1-cusps M_W^{seg} and M_W^{nn} are both non-naked.*

Proof The description of $\overline{M_W}^{\mathrm{SBB}}$ as a GIT quotient [56, Theorem 7.9] allows us to apply [12, Corollary 3] to confirm there is an effective $\mathbb{Q}$-divisor D on $\overline{M_W}^{\mathrm{SBB}}$ such that $-K_{\overline{M_W}^{\mathrm{SBB}}} - D$ is ample. Therefore, $-K_{\overline{M_W}^{\mathrm{SBB}}}$ is big. On the other hand, $\overline{M_W}^{\mathrm{SBB}}$ has Picard rank 1 because of the same GIT quotient description. Hence, the bigness of $-K_{\overline{M_W}^{\mathrm{SBB}}}$ implies it is actually even ample i.e., $\overline{M_W}^{\mathrm{SBB}}$ is a Fano variety.

The fact that both 1-cusps are non-naked are follows from Corollary 2.8, because $\overline{M_W}^{\mathrm{SBB}}$ is log terminal as proven in [57, Part I, Sect. 2]. (The log terminality also follows from [12, Theorem1] combined again with the fact that $\overline{M_W}^{\mathrm{SBB}}$ has Picard rank 1.) As for the rationality of M_W, [44] proved it, based on more classical rationality result of the moduli space of hyperelliptic curves (of genus 5).

The only remained thing to prove in the above theorem is that $\overline{M_W}^{\mathrm{SBB}}$ is not a weighted projective space. From the analysis of singularity type along 1-cusp M_W^{nn} in [57, Part I, Theorem 2.2], it easily follows that the local fundamental group along the transversal slice is $(\mathbb{Z}/2\mathbb{Z})^4$ hence not cyclic. In particular, $\overline{M_W}^{\mathrm{SBB}}$ can not be a weighted projective space. We complete the proof of Theorem 3.6. □

As a corollary, we also observe the following.

Corollary 3.7 *On the orthogonal modular variety $\overline{M_W}^{\mathrm{SBB}}$, there are special reflective modular forms which satisfy Assumption 2.2(i) (of Sect. 2.2) for some $s(X) > 1$ and sufficiently divisible $N \in \mathbb{Z}_{>0}$.*

Proof By the above theorem 3.6, we can apply Theorem 2.11 to complete the proof. □

3.3 Orthogonal Modular Cases, Part II

From here, we use the Borcherds products to show that various Satake-Baily-Borel compactifications of orthogonal modular varieties are Fano varieties or log canonical models.

Notation. Let

$$\mathcal{H}(\ell) := \{v \in \mathcal{D}_\Lambda \mid (v, \ell) = 0\}$$

be the special divisor with respect to $\ell \in \Lambda$ with $(\ell, \ell) < 0$. For any primitive element $r \in \Lambda$ satisfying $(r, r) < 0$, we define the *reflection* $\sigma_r \in O^+(\Lambda)(\mathbb{Q})$ with respect to r as follows:

$$\sigma_r(\ell) := \ell - \frac{2(\ell, r)}{(r, r)} r.$$

Then, the union of ramification divisors of $\pi_\Gamma \colon \mathcal{D}_\Lambda \to \Gamma\backslash\mathcal{D}_\Lambda$ is

$$\bigcup_{\substack{r\in\Lambda/\pm:\text{primitive}\\ \sigma_r\in\Gamma \text{ or } -\sigma_r\in\Gamma}} \mathcal{H}(r)$$

by [30] for $\Gamma \subset O^+(\Lambda)$ and $n > 2$. They also showed that the ramification degrees are 2. We sometimes denote π_Γ as π. We also define

$$\mathcal{H}_{-2} := \bigcup_{\ell\in\Lambda,\ \ell^2=-2} \mathcal{H}(\ell)$$
$$\mathcal{H}_{-4} := \bigcup_{\ell\in\Lambda,\ \ell^2=-4} \mathcal{H}(\ell)$$
$$\mathcal{H}_{-4,special-even} := \bigcup_{\ell\in\Lambda:special-even,\ \ell^2=-4} \mathcal{H}(\ell).$$

Here we say a vector $r \in \Lambda$ is special-even (also called even type e.g., in [43]) if $(\ell.r)$ is even for any $\ell \in \Lambda$, i.e., $\mathrm{div}(r)$ is even integer, so that the corresponding reflection lies in Γ. We define $\mathrm{div}(r)$ is the positive generator of the ideal

$$\{(\ell, r) \mid \ell \in \Lambda\}.$$

Remark 3.8 Below, for orthogonal cases, if f is a modular form corresponding to a section satisfying Assumption 2.2(i), we can compute $s(X) = \frac{k}{2mn}$. Here, k is the weight of f and m is the multiplicity of $\mathrm{div} f$, and $n = \dim X$.

Example 3.9 Let $II_{2,26} = U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus E_8(-1)$ be an even unimodular lattice of signature (2, 26). We consider the case $\Gamma = O^+(\Lambda)$. There is the modular form Φ_{12} of weight 12 on $\mathcal{D}_{II_{2,26}}$ by Borcherds [9] with

$$\mathrm{div}\Phi_{12} = \mathcal{H}_{-2}. \tag{4}$$

On the other hand, the ramification divisors of the map $\pi \colon II_{2,26} \to X := O^+(II_{2,26})\backslash\mathcal{D}_{II_{2,26}}$ are $\mathcal{H}_{-2}$ by the even unimodularity of Λ and [30].

Now $\Phi_{12}^{2\times 26}$ satisfies Assumption 1.2(i) with $s(X) = \frac{3}{13}$ and by Theorem 2.4(iii) so that the Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of the 26-dimensional orthogonal modular variety $X = O^+(II_{2,26})\backslash\mathcal{D}_{II_{2,26}}$ is a log canonical model i.e., with ample canonical divisor $K_{\overline{X}^{\mathrm{SBB}}}$ and at worst log canonical singularities. Let us specify and study the non-log-terminal locus or the log canonical center.

First, recall that there are exactly 24 1-cusps, which correspond to Niemeier lattices and all intersect at a common closed point (cf., e.g., [26, 1.1]). In particular, there is a 1-cusp which is the compactification of the modular curve $SL(2,\mathbb{Z})\backslash\mathbb{H}$ corresponding to the Leech lattice. We denote the particular 1-cusp as C_{Leech}.

For the Harish-Chandra-Borel embedding

$$\mathcal{D}_{II_{2,26}} \subset \mathcal{D}^c_{II_{2,26}} \subset \mathbb{P}(II_{2,26} \otimes \mathbb{C}),$$

$\mathcal{O}_{\mathbb{P}(II_{2,26}\otimes\mathbb{C})}(1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)|_{\mathbb{H}}$ for any 1-cusp $\mathbb{H} \subset \mathbb{P}^1$. For instance, by [9, Sect. 10], [26, 1.2], Φ_{12} restricts to the Ramanujan cusp form $\Delta_{12}(q) := q\prod_{n\geq 1}(1-q^n)^{24}$ of weight 12 on C_{Leech}. Since the only modular branch divisor is $\mathcal{H}_{-2}$, together with (4) and Lemma 2.9, it implies that the only log canonical center is the C_{Leech}. Recall that through the well-known isomorphism $SL(2,\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{A}^1(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C})$, the elliptic modular forms of weight $12k$ can be regarded with a section of $\mathcal{O}_{\mathbb{P}^1}(k)$, at the level of coarse moduli. In other words, $\mathcal{O}_{\mathbb{P}^1}(12k)|_{\mathbb{H}}$ descends to a line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ on $\mathbb{P}^1 \simeq SL(2,\mathbb{Z})\backslash\overline{\mathbb{H}}$ where $\overline{\mathbb{H}}$ denotes the rational closure of $\mathbb{H}$.

In particular, $(2s(X)L.C_{\mathrm{Leech}}) = 1$, where L follows the notation of Sect. 2.1. Equivalently $(K_{\overline{X}^{\mathrm{SBB}}}.C_{\mathrm{Leech}}) = \frac{5}{3}$, $(\overline{B}.C_{\mathrm{Leech}}) = 1$ as $s(X) = \frac{3}{13}$. We summarize our conclusion in this case neatly as $II_{2,26}$ attracts special attention.

Corollary 3.10 ($II_{2,26}$ case) *The Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of the 26-dimensional orthogonal modular variety $X = O^+(II_{2,26})\backslash\mathcal{D}_{II_{2,26}}$ is a log canonical model i.e., with ample canonical divisor $K_{\overline{X}^{\mathrm{SBB}}}$ and at worst log canonical singularities. Further, the non-log-terminal locus is the single $C_{\mathrm{Leech}} \simeq \mathbb{P}^1$ in the boundary $\partial X^{\mathrm{SBB}}$ which compactifies 1-cusp $SL(2,\mathbb{Z})\backslash\mathbb{H}$ and is characterized by that the corresponding isotropic plane $p \subset II_{2,26}\otimes\mathbb{R}$ satisfies that $(p^\perp \cap II_{2,26})/(p \cap II_{2,26})$ is the Leech lattice i.e., contains no roots. Its degree is $(K_{\overline{X}^{\mathrm{SBB}}}.C_{\mathrm{Leech}}) = \frac{5}{3}$. (resp., $(\overline{B}.C_{\mathrm{Leech}}) = 1$).*

Later in Example 3.32, we also construct a 13-dimensional unitary modular subvariety which also compactifies with ample canonical class as the Satake-Baily-Borel compactification.

Example 3.11 Let $\Lambda := U \oplus U \oplus E_8(-1)$ be an even unimodular lattice of signature (2, 10). We again consider the case $\Gamma = O^+(\Lambda)$. Borcherds constructed a reflective modular form on $\mathcal{D}_\Lambda$.

Theorem 3.12 ([9, 10.1, 16.1]) *There is a reflective modular form Φ_{252} of weight 252 on $\mathcal{D}_\Lambda$ such that*

$$\mathrm{div}\Phi_{252} = \mathcal{H}_{-2}.$$

Here, by the map $\pi : \mathcal{D}_\Lambda \to X := O^+(\Lambda)\backslash\mathcal{D}_\Lambda$, the divisors $\mathcal{H}_{-2}$ maps to the unique branch divisors (cf., [30, Sect. 2]). Hence Φ_{252}^{10t} satisfies Assumption 2.2(i) with $s(X) = \frac{63}{5}$ for some $t \in \mathbb{Z}$, and by Theorem 2.4(i), the compactified Shimura variety $\overline{X}^{\mathrm{SBB}}$ is a Fano variety. Actually, [34, 1.1], [17, 4.1] (also attributed to Shiga and [46]) shows it is the weighted projective space $\mathbb{P}(2, 5, 6, 8, 9, 11, 12, 14, 15, 18, 21)$.

Example 3.13 (*Moduli of Enriques surfaces*) The well-studied moduli space M_{Enr} of (unpolarized) Enriques surfaces (cf., e.g., [10, 41, 54, 59]) also fit into our setting. Let $\Lambda_{Enr} := U \oplus U(2) \oplus E_8(-2)$ be an even lattice of signature (2, 10). Then the Shimura variety

$$M_{Enr} := O^+(\Lambda_{Enr})\backslash\mathcal{D}_{\Lambda_{Enr}}$$

is a 10-dimensional quasi-projective variety. Now we review the ramification divisors of the natural map $\pi : \mathcal{D}_{\Lambda_{\mathrm{Enr}}} \to M_{\mathrm{Enr}}$ and moduli description. From [29, 30], the ramification divisors are

$$\mathcal{H}_{-2} \cup \mathcal{H}_{-4,special-even}.$$

On the other hand, let

$$\widetilde{M_{Enr}} := \widetilde{O}^+(\Lambda_{Enr})\backslash\mathcal{D}_{L_{Enr}}$$

be a finite cover of M_{Enr}. Then the following are known.

Proposition 3.14 *(i)* *$M_{Enr}\backslash\pi(\mathcal{H}_{-2})$ is the so-called moduli space of Enriques surfaces (cf., e.g., [54]). Moreover this is rational (Kondo [41]).*
(ii) *$\widetilde{M_{Enr}}\backslash\pi(\mathcal{H}_{-2})$, which is a finite cover of M_{Enr}, is the moduli space of Enriques surfaces with a certain level-2 structure. Moreover $\widetilde{M_{Enr}}$ and $\widetilde{M_{Enr}}\backslash\pi(\mathcal{H}_{-2})$ are of general type (Gritsenko-Hulek cf., [29]).*
(iii) *$M_{Enr}\backslash(\pi(\mathcal{H}_{-2}) \cup \pi(\mathcal{H}_{-4,special-even}))$ is the moduli space of non-nodal Enriques surfaces.*

Going back to our situation, we need special reflective modular forms satisfying Assumption 2.2(i). Our input here is the following.

Lemma 3.15 ([10, 43]) *There exist two reflective modular forms Φ_4 and Φ_{124} on $\mathcal{D}_{L_{Enr}}$ of weights* 4, 124 *respectively such that;*

$$\mathrm{div}\Phi_4 = \mathcal{H}_{-2},$$
$$\mathrm{div}\Phi_{124} = \mathcal{H}_{-4,\textit{special-even}}.$$

We put $F_{128} := \Phi_4\Phi_{124}$. Then this is a weight 128 modular form on $\mathcal{D}_{L_{Enr}}$ and $\mathrm{div}(F_{128})$ is exactly the ramification divisors of the map $\pi : \mathcal{D}_{L_{Enr}} \to M_{Enr}$ with coefficients 1. Now F_{128}^2 has a trivial character and satisfies Assumption 2.2(i) with $s(X) = \frac{32}{5}$ and by Theorem 2.4(i), $\overline{M_{Enr}}^{\mathrm{SBB}}$ is a log canonical Fano variety.

Actually, it is even log terminal without naked cusps as we confirm in the following. By [59, 3.3, 4.5], there are only two 0-cusps which correspond to an isotropic vector e in the first summand U and an isotropic vector e' the second summand $U(2)$ of Λ_{Enr}. They belong to the same 1-cusp which corresponds to isotropic plane $\mathbb{Q}e \oplus \mathbb{Q}e'$. That 1-cusp is contained in the closure of $\mathcal{H}_{-4,special-even}$ since e and e' are orthogonal to the (norm-doubled) root of $E_8(-2)$, the third summand of L_{Enr}. By *loc.cit*, the only other 1-cusp corresponds to another isotropic plane

$$p = \mathbb{Q}e' \oplus \mathbb{Q}(2e + 2f + \alpha)$$

where e, f is the standard basis of the first summand U and α is norm -8 integral vector in the third summand $E_8(-2)$. Since p is obviously orthogonal to the -2 vector $e - f \in U$, the corresponding 1-cusp is also contained in the closure of the Coble locus $\mathcal{H}_{-2}$. Hence there are no naked cusps so that we conclude the following.

Corollary 3.16 *The Satake-Baily-Borel compactification* $\overline{M_{Enr}}^{\mathrm{SBB}}$ *of the moduli of Enriques surfaces* M_{Enr} *is a log terminal Fano variety.*

Example 3.17 (*Moduli of log Enriques surfaces*) For each $1 \leq k \leq 10$ $(k \neq 2)$, let $\Lambda_{\mathrm{logEnr},k} := U(2) \oplus A_1 \oplus A_1(-1)^{\oplus 9-k}$ be an even lattice of signature $(2, 10-k)$. Then the associated Shimura variety $O^+(\Lambda_{logEnr,k})\backslash\mathcal{D}_{L_{logEnr}}$ is a (partial compactification of) the moduli space of log Enriques surface with k $\frac{1}{4}(1,1)$ singularities. For the definition of log Enriques surfaces with $\frac{1}{4}(1,1)$ singularities, see [16, Definition 2.1, 2.6]. Yoshikawa [63] and Ma [47] constructed reflective modular forms on $\mathcal{D}_{L_{logEnr}}$ for $k \leq 7$ which we use.

Theorem 3.18 ([63, Theorem 4.2(i)]) *There is a reflective modular form* Ψ_4 *of weight* $4 + k$ *on* $\mathcal{D}_{\Lambda_{logEnr,k}}$ *with*

$$\mathrm{div}\Psi_{4+k} = \mathcal{H}_{-2}.$$

Theorem 3.19 ([47, Appendix by Yoshikawa; A.4, proof of A.5]) *There is a reflective modular form* $\Psi_{124,k}$ *of weight* $-k^2 - 9k + 124$ *on* $\mathcal{D}_{\Lambda_{logEnr,k}}$ *with*

$$\mathrm{div}\Psi_{124,k} = \mathcal{H}_{-4}.$$

On the other hand, the ramification divisors of the map $\pi \colon \mathcal{D}_{L_{logEnr,k}} \to X := O^+(L_{logEnr,k})\backslash\mathcal{D}_{L_{logEnr,k}}$ is the union of special divisors with respect to (-2)-vectors and (-4)-vectors by the same discussion. As $(\Psi_{4+k}\Psi_{124,k})^{t(10-k)}$ with $t \in \mathbb{Z}_{>0}$ satisfies Assumption 2.2(i) with $s(X) = \frac{-k^2-8k+128}{2(10-k)}$ for $k \leq 7$, by Theorem 1.3 (i), we conclude the following.

Corollary 3.20 *For the above (partially compactified) moduli spaces of log Enriques surface with* k $\frac{1}{4}(1,1)$ *singularities with* $1 \leq k \leq 7$ $(k \neq 2)$ $X = O^+(\Lambda_{logEnr,k})\backslash \mathcal{D}_{L_{logEnr}}$*, the Satake-Baily-Borel compactifications* $\overline{X}^{\mathrm{SBB}}$ *are Fano varieties.*

Actually, they are also unirational, by [47].

Example 3.21 (*Simple lattices case*) Let Λ be a quadratic lattice over $\mathbb{Z}$ of signature $(2, n)$. We recall from [15] that Λ is called *simple* if the space of cusp forms of weight $1 + \frac{n}{2}$ associated with a finite quadratic form $\Lambda^\vee/\Lambda$ is zero. Then the special divisors on $\mathcal{D}_\Lambda$ are all given by the divisors of Borcherds lift, so that we can apply Theorem 2.4.

In fact, Wang-Williams [65] showed that for every simple lattice Λ of signature $(2, n)$ with $3 \le n \le 10$, the graded algebra of modular forms for certain subgroups of the orthogonal group is freely generated. From this, we have the associated Shimura varieties are weighted projective spaces, in particular, log terminal $\mathbb{Q}$-Fano.

From Theorem 2.4, all Borcherds product satisfying Assumption 2.2(i) should have $s(X) > 1$. Also from Corollary 2.8, the boundary of the Satake-Baily-Borel compactification is in the closure of the branch divisors. See the tables of examples in [65].

We remark that before [15, 65] showed there are only finitely many isometry classes of even simple lattices Λ of signature $(2, n)$.

3.4 Preparation for Unitary Case—Hermitian Lattice

Here, we recall some material on Hermitian lattices treated in [35] to prepare for constructing some examples of unitary modular varieties from the next subsection. There, we similarly apply Theorem 2.4 to certain restriction of Borcherds products to explore their birational properties.

Here is the setup. Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field for a square-free negative integer d, and $\mathcal{O}_F$ be its ring of integers. Let δ be the inverse different of F, i.e.,

$$\delta := \begin{cases} \frac{1}{2\sqrt{d}} & (d \equiv 2, 3 \bmod 4), \\ \frac{1}{\sqrt{d}} & (d \equiv 1 \bmod 4). \end{cases}$$

Let $(\Lambda, \langle\ ,\ \rangle)$ be a Hermitian lattice of signature $(1, n)$ over $\mathcal{O}_F$ in the sense of [35] i.e., a finite free $\mathcal{O}_F$-module with an Hermitian form which is $\delta\mathcal{O}_F$-valued. We define the dual lattice $\Lambda^\vee$ as

$$\Lambda^\vee := \{v \in \Lambda \otimes_{\mathcal{O}_F} F \mid \langle v, w\rangle \in \delta\mathcal{O}_F\ (\forall w \in \Lambda)\}.$$

We say Λ is unimodular if $\Lambda = \Lambda^\vee$ and Λ is even if $\langle v, v\rangle \in \mathbb{Z}$ for all $v \in \Lambda$. The latter means the associated quadratic form is even. It is also known that the quotient $A_\Lambda = \Lambda^\vee/\Lambda$ is a finite $\mathcal{O}_F$-module, called the discriminant group. Then, $\widetilde{U}(\Lambda) := \{g \in U(\Lambda) \mid g|_{A_\Lambda} = 1_{A_\Lambda}\}$ is the so-called discriminant kernel or the stable unitary group. For a Hermitian lattice Λ, we define $\Lambda(a) := (\Lambda, a\langle\ ,\ \rangle)$ for $a \in \delta\mathcal{O}_F$. Analogously to quadratic forms, we also have the following proposition.

Proposition 3.22 *There exists a unimodular Hermitian lattice M and an element $b \in \mathcal{O}_F$ such that $\Lambda = M(b)$ if and only if the ideal $\{\langle v, w\rangle \in \delta\mathcal{O}_F \mid w \in \Lambda\}$ with respect to $v \in \Lambda$ is equal $b\delta\mathcal{O}_F$ for every primitive element $v \in \Lambda$.*

Let D_Λ be the Hermitian symmetric domain (complex ball) with respect to $U(\Lambda)(\mathbb{R})$, equivalently,

$$D_\Lambda := \{v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid \langle v, v\rangle > 0\}$$

and $H(v)$ be the special divisor with respect to $v \in \Lambda$. For any element $r \in \Lambda$ satisfying $\langle r, r\rangle < 0$ and $\xi \in \mathcal{O}_F^\times \backslash \{1\}$, we define the *quasi-reflection* $\tau_{r,\xi} \in U(\Lambda)(\mathbb{Q})$ with respect to r, ξ as follows:

$$\tau_{r,\xi}(\ell) := \ell - (1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} r.$$

Note that for $\xi = -1$, we have the usual reflection. See also [2]. We also remark that, for example, for $F = \mathbb{Q}(\sqrt{-1})$, we get $\tau_{r,\sqrt{-1}}^2 = \tau_{r,-1}$ and for $F = \mathbb{Q}(\sqrt{-3})$, we get $\tau_{r,\omega}^2 = \tau_{r,\overline{\omega}}$ for any $r \in \Lambda$ where ω is a primitive third root of unity.

The union of ramification divisors of $\pi_\Gamma\colon D_\Lambda \to \Gamma \backslash D_\Lambda$ is

$$\bigcup_r H(r)$$

by [8, Corollary 3] for $\Gamma \subset U(\Lambda)$ and $n > 1$. Here, the union runs thorough primitive elements $r \in \Lambda/\mathcal{O}_F^\times$ with $\langle r, r\rangle < 0$ such that $\eta\tau_{r,\xi} \in \Gamma$ for some $\eta \in \mathcal{O}_F^\times$ and $\xi \in \mathcal{O}_F^\times \backslash \{1\}$. We consider the natural embedding of the type I domain to the type IV domain

$$\iota\colon D_\Lambda \hookrightarrow \mathcal{D}_{\Lambda_Q}$$

where $(\Lambda_Q, (\ ,\ ))$ is the quadratic lattice associated with $(\Lambda, \langle\ ,\ \rangle)$, i.e., $\Lambda_Q := \Lambda$ as a $\mathbb{Z}$-module and $(\ ,\ ) := \mathrm{Tr}_{F/\mathbb{Q}}\langle\ ,\ \rangle$. For the analysis of ramification divisors on D_Λ, we first prepare the following lemma.

Lemma 3.23 *For $F = \mathbb{Q}(\sqrt{d})$, assume $d \equiv 2, 3 \bmod 4$ or $d = -3$. Then*

$$\iota\Bigg(\bigcup_{\substack{r\in\Lambda/\mathcal{O}_F^\times:\text{primitive}\\ \eta\tau_{r,\xi}\in U(\Lambda)\text{ for }\exists\eta\in\mathcal{O}_F^\times,\ \exists\xi\in\mathcal{O}_F^\times\backslash\{1\}}} H(r)\Bigg) \subset \bigcup_{\substack{r\in\Lambda_Q/\pm:\text{primitive}\\ \sigma_r\in O^+(\Lambda_Q)\text{ or }-\sigma_r\in O^+(\Lambda_Q)}} \mathcal{H}(r) \cap \iota(D_\Lambda).$$

Proof For $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, it suffices to show that if

$$\frac{2\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathcal{O}_F,$$

then

$$\alpha := \frac{2(\ell, r)}{(r, r)} = \frac{2\,\mathrm{Tr}_{F/Q}\langle \ell, r\rangle}{\mathrm{Tr}_{F/\mathbb{Q}}\langle r, r\rangle} \in \mathbb{Z}.$$

Since $\langle r, r\rangle \in \mathbb{Q}$, we have

$$\alpha = \Re \frac{2\langle \ell, r\rangle}{\langle r, r\rangle}.$$

Hence for $d \equiv 2, 3 \bmod 4$ with $d \neq -1$, this concludes lemma.

For $F = \mathbb{Q}(\sqrt{-1})$, it needs to show that if

$$(1-\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathcal{O}_F \text{ or } (1+\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathcal{O}_F,$$

then $\alpha \in \mathbb{Z}$. In the following, let a, b be rational integers. First, we assume

$$(1-\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = a + \sqrt{-1}b \in \mathcal{O}_F.$$

Then $\alpha = a - b \in \mathbb{Z}$. Second, we assume

$$(1+\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = a + \sqrt{-1}b \in \mathcal{O}_F.$$

Then $\alpha = a + b \in \mathbb{Z}$. This concludes lemma for $F = \mathbb{Q}(\sqrt{-1})$.

For $F = \mathbb{Q}(\sqrt{-3})$, assume that one of the following holds.

$$(1 \pm \omega)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathcal{O}_F, \tag{5}$$

$$(1 \pm \omega^2)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathcal{O}_F, \tag{6}$$

$$2\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathcal{O}_F. \tag{7}$$

Through some simple computation, when (5) or (6) hold, then we have $\alpha \in \mathbb{Z}$. Finally, we assume (7). Let

$$\alpha = \alpha_1 = \frac{2(\ell, r)}{(r, r)} = a - \frac{b}{2},$$

$$\alpha_2 = \frac{2(\ell, \omega r)}{(\omega r, \omega r)} = -\frac{a}{2} + b,$$

$$\alpha_3 = \frac{2(\ell, \omega^2 r)}{(\omega^2 r, \omega^2 r)} = -\frac{a+b}{2}.$$

Hence, the assumption $a + \omega b \in \mathcal{O}_F$ implies one of α_i for $i = 1, 2, 3$ is an element of $\mathbb{Z}$. On the other hand, we have $H(r) = H(\omega r) = H(\omega^2 r)$ and $\iota(H(r)) \subset \mathcal{H}(r)$, thus this concludes lemma for $F = \mathbb{Q}(\sqrt{-3})$. □

For the computation of multiplicities of unitary modular forms later, we need the following converse to [35, Remark after 6.1].

Lemma 3.24 *Let $r \in \Lambda$ be a primitive element with $\langle r, r\rangle < 0$.*

(i) The special divisor $H(r)$ is contained in exactly $\frac{\#\mathcal{O}_F^\times}{2}$ special divisors of the form $\mathcal{H}(r') \subset \mathcal{D}_{\Lambda_Q}$ for some primitive $r' \in \Lambda_Q$.
(ii) The restriction of the special divisor $\mathcal{H}_r|_{D_\Lambda}$ is $H(r)$ with multiplicity 1 i.e., reduced.

Proof We fix $\sqrt{d} \in \mathbb{C}$ and the corresponding embedding $F \hookrightarrow \mathbb{C}$. First, we prove (i). Note $\mathcal{H}(r)|_{D_\Lambda} = \mathcal{H}(r')|_{D_\Lambda}$ if and only if $\mathbb{C}r' = \mathbb{C}r$ for $r, r' \in \Lambda$. This implies $r = ar'$ for some $a \in \mathbb{C}^\times$. Since r is primitive, we have $a \in \mathcal{O}_F^\times$. On the other hand, as $\mathcal{H}(r')$ only depends on $\mathbb{R}r'$ so that $\mathcal{H}(r') = \mathcal{H}(-r')$, the number we concern is $\frac{\#\mathcal{O}_F^*}{2}$.

The proof of (ii) is as follows. Since $\langle r, r\rangle < 0$, $\mathcal{H}(r)$ is again an orthogonal symmetric domain which is an (analytic) open subset of a quadric hypersurface, say $Q^{n-1} \subset Q^n \subset \mathbb{P}^{n+1}$. Thus the restriction of the Cartier divisor $r = 0$ to Q^n is reduced and $\mathcal{H}(r)$ is its open subset. $H(r)$ is also an open subset of the restriction of $r = 0$ to the linear subspace, which is also clearly reduced. Hence the assertion follows. □

3.5 Unramifiedness of Unitary Modular Varieties

Theorem 3.25 *Let $F = \mathbb{Q}(\sqrt{d})$ $(d \neq -1)$ be an imaginary quadratic field and Λ be a Hermitian unimodular lattice over $\mathcal{O}_F$ of signature $(1, n)$ for $n > 1$. We assume $d \equiv 2, 3 \bmod 4$. Then for any arithmetic subgroup $\Gamma \subset U(\Lambda)$, the canonical map $\pi_\Gamma : D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1, so that $\overline{X}^{\mathrm{SBB}}$ is a log canonical model.*

Proof It suffices to show the claim for $\Gamma = U(\Lambda)$. The ramification divisors are defined by $\tau_{r,\xi}$ for some primitive $r \in \Lambda$ and $\xi \in \mathcal{O}_F^\times\backslash\{1\}$ and by Lemma 3.23, they are included in the set

$$\bigcup_{r\in\Lambda, b\in\mathbb{Z}, \xi\in\mathcal{O}_F^\times\backslash\{1\}} \quad \bigcup_{\substack{r\in\Lambda/\mathcal{O}_F^\times \\ \langle r,r\rangle = -\frac{b}{2},\ \tau_{r,\xi}\in U(\Lambda)}} H(r).$$

Now

$$\tau_{r,\xi}(\ell) = \ell - (1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} r.$$

We assume that $r \in \Lambda$ is a reflective element, that is, $\tau_{r,\xi} \in U(\Lambda)$ for some $\xi \in \mathcal{O}_F^\times\backslash\{1\}$. Then

$$(1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = -\frac{2(1-\xi)\langle \ell, r\rangle}{b}.$$

Since r is primitive and Λ is unimodular, by Proposition 3.22, there exists an $\ell \in \Lambda$ such that $\langle \ell, r\rangle = \frac{1}{2\sqrt{d}}$, so we have

$$(1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = -\frac{1-\xi}{b\sqrt{d}} \notin \mathcal{O}_F$$

for $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$. This implies $\tau_{r,\xi} \notin U(\Lambda)$ and this is contradiction. The last assertion then follows from [51] (or as a special case of our Theorem 2.4(iii)). $\square$

Note that we can also deduce this result from [66, Lemma 2.2].

Corollary 3.26 *Let $F = \mathbb{Q}(\sqrt{d})$ $(d \neq -1)$ be an imaginary quadratic field and $(\Lambda, \langle\ ,\ \rangle) = M(b)$ be a Hermitian lattice over $\mathcal{O}_F$ of signature $(1, n)$ for $n > 1$ where M is a unimodular Hermitian lattice and $b \in \mathcal{O}_F$. We assume $d \equiv 2, 3 \bmod 4$, and $\frac{b}{\sqrt{d}} \notin \mathcal{O}_F$. Then for any arithmetic subgroup $\Gamma \subset U(\Lambda)$, the canonical map $\pi_\Gamma\colon D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1.*

3.6 Unitary Modular Cases, Part I—Fano Cases

Below, for the definition of Hermitian lattices; see Appendix 3.8.

Remark 3.27 We can estimate the value $s(X)$ as orthogonal modular varieties and use it to determine the birational types of ball quotients. Note that the ramification degrees arising from unitary cases may differ from orthogonal ones [8], so we have to pay attention to the computation of a; compare with Remark 3.8.

For $F = \mathbb{Q}(\sqrt{-1})$, let B_2 (resp. B_4) be a union of ramification divisor with ramification degree 2 (resp. 4). If a modular form f of weight k vanishes on B_2 (resp. B_4) with order $2m$ (resp. $3\,m$) for some $m \in \mathbb{Z}_{>0}$, then f satisfies Assumption 2.1(i) and $s(X) = \frac{k}{4mn}$.

Example 3.28 For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}$ be an even unimodular Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ of signature $(1, 5)$ whose associated quadratic lattice is $\Lambda_Q = U \oplus U \oplus E_8(-1)$.

The only ramification divisors of the map $D_\Lambda \to X := U(\Lambda)\backslash D_\Lambda$ are

$$\bigcup_{\substack{r\in\Lambda/\mathcal{O}_F^\times:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

with ramification degree 2. For more details, see Example 3.32.

By Example 3.11, $f := \Phi_{252}|_{D_\Lambda}$ is a weight 252 modular form with

$$\operatorname{div} f = 2 \sum_{\substack{r \in L/\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r)$$

whose coefficient comes from Lemma 3.24. Therefore applying Theorem 2.4(i) for f^{12} with $s(X) = \frac{21}{2}$, we have the following.

Corollary 3.29 *The Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of the Shimura variety $X := U(\Lambda)\backslash D_\Lambda$ is a Fano variety, where $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}$ for $F = \mathbb{Q}(\sqrt{-1})$.*

Example 3.30 For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U\oplus U(2)} \oplus \Lambda_{E_8(-1)}(2)$ be an even Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ of signature $(1, 5)$ whose associated quadratic lattice is $\Lambda_Q = U \oplus U(2) \oplus E_8(-2)$. The ramification divisors on $\mathcal{D}_{\Lambda_Q}$ with respect to $O^+(\Lambda_Q)$ is the union of special divisors with respect to (-2)-vectors and special-even (-4)-vectors, so the ramification divisors on D_Λ with respect to $U(\Lambda)$ are included in the union of special divisors with respect to (-1)-vectors and special-even (-2)-vectors since $\langle v, v\rangle$ is real for all $v \in \Lambda$. Here we say a vector $r \in \Lambda$ is special-even if $\Re\langle r, v\rangle \in \mathbb{Z}$ for any $v \in \Lambda$. The only ramification divisors of π are

$$\bigcup_{\substack{r \in L/\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1,\ \tau_{r,-1}\in U(\Lambda)}} H(r) \quad \cup \bigcup_{\substack{r \in L/\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive} \\ \langle r,r\rangle=-2,\ \tau_{r,-1}\in U(\Lambda)}} H(r)$$

with ramification degree $d_i = 2$ and

$$\bigcup_{\substack{r \in \Lambda/\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1,\ \tau_{r,\sqrt{-1}}\in U(\Lambda)}} H(r) \quad \cup \bigcup_{\substack{r \in \Lambda/\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive} \\ \langle r,r\rangle=-2,\ \tau_{r,\sqrt{-1}}\in U(\Lambda)}} H(r).$$

with ramification degree $d_i = 4$. For any primitive element $r \in \Lambda$ with $\langle r, r\rangle = -1$, we have

$$\tau_{r,-1}(\ell) = \ell + 2\langle \ell, r\rangle r.$$

By the description of Hermitian lattices $\Lambda_{U\oplus U(2)}$ and $\Lambda_{E_{8(-1)}(2)}$,

$$2\langle \ell, r\rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}.$$

Hence $\tau_{r,-1} \in U(\Lambda)$ for any (-1)-primitive element $r \in \Lambda$. For any special-even primitive element $r \in \Lambda$ with $\langle r, r\rangle = -2$, we have

$$\tau_{r,-1}(\ell) = \ell + \langle \ell, r\rangle r.$$

By the definition of $\Lambda_{U\oplus U(2)}$, if $\Re\langle \ell, r\rangle \in \mathbb{Z}$, then $\Im\langle \ell, r\rangle \in \mathbb{Z}$ for any $\ell \in \Lambda$. Also by the definition of $\Lambda_{E_8}(-2)$, we have $\langle \ell, r\rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ for any $\ell \in \Lambda$. Hence $\tau_{r,-1} \in$

$U(\Lambda)$ for any special-even (-2)-primitive vector $r \in \Lambda$. Therefore the map $D_\Lambda \to X := U(\Lambda)\backslash D_\Lambda$ ramifies along

$$\bigcup_{\substack{r \in L/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}\text{:primitive} \\ \langle r,r\rangle=-1}} H(r) \quad \cup \bigcup_{\substack{r \in L/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}\text{:special-even, primitive} \\ \langle r,r\rangle=-2}} H(r).$$

For (-1)-primitive vector $r \in \Lambda$,

$$\tau_{r,\sqrt{-1}}(\ell) = \ell + (1-\sqrt{-1})\langle \ell, r\rangle r.$$

If $r \in \Lambda_{E_8(-1)}(2)$, then by the description of the Hermitian matrix defining $\Lambda_{E_8(-2)}$, we have $\langle \ell, r\rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, so $(1-\sqrt{-1})\langle \ell, r\rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. If $r \in \Lambda_{U\oplus U(2)}$, then the ideal $\{\langle \ell, r\rangle \mid \ell \in \Lambda_{U\oplus U(2)}\}$ is generated by $\frac{1+\sqrt{-1}}{2}$ since $\det(L_{U\oplus U(2)}) = \frac{1}{2}$, so $(1-\sqrt{-1})\langle \ell, r\rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. From a similar discussion as above, we have $\tau_{r,\sqrt{-1}} \in U(\Lambda)$ for any (-1)-primitive vector $r \in \Lambda$.

For special-even (-2)-primitive vector $r \in \Lambda$,

$$\tau_{r,\sqrt{-1}}(\ell) = \ell + \frac{(1-\sqrt{-1})}{2}\langle \ell, r\rangle r.$$

If $r \in \Lambda_{E_8(-1)}(2)$, then there exists an $\ell \in \Lambda_{E_8(-1)}(2)$ such that $\langle \ell, r\rangle = 1$, so we have $\frac{(1-\sqrt{-1})\langle \ell,r\rangle}{2} = \frac{1-\sqrt{-1}}{2} \notin \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. If $r \in \Lambda_{U\oplus U(2)}$, then there exists an $\ell \in \Lambda_{U\oplus U(2)}$ such that $\langle \ell, r\rangle = \frac{1+\sqrt{-1}}{2}$, so we have $\frac{(1-\sqrt{-1})\langle \ell,r\rangle}{2} = \frac{1}{2} \notin \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. Thus, we have $\tau_{r,\sqrt{-1}} \notin U(\Lambda)$ for any special-even (-2)-primitive vector $r \in \Lambda$.

Therefore, the ramification in codimension 1 only occurs along

$$\bigcup_{\substack{r \in \Lambda/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}\text{:primitive} \\ \langle r,r\rangle=-1}} H(r)$$

with ramification degree 2, and along

$$\bigcup_{\substack{r \in \Lambda/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}\text{:special-even, primitive} \\ \langle r,r\rangle=-2}} H(r)$$

with ramification degree 4.

This example implies Theorem 3.25 does not hold for non-unimodular lattices and $F = \mathbb{Q}(\sqrt{-1})$.

By Example 3.13, we have modular forms $\Phi_4|_{D_\Lambda}$ and $\Phi_{124}|_{D_\Lambda}$ such that

$$\mathrm{div}\Phi_4|_{D_\Lambda} = 2 \sum_{\substack{r\in\Lambda/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

$$\mathrm{div}\Phi_{124}|_{D_\Lambda} = 2 \sum_{\substack{r\in\Lambda/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive}\\ \langle r,r\rangle=-2}} H(r)$$

whose coefficient again comes from Lemma 3.24.

Hence, applying Theorem 2.4(i) to $(\Phi_4|^2_{D_\Lambda}\Phi_{124}|^3_{D_\Lambda})^{12}$ with $s(X) = 62$, we have the following.

Corollary 3.31 *The Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of the Shimura variety $X := U(\Lambda)\backslash D_\Lambda$ is a Fano variety, where $\Lambda := \Lambda_{U\oplus U(2)} \oplus \Lambda_{E_8(-1)}(2)$ for $F = \mathbb{Q}(\sqrt{-1})$.*

3.7 Unitary Modular Cases, Part II—with Ample Canonical Class

Example 3.32 For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)}$ be an even unimodular Hermitian lattice of signature $(1, 13)$ whose associated quadratic lattice is $\Lambda_Q = II_{2,26} = U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus E_8(-1)$. The ramification divisors on $\mathcal{D}_{\Lambda_Q}$ with respect to $O^+(\Lambda_Q)$ is the union of special divisors with respect to (-2)-vectors, so the ramification divisors on D_Λ with respect to $U(\Lambda)$ are included in the union of special divisors with respect to (-1)-vectors as $\langle v, v\rangle$ is real for all $v \in \Lambda$. There exist possibly double ramification divisors i.e., those with $d_i = 2$, and quadruple ramification divisors i.e., those with $d_i = 4$, of the natural morphism $\pi : D_\Lambda \to X := U(\Lambda)\backslash D_\Lambda$. It ramifies in codimension 1 along

$$\bigcup_{\substack{r\in\Lambda/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1,\ \tau_{r,-1}\in U(\Lambda)}} H(r)$$

with ramification degree 2, and

$$\bigcup_{\substack{r\in\Lambda/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1,\ \tau_{r,\sqrt{-1}}\in U(\Lambda)}} H(r)$$

with ramification degree 4.

For any primitive element $r \in \Lambda$ with $\langle r, r\rangle = -1$, we have

$$\tau_{r,\sqrt{-1}}(\ell) = \ell + (1 - \sqrt{-1})\langle \ell, r\rangle r,$$

but by Proposition 3.22 and unimodularity of Λ, $\langle \ell, r\rangle = \frac{1}{2\sqrt{-1}}$ for some $\ell \in \Lambda$. Hence $\tau_{r,-1} \notin U(\Lambda)$ for any (-1)-primitive element $r \in \Lambda$, that is, there is no quadruple ramification divisors.

For any primitive element $r \in \Lambda$ with $\langle r, r\rangle = -1$, we have

$$\tau_{r,-1}(\ell) = \ell + 2\langle \ell, r\rangle r.$$

Here,

$$\langle \ell, r\rangle \in \delta\mathcal{O}_F = \frac{1}{2\sqrt{-1}}\mathcal{O}_{\mathbb{Q}(\sqrt{-1})},$$

so $2\langle \ell, r\rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. Thus, $\tau_{r,-1} \in U(\Lambda)$ for any (-1)-primitive element $r \in \Lambda$, that is, there are only double ramification divisors along

$$\bigcup_{\substack{r\in\Lambda/\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

with ramification degree 2. By Example 3.9, $f := \Phi_{12}|_{D_\Lambda}$ is a weight 12 modular form whose divisors are equal to double ramification divisors;

$$\operatorname{div} f = 2\sum_{\substack{r\in\Lambda/\mathcal{O}^{\times}_{F}:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

whose coefficient again comes from Lemma 3.24. Therefore applying Theorem 2.4(iii) to f^{28} with $s(X) = \frac{3}{14}$, we have the following the following.

Corollary 3.33 *The Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of the Shimura variety $X := U(\Lambda)\backslash D_\Lambda$ is a log canonical model, where $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)}$ for $F = \mathbb{Q}(\sqrt{-1})$. Recall from Terminology after Theorem 2.4 that a log canonical model in this paper means it has only log canonical singularities and ample canonical class.*

Example 3.34 For $F = \mathbb{Q}(\sqrt{-2})$, let $\Lambda := \Lambda'_{U\oplus U(2)} \oplus \Lambda'_{E_8(-1)}(2)$ be an even Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ of signature $(1, 5)$. The union of ramification divisors of the map $\pi : D_\Lambda \to X := U(\Lambda)\backslash D_\Lambda$ are the union of special divisors with respect to (-1)-vectors only, unlike $F = \mathbb{Q}(\sqrt{-1})$ case. Of course, these divisors ramify with ramification degree 2, so we can also show $\overline{X}^{\mathrm{SBB}}$ is a log canonical model. (Applying Theorem 2.4(iii) to f^{12} with $s(X) = \frac{1}{6}$, where $f := \Phi_4|_{D_\Lambda}$.) This example implies Theorem 3.25 does not hold for non-unimodular lattices and there exist Hermitian lattices, whose quadratic lattices are the same, admitting Shimura varieties with various birational types according to imaginary quadratic fields.

Corollary 3.35 *The Satake-Baily-Borel compactification $\overline{X}^{\mathrm{SBB}}$ of the Shimura variety $X := U(\Lambda)\backslash D_\Lambda$ is a log canonical model, where $\Lambda := \Lambda'_{U\oplus U(2)} \oplus \Lambda'_{E_8(-1)}(2)$ for $F = \mathbb{Q}(\sqrt{-2})$.*

Remark 3.36 For $F = \mathbb{Q}(\sqrt{-2})$, let $\Lambda := \Lambda'_{U\oplus U} \oplus \Lambda'_{E_8(-1)} \oplus \Lambda'_{E_8(-1)} \oplus \Lambda'_{E_8(-1)}$ be an even unimodular Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ of signature $(1, 13)$, whose associated quadratic lattice Λ_Q is $U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus E_8(-1)$.

Now, we know that for any arithmetic subgroup $\Gamma \subset U(\Lambda)$, the map $\pi : D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1. This is exactly an example of Theorem 3.25. Thus the Satake-Baily-Borel compactification $\overline{\Gamma\backslash D_\Lambda}^{\mathrm{SBB}}$ is a log canonical model.

Remark 3.37 For any imaginary quadratic field with class number 1, we can construct $\Lambda_{U\oplus U}$ and Λ_{E_8}; see [49, Appendix A]. As in Theorem 3.25, we can show that the corresponding map does not ramify in codimension 1 for any arithmetic subgroup so that the Satake-Baily-Borel compactification is log canonical model again.

Remark 3.38 By the same reason as Remark 3.36, for $F \neq \mathbb{Q}(\sqrt{-1})$, the map $\pi : D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1, where $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}$ and $\Gamma \subset U(\Lambda)$ is any arithmetic subgroup. This is also an example of Theorem 3.25 and $\overline{\Gamma\backslash D_\Lambda}^{\mathrm{SBB}}$ is a log canonical model.

3.8 More Examples

For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda_{-1} := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}(2)$. Then, the map $\pi : D_{\Lambda_{-1}} \to U(\Lambda_{-1})\backslash D_{\Lambda_{-1}}$ ramifies at the union of special divisors with respect to (-1)-vectors and (-2)-special-even vectors. By [64, Theorem 8.1], there exists a reflective modular form Ψ_{12} of weight 12 on $\mathcal{D}_{(\Lambda_{-1})_Q}$ such that

$$\mathrm{div}\Psi_{12}|_{D_\Lambda} = 2 \sum_{\substack{r\in\Lambda_{-1}/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r)$$

whose coefficient again comes from Lemma 3.24. Thus, $\iota^\star\Psi_{12} = \Psi_{12}|_{D_{\Lambda_{-1}}}$ is a reflective modular form on $D_{\Lambda_{-1}}$, but this does not satisfy Assumption 2.2(ii) because the ramification divisors properly include the divisors of $\Psi_{12}|_{D_{\Lambda_{-1}}}$, i.e.,

$$\mathrm{Supp}(\mathrm{div}\Psi_{12}|_{D_\Lambda}) \subsetneq \bigcup_{\substack{r\in L/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r) \;\cup \bigcup_{\substack{r\in L/\mathcal{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive} \\ \langle r,r\rangle=-2}} H(r),$$

where the right-hand side is the ramification divisor. Hence, we can not show the Fano-ness of $\overline{(U(\Lambda_{-1})\backslash D_{\Lambda_{-1}})}^{\mathrm{SBB}}$ in this way (but we can show the uniruledness or more strongly, rationally-chain-connectedness of $U(\Lambda_{-1})\backslash D_{\Lambda_{-1}}$ by [49, Theorem 5.1]).

On the other hand, for $F=\mathbb{Q}(\sqrt{-2})$, let Λ_{-2} be the Herimtian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ of signature $(1,5)$ whose associated quadratic lattice is $U\oplus U\oplus E_8(-2)$. Then the map $\pi : D_{\Lambda_{-2}} \to U(\Lambda_{-2})\backslash D_{\Lambda_{-2}}$ has no ramification divisors, so we can not even show the uniruledness.

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Appendix: Matrix Definitions

The following matrices are taken from [49, Appendix].

A.1 $\mathbb{Q}(\sqrt{-1})$ *Cases*

$\mathbb{Q}(\sqrt{-1})$ Cases Let $\Lambda_{U\oplus U}$ be an even unimodular Hermitian lattice of signature $(1,1)$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ defined by the matrix

$$\frac{1}{2\sqrt{-1}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda_{U\oplus U})_Q$ is $U\oplus U$.

Let $\Lambda_{U\oplus U(2)}$ be an even Hermitian lattice of signature $(1,1)$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ defined by the matrix

$$\frac{1}{2}\begin{pmatrix} 0 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda_{U\oplus U(2)})_Q$ is $U\oplus U(2)$.

Let $\Lambda_{E_8(-1)}$ be an even unimodular Hermitian lattice of signature $(0,4)$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ defined by the matrix

$$-\frac{1}{2}\begin{pmatrix} 2 & -\sqrt{-1} & -\sqrt{-1} & 1 \\ \sqrt{-1} & 2 & 1 & \sqrt{-1} \\ \sqrt{-1} & 1 & 2 & 1 \\ 1 & -\sqrt{-1} & 1 & 2 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda_{E_8(-1)})_Q$ is $E_8(-1)$. This matrix is called Iyanaga's matrix.

A.2 $\mathbb{Q}(\sqrt{-2})$ Cases

$\mathbb{Q}(\sqrt{-2})$ Cases Let $\Lambda'_{U\oplus U}$ be an even unimodular Hermitian lattice of signature $(1, 1)$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$\frac{1}{2\sqrt{-2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda'_{U\oplus U})_Q$ is $U \oplus U$.

Let $\Lambda'_{U\oplus U(2)}$ be a Hermitian lattice of signature $(1, 1)$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda'_{U\oplus U(2)})_Q$ is $U \oplus U(2)$.

Let $\Lambda'_{E_8(-1)}$ be an even unimodular Hermitian lattice of signature $(0, 4)$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$-\frac{1}{2}\begin{pmatrix} 2 & 0 & \sqrt{-2}+1 & \frac{1}{2}\sqrt{-2} \\ 0 & 2 & \frac{1}{2}\sqrt{-2} & 1-\sqrt{-2} \\ 1-\sqrt{-2} & -\frac{1}{2}\sqrt{-2} & 2 & 0 \\ -\frac{1}{2}\sqrt{-2} & \sqrt{-2}+1 & 0 & 2 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda'_{E_8(-1)})_Q$ is $E_8(-1)$.

References

1. Alexeev, V.: Log canonical singularities and complete moduli of stable pairs. arXiv:9608013
2. Allcock, D., Freitag, E.: Cubic surfaces and Borcherds products. Comment. Math. Helv. **77**(2), 270–296 (2002)
3. Allcock, D., Carlson, J.A., Toledo, D.: The complex hyperbolic geometry of the moduli space of cubic surfaces. J. Algebr. Geom. **11**, 659–724 (2002)
4. Ambro, F.: Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. **1**(240), 214–233 (2003)
5. Ash, A., Mumford, D., Rapoport, M., Tai, Y.-S.: Smooth compactification of locally symmetric varieties. Cambridge Mathematical Library
6. Baily, W., Borel, A.: Compactification of arithmetic quotients of bounded symmetric domains. Ann. of Math. **84**(12), 442–528 (1966)

7. Barth, W., Nieto, I.: Abelian surfaces of type $(1, 3)$ and quartic surfaces with 16 skew lines. J. Alg. Geom. **3**, 173–222 (1994)
8. Behrens, N.: Singularities of ball quotients. Geom. Dedicata **159**, 389–407 (2012)
9. Borcherds, R.: Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products. Invent. Math. **120**(1), 161–213 (1995)
10. Borcherds, R.: The moduli space of Enriques surfaces and the fake Monster Lie superalgebra. Topology **35**(3), 699–710 (1996)
11. Borcherds, R.: Automorphic forms with singularities on Grassmannians. Invent. Math. **132**(3), 491–562 (1998)
12. Braun, L., Greb, D., Langlois, K., Moraga, J.: Reductive quotients of KLT singularities. arXiv:2111.02812
13. Bruinier, J.H.: Borcherds products on $O(2, l)$ and Chern classes of Heegner divisors. In: Lecture Notes in Mathematics, vol. 1780. Springer, Berlin (2002)
14. Bruinier, J.H.: On the converse theorem for Borcherds products. J. Algbr. **397**, 315–342 (2014)
15. Bruinier, J.H., Ehlen, S., Freitag, E.: Lattices with many Borcherds products. Math. Comp. **85**(300), 1953–1981 (2016)
16. Dai, X., Yoshikawa, K.: Analytic torsion for log-Enriques surfaces and Borcherds product. arXiv:2009.10302
17. Dieckmann, C., Krieg, A., Woitalla, M.: The graded ring of modular forms on the Cayley half-space of degree two. Ramanujan J. **48**(2), 385–398 (2019)
18. Drezet, J.-M., Narasimhan, M.S.: Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. Invent. Math. **97**(1), 53–94 (1989)
19. Freitag, E.: Siegelsche Modulfunktionen. Springer, Berlin (1983)
20. Freitag, E., Salvati Manni, R.: Some Siegel threefolds with a Calabi-Yau model. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), 833–850 (2010)
21. Fujino, O.: Some problems on Fano varieties. In: Proceeding to the Conference “Fukuso-kikagaku no shomondai (2010)” in Japanese
22. Fujino, O.: Fundamental theorems for the log minimal model program. Publ. Res. Inst. Math. Sci. **47**(3), 727–789 (2011)
23. Fujino, O.: Foundations of the minimal model program. Mathematical Society of Japan, MSJ Memoirs, Tokyo (2017)
24. Fujino, O., Gongyo, Y.: On canonical bundle formulas and subadjunctions. Michigan Math. J. **61**(2), 255–264 (2012)
25. Gritsenko, V.: Reflective modular forms in algebraic geometry. arXiv:1012.4155
26. Gritsenko, V.: 24 faces of the Borcherds modular forms Φ_{12}. arXiv:1203.6503
27. Gritsenko, V.: Reflective modular forms and applications. Russ. Math. Surv. **73**, 797–864 (2018)
28. Gritsenko, V., Hulek, K.: Uniruledness of orthogonal modular varieties. J. Alg. Geom. **23**, 711–725 (2014)
29. Gritsenko, V., Hulek, K.: Moduli of polarized Enriques surfaces, in K3 surfaces and their moduli. Progr. Math. **315**, 55–72 (2016)
30. Gritsenko, V., Hulek, K., Sankaran, G.: The Kodaira dimension of the moduli spaces of K3 surfaces. Invent. math. **169**, 519–567 (2007)
31. Gritsenko, V., Nikulin, V.: Automorphic forms and Lorentzian Kac-Moody algebras. I, Int. J. Math. **9**(2), 153–199 (1998)
32. Gritsenko, V., Nikulin, V.: Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections. Proc. Lond. Math. Soc. **116**(3), 485–533 (2018)
33. Hacon, C., Mckernan, J.: On Shokurov’s rational connectedness conjecture. Duke Math. J. **138**(1), 119–136, 15 (2007)
34. Hashimoto, K., Ueda, K.: The ring of modular forms for the even unimodular lattice of signature $(2, 10)$. Proc. Am. Math. Soc. **150**(2), 547–558 (2022)
35. Hofmann, E.: Borcherds products on unitary groups. Math. Ann. **358**(3–4), 799–832 (2014)
36. Huybrechts, D.: Complete curves in moduli spaces of stable bundles on surfaces. Math. Ann. **298**(1), 67–78 (1994)

37. Ichikawa, T.: Siegel modular forms of degree 2 over rings. J. Number Theory **129**(4), 818–823 (2009)
38. Igusa, J.: On Siegel modular forms of genus two. II, Am. J. Math. **86**, 392–412 (1964)
39. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties, Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1998)
40. Kondō, S.: On the Kodaira dimension of the moduli spaces of K3 surfaces. Compositio Math. **89**, 251–299 (1993)
41. Kondō, S.: The rationality of the moduli space of Enriques surfaces. Compositio Math. **91**, 159–173 (1994)
42. Kondō, S.: On the Kodaira dimension of the moduli spaces of K3 surfaces. II, Compositio Math. **116**, 111–117 (1999)
43. Kondō, S.: The moduli space of Enriques surfaces and Borcherds products. J. Alg. Geom. **11**, 601–627 (2002)
44. Lejarraga, P.: The moduli of Weierstrass fibrations over $\mathbb{P}^1$: rationality. Rocky Mountain J. Math. **23**(2), 649–650 (1993)
45. Li, J.: Kodaira dimension of moduli space of vector bundles on surfaces. Invent. Math. **115**(1), 1–40 (1994)
46. Looijenga, E.: The smoothing components of a triangle singularity. II, Math. Ann. **269**(3), 357–387 (1984)
47. Ma, S.: The unirationality of the moduli spaces of 2-elementary K3 surfaces. With an appendix by Ken-Ichi Yoshikawa. Proc. Lond. Math. Soc. (3) **105**(4), 757–786 (2012)
48. Ma, S.: On the Kodaira dimension of orthogonal modular varieties. Invent. Math. 859–911 (2018)
49. Maeda, Y.: Uniruledness of unitary Shimura varieties associated with Hermitian forms of signatures $(1, 3)$, $(1, 4)$, $(1, 5)$. Master thesis (Department of Mathematics, Kyoto university). arXiv:2008.13106
50. Maeda, Y.: Reflective obstructions of unitary modular varieties. arXiv:2204.01128
51. Mumford, D.: Hirzebruch's proportionality theorem in the noncompact case. Invent Math. **42**, 239–277 (1977)
52. Mumford, D.: On the Kodaira dimension of the Siegel modular variety. In: Algebraic Geometry-Open Problems. Lecture Notes in Mathematics, vol. 997, pp. 348–375. Springer, Berlin (1983)
53. Nagano, A., Ueda, K.: The ring of modular forms for the even unimodular lattice of signature $(2, 18)$. Hiroshima Math. J. **52**(1), 43–51 (2022)
54. Namikawa, Y.: Periods of Enriques surfaces. Math. Ann. **270**(2), 201–222 (1985)
55. Odaka, Y., Spotti, C., Sun, S.: Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics. J. Differ. Geom. **102**(1), 127–172 (2016)
56. Odaka, Y., Oshima, Y.: Collapsing K3 Surface, Tropical Geometry and Moduli Compactifications of Satake, Morgan-Shalen Type, MSJ Memoirs. Mathematical Society of Japan, Tokyo (2021)
57. Odaka, Y.: PL density invariant for type II degenerating K3 surfaces, Moduli compactification and hyperKahler metrics. Nagoya Math, J. (2020)
58. Satake, I.: On compactifications of the quotient spaces for arithmetically defined discontinuous groups. Ann. Math. **72**(2), 555–580 (1960)
59. Sterk, H.: Compactifications of the period space of Enriques surfaces I. Math. Z. **207**(1), 1–36 (1991)
60. Tai, Y.S.: On the Kodaira dimension of the moduli space of abelian varieties. Invent. Math. **68**, 425–439 (1982)
61. van der Geer, G.: On the geometry of a Siegel modular threefold. Math. Ann. **260**(3), 317–350 (1982)
62. van der Geer, G.: Siegel modular forms of degree two and three and invariant theory. arXiv:2102.02245
63. Yoshikawa, K.: Calabi-Yau threefolds of Borcea-Voisin, analytic torsion, and Borcherds products. Astérisque No. **328**(2009), 355–393 (2010)

64. Yoshikawa, K.: K3 surfaces with involution, equivalent analytic torsion, and automorphic forms on the moduli spaces, II: a structure theorem for $r(M)>10$. J. Reine Angew. Math. **677**, 15–70 (2013)
65. Wang, H., Williams, B.: Simple lattices and free algebras of modular forms. arXiv:2009.13343
66. Wang, H., Williams, B.: Free algebras of modular forms on ball quotients. arXiv:2105.14892
67. Zhang, D.-Q.: Logarithmic Enriques surfaces. J. Math. Kyoto Univ. **31**, 419–466 (1991)
68. Zhang, Q.: Rational connectedness of log Q-Fano varieties. J. Reine Angew. Math. **590**, 131–142 (2006)

Simply Connected Sasaki-Einstein 5-manifolds: Old and New

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Abstract We present the state of the art concerning classification of closed simply connected Sasaki-Einstein 5-manifolds.

Keywords Sasaki-Einstein metrics · K-stability · Del Pezzo surfaces

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1 Introduction

1.1 Sasaki-Einstein 5-Manifold

A Riemannian manifold (M, g) is called Sasakian if the cone metric $r^2g + dr^2$ defines a Kähler metric on $M \times \mathbb{R}^+$. If the metric g satisfies the Einstein condition, i.e., $\mathrm{Ric}_g = \lambda g$ for some constant λ, then the metric g is called Einstein.

In this note, we briefly explain how to find closed simply connected 5-manifolds that allow Sasaki-Einstein metrics. We then list closed simply connected 5-manifolds that are known so far to admit Sasaki-Einstein metrics. We also present possible candidates for Sasaki-Einstein 5-manifolds to complete the classification of closed simply connected Sasaki-Einstein 5-manifolds.

A numerous number of closed simply connected Sasaki-Einstein manifolds, in particular 5-manifolds, have been mined based on the method introduced by

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Kobayashi [23] and developed by Boyer, Galicki, and Kollár [5, 10, 25]. Their method may be summarized as follows. A quasi-regular Sasakian structure on a manifold L can be written as the unit circle subbundle of a holomorphic Seifert $\mathbb{C}^*$-bundle over a complex algebraic orbifold (S, Δ), where $\Delta = \sum \left(1 - \frac{1}{m_i}\right) D_i$, m_i's are positive integers, and D_i's are distinct irreducible divisors. A simply connected Sasakian manifold L is Einstein if and only if $-(K_S + \Delta)$ is ample, the first Chern class of $c_1(L/X)$ is a rational multiple of $-(K_S + \Delta)$, and there is an orbifold Kähler-Einstein metric on the orbifold (S, Δ).

Closed simply connected 5-manifolds are completely classified by Barden and Smale [1, 37]. Since every closed simply connected Sasaki-Einstein 5-manifold is spin, for the purpose of the present note it is enough to explain the classification of closed simply connected spin 5-manifolds done by Smale [37].

Theorem 1.1 ([37]) *For a positive integer m, up to diffeomorphisms, there is a unique closed simply connected spin 5-manifold M_m with $\mathrm{H}_2(M_m, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. Furthermore, a closed simply connected spin 5-manifold M is of the form*

$$M = kM_\infty \# M_{m_1} \# \dots \# M_{m_r},$$

where kM_∞ is the k-fold connected sum of $S^2 \times S^3$ for a non-negative integer k and m_i is a positive integer greater than 1 with m_i dividing m_{i+1}. In particular, for any such M,

$$\begin{aligned}\mathrm{H}_2(M, \mathbb{Z}) &= \mathbb{Z}^{\oplus k} \oplus \mathrm{H}_2(M_{m_1}, \mathbb{Z}) \oplus \dots \oplus \mathrm{H}_2(M_{m_r}, \mathbb{Z}) \\ &= \mathbb{Z}^{\oplus k} \oplus (\mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_1\mathbb{Z}) \oplus \dots \oplus (\mathbb{Z}/m_r\mathbb{Z} \oplus \mathbb{Z}/m_r\mathbb{Z}).\end{aligned}$$

A closed simply connected spin 5-manifold is called a Smale 5-manifold.

Besides spinnability, closed simply connected 5-manifolds have topological conditions in the second integral homology groups to carry Sasaki-Einstein structures. The following topological properties of closed simply connected Sasaki-Einstein 5-manifolds are verified in [25, Theorems 1.4, 1.6, 1.8] and [27, Theorem 7].

Theorem 1.2 *Let M be a closed simply connected Sasaki-Einstein 5-manifold.*

(1) The torsion part of $\mathrm{H}_2(M, \mathbb{Z})$ is one of the following:

- *(a) $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$, where m is a positive integer;*
- *(b) $(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^{\oplus 2}$;*
- *(c) $(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})^{\oplus 2}$;*
- *(d) $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 2}$, $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 3}$, $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 4}$;*
- *(e) $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^{\oplus n}$, where $n \geq 1$.*

(2) If $\mathrm{H}_2(M, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ for a positive integer m, then m is not divisible by 30.

(3) If $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^{\oplus 2}$ or $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 4}$, then $\mathrm{rank}(\mathrm{H}_2(M, \mathbb{Z})) = 0$.

(4) If $\mathrm{H}_2(M,\mathbb{Z})_{\mathrm{tor}} = (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})^{\oplus 2}$*, then* $\mathrm{rank}\,(\mathrm{H}_2(M,\mathbb{Z})) \leq 1$.
(5) If $\mathrm{H}_2(M,\mathbb{Z})_{\mathrm{tor}} = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ *for an integer* $m \geq 12$*, then* $\mathrm{rank}\,(\mathrm{H}_2(M,\mathbb{Z})) \leq 8$.

Therefore, to classify Samle 5-manifolds that allow Sasaki-Einstein metrics, we have only to consider 5-manifolds with the second integral homology groups described in Theorem 1.2. For such a 5-manifold M, we determine whether it has a quasi-regular Sasakian structure, which is given by a Seifert bundle $L \to (S, \Delta)$. We then determine whether the orbifold del Pezzo surface (S, Δ) allows an orbifold Kähler-Einstein metric.

1.2 Kähler-Einstein Orbifold

The theory on Kähler-Einstein metrics and K-stability of Fano varieties and the theory on valuative criterions for K-stability have developed dramatically for the last ten years.

To briefly explain a classical method to determine existence of Kähler-Einstein metrics on Fano orbifolds, we introduce the α-invariants of Fano varieties. Let X be a projective $\mathbb{Q}$-factorial normal variety and Ω be a $\mathbb{Q}$-divisor on X such that the log pair (X, Ω) has at worst Kawamata log terminal singularities. We suppose that (X, Ω) is a log $\mathbb{Q}$-Fano variety, i.e., the divisor $-(K_X + \Omega)$ is ample.

Definition 1.3 The α-invariant of the log $\mathbb{Q}$-Fano variety (X, Ω) is defined by

$$\alpha(X,\Omega) = \sup\left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, \Omega + \lambda D) \text{ is log canonical for every} \\ \text{effective } \mathbb{Q}\text{-divisor } D \text{ numerically equivalent to } -(K_X + \Omega) \end{array} \right\}.$$

One of the roles of the α-invariant on Fano orbifolds is given by the following statement that has been gradually improved.

Theorem 1.4 ([20, 34, 39]) *Let* (X, Ω) *be a Fano orbifold. If*

$$\alpha(X,\Omega) > \frac{\dim(X)}{\dim(X)+1},$$

then (X, Ω) *admits an orbifold Kähler-Einstein metric.*

Boyer, Galicki, Kollár, and Nakamaye applied the α-invariant method to various Fano orbifolds, in particular, del Pezzo orbifolds, and they found A numerous number of closed simply connected Sasaki-Einstein manifolds [6, 7, 10–13, 25, 26].

In 2016 Fujita and Odaka introduced a new invariant of a Fano variety, so-called δ-invariant, which has evolved into a criterion for K-stability through the work of Blum and Jonsson. The δ-invariant measures how singular the average divisors of sections that form a basis for plurianticanonical linear systems are, using their log canonical thresholds.

Definition 1.5 Let m be a positive integer such that $|-m(K_X+\Omega)|$ is non-empty. Set $\ell_m = h^0(X, \mathcal{O}_X(-m(K_X+\Omega)))$. For a section s in $\mathrm{H}^0(X, \mathcal{O}_X(-m(K_X+\Omega)))$, we denote the effective divisor of the section s by $D(s)$. If ℓ_m sections $s_1, \dots, s_{\ell_m}$ form a basis of the space $\mathrm{H}^0(X, \mathcal{O}_X(-m(K_X+\Omega)))$, then the anticanonical $\mathbb{Q}$-divisor

$$D := \frac{1}{\ell_m}\sum_{i=1}^{\ell_m}\frac{1}{m}D(s_i)$$

is said to be of m-basis type. We set

$$\delta_m(X,\Omega) = \sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (X,\Omega+\lambda D) \text{ is log canonical for}\\ \text{every effective } \mathbb{Q}\text{-divisor } D \text{ of } m\text{-basis type}\end{array}\right\}.$$

The δ-invariant of (X, Ω) is defined by the number

$$\delta(X,\Omega) = \limsup_m \delta_m(X,\Omega).$$

Using the δ-invariant, Fujita and Odaka set up a criterion for K-(semi)stability in an algebro-geometric way. Indeed, the following assertion has been verified to be true by Blum-Jonsson [4] and Fujita-Odaka [21].

Theorem 1.6 *A log $\mathbb{Q}$-Fano variety (X, Ω) is K-stable (resp. K-semistable) if and only if $\delta(X,\Omega) > 1$ (resp. ≥ 1).*

Note that uniform K-stability is equivalent to K-stability ([31, Theorem 1.5]).

The bridge between K-polystability and existence of Kähler-Einstein metrics has been completely established for log Fano pairs [2, 3, 17–19, 29–31, 40, 41].

Theorem 1.7 *A Fano orbifold (X, Ω) is K-polystable if and only if it allows an orbifold Kähler-Einstein metric.*

For new Sasaki-Einstein 5-manifolds, this newly established δ-invariant method has been applied to certain del Pezzo hypersurfaces in [35]. This application brings us new Sasaki-Einstein rational homology 5-spheres, nM_2, $n \geq 2$.

1.3 Tools for α-Invariant

The tools to estimate the α-invariants in this note are presented in this section.

Let S be a surface with at most cyclic quotient singularities and D an effective $\mathbb{Q}$-divisor on the surface S. Also let p be a point of S.

Lemma 1.8 *Suppose that p is a smooth point of S. If the log pair (S, D) is not log canonical at p, then* $\mathrm{mult}_p(D) > 1$.

Proof See [28, Proposition 9.5.13], for instance. □

Let C be an integral curve on S that passes through the point p. Suppose that C is not contained in the support of the divisor D. If p is a smooth point of the surface S and the log pair (S, D) is not log canonical at p, then it follows from Lemma 1.8 that $D \cdot C > 1$.

Now suppose that p is a singular point of S. Recall that S has a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ at p, where a and b are coprime positive integers that are also coprime to n.

Lemma 1.9 *If the log pair (S, D) is not log canonical at p and C is not contained in the support of the divisor D, then*

$$D \cdot C > \frac{1}{n}.$$

Proof This follows from [24, Proposition 3.16], Lemma 1.8, and [15, Lemma 2.2]. □

In general, the curve C may be contained in the support of the divisor D. In this case, we write

$$D = aC + \Omega,$$

where a is a positive rational number and Ω is an effective $\mathbb{Q}$-divisor on S whose support does not contain the curve C.

We first suppose that both the surface S and the curve C are smooth at p.

Lemma 1.10 *Suppose that $a \leq 1$. If the log pair (S, D) is not log canonical at p, then*

$$C \cdot \Omega \geq (C \cdot \Omega)_p > 1,$$

where $(C \cdot \Omega)_p$ is the local intersection number of C and Ω at p.

Proof This immediately follows from the inversion of adjunction (see [36, Corollary 3.12]). □

We now suppose that S has a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ at p as before.

Lemma 1.11 *Suppose that $a \leq 1$. If the curve C is smooth at p and the log pair (S, D) is not log canonical at p, then*

$$C \cdot \Omega > \frac{1}{n}.$$

Proof The required inequality follows from the inverse of adjunction, [24, Proposition 3.16], and [15, Lemma 2.2]. □

2 Link of a Quasi-Homogeneous Hypersurface Singularity

2.1 *From Kähler-Einstein to Sasaki-Einstein*

Links of quasi-homogeneous hypersurface singularities are Seifert bundles over the corresponding projective hypersurfaces in weighted projective spaces.

Let X be a quasi-smooth hypersurface in a weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(a_0, a_1, \ldots, a_n)$ defined by a quasi-homogeneous polynomial $F(z_0, z_1, \ldots, z_n)$ in variables $z_0, \ldots, z_n$ with weights $\mathrm{wt}(z_i) = a_i$. The equation $F(z_0, z_1, \ldots, z_n) = 0$ also defines a hypersurface $\widehat{X}$ in $\mathbb{C}^{n+1}$ that is smooth outside the origin. The link of X is defined by the intersection

$$L_X = S^{2n+1}_{\mathbf{w}} \cap \widehat{X},$$

where $S^{2n+1}_{\mathbf{w}}$ is the unit sphere centred at the origin in $\mathbb{C}^{n+1}$ with the Sasakian structure induced from the weight $\mathbf{w} = (a_0, a_1, \ldots, a_n)$ (see [6, Sect. 1] [38, Example]). This is a smooth compact manifold of real dimension $2n - 1$. It is simply-connected if $n \geq 3$ ([32, Theorem 5.2]). The situation can be diagrammed as follows [13]:

$$\begin{tikzcd} L_X \arrow[r, hook] \arrow[d] & S^{2n+1}_{\mathbf{w}} \arrow[d] \\ X \arrow[r, hook] & \mathbb{P}(\mathbf{w}) \end{tikzcd}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are S^1 orbibundles and orbifold Riemannian submersions.

Put $m = \gcd(a_1, \ldots, a_n)$. Suppose that $m > 1$ and $\gcd(a_0, a_1, \ldots, a_{i-1}, \widehat{a_i}, a_{i+1}, \ldots, a_n) = 1$ for each $i = 1, \ldots, n$. Also set $b_0 = a_0$ and $b_i = \frac{a_i}{m}$ for $i = 1, \ldots, n$. The weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$ is not well-formed, while the weighted projective space $\mathbb{P}(b_0, b_1, \ldots, b_n)$ is well-formed (see [22, Definition 5.11]).

There is a quasi-homogeneous polynomial $G(x_0, \ldots, x_n)$ in variables $x_0, \ldots, x_n$ with weights $\mathrm{wt}(x_i) = b_i$ such that $F(z_0, z_1, \ldots, z_n) = G(z_0^m, z_1, \ldots, z_n)$. The equation $G(x_0, \ldots, x_n) = 0$ defines a quasi-smooth hypersurface Y in $\mathbb{P}(b_0, b_1, \ldots, b_n)$. We suppose that $\deg_{\mathbf{w}}(F) - \sum a_i < 0$ and Y is well-formed in $\mathbb{P}(b_0, b_1, \ldots, b_n)$ (see [22, Definition 6.9]). Denote by D the divisor on Y cut by $x_0 = 0$. Note that $(Y, \frac{m-1}{m}D)$ is a Fano orbifold since $\deg_{\mathbf{w}}(F) - \sum a_i < 0$.

The method by Kobayashi has evolved into the following assertion through the works of Boyer, Galicki, and Kollár

Theorem 2.1 ([5, Theorem 2.1], [23, Theorem 5]) *If there is a Kähler-Einstein edge metric on Y with angle $\frac{2\pi}{m}$ along the divisor D, then there is a Sasaki-Einstein metric on the link L_X of X.*

In the following three sections, we apply the method to certain del Pezzo hypersurfaces in three dimensional weighted projective spaces. This application brings us new Sasakin-Einstein Smale 5-manifolds, $3M_\infty\#3M_2$, $2M_\infty\#3M_2$, and $2M_\infty\#4M_2$, which support Conjecture 3.5. In order to verify existence of orbifold Kähler-Einstein metrics, the α-invariant method will be applied to the del Pezzo hypersurfaces corresponding to the Smale 5-manifolds.

From now on, for a given weighted projective space $\mathbb{P}(a_0, a_1, a_2, a_3)$, we always use the quasi-homogeneous coordinates x, y, z, w with $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(w) = a_3$.

2.2 Smale 5-Manifold $3M_\infty\#3M_2$

For a positive integer k, let $\widehat{S}_1$ be a quasi-smooth hypersurface of degree $8k$ in $\mathbb{P}(2, 2k, 2k, 4k-1)$. We may assume that it is defined by a quasi-homogeneous equation

$$xw^2 + f_4(y,z) + x^k f_3(y,z) + x^{2k} f_2(y,z) + x^{3k} f_1(y,z) + ax^{4k} = 0,$$

where f_d is a homogenous polynomial of degree d and a is a constant. Since $\widehat{S}_1$ is quasi-smooth, the polynomial f_4 must be reduced. As an orbifold, $\widehat{S}_1$ can be regarded as the log del Pezzo surface $\left(S_1, \frac{1}{2}C_w\right)$, where S_1 is the quasi-smooth hypersurface of degree $4k$ in $\mathbb{P}(1, k, k, 4k-1)$ defined by the quasi-homogeneous equation

$$xw + f_4(y,z) + x^k f_3(y,z) + x^{2k} f_2(y,z) + x^{3k} f_1(y,z) + ax^{4k} = 0$$

and C_w is the curve on S_1 cut by $w = 0$. The curve C_w is isomorphic to a smooth plane quartic curve. Meanwhile, it follows from [33, Corollary] that the link of $\widehat{S}_1$ has the second Betti number 3. Therefore, the link of $\widehat{S}_1$ is diffeomorphic to $3M_\infty\#3M_2$ by [25, Theorem 5.7] and Theorem 1.1.

Theorem 2.2 *For $k \geq 2$, $\left(S_1, \frac{1}{2}C_w\right)$ allows an orbifold Kähler-Einstein metric.*

Proof By Theorem 1.4, it is enough to show

$$\alpha\left(S_1, \frac{1}{2}C_w\right) > \frac{2}{3}.$$

Let D be an effective $\mathbb{Q}$-divisor such that $D \equiv -\left(K_{S_1} + \frac{1}{2}C_w\right)$. We then suppose that $\left(S_1, \frac{1}{2}C_w + \frac{3}{4}D\right)$ is not log canonical at a point p on S_1.

We first write $D = \alpha C_w + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve C_w. Let E be a general member in $|\mathcal{O}_{S_1}(k)|$. Then

$$\frac{2}{4k-1} = D \cdot E \geq \alpha C_w \cdot E = 4\alpha,$$

which implies $\alpha \leq \frac{2}{4(4k-1)} \leq \frac{1}{14}$. Therefore, $\left(S_1, C_w + \frac{3}{4}\Omega\right)$ is not log canonical at p. However,

$$\frac{3}{4}C_w \cdot \Omega = \frac{3-6(4k-1)\alpha}{2k} \leq \frac{3}{2k}.$$

It then follows from Lemmas 1.8 and 1.10 that p is either a singular point of S_1 on C_w or located outside C_w.

The hyperplane section C_x by $x = 0$ consists of four distinct curves $C_1, \dots, C_4$ that meet only at $[0, 0, 0, 1]$. Each curve C_i passes through exactly one of the four singular points of type $\frac{1}{k}(1, -1)$ on S_1. Suppose that p lies on C_i for some i. We write $D = \alpha C_w + \beta C_i + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither C_w nor C_i. It follows from [16, Lemma 2.2] that we may assume that the support of D does not contain the support of C_x. This implies that either $\beta = 0$ or the support of Δ does not contain C_j for some $j \neq i$. In the latter case, we have

$$\frac{1}{2k(4k-1)} = D \cdot C_j \geq \beta C_i \cdot C_j = \beta \frac{1}{4k-1}.$$

Therefore, $\beta \leq \frac{1}{2k}$, and hence $\left(S_1, C_i + \left(\frac{1}{2} + \frac{3}{4}\alpha\right) C_w + \frac{3}{4}\Delta\right)$ is not log canonical at p. However,

$$\begin{aligned}\frac{3}{4}\Delta \cdot C_i &= \frac{3}{4}\left(D - \alpha C_w - \beta C_i\right) \cdot C_i \\ &\leq \frac{3}{4}\left(\frac{1}{2k(4k-1)} + \beta\frac{3k-1}{k(4k-1)}\right) \leq \frac{3}{8k(4k-1)} + \frac{3(3k-1)}{8k^2(4k-1)} = \frac{3}{8k^2} \leq \frac{1}{4k-1},\end{aligned}$$

$$\begin{aligned}\left(\left(\frac{1}{2} + \frac{3}{4}\alpha\right) C_w + \frac{3}{4}\Delta\right) \cdot C_i &= \frac{1}{k}\left(\frac{1}{2} + \frac{3}{4}\alpha\right) + \frac{3}{4}\Delta \cdot C_i \\ &\leq \frac{1}{k}\left(\frac{1}{2} + \frac{3}{4}\alpha\right) + \frac{3}{8k^2} \leq \frac{1}{2k} + \frac{3}{8k(4k-1)} + \frac{3}{8k^2} \leq \frac{1}{k}.\end{aligned}$$

By Lemmas 1.10 and 1.11, these two inequalities implies that p must be outside C_x.

Let C be a general member in $|\mathcal{O}_{S_1}(k)|$ passing through p and we write $D = \gamma C + \Lambda$, where Λ is an effective $\mathbb{Q}$-divisor whose support does not contain C. Then

$$\frac{2}{4k-1} = D \cdot C \geq \gamma C^2 = \frac{4k\gamma}{4k-1},$$

which implies $\gamma \leq \frac{1}{2k}$. Therefore, $\left(S_1, C + \frac{1}{2}C_w + \frac{3}{4}\Lambda\right)$ is not log canonical at p. However,

$$\frac{3}{4}\Lambda \cdot C = \frac{3}{4}(D - \gamma C) \cdot C \leq \frac{3}{4}D \cdot C = \frac{3}{2(4k-1)}.$$

This contradicts to Lemma 1.10. Therefore, $(S_1, \frac{1}{2}C_w + \frac{3}{4}D)$ must be log canonical. Consequently, we obtain

$$\alpha\left(S_1, \frac{1}{2}C_w\right) \geq \frac{3}{4}.$$

$\square$

Corollary 1 *The Smale 5-manifold $3M_\infty\#3M_2$ admits a Sasaki-Einstein metric.*

Proof Since the link of $\widehat{S}_1$ is diffeomorphic to $3M_\infty\#3M_2$, the statement follows from Theorems 2.1 and 2.2. $\square$

2.3 *Smale* 5-*Manifolds* $2M_\infty\#4M_2$

For a positive integer k, let $\widehat{S}_2$ be a quasi-smooth hypersurface of degree $12k$ in $\mathbb{P}(2, 2k, 4k, 6k-1)$. We may assume that it is defined by a quasi-homogeneous equation

$$xw^2 + z^3 + y^6 + ax^{6k} + \sum_{\ell=1}^{5} x^{\ell k} g_{6-\ell}(y, z) = 0,$$

where g_d is a quasi-homogenous polynomial of degree dk and a is a constant. As an orbifold, $\widehat{S}_2$ can be regarded as the log del Pezzo surface $\left(S_2, \frac{1}{2}C_w\right)$, where S_2 is the quasi-smooth hypersurface of degree $6k$ in $\mathbb{P}(1, k, 2k, 6k-1)$ defined by the quasi-homogeneous equation

$$xw + z^3 + y^6 + ax^{6k} + \sum_{\ell=1}^{5} x^{\ell k} g_{6-\ell}(y, z) = 0$$

and C_w is the curve on S_2 cut by $w = 0$. Since the hypersurface $\widehat{S}_2$ is quasi-smooth, the curve C_w must be smooth.

Theorem 2.3 *For $k \geq 2$, $\left(S_2, \frac{1}{2}C_w\right)$ allows an orbifold Kähler-Einstein metric.*

Proof Since the proof of Theorem 2.2 works verbatim, we omit the proof. $\square$

Corollary 2 *The Smale 5-manifold $2M_\infty\#4M_2$ admits a Sasaki-Einstein metric.*

Proof The curve C_w is isomorphic to a smooth curve of degree 6 in $\mathbb{P}(1, 1, 2)$, and hence its genus is 4. Meanwhile, it follows from [33, Corollary] that the link of $\widehat{S}_2$ has the second Betti number 2. Therefore, the link of $\widehat{S}_2$ is diffeomorphic to $2M_\infty\#4M_2$ by [25, Theorem 5.7]. Then the statement follows from Theorems 2.1 and 2.3. $\square$

2.4 Smale 5-Manifolds $2M_\infty\#3M_2$

Let $\widehat{S}_3$ be a quasi-smooth hypersurface of degree 18 in $\mathbb{P}(2, 4, 6, 7)$. As an orbifold, $\widehat{S}_3$ can be regarded as the log del Pezzo surface $\left(S_3, \frac{1}{2}C_w\right)$, where S_3 is a quasi-smooth hypersurface of degree 9 in $\mathbb{P}(1, 2, 3, 7)$ and C_w is the curve on S_3 cut by $w = 0$. The curve C_w is quasi-smooth in $\mathbb{P}(1, 2, 3)$.

We suppose that the curve C_x cut by $x = 0$ intersects C_w at a point other than $[0 : 1 : 0 : 0]$. We may then assume that the surface S_3 is defined by

$$yw + y^3z + z^3 + xg_8(x, y, z) = 0,$$

where g_8 is a quasi-homogenous polynomial of degree 8 in x, y, z.

Theorem 2.4 *The log pair $\left(S_3, \frac{1}{2}C_w\right)$ admits an orbifold Kähler-Einstein metric.*

Proof Let D be an effective $\mathbb{Q}$-divisor such that $D \equiv -\left(K_{S_3} + \frac{1}{2}C_w\right)$. It is enough to show that $\left(S_3, \frac{1}{2}C_w + \frac{3}{4}D\right)$ is log canonical.

Suppose that $\left(S_3, \frac{1}{2}C_w + \frac{3}{4}D\right)$ is not log canonical at a point p on S.

Write $D = \alpha C_w + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain C_w. Let E be a general member in $|\mathcal{O}_{S_3}(2)|$. Then

$$\frac{3}{14} = D \cdot E \geq \alpha C_w \cdot E = 3\alpha,$$

which implies $\alpha \leq \frac{1}{14}$. Therefore, $\left(S_3, C_w + \frac{3}{4}\Omega\right)$ is not log canonical at p. However,

$$\frac{3}{4}C_w \cdot \Omega = \frac{9(1 - 14\alpha)}{16} \leq \frac{9}{16}.$$

This means that p is either $[0 : 1 : 0 : 0]$ or located outside C_w.

Now we write $D = \alpha C_w + \beta C_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither C_w nor C_x. We have $\beta \leq \frac{1}{2}$ since

$$\frac{3}{14} = D \cdot E \geq \beta C_x \cdot E = \frac{3\beta}{7}.$$

Therefore, $\left(S_3, \left(\frac{1}{2} + \frac{3\alpha}{4}\right) C_w + C_x + \frac{3}{4}\Delta\right)$ is not log canonical at p. The inequality

$$\frac{3}{4}\Delta \cdot C_x = \frac{3}{4}\left(D - \alpha C_w - \beta C_x\right) \cdot C_x \leq \frac{9}{112}$$

implies that p is either $[0 : 1 : 0 : 0]$ or located outside C_x. Suppose that $p = [0 : 1 : 0 : 0]$. Note that C_x intersects C_w at a point other than p. Then

$$\left(\frac{1}{2} + \frac{3\alpha}{4}\right)(C_w \cdot C_x)_p + \frac{3}{4}\Delta \cdot C_x \leq \frac{1}{2}\left(\frac{1}{2} + \frac{3\alpha}{4}\right) + \frac{9}{112} \leq \frac{1}{2}$$

yields a contradiction. Therefore, the point p must be outside $C_x \cup C_w$.

The hyperplane section C_y by $y = 0$ consists of three distinct curves C_1, C_2, C_3 that meet only at $[0 : 0 : 0 : 1]$. Suppose that p lies on C_i for some i. We write $D = \alpha C_w + \gamma C_i + \Gamma$, where Γ is an effective $\mathbb{Q}$-divisor whose support contains neither C_w nor C_i. It follows from [16, Lemma 2.2] that we may assume that the support of D does not contain the support of C_y. This implies that either $\gamma = 0$ or the support of Γ does not contain C_j for some $j \neq i$. In the latter case, we have

$$\frac{1}{14} = D \cdot C_j \geq \gamma C_i \cdot C_j = \frac{3\gamma}{7}.$$

Therefore, $\gamma \leq \frac{1}{6}$. This implies that $\left(S_3, C_i + \left(\frac{1}{2} + \frac{3}{4}\alpha\right) C_w + \frac{3}{4}\Gamma\right)$ is not log canonical at p. However, the inequality

$$\frac{3}{4}\Gamma \cdot C_i = \frac{3}{4}(D - \alpha C_w - \gamma C_i) \cdot C_i \leq \frac{3}{56} + \frac{3\gamma}{7} < 1$$

shows a contradiction. Therefore, p must be outside C_y.

Then there is a unique curve C in $|\mathcal{O}_{S_3}(2)|$ passing through p. Since p is located outside $C_x \cup C_y \cup C_w$, the curve C must be irreducible. Write $D = \delta C + \Lambda$, where Λ is an effective $\mathbb{Q}$-divisor whose support does not contain C. Then

$$\frac{3}{14} = D \cdot C \geq \delta C^2 = \frac{6\delta}{7},$$

which implies $\delta \leq \frac{1}{4}$. Therefore, $\left(S_3, C + \frac{1}{2}C_w + \frac{3}{4}\Lambda\right)$ is not log canonical at p. However,

$$\frac{3}{4}\Lambda \cdot C = \frac{3}{4}(D - \delta C) \cdot C \leq \frac{3}{4}D \cdot C = \frac{9}{56},$$

which yields a contradiction. Therefore, $(S_3, \frac{1}{2}C_w + \frac{3}{4}D)$ must be log canonical. □

Corollary 3 *The Smale 5-manifold $2M_\infty \# 3M_2$ admits a Sasaki-Einstein metric.*

Proof The genus of the curve C_w is 3. Therefore, [33, Corollary] and [25, Theorem 5.7] imply that the link of $\widehat{S}$ is diffeomorphic to $2M_\infty \# 3M_2$. The statement then follows from Theorems 2.1 and 2.4. □

3 Sasaki-Einstein 5-Manifolds: Old and New

3.1 Known Sasaki-Einstein 5-Manifolds

For every smooth del Pezzo surface S, every smooth member C in $|-K_S|$, and $\beta \in (0, 1]$, Cheltsov and Martinez-Garcia compute the α-invariant of $(S, (1-\beta)C)$. For our purpose, their result can be summarized as follows.

Theorem 3.1 *Let S_r be the blow up of $\mathbb{P}^2$ along r points in general position and C_r be a smooth member in its anticanonical linear system.*

- *If $r \geq 3$, then*
$$\alpha\left(S_r, \frac{m-1}{m}C_r\right) > \frac{2}{3}$$
for each integer $m \geq 2$.
- *If $r = 2$, then*
$$\alpha\left(S_r, \frac{m-1}{m}C_r\right) > \frac{2}{3}$$
for each integer $m \geq 3$.
- *If $r = 1$, then*
$$\alpha\left(S_r, \frac{m-1}{m}C_r\right) > \frac{2}{3}$$
for each integer $m \geq 4$.

Proof This immediately follows from [14, Sect. 2] and [14, Theorem 4.1]. □

This result of Cheltsov and Martinez-Garcia slightly improves [27, Corollary 21].

Theorem 3.2 *The Smale 5-manifold $rM_\infty \# M_m$ allows a Sasakin-Einstein metric if $m \geq 2$ for $3 \leq r \leq 8$, $m \geq 3$ for $r = 2$, and $m \geq 4$ for $r = 1$.*

Proof Let S_r be the blow up of $\mathbb{P}^2$ along r points in general position. It is a smooth del Pezzo surface of degree $9-r$. Let C_r be a smooth member in its anticanonical linear system. Note that the curve C_r is an elliptic curve. For $1 \leq r \leq 8$ and $m \geq 2$, [27, Corollary 21] provides a simply connected Seifert bundle L over the orbifold $\left(S_r, \frac{m-1}{m}C_r\right)$ whose first Chern class is a rational multiple of $-\left(K_{S_r} + \frac{m-1}{m}C_r\right)$. It follows from [25, Theorem 5.7] that L is diffeomorphic to the Samle 5-manifold $rM_\infty \# M_m$.

Meanwhile, Theorems 1.4 and 3.1 imply that $\left(S_r, \frac{m-1}{m}C_r\right)$ allows an orbifold Kähler-Einstein metric if $m \geq 2$ for $3 \leq r \leq 8$, $m \geq 3$ for $r = 2$, and $m \geq 4$ for $r = 1$. Therefore, in such a case, the Seifert bundle L allows a Sasaki-Einstein metric. □

Theorem 3.2 answers [9, Question 11.4.1 (iii)].

The following table lists all the Smale 5-manifolds that are known to allow Sasaki-Einstein metrics.

Smale 5-manfolds	Sasaki-Einstein	References
kM_∞	every $k \geq 0$	[7, Theorem 1.2] for $k \leq 7$ [12, Theorem A] for $k = 8$ [11, Theorem A] for $k = 9$ [26, Theorem 1] for $k \geq 6$
$8M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 2$
$7M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 2$
$6M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 2$
$5M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 2$
$4M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 2$
$3M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 2$
$2M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 3$ [13, Theorem A] for $m = 2$
$M_\infty \# M_m$	every $m \geq 2$	[27, Corollary 21] for $m \geq 7$ Theorem 3.2 for $m \geq 4$ [13, Theorem A] for $m = 3$ [13, Table 3] for $m = 2$
$M_\infty \# 2M_4$	Yes	[9, Theorem 11.4.13]
$M_\infty \# 2M_3$	Yes	[13, Theorem A]
$M_\infty \# 3M_3$	Yes	[13, Theorem A]
$M_\infty \# nM_2$	every $n \geq 2$	[25, Proof 9.6]
M_m	every $m \geq 2$ with $30 \nmid m$	[8, Theorem 21] for $m \geq 3$ with $6 \nmid m$ [25, Theorem 1.6] for $m \geq 12$ with $30 \nmid m$ [9, Theorem 11.4.12] for $m = 6$ [13, Table 2] for $m = 2$
$2M_5$	Yes	[25, Proof 9.6]
$2M_4$	Yes	[25, Proof 9.6]
$4M_3$	Yes	[25, Proof 9.6]
$3M_3$	Yes	[8, p. 359]
$2M_3$	Yes	[27, Example 19]
nM_2	every $n \geq 2$	[13, Theorem A] for $n = 2, 3, 5, 6, 7$ [35, Main Theorem] for $n \geq 4$

3.2 Remaining Candidates for Sasaki-Einstein 5-manifolds

The table in the previous section and Theorem 1.2 show that the following 5-manifolds remain as candidates for closed simply connected Sasakin-Einstein 5-manifolds.

Smale 5-manfolds	Candidates	Partial results
$kM_\infty\#2M_3$	$k \geq 2$	
$kM_\infty\#3M_3$	$k \geq 2$	
$kM_\infty\#nM_2$	$k \geq 3, n \geq 2$	[13, Theorem A] for $5M_\infty\#2M_2$ [13, Theorem A] for $4M_\infty\#2M_2$ Corollary 1 for $3M_\infty\#3M_2$ Corollary 2 for $2M_\infty\#4M_2$ Corollary 3 for $2M_\infty\#3M_2$
$kM_\infty\#M_m$	$k \geq 9$ and $2 \leq m \leq 11$	

To complete the classification of Sasaki-Einstein Smale 5-manifolds, we propose the following three conjectures.

Conjecture 3.3 ([9, Question 11.4.1 (i)]) Let M be a Smale 5-manifold of the second Betti number at least 9. It admits a Sasaki-Einstein metric if and only if $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}} = 0$.

Conjecture 3.4 Let M be a Smale 5-manifold of the second Betti number at least 2. If it admits a Sasaki-Einstein metric, then $\mathrm{H}_2(M, \mathbb{Z})_{\mathrm{tor}}$ can be neither $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ nor $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})^{\oplus 3}$.

Conjecture 3.5 For each integer $k \leq 8$ and $n \geq 2$, the Smale 5-manifold $kM_\infty\#nM_2$ admits a Sasaki-Einstein metric.

Note that the last conjecture holds for $k = 0, 1$ (see the table in Sect. 3.1).

Consequently, if the three conjectures above are true, then we are able to complete the classification of closed simply connected Sasaki-Einstein 5-manifolds by adding the Smale 5-manifold $kM_\infty\#nM_2$, $2 \leq k \leq 8$ and $n \geq 2$ to the table in Sect. 3.1.

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References

1. Barden, D.: Simply connected five-manifolds. Ann. Math. **82**(2), 365–385 (1965)
2. Berman, Robert, Boucksom, Sébastien., Jonsson, Mattias: A variational approach to the Yau-Tian-Donaldson conjecture. J. Am. Math. Soc. **34**(3), 605–652 (2021)
3. Robert, J.: Berman, K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics. Invent. Math. **203**(3), 973–1025 (2016)

4. Blum, H., Jonsson, M.: Thresholds, valuations, and K-stability. Adv. Math. **365**(57), 107062 (2020)
5. Boyer, C.P., Galicki, K.: On Sasakian-Einstein geometry. Int. J. Math. **11**(7), 873–909 (2000)
6. Boyer, C.P., Galicki, K.: New Einstein metrics in dimension five. J. Differ. Geom. **57**(3), 443–463 (2001)
7. Boyer, C.P., Galicki, K.: New Einstein metrics on $8\#(S^2 \times S^3)$. Differ. Geom. Appl. **19**(2), 245–251 (2003)
8. Boyer, C.P., Galicki, K.: Einstein metrics on rational homology spheres. J. Differ. Geom. **74**(3), 353–362 (2006)
9. Boyer, C.P., Galicki, K.: Sasakian Geometry. Oxford Mathematical Monographs, Oxford University Press, Oxford (2008)
10. Boyer, C.P., Galicki, K., Kollár, J.: Einstein metrics on spheres. Ann. Math. (2) **162**(1), 557–580 (2005)
11. Boyer, C.P., Galicki, K., Nakamaye, M.: Sasakian-Einstein structures on $9\#(S^2 \times S^3)$. Trans. Am. Math. Soc. **354**(8), 2983–2996 (2002)
12. Boyer, C.P., Galicki, K., Nakamaye, M.: On the geometry of Sasakian-Einstein 5-manifolds. Math. Ann. **325**(3), 485–524 (2003)
13. Boye, C.P., Nakamaye, M.: On Sasaki-Einstein manifolds in dimension five. Geom. Dedicata **144**, 141–156 (2010)
14. Cheltsov, I., Martinez-Garcia, J.: Dynamic alpha-invariants of del Pezzo surfaces. Int. Math. Res. Not. IMRN (10), 2994–3028 (2016)
15. Cheltsov, Ivan, Park, Jihun, Shramov, Constantin: Exceptional del Pezzo hypersurfaces. J. Geom. Anal. **20**(4), 787–816 (2010)
16. Cheltsov, Ivan, Park, Jihun, Won, Joonyeong: Affine cones over smooth cubic surfaces. J. Eur. Math. Soc. (JEMS) **18**(7), 1537–1564 (2016)
17. Chen, X., Donaldson, S., Sun, S.: Kähler-Einstein metrics on Fano manifolds. I: approximation of metrics with cone singularities. J. Am. Math. Soc. **28**(1), 183–197 (2015)
18. Chen, X., Donaldson, S., Sun, S.: Kähler-Einstein metrics on Fano manifolds. II: limits with cone angle less than 2π. J. Am. Math. Soc. **28**(1), 199–234 (2015)
19. Chen, X., Donaldson, S., Sun, S.: Kähler-Einstein metrics on Fano manifolds. III: limits as cone angle approaches 2π and completion of the main proof. J. Am. Math. Soc. **28**(1), 235–278 (2015)
20. Demailly, J.-P., Kollár, J.: Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. Ann. Sci. École Norm. Sup. **34**(4), 525–556 (2001)
21. Fujita, K., Odaka, Y.: On the K-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. (2) **70**(4), 511–521 (2018)
22. Iano-Fletcher, A.R.: Working with weighted complete intersections. In: Explicit Birational Geometry of 3-Folds. London Mathematical Society Lecture Note series, vol. 281, pp. 101–173. Cambridge University Press, Cambridge (2000)
23. Kobayashi, S.: Topology of positively pinched Kaehler manifolds. Tohoku Math. J. **15**(2), 121–139 (1963)
24. Kollár, J.: Singularities of pairs, Algebraic geometry–Santa Cruz. In: Proceedings of Symposia in Pure Mathematics, vol. 62, American Mathematical Society Providence, RI 1997, pp. 221–287 (1995)
25. Kollár, János.: Einstein metrics on five-dimensional Seifert bundles. J. Geom. Anal. **15**(3), 445–476 (2005)
26. Kollár, János.: Einstein metrics on connected sums of $S^2 \times S^3$. J. Differ. Geom. **75**(2), 259–272 (2007)
27. Kollár, J.: Positive Sasakian structures on 5-manifolds. In: Riemannian Topology and Geometric Structures on Manifolds. Progress in Mathematics, vol. 271, Birkhäuser Boston, Boston, MA, pp. 93–117 (2009)
28. Lazarsfeld, R.: Positivity in algebraic geometry. In: II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer, Berlin, 2004, Positivity for vector bundles, and multiplier ideals

29. Li, C.: $\mathbb{G}$-uniform stability and Kähler-Einstein metrics on Fano varieties. Invent. math. (2021). https://doi.org/10.1007/s00222-021-01075-9
30. Li, C., Tian, G., Wang, F.: The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties. Peking Math. J. (2021). https://doi.org/10.1007/s42543-021-00039-5
31. Liu, Y., Xu, C., Zhuang, Z.: Finite generation for valuations computing stability thresholds and applications to K-stability. arXiv:2102.09405
32. Milnor, J.: Singular points of complex hypersurfaces. In: Annals of Mathematics Studies, vol. 61, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1968)
33. Milnor, John, Orlik, Peter: Isolated singularities defined by weighted homogeneous polynomials. Topology **9**, 385–393 (1970)
34. Nadel, A.M.: Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. Ann. Math. **132**(3), 549–596 (1990)
35. Park, Jihun, Joonyeong, Won: Simply connected Sasaki-Einstein rational homology 5-spheres. Duke Math. J. **170**(6), 1085–1112 (2021)
36. Vyacheslav, V.: Shokurov, three-dimensional log perestroikas. Izv. Ross. Akad. Nauk Ser. Mat. **56**(1), 105–203 (1992)
37. Smale, S.: On the structure of 5-manifolds. Ann. Math. **75**, 38–46 (1962)
38. Takahashi, T.: Deformations of Sasakian structures and its application to the Brieskorn manifolds. Tohoku Math. J. **30**(1), 37–43 (1978)
39. Tian, Gang: On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M)>0$. Invent. Math. **89**(2), 225–246 (1987)
40. Tian, Gang: K-stability and Kähler-Einstein metrics. Comm. Pure Appl. Math. **68**(7), 1085–1156 (2015)
41. Chenyang, Xu.: K-stability of Fano varieties: an algebro-geometric approach. EMS Surv. Math. Sci. **8**(1–2), 265–354 (2021)

Singularities of Pluri-Fundamental Divisors on Gorenstein Fano Varieties of Coindex 4

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Abstract Let X be a Gorenstein canonical Fano variety of coindex 4 and dimension n with H fundamental divisor. Assume $h^0(X, H) \geq n-2$. We prove that a general element of the linear system $|mH|$ has at worst canonical singularities for any integer $m \geq 1$. When X has terminal singularities and $n \geq 5$, we show that a general element of $|mH|$ has at worst terminal singularities for any integer $m \geq 1$. When $n = 4$, we give an example of Gorenstein terminal Fano fourfold X such that a general element of $|H|$ does not have terminal singularities.

Keywords Fano variety · Fundamental divisor · Singularity of a pair · Anticanonical divisor

2010 Mathematics Subject Classification 14J45

1 Introduction

Throughout the paper, we work over the field $\mathbb{C}$ of complex numbers. Let X be a Gorenstein Fano variety of dimension n with canonical singularities. The *index* of X is

$$i_X := \max\{t \in \mathbb{Z} \mid -K_X \sim tH \text{ where } H \text{ is an ample Cartier divisor}\}.$$

It is well known that

$$1 \leq i_X \leq n+1.$$

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The *coindex* of X is $n+1-i_X$. An ample Cartier divisor H on X with $-K_X \sim i_X H$ is called the *fundamental divisor* of X. Since $\mathrm{Pic}(X)$ is torsion-free, H is uniquely determined up to linear equivalence. It is a natural problem to study singularities of general members in pluri-fundamental linear systems $|mH|$ for all integers $m \geq 1$.

By Kobayashi–Ochiai, X is a projective space if $i_X = n+1$, and X is a hyperquadric if $i_X = n$. A Gorenstein canonical Fano variety X with $i_X = n-1$ is a *del Pezzo variety*, and del Pezzo varieties were classified by Fujita [8, 10]. If X is a del Pezzo variety, then the base locus $\mathrm{Bs}\,|H|$ is empty or consists of a single point neither in $\mathrm{Sing}\,X$ nor in $\mathrm{Sing}\,Y$, where $Y \in |H|$ is a general member. Thus Y has canonical/terminal singularities if X has canonical/terminal singularities. A Gorenstein canonical Fano variety X with $i_X = n-2$ is a *Mukai variety*, and smooth Mukai varieties were classified by Mukai [21] under the assumption that $|H|$ contains a smooth divisor. Mella [20] verified this assumption, and moreover, he also proved that if X is a Gorenstein Mukai variety with canonical/terminal singularities, then a general member in $|H|$ has canonical/terminal singularities except when X is a complete intersection in $\mathbb{P}(1,1,1,1,2,3)$ of a quadric defined in the first four linear variables and a sextic. Finally, note that if $i_X \geq n-2$, then $|mH|$ is base point free for every integer $m \geq 2$ (see [18, Remark 4.5]). We can conclude that singularities of general members in $|mH|$ with $m \geq 1$ are well understood when $i_X \geq n-2$.

In this paper, we consider the case $i_X = n-3$, i.e., X has coindex 4. Floris [4] proved that a general member of the linear system $|H|$ has canonical singularities if X is a Gorenstein canonical Fano variety of coindex 4 and $h^0(X, H) \neq 0$. However, in contrasts to the smaller coindex cases, there is a smooth Fano fourfold X of coindex 4 such that every member in $|H|$ is singular (see [14, Example 2.12]). Heuberger [12] proved that if X is a smooth Fano fourfold, then a general member in $|-K_X|$ has only terminal singularities. This is a natural generalization of a classical result of Shokurov [23] for smooth Fano threefolds. Heuberger's theorem together with aforementioned results implies that a general member in $|H|$ has only terminal singularities if X is smooth.

The main result of this paper is the following.

Theorem 1.1 *Let X be a Gorenstein canonical Fano variety of coindex 4 and dimension $n \geq 4$, and H be the fundamental divisor of X.*

(1) Assume that $h^0(X, H) \geq n-2$. Then a general member of the linear system $|mH|$ has only canonical singularities for every integer $m \geq 1$.
(2) Assume that X has terminal singularities and $h^0(X, H) \geq n-2$. Then a general member of the linear system $|mH|$ has only terminal singularities for every integer $m \geq 1$ unless $(n, m) = (4, 1), (4, 2), (4, 3)$.
(3) Assume that X is smooth. Then $h^0(X, H) \geq n-2$, and a general member of the linear system $|mH|$ has only terminal singularities for every integer $m \geq 1$ unless $(n, m) = (4, 2)$.

If X is a smooth Fano variety of coindex 4 and dimension n, then Floris [4, Theorem 1.2] and Liu [18, Theorem 1.2] showed that $h^0(X, H) \geq n-2$. If X is singular, then we do not know whether $H^0(X, H) \neq 0$. This nonvanishing follows from the following:

Conjecture 1.2 (*Ambro–Kawamata effective nonvanishing conjecture* [1, 16]) Let (X, Δ) be a klt pair, and D be a Cartier divisor on X. If D is nef and $D - (K_X + \Delta)$ is nef and big, then $H^0(X, D) \neq 0$.

This conjecture has been verified for low dimensional varieties [16] and Fano weighted complete intersections [22]. Especially, [16, Proposition 4.1 and Theorem 5.2] say that if X is a Gorenstein Fano fourfold with canonical singularities, then $h^0(X, H) \geq 2$. Although the methods of the present paper do not yield the results for higher coindex cases directly, we may still expect that Theorem 1.1 for higher coindex would follow from the effective nonvanishing conjecture (cf. [13]).

In Theorem 1.1 (2), when $n = 4$, one cannot expect that a general member in $|H|$ has terminal singularities. We give an example of Gorenstein terminal Fano fourfold X such that a general member of the linear system $|H|$ does not have terminal singularities (see Example 2.4 (1)). In Theorem 1.1 (3), we do not know whether there is an example of a smooth Fano fourfold such that a general element in $|2H|$ does not have terminal singularities. See Remark 3.5 for some partial result.

By [19, Corollary 3], if X is a Gorenstein Fano variety of coindex 4 and dimension n with canonical singularities and $h^0(X, H) \geq n - 2$, then $|mH|$ is base point free for any integer $m \geq 4$ (see Remark 2.2). In particular, if X is a smooth Fano variety of coindex 4, then a general member in $|mH|$ is smooth for any integer $m \geq 4$.

One may expect that if a general member in $|H|$ has only mild singularities, then so does a general member in $|mH|$ for any $m \geq 2$. More generally, we may ask the following:

Question 1.3 Let X be a smooth projective variety, and L, M be divisors on X. Suppose that general members of $|L|$ and $|M|$ have only canonical/terminal singularities. Then does a general member of $|L + M|$ have also canonical/terminal singularities?

The answer is "NO" of course. Some counterexamples are given in Example 3.1.

Organization. The paper is organized as follows. Section 2 is devoted to proving Theorem 1.1 (1) and (2). We also give some examples of terminal Fano fourfolds in which a general member of the fundamental linear system does not have terminal singularities (see Example 2.4). In Sect. 3, we negatively answer Question 1.3 in Example 3.1, and we prove Theorem 1.1 (3).

2 Pluri-fundamental Divisors on Singular Fano Varieties

In this section, we prove Theorem 1.1 (1) and (2). For the definitions and basic properties of singularities of pairs, we refer to [17]. We begin with fixing some notations. Let X be a Gorenstein Fano variety of coindex 4 and dimension $n \geq 4$ with canonical singularities, and H be the fundamental divisor on X. We have $-K_X = (n - 3)H$. Assume that $h^0(X, H) \geq n - 2$. Then $|mH| \neq \emptyset$ for each integer $m \geq 1$. Take a log resolution

$$f_m : X_m \longrightarrow X$$

of the ideal of the base locus $\mathrm{Bs}\,|mH|$. We may assume that f_m is obtained by a sequence of blow-ups along smooth centers. We write

$$K_{X_m} = f_m^* K_X + \sum_i a_{m,i} E_{m,i} \quad \text{and} \quad |f_m^* mH| = |M_m| + \sum_i r_{m,i} E_{m,i},$$

where all $E_{m,i}$ are prime divisors, $|M_m|$ is the free part of $|f_m^* mH|$, and $\sum_i r_{m,i} E_{m,i}$ is the fixed part of $|f_m^* mH|$. By [4, Theorem 1.1], a general member of $|H|$ is irreducible. Since $h^0(X, H) \geq 2$, it follows that $\dim \mathrm{Bs}\,|H| \leq n-2$; thus $\dim \mathrm{Bs}\,|mH| \leq n-2$ for each integer $m \geq 1$. Hence every $E_{m,i}$ is an f_m-exceptional divisor. Since H is a Cartier divisor, all $a_{m,i}$ and $r_{m,i}$ are nonnegative integers.

Lemma 2.1 $\dim \mathrm{Bs}\,|mH| \leq 2$ *for any integer* $m \geq 1$.

Proof By [4, Proposition 4.1], (X, X_{n-1}) is a plt(=purely log terminal) pair, where $X_{n-1} \in |H|$ is a general member. As X_{n-1} is connected, [17, Proposition 5.51] shows that X_{n-1} is irreducible and normal. By [17, Theorem 5.50], X_{n-1} has Gorenstein canonical singularities. Note that $-K_{X_{n-1}} = ((n-1)-3)H_{n-1}$, where $H_{n-1} := H|_{X_{n-1}}$. If $n \geq 5$, then X_{n-1} is an $(n-1)$-dimensional Gorenstein canonical Fano variety of index $i_{X_{n-1}} \geq (n-1)-3$ with $h^0(X_{n-1}, H_{n-1}) \geq (n-1)-2$. If $i_{X_{n-1}} > (n-1)-3$, then $|H_{n-1}|$ is base point free (cf. [18, Remark 4.5]) so that (X_{n-1}, X_{n-2}) is a plt pair, where $X_{n-2} \in |H_{n-1}|$ is a general member. If $i_{X_{n-1}} = (n-1)-3$, then by [4, Proposition 4.1], (X_{n-1}, X_{n-2}) is also a plt pair. Continuing this process, we finally obtain a Calabi–Yau threefold X_3 with canonical singularities and $h^0(X_3, H_4|_{X_3}) \geq 1$. Notice that

$$\mathrm{Bs}\,|H| = \mathrm{Bs}\,|H_{n-1}| = \cdots = \mathrm{Bs}\,|H_4| = \mathrm{Bs}\,|H_4|_{X_3}|.$$

This shows $\dim \mathrm{Bs}\,|H| \leq 2$. Note that $\mathrm{Bs}\,|mH| \subseteq \mathrm{Bs}\,|H|$ for any $m \geq 2$. Then the lemma follows. □

Remark 2.2 (1) If the Ambro–Kawamata effective nonvanishing conjecture is true for Gorenstein Fano variety of coindex 4 with canonical singularities, then [4, Proposition 4.1] and the "ladder" argument as in the proof of Lemma 2.1 show that $h^0(X, H) \geq n-2$.

(2) If X is a Gorenstein Fano variety of coindex 4 with canonical singularities and $h^0(X, H) \geq n-2$, then the "ladder" argument and [19, Theorem 2] show that $|mH|$ is base point free for every integer $m \geq 4$.

The following proposition, inspired by [12, Proposition 9], is the key ingredient of the proof of Theorem 1.1.

Proposition 2.3 *For an integer* $m \geq 1$*, we have*

$$a_{m,i} \geq \frac{m+n-3}{m} r_{m,i} - 1 \quad \textit{for all } i.$$

Proof Suppose that $a_{m,i} - \frac{m+n-3}{m} r_{m,i} < -1$ for some i. Let

$$c_0 := \inf\{c \mid a_{m,i} - c r_{m,i} \leq -1 \text{ for some } i\}.$$

Then $0 < c_0 < \frac{m+n-3}{m}$. For an integer $k > c_0$, choose k general members $D_1, \dots, D_k \in |mH|$, and let $\Delta := c_0 \cdot \frac{D_1 + \cdots + D_k}{k}$. Then the pair (X, Δ) is lc(=log canonical) but not klt(=Kawamata log terminal). Let W be a minimal lc center of the lc pair (X, Δ). Since $D_1, \dots, D_k \in |mH|$ are general, (X, Δ) is klt outside the base locus $\mathrm{Bs}\,|mH|$ (cf. [1, Lemma 5.1]). Thus W is contained in $\mathrm{Bs}\,|mH|$, so $\dim W \leq 2$ by Lemma 2.1.

By the generalization of Kawamata's subadjunction formula [7, Theorem 1.2], there exists an effective divisor Γ on W such that

$$(K_X + \Delta)|_W \sim_{\mathbb{Q}} K_W + \Gamma$$

and the pair (W, Γ) is klt. Note that

$$mH - (K_X + \Delta) \sim_{\mathbb{Q}} (m + n - 3 - c_0 m)H.$$

Since $c_0 m < m + n - 3$, it follows that $mH - (K_X + \Delta)$ is ample. Then $mH|_W - (K_W + \Gamma)$ is ample. Recall that $\dim W \leq 2$. By [16, Theorem 3.1], $H^0(W, mH|_W) \neq 0$. Now, since $mH - (K_X + \Delta)$ is ample and W is an lc center of the lc pair (X, Δ), we can apply [6, Theorem 2.2] to see that the restriction map

$$H^0(X, mH) \longrightarrow H^0(W, mH|_W)$$

is surjective. However, $W \subseteq \mathrm{Bs}\,|mH|$, so this restriction map is the zero map. We obtain $H^0(W, mH|_W) = 0$, which is a contradiction. Thus the proposition holds. □

We are ready to prove Theorem 1.1 (1) and (2).

Proof of Theorem 1.1 (1) and (2) Recall that X is a Gorenstein Fano variety of coindex 4 and dimension n with canonical singularities. We assume that $h^0(X, H) \geq n - 2$. Let $Y_m \in |mH|$ be a general element for an integer $m \geq 1$.

(1) We want to prove that Y_m has canonical singularities. If (X, Y_m) is a plt pair, then [17, Theorem 5.50 and Proposition 5.51] imply that Y_m has canonical singularities since Y_m is Gorenstein. Thus it is enough to show that the pair (X, Y_m) is plt. The birational morphism $f_m \colon X_m \to X$ is a log resolution of (X, Y_m). We have

$$K_{X_m} + f_{m,*}^{-1} Y_m = f_m^*(K_X + Y_m) + \sum_i (a_{m,i} - r_{m,i}) E_{m,i}.$$

If $r_{m,i} = 0$, then $a_{m,i} - r_{m,i} \geq 0 > -1$. If $r_{m,i} \geq 1$, then Proposition 2.3 implies that

$$a_{m,i} - r_{m,i} \geq \frac{n-3}{m} r_{m,i} - 1 > -1.$$

Thus (X, Y_m) is a plt pair.
(2) Assume that X has terminal singularities and $n = \dim X \geq 5$ or $m \geq 4$. We want to show that Y_m has terminal singularities. If $m \geq 4$, then [19, Theorem 2] (see also Remark 2.2 (2)) implies that $|mH|$ is base point free; hence Y_m has terminal singularities. From now on, assume that $n \geq 5$ and $1 \leq m \leq 3$. We know that Y_m is a normal projective variety with canonical singularities. Let $Y'_m := f^{-1}_{m,*} Y_m$ be the strict transform of Y_m under f_m. Since $Y'_m \in |M_m|$ is a general element, Y'_m is smooth. Then

$$f'_m := f_m|_{Y'_m} : Y'_m \longrightarrow Y_m$$

is a log resolution of Y_m. We have

$$K_{Y'_m} = f'^*_m K_{Y_m} + \sum_i (a_{m,i} - r_{m,i}) E_{m,i}|_{Y'_m}.$$

Since X has terminal singularities, we have $a_{m,i} \geq 1$ for all i.

Consider the case $m = 1$. Note that $\frac{m+n-3}{m} = n - 2 \geq 3$. If $r_{1,i} \geq 1$, then Proposition 2.3 implies that $a_{1,i} - r_{1,i} \geq 2r_{1,i} - 1 > 0$. If $r_{1,i} = 0$, then $a_{1,i} - r_{1,i} > 0$. Thus Y_1 has terminal singularities.

Suppose now that Y_m does not have terminal singularities for some $2 \leq m \leq 3$. Then there is some i_0 such that $a_{m,i_0} = r_{m,i_0} \geq 1$ and $E_{m,i_0}|_{Y'_m}$ is an $f_m|_{Y'_m}$-exceptional divisor. Since Y_m has terminal singularities outside $\mathrm{Bs}\,|mH|$, we see that $f_m(E_{m,i_0}) \subseteq \mathrm{Bs}\,|mH|$. Since $n \geq 5$ and $2 \leq m \leq 3$, we have $\frac{m+n-3}{m} \geq \frac{5}{3}$. If $r_{m,i} \geq 2$, then Proposition 2.3 implies that $a_{m,i} - r_{m,i} \geq \frac{2}{3} r_{m,i} - 1 > 0$. Thus $a_{m,i_0} = r_{m,i_0} = 1$. If $f_m(E_{m,i_0}) \nsubseteq \mathrm{Sing}\, X$, then $\dim f_m(E_{m,i_0}) = n - 2$ since f_m is a composition of smooth center blow-ups. This means that $f_m(E_{m,i_0})$ is a divisor on Y_m and $E_{m,i_0}|_{Y'_m}$ is not an $f_m|_{Y'_m}$-exceptional divisor. Thus $f_m(E_{m,i_0}) \subseteq \mathrm{Sing}\, X$, and $\dim f_m(E_{m,i_0}) \leq n - 3$ because X has terminal singularities. By taking further blow-ups, we may assume that $f_1 = f_m$ and $X_1 = X_m$. Then there is an i_1 such that $E_{1,i_1} = E_{m,i_0}$. We have $a_{1,i_1} = a_{m,i_0} = 1$. Now, since $f_m(E_{m,i_0}) \subseteq \mathrm{Bs}\,|mH| \subseteq \mathrm{Bs}\,|H|$, it follows that $r_{1,i_1} \geq 1$. Thus $a_{1,i_1} - r_{1,i_1} \leq 0$. Note that $E_{1,i_1}|_{Y'_1}$ is an $f_1|_{Y'_1}$-exceptional divisor. We get a contradiction to that Y_1 has terminal singularities. Hence Y_m has terminal singularities for any $2 \leq m \leq 3$. □

Finally, we provide some examples of terminal Fano fourfolds in which a general element in the fundamental linear system does not have terminal singularities.

Example 2.4 (1) Let $Z := X_{2,6}$ be a complete intersection in $\mathbb{P}(1, 1, 1, 1, 2, 3)$ of a general quadric defined in the first four linear variables $x_0, \ldots, x_3$ and a general sextic (cf. [20, Theorem 1]). Then Z is a Gorenstein terminal Fano threefold of index 1, and $\mathrm{Sing}\, Z = \{p = (0 : 0 : 0 : 0 : -1 : 1)\}$. A general member of $|H_Z|$ is singular at p, where $H_Z = -K_Z$ is the fundamental divisor of Z. Let $X := Z \times \mathbb{P}^1$ so that X is a Gorenstein Fano fourfold of index 1 with terminal singularities. Note that $H = -K_X = \pi_1^*(-K_Z) + \pi_2^*(-K_{\mathbb{P}^1})$ is the fundamental divisor of X, where

$\pi_1 : X \to Z$ and $\pi_2 : X \to \mathbb{P}^1$ are projections. A general element Y in $|H|$ has one dimensional singular locus $\{p\} \times \mathbb{P}^1$. Since $\dim Y = 3$, it follows that Y does not have terminal singularities. Here Y is a Gorenstein Calabi–Yau threefold with canonical singularities.

(2) Let $X := X_9$ be a weighted hypersurface in $\mathbb{P}(1, 1, 1, 1, 3, 3)$ of degree 9 (cf. quasismooth Fano 4-fold hypersurfaces ID 8 in [11] based on [2]). Then X is a non-Gorenstein $\mathbb{Q}$-Fano fourfold with terminal singularities such that $-K_X$ is a hyperplane with $(-K_X)^4 = 1$. Note that $\operatorname{Sing} X$ consists of three terminal singular points of the type $\frac{1}{3}(1, 1, 1, 1)$. A general element in $|-K_X|$ is a weighted hypersurface S_9 in $\mathbb{P}(1, 1, 1, 3, 3)$ of degree 9, and S_9 is a Gorenstein canonical Calabi–Yau threefold. Note that $\operatorname{Sing} S_9$ consists of three (non-terminal) canonical singular points of the type $\frac{1}{3}(1, 1, 1)$.

3 Pluri-fundamental Divisors on Smooth Fano Varieties

In this section, we first answer Question 1.3 by constructing smooth projective varieties X and divisors M such that general members in $|M|$ are smooth but all members in $|mM|$ are not normal for some $m \geq 2$, and then prove Theorem 1.1 (3).

Example 3.1 (1) If E is an exceptional divisor on a smooth projective variety, then $|mE| = \{mE\}$ for all $m \geq 1$. Now, E is smooth, but mE is non-reduced for any $m \geq 2$.

(2) Let C be a smooth projective curve of genus 2. There are two distinct points P, Q on C such that $2P \sim 2Q \sim K_C$. In particular, $Q - P \in \operatorname{Pic}^0(C)$ is a 2-torsion. We can also find $\tau \in \operatorname{Pic}^0(C)$ such that $H^0(C, P + \tau) = H^0(C, Q - P + \tau) = H^0(C, 2\tau) = 0$. Let $E := \mathcal{O}_C(P) \oplus \mathcal{O}_C(\tau)$, and $S := \mathbb{P}(E)$ with the natural projection $\pi : S \to C$ and the tautological divisor H, i.e., $\mathcal{O}_S(H) = \mathcal{O}_{\mathbb{P}(E)}(1)$. Let $A := H$ and $B := H + \pi^*(Q - P)$. Then A, B are sections of π, so they are smooth irreducible curves isomorphic to C. Furthermore, A, B satisfy the following:

- $A^2 = B^2 = A.B = 1$,
- $A \nsim B$ but $2A \sim 2B$,
- $h^0(S, A) = h^0(S, B) = 1$, and
- $h^0(S, 2A) = h^0(S, 2B) = 2$.

Notice that A, B meet at one point p on S and every member of $|2A| = |2B|$ has multiplicity at least 2 at p. Thus every member in $|2A| = |2B|$ is not normal.

(3) [22, Example 5.9] For an integer $m \geq 1$, let

$$X = X_{(2m+1)(2m+2)} \subseteq \mathbb{P}(\underbrace{1, \dots, 1}_{1+2m(2m+1)}, 2m + 1, 2m + 2)$$

be a weighted hypersurface of degree $(2m + 1)(2m + 2)$. Then X is a smooth Fano variety of index 2. If H is the fundamental divisor of X, then a general member of $|H|$ is smooth. However, $|-2iH|$ does not contain a smooth member for any $1 \leq i \leq m$. In this case, a general member in $|-2iH|$ has terminal singularities.

We now turn to the proof of Theorem 1.1 (3).

Proof of Theorem 1.1 (3) except the case $(n, m) = (4, 3)$ Let X be a smooth Fano variety of coindex 4 and dimension n with fundamental divisor H. By Theorem 1.1 (2), we only have to consider the cases $(n, m) = (4, 1), (4, 3)$. If $(n, m) = (4, 1)$, then $H = -K_X$. Now, [12, Theorem 2] says that a general element in $|-K_X|$ has terminal singularities. □

Remark 3.2 Let X be a smooth Fano variety of coindex 4, and H be the fundamental divisor of X. By [19, Theorem 4], $|mH|$ is base point free for any integer $m \geq 4$; hence a general element $Y_m \in |mH|$ is smooth in this case. But there is a smooth Fano fourfold X of coindex 4 such that every member in $|H|$ is singular (see [14, Example 2.12]).

To finish the proof of Theorem 1.1, it only remains to prove that if H is the fundamental divisor of a smooth Fano fourfold X of coindex 4, then a general element $Y \in |3H|$ has terminal singularities. We know that Y has canonical singularities. As in Sect. 2, take a log resolution $f: X_3 \to X$ of the ideal of the base locus $\mathrm{Bs}\,|3H|$. We may assume that f is isomorphic outside $\mathrm{Bs}\,|3H|$ and it is obtained by a sequence of blow-ups along smooth centers. We write

$$K_{X_3} = f^*K_X + \sum_i a_i E_i \quad \text{and} \quad |f^*3H| = |M| + \sum_i r_i E_i,$$

where all E_i are f-exceptional prime divisors and $|M|$ is the free part of $|f^*3H|$ and $\sum_i r_i E_i$ is the fixed part of $|f^*3H|$. We may assume that $f(E_i) \subseteq \mathrm{Bs}\,|3H|$ for all i. All a_i and r_i are positive integers.

Lemma 3.3 *If a general element Y in $|3H|$ has at worst isolated singularity at x and* $\mathrm{mult}_x Y \leq 2$, *then Y has terminal singularity at x.*

Proof We may assume that f factors through the blow-up of X at x with exceptional divisor E_{i_0}. We have $a_{i_0} = 3$ and $r_{i_0} \leq 2$, so $a_{i_0} - r_{i_0} > 0$. For every f-exceptional divisor E_i with $f(E_i) = \{x\}$ but $E_i \neq E_{i_0}$, we have $a_i \geq 4$ since f is a composition of smooth center blow-ups. It is impossible that $a_i = r_i \geq 4$ because Proposition 2.3 says that $a_i \geq \frac{4}{3}r_i - 1 > r_i$ when $r_i \geq 4$. Thus $a_i - r_i > 0$, and hence, Y has terminal singularity at x. □

Lemma 3.4 $\dim \mathrm{Bs}\,|mH| \leq 1$ *for any integer* $m \geq 2$. *In particular,* $\dim \mathrm{Sing}\, Y \leq 1$.

Proof Suppose that $\dim \mathrm{Bs}\,|mH| \geq 2$ for some integer $m \geq 2$. By Lemma 2.1, we have $\dim \mathrm{Bs}\,|mH| = 2$, so there is an irreducible surface $S \subseteq \mathrm{Bs}\,|mH| \subseteq \mathrm{Bs}\,|H|$. Now, take two general elements $D_1, D_2 \in |H|$. By Proposition 2.3, $(X, D_1 + D_2)$ is an lc pair, and S is an lc center of $(X, D_1 + D_2)$. There is a minimal lc center C of

$(X, D_1 + D_2)$ contained in S. By [7, Theorem 1.2], there is an effective divisor Γ on C such that

$$(K_X + D_1 + D_2)|_C \sim_{\mathbb{Q}} K_C + \Gamma$$

and (C, Γ) is a klt pair. By [16, Theorem 3.1], $H^0(C, mH|_C) \neq 0$ since $mH - (K_X + D_1 + D_2) \sim (m-1)H$ is ample. Now, by [6, Theorem 2.2], the restriction map

$$H^0(X, mH) \longrightarrow H^0(S, mH|_S)$$

is surjective. However, $S \subseteq \mathrm{Bs}\,|mH|$, so this restriction map is the zero map. We get a contradiction. Therefore, $\dim \mathrm{Bs}\,|mH| \leq 1$ for any integer $m \geq 2$. Now, since $\mathrm{Sing}\, Y \subseteq \mathrm{Bs}\,|3H|$, it follows that $\dim \mathrm{Sing}\, Y \leq 1$. □

Proof of Theorem 1.1 (3) for the case $(n, m) = (4, 3)$ We want to prove that a general element $Y \in |3H|$ has terminal singularities. Note that $H = -K_X$ and $\dim \mathrm{Bs}\,|3H| \leq 1$ by Lemma 3.4. We know that $h^0(X, H) \geq 2$.

First, assume that $H^4 \geq 2$. Take a general element $Z \in |H|$, which is a Gorenstein Calabi–Yau threefold with terminal singularities. Suppose that $\dim \mathrm{Bs}\,|3H| = 1$. Then Z is nonsingular at a general point x in $\mathrm{Bs}\,|3H|$. By [15, Theorem 3.1], $|3H|_Z|$ is base point free at x. But $x \in \mathrm{Bs}\,|3H| = \mathrm{Bs}\,|3H|_Z|$, so we get a contradiction. Thus $\dim \mathrm{Bs}\,|3H| \leq 0$. Suppose that Y has non-terminal singularity at x. By Lemma 3.3, $\mathrm{mult}_x\, Y \geq 3$. Now, by Proposition 2.3, $(X, Z + Y)$ is an lc pair. Thus $\mathrm{mult}_x\, Z = 1$, so Z is nonsingular at x. By [15, Theorem 3.1], $3H|_Z$ is base point free at x, so we get a contradiction as before. Hence Y has at worst terminal singularities.

Next, assume that $H^4 = 1$. The sectional genus of the polarized pair (X, H) is $g(X, H) = \frac{(K_X+3H).H^3}{2} + 1 = 2$. By Fujita's classification [9, Proposition C], we have $2 \leq h^0(X, H) \leq 4$, and the following hold:

- $h^0(X, H) = 4 \Leftrightarrow X = X_{10} \subseteq \mathbb{P}(1, 1, 1, 1, 2, 5)$ is a hypersurface of degree 10.
- $h^0(X, H) = 3 \Leftrightarrow X = X_{6,6} \subseteq \mathbb{P}(1, 1, 1, 2, 2, 3, 3)$ is a complete intersection of type (6,6).

If $h^0(X, H) = 4$, then $\mathrm{Bs}\,|H| = \mathrm{Bs}\,|3H| = \{x\}$ and $|2H|$ is base point free. In this case, $\mathrm{mult}_x\, Y = 1$, so Y is smooth. If $h^0(X, H) = 3$, then $|3H|$ is base point free so that Y is smooth. We now suppose that $h^0(X, H) = 2$.[1] By Riemann–Roch formula, we have

$$h^0(X, mH) = \frac{m^2(m+1)^2}{24} H^4 + \frac{m(m+1)}{24} H^2.c_2(X) + 1.$$

Then $H^2.c_2(X) = 10$, and $h^0(X, 2H) = 5$, $h^0(X, 3H) = 12$. Let $Z_1, Z_2 \in |H|$ and $W \in |2H|$ be general members. Then $S := Z_1 \cap Z_2$ is an irreducible Gorenstein surface with $K_S = H|_S$, and $C := Z_1 \cap Z_2 \cap W$ is a Gorenstein curve with $K_C = 3H|_C$. We have $H^i(X, \ell H) = 0$ for $1 \leq i \leq 3$ and $\ell \in \mathbb{Z}$, so we get

[1] It is unknown whether there is a smooth Fano fourfold X with $h^0(X, -K_X) = 2$ (cf. [19, Question 5]).

$$h^0(Z_1, H|_{Z_1}) = 1,\ h^0(Z_1, 2H|_{Z_1}) = 3,\ h^0(Z_1, 3H|_{Z_1}) = 7$$
$$h^0(S, H|_S) = 0,\ h^0(S, 2H|_S) = 2,\ h^0(S, 3H|_S) = 4$$
$$h^0(C, H|_C) = 0,\ h^0(C, 2H|_C) = 1,\ h^0(C, 3H|_C) = 4.$$

Thus $p_a(C) = h^0(C, 3H|_C) = 4$. As $C.H|_S = 2$, we see that C has at most two irreducible components. If C is non-reduced, then $C = 2H'$ for some $H' \in |H|_S|$. However, since $h^0(S, H|_S) = 0$, it follows that C is reduced.

Suppose that there is an irreducible curve A on X with $A \subseteq \mathrm{Bs}\,|2H| \cap \mathrm{Bs}\,|3H|$. Since $h^0(S, C) = h^0(S, 2H|_S) = 2$, it follows that C has two irreducible components. We write $C = A + B$ on S. Since the restriction map $H^0(X, 3H) \to H^0(C, K_C)$ is surjective, we have $A \subseteq \mathrm{Bs}\,|K_C|$. Note that

$$\deg_A(K_C) = \deg_B(K_C) = 3$$

since

$$\deg_A(H|_C) = A.H = 1 \quad \text{and} \quad \deg_B(H|_C) = B.H = 1.$$

By [5, Definition 2.1 and Formula (3)], we have

$$4 = p_a(C) = p_a(A) + p_a(B) + A \cdot B - 1,$$

where

$$A \cdot B := \deg_A(K_C) - 2p_a(A) + 2 = \deg_B(K_C) - 2p_a(B) + 2.$$

If $A \cdot B \geq 2$, then C is numerically 2-connected in the sense of [3, Definition 3.1]. In this case, by [3, Theorem 3.3], $|K_C|$ is base point free, so we get a contradiction to that $A \subseteq \mathrm{Bs}\,|K_C|$. Thus $A \cdot B = 1$, and then, $p_a(A) = p_a(B) = 2$. Consider an exact sequence

$$0 \longrightarrow \omega_B \longrightarrow \omega_C \longrightarrow \omega_C|_A \longrightarrow 0,$$

which induces the following exact sequence

$$0 \longrightarrow H^0(B, K_B) \longrightarrow H^0(C, K_C) \longrightarrow H^0(A, K_C|_A)$$

Then $h^0(A, K_C|_A) \geq 2$, which is a contradiction to that $A \subseteq \mathrm{Bs}\,|K_C|$. Thus we obtain $\dim \mathrm{Bs}\,|2H| \cap \mathrm{Bs}\,|3H| \leq 0$.

Recall that Z_1 is a Gorenstein Calabi–Yau threefold with terminal singularities. Then $\dim \mathrm{Sing}\, Z_1 \leq 0$. If Y is singular along a curve D, then $D \subseteq \mathrm{Bs}\,|3H|$ and $D \nsubseteq \mathrm{Bs}\,|2H|$. For a general point $x \in D$, we have $\mathrm{mult}_x\,|H| = 1$ and $\mathrm{mult}_x\,|2H| = 0$, so $\mathrm{mult}_x\, Y = 1$ by the upper semicontinuity of the multiplicity. We get a contradiction because Y is singular at x. This means that Y cannot be singular along a curve. By Bertini's theorem, we see that $\mathrm{mult}_x\, Y \leq 2$ for all $x \notin \mathrm{Bs}\,|2H| \cup \mathrm{Sing}\, Z_1$. Now, suppose that Y has an isolated non-terminal singular point x. By Lemma 3.3, $\mathrm{mult}_x\, Y \geq 3$, which implies that $x \in \mathrm{Bs}\,|2H| \cup \mathrm{Sing}\, Z_1$. Note that every general element $Y' \in |3H|$ has $\mathrm{mult}_x Y' \geq 3$. By Proposition 2.3, $(X, Y + Z_1)$ is an lc pair, so $\mathrm{mult}_x Z_1 = 1$. If $\mathrm{mult}_x W = 1$, then the upper semicontinuity of the multiplicity shows that $\mathrm{mult}_x Y' \leq 2$, which is a contradiction. Thus $\mathrm{mult}_x W \geq 2$. Now, $\dim \mathrm{Bs}\, |2H| \cap \mathrm{Bs}\, |3H| \leq 0$ implies that $C \cap Y'$ has dimension zero. Notice that $C \cap Y' = Z_1 \cap Z_2 \cap W \cap Y'$ has length 6, and recall that $\mathrm{mult}_x W \geq 2$ and $\mathrm{mult}_x Y' \geq 3$. Hence $C \cap Y'$ is indeed supported at a single point x. Since $H^0(X, 3H) \to H^0(C, 3H|_C)$ is surjective, every element in $|3H|_C|$ has a single support x. But this is impossible since $h^0(C, 3H|_C) \geq 2$. We can conclude that Y has at worst terminal singularities. □

Remark 3.5 Let X be a Fano fourfold of coindex 4 with fundamental divisor $H = -K_X$. Suppose that $H^4 \geq 4$, $H^2.S \geq 3$ for every irreducible surface S, and $H^3.C \geq 2$ for every irreducible curve C. Take a general element $Z \in |H|$. By [16, Theorem 3.1], $|2H|_Z|$ is base point free at every nonsingular point in Z. This implies that $\dim \mathrm{Bs}\, |2H| \leq 0$. In this case, we can easily show that a general member in $|2H|$ has terminal singularities.

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References

1. Ambro, F.: Ladders on Fano varieties. J. Math. Sci. (New York) **94**, 1126–1135 (1999)
2. Brown, G., Kasprzyk, A.: Four-dimensional projective orbifold hypersurfaces. Exp. Math. **25**, 176–193 (2016)
3. Catanese, F., Franciosi, M., Hulek, K., Reid, M.: Embeddings of curves and surfaces. Nagoya Math. J. **154**, 185–220 (1999)
4. Floris, E.: Fundamental divisors on Fano varieties of index $n - 3$. Geom. Dedicata **162**, 1–7
5. Franciosi, M., Tenni, E.: On Clifford's theorem for singular curves. Proc. Lond. Math. Soc. **108** (2014), 225–252
6. Fujino, O.: Fundamental theorems for the log minimal model program. Publ. Res. Inst. Math. Sci. **47**, 727–789 (2011)
7. Fujino, O., Gongyo, Y.: On canonical bundle formulas and subadjunctions. Michigan Math. J. **61**, 255–264 (2012)
8. Fujita, T.: On the structure of polarized manifolds with total deficiency one I, II, and III. J. Math. Soc. Japan **32**, 709–725 (1980); **33**, 415–434 (1981); **36**, 75–89 (1984)
9. Fujita, T.: On polarized manifolds of sectional genus two. Proc. Japan Acad. **52**, Ser. A, 69–72 (1986)
10. Fujita, T.: On singular del Pezzo varieties. Lect. Notes Math. **1417**, 117–128
11. Graded Ring Database: http://www.grdb.co.uk
12. Heuberger, L.: Anticanonical divisors on Fano fourfolds (2015). arXiv:1505.02437
13. Höring, A., Śmiech, R.: Anticanonical system of Fano fivefolds. Math. Nachr. **293**, 115–119 (2020)
14. Höring, A., Voisin, C.: Anticanonical divisors and curve classes on Fano manifolds. Pure Appl. Math. Q. **7**, Special Issue: In memory of Echart Viewhweg, 1371–1393 (2011)
15. Kawamata, Y.: On Fujita's freeness conjecture for 3-folds and 4-folds. Math. Ann. **308**, 491–505 (1997)

16. Kawamata, Y.: On effective non-vanishing and base-point-freeness. Asian J. Math. **4**, Kodaira's issue, 173–181 (2000)
17. Kollár, J., Mori, S.: Birational geometry of algebraic varieties. Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998)
18. Liu, J.: Second Chern class of Fano manifolds and anti-canonical geometry. Math. Ann. **375**, 655–669 (2019)
19. Liu, J.: Note on quasi-polarized canonical Calabi-Yau threefolds. C. R. Math. Acad. Sci. Paris **358**, 415–420 (2020)
20. Mella, M.: Existence of good divisors on Mukai varieties. J. Algebr. Geom. **8**, 197–206 (1999)
21. Mukai, S.: Biregular classification of Fano threefolds and Fano manifolds of coindex 3. Proc. Nat. Acad. Sci. **86**, 3000–3002 (1989)
22. Pizzato, M., Sano, T., Tasin, L.: Effective nonvanishing for Fano weighted complete intersections. Algebr. Number Theory **11**, 2369–2395 (2017)
23. Shokurov, V.V.: Smoothness of a general anticanonical divisor on a Fano variety. Izv. Akad. Nauk. SSSR Ser. Mat. **43**, 430–441 (1972)

A Database of Group Actions on Riemann Surfaces

Jennifer Paulhus

Abstract The automorphism group of a Riemann surface is an important object in a number of different mathematical fields. An algorithm of Thomas Breuer determines all such groups for a fixed genus given a complete classification of groups up to a sufficiently large order, but data generated from this algorithm did not include the generators of the corresponding monodromy group, another crucial piece of information for researchers. This paper describes modifications the author made to Breuer's code to add the generators, as well as other new code to compute additional information about a given Riemann surface. Data from this project has been incorporated into the *L-functions and Modular Forms Database* (http://www.lmfdb.org) and we also describe the relevant data which may be found there.

Keywords Riemann surfaces · Automorphism groups · Group actions · Surface kernel epimorphisms · Fuchsian groups · Algebraic curves · Automorphisms

AMS classification 14H37 · 20H10 · 30F20

1 Introduction

The study of groups acting on Riemann surfaces and the corresponding branched coverings is classical, dating back to work of Klein, Hurwitz, and others [26, 27, 32]. The field saw a revival in the mid 19th century [21, 22, 36, 51], while the advent of computer algebra programs led to many new advances since the beginning of this century. Of particular interest for this paper is work of Breuer who created an algorithm and wrote computer code to determine all groups acting on Riemann surfaces of a given genus [5]. He ran the code up to genus 48, and recorded the groups

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along with limited information about the ramification of the mapping $X \to X/G$ from a Riemann surface X to the orbit space of X by the group G acting on it.

Groups acting on Riemann surfaces are also important to mathematicians working in other areas, from the Galois theory of extensions of $\mathbb{C}(z)$ [55], to Jacobian variety decompositions [33, 41], to Galois covers of the projective line corresponding to Shimura varieties [15], to questions about indecomposable rational functions [16]. Most of these topics utilize the generators of the monodromy group of the covering corresponding to the mapping $X \to X/G$. Within Breuer's code, generators of the monodromy group were also computed, but not recorded. We added functionality to Breuer's code to fully compute these generators, and wrote new code to compute additional information about Riemann surfaces. As this data will aid other researchers, we are creating a publicly visible, easily accessible database containing this data.

Enter the *L-functions and Modular Forms Database* (LMFDB), a huge database of mathematical objects. As an established database with a strong infrastructure, LMFDB is an ideal location to post this data. Part of its goal is to provide opportunities for unexpected connections between mathematical concepts. This paper describes the modifications we made to Breuer's code, as well as additional computations we use to generate data on LMFDB (such as which actions correspond to full automorphism groups, and which correspond to hyperelliptic curves). The relevant code may be found at http://github.com/jenpaulhus/group-actions-RS and the database is at http://www.lmfdb.org/HigherGenus/C/Aut.

Section 2 is an overview of the necessary mathematical background on groups acting on Riemann surfaces, and in Sect. 3 we describe the theoretical underpinnings of the original code of Breuer. In Sect. 4 we explain the new mathematical information added to the data and discuss the organization of the data on LMFDB. Finally, in Sect. 5 we enumerate planned future additions to the database.

2 Background on Riemann Surfaces

Let X be a compact Riemann surface of genus $g \geq 2$ (also referred to as a "curve"), and let G =Aut(X), the group of biholomorphic maps from X to itself. It is well known that this group is finite and bounded in size by $84(g-1)$. There is a natural mapping $\phi : X \to Y = X/G$ where Y is the orbit space of X under the action of G (ϕ sends $x \in X$ to the orbit of x under the action of G), and g_0 denotes the genus of the quotient Y. It is possible that this mapping branches at several points of Y, say on a set $\mathcal{B} \subset Y$ of size r. Letting $\phi^{-1}(\mathcal{B}) \subset X$ be the inverse image of these points, the mapping from $X - \phi^{-1}(\mathcal{B})$ to $Y - \mathcal{B}$ is a degree d covering for some positive integer d. For details on the covering space theory used in the paper, we recommend [35, Chaps. 11 and 12]. For our specific situation, we recommend [17] or [5].

Fix a base point $y_0 \in Y - \mathcal{B}$. Then $\phi^{-1}(y_0)$ consists of d points in $X - \phi^{-1}(\mathcal{B})$, say $\phi^{-1}(y_0) = \{x_1, \ldots, x_d\} \subset X$. Now consider a loop starting at y_0 and traveling once around one branch point in $\mathcal{B}$. For each element x_i in $\phi^{-1}(y_0)$ this loop lifts uniquely to a path in X which starts at x_i and ends at some $x_j \in \phi^{-1}(y_0)$, thus defining a permutation on the d elements of $\phi^{-1}(y_0)$: send i to the number of the endpoint of the corresponding lift starting at x_i. There is one such permutation for each element of $\mathcal{B}$ and these r permutations induce a map $\rho : \pi_1(Y - \mathcal{B}, y_0) \to S_d$ where S_d is the symmetric group on d elements, and the image of ρ is called the *geometric monodromy group* which is isomorphic to the Galois group of the covering which in our setting is $\mathrm{Aut}(X)$. The order of each permutation corresponding to a loop around one element of $\mathcal{B}$ is denoted m_i for $1 \leq i \leq r$. When X and Y are connected, the image of ρ is a transitive subgroup of S_d.

The universal cover of a compact Riemann surface of genus greater than 1 is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ which has automorphism group $\mathrm{PSL}(2, \mathbb{R})$, and so X may be described as the orbit space of $\mathbb{H}$ by a torsion free subgroup of $\mathrm{Aut}(\mathbb{H})$ (see [5, Theorem 3.9] or [30, 4.19.8]). Call that torsion free subgroup K. It is isomorphic to $\pi_1(X, x_0)$, for any base point $x_0 \in X$ since Riemann surfaces are path-connected.

Similarly, Y is equivalent to the orbit space of $\mathbb{H}$ by a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ called a *Fuchsian group*. These Fuchsian groups have an explicit presentation which considers $\mathcal{B} \subset Y$[5, Theorem 3.2]:

$$\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1, \gamma_j^{m_j} = 1 \rangle \tag{1}$$

where $[\alpha_i, \beta_i]$ is the commutator of α_i and β_i. The list of non-negative integers $[g_0; m_1, \ldots, m_r]$ is called the *signature* of Γ and is uniquely determined for each Fuchsian group. The action of Γ on $\mathbb{H}$ induces an action of Γ/K on $\mathbb{H}/K$, so $G \cong \Gamma/K$. As such we have an exact sequence

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\eta} G \to 1. \tag{2}$$

Then, $G = \mathrm{Aut}(X)$ may also be defined as the image of a *surface kernel epimorphism*, a surjection $\eta : \Gamma \to G$.

Observe that different surface kernel epimorphisms may exist for fixed groups Γ and G so to classify actions it is not sufficient to only give the group and signature. We also need to describe the map η via, say, a description of where η sends the generators. Due to the structure of Γ, the Galois group of the covering is completely defined by $2g_0$ *hyperbolic* generators $a_1, b_1, \ldots, a_{g_0}, b_{g_0}$ and r *elliptic* generators $c_1, \ldots, c_r$ such that the c_i have order m_i and the product $\prod_{i=1}^{g_0} [a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$ where 1_G is the identity element of G. We call this list of $2g_0 + r$ generators of G a *generating vector*.

Conversely, suppose G is any transitive subgroup of some symmetric group S_d with $2g_0 + r$ generators $\{a_1, b_1, \ldots, a_{g_0}, b_{g_0}, c_1, \ldots, c_r\}$ such that the c_i have order m_i and $\prod_{i=1}^{g_0} [a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$. We say such a group has *product one generators*, and a set of $2g_0 + r$ generators is a *product one generator*. Then any surjection

$\eta : \Gamma \to G$ defined as $\eta(\alpha_i) = a_i$, $\eta(\beta_i) = b_i$, and $\eta(\gamma_i) = c_i$ has a corresponding kernel K, and G acts on the compact Riemann surface X defined as the orbits of K acting on $\mathbb{H}$.

Hence there is a one-to-one correspondence between surjective maps $\eta : \Gamma \to G$ with $\ker(\eta)$ a torsion free group and finite groups which have product one generators. This is the beautiful existence theorem of Riemann (really a generalization of it) and it gives a way to translate the topological language of ramified coverings to the world of generators of finite groups. There are several very good sources on Riemann's existence theorem, particularly [17]. For a brief survey with generalizations and historical perspectives, see [20]. The topic is also treated briefly in [39, pp. 90–94], or in relation to function fields and the Inverse Galois Problem in [55].

There are a number of different equivalence relations that may be placed on the surface kernel epimorphisms and we must make choices about which equivalence relation to classify group actions up to in the database. For more information on classifications of automorphism groups of Riemann surfaces up to other equivalence classes see Sects. 4 and 5. Breuer's algorithm computes epimorphisms up to an equivalence relation which is slightly weaker than topological or conformal equivalence, meaning two distinct group actions in his data may actually be topologically (or even conformally) equivalent.

Let G be a finite group which is the image of a surface kernel epimorphism $\eta : \Gamma \to G$, with $[g_0; m_1, \ldots, m_r]$ the signature of Γ. We do not want to consider two actions to be distinct if they merely come from a permuting of the m_i in the corresponding signature. As such we will assume that the m_i in the signature are in non-decreasing order. We denote by $\mathcal{C} = (C_1, \ldots, C_r)$ a list of r conjugacy classes in G (not necessarily distinct) each containing elements of order m_i. Define S to be a set of r-tuples $\{(s_1, \ldots, s_r) : s_i \in C_i\}$. Then G acts on S by component-wise conjugation called *simultaneous conjugation*. We note for later that this is precisely the action of the inner automorphisms of G on the generating vectors.

The properties of a tuple in S being a product one generator are invariant under simultaneous conjugation. In the special case when these tuples are generating vectors, any two vectors in the same orbit under simultaneous conjugation represent conformally equivalent actions in the Riemann surface (although the converse is not always true). This follows from the definition of conformal equivalence (see Sect. 5.2) and the fact that conjugation is an element of $\mathrm{Aut}(G)$. We classify our actions up to simultaneous conjugation.

Given a Riemann surface X of genus g, a group G acting on X with signature $[g_0; m_1, \ldots, m_r]$, a tuple $\mathcal{C} = (C_1, \ldots, C_r)$ of conjugacy classes of G, and a generating vector $(s_1, \ldots, s_r)$ with s_i in C_i, then the tuple $(g, G, \mathcal{C})$ is called a *refined passport* [50] (alternatively that *X is of ramification type* $(g, G, \mathcal{C})$ [37] or that the action is defined by the *geometric signature* $[g_0; [m_1, C_1], [m_2, C_2], \ldots, [m_r, C_r]]$ [47]). A *passport* is a similar tuple of information, but the conjugacy classes are only considered in S_d, so the actions are only classified up to the cycle type of the generators of G.

3 Breuer's Code

Breuer's contribution to this topic was to devise an algorithm to generate a list of all groups and corresponding signatures for which there is a surface kernel epimorphism $\eta : \Gamma \to G$ for a fixed genus. We only give a brief overview of his algorithm here (see [5] for more details).

Breuer's algorithm first generates a list of all possible signatures for Fuchsian groups Γ for a given genus g and given order n of the automorphism group, using combinatorial restrictions on possible m_i values as well as the Riemann-Hurwitz formula.

Next the algorithm searches the small group database in [18] and uses group theoretic results to construct a list of groups G of order n which could have one of the determined admissible signatures for that n. If a group of order n does not have elements of orders corresponding to the values in the signature, it is removed from the list of potential automorphism groups.

Finally, the algorithm determines which possible groups G satisfy the condition that there is a surjective morphism $\eta : \Gamma \to G$. This step in the algorithm utilizes several different group theoretic results concerning the structure of conjugacy classes. The algorithm first attempts to show no such surjection exists. It determines all possible lists of conjugacy classes $\mathcal{C} = (C_1, \dots, C_r)$ such that the order of elements in C_i is m_i (i.e., potential refined passports for a given genus and group). Breuer then computes the size of $\mathrm{Hom}_{\mathcal{C}}(g_0, G)$, the set of homomorphisms from the Fuchsian group corresponding to the given signature to the group G, using the following theorem.

Theorem 3.1 (Theorem 3, [29]) *With $\mathcal{C} = (C_1, \dots, C_r)$ as above,*

$$|Hom_{\mathcal{C}}(g_0, G)| = |G|^{2g_0-1} \sum_{\chi \in Irr(G)} \chi(1)^{2-2g-r} \prod_{i=1}^{r} \sum_{\sigma_i \in C_i} \chi(\sigma_i).$$

When this value is 0, there cannot be a surface kernel epimorphism for that refined passport. In the case where $g_0 = 0$ a result in [48, Theorem 1] gives a sufficient condition on the irreducible characters of a group G to show there is not a surjective homomorphism $\eta : \Gamma \to G$.

Conversely, to show there is an epimorphism $\eta : \Gamma \to G$, a specific generating vector defining the particular surface kernel epimorphism must be found (as the images in G of $\alpha_i, \beta_i, \gamma_j$ from (1) under the mapping η). A brute search of all possible generating vectors for a given refined passport is not feasible, especially for large signatures or large groups.

Instead, for the case $g_0 = 0$, Breuer considers a possible list of conjugacy classes $\mathcal{C} = (C_1, \dots, C_r)$ which has not been ruled out by the results outlined above, and uses the following proposition to quickly generate one element of each orbit under the action of simultaneous conjugation.

Proposition 3.2 (Lemma 15.27, [5]) *Fix elements $\sigma_i \in C_i$ for each $1 \leq i \leq r$. Then the following set T gives us precisely one representative for each orbit of the action of G on $S = \{(s_1, \ldots, s_r) : s_i \in C_i\}$ by simultaneous conjugation:*

$$T = \{(\sigma_1, \sigma_2^{b_2}, \ldots, \sigma_r^{b_r}) : b_i \in R(b_1, \ldots, b_{i-1}) \text{ for } 2 \leq i \leq r\}$$

where $R(b_1, \ldots, b_{i-1})$ is a set of representatives of the double coset

$$C_G(\sigma_i)\backslash G/C_G(\sigma_1, \sigma_2^{b_2}, \ldots, \sigma_{i-1}^{b_{i-1}}),$$

defined iteratively and where $C_G(g_1, g_2, \ldots, g_k)$ means the intersection of the centralizers of $g_i \in G$ for $1 \leq i \leq k$.

The set T may or may not contain product one generators, but if it does these will represent all possible generating vectors up to simultaneous conjugation (again, for the case where $g_0 = 0$). Thus each element of T is tested to see if it is a product one generator.

The $g_0 > 0$ case follows similarly by considering $\mathrm{Hom}_C(g_0, G)$ as a disjoint union and using Proposition 3.2 on certain lists of conjugacy classes. More details may be found in the proof of Theorem 3.1 in [29].

Breuer did not record these generating vectors in his original data, though. His goal was to list group and signature pairs only.

4 New Additions

As mentioned above, one way to fully classify group actions on Riemann surfaces, is to produce a generating vector for each action. We converted Breuer's code to the computer algebra language Magma [4] to align the code with other programs written by the author. We also added functionality which, given a group and signature, outputs the generating vector(s) for each refined passport up to simultaneous conjugation, generated via Proposition 3.2 (see [42], specifically the file `genvectors.mag`). With this code we do not need to reproduce all of Breuer's program. We use his already generated group and signature pairs as a starting point, and then add the generating vectors using the modified version of his code.

There is a software package in GAP called `MapClass`, which, among other computations, finds the generating vectors given a group and list of conjugacy classes corresponding to a refined passport [28]. Quotients of all triangle groups (actions such that $g_0 = 0$ and $r = 3$ or 4) acting on surfaces of genus up to 101, giving one generating vector per group and signature pair may be found at [10]. We also note that lists of actions with monodromy up to genus 21 were independently computed and posted online [31].

4.1 Full Actions

One important piece of information which is not determined in Breuer's original code is whether the group action described is the full automorphism group for the family of curves with corresponding data. Suppose we have an exact sequence

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\eta} G \to 1$$

as in (2), and a corresponding generating vector from our modified version of Breuer's code. It is possible that there is some group $\tilde{G}$ so that $G < \tilde{G}$, a Fuchsian group Γ_0, a mapping $j : \Gamma \to \Gamma_0$, and an exact sequence

$$1 \to K \xrightarrow{\iota_o} \Gamma_0 \xrightarrow{\eta_0} \tilde{G} \to 1$$

so that $\eta = \eta_0 \circ j$. In this case, the generic element of this family of Riemann surfaces has automorphism group $\tilde{G}$ and signature that of Γ_0.

In [46] there are conditions for determining exactly when this situation occurs. (Identical results were independently discovered in [8].) Given G and Γ, the paper also describes explicitly how to compute $\tilde{G}$ and Γ_0. The cases where $G \lhd \tilde{G}$ are covered in [46, Theorem p. 390], while the remaining cases are covered in Tables 1 and 2 of that paper. First, the signature of Γ must match one of only a handful of signatures for which this scenario can happen, originally listed in [52]. For example, if $g_0 = 0$ and there are more than 4 branch points, the given group G is always the full automorphism group of the generic point of the family (η in this case never satisfies the conditions outlined in [46]). In the cases where $G \lhd \tilde{G}$, if the signature is one of the few that might lead to a larger automorphism group there must also exist an element of the automorphism group of G that behaves in a certain way on the generating vector corresponding to the action η.

We have written code [43] which takes the output of the modified Breuer program and determines if the mapping η defined by a generating vector satisfies one of the conditions outlined in Ries. When such an example is found, the group $\tilde{G}$ and signature of Γ_0 are also recorded. One caveat: the code only determines the group $\tilde{G}$ and signature of Γ_0, it does not determine exactly which refined passport (if there is more than one) the original group G and signature correspond to. This should be possible to determine using information in the proof of the Theorem on p. 390 in [46].

In the special case when the signature of the action is $[0; k, k, k]$ or $[0; k, k, k, k]$, we must determine if there exists an automorphism of G which acts in a certain way on a generating vector *up to applying an element of* $\mathrm{Aut}^+(\Gamma)$ *to the elements of the generating vector*, where $\mathrm{Aut}^+(\Gamma)$ is orientation preserving automorphisms of Γ. In the two cases when this happens, $g_0 = 0$ so the group $\mathrm{Aut}^+(\Gamma)$ is the Artin braid group. This group is an infinite (but finitely generated) group generated by $Q_1, \ldots, Q_{r-1}$ where Q_i is the mapping sending one generating vector $(s_1, s_2, \ldots, s_r)$

to $(s_1, \ldots, s_{i-1}, s_{i+1}, s_{i+1}^{-1} s_i s_{i+1}, s_{i+2}, \ldots, s_r)$ [38, Sect. 3.7]. We call two generating vectors which are equivalent up to the action of this group *braid equivalent*.

Even though the braid group is infinite, the orbit of a given generating vector under the action of the elements of the braid group is finite (since the group G is finite there are only a finite number of generating vectors). To exhaustively determine whether the action corresponds to the full group, we need to generate the whole orbit of a given generating vector and test if there is an element of $\mathrm{Aut}(G)$ which acts on one of the generating vectors in that orbit in such a way to satisfy the conditions as described in Ries's paper. To do this, given a generating vector and all cycles of it (or permutations if the group is abelian), we apply the braids Q_1, Q_2 (and Q_3 in the case of $[0; k, k, k, k]$) to the list of generating vectors and test all of the elements in this list against the condition set out in [46, Theorem p. 390]. If we find an automorphism satisfying the conditions in this theorem, we have a candidate for the full automorphism group. If not, we apply the braids to the new larger set and repeat the process. Eventually the whole orbit is generated (if it doesn't find, along the way, a generating vector in the orbit which satisfies the condition mentioned above) and the program will terminate since the orbit is finite. If it terminates without finding a generating vector satisfying the conditions, the action represented by the initial generating vector must be the full automorphism group.

4.2 Special Properties

Once we determine whether an action represents the full automorphism group, we compute additional information connected to the given refined passports. For example, Riemann surfaces described by these actions might be hyperelliptic curves or cyclic trigonal curves. A hyperelliptic curve of genus g is defined by the presence in its automorphism group of a central involution with $2g + 2$ fixed points, while a cyclic trigonal curve of genus g is defined by the presence of an automorphism of order 3 which fixes $g + 2$ points. Using [53], given a generating vector we compute the number of fixed points of a given automorphism (also see [5, Lemma 10.4]), and then determine if the curve is hyperelliptic or cyclic trigonal. The code also computes the hyperelliptic involution or trigonal automorphism, which we include in the database.

Work of the author gives a method to use the automorphism group of a curve (and the generating vectors of the action) to produce a decomposition of its Jacobian variety [41]. The code to implement this method may be found at [44] and we use that code on our compiled list of generating vectors. An entry such as $E \times E^3 \times A_4 \times A_5^2$ in the database means the decomposition consists of four factors: an elliptic curve, three isogenous copies of (possibly) another elliptic curve, one dimension four abelian variety, and two isogenous copies of a dimension five abelian variety. Each factor corresponds to a particular irreducible $\mathbb{C}$-representation of G and we also record the corresponding irreducible $\mathbb{C}$-character as determined by Magma's character table for the group.

While generating vectors themselves are enough to define group actions on Riemann surfaces, the equation(s) for the curves in a given family are valuable to know as well. Determining an equation for a curve given an automorphism group and signature is, in general, a very hard problem and there are many papers in which equations for certain families of curves with group actions are found, for example [23, 24, 45, 54]. For this database we add known equations for hyperelliptic curves [49], genus 3 curves with automorphisms [37], and genus 4–7 curves with "large" automorphism groups (the size of the automorphism group is at least $4(g-1)$) [53] with one small exception. In [49] the equations are classified up to passports, not up to refined passports (the cycle structure of the generating vectors instead of the conjugacy classes in G). In two cases (if $G \cong C_2 \times C_2$, and if $G \cong C_4 \times C_2$ and the quotient of G by the hyperelliptic involution is $C_2 \times C_2$) there is more than one equation listed in [49] but in our data there are distinct refined passports which are in the same passport. The author does not know a way to determine which equation(s) correspond to which refined passport.

4.3 Equivalence Relations

As we mentioned earlier, distinct generating vectors may well produce actions which are the same up to certain equivalence relations. Breuer's code already only produces actions up to simultaneous conjugation, but we also compute equivalence classes for two other equivalence relations.

Two actions η_1 and η_2 are *topologically equivalent* if there exists an $\omega \in \mathrm{Aut}(G)$ and $\phi \in \mathrm{Aut}^+(\Gamma)$ so that the following diagram commutes [7].

$$\begin{tikzcd} \Gamma \arrow[r, "\eta_1"] \arrow[d, "\phi"] & G \arrow[d, "\omega"] \\ \Gamma \arrow[r, "\eta_2"] & G \end{tikzcd}$$

Notice this means that $\eta_2 = \omega \circ \eta_1 \circ \phi^{-1}$. As such, two actions are topologically equivalent precisely when they are in the same orbit under the action of $\mathrm{Aut}(G) \times \mathrm{Aut}^+(\Gamma)$ [7, Proposition 2.2]. This last statement translates the definition of topological equivalence to an algebraic condition which is computationally feasible to check in many cases. Based on Sage code described in [3] we wrote Magma code which, in the case when $g_0 = 0$, inputs all generating vectors (up to simultaneous conjugation) for a fixed group and signature and returns a representative (and the corresponding orbit) of each equivalence class of generating vectors. We restrict to $g_0 = 0$ because $\mathrm{Aut}^+(\Gamma)$ is much easier to work with in this case.

In the study of Hurwitz spaces (and the related inverse Galois problem) generating vectors up to the action of $\mathrm{Inn}(G) \times \mathrm{Aut}^+(\Gamma)$ are instead used. Since Breuer's code already computes one representative per equivalence class under the action of inner automorphisms (which is precisely simultaneous conjugation) and since the actions of each group in this direct product commute with each other, to find the orbits under the action we only need consider the action of $\mathrm{Aut}^+(\Gamma)$ on the output of the modified Breuer code for each group and signature pair. When $g_0 = 0$, this action is exactly the *braid action* we described in Sect. 4.1 and we use the same technique described there to compute equivalence classes under the braiding action and assign a representative generating vector for each orbit.

4.4 Summary

One note about our presentation of groups. Breuer's original code outputs a group as labeled in Magma or GAP, so as a pair (a, b) which indicates the group is of order a and is the bth group of that order in the database of small groups. Our Magma version of Breuer's code requires the group to be a permutation group to compute double coset representatives as in Proposition 3.2. However, in Magma many groups of the form `SmallGroup(a,b)` are not permutation groups. Also, to correspond to the mapping $\rho : \pi_1(Y - \mathcal{B}, y_0) \to S_d$ from Sect. 2, the group G must be transitive and satisfy the Riemann-Hurwitz formula. So we first convert the group to a minimum degree transitive permutation group. The code to do this is at [43]. In doing so, we are specifying that our covers are Galois.

Putting everything together, the final process to create the database at http://www.lmfdb.org/HigherGenus/C/Aut is:

- For a fixed genus, load all the signature and group pairs computed with Breuer's original program and loop over this data.
- Convert groups of the form `SmallGroup(a, b)` in Breuer's data to permutation groups.
- Use our modified version of Breuer's code to determine the refined passports, and compute generating vector(s) for each.
- Determine if the action on each refined passport describes the full automorphism group of the family.
- Compute the Jacobian variety decomposition.
- If the action is the full action, check if the family consists of hyperelliptic or cyclic trigonal curves. In special cases we add equations.
- In the case of $g_0 = 0$, determine equivalence classes and representatives up to braid and topological equivalence.
- Future additional information will be computed at this point.

4.5 Organization of the Data on LMFDB

As of publication of this paper, the database contains complete data up to genus 15 when the quotient X/G is the Riemann sphere ($g_0 = 0$) and up to genus 7 when $g_0 > 0$.

Each tuple of information: (genus, group, signature) has its own page on LMFDB. On each such page there is a list of the different refined passports corresponding to the given genus, group, and signature, and links to individual pages for each refined passport. Up to genus 7, every page also gives an option to only view actions up to topological equivalence. Clicking on the label of the given representative for an equivalence class leads to a page which lists all the refined passports in the given equivalence class (and further delineated according to which are braid equivalent to each other).

The individual pages of each refined passport list all generating vectors corresponding to this passport. We also list which conjugacy classes the refined passport corresponds to (as labeled by Magma when we initially generate the data–see Sect. 5.1). These pages also contain information about whether the action represents the full automorphism group of the family of Riemann surfaces. If the example is not the full automorphism group, a link to the action which does correspond to the full automorphism group is also included. We note if a refined passport of a full automorphism group corresponds to a hyperelliptic curve or a cyclic trigonal curve, and list the corresponding hyperelliptic involution or trigonal automorphism. Known equations are also displayed on these pages. Up to genus 7 if there is more than one generating vector on a page, there is an option to list only the representatives of each orbit under the braid action instead of all generating vectors. This feature is of particular value as the genus gets large, as there are examples of refined passports with thousands of distinct generating vectors up to simultaneous conjugation but only a small handful up to braid action.

On both types of pages, a download button is available which downloads a Magma or GAP record with information for the given refined passport (or several records representing all the refined passports corresponding to a specific group and signature). For researchers working on questions requiring computations of generating vectors this feature should be the most useful as these files can simply be downloaded and then loaded into Magma or GAP for immediate access to the generating vectors. Also, a variety of search fields such as signature, or dimension of the family, or whether the family is hyperelliptic add to the functionality of the pages, and all search results may also be downloaded as Magma or GAP files.

A variety of statistics about the data currently in the database reside at https://www.lmfdb.org/HigherGenus/C/Aut/stats. The statistics list the maximum order of a group acting for each genus and all the unique groups which act for a fixed genus. The number of distinct refined passports and distinct generating vectors for each genus are also calculated, as well as the distribution of generating vectors in the database by dimension.

5 Future Work

We plan to add additional information to the database. Here are a few examples.

5.1 *Better Representation of Groups*

One issue with the current data is that different representations of isomorphic groups can create different lists of generating vectors as the representatives of each orbit under the equivalence relations we have discussed. Also the labeling of the irreducible characters or conjugacy classes is dependent on Magma's labeling for that particular representation of the group (so the 2nd conjugacy class may not represent the same conjugacy class for distinct isomorphic groups).

Recently a database of small groups has been incorporated into LMFDB (see https://www.lmfdb.org/Groups/Abstract/). Among many other pieces of information for each group, particular elements of the group are fixed as generators (as are relations defining the group) and the conjugacy classes and irreducible characters of the group have a fixed labeling, all assigned in a deterministic way.

We can redo the computations from scratch (i.e., follow the steps outlined in Sect. 4.4) but now starting from the fixed representation of the group as defined in the small group database. Doing so ensures that labeling of generating vectors, conjugacy classes, and irreducible characters will be deterministic. No more debate over what is meant by the 2nd irreducible character or the 2nd conjugacy class of the group! The group pages also produce character tables and we will be able to link the irreducible characters listed on our pages directly to the corresponding row of the character table presented on the group's page.

5.2 *Equivalence Relations*

Some researchers only requires knowledge about distinct actions up to conformal (or analytic) equivalence. Two actions $\eta_1 : \Gamma \to G$ and $\eta_2 : \Gamma \to G$ are *conformally equivalent* if there is some $\omega \in \mathrm{Aut}(G)$ and $\widetilde{h} \in \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$ so that the following diagram commutes

$$\begin{array}{ccccc} K & \longrightarrow & \Gamma & \xrightarrow{\eta_1} & G \\ \big\downarrow{\scriptstyle \widetilde{h}^*} & & \big\downarrow{\scriptstyle \widetilde{h}^*} & & \big\downarrow{\scriptstyle \omega} \\ K & \longrightarrow & \Gamma & \xrightarrow{\eta_2} & G \end{array}$$

where $\widetilde{h}^*$ is the map that takes some $\gamma \in K$ (or in Γ) and sends it to $\widetilde{h}\gamma\widetilde{h}^{-1}$ [7]. This definition induces a conformal mapping $h : X \to X$ where $X = \mathbb{H}/K$. We hope to find a way to efficiently compute equivalence classes of generating vectors up to conformal equivalence, and then provide options on the LMFDB pages to only show generating vectors up to conformal equivalence.

5.3 Higher Genus Data

Breuer computed all group and signature pairs up to genus 48, and Conder computed group and signature pairs for large groups (those with $|G| > 4(g-1)$) up to much higher genus [11]. We plan to use the steps described in Sect. 4.4 to compute and then upload higher genus data to the database, although first some current code will need to be made more efficient to effectively compute data in higher genus.

As one particular example, the code to compute orbits of actions under topological equivalence is very slow for particular families of groups as the genus increase. There are several theoretical results and computational techniques that will speed up these computations. In addition, for $g_0 > 0$ the action of $\mathrm{Aut}^+(\Gamma)$ is more complicated than in the case where the quotient genus is the Riemann sphere and so we don't currently provide the option to list actions with $g_0 > 0$ up to topological equivalence. The code we use to compute topological equivalence would need to be rewritten to be able to do so.

5.4 Other Topics

- There is much current research on superelliptic curves, and we could incorporate known data about these families into LMFDB.
- A new section in the LMFDB provides a database of Belyĭ maps [40]. There are many connections that could be made between that database and the one described in this paper.
- Let $H < G$, then there are methods for determining both coverings $X \to X/H$ and $X/H \to X/G$ [47]. These methods were programmed in SAGE [3] and we intend to add this feature to our code.
- The group and signature pairs which show up for a fixed genus create a poset. We could display such a diagram to emphasize connections among families of curves in the moduli space $\mathcal{M}_g$. Several papers describe the branch locus of $\mathcal{M}_g$ for low genus g, for example [1, 6, 12]. Our database provides the information and tools to attempt a description in higher genus.
- The Riemann matrix and corresponding period matrix are crucial objects for understanding certain computational properties of Riemann surfaces. If an equation for the curve is known, there are some computational ways to compute these matrices [13, 14, 19]. If only the group action is known, there are algorithms that work well in low genus that could be added to our computations [2, 34].

- It would be nice to determine the fields of definition of these curves. Some work in this direction includes [9, 25], as well as [24] which produces an algorithm to find an algebraic model of a given curve in its (minimal) definition field. It is worth trying to program this algorithm.

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References

1. Bartolini, G., Costa, A.F., Izquierdo, M.: On the connectivity of branch loci of moduli spaces. Ann. Acad. Sci. Fenn. Math. **38**(1), 245–258 (2013)
2. Behn, A., Rodríguez, R.E., Rojas, A.M.: Adapted hyperbolic polygons and symplectic representations for group actions on Riemann surfaces. J. Pure Appl. Algebra **217**(3), 409–426 (2013)
3. Behn, A., Rojas, A.M., Tello-Carrera, M.: A SAGE package for n-gonal equisymmetric stratification of. Exp. Math. 1–16 (2020)
4. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. J. Symb. Comput. **24**(3–4), 235–265 (1997). Computational algebra and number theory (London, 1993)
5. Breuer, T.: Characters and automorphism groups of compact Riemann surfaces. London Mathematical Society Lecture Note Series, vol. 280. Cambridge University Press, Cambridge (2000)
6. Broughton, S.A.: The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups. Topol. Appl. **37**(2), 101–113 (1990)
7. Broughton, S.A.: Classifying finite group actions on surfaces of low genus. J. Pure Appl. Algebra **69**(3), 233–270 (1991)
8. Bujalance, E., Cirre, F.J., Conder, M.D.: On extendability of group actions on compact Riemann surfaces. Trans. Am. Math. Soc. **355**(4), 1537–1557 (2003)
9. Cardona, G., Quer, J.: Field of moduli and field of definition for curves of genus 2. In: Computational Aspects of Algebraic Curves. Lecture Notes in Computer Science, vol. 13, pp. 71–83. World Scientific Publishing, Hackensack, NJ (2005)
10. Conder, M.: Quotients of triangle groups acting on surfaces of genus 2–101 (2007). https://www.math.auckland.ac.nz/~conder/TriangleGroupQuotients101.txt
11. Conder, M.D.E.: Large group actions on surfaces. In: Riemann and Klein surfaces, automorphisms, symmetries and moduli spaces. Contemporary Mathematics, vol. 629, pp. 77–97. American Mathematical Society, Providence, RI (2014)

12. Costa, A.F., Izquierdo, M.: On the connectedness of the branch locus of the moduli space of Riemann surfaces of genus 4. Glasg. Math. J. **52**(2), 401–408 (2010)
13. Costa, E., Mascot, N., Sijsling, J., Voight, J.: Rigorous computation of the endomorphism ring of a Jacobian. Math. Comp. **88**(317), 1303–1339 (2019)
14. Bernard, D., Van Hoeij, M.: Computing Riemann matrices of algebraic curves. Adv. Nonlinear Math. Sci. **152/153**, 28–46 (2001)
15. Frediani, P., Ghigi, A., Penegini, M.: Shimura varieties in the Torelli locus via Galois coverings. Int. Math. Res. Not. IMRN **20**, 10595–10623 (2015)
16. Fried, M.: The field of definition of function fields and a problem in the reducibility of polynomials in two variables. Illinois J. Math. **17**, 128–146 (1973)
17. Fried, M.: Exposition on an arithmetic-group theoretic connection via Riemann's existence theorem. In: The Santa Cruz Conference on Finite Groups (University California, Santa Cruz, California, 1979). Proceedings of Symposia in Pure Mathematics, vol. 37, pp. 571–602. American Mathematical Society, Providence, R.I. (1980)
18. The GAP Group.: GAP—Groups, Algorithms, and Programming, Version 4.4 (2006). http://www.gap-system.org
19. Gianni, P., Seppälä, M., Silhol, R., Trager, B.: Riemann surfaces, plane algebraic curves and their period matrices. Symb. Numer. Algebr. Polynomials **26**, 789–803 (1998)
20. Harbater, D.: Riemann's existence theorem. In: The Legacy of Bernhard Riemann After 150 Years, pp. 275–286. Higher Education Press and International Press (2015)
21. Harvey, W.J.: Cyclic groups of automorphisms of a compact Riemann surface. Quart. J. Math. Oxford Ser. **2**(17), 86–97 (1966)
22. Harvey, W.J.: On branch loci in Teichmüller space. Trans. Am. Math. Soc. **153**, 387–399 (1971)
23. Hidalgo, R.A., Jiménez, L., Quispe, S., Reyes-Carocca, S.: Quasiplatonic curves with symmetry group $\mathbb{Z}_2^2 \times \mathbb{Z}_m$ are definable over $\mathbb{Q}$. Bull. Lond. Math. Soc. **49**(1), 165–183 (2017)
24. Hidalgo,R.A., Reyes-Carocca, S.: Weil's Galois descent theorem from a computational point of view. In: Geometry at the frontier—symmetries and moduli spaces of algebraic varieties. Contemporary Mathematics, vol. 766, pp. 217–228. American Mathematical Society, [Providence], RI (2021)
25. Huggins, B.S.: Fields of moduli and fields of definition of curves. ProQuest LLC, Ann Arbor, MI (2005). Thesis (Ph.D.)–University of California, Berkeley
26. Hurwitz, A.: Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. Math. Ann. **41**(3), 403–442 (1892)
27. Hurwitz, A.: Ueber die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten. Math. Ann. **55**(1), 53–66 (1901)
28. James, A., Magaard, K., Shpectorov, S., Völklein, H.: MapClass (2012). http://www.gap-system.org/Packages/mapclass.html
29. Jones, G.A.: Enumeration of homomorphisms and surface-coverings. Quart. J. Math. Oxford Ser. **46**(184), 485–507 (1995)
30. Jones, G.A., Singerman, D.: Complex functions: an algebraic and geometric viewpoint. Cambridge University Press, Cambridge (1987)
31. Karbas, J., Nedela, R.: Actions of finite groups on Riemann surfaces of higher genera (2013). https://www.savbb.sk/~karabas/finacts.html
32. Klein, F.: Ueber die transformation siebenter Ordnung der elliptischen Functionen. Math. Ann. **14**(3), 428–471 (1878)
33. Lange, H., Recillas, S.: Abelian varieties with group action. J. Reine Angew. Math. **575**, 135–155 (2004)
34. Lange, H., Rodríguez, R.E., Rojas, A.M.: Polarizations on abelian subvarieties of principally polarized abelian varieties with dihedral group actions. Math. Z. **276**(1–2), 397–420 (2014)
35. Lee, J.: Introduction to Topological Manifolds. Graduate Texts in Mathematics, vol. 202, 2nd edn. Springer, Berlin (2011)
36. Macbeath, A.M.: On a theorem of Hurwitz. Proc. Glasgow Math. Assoc. **5**(90–96), 1961 (1961)
37. Magaard, K., Shaska, T., Shpectorov, S., Völklein, H.: The locus of curves with prescribed automorphism group. Sūrikaisekikenkyūsho Kōkyūroku (1267), 112–141 (2002). Communications in arithmetic fundamental groups (Kyoto, 1999/2001)

38. Magnus, W., Karrass, A., Solitar, D.: Combinatorial group theory: presentations of groups in terms of generators and relations. Interscience Publishers [Wiley, Inc.], New York-London-Sydney (1966)
39. Miranda, R.: Algebraic curves and Riemann surfaces. Graduate Studies in Mathematics, vol. 5. American Mathematical Society, Providence, RI (1995)
40. Musty, M., Schiavone, S., Sijsling, J., Voight, J.: A database of Belyi maps. In: Proceedings of the Thirteenth Algorithmic Number Theory Symposium. Open Book Series, vol. 2, pp. 375–392. Mathematical Sciences Publishers, Berkeley, CA (2019)
41. Paulhus, J.: Decomposing Jacobians of curves with extra automorphisms. Acta Arith. **132**(3), 231–244 (2008)
42. Paulhus, J.: Magma code to find generating vectors for groups acting on Riemann surfaces (2015). https://github.com/jenpaulhus/breuer-modified/
43. Paulhus, J.: Code for a database of group actions on Riemann surfaces (2016). https://github.com/jenpaulhus/group-actions-RS
44. Paulhus, J., Rojas, A.M.: Magma code to determine Jacobian variety decompositions (2016). https://github.com/jenpaulhus/decompose-jacobians
45. Reyes-Carocca, S., Rojas, A.M.: Rojas. On large prime actions on Riemann surfaces. J. Group Theory (2022)
46. Ries, J.F.X.: Subvarieties of moduli space determined by finite groups acting on surfaces. Trans. Am. Math. Soc. **335**(1), 385–406 (1993)
47. Rojas, A.M.: Group actions on Jacobian varieties. Rev. Mat. Iberoam. **23**(2), 397–420 (2007)
48. Scott, L.L.: Matrices and cohomology. Ann. of Math. **105**(3), 473–492 (1977)
49. Shaska, T.: Determining the automorphism group of a hyperelliptic curve. In: Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, pp. 248–254 (electronic). ACM, New York (2003)
50. Sijsling, J., Voight, J.: On computing Belyi maps. In: Numéro consacré au trimestre "Méthodes arithmétiques et applications, automne 2013. Publicationes Mathematicae Besançon Algèbre Théorie Nr., vol. 2014/1, pp. 73–131. Presses University, Franche-Comté, Besançon (2014)
51. Singerman, D.: Subgroups of Fuchsian groups and finite permutation groups. Bull. London Math. Soc. **2**, 319–323 (1970)
52. Singerman, D.: Finitely maximal Fuchsian groups. J. London Math. Soc. **2**(6), 29–38 (1972)
53. Swinarski, D.: Software for computing conformal block divisors on $\overline{M}_{0,n}$. J. Softw. Algebra Geom. **8**, 81–86 (2018)
54. Turbek, P.: Computing equations, automorphisms and symmetries of Riemann surfaces. In: Riemann and Klein Surfaces, Automorphisms, Symmetries and Moduli Spaces. Contemporary Mathematics, vol. 629, pp. 335–348. American Mathematical Society, Providence, RI (2014)
55. Völklein, H.: Groups as Galois groups. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1996). An introduction

A 1-Dimensional Component of K-Moduli of del Pezzo Surfaces

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Abstract We explicitly construct a component of the K-moduli space of K-polystable del Pezzo surfaces which is a smooth rational curve.

Keywords K-stability · Del pezzo surfaces · K-moduli

1 Introduction

One of the most important and recent results in K-stability and in the theory of Fano varieties is the construction of K-moduli [3, 8–10, 15, 18, 22, 35, 37]. It has been proved that, for every positive integer n and every positive rational number V, $\mathbb{Q}$-Gorenstein families of K-semistable Fano varieties over $\mathbb{C}$ of dimension n and anticanonical volume V form an algebraic stack $\mathcal{M}^{\mathrm{Kss}}_{n,V}$ of finite type over $\mathbb{C}$. Moreover, this stack admits a good moduli space $M^{\mathrm{Kps}}_{n,V}$, which is a projective scheme over $\mathbb{C}$, and the set of closed points of $M^{\mathrm{Kps}}_{n,V}$ coincides with the set of K-polystable Fano varieties over $\mathbb{C}$ of dimension n and anticanonical volume V. We refer the reader to [36] for a survey on these topics.

The case of smoothable del Pezzo surfaces has been extensively studied [23, 25, 26]. Moreover, K-moduli are understood for cubic 3-folds [21], cubic 4-folds [19], and for certain pairs (S, C) where S is a surface and C is a curve on S [5, 6].

The goal of this note is to show how toric geometry and deformation theory can help understanding the geometry of explicit components of K-moduli. Similar ideas were used in [16] to construct examples of reducible or non-reduced K-moduli of Fano 3-folds (see also [28, 29]), in [20] to study the K-stability of certain del Pezzo surfaces with Fano index 2, and in [24] to study the dimension of K-moduli. In this note we analyse a specific example of K-polystable toric del Pezzo surface and we prove the following:

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Theorem 1.1 *There exists a connected component of $M^{\mathrm{Kps}}_{2,\frac{22}{15}}$ which is isomorphic to $\mathbb{P}^1$.*

It is natural to wonder about the following:

Question 1.2 Does there exist $V \in \mathbb{Q}_{>0}$ such that a connected component of $M^{\mathrm{Kps}}_{2,V}$ is a smooth curve of positive genus?

Outline. In Sect. 2.1 we briefly recall the deformation theory of the surface singularity given by the cone over the rational normal curve of degree 4. In Sect. 2.2 we introduce a Fano polygon P and a K-polystable toric del Pezzo surface X, and we analyse its deformation theory; in particular, we show that the connected component of the K-moduli space of K-polystable del Pezzo surfaces that contains X is smooth and 1-dimensional. In Sect. 2.4 we prove that X is a hypersurface in a toric 3-fold Y and in Sect. 2.5 we prove that deforming X inside the linear system $|\mathcal{O}_Y(X)|$ on Y gives the versal deformation of X. This gives a non-constant morphism from an open subset of $|\mathcal{O}_Y(X)|$ to the K-moduli space. In Sect. 2.3 we conclude the proof of Theorem 1.1. In Sect. 3 we sketch what mirror symmetry says in this context.

Notation and conventions. We work over an algebraically closed field of characteristic zero, which is denoted by $\mathbb{C}$. A *Fano* variety is a normal projective variety over $\mathbb{C}$ such that its anticanonical divisor is $\mathbb{Q}$-Cartier and ample. A *del Pezzo* surface is a Fano variety of dimension 2. We assume that the reader is familiar with toric geometry [14]. Every toric variety we consider is normal.

If $r, a_1, \dots, a_n$ are integers and $r \geq 1$, then the symbol $\frac{1}{r}(a_1, \dots, a_n)$ stands for the quotient of $\mathbb{A}^n$ under the action of the cyclic group $\boldsymbol{\mu}_r$ defined by $\zeta \cdot (x_1, \dots, x_n) = (\zeta^{a_1}x_1, \dots, \zeta^{a_n}x_n)$ for every $\zeta \in \boldsymbol{\mu}_r$. We use the same symbol to indicate the étale-equivalence class of the singularity of this quotient variety at the image of the origin of $\mathbb{A}^n$.

2 Proof

2.1 Deformations of $\frac{1}{4}(1, 1)$

The cyclic quotient singularity $\frac{1}{4}(1, 1)$ is the affine cone over the 4th Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^4$. The deformations of this singularity have been studied by Pinkham [32, Sect. 4]. Here we concentrate on the $\mathbb{Q}$-Gorenstein deformations—see [29, Sect. 2] for a quick recap.

The singularity $\frac{1}{4}(1, 1)$ has Gorenstein index 2. Its index 1 cover is $\frac{1}{2}(1, 1)$, which is the hypersurface singularity $(xy - z^2 = 0)$ in $\mathbb{A}^3_{x,y,z} = \operatorname{Spec} \mathbb{C}[x, y, z]$. Therefore $\frac{1}{4}(1, 1)$ is the closed subscheme of the 3-fold quotient singularity $\frac{1}{2}(1, 1, 1)_{x,y,z}$ given, with respect to the orbifold coordinates x, y, z, by the equation $xy - z^2 = 0$.

Since the miniversal deformation of $\frac{1}{2}(1, 1)$ is given by $xy - z^2 + t = 0$ in $\mathbb{A}^3_{x,y,z}$ over $\mathbb{C}[\![t]\!]$, we have that the miniversal $\mathbb{Q}$-Gorenstein deformation of $\frac{1}{4}(1, 1)$ is given by

$$xy - z^2 + t = 0 \tag{1}$$

inside $\frac{1}{2}(1, 1, 1)_{x,y,z}$ over $\mathbb{C}[\![t]\!]$. This specifies a formal morphism

$$\mathrm{Spf}(\mathbb{C}[\![t]\!]) \longrightarrow \mathrm{Def}^{\mathrm{qG}}\left(\frac{1}{4}(1, 1)\right), \tag{2}$$

which is smooth and induces an isomorphism on tangent spaces. Here Spf denotes the formal spectrum of a local noetherian $\mathbb{C}$-algebra. We will always use this morphism when considering the $\mathbb{Q}$-Gorenstein deformation functor of the singularity $\frac{1}{4}(1, 1)$.

Now we make a calculation which will be useful in Sect. 2.5. Consider the 2-parameter deformation

$$xy - z^2 + s_1 + s_2 z^4 = 0 \tag{3}$$

in $\frac{1}{2}(1, 1, 1)_{x,y,z}$ over $\mathbb{C}[\![s_1, s_2]\!]$. By versality this deformation comes from the miniversal deformation (1) via pull-back along a formal morphism

$$\mathrm{Spf}(\mathbb{C}[\![s_1, s_2]\!]) \longrightarrow \mathrm{Spf}(\mathbb{C}[\![t]\!]) \xrightarrow{(2)} \mathrm{Def}^{\mathrm{qG}}\left(\frac{1}{4}(1, 1)\right), \tag{4}$$

which is induced by a local $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\![t]\!] \to \mathbb{C}[\![s_1, s_2]\!]$. Via the automorphism of $\frac{1}{2}(1, 1, 1)_{x,y,z} \times \mathrm{Spf}(\mathbb{C}[\![s_1, s_2]\!])$ given by

$$z \mapsto z\sqrt{1 - s_2 z^2} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (1 - 2n)} s_2^n z^{n+1}$$

we get an isomorphism of the deformation (3) with $xy - z^2 + s_1 = 0$, which is exactly the miniversal deformation (1) once we use the equality $t = s_1$. Therefore the morphism in (4) is induced by the local $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\![t]\!] \to \mathbb{C}[\![s_1, s_2]\!]$ given by $t \mapsto s_1$.

2.2 The Surface X

In the lattice $N = \mathbb{Z}^2$ consider the polygon P which is the convex hull of the points

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and is depicted in Fig. 1. (The meaning of the red segments in this figure will be clear in Sect. 2.4.) It is clear that P is a *Fano polytope*, i.e. it is a lattice polytope such that

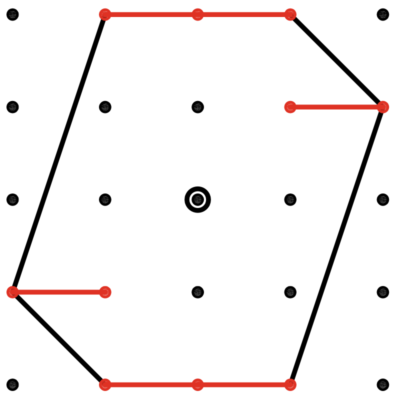

Fig. 1 The polygon P in Sect. 2.2

the origin is in the interior and the vertices are primitive lattice points. Because of this we can consider the face fan (also called spanning fan) of P: this is the collection of cones (with apex at the origin) over the faces of P; it is made up of 6 rational cones in N.

Proposition 2.1 *Let X be the toric variety associated to the face fan of P. Then:*

(1) X is a K-polystable toric del Pezzo surface with anticanonical volume $\frac{22}{15}$;
(2) the surface X has exactly 6 singular points: 2 points of type $\frac{1}{3}(1,1)$, 2 points of type $\frac{1}{4}(1,1)$, 2 points of type $\frac{1}{5}(1,2)$;
(3) the automorphism group $\mathrm{Aut}(X)$ is isomorphic to $(\mathbb{C}^*)^2 \rtimes C_2$, where C_2 is the cyclic group of order 2 and the non-trivial element of C_2 acts on $(\mathbb{C}^*)^2$ via $(z,w) \mapsto (z^{-1}, w^{-1})$.

Proof (1) Since N has rank 2, the dimension of X is 2. By a slight modification of [14, Theorem 8.3.4], since the fan of X is the face fan of a Fano polytope, we have that X is Fano.

Let P° denote the polar of P (see [16, Sect. 2.4]); it is a rational polytope in the dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and is the moment polytope of the toric boundary of X, which is the reduced sum of the torus invariant prime divisors of X and is an anticanonical divisor. The anticanonical volume of X is the normalised volume of P°, which is $\frac{22}{15}$. Here the normalised volume is the double of the Lebesgue measure: in this way the normalised volume of a unimodular simplex is 1. Since P is centrally symmetric (i.e. $P = -P$), also P° is centrally symmetric, hence the barycentre of P° is the origin. Therefore X is K-polystable by [7].

In order to prove (2) one needs to analyse the six 2-dimensional cones of the face fan of P and apply [14, Sect. 10.1]. For instance, the two horizontal edges of P give the two $\frac{1}{4}(1,1)$ singularities.

(3) let T_N denote the 2-dimensional torus $N \otimes_{\mathbb{Z}} (\mathbb{C}^*)^2 = \mathrm{Spec}\, \mathbb{C}[M]$ which acts on X. Let $\mathrm{Aut}(P)$ be the group of the symmetries of P: it is generated by $-\mathrm{id}_N$. Since every facet of P° has no interior lattice points, by [16, Proposition 2.8] $\mathrm{Aut}(X)$ is the semidirect product $T_N \rtimes \{\pm \mathrm{id}_N\}$. □

The points of type $\frac{1}{3}(1,1)$ and $\frac{1}{5}(1,2)$ are $\mathbb{Q}$-Gorenstein rigid, i.e. they do not deform $\mathbb{Q}$-Gorensteinly. The $\mathbb{Q}$-Gorenstein deformations of $\frac{1}{4}(1,1)$ have been considered in Sect. 2.1.

By [1, Lemma 6] there are no local-to-global obstructions for $\mathbb{Q}$-Gorenstein deformations of X, so the $\mathbb{Q}$-Gorenstein smoothings of the two $\frac{1}{4}(1,1)$ points of X, which we denote p_1 and p_2, can be realised globally and simultaneously. More precisely, since $\mathrm{H}^i(X, T_X) = 0$ for $i \geq 1$ by [30], the product of the restriction morphisms to the germs $(p_i \in X)$

$$\mathrm{Def}^{\mathrm{qG}}(X) \longrightarrow \mathrm{Def}^{\mathrm{qG}}(p_1 \in X) \times \mathrm{Def}^{\mathrm{qG}}(p_2 \in X) \tag{5}$$

is smooth and induces an isomorphism on tangent spaces. So $\mathbb{C}[\![t_1, t_2]\!]$ is the hull of $\mathrm{Def}^{\mathrm{qG}}(X)$ and t_i is the $\mathbb{Q}$-Gorenstein smoothing parameter of $(p_i \in X)$. Here the parameter t_i is defined through (2). In the next section we will realise the miniversal $\mathbb{Q}$-Gorenstein deformation of X in a linear system in a toric Fano 3-fold.

Proposition 2.2 *Let $\mathcal{M}$ (resp. M) be the connected component of the K-moduli stack $\mathcal{M}^{\mathrm{Kss}}_{2,\frac{22}{15}}$ (resp. the K-moduli space $M^{\mathrm{Kps}}_{2,\frac{22}{15}}$) which contains the point corresponding to X. Then M is a smooth projective irreducible curve.*

Proof Since $\mathbb{Q}$-Gorenstein deformations of del Pezzo surfaces are unobstructed by [1, Lemma 6], by [16, Remark 2.4] we get that $\mathcal{M}$ is smooth and M is normal. Moreover M is projective by [22].

The automorphism group $\mathrm{Aut}(X)$ acts on the hull $\mathbb{C}[\![t_1, t_2]\!]$. The weights of t_1 (resp. t_2) in M is $(0, 1)$ (resp. $(0, -1)$). Therefore the invariant subring of the formal action of $T_N = (\mathbb{C}^*)^2$ on $\mathbb{C}[\![t_1, t_2]\!]$ is $\mathbb{C}[\![t_1 t_2]\!]$. The group C_2 swaps t_1 and t_2, so it leaves $t_1 t_2$ invariant. Therefore the invariant subring of the formal action of $\mathrm{Aut}(X)$ on $\mathbb{C}[\![t_1, t_2]\!]$ is $\mathbb{C}[\![t_1 t_2]\!]$.

By the Luna étale slice theorem for algebraic stacks [4] the local structure of $\mathcal{M} \to M$ is given by the commutative square

$$\begin{tikzcd}
\left[\mathrm{Spf}\, \mathbb{C}[\![t_1, t_2]\!] \ / \ \mathrm{Aut}(X)\right] \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\
\mathrm{Spf}\, \mathbb{C}[\![t_1 t_2]\!] \arrow[r] & M
\end{tikzcd}$$

where the horizontal maps are formally étale and maps the closed point to $[X]$. This implies that M has dimension 1. Hence M is a smooth projective curve. □

2.3 The 3-Fold Y and the Proof of Theorem 1.1

Consider $\mathbb{A}^6$ with coordinates $x_1, x_2, y_1, y_2, z_1, z_2$. Consider the toric 3-fold Y given by the GIT quotient $\mathbb{A}^6 /\!\!/ (\mathbb{C}^*)^3$ where the linear action of $(\mathbb{C}^*)^3$ on $\mathbb{A}^6$ is specified by the weights

$$\begin{array}{cccccc|c} x_1 & x_2 & y_1 & y_2 & z_1 & z_2 & \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & L_1 \\ 0 & 1 & 3 & 1 & 0 & 6 & L_2 \\ 1 & 0 & 1 & 3 & 6 & 0 & L_3 \end{array}$$

and by the stability condition whose unstable locus is the vanishing locus of the ideal

$$(x_1, x_2, z_1) \cdot (x_1, x_2, z_2) \cdot (y_1, y_2) \cdot (y_1, z_2) \cdot (y_2, z_1) \tag{6}$$

in the polynomial ring $\mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]$. Now L_1, L_2, L_3 are the $\mathbb{Q}$-line bundles on Y which come from the standard basis of the character lattice of $(\mathbb{C}^*)^3$. They form a $\mathbb{Z}$-basis of the divisor class group of Y.

We see that $\mathrm{H}^0(Y, 2L_1 + 6L_2 + 6L_3)$ has dimension 4 and its monomial basis is made up of the monomials

$$z_1z_2, \ \ y_1y_2x_1^2x_2^2, \ \ x_1^4y_1^2, \ \ x_2^4y_2^2.$$

We consider a special affine subspace of $\mathrm{H}^0(Y, 2L_1 + 6L_2 + 6L_3)$ and we relate to the surface X considered in Sect. 2.2:

Proposition 2.3 *Let Y be the toric 3-fold defined above. Let X be the toric del Pezzo surface considered in Sect. 2.2. Consider the flat family $\mathscr{X} \to \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[s_1, s_2]$ of hypersurfaces in the linear system $|2L_1 + 6L_2 + 6L_3|$ on Y defined by the equation*

$$z_1z_2 - y_1y_2x_1^2x_2^2 + s_1x_1^4y_1^2 + s_2x_2^4y_2^2 = 0. \tag{7}$$

Then:

(A) the fibre of $\mathscr{X} \to \mathbb{A}^2$ over the origin $0 \in \mathbb{A}^2$ is the toric surface X;
(B) the base change of $\mathscr{X} \to \mathbb{A}^2$ to $\mathbb{C}[[s_1, s_2]]$ is the miniversal $\mathbb{Q}$-Gorenstein deformation of X.

We postpone the proof of this proposition: the proof of (A) is given in Sect. 2.4 and the proof of (B) is given in Sect. 2.5. Now we show how this proposition implies our main result.

Proof of Theorem 1.1. Let $\mathcal{M}$ and M be as in Proposition 2.2. We have that M is a smooth projective irreducible curve. We want to show that M is isomorphic to $\mathbb{P}^1$.

Let $\mathscr{X} \to \mathbb{A}^2$ be the $\mathbb{Q}$-Gorenstein family considered in Proposition 2.3. Since the central fibre is K-polystable, by openness of K-semistability [9], there exists an open neighbourhood U of the origin in $\mathbb{A}^2$ such that the fibred product $\mathscr{X} \times_{\mathbb{A}^2} U \to U$ induces a morphism $U \to \mathcal{M}$, which is formally smooth at the origin.

By looking at the action of $\operatorname{Aut}(X)$ on the base of the miniversal $\mathbb{Q}$-Gorenstein deformation of X (see the proof of Proposition 2.2), we see that there are K-polystable surfaces in U non-isomorphic to X. Therefore, by composing $U \to \mathcal{M}$ with $\mathcal{M} \to M$, we get a non-constant morphism $U \to M$. By restricting to a general line passing through the origin in $U \subseteq \mathbb{A}^2$, we get that M is unirational. Therefore M is rational by Lüroth's theorem. This concludes the proof of Theorem 1.1. □

2.4 Proof of Proposition 2.3(A)

We need to prove that the surface X is the hypersurface in the 3-fold Y defined by the equation $z_1 z_2 - y_1 y_2 x_1^2 x_2^2 = 0$. We apply the Laurent inversion method [13, 33, 34].

Let e_1, e_2 be the standard basis of $N = \mathbb{Z}^2$. Consider the decomposition

$$N = \overline{N} \oplus N_U$$

where $\overline{N} = \mathbb{Z}e_1$ and $N_U = \mathbb{Z}e_2$. Let $\overline{M}$ be the dual lattice of $\overline{N}$. Let Z be the $T_{\overline{M}}$-toric variety associated to complete fan in the lattice $\overline{M}$ with rays generated by e_1^* and $-e_1^*$. It is clear that Z is isomorphic to $\mathbb{P}^1$. Let $\mathrm{Div}_{T_{\overline{M}}}(Z)$ be the rank-2 lattice consisting of the torus invariant divisors on Z: a basis of $\mathrm{Div}_{T_{\overline{M}}}(Z)$ is given by the torus invariant prime divisors on Z, namely E_+, E_-, which are associated to the rays $e_1^*, -e_1^*$ respectively. The divisor sequence [14, Theorem 4.1.3] of Z is

$$0 \longrightarrow \overline{N} = \mathbb{Z}e_1 \xrightarrow{\rho^\star = \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathrm{Div}_{T_{\overline{M}}}(Z) = \mathbb{Z}E_+ \oplus \mathbb{Z}E_- \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathrm{Pic}(Z) = \mathbb{Z} \longrightarrow 0.$$

We consider the following ample torus invariant divisors on Z

$$\begin{aligned} D_{x_1} &= E_+ + E_- & \qquad D_{y_1} &= -E_+ + 2E_- \\ D_{x_2} &= E_+ + E_- & \qquad D_{y_2} &= 2E_+ - E_- \end{aligned}$$

and their corresponding moment polytopes in $\overline{N}$:

$$\begin{aligned} P_{D_{x_1}} &= \mathrm{conv}\,\{-e_1, e_1\} & \qquad P_{D_{y_1}} &= \mathrm{conv}\,\{e_1, 2e_1\}, \\ P_{D_{x_2}} &= \mathrm{conv}\,\{-e_1, e_1\} & \qquad P_{D_{y_2}} &= \mathrm{conv}\,\{-2e_1, -e_1\}. \end{aligned}$$

Now consider the following elements in the lattice $N_U = \mathbb{Z}e_2$:

$$\begin{aligned} \chi_{x_1} &= 2e_2 & \qquad \chi_{y_1} &= e_2 \\ \chi_{x_2} &= -2e_2 & \qquad \chi_{y_2} &= -e_2. \end{aligned}$$

The polytopes

$$\begin{array}{ll} P_{D_{x_1}} + \chi_{x_1} & P_{D_{y_1}} + \chi_{y_1} \\ P_{D_{x_2}} + \chi_{x_2} & P_{D_{y_2}} + \chi_{y_2} \end{array}$$

in $N = \overline{N} \oplus N_U$ are the four red segments in Fig. 1. Clearly the polygon P is the convex hull of these four segments. By [13, Definition 3.1] the set

$$S = \left\{ \left(D_{x_1}, \chi_{x_1}\right), \ \left(D_{x_2}, \chi_{x_2}\right), \ \left(D_{y_1}, \chi_{y_1}\right), \ \left(D_{y_2}, \chi_{y_2}\right) \right\}$$

is a 'scaffolding' on the Fano polygon P.

Consider the rank-3 lattice $\tilde{N} := \mathrm{Div}_{T_{\overline{M}}}(Z) \oplus N_U = \mathbb{Z}E_+ \oplus \mathbb{Z}E_- \oplus \mathbb{Z}e_2$. Let $\tilde{M}$ be the dual lattice of $\tilde{N}$ and let $\langle \cdot, \cdot \rangle \colon \tilde{M} \times \tilde{N} \to \mathbb{Z}$ be the duality pairing. Following [13, Definition A.1] we consider the polytope $Q_S \subseteq \tilde{M}_{\mathbb{R}}$ defined by the following inequalities:

$$\begin{aligned} \langle \cdot , -D_{x_1} + \chi_{x_1} \rangle &\geq -1, \\ \langle \cdot , -D_{x_2} + \chi_{x_2} \rangle &\geq -1, \\ \langle \cdot , -D_{y_1} + \chi_{y_1} \rangle &\geq -1, \\ \langle \cdot , -D_{y_2} + \chi_{y_2} \rangle &\geq -1, \\ \langle \cdot , E_+ \rangle &\geq 0, \\ \langle \cdot , E_- \rangle &\geq 0. \end{aligned}$$

Let Σ_S be the normal fan of Q_S. One can see that Σ_S is the complete simplicial fan in $\tilde{N} = \mathrm{Div}_{T_{\overline{M}}}(Z) \oplus N_U$ with rays generated by the following vectors:

$$\begin{aligned} x_1 &= -D_{x_1} + \chi_{x_1} = -E_+ - E_- + 2e_2 \\ x_2 &= -D_{x_2} + \chi_{x_2} = -E_+ - E_- - 2e_2 \\ y_1 &= -D_{y_1} + \chi_{y_1} = E_+ - 2E_- + e_2 \\ y_2 &= -D_{y_2} + \chi_{y_2} = -2E_+ + E_- - e_2 \\ z_1 &= E_+, \\ z_2 &= E_-. \end{aligned}$$

Let Y be the $T_{\tilde{N}}$-toric variety associated to the fan Σ_S. Thus Y is a $\mathbb{Q}$-factorial Fano 3-fold with Cox coordinates $x_1, x_2, y_1, y_2, z_1, z_2$. With respect to the basis of $\tilde{N}$ given by E_+, E_-, e_2, the rays of the fan Σ_S are the columns of the matrix

$$\begin{pmatrix} -1 & -1 & 1 & -2 & 1 & 0 \\ -1 & -1 & -2 & 1 & 0 & 1 \\ 2 & -2 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

The transpose of this matrix gives an injective $\mathbb{Z}$-linear homomorphism $\tilde{M} \to \mathbb{Z}^6$. By [14, Theorem 4.1.3] the cokernel of this is the divisor map of Y and is isomorphic to the divisor class group of Y. In this case, one finds that the divisor map of Y is the $\mathbb{Z}$-linear homomorphism $\mathbb{Z}^6 \to \mathrm{Cl}(Y) \simeq \mathbb{Z}^3$ given by the following matrix.

$$\begin{array}{cccccc|c} x_1 & x_2 & y_1 & y_2 & z_1 & z_2 & \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & L_1 \\ 0 & 1 & 3 & 1 & 0 & 6 & L_2 \\ 1 & 0 & 1 & 3 & 6 & 0 & L_3 \end{array}$$

Here L_1, L_2, L_3 are the elements of the chosen $\mathbb{Z}$-basis of $\mathrm{Cl}(Y)$. This 3×6 matrix gives the weights of a linear action of the torus $(\mathbb{C}^*)^3$ on $\mathbb{A}^6$. By [14, Sect. 5.1] Y is the GIT quotient of this action with respect to the stability condition given by the irrelevant ideal

$$(x_1, x_2, z_1) \cdot (x_1, x_2, z_2) \cdot (y_1, y_2) \cdot (y_1, z_2) \cdot (y_2, z_1).$$

Therefore Y is the toric 3-fold considered in Sect. 2.3.

We now consider the injective linear map

$$\theta := \rho^\star \oplus \mathrm{id}_{N_U} : N = \overline{N} \oplus N_U \longrightarrow \tilde{N} = \mathrm{Div}_{T_{\overline{M}}}(Z) \oplus N_U.$$

By [13, Theorem 5.5] θ induces a toric morphism $X \to Y$ which is a closed embedding. We want to understand the ideal of this closed embedding in the Cox ring of Y by using the map θ.

We follow [34, Remark 2.6]. We see that $\theta(N)$ is the hyperplane defined by the vanishing of $h = E_+^* + E_-^* \in \tilde{M}$. Now we compute the duality pairing between h and the primitive generators of the rays of Σ_S: $\langle h, x_1\rangle = \langle h, x_2\rangle = -2$, $\langle h, y_1\rangle = \langle h, y_2\rangle = -1$, $\langle h, z_1\rangle = \langle h, z_2\rangle = 1$. We get that the polynomial

$$z_1 z_2 - y_1 y_2 x_1^2 x_2^2 \tag{8}$$

is the generator of the ideal of the closed embedding $X \hookrightarrow Y$ in the Cox ring of Y. In other words, X is the hypersurface in Y defined by the vanishing of the polynomial (8) in the Cox coordinates of Y. This concludes of (A) in Proposition 2.3.

2.5 *Proof of Proposition 2.3(B)*

We want to show that, after base change to $\mathbb{C}[\![s_1, s_2]\!]$, the family of hypersurfaces in Y defined by the vanishing of (7) is the miniversal $\mathbb{Q}$-Gorenstein deformation of X. Since the map in (5) is smooth and induces an isomorphism on tangent spaces, we need to check that locally this family induces the miniversal deformations of the singularity germs of X. Let t_1 and t_2 be the two smoothing parameters of the two $\frac{1}{4}(1, 1)$ singularities of X, as fixed in Sect. 2.1. We proceed by analysing each chart of the affine open cover of Y given by the fan Σ_S.

- The cone σ_{x_1,z_1,z_2} gives the isolated singularity $\frac{1}{2}(1,1,1)_{x_1,z_1,z_2}$ on Y. In this chart, by dehomogenising (7), we get the equation $z_1z_2 - x_1^2 + s_1x_1^4 + s_2 = 0$ in the orbifold coordinates. This is exactly the $\mathbb{Q}$-Gorenstein smoothing of $\frac{1}{4}(1,1)$ described at the end of Sect. 2.1. So we have $t_2 = s_2$.
- The cone σ_{x_2,z_1,z_2} gives the isolated singularity $\frac{1}{2}(1,1,1)_{x_2,z_1,z_2}$ on Y. In this chart we get the equation $z_1z_2 - x_2^2 + s_1 + s_2x_2^4 = 0$. We are in a completely analogous situation as the previous case, so $t_1 = s_1$.
- The cone σ_{x_1,y_2,z_2} gives the isolated singularity $\frac{1}{5}(2,1,4)_{x_1,y_2,z_2}$ on Y. In this chart we get the equation $z_2 - y_2x_1^2 + s_1x_1^4 + s_2y_2^2 = 0$, which is quasi-smooth because there is no constant term and z_2 appears with degree 1. So all fibres of $\mathscr{X} \to \mathring{A}^2$ have a $\frac{1}{5}(1,2)$ singularity at the 0-stratum of this chart of Y.
- The cone σ_{x_2,y_1,z_1} gives the isolated singularity $\frac{1}{5}(1,2,4)_{x_2,y_1,z_1}$ on Y. The equation is $z_1 - y_1x_2^2 + s_1y_1^2 + s_2x_2^4 = 0$ and, in a way analogous to the previous case, we get a $\frac{1}{5}(1,2)$ singularity on every fibre of $\mathscr{X} \to \mathring{A}^2$ at the 0-stratum of this chart of Y.
- The cone σ_{x_1,y_1,z_1} gives the non-isolated singularity $\frac{1}{3}(1,1,0)_{x_1,y_1,z_1}$. The equation is $z_1 - y_1x_1^2 + s_1x_1^4y_1^2 + s_2 = 0$. Since it is quasi-smooth, this gives a $\frac{1}{3}(1,1)$ singularity on every fibre of $\mathscr{X} \to \mathring{A}^2$ at a point on the curve $(x_1 = y_1 = 0) \subset Y$.
- The cone σ_{x_2,y_2,z_2} gives the non-isolated singularity $\frac{1}{3}(1,1,0)_{x_2,y_2,z_2}$. The equation is $z_2 - y_2x_2^2 + s_1 + s_2x_2^4y_2^2 = 0$ and, similarly to the previous case, we have a $\frac{1}{3}(1,1)$ singularity on every fibre of $\mathscr{X} \to \mathring{A}^2$ at a point on the curve $(x_2 = y_2 = 0) \subset Y$.
- In the fan Σ_S there are two 3-dimensional cones which we have not been analysed yet: these are σ_{x_1,x_2,y_1}, whose corresponding chart on Y is the non-isolated singularity $\frac{1}{12}(3,1,4)_{x_1,x_2,y_1}$, and σ_{x_1,x_2,y_2}, which gives the non-isolated singularity $\frac{1}{12}(4,1,3)_{x_1,x_2,y_2}$ on Y. We want to show that it is useless to analyse these cones. Let V denote the complement in Y of the union of the already analysed charts; V is made up of 3 torus-orbits: the 0-stratum corresponding to σ_{x_1,x_2,y_1}, the 0-stratum corresponding to σ_{x_1,x_2,y_2}, and the 1-stratum corresponding to σ_{x_1,x_2}. In other words V is the projective curve $(x_1 = x_2 = 0)$ in Y. By looking at the Eq. (7) and at the irrelevant ideal (6) it is clear that V does not intersect any fibre of $\mathscr{X} \to \mathring{A}^2$.

To sum up, we have that the family $\mathscr{X} \to \mathring{A}^2$ realises the $\mathbb{Q}$-Gorenstein smoothings of the two $\frac{1}{4}(1,1)$ points on X and leaves the $\frac{1}{3}(1,1)$ points and $\frac{1}{5}(1,2)$ points untouched (i.e. the deformation is formally isomorphic to a product around these points of the central fibre). By versality the family $\mathscr{X} \to \mathring{A}^2$ induces a morphism $\mathrm{Spf}(\mathbb{C}[\![s_1,s_2]\!]) \to \mathrm{Def}^{\mathrm{qG}}(X)$, which is associated to the isomorphism $\mathbb{C}[\![s_1,s_2]\!] \simeq \mathbb{C}[\![t_1,t_2]\!]$, where $s_1 = t_1$ and $s_2 = t_2$. In other words, the base change of $\mathscr{X} \to \mathring{A}^2$ to $\mathbb{C}[\![s_1,s_2]\!]$ is the miniversal $\mathbb{Q}$-Gorenstein deformation of X. This concludes the proof of Proposition 2.3(B).

3 Mirror Symmetry

In [1] some conjectures for del Pezzo surfaces were formulated. In this section we sketch some evidence for these conjectures in the case of the toric del Pezzo surface X and of its $\mathbb{Q}$-Gorenstein deformations. In addition to [1], we refer the reader to [11, 12, 31] and to the references therein for more details about the notions introduced below.

3.1 Combinatorial Avatars of Connected Components of Moduli of Del Pezzo Surfaces

According to [1, Conjecture A] there is a 1-to-1 correspondence between

- connected components of the moduli stack of del Pezzo surfaces (with a toric degeneration) and
- mutation equivalence classes of Fano polygons.

Here a *Fano polygon* is a lattice polygon whose face fan defines a del Pezzo surface (an example is P in Sect. 2.2); and *mutation* is a certain equivalence relation on Fano polygons introduced in [2]—we do not give further details here and we refer the reader to [1, 17].

The correspondence works in the following way: to (the mutation equivalence class of) the Fano polygon P one associates the connected component $\mathscr{M}$ of the moduli stack of del Pezzo surfaces which contains the surface X, which is the toric del Pezzo surface associated to the face fan of P. One has that $\mathscr{M}$ is smooth and contains $\mathcal{M}$ (the connected component of the K-moduli stack parametrising K-semistable del Pezzo surfaces and containing X) as an open substack.

3.2 Classical Period

Consider the family of *maximally mutable* Laurent polynomials with Newton polytope P and with *T-binomial edge coefficients* [1, Definition 4]. This is the 6-dimensional family

$$\begin{aligned} f &= x^2y + x^{-2}y^{-1} + (x + 2 + x^{-1})(y^2 + y^{-2}) \\ &\quad + a_1xy + a_2x^{-1}y^{-1} + b_1x + b_2x^{-1} + c_1xy^{-1} + c_2x^{-1}y \end{aligned}$$

in $\mathbb{Q}[a_1, a_2, b_1, b_2, c_1, c_2][x^{\pm}, y^{\pm}]$, where $a_1, a_2, b_1, b_2, c_1, c_2$ are indeterminates. In Fig. 2 the coefficients of f are written next to the corresponding lattice points of P.

The *classical period* of f is the power series

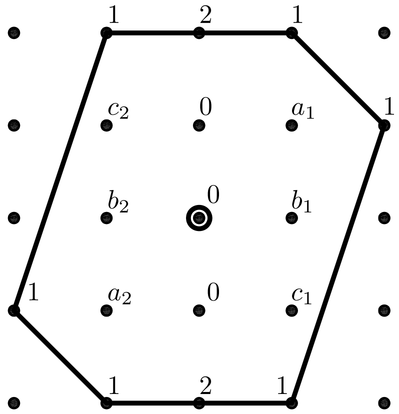

Fig. 2 The coefficients of the maximally mutable Laurent polynomials with Newton polytope P and with T-binomial edge coefficients (see Sect. 3.2)

$$\pi_f(t) = \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\{(x,y)\in(\mathbb{C}^*)^2 \,|\, |x|=|y|=\varepsilon\}} \frac{1}{1-tf(x,y)} \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

in $\mathbb{Q}[a_1, a_2, b_1, b_2, c_1, c_2][\![t]\!]$, for some $0 < \varepsilon \ll 1$. The first coefficients of π_f are:

$$\begin{aligned}\pi_f(t) = 1 + 2(a_1a_2 + b_1b_2 + c_1c_2 + 7)t^2 + \\ + 6(a_1b_1 + 2a_1c_2 + a_2b_2 + 2a_2c_1 + 4b_1 + 4b_2 + c_1 + c_2)t^3 + \cdots .\end{aligned}$$

3.3 Quantum Period

Let X' be the surface corresponding to a general point in $\mathscr{M}$; in other words, X' is a general $\mathbb{Q}$-Gorenstein deformation of the toric surface X. The *quantum period* of X' [27, Definition 3.2] is a certain generating function for genus zero Gromov–Witten invariants of X' which depends on certain parameters related to the singularities of X'. In this case there are 6 parameters because the singular locus of X' is made up of 2 points of type $\frac{1}{3}(1, 1)$ and 2 points of type $\frac{1}{5}(1, 2)$.

In general it is very difficult to compute the quantum period of a Fano orbifold. Since X' is a hypersurface in the toric Fano Y, one can use the quantum Lefschetz theorem to compute a specialisation of the quantum period of X', i.e. the power series $G_{X'} \in \mathbb{Q}[\![t]\!]$ obtained from the quantum period by setting the parameters equal to some numbers. This can be done as follows. We use the notation as in Sect. 2.4. One can see that the nef cone of Y is spanned by the divisor classes

$$
\begin{aligned}
L_1 + 3L_2 + 3L_3,\\
4L_1 + 9L_2 + 9L_3,\\
5L_1 + 9L_2 + 15L_3,\\
5L_1 + 15L_2 + 9L_3.
\end{aligned}
$$

We consider the cone $\Lambda \subseteq \mathbb{R}^3$ defined by the inequalities

$$
\begin{aligned}
l_1 + 3l_2 + 3l_3 &\geq 0,\\
4l_1 + 9l_2 + 9l_3 &\geq 0,\\
5l_1 + 9l_2 + 15l_3 &\geq 0,\\
5l_1 + 15l_2 + 9l_3 &\geq 0
\end{aligned}
$$

and by the inequalities

$$
\begin{aligned}
l_3 &\geq 0,\\
l_2 &\geq 0,\\
l_1 + 3l_2 + l_3 &\geq 0,\\
l_1 + l_2 + 3l_3 &\geq 0,\\
l_1 + 6l_3 &\geq 0,\\
l_1 + 6l_2 &\geq 0.
\end{aligned}
$$

The first inequalities say that we are taking (the closure of) the cone of the effective curves in $\mathrm{N}_1(Y)_{\mathbb{R}}$, i.e. we are taking the dual of the nef cone of Y; with the second inequalities we are taking the curve classes on which the prime torus-invariant divisors of Y have non negative degrees.

By using methods similar to [27], one can prove that a specialisation of the quantum period of X' is the power series $G_{X'}(t) \in \mathbb{Q}[\![t]\!]$ equal to

$$
\sum_{(l_1,l_2,l_3)\in\Lambda\cap\mathbb{Z}^3} \frac{(2l_1 + 6l_2 + 6l_3)!}{l_3!\, l_2!\, (l_1 + 3l_2 + l_3)!\, (l_1 + l_2 + 3l_3)!\, (l_1 + 6l_3)!\, (l_1 + 6l_2)!} t^{2l_1+5l_2+5l_3}.
$$

Notice the following numerology: at the denominator there are the factorial of the degrees of the prime torus-invariant divisors of Y, the numerator is the factorial of the degree of the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(X') = 2L_1 + 6L_2 + 6L_3$, the exponent of t is the degree of the $\mathbb{Q}$-line bundle $-K_Y - X' = 2L_1 + 5L_2 + 5L_3$, which by adjunction restricts to $-K_{X'}$ on X'.

If $\sum_{d\geq 0} C_d t^d$ is the quantum period of X', then the *regularised quantum period* of X' is $\sum_{d\geq 0} d!\, C_d t^d$. From the computation above one computes the first coefficients of a specialisation of the regularised quantum period of X':

$$\widehat{G}_{X'}(t) = 1 + 16t^2 + 936t^4 + 520t^5 + 76840t^6 + 131880t^7 + 7360920t^8 + \\ + 22806000t^9 + 770459256t^{10} + 3451657440t^{11} + 85553394696t^{12} + \cdots .$$

3.4 Equality of Periods

A second mirror-symmetric expectation [1, Conjecture B] is that there is an equality between

- the regularised quantum period of a general surface X' in $\mathscr{M}$ and
- the classical period of the family of maximally mutable Laurent polynomials with Newton polytope P and with T-binomial edge coefficients.

Notice that in our case both periods depend on 6 parameters which should be identified.

Combining Sects. 3.2 and 3.3 one can verify the equality between a specialisation of the regularised quantum period of X' and the classical period of the Laurent polynomial obtained from f by setting $a_1 = a_2 = 1$ and $b_1 = b_2 = c_1 = c_2 = 0$:

$$\widehat{G}_{X'}(t) = \pi_f(t)|_{a_1=a_2=1,\ b_1=b_2=c_1=c_2=0}.$$

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References

1. Akhtar, M., Coates, T., Corti, A., Heuberger, L., Kasprzyk, A., Oneto, A., Petracci, A., Prince, T., Tveiten, K.: Mirror symmetry and the classification of orbifold del Pezzo surfaces. Proc. Am. Math. Soc. **144**(2), 513–527 (2016)
2. Akhtar, M., Coates, T., Galkin, S., Kasprzyk, A.M.: Minkowski polynomials and mutations. SIGMA Symmetry Integr. Geom. Methods Appl. **8**, 094, 17 (2012)
3. Alper, J., Blum, H., Halpern-Leistner, D., Xu, C.: Reductivity of the automorphism group of K-polystable Fano varieties. Invent. Math. **222**(3), 995–1032 (2020)
4. Alper, J., Hall, J., Rydh, D.: A Luna étale slice theorem for algebraic stacks. Ann. of Math. **191**(3), 675–738 (2020)
5. Ascher, K., DeVleming, K., Liu, Y.: K-moduli of curves on a quadric surface and K3 surfaces. To appear in J. Inst. Math. Jussieu. arxiv:2006.06816
6. Ascher, K., DeVleming, K., Liu, Y.: Wall crossing for K-moduli spaces of plane curves. arxiv:1909.04576
7. Berman, R.: K-polystability of $\mathbb{Q}$ -Fano varieties admitting Kähler-Einstein metrics. Invent. Math. **203**(3), 973–1025 (2016)
8. Blum, H., Halpern-Leistner, D., Liu, Y., Xu, C.: On properness of K-moduli spaces and optimal degenerations of Fano varieties. Selecta Math. (N.S.) **27**(4), 73, 39 (2021)
9. Blum, H., Liu, Y., Xu, C.: Openness of K-semistability for Fano varieties. Duke Math. J. **171**(13), 2753-2797 (2022)

10. Liu, y., Petracci, A.: On K-stability of some del Pezzo surfaces of Fano index 2. Bull. London Math. Soc. **54**(2), 517-525 (2022) https://doi.org/10.1112/blms.12581
11. Coates, T., Corti, A., Galkin, S., Golyshev, V., Kasprzyk, A.: Mirror symmetry and Fano manifolds. In: European Congress of Mathematics, pp. 285–300 (2013)
12. Coates, T., Corti, A., Galkin, S., Kasprzyk, A.: Quantum periods for 3-dimensional Fano manifolds. Geom. Topol. **20**(1), 103–256 (2016)
13. Coates, T., Kasprzyk, A., Prince, T.: Laurent inversion. Pure Appl. Math. Q. **15**(4), 1135–1179 (2019)
14. Petracci, A.: On deformations of toric Fano varieties. In: A.M. Kasprzyk and B. Nill (eds), Interactions with Lattice Polytopes. Springer Proceedings in Mathematics & Statistics, vol, 386. Springer. https://doi.org/10.1007/978-3-030-98327-7_14
15. Jiang, C.: Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below. Ann. Sci. Éc. Norm. Supér. **53**(5), 1235–1248 (2020)
16. Kaloghiros, A.-S., Petracci, A.: On toric geometry and K-stability of Fano varieties. Trans. Am. Math. Soc. Ser. B **8**, 548–577 (2021)
17. Kasprzyk, A., Nill, B., Prince, T.: Minimality and mutation-equivalence of polygons. Forum Math. Sigma **5**, e18, 48 (2017)
18. Li, C., Wang, X., Xu, C.: Algebraicity of the metric tangent cones and equivariant K-stability. J. Am. Math. Soc. **34**(4), 1175–1214 (2021)
19. Liu, Y.: K-stability of cubic fourfolds. J. Reine Angew. Math. **786**, 55–77 (2022)
20. Liu, Y., Petracci, A.: On K-stability of some del Pezzo surfaces of Fano index 2. Bull. Lond. Math. Soc. **54**(2), 517–525 (2022)
21. Liu, Y., Xu, C.: K-stability of cubic threefolds. Duke Math. J. **168**(11), 2029–2073 (2019)
22. Liu, Y., Xu, C., Zhuang, Z.: Finite generation for valuations computing stability thresholds and applications to K-stability. Ann. of Math. 196(2), 507–566 (2022)
23. Mabuchi, T., Mukai, S.: Stability and Einstein-Kähler metric of a quartic del Pezzo surface. In: Einstein Metrics and Yang-Mills Connections (Sanda, 1990), vol. 145, pp. 133–160 (1993)
24. Martinez-Garcia, J., Spotti, C.: Some observations on the dimension of Fano K-moduli. Springer Proceedings in Mathematics & Statistics, vol, 409, pp. 607–614. https://doi.org/10.1007/978-3-030-98327-7_30
25. Odaka, Y.: Compact moduli spaces of Kähler-Einstein Fano varieties. Publ. Res. Inst. Math. Sci. **51**(3), 549–565 (2015)
26. Odaka, Y., Spotti, C., Sun, S.: Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics. J. Differ. Geom. **102**(1), 127–172 (2016)
27. Oneto, A., Petracci, A.: On the quantum periods of del Pezzo surfaces with $\frac{1}{3}(1, 1)$ singularities. Adv. Geom. **18**(3), 303–336 (2018)
28. Petracci, A.: K-moduli of Fano 3-folds can have embedded points. arxiv:2105.02307
29. Petracci, A.: On deformation spaces of toric singularities and on singularities of K-moduli of Fano varieties. Trans. Amer. Math. Soc. **375**(8), 5617–5643 (2022). https://doi.org/10.1090/tran/8636
30. Petracci, A.: On deformations of toric Fano varieties. In: A.M. Kasprzyk, B. Nill (eds), Interactions with Lattice Polytopes. Springer Proceedings in Mathematics & Statistics, **386**. Springer. https://doi.org/10.1007/978-3-030-98327-7_14
31. Petracci, A.: An example of mirror symmetry for Fano threefolds. In: Birational Geometry and Moduli Spaces, pp. 173–188 (2020)
32. Pinkham, H.C.: Deformations of cones with negative grading. J. Algebra **30**, 92–102 (1974)
33. Prince, T.: Cracked polytopes and Fano toric complete intersections. Manuscripta Math. **163**(1–2), 165–83 (2020)
34. Prince, T.: From cracked polytopes to Fano threefolds. Manuscripta Math. **164**(1–2), 267–320 (2021)
35. Xu, C.: A minimizing valuation is quasi-monomial. Ann. Math. **191**(3), 1003–1030 (2020)
36. Xu, C.: K-stability of Fano varieties: an algebro-geometric approach. EMS Surv. Math. Sci. **8**(1–2), 265–354 (2021)
37. Xu, C., Zhuang, Z.: On positivity of the CM line bundle on K-moduli spaces. Ann. Math. **192**(3), 1005–1068 (2020)

A Non-standard Bezout Theorem for Curves

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Abstract This paper provides a non-standard analogue of Bezout's theorem for algebraic curves. We achieve this by showing that, in all characteristics, the notion of Zariski multiplicity coincides with intersection multiplicity when we consider the full families of projective degree d and degree e curves in $P^2(L)$. The result is particularly interesting in that it holds even when we consider intersections at singular points of curves or when the curves contain non-reduced components.

Keywords Bezout theorem · Zariski multiplicity · Intersection multiplicity

The techniques of non-standard analysis, originally developed for the real numbers, were recently introduced by Zilber in the context of Zariski structures. In [17], he gives a rigorous notion of Zariski multiplicity, which, in the case of 2 curves C_1 and C_2, intersecting in a point a, can count the number of intersections of the 2 curves in an infinitely small neighborhood of a after moving one of the curves. This idea was used intuitively in the work of the Italian school of algebraic geometry, in particular by Severi. One advantage of this approach is that it avoids an over reliance on algebra, in favour of a more geometric approach. The successes of their work are well known; the development of the notion of genus for algebraic curves, building on the original ideas of Plucker, and the classification of algebraic surfaces. This paper sets out to show that this non-standard analysis can be useful in algebraic geometry, by providing a more geometric framework for understanding intersections of algebraic curves in the plane. In particular, the main result of the paper, a geometric proof of Bezout's theorem, enhances an important idea in the foundational work of the Italian school. We assume some familiarity with certain notions from algebraic and analytic geometry, as well as the material from Sects. 1–5 of [7]. We summarise the relevant facts, for the proofs of the paper, in the following three sections.

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1 Etale Morphisms and Algebraic Multiplicity

Definition 1.1 A morphism f of finite type between varieties X and Y is said to be etale if for all $x \in X$ there are open affine neighborhoods U of x and V of $f(x)$ with $f(U) \subset V$ such that restricted to these neighborhoods the pull back on functions is given by the inclusion;

$$f^* : L[V] \to L[V]\frac{[x_1, \dots, x_n]}{< f_1, \dots, f_n >}$$

and

$$det(\frac{\partial f_i}{\partial x_j})(x) \neq 0 \text{ , } (*)$$

The coordinate free definition of etale is that f should be flat and unramified, where a morphism f is unramified if the sheaf of relative differentials $\Omega_{X/Y} = 0$, clearly this last confition is satisfied using the condition $(*)$. If we tensor the exact sequence,

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

with $L(x)$ the residue field of x, we obtain an isomorphism

$$f^*\Omega_Y \otimes L(x) \to \Omega_X \otimes L(x).$$

Identifying $\Omega_X \otimes L(x)$ with $T^*_{x,X}$ gives that

$$df : (m_x/m_x^2)^* \to (m_{f(x)}/m^x_{f(x)})^*$$

is an isomorphism of tangent spaces or dually $f^*(m_{f(x)}) = m_x$. Call this property of etale morphisms $(**)$.

We will also require some facts about the etale topology on an algebraic variety Y, see [14] for more details. We consider a category Y_{et} whose objects are etale morphisms $U \to Y$ and whose arrows are Y-morphisms from $U \to V$. This category has the following 2 desirable properties. First given $y \in Y$, the set of objects of the form $(U, x) \to (Y, y)$ form a directed system, namely $(U, x) \subset (U', x')$ if there exists an etale morphism $U \to U'$ taking x to x'. Secondly, we can take "intersections" of open sets U_i and U_j by considering $U_{ij} = U_i \times_Y U_j$; the projection maps are easily show to be etale and the composition of etale maps is etale, so $U_{ij} \to Y$ still lies in Y_{et}. If Y is an irreducible variety over L, then all etale morphisms into Y must come from reduced schemes of finite type over L, though they may well fail to be irreducible considered as algebraic varieties. Now we can define the local ring of Y in the etale toplogy to be;

$$O^{\wedge}_{y,Y} = lim_{\to, y \in U} O_U(U)$$

As any open set U of Y clearly induces an etale morphism $U \rightarrow_i Y$ of inclusion, we have that $O_{y,Y} \subset O^{\wedge}_{y,Y}$. We want to prove that $O^{\wedge}_{y,Y}$ is a Henselian ring and in fact the smallest Henselian ring containing $O_{y,Y}$. We need the following lemma about Henselian rings, which can be found in [15].

Lemma 1.2 *Let R be a local ring with residue field L, and maximal ideal $\mathfrak{m}$. Suppose that R satisfies the following condition.*

If $f_1, \ldots f_n \in R[x_1, \ldots x_n]$ and the reductions modulo the maximal ideal $\mathfrak{m}$, $\bar{f}_1 \ldots \bar{f}_n$ have a common root $\bar{a}$ in L^n, for which $Jac(\bar{f})(\bar{a}) = (\frac{\partial \bar{f}_i}{\partial x_j})_{ij}(\bar{a}) \neq 0$, then $\bar{a}$ lifts to a common root in R^n (*).

Then R is Henselian.

It remains to show that $O^{\wedge}_{y,Y}$ satisfies $(*)$.

Proof Given $f_{1,} \ldots f_n$ satisfying the condition of $(*)$, we can assume the coefficients of the f_i belong to $O_{U_i}(U_i)$ for covers $U_i \rightarrow Y$; taking the intersection $U_{1 \ldots i \ldots n}$ we may even assume the coefficients define functions on a single etale cover U of Y. By the remarks above we can consider U as an algebraic variety over L, and even an affine algebraic variety after taking the corresponding inclusion. We then consider the variety $V \subset U \times A^n$ defined by $Spec(\frac{R(U)[x_1,\ldots,x_n]}{f_1,\ldots f_n})$. Letting $u \in U$ denote the point in U lying over $y \in Y$, the residue of the coefficients of the f_i at u corresponds to the residue in the local ring R, which tells us exactly that the point $(u, \bar{a})$ lies in V. By the Jacobian condition, we have that the projection $\pi : V \rightarrow U$ is etale at the point $(u, \bar{a})$, and hence on some open neighborhood of $(u, \bar{a})$, using Nakayama's Lemma applied to $\Omega_{V/U}$. Therefore, replacing V by the open subset $U' \subset V$ gives an etale cover of U and therefore of Y, lying over y. Now clearly the coordinate functions $x_1, \ldots x_n$ restricted to U' lie in $O^{\wedge}_{y,Y}$ and lift the root $\bar{a}$ to a root in $O^{\wedge}_{y,Y}$ $\square$

We define the Henselization of a local ring R to be the smallest Henselian ring $R' \supset R$, with $R' \subset Frac(R)^{alg}$. We have in fact, see [14], that;

Theorem 1.3 *Given an algebraic variety Y, $O^{\wedge}_{y,Y}$ is the Henselization of $O_{y,Y}$*

We recall the following Definition 3.6.7 from [17];

Definition 1.4 Let $F \subset D \times \mathcal{M}^k$ be a finite covering of D and $(a, b) \in F$, then;

$$Mult_b(a, F/D) = Card(F(a', {}^*\mathcal{M}^k)) \cap \mathcal{V}_b$$

for $a' \in \mathcal{V}_a$ generic in D over $\mathcal{M}$, where;

$$\mathcal{V}_a = \{a' \in {}^*\mathcal{D} : \pi(a') = a\}$$

$\mathcal{M} \prec {}^*\mathcal{M}$ and $\pi : {}^*\mathcal{M} \rightarrow \mathcal{M}$ is a universal specialisation.

Definition 1.5 If F is a finite covering of D, we say that F is unramified in the sense of Zariski structures if for all $(a, b) \in F$, $mult_b(a, F/D) = 1$.

The following theorem requires some knowledge of Zariski structures, see Sects. 1–4 of [7], or Sect. 2 of this paper.

Theorem 1.6 *Zariski multiplcity is preserved by etale morphisms Let $\pi : X \to Y$ be an etale morphism with Y smooth, then any $(ab) \in graph(\pi) \subset X \times Y$ is unramified in the sense of Zariski structures.*

For this we need the following fact whose algebraic proof relies on the fact that etale morphisms are flat, see [13];

Fact 1.7 *Any etale morphism can be locally presented in the form*

$$\begin{array}{ccc} V & \xrightarrow{g} & Spec((A[T]/f(T))_d) \\ \downarrow \pi & & \downarrow \pi' \\ U & \xrightarrow{h} & Spec(A) \end{array}$$

where $f(T)$ is a monic polynomial in $A[T]$, the derivative $f'(T)$ is invertible in $(A[T]/f(T))_d$, g, h are isomorphisms and $(A[T]/f(T))_d = \{\frac{h}{d^n} : h \in A[T], n \in \mathcal{Z}_{\geq 0}\}$.

Using Lemma 4.6 of [7] and the fact that the open set V is smooth, we may safely replace $graph(\pi)$ by $\overline{graph(\pi')} \subset F'' \times F$ where F'' is the projective closure of $Spec((A[T]/f(T))$, F is the projective closure of $Spec(A)$ and $\overline{graph(\pi')}$ is the projective closure of $graph(\pi')$ and show that $(g(b)a)$ is Zariski unramified. Note that over the open subset $U = Spec(A) \subset F$, $\overline{graph(\pi')} = Spec(A[T]/f(T))$ as this is closed in $U \times F''$. For ease of notation, we replace $(g(b)a)$ by (ba).

Suppose that f has degree n. Let $\sigma_1 \ldots \sigma_n$ be the elementary symmetric functions in n variables $T_1, \ldots T_n$. Consider the equations

$\sigma_1(T_1, \ldots, T_n) = a_1$

$\ldots$

$\sigma_n(T_1, \ldots, T_n) = a_n$ (*)

where $a_1, \ldots a_n$ are the coefficients of f with appropriate sign. These cut out a closed subscheme $C \subset Spec(A[T_1 \ldots T_N])$. Suppose $(ba) \in graph(\pi') = Spec(A[T]/f(T))$ is ramified in the sense of Zariski structures, then I can find $(a'b_1b_2) \in \mathcal{V}_{abb}$ with $(a'b_1)$,$(a'b_2) \in Spec(A(T)/f(T))$ and b_1, b_2 distinct. Then complete (b_1b_2) to an n-tuple $(b_1b_2c'_1 \ldots c'_{n-2})$ corresponding to the roots of f over a'. The tuple $(a'b_1b_2c'_1 \ldots c'_{n-2})$ satisfies C, hence so does the specialisation $(abbc_1 \ldots c_{n-2})$. Then the tuple $(bbc_1 \ldots c_{n-2})$ satisfies $(*)$ with the coefficients evaluated at a. However such a solution is unique up to permutation and corresponds to the roots of f over a. This shows that f has a double root at (ab) and therefore $f'(T)|_{ab} = 0$. As (ab) lies inside $Spec(A[T]/f(T))_d$, this contradicts the fact that f' is invertible in $A[T]/f(T))_d$.

We also review some facts about algebraic multiplicity and show that algebraic multiplicity is preserved by etale morphisms.

Definition 1.8 Given projective varieties X_1, X_2 and a finite morphism $f : X_1 \to X_2$, the algebraic multiplicity $mult^{alg}_{af(a)}(X_1/X_2)$ of f at $a \in X_1$ is $length(O_{a,X_1}/f^*m_{f(a)})$ where $m_{f(a)}$ is the maximal ideal of the local ring $O_{f(a)}$.

Remark 1.9 Note that this is finite, by the fact that finite morphisms have finite fibres and the ring $O_{a,X_1}/f^*m_{f(a)}$ is a localisation of the fibre $f^{-1}(f(a)) \cong R(f^{-1}(U)) \otimes_{R(U)} L \cong R(f^{-1}(U))/m_{f(a)}$ where U is an affine subset of X_2 containing $f(a)$.

We now have the following.

Theorem 1.10 (Algebraic multiplicity is preserved by etale morphisms) *Given finite morphisms $f : X_3 \to X_2$ and $g : X_2 \to X_1$ with f etale. If $a \in X_3$, then $mult^{alg}_{a,gf(a)}(X_3/X_1) = mult^{alg}_{f(a),gf(a)}(X_2/X_1)$.*

Proof This result is essentially given in [15]. Let $O^{\wedge}_{f(a),X_2}$ be the Henselisation of the local ring at $f(a)$. By base change, we have an etale morphism $f' : X' = X_3 \times_{X_2} Spec(O^{\wedge}_{f(a)}) \to Spec(O^{\wedge}_{f(a)})$. By the definition of an etale morphism given above, we may write this cover locally in the form $Spec(O^{\wedge}_{f(a)}\frac{[x_1,\ldots,x_n]}{f_1,\ldots,f_n})$, with $det(\frac{\partial f_i}{\partial x_j}) \neq 0$ at each closed point in the fibre over $f(a)$. At the closed point a, let a_i be the residues of the x_i in L, then we have that $(a_1, \ldots a_n)$ is a common root for $\{\bar{f}_1, \ldots, \bar{f}_n\}$ where $\bar{f}_i$ is obtained by reducing f_i with respect to the maximal ideal $m_{f(a),X_2}$ of $O^{\wedge}_{f(a)}$. As $O^{\wedge}_{f(a)}$ is Henselian, by the above, and the determinant condition, we can lift the roots a_i to roots α_i of the f_i in $O^{\wedge}_{f(a)}$. We therefore obtain a subscheme $Z = Spec(O^{\wedge}_{f(a)}\frac{[x_1,\ldots,x_n]}{<x_1-\alpha_1,\ldots,x_n-\alpha_n} >)$ of X' which is isomorphic to $Spec(O^{\wedge}_{f(a)})$ under the restriction of f. Let Q be the $O_{X'}$ ideal defining Z, we then have that $m_{a,X'} = f^*m_{f(a),X_2} \oplus Q_a$. As f is etale, by $(**)$ after Definition 1.1 above, $m_{a,X'} = f^*m_{f(a),X_2}$, therefore $Q_a = 0$ and by Nakayama's lemma $Q = 0$ in an open neighborhood of a in X'. This gives that $Z = X'$ in an open neighborhood of a. Hence we obtain the sequence $O_{f(a),X_2} \to_{f^*} O_{a,X_3} \to_{i^*} O_{a,X'}$ $(***)$ where the map i^*f^* is the inclusion of $O_{f(a),X_2}$ inside $O^{\wedge}_{f(a),X_2}$. Now if $n \subset m_{f(a),X_2}$ is the pullback $g^*m_{gf(a),X_1}$, we have that $length(O_{f(a),X_2}/n) = length(O^{\wedge}_{f(a),X_2}/n)$, hence the result follows by $(***)$ as required. $\square$

2 Zariski Multiplicity

We work in the context of Theorem 3.3 in [7]. Namely, W (we used the notation V in [7]) will denote a smooth projective variety defined over an algebraically closed field L, considered as a Zariski structure with closed sets given by algebraic subvarieties defined over L. All notions connected to the definition of Zariski multiplicity will come from a fixed specialisation map $\pi : W(K_\omega) \to W(L)$ where K_ω denotes a "universal" algebraically closed field containing $L = K_0$. We consider D a smooth subvariety of some cartesian power W^m and a finite cover, with respect to projection onto the first coordinate, $F \subset D \times W^k$, all defined over L (*). This allows us to make sense of Zariski multiplicity. In general, we can move freely between Zariski structure notation and algebraic geometry notation. Clearly $(*)$ makes sense algebraically. Conversely, if X and Y denote fixed projective varieties defined over L with Y smooth and a finite morphism $f : X \to Y$ over L is given , then we can reduce to the situation of $(*)$ by taking F to be $graph(f) \subset X \times Y$ with the projection map onto the second factor and W to be the corresponding projective space $P^n(L)$ where $X, Y \subset P^n(L)$. We can even take W to be the 1-dimensional Zariski structure $P^1(L)$ by using the embedding of $P^n(L)$ into the N'th Cartesian power of $P^1(L)$ for sufficiently large N.

We use the definition of Zariski multiplicity for irreducible finite covers, see Definition 1.4 and also given in 4.1 of [7]. We will also require the following generalisation.

Definition 2.1 Let $F \subset D \times W^k$ be an equidimensional, finite cover of smooth D, with irreducible components $C_1, \dots, C_n$. Then for $(ab) \in F$, we define $Mult_{ab}(F/D) = \sum_{(ab)\in C_i} Mult_{ab}(C_i/D)$.

Clearly this is well defined using the definition of Zariski multiplicity for irreducible covers. However, until Lemma 2.10, the assumption that F is irreducible will be in force.

Lemma 2.2 (Zariski multiplicity is multiplicative over composition) *Suppose that F_1, F_2 and F_3 are smooth, irreducible, with $F_2 \subset F_1 \times W^k$ and $F_3 \subset F_2 \times W^l$ finite covers. Let $(abc) \in F_3 \subset F_1 \times W^k \times W^l$. Then $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$ $mult_{abc}(F_3/F_2)$.*

Proof To see this, let $m = mult_{ab}(F_2/F_1)$ and $n = mult_{abc}(F_3/F_2)$. Choose $a' \in \mathcal{V}_a \cap F_1(K_\omega)$ generic over L. By definition, we can find distinct $b_1 \dots b_m$ in $W^k(K_\omega) \cap \mathcal{V}_b$ such that $F_2(a', b_i)$ holds. As F_2 is a finite cover of F_1, we have that $dim(a'b_i/L) = dim(a'/L) = dim(F_1) = dim(F_2)$, so each $(a'b_i) \in \mathcal{V}_{ab} \cap F_2$ is generic over L. Again by definition, we can find distinct $c_{i1} \dots c_{in}$ in $W^l(K_\omega) \cap \mathcal{V}_c$ such that $F_3(a'b_ic_{ij})$ holds. Then the mn distinct elements $(a'b_ic_{ij})$ are in $\mathcal{V}_{abc}$, so by definition of multiplicity $mult_{abc}(F_3/F_1) = mn$ as required. $\square$

Lemma 2.3 *Let hypotheses be as in the above lemma with the extra condition that the cover F_3/F_2 is etale. Then for $(abc) \in F_3$, $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$*

Proof This is an immediate consequence of Lemma 2.2 and Theorem 1.6. □

Lemma 2.4 (Zariski multiplicity is summable over specialisation) *Suppose that $F \subset D \times W^k$ is a finite irreducible cover with D smooth. Suppose $(ab) \in F$, $a' \in \mathcal{V}_a \cap D$ and $a'' \in \mathcal{V}_{a'} \cap D$ with a'' generic over L. Then*

$$Mult_{ab}(F/D) = \Sigma_{b' \in \mathcal{V}_b \cap F(a')} Mult_{a'b'}(F/D)$$

where $F(a') = \{y \in F : pr(y) = a'\}$ and $pr : F \to D$ is a projection.

Proof Suppose $F(a''b_1), \ldots F(a''b_n)$ hold with $b_i \in \mathcal{V}_b$, so $\{b_1, \ldots, b_n\}$ witness the fact that $Mult_{ab}(F/D) = n$. Write $\{b_1, \ldots b_n\}$ as $\{b_{11}, \ldots, b_{1m_1}, b_{21}, \ldots, b_{2m_2}, \ldots, b_{i1}, \ldots b_{ij}, \ldots, b_{im_i}, \ldots, b_{nm_n}\}$ (*), where b_{ij} maps to a_i in the specialisation taking a'' to a'. To prove the lemma, it is sufficient to show that $F(a'y) \cap \mathcal{V}_b = \{a_1, \ldots, a_n\}$ and $Mult_{(a'a_i)}(F/D) = m_i$. The second statement just follows from the fact that a'' is generic in D over L in $\mathcal{V}_{a'}$. To prove the first statement, suppose we can find a_{n+1} with $F(a'a_{n+1})$ and $a_{n+1} \in \mathcal{V}_b$ but $a_{n+1} \notin \{a_1, \ldots a_n\}$. By Theorem 3.3 in [7], we can find c with $F(a''c)$ and $(a''c)$ specialising to $(a'a_{n+1})$. As $a_{n+1} \in \mathcal{V}_b$, $(a'a_{n+1})$ specialises to (ab), hence so does $(a''c)$. Therefore, c must witness the fact that $Mult_{ab}(F/D) = n$ and appear in the set $\{b_1, \ldots, b_n\}$. This clearly contradicts the arrangement of $\{b_1, \ldots, b_n\}$ given in $(*)$. □

Definition 2.5 Let $F \subset U \times V \times W^k$ be an irreducible finite cover of $U \times V$ with U and V smooth.

Given $(u, v, x) \in F$ we define;

$LeftMult_{u,v,x}(F/D) = Card(\mathcal{V}_x \cap F(u', v))$ for $u' \in \mathcal{V}_u \cap U$ generic over L.

$RightMult_{u,v,x}(F/D) = Card(\mathcal{V}_x \cap F(u, v'))$ for $v' \in \mathcal{V}_v \cap V$ generic over L.

We first show that both left and right multiplicity are well defined. In order to see this, observe that the fibres $F(u, V)$ and $F(U, v)$ are finite covers of V and U respectively with U and V smooth. Moreover, the fibres $F(u, V)$ and $F(U, v)$ are equidimensional covers of V and U respectively. In order to see this, as U is smooth, it satisfies the presmoothness axiom with the smooth projective variety W^k given in Definition 1.1 of [7]. The fibre $F(u, V) = F \cap (W^k \times \{u\} \times V)$. By presmoothness, each irreducible component of the intersection has dimension at least $dim(F) + dim(W^k \times V) - dim(U \times V \times W^k) = dim(F) - dim(U) = dim(V)$. As $F(u, V)$ is a finite cover of V, it has exactly this dimension. Now we can use the definition of Zariski multiplicity given in Definition 2.1.

We then claim the following.

Lemma 2.6 (Factoring Multiplicity) *In the situation of the above definition, we have that;*

$Mult_{u,v,x}(F/U \times V) = \Sigma_{x' \in (\mathcal{V}_x \cap F(y,u',v))} RightMult_{x',u',v}(F/U \times V)$ for u' generic in U over L.

$Mult_{u,v,x}(F/U \times V) = \Sigma_{x' \in (\mathcal{V}_x \cap F(y,u,v'))} LeftMult_{x',u,v'}(F/U \times V)$ for v' generic in V over L.

Proof We just prove the first statement, the proof of the second is apart from notation identical. By the construction in Sect. 2 and Lemma 3.2 of [7], we can choose algebraically closed fields $L = K_0 \subset K_{n_1} \subset K_{n_2} \subset K_\omega$, and tuples $u' \in K_{n_1}$, $v' \in K_{n_2}$ such that u' is generic in U over L, v' is generic in V over K_{n_1} with specialisations $\pi_1 : P^n(K_{n_1}) \to P^n(L)$ and $\pi_2 : P^n(K_{n_2}) \to P^n(K_1)$ such that $\pi_2(u'v') = (u'v)$ and $\pi_1(u'v) = (uv)$. Now $dim(u'v'/L) = dim(v'/L(u')) + dim(u'/L) = dim(V) + dim(U)$, hence $u'v'$ is generic in $U \times V$ over L. Therefore $Mult_{u,v,x} = Card(\mathcal{V}_x \cap F(u'v'))$. Let $S = \{y_{11}, \ldots, y_{1m_1}, \ldots, y_{ij_i}, \ldots, y_{n1}, \ldots, y_{nm_n}\}$ be distinct elements in $\mathcal{V}_x \cap W^k$ witnessing this multiplicity such that for $1 \leq j_i \leq m_i, \pi_2(y_{ij_i}) = z_i \in \mathcal{V}_x \cap W^k$. It is sufficient to show that $RightMult_{u'v,z_i}(F/U \times V) = m_i$ and $\{z_1, \ldots z_n\}$ enumerates $\mathcal{V}_x \cap F(y, u', v)$. The first statement follows as $v' \in \mathcal{V}_v \cap V$ is generic in V over $L(u')$. For the second statement, suppose that we can find $z_{n+1} \in \mathcal{V}_x \cap F(y, u', v)$ with $z_{n+1} \notin \{z_1, \ldots z_n\}$. Consider $F(u', V)$ as a finite cover of V, defined over $L(u')$, so by the above $F(u', V)$ is an equidimensional, see Definition 2.9 finite cover of V. Then, as v' was chosen to be generic in V over $L(u')$, choosing an irreducible component of $F(u', V)$ passing through $(z_{n+1}, u'v)$, by the lifting result of Theorem 3.3 in [7], we can find $y_{n+1} \in \mathcal{V}_{z_{n+1}} \cap W^k$ such that $F(y_{n+1}, u', v')$. Clearly, $y_{n+1} \in S$ which contradicts the definition of S. □

Theorem 3.3 of [7] does not hold in the case when D fails to be smooth. However, in the case of etale covers, we still have the following result;

Lemma 2.7 *Lifting Lemma for Etale Covers*

Let $F \subset D \times W^k$ be an etale cover of D defined over L, with the projection map denoted by f. Then given $a \in D$, $(ab) \in F$ and $a' \in \mathcal{V}_a \cap D$ generic over L, we can find $b' \in \mathcal{V}_b$ such that $F(a', b')$ holds. Moreover b' is unique, hence $Mult_{ab}(F/D) = 1$. Moreover, in the situation of Lemma 2.3, without requiring that F_2 is smooth, we have that for $(abc) \in F_3$, $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$.

Proof Using the definition of etale given in Sect. 1 above, we can assume that the cover is given algebraically in the form $f^* : L[D] \to L[D]\frac{[x_1,\ldots,x_n]}{f_1,\ldots,f_n}$ with $det(\frac{\partial f_i}{\partial x_j})_{ij}(x) \neq 0$ for all $x \in F$. So we can present the cover in the form $f_1(x, y) = 0$, $f_2(x, y) = 0, \ldots, f_n(x, y) = 0$, with y in D and x in $A^n(L)$. Let L_m be the algebraic closure of the field generated by L and $\bar{g}(a)$ where $\bar{g}$ is a tuple of functions defining D locally. Consider the system of equations $f_1(x, a) = f_2(x, a) = \ldots = f_n(x, a) = 0$ defined over L_m. Then this system is solved by b in L_m with the property that $det(\frac{\partial f_i}{\partial x_j})_{ij}(b) \neq 0$ (*). Now suppose that $a' \in \mathcal{V}_a \cap D$ is chosen to be generic over L. By the construction given in Lemma 2.2 of [7], we may assume that a' lies in $L_s[[t^{1/r}]]$, the formal power series in the variable $t^{1/r}$ for some algebraically closed field L_s extending L_m. This is a henselian ring, hence if we consider the system of equations $f_1(x, a') = f_2(x, a') = \ldots = f_n(x, a') = 0$ with coefficients in $L_s[[t^{1/r}]]$, by the fact that the system specialises to a solution in L_s with the condition (*) we can find a solution b' in $L_s[[t^{1/r}]]$. Then $(a'b')$ lies in F and by construction $b' \in \mathcal{V}_b$. The uniqueness result follows from the proof of Theorem 1.6. For the last part, suppose that $mult_{ab}(F_2/F_1) = n$, then we can find $a' \in \mathcal{V}_a \cap F_1$ generic over L and $\{b_1, \ldots b_n\} \in \mathcal{V}_b \cap W^k$ distinct such that $F(a', b_i)$ holds. Each $(a'b_i)$ is generic in F_2 over L, hence by the previous part of the lemma, we can find a unique $c_i \in \mathcal{V}_c \cap W^l$ such that $F_3(a'b_ic_i)$ holds. This show that $mult_{abc}(F_3/F_1) = n$ as required. □

Lemma 2.8 (Lifting Lemma for Etale Covers with Right(Left) Multiplicity) *Let hypotheses be as in Lemma 2.2, with the additional assumption that* $F_1 = U \times V$, F_2 *is a smooth irreducible cover of* F_1 *and* F_3 *is an irreducible etale cover of* F_2. *Then, with notion as in Definition 2.5, given* $(uvbc) \in F_3$, $RightMult_{uvbc}(F_3/F_1) = RightMult_{uvb}(F_2/F_1)$. *Similarly for left multiplicity.*

Proof Suppose that $RightMult_{uvb}(F_2/F_1) = n$, then for $v' \in \mathcal{V}_b$ generic in V over L, we can find $\{b_1, \ldots, b_i, \ldots b_n\} \in \mathcal{V}_b$ with $F_2(uv'b_i)$ holding. For each b_i we claim that there exists a unique $c_i \in \mathcal{V}_c$ such that $F_3(uv'b_ic_i)$ holds. For the existence, we can use Lemma 2.7, with the simple modification that, with the notation there, if L_m is the algebraic closure of the field generated by $\bar{g}(uv)$, then provided $dim(V) \geq 1$, we can find $v' \in \mathcal{V}_v \cap V$ generic over L with $uv' \in L_s[[t^{1/r}]]$ for some algebraically closed field L_s containing L_m. For the uniqueness, we can use the fact that Zariski multiplicity is summable over specialisation, see Lemma 2.4, and the fact that for generic $(u'v'b_i') \in \mathcal{V}_{uvb} \cap F_2$, we can find a unique $c_i' \in \mathcal{V}_c$ such that $F_3(u'v'b_i'c_i')$ holds. Finally, we claim that $\{b_1c_1, \ldots, b_nc_n\}$ enumerate $F_3(uv'xy) \cap \mathcal{V}_{bc}$. This is clear by the above proof and the fact that $\{b_1, \ldots, b_n\}$ enumerates $F_2(uv'x) \cap \mathcal{V}_b$.□

Definition 2.9 We say that $g : F \to D$ is an equidimensional finite cover of D if $F = \bigcup_{1 \leq i \leq k} F_i$ with F_i irreducible, $dim(F) = dim(F_i)$, and $g : F_i \to D$ finite.

Lemma 2.10 *The following versions of the above properties hold when we consider finite equidimensional covers, possibly with components, with the definition of Zariski multiplicity given in Definition 2.1.*

Proof For Lemma 2.3, we replace the hypotheses with F_1 is smooth irreducible, F_2 is an equidimensional finite cover of F_1 and F_3 is an etale cover of F_2. We then claim, using notation as in Lemma 2.2, that $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1)$. By definition $mult_{abc}(F_3/F_1) = \sum_{(abc) \in C_i}(mult_{abc}(C_i/F_1))$, where C_i are the irreducible components of F_3 passing through (abc). As F_3 is an etale cover of F_2, the images of the C_i are precisely the irreducible components D_i of F_2 passing through (ab), each C_i is an etale cover of D_i and $mult_{ab}(F_2/F_1) = \sum_{(ab) \in D_i}(mult_{ab}(D_i/F_1))$. Hence, it is sufficient to prove the result in the case when F_2 and F_3 are irreducible. This is just Lemma 2.3.

For Lemma 2.4, we replace the hypothesis with F is an equidimensional finite cover of D. The proof then goes through exactly as in the lemma with the observation that if we find $a_{n+1} \in \mathcal{V}_b$ and $F(a'a_{n+1})$ then we can find an irreducible component C passing through $(a'a_{n+1})$ which allows us to apply Theorem 3.3 in [7] to obtain c with $C(a''c)$ and $(a''c)$ specialising to $(a'a_{n+1})$.

For Definition 2.5, we alter the hypothesis to F is an equidimensional finite cover of $U \times V$. Again, we can use an identical proof to show that left multiplicity and right multiplicity are well defined. The proof of Lemma 2.6 with the new hypothesis on F is identical.

We don't require a modified version of Lemma 2.7, the result we need is contained in the modified proof of Lemma 2.3.

For Lemma 2.8, we alter the hypotheses to F_2 is an equidimensional cover of F_1 and F_3 is an etale cover of F_2. We then claim that for (uvb) a non-singular point of F_2 and $(uvbc) \in F_3$, necessarily non-singular as well, that $RightMult_{uvbc}(F_3/F_1) = RightMult_{uvb}(F_2/F_1)$ and similarily for left multiplicity. To prove this, note that as (uvb) and $(uvbc)$ are non-singular points, there exist unique components C and D passing through (uvb) and $(uvbc)$ respectively. Now replacing C and D by the open subsets C' and D' of smooth points, we can apply the definition of Right Multiplicity and the proof of Lemma 2.8. $\square$

3 Analytic Methods

In order to use the method of etale morphisms, which preserve Zariski multiplicity, we need to work inside the Henselisation of local rings $L[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$. In the next section, we will only need the result for the local ring in 2 variables $L[x, y]_{(x,y)}$.

We let $L[[x_1, \dots, x_n]]$ denote the ring of formal power series in n variables, which is the formal completion of $L[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ with respect to the canonical order valuation, see for example Sect. 2 of [7]. The following is a classical result, requiring the fact that etale morphisms are flat, used in the proof of the Artin approximation theorem. This relates the henselisation of the ring $L\{x_1, \dots, x_n\}$ of strictly convergent power series in several variables with its formal completion $L[[x_1, \dots, x_n]]$, see [3] or [16]. Namely, that the henselisation of $(L[x_1, \dots x_n]_{(x_1,\dots x_n)})$ is equal to $L[[x_1, \dots x_n]] \cap L(x_1, \dots x_n)^{alg}$, where $L(x_1, \dots x_n)^{alg}$ is the algebraic closure of the function field $L(x_1, \dots x_n)$.

This implies that

$$O^{\wedge}_{\bar{0},A^n} \cong L[[x_1, \dots x_n]] \cap L(x_1, \dots x_n)^{alg}$$

The following result, which can be found in [4], is essential for the next section.

Lemma 3.1 (Weierstrass Preparation) *Let $F(x_1, \ldots x_n)$ be a polynomial in $L[x_1, \ldots, x_n]$ which is regular in the variable x_n. Then we have $F(x_1, \ldots, x_n) = U(x_1, \ldots, x_n)G(x_1, \ldots, x_n)$ where $U(x_1, \ldots, x_n)$ is a unit in the local ring $L[[x_1, \ldots, x_n]]$ and*
$G(x_1, \ldots, \ldots x_n)$ is a Weierstrass polynomial in x_n with coefficients in $L[[x_1, \ldots, x_{n-1}]]$

We will require the Weierstrass decomposition to hold inside the henselisation of $(L[x_1, \ldots, x_n])$, therefore we need to show that the Weierstass data can be found inside $L(x_1, \ldots, x_n)^{alg}$. This is achieved by the following lemma.

Lemma 3.2 (Definability of Weierstrass data) *Let $F(x_1, \ldots, x_n)$ be a polynomial with coefficients in L such that F is regular in x_n, then if $F(x_1, \ldots, x_n) = U(x_1, \ldots, x_n)G(x_1, \ldots, x_n)$ is the Weierstrass decomposition of F with $G(x_1, \ldots, x_n) = x_n^m + a_1(x_1, \ldots, x_{n-1})x_n^{m-1} + \cdots + a_m(x_1, \ldots, x_{n-1})$, and $a_i \in L[[x_1, \ldots, x_{n-1}]]$, $U(x_1, \ldots, x_n) \in L[[x_1, \ldots, x_n]]$, then $a_i(x_1, \ldots, x_{n-1}) \in L(x_1, \ldots, x_{n-1})^{alg}$ and $U(x_1, \ldots, x_n) \in L(x_1, \ldots, x_n)^{alg}$.*

Proof This can be proved by rigid analytic methods. Equip L with a complete non-trivial non-archimedean valuation v and corresponding norm $||.||_v$, this can be done for example by assuming that L is a power series field of large transcendence degree with a non-archimidean valuation, see [4, 6]. Let $T_{n-1}(L)$ be the free Tate algebra in the indeterminate variables $x_1, \ldots, x_{n-1}$ over L, that is the subalgebra of strictly convergent power series in $L[[x_1, \ldots, x_{n-1}]]$. By the proof of Weierstrass preparation in [4], as $F \in T_{n-1}(L)[x_n]$, the coefficients a_i lie in $T_{n-1}(L)$ and $U(x_1, \ldots, x_n) \in T_{n-1}(L)[x_n]$. Now choose $(u_1, \ldots u_{n-1}) \subset L$ transcendental over the coefficients of F with $max(\{||u_i||\}) \leq 1$. Then if $s_1(\bar{u}), \ldots, s_m(\bar{u})$ denote the roots of $F(\bar{u}, x_n)$ with $||s_i(\bar{u})|| \leq 1$, then both $U(\bar{u}, s_i(\bar{u}))$ and $G(\bar{u}, s_i(\bar{u}))$ define elements of L and moreover, by a theorem in [16], we have that the coefficients $a_i(\bar{u})$ are symmetric functions of the $s_i(\bar{u})$. Hence the $a_i(\bar{u})$ belong to $L(\bar{u})^{alg}$. As $\bar{u}$ was transcendental, we have that each $a_i \in L[x_1, \ldots, x_{n-1}]^{alg}$. As $U(x_1, \ldots x_n) = F/G(x_1, \ldots, x_n)$, we clearly have that $U(x_1, \ldots, x_n) \in L[x_1, \ldots, x_n]^{alg}$ as well. $\square$

4 Families of Curves in $P^2(L)$

We consider the family Q_d of projective curves in $P^2(L)$ with degree d. An element of Q_d may be written;

$$\sum_{0 \leq i+j \leq d} a_{ij}(X/Z)^i(Y/Z)^j = 0$$

which, rewriting in homogenous form, becomes;

$$\sum_{0 \leq i+j \leq d} a_{ij}X^iY^jZ^{d-(i+j)} = 0$$

For ease of notation, we will use affine coordinates $x = X/Z$ and $y = Y/Z$. More generally, if we give an affine cover, we implicitly assume that it can be projectivized by taking $\bar{y} = (y_1, \dots, y_n) = (Y_1/Z, \dots, Y_n/Z)$. As the notion of Zariski multiplicity is local, this will not effect our calculations.

Now consider two such families Q_d and Q_e. Then we have the cover obtained by intersecting degree d and degree e curves

$$Spec(L[x, y, u_{ij}, v_{ij}]/ < s(u_{ij}, x, y), t(v_{ij}, x, y) >) \rightarrow Spec(L[u_{ij}, v_{ij}]).(*)$$

where

$$s(u_{ij}, x, y) = \sum_{0 \le i+j \le d} u_{ij} x^i y^j$$

$$t(v_{ij}, x, y) = \sum_{0 \le i+j \le e} v_{ij} x^i y^j$$

We denote the parameter space for degree d curves by U and the parameter space for degree e curves by V. These are affine spaces of dimension $(d+1)(d+2)/2$ and $(e+1)(e+2)/2$ respectively. Both Q_d and Q_e are irreducible. The cover (*) is generically finite, that is there exists an open subset $U' \subset Sp(L[u_{ij}, v_{ij}])$ for which the restricted cover has finite fibres. Throughout this section, we will denote the base space of the cover by $U \times V$, bearing in mind that we implicitly mean by this $(U \times V) \cap U'$. Now, given 2 fixed parameters sets $\bar{u}$ and $\bar{v}$, with $(\bar{u}, \bar{v}) \in U'$, corresponding to curves $C_{\bar{u}}$ and $C_{\bar{v}}$, the algebraic multiplicity of the cover $(*)$ at $(00, \bar{u}, \bar{v})$ is exactly the intersection multiplicity $I(C_{\bar{u}}, C_{\bar{v}}, 00)$ of the curves at (00). The cover (*) is equidimensional as $U \times V$ satisfies the presmoothness axiom with the smooth projective variety $P^2(L)$. Restricting to a finite cover over U', by definition 2.1 we can also define the Zariski multiplicity of the cover at the point $(00, \bar{u}, \bar{v})$. The main result that we shall prove in this paper is the following, which generalises an observation given in [12].

Theorem 4.1 *In all characteristics, the algebraic multiplicity and Zariski multiplicity of the cover* $(*)$ *coincide at* $(00, \bar{u}, \bar{v})$.

Definition 4.2 We say that a monic polynomial $p(x, \bar{y})$ is Weierstrass in x if $p(x, \bar{y}) = x^n + \cdots + q_j(\bar{y}) x^{n-j} + \cdots + q_n(\bar{y})$ with $q_j(\bar{0}) = 0$.

Definition 4.3 Let $F(x, \bar{y})$ be a polynomial in x with coefficients in $L[\bar{y}]$. We say the cover

$$Spec(L[x\bar{y}]/ < F(x, \bar{y}) >) \rightarrow Spec(L[\bar{y}])$$

is generically reduced if for generic $\bar{u} \in Spec(L[\bar{y}])$, $F(x, \bar{u})$ has no repeated roots.

Definition 4.4 Let $F \to U \times V$ be a finite cover with U and V smooth, such that for $(\bar{u}, \bar{v}) \in U \times V$ the fibre $F(\bar{u}, \bar{v})$ consists of the intersection of algebraic curves $F_{\bar{u}}, F_{\bar{v}}$. We call the family sufficiently deformable at $(\bar{u}_0, \bar{v}_0)$ if there exists $\bar{u}' \in U$ generic over L such that $F_{\bar{u}'}$ intersects $F_{\bar{v}_0}$ transversely at simple points.

We now require a series of lemmas.

Lemma 4.5 *Let $F(x, \bar{y})$ be a Weierstrass polynomial in x with $F(0, \bar{0}) = 0$ then algebraic multiplicity and Zariski multiplicity coincide at $(0, \bar{0})$ if the cover*

$$Spec(L[x\bar{y}]/ < F(x, \bar{y}) >) \to Spec(L[\bar{y}])$$

is generically reduced.

Proof We have that $F(x, \bar{y}) = x^n + q_1(\bar{y})x^{n-1} + \ldots + q_n(\bar{y})$ where $q_i(\bar{0}) = 0$. The algebraic multiplicity is given by $length(L[x]/F(x, \bar{0})) = ord(F(x, \bar{0}) = n$ in the ring $L[x]$ with the canonical valuation. We first claim that the Zariski multiplicity is the number of solutions to $x^n + q_1(\bar{\epsilon})x^{n-1} + \ldots + q_n(\bar{\epsilon}) = 0$ (†), where $\bar{\epsilon}$ is generic in $\mathcal{V}_{\bar{0}}$. For suppose that $(a, \bar{\epsilon})$ is such a solution, then $F(a, \bar{\epsilon}) = 0$ and by specialisation $F(\pi(a), \bar{0}) = 0$. As F is a Weierstrass polynomial in x, $\pi(a) = 0$, hence $a \in \mathcal{V}_0$, giving the claim. We have that $Disc(F(x, \bar{y})) = Res_{\bar{y}}(F, \frac{\partial F}{\partial x})$ is a regular polynomial in $\bar{y}$ defined over L. By the fact that the cover is generically reduced, this defines a proper closed subset of $Spec(L[\bar{y}])$. Therefore, $Disc(F(x, \bar{y}))|\bar{\epsilon} \neq 0$, hence (†) has no repeated roots. This gives the lemma. □

Lemma 4.6 *Let $F(x, \bar{y})$ be any polynomial with $F(x, \bar{0}) \neq 0$ and $F(0, \bar{0}) = 0$. Then if the cover $Spec(L[x, \bar{y}]/ < F(x, \bar{y}) >) \to Spec(L[\bar{y}])$ is generically reduced, the Zariski multiplicity at $(0, \bar{0})$ equals $ord(F(x, \bar{0}))$ in $L[x]$.*

Proof By the Weierstrass Preparation Theorem, Lemma 3.1, we can write $F(x, \bar{y}) = U(x, \bar{y})G(x, \bar{y})$ with $U(x, \bar{y}), G(x, \bar{y}) \in L[[x, \bar{y}]]$, $G(x, \bar{y})$ a Weierstrass polynomial in x and $deg(G) = ord(F(x, \bar{0}))$, see also the more closely related statement given in [2]. By Lemma 3.2, we may take the new coefficients to lie inside the Henselized ring $L[x, \bar{y}]^{\wedge}_{\bar{0}}$, hence inside some finite etale extension $L[x, \bar{y}]^{ext}$ of $L[x, \bar{y}]$ (possibly after localising $L[x, \bar{y}]$ corresponding to an open subset of $Spec(L[x, \bar{y}])$ containing $(0, \bar{0})$). Now we have the sequence of morphisms;

$$Sp(L[x, \bar{y}]^{ext}/UG) \to Spec(L[x, \bar{y}]/F) \to Spec(L[\bar{y}])$$

The left hand morphism is etale at $\bar{0}$, hence by Lemma 2.3 or Lemma 2.7, to compute the Zariski multiplicity of the right hand morphism, we need to compute the Zariski multiplicity of the cover

$$Spec(L[x, \bar{y}]^{ext}/UG) \to Spec(L[\bar{y}])$$

at $(0, \bar{0})^{lift}$, the marked point in the cover above $(0, \bar{0})$. Choose $\bar{\epsilon} \in \mathcal{V}_{\bar{0}}$, the fibre of the cover is given formally analytically by $L[[x, \bar{y}]]/ < UG > \otimes_{L[\bar{y}], \bar{y} \mapsto \bar{\epsilon}} L$, hence by solutions to $U(x,\bar{\epsilon})G(x,\epsilon)$. By definition of Zariski multiplicity, we consider only solutions $(x\bar{\epsilon})$ in $\mathcal{V}_{(0,\bar{0})^{lift}}$. As $U(x,\bar{y})$ is a unit in the local ring $L[x,\bar{y}]^{ext}_{(0,\bar{0})^{lift}}$, we must have $U(x,\bar{\epsilon}) \neq 0$ for such solutions, otherwise by specialisation $U((0,\bar{0})^{lift}) = 0$. Hence, the solutions are given by $G(x,\bar{\epsilon}) = 0$. Now, we use the previous lemma to give that the Zariski multiplicity is exactly $deg(G)$ as required. □

Now return to the cover

$$Sp(L[x,y,u_{ij},v_{ij}]/ < s(u_{ij},x,y), t(v_{ij},x,y) >) \rightarrow Sp(L[u_{ij},v_{ij}])\ (*)$$

We will show below, Lemma 4.12, that this is a sufficiently deformable family at $(\bar{u}_0,\bar{v}_0)$ when $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ define reduced curves. We claim the following.

Lemma 4.7 *Suppose parameters $\bar{u}^0$ and $\bar{v}^0$ are chosen such that $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are reduced Weierstrass polynomials in x. Then the Zariski multiplicity of the cover $(*)$ at $(0,0,\bar{u}^0,\bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0},C_{\bar{v}^0},(0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

Proof Introduce new parameters $\bar{u}'$ and $\bar{v}'$. Let $C^{\bar{u}'}_{\bar{u}^0}$ and $C^{\bar{v}'}_{\bar{v}^0}$ denote the curves $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ deformed by the parameters $\bar{u}'$ and $\bar{v}'$ respectively. That is $C^{\bar{u}'}_{\bar{u}^0}$ is given by the new equation $\Sigma_{1\leq i+j\leq d}(u^0_{ij}+u'_{ij})x^iy^j$. Let $F(y,\bar{u}',\bar{v}') = Res(C^{\bar{u}'}_{\bar{u}^0}, C^{\bar{v}'}_{\bar{v}^0})$. Then,

$$F(0,\bar{0},\bar{0}) = Res(s(u^0_{ij},x,0), t(v^0_{ij},x,0)) = 0$$

as $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are Weierstrass in x and share a common solution at $(0,0)$. By a result due to Abhyankar, see for example [1], $ord_y(F(y,\bar{0},\bar{0}))=\Sigma_x I(C_{\bar{u}^0},C_{\bar{v}^0},(x0))$ at common solutions $(x,0)$ to $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ over $y=0$. As $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ are Weierstrass polynomials in x, this is just $I(C_{\bar{u}^0},C_{\bar{v}^0},(0,0))$. By the previous lemma and the fact that $F(y,\bar{u},\bar{v})$ is generically reduced (see argument (†) below), it is therefore sufficient to prove that the Zariski multiplicity of the cover $(*)$ at $(00,\bar{u}^0,\bar{v}^0)$ equals the Zariski multiplicity of the cover $Spec(L[y,\bar{u}',\bar{v}']/ < F(y,\bar{u}',\bar{v}') >) \rightarrow Spec(L[\bar{u}',\bar{v}'])$ $(**)$ at $(0,\bar{0},\bar{0})$. Suppose the Zariski multiplity of $(**)$ equals n. Then there exist distinct $y_1,\ldots,y_n \in \mathcal{V}_0$ and $(\bar{\delta},\bar{\epsilon})$ generic in $\mathcal{V}_{(\bar{0},\bar{0})}\cap U\times V$ such that $F(y_i,\bar{\delta},\bar{\epsilon})$ holds. Consider $Q(\bar{u}',\bar{v}') = Res(F(y,\bar{u}',\bar{v}'), \partial F/\partial y(y,\bar{u}',\bar{v}'))$. This defines a closed subset of $U\times V$ defined over L, we claim that this in fact proper closed (†). By the fact that the family is sufficently deformable at $(\bar{u}_0,\bar{v}_0)$, we can find $(\bar{u},\bar{v}_0)$ such that $C_{\bar{u}}$ intersects $C_{\bar{v}_0}$ transversely at simple points. Without loss of generality, making a linear change of coordinates, we may suppose that for there do not exists points of intersection of the form (x_1y) and (x_2y) for $x_1\neq x_2$. By Abhyankar's result, this implies that $F(y,\bar{u}',\bar{v}_0)$ has no repeated roots. Then, by genericity of $(\bar{\delta},\bar{\epsilon})$, we have that $Q(\bar{\delta},\bar{\epsilon})\neq 0$. Hence $F(y_i,\bar{\delta},\bar{\epsilon})$ is a non-repeated root. By Abhyankar's result, we can find a unique x_i with (x_iy_i) a common solution to the deformed curves $C^{\bar{\delta}}_{\bar{u}^0}$ and $C^{\bar{\epsilon}}_{\bar{v}^0}$. We claim that each $(x_iy_i)\in\mathcal{V}_{00}$. As $C^{\bar{\delta}}_{\bar{u}^0}(x_iy_i)=0$, by the fact $(\bar{u}^0,\bar{\delta},y_i)$ specialises to $(\bar{u}^0,\bar{0},0)$ and $C_{\bar{u}^0}$ is a Weierstrass polynomial in x, we have that $\pi(x_i)=0$ as well. This shows that the Zariski multiplicity of the cover $(*)$, referred to at the beginning of Sect. 4, in this case, is at least n. Suppose the Zariski multiplicity of the cover $(*)$ is strictly bigger than n, then we can find generic parameters $\{\overline{u}', \overline{v}'\} \subset \mathcal{V}_{\overline{0},\overline{0}}$ and distinct $\{(x_1 y_1), \ldots, (x_{n+1} y_{n+1})\} \subset \mathcal{V}_{(0,0)} \cap C^{\overline{u}}_{\overline{u}^0} \cap C^{\overline{v}}_{\overline{v}^0}$. If, for some y_i, there exist at least 2 distinct x_{j_i}, with $(x_{j_i}, y_i) \in \{(x_1 y_1), \ldots, (x_{n+1} y_{n+1})\}$, then $ord_{y_i} F(y, \overline{u}', \overline{v}') \geq 2$, contradicting the fact that F is generically reduced. Otherwise, there exist at least $n+1$ distinct y_i, corresponding to solutions $F(y_i, \overline{u}', \overline{v}') = 0$, with $y_i \in \mathcal{V}_0$, $(\dagger\dagger)$. Using the fact that $ord_y F(y, \overline{0}, \overline{0}) = n$, and using Lemma 4.6, the Zariski multiplicity of the cover $Spec(L[\bar{u}, \bar{v}, y]/ < F(y, \bar{u}, \bar{v}) >) \rightarrow Spec(L[\bar{u}\bar{v}])$ at $(0, \bar{0}\bar{0})$ is n, contradicting $(\dagger\dagger)$. □

We now have the following result;

Lemma 4.8 *Let $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ be reduced curves, having finite intersection, then the Zariski multiplicity, see Definition 1.4, of the cover $(*)$ at $((0,0), \bar{u}^0, \bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0}, C_{\bar{v}^0}, (0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

Proof We have $C_{\bar{u}^0} = s(u^0_{ij}, x, y)$ and $C_{\bar{v}^0} = t(v^0_{ij}, x, y)$. By making the substitutions $\bar{U} = \bar{u}^0 + \bar{u}$ and $\bar{V} = \bar{v}^0 + \bar{v}$, we may assume that $\bar{u}^0 = \bar{v}^0 = \bar{0}$. Moreover, we can suppose that;

$$s(\bar{0}_{ij}, x, 0) \neq 0 \text{ and } \qquad t(\bar{0}_{ij}, x, 0) \neq 0. (**)$$

This can be achieved by making the invertible linear change of variables $(x' = x, y' = \lambda x + \mu y)$ with $(\lambda, \mu) \in L^2$ and $\mu \neq 0$, noting that as $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ are curves, for some choice of (λ, μ), the corresponding polynomials $s(u^0_{ij}, x, y)$ and $t(v^0_{ij}, x, y)$ do not vanish identically on the line $\lambda x + \mu y = 0$. It is trivial to check that the transformation preserves both Zariski multiplicity and intersection multiplicity, so our calculations are not effected.

We may then apply the Weierstrass preparation theorem, Lemma 3.1, in the ring $L[[u_{ij}, v_{ij}, x, y]]$, obtaining factorisations $s(u_{ij}, x, y) = U_1(u_{ij}, x, y)S(u_{ij}, x, y)$ and $t(v_{ij}, x, y) = U_2(v_{ij}, x, y)T(v_{ij}, x, y)$ where U_1 and U_2 are units in the local rings $L[[u_{ij}, x, y]]$ and $L[[v_{ij}, x, y]]$, S, T are Weierstrass polynomials in x with coefficients in $L[[u_{ij}, y]]$ and $L[[v_{ij}, y]]$ respectively. A close inspection of the Weierstrass preparation theorem, see [2], shows that we can obtain the following uniformity in the parameters $\bar{u}$ and $\bar{v}$.

Namely, if $U = \{u_{ij} : s(u_{ij}, x, 0) \neq 0\}$ and $V = \{v_{ij} : t(v_{ij}, x, 0) \neq 0\}$, are the constructible sets for which $(*)$ holds, then if we let R_U and R_V denote the coordinate rings of U and V, we may assume U_1, U_2 lie in $R_U[[x, y]]$ and the coefficients of S, T lie in $R_U[[y]]$ and $R_V[[y]]$ respectively. By Lemma 3.2, we may assume that U_1, U_2, S and T lie in a finite etale extension $R_{U\times V}[x, y]^{ext}$ of the algebra $A = R_{U\times V}[x, y]$ (again, possibly after localisation corresponding to an open subvariety of $Spec(A)$. Now we have the sequence of morphisms.

$Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1 S, U_2 T>}) \rightarrow Spec(\frac{R_{U\times V}[x,y]}{<s,t>}) \rightarrow Spec(R_{U\times V})$.

We claim that the left hand morphism is etale at the point $(\bar{0}, \bar{0}, (00)^{lift})$. This follows from the fact that $R_{U\times V}[x, y]^{ext}$ is an etale extension of $R_{U\times V}[x, y]$ and the maximal ideal given by $(\bar{0}, \bar{0}, (00)^{lift})$ contains $< U_1S, U_2T >$. Now consider the cover;

$$Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1S,U_2T>}) \rightarrow Spec(R_{U\times V}) \ (***)$$

For $\bar{u}, \bar{v}$ in $U \times V$, the fibre of this cover over $\bar{u}, \bar{v}$ corresponds exactly to the intersection of the reducible curves $C'_{\bar{u}}$ and $C'_{\bar{v}}$ which lift the original curves $C_{\bar{u}}$ and $C_{\bar{v}}$ to an etale cover of $Spec(L[xy])$. By Theorem 1.10 and Lemma 2.3, in the case when $C_{\bar{u}_0}, C_{\bar{v}_0}$ intersect at simple points, or Lemma 2.7, for singular points of intersection, and the corresponding Lemma 2.10 for reducible covers, it is sufficient to show that the Zariski multiplicity of the cover (***) at $(\bar{0}, \bar{0}, (00)^{lift})$ corresponds to the intersection multiplicity of the curves $C'_{\bar{u}_0}, C'_{\bar{v}_0}$ at $(00)^{lift}$. The idea now is to apply Lemma 4.7 to the Weierstrass factors of $C'_{\bar{u}}$ and $C'_{\bar{v}}$. This will be achieved by the "unit removal" lemma below, Lemma 4.15. □

In order to prove the "unit removal lemma", we first require some more definitions and a moving lemma for curves;

Definition 4.9 Let $X \rightarrow Spec(L[x, y])$ be an etale cover in a neighboorhood of $(0, 0)$, with distiguished point $(0, 0)^{lift}$. We call a curve C on X passing through $(0, 0)^{lift}$ Weierstrass if, in the power series ring $L[[x, y]]$, the defining equation of C may be written as a Weierstrass polynomial in x with coefficients in $L[[y]]$.

Definition 4.10 Let $F \rightarrow U \times V$ be a finite equidimensional cover of a smooth base of parameters $U \times V$ with a section $s : U \times V \rightarrow F$. We call the cover Weierstrass with units if the fibres $F(\bar{u}, \bar{v})$ can be written as the intersection of reducible curves $C'_{\bar{u}}$ and $C'_{\bar{v}}$ in an etale cover $A_{\bar{u},\bar{v}}$ of $U_{\bar{u},\bar{v}} \subset Spec(L[x, y])$ with the distinguished point $s(\bar{u}, \bar{v})$ lying above $(0, 0)$ and $C'_{\bar{u}}, C'_{\bar{v}}$ factoring as $U_{\bar{u}}F_{\bar{u}}$ and $U_{\bar{v}}F_{\bar{v}}$ with $U_{\bar{u}}, U_{\bar{v}}$ units in the local ring $O_{s(\bar{u},\bar{v}),A_{\bar{u},\bar{v}}}$ and $F_{\bar{u}}, F_{\bar{v}}$ Weierstrass curves in $A_{\bar{u},\bar{v}}$.

Let hypotheses on F, U and V be as above. We call the cover Weierstrass if the fibres $F(\bar{u}, \bar{v})$ can be written as above but with $C'_{\bar{u}}, C'_{\bar{v}}$ Weierstrass curves in $A_{\bar{u},\bar{v}}$.

We say that a Weierstrass cover (with units) factors through the family of projective degree d and degree e curves if the cover $F \rightarrow U \times V$ factors as $F \rightarrow F' \rightarrow U \times V$ where $F' \rightarrow U \times V$ is the finite equidimensional cover obtained by intersecting the families Q_d and Q_e restricted to U and V.

Lemma 4.11 *The cover (***) in Lemma 4.8 is a Weierstrass cover with units factoring through the family of projective degree d and degree e curves.*

Proof Clear by the above definitions. □

Lemma 4.12 *Moving Lemma for Reduced Curves*
Let Q_d and Q_e be the families of all *projective degree d and degree e curves. That is, with the usual coordinate convention $x = X/Z$, $y = Y/Z$, Q_d consists of all curves of the form $s(\bar{u}, x, y) = \sum_{0\leq i+j\leq d} u_{ij}x^i y^j$. Then, if $\bar{u}, \bar{v}$ are chosen in L, so that the*

reduced curves $C_{\bar{u}}$ and $C_{\bar{v}}$ are defined over L, if the tuple $\bar{u}'$ is chosen to be generic in U over L, the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}$ transversely at simple points.

Proof We can give an explicit calculation;

Let $C_{\bar{u}}^{\bar{u}'}$ be defined by the equation $s(\bar{u}', x, y) = \sum_{0\leq i+j\leq d} u'_{ij}x^i y^j$ and $C_{\bar{v}}$ by $t(\bar{v}, x, y) = \sum_{0\leq i+j\leq e} v_{ij}x^i y^j$ with $\{v_{ij} : 0 \leq i + j \leq e\} \subset L$ and $\{u'_{ij} : 0 \leq i + j \leq d\}$ algebraically independent over L. Let $(x_0 y_0)$ be a point of intersection, then $dim(x_0 y_0/L) = 1$, otherwise $dim(x_0 y_0/L) = 0$ and, as L is algebraically closed, we must have that $x_0, y_0 \in L$. Substituting $(x_0 y_0)$ into the equation $s(\bar{u}', x, y) = 0$, we get a non trivial linear dependence over L between u'_{00} and u'_{ij} for $1 \leq i + j \leq d$ which is impossible. Now, the locus of singular points for $C_{\bar{v}}$ is defined over L and hence $(x_0 y_0)$ is a simple point of $C_{\bar{v}}$. Now we further claim that $s(\bar{u}', x, y) = 0$ defines a non-singular curve in $P^2(K_\omega)$ with transverse intersection to $C_{\bar{v}}$ Consider the conditions Sing$(\bar{u})$ given by $\exists x_0 \exists y_0((\frac{\partial s}{\partial x}(x_0 y_0), \frac{\partial s}{\partial y}(x_0 y_0)) = (0, 0))$ and Non-Transverse$(\bar{u})$ by $\exists x_0 \exists y_0(\frac{\partial s}{\partial x}(x_0 y_0)\frac{\partial t}{\partial y}(x_0 y_0) - \frac{\partial s}{\partial y}(x_0 y_0)\frac{\partial t}{\partial x}(x_0 y_0) = 0)$ By the properness of $P^2(K_\omega)$, these conditions define closed subsets of the parameter space U defined over L. We claim that this in fact a proper closed subset. This can be proved in a number of ways. In the case where we restrict ourselves to affine curves, the result follows from a classical result of Kleiman, see [10], as affine space $A^2(K_\omega)$ is homogenous for the action of the additive group $(A^2(K_\omega), +)$. More generally, we can use the moving lemma, given in [9], by observing that the class of all degree d projective curves is closed under rational equivalence. We can also give an explicit proof using Bertini's theorem;

Observe that the curve $C_{\bar{u}}$ defines a complete linear system $|C_{\bar{u}}|$ corresponding exactly to the zero loci of sections σ of the bundle $\mathcal{O}_{P^2}(d)$. We claim the following;

(i). The system $|C_{\bar{u}}|$ is base point free.

(ii). The system $|C_{\bar{u}}|$ separates points.

Now we can define a morphism $\Phi_d : P^2(K) \to P^{d(d+3)/2}(K)$, by sending $x \in P^2$ to the hyperplane $H_x \subset U$ of curves of degree d, passing through x. By (i) and (ii), the restriction of Φ_d to $C_{\bar{v}}$ is injective. By arguments on Frobenius for curves, given in [7], we can assume that Φ_d is an immersion. Using Bertini's Theorem, a generic hyperplane $\mathcal{H}_{\bar{u}'}$ of $P^{d(d+3)/2}(K)$ will intersect $Im(C_{\bar{v}})$ transversely in simple points. By definition of the morphism Φ_d, and the fact that it is an immersion, the corresponding curve $C_{\bar{u}}^{\bar{u}'}$ also intersects $C_{\bar{v}}$ transversely in simple points.

One can also give an enumerative calculation, which was done in an older version of this paper, see [5], but it seems unnecessary. □

Remark 4.13 If we restrict the family of curves, the result in general fails. A simple example is given by the family of all projective degree 3 curves $Q_3^{0,0}$ passing through $(0,0)$ with $x = X/Z$ and $y = Y/Z$. If we take $C_{\bar{v}}$ to be the cusp $x^2 - y^3$, then any curve in $Q_3^{0,0}$ will have a non-transverse intersection with $C_{\bar{v}}$ at the origin.

Lemma 4.14 *Moving Lemma for Curves with Finitely Many Marked Points*
Let hypotheses be as in the previous lemma with $C_{\bar{u}}$ and $C_{\bar{v}}$ defining reduced curves. Suppose also that there exists finitely many marked points $\{p_1, \ldots, p_n\}$ on $C_{\bar{v}}$ defined over L. Then for $\bar{u}' \in U$ generic over L the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}$ transversely at finitely many simple points excluding the set $\{p_1, \ldots, p_n\}$.

Proof As before, the condition that $\bar{u}'$ defines a curve $C_{\bar{u}}^{\bar{u}'}$ either with non-transverse intersection to $C_{\bar{v}}$ or passing through at least one of the points $\{p_1, \ldots, p_n\}$ is a closed subset of U defined over L. Using the above proof and the obvious fact that we can find a curve $C_{\bar{u}}^{\bar{u}'}$ not passing through any of the points $\{p_1, \ldots, p_n\}$, we see that it is proper closed. □

Lemma 4.15 *Unit Removal for Reduced Curves*
Let $(\pi, s) : F \to U \times V$ be a Weierstrass cover with units factoring through projective degree d and degree e curves. Let $(\bar{u}, \bar{v}) \in U \times V$, then there exists a Weierstrass cover $(\pi', s') : F^- \to U' \times V'$ with $U' \subset U$ and $V' \subset V$ open subsets, $(\bar{u}, \bar{v}) \in U' \times V'$, such that $Mult_{(\bar{u},\bar{v},s(\bar{u},\bar{v}))}(F/U \times V) = Mult_{\bar{u},\bar{v},s'(\bar{u},\bar{v}))}(F^-/U' \times V')$.

Proof Let $C'_{\bar{u}}$ and $C'_{\bar{v}}$ be the Weierstrass curves with units in $A_{\bar{u},\bar{v}}$ lifting the curves $C_{\bar{u}}$ and $C_{\bar{v}}$. Now suppose that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F/U \times V) = n$. Then we can find $(\bar{u}', \bar{v}') \in \mathcal{V}_{\bar{u}\bar{v}} \cap U \times V$ generic over L such that the deformed curve $C_{\bar{u}}^{\bar{u}'}$ intersects $C_{\bar{v}}^{\bar{v}'}$ at the n distinct points $x_1, \ldots, x_n$ in $\mathcal{V}_{s(\bar{u},\bar{v})}$. Now using the Weierstrass factorisations of $C_{\bar{u}}^{\bar{u}'}$ and $C_{\bar{v}}^{\bar{v}'}$, we claim that $U_{\bar{u}}^{\bar{u}'}(x_i) \neq 0$ and $U_{\bar{v}}^{\bar{v}'}(x_i) \neq 0$. Suppose not, then $U_{\bar{u}}^{\bar{u}'}(x_i) = U_{\bar{v}}^{\bar{v}'}(x_i) = 0$ and as $(\bar{u}', \bar{v}', x_i)$ specialises to $(\bar{u}, \bar{v}, s(\bar{u}, \bar{v}))$, then $U_{\bar{u}}(s(\bar{u}, \bar{v})) = U_{\bar{v}}(s(\bar{u}, \bar{v})) = 0$. This contradicts the fact that $U_{\bar{u}}$ and $U_{\bar{v}}$ are units in the local ring $O_{s(\bar{u},\bar{v}),A_{\bar{u},\bar{v}}}$. Therefore, we must have that $F_{\bar{u}}^{\bar{u}'}(x_i) = F_{\bar{v}}^{\bar{v}'}(x_i) = 0$. This shows that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F^-/U \times V) \geq n$ where $F^- \to U \times V$ is the cover of $U \times V$ obtained by taking as fibres $F^-(\bar{u}, \bar{v})$ the intersection of the Weierstrass factors $F_{\bar{u}}$ and $F_{\bar{v}}$. Formally, if F is defined by $Spec(\frac{R_{U\times V}[x,y]^{ext}}{<U_1 S, U_2 T>})$ then F^- is defined by $Spec(\frac{R_{U\times V}[x,y]^{ext}}{<S,T>})$. Clearly as $F^- \subset F$ is a union of components of F, we have that $Mult_{\bar{u},\bar{v},s(\bar{u},\bar{v})}(F^-/U \times V) \leq n$ as well. This proves the lemma. □

We now complete the proof of Lemma 4.8. By unit removal, it is sufficient to compute the Zariski multiplicity of the cover

$$Spec(\frac{R_{U\times V}[x, y]^{ext}}{< S, T >}) \to Spec(R_{U\times V})$$

The fibre over $(\bar{u}, \bar{v})$ of this cover corresponds exactly to the intersection of the Weierstrass curves $F_{\bar{u}}$ and $F_{\bar{v}}$ lifting $C_{\bar{u}}$ and $C_{\bar{v}}$. We then use Lemma 2.7, noting that the Weierstrass factors are still reduced, see [2], to finish the result, with the straightforward modification that we work in a uniform family of etale covers.

We now turn to the problem of non-reduced curves. We will show the following stronger version of Lemma 4.8.

Lemma 4.16 *Let $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ be non-reduced curves having finite intersection, then the Zariski multiplicity of the cover (*) at $((0,0), \bar{u}^0, \bar{v}^0)$ equals the intersection multiplicity $I(C_{\bar{u}^0}, C_{\bar{v}^0}, (0,0))$ of $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ at $(0,0)$.*

First, we will require some more lemmas.

Lemma 4.17 *Let $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ be reduced curves intersecting transversely at $(0,0)$. Then the Zariski multiplicity, left multiplicity and right multiplicity of the cover (*) at $((0,0), \bar{u}^0, \bar{v}^0)$ equals 1.*

Proof First note that by Lemma 2.6, and the corresponding Lemma 2.10, and the fact that a generic deformation $C_{\bar{v}_0}^{\bar{v}'}$ will still intersect $C_{\bar{u}_0}$ transversely by Lemma 4.12, it is sufficient to prove the result for right multiplicity.

In order to show this we require the following result, given for analytic curves in [2], we will only need the result for polynomials.

Implicit Function Theorem:

If $G(X,Y)$ is a power series with $G(0,0)=0$ then $G_Y(0,0) \neq 0$ implies there exists a power series $\eta(X)$ with $\eta(0)=0$ such that $G(X,\eta(X))=0$.

In order to show that $RightMult_{(0,0),\bar{u}^0,\bar{v}^0}(F'/U \times V) = 1$, where F' is the family obtained by intersecting degree d and degree e curves, we apply the implicit function theorem to the curve $C_{\bar{u}^0}$ at the point $(0,0)$ of intersection with $C_{\bar{v}^0}$. Let $G(X,Y)$ and $H(X,Y)$ denote the polynomials defining the curves. We have that $G(0,0) = H(0,0) = 0$. Moreover, as the first curve is non-singular at $(0,0)$, we may also assume that $G_Y(0,0) \neq 0$. Now let $\eta(X)$ be given by the theorem. As the intersection of the curves $C_{\bar{u}^0}$ and $C_{\bar{v}^0}$ is transverse, $ord_X H(X,\eta(X)) = 1$.Now we have the sequence of maps;

$$L[\bar{v}] \rightarrow \frac{L[X,Y][\bar{v}]}{< G(u^0,X,Y), H(\bar{v},X,Y) >} \rightarrow \frac{L[X]^{ext}[Y][\bar{v}]}{< Y-\eta(X), H(\bar{v},X,Y) >}.$$

where $L[X]^{ext}$ is an etale extension of $L[X]$ containing $\eta(X)$. Note that $\eta(X)$ is trivially algebraic over $L(X)$. This corresponds to a sequence of finite covers $F_1 \rightarrow F'(u_0,V) \rightarrow Spec(L[\bar{v}])$. The left hand morphism is trivially etale at $(\bar{v}^0,(00)^{lift})$, hence it is sufficient to compute the Zariski multiplicity of $F' \rightarrow Spec(L[\bar{v}])$ at $(\bar{v}^0,(00)^{lift})$ by Lemma 2.3, or the corresponding Lemma 2.10. This is a straightforward calculation, the fibre over $\bar{v}^0$ consists of the scheme $Spec(\frac{L[X,\eta(X)]}{G(X,\eta(X))}) = Spec(L)$ as $ord_X(H(X,\eta(X))) = 1$, hence is etale at the point $(\bar{v}^0,(00)^{lift})$. By Theorem 1.6, the Zariski multiplicity is 1. □

Lemma 4.18 *Let hypotheses be as in Lemma 4.17, then for any $(\bar{u}',\bar{v}') \in \mathcal{V}_{(\bar{u}^0,\bar{v}^0)}$, we have that $Card(F'(\bar{u}',\bar{v}') \cap \mathcal{V}_{0,0}) = 1$*

Proof This follows immediately from Lemmas 4.17 and 2.4. □

Definition 4.19 For ease of notation, given curves $C_{\bar{u}}$ and $C_{\bar{v}}$ of degree d and degree e intersecting at $x \in P^2(K_\omega)$, we define $Mult_x(C_{\bar{u}}, C_{\bar{v}})$ to be the corresponding Zariski multiplicity of the cover $F' \to U \times V$ at the point $(x, \bar{u}, \bar{v})$. Similarly for left/right multiplicity.

We can now give the proof of Lemma 4.16.

Proof Case 1. $C_{\bar{v}_0}$ is a reduced curve (possibly having components). Write $C_{\bar{u}^0}$ as $G_1^{n_1}(X,Y)\ldots G_m^{n_m}(X,Y) = 0$ with G_i the reduced irreducible components of $C_{\bar{u}_0}$ with degree d_i passing through $(0,0)$. Choose $\bar{\epsilon}_1^1, \ldots \bar{\epsilon}_1^{n_1}, \ldots \bar{\epsilon}_i^j, \ldots \bar{\epsilon}_m^1, \ldots, \bar{\epsilon}_m^{n_m}$ independent generic in U_i, the parameter space for degree d_i projective curves with $\bar{\epsilon}_i^j \in \mathcal{V}_{\bar{u}_i^0}$, where $\bar{u}_i^0$ defines G_i. By repeated application of Lemma 4.14, the deformed curves $G_i^{\bar{\epsilon}_i^j} = 0$ intersect $C_{\bar{v}_0}$ transversely at disjoint sets of points We denote by $Z_{\bar{\epsilon}_i^j}$ those points lying in $\mathcal{V}_{00}$. Now the curve defined by $\prod_{ij} G_i^{\bar{\epsilon}_i^j} = 0$ is a deformation $C_{\bar{u}^0}^{\bar{\epsilon}}$ of $C_{\bar{u}^0}$. We let $Z_{\bar{\epsilon}}$ denote the points of intersection of $C_{\bar{u}^0}^{\bar{\epsilon}}$ with $C_{\bar{v}_0}$ in $\mathcal{V}_{00}$. Then we have;

$$Z_{\bar{\epsilon}} = \bigcup_{ij} Z_{\bar{\epsilon}_i^j}$$

$$Card(Z_{\bar{\epsilon}}) = \sum_{ij} Card(Z_{\bar{\epsilon}_i^j})$$

By Lemma 2.4, we have that

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{x \in Z_{\bar{\epsilon}}} LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0})$$

$$= \sum_{i,j} \sum_{x \in Z_{\bar{\epsilon}_i^j}} LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0})(*)$$

We now claim that for a point $x \in Z_{\bar{\epsilon}_i^j}$,

$$LeftMult_x(C_{\bar{u}^0}^{\bar{\epsilon}}, C_{\bar{v}^0}) = LeftMult_x(G_i^{\bar{\epsilon}_i^j}, C_{\bar{v}_0})(**)$$

This follows as both the reduced curves $C_{\bar{u}_0}^{\bar{\epsilon}}$ and $G_i^{\bar{\epsilon}_i^j}$ intersect $C_{\bar{v}_0}$ transversely at x. Hence, in both cases the left multiplicity is 1, by Lemma 4.17.

Combining $(*)$ and $(**)$, we obtain;

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i,j} \sum_{x \in Z_{\bar{\epsilon}_i^j}} LeftMult_x(G_i^{\bar{\epsilon}_i^j}, C_{\bar{v}^0})$$

Now using Lemma 2.4 again gives that;

$$LeftMult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} n_i LeftMult_{(00)}(G_i, C_{\bar{v}^0})(***)$$

If we go through exactly the same calculation with Mult replacing Left Mult, we see as well that

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} n_i Mult_{(00)}(G_i, C_{\bar{v}^0})$$

By Lemma 4.8, this gives

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} n_i I(G_i, C_{\bar{v}^0}, (00))$$

By a straightforward algebraic calculation, see the references below at the end of the proof for the required more general result, this gives

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = I(C_{\bar{u}^0}, C_{\bar{v}^0}, (00))$$

as required.

Case 2. Both $C_{\bar{u}_0}$ and $C_{\bar{v}_0}$ define non-reduced curves. Write $C_{\bar{u}_0}$ as above and $C_{\bar{v}_0}$ as $H_1^{e_1} \dots H_n^{e_n}$ with H_i the reduced compoments with degree c_i of $C_{\bar{v}_0}$ passing through (00). Then $H_1 \dots H_n = 0$ defines a reduced curve passing through (00). Now repeat the argument in Case 1 for the curves $C_{\bar{u}_0}$ and $H_1 \dots H_n = 0$. Again let $Z_{\bar{\epsilon}}$ be the intersection points of the deformed curve $C_{\bar{u}_0}^{\epsilon}$ with $H_1 \dots H_n = 0$ in $\mathcal{V}_{(00)}$. By (***) of Case 1, Lemmas 2.4 and 4.18 with the fact that the intersection of $C_{\bar{u}^0}^{\bar{\epsilon}}$ with $H_1 \dots H_n$ is transverse, we have;

$$Card(Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i Mult_{(00)}(G_i, H_1 \dots H_n)$$

Now using the argument in Case 1 applied to the reduced curves G_i and $H_1 \dots H_n$, we have;

$$Card(Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i \sum_{j=1}^{n} I(G_i, H_j, (00))(*)$$

We claim that for any component H_j

$$Card(H_j \cap Z_{\bar{\epsilon}}) = \sum_{i=1}^{m} n_i I(G_i, H_j, (00))$$

This follows as the deformed curve $C^{\bar{\epsilon}}_{\bar{u}_0}$ a fortiori intersects H_j transversely at simple points. Therefore, again by Case 1, gives the expected multiplicity. Now, using this together with (*), we write $Z_{\bar{\epsilon}}$ as $\cup_j Z^j_{\bar{\epsilon}}$ where $Z^j_{\bar{\epsilon}}$ are the disjoint sets consisting of the intersection of $C^{\bar{\epsilon}}_{\bar{u}_0}$ with H_j. Then by Lemma 2.6, we have that

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_j \sum_{x \in Z^j_{\bar{\epsilon}}} RightMult_x(C^{\bar{\epsilon}}_{\bar{u}^0}, C_{\bar{v}^0})$$

We can now calculate the Right Mult term by applying Case 1 to the intersection of $C_{\bar{v}_0}$ with the reduced curve $C^{\bar{\epsilon}}_{\bar{u}_0}$ at the points of intersection $x \in Z^j_{\bar{\epsilon}}$. At a point $x \in Z^j_{\bar{\epsilon}}$, we have that

$$RightMult_x(C^{\bar{\epsilon}}_{\bar{u}^0}, C_{\bar{v}^0}) = e_j I(C^{\bar{\epsilon}}_{\bar{u}^0}, H_j, x) = e_j$$

as the intersection is transverse. Finally this gives;

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = \sum_{i=1}^{m} \sum_{j=1}^{n} n_i e_j I(G_i, H_j, (00))$$

By an algebraic result, see [11] for the case of complex algebraic curves, or [8] for its generalisation to algebraic curves in arbitrary characteristics, we have

$$Mult_{(00)}(C_{\bar{u}^0}, C_{\bar{v}^0}) = I(C_{\bar{u}^0}, C_{\bar{v}^0}, (00))$$

as required. □

The following version of Bezout's theorem in all characteristics is now an easy generalisation from the above lemma. For curves C_1 and C_2 in $P^2(L)$, we let $M(C_1, C_2, x)$ denote the intersection multiplicity or the Zariski multiplicity, we know from the above that the two are equivalent.

Theorem 4.20 (Non-Standard Bezout)
Let C_1 and C_2 be projective curves of degree d and degree e in $P^2(L)$, possibly with non-reduced components, intersecting at finitely many points $\{x_1, \ldots, x_i, \ldots x_n\}$, then we have;

$$\sum_{i=1}^{n} M(C_1, C_2, x_i) = de$$

.

Of course, we could just quote the algebraic result given in [10] (though this in fact only holds for reduced curves). Instead we can give a non-standard proof, which in many ways is conceptually simpler and doesn't involve any algebra.

Proof Let Q_d and Q_e be the families of all projective degree d and degree e curves. Then we have the cover $F \rightarrow U \times V$ with $F \subset U \times V \times P^2(L)$ obtained by intersecting the families Q_d and Q_e. We have that

$$\sum_{i=1}^{n} M(C_1, C_2, x_i) = \sum_{i=1}^{n} Mult_{x_i \in F(\bar{u}_0, \bar{v}_0)}(F/U \times V)$$

where $(\bar{u}_0, \bar{v}_0)$ define C_1 and C_2. By Lemma 4.3 in [7], this equals

$$\sum_{x \in F(\bar{u}, \bar{v})} Mult_{x, \bar{u}, \bar{v}}(F/U \times V)$$

where $(\bar{u}, \bar{v})$ is generic in $U \times V$. Using, for example, the proof of Lemma 4.12, generically independent curves $C_{\bar{u}}$ and $C_{\bar{v}}$ intersect transversely at a finite number of simple points. Hence, by Lemma 4.17, the Zariski multiplicity calculated at these points is 1. As the cover F has degree de, there is a total number de of these points as required. □

References

1. Abhyankar, S.S.: Algebraic Space Curves. Les Presses de l'Universite de Montreal, Montreal, Canada (1971)
2. Abhyankar, S.S.: Algebraic geometry for scientists and engineers. AMS Math. Surv. **35** (1990)
3. Artin, M.: On the solutions of analytic equations. Invent. Math. **5**, 277–291 (1968)
4. Bosch, S., Guntzer, U., Remmert, R.: Grundlehren der mathematischen Wissenscaften, vol. 261. Springer (1984)
5. de Piro, T.: A Non-Standard Bezout Theorem AG/LO (0406176)
6. de Piro, T.: Infinitesimals in a Recursively Enumerable Prime Model (2007). http://www.curvalinea.net
7. de Piro, T.: Zariski Structures and Algebraic Curves (2007). http://www.curvalinea.net. old version "Zariski Structures and Algebraic Geometry" on AG arXiv (0402301) (2004)
8. Fulton, W.: Algebraic Curves. Mathematics Lecture Notes Series, Bejamin/Cummings (1969)
9. Fulton, W.: Intersection Theory. Springer (1998)
10. Hartshorne, R.: Algebraic Geometry. Springer (1977)
11. Kirwan, F.: Complex Algebraic Curves. London Mathematical Society, Student Texts, vol. 23 (1992)
12. Marker, D.: Zariski Geometries, Model Theory and Algebraic Geometry. Lecture Notes in Mathematics, vol. 1696. Springer (1998)
13. Milne, J.: Etale Cohomology. Princeton UP (1980)
14. Milne, J.: Lectures on Etale Cohomology (1988). www.lsa.umich.edu/jmilne
15. Mumford, D.: Red Book of Varieties and Schemes. Springer (1999)
16. Robinson, Z.: A rigid analytic approximation theorem. Asterisque **264** (2000)
17. Zilber, B.: Zariski Geometries. Geometry from the Logician's Point of View, LMS (2010)

Embeddings of the Symmetric Groups to the Space Cremona Group

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Abstract We study embeddings of symmetric groups to the space Cremona group.

Keywords Cremona group · Fano variety · Terminal singularity

1 Introduction

The classification of finite subgroups in the Cremona groups $\mathrm{Cr}_n(\mathbb{C})$ is an old-standing problem that goes back to works of classics of Italian algebraic geometry. In the last two decades the interest to this problem was reactivated; see for example [6, 15], and references therein. In particular, now there is a basically complete classification of finite subgroups in the plane Cremona group $\mathrm{Cr}_2(\mathbb{C})$ [6].

In this paper we are interested in embeddings of symmetric groups $\mathfrak{S}_N$ to Cremona group $\mathrm{Cr}_3(\mathbb{C})$ and, more generally, to groups of birational self-maps of three-dimensional rationally connected varieties. This problem is interesting not only in its own sake but also in relation with computation of essential dimension of $\mathfrak{S}_N$ (cf. [7]).

Proposition 1.1 *Let X be a rationally connected threefold and let* $\mathrm{Bir}(X)$ *be the group of its birational self-maps.*

(i) For $n \geq 8$ the symmetric group $\mathfrak{S}_n$ does not admit any embedding to $\mathrm{Bir}(X)$.
(ii) Any embedding $\mathfrak{S}_7 \subset \mathrm{Bir}(X)$ up to conjugation is induced by the action of $\mathfrak{S}_7$ on the smooth variety $X_6' \subset \mathbb{P}^5 \subset \mathbb{P}^6$ given by the equations

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$$\sum_{i=1}^{7} x_i = \sum_{i=1}^{7} x_i^2 = \sum_{i=1}^{7} x_i^3 = 0 \tag{1.1.1}$$

with natural action of $\mathfrak{S}_7$ on $x_1, \dots, x_7$ by permutations. Moreover, any three-dimensional $\mathfrak{S}_7$-Mori fiber space over a rationally connected base is equivariantly isomorphic to the variety (1.1.1).

In particular, $\mathfrak{S}_n$ is non embeddable to $\mathrm{Cr}_3(\mathbb{C})$ for $n \geq 7$ (because the variety (1.1.1) is not rational [2]).

Note that embeddings to $\mathrm{Cr}_3(\mathbb{C})$ of some other classes of "large" finite groups were studied in [1, 10].

Our second main result is related to the symmetric group $\mathfrak{S}_6$. Unfortunately it is not complete.

Proposition 1.2 *Let Y be a rationally connected threefold. Then for any embedding $\mathfrak{S}_6 \subset \mathrm{Bir}(Y)$ there exists a $\mathfrak{S}_6$-equivariant birational map $Y \dashrightarrow X$ such that one of the following holds:*

(i) X is a Fano threefold with at worst terminal Gorenstein $\mathfrak{S}_6\mathbb{Q}$-factorial singularities and $\rho(X) = 1$;
(ii) X is a Fano threefold with terminal $\mathfrak{S}_6\mathbb{Q}$-factorial singularities and $\rho(X) = 1$ such that all the non-Gorenstein points of X are cyclic quotients of index 2. The number n of these points equals 12 or 15 and there are two possibilities:
   (A) $n = 12$, $-K_X^3 = 2g + 4$, $g \geq -1$, $\dim |-K_X| = g + 1$;
   (B) $n = 15$, $-K_X^3 = 2g + 11/2$, $g \geq -2$, $\dim |-K_X| = g + 1$.

Moreover, any $\mathfrak{S}_6$-Mori fiber space is equivariantly isomorphic to one of the above cases.

Recall that a normal G-variety X is said to be *$G\mathbb{Q}$-factorial* if some multiple nD of any G-invariant divisor is Cartier [14].

We do not know any examples of non-Gorenstein Fano threefolds admitting an $\mathfrak{S}_6$-action. We expect that the case 1.2(ii) does not occur. In contrast there are a lot of examples of actions of $\mathfrak{S}_6$ on Gorenstein Fano threefolds (case 1.2(i)). In Sect. 6 we collect known ones. However we do not assert that our collection is complete. The particular the case of actions of $\mathfrak{S}_6$ on del Pezzo threefolds is completely studied in Sect. 7.

2 Preliminaries

We work over a complex number field $\mathbb{C}$ throughout.

Notation 2.1 • $\mathfrak{S}_n$ and $\mathfrak{A}_n$ denote the symmetric and the alternating groups, respectively.

- As usual, $\mathrm{Pic}(X)$ denotes the Picard group of a variety X and $\rho(X)$ is the rank of $\mathrm{Pic}(X)$.
- $\mathrm{Cl}(X)$ denotes the Weil divisor class group of a normal variety X and $\mathrm{r}(X) := \mathrm{rk}\mathrm{Cl}(X)$.
- If a group G acts on an object A, then A^G is the set of G-invariant elements.
- If a group G acts on a variety X, then $\rho(X)^G := \mathrm{rk}\mathrm{Pic}(X)^G$ and $\mathrm{r}(X)^G := \mathrm{rk}\mathrm{Cl}(X)^G$.

Throughout this paper we use the terminology and notation of the equivariant minimal model program [14]. In particular, a *$G\mathbb{Q}$-Fano variety* X is a variety equipped with an action of a finite group G such that X has only terminal $G\mathbb{Q}$-factorial singularities, $\mathrm{rk}\mathrm{Pic}(X) = 1$, and the anticanonical divisor $-K_X$ is ample. In this situation, we say that X is a *G-Fano variety* if X is Gorenstein or, equivalently, K_X is a Cartier divisor.

For any (possibly singular) Fano threefold X we define its *genus* as follows

$$g = \mathrm{g}(X) := \dim|-K_X| - 1 = \dim H^0(X, -K_X) - 2.$$

Thus $\mathrm{g}(X)$ is an integer ≥ -2. This definition agrees with usual definition of genus for smooth Fano threefolds [8].

We need some standard information about groups $\mathfrak{S}_6$ and $\mathfrak{A}_6$ and their actions on finite sets and lower-dimensional algebraic varieties.

Lemma 2.2 *(i) Let ρ be a faithful irreducible representation of the group $\mathfrak{S}_6$. Then* $\dim(\rho) \in \{5, 9, 10, 16\}$.

(ii) Let ψ be a faithful irreducible representation of the group $\mathfrak{A}_6$. Then $\dim(\psi) \in \{5, 8, 9, 10\}$.

Recall that $[\mathrm{Aut}(\mathfrak{S}_6) : \mathrm{Inn}(\mathfrak{S}_6)] = 2$, where $\mathrm{Inn}(\mathfrak{S}_6)$ is the subgroup of inner automorphisms. Let υ be an outer automorphism.

Proposition 2.3 (see e.g. [19, Sect. 2.6, Theorem 2.4], [4])

(i) Up to conjugacy, a maximal subgroup of $\mathfrak{S}_6$ is one of the following:

$$\mathfrak{A}_6,\ \ \mathfrak{S}_5,\ \ \upsilon(\mathfrak{S}_5),\ \ \mathrm{N}(\mathfrak{S}_3 \times \mathfrak{S}_3) \simeq (\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \boldsymbol{\mu}_2,\ \ \mathfrak{S}_4 \times \mathfrak{S}_2,\ \ \upsilon(\mathfrak{S}_4 \times \mathfrak{S}_2).$$

(ii) Up to conjugacy, a maximal subgroup of $\mathfrak{A}_7$ is one of the following:

$$\mathfrak{A}_6,\ \ \mathfrak{S}_5,\ \ H_1,\ \ H_2,\ \ \mathrm{N}(\mathfrak{A}_4 \times \mathfrak{A}_3)$$

where $H_1 \simeq H_2 \simeq \mathrm{PSL}_2(\mathbf{F}_7)$ and $|\mathrm{N}(\mathfrak{A}_4 \times \mathfrak{A}_3)| = 72$.

Corollary 2.4 *Let $G := \mathfrak{S}_6$ act transitively on a set Ω with $|\Omega| \le 16$. Let G_P be the stabilizer of $P \in \Omega$. There are only the following cases:*

No.	1°	2°	3°	4°	5°	6°
$\lvert\Omega\rvert$	1	2	6	12	10	15
G_P	$\mathfrak{S}_6$	$\mathfrak{A}_6$	$\mathfrak{S}_5$	$\mathfrak{A}_5$	$(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \boldsymbol{\mu}_2$	$\mathfrak{S}_4 \times \mathfrak{S}_2$

Lemma 2.5 *The groups $\mathfrak{S}_6$ and $\mathfrak{A}_7$ do not admit embeddings into* $\mathrm{Cr}_2(\mathbb{C})$ *nor into* $\mathrm{Bir}(S)$*, where S is an elliptic ruled surface.*

Proof See e.g. [6]. □

Lemma 2.6 ([6])

(i) The group $\mathfrak{A}_6$ does not admit embeddings into $\mathrm{Bir}(S)$*, where S is an elliptic ruled surface.*
(ii) Up to conjugacy and automorphisms of $\mathfrak{A}_6$ there exists exactly one embedding $\mathfrak{A}_6 \subset \mathrm{PGL}_3(\mathbb{C})$.

The image of $\mathfrak{A}_6 \hookrightarrow \mathrm{PGL}_3(\mathbb{C})$ whose image is called the *Valentiner group* (see [6] and references therein).

Lemma 2.7 ([12, Lemma B.2]) *Let X be a threefold with at worst terminal singularities such that* $\mathrm{Aut}(X)$ *has a subgroup G isomorphic to $\mathfrak{A}_6$. Then X contains no G-invariant points.*

3 Main Reduction

The following assertion is well known (see e.g. [14, Sect. 14])

Proposition 3.1 *Let Y be a rationally connected variety and let $G \subset$* $\mathrm{Bir}(Y)$ *be a finite subgroup. Then there there exists a G-equivariant birational map $Y \dashrightarrow X$, where X is a projective variety having a G-Mori fiber space structure $f : X \to Z$.*

Proposition 3.2 (cf. [10, Sect. 4.2]) *Let $G = \mathfrak{S}_N$ with $N \ge 6$ and let $f : X \to Z$ be a G-Mori fiber space, where X is rationally connected threefold. Then Z is a point, i.e. X is a $G\mathbb{Q}$-Fano threefold.*

Proof Assume that $\dim(Z) \ge 1$. Then f is either $G\mathbb{Q}$-del Pezzo fibration and $Z \simeq \mathbb{P}^1$ or a $G\mathbb{Q}$-conic bundle and Z a rational surface (see [14, Sect. 10]). The map f induces a homomorphism

$$\Phi : G \longrightarrow \mathrm{Aut}(Z).$$

Assume that $\dim(Z) = 2$. Then the generic fiber X_η of f is a smooth rational curve. The kernel of Φ acts on X_η faithfully. Hence $\ker(\Phi) \not\supset \mathfrak{A}_6$ and so Φ is injective. This contradicts Lemma 2.5.

Thus we may assume that f is a $G\mathbb{Q}$-del Pezzo fibration. Again by Lemma 2.5 the image of Φ is either trivial or a cyclic group of order two. In both cases $\Phi(G)$ has a fixed point $o \in Z$. Let F be the scheme fiber over o. Then the proof of [12, Lemma B.5] works without any changes. Thus $F \simeq \mathbb{P}^2$. Since $\mathfrak{S}_N$, for $N \geq 6$, does not act faithfully on $\mathbb{P}^2$, the alternating subgroup $\mathfrak{A}_N$ acts on F trivially. This again contradicts Lemma 2.7. □

Corollary 3.3 *Let Y be a rationally connected algebraic threefold. Suppose that* $\mathrm{Bir}(Y)$ *contains a subgroup $G \simeq \mathfrak{S}_N$ with $N \geq 6$. Then there exists a G-equivariant birational map $Y \dashrightarrow X$, where X is a $G\mathbb{Q}$-Fano threefold.*

4 Non-gorenstein Fano Threefolds

The goal of this section is to prove the following result.

Proposition 4.1 (cf. [10, Lemma 6.1]) *Let X be a non-Gorenstein Fano threefold with terminal singularities such that* $\mathrm{Aut}(X) \supset G \simeq \mathfrak{S}_6$. *Let Ω be the set of all non-Gorenstein points. Then any point $P \in \Omega$ is a cyclic quotient singularity of type* $\frac{1}{2}(1, 1, 1)$, *the action of G on Ω is transitive, and one of the following holds:*

(i) $|\Omega| = 12$, $G_P \simeq \mathfrak{A}_5$,
(ii) $|\Omega| = 15$, $G_P \simeq \mathfrak{S}_4 \times \mathfrak{S}_2$.

Proof Fix a non-Gorenstein point $P \in X$ and let $P_1 = P, \dots, P_k$ be its orbit. Let $r \geq 2$ be the Gorenstein index of $(X \ni P)$, let

$$\pi\colon (X^\sharp, P^\sharp) \longrightarrow (X, P)$$

be the index one cover (see [17, Sect. 3.5]), where $P^\sharp = \pi^{-1}(P)$. Then π is the topological universal cover (of degree r). Thus, there is an exact sequence of groups

$$1 \longrightarrow M \xrightarrow{\alpha} G_P^\sharp \xrightarrow{\beta} G_P \longrightarrow 1, \tag{4.1.1}$$

where $M \simeq \boldsymbol{\mu}_r$ and $G_P^\sharp$ is a finite subgroup in $\mathrm{Aut}(X^\sharp, P^\sharp)$. The group $G_P^\sharp$ faithfully acts on the Zariski tangent space $T := T_{P^\sharp, X^\sharp}$ to $X^\sharp$ at the point $P^\sharp$. Recall that $(X^\sharp, P^\sharp)$ is a hypersurface singularity [17]. Hence, we have $\dim(T) \leq 4$. By the classification of three-dimensional terminal singularities (see [17, Sect. 6.1]) the action of the group $\alpha(M)$ on T in some coordinate system has one of the following forms:

(a) $\dim(T) = 3$, $(x_1, x_2, x_3) \longmapsto (\zeta_r x_1, \zeta_r^{-1} x_2, \zeta_r^a x_3)$,
(b) $\dim(T) = 4$, $(x_1, x_2, x_3, x_4) \longmapsto (\zeta_r x_1, \zeta_r^{-1} x_2, \zeta_r^a x_3, x_4)$,
(c) $\dim(T) = 4$, $(x_1, x_2, x_3, x_4) \longmapsto (\zeta_4 x_1, -\zeta_4 x_2, \zeta_4 x_3, -x_4)$, $r = 4$,

where ζ_r is a primitive r-th root of unity and $\gcd(r, a) = 1$.

Denote by $T' \subset T$ the subspace generated by the M-eigenspaces on which M acts faithfully. Thus $\dim(T') = 3$ and $T' = T$ in the case (a). The subspace T' is $G_P^\sharp$-invariant and so $G_P^\sharp \subset \mathrm{GL}(T') = \mathrm{GL}_3(\mathbb{C})$.

Recall that we can associate with $(X \ni P)$ a basket $\mathbf{B}(X, P)$, that is, a finite collection of cyclic quotient terminal singularities $(X_\alpha \ni P_\alpha)$ [17, Sect. 6.1 and Theorem 9.1 (III)]. Moreover, $\mathbf{B}(X, P) = \{(X \ni P)\}$ if and only if $(X \ni P)$ is a cyclic quotient singularity (case (a)). In the case (b) all the singularities in $\mathbf{B}(X, P)$ are of index r. In the case (c) the basket $\mathbf{B}(X, P)$ contains a point of index 4 and at least one point of index 2. Denote

$$\mathbf{B}(X) := \bigcup_{P \in X} \mathbf{B}(X, P).$$

According to [9] we have

$$\sum_{Q \in \mathbf{B}(X)} \left(r_Q - \frac{1}{r_Q} \right) \le 24. \tag{4.1.2}$$

In particular, $k \le |\Omega| \le 16$.

We use Corollary 2.4. First we consider the case Corollary 2.4.2^o, i.e. $G \simeq \mathfrak{A}_6$. Since $\mathfrak{A}_6$ has no non-trivial three-dimensional representations of dimension ≤ 4 (see Lemma 2.2), the sequence (4.1.1) does not split. In particular, the representation $G_P^\sharp \hookrightarrow \mathrm{GL}(T')$ is irreducible (otherwise the restriction to a two-dimensional $G_P^\sharp$-invariant subspace $T_2 \subset T'$ would be a faithful representation of $G_P^\sharp$). Then M acts on T' by scalar multiplications and so $r = 2$. But in this case the determinant map $\det : G_P^\sharp \to \mathbb{C}^*$ splits the sequence (4.1.1). Cases Corollary 2.4.1^o and 3^o are similar. Thus $k \ge 10$ and we may assume that any G-orbit on Ω has length at least 10.

Then $r = 2$ by (4.1.2). Moreover, $P \in X$ is a cyclic quotient singularity and $|\Omega| = k$, i.e. $P = P_1, \dots, P_k$ are all non-Gorenstein points of X. We have $T' := T \simeq \mathbb{C}^3$. Since any normal subgroup of order 2 is contained in the center, $G_P^\sharp$ is a central extension of G_P. Let $H^\sharp = G_P^\sharp \cap \mathrm{SL}(T')$. Then $H^\sharp$ is a normal subgroup of $G_P^\sharp$ and $G_P^\sharp / H^\sharp \simeq \boldsymbol{\mu}_m$ for some m. Moreover, $H^\sharp$ does not contain M (because $H^\sharp \subset \mathrm{SL}(T')$ and M is a group of order 2 acting by scalar multiplication). Hence, m is even. Let $H := \beta(H^\sharp)$. Then $H \simeq H^\sharp$ and $G_P / H \simeq \boldsymbol{\mu}_{m/2}$.

Consider the case Corollary 2.4.5^o. Then $G_P \simeq (\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \boldsymbol{\mu}_2$. The commutator subgroup $[G_P, G_P]$ is a group of order 18 and

$$G_P / [G_P, G_P] \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2.$$

Since G_P / H is cyclic, H either coincides with G_P or $[G_P : H] = 2$. If $H = G_P$, then $G_P^\sharp = H^\sharp \times M \simeq G_P \times \boldsymbol{\mu}_2$. On the other hand, G_P has no three-dimensional faithful representation. Hence, H is an index 2 subgroup in G_P and $G_P^\sharp / H^\sharp \simeq \boldsymbol{\mu}_4$. Note that the center of $H \simeq H^\sharp$ is trivial and H has no faithful two-dimensional representations. Hence, the representation of H on T' is irreducible. Let $\mathrm{Syl}_3(H)$ be the Sylow 3-subgroup. Then $\mathrm{Syl}_3(H)$ is normal in H and $H/\mathrm{Syl}_3(H)$ transitively permute $\mathrm{Syl}_3(H)$-eigenspaces. This is impossible. $\square$

Corollary 4.2 *Let X be a Fano threefold with terminal singularities such that $\mathrm{Aut}(X) \supset G \simeq \mathfrak{S}_7$. Then X is Gorenstein.*

Proof Easily follows from Proposition 4.1. We have to note only that $\mathfrak{S}_7$ contains no subgroups of index 12 or 15, hence $\mathfrak{S}_7$ cannot act on the set Ω as in Proposition 4.1. $\square$

Proof of Proposition 1.2 Let $G = \mathfrak{S}_6$ be a subgroup in $\mathrm{Bir}(Y)$, where Y is a rationally connected threefold. By Proposition 3.1 there exists an equivariant birational map $Y \dashrightarrow X$, where X is a variety having a G-Mori fiber space structure $X \to Z$. By Proposition 3.2 the base Z is a point, i.e. X is a $G\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein. By Proposition 4.1 we have only one of the cases Propositions 4.1(i) or 4.1(ii). It remains to prove the last assertion of 1.2(ii). For this, we apply the orbifold Riemann-Roch [17]:

$$g + 1 = \dim |-K_X| = \frac{1}{2}(-K_X)^3 - \frac{1}{4}k + 2.$$

Hence, $-K_X^3 = 2g - 2 + k/2$, where $k = 12$ or 15. $\square$

5 Proof of Proposition 1.1

In this section we prove Proposition 1.1. Note that the proof also follows from [1, Theorem 4.3]. Let X be a rationally connected variety such that $\mathrm{Bir}(X) \supset G \simeq \mathfrak{S}_n$, $n \geq 7$. Let $G_0 := \mathfrak{A}_n \subset G$. By [10, Theorem 1.5] we have $n = 7$. By Proposition 3.1 we may assume that X is a variety having a G-Mori fiber space structure $X \to Z$. By Proposition 3.2 the base Z is a point, i.e. X is a $G\mathbb{Q}$-Fano threefold and by Corollary 4.2 the singularities of X are Gorenstein.

Lemma 5.1 *The linear system $|-K_X|$ is very ample and defines an embedding*

$$X = X_{2g-2} \subset \mathbb{P}\left(H^0(X, -K_X)^\vee\right) = \mathbb{P}^{g+1}.$$

If $g \geq 5$, then the image X_{2g-2} is an intersection of quadrics in $\mathbb{P}^{g+1}$.

Proof See [10, Lemmas 5.3 and 5.4]. $\square$

Lemma 5.2 *If $g \leq 4$, then $g = 4$ and we have the case 1.1(ii).*

Proof The embedding $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ is G-equivariant and $\mathbb{P}^{g+1}$ is naturally identified with $\mathbb{P}(H^0(X, -K_X)^\vee)$. Since $\mathfrak{S}_7$ has no non-trivial representations of dimension ≤ 5, we have $g \geq 4$. If $g = 4$, then by [10, Lemma 5.4] $X = X_6 \subset \mathbb{P}^5$ is an intersection of a quadric and a cubic. Here the representation of G on $H^0(X, -K_X)$ is irreducible and there exists exactly one invariant quadric. We obtain the case 1.1(ii). This variety X'_6 is not rational according to [2]. □

Assume that $g = 5$. In this case $\dim H^0(X, -K_X) = 7$. The variety $X \subset \mathbb{P}^6$ is a complete intersection of three quadrics, say Q_1, Q_2, Q_3 (see Lemma 5.1). They generate a two-dimensional linear system (net) $\mathscr{Q}$. The induced action of $G_0 = \mathfrak{A}_7$ on $\mathscr{Q}$ is trivial by Lemma 2.5. Therefore, all the quadrics in the net $\mathscr{Q}$ are G_0-invariant. The faithful G_0-representation $H^0(X, -K_X)^\vee$ is reducible:

$$H^0(X, -K_X)^\vee = W_1 \oplus W_6,$$

where $\dim(W_1) = 1$ and $\dim(W_6) = 6$. Here W_6 is again a (unique) standard irreducible representation having exactly one invariant quadric, say Q'. On the other hand, the surface $\mathbb{P}(W_6) \cap X$ is a complete intersection of (invariant) quadrics $Q_i \cap \mathbb{P}(W_6)$. Again this gives a contradiction.

From now on we assume that $g \geq 6$. Let $G_0 := \mathfrak{A}_7 \subset G$. If $\mathrm{r}(X)^{G_0} = 1$, then by [10, Theorem 1.5] we have $X \simeq \mathbb{P}^3$. In this case the action of $\mathfrak{S}_7$ on $\mathbb{P}^3$ lifts to a faithful action of a central extension $\tilde{\mathfrak{S}}_7$ of $\mathfrak{S}_7$ on $H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$. Since the Schur multiplier of $\mathfrak{S}_7$ has order 2 (see e.g. [4]), we may assume that $\tilde{\mathfrak{S}}_7$ is either $\mathfrak{S}_7$ itself or a double cover of $\mathfrak{S}_7$. However such $\tilde{\mathfrak{S}}_7$ has no faithful four-dimensional representations, a contradiction.

Thus we assume that $\mathrm{r}(X)^{G_0} > 1$. We claim that X contains no planes. Indeed, in the case $\rho(X) = 1$ this is the statement of [13, Theorem 1.1]. For general case, we just note that the proof of [13, Theorem 1.1] works without changes. It uses only that X is a G-Fano threefold with $g \geq 6$ such that $-K_X$ is very ample and the image of the anticanonical map is an intersection of quadrics (Lemma 5.1). Thus X contains no planes.

Let $X_1 \to X$ be a small $G_0\mathbb{Q}$-factorialization (we take $X_1 = X$ if X is $G_0\mathbb{Q}$-factorial). Then $\rho(X_1)^{G_0} > 1$. Run G_0-equivariant MMP on X_1:

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{N-2}} X_{N-1} \xrightarrow{\varphi_{N-1}} X_N$$

Since X contains no planes, the variety X_1 contains no surfaces S_1 such that $(-K_{X_1})^2 \cdot S_1 = 1$. Then by the classification of G-extremal contractions (see e.g. [14, Theorems 7.1.1 and 8.2.4]) the variety X_2 has at worst terminal Gorenstein singularities whose anticanonical class $-K_{X_2}$ is nef, big, and does not contract divisors. Moreover, $\dim|-K_{X_2}| \geq \dim|-K_{X_1}|$ and X_2 contains no G_0-invariant effective divisors S_2 such that $(-K_{X_2})^2 \cdot S_2 = 1$. Continuing the process we end up with a G_0-Mori fiber space X_N/Z with

$$\dim|-K_{X_N}| \geq \dim|-K_X| = g + 1 \geq 7.$$

Then by [10, Theorem 1.5] we have $X_N \simeq \mathbb{P}^3$. Denote $Y := X_{N-1}$ and consider the last contraction

$$\varphi = \varphi_{N-1} : Y = X_{N-1} \longrightarrow X_N = \mathbb{P}^3.$$

Let $E \subset Y$ be its exceptional divisor. Assume that $\dim \varphi(E) = 0$. Again by the classification of extremal contractions [14, Theorems 7.1.1 and 8.2.4] φ is the blowup of a G_0-invariant set of distinct points $P_1, \ldots, P_r$. By Lemma 2.7 the stabilizer of any point on $\mathbb{P}^3$ is not isomorphic to $\mathfrak{A}_6$. Hence, $r \geq 8$ and so $(-K_Y)^3 = (-K_{\mathbb{P}^3})^3 - 8r \leq 0$ a contradiction. Therefore, $C := \varphi(E)$ is a curve. Clearly, C is not contained in a plane. Assume that C is irreducible. Let $C' \to C$ be the normalization. Since $\mathrm{Aut}(\mathbb{P}^1)$ contains no subgroups isomorphic to $\mathfrak{A}_7$, C' is not rational. Also, one can see that C' is not an elliptic curve. We have

$$(-K_{X_N})^3 - (-K_{X_{N-1}})^3 = 2(-K_{X_{N-1}})^2 \cdot E + 2\mathrm{p_a}(C) - 2.$$

(see e.g. [13, Proposition 5.1]).

Thus $2\mathrm{p_a}(C) - 2 < 64 - 10$ and $\mathrm{p_a}(C) \leq 27$. On the other hand, by the Hurwitz bound the the order of the automorphism group of C' is at most $84(\mathrm{g}(C') - 1) \leq 2184$. Since $2184 < |\mathfrak{A}_7|$, the group $G_0 = \mathfrak{A}_7$ cannot act effectively on C, a contradiction.

Thus, the curve C is reducible. Let $C_1, \ldots, C_r$ be its irreducible components and let G_1 be the stabilizer of C_1 in $G_0 = \mathfrak{A}_7$. By [10, Lemma 5.3] the linear system $|-K_{X_{N-1}}|$ is base point free. Let $S_1, S_2 \in |-K_{X_{N-1}}|$ be two general members. Then $C \subset \varphi(S_1) \cap \varphi(S_2)$, where $\varphi(S_i) \in |-K_{\mathbb{P}^3}|$. Hence $r \leq \deg C \leq 16$. Since the group G_0 permutes $C_1, \ldots, C_r$ transitively, by Proposition 2.3(ii) we have only two possibilities:

(i) $G_1 \simeq \mathrm{PSL}_2(\mathbf{F}_7)$, $r = 15$;
(ii) $G_1 \simeq \mathfrak{A}_6$, $r = 7$.

In both cases $\deg(C_1) \leq 2$. Hence $C_1 \simeq \mathbb{P}^1$. On the other hand, the groups $\mathrm{PSL}_2(\mathbf{F}_7)$ and $\mathfrak{A}_6$ cannot act on $\mathbb{P}^1$ effectively, a contradiction.

6 Examples

In this section we collect examples of $\mathfrak{S}_6$-Fano threefolds. Note that a faithful action of a group G on an algebraic variety X induces an embedding $G \subset \mathrm{Bir}(X)$ and an embedding $G \subset \mathrm{Cr}_n(\mathbb{C})$ if X is rational.

Example 6.1 Let $\tilde{\mathfrak{S}}_6$ be the pull-back of $\mathfrak{S}_6 \subset \mathrm{SO}_6(\mathbb{R})$ under the double cover

$$\mathrm{SU}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{R}).$$

Then $\tilde{\mathfrak{S}}_6$ is a non-trivial central extension of $\mathfrak{S}_6$ by $\boldsymbol{\mu}_2$. This defines an embedding $\mathfrak{S}_6 \subset \mathrm{PGL}_4(\mathbb{C})$, so $\mathfrak{S}_6$ acts on $\mathbb{P}^3$.

In all examples below the group $\mathfrak{S}_6$ is supposed to act on $x_1, \dots, x_6$ by permutations.

Example 6.2 Let X be a smooth quadric threefold given in $\mathbb{P}^5$ by

$$\sum_{i=1}^{6} x_i = \sum_{i=1}^{6} x_i^2 = 0.$$

Then X admits a $\mathfrak{S}_6$-action by permutations of coordinates.

Example 6.3 The *Segre cubic* X_3^{s} is a subvariety in $\mathbb{P}^5$ given by the equations

$$\sum_{i=1}^{6} x_i = \sum_{i=1}^{6} x_i^3 = 0.$$

The singular locus of this cubic consists of 10 nodes and $\operatorname{Aut} X_3^{\mathrm{s}} \simeq \mathfrak{S}_6$. Moreover, the quotient $X_3^{\mathrm{s}}/\mathfrak{S}_6 \subset \mathbb{P}^5/\mathfrak{S}_6$ is isomorphic to the weighted projective space $\mathbb{P}(2, 4, 5, 6) \subset \mathbb{P}(1, 2, 3, 4, 5, 6)$. Therefore, $\mathrm{r}(X_3^{\mathrm{s}})^{\mathfrak{S}_6} = 1$ and so X_3^{s} is a $\mathfrak{S}_6$-Fano threefold. Obviously, X_3^{s} is rational.

Example 6.4 ([18, Sect. 4]) Let $X = X_4(\lambda) \subset \mathbb{P}^4 \subset \mathbb{P}^5$ is a quartic given by the equations

$$\sum_{i=1}^{6} x_i = \sum_{i=1}^{6} x_i^4 + \lambda\Big(\sum_{i=1}^{6} x_i^2\Big)^2 = 0,$$

where λ is a constant $\neq -1/4$. The hypersurface $X_4(\lambda)$ is singular at the 30 points of the $\mathfrak{S}_6$-orbit of $(1 : \omega : \omega^2 : 1 : \omega : \omega^2)$ with $\omega := e^{2\pi i/3}$. Additional isolated singularities occur only in the following cases

- $\lambda = -1/2$, $|\mathrm{Sing}(X)| = 45$,
- $\lambda = -7/10$, $|\mathrm{Sing}(X)| = 36$,
- $\lambda = -1/6$, $|\mathrm{Sing}(X)| = 40$.

In the case $\lambda = -1/4$, the singularities are not isolated and $X_4(\lambda)$ is so-called *Igusa quartic*.

As in Example 6.3 one can see that $\mathrm{r}(X_4(\lambda))^{\mathfrak{S}_6} = 1$. For $\lambda \notin \{-1/2, -7/10, -1/6\}$ the variety $X_4(\lambda)$ is not rational [3]. For $\lambda = -1/2$ the variety $X_4(\lambda)$ is called *Burkhardt quartic*. It is rational. For $\lambda = -7/10$ and $-1/6$, $X_4(\lambda)$ is also rational [5].

Example 6.5 (double quadric). Let $X = X_{2\cdot 4} \subset \mathbb{P}(1^6, 2)$ is given by

$$\sum_{i=1}^{6} x_i = \sum_{i=1}^{6} x_i^2 = y^2 - \sum_{i=1}^{6} x_i^4 = 0,$$

where x_i and y are coordinates in $\mathbb{P}(1^6, 2)$ with $\deg x_i = 1$, $\deg y = 2$. This $X_{2\cdot 4}$ has 30 nodes and $\mathrm{r}(X_{2\cdot 4}) = 1$. This variety was studied in [16]. It is not rational.

Example 6.6 (cubic complex). Consider the following complete intersections of a quadric and a cubic $X_6 \subset \mathbb{P}^5$:

$$X_6(\lambda) : \sum_{i=1}^6 x_i^2 = \sum_{i=1}^6 x_i^3 - \lambda\Big(\sum_{i=1}^6 x_i\Big)^3 = 0,$$

$$X_6' : 6\sum_{i=1}^6 x_i^2 - \Big(\sum_{i=1}^6 x_i\Big)^2 = \sum_{i=1}^6 x_i^3 = 0.$$

Then these X_6's are Fano threefolds with at worst Gorenstein terminal singularities. Let $\lambda_0 := -1/18$, let λ_1 be a root of $180\lambda^2 + 20\lambda + 1$, and let λ_2 be a root of $288\lambda^2 + 32\lambda + 1$. Then

(i) X_6' and $X_6(\lambda)$ for $\lambda \notin \{\lambda_0, \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2\}$ are smooth,
(ii) $|\mathrm{Sing}(X(\lambda_1))| = |\mathrm{Sing}(X(\bar{\lambda}_1))| = 6$,
(iii) $|\mathrm{Sing}(X(\lambda_2))| = |\mathrm{Sing}(X(\bar{\lambda}_2))| = 15$,
(iv) $|\mathrm{Sing}(X(\lambda_0))| = 20$.

In all cases $\mathrm{r}(X)^{\mathfrak{S}_6} = 1$. The general variety $X_6(\lambda)$ in the pencil is not rational [2].

7 del Pezzo Threefolds

Recall that a *del Pezzo variety* is a Fano variety X with at worst canonical Gorenstein singularities such that the canonical class K_X is divisible by $\dim(X) - 1$ in the group $\mathrm{Pic}(X)$. In this section, we consider only del Pezzo threefolds with at worst terminal Gorenstein singularities. In this case, $-K_X = 2H$, where H is am ample Cartier divisor. Usually, $\mathbb{P}^3$ is not considered as a del Pezzo threefold. The self-intersection number $d := H^3 = -K_X^3/8$ is called the *degree* of X.

Example 7.1 Let $G = \mathfrak{A}_6 \subset \mathrm{PGL}_3(\mathbb{C})$ be the Valentiner group (see Lemma 2.6). Then G acts faithfully on the variety X_6 of complete flags on $\mathbb{P}^2$. This X_6 is a smooth del Pezzo variety of degree 6 [8, 11]. Note however that $\rho(X_6) = 2$ and the induced action of $\mathfrak{A}_6$ on $\mathrm{Pic}(X_6)$ is trivial. Therefore, X_6 is not an $\mathfrak{A}_6$-Fano threefold.

Proposition 7.2 *Assume that $\mathfrak{A}_6$ faithfully acts on a del Pezzo threefold X (here we do not assume that X is a G-Fano). Then X is equivariantly isomorphic to one of the following varieties:*

(i) $X = X_3 \subset \mathbb{P}^4$ is the Segre cubic (see Example 6.3);
(ii) X is the variety of complete flags on $\mathbb{P}^2$ (see Example 7.1).

Proof Let d be the degree of X. It is easy to see by Riemann-Roch that

$$\dim H^0\big(X, -\tfrac{1}{2}K_X\big) = d + 2.$$

The action of the group $\mathfrak{A}_6$ on X lifts to an action of its central extension (double cover) $\tilde{\mathfrak{A}}_6$ by $\boldsymbol{\mu}_2$ on $H^0(X, -\frac{1}{2}K_X)$. Recall that there exists an exceptional isomorphism

$$\mathfrak{A}_6 \simeq \mathrm{PSL}_2(\mathbf{F}_9)$$

(see e.g. [19, Sect. 3.3.5]). Thus a unique central extension $\tilde{\mathfrak{A}}_6$ of $\mathfrak{A}_6$ by $\boldsymbol{\mu}_2$ can be identified with $\mathrm{SL}_2(\mathbf{F}_9)$.

Lemma 7.3 *Let X be a del Pezzo threefold that admit a faithful action of $\mathfrak{A}_6$. Then the linear system $|-\frac{1}{2}K_X|$ contains no invariant members.*

Proof Suppose that $S \in |-\frac{1}{2}K_X|$ is an invariant divisor. Then $-(K_X + S)$ is ample, we can apply quite standard connectedness arguments of Shokurov (see, e.g., [12, Lemma B.5]): for a suitable G-invariant boundary D, the pair (X, D) is lc, the divisor $-(K_X + D)$ is ample, and the minimal locus V of log canonical singularities is also G-invariant. Moreover, V is either a point or a smooth rational curve. By Lemma 2.7 the group $\mathfrak{A}_6$ has no fixed points. Hence, $\mathfrak{A}_6 \subset \mathrm{Aut}(\mathbb{P}^1)$, a contradiction. □

The dimensions of irreducible representations of $\tilde{\mathfrak{A}}_6 = \mathrm{SL}_2(\mathbf{F}_9)$ are as follows: 1, 4, 5, 8, 9, 10. By Lemma 7.3 $H^0(X, -\frac{1}{2}K_X)$ has no one-dimensional subrepresentations. Note that $1 \le d \le 7$ (see e.g. [11]). Therefore, $d \in \{2, 3, 6, 7\}$. Consider these possibilities one by one.

Case $d = 2$. In this case the half-canonical map is a double cover $\pi : X \to \mathbb{P}^3 = \mathbb{P}(H^0(X, -\frac{1}{2}K_X)^\vee)$ whose branch divisor $B \subset \mathbb{P}^3$ is a quartic.

But it is easy to compute that the group $\tilde{\mathfrak{A}}_6 = \mathrm{SL}_2(\mathbf{F}_9)$ has no invariants $0 \ne \phi \in \mathrm{S}^4 H^0(X, -\frac{1}{2}K_X)^\vee$.

Case $d = 3$. In this case $X = X_3 \subset \mathbb{P}^4$ is a cubic. The action of $\tilde{\mathfrak{A}}_6$ on $H^0(X, -\frac{1}{2}K_X)$ is not faithful; it is induced from the standard representation of $\mathfrak{A}_6$ on $\mathbb{C}^5$. Then there is exactly one invariant hypersurface of degree 3 (see 7.2(i)).

Case $d = 6$. By [11] X has at most one singular point. Then by Lemma 2.7 X is smooth. Assume that $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then the induced representation of $\mathfrak{A}_6$ on $\mathrm{Pic}(X)$ is trivial. Hence the group $\mathfrak{A}_6$ effectively acts on the factors of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Clearly, this is impossible. Therefore, $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then by the classification [8] X is unique up to isomorphism, and it can be realized as the variety of complete flags on $\mathbb{P}^2$. We get the case 7.2(ii).

Case $d = 7$. In this case the variety X is smooth and isomorphic to the blowup of a point on $\mathbb{P}^3$ (see e.g. [11]). Thus the action of $\mathfrak{A}_6$ descends to $\mathbb{P}^3$. On the other hand, this action has no fixed points by Lemma 2.7, a contradiction. □

Corollary 7.4 *Assume that the group $\mathfrak{S}_6$ faithfully acts on a del Pezzo threefold X of degree d. Then X is equivariantly isomorphic to the Segre cubic (Example 6.3).*

Proof The action of the subgroup $\mathfrak{A}_6 \subset \mathfrak{S}_6$ is described in Proposition 7.2. It is sufficient to show that the case 7.2(ii) does not occur. Assume that X is isomorphic to the variety of complete flags on $\mathbb{P}^2$. In this case the linear system $|-\frac{1}{2}K_X|$ defines an embedding $X = X_6 \subset \mathbb{P}^7$, where the space $\mathbb{P}^7$ can be identified with the projectivization of $H^0(X, -\frac{1}{2}K_X)^\vee$. The action of the group $\mathfrak{S}_6$ on $X = X_6 \subset \mathbb{P}^7$ lifts to an action of a double cover $\tilde{\mathfrak{S}}_6$ on $H^0(X, -\frac{1}{2}K_X)^\vee$. By Lemma 7.3 the representation of $\tilde{\mathfrak{S}}_6$ on $H^0(X, -\frac{1}{2}K_X)^\vee$ has no one-dimensional subrepresentations. Then $H^0(X, -\frac{1}{2}K_X)^\vee = W' \oplus W''$, where W' and W'' are four-dimensional irreducible faithful representations. Thus $\mathbb{P}(H^0(X, -\frac{1}{2}K_X)^\vee)$ contains two disjoint $\mathfrak{S}_6$-invariant three-dimensional subspaces $\mathbb{P}(W')$ and $\mathbb{P}(W'')$. Recall that $X = X_6 \subset \mathbb{P}^7$ is an intersection of quadrics. Assume that $\mathbb{P}(W') \cap X$ is not empty. If $\dim(\mathbb{P}(W') \cap X) = 2$, then $\mathbb{P}(W') \cap X$ is a $\mathfrak{S}_6$-invariant quadric in $\mathbb{P}(W') = \mathbb{P}^3$. Clearly, this is impossible. If $\dim(\mathbb{P}(W') \cap X) = 1$, then $\mathbb{P}(W') \cap X$ contains a $\mathfrak{S}_6$-invariant curve of degree ≤ 4 and the genus of this curve is at most 1. Again this is impossible. Let $\dim(\mathbb{P}(W') \cap X) = 0$. Then $\mathbb{P}(W') \cap X = \{P_1, \ldots, P_k\}$, where $k \leq 8$. By Corollary 2.4 and Lemma 2.7 the stabilizer $G_1 \subset \mathfrak{S}_6$ of P_1 is isomorphic to $\mathfrak{S}_5$. Note that the representation of G_1 in the tangent space $T_{P_1,X}$ is faithful. On the other hand, the group $\mathfrak{S}_5$ has no faithful three-dimensional representations, a contradiction.

Therefore, $\mathbb{P}(W') \cap X = \varnothing$. Then the projection $p : X \to \mathbb{P}(W'')$ from $\mathbb{P}(W')$ must be a $\mathfrak{S}_6$-equivariant finite morphism. By the Hurwitz formula

$$K_X = p^* K_{\mathbb{P}(W'')} - R$$

where R is the ramification divisor. Since $p^* K_{\mathbb{P}(W'')} \sim -4H \sim 2K_X$, where H is a hyperplane section of X, the divisor R cannot be effective, a contradiction. □

References

1. Blanc, J., Cheltsov, I., Duncan, A., Prokhorov, Y.: Finite quasisimple groups acting on rationally connected threefolds. Math. Proc. Cambridge Phil. Soc., published online. https://doi.org/10.1017/S030500412200041X
2. Beauville, A.: Non-rationality of the symmetric sextic Fano threefold. In: Geometry and Arithmetic, EMS Ser. Congr. Rep., pp. 57–60. European Mathematical Society, Zürich (2012)
3. Beauville, A.: Non-rationality of the $\mathfrak{S}_6$-symmetric quartic threefolds. Rend. Semin. Mat. Univ. Politec. Torino **71**(3-4), 385–388 (2013)
4. Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With Computer Assisted from J.G. Thackray, vol. XXXIII, 252 p. Clarendon Press, Oxford (1985)
5. Cheltsov, I., Shramov, C.: Two rational nodal quartic 3-folds. Q. J. Math. **67**(4), 573–601 (2016)
6. Dolgachev, I.V., Iskovskikh, V.A.: Finite subgroups of the plane Cremona group. In: Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin, volume 269 of Progress in Mathematics, vol. I, pp. 443–548. Birkhäuser Boston Inc., Boston, MA (2009)
7. Duncan, A.: Essential dimensions of A_7 and S_7. Math. Res. Lett. **17**(2), 263–266 (2010)
8. Iskovskikh, V.A., Prokhorov, Y.: Fano varieties. Algebraic geometry V, volume 47 of Encyclopaedia of Mathematical Sciences. Springer, Berlin (1999)
9. Kollár, J., Miyaoka, Y., Mori, S., Takagi, H.: Boundedness of canonical **Q**-Fano 3-folds. Proc. Jpn. Acad. Ser. A Math. Sci. **76**(5):73–77 (2000)

10. Prokhorov, Yu.: Simple finite subgroups of the Cremona group of rank 3. J. Algeb. Geom. **21**(3), 563–600 (2012)
11. Prokhorov, Y.: G-Fano threefolds. I. Adv. Geom. **13**(3), 389–418 (2013)
12. Prokhorov, Y.: Del Pezzo fibrations. Appendix to "Five embeddings of one simple group" by I. Cheltsov and C. Shramov. Trans. Am. Math. Soc. **366**, 1289–1331 (2014)
13. Prokhorov, Yu.: On G-Fano threefolds. Izvestiya: Math **79**(4), 795–808 (2015)
14. Prokhorov, Y.: Equivariant minimal model program. Russian Math. Surv. **76**(3), 461–542 (2021)
15. Prokhorov, Y.: Finite groups of birational transformations (2021). to appear in the Proceedings of the 8th ECM. arxiv:2108.13325
16. Przyjalkowski, V., Shramov, C.: Double quadrics with large automorphism groups. Proc. Steklov Inst. Math. **294**, 154–175 (2016)
17. Reid, M.: Young person's guide to canonical singularities. In: Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), volume 46 of Proceedings of Symposia in Pure Mathematics, pp. 345–414. American Mathematical Society, Providence, RI (1987)
18. van der Geer, G.: On the geometry of a Siegel modular threefold. Math. Ann. **260**(3), 317–350 (1982)
19. Wilson, R.A.: The finite simple groups. Graduate Texts in Mathematics, vol. 251. Springer, London Ltd, London (2009)

On Hodge-Riemann Cohomology Classes

Julius Ross and Matei Toma

Abstract We prove that Schur classes of nef vector bundles are limits of classes that have a property analogous to the Hodge-Riemann bilinear relations. We give a number of applications, including (1) new log-concavity statements about characteristic classes of nef vector bundles (2) log-concavity statements about Schur and related polynomials (3) another proof that normalized Schur polynomials are Lorentzian.

Keywords Schur classes · Vector bundles · Hodge-Riemann relations

Mathematics Subject Classification: 14C17 · 14J60 · 32J27 · 52A40

1 Introduction

Since the dawn of time, human beings have asked some fundamental questions: who are we? why are we here? is there life after death? Unable to answer any of these, in this paper we will consider cohomology classes on a complex projective manifold that have a property analogous to the Hard-Lefschetz Theorem and Hodge-Riemann bilinear relations.

To state our results let X be a projective manifold of dimension $d \geq 2$. We say that a cohomology class $\Omega \in H^{d-2,d-2}(X;\mathbb{R})$ has the *Hodge-Riemann property* if the intersection form

$$Q_\Omega(\alpha,\alpha') := \int_X \alpha\Omega\alpha' \text{ for } \alpha,\alpha' \in H^{1,1}(X;\mathbb{R})$$

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has signature $(+, -, -, \dots, -)$. We write

$$\mathrm{HR}(X) = \{\Omega \text{ with the Hodge Riemann property}\}$$

and $\overline{\mathrm{HR}}(X)$ for its closure.

This definition is made in light of the fact that the classical Hodge-Riemann bilinear relations say precisely that if L is an ample line bundle on X, then $c_1(L)^{d-2}$ is in $\mathrm{HR}(X)$. A natural question, initiated by Gromov [12], is if there are other cohomology classes that have this property, and our first result answers this in terms of certain characteristic classes of vector bundles.

Theorem ($\subseteq$ Theorem 7.2) *Let E be a nef vector bundle on X and λ be a partition of $d-2$. Then the Schur class $s_\lambda(E)$ lies in $\overline{HR}(X)$.*

In fact we can do better; for each i define the *derived Schur polynomials* $s_\lambda^{(i)}$ by requiring that

$$s_\lambda(x_1 + t, \dots, x_e + t) = \sum_{i=0}^{|\lambda|} s_\lambda^{(i)}(x_1, \dots, x_e) t^i.$$

Theorem ($\subseteq$ Theorem 7.2) *Let E be a nef vector bundle on X and λ be a partition of $d-2+i$. Then the derived Schur class $s_\lambda^{(i)}(E)$ lies in $\overline{HR}(X)$.*

We prove moreover:

- Analogous statements hold for monomials of derived Schur classes of possibly different nef vector bundles (Theorem 7.4).
- If E is perturbed by adding a sufficiently small ample class, then $s_\lambda(E)$ lies in $\mathrm{HR}(X)$ (rather than in just the closure) (Remark 7.3).
- The above holds even in the setting of compact Kähler manifolds, where nefness of E is taken in the metric sense following Demailly-Peternell-Schneider (Theorem 8.3).

*

Our above result is interesting even in the case that $E = \oplus_{i=1}^e L_i$ is a direct sum of ample line bundles, from which we deduce that the Schur polynomial $s_\lambda(c_1(L_1), \dots, c_1(L_e))$ lies in $\overline{\mathrm{HR}}(X)$. As a concrete example, $s_{(1,1)}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$, so if L_1 and L_2 are ample line bundles on a fourfold the class

$$c_1(L_1)^2 + c_1(L_1)c_1(L_2) + c_1(L_2)^2 \in \overline{\mathrm{HR}}(X). \tag{1.1}$$

As already noted, the classical Hodge-Riemann bilinear relations tell us that the classes $c_1(L_1)^2$ and $c_1(L_2)^2$ both lie in $\mathrm{HR}(X)$, and it was proved by Gromov [12] that the mixed term $c_1(L_1)c_1(L_2)$ also lies in $\mathrm{HR}(X)$. However in general having the Hodge-Riemann property is not preserved under taking convex combinations, and thus (1.1) is new.

From these considerations it is natural to ask which universal combinations of characteristic classes of ample (resp. nef) vector bundles lie in $\mathrm{HR}(X)$ (resp. $\overline{\mathrm{HR}}(X)$). Although we do not know the full answer to this, the following is a contribution in this direction.

Theorem ($\subseteq$ Theorem 9.3) *Let E be a nef vector bundle on a projective manifold of dimension d, and λ be a partition of $d-2$. Suppose $\mu_0, \ldots, \mu_{d-2}$ is a Pólya frequency sequence of non-negative real numbers. Then the combination*

$$\sum_{i=0}^{d-2} \mu_i s_\lambda^{(i)}(E) c_1(E)^i$$

lies in $\overline{HR}(X)$.

*

As an application of these results we are able to give various new inequalities between characteristic classes of nef vector bundles. Continuing to assume X is projective of dimension d, let λ and μ be partitions of length $|\lambda|$ and $|\mu|$ respectively and assume $|\lambda| + |\mu| \geq d$.

Theorem (= Theorem 10.5) *Assume E, F are nef vector bundles on X. Then the sequence*

$$i \mapsto \int_X s_\lambda^{(|\lambda|+|\mu|-d-i)}(E) s_\mu^{(i)}(F) \tag{1.2}$$

is log-concave

As a particular case, we get that if E is a nef vector bundle and λ a partition of d, then

$$j \mapsto \int_X s_\lambda^{(j)}(E) c_1(E)^j$$

is log-concave, which as a special case says the map

$$i \mapsto \int_X c_i(E) c_1(E)^{d-i}$$

is also log-concave. One should think of these statements as higher-rank analogs of the Khovanskii-Tessier inequalities. We even get combinatorial applications of this, such as the following:

Corollary (= Corollary 10.10) *Let λ and μ be partitions, and let d be an integer with $d \leq |\lambda| + |\mu|$. Assume $x_1, \ldots, x_e, y_1, \ldots, y_f \in \mathbb{R}_{\geq 0}$. Then the sequence*

$$i \mapsto s_\lambda^{(|\lambda|+|\mu|-d+i)}(x_1, \ldots, x_e) s_\mu^{(i)}(y_1, \ldots, y_f)$$

is log concave.

Corollary (= Corollary *10.12*) *Let λ be a partition and $x_1, \ldots, x_e \in \mathbb{R}_{\geq 0}$. Then the sequence*

$$i \mapsto s_\lambda^{(i)}(x_1, \ldots, x_e)$$

is log-concave.

This last statement has been known for a long time for the partition $\lambda = (e)$, for then the derived Schur polynomials become the elementary symmetric polynomials c_i (see Example 3.2). Then more is true namely, $i \mapsto c_i(x_1, \ldots, x_e)$ is ultra-log concave—a result which is due to Newton [18] (see, for example, [5, Chap. 11] for a modern treatment).

As a final application we show how knowing that Schur classes of nef bundles lie in $\overline{\mathrm{HR}}(X)$ gives another proof of a result of Huh-Matherne-Mészáros-Dizier [13] that the normalized Schur polynomials are Lorentzian.

1.1 Comparison with Previous Work

There is some overlap between Theorem 7.2 and our original work on the subject in [21]. A principal difference is that in [21] we show that derived Schur classes of ample bundles have the Hodge-Riemann property, whereas here we settle in merely showing these classes are limits of classes with this property. So even though logically many of our results follow from [21], the proofs we give here are simpler and substantially shorter. In fact, our account here does not depend on any of the details of [21] and is self-contained relying only on a few standard techniques in the field (as contained say in [16]). The main tools we use are the Bloch-Gieseker theorem, and the cone classes of Fulton-Lazarsfeld that express Schur classes as pushforwards of certain Chern classes (which builds on the determinantal formula of Kempf-Laksov [14]) . The material on the non-projective case in §8, on convex combinations in §9 and on inequalities in §10 is all new.

We refer the reader to [21] for a survey of other works concerning Hodge-Riemann classes. Although there are many places in which log-convexity and Schur polynomials meet (e.g. [4, 10, 13, 15, 19, 20]) we are not aware of any previous inequalities that cover precisely those studied here.

1.2 Organization of the Paper

Sections 2, 3 and 4 contain preliminary material on Schur polynomials, derived Schur polynomials and cone classes. We also include in Sect. 5 a self-contained proof of a theorem of Fulton-Lazarsfeld concerning positivity of (derived) Schur polynomials. The main theorems about derived Schur classes having the Hodge-Riemann property are proved in Sect. 7, and in Sect. 8 we explain how this extends to the non-projective case. In Sect. 9 we consider convex combinations of Hodge-Riemann classes, and in Sect. 10 we give our application to inequalities and our proof that normalized Schur polynomials are Lorentzian.

2 Notation and Convention

We work throughout over the complex numbers. For the majority of the paper we will take X to be a projective manifold (which we always assume is connected), and E a vector bundle (which we always assume to be algebraic). Given such a vector bundle E we denote by $\pi : \mathbb{P}(E) \to X$ the space of one-dimensional quotients of E, and by $\pi : \mathbb{P}_{sub}(E) \to X$ the space of one-dimensional subspaces of E. We say that a vector bundle E is ample (resp. nef) if the hyperplane bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is ample (resp. nef).

We will make use of the formalism of $\mathbb{Q}$-twisted bundles (see [16, Sect. 6.2, 8.1.A], [17, p. 457]). Given a vector bundle E on X of rank e and an element $\delta \in N^1(X)_{\mathbb{Q}}$ the $\mathbb{Q}$-twisted bundle denoted $E\langle\delta\rangle$ is a formal object understood to have Chern classes defined by the rule

$$c_p(E\langle\delta\rangle) := \sum_{k=0}^{p} \binom{e-k}{p-k} c_k(E)\delta^{p-k} \text{ for } 0 \leq p \leq e. \tag{2.1}$$

Here and henceforth we abuse notation and write δ also for its image under $N^1(X)_{\mathbb{Q}} \to H^2(X;\mathbb{Q})$, so the above intersection is taking place in the cohomology ring $H^*(X)$.

By the rank of $E\langle\delta\rangle$ we mean the rank of E. The above definition is made so if $\delta = c_1(L)$ for a line bundle L on X then

$$c_p(E\langle c_1(L)\rangle) = c_p(E \otimes L).$$

The splitting principle provides for any vector bundle E a morphism $p : X' \to X$ such that $p^*H^*(X)$ injects into $H^*(X')$ and so that $p^*E = \bigoplus L_i$ is a direct sum of line bundles. In this situation we call $x_i := c_1(L_i)$ the *Chern roots* of E. So, if E has Chern roots given by $x_1, \ldots, x_e$ then $E\langle\delta\rangle$ is understood to have Chern roots $x_1 + \delta, \ldots, x_e + \delta$. The twist of an $\mathbb{Q}$-twisted bundle is given by the rule $E\langle\delta\rangle\langle\delta'\rangle = E\langle\delta + \delta'\rangle$. That (2.1) continues to hold when E is an $\mathbb{Q}$-twisted bundle is an elementary calculation - for convenience of the reader we omit the proof.

We say that $E\langle\delta\rangle$ is ample (resp. nef) if the class $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \pi^*\delta$ is ample (resp. nef) on $\mathbb{P}(E)$.

Suppose $p(x_1, \ldots, x_e)$ is a homogeneous symmetric polynomial of degree d' and E is a $\mathbb{Q}$-twisted vector bundle of rank E on X with Chern roots $\tau_1, \ldots, \tau_e$. Then we have the well-defined characteristic class

$$p(E) := p(\tau_1, \dots, \tau_e) \in H^{d',d'}(X; \mathbb{R}).$$

By abuse of notation we let c_i denote the ith elementary symmetric polynomial, so $c_i(E) \in H^{i,i}(X; \mathbb{R})$ is unambiguously defined as the ith-Chern class of E.

3 Derived Schur Classes

By a partition λ of an integer $b \geq 1$ we mean a sequence $0 \leq \lambda_N \leq \cdots \leq \lambda_1$ such that $|\lambda| := \sum_i \lambda_i = b$. For such a partition, the Schur polynomial s_λ is the symmetric polynomial of degree $|\lambda|$ in $e \geq 1$ variables given by

$$s_\lambda = \det \begin{pmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+N-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+N-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{\lambda_N-N+1} & c_{\lambda_N-N+2} & \cdots & c_{\lambda_N} \end{pmatrix}$$

where c_i denotes the i-th elementary symmetric polynomial. The coefficients of s_λ count the number of certain semi-standard Young tableau (the reader is referred to [7] for more background concerning Schur polynomials). In particular, Schur polynomials are monomial positive by which we mean when written as a sum of monomials each (non-trivial) coefficient is strictly positive.

We will have use for the following symmetric polynomials associated to Schur polynomials.

Definition 3.1 (*Derived Schur polynomials*) Let λ be a partition. For any $e \geq 1$ we define $s_\lambda^{(i)}(x_1, \dots, x_e)$ for $i = 0, \dots, |\lambda|$ by requiring that

$$s_\lambda(x_1 + t, \dots, x_e + t) = \sum_{i=0}^{|\lambda|} s_\lambda^{(i)}(x_1, \dots, x_e) t^i \text{ for all } t \in \mathbb{R}.$$

In fact $s_\lambda^{(i)}$ depends also on e but we drop that from the notation. By convention we set $s_\lambda^{(i)} = 0$ for $i \notin \{0, \dots, |\lambda|\}$. For $0 \leq i \leq |\lambda|$, clearly $s_\lambda^{(i)}$ is a homogeneous symmetric polynomial of degree $|\lambda| - i$ and $s_\lambda^{(0)} = s_\lambda$.

Thus for any $\mathbb{Q}$-twisted vector bundle E of rank e we have classes

$$s_\lambda^{(i)}(E) \in H^{|\lambda|-i,|\lambda|-i}(X; \mathbb{R}),$$

and by construction if $\delta \in N^1(X)_{\mathbb{Q}}$ then

$$s_\lambda(E\langle\delta\rangle) = \sum_{i=0}^{|\lambda|} s_\lambda^{(i)}(E)\delta^i.$$

Example 3.2 (Chern classes) Consider the partition of $\lambda = (p)$ consisting of just one integer. Then $s_\lambda = c_p$, and from standard properties of Chern classes of a tensor product if rk $E = e \geq p$ then

$$s_\lambda^{(i)}(E) = \binom{e-p+i}{i} c_{p-i}(E) \text{ for all } 0 \leq i \leq p.$$

Example 3.3 (Derived Schur polynomials of Low degree) We list some of the derived Schur classes of low degree for a bundle E of rank e. First

$$s_{(1)} = c_1, \quad s_{(1)}^{(1)} = e$$

and for $e \geq 2$,

$$\begin{array}{lll} s_{(2,0)} = c_2 & s_{(2,0)}^{(1)} = (e-1)c_1 & s_{(2,0)}^{(2)} = \binom{e}{2} \\ s_{(1,1)} = c_1^2 - c_2, & s_{(1,1)}^{(1)} = (e+1)c_1 & s_{(1,1)}^{(2)} = \binom{e+1}{2} \end{array}$$

and for $e \geq 3$,

$$\begin{array}{lll} s_{(3,0,0)} = c_3 & s_{(3,0,0)}^{(1)} = (e-2)c_2 & s_{(3,0,0)}^{(2)} = \binom{e-1}{2} c_1 \\ & & s_{(3,0,0)}^{(3)} = \binom{e}{3} \\ s_{(2,1,0)} = c_1c_2 - c_3 & s_{(2,1,0)}^{(1)} = 2c_2 + (e-1)c_1^2 & s_{(2,1,0)}^{(2)} = (e^2-1)c_1 \\ & & s_{(2,1,0)}^{(3)} = 2\binom{e+1}{3} \\ s_{(1,1,1)} = c_1^3 - 2c_1c_2 + c_3 & s_{(1,1,1)}^{(1)} = (e+2)(c_1^2 - c_2) & s_{(1,1,1)}^{(2)} = \binom{e+2}{2} c_1 \\ & & s_{(1,1,1)}^{(3)} = \binom{e+2}{3} \end{array}$$

Example 3.4 (Lowest Degree Derived Schur Classes) Suppose $e \geq \lambda_1$. Then we can write the Schur polynomial as a sum of monomials

$$s_\lambda(x_1, \ldots, x_e) = \sum_{|\alpha|=|\lambda|} c_\alpha x_1^{\alpha_1} \cdots x_e^{\alpha_e}$$

where $c_\alpha \geq 0$ for all α (in fact the c_α count the number of semistandard Young tableaux of weight α whose shape is conjugate to λ). Since $e \geq \lambda_1$, s_λ is not identically zero, so at least one of the c_α is strictly positive. Thus in the expansion

$$s_\lambda(x_1+t,\dots,x_e+t)=\sum_{i=0}^{|\lambda|} s_\lambda^{(i)}(x_1,\dots,x_e)t^i$$

the coefficient in front of $t^{|\lambda|}$ is strictly positive, i.e. $s_\lambda^{(|\lambda|)}>0$.

So, in terms of characteristic classes, if E has rank at least λ_1 then

$$s_\lambda^{(|\lambda|)}(E)\in H^0(X;\mathbb{R})=\mathbb{R}$$

is strictly positive.

4 Cone Classes

We will rely on a construction exploited by Fulton-Lazarsfeld that express Schur classes as the pushforward of Chern classes, and we include a brief description here. Let E be a vector bundle of rank e on X of dimension d and suppose $0\leq\lambda_N\leq\lambda_{N-1}\leq\cdots\leq\lambda_1$ is a partition of length $|\lambda|=b\geq 1$ and $\lambda_1\leq e$. Set $a_i:=e+i-\lambda_i$ and fix a vector space V of dimension $e+N$. Then it is possible to find a nested sequence of subspaces $0\subsetneq A_1\subsetneq A_2\subsetneq\cdots\subsetneq A_N\subset V$ with $\dim(A_i)=a_i$.

By abuse of notation we also let V denote the trivial bundle over X. We set $F:=V^*\otimes E=\mathrm{Hom}(V,E)$ and let $f+1=\mathrm{rk}(F)=e(e+N)$. Then inside F define

$$\hat{C}:=\{\sigma\in\mathrm{Hom}(V,E):\dim\ker(\sigma(x))\cap A_i\geq i \text{ for all } i=1,\dots,N \text{ and } x\in X\}$$

which is a cone in F. Finally set

$$C=[\hat{C}]\subset\mathbb{P}_{\mathrm{sub}}(F).$$

Proposition 4.1 *C has codimension b and dimension $d+f-b$, has irreducible fibers over X and is flat over X (in fact it is locally a product). Moreover if*

$$0\to\mathcal{O}_{\mathbb{P}_{\mathrm{sub}}(F)}(-1)\to\pi^*F\to U\to 0 \tag{4.1}$$

is the tautological sequence then

$$s_\lambda(E)=\pi_*c_f(U|_C). \tag{4.2}$$

Proof This is described by Fulton-Lazarsfeld in [9]. An account (that is written for the the case $|\lambda|=d$) can be found in [16, (8.12)] and an account for general $|\lambda|$ is given in [21, Proposition 5.1] that is based on [8]. We remark that in [21, Proposition 5.1] we made the additional assumption that $N\geq b$ and $e\geq 2$, but have since realized these are not necessary (we used this to ensure that $f \geq b$, but this actually follows immediately from $e \geq \lambda_1$). □

This extends to $\mathbb{Q}$-twisted bundles $E' = E\langle\delta\rangle$. Here we identify

$$P' := \mathbb{P}_{\rm sub}(F\langle\delta\rangle) \xrightarrow{\pi} X$$

with $\mathbb{P}_{\rm sub}(F) \xrightarrow{\pi} X$ but the quotient bundle U on P' is replaced by $U' := U\langle\pi^*\delta\rangle$. We consider the same cone $[C] \subset P'$. Then (4.2) still holds in the sense that

$$s_\lambda(E') = \pi_* c_f(U'|_C). \tag{4.3}$$

To see this, observe that as $\delta \in N^1(X)_{\mathbb{Q}}$ we have $\delta = \frac{1}{m}c_1(L)$ for some $m \in \mathbb{Z}$ and line bundle L. Then for t divisible by m

$$\pi_*(c_f(U\langle t\pi^*\delta\rangle|_C) = \pi_* c_f(U \otimes \pi^* L^{t/m}|_C) = s_\lambda(E \otimes L^{t/m}) = s_\lambda(E\langle t\delta\rangle) \tag{4.4}$$

where the second equality uses (4.2). But both sides of (4.4) are polynomials in t, so since this equality holds for infinitely many t it must hold for all $t \in \mathbb{Q}$, in particular when $t = 1$ which gives (4.3).

A key feature we will rely on is that if E' is assumed to be nef then so is U'. For if E' is nef then so is $F' := F\langle\delta\rangle$ and the formal surjection $F' \to U'$ coming from (4.1) implies that U' is also nef (see [16, Lemma 6.2.8] for these properties of nef $\mathbb{Q}$-twisted bundles).

Another extension is to the product of Schur classes of possibly different vector bundles $E_1, \ldots, E_p$ on X. Let $\lambda^1, \ldots, \lambda^p$ be partitions and assume $\mathrm{rk}(E_j) \geq \lambda_1^j$ for $j = 1, \ldots, p$. We consider again the corresponding cones C_i that sit inside $F_i := \mathrm{Hom}(V_i, E_i)$ for some vector space V_i. We may consider the fiber product $C := C_1 \times_X C_2 \times_X \cdots \times_X C_p$ inside $\bigoplus_j \mathrm{Hom}(V_i, E_i) =: F$ and its projectivization $[C] \subset \mathbb{P}_{\rm sub}(F)$. Then, using that each C_i is flat over X, if U is the tautological vector bundle on $\mathbb{P}_{\rm sub}(F)$ of rank f we have

$$\pi_* c_f(U|_C) = \prod_j s_{\lambda^j}(E_j) \tag{4.5}$$

(see [16, 8.1.19], [9, Sect. 3c]).

5 Fulton-Lazarsfeld Positivity

Using the cone construction we quickly get the following positivity statement, which is essentially a weak version of a result of Fulton-Lazarsfeld [9]. For the reader's convenience we include the short proof here.

Proposition 5.1 *Let X be smooth and projective of dimension d, λ be a partition of length $d+i$ for some $i \geq 0$ and E be an $\mathbb{Q}$-twisted nef vector bundle. Then $\int_X s_\lambda^{(i)}(E) \geq 0$.*

Proof We first claim that if E is a nef $\mathbb{Q}$-twisted bundle of rank d on an irreducible projective variety X of dimension d then $\int_X c_d(E) \geq 0$. By taking a resolution of singularities we may assume X is smooth. Let h be an ample class on X. By the Bloch-Gieseker Theorem [2] we have $\int_X c_d(E\langle th\rangle) \neq 0$ for all $t > 0$ since $E\langle th\rangle$ is ample (here we allow t to be irrational extending the notation in the obvious way, and observe that although the original Bloch Gieseker result is not stated for twisted bundles the same proof works in this setting, see [16, p. 113] or Sect. 8). Expanding this as a polynomial in t gives

$$0 \neq \int_X c_d(E) + tc_{d-1}(E)h + \cdots + t^d h^d \text{ for all } t \in \mathbb{R}_{>0}.$$

Clearly this polynomial is strictly positive for $t \gg 0$, and hence since it is nowhere-vanishing, is strictly positive for all $t > 0$. In particular $\int_X c_d(E) \geq 0$ as claimed.

To prove the Proposition, we may assume $e := \mathrm{rk}(E) \geq \lambda_1$ else $s_\lambda(E) = 0$ and the statement is trivial. When $|\lambda| = d$, (4.3) gives a map $\pi : C \to X$ from an irreducible variety C of dimension n and a nef $\mathbb{Q}$-twisted bundle U of rank n so that $\pi_* c_n(U) = s_\lambda(E)$. So by the previous paragraph $\int_X s_\lambda(E) = \int_C c_n(U) \geq 0$.

Finally suppose $i \geq 0$ and $|\lambda| = d+i$. Set $\hat{X} = X \times \mathbb{P}^i$ and $\tau = c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Since $|\lambda| = \dim(\hat{X})$ we have

$$0 \leq \int_{\hat{X}} s_\lambda(E\langle\tau\rangle) = \int_{\hat{X}} \sum_{j=0}^{|\lambda|+i} s_\lambda^{(j)}(E)\tau^j = \int_X s_\lambda^{(i)}(E) \int_{\mathbb{P}^i} \tau^i = \int_X s_\lambda^{(i)}(E).$$

□

Corollary 5.2 *Let X be smooth and projective of dimension d, λ be a partition of length $d+i-2$, let E be a nef $\mathbb{Q}$-twisted bundle of rank $e \geq \lambda_1$ and h be an ample class on X. Then $\int_X s_\lambda^{(i)}(E)h^2 \geq 0$.*

Proof Rescale so h is very ample, and apply the previous theorem to the restriction of E to the intersection of two general elements in the linear series defined by h. □

Remark 5.3 By passing to a resolution of singularities, one sees that the statement of Propositon 5 and Corollary 5.2 extend to the case that X is irreducible and projective but not necessarily smooth.

Remark 5.4 (Derived Schur Polynomials are Numerically Positive) If $|\lambda| = d+i$ then $\int_X s_\lambda^{(i)}(E) \geq 0$ for all nef vector bundles E on any irreducible projective variety X of dimension d. That is, $s_\lambda^{(i)}$ is a numerically positive polynomial in the sense of Fulton-Lazarsfeld, and hence by their main result [9, Theorem I] we deduce $s_\lambda^{(i)}$ can be written as a non-negative linear combination of the Schur polynomials $\{s_\mu : |\mu| = d\}$. This answers a question of Xiao [22, p. 10].

Remark 5.5 (Monomials of Derived Schur Classes) It is easy to extend this to monomials of derived Schur polynomials. That is, if $E_1, \ldots, E_p$ are nef bundles on X and $\lambda^1, \ldots, \lambda^p$ are partitions such that $\sum_j |\lambda^j| = d$ then

$$\int_X \prod_j s_{\lambda^j}(E_j) \geq 0. \tag{5.1}$$

We simply repeat the proof of Proposition 5.1 using (4.5) in place of (4.3)). For the derived case suppose we also have integers $i_1, \ldots, i_p$ and that our partitions are such that $\sum_j |\lambda^{(j)}| - i_j = d$. Then

$$\int_X \prod_j s_{\lambda^j}^{(i_j)}(E_j) \geq 0. \tag{5.2}$$

To see this consider the product $\hat{X} := X \times \prod_j \mathbb{P}^{i_j}$ and let τ_j be the pullback of the hyperplane class in $\mathbb{P}^{i_j}$ to $\hat{X}$. Then (5.1) applies to the class $\prod_j s_{\lambda^j}(E_j(\tau_j))$. Expanding this as a symmetric polynomial in the τ_j the coefficient of $\prod_j \tau_j^{i_j}$ is precisely $\prod_j s_{\lambda^j}^{(i_j)}(E_j)$ so (5.2) follows. The analog of Corollary 5.2 also holds for monomials of derived Schur polynomials.

6 Hodge-Riemann Classes

Let X be a projective smooth variety dimension d and let $\Omega \in H^{d-2,d-2}(X;\mathbb{R})$. This defines an intersection form

$$Q_\Omega(\alpha, \alpha') = \int_X \alpha\Omega\alpha' \text{ for } \alpha, \alpha' \in H^{1,1}(X;\mathbb{R}).$$

Definition 6.1 *(Hodge-Riemann Property)* We say that a bilinear form Q on a finite dimensional vector space has the *Hodge-Riemann property* if Q is non-degenerate and has precisely one positive eigenvalue. We say that $\Omega \in H^{d-2,d-2}(X;\mathbb{R})$ has the Hodge-Riemann property if Q_Ω does, and denote by $\mathrm{HR}(X)$ denote the set of all Ω with this property.

Definition 6.2 *(Weak Hodge-Riemann Property)* A bilinear form Q on a finite dimensional vector space is said to have the *weak Hodge-Riemann property* if it is a limit of bilinear forms that have the Hodge-Riemann property. We say that Ω has the weak Hodge-Riemann property if Q_Ω does, and denotes by $\mathrm{HR}_w(X)$ the set of Ω with this property.

So Q has the weak Hodge-Riemann property if and only if it has one eigenvalue that is non-negative, and all the others are non-positive. Clearly

$$\overline{\mathrm{HR}}(X) \subset \mathrm{HR}_w(X)$$

but we do not claim these are equal (the issue being that in principle Q_Ω could be the limit of bilinear forms with the Hodge-Riemann property that do not come from classes in $H^{d-2,d-2}(X;\mathbb{R})$). If h is ample then by the classical Hodge-Riemann bilinear relations $h^{d-2} \in \mathrm{HR}(X)$, and so $\mathrm{HR}_w(X)$ is a non-empty closed cone inside $H^{d-2,d-2}(X;\mathbb{R})$.

It is convenient to work with $\mathrm{HR}_w(X)$ as it behaves well with respect to pullbacks and pushforwards. This is captured by the following simple piece of linear algebra.

Lemma 6.3 *Let $f : V \to W$ be a linear map of vector spaces and Q_V and Q_W be bilinear forms on V and W respectively such that*

$$Q_W(f(v), f(v')) = Q_V(v, v') \text{ for all } v, v' \in V.$$

Suppose that Q_W has the weak Hodge-Riemann property and there is a $v_0 \in V \setminus \{0\}$ with $Q_V(v_0, v_0) \geq 0$. Then Q_V has the weak Hodge-Riemann property.

Proof Let $N = \ker(f)$. Then N is orthogonal to all of V with respect to Q_V. The signature on a complementary subspace to N is induced by Q_W. Thus Q_V can only be negative semi-definite, or have the weak Hodge-Riemann property, and the assumption that $Q_V(v_0, v_0) \geq 0$ means it is the latter case that occurs. □

Lemma 6.4 (Pullbacks) *Let $\pi : X' \to X$ be a surjective map between smooth varieties of dimension d. Let $\Omega \in H^{d-2,d-2}(X,\mathbb{R})$ and suppose there is an $h \in H^{1,1}(X;\mathbb{R}) \setminus \{0\}$ with $\int_X \Omega h^2 \geq 0$ and that $\pi^*\Omega \in \mathrm{HR}_w(X')$. Then $\Omega \in \mathrm{HR}_w(X)$.*

Proof This follows from Lemma 6.3 applied to $\pi^* : H^{1,1}(X;\mathbb{R}) \to H^{1,1}(X';\mathbb{R})$ since $Q_{\pi^*\Omega}(\pi^*\alpha, \pi^*\alpha') = \int_{X'} \pi^*(\Omega\alpha\alpha') = \deg(\pi) \int_X \Omega\alpha\alpha' = \deg(\pi)\, Q_\Omega(\alpha, \alpha')$. □

Lemma 6.5 (Pushforwards) *Let $\pi : X' \to X$ be a surjective map between smooth varieties. Let $\Omega' \in \mathrm{HR}_w(X')$ and suppose there is an $h \in H^{1,1}(X;\mathbb{R}) \setminus \{0\}$ with $\int_X (\pi_*\Omega')h^2 \geq 0$. Then $\pi_*\Omega' \in \mathrm{HR}_w(X)$.*

Proof This follows from Lemma 6.3 applied to $\pi^* : H^{1,1}(X;\mathbb{R}) \to H^{1,1}(X';\mathbb{R})$ since from the projection formula,

$$Q_{\Omega'}(\pi^*\alpha, \pi^*\alpha') = \int_{X'} \Omega'(\pi^*\alpha)(\pi^*\alpha') = \int_X \pi_*\Omega'\alpha\alpha' = Q_{\pi_*\Omega}(\alpha, \alpha').$$

□

We will need the following variant that allows for an intermediate space that might not be smooth.

Lemma 6.6 *Let X, Y, Z be irreducible projective varieties with morphisms $Z \xrightarrow{\sigma} Y \xrightarrow{\pi} X$ and assume that Z and X are smooth. Let $d = \dim X$ and assume Z and Y are of the same dimension n and that σ is surjective. Let $\Omega \in H^{2n-4}(Y;\mathbb{R})$ be such that $\Omega' := \pi_*\Omega \in H^{d-2,d-2}(X;\mathbb{R})$. Assume*

(i) $\sigma^*\Omega \in \mathrm{HR}_w(Z)$.
(ii) There exists an $h \in H^{1,1}(X;\mathbb{R}) \setminus \{0\}$ *such that* $\int_X (\pi_*\Omega)h^2 \geq 0$.

Then $\pi_*\Omega \in \mathrm{HR}_w(X)$.

Proof Let $p = \pi \circ \sigma : Z \to X$. By the projection formula

$$\begin{aligned} Q_{\sigma^*\Omega}(p^*\alpha, p^*\alpha') &= \int_Z \sigma^*\Omega p^*\alpha p^*\alpha' = \int_Z \sigma^*\Omega\sigma^*\pi^*\alpha\sigma^*\pi^*\alpha' \\ &= \deg(\sigma)\int_Y \Omega\pi^*\alpha\pi^*\alpha' = \deg(\sigma)\int_X (\pi_*\Omega)\alpha\alpha' = \deg(\sigma)Q_{\pi_*\Omega}(\alpha,\alpha'). \end{aligned}$$

Thus the result follows from Lemma 6.3 applied to $p^*: H^{1,1}(X;\mathbb{R}) \to H^{1,1}(Z;\mathbb{R})$. □

7 Schur Classes Are in $\overline{HR}$

Lemma 7.1 *Let X be a smooth projective manifold of dimension $d \geq 4$, and E be a nef $\mathbb{Q}$-twisted bundle of rank $d-2$. Then $c_{d-2}(E) \in \mathrm{HR}_w(X)$.*

Proof This is exactly as in [21, Proposition 3.1]. First assume that E is ample and X is smooth. By a consequence of the Bloch-Gieseker Theorem for all $t \in \mathbb{R}_{\geq 0}$ the intersection form

$$Q_t(\alpha) := \int_X \alpha c_{d-2}(E\langle th\rangle)\alpha \text{ for } \alpha \in H^{1,1}(X;\mathbb{R})$$

is non-degenerate (we remark that we are allowing possibly irrational t here, and then $c_{d-2}(E\langle th\rangle)$ is to be understood as being defined as in (2.1)). Now for small t we have

$$c_{d-2}(E\langle th\rangle) = t^{d-2}h^{d-2} + O(t^{d-3}).$$

Observe that for an intersection form Q, having signature $(+,-\ldots,-)$ is invariant under multiplying Q by a positive multiple, and is an open condition as Q varies continuously. Thus since we know that h^{d-2} has the Hodge-Riemann property, the intersection form $(\alpha,\beta) \mapsto \int_X \alpha h^{d-2}\beta$ has signature $(+,-\ldots,-)$, and hence so does Q_t for t sufficiently large. But Q_t is non-degenerate for all $t \geq 0$, and hence Q_t must have this same signature for all $t \geq 0$. Thus $c_{d-2}(E) \in \mathrm{HR}(X)$.

Since any $\mathbb{Q}$-twisted nef bundle E can be approximated by an $\mathbb{Q}$-twisted ample vector bundle we deduce that $c_{d-2}(E) \in \overline{\mathrm{HR}}(X) \subset \mathrm{HR}_w(X)$. □

Theorem 7.2 (Derived Schur Classes are in $\overline{HR}$) *Let X be smooth and projective of dimension $d \geq 2$, let λ be a partition of length $d + i - 2$ and let E be a $\mathbb{Q}$-twisted nef vector bundle on X. Then*

$$s_\lambda^{(i)}(E) \in \overline{HR}(X).$$

Proof The statement is trivial unless $e := \mathrm{rk}(E) \geq \lambda_1$ and $d \geq 2$ which we assume is the case. When $d = 3$, $s_\lambda^{(i)}$ is a positive multiple of c_1 and then the result we want follows from the classical Hodge-Riemann bilinear relations. So we can assume from now on that $d \geq 4$.

Fix an ample class h on X. We first prove that $s_\lambda(E) \in \mathrm{HR}_w(X)$. Consider the case $i = 0$ so $|\lambda| = d - 2$. By Corollary 5.2 $\int_X s_\lambda(E)h^2 \geq 0$. Also, the cone construction described in §4 (particularly (4.3)) gives an irreducible variety $\pi : C \to X$ of dimension n and a nef $\mathbb{Q}$-twisted vector bundle U of rank $n - 2$ such that

$$\pi_* c_{n-2}(U) = s_\lambda(E).$$

Since C is irreducible we can take a resolution of singularities $\sigma : C' \to C$. Then $\sigma^* U$ is also nef, and Lemma 7.1 gives $c_{n-2}(\sigma^* U) \in \mathrm{HR}_w(C')$. Thus Lemma 6.6 implies $s_\lambda(E) \in \mathrm{HR}_w(X)$.

Consider next the case $i \geq 1$, so $|\lambda| = d + i - 2$. Again by Corollary 5.2, $\int_X s_\lambda^{(i)}(E)h^2 \geq 0$. Consider the product $\hat{X} = X \times \mathbb{P}^i$ and set $\tau = c_1(\mathcal{O}_{\mathbb{P}^i}(1))$. Suppressing pullback notation, the $\mathbb{Q}$-twisted bundle $E\langle\tau\rangle$ on $\hat{X}$ is nef, so by the previous paragraph $s_\lambda(E\langle\tau\rangle) \in \mathrm{HR}_w(\hat{X})$. Now

$$s_\lambda(E\langle\tau\rangle) = \sum_{j=0}^{|\lambda|} s_\lambda^{(j)}(E)\tau^j$$

so if $\pi : \hat{X} \to X$ is the projection

$$\pi_* s_\lambda(E\langle\tau\rangle) = s_\lambda^{(i)}(E).$$

Thus by Lemma 6.5 we get also $s_\lambda^{(i)}(E) \in \mathrm{HR}_w(X)$.

To complete the proof define

$$\Omega_t = s_\lambda^{(i)}(E\langle th\rangle) \text{ for } t \in \mathbb{Q}_{\geq 0}$$

and

$$f(t) = \det(Q_{\Omega_t}).$$

Note that the leading term of Ω_t is a positive multiple of h^{d-2} (this is Example 3.4 and it is here we use that $e \geq \lambda_1$). In particular, for t sufficiently large Q_{Ω_t} is non-degenerate (in fact it has the Hodge-Riemann property). Thus f is not identically zero, and since it is a polynomial in t this implies $f(t) \neq 0$ for all but finitely many t. Thus there is an $\epsilon > 0$ so that $f(t) \neq 0$ for rational $0 < t < \epsilon$ and we henceforth consider only t in this range. Then Q_{Ω_t} is non-degenerate, and as $Q_{\Omega_t}(h, h) \geq 0$ it cannot be negative definite. The previous paragraph gives $\Omega_t \in \mathrm{HR}_w(X)$, so we must actually have $\Omega_t \in \mathrm{HR}(X)$ for small $t \in \mathbb{Q}_{>0}$. Thus $\Omega_0 = s_\lambda^{(i)}(E) \in \overline{\mathrm{HR}}(X)$ as claimed. □

Remark 7.3 Note the above proof gives more, namely that if h is an ample class and E is nef and $\lambda_1 \leq \mathrm{rk}(E)$ we have

$$s_\lambda^{(i)}(E\langle th\rangle) \in \mathrm{HR}(X) \text{ for all but possibly finitely many } t \in \mathbb{Q}_{>0}.$$

As mentioned in the introduction, the main result of [21] says more namely that if E is ample of rank at least λ_1 then $s_\lambda^{(i)}(E) \in \mathrm{HR}(X)$, but the proof of that statement is significantly harder.

Theorem 7.4 (Monomials of Schur Classes are in $\overline{HR}$) *Let X be smooth and projective of dimension d and $E_1, \ldots, E_p$ be nef vector bundles on X. Let $\lambda^1, \ldots, \lambda^p$ be partitions such that*

$$\sum_i |\lambda^i| = d - 2.$$

Then the monomial of Schur polynomials

$$\prod_i s_{\lambda^i}(E_i)$$

lies in $\overline{HR}(X)$.

Proof The proof is similar to what has already been said, so we merely sketch the details. Set $\Omega = \prod_i s_{\lambda^i}(E_i)$. Then (4.5) gives a map $\pi : C \to X$ from an irreducible variety of dimension n and nef bundle bundle U on C so $\pi_* c_{n-2}(U) = \Omega$. A small modification of the proof of Proposition 5.1 and Corollary 5.2 means that if h is ample $\int_X \Omega h^2 \geq 0$.

Consider

$$\Omega_t := \pi_* c_{n-2}(U\langle t\pi^* h\rangle)$$

and take a resolution $\sigma : C' \to C$. Then $\sigma^* U\langle \pi^* h\rangle$ remains nef, so Lemma 6.6 implies $\Omega_t \in \mathrm{HR}_w(X)$.

Now we can equally apply this construction replacing each E_i with $E_i \otimes \mathcal{O}(th)$ for $t \in \mathbb{N}$ (which one can check does not change $\pi : C \to X$) giving

$$\pi_* c_{n-2}(U\langle th\rangle) = \prod_i s_{\lambda^i}(E_i\langle th\rangle) \text{ for } t \in \mathbb{N}.$$

In particular applying Example 3.4 to each factor on the right hand side, the highest power of t is a positive multiple of h^{d-2}. Thus for almost all $t \in \mathbb{Q}_{>0}$ we have Q_{Ω_t} is non-degenerate, and so in fact $Q_{\Omega_t} \in \mathrm{HR}(X)$. Taking the limit as $t \to 0$ gives the result we want. □

8 The Kähler Case

The main place in which projectivity has been used so far is in the application of the Bloch-Gieseker Theorem, and here we explain how this projectivity assumption can be relaxed. Following Demailly-Peternell-Schneider [6] we say a line bundle L on a compact Kähler manifold X is *nef* if for all $\epsilon > 0$ and all Kähler forms ω on X there exists a hermitian metric h on L with curvature $dd^c \log h \geq -\epsilon\omega$. We say that a vector bundle E on X is nef if the hyperplane bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef.

For the rest of this section let (X, ω) be a compact Kähler manifold of dimension d. Given a vector bundle E and $\delta \in H^{1,1}(X; \mathbb{R})$ we can consider the $\mathbb{R}$-twisted bundle $E\langle\delta\rangle$ whose Chern classes are defined just as in the case of $\mathbb{Q}$-twists in the projective case. We identify $\mathbb{P}(E\langle\delta\rangle)$ with $\mathbb{P}(E)$, and say that $E\langle\delta\rangle$ is nef if for any Kähler metric ω' on $\mathbb{P}(E)$, any $\epsilon > 0$, and any closed $(1, 1)$ form δ' on X such that $[\delta'] = \delta$, there exists a hermitian metric h on $\mathcal{O}_{\mathbb{P}(E)}(1)$ such that

$$dd^c \log h + \pi^*\delta' \geq -\epsilon\omega'.$$

We refer the reader to [6] for the fundamental properties of nef bundles on compact Kähler manifolds, in particular to the statement that a quotient of a nef bundle is again nef, and the direct sum of two nef bundles is again nef (and each of these statements extend to the case of $\mathbb{R}$-twisted nef bundles with minor modifications of the proofs involved).

Theorem 8.1 (Bloch-Gieseker for Kähler Manifolds) *Let E be a nef $\mathbb{R}$-twisted vector bundle of rank $e \leq d$ and $t > 0$. Let $e + j \leq d$ and consider*

$$\Omega := c_e(E\langle t\omega\rangle) \wedge \omega^j.$$

Then then map

$$H^{d-e-j}(X) \xrightarrow{\wedge\Omega} H^{d+e+j}(X)$$

is an isomorphism.

Proof Write $E = E'\langle\delta\rangle$ where E' is a genuine vector bundle. Fix $t > 0$ and set $E_t := E\langle t\omega\rangle = E'\langle\delta + t\omega\rangle$. Set $\pi : \mathbb{P}(E') \to X$ and define $\zeta' = c_1(\mathcal{O}_{\mathbb{P}(E')}(1))$ and $\zeta := \zeta' + \pi^*(\delta + t[\omega])$. Then $\zeta^e - c_1(E_t)\zeta^{e-1} + \cdots + (-1)^e c_e(E_t) = 0$ where we supress pullback notation for convenience.

Suppose $a \in H^{d-e-j}(X)$ has $ac_e(E_t)\omega^j = 0$, and we will show that $a = 0$. To this end define

$$b = a.(\zeta^{e-1} - c_1(E_t)\zeta^{e-2} + \cdots + (-1)^{e-1}c_{e-1}(E_t))$$

so by construction

$$\zeta b\omega^j = \pm ac_e(E_t)\omega^j = 0.$$

We claim that ζ is a Kähler class. Given this for now, the Hard-Lefschetz property for ζ then gives $b\omega^j = 0$ and hence $a\omega^j = \pi_*(b\omega^j) = 0$ and hence $a = 0$ by the Hard-Lefschetz property of ω^j

It remains to show that ζ is Kähler, and the following is essentially what is described in [6, Proof of Theorem 1.12]. Fix ω' a Kähler metric on $\mathbb{P}(E')$, and fix a hermitian metric on E' which induces a hermitian metric $\hat{h}$ on $\mathcal{O}_{\mathbb{P}(E')}(1)$. Then $dd^c \log \hat{h}$ is strictly positive in the fiber directions, so there is a constant $C > 0$ with

$$dd^c \log \hat{h} + C\pi^*\omega \geq C^{-1}\omega'.$$

Let δ' be a closed $(1,1)$-form on X with $[\delta'] = \delta$, and choose $\epsilon > 0$ sufficiently small that $(t - C^2\epsilon)\omega + C\epsilon\delta' > 0$. Then as E is assumed to be nef there is a hermitian metric h on $\mathcal{O}_{\mathbb{P}(E')}(1)$ such that $dd^c \log h + \pi^*\delta' \geq -\epsilon\omega'$.

Then the class $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E')}(1)) + \pi^*[\delta + t\omega]$ is represented by the form

$$(1 - C\epsilon)dd^c \log h + C\epsilon dd^c \log \hat{h} + \pi^*(\delta' + t\omega)$$

which is bounded from below by

$$\begin{aligned}(1 - C\epsilon)(-\epsilon\omega' - \pi^*\delta') + C\epsilon(C^{-1}\omega' - C\pi^*\omega) + \pi^*(t\omega + \delta')&\\ = C\epsilon^2\omega' + (t - C^2\epsilon)\pi^*\omega + C\epsilon\pi^*\delta'&\\ \geq C\epsilon^2\omega' > 0.&\end{aligned}$$

Thus ζ is a Kähler class as claimed. □

Corollary 8.2 *Let E be a nef $\mathbb{R}$-twisted vector bundle of rank $e \leq d$ and $j = d - e$. Then*

$$\int_X c_e(E)\omega^j \geq 0.$$

Proof Let $f(t) = \int_X c_e(E\langle t\omega\rangle)\omega^j$. The Bloch-Gieseker theorem implies $f(t) \neq 0$ for all $t > 0$, and since it is clearly positive for $t \gg 0$ f is not identically zero. Since f is polynomial in t we get $f(t) > 0$ for $t > 0$ sufficiently small, which proves the statement. □

From here almost all the results in this paper extend to the Kähler case, and the proofs have only trivial modifications. We state only one and leave the rest to the reader.

Theorem 8.3 (Derived Schur classes of nef vector bundles on Kähler manifolds are in $\overline{\mathrm{HR}}$) *Let X be a compact Kähler manifold of dimension $d \geq 2$, let λ be a partition of length $d+i-2$ and let E be an $\mathbb{R}$-twisted nef vector bundle on X. Then*

$$s_\lambda^{(i)}(E) \in \overline{HR}(X).$$

9 Combinations of Derived Schur Classes

An interesting feature of the Hodge-Riemann property for bilinear forms is that it generally is not preserved by taking convex combinations, and so there is no reason to expect that a convex combination of classes with the Hodge-Riemann property again has the Hodge-Riemann property. In fact this phenomena occurs even for combinations of Schur classes of an ample vector bundle as the following example shows

Example 9.1 ([21, Sect. 9.2]) Let $X = \mathbb{P}^2 \times \mathbb{P}^3$ Then $N^1(X)$ is two-dimensional, with generators a, b that satisfy $a^3 = 0, a^2b^3 = 1$. Set $\mathcal{O}_X(a, b) = \mathcal{O}_{\mathbb{P}_2}(a) \boxtimes \mathcal{O}_{\mathbb{P}^3}(b)$ and consider the nef vector bundle

$$E = \mathcal{O}(1,0) \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1).$$

One computes that the form

$$(1-t)c_3(E) + ts_{(1,1,1)}(E)$$

gives an intersection form on $N^1(X)$ with matrix

$$Q_t := \begin{pmatrix} t & 2t \\ 2t & 1+2t \end{pmatrix}.$$

For $t \in (0, 1/2)$ the matrix Q_t has two strictly positive eigenvalues. Thus fixing $t \in (0, 1/2)$, any small pertubation of E by an ample class gives an ample $\mathbb{Q}$-twisted bundle E' so that $(1-t)c_3(E') + ts_{(1,1,1)}(E')$ does not have the Hodge-Riemann property.

Given this it is interesting to ask if there are particular convex combinations of (derived) Schur classes that do retain the Hodge-Riemann property. To state one such result we need the following definition, for which we recall a matrix is said to be *totally positive* if all its minors have non-negative determinant,.

Definition 9.2 *(Pólya Frequency Sequence)* Let $\mu_0, \ldots, \mu_N$ be non-negative numbers, and set $\mu_i = 0$ for $i < 0$. We say $\mu_0, \ldots, \mu_N$ is a *Pólya frequency sequence* if the matrix

$$\mu := (\mu_{i-j})_{i,j=0}^{N}$$

is totally positive.

Theorem 9.3 *Suppose that X has dimension $d \geq 4$ that h is an nef class on X and E is a nef vector bundle. Let $|\lambda| = d-2$ and $\mu_0, \ldots, \mu_{d-2}$ be a Pólya frequency sequence. Then the class*

$$\sum_{i=0}^{d-2} \mu_i s_\lambda^{(i)}(E) h^i \tag{9.1}$$

lies in $\overline{HR}(X)$.

Theorem 9.3 follows quickly from the following statement, for which we recall c_i denotes the i-th elementary symmetric polynomial.

Proposition 9.4 *Suppose that X has dimension $d \geq 4$ and E is a nef vector bundle. Let λ be a partition of $d-2$. Let $D_1, \ldots, D_q$ be ample $\mathbb{Q}$-divisors on X for some $q \geq 1$. Then for any $t_1, \ldots, t_q \in \mathbb{Q}_{>0}$ the class*

$$\sum_{i=0}^{d-2} s_\lambda^{(i)}(E) c_i(t_1 D_1, \ldots, t_q D_q)$$

lies in $\overline{HR}(X)$.

Proof of Theorem 9.3 If all the μ_i vanish the statement is trivial, so we assume this is not the case. From the Aissen-Schoenberg-Whitney Theorem [1], the assumption that μ_i is a Pólya frequency sequence implies that the generating function

$$\sum_{i=0}^{d-2} \mu_i z^i$$

has only real roots, and since each μ_i is non-negative these roots are then necessarily non-positive. Writing these roots as $\{-t_j\}$ for $t_j \in \mathbb{R}_{\geq 0}$ means

$$\sum_{i=0}^{d-2} \mu_i z^i = \kappa \prod_{j=0}^{N} (z + t_j) \text{ where } \kappa > 0$$

which implies

$$\mu_i = \kappa c_i(t_1, \ldots, t_N) \text{ for all } i.$$

Now for each j let $t_j^{(n)} \in \mathbb{Q}_{>0}$ tend to t_j as $n \to \infty$. Fix an ample divisor h'' and consider the class $h' := h + \frac{1}{n}h''$. Proposition 9.4 (applied with $q = N$ and $D_1 = \cdots = D_q = h'$) implies

$$\sum_{i=0}^{d-2} s_\lambda^{(i)}(E) c_i(t_1^{(n)}, \ldots, t_N^{(n)})(h')^i$$

lies in $\overline{\mathrm{HR}}(X)$. Taking the limit as $n \to \infty$ gives the statement we want. □

Proof of Proposition 9.4 Set

$$\Omega := \Omega(D_1, \ldots, D_p) := \sum_{i=0}^{d-2} s_\lambda^{(i)}(E) c_i(D_1, \ldots, D_p).$$

Without loss of generality we may assume all the D_i are integral and very ample. Write $t_j = r_j/s$ for some positive integers r_j and s. By an iterated application of the Bloch-Gieseker covering construction, we find a finite $u : Y \to X$ and line bundles η_j on X' such that that $\eta_j^{\otimes s} = u^*\mathcal{O}(D_j)$. Thus

$$r_j c_1(\eta_j) = t_j u^* D_j.$$

Set $E' = u^*E$. Consider the cone construction for E' as described in §4. That is, there is a surjective $\pi : C \to Y$ from an irreducible variety C of dimension n, and a nef vector bundle U on C' of rank $n-2$ such that $\pi_* c_{n-2}(U) = s_\lambda(E')$. In fact more is true namely;

Lemma 9.5

$$\pi_* c_{n-2-i}(U|_C) = s_\lambda^{(i)}(E') \text{ for } 0 \le i \le |\lambda|. \tag{9.2}$$

Sketch Proof. Formally this is clear: for if $\delta' \in H^{1,1}(X;\mathbb{R})$ then $c_{n-2}(U\langle \pi^*\delta'\rangle) = \sum c_{n-2-i}(U)(\pi^*\delta')^i$ and pushing this forward to X gives a polynomial in δ' of classes on X whose coefficients are the derived Schur classes $s_\lambda^{(i)}(E')$. For a full proof we refer the reader to [21, Proposition 5.2]. □

Continuing with the proof of the Proposition, set

$$F = \bigoplus_{i=1}^{p} \eta_i^{\otimes r_i}$$

so

$$c_j(F) = c_j(r_1 c_1(\eta_1), \cdots, r_p c_1(\eta_p)) = u^* c_j(t_1 D_1, \ldots, t_p D_p).$$

Then on C' the bundle

$$\tilde{U} := U \oplus \pi^* F$$

is nef. Take a resolution $\sigma : C \to C'$, the vector bundle $\sigma^* U$ remains nef and so using Theorem 7.2 and Lemma 6.6

$$\pi_* c_{n-2}(\tilde{U}) \in \mathrm{HR}_w(Y).$$

But

$$\begin{aligned}\pi_* c_{n-2}(\tilde{U}) &= \pi_*(c_{n-2}(U) + c_{n-3}(U)\pi^* c_1(F) + \cdots + c_{n-2-d}(U)\pi^* c_d(F)) \\ &= s_\lambda(E') + s_\lambda^{(1)}(E')c_1(F) + \cdots + s_\lambda^{(d-2)}(E')c_{d-2}(F) \\ &= u^*\Omega.\end{aligned}$$

So by Lemma 6.4 applied to $u : Y \to X$ we conclude that $\Omega \in \mathrm{HR}_w(X)$.

To show that in fact $\Omega \in \overline{\mathrm{HR}}(X)$ we consider the effect of replacing each D_i with $D_i + th$. Let $\Omega_t := \Omega(D_1 + th, \ldots, D_p + th)$ which is a polynomial in t whose t^{d-2} term is some positive multiple of h^{d-2}. Setting $f(t) = \det(Q_{\Omega_t})$ we conclude exactly as in the end of the proof of Theorem 7.2 that $\Omega_t \in \mathrm{HR}(X)$ for $t \in \mathbb{Q}_+$ sufficiently small, and thus $\Omega \in \overline{\mathrm{HR}}(X)$ as required. □

Question 9.6 *Suppose that $\mu_1, \ldots, \mu_{d-2}$ is a Pólya frequency sequence with each μ_i strictly positive, and that h and E are ample. Is it then the case that the class in (9.1) is actually in $HR(X)$? The difficulty here is that to follow the proof we have given above one needs to address the possibility that some of the t_j are irrational.*

10 Inequalities

10.1 Hodge-Index Type Inequalities

The simplest and most fundamental inequality obtained from the Hodge-Riemann property is the Hodge-index inequality.

Theorem 10.1 (Hodge-Index Theorem) *Let X be a manifold of dimension d and $\Omega \in \mathrm{HR}_w(X)$. If $\beta \in H^{1,1}(X)$ is such that $\int_X \beta^2\Omega \geq 0$ then for any $\alpha \in H^{1,1}(X)$ it holds that*

$$\int_X \alpha^2\Omega \int_X \beta^2\Omega \leq \left(\int_X \alpha\beta\Omega\right)^2. \tag{10.1}$$

Moreover if $\Omega \in HR(X)$ and $\int_X \beta^2\Omega > 0$ then equality holds in (10.1) if and only if α and β are proportional.

Proof The statement is about symmetric bilinear forms with the given signature and its proof is standard. Indeed, the case when $\int_X \beta^2\Omega = 0$ is trivial and the case when the intersection form is nondegenerate and $\int_X \beta^2\Omega > 0$ is classical. Finally, the case when the intersection form is degenerate and $\int_X \beta^2\Omega > 0$ reduces itself to the previous one by modding out the kernel of the intersection form. $\square$

In particular (namely Theorem 7.2) the inequality (10.1) applies when $\Omega = s_\lambda(E)$ whenever λ is a partition of $d-2$, E is a nef $\mathbb{Q}$-twisted bundle on X and β is nef. We now prove a variant of this that gives additional information.

Theorem 10.2 *Let X be a projective manifold of dimension $d \geq 4$ and let E be a $\mathbb{Q}$-twisted nef vector bundle and $h \in H^{1,1}(X;\mathbb{R})$ be nef. Also let λ be a partition of length $|\lambda| = d-1$. Then for all $\alpha \in H^{1,1}(X;\mathbb{R})$,*

$$\int_X \alpha^2 s_\lambda^{(1)}(E) \int_X h s_\lambda(E) \leq 2\int_X \alpha h s_\lambda^{(1)}(E) \int_X \alpha s_\lambda(E). \tag{10.2}$$

Remark 10.3 (i) In the case that $\lambda = (d-1)$ and $\mathrm{rk}(E) = d-1$ the inequality (10.2) becomes

$$\int_X \alpha^2 c_{d-2}(E) \int_X h c_{d-1}(E) \leq 2\int_X \alpha h c_{d-2}(E) \int_X \alpha c_{d-1}(E). \tag{10.3}$$

This was previously proved in [21, Theorem 8.2]. In fact (10.3) was shown to hold for all nef vector bundles of rank at least $d-1$ and if E, h are assumed ample then equality holds in (10.3) if and only if $\alpha = 0$. We imagine a similar statement holds in the context of Theorem 10.2.

(ii) Assume in the setting of Theorem 10.2 that $\int_X s_\lambda(E)h > 0$ and let W be the kernel of the map $H^{1,1}(X) \to \mathbb{R}$ given by $\alpha \mapsto \int_X \alpha s_\lambda(E)$. Then W has codimension 1, and (10.2) says that the intersection form $Q_{s_\lambda(E)}$ is negative semidefinite on W. This is different information to the Hodge-Index inequality which is essentially a reformulation of the fact that this intersection form is negative semidefinite on the orthogonal complement of h.

(iii) The inequality (10.2) generalizes to any homogeneous symmetric polynomial p in e variables with the property that $p(E) \in \overline{\mathrm{HR}}(X)$ for all $\mathbb{Q}$-twisted nef vector bundles E of rank e (with the obvious definition for the derived polynomials $p^{(i)}$).

Proof of Theorem 10.2 If $e := \mathrm{rk}(E) < \lambda_1$ the statement is trivial, so we assume $e \geq \lambda_1$. We start with some reductions. By continuity, it is sufficient to prove this under the additional assumption that h is ample. Also replacing E with $E\langle th\rangle$ for $t \in \mathbb{Q}_{>0}$ sufficiently small we may assume that $\int_X s_\lambda(E)h > 0$.

Now set $\hat{X} = X \times \mathbb{P}^1$ and $\hat{E} = E \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. Observe $\hat{E}$ is nef on $\hat{X}$ and $|\lambda| = \dim(\hat{X}) - 2$. So Theorem 7.2 implies

$$s_\lambda(\hat{E}) \in \overline{\mathrm{HR}}(\hat{X}).$$

Let $\alpha \in H^{1,1}(X;\mathbb{R})$ and denote by τ the hyperplane class on $\mathbb{P}^1$. Also to ease notation define

$$\Omega := s_\lambda(E) \in H^{d-1,d-1}(X;\mathbb{R}) \text{ and } \Omega' := s_\lambda^{(1)}(E) \in H^{d-2,d-2}(X;\mathbb{R})$$

so $s_\lambda(\hat{E}) = \Omega + \Omega'\tau$.

Now define

$$\hat{\alpha} := \alpha - \kappa\tau \text{ where } \kappa := \frac{\int_X \alpha\Omega' h}{\int_X \Omega h}$$

so

$$\hat{\alpha} s_\lambda(\hat{E}) h = \hat{\alpha}(\Omega + \tau\Omega')h = 0.$$

Also observe

$$\int_{\hat{X}} s_\lambda(\hat{E})h^2 = \int_X \Omega' h^2 > 0$$

so the Hodge-Index inequality applied to $s_\lambda(\hat{E})$ yields

$$0 \geq \int_{\hat{X}} \hat{\alpha}^2 s_\lambda(\hat{E}) = \int_{\hat{X}} (\alpha^2 - 2\kappa\alpha\tau)(\Omega + \tau\Omega') = \int_X \alpha^2\Omega' - 2\kappa \int_X \alpha\Omega.$$

Rearranging this gives (10.2). $\square$

10.2 Khovanskii-Tessier-Type Inequalities

Let X be smooth and projective of dimension d. Suppose that E, F are vector bundles on X, and let λ and μ be partitions of length $|\lambda|$ and $|\mu|$ respectively, and to avoid trivialities we assume $|\lambda| + |\mu| \geq d$.

Definition 10.4 We say a sequence $(a_i)_{i\in\mathbb{Z}}$ of non-negative real numbers is *log concave* if

$$a_{i-1}a_{i+1} \leq a_i^2 \text{ for all } i \tag{10.4}$$

We note that for a finite sequence, say $a_i = 0$ for $i < 0$ and for $i > n$, log-concavity is equivalent to (10.4) holding in the range $i = 1, \ldots, n-1$.

Theorem 10.5 *Assume E, F are nef. Then the sequence*

$$i \mapsto \int_X s_\lambda^{(|\lambda|+|\mu|-d-i)}(E) s_\mu^{(i)}(F) \tag{10.5}$$

is log-concave

Before giving the proof, some special cases are worth emphasising.

Corollary 10.6 *Suppose that $|\lambda| = |\mu| = d$. Then the sequence*

$$i \mapsto \int_X s_\mu^{(d-i)}(E) s_\lambda^{(i)}(F)$$

is log-concave

Corollary 10.7 *Suppose that $|\lambda| = d$ and let h be a nef class on X. Then the sequence*

$$i \mapsto \int_X s_\lambda^{(d-i)}(E) h^{d-i} \tag{10.6}$$

is log-concave. In particular the map

$$i \mapsto \int_X c_i(E) h^{d-i} \tag{10.7}$$

is log-concave.

Proof of Corollary 10.7 By continuity we may assume that h is ample. Let L be a line bundle with $c_1(L) = h$. By rescaling h we may, without loss of generality, assume L is globally generated giving a surjection

$$\mathcal{O}^{\oplus f+1} \to L \to 0$$

for some integer f. Let F^* be the kernel of this surjection. Then F is a vector bundle of rank f that is globally generated and hence nef. Now set $\mu = (f)$, so $s_\mu^{(j)}(F) = c_{f-j}(F) = h^{f-j}$. We now replace i with $f - d + i$ in (10.5) (which is an affine linear transformation so does not affect log-concavity). Note that

$$|\lambda| + |\mu| - d - (f - d + i) = |\lambda| - i,$$

so Theorem 10.5 gives (10.6)

Finally (10.7) follows upon letting $e := \mathrm{rk}(E)$ and putting $\lambda = (e)$ so $s_\lambda^{(j)}(E) = c_{e-j}(E)$ so $s_\lambda^{(|\lambda|-i)}(E) = c_i(E)$. $\square$

Proof of Theorem 10.5 The first thing to note is that all the quantities in (10.5) are non-negative (see Remark 5.5). Also, we may as well assume $\mathrm{rk}(E) \geq \lambda_1$ and $\mathrm{rk}(F) \geq \mu_1$ else the statement is trivial.

Set

$$j = |\lambda| + |\mu| - d - i$$

and define

$$a_i := \int_X s_\lambda^{(j)}(E) s_\mu^{(i)}(F)$$

so the task is to show that (a_i) is log-concave. We observe that $a_i = 0$ if either i or j are negative, or $i > |\mu|$ or $j > |\lambda|$. Thus the range of interest is

$$\underline{i} := \max\{0, |\mu| - d\} \leq i \leq \min\{|\mu|, |\lambda| + |\mu| - d\} =: \overline{i}.$$

Fix such an i in this range and consider

$$\hat{X} = X \times \mathbb{P}^{j+1} \times \mathbb{P}^{i+1}.$$

Let τ_1 be the pullback of the hyperplane class on $\mathbb{P}^{j+1}$ and τ_2 the pullback of the hyperplane class on $\mathbb{P}^{i+1}$ and consider

$$\Omega = s_\lambda(E(\tau_1)) \cdot s_\mu(F(\tau_2)).$$

Observe that by construction $|\lambda| + |\mu| = d + i + j = \dim \hat{X} - 2 =: \hat{d} - 2$. Expanding Ω as a polynomial in τ_1, τ_2 one sees that the coefficient of $\tau_1^j \tau_2^i$ is precisely $s_\lambda^{(j)} s_\mu^{(i)}$. Thus

$$\int_{\hat{X}} \Omega \tau_1 \tau_2 = \int_X s_\lambda^{(j)} s_\mu^{(i)} \int_{\mathbb{P}^{j+1}} \tau_1^{j+1} \int_{\mathbb{P}^{i+1}} \tau_2^{i+1} = \int_X s_\lambda^{(j)} s_\mu^{(i)} = a_i.$$

Similarly $\int_{\hat{X}} \Omega \tau_1^2 = a_{i-1}$ and $\int_{\hat{X}} \Omega \tau_2^2 = a_{i+1}$.

Now, since $E(\tau_1)$ and $F(\tau_2)$ are nef on $\hat{X}$ we know from Theorem 7.4 that $\Omega \in \overline{\mathrm{HR}}(\hat{X})$. Thus the Hodge-Index inequality (10.1) applies with respect to the classes τ_1, τ_2 which is

$$\int_{\hat{X}} \Omega \tau_1^2 \int_{\hat{X}} \Omega \tau_2^2 \leq \left(\int_{\hat{X}} \Omega \tau_1 \tau_2 \right)^2 \tag{10.8}$$

giving the log-concavity we wanted. □

Remark 10.8 In [21] we gave a slightly different proof of (10.6) which gave more, namely that if X is smooth and E and h are ample then the map in question is strictly log-concave. We expect that an analogous improvement can be made to Theorem 10.5, but it is not clear how this can be proved using the methods we have given here, since the bundle F constructed in the above proof is only nef.

Question 10.9 *Is there a natural statement along the lines of Theorem 10.5 that applies to three or more nef vector bundles? For instance perhaps it is possible to package characteristic numbers into a homogeneous polynomial that can be shown to be Lorentzian in the sense of Brändén-Huh [3].*

Corollary 10.10 *Let λ and μ be partitions, and let d be an integer with $d \leq |\lambda| + |\mu|$. Assume $x_1, \ldots, x_e, y_1, \ldots, y_f \in \mathbb{R}_{\geq 0}$. Then the sequence*

$$i \mapsto s_\lambda^{(|\lambda|+|\mu|-d+i)}(x_1, \ldots, x_e) s_\mu^{(i)}(y_1, \ldots, y_f)$$

is log concave.

Proof By continuity we may assume the x_i and y_i are rational. Furthermore, by clearing denominators, we may suppose they all lie in $\mathbb{N}$. Then take $X = \mathbb{P}^d$ and $E = \bigoplus_{i=1}^{e} \mathcal{O}_{\mathbb{P}^d}(x_i)$ and $F = \bigoplus_{i=1}^{f} \mathcal{O}_{\mathbb{P}^d}(y_i)$. Then for any symmetric polynomial p of degree δ we have $p(E) = p(x_1, \ldots, x_e)\tau^{\delta}$ and similarly for F. Thus what we want follows from Theorem 10.5. □

Putting $e = f$ we can consider

$$u_i := s_\lambda^{(|\lambda|+|\mu|-d+i)} s_\mu^{(i)}$$

as a polynomial in $x_1, \ldots, x_e$. Still assuming $d \leq |\lambda| + |\mu|$, Corollary 10.10 says that

$$(u_i^2 - u_{i+1}u_{i-1})(x_1, \ldots, x_e) \geq 0 \text{ for any } x_1, \ldots, x_e \in \mathbb{R}_{\geq 0}.$$

Question 10.11 *Is $u_i^2 - u_{i+1}u_{i-1}$ monomial-positive (i.e. a sum of monomials with all non-negative coefficients)?*

Corollary 10.12 *Let λ be a partition and $x_1, \ldots, x_e \in \mathbb{R}_{\geq 0}$. Then the sequence*

$$i \mapsto s_\lambda^{(i)}(x_1, \ldots, x_e)$$

is log-concave.

Proof By continuity we may assume $x_i \in \mathbb{Q}_{>0}$, and then by clearing denominators that they are all in $\mathbb{N}$. Set $d = |\lambda|$ and $X = \mathbb{P}^d$ and $E = \bigoplus_{j=1}^{e} \mathcal{O}_{\mathbb{P}^d}(x_i)$ and $h = c_1(E)$ which are both ample. Then for any symmetric polynomial p of degree d in e variables we have $\int_X p(E) = p(x_1, \ldots, x_e)$. Thus Corollary 10.7 tells us that the map

$$i \mapsto s_\lambda^{(d-i)}(x_1, \ldots, x_e)(x_1 + \cdots x_e)^{d-i} =: a_i$$

is log-concave That is $a_{i-1}a_{i+1} \leq a_i^2$, and dividing both sides of this inequality by $(x_1 + \ldots + x_e)^{2d-2i}$ gives that $i \mapsto s_\lambda^{(d-i)}(x_1, \ldots, x_e)$ is log-concave. Replacing $d - i$ with i does not change the log-concavity, so we are done. □

Question 10.13 *Do Corollary 10.10 or Corollary 10.12 have a purely combinatorial proof?*

10.3 Lorentzian Property of Schur Polynomials

We end with a discussion on how our results relate to those of Huh-Matherne-Mészáros-Dizier [13]. To do so we need some definitions that come from [3]. A symmetric homogeneous polynomial $p(x_1, \ldots, x_e)$ of degree d is said to be *strictly Lorentzian* if all the coefficients of p are positive and for any $\alpha \in \mathbb{N}^e$ with $\sum_j \alpha_j = d - 2$ we have

$$\frac{\partial^\alpha p}{\partial x^\alpha} \text{ has signature } (+, -, \ldots, -).$$

We say p is *Lorentzian* if it is the limit of strictly Lorentzian polynomials.

Any homogeneous polynomial p of degree d can be written as $p = \sum_\mu a_\mu x^\mu$ where the sum is over $\mu \in \mathbb{Z}^e_{\geq 0}$ with $\sum \mu_j = d$. We write $[p]_\mu := a_\mu$ for the coefficient of x^μ. The *normalization* of p is defined by

$$N(p) := \sum_\mu \frac{a_\mu}{\mu!} x^\mu.$$

Theorem 10.14 (Huh-Matherne-Mészáros-Dizier *[13, Theorem 3]*) *The normalized Schur polynomials $N(s_\lambda)$ are Lorentzian.*

Our proof needs a preparatory statement. For this we set

$$t_j(x_1, \ldots, x_e) = x_j \text{ for each } j = 1, \ldots, e.$$

Lemma 10.15 *Let $p(x_1, \ldots, x_e)$ be a homogeneous polynomial of degree d, let e' be any integer satisfying $e' \geq \max_{1 \leq j \leq e} \deg_{x_j}(p)$, where $\deg_{x_j}(p)$ is the degree of p with respect to the indeterminate x_j, and set*

$$q(x_1, \ldots, x_e) := x_1^{e'} \cdots x_e^{e'} p(x_1^{-1}, \ldots, x_e^{-1}).$$

Let $\alpha \in \mathbb{Z}^e_{\geq 0}$ with $\sum_j \alpha_j = d - 2$ and set $\beta_j := e' - \alpha_j$. Then

$$\frac{\partial^\alpha}{\partial x^\alpha} N(p) = \frac{1}{2} \sum_{1 \leq i, j \leq e} [q t_i t_j]_\beta x_i x_j.$$

Proof For $1 \leq i \leq e$ set $\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^e$ with 1 at the i-th position. Then if p is written as $p = \sum_\mu a_\mu x^\mu$, we get

$$\frac{\partial^\alpha}{\partial x^\alpha} N(p) = \frac{1}{2} \sum_{1 \leq i, j \leq e} a_{\alpha + \delta_i + \delta_j} x_i x_j = \frac{1}{2} \sum_{1 \leq i, j \leq e} [q t_i t_j]_\beta x_i x_j,$$

as one can check by expanding p in monomials. □

Proof of Theorem 10.14 Take a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ of $d := |\lambda|$ with $0 \leq \lambda_N \leq \cdots \leq \lambda_1$ and assume $\lambda_1 \leq e$ else the statement is trivial. Then d is the degree of $s_\lambda(x_1, \ldots, x_e)$. Note that by adding zero members to the partition λ we may increase N without changing the value of s_λ. We may therefore suppose that in our case $N \geq e$. The dual partition to λ is defined by

$$\overline{\lambda}_i := e - \lambda_{N-i} \text{ for } i = 1, \ldots, N$$

so $|\overline{\lambda}| = Ne - |\lambda| = Ne - d$.

Applying the definition

$$s_\lambda = \det \begin{pmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+N-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+N-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{\lambda_N-N+1} & c_{\lambda_N-N+2} & \cdots & c_{\lambda_N} \end{pmatrix}$$

to

$$x_1^N \cdots x_e^N s_\lambda(x_1^{-1}, \ldots, x_e^{-1})$$

and multiplying each row of the matrix defining

$$s_\lambda(x_1^{-1}, \ldots, x_e^{-1})$$

with $x_1 \cdots x_e$, we get

$$x_1^N \cdots x_e^N s_\lambda(x_1^{-1}, \ldots, x_e^{-1}) =$$

$$\det \begin{pmatrix} c_{e-\lambda_1} & c_{e-\lambda_1-1} & \cdots & c_{e-\lambda_1-N+1} \\ c_{e-\lambda_2+1} & c_{e-\lambda_2} & \cdots & c_{e-\lambda_2-N+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{e-\lambda_N+N-1} & c_{e-\lambda_N+N-2} & \cdots & c_{e-\lambda_N} \end{pmatrix} = s_{\overline{\lambda}}(x_1, \ldots, x_e).$$

Thus

$$s_{\overline{\lambda}}(x_1, \ldots, x_e) = x_1^N \cdots x_e^N s_\lambda(x_1^{-1}, \ldots, x_e^{-1})$$

and, equivalently,

$$s_\lambda(x_1, \ldots, x_e) = x_1^N \cdots x_e^N s_{\overline{\lambda}}(x_1^{-1}, \ldots, x_e^{-1}).$$

It is tempting to now apply Lemma 10.15, but before doing that we introduce a small perturbation. For $\epsilon > 0$ set $\tilde{x}_j := x_j + \epsilon \sum_p x_p$ and let

$$q_\epsilon(x_1, \ldots, x_e) := s_{\overline{\lambda}}(\tilde{x}_1, \ldots, \tilde{x}_e)$$

and

$$p_\epsilon(x_1, \ldots, x_e) := x_1^N \cdots x_e^N q_\epsilon(x_1^{-1}, \ldots, x_e^{-1}),$$

so

$$q_\epsilon(x_1, \ldots, x_e) = x_1^N \cdots x_e^N p_\epsilon(x_1^{-1}, \ldots, x_e^{-1}). \tag{10.9}$$

We will show that $N(p_\epsilon)$ is strictly Lorentzian for small $\epsilon > 0$, which completes the proof since p_ϵ tends to s_λ as ϵ tends to zero.

To this end, let $\alpha \in \mathbb{Z}^e_{\geq 0}$ with $\sum_j \alpha_j = d-2$ and set $\beta_j := N - \alpha_j$ and

$$X := \prod_{j=1}^{e} \mathbb{P}^{\beta_j}.$$

Let τ_j denote the pulback of the hyperplane class on $\mathbb{P}^{\beta_j}$ to X, and set $h := \sum_j \tau_j$ which is ample. Next set

$$E := \bigoplus_{j=1}^{e} \pi_j^* \mathcal{O}_{\mathbb{P}^{\beta_j}}(1) \text{ and } E' := E\langle \epsilon h\rangle.$$

Then E is a nef vector bundle on X and by construction $\dim X = Ne - d + 2 = |\overline{\lambda}| + 2$. So from Theorem 7.2 we know $s_{\overline{\lambda}}(E) \in \overline{\mathrm{HR}}(X)$. In fact by Remark 7.3 we actually have $s_{\overline{\lambda}}(E') \in \mathrm{HR}(X)$ for sufficiently small $\epsilon > 0$ and we assume henceforth this is the case.

Now by (10.9) and Lemma 10.15,

$$\frac{\partial^\alpha}{\partial x^\alpha} N(p_\epsilon) = \frac{1}{2} \sum_{1 \leq i,j \leq e} [q_\epsilon t_i t_j]_\beta x_i x_j \tag{10.10}$$

and our goal is to show that this has the desired signature. But this is precisely what we already know, since thinking of $s_{\overline{\lambda}}(E')\tau_i\tau_j$ as a homogeneous polynomial in $\tau_1, \ldots, \tau_e$, integrating over X picks out precisely the coefficient of τ^β, and as E' has Chern roots $\tau_1 + \epsilon h, \cdots, \tau_e + \epsilon h$ this becomes

$$\int_X s_{\overline{\lambda}}(E')\tau_i\tau_j = [q_\epsilon t_i t_j]_\beta.$$

Hence the quadratic form in (10.10) is precisely the intersection form $\frac{1}{2} Q_{s_{\overline{\lambda}}(E')}$ on $H^{1,1}(X)$, which has signature $(+, -, \ldots, -)$ and we are done. □

Remark 10.16 There is a lot of overlap between what we have here and the original proof in [13]. For instance we rely here on our Theorem that Schur classes of (certain) ample vector bundles have the Hodge-Riemann property, which in turn relies on the Bloch-Gieseker theorem and thus on the classical Hard-Lefschetz Theorem. On the other hand, [13] relies on the fact that the volume function on a projective variety is Lorentzian, which is a facet of the Hodge-index inequalities (that are a consequence of the Hodge-Riemann bilinear relations).

Also, instead of our cone classes discussed in Sect. 4, the authors in [13] use a different aspect of Schur classes that is also a degeneracy locus. Finally we remark the use of the dual partition $\overline{\lambda}$ also appears crucially in [13]. Nevertheless there is a slightly different feel to the two proofs, and we leave it to the readers to decide if they consider them "essentially the same" [11].

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References

1. Aissen, M., Schoenberg, I.J., Whitney, A.M.: On the generating functions of totally positive sequences. I. J. Anal. Math. **2**, 93–103 (1952)
2. Bloch, S., Gieseker, D.: The positivity of the Chern classes of an ample vector bundle. Invent. Math. **12**, 112–117 (1971)
3. Brändén, P., Huh, J.: Lorentzian polynomials. Ann. Math. **192**(3)(2), 821–891 (2020)
4. Chen, W.Y.C., Wang, L.X.W., Yang, A.L.B.: Schur positivity and the q-log-convexity of the Narayana polynomials. J. Algebr. Combin. **32**(3), 303–338 (2010)
5. Cvetkovski, Z.: Inequalities. Theorems, techniques and selected problems. Springer, Heidelberg (2012)
6. Demailly, J.-P., Peternell, T., Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. J. Algebr. Geom. **3**(2), 295–345 (1994)
7. Fulton, W.: Young tableaux, volume 35 of London Mathematical Society Student Texts. With applications to representation theory and geometry. Cambridge University Press, Cambridge (1997)
8. Fulton, W.: Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2nd ed. Springer, Berlin (1998)
9. Fulton, W., Lazarsfeld, R.: Positive polynomials for ample vector bundles. Ann. Math. **118**(1)(2), 35–60 (1983)
10. Gao, A.L.L., Xie, M.H.Y., Yang, A.L.B.: Schur positivity and log-concavity related to longest increasing subsequences. Disc. Math. **342**(9), 2570–2578 (2019)
11. Gowers, T.: When are two proofs essentially the same?
12. Gromov, M.: Convex sets and Kähler manifolds. In: Advances in Differential Geometry and Topology, pp. 1–38. World Scientific Publishing, Teaneck, NJ (1990)
13. Huh, J., Matherne, J.P., Mészáros, K., St. Dizier, A.: Logarithmic concavity of Schur and related polynomials. Trans. Amer. Math. Soc. Trans. Am. Math. Soc. **375**(6), 4411–4427 (2022). https://doi.org/10.1090/tran/8606. https://doi-org.proxy.cc.uic.edu/10.1090/tran/8606
14. Kempf, G., Laksov, D.: The determinantal formula of Schubert calculus. Acta Math. **132**, 153–162 (1974)
15. Lam, T., Postnikov, A., Pylyavskyy, P.: Schur positivity and Schur log-concavity. Am. J. Math. **129**(6), 1611–1622 (2007)
16. Lazarsfeld, R.: Positivity in algebraic geometry. II, volume 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Positivity for vector bundles, and multiplier ideals. Springer, Berlin (2004)
17. Miyaoka, Y.: The Chern classes and Kodaira dimension of a minimal variety. In: Algebraic geometry, Sendai, 1985, volume 10 of Advanced Studies in Pure Mathematics, pp. 449–476. North-Holland, Amsterdam (1987)
18. Newton, I.: Arithmetica universalis: sive de compositione et resolutione arithmetica liber (1707)

19. Okounkov, A.: Why would multiplicities be log-concave? In: The orbit method in geometry and physics (Marseille, 2000), volume 213 of Progress in Mathematics, pp. 329–347. Birkhäuser Boston, Boston, MA (2003)
20. Richards, D.S.P.: Log-convexity properties of Schur functions and generalized hypergeometric functions of matrix argument. Ramanujan J. **23**(1–3), 397–407 (2010)
21. Ross, J., Toma, M.: Hodge-Riemann bilinear relations for Schur classes of ample vector bundles (2019). arXiv:1905.13636. to appear in Ann. Sci. École Norm. Sup
22. Xiao, J.: On the positivity of high-degree Schur classes of an ample vector bundle. Sci. China Math. **65**(1), 51–62 (2022). https://doi.org/10.1007/s11425-020-1868-7. https://doi-org.proxy.cc.uic.edu/10.1007/s11425-020-1868-7

On Large Deviation Principles and the Monge–Ampère Equation (Following Berman, Hultgren)

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Abstract This is mostly an exposition, aimed to be accessible to geometers, analysts, and probabilists, of a fundamental recent theorem of R. Berman with recent developments by J. Hultgren, that asserts that the second boundary value problem for the real Monge–Ampère equation admits a probabilistic interpretation, in terms of many particle limit of permanental point processes satisfying a large deviation principle with a rate function given explicitly using optimal transport. An alternative proof of a step in the Berman–Hultgren Theorem is presented allowing to to deal with all "tempratures" simultaneously instead of first reducing to the zero-temperature case.

Keywords Probability theory · Monge-Ampere equations · Optimal transport problem

1 Introduction

The purpose of this exposition is to present one particularly beautiful connection between the Monge–Ampère equation and probability, specifically, a large deviation principle, discovered by Berman. Since the original work by Berman is still unpublished [4], and moreover deals with the more technically involved case where the gradient image is a polytope (that arises from toric varieties), it seemed more pedagogical to give an exposition that concentrates on subsequent work of Hultgren [17] that elaborates Berman's ideas in the case the gradient image has no boundary (that arises from Abelian varieties) as many of the key ideas are present already in the latter setting.

Dedicated to Steve Zelditch on the occasion of his $68th$ *birthday.*

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It is worth pointing out that while the lack of boundary is a simplification, Hultgren beautifully deals with a different set of technicalities that arises in the Abelian setting that is absent from Berman's toric setting: theta function analysis.

We completely skip the connection to Abelian varieties in this exposition as our goal was to present strictly the Monge–Ampère to LDP connection, stripping away the underlying geometry. Our exposition culminates in Theorem 8.8, due to Hultgren.

We take the opportunity to present an alternative proof to Theorem 8.8 that deals with all "tempratures" simultaneously instead of first reducing to the zero-temperature case as in the work of Berman and Hultgren (cf. [16, Remark 24, p. 59]). Basically this amounts to replacing an application of the Gärtner–Ellis theorem with a direct computation (we still use Gärtner–Ellis theorem in several other places). The proof we present culminates in Sect. 8.1 and is self-contained in the sense that we present essentially all the basic prerequisites from large deviations theory and optimal transport.

The family of Monge–Ampère equations Berman originally considers actually corresponds to and is inspired by the Ricci continuity method introduced by the author in 2008 [28] in connection with the Ricci flow and the search for Kähler–Einstein metrics. The idea there, explained in detail in the survey [29, Sect. 6], is to extend Aubin's continuity method originally defined for parameter values $t \in [0, 1]$ all the way 'back' in time to $t \in (-\infty, 1]$. This is motivated by the Ricci flow [28, Sect. 3], [29, Sect. 6] and is exploited heavily in subsequent work on existence of singular Kähler–Einstein metrics where the standard continuity method of Aubin cannot be readily used, but where the asymptotic analysis of the limit $t \to -\infty$ allows to bypass the difficulty in getting the continuity method 'started' [18, Sect. 9]. Berman [4] discovered a physics interpretation for this analytical gadget where the temperature corresponds precisely to $-1/t$, and so the limit $t \to -\infty$ becomes for him a 'zero-temperature limit'. Making this connection to physics proved extremely fruitful as it led Berman to several observations, including the LDP result we describe in this survey.

Goal of present work. The purpose of these lectures is to give a detailed exposition of some of Berman's ideas [4] in the setting of Hultgren's work [17] hopefully with some simplification (in particular the alternative proof mentioned above). We give some additional background in probability, hopefully to allow the dissemination of this beautiful piece of mathematics to a wider audience, given that the necessary background from probability might not be standard for most students in geometric analysis. We learned the little probability that we were able to present here from reading Berman and Hultgren [4, 17] as well as using the classic reference of Dembo–Zeitouni [8] and the more recent textbook of Rassoul-Agha–Seppäläinen [25] where thorough, and probably more accurate, presentations of the results in Sects. 3–6 can be found. For the results on optimal transport our main reference is Ambrosio–Gigli [1].

Organization. We start by giving some anecdotes from the history of relations between Monge–Ampère equations and probability in Sect. 2. This by no means is even an attempt at an exhaustive historical account. Rather, it is for the sake of placing the idea of Berman in a broader historical context. Section 3 serves as a gentle crash (oxymoron alert) course on large deviation principles (LDP). Section 4 discusses moment generating functions and the associated LDP with rate function coming from the Legendre transform of the function: this is the Gärtner–Ellis Theorem. Section 5 completes the proof of the Gärtner–Ellis Theorem. Section 6 discusses a general criterion for the existence of an LDP without using a moment generating function. Section 7 briefly reviews the fundamentals of optimal transportation, and computes, following Berman [4], the Legendre transform of Wasserstein distance as well as identifies the candidate rate function for a family of Monge–Ampère equations (34) related to the Ricci continuity method [28, 29]. Section 8 presents the proof of Berman–Hultgren's Theorem 8.8, showing an LDP for a sequence of empirical measures arising from theta functions on Abelian varieties (although we do not go into any of the underlying complex geometry, which is beautifully presented in Hultgren's work and in fact is one of the novelties of his work [17]). The rate function is related to optimal transport, and the whole construction is intimately related to solutions of the "master equation" (34). Most of Sect. 8 is devoted to our approach described above to the proof of Theorem 8.8, culminating in Sect. 8.1, and in Sect. 8.2 we present the original proof of Berman and Hultgren, and briefly compare the two approaches.

2 Monge–Ampère and Probability

The Laplace and Poisson equations have myriad probabilistic connections and interpretations, e.g., through Brownian motion, eigenfunctions, nodal sets [6, 19, 35]. Being the higher-dimensional analogue of these equations, one would expect similar, albeit more complicated, relations between the (homogeneous or non-homogeneous) Monge–Ampère equation and probability.

Perhaps Gaveau was the first to pioneer such relations, when he discovered that the solution to the complex Monge–Ampère equation can be expressed as the value function of a stochastic optimal control problem and found a semi-group that can be studied in relation to a parabolic version of the Monge–Ampère equation [12–14]. This generalized the classical probabilistic representation of the solution of the Laplace equation in one complex dimension. Another fundamental relation was discovered by Krylov who proved $C^{1,1}$ a priori estimates for the real Monge–Ampère equation, among other results [20–23]. We refer to Delarue [7] for excellent lecture notes that survey, expand, and give a pedagogical point of view on both Gaveau's and Krylov's achievements (and also cover the complex case for the latter). Another type of relation between Monge–Ampère and probability arises in the theory of optimal transportation (see, e.g., Villani [31, 32]) where one seeks a map pushing forward one probability measure into another. Indeed, the optimal (cost-minimizing) map can be expressed as the gradient of a convex function and the push-forward equation becomes a real Monge–Ampère equation.

Another classical connection appears through the theory of Markov semigroups, which in turn are closely related to the heat kernel. In this context hypercontractivity plays a central role and leads to (logarithmic and regular) Sobolev inequalities. This goes back to work of Gross [15]. For our anecdotal storytelling we mention that Bakry and Bakry–Ledoux [2, 3] showed how to use these ideas to establish Sobolev and diameter estimates in the presence of positive Ricci curvature. This was then applied in the setting of a degenerate complex Monge–Ampère equation [18, Proposition 6.2] to by-pass standard "Riemannian" proofs that do not readily apply in the degenerate setting.

A spectacular relation between Monge–Ampère equations and probability was discovered by Zelditch who together with collaborators studied several instances where large deviation principles (LDP) make their appearance in complex geometry [11, 33, 34]. In particular, Song–Zelditch found a large deviation principle that underlies the canonical Bergman approximation scheme [9, 24] for the Cauchy problem for the homogeneous real Monge–Ampère equation (this equation governs initial value geodesics in the space of Kähler metrics with toric symmetry) [30].

Berman subsequently discovered that an LDP holds also in the quite distinct setting of the second boundary value problem for non-homogeneous real Monge–Ampère equation [4] that appears naturally in the setting of toric Kähler manifolds as well as optimal transport. Since the gradient image in this setting is a polytope, there are issues with the corners and the boundary that render the computations more involved. For that reason, the subsequent follow-up work of Hultgren is more appropriate for our exposition, as in the setting of Abelian varieties that he studies the gradient image is a torus, while the main features of Berman's work are still present. The sections of the line bundles over Abelian varieties, theta functions, are more complicated to represent than the simple toric monomials that appear in the case of toric varieties, but that is not a steep price to pay for the lack of boundary. Although outside the scope of these notes, we mention that Berman also discovered an LDP in a sort of complex version of his aforementioned toric result in the case $\beta > 0$, where the toric variety is replaced by a polarized complex manifold and the role of the permanental point process is played by a determinantal point process [5].

3 Large Deviation Principles

We will be interested in asymptotic behavior, or more precisely, the asymptotic concentration, of a sequence of probability spaces

$$(\mathcal{X}, \mathcal{A}, \mu_n),$$

indexed by $n \in \mathbb{N}$. Here, $(\mathcal{X}, \mathcal{A})$ is a measure space, i.e., $\mathcal{X}$ is a set, also called the sample space, consisting of all possible outcomes, and $\mathcal{A}$ is a σ-algebra (the collection of measurable sets in $\mathcal{X}$), and $\{\mu_n\}$ are probability measures on $(\mathcal{X}, \mathcal{A})$, i.e., functions $\mu_n : \mathcal{A} \to [0, \infty]$ satisfying $\mu_n(\emptyset) = 0$ and $\mu_n(\cup_i A_i) = \sum_i \mu_n(A_i)$

whenever $A_k \cap A_l = \emptyset$ for all k, l, and with total mass $\mu_n(\mathcal{X}) = 1$. The last property is what makes a general measure a probability measure. Often, it is customary to omit $\mathcal{A}$ from the notation and refer to the triple $(\mathcal{X}, \mathcal{A}, \mu_n)$ simply by the notation $\mu_n \in P(\mathcal{X})$, where $P(\mathcal{X})$ denotes the space of probability measures on $(\mathcal{X}, \mathcal{A})$. Since the next definition requires a topology, we will always assume our measure space is Borel (i.e., the measurable sets are generated by the open ones).

Definition 3.1 We say that $\{(\mathcal{X}, \mathcal{A}, \mu_n)\}_n$ (or, for brevity, sometimes just $\{\mu_n\}_n$) satisfies a large deviation principle with normalization $r_n \nearrow \infty$ (and denote this statement by LDP(μ_n, r_n)) if there exists a lower semicontinuous function $I : \mathcal{X} \to [0, \infty]$ such that

$$\liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(O) \geq -\inf_O I, \qquad \forall\, O \quad \text{open in } \ \mathcal{X},$$

and

$$\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(C) \leq -\inf_C I, \qquad \forall\, C \quad \text{closed in } \ \mathcal{X}.$$

Under mild assumptions on $\mathcal{X}$, there is actually no ambiguity in the *rate function* I [25, Theorem 2.13]. Here is where the stipulation that the rate function be lower semicontinuous is relevant.

Lemma 3.2 *Suppose $\mathcal{X}$ is a regular topological space and that the sequence of probability spaces $(\mathcal{X}, \mathcal{A}, \mu_n)$ satisfies $LDP(\mu_n, r_n)$. Then,*

$$I(x) = \sup\left\{ -\liminf \frac{1}{r_n} \log \mu_n(O) : O \ni x, O \quad \textit{is an open set in } \mathcal{X} \right\}.$$

Remark 3.3 For a sort of converse see Proposition 6.1.

Remark 3.4 The definition of a regular topological space will be given in the proof shortly.

Proof Define I by the above formula. Suppose that LDP(μ_n, r_n) holds with rate function F. By Definition 3.1, whenever O is an open set in $\mathcal{X}$ with $O \ni x$,

$$F(x) \geq \inf_O F \geq -\liminf \frac{1}{r_n} \log \mu_n(O).$$

Taking the supremum over all such O does not change the left hand side, while the right hand side becomes $I(x)$. Thus, $F \geq I$.

Conversely, fix x and let c be such that $c < F(x)$. It suffices to show that $c \leq I(x)$. Indeed, the assumption on $\mathcal{X}$ means that any point can be separated from any closed set not containing it by means of disjoint open sets. Thus, we can separate x from the closed set $\{F \leq c\} \not\ni x$, i.e., choose open $G \ni x$ with $\overline{G} \cap \{F \leq c\} = \emptyset$, i.e., $\overline{G} \subset \{F > c\}$. Now, since F is lower semicontinuous its inf over any closed set is attained. In particular, $\inf_{\overline{G}} F \geq c$. (Note that we could not otherwise conclude $c \leq \inf_{\{F>c\}} F$!) Thus,

$$c \leq \inf_{\overline{G}} F \leq -\limsup \frac{1}{r_n} \log \mu_n(\overline{G}) \leq -\liminf \frac{1}{r_n} \log \mu_n(G) \leq I(x),$$

so $F(x) \leq I(x)$, concluding the proof. □

The fact that the function I is nonnegative is crucial: it means probability of events (an event is an element of $\mathcal{A}$) is exponentially decaying in general, with *rate*

$$`r_n \times \text{infimum of } I \text{ over the closure of the event}'.$$

Of particular interest are therefore the zeros of the rate function, i.e., the events $I^{-1}(0)$ (when this set is non-empty; it is always non-empty if the rate function is *good* [8, p. 4], i.e., I has compact sub-level sets in $\mathcal{X}$). The significance of zeros is nicely captured in terms of random variables, which we now turn to discuss.

Remark 3.5 Note that by setting $O = C = \mathcal{X}$ it follows that $\inf I = 0$.

⋆ ⋆ ⋆

Probability measures and random variables can often be interchanged in the discussion, and by abuse of terminology this will sometimes be the case. Let us briefly discuss the terminology involved. Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a probability space and let $(\mathcal{Y}, \mathcal{B})$ be a measure space (a typical example is $\mathbb{R}$ with $\mathcal{B}$ being the usual Borel sets). A *random variable* with values in $\mathcal{Y}$ is a measurable function $X : \mathcal{X} \to \mathcal{Y}$ (here, measurable takes into account both $\mathcal{A}$ and $\mathcal{B}$). To such an X one may associate a probability space $(\mathcal{Y}, \mathcal{B}, \nu)$ defined by

$$\nu(B) := \mu(\{x \in \mathcal{X} \; : \; X(x) \in B\}), \quad \text{for } B \in \mathcal{B}$$

(the previous formula is often written, with some abuse, as $\nu(B) := \mu\{X \in B\}$). In other words, $\nu(B) := \mu(X^{-1}(B))$, or,

$$\nu = X_{\#}\mu.$$

One also refers to ν as the *law* of X. Note that often when discussing the random variable X the "background space" $(\mathcal{X}, \mathcal{A}, \mu)$ is completely auxiliary/irrelevant since one is completely focused on $(\mathcal{Y}, \mathcal{B}, \nu)$. Thus, often one does not distinguish between X and the resulting or pushed-forward measure ν.

⋆ ⋆ ⋆

When the sequence of probability spaces

$$(\mathcal{X}, \mathcal{A}, \mu_n)$$

happen to correspond to the laws of a sequence of random variables X_n (on some auxiliary probability space $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{A}}, \widetilde{\mu})$) with values in $\mathcal{X}$, the zeros of the rate function have the following meaning. We say that $x \in \mathcal{X}$ is a limiting value of $\{X_n\}$ with probability c if

$$\lim_n \widetilde{\mu}(\{|X_n - x| < \epsilon\}) = \lim_n \mu_n(B_\epsilon(x)) = c,$$

for all $\epsilon > 0$ small.

Lemma 3.6 *Consider a sequence of random variables X_n whose laws μ_n satisfy LDP(μ_n, r_n). Suppose that $x \in \mathcal{X}$ is a limiting value of $\{X_n\}$ with probability $c > 0$ (independent of n). Then $x \in I^{-1}(0)$. In fact, $x \in I^{-1}(0)$ whenever $\frac{1}{r_n} \log \mu_n(\{|X_n - x| < \epsilon\}) \to 0$.*

To give an intuitive idea of Lemma 3.6, at least in the situations that we will be interested in, $x \in I^{-1}(0)$ whenever $x \in \mathcal{X}$ is a "limiting value of X_n with probability decaying slower than e^{-Cr_n} (for all $C > 0$)."

Proof One has

$$\inf_{\overline{B_\epsilon(x)}} I \leq -\limsup \frac{1}{r_n} \log \mu_n(X_n \in \overline{B_\epsilon(x)}) = -\limsup \frac{1}{r_n} \log \widetilde{\mu}_n(X_n^{-1}(\overline{B_\epsilon(x)})) = 0.$$

Letting $\epsilon \to 0$ and using $I \geq 0$ and lower semicontinuity guarantees $I(x) = 0$ since $I \geq 0$. □

One particular situation of practical interest is when $I^{-1}(0)$ is a singleton. In that case a sort of converse of the previous statement holds.

Corollary 3.7 *Suppose that the sequence of probability spaces $(\mathcal{X}, \mathcal{A}, \mu_n)$ satisfies LDP(μ_n, r_n) with a good rate function I (i.e., I has compact sub-level sets in $\mathcal{X}$) and that $I^{-1}(0) = \{x\} \subset \mathcal{X}$. Then $\mu_n \to \delta_x$ weakly.*

Proof First, note that. □

Claim 3.8 *For all closed $C \subset \mathcal{X}$,*

$$\limsup \mu_n(C) \leq \delta_x(C).$$

Proof Indeed, this is trivial when $x \in C$ since then the right-hand side equals 1. Otherwise,

$$\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(C) \leq -\inf_C I =: -\epsilon < 0,$$

since I is good and so the infimum is attained, and by assumption it cannot be zero as x is the only zero of the nonnegative function I. Thus, for n large

$$\frac{1}{r_n}\log\mu_n(C) \le -\inf_C I = -\epsilon,$$

so $\mu_n(C) < e^{-r_n\epsilon}$ and recalling that $r_n \nearrow \infty$ (Definition 3.1), $\limsup \mu_n(C) = 0 = \delta_x(C)$ as desired. $\square$

The proof now follows from the so-called portmanteau theorem, but we give the quick proof for completeness. Note that Claim 3.8 implies

$$\liminf \mu_n(O) \ge \delta_x(O),$$

for all open $O \subset \mathcal{X}$ (by looking at $C = \mathcal{X}\setminus O$). To show the convergence is equivalent to showing

$$\lim \int_{\mathcal{X}} f\mu_n = \int_{\mathcal{X}} f\delta_x,$$

for all bounded continuous functions f. Indeed, if $0 \le a \le f \le b$,

$$\begin{aligned}
\liminf \int_{\mathcal{X}} f\mu_n &= \liminf \int_a^b \mu_n\{f > t\}dt \\
&\ge \int_a^b \liminf \mu_n\{f > t\}dt \\
&\ge \int_a^b \delta_x\{f > t\}dt \\
&= \int_{\mathcal{X}} f\delta_x.
\end{aligned}$$

Repeating the above computation for $-f$ gives $\limsup \int_{\mathcal{X}} f\mu_n \le \int_{\mathcal{X}} f\delta_x$, so Corollary 3.7 follows.

4 Moment Generating Functions

From now on we will further impose that $\mathcal{X}$ has the structure of a topological vector space so that we have notions of a dual space $\mathcal{X}^*$ consisting of linear functionals on $\mathcal{X}$ and a corresponding pairing denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$.

The logarithm of the moment generating function of the sequence of probability spaces $(\mathcal{X}, \mathcal{A}, \mu_n)$ with normalization $\{r_n\}$ is defined by

$$p(\theta) := \lim \frac{1}{r_n}\log\int_{\mathcal{X}} e^{r_n\langle\theta,x\rangle}\mu_n(x),$$

assuming the limit exists and is finite, for each $\theta \in \mathcal{X}^*$. (For example, if $\mathcal{X} = \mathbb{R}^n$ then $\mathcal{X}^* = \mathbb{R}^n$, and if $\mathcal{X} = P(\mathbb{R}^d)$ then $\mathcal{X}^* = C_b^0(\mathbb{R}^d)$.)

A generating function encodes a lot of information, as discovered by Gärtner, and rediscovered by Ellis. The following theorem holds for rather general $\mathcal{X}$ but we will prove it for $\mathcal{X} = \mathbb{R}^n$ although the proof essentially works verbatim for any locally convex topological vector space $\mathcal{X}$ under the assumption of exponential tightness of the sequence of measures (defined in Sect. 5.2). Being a novice in the field, the author will follow Berman and Zelditch and refer to the next theorem as the Gärtner–Ellis Theorem, although a more accurate attribution of credit is given in the historical notes in Dembo–Zeitouni to which the reader is warmly referred to for much more accurate statements of all the results on LDPs that we discuss in these notes [8, Sect. 2.3].

In general, exponential tightness is a necessary assumption (which will hold in our specific setting in $\mathcal{X} = \mathbb{R}^n$ (see Lemma 5.5) and $\mathcal{X} = P(X)$ for X a compact manifold). In the following we denote by $f^\star$ the Legendre dual of f [27, p. 104],

$$f^\star(y) := \sup_x [\langle x, y\rangle - f(x)]. \tag{1}$$

(We will later use this same definition more generally for functions on abstract spaces where the pairing will be taken to be the natural one in each setting.)

Theorem 4.1 *Suppose that the moment generating function p of the sequence of probability spaces $(\mathcal{X}, \mathcal{A}, \mu_n)$ with normalization $\{r_n\}$ is well defined (in particular, finite) and Gateaux differentiable. Then LDP(μ_n, r_n) with rate function $p^\star$.*

Before proving the Gärtner–Ellis Theorem let us prove two famous corollaries thereof.

4.1 Cramér's Theorem

Let $\{X_i\}_{i=1}^n$ be independent, identically distributed, random variables (i.i.d.r.v.) on $\widetilde{\mathcal{X}}$ with values in $\mathbb{R}$. This means that the law of X_i is equal to some $\mu \in P(\mathbb{R})$ regardless of i. The *sample mean* is by definition the random variable with values in $\mathbb{R}$,

$$S_n := \sum X_i/n.$$

This is the "probability" notation. Recalling that a random variable is really a function leads to a more precise notation. The random variable S_n is the measurable function $S_n : \widetilde{\mathcal{X}}^n \to \mathbb{R}$

$$(x_1, \ldots, x_n) \mapsto [X_1(x_1) + \ldots X_n(x_n)]/n.$$

This is really the composition

$$(x_1, \ldots, x_n) \mapsto (X_1(x_1), \ldots, X_n(x_n)) \mapsto [X_1(x_1) + \ldots X_n(x_n)]/n,$$

and so the law of S_n is the push forward of the law of the $\mathbb{R}^n$-valued random variable

$$X_1 \otimes \cdots \otimes X_n := (X_1, \ldots, X_n)$$

under the "mean map"

$$s_n : (k_1, \ldots, k_n) \mapsto \sum k_i/n.$$

The law of $X_1 \otimes \cdots \otimes X_n$ is $\mu \otimes \cdots \otimes \mu$. Thus, the law of S_n is $\mu_n := (s_n)_\# \mu^{\otimes n}$.

Corollary 4.2 *Suppose that $\mu = f dx$ with $f \in C^0(\mathbb{R})$ and with compact support. Then LDP(μ_n, n) with rate function $p^\star$.*

Proof Let $\theta \in \mathbb{R}$. The moment generating function is

$$\begin{aligned} p(\theta) &= \lim \frac{1}{n} \log \int_{\mathbb{R}} e^{nx\theta} \mu_n(x) \\ &= \lim \frac{1}{n} \log \int_{\mathbb{R}^n} e^{n\, s_n(a_1,\ldots,a_n)\theta} \mu^{\otimes n}(a_1, \ldots, a_n) \\ &= \lim \frac{1}{n} \log \int_{\mathbb{R}} e^{a_1\theta} \mu(a_1) \cdots \int_{\mathbb{R}} e^{a_n\theta} \mu(a_n) \\ &= \lim \frac{1}{n} \log \Big(\int_{\mathbb{R}} e^{a\theta} \mu(a)^n \Big)^n \\ &= \log \int_{\mathbb{R}} e^{a\theta} \mu(a) \end{aligned} \tag{2}$$

This is C^1 because of the assumption on μ so we are done by Theorem 4.1. $\square$

4.2 Sanov's Theorem

The projection via the mean map gives rather crude information. Set $X^n := X \otimes \cdots \otimes X$. Another sequence of measures that can be obtained from the n-fold product via the "empirical" map $\delta^n : X^n \to P(X)$,

$$\delta^n(x_1, \ldots, x_n) := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}.$$

The measures

$$\Gamma_n := (\delta^n)_\# \mu^{\otimes n} \in P(P(X)), \tag{3}$$

all live on the same space and therefore can be studied via a large deviation principle, if the associated moment generating function exists. Note that Γ_n is the law of the random variable $\delta^n : (X^n, \mu^{\otimes n}) \to P(X)$, i.e., of the *random measure* δ^n (where the randomness is determined by $\mu^{\otimes n}$, i.e., by sampling n points in X independently, each according to μ).

Define the entropy functional $\mathrm{Ent} : P(X) \times P(X) \to \mathbb{R}$,

$$\mathrm{Ent}(\mu, \nu) := \int_X \log \frac{\nu}{\mu} \nu, \tag{4}$$

whenever ν is absolutely continuous with respect to μ, and ∞ otherwise.

Corollary 4.3 *Suppose that $\mu = f dx$ with $f \in C^0(X)$ and with compact support. LDP(Γ_n, n) with rate function* $\mathrm{Ent}(\mu, \cdot)$.

Proof Now $\mathcal{X} = P(X)$. Let $\theta \in C_b^0(X) = \mathcal{X}^*$. The moment generating function is

$$\begin{aligned} p(\theta) &= \lim \frac{1}{n} \log \int_{P(X)} e^{n\langle \theta, \nu \rangle} \Gamma_n(\nu) \\ &= \lim \frac{1}{n} \log \int_{P(X)} e^{n\langle \theta, \nu \rangle} (\delta_n)_{\#} \mu^{\otimes n} \\ &= \lim \frac{1}{n} \log \int_{X^n} e^{n\langle \theta, \delta^n(x_1, \ldots, x_n) \rangle} \mu(x_1) \otimes \cdots \mu(x_n) \\ &= \lim \frac{1}{n} \log \int_{X^n} e^{n\langle \theta, \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \rangle} \mu(x_1) \otimes \cdots \mu(x_n) \\ &= \lim \frac{1}{n} \log \int_{X^n} e^{\sum_{i=1}^n \theta(x_i)} \mu(x_1) \otimes \cdots \mu(x_n) \\ &= \lim \frac{1}{n} \log \left(\int_X e^{\theta} \mu \right)^n \\ &= \log \int_X e^{\theta} \mu. \end{aligned} \tag{5}$$

This is C^1 because of the assumption on μ so we are done by Theorem 4.1 and Lemma 4.4 below. □

Define $I : C^0(X) \to \mathbb{R}$,

$$I_\mu(\theta) := \log \int_X e^{\theta} \mu. \tag{6}$$

Recall the definition of the Legendre transform (1), where in the following lemma the pairing is taken to be the usual "integration pairing"' between functions and measures.

Lemma 4.4 *The Legendre transform of* $\mathrm{Ent}(\mu, \cdot)$ *is* I_μ *and vice versa.*

Proof First, I_μ is convex on $C(X)$ since it is a moment generating function (see Lemma 4.5). Alternatively, the arguments in the proof of Lemma 4.5 show convexity.

We will show that Legendre transform of I_μ is $\mathrm{Ent}(\mu,\ \cdot\ )$ which therefore will imply that the latter is also convex. We claim that

$$\mathrm{Ent}(\mu, \nu) + I_\mu(\theta) \geq \langle \theta, \nu \rangle. \tag{7}$$

Indeed,

$$\begin{aligned} I_\mu(\theta) - \langle \theta, \nu \rangle &= \log \int_X e^\theta \mu - \langle \theta, \nu \rangle \\ &= \log \int_X e^\theta \frac{\mu}{\nu} \nu - \langle \theta, \nu \rangle \\ &\geq \int_X \log \left(e^\theta \frac{\mu}{\nu} \right) \nu - \langle \theta, \nu \rangle \\ &= \int_X \left(\theta + \log \frac{\mu}{\nu} \right) \nu - \langle \theta, \nu \rangle = -\mathrm{Ent}(\mu, \nu), \end{aligned}$$

with equality if and only if $\nu = e^\theta \mu / \int e^\theta \mu$ (so that $\nu \in P(X)$). Thus, $\mathrm{Ent}(\mu, \nu) \geq I_\mu^\star(\nu) := \sup_\theta [\langle \theta, \nu \rangle - I_\mu(\theta)]$. On the other hand, putting $\theta = \log \frac{\nu}{\mu}$,

$$\left\langle \log \frac{\nu}{\mu}, \nu \right\rangle - I_\mu \left(\log \frac{\nu}{\mu} \right) = \mathrm{Ent}(\mu, \nu) - 0,$$

so $\mathrm{Ent}(\mu, \nu) \leq \sup_\theta [\langle \theta, \nu \rangle - I_\mu(\theta)]$. Thus, $\mathrm{Ent}(\mu, \nu) = I_\mu^\star(\nu)$ and so in particular from the general property of the Legendre transform (1) (see also [27, p. 104]),

$$f(x) + f^\star(y) \geq \langle x, y \rangle, \tag{8}$$

it follows that (7) holds, as claimed. Equation (7) gives,

$$\sup_\nu \big[\langle \theta, \nu \rangle - \mathrm{Ent}(\mu, \nu) \big] \leq I_\mu(\theta),$$

and now putting $\nu = e^\theta \mu / \int e^\theta \mu$ we see equality is attained, so I_μ is the Legendre transform of $\mathrm{Ent}(\mu,\ \cdot\ )$, concluding the proof. □

4.3 Properties of the Moment Generating Function

Lemma 4.5 *The moment generating function is convex.*

Proof The pointwise limit of a sequence of convex functions is convex (one way to think about it is in terms of the epigraphs— and clearly the limits of convex sets is a convex sets, and the limits of epigraphs is moreover an epigraph). Thus, it suffices to show that

$$\frac{1}{r_n}\log\int_{\mathcal{X}} e^{r_n\langle\theta,x\rangle}\mu_n(x)$$

is a convex function. Indeed, since $||fg||_{L^1(\mu)} \le ||f||_{L^2(\mu)}||g||_{L^2(\mu)}$,

$$\begin{aligned}\frac{1}{r_n}\log\int_{\mathcal{X}} e^{r_n\langle(\theta_1+\theta_2)/2,x\rangle}\mu_n(x) &= \frac{1}{r_n}\log\int_{\mathcal{X}}\sqrt{e^{r_n\langle\theta_1,x\rangle}}\sqrt{e^{r_n\langle\theta_2,x\rangle}}\mu_n(x)\\ &\le \frac{1}{r_n}\log\left(\sqrt{\int_{\mathcal{X}} e^{r_n\langle\theta_1,x\rangle}\mu_n(x)}\sqrt{\int_{\mathcal{X}} e^{r_n\langle\theta_2,x\rangle}\mu_n(x)}\right)\\ &= \frac{1}{2}\left(\frac{1}{r_n}\log\int_{\mathcal{X}} e^{r_n\langle\theta_1,x\rangle}\mu_n(x) + \frac{1}{r_n}\log\int_{\mathcal{X}} e^{r_n\langle\theta_2,x\rangle}\mu_n(x)\right),\end{aligned}$$

as desired. □

Lemma 4.6 *$p^\star$ is convex and nonnegative.*

Proof By definition

$$p^\star(x) := \sup_\theta[\langle x,\theta\rangle - p(\theta)]$$

is a supremum of affine functions, hence it is convex. Plugging in $\theta = 0$ and using that $p(0) = 0$ it follows that $p^\star(x) \ge 0$. □

The reader that compares the statement of Theorem 4.1 to that in some books might note that one does not really need to assume the moment generating function is differentiable and certain weaker assumptions are enough. One of them though is automatic from convexity:

Lemma 4.7 *If there exists a small ball B about the origin on which $p < \infty$ then $p > -\infty$ everywhere.*

Proof This is a general fact about convex functions that can be proved as follows:

$$p(0) \le \frac{1}{1+C}p(-C\theta) + \frac{C}{1+C}p(\theta),$$

so

$$\frac{C}{1+C}p(\theta) \ge p(0) - \frac{1}{1+C}p(-C\theta).$$

Now, choose $C > 0$ small enough so that $C\theta \in B$ and note $p(0) = 0$. □

5 Proof of the Gärtner–Ellis Theorem

The goal of this section is to give a proof of Theorem 4.1. The proof of the upper bound (the one about close sets) is a little easier and so we will go over it at first.

There are two main steps: first, prove the upper bound for compact sets; second, show that compact sets capture the general case.

5.1 The Upper Bound for Compact Sets

The upper bound for compact sets is sometimes called the weak upper bound LDP. Let $C \subset \mathcal{X}$ then be a compact set. We claim that

$$\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(C) \leq -\inf_C p^\star.$$

Fix $\delta > 0$. For each $x \in \mathcal{X}$ let $y(x) \in \mathcal{X}^*$ satisfy $\langle x, y(x)\rangle - p(y(x)) > p^\star(x) - \delta$ and let $B_x^{\delta,y(x)}$ be a neighborhood of $x \in \mathcal{X}$ defined as follows

$$B_x^{\delta,y(x)} := \{z \in \mathcal{X} \,:\, |\langle z, y(x)\rangle - \langle x, y(x)\rangle| < \delta\}.$$

Finitely many neighborhoods of affine subspaces $B_{x_1}^{\delta,y_1(x_1)}, \ldots, B_{x_m}^{\delta,y_m(x_m)}$ cover C by compactness. Observe that asymptotically we reduce the calculations for the "left-hand side" to one ball:

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(C) &\leq \limsup_{n\to\infty} \frac{1}{r_n} \log \sum_{i=1}^m \mu_n(B_{x_i}^{\delta,y_i(x_i)}) \\
&= \limsup_{n\to\infty} \frac{1}{r_n} \log \Big(m \sup_{i\in\{1,\ldots m\}} \mu_n(B_{x_i}^{\delta,y_i(x_i)})\Big) \\
&= \limsup_{n\to\infty} \left[\frac{1}{r_n} \log m + \frac{1}{r_n} \log \sup_{i\in\{1,\ldots m\}} \mu_n(B_{x_i}^{\delta,y_i(x_i)})\right] \\
&= \limsup_{n\to\infty} \frac{1}{r_n} \log \sup_{i\in\{1,\ldots m\}} \mu_n(B_{x_i}^{\delta,y_i(x_i)}).
\end{aligned}$$

Now,

$$\begin{aligned}
\mu_n(B_{x_i}^{\delta,y_i(x_i)}) &= \int_{B_{x_i}^{\delta,y_i(x_i)}} \mu_n(z) \\
&= \int_{B_{x_i}^{\delta,y_i(x_i)}} e^{r_n\langle z,y_i(x_i)\rangle} e^{-r_n\langle z,y_i(x_i)\rangle} \mu_n \\
&\leq e^{r_n\delta - r_n\langle x_i,y_i(x_i)\rangle} \int_{B_{x_i}^{\delta,y_i(x_i)}} e^{r_n\langle z,y_i(x_i)\rangle} \mu_n \\
&\leq e^{r_n\delta - r_n\langle x_i,y_i(x_i)\rangle} \int_{\mathcal{X}} e^{r_n\langle z,y_i(x_i)\rangle} \mu_n
\end{aligned}$$

(integration in the z variable). Taking log and the limit,

$$\limsup \frac{1}{r_n} \log \mu_n(B_{x_i}^{\delta, y_i(x_i)}) \leq -\langle x_i, y_i(x_i)\rangle + \delta + p(y_i(x_i)) \leq -p^\star(x_i) + 2\delta \leq -\inf_C p^\star + 2\delta.$$

Letting δ tend to zero completes the argument.

5.2 *Exponential Tightness*

Definition 5.1 A sequence of probability measures $\{\mu_n\} \subset P(\mathcal{X})$ is exponentially tight with normalization r_n if for each $b \in (0, \infty)$ there exists a compact set $K_b \subset \mathcal{X}$ such that

$$\limsup \frac{1}{r_n} \log \mu_n(\mathcal{X} \backslash K_b) \leq -b.$$

Remark 5.2 The point of course is that K_b is independent of n.

Remark 5.3 Note, of course, that exponential tightness is automatic if $\mathcal{X}$ is compact (or if there is a compact set containing the support of all the μ_n)! In particular, note that $\mathcal{X} = P(B)$ (for some compact (finite-dimensional) manifold B) is a compact set by Prokhorov's theorem [32, p. 43], [1, Theorem 1.3]: Let $\mathcal{X}$ be a Polish space and $P \subset P(\mathcal{X})$; then P is pre-compact for the week topology if and only if for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset \mathcal{X}$ such that $\mu(\mathcal{X} \backslash K_\epsilon) \leq \epsilon$ for all $\mu \in P$.

Lemma 5.4 *Suppose that the large deviation upper bound inequality holds for* (μ_n, r_n) *for all compact sets. Suppose also that sequence of probability measures* $\{\mu_n\} \subset P(\mathcal{X})$ *is exponentially tight with normalization* r_n*. Then the large deviation upper bound inequality holds.*

Proof Then equality hold for all compact sets by assumption. So consider a closed set F that is not necessarily compact. Of course, $\mu_n(F) \leq \mu_n(F \cap K_b) + \mu_n(\mathcal{X} \backslash K_b)$. This is a very coarse inequality (since $\mathcal{X} \backslash K_b$ is a large set!), but it does the job since $\mu_n(\mathcal{X} \backslash K_b)$ is uniformly small and $F \cap K_b$ is compact (so we can apply the large deviation upper bound to it):

$$\begin{aligned}
\limsup \frac{1}{r_n} \log \mu_n(F) &\leq \limsup \frac{1}{r_n} \log[\mu_n(F \cap K_b) + \mu_n(\mathcal{X} \backslash K_b)] \\
&\leq \max\{-b, \limsup \frac{1}{r_n} \log \mu_n(F \cap K_b)\} \\
&\leq \max\{-b, -\inf_{F \cap K_b} I\} \\
&\leq \max\{-b, -\inf_F I\},
\end{aligned}$$

and by choosing $b > \inf_F I$ (recall $I \geq 0$ by assumption) we get $\limsup \frac{1}{r_n} \log \mu_n(F) \leq -\inf_F I$. □

Thus, to complete the proof of the upper bound it remains to show:

Lemma 5.5 *The sequence of probability measures* $\{\mu_n\} \subset P(\mathcal{X})$ *is exponentially tight with normalization* r_n.

Proof For this proof we assume $\mathcal{X} = \mathbb{R}^n$ (otherwise, one needs to incorporate the exponential tightness assumption into the assumptions of Theorem 4.1). This follows directly from the assumption that a moment generating function exists. Indeed, choose a coordinate x_i and bound the tail in that direction:

$$\begin{aligned}\mu_n\{x_i \geq b\} &= \int_{\{x_i \geq b\}} e^{-r_n\langle\theta,x\rangle} e^{r_n\langle\theta,x\rangle} \mu_n(x) \\ &\leq e^{-r_n b|\theta|} \int_{\{x_i \geq b\}} e^{r_n\langle\theta,x\rangle} \mu_n(x) \\ &\leq e^{-r_n b|\theta|} \int_{\mathcal{X}} e^{r_n\langle\theta,x\rangle} \mu_n(x) \\ &= e^{-r_n b|\theta|} p_n(\theta)\end{aligned}$$

Now,

$$\frac{1}{r_n} \log p_n(\theta) = p(\theta) + o(1),$$

so

$$p_n(\theta) = e^{r_n(p(\theta)+o(1))}.$$

So, fixing θ and then choosing $b > 0$ sufficiently large ($p(\theta)$ is finite!), and summing over all coordinate directions concludes the proof. $\square$

5.3 The Lower Bound

Our goal is to show that

$$\liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(O) \geq -\inf_O p^\star, \qquad \forall\ O \quad \text{open in } \mathcal{X}.$$

Fix an open set O and a point $z \in O$ where $\inf_O p^\star$ is attained up to some ϵ. First, as in the proof of the upper bound, we will show that $\frac{1}{r_n} \log \mu_n(O)$ is essentially equal to $\frac{1}{r_n} \log \mu_n(B_\delta)$ for some ball B_δ containing z. Indeed, for any B and any $\sigma \in \mathcal{X}^*$,

$$\frac{1}{r_n} \log \mu_n(B) = \frac{1}{r_n} \log \int_B e^{-r_n\langle z,\sigma\rangle} e^{r_n\langle z,\sigma\rangle} \mu_n(z) \tag{9}$$

We want to do essentially the same computation as for the upper bound, except that now of course the inequality

$$\int_B e^{r_n\langle z,\sigma\rangle}\mu_n \le \int_{\mathcal{X}} e^{r_n\langle z,\sigma\rangle}\mu_n$$

goes in the wrong direction. To remedy that, we need to identify some way of localizing the integral so that, at least asymptotically, the integrals are equal. The key is to notice that the relevant point for localizing is

$$\sigma := (\nabla p)^{-1}(z).$$

Consider

$$e^{\langle\sigma,\,\cdot\,\rangle}\mu_n,$$

or, rather, the associated probability measures

$$\nu_{\sigma,n} := e^{\langle\sigma,\,\cdot\,\rangle}\mu_n \Big/ \int_{\mathcal{X}} e^{\langle\sigma,y\rangle}\mu_n(y).$$

Lemma 5.6 *The sequence of probability measures* $\{\nu_{\sigma,n}\}$ *localizes around* $z = \nabla p(\sigma)$*. More precisely, we have the upper bound large deviation inequality for* $\{(\nu_{\sigma,n}, r_n)\}$ *with rate function* $p^\star - \langle\sigma,\,\cdot\,\rangle + p(\sigma)$*.*

Remark 5.7 Recall that

$$p^\star(z) = \langle\sigma, z\rangle - p(\sigma), \tag{10}$$

and

$$p^\star(\,\cdot\,) + p(\sigma) > \langle\sigma,\,\cdot\,\rangle \qquad \text{away from } z. \tag{11}$$

We postpone the proof of Lemma 5.6 to the end of the section.

Thus, the rate function in the statement is nonnegative with a unique zero at z. Thus the desired localization:

Corollary 5.8 *For any* $\delta > 0$*,*

$$\lim_n \nu_{\sigma,n}(B_\delta^z) = 1.$$

Proof It suffices to show that

$$\limsup_n \frac{1}{r_n}\log \nu_{\sigma,n}(\mathcal{X}\backslash B_\delta^z) < 0.$$

By the large deviation upper bound inequality for $\{(\nu_{\sigma,n}, r_n)\}$ and (11) (note $\mathcal{X}\backslash B_\delta^z$ is closed!),

$$\limsup_n \frac{1}{r_n}\log \nu_{\sigma,n}(\mathcal{X}\backslash B_\delta^z) \le -\inf_{x\notin B_\delta^z}[p^\star(x) - \langle\sigma, x\rangle + p(\sigma)] \le -C,$$

for some $C = C(\delta)$, as desired. $\square$

Thus, as we wished,

$$\int_{B_\delta^z} \nu_{\sigma,n} = \int_{\mathcal{X}} \nu_{\sigma,n} + o(1) = 1 + o(1).$$

Now we are ready to go back to (9),

$$\begin{aligned}
\frac{1}{r_n}\log \mu_n(B_\delta^z) &= \frac{1}{r_n}\log \int_{B_\delta^z} e^{-r_n\langle\sigma,x\rangle} e^{r_n\langle\sigma,x\rangle}\mu_n(z)\\
&= \frac{1}{r_n}\log \int_{B_\delta^z} e^{-r_n\langle\sigma,x\rangle} p_n(\sigma)\nu_{\sigma,n}(x)\\
&= p(\sigma) + o(1) + \frac{1}{r_n}\log \int_{B_\delta^z} e^{-r_n\langle\sigma,x\rangle}\nu_{\sigma,n}(x)\\
&\geq p(\sigma) + o(1) + \frac{1}{r_n}\log \inf_{B_\delta^z} e^{-r_n\langle\sigma,x\rangle} + \frac{1}{r_n}\log \int_{B_\delta^z} \nu_{\sigma,n}(x)\\
&\geq p(\sigma) + o(1) - \langle\sigma, z\rangle - \delta + o(1)\\
&= -p^\star(z) + o(1) - \delta,
\end{aligned}$$

where we used Corollary 5.8 and (10). Letting first n go to infinity and then δ go to zero concludes the proof.

Proof (Proof of Lemma 5.6) First, observe that the Legendre transform of

$$p(\cdot\, + \sigma) - p(\sigma)$$

is

$$p^\star - \langle\sigma, \cdot\rangle + p(\sigma). \tag{12}$$

Thus, it suffices to show that the moment generating function of $\{\nu_{\sigma,n}\}$ is $p(\cdot\, + \sigma) - p(\sigma)$. Indeed,

$$\begin{aligned}
\lim \frac{1}{r_n}\log \int_{\mathcal{X}} e^{r_n\langle\theta,x\rangle}\nu_{\sigma,n}(x) &= \lim \frac{1}{r_n}\log \int_{\mathcal{X}} e^{r_n\langle\theta,x\rangle} e^{\langle\sigma,x\rangle}\mu_n(x) \Big/ p_n(\sigma)\\
&= -p(\sigma) + \lim \frac{1}{r_n}\log \int_{\mathcal{X}} e^{r_n\langle\theta+\sigma,x\rangle}\mu_n(x)\\
&= -p(\sigma) + p(\theta + \sigma).
\end{aligned}$$

All the assumptions of Theorem 4.1 are met for this generating function since they are met for p. Under those assumptions we have already established the upper bound inequality in Theorem 4.1. Therefore, we have the upper bound large deviation inequality for $\{(\nu_{\sigma,n}, r_n)\}$ with rate function (12). $\square$

6 LDP Without Moment Generating Functions

Sometimes, a large deviation principle holds even when the assumptions of Theorem 4.1 are not satisfied. For one, a moment generating function may not exist. Also, the rate function can sometimes be nonconvex (not that Theorem 4.1 guarantees the rate function will be convex, being the Legendre transform of the moment generating function). The following result nevertheless characterizes large deviation principles and gives a useful tool to show their existence. It is sort of a converse for Lemma 3.2.

Proposition 6.1 *Let $\mathcal{X}$ be a compact metric space. LDP (μ_n, r_n) if and only if*

$$\lim_{d\to 0} \limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_d(x)) = \lim_{d\to 0} \liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_d(x)), \quad \forall x \in \mathcal{X}. \tag{13}$$

The rate function is then equal to minus (13).

Proof $\Longleftarrow$: Suppose (13) holds, and denote either side by $-g(x)$. Note that g is indeed a rate function: it is evidently not negative, and it is lower semicontinuous because the super level sets $\{g > a\}$ are open as if $g(x) > a$, i.e., $-g(x) < -a$, then because the limits in (13) are decreasing in d, then for some $B_d(x)$,

$$\liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_d(x)) < -a,$$

so for every $y \in B_d(x)$ choosing d' so that $B_{d'}(y) \subset B_d(x)$,

$$\liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_{d'}(y)) \leq \liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_d(x)) < -a,$$

so $-g(y) < -a$.

To prove the lower bound, let $O \in \mathcal{X}$ be open, fix $\epsilon > 0$, and let $x \in O$ be such that $g(x) \leq \inf_O g + \epsilon$, and let $d > 0$ be such that $B_d(x) \subset O$. Then

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(O) &\geq \liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_\delta(x)) \\ &\geq \lim_{d\to 0} \liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_\delta(x)) \\ &= -g(x) =\geq -\inf_O g - \epsilon.\end{aligned}$$

Now let ϵ tend to zero.

To prove the upper bound, let $C \subset \mathcal{X}$ be closed. Because $\mathcal{X}$ is compact, so is C. Similar to the proof of Sect. 5.1 we cover C with finitely many balls and then

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(C) &\le \limsup_{n\to\infty} \frac{1}{r_n} \log \sum_{i=1}^{m} \mu_n(B_{d_i}(x_i)) \\
&= \limsup_{n\to\infty} \frac{1}{r_n} \log \big(m \sup_{i\in\{1,\dots m\}} \mu_n(B_{d_i}(x_i))\big) \\
&= \limsup_{n\to\infty} \left[\frac{1}{r_n} \log m + \frac{1}{r_n} \log \sup_{i\in\{1,\dots m\}} \mu_n(B_{d_i}(x_i))\right] \\
&= \limsup_{n\to\infty} \frac{1}{r_n} \log \sup_{i\in\{1,\dots m\}} \mu_n(B_{d_i}(x_i)) \\
&= \max_{i\in\{1,\dots m\}} \limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_{d_i}(x_i)) \\
&= \limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_{d_1}(x_1)),
\end{aligned}$$

where the last equality is without loss of generality. Fix $\epsilon > 0$. By (13), for each $x \in \mathcal{X}$ there exists $d(x,\epsilon) > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_{d(x,\epsilon)}(x)) \le \max\{-g(x)+\epsilon, -1/\epsilon\}$$

(note that if g is finite one can simplify the right-hand side to $-g(x)+\epsilon$). Now, cover C with the balls $\cup_{x\in C} B_{d(x,\epsilon)}(x)$; then, by compactness of C we may choose a finite sub-cover, and thus, we may assume that we have chosen $d_1 = d(x_1,\epsilon)$! Thus,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(C) &\le \limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_{d_1}(x_1)) \\
&\le \max\{-g(x_1)+\epsilon, -1/\epsilon\},
\end{aligned}$$

and letting ϵ tend to zero,

$$\limsup_{n\to\infty} \frac{1}{r_n} \log \mu_n(C) \le -g(x_1) \le -\inf_C g.$$

$\underline{\Longrightarrow}$: Conversely, suppose LDP (μ_n, r_n) with rate function I. By the large deviation lower bound,

$$I(x) \ge \inf_O I \ge -\liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(O),$$

for any open set O containing x, so putting $O = B_d(x)$ and supping over d,

$$I(x) \ge -\lim_{d\to 0} \liminf_{n\to\infty} \frac{1}{r_n} \log \mu_n(B_d(x)).$$

By the large deviation upper bound,

$$-\limsup_{n\to\infty}\frac{1}{r_n}\log\mu_n(\overline{B_d(x)})\geq\inf_{\overline{B_d(x)}}I,$$

so

$$I(x)\geq-\lim_{d\to0}\liminf_{n\to\infty}\frac{1}{r_n}\log\mu_n(B_d(x))\geq-\lim_{d\to0}\limsup_{n\to\infty}\frac{1}{r_n}\log\mu_n(\overline{B_d(x)})\geq\lim_{d\to0}\inf_{\overline{B_d(x)}}I,$$

and it suffices to show that

$$\lim_{d\to0}\inf_{\overline{B_d(x)}}I\geq I(x).$$

If not, there exists x_k such that $\lim_k I(x_k)<I(x)$ and $x_k\to x$, contradicting lower semicontinuity of I. □

7 Optimal Transport

The problem of optimally transporting a given probability measure $\mu\in P(X)$ (the source measure) to another given probability measure $\nu\in P(Y)$ (the target measure) has a long history, going back to Monge in the 18th century. It is the problem of finding a measurable map $T:X\to Y$ satisfying

$$T_{\#}\mu=\nu \tag{14}$$

and minimizing

$$\int_X c(x,T(x))\mu(x),$$

where $c:X\times Y\to\mathbb{R}$ is some given *cost function*, typically $c(x,y)=|x-y|^2$. This integral is the total cost, and $c(x,T(x))\mu(dx)$ is the infinitesimal cost of transporting x to $T(x)$, with $\mu(dx)$ measuring the amount of mass at the source point x. By abuse of notation we denote the latter by $\mu(x)$ and not $d\mu(x)$ or $\mu(dx)$.

As in previous sections, one should think of $X=Y=\mathbb{R}^n$ or $X=Y=P(\mathbb{R}^d)$ as the typical examples in our course for the underlying spaces. Typical examples for the measures to be transported include uniform measures 1_Ω (for a unit-volume set Ω) and the empirical measures

$$\delta^n(x_1,\dots,x_n):=\frac{1}{n}\sum_{i=1}^n\delta_{x_i}.$$

Choosing both the source and the target measures to be empirical measures with the same number of point masses (i.e., choosing μ and ν in the image of δ^n for the same n) gives rise to the so-called *discrete optimal transport problem.* The solution is then given by a permutation $\sigma \in S_n$ on the set of n letters, satisfying

$$\sum_{i=1}^{n} c(x_i, y_i) \le \sum_{i=1}^{n} c(x_i, y_{\sigma(i)}). \tag{15}$$

In the prototypical case of squared distance cost some cancellations give that

$$\sum_{i=1}^{n} |x_i - y_i|^2 \le \sum_{i=1}^{n} |x_i - y_{\sigma(i)}|^2$$

can be rewritten as

$$\sum_{i=1}^{n} -\langle x_i, y_i\rangle \le \sum_{i=1}^{n} -\langle x_i, y_{\sigma(i)}\rangle. \tag{16}$$

So, the cost $|x-y|^2$ is really 'equivalent' to the cost $-\langle x, y\rangle$. More generally, if $c(x, y) = d(x, y) + f(x) + g(y)$ then c and d are equivalent:

$$\int_X c(x, T(x))\mu(x) = \int_X d(x, T(x))\mu(x) + \int_X f\mu + \int_X g(T(x))\mu = \int_X d(x, T(x))\mu(x) + \int_X f\mu + \int_Y g\nu$$

(as $\int_X f\mu + \int_Y g\nu$ is a constant completely determined by the "data" μ, ν), where in the last equation we used that

$$\int_X g \circ T\mu = \int_Y g\nu,$$

by (14) [31, (9)].

More generally, one can search for an optimal transportation *plan*. Given a product space, say $X \times Y$, equipped with projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, the marginals of a measure $\gamma \in P(X \times Y)$ are $(\pi_X)_\#\gamma$ and $(\pi_Y)_\#\gamma$.

Definition 7.1 A transportation plan is a probability measure $\gamma \in P(X \times Y)$ whose marginals are μ and ν. We denote this by $\gamma \in \Pi(\mu, \nu)$.

I.e., $\gamma(A \times Y) = \mu(A)$ and $\gamma(X \times B) = \nu(B)$ for all Borel $A \subset X$ and $B \subset Y$

Definition 7.2 An *optimal transportation plan* is a transportation plan minimizing

$$\int_{X\times Y} c\gamma.$$

The "best" transport plan is the one coming from transport map $T : X \to Y$. Denote by $\mathrm{Id} \otimes T : X \to X \times Y$ the map $x \mapsto (x, T(x))$. Indeed, $\gamma := (\mathrm{Id} \otimes T)_\#\mu \in \Pi(\mu, \nu)$ since $(\mathrm{Id} \otimes T)_\#\mu(A \times Y) = \mu\big((\mathrm{Id} \otimes T)^{-1}(A \times Y)\big) = \mu(A)$ and $(\mathrm{Id} \otimes T)_\#\mu(X \times B) = \mu\big((\mathrm{Id} \otimes T)^{-1}(X \times B)\big) = \mu(T^{-1}(B)) = \nu(B)$ since

$T_{\#}\mu = \nu$. Our goal will be to show that under some natural assumptions the *optimal* plan must be of such a form, i.e., supported on the graph of a map.

For example, in the case of empirical measures, a transportation plan must be coming from a map represented by a permutation (in other words, it must be supported on the graph of a permutation in the product space): if one of the source points is not in supp γ then the first marginal condition is violated ($\gamma(\{x_i\} \times Y) = 0$ while $\mu\{x_i\} = 1/n$ but $\gamma(A \times Y) = \mu(A)$), while if one of the target points is not in supp γ then the second marginal condition is violated. Thus $\sigma \in S_n$, or more precisely,

$$\gamma := \frac{1}{n} \sum \delta_{(x_i, y_{\sigma(i)})} = \delta_{\mathrm{gr}(T)},$$

(where $T : x_i \mapsto y_{\sigma(i)}$) is optimal if and only if (16) holds.

7.1 From the Discrete Problem to the General One

A beautiful part of the story is that the discrete problem actually is the key to understanding the general transport problem. Equation (16) leads to the following definition (we replace n in (16) with m for the following discussion).

Definition 7.3 A set $A \subset X \times Y$ is cyclically monotone if (16) holds for any $\{(x_i, y_i)\}_{i=1}^m \subset A$, $m \in \mathbb{N}$, and any $\sigma \in S_m$.

Cyclical monotonicity essentially characterizes convexity. More precisely, the graph of the sub differential of a convex function f is cyclically monotone: if $y_i \in \partial f(x_i)$

$$f(z) \geq f(x_i) + \langle z - x_i, y \rangle, \quad \forall z,$$

so taking $z = x_{i+1}$ (with $x_{m+1} = x_1$) and adding up the equations yields (16). Conversely, to a cyclically monotone set A we can associate a convex function f_A such that $A \subset \mathrm{gr}(\partial f_A)$ as follows. Fix $(x_0, y_0) \in A$ and set

$$f_A(x) := \sup_{m \in \mathbb{N}} \sup_{\{(x_i, y_i)\}_{i=1}^m \subset A} \Big\{ \langle x - x_m, y_m \rangle + \langle x_m - x_{m-1}, y_{m-1} \rangle + \ldots + \langle x_1 - x_0, y_0 \rangle \Big\}.$$

Note that f_A is not $\pm\infty$:

Claim 7.4 $f_A(x_0) = 0$.

Remark 7.5 We will eventually prove much more, namely, that f_A is nowhere $\pm\infty$.

Proof First, $f_A(x_0) \leq 0$ from (16) with m replaced by $m + 1$ and $\sigma(i) = i + 1$. Second, $f_A(x_0) \geq 0$ by putting $m = 1$ and $(x_1, y_1) = (x_0, y_0)$ in the definition of f_A. □

Finally, if $(a, b) \in A$, want to show $b \in \partial f_A(a)$ (the *sub-differential* of f_A), i.e.,

$$f_A(z) \geq f_A(a) + \langle z - a, b \rangle, \quad \forall z. \tag{17}$$

Given any $\epsilon > 0$ there is some $m \in \mathbb{N}$ and some $\{(x_i, y_i)\}_{i=1}^m \subset A$ such that

$$f_A(a) - \epsilon = \langle a - x_m, y_m \rangle + \langle x_m - x_{m-1}, y_{m-1} \rangle + \ldots + \langle x_1 - x_0, y_0 \rangle. \tag{18}$$

Thus,

$$f_A(a) + \langle z - a, b \rangle = \epsilon + \langle z - a, b \rangle + \langle a - x_m, y_m \rangle + \langle x_m - x_{m-1}, y_{m-1} \rangle + \ldots + \langle x_1 - x_0, y_0 \rangle \leq \epsilon + f_A(z),$$

putting $m + 1$ and $\{(x_i, y_i)\}_{i=1}^m \cup \{(a, b)\}$ in the definition of f_A. Letting $\epsilon \to 0$ concludes the proof of (17).

Exercise 7.6 Find the mistake in the previous argument.

Solution. The problem was that we were implicitly assuming that $f_A(a) < \infty$. Indeed, if $f_A(a) = \infty$ one cannot, given any $\epsilon > 0$, find $\{(x_i, y_i)\}_{i=1}^m \subset A$ such that (18) holds. Instead, let $t \in \mathbb{R}$ be any number satisfying $t < f_A(a)$ (possibly, $f_A(a) = \infty$). Now, there do exist, by definition, $\{(x_i, y_i)\}_{i=1}^m \subset A$ such that

$$t < \langle a - x_m, y_m \rangle + \langle x_m - x_{m-1}, y_{m-1} \rangle + \ldots + \langle x_1 - x_0, y_0 \rangle. \tag{19}$$

Thus, for all z,

$$t + \langle z - a, b \rangle < \langle z - a, b \rangle + \langle a - x_m, y_m \rangle + \langle x_m - x_{m-1}, y_{m-1} \rangle + \ldots + \langle x_1 - x_0, y_0 \rangle \leq f_A(z), \tag{20}$$

by putting $m + 1$ and $\{(x_i, y_i)\}_{i=1}^m \cup \{(a, b)\}$ in the definition of f_A. Supping over all t in (20),

$$f_A(a) + \langle z - a, b \rangle = \sup_{t < f_A(a)} t + \langle z - a, b \rangle \leq f_A(z),$$

as desired, i.e., $b \in \partial f_A(a)$, unless $f_A(a) = \infty$ (in which case $\partial f_A(a) = \emptyset$ by definition). To exclude this, i.e., to show f_A is always finite, we put $z = 0$ in (20), and use Claim 7.4,

$$t + \langle 0 - a, b \rangle < f_A(0) = 0,$$

so we get the a priori estimate

$$t < \langle a, b \rangle, \qquad \text{for any } t < f_A(a).$$

Hence, $f_A(a) = \sup_{t < f_A(a)} t \leq \langle a, b \rangle$ and in particular $f_A(a)$ is finite (obviously $f_A > -\infty$ since it is a supremum of finite quantities over a nonempty set).

Recall the definition of the sub-differential ∂f (17) of a convex function $f : \mathbb{R}^n \to \mathbb{R}$. The graph of ∂f is defined as $\mathrm{gr}(\partial f) := \{(x, y) : x \in \mathbb{R}^n, y \in \partial f(x)\}$. Thus, we have shown:

Theorem 7.7 (Rockafellar's Theorem) *A set* $A \subset \mathbb{R}^n \times \mathbb{R}^n$ *is cyclically monotone if and only if* $A \subset \mathrm{gr}(\partial f)$ *for a convex function* $f : \mathbb{R}^n \to \mathbb{R}$.

Remark 7.8 Note, again, that part of the conclusion of the theorem is that f is finite (which should not come as a surprise, as also the set A is "finite-valued" by assumption; the converse direction is obvious since if f is finite-valued then so is ∂f). In fact,

$$f_A(x) \leq \inf_{(x,y)\in A} \langle x, y\rangle \leq h_{\{x\}\times\mathbb{R}^n\cap A}(x),$$

where h_K is the support function of the set K with equality in the last inequality if A is graphical (i.e., of the form $\mathrm{gr}(\partial f)$ with f as above).

The Fundamental Theorem of OT gives the final chain connecting the discrete problem to the continuous problem: the support of an optimal transport plan is cyclically monotone. Thus, by a previous observation it is contained in the graph of the sub differential of a convex function. Since a convex function is differentiable away from a (Lesbegue) measure zero set as measures $(\mathrm{Id} \otimes \partial f)_{\#}\mu = (\mathrm{Id} \otimes \nabla f)_{\#}\mu$ where on the right hand side by ∇f we mean (by some abuse of notation) the gradient *map* restricted to those $x \in \mathbb{R}^n$ where $\partial f(x)$ is a singleton (that we then denote by $\nabla f(x)$); note also that assuming μ is absolutely continuous, Lebesgue measure zero sets are also μ-measure zero sets. Thus, we have come full circle, and solved the original transportation problem in terms of a *map*.

Theorem 7.9 (Fundamental Theorem of Optimal Transport) *Let* $\gamma \in \Pi(\mu, \nu)$. *Then,*
γ *optimal* $\Leftrightarrow$ $\mathrm{supp}\,\gamma$ *is cyclically monotone* $\Leftrightarrow$ *exists* f *convex such that* $\mathrm{supp}\,\gamma \subset \mathrm{gr}(\partial f)$.

Proof By Theorem 7.7, it suffices to show the first equivalence, but we will actually only use the hard part of Theorem 7.7 and proceed to prove the implications cyclically. First, suppose that for some $\gamma \in \Pi(\mu, \nu)$, $\mathrm{supp}\,\gamma$ is cyclically monotone. By Theorem 7.7, there exists f convex such that $\mathrm{supp}\,\gamma \subset \mathrm{gr}(\partial f)$. Thus, for every $\tilde{\gamma} \in \Pi(\mu, \nu)$,

$$\begin{aligned}
\int_{X\times Y} -\langle x, y\rangle\gamma &= \int_{\mathrm{supp}\,\gamma} -\langle x, y\rangle\gamma \\
&= \int_{\mathrm{supp}\,\gamma} [-f(x) - f^{\star}(y)]\gamma \\
&= -\int_X f\mu - \int_Y f^{\star}\nu. \qquad (21)\\
&= \int_{X\times Y} [-f(x) - f^{\star}(y)]\tilde{\gamma} \\
&\leq \int_{X\times Y} -\langle x, y\rangle\tilde{\gamma},
\end{aligned}$$

so by definition γ is optimal.

Next, assume that γ is optimal. We want to show that $\operatorname{supp}\gamma$ is cyclically monotone. Fix $\{(x_i, y_i)\}_{i=1}^m \subset \operatorname{supp}\gamma$. To that end, we carefully construct $\tilde{\gamma} \in \Pi(\mu, \nu)$ of the form $\tilde{\gamma} := \gamma + \eta$ with

$$0 \leq \int_{X\times Y} -\langle x, y\rangle\tilde{\gamma} - \int_{X\times Y} -\langle x, y\rangle\gamma \approx \sum_{i=1}^{n} -\langle x_i, y_{\sigma(i)}\rangle - \sum_{i=1}^{n} -\langle x_i, y_i\rangle. \quad (22)$$

Of course, the idea is to construct the positive part of η to be concentrated near $\{(x_i, y_{\sigma(i)})\}_{i=1}^m$ and the negative part of η to be concentrated near $\{(x_i, y_i)\}_{i=1}^m$. We have to do this in such a way that $\gamma + \eta$ is still admissible (i.e., a transport plan). Equivalently, $(\pi_X)_\# \eta = 0,\ (\pi_Y)_\# \eta = 0$.

For the construction, we fix $\epsilon > 0$. Set

$$\Gamma := \prod_{i=1}^{m} \frac{\gamma|_{B_\epsilon(x_i)\times B_\epsilon(y_i)}}{|\gamma\big(B_\epsilon(x_i) \times B_\epsilon(y_i)\big)|}$$

This is an auxiliary probability measure on $P\big((X \times Y)^m\big)$. It is useful, because it's marginals allow us to cyclically modify the way γ transports: in order to transport $B_\epsilon(x_i)$ to $B_\epsilon(y_{\sigma(i)})$ instead of to $B_\epsilon(y_i)$ we would add

$$(\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_{\sigma(i)})})_\#\Gamma - (\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_i)})_\#\Gamma.$$

to γ. So, overall, we set

$$\eta := \frac{\min_i |\gamma\big(B_\epsilon(x_i) \times B_\epsilon(y_i)\big)|}{m} \sum_{i=1}^{m} \Big[(\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_{\sigma(i)})})_\#\Gamma - (\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_i)})_\#\Gamma\Big]. \quad (23)$$

The constant $\min_i |\gamma\big(B_\epsilon(x_i) \times B_\epsilon(y_i)\big)|/m$ in front of some ensures that $\gamma + \eta$ is still a positive measure (recalling that $|\gamma\big(B_\epsilon(x_i) \times B_\epsilon(y_i)\big)|$ appears in the denominator of Γ, so the largest negative term inside the brackets in (23) is $\min_i |\gamma\big(B_\epsilon(x_i) \times B_\epsilon(y_i)\big)|$ and there are at most m of these negative terms). Next, to show $(\pi_Y)_\#\eta = 0$ amounts to $\eta(X \times B) = 0$ for each B, and indeed, up to a positive factor $(\pi_Y)_\#\eta(B) = \eta(X \times B)$ equals

$$\begin{aligned}
&\sum_{i=1}^{m} \Big[(\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_{\sigma(i)})})_\#\Gamma(X \times B) - (\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_i)})_\#\Gamma(X \times B)\Big] \\
&= \sum_{i=1}^{m} \Gamma(X \times Y \times \cdots \times \overset{\sigma \triangleright i \triangleleft\text{-th slot}}{X \times B} \times \cdots \times X \times Y) \\
&\quad - \sum_{i=1}^{m} \Gamma(X \times Y \times \cdots \times \overset{i\text{-th slot}}{X \times B} \times \cdots \times X \times Y) = 0
\end{aligned}$$

Finally, $(\pi_X)_\# \eta = 0$ is easier as

$$\begin{aligned}
&\sum_{i=1}^{m} \Big[(\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_{\sigma(i)})})_\# \Gamma(A \times Y) - (\pi_{B_\epsilon(x_i)}, \pi_{B_\epsilon(y_i)})_\# \Gamma(A \times Y) \Big] \\
&= \sum_{i=1}^{m} \Gamma(X \times Y \times \cdots \times \overset{\sigma \triangleright i \triangleleft\text{-th slot}}{A \times Y} \times \cdots \times X \times Y) \\
&\quad - \sum_{i=1}^{m} \Gamma(X \times Y \times \cdots \times \overset{i\text{-th slot}}{A \times Y} \times \cdots \times X \times Y) = \sum_{i=1}^{m} 0 = 0
\end{aligned}$$

(i.e., is term-by-term zero). Thus, we have shown (22) up to $o(\epsilon)$. Letting $\epsilon \to 0$ proves (16), as claimed. □

7.2 *Dual Formulation*

A rather immediate consequence of Theorems 7.7 and 7.9 is the following dual formulation of the optimal transportation problem in terms of an optimization problem on *functions* instead of measures.

Theorem 7.10 *Let* $c(x, y) = -\langle x, y \rangle$.

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{X \times Y} c\gamma = \sup_{f(x)+g(y) \le c(x,y)} \Big[\int_X f\mu + \int_Y g\nu \Big]. \tag{24}$$

Proof According to Lemma 7.12 there exists γ realizing the infimum on the left-hand side. Let f, g be such that $f(x) + g(y) \le -\langle x, y \rangle$. Then

$$\int_{X \times Y} c\gamma \ge \int_{X \times Y} (f(x) + g(y))\gamma = \int_X f\mu + \int_Y g\nu.$$

It thus remains to show

$$\int_{X \times Y} c\gamma \le \sup_{f(x)+g(y) \le c(x,y)} \Big[\int_X f\mu + \int_Y g\nu \Big]. \tag{25}$$

Theorems 7.7 and 7.9 imply that supp $\gamma \subset \mathrm{gr}(\nabla\phi)$ for some convex function ϕ. Since $\phi(x) + \phi^\star(y) \ge -c(x, y)$ with the quality if and only if $(x, y) \in \mathrm{gr}(\partial\phi)$,

$$\begin{aligned}\int_{X\times Y} c\gamma &= \int_{\operatorname{supp}\gamma} c\gamma\\ &= \int_{\operatorname{gr}(\partial\phi)} c\gamma\\ &= \int_{X\times Y} [-\phi(x)-\phi^\star(y)]\gamma\\ &= \int_{X\times Y} [-\phi(x)-\phi^\star(y)]\gamma\\ &= -\int_X \phi\mu - \int_Y \phi^\star\nu.\end{aligned}$$

Thus, (25) holds as already the pair $(f, g) = (-\phi, -\phi^\star)$ equals the left-hand side.□

In fact, we saw that the dual formulation can be given in terms of a single (and convex) function. Also, we play a bit with the signs, to get:

Corollary 7.11 (Dual formulation of optimal transportation)

$$\sup_{\gamma\in\Pi(\mu,\nu)} \int_{X\times Y} \langle x, y\rangle\gamma = \inf_{f\in C(X)} \Big[\int_X f\mu + \int_Y f^\star\nu\Big] = \inf_{f\in \mathrm{Cvx}(X)} \Big[\int_X f\mu + \int_Y f^\star\nu\Big].$$

Lemma 7.12 *The infimum on the left hand side of* (24) *is attained.*

Proof The proof follows Ambrosio–Gigli who work in a more general setting [1, Sect. 1.1]. Since $c : X \times Y \to \mathbb{R}$ is continuous (in fact, lower semicontinuous is enough, with slightly more work [1, Theorem 1.2]) then $\gamma \mapsto \int c\gamma$ is continuous with respect to the weak topology. Since

$$\gamma(X \times Y\backslash K_1 \times K_2) \le \mu(X\backslash K_1) + \mu(Y\backslash K_2), \tag{26}$$

for *any* $\gamma \in \Pi(\mu, \nu)$, it follows that $\Pi(\mu, \nu)$ satisfies the assumptions of Prokhorov's Theorem (see [1, Theorem 1.3]): indeed, the right-hand side of (26) can be made arbitrarily small by Ulam's Theorem (any Borel probability measure on a Polish space is concentrated on a compact set up to an arbitrarily small error) applied to the Polish measure spaces (X, μ) and (Y, ν). Thus, $\Pi(\mu, \nu)$ is pre-compact. The closure of a pre-compact set is by definition compact, it suffices to show that $\Pi(\mu, \nu)$ is actually closed (with respect to the weak topology), but this is immediate since $\int f\mu = \int f(x)\gamma_n \to \int f(x)\gamma = \int f\mu$ (note that $(x, y) \mapsto f(x)$ for $f \in C(X)$ is in $C(X \times Y)$ so then $\alpha \mapsto \int f(x)\alpha$ is continuous with respect to the weak topology) and similarly for the other marginal. Since a lower semicontinuous functional attains its infimum on a compact set, we are done. □

7.3 The Legendre Transform of Wasserstein Distance

Denote by

$$C(X) := C^0(X) \cap L^\infty(X) \tag{27}$$

the continuous and bounded functions on X.

Let $J_\nu : C(X) \to \mathbb{R}$ be

$$J_\nu(f) := \int_Y f^\star \nu. \tag{28}$$

Denote by $W_2^2 : P(X) \times P(Y) \to \mathbb{R}$

$$W_2^2(\mu, \nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \int_{X \times Y} c\gamma \tag{29}$$

the *Wasserstein distance between* μ *and* ν. It will be convenient to extend this functional to $\mathcal{M}(X) \times \mathcal{M}(Y)$ "by infinity," namely

$$W_2^2(\mu, \nu) := \infty$$

if either $\mu \notin P(X)$ or $\nu \notin P(Y)$.

Our work so far can be summarized in terms of Legendre duality these two functionals: indeed, for any $\mu \in P(X)$,

$$\begin{aligned} -W_2^2(\mu, \nu) &= \inf_{f \in C(X)} \Big[\int_X f\mu + \int_Y f^\star \nu \Big] \\ &= - \sup_{f \in C(X)} \Big[\langle f, -\mu \rangle - J_\nu(f) \Big] = -J_\nu^\star(-\mu), \end{aligned}$$

while if $\mu \in \mathcal{M}(X)$ but $\mu \notin P(X)$ then since $(f + C)^\star = f^\star - C$ we see $\langle f + C, -\mu \rangle - J(f + C) = \langle f, -\mu \rangle - J_\nu(f) + C(1 - \mu(X))$ which can be made arbitrarily large if $\mu(X) \neq 1$, which is to say that $J_\nu^\star(-\mu) = \infty$, i.e., we have

$$W_2^2(\mu, \nu) = J_\nu^\star(-\mu). \tag{30}$$

The following theorem summarizes this and more. This is the first time the *Monge–Ampère operator*

$$\mathrm{MA}_\nu\, f := (\nabla f^\star)_\# \nu \tag{31}$$

makes its appearance.

Remark 7.13 When $\nu = dx$ this reduces to the well-known Monge–Ampère operator

$$\mathrm{MA}\, f = d\nabla f := d\frac{\partial f}{\partial x_1} \wedge \cdots \wedge d\frac{\partial f}{\partial x_n},$$

since

$$\int_X F(\nabla f^\star)_\# dx = \int_Y F\circ(\nabla f^\star)dx = \int_Y F\circ(\nabla f)^{-1}dx = \int_X Fd\nabla f(x) = \int_X F\det\nabla f(x).$$

When $f \in C^2$ then $\mathrm{MA}\, f = \det \nabla^2 f$.

Theorem 7.14 *J_ν and $W_2^2(-\cdot,\nu)$ are convex, lower semicontinuous, and Legendre dual to each other. J_ν is Gateaux differentiable and*

$$dJ_\nu|_f = -\mathrm{MA}_\nu\, f. \tag{32}$$

Proof • First, note that J_ν is actually continuous: if $C(X) \ni f_j \to f$ in C^0 then $f_j^\star \to f^\star$ pointwise and hence uniformly [27, Theorem 10.8] so J is continuous.

• Convexity of J is elementary:

$$\begin{aligned} J_\nu\Big(\frac{\theta+\chi}{2}\Big) &= \int_Y \sup_x \Big[\langle x,y\rangle - \frac{\theta(x)+\chi(x)}{2}\Big]\nu \\ &\le \frac{1}{2}\int_Y \sup_x[\langle x,y\rangle - \theta(x)]\nu + \frac{1}{2}\int_Y \sup_x[\langle x,y\rangle - \chi(x)]\nu \\ &= \frac{1}{2}J_\nu(\theta) + \frac{1}{2}J_\nu(\chi). \end{aligned}$$

• Convexity of $W_2^2(-\cdot,\nu)$ then follows from (30), being the supremum of affine functionals, which also implies lower semi-continuity (the supremum of continuous functions).

• Legendre duality was already proven in (30).

• Legendre duality implies that

$$J_\nu(f) + W_2^2(\mu,\nu) \ge -\langle f,\mu\rangle. \tag{33}$$

Fix f. To show Gateaux differentiability and (32) it suffices to show that there exists exactly one μ for which equality is attained, and that then $\mu = \mathrm{MA}_\nu\, f$. Define $\mu =: (\nabla f^\star)_\# \nu$. By Theorem 7.9 and Corollary 7.11 then $(\mathrm{Id} \otimes \nabla f)_\# \mu \in \Pi(\mu,\nu)$ is an optimal transport plan and

$$W_2^2(\mu,\nu) = -\langle f,\mu\rangle - \langle f^\star,\nu\rangle = -\langle f,\mu\rangle - J_\nu(f).$$

Now, $\mu = (\nabla f^\star)_\#\nu = \mathrm{MA}_\nu\, f$ by definition. Thus, it suffices to show that this μ is the only one attaining equality in (33), i.e., it suffices to show that J is strictly convex. Suppose α is another such measure, i.e.,

$$W_2^2(\alpha,\nu) = -\langle f,\alpha\rangle - \langle f^\star,\nu\rangle.$$

From Theorem 7.9 and Corollary 7.11 $(\mathrm{Id} \otimes \nabla f)_\#\alpha \in \Pi(\alpha,\nu)$ is an optimal transport plan and $\alpha = (\nabla f^\star)_\#\nu$ so $\alpha = \mu$. □

7.4 Rate Function for Monge–Ampère

Let $\beta \in \mathbb{R}$ and let

$$\mu_0 \in P(X)$$

be a fixed reference probability measure. We are interested in the Monge–Ampère equation

$$\mathrm{MA}_\nu\, f = e^{\beta f}\mu_0 \Big/ \int_X e^{\beta f}\mu_0. \tag{34}$$

Define $F_{\beta,\nu} : C(X) \to \mathbb{R}$ by

$$F_{\beta,\mu_0,\nu}(\theta) := \frac{1}{\beta} I_{\mu_0}(\beta\theta) + J_\nu(\theta). \tag{35}$$

By (32) and the proof of Lemma 4.4:

Lemma 7.15 *$F_{\beta,\mu_0,\nu}$ is Gateaux differentiable and*

$$dF_{\beta,\mu_0,\nu}|_\theta = e^{\beta f}\mu_0 \Big/ \int_X e^{\beta f}\mu_0 - \mathrm{MA}_\nu\, f. \tag{36}$$

Finally, we can define the rate function underlining the Monge–Ampère equation:

$$G_{\beta,\mu_0,\nu} := \beta W_2^2(\,\cdot\,,\nu) + \mathrm{Ent}(\mu_0,\ \cdot\,) + C, \tag{37}$$

where C is a constant that will guarantee the function is nonnegative and zero at its minimum.

Proposition 7.16 *Assume that $F_{\beta,\mu_0,\nu}$ admits a unique (up to a constant) minimizer $\phi_{\min}$. Then $G_{\beta,\mu_0,\nu}$ admits a unique minimizer $\mu = \mathrm{MA}_\nu\,\phi_{\min}$. The converse is also true.*

Before going into the proof, let us motivate it with a general observation about Legendre duals in finite dimensions. If

$$F = f_1 + f_2$$

is the sum of two differentiable strictly convex functions, and x is the unique minimum of F then

$$G(y) := f_1^\star(y) + f_2^\star(-y)$$

has a unique minimum at $df_1(x)$. Indeed,

$$df_1(x) = -df_2(x)$$

while (G is differentiable since $f_1^\star$ and $f_2^\star$ are by the strict convexity of f_1, f_2 [27, Theorem 26.3])

$$dG(y) = df_1^\star(y) - df_2^\star(-y) = (df_1)^{-1}(y) - (df_2)^{-1}(-y)$$

and setting $y = df_1(x) = -df_2(x)$

$$dG(y) = (df_1)^{-1}(df_1(x)) - (df_2)^{-1}(- - df_2(x)) = x - x = 0.$$

Thus, y is a critical point of the convex function G, hence a minimum point. This is the only minimum point since the proof is reversible: if $dG(\tilde{y}) = 0$ we get $df_1^\star(\tilde{y}) = df_2^\star(-\tilde{y})$ so $df_1(\tilde{x}) = -df_2(\tilde{x})$ for $\tilde{x} = df_1^\star(\tilde{y})$, so then $\tilde{x}$ is a critical point of F, hence a minimum, so then $x = \tilde{x}$ since by assumption x was the unique minimum. Thus $df_1^\star(y) = df_1^\star(\tilde{y})$ implying $y = \tilde{y}$ if $f_1^\star$ is strictly convex, but this follows from differentiability of f_1 [27, Theorem 26.3].

Proof Essentially, the conclusion of the finite-dimensional discussion above holds also in our situation by chasing through the definitions and avoiding the use of [27, Theorem 26.3]. Here goes.

First, by Theorem 7.14 and (8),

$$W_2^2(\mu, \nu) + J_\nu(f) \geq -\langle f, \mu\rangle, \qquad \text{equality if and only if } \mu = \mathrm{MA}_\nu\, f.$$

Second, by Lemma 4.4 and (8),

$$\mathrm{Ent}(\mu_0, \mu) + I_{\mu_0}(f) \geq \langle f, \mu\rangle, \qquad \text{equality if and only if } \mu = e^f \mu_0 \Big/ \int_X e^f \mu_0.$$

Let $\phi_{\min}$ be the minimizer of $F_{\beta,\mu_0,\nu}$. By Lemma 7.15

$$\mathrm{MA}_\nu\, \phi_{\min} = e^{\beta\phi_{\min}}\mu_0 \Big/ \int_X e^{\beta\phi_{\min}}\mu_0. \tag{38}$$

• Assume first $\beta > 0$. Then setting $f = \phi_{\min}$ and $f = \beta\phi_{\min}$, respectively, in the inequalities above

$$\begin{aligned} G_{\beta,\mu_0,\nu}(\mu) &= \beta W_2^2(\mu, \nu) + \mathrm{Ent}(\mu_0, \mu) + C, \\ &\geq -\beta\langle \phi_{\min}, \mu\rangle - \beta J_\nu(\phi_{\min}) - I_{\mu_0}(\beta\phi_{\min}) + \langle \beta\phi_{\min}, \mu\rangle \\ &= -\beta J_\nu(\phi_{\min}) - I_{\mu_0}(\beta\phi_{\min}) \\ &= -\beta F_{\beta,\mu_0,\nu}(\phi_{\min}), \end{aligned}$$

with equality if and only if $\mu = e^{\beta\phi_{\min}}\mu_0 / \int_X e^{\beta\phi_{\min}}\mu_0 = \mathrm{MA}_\nu\, \phi_{\min}$. Note that $-\beta F_{\beta,\mu_0,\nu}(\phi_{\min})$ is some constant independent of μ. Thus, $\mu = \mathrm{MA}_\nu\, \phi_{\min}$ is the unique minimizer of $G_{\beta,\mu_0,\nu}$.

• Assume now that $\beta < 0$. Fix $\mu \in \mathcal{M}(X)$. Let $\phi \in C(X)$ be such that equality hold in (33). Now applying above argument to $f = \phi$ and $f = \beta\phi$, respectively, gives

$$G_{\beta,\mu_0,\nu}(\mu) \geq -\beta F_{\beta,\mu_0,\nu}(\phi) \geq -\beta F_{\beta,\mu_0,\nu}(\phi_{\min}),$$

(the last inequality simply because $\phi_{\min}$ is a minimizer of $F_{\beta,\mu_0,\nu}$ and $\beta < 0$) with equality in the first inequality if and only if $\mu = e^{\beta\phi}\mu_0 / \int_X e^{\beta\phi}\mu_0$ and in the second inequality if and only if $\phi = \phi_{\min}$ so overall $\mu = e^{\beta\phi_{\min}}\mu_0 / \int_X e^{\beta\phi_{\min}}\mu_0 = \mathrm{MA}_\nu\, \phi_{\min}$ by (38). □

Remark 7.17 We leave the details for the simpler case $\beta = 0$ to the reader (in this special case the Wasserstein distance does not even appear, and one is basically reduced to Sanov's Theorem (Corollary 4.3)). Of course, one has to also define $F_{0,\mu_0,\nu}$ appropriately by taking the derivative at $\beta = 0$ of (35).

8 Moment Generating Function for Monge–Ampère

Our goal is now to construct a sequence of probability measures on $P(X)$ (i.e., random measures, or elements of $P(P(X))$) whose moment generating function (for some normalization) is precisely $G_{\beta,\mu_0,\nu}$.

Naturally, in view of Sanov's Theorem (Corollary 4.3), the entropy term in $G_{\beta,\mu_0,\nu}$ will come from $\mu_0^{\otimes n}$. To obtain the Wasserstein distance term we will need to multiply the symmetric measure by a symmetric function that captures discrete optimal transport distance. Here is the key observation [10, Theorem 3.2]. Let $H_{n^d} : X^{n^d} \to \mathbb{R}$ (the reader can basically consider the examples (40) and (41) although the next lemma is more general). Set

$$\Gamma_{\beta,n} := \delta_\#^{n^d}\left(e^{-\beta H_{n^d}}\mu_0^{\otimes n^d}\right)\Big/ Z_{\beta,n} \in P(P(X)),$$

with $Z_{\beta,n} := \int_{X^{n^d}} e^{-\beta H_{n^d}}\mu_0^{\otimes n^d}$ is the normalizing constant guaranteeing that $\Gamma_{\beta,n}$ is a probability measure; let

$$C_\beta := \lim_n \frac{1}{n^d}\log Z_{\beta,n}, \tag{39}$$

where the limit exists and is finite according to Claim 8.3 below.

Lemma 8.1 *Let $E : P(X) \to \mathbb{R}$ be continuous and let $H_{n^d} : X^{n^d} \to \mathbb{R}$. Suppose that* $\lim_{n\to\infty} ||H_{n^d}/n^d - E \circ \delta^{n^d}||_{L^\infty(X^{n^d})} = 0$.
Then LDP$(\Gamma_{\beta,n}, n^d)$ *with rate function* $\beta E + \mathrm{Ent}(\mu_0, \cdot) + C_{\beta,n}$, *where*

$$C_\beta = -\inf_\mu \left[\beta E(\mu) + \mathrm{Ent}(\mu_0, \mu)\right].$$

Remark 8.2 In some sense, the constant C_β has to equal this value if this function is to be a rate function, indeed this way the infimum is equal to zero, as it must by Remark 3.5.

Proof This does not seem to follow easily from a moment generating function computation. Instead, we use the more direct criterion given by Proposition 6.1. Now $\mathcal{X} = P(X)$, with balls taken with respect to the p-Wasserstein distance ($p \in [1, \infty)$). Here we need the fact that when X is compact, $P(X)$ equipped with the p-Wasserstein distance function is a compact metric space [1, Sect. 2], [32] (the point is that "p-Wasserstein distance metrizes the weak topology" and that $P(X)$ is compact with respect to the weak topology). We compute (and apply Claim 8.3 below),

$$
\begin{aligned}
\lim_{e\to 0}\limsup_{n\to\infty}\frac{1}{n^d}\log\Gamma_{\beta,n}(B_e(\mu)) &= \lim_{e\to 0}\limsup_{n\to\infty}\frac{1}{n^d}\log\int_{(\delta^{n^d})^{-1}(B_e(\mu))} e^{-\beta H_{n^d}}\mu_0^{\otimes n^d} - C_{\beta,n}\\
&= \lim_{e\to 0}\limsup_{n\to\infty}\frac{1}{n^d}\log\int_{(\delta^{n^d})^{-1}(B_e(\mu))} e^{-\beta n^d(E\circ\delta^n+o(1))}\mu_0^{\otimes n^d} - C_{\beta,n}\\
&= -\beta E(\mu) + \lim_{e\to 0}\limsup_{n\to\infty}\frac{1}{n^d}\log\int_{(\delta^{n^d})^{-1}(B_e(\mu))} \mu_0^{\otimes n^d} - C_{\beta,n}\\
&= -\beta E(\mu) - \operatorname{Ent}(\mu_0,\mu) - C_{\beta,n},
\end{aligned}
$$

by Corollary 4.3 and Proposition 6.1 (remembering the minus sign in the latter). Similarly for the liminf. Applying Proposition 6.1 again, we are done. □

Claim 8.3 *The limit* (39) *exists and is finite. In fact,*

$$C_{\beta,n} = -\inf_\mu\big[\beta E(\mu) + \operatorname{Ent}(\mu_0,\mu)\big].$$

Proof By our previous computation the limit is bounded below, indeed, for any $e > 0$ and any μ,

$$
\begin{aligned}
\liminf_n \frac{1}{n^d}\log Z_{\beta,n} &= \liminf_n \frac{1}{n^d}\log\int_{X^{n^d}} e^{-\beta H_{n^d}}\mu_0^{\otimes n^d}\\
&\geq \liminf_n \frac{1}{n^d}\log\int_{(\delta^{n^d})^{-1}(B_e(\mu))} e^{-\beta H_{n^d}}\mu_0^{\otimes n^d},
\end{aligned}
$$

so for every μ,

$$
\begin{aligned}
\liminf_n \frac{1}{n^d}\log Z_{\beta,n} &\geq -\beta E(\mu) - \operatorname{Ent}(\mu_0,\mu)\\
&\geq \sup_\mu\big[-\beta E(\mu) - \operatorname{Ent}(\mu_0,\mu)\big]\\
&= -\inf_\mu\big[\beta E(\mu) + \operatorname{Ent}(\mu_0,\mu)\big],
\end{aligned}
$$

and so,

$$\limsup_n \frac{1}{n^d}\log Z_{\beta,n} \geq -\inf_\mu\big[\beta E(\mu) + \operatorname{Ent}(\mu_0,\mu)\big].$$

Finally, by compactness of $P(X)$, fix $e > 0$ and cover the space with finitely-many balls $B_e(\mu_1), \dots, B_e(\mu_k)$. Then, of course,

$$\begin{aligned}
Z_{\beta,n} &= \int_{P(X)} (\delta^{n^d})_\# \Big(e^{-\beta H_{n^d}} \mu_0^{\otimes n^d} \Big) \\
&= \int_{(\delta^{n^d})^{-1}(P(X))} e^{-\beta H_{n^d}} \mu_0^{\otimes n^d} \\
&\le \sum_{j=1}^{k} \int_{(\delta^{n^d})^{-1}(B_e(\mu_j))} e^{-\beta H_{n^d}} \mu_0^{\otimes n^d} \\
&\le k \sup_j \int_{(\delta^{n^d})^{-1}(B_e(\mu_j))} e^{-\beta H_{n^d}} \mu_0^{\otimes n^d},
\end{aligned}$$

so

$$\begin{aligned}
\liminf_n \frac{1}{n^d} \log Z_{\beta,n} &\le \liminf_n \frac{1}{n^d} \log k + \liminf_n \frac{1}{n^d} \log \sup_j \int_{(\delta^{n^d})^{-1}(B_e(\mu_j))} e^{-\beta H_{n^d}} \mu_0^{\otimes n^d} \\
&= \liminf_n \frac{1}{n^d} \log \sup_j \int_{(\delta^{n^d})^{-1}(B_e(\mu_j))} e^{-\beta H_{n^d}} \mu_0^{\otimes n^d} \\
&= \sup_j \big[-\beta E(\mu_j) - \mathrm{Ent}(\mu_0, \mu_j) \big] \\
&= -\inf_j \big[\beta E(\mu_j) + \mathrm{Ent}(\mu_0, \mu_j) \big] \\
&\le -\inf_\mu \big[\beta E(\mu) + \mathrm{Ent}(\mu_0, \mu) \big].
\end{aligned}$$

Similarly,

$$\limsup_n \frac{1}{n^d} \log Z_{\beta,n} \le -\inf_\mu \big[\beta E(\mu) + \mathrm{Ent}(\mu_0, \mu) \big],$$

so we conclude $C_{\beta,n} = \lim_n \frac{1}{n^d} \log Z_{\beta,n}$ exists and equals $-\inf_\mu \big[\beta E(\mu) + \mathrm{Ent}(\mu_0, \mu) \big]$. □

8.1 Finite-dimensional Approximations of Wasserstein Distance

In view of Lemma 8.1, Proposition 7.16 (and (37)), it remains for us to construct functions $\{H_n\}$ that approximate the (pull-back under the empirical map of the) Wasserstein distance $W_2^2(\cdot, \nu)$. We will do this in the special case $\nu = dx$.

Let $n \in \mathbb{N}$. There are n^d $1/n$-lattice points of the cube $[0, 1]^d$, and we denote them by $p_1, \dots, p_{n^d}$. Set

$$\phi_i^{(n)}(x) := \sum_{m \in \mathbb{Z}^d} e^{-n|x - p_i - m|^2}.$$

A sort of "theta function" for the real torus

$$\mathbb{T} := \mathbb{R}^d/\mathbb{Z}^d.$$

There are two sorts of symmetric functions on $\mathbb{T}^{n^d}$ one may cook up from the ϕ_i's. First, consider the matrix

$$\Phi(x_1, \ldots, x_n) := [\phi_i^{(n)}(x_j)]_{i,j=1}^{n^d}.$$

Which functions f of $\Phi(x_1, \ldots, x_n)$ are invariant under permutations, i.e., satisfy

$$f(\Phi(x_1, \ldots, x_n)) = f(\Phi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}))?$$

In other words, which functions of a matrix are invariant under permutations of rows? Note that the determinant is only invariant up to a sign. However, the permanent is fully invariant. First,

$$H_n(x_1, \ldots, x_n) := -\frac{1}{n} \log \operatorname{per} \Phi(x_1, \ldots, x_n). \tag{40}$$

Here,

$$\operatorname{per} A := \sum_{\sigma \in S_{n^d}} \prod_{i=1}^{n^d} A_{i\sigma(i)}.$$

Second,

$$H_n(x_1, \ldots, x_n) := -\frac{1}{n} \log \operatorname{per}_{\text{strop}} \Phi(x_1, \ldots, x_n). \tag{41}$$

Here, the *semi-tropical permanent* is obtained from the permanent by replacing summation by supremum,

$$\operatorname{per}_{\text{strop}} A := \sup_{\sigma \in S_{n^d}} \prod_{i=1}^{n^d} A_{i\sigma(i)}.$$

Lemma 8.4 *For both* (40) *and* (41), $\lim_{n\to\infty} ||H_n/n^d - W_2^2\big(dx, \delta^n(\cdot)\big)||_{L^\infty(X^n)} = 0$.

Proof Since $\delta^{n^d}(p_1, \ldots, p_{n^d}) \to dx$ weakly (the points are dense and uniformly distributed) then in view of Claim 8.5 below, it suffices to show that

$$\lim_{n\to\infty} ||H_n/n^d - W_2^2\big(\delta^{n^d}(p_1, \ldots, p_{n^d}), \delta^{n^d}(\cdot)\big)||_{L^\infty(X^{n^d})} = 0.$$

This is a nice simplification since we have an explicit formula for the Wasserstein distance on the image of the empirical map! Indeed [31, p. 5],

$$W_2^2\big(\delta^{n^d}(p_1, \ldots, p_{n^d}), \delta^{n^d}(x_1, \ldots, x_{n^d})\big) = \inf_{\sigma \in S_{n^d}} \sum d(p_i, x_{\sigma(i)})^2.$$

It is now a simple exercise to complete the proof using Claims 8.6 and 8.7 below.□

Claim 8.5 *Let M be compact manifold. Let $x_1, \ldots, x_k \in M$ and $p_1, \ldots, p_k \in M$. Suppose that $\delta^k(p_1, \ldots, p_k) \to \nu$ weakly. Then*

$$\lim_k ||W_2^2\big(\delta^k(p_1, \ldots, p_k), \delta^k(\cdot)\big) - W_2^2\big(\nu, \delta^k(\cdot)\big)||_{L^\infty(X^k)} = 0.$$

Proof Wasserstein distance is a distance function (see [31, 32]), hence,

$$\big|W_2\big(\delta^k(p_1, \ldots, p_k), \delta^k(x_1, \ldots, x_k)\big) - W_2\big(\nu, \delta^k(x_1, \ldots, x_k)\big)\big| \le W_2\big(\delta^k(p_1, \ldots, p_k), \nu\big),$$

with the right-hand side independent of $x_1, \ldots, x_k$. On a compact manifold weak convergence implies convergence in the Wasserstein distance, hence the right-hand side converges to zero as k tends to infinity. Finally,

$$\begin{aligned}
&\big|W_2^2\big(\delta^k(p_1, \ldots, p_k), \delta^k(x_1, \ldots, x_k)\big) - W_2^2\big(\nu, \delta^k(x_1, \ldots, x_k)\big)\big| \\
&\quad \le \big(W_2(\delta^k(p_1, \ldots, p_k), \delta^k(x_1, \ldots, x_k)) + W_2\big(\nu, \delta^k(x_1, \ldots, x_k)\big)\big) W_2\big(\delta^k(p_1, \ldots, p_k), \nu\big) \\
&\quad \le \big(2W_2\big(\delta^k(p_1, \ldots, p_k), \delta^k(x_1, \ldots, x_k)\big) + o(1)\big) W_2(\delta^k(p_1, \ldots, p_k), \nu) \\
&\quad \le \big(2\frac{1}{k}\sum d(x_i, p_i)^2 + o(1)\big) W_2\big(\delta^k(p_1, \ldots, p_k), \nu\big) \\
&\quad \le \big(C(M) + o(1)\big) W_2\big(\delta^k(p_1, \ldots, p_k), \nu\big),
\end{aligned}$$

since compactness implies the diameter is bounded. This concludes the proof. □

Claim 8.6 *Let $F : S_{n^d} \to (0, \infty)$. Then*

$$\frac{1}{n^{d+1}} \log \sup_\sigma F(\sigma) = \frac{1}{n^{d+1}} \log \sum_\sigma F(\sigma) + o(1).$$

Proof Of course,

$$\frac{1}{n^{d+1}} \log \sup_\sigma F(\sigma) \le \frac{1}{n^{d+1}} \log \sum_\sigma F(\sigma).$$

Conversely,

$$\frac{1}{n^{d+1}} \log \sum_\sigma F(\sigma) \le \frac{1}{n^{d+1}} \log n^d \sup_\sigma F(\sigma).$$

and by Stirling $\frac{1}{n^{d+1}} \log n^d = o(1)$. □

Claim 8.7 $-\frac{1}{n} \log \phi_i^{(n)}(x) = d(p_i, x)^2 + o(1)$.

Proof By definition,

$$d(p_i, x)^2 = \inf_{m\in\mathbb{Z}^d} |x - p_i - m|^2.$$

Now, of course,

$$\frac{1}{n}\log\phi_i^{(n)}(x) = \frac{1}{n}\log\sum_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2} \geq \frac{1}{n}\log\sup_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2} = -d(p_i, x)^2.$$

Conversely, for every $\epsilon > 0$ there exists $C, R > 0$ such that

$$\sum_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2} \leq \sum_{m\in B_R(0)\cap\mathbb{Z}^d} e^{-n|x-p_i-m|^2} + \epsilon \leq C \sum_{m\in B_R(0)\cap\mathbb{Z}^d} e^{-n|x-p_i-m|^2}.$$

Assuming, without loss of generality, that $\sup_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2}$ is obtained in $B_R(0)$, we have

$$\sum_{m\in B_R(0)\cap\mathbb{Z}^d} e^{-n|x-p_i-m|^2} \leq CR^d \sup_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2},$$

so

$$\begin{aligned}\frac{1}{n}\log\phi_i^{(n)}(x) &= \frac{1}{n}\log\sum_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2}\\ &\leq \frac{1}{n}\log C + \frac{1}{n}\log\sup_{m\in B_R(0)\cap\mathbb{Z}^d} e^{-n|x-p_i-m|^2}\\ &\leq o(1) + \frac{1}{n}\log CR^d \sup_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2}\\ &= o(1) + \frac{1}{n}\log\sup_{m\in\mathbb{Z}^d} e^{-n|x-p_i-m|^2}\\ &= o(1) - d(p_i, x)^2,\end{aligned}$$

where $o(1)$ depends on R, but goes to zero as n tends to infinity (for R fixed). Letting n tend to infinity concludes the proof. □

Finally, we obtain the following theorem due to Berman [4, Theorem 1.1] and Hultgren [17, Theorem 3.2].

Theorem 8.8 *For both* (40) *and* (41), *LDP*$(\Gamma_{\beta,n}, n^d)$ *with rate function* $G_{\beta,\mu_0,dx}$. *The set* $G^{-1}_{\beta,\mu_0,dx}(0)$ *is a singleton precisely when* (34) *has a unique solution.*

Proof The first statement follows from Lemmas 8.1 and 8.4. The second statement follows from Proposition 7.16. □

8.2 An Alternative Proof—Zero Temperature Approach

We were lucky enough to find finite-dimensional approximations to the main functional we were interested in (for all β at once). Here is an alternative approach, due to Berman in the permanental/Monge–Ampère setting, which allows to 'reverse engineer' the main functional by computing the limit of the moment generating functions when $\beta = \beta_n = n \to \infty$. This is an easier task because in this "zero-temprature limit" the entropy contribution disappears. This is a standard method in the field and its benefit in this setting is that when writing out the moment generating functions explicitly, the symmetry in the Hamiltonians can be exploited to reduce much of the complexity. Once an explicit formula for the limit is attained [4, Proposition 5.3], [17, Lemma 3.8], the Gärtner–Ellis Theorem can be invoked to deduce an LDP for this "zero-temperature" case (stated as a part of Theorem 1.1 in Berman's paper [4] and as Theorem 3.6 in Hultgren's article [17]). This LDP can then be used to deduce Lemma 8.4 above (corresponding to Lemma 4.9 in [4] and Lemma 3.14 in [17]), after which Theorem 8.8 is proved as in the previous section. This original approach of Berman and Hultgren to proving Theorem 8.8 also can be made to work in the case we are no longer on the torus, but rather on a non-compact manifold, as in the toric setting. The reason we chose the proof we presented above is that we found it slightly more pedagogical to directly deal with all β at once and avoid this extra use of the Gartner–Ellis theorem.

We will now explain the main part of the alternative argument for Theorem 8.8, that as just mentioned, was the original proof. Namely, how to prove the LDP when $\beta = \beta_k = k \to \infty$. Here is the main observation:

Lemma 8.9 *Let $H_n : X^{n^d} \to \mathbb{R}$ be given by (40) or (41). Set*

$$\Gamma_{n,n} := \delta_{\#}^{n^d}\left(e^{-nH_n}\mu_0^{\otimes n^d}\right)\Big/ Z_{n,n} \in P(P(X)).$$

Then $LDP(\Gamma_{n,n}, n^{d+1})$ with rate function $W_2^2(dx,\ \cdot)$.

Proof This time, we can use Theorem 4.1. By Claim 8.10 below, $\lim \frac{1}{n^{d+1}} \log Z_{n,n} = 0$. Thus, assuming (40), the moment generating function simplifies as follows,

$$
\begin{aligned}
p(\theta) &= \lim \frac{1}{n^{d+1}} \log \int_{P(X)} e^{n^{d+1}\langle\theta,\nu\rangle} \Gamma_{n,n}(\nu) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} e^{n^{d+1}\langle\theta,\delta^{n^d}(\,\cdot\,)\rangle} \Big(e^{-nH_n}\mu_0^{\otimes n^d}\Big) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} e^{n^{d+1}\langle\theta,\delta^{n^d}(x_1,\ldots,x_{n^d})\rangle} e^{-nH_n(x_1,\ldots,x_{n^d})} \mu_0(x_1)\otimes\cdots\mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} e^{n^{d+1}n^{-d}\sum_{i=1}^{n^d}\theta(x_i)} e^{-nH_n(x_1,\ldots,x_{n^d})} \mu_0(x_1)\otimes\cdots\mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} e^{n\sum_{i=1}^{n^d}\theta(x_i)} \sum_\sigma \prod \phi_i^{(n)}(x_{\sigma(i)}) \mu_0(x_1)\otimes\cdots\mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} \sum_\sigma \prod \Big[e^{n\theta(x_{\sigma(i)})}\phi_i^{(n)}(x_{\sigma(i)})\Big] \mu_0(x_1)\otimes\cdots\mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} n^d! \prod \Big[e^{n\theta(x_i)}\phi_i^{(n)}(x_i)\Big] \mu_0(x_1)\otimes\cdots\mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} \prod \Big[e^{n\theta(x_i)}\phi_i^{(n)}(x_i)\Big] \mu_0(x_1)\otimes\cdots\mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \prod \Big[\int_X e^{n\theta}\phi_i^{(n)}\mu_0\Big] \\
&= \lim \frac{1}{n^d} \sum_{i=1}^{n^d} \frac{1}{n} \log \Big[\int_X e^{n\theta}\phi_i^{(n)}\mu_0\Big].
\end{aligned}
$$

By Claim 8.7, $-\frac{1}{n}\log\phi_i^{(n)}(x) = d(p_i,x)^2 + o(1)$. and by Claim 8.13 below we thus have

$$
p(\theta) = \lim \frac{1}{n^d} \sum_{i=1}^{n^d} \big[(-\theta)^\star(p_i) + o(1)\big] = \lim \big\langle \delta^{n^d}(p_1,\ldots,p_{n^d}), (-\theta)^\star \big\rangle = \int_X ((-\theta)^\star dx,
$$

since $\delta^k(p_1,\ldots,p_k) \to dx$. Thus, by Theorems 4.1 and 7.14 we are done. □

Claim 8.10 $\lim \frac{1}{n^{d+1}} \log Z_{n,n} = 0$.

In fact, we will give a rate of decay, $\frac{1}{n^{d+1}} \log Z_{n,n} = O(1/n)$.

Remark 8.11 It actually suffices to show that $\lim \frac{1}{n^{d+1}} \log Z_{n,n}$ exists—it then must be zero: by Theorem 4.1, once we have a large deviation principle and we know that the rate function is $W_2^2(dx,\,\cdot\,)$ up to a constant, then we can determine that constant by the fact that the infimum of the rate function must be zero (Remark 3.5). Since $\inf W_2^2(dx,\,\cdot\,) = 0$ (attained for dx), we get the constant must be zero. At any rate, we will prove Claim 8.10 directly.

Remark 8.12 In fact, here is a quick proof: $\lim_n \frac{1}{n^{d+1}} \log Z_{n,n} = p(0)$, which, by the previous computation, equals $\int 0^\star = 0$.

Proof We compute,

$$\begin{aligned}
\lim \frac{1}{n^{d+1}} \log Z_{n,n} &= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} \text{per}[\phi_i^{(n)}(x_j)]\mu_0^{\otimes n^d} \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} \sum_\sigma \prod_{i=1}^{n^d} \phi_i^{(n)}(x_{\sigma(i)})\mu_0(x_1) \otimes \cdots \mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} \sum_\sigma \prod_{i=1}^{n^d} \phi_i^{(n)}(x_i)\mu_0(x_1) \otimes \cdots \mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} n^d! \prod \phi_i^{(n)}(x_i)\mu_0(x_1) \otimes \cdots \mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \int_{X^{n^d}} \prod \phi_i^{(n)}(x_i)\mu_0(x_1) \otimes \cdots \mu_0(x_{n^d}) \\
&= \lim \frac{1}{n^{d+1}} \log \prod \int_X \phi_i^{(n)}\mu_0 \\
&= \lim \frac{1}{n^d} \sum_{i=1}^{n^d} \frac{1}{n} \log ||\phi_i^{(n)}||_{L^1(\mu_0)} \leq C_n,
\end{aligned}$$

where $C_n := \sup_{i=1,\ldots n^d} \frac{1}{n} \log ||\phi_i^{(n)}||_{L^1(\mu_0)}$. Now, it remains to estimate C_n. By Claim 8.7, $-\frac{1}{n}\log \phi_i^{(n)}(x) = d(p_i, x)^2 + o(1)$, so

$$\frac{1}{n} \log ||\phi_i^{(n)}||_{L^1(\mu_0)} \leq \frac{1}{n} \log ||e^{-n(d(p_i,x)^2+o(1))}||_{L^1(\mu_0)} = O(1/n).$$

Since i was arbitrary, $C_n = O(1/n)$ and we are done. □

Claim 8.13 $\lim_{k\to\infty} \frac{1}{k} \log \int e^{k(d(x,y)^2 - f(x))}dx = f^\star(y)$.

Proof Of course, $\lim_{k\to\infty} ||F||_{L^k(X,\mu)} = ||F||_{L^\infty(X)}$ for continuous F and compact X and probability μ. By definition, $\sup_x[d(x,y)^2 - f(x)] = f^\star(y)$. So,

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k} \log \int e^{k(d(x,y)^2-f(x))}dx &= \lim_{k\to\infty} \log ||e^{d(x,y)^2-f(x)}||_{L^k(dx)} \\
&= \log ||e^{d(x,y)^2-f(x)}||_{L^\infty} \\
&= ||d(x,y)^2 - f(x)||_{L^\infty} \\
&= f^\star(y),
\end{aligned}$$

as desired. □

The alternative proof of Theorem 8.8 (that is actually the original proof in [17]) is now a consequence. The point is that once we know there is a large deviation principle for $\beta \to \infty$ we can use Proposition 6.1 and Sanov's Corollary 4.3 to deduce the convergence in Lemma 8.4 in an argument which provide a formal converse of Lemma 8.1 above, valid in the $\beta \to \infty$ case (see Lemma 4.9 in [4] or the proof of Theorem 3.2 in [17]). After that we get the LDP in Theorem 8.8 by applying Lemma 8.4 as above.

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References

1. Ambrosio, L., Gigli, N.: A user's guide to optimal transport, version downloaded in 2011 from https://cvgmt.sns.it/paper/195/
2. Bakry, D.: L'hypercontractivité et son utilisation en théorie des semigroupes, in Lectures on Probability Theory (Saint-Flour, 1992). Lect. Notes Math. **1581**, 1–114 (1994)
3. Bakry, D., Ledoux, M.: Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator. Duke Math. J. **85**, 253–270 (1996)
4. Berman, R.: Statistical mechanics of permanents, real Monge–Ampère equations and optimal transport. arxiv:1302.4045
5. Berman, R.: Large deviations for Gibbs measures with singular Hamiltonians and emergence of Kähler-Einstein metrics. Commun. Math. Phys. **354**, 1133–1172 (2017)
6. Chung, K.L.: Green, Brown, and Probability and Brownian Motion on the Line. World Scientific (2002)
7. Delarue, F.: Probabilistic approach to regularity. In: Guedj, V. (ed.), Complex Monge–Ampère Equations and Geodesics in the Space of Kähler metrics. Lecture Notes in Mathematics, vol. 2038, pp. 55–198 (2011)
8. Dembo, A., Zeitouni, O.: Large Deviation Techniques and Applications, 2nd ed. Springer (1998)
9. Donaldson, S.K.: Scalar curvature and projective embeddings I. J. Differ. Geom. **59**, 479–522 (2001)
10. Ellis, R.S., Haven, K., Turkington, B.: Large Deviation Principles and complete equivalence and nonequivalence results for pure and mixed ensembles. J. Statist. Phys. **101**, 999–1064 (2000)
11. Feng, R., Zelditch, S.: Large deviations for zeros of $P(\phi)_2$ random polynomials. J. Stat. Phys. **143**, 619–635 (2011)
12. Gaveau, B.: Méthodes de contrôle optimale en analyse complexe. I. Résolution d'équations de Monge–Ampère. J. Funct. Anal. **25**, 391–411 (1977)
13. Gaveau, B.: Méthodes de contrôle optimale en analyse complexe. II. Équation de Monge–Ampère dans certains domaines faiblement pseudoconvexes. Bull. Soc. Math. France **102**, 101–128 (1978)
14. Gaveau, B.: Méthodes de contrôle optimale en analyse complexe ou réelle. III. Diffusion d'équations des fonctions plurisousharmoniques ou convexes. J. Funct. Anal. **32**, 228–253 (1979)
15. Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1976)

16. Hultgren, J.: Real and complex Monge–Ampère equations, statistical mechanics and canonical metrics, Ph.D. thesis, Chalmers University of Technology and University of Gothenburg (2018)
17. Hultgren, J.: Permanental point processes on real tori, theta functions and Monge-Ampére equations. Ann. Fac. Sci. Toulouse: Math. **28**, 11–65 (2019)
18. Jeffres, T., Mazzeo, R., Rubinstein, Y.A.: Kahler-Einstein metrics with edge singularities, (with an appendix by C. Li and Y.A. Rubinstein). Ann. Math. **183**, 95–176 (2016)
19. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus, 2nd ed. Springer (1991)
20. Krylov, N.V.: Nonlinear Elliptic and Parabolic Equations of the Second Order. D. Reidel Publishing (1987)
21. Krlylov, N.V.: Smoothness of the payoff function for a controllable diffusion process in a domain, Izv. Akad. Nauk SSSR Ser. Mat. **53**, 66–96; translation in Math. USSR-Izv. **34**(1990), 65–95 (1989)
22. Krylov, N.V.: Moment estimates for the quasiderivatives, with respect to the initial data, of solutions of stochastic equations and their application. Math. USSR-Sb. **64**, 505–526 (1989)
23. Krylov, N.V.: On control of diffusion processes on a surface in Euclidean space. Math. USSR-Sb. **65**, 185–203 (1990)
24. Phong, D.H., Sturm, J.: The Monge-Ampère operator and geodesics in the space of Kähler potentials. Invent. Math. **166**, 125–149 (2006)
25. Rassoul-Agha, F., Seppäläinen, T.: A course on large deviations with an introduction to gibbs measures. Am. Math. Soc. (2015)
26. Rauch, J., Taylor, B.A.: The Dirichlet problem for the multidimensional Monge-Ampère equation. Rocky Mt. J. Math. **7**, 345–364 (1977)
27. Rockafellar, R.T.: Convex Analysis, Princeton University Press (1970)
28. Rubinstein, Y.A.: Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics. Adv. Math. **218**, 1526–1565 (2008)
29. Rubinstein, Y.A.: Smooth and singular Kähler–Einstein metrics. In: Albin, P. et al. (eds.) Geometric and Spectral Analysis. Contemporary Mathematics—American Mathematical Society and Centre de Recherches Mathématiques, vol. 630, pp. 45–138 (2014)
30. Song, J., Zelditch, S.: Test configurations, large deviations and geodesic rays on toric varieties. Adv. Math. **229**, 2338–2378 (2012)
31. Villani, C.: Topics in optimal transportation. Am. Math. Soc. (2003)
32. Villani, C.: Optimal Transport. Springer (2009)
33. Zeitouni, O., Zelditch, S.: Large deviations of empirical measures of zeros of random polynomials. Int. Math. Res. Not. 3935–3992 (2020)
34. Zelditch, S.: Large deviations of empirical measures of zeros on Riemann surfaces. Int. Math. Res. Not. 592–664 (2013)
35. Zelditch, S.: Eigenfunctions of the Laplacian on a Riemannian Manifold. Am. Math. Soc. (2017)

On Birational Boundedness of Some Calabi–Yau Hypersurfaces

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Abstract We show the birational boundedness of anti-canonical irreducible hypersurfaces which form 3-fold plt pairs. We also treat a collection of Du Val K3 surfaces which is birationally bounded but unbounded.

Keywords Calabi-Yau varieties · Boundedness

1 Introduction

In the classification of algebraic varieties, Calabi–Yau manifolds (CY manifolds for short) form an important class. It is not known whether n-dimensional CY manifolds form a bounded family for a fixed $n \geq 3$.

On the other hand, in the 2-dimensional case, there are infinitely many projective families of K3 surfaces although they are analytically deformation equivalent. Reid observed that there are only 95 families of weighted K3 hypersurfaces ([31, pp.300], [18, 13.3]). Inspired by this, we ask whether K3 surfaces in a 3-fold are bounded or not. We show the following statement in this note.

Theorem 1.1 *Let* (X, D) *be a plt pair such that* $\dim X = 3$*,* D *is irreducible and reduced, and* $K_X + D \sim 0$*. Then* D *forms a birational bounded family.*

An interesting feature is that X can be unbounded as in Example 2.11. In fact, we study the birational boundedness of a prime divisor D for a 3-fold plt pair (X, D) such that $K_X + D \equiv 0$ in Theorem 2.12. It turns out that D is birationally bounded unless X is birational to a conic bundle over a Du Val surface S with $K_S \sim 0$. The divisor D can be unbounded as in the exceptional case as in Example 2.15. The pair as above is called a plt CY pair in this note (Definition 2.1). CY pairs have been studied in several contexts of algebraic geometry (cf. [4], [9], [26], etc).

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The following example due to Oguiso forces us to use ‘birational boundedness’ rather than ‘boundedness’ in Theorem 1.1.

Theorem 1.2 (= Example *3.2*) *Fix any positive integer d. Then we have an unbounded collection of Du Val K3 surfaces which are birational contractions of smooth K3 surfaces of degree $2d$.*

When $d = 2$, the examples are birational contractions of some smooth quartic surfaces and infinitely many of them can be embedded into rational 3-folds (Remark 3.3). Thus the statement in Theorem 1.1 is optimal in a sense.

Classically, examples of CY 3-folds are constructed by taking weighted or toric hypersurfaces. In Sect. 4, we ask whether CY hypersurfaces in rationally connected varieties form a bounded family. We confirm that toric hypersurfaces form a bounded family in Corollary 4.6.

Throughout this paper, we work over the complex number field $\mathbb{C}$.

2 Finiteness of Anticanonical Calabi–Yau Surfaces in a 3-fold

We follow the notation in [22].

Definition 2.1 We say that (X, D) is a *plt Calabi–Yau (CY) pair* if (X, D) is a plt pair such that $K_X + D \equiv 0$. A plt CY pair (X, D) is called a *reduced plt CY pair* if D is a reduced divisor.

Note that X can be non-$\mathbb{Q}$-Gorenstein, but the support of the round down $\lfloor D \rfloor$ of D is normal (cf. [22, Proposion 5.51]). Note also that X is $\mathbb{Q}$-factorial in codimension 2 (cf. [12, Proposition 9.1]) and $K_X + D$ is torsion (cf. [21, Corollary 10], [13, Theorem 1.2]).

When $K_X + D \sim 0$ and D is reduced, we have the following.

Proposition 2.2 *Let (X, D) be a reduced plt CY pair such that $K_X + D \sim 0$.*

Then D has only canonical singularities. If X is $\mathbb{Q}$-Gorenstein, then X has only canonical singularities.

Proof We can take a log resolution $\mu\colon \tilde{X} \to X$ of (X, D) such that

$$K_{\tilde{X}} + \tilde{D} = \mu^*(K_X + D) + \sum a_i E_i$$

for some integers $a_i \geq 0$, where $\tilde{D}$ is the strict transform of D and E_i is the exceptional divisor. Note that $a_i \geq 0$ since $K_X + D$ is Cartier. This implies that X has only canonical singularities in codimension 2 (outside the non-$\mathbb{Q}$-Gorenstein locus). In particular, we see that K_X is Cartier in codimension 2 and $(K_X + D)|_D = K_D$ is trivial. Thus, by restricting the equality to $\tilde{D}$, we see that D has only canonical singularities. □

The plt CY property is preserved by steps of the K_X-MMP as follows.

Proposition 2.3 *Let (X, D) be a reduced plt CY pair such that X is projective and $\mathbb{Q}$-factorial. Let $\phi\colon X \dashrightarrow Y$ be a birational map which is a step of a K_X-MMP, that is, ϕ is either a divisorial contraction or a flip. Let $D_Y := \phi_* D$.*

Then the pair (Y, D_Y) is also a plt CY pair.

Remark 2.4 We can not hope that (Y, D_Y) is dlt when (X, D) is so. Consider the pair $(\mathbb{P}^3, D)$ for a quartic surface D with a simple elliptic singularity $p \in D$ and its blow-up $X_1 \to \mathbb{P}^3$ at p. Let $D_1 \subset X_1$ be the strict transform of D and E_1 be the exceptional divisor. Then $(X_1, D_1 + E_1)$ is a dlt CY pair, $X_1 \to \mathbb{P}^3$ is a K_X-negative divisorial contraction and $(\mathbb{P}^3, D)$ is lc and not dlt.

Proof Since we have $D_Y = \phi_*(D) \in |-K_Y|_{\mathbb{Q}}$, it is enough to show that (Y, D_Y) is plt. Let E be an exceptional divisor over Y (hence over X). If $\phi\colon X \to Y$ is a divisorial contraction and E is the ϕ-exceptional prime divisor, we see that $E \not\subset \operatorname{Supp} D$ by the negativity lemma (cf. [3, Lemma 3.6.2]) since D is ϕ-ample. Hence we have $-1 < a(E, X, D) = a(E, Y, D_Y)$ since both $K_X + D$ and $K_Y + D_Y$ are trivial. Also when ϕ is a flip, we have the same equality by the same reason. Hence we see that both discrepancies are greater than -1, thus (Y, D_Y) is also plt. $\square$

The following is based on the argument in the e-mail from Chen Jiang.

Proposition 2.5 *Let $n \in \mathbb{Z}_{>0}$ and $I \subset [1, 0] \cap \mathbb{Q}$ be a DCC set. Let (X, D) be an n-dimensional projective plt CY pair such that the coefficients of D belong to I. Then we have the following.*

(i) (X, D) is ϵ-plt for some $\epsilon > 0$ which only depends on n and I, that is, for an exceptional divisor E over X, the discrepancy $a(E; X, D) > -1 + \epsilon$.

(ii) Assume that $\dim X = 3$ *and D is reduced.*
Then D is bounded except when D has only Du Val singularities and X is smooth in codimension 2 around D.
We have $(K_X + D)|_D = K_D$ in the exceptional case.

Proof (i) This can be shown by the same argument as [11, Corollary 2.9] (In fact, (i) follows from [4, Lemma 2.48]). Suppose that there exists a plt CY pair (X_n, D_n) which is ϵ_n-plt for some $\epsilon_n > 0$ such that $(\epsilon_n)_n$ is a decreasing sequence and $\lim_{n\to\infty} \epsilon_n = 0$. Then there is an extraction $\tilde{X}_n \to X_n$ of a divisor E_n with $a(E_n; X_n, D_n) = -1 + \epsilon_n$ so that $(\tilde{X}_n, \tilde{D}_n + (1 - \epsilon_n)E_n)$ satisfies the assumption of the global ACC [15, Theorem 1.5] since $I \cup \{1 - \epsilon_n \mid n \in \mathbb{N}\}$ is a DCC set. Thus $\{1 - \epsilon_n \mid n \in \mathbb{N}\}$ is a finite set and this is a contradiction.

(ii) By the adjunction using the different, we have an equality

$$K_X + D|_D = K_D + \sum_{i=1}^{l} b_i B_i$$

as $\mathbb{Q}$-divisors for some prime divisors $B_1, \ldots, B_l$. Note that b_i belongs to some finite set I_0 by the global ACC [15, Theorem 1.5] since b_i belongs to a DCC set $\{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Suppose that $b_i \neq 0$ for some i. Then we see that $(D, \sum b_i B_i)$ is ϵ-lc for some ϵ independent of X. By [2, Theorem 6.9], we see that D belongs to a bounded family.

Hence the problem is reduced to the case $K_X + D|_D = K_D$. This implies that X is smooth at all codimension 1 points of D by the local computation of the different (cf. [23, Proposition 4.5 (1)]). Thus we see that $K_D \equiv 0$. Such surfaces are bounded except when D has only Du Val singularities by [2, Theorem 6.9]. □

If a plt CY pair (X, D) admits a del Pezzo fibration $X \to C$ over a curve, then D belongs to a bounded family as follows. (Note that C is either $\mathbb{P}^1$ or an elliptic curve by the canonical bundle formula.)

Proposition 2.6 *Let (X, D) be a projective $\mathbb{Q}$-Gorenstein 3-fold plt CY pair with a fiber space $\phi\colon X \to C$ over a smooth curve C such that D is irreducible, reduced and ϕ-ample.*

Then there exist a positive integer N and an ample line bundle $\mathcal{H}$ on D such that N is independent of X and $\mathcal{H}^2 \leq N$, thus such D's form a bounded family.

Proof Note first that (X, D) is ϵ-plt by Proposition 2.5 (i) for some $\epsilon > 0$ and the general fiber X_p over $p \in C$ of ϕ is an ϵ-lc log del Pezzo surface. By Proposition 2.5(ii), it is enough to consider the case where D has only Du Val singularities and X is smooth in codimension 2 around D. By this, the restriction $-K_X|_D$ is determined as a Weil divisor. □

Claim 2.7 There exists a positive integer m such that m is independent of X and mL is a Cartier divisor for all Weil divisor L on D.

Proof of Claim. The claim follows since there are finitely many possibilities for the singularities on D (cf. [1, (4.8.1)]). Let $\nu_D\colon \tilde{D} \to D$ be the minimal resolution. If D is singular, then $\tilde{D}$ is either a K3 surface or an Enriques surface. Then the number of the ν_D-exceptional (-2)-curves is less than $\rho(\tilde{D}) \leq 20$ (or < 10 if $\tilde{D}$ is Enriques) since the exceptional curves are linearly independent in Pic $\tilde{D}$.

We shall find an ample divisor of the form $m(-K_X + aF)|_D$ for a fiber $F := \phi^{-1}(p)$. The point is that a can be unbounded as in Example 2.11, but the degree of the divisor is bounded.

Let $\phi_D := \phi|_D\colon D \to C$ and $F_D := \phi_D^{-1}(p)$ be its fiber over $p \in C$. Then ϕ_D is an elliptic fibration since, for a general $p \in C$, we have $F_D \in |-K_F|_{\mathbb{Q}}$ for a log del Pezzo surface F and we check $h^0(F_D, \mathcal{O}_{F_D}) \simeq \mathbb{C}$.

Let $\mathcal{L}_D := m(-K_X|_D)$ be the restricted divisor which is ϕ_D-ample. Let

$$\alpha := \min\{a \in \mathbb{Z} \mid h^0(D, \mathcal{L}_D + aF_D) \neq 0\}.$$

Then we have an exact sequence

$$0 = H^0(D, \mathcal{L}_D + (\alpha - 1)F_D) \to H^0(D, \mathcal{L}_D + \alpha F_D) \to H^0(F_D, (\mathcal{L}_D + \alpha F_D)|_{F_D}).$$

Note that $(\mathcal{L}_D + \alpha F_D)|_{F_D} = -mK_X|_{F_D}$ and its degree is $m(-K_F^2) =: md$, where F is a general fiber of ϕ which is an ϵ-lc del Pezzo surface of degree d. Indeed, we have

$$-K_X \cdot F_D = -K_X \cdot D \cdot F = (-K_X)^2 \cdot F = (-K_F)^2 = d.$$

Note that $d \leq \delta$ for some integer $\delta = \delta_\epsilon$ determined by ϵ (the maximal integer degree of ϵ-lc del Pezzo surfaces. See [19] for the optimal bound.). Since F_D is an elliptic curve, we have $h^0(F_D, (\mathcal{L}_D + \alpha F_D)|_{F_D}) = md$. Thus, by the above exact sequence, we see that

$$h^0(D, \mathcal{L}_D + \alpha F_D) \leq md. \tag{1}$$

Claim 2.8 The Cartier divisor $\mathcal{L}_D + (k + \alpha)F_D$ is ample for $k > 2\delta m$.

Proof of Claim. Let $\nu_D\colon \tilde{D} \to D$ be the minimal resolution of D and $F_{\tilde{D}} := \nu_D^*(F_D)$ be the pull-back, and $\phi_{\tilde{D}} := \nu_D \circ \phi_D\colon \tilde{D} \to C$ be the composition.

Let $\nu_D^*(\mathcal{L}_D + \alpha F_D) = M + E$ be the decomposition to the mobile part M and fixed part E. We can write $E = \sum_{i=1}^{l} a_i C_i$ for some $a_i \geq 0$ and (-2)-curves $C_1, \ldots, C_l$ so that $C_1, \ldots, C_{l'}$ are $\phi_{\tilde{D}}$-horizontal and $C_{l'+1}, \ldots, C_l$ are $\phi_{\tilde{D}}$-vertical. Note that

$$md = \nu_D^*(\mathcal{L}_D + \alpha F_D) \cdot F_{\tilde{D}} \geq E \cdot F_{\tilde{D}} = (\sum_{i=1}^{l} a_i C_i) \cdot F_{\tilde{D}} \geq \sum_{i=1}^{l'} a_i$$

since C_i is vertical for $i > l'$. Hence we obtain

$$a_i \leq \delta m \ \ (i = 1, \ldots, l').$$

In order to check $\mathcal{L}_D + (k + \alpha)F_D$ is nef, it is enough to check

$$\nu_D^*(\mathcal{L}_D + (\alpha + k)F_D) \cdot C_i \geq 0$$

for $i = 1, \ldots, l'$ since $\mathcal{L}_D$ is ϕ_D-ample. For $k \geq 2\delta m$, we have

$$\begin{aligned} \nu_D^*(\mathcal{L}_D + (\alpha + k)F_D) \cdot C_i &= (M + E + kF_{\tilde{D}}) \cdot C_i \\ \geq (a_i C_i + kF_{\tilde{D}}) \cdot C_i = -2a_i + k(F_{\tilde{D}} \cdot C_i) &\geq -2\delta m + k(F_{\tilde{D}} \cdot C_i) \geq 0. \end{aligned}$$

since C_i is horizontal and $F_{\tilde{D}} \cdot C_i \geq 1$. Thus $\mathcal{L}_D + (\alpha + k)F_D$ is nef for $k \geq 2\delta m$, thus ample when $k > 2\delta m$. □

For a positive integer β and a divisor $\mathcal{L}_\beta := \mathcal{L}_D + (\alpha+\beta)F_D$, we have an exact sequence

$$0 \to H^0(D, \mathcal{L}_\beta) \to H^0(D, \mathcal{L}_{\beta+1}) \to H^0(F_D, \mathcal{L}_{\beta+1}|_{F_D}).$$

By $h^0(F_D, \mathcal{L}_{\beta+1}|_{F_D}) = h^0(F_D, \mathcal{L}_D|_{F_D}) = md$ as before, we have

$$h^0(D, \mathcal{L}_{\beta+1}) \le h^0(D, \mathcal{L}_\beta) + md.$$

By this and (1), we obtain

$$h^0(D, \mathcal{L}_{2\delta m+1}) \le md + (2\delta m + 1)md = 2\delta m^2 d + 2md \le 2m^2\delta^2 + 2m\delta.$$

Since $\mathcal{L}_{2\delta m+1}$ is ample, we have $h^i(D, \mathcal{L}_{2\delta m+1}) = 0$ for $i = 1, 2$. Since $\mathcal{L}_{2\delta m+1}$ is Cartier, we obtain

$$h^0(D, \mathcal{L}_{2\delta m+1}) = \chi(D, \mathcal{L}_{2\delta m+1}) = \chi_D + \frac{(\mathcal{L}_{2\delta m+1})^2}{2},$$

where $\chi_D := \chi(D, \mathcal{O}_D) = 0, 1, 2$ since D is either a (Du Val) K3 surface, Enriques surface or abelian surface.

Thus we see that $\mathcal{L}^2_{2\delta m+1}$ is bounded by the constant $2(2m^2\delta^2 + 2m\delta - \chi_D)$ and $\mathcal{H} := \mathcal{L}_{2\delta m+1}$ has the required property. By [2, Lemma 3.7 (1)], we see that D forms a bounded family. □

Remark 2.9 When D is an abelian surface, we have the same statement as Claim 2.8 for $k > 0$ since an effective divisor on D is nef.

Example 2.10 There are infinitely many examples of conic bundles with smooth anticanonical members in [24, Example 20]. Let $\mathbb{P} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(c))$ for $c \ge 3$ and $X := X_c \in |\mathcal{O}_{\mathbb{P}}(2)|$ be a smooth member. Then $\phi\colon X \to \mathbb{P}^2$ is a conic bundle and $|-K_X|$ contains a smooth member D. Since D is also an anticanonical member of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3))$, we see that D is bounded with a polarization $\mathcal{O}_{\mathbb{P}}(1)|_D$ of degree 18. We see that $\rho(X) = 2$ by the Lefschetz type theorem [32, Theorem 2] and check that the collection $\{X_c\}_{c=1,2,\dots}$ is unbounded. Indeed, a nef and big divisor on X can be written as

$$H_{a,b} := -aK_X + bF = a(-K_X + cF) + (b - ca)F,$$

where $a, b \in \mathbb{Z}_{>0}$ satisfy $b \ge ca$ and $F := \phi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Thus we compute

$$H^3_{a,b} \ge (-K_X + cF)^3 = 2(\mathcal{O}_{\mathbb{P}}(1)^4) = 2(c^2 + 3c + 9)$$

by using $H \cdot (H - 3f) \cdot (H - cf) = 0$ for $H := \mathcal{O}_{\mathbb{P}}(1)$ and $f := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ for $\pi\colon \mathbb{P} \to \mathbb{P}^2$. Indeed, since we have $H^3 = (c+3)H^2 \cdot f - 3cH \cdot f^2$ and $H^2 \cdot f^2 = 1$, we obtain

$$H^4 = H(H^3) = H((c+3)H^2 \cdot f - 3cH \cdot f^2) = (c+3)H^3 \cdot f - 3c(H^2 f^2)$$
$$= (c+3)((c+3)H^2 \cdot f - 3cH \cdot f^2) \cdot f - 3c \cdot 1 = (c+3)^2 - 3c = c^2 + 3c + 9.$$

Hence we see the unboundedness of X_c.

Moreover, we check that the collection $\{X_c \mid c \geq 3\}$ is birationally unbounded by the same argument as [27]. Indeed, the discriminant curve $B_c \subset \mathbb{P}^2$ of $\phi_c \colon X_c \to \mathbb{P}^2$ has degree $2c+6$ as [24, Example 20], thus $4K_{\mathbb{P}^2} + B_c$ is effective when $c \geq 3$. Hence the conic bundle $\phi_c \colon X_c \to \mathbb{P}^2$ is birationally rigid (cf. [10, Theorem 4.2]). Then we can use the argument in [27, Sect. 3] to show that $\{X_c \mid c \geq 3\}$ is birationally unbounded.

Example 2.11 There also exist infinitely many examples of del Pezzo fibrations $X \to \mathbb{P}^1$ such that X is smooth and $|-K_X|$ contains a smooth member. Let

$$X := X_n \subset \mathbb{P} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2) \oplus \mathcal{O}(n))$$

be a smooth member of $|\mathcal{O}_{\mathbb{P}}(3)|$. Then the induced projection $\phi \colon X \to \mathbb{P}^1$ is a del Pezzo fibration and $|-K_X| = |\mathcal{O}_{\mathbb{P}}(1) \otimes \phi^*\mathcal{O}(-n)|$ contains a smooth member S. We see that S is isomorphic to an anticanonical member of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$ and has a polarization of the degree independent of n. However, the collection $\{X_n\}_{n \in \mathbb{N}}$ is not bounded. Indeed, we see that $\operatorname{Pic} X = \mathbb{Z}(-K_X) \oplus \mathbb{Z}(F)$ for $F := \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$ as above, and a nef and big line bundle

$$G_{a,b} := a(-K_X) + bF = a(-K_X + nF) + (b - na)F$$

should satisfy $b \geq na$. Thus we see the unboundedness of X_n by computing

$$G_{a,b}^3 \geq (-K_X + nF)^3 = 3n + 6$$

since $0 = H^2 \cdot (H - 2f)(H - nf) = H^2(H^2 - (n+2)H \cdot f + 2nf^2) = H^4 - (n+2)H^3 \cdot f = H^4 - (n+2)$, where $H := \mathcal{O}_{\mathbb{P}}(1)$ and f is the fiber class.

For an elliptic curve C and a positive integer d, consider $\mathbb{P}_C := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C(dP))$ and a smooth member $X_d \in |\mathcal{O}_{\mathbb{P}_C}(3)|$. Then $X_d \to C$ is a del Pezzo fibration and $S_d \in |-K_{X_d}|$ is an abelian surface with a bounded polarization. We check the unboundedness of X_d by a similar calculation as above.

The following implies Theorem 1.1.

Theorem 2.12 *Let (X, D) be a projective 3-fold plt CY pair such that D is irreducible and reduced. Then D is birationally bounded unless all of the following hold:*

(1) $K_X + D \not\sim 0$, but $2(K_X + D) \sim 0$.
(2) X is birational to a conic bundle $Y \to S$ such that S is either a Du Val K3 surface or an abelian surface.

(3) *For the strict transform $D_Y \subset Y$ of D, the induced morphism $D_Y \to S$ is étale in codimension 1*

In particular, Theorem 1.1 holds.

Proof By taking a small $\mathbb{Q}$-factorial modification (cf. [23, Corollary 1.37]), we may assume that X is $\mathbb{Q}$-factorial.

Let $\phi\colon X \dashrightarrow X_m$ be a birational map induced by a K_X-MMP and $\phi_D\colon D \dashrightarrow D_m$ be the birational map induced by ϕ. We also have a Mori fiber space $\phi_m\colon X_m \to S$. Note that (X_m, D_m) is also a plt CY pair by Proposition 2.3. It is enough to consider the case where D_m has only Du Val singularities by Proposition 2.5(ii). The problem is to bound such D_m.

Consider the case $\dim S = 0$. Then X_m is a ϵ-lc Fano 3-fold for some $\epsilon > 0$ by Proposition 2.5, thus it is bounded by [5, Theorem 1.1] and D_m is also bounded.

Next consider the case $\dim S = 1$. Then $X_m \to S$ is a del Pezzo fibration and D_m is bounded by Proposition 2.6.

Next consider the case where $\dim S = 2$ and the induced morphism $D_m \to S$ is of degree 2 and branched along a curve. Then $(S, \frac{1}{2}R)$ is a $\frac{1}{2}$-lc CY pair (cf. [22, Proposition 5.20]), where $R \in |-2K_S|$ is the branch divisor of the double cover $\pi_m\colon D_m \to S$ (or its Stein factorization). Then $(S, \frac{1}{2}R)$ is log bounded by [2, Theorem 6.9]. Thus D_m is also bounded since it is a crepant modification of the double cover of S branched along R (For a polarization H on S with the bounded degree, $\pi_m^* H$ gives a quasi-polarization on D_m with the bounded degree).

Finally consider the case where $\dim S = 2$ and $\pi_m\colon D_m \to S$ is étale in codimension 1. Then we see that $K_S \equiv 0$. Thus S and D_m are bounded unless S has only Du Val singularities by [2, Theorem 6.8]. Since we are interested in the birational boundedness of D, it is enough to assume $K_S \sim 0$, that is, S is either a Du Val K3 surface or an abelian surface since Enriques surfaces and bielliptic surfaces are bounded. Hence the problem is reduced to the following claim.

Claim 2.13 In the above setting, assume that S is a Du Val K3 surface or an abelian surface. Then we have the following.

(i) $K_{X_m} + D_m \not\sim 0$.
(ii) $2(K_{X_m} + D_m) \sim 0$.

Proof of Claim. Let $X := X_m$ and $D := D_m$ with a conic bundle $\phi\colon X \to S$. Note that $\phi_D := \phi|_D$ is étale in codimension 1 and, if S is an abelian surface, then ϕ_D is étale by the purity of the branch locus.
(i) Suppose that $K_X + D \sim 0$ and we shall find a contradiction. Since we have the usual adjunction $K_X + D|_D = K_D$ and $\mathcal{O}_X(K_X)$ is S_2, we obtain an exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D) \to \mathcal{O}_D(K_D) \to 0.$$

Since the restriction $H^0(X, K_X + D) \to H^0(D, K_D)$ is surjective, we obtain the exact sequence

$$0 \to H^1(X, K_X) \to H^1(X, \mathcal{O}_X) \xrightarrow{\alpha} H^1(D, \mathcal{O}_D).$$

By the Serre duality and the Leray spectral sequence, we obtain

$$H^1(X, K_X) \simeq H^2(X, \mathcal{O}_X)^* \simeq H^2(S, \mathcal{O}_S)^* \simeq \mathbb{C}$$

and $H^1(X, \mathcal{O}_X) \simeq H^1(S, \mathcal{O}_S)$. Note that $R^i\phi_*\mathcal{O}_X = 0$ by the Kawamata-Viehweg vanishing since $-K_X$ is ϕ-ample. If S is a Du Val K3 surface, then we have $H^1(S, \mathcal{O}_S) = 0$ and this contradicts the above exact sequence. If S is an abelian surface, then we check that α in the exact sequence is injective. Indeed, α can be regarded as $\phi_D^*\colon H^1(S, \mathcal{O}_S) \to H^1(D, \mathcal{O}_D)$ and this is an isomorphism since ϕ_D is étale. This again contradicts the above exact sequence. Thus we see that $K_X + D$ is not trivial.

(ii) Let $m \in \mathbb{Z}_{>1}$ be a minimal integer such that $m(K_X + D) \sim 0$ and let $\Pi\colon X' := \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}_X(i(K_X + D)) \to X$ be the cyclic cover defined by an isomorphism $\mathcal{O}_X(m(K_X + D)) \simeq \mathcal{O}_X$. Then $D' := \Pi^{-1}(D)$ satisfies that $D' \simeq \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}_X(i(K_X + D)|_D) \simeq \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}_X(iK_D)$. By $K_D \sim 0$, we see that D' is a disjoint union of m copies of D. By $K_{X'} + D' \sim 0$ and [23, Proposition 4.37 (3)], we see that $m = 2$, that is, $2(K_X + D) \sim 0$. □

This finishes the proof of Theorem 2.12. □

The case where $D_m \to S$ is étale really occurs as follows. We also have examples where D_m can be any abelian surface, thus gives examples of birationally unbounded D in Theorem 2.12 by Claim 2.17.

Example 2.14 Let S be an Enriques surface and $X := \mathbb{P}_S(\mathcal{O}_S \oplus \omega_S)$. Then the linear system $|-K_X| = |\mathcal{O}_{\mathbb{P}}(2)|$ is free. Indeed, it contains two members $2\sigma_0, 2\sigma_\infty$ with disjoint support, where σ_0, σ_∞ are the sections corresponding to two surjections $\mathcal{O} \oplus \omega_S \twoheadrightarrow \mathcal{O}$, $\mathcal{O} \oplus \omega_S \twoheadrightarrow \omega_S$. Then we see that a general member $D \in |\mathcal{O}_{\mathbb{P}}(2)|$ is irreducible since we have an exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}}) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_{\mathbb{P}}(-2))$$

and obtain $H^0(D, \mathcal{O}_D) \simeq \mathbb{C}$ by

$$H^1(\mathcal{O}_{\mathbb{P}}(-2)) = H^1(\omega_{\mathbb{P}}) \simeq H^2(\mathcal{O}_{\mathbb{P}})^* \simeq H^2(\mathcal{O}_S) = 0.$$

Then, since there is an étale double cover $D \to S$, we see that D is a K3 surface. It is well-known that Enriques surface has a polarization H such that $H^2 = 2$, thus Enriques surfaces form a bounded family.

We can construct a similar example from any abelian surface A and its translation $\tau \in \operatorname{Aut} A$ by a 2-torsion point on A. Note that the quotient morphism $q\colon A \to A/\tau$ is étale and $\bar{A} := A/\tau$ is also an abelian surface. Let $Y := \mathbb{P}_{\bar{A}}(\mathcal{O} \oplus \mathcal{L})$, where $q_*\mathcal{O}_A \simeq \mathcal{O}_{\bar{A}} \oplus \mathcal{L}$. Then $|\mathcal{O}_{\mathbb{P}}(2)|$ is free and contains a smooth member $\Delta \simeq A$ as above. Note that $-K_Y = \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \mathcal{L}$, thus $-K_Y \equiv \mathcal{O}_{\mathbb{P}}(2)$ but $-K_Y \not\sim \mathcal{O}_{\mathbb{P}}(2)$. Note also that A forms a birationally unbounded family by Claim 2.17.

The following gives unbounded examples in the case where $D_m \to S$ is étale in codimension 1 and S is singular.

Example 2.15 Let D be a smooth K3 surface with a Nikulin involution $\iota \in \operatorname{Aut} D$, that is, ι is a symplectic involution so that $S := D/\iota$ is a Du Val K3 surface with 8 A_1-singularities $p_1, \ldots, p_8$. There are infinitely many components of the moduli space which parametrize K3 surfaces with Nikulin involutions as in [34, Proposition 2.3]. Let $\pi : D \to S$ be the quotient morphism and $S' := S \setminus \operatorname{Sing} S$ be the smooth part. Note that $\pi_* \mathcal{O}_D \simeq \mathcal{O}_S \oplus \mathcal{L}$ for some reflexive sheaf $\mathcal{L}$ of rank 1 such that $\mathcal{L}^{[2]} := (\mathcal{L}^{\otimes 2})^{**} \simeq \mathcal{O}_S$.

We can construct a $\mathbb{Q}$-conic bundle

$$\mathbb{P} := \mathbb{P}_S(j_*(\operatorname{Sym}(\mathcal{O}_{S'} \oplus \mathcal{L}|_{S'})) \to S,$$

where $j : S' \to S$ is an open immersion and Sym is the symmetric algebra. We check that $\mathbb{P}$ has at most $1/2(1, 1, 1)$-singularities by local computation. We also check that $|\mathcal{O}_{\mathbb{P}}(2)|$ is a free linear system and contains a smooth irreducible member Δ as in Example 2.14. We see that Δ is a K3 surface which can be isomorphic to the original D. Then the pair $(\mathbb{P}, \Delta)$ is a plt CY pair such that $K_{\mathbb{P}} + \Delta$ is 2-torsion. We expect that the set of D with Nikulin involutions form a birationally unbounded family.

We can do the same construction starting from any abelian surface A and its (-1)-involution $\iota \in \operatorname{Aut} A$. That is, we can construct a $\mathbb{Q}$-conic bundle $X \to T := A/\iota$ with $\Delta \subset X$ so that (X, Δ) is plt, $K_X + \Delta \equiv 0$ and $\Delta \simeq A$ is an abelian surface.

Remark 2.16 Without the assumption that D is irreducible, the statement is false. For example, consider the product $X = S \times \mathbb{P}^1$ of a K3 surface (or an abelian surface) S and $\mathbb{P}^1$. Note that families of K3 surfaces and abelian surfaces are algebraically unbounded although they are analytically bounded.

We can also show that the collection of projective K3 surfaces (or abelian surfaces) is birationally unbounded as follows. (This may be well-known, but we include the explanation for the possible convenience of the reader.)

Claim 2.17 Let $\mathcal{C} := \{S_d \mid d \in \mathbb{Z}_{>0}\}$ be the collection of smooth projective K3 surfaces (or abelian surfaces), where S_d satisfies $\operatorname{Pic} S_d = \mathbb{Z} \cdot H_d$ and H_d is an ample line bundle of degree $H_d^2 = 2d$. Then $\mathcal{C}$ is birationally unbounded.

Proof of Claim. The argument is similar as that of [27, Sect. 3].

Suppose that $\mathcal{C}$ is birationally bounded. Then there exists a projective morphism of algebraic schemes $\phi : \mathcal{S} \to T$ such that, for $d \in \mathbb{Z}_{>0}$, there exist $t_d \in T$ and a birational map $\mu_d : \mathcal{S}_{t_d} \dashrightarrow S_d$ from the fiber $\mathcal{S}_{t_d} := \phi^{-1}(t_d)$. Let $T' := \{t_d \mid d \in \mathbb{Z}_{>0}\} \subset T$ and $Z := \overline{T'} \subset T$ be its closure. Then there exists an irreducible component $Z_i \subset Z$ containing infinitely many t_d's. By considering the base change to Z_i, we may assume that T is irreducible and that $T' \subset T$ is dense and contains infinitely many t_d's.

Let $\eta \in T$ be the generic point and $\mathcal{S}_\eta$ be the generic fiber of ϕ. By taking a resolution of $\mathcal{S}_\eta$ and replacing T by an open subset, we may assume that $\phi\colon \mathcal{S} \to T$ is a smooth family of projective surfaces with birational maps $\mu_d\colon \mathcal{S}_{t_d} \dashrightarrow S_d$ for infinitely many d. By running a $K_{\mathcal{S}}$-MMP over T, we may assume that $K_{\mathcal{S}/T}$ is ϕ-nef, thus μ_d is an isomorphism for d with $t_d \in T$. Let $\mathcal{H}$ be a ϕ-ample line bundle on $\mathcal{S}$ and $M := (\mathcal{H}_t)^2 > 0$ be its degree. We can take $d \gg 0$ such that $2d > M$ and $t_d \in T$. Since $\operatorname{Pic} S_d = \mathbb{Z}H_d \ni \mathcal{H}_{t_d}$ and $H_d^2 = 2d > \mathcal{H}_{t_d}^2 = M > 0$, this is a contradiction. Hence we see that $\mathcal{C}$ is birationally unbounded. $\square$

Remark 2.18 If we only assume that (X, D) is a log canonical pair such that D is irreducible and $K_X + D \sim 0$, then such (X, D) forms an unbounded family. For example, we can consider a polarized K3 surface (S, L) of any degree and its projective cone $X := C_p(S, L)$.

Remark 2.19 For any $d > 0$, there exists an abelian variety A of dimension $n \geq 2$ with a primitive ample divisor L of type $(1, \ldots, 1, d)$ such that $h^0(A, L) = d$ (and $L^n = n!d$). A general abelian variety of type $(1, 1, \ldots, d)$ has the Picard rank 1. Hence abelian varieties of dimension ≥ 2 are algebraically unbounded (cf. [6, 8.11(1)]).

The statement in Theorem 1.1 does not hold when $\dim X \geq 4$. Let $X_d := A_d \times \mathbb{P}^2$, where (A_d, L_d) is a general abelian variety with a primitive polarization L_d of type $(1, \ldots, 1, d)$ as above. Then there exists a smooth member $D_d := A_d \times C \in |-K_{X_d}|$ so that (X_d, D_d) is a plt CY pair and D_d is irreducible and reduced, where $C \subset \mathbb{P}^2$ is an elliptic curve. Such D_d forms an unbounded family since there is no non-constant map $C \to A_d$ and $\operatorname{Pic}(A_d \times C) \simeq \operatorname{Pic} A_d \times \operatorname{Pic} C$.

We can also show such D_d forms a birationally unbounded family by a similar argument as Claim 2.17 using the relative MMP guaranteed by [16, Theorem 1.2] as follows. Suppose that $\{D_d \mid d \in \mathbb{Z}_{>0}\}$ is birationally bounded. Then, as in Claim 2.17, we can construct a smooth family $\phi\colon \mathcal{A} \to T$ over a smooth variety T with infinitely many points $t_d \in T$ and a birational map $\mu_d\colon \mathcal{A}_{t_d} \dashrightarrow D_d$. By [16, Theorem 1.2], we obtain a birational map $\mathcal{A} \dashrightarrow \mathcal{A}'$ to a good minimal model $\mathcal{A}'$ of $\mathcal{A}$ over T with a morphism $\phi'\colon \mathcal{A}' \to T$. Since an abelian variety contains no rational curve and there is no flop on it, we see that $\mathcal{A}'_{t_d} \simeq D_d$ for $t_d \in T$. Let $\mathcal{H}'$ be a ϕ'-ample line bundle on $\mathcal{A}'$. By considering $\mathcal{H}'|_{\mathcal{A}'_{t_d}}$ and its pull-back to A_d for sufficiently large d, we obtain a contradiction as before. Hence we obtain the required birational unboundedness.

3 Birational Bounded Family of Du Val K3 Surfaces which are Unbounded

We consider the following problem in this section.

Problem 3.1 *Let S be a smooth K3 surface with an ample line bundle L with $L^2 = 2d$ for a fixed $d > 0$. Let $S \to S'$ be a birational morphism onto a normal surface S'*

(which is a Du Val K3 surface). Does there exist an ample line bundle L' on S' with $L'^2 \leq N_d$ for some N_d determined by d?

The following example in the e-mail from Keiji Oguiso is a counterexample to the problem and shows that a birational bounded family of Du Val K3 surfaces can be unbounded.

Example 3.2 Let d, m be any positive integers. Let S be a polarized K3 surface of degree $2d$ of Picard number 2 such that $\mathrm{Pic}\,(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ with intersection form

$$(H^2) = 2d \ , \ (H.C) = m \ , \ (C^2) = -2$$

which is constructed in [30, Theorem 3] (The $d = 2$ case is treated in [29]). We know that H is very ample when $d \geq 2$ and C is a (-2)-curve by [30, Lemma 1.2].

We have a contraction $\pi : S \to T$ of C to the rational double point of type A_1. Let L be an ample Cartier divisor on T. (Note that the local class group of A_1 is $\mathbb{Z}/2$ so that $2L'$ is Cartier for any Weil divisor L' on T). Then

$$\pi^* L = aH + bC,$$

where a and b are integers and moreover $a > 0$, as $\pi^* L$ is a nef and big Cartier divisor. Since π is the contraction of C, it follows that $(\pi^* L.C) = 0$. Hence, by $\pi^* L = aH + bC$, we have

$$a(H.C) + b(C^2) = 0.$$

Substituting $(H.C) = m$ and $(C^2) = -2$ into the equation above, it follows that

$$b = \frac{am}{2}.$$

Also, from $\pi^* L - bC = aH$ with $(\pi^* L.C) = 0$, $(C^2) = -2$ and $(\pi^* L)^2 = (L^2)$ (as π is a birational morphism), we have

$$(L^2) - 2b^2 = (\pi^* L - bC)^2 = a^2(H^2) = 2da^2.$$

Hence, for any ample Cartier divisor on T, we have

$$(L^2) = 2da^2 + 2b^2 = a^2(2d + \frac{m^2}{2}) \geq 2d + \frac{m^2}{2}.$$

Since m can be taken any positive integer, it follows that the degree of the polarizations on the birational contractions of polarized K3 surfaces of degree $2d$ is unbounded. Hence contractions of polarized K3 surfaces of a fixed degree do not necessarily form a bounded projective family.

Remark 3.3 In this remark, we ask whether the surface S and T in Example 3.2 can be embedded in a rationally connected 3-fold when $d = 2$.

We have an embedding $S \subset \mathbb{P}^3$ as a quartic surface. When $m = 2l$ is even, we can construct a 3-fold $\bar{X}$ which contains T as an anticanonical hypersurface so that $(\bar{X}, T)$ is a plt CY pair as follows.

Assume that $m = 2l$ is even. By the above consideration, the effective cone $\overline{\mathrm{NE}}(S) \subset \mathrm{Pic}(S) \otimes \mathbb{R}$ of S is generated by (-2)-curves by [25, Theorem 2]. Hence we can write $\overline{\mathrm{NE}}(S) = \mathbb{R}_{\geq 0}[C] + \mathbb{R}_{\geq 0}[\Gamma]$ for some (-2)-curve Γ. Note that $lH - C$ is effective since $(lH - C)^2 = -2$ and $(lH - C) \cdot H = 2l > 0$. Note that such classes can be reducible in general.

We show that $\Gamma \sim lH - C$ as follows. Note that we can write $\Gamma = aH - bC$ for some $a, b \in \mathbb{Z}_{>0}$. Since we have

$$-2 = (aH - bC)^2 = 4a^2 - 2mba - 2b^2,$$

we obtain $a(2a - mb) = 2a^2 - mba = b^2 - 1 = (b-1)(b+1)$. If $b > 1$, then we have $2a - mb = 2(a - lb) > 0$ and

$$aH - bC = (a - lb)H + b(lH - C) \in \mathbb{R}_{>0}C + \mathbb{R}_{>0}(lH - C) \subset \overline{\mathrm{NE}}(S).$$

Hence $aH - bC$ is not on the boundary of $\overline{\mathrm{NE}}(S)$. Thus we see that $b = 1$ and $a = l$, that is, $\Gamma \sim lH - C$.

Now let $\mu \colon X \to \mathbb{P}^3$ be the blow-up along Γ. Let $E_\Gamma := \mu^{-1}(\Gamma)$ be the exceptional divisor and $\tilde{S} \subset X$ be the strict transform of S. Let

$$L := \mu^* \mathcal{O}_{\mathbb{P}^3}(l^2 + 1) - lE_\Gamma.$$

We see that the restriction $L|_{\tilde{S}} = (l^2 + 1)H - l(lH - C) = H + lC$ is the line bundle which induces the birational contraction $\pi \colon S \to T$ in Example 3.2.

Now assume that $l > 4$. We see that L is base point free and induces a birational morphism as follows. Note that

$$L = \mu^* \mathcal{O}(1) + l(\mu^* \mathcal{O}(l) - E_\Gamma) \tag{2}$$

and the linear system $|\mu^* \mathcal{O}(l) - E_\Gamma|$ contains $\tilde{S} + S_{l-4}$ for all $S_{l-4} \in |\mu^* \mathcal{O}(l-4)|$. Hence the base locus $\mathrm{Bs}\, L$ of $|L|$ is contained in $\tilde{S}$. Since $L|_{\tilde{S}}$ is base point free, we see that L is nef. By (2), we see that L is big. Finally, we have an exact sequence

$$H^0(X, L) \to H^0(\tilde{S}, L|_{\tilde{S}}) \to H^1(X, L - S) = 0$$

by the Kawamata-Viehweg vanishing since $\tilde{S} \in |-K_X|$, $L - \tilde{S} = K_X + L$ and L is nef and big. This implies that $|L|$ is base point free and induces a birational contraction $\Phi_L \colon X \to \bar{X}$. We see that $\Phi_L(\tilde{S}) \simeq T$.

We see that $(\bar{X}, T)$ is a plt CY pair although $\bar{X}$ is not $\mathbb{Q}$-Gorenstein.

Remark 3.4 Let $S \subset \mathbb{P}^3$ be as in Remark 3.3 for an odd $m \geq 3$. As in Remark 3.3, we see that $\overline{\mathrm{NE}}(S)$ is generated by C and another -2-curve Γ. However, it seems difficult to describe Γ explicitly. In order to find such a class, we need to find an integer solution (a, b) of the quadratic equation

$$4a^2 - 2mab - 2b^2 = -2$$

with $a, b > 0$. By a computer program in [28], we find solutions for an explicit m. For $m = 15$, the solutions are $(a, b) = (2G, F - 15G)$, where

$$F + G\sqrt{233} = (2144801346/2 + 140510608/2\sqrt{233})^n \quad \text{for } n \geq 0.$$

4 Some Results in Higher Dimensional Case

We consider the following problem in this section.

Problem 4.1 *Let $n > 0$ and X be a normal projective rationally connected n-fold with an irreducible $D \in |-K_X|$ such that (X, D) is a plt pair (and D is a strict CY variety with only canonical singularities). Does such D form a birationally bounded family?*

Remark 4.2 If $\dim X = 4$, then D is a CY 3-fold. By taking a small $\mathbb{Q}$-factorial modification and running K_X-MMP as before, we may assume that there is a Mori fiber space $\phi \colon X \to S$ which induces a surjective morphism $\phi_D := \phi|_D \colon D \to S$. The problem is to bound this D.

If $\dim S = 0$, then X is a $\mathbb{Q}$-Fano 4-fold with canonical singularities and it is bounded (cf. [5]), thus D is also bounded. If $\dim S = 2$, then $\phi_D \colon D \to S$ is an elliptic fibration. Indeed, we check this as in the proof of Proposition 2.6 since its general fiber is an anticanonical member of the general fiber of ϕ which is a log del Pezzo surface. Hence D is birationally bounded by Gross' theorem [14]. If $\dim S = 3$, then $D \to S$ is a generically 2:1-cover and $D \to S$ is branched along a divisor $R \subset S$ since S is rationally connected. Thus $(S, \frac{1}{2}R)$ is a klt CY pair and $R \in |-2K_S|$. S is birationally bounded by [7, Theorem 1.6] and D is also birationally bounded.

Hence the problem is reduced to the case $\dim S = 1$. However, we don't know how to show the boundedness in these cases.

Chen Jiang also pointed out the following.

Proposition 4.3 *Let $n, m > 0$. Let (X, D) be a n-dimensional reduced plt CY pair such that X is of Fano type. Assume that there exists a $\mathbb{Q}$-divisor $B \neq D$ such that mB is integral, $K_X + B \equiv 0$ and (X, B) is lc.*

Then the pair (X, D) is log bounded.

Proof We see that (X, D) is ϵ-plt for some $\epsilon > 0$ by Proposition 2.5(i). Then we see that $(X, \frac{1}{2}(B + D))$ is ϵ'-lc for $\epsilon' := \min\{1/2m, \epsilon/2\}$. By [17, Theorem 1.3], we see that (X, D) forms a log bounded family. □

Remark 4.4 The following is pointed out by Yoshinori Gongyo and Roberto Svaldi after the submission to arXiv.

Proposition 4.5 *Let (X, D) be a reduced plt CY pair such that X is of Fano type.*

(i) Then (X, D) is log bitarionally bounded.
(ii) Assume that X is $\mathbb{Q}$-factorial. Then (X, D) is log bounded.

Proof (i) By taking a small $\mathbb{Q}$-factorial modification, we may assume that X is $\mathbb{Q}$-factorial. Let $\mu \colon X \dashrightarrow X'$ be a birational map induced by a $(-K_X)$-MMP which exists since X is a Mori dream space by [3]. Then we see that $-K_{X'}$ is nef and big. Let $D' := \mu_* D$. Note that μ does not contract D since D is big. Then the pair (X', D') is also a plt CY and ϵ-plt for some $\epsilon > 0$ by Proposition 2.5. By these, we see that X' is an ϵ-lc weak Fano variety, thus it is bounded by [5]. Hence $D' \equiv -K_{X'}$ is also bounded.

(ii) We also use the notation in (i). Then (X', D') is ϵ-plt and $-K_{X'}$ is nef and big. Thus we can take a positive integer m determined by dim X such that $-mK_{X'}$ is base point free. Then, by taking a general member of $A' \in |-mK_{X'}|$ and putting $B' := \frac{1}{m}A'$, we obtain a $\frac{1}{m}$-lc CY pair (X', B'). Moreover, $K_{X'} + B'$ is an m-complement (cf. [4, 2.18]). Then we obtain an m-complement $K_X + B$ as in [4, 6.1(3)], where B is the sum of the strict transform of B' and some effective divisor supported on the μ-exceptional divisors. Hence, by Proposition 4.3, we see that (X, D) is log bounded. □

Johnson–Kollár [20] proved that there are only finitely many quasismooth weighted CY hypersurfaces of fixed dimension. Chen [8] proved that there are only finitely many families of CY weighted complete intersections. CY varieties in toric varieties are often considered in mirror symmetry and so on. Although toric varieties are unbounded, we can show the following.

Corollary 4.6 *Let X be a normal projective toric variety with $D \in |-K_X|$ with only canonical singularities. Then (X, D) form a log bounded family. (Thus both X and D are bounded.)*

Proof Note that, since X is toric and $\mathbb{Q}$-Gorenstein in codimension 2, we see that X has only canonical singularities in codimension 2 by [33, Theorem 5]. Thus D is Cartier in codimension 2 and (X, D) is plt by inversion of adjunction (cf. [22, Theorem 5.50]).

Let $\Delta \subset X$ be the union of toric divisors. Then we see that (X, Δ) is lc. By applying Proposition 4.3, we obtain the claim. □

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References

1. Altınok, S., Brown, G., Reid, M.: Fano 3-folds, $K3$ surfaces and graded rings, topology and geometry: commemorating SISTAG. In: Contemporary Mathematics, vol. 314, pp. 25–53. American Mathematical Society, Providence, RI (2002)
2. Alexeev, V.: Boundedness and K^2 for log surfaces. Internat. J. Math. **5**(6), 779–810 (1994)
3. Birkar, C., Cascini, P., Hacon, C.D., McKernan, J.: Existence of minimal models for varieties of log general type. J. Am. Math. Soc. **23**(2), 405–468 (2010)
4. Birkar, C.: Anti-pluricanonical systems on Fano varieties. Ann. Math. **190**(2), 345–463 (2019)
5. Birkar, C.: Singularities of linear systems and boundedness of Fano varieties. Ann. Math. **193**(2), 347–405 (2021)
6. Birkenhake, C., Lange, H.: Complex Abelian Varieties, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302. Springer, Berlin (2004)
7. Chen, W., Di Cerbo, G., Han, J., Jiang, C., Svaldi, R.: Birational boundedness of rationally connected Calabi-Yau 3-folds. Adv. Math. **378**, 107541 (2021)
8. Chen, J.-J.: Finiteness of Calabi-Yau quasismooth weighted complete intersections. Int. Math. Res. Not. IMRN (12), 3793–3809 (2015)
9. Corti, A., Kaloghiros, A.-S.: The Sarkisov program for Mori fibred Calabi-Yau pairs. Algebr. Geom. **3**(3), 370–384 (2016)
10. Corti, A.: Singularities of linear systems and 3-fold birational geometry, explicit birational geometry of 3-folds. In: London Mathematical Society. Lecture Note Series, vol. 281, pp. 259–312. Cambridge University Press, Cambridge (2000)
11. Di Cerbo, G., Svaldi, R.: Birational boundedness of low dimensional elliptic Calabi-Yau varieties with a section. https://arxiv.org/abs/1608.02997
12. Greb, D., Kebekus, S., Kovács, S.J., Peternell, T.: Differential forms on log canonical spaces. Publ. Math. Inst. Hautes Études Sci. (114), 87–169 (2011)
13. Gongyo, Y.: Abundance theorem for numerically trivial log canonical divisors of semi-log canonical pairs. J. Algebr. Geom. **22**(3), 549–564 (2013)
14. Gross, M.: A finiteness theorem for elliptic Calabi-Yau threefolds. Duke Math. J. **74**(2), 271–299 (1994)
15. Hacon, C.D., McKernan, J., Xu, C.: ACC for log canonical thresholds. Ann. Math. **180**(2), 523–571 (2014)
16. Hacon, C.D., McKernan, J., Xu, C.: Boundedness of moduli of varieties of general type. J. Eur. Math. Soc. (JEMS) **20**(4), 865–901 (2018)
17. Hacon, C.D., Xu, C.: Boundedness of log Calabi-Yau pairs of Fano type. Math. Res. Lett. **22**(6), 1699–1716 (2015)
18. Iano-Fletcher, A.R.: Working with weighted complete intersections, explicit birational geometry of 3-folds. In: London Mathematical Society. Lecture Notes Series, vol. 281, pp. 101–173. Cambridge University Press, Cambridge (2000)
19. Jiang, C.: Bounding the volumes of singular weak log del Pezzo surfaces. Internat. J. Math. **24**(13), 1350110, 27 (2013)
20. Johnson, J.M.: Fano hypersurfaces in weighted projective 4-spaces. Experiment. Math. **10**(1), 151–158 (2001)

21. Kawamata, Y.: On the abundance theorem in the case of numerical Kodaira dimension zero. Am. J. Math. **135**(1), 115–124 (2013)
22. Kollár, J., Mori, S.: Birational geometry of algebraic varieties. Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998). With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original
23. János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, With a collaboration of Sándor Kovács
24. Kollár, J.: Conic bundles that are not birational to numerical Calabi-Yau pairs, Épijournal Géom. Algébrique **1**, Art. 1, 14 (2017)
25. Sándor, J.S.: The cone of curves of a $K3$ surface. Math. Ann. **300**(4), 681–691 (1994)
26. Kollár, J., Chenyang, X.: The dual complex of Calabi-Yau pairs. Invent. Math. **205**(3), 527–557 (2016)
27. Lin, J.: Birational unboundedness of $\mathbb{Q}$-Fano threefolds. Int. Math. Res. Not. (6), 301–312 (2003)
28. Matthews, K.: (2015). http://www.numbertheory.org/php/binarygen.html
29. Mori, S.: On degrees and genera of curves on smooth quartic surfaces in P^3. Nagoya Math. J. **96**, 127–132 (1984)
30. Oguiso, K.: Two remarks on Calabi-Yau Moishezon threefolds. J. Reine Angew. Math. **452**, 153–161 (1994)
31. Reid, M.: Canonical 3-folds, Journées de Géometrie Algébrique d'Angers, Juillet 1979, Algebraic Geometry, Angers,: Sijthoff & Noordhoff. Alphen aan den Rijn-Germantown, Md. **1980**, 273–310 (1979)
32. Ravindra, G.V., Srinivas, V.: The Grothendieck-Lefschetz theorem for normal projective varieties. J. Algebr. Geom. **15**(3), 563–590 (2006)
33. Stevens, J.: On canonical singularities as total spaces of deformations. Abh. Math. Sem. Univ. Hamburg **58**, 275–283 (1988)
34. van Geemen, B., Sarti, A.: Nikulin involutions on $K3$ surfaces. Math. Z. **255**(4), 731–753 (2007)

Abelian Varieties, Quaternion Trick and Endomorphisms

Yuri G. Zarhin

Abstract The quaternion trick is an explicit construction that associates to a polarized abelian variety X a principal polarization of $(X \times X^t)^4$. The aim of this note is to show that this construction is compatible with endomorphisms of X and X^t. See Theorem 1.1 for a precise statement.

Keywords Abelian varieties · Quaternion trick · Endomorphisms

1 Introduction

Throughout this paper, K is a field. If X is an abelian variety over K then we write $\mathrm{End}(X)$ for the ring of all K-endomorphisms of X. If m is an integer then we write m_X for the multiplication by m in X; in particular, 1_X is the identity map. (Sometimes we will use notation m instead of m_X.)

If Y is an abelian variety over K then we write $\mathrm{Hom}(X, Y)$ for the group of all K-homomorphisms $X \to Y$.

Let X be an abelian variety over a field K and let X^t be its dual. If u is an endomorphism of X then we write u^t for the dual endomorphism of X^t. The corresponding map of the endomorphism rings

$$\mathrm{End}(X) \to \mathrm{End}(X^t),\ u \mapsto u^t$$

is an *antiisomorphism* of rings. If m is a positive integer then there are the natural "diagonal" ring embeddings

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$$\Delta_{m,X} : \mathrm{End}(X) \hookrightarrow \mathrm{Mat}_m(\mathrm{End}(X)) = \mathrm{End}(X^m),$$

$$\Delta_{m,X^t} : \mathrm{End}(X^t) \hookrightarrow \mathrm{Mat}_m(\mathrm{End}(X^t)) = \mathrm{End}((X^t)^m) = \mathrm{End}((X^m)^t),$$

which send 1_X (resp. 1_{X^t}) to the identity automorphism of X^m (resp. of $(X^t)^m$). Namely, if u is an endomorphism of X (resp. of X^t) then $\Delta_m(u)$ sends $(x_1, \dots x_m)$ to $(ux_1, \dots, ux_m)$ for all $(x_1, \dots, x_m)$ in X^m (resp. in $(X^t)^m$). Clearly, the subring $\Delta_{m,X}(\mathrm{End}(X))$ of $\mathrm{End}(X^m)$ commutes with the subring

$$\mathrm{Mat}_m(\mathbb{Z}) \subset \mathrm{Mat}_m(\mathrm{End}(X)) = \mathrm{End}(X^m).$$

Let $\lambda : X \to X^t$ be a *polarization* on X that is defined over K. For every positive integer m we consider the polarization

$$\lambda^m : X^m \to (X^m)^t = (X^t)^m, \ (x_1, \dots, x_m) \mapsto (\lambda(x_1), \dots, \lambda(x_m)) \tag{1}$$

of X^m that is also defined over K. We have

$$\dim(X^m) = m \cdot \dim(X), \ \deg(\lambda^m) = \deg(\lambda)^m, \ \ker(\lambda^m) = \ker(\lambda)^m \subset X^m. \tag{2}$$

The aim of this note is to prove the following assertion.

Theorem 1.1 *Let $\lambda : X \to X^t$ be a polarization on X that is defined over K. Let O be an associative ring with* 1 *endowed with an involutive antiautomorphism*

$$O \to O, \ e \mapsto e^*.$$

Suppose that we are given a ring embedding $\iota : O \to \mathrm{End}(X)$ that sends 1 *to* 1_X *and such that in* $\mathrm{Hom}(X, X^t)$

$$\lambda \circ \iota(e) = \iota(e^*)^t \circ \lambda, \ \forall e \in O. \tag{3}$$

Then:

(i) The map

$$\iota^* : O \to \mathrm{End}(X^t), e \mapsto \left(\iota(e^*)\right)^t \tag{4}$$

is a ring embedding that sends 1 *to* 1_{X^t}.

(ii) Let us consider the ring embeding

$$\kappa_4 = \Delta_{4,X^t} \circ \iota^* \oplus \Delta_{4,X} \circ \iota : O \hookrightarrow \mathrm{End}((X^t)^4) \oplus \mathrm{End}(X^4) \subset \mathrm{End}\left((X^t)^4 \times X^4\right) = \mathrm{End}((X \times X^t)^4), \tag{5}$$

$$e \mapsto \left(\Delta_{4,X^t}(\iota^*(e)), \Delta_{4,X}(\iota(e)\right) \in \mathrm{End}((X^t)^4) \oplus \mathrm{End}(X^4) \subset \mathrm{End}\left((X^t)^4 \times X^4\right).$$

Then there exists a principal polarization μ on $X^4 \times (X^t)^4 = (X \times X^t)^4$ that is defined over K and enjoys the following properties.

$$\mu \circ \kappa_4(u) = \kappa_4(u^*)^t \circ \mu \ \forall u \in O. \tag{6}$$

Remark 1.2 Clearly, $(1_X)^t = 1_{X^t}$. Since both maps

$$\mathrm{End}(X) \to \mathrm{End}(X^t), \ u \mapsto u^t \ \text{ and } \ O \to O, \ e \mapsto e^*$$

are ring antiisomorphisms, and $\iota : O \to \mathrm{End}(X)$ is a ring embedding that sends 1 to 1_X, the composition

$$O \to \mathrm{End}(X^t), \ e \mapsto \big(\iota(e^*)\big)^t$$

is obviously a ring embedding that sends 1 to 1_{X^t}, which proves Theorem 1.1(i). The rest of the paper is devoted to the proof of Theorem 1.1(ii).

Remarks 1.3 We keep the notation and assumptions of Theorem 1.1.

- Formula (3) implies that for every positive integer m

$$\lambda^m \circ \Delta_{m,X}(\iota(e)) = (\Delta_{m,X^t}(\iota(e^*))^t \circ \lambda^m \ \forall e \in O. \tag{7}$$

- It is well known [4] that the additive group of $\mathrm{End}(X)$ is a free $\mathbb{Z}$-module of finite rank. Since ι is an embedding, the additive group of O is also a free $\mathbb{Z}$-module of finite rank.

Remark 1.4 When $O = \mathbb{Z}$, the assertion of Theorem 1.1 is a well known *quaternion trick* [9, Lemma 2.5], [10, Sect. 5], [11, Sect. 1.13 and 7] (See [3, Chap. IX, Sect. 1] where Deligne's proof is given).

This note may be viewed as a natural continuation of [11]. In particular, we freely use a (more or less) standard notation from [11].

The paper is organized as follows. Section 2 contains auxiliary results that deal with interrelations between polarizations, isogenies and endomorphisms of abelian varieties. In Sect. 3 we compare the situation over K and over its fixed algebraic closure $\bar{K}$. We prove Theorem 1.1 (ii) in Sect. 4.

2 Polarizations and Isogenies

Proposition 2.1 *Let (Y, λ) be a polarized abelian variety over K. Let $\pi : Y \to Z$ be a K-isogeny of abelian varieties over K. Suppose that there exists a polarization $\mu : Z \to Z^t$ that is defined over K and such that*

$$\lambda = \pi^t \circ \mu \circ \pi. \tag{8}$$

Let D be an associative ring with 1 *endowed with an involutive antiautomorphism*

$$D \to D, \ e \mapsto e^*.$$

Suppose that we are given an injective ring embedding $j : D \to \mathrm{End}(Y)$ that sends 1 *to* 1_Y *and such that in* $\mathrm{Hom}(Y, Y^t)$

$$\lambda \circ j(e) = j(e^*)^t \circ \lambda \ \forall e \in D. \tag{9}$$

Suppose that there exists a ring embedding $j_\pi : D \hookrightarrow \mathrm{End}(Z)$ that sends 1 *to* 1_Z *and such that*

$$j_\pi(e) \circ \pi = \pi \circ j(e) \ \forall e \in D. \tag{10}$$

Then

$$\mu \circ j_\pi(e) = j_\pi(e^*)^t \circ \mu \ \forall e \in D. \tag{11}$$

Proof Let $u \in D$. Plugging in formula (8) for λ into (9), we obtain

$$\pi^t \circ \mu \circ \pi \circ j(e) = j(e^*)^t \circ \pi^t \circ \mu \circ \pi.$$

Formula (10) allows us to replace $\pi \circ j(e)$ by $j_\pi(e) \circ \pi$ and get

$$\pi^t \circ \mu \circ j_\pi(e) \circ \pi = j(e^*)^t \circ \pi^t \circ \mu \circ \pi.$$

Dividing both sides by isogeny π from the right, we get

$$\pi^t \circ \mu \circ j_\pi(e) = j(e^*)^t \circ \pi^t \circ \mu.$$

Taking into account that $j(e^*)^t \circ \pi^t = (\pi \circ j(e^*))^t$, we get

$$\pi^t \circ \mu \circ j_\pi(e) = (\pi \circ j(e^*))^t \circ \mu. \tag{12}$$

Applying (10) to e^* instead of e, we get $j_\pi(e^*) \circ \pi = \pi \circ j(e^*)$. Combining this with (12), we obtain

$$\pi^t \circ \mu \circ j_\pi(e) = (j_\pi(e^*) \circ \pi)^t \circ \mu = \pi^t \circ j_\pi(e^*)^t \circ \mu$$

and therefore

$$\pi^t \circ \mu \circ j_\pi(e) = \pi^t \circ j_\pi(e^*)^t \circ \mu.$$

Dividing both sides by isogeny π^t from the left, we get

$$\mu \circ j_\pi(e) = j_\pi(e^*)^t \circ \mu,$$

which proves the desired formula (11). $\square$

3 Base Change

In what follows, $\bar{K}$ stands for a fixed algebraic closure of K. If X (resp. W) is an algebraic variety (resp. group scheme) over K then we write $\bar{X}$ (resp. $\bar{W}$) for the corresponding algebraic variety (resp. group scheme) over $\bar{K}$. Similarly, if f is a morphism of K-varieties (resp. group schemes) then we write $\bar{f}$ for the corresponding morphism of algebraic varieties (resp. group schemes) over $\bar{K}$. In particular, if X is an abelian variety with K-polarization $\lambda : X \to X^t$ then

$$\bar{\lambda} : \bar{X} \to \overline{X^t} = \bar{X}^t \tag{13}$$

is a polarization of $\bar{X}$, and

$$\deg(\bar{\lambda}) = \deg(\lambda), \ \ker(\bar{\lambda}) = \overline{\ker(\lambda)}, \tag{14}$$

$$\overline{X^m} = \bar{X}^m, \ \overline{X^m}^t = (\overline{X^t})^m, \ \overline{\lambda^m} = \bar{\lambda}^m$$

for all positive integers m.

If W is a finite commutative group scheme then $\bar{W}$ is a finite commutative group scheme over $\bar{K}$ and the *orders* of W and $\bar{W}$ coincide. We have

$$\bar{W}^m = \overline{W^m} \ \ \forall m. \tag{15}$$

In addition, if d is the order of W then the orders of $\bar{W}^m$ and W^m both equal d^m. (See [1, 2, 5, 7, 8, 11] for a furher discussion of commutative finite group schemes over fields.)

4 Quaternion Trick

In what follows, we freely use the notation and assertions of Sect. 3.

Proof of Theorem 1.1 Let us put $g := \dim(X)$. We may assume that $g \geq 1$. Recall that λ is an isogeny and therefore $\ker(\lambda)$ is a finite group subscheme in X. Let $n := \deg(\lambda)$. Then $\ker(\lambda)$ has order n and therefore is killed by multiplication by n, see [6].

Choose a quadruple of integers a, b, c, d such that

$$s := a^2 + b^2 + c^2 + d^2$$

is congruent to -1 modulo n. (In particular, $s \neq 0$.) We denote by $\mathcal{I}$ the "quaternion"

$$\mathcal{I} = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{Z}) \subset \mathrm{Mat}_4(\mathrm{End}(X)) = \mathrm{End}(X^4).$$

Following [11, pp. 330–331], let us consider a finite group subscheme

$$V \subset \ker(\lambda^4) \times \ker(\lambda^4) \subset X^4 \times X^4 = X^8$$

that is the graph of

$$\mathcal{I} : \ker(\lambda^4) \to \ker(\lambda^4).$$

In particular, V and $\ker(\lambda^4)$ are isomorphic finite group schemes over K and therefore have the same order, namely, n^4. Clearly,

$$\bar{V} \subset \ker(\bar{\lambda}^4) \times \ker(\bar{\lambda}^4) = \ker(\bar{\lambda}^8),$$

the orders of isomorphic finite group $\bar{K}$-schemes $\bar{V}$ and $\ker(\bar{\lambda}^4)$ coincide and also equal n^4. It is checked in [11, pp. 330–331] that $\bar{V}$ is an *isotropic* finite group subscheme in $\ker(\bar{\lambda}^8)$ with respect to the *Riemann form* [4, Sect. 23]

$$e_{\bar{\lambda}^8} : \ker(\bar{\lambda}^8) \times \ker(\bar{\lambda}^8) \to \mathbb{G}_{\mathrm{m}}$$

attached to the polarization

$$\bar{\lambda}^8 : \bar{X}^8 \to \left(\bar{X}^8\right)^t$$

of $\bar{X}^8 = \overline{X^8}$. (Here $\mathbb{G}_{\mathrm{m}}$ is the *multiplicative* group scheme over $\bar{K}$.) Since the order of $\bar{V}$ is $n^4 = \sqrt{n^8}$, it is the *square root* of the order of $\ker(\bar{\lambda}^8)$. This means that $\bar{V}$ is a *maximal isotropic* finite group subscheme of $\ker(\bar{\lambda}^8)$.

Let us consider a K-morphism of $8g$-dimensional abelian varieties

$$\pi : X^8 = X^4 \times X^4 \to (X^t)^4 \times X^4,\ (x_4, y_4) \mapsto (\lambda^4(x_4), \mathcal{I}(x_4) - y_4)\ \forall x_4, y_4 \in X^4. \tag{16}$$

Clearly, $\ker(\pi) = V$, which is a finite group scheme, hence π is an *isogeny* and

$$X^4 \times (X^t)^4 \cong X^8/V.$$

In light of *descent theory* [11, Sect. 1.13] (applied to X^8, λ^8, $(X^t)^4 \times X^4$ instead of X, λ, Y), the *maximal isotropy* of $\bar{V}$ implies that there exists a *principal polarization*

$$\mu : (X^t)^4 \times X^4 \to \left((X^t)^4 \times X^4\right)^t$$

on $(X^t)^4 \times X^4$ that is defined over K and such that

$$\pi^t \circ \mu \circ \pi = \lambda^8 : X^8 \to (X^8)^t = (X^t)^8. \tag{17}$$

Let us consider the ring embeddings

$$j = \Delta_{8,X} \circ \iota : \mathcal{O} \to \mathrm{End}(X^8) = \mathrm{Mat}_2(\mathrm{End}(X^4)),\ e \mapsto \begin{pmatrix} \Delta_{4,X}(\iota(e)) & 0 \\ 0 & \Delta_{4,X}(\iota(e)) \end{pmatrix}$$

and

$$j_\pi = \kappa_4 : \mathcal{O} \to \mathrm{Mat}_4(\mathrm{End}(X^t) \oplus \mathrm{Mat}_4(\mathrm{End}(X)) = \mathrm{End}\left((X^t)^4\right) \oplus \mathrm{End}(X^4) \subset \mathrm{End}\left((X^t)^4 \times X^4\right),$$

$$e \mapsto \left(\Delta_{4,X^t}(\iota^*(e)), \Delta_4(\iota(e))\right) = \begin{pmatrix} \Delta_{4,X^t}(\iota^*(e)) & 0 \\ 0 & \Delta_{4,X}(\iota(e)) \end{pmatrix}.$$

Let us put $Y = X^8$, $Z = (X^t)^4 \times X^4$, $D = \mathcal{O}$, $m = 4$ and check that j and j_π enjoy property (10). First, notice that the matrix

$$\mathcal{I} \in \mathrm{Mat}_4(\mathbb{Z}) \subset \mathrm{Mat}_4(\mathrm{End}(X)) = \mathrm{End}(X^4)$$

commutes with the "scalar" matrix

$$\Delta_{4,X}(\iota(e)) = \begin{pmatrix} \iota(e) & 0 & 0 & 0 \\ 0 & \iota(e) & 0 & 0 \\ 0 & 0 & \iota(e) & 0 \\ 0 & 0 & 0 & \iota(e) \end{pmatrix} \in \mathrm{Mat}_4\left(\mathrm{End}(X)\right) = \mathrm{End}\left(X^4\right),$$

i.e.,

$$\mathcal{I} \circ \Delta_{4,X}(\iota(e)) = \Delta_{4,X}(\iota(e)) \circ \mathcal{I}. \tag{18}$$

Second, plugging $m = 4$ in (7), we get

$$\lambda^4 \circ \Delta_{4,X}(\iota(e)) = \Delta_{4,X^t}(\iota^*(e)) \circ \lambda^4\ \forall e \in \mathcal{O}. \tag{19}$$

Third, if

$$e \in \mathcal{O},\ (x_4, y_4) \in X^4(\bar{K}) \times X^4(\bar{K}) = X^8(\bar{K})$$

then

$$\pi \circ j(e)(x_4, y_4) = \pi\left(\Delta_{4,X}(\iota(e))(x_4), \Delta_{4,X}(\iota(e))(y_4)\right) =$$

$$\left(\lambda^4 \circ \Delta_{4,X}(\iota(e))(x_4), \mathcal{I} \circ \Delta_{4,X}(\iota(e))(x_4) - \Delta_{4,X}(\iota(e))(y_4)\right).$$

Taking into account equalities (19) and (18), we obtain that

$$\pi \circ j(e)(x_4, y_4) = \left(\Delta_{4,X'}(\iota^*(e)) \circ \lambda^4(x_4), \Delta_{4,X}(\iota(e))\mathcal{I}(x_4) - \Delta_{4,X}(\iota(e))(y_4)\right)$$

$$= \begin{pmatrix} \Delta_{4,X'}(\iota^*(e)) & 0 \\ 0 & \Delta_{4,X}(\iota(e)) \end{pmatrix} \left(\lambda^4(x_4), \mathcal{I}(x_4) - y_4\right) = \begin{pmatrix} \Delta_{4,X'}(\iota^*(e)) & 0 \\ 0 & \Delta_{4,X}(\iota(e)) \end{pmatrix} \circ \pi\,(x_4, y_4)$$

$$= j_\pi(e) \circ \pi\,(x_4, y_4)\,.$$

This means that

$$j_\pi(e) \circ \pi = \pi \circ j(e) \;\forall e \in O.$$

Now the desired result follows from (17) combined with Proposition 2.1 applied to λ^8 (instead of λ). □

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References

1. Demazure, M., Gabriel, P.: Groupes algébriques. Tome I. North Holland, Amsterdam (1970)
2. Hoobler, R., Magid, A.: Finite group schemes over fields. Proc. Am. Math. Soc. **33**, 310–312 (1972)
3. Moret-Bailly, L.: Pinceaux de variétés abéliennes, Astérisque, vol. 129 (1985)
4. Mumford, D.: Abelian Varieties, 2nd edn. Oxford University Press (1974)
5. Oort, F., Strooker, J.R.: The category of finite groups over a field. Indag. Math. **29**, 163–169 (1967)
6. Oort, F., Tate, J.: Group schemes of prime order. Ann. Sci. École Norm. Sup. 4^e série, **3**, 1–21 (1970)
7. Pink, R.: Finite group schemes. Lecture course in WS 2004/05 ETH Zürich. http://www.math.ethz.ch/~pink/ftp/FGS/CompleteNotes.pdf
8. Shatz, S.S.: Group schemes, Formal groups and p-divisible groups. In: Cornell, G., Silverman, J.H. (eds.), Chapter III In: Arithmetic Geometry. Springer, New York (1986)
9. Zarhin, Y.G.: Endomorphisms of abelian varieties and points of finite order in characteristic P. Mat. Zametki, **21**, 737–744. Math. Notes. **21**(1978), 415–419 (1977)
10. Zarhin, Yu.G.: A finiteness theorem for unpolarized Abelian varieties over number fields with prescribed places of bad reduction. Invent. Math. **79**, 309–321 (1985)
11. Zarhin, Y.G.: Homomorphisms of abelian varieties over finite fields. In: Kaledin, D., Tschinkel, Y. (eds.), Higher-dimensional Geometry Over Finite Fields, pp. 315–343. IOS Press, Amsterdam (2008). arXiv:0711.1615v5 [math.AG]

On the Cheltsov–Rubinstein Conjecture

Kento Fujita, Yuchen Liu, Hendrik Süss, Kewei Zhang, and Ziquan Zhuang

Abstract In this note we investigate the Cheltsov–Rubinstein conjecture. We show that this conjecture does not hold in general and some counterexamples will be presented.

Keywords K-stability · Asymptotically log Fano vartieties · Asymptotically log del Pezzo surfaces

1 Introduction

In the study of canonical metrics on Fano type manifolds, Kähler-Einstein edge (KEE) metrics are a natural generalization of Kähler-Einstein metrics: they are smooth metrics on the complement of a divisor, and have a conical singularity of angle $2\pi\beta$

We work over the complex number field $\mathbb{C}$.

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transverse to that complex edge (see [21] for a survey, precise definition and references). Considerable amount of work on KEE metrics in recent years has concerned the behavior of such metrics when the cone angle is relatively large (e.g., close to 2π).

In 2013, Cheltsov–Rubinstein [4] initiated a systematic study of the behavior in the other extreme when the *cone angle* β goes to zero. To explore this small cone angle world, it is natural to work on *asymptotically log Fano varieties*, a class of varieties introduced in op. cit.

Definition 1.1 ([4]) Let X be a normal projective variety over $\mathbb{C}$. Let $D = \sum D_i$ be an effective divisor on X, where each D_i is a prime divisor. We say the pair (X, D) is (strongly) asymptotically log Fano if the log pair $(X, \sum(1-\beta_i)D_i)$ is log Fano for (all) sufficiently small $\beta_i \in (0, 1]$.

In dimension 2, we also use *log del Pezzo* to stand for log Fano. Note that, if D has only one component, then (X, D) being (strongly) asymptotically log Fano just means that $(X, (1-\beta)D)$ is log Fano for sufficiently small β. Regarding the existence of KEE metrics on such pairs, Cheltsov–Rubinstein proposed the following conjecture.

Conjecture 1.2 ([4]) Let (X, D) be a smooth asymptotically log Fano pair where D is a smooth divisor. Then $(X, (1-\beta)D)$ admits a KEE metric with sufficiently small cone angle β along D if and only if $(K_X + D)^{\dim X} = 0$.

One direction of this conjecture (the necessary part) has been verified by Cheltsov–Rubinstein [5] (for dimension 2) and subsequently by Fujita [11] (for higher dimensions). Moreover, Cheltsov–Rubinstein [4, 5] confirmed the conjecture for all pairs in dimension 2 except one infinite family of pairs, and recently Cheltsov–Rubinstein–Zhang [6] confirmed the conjecture for all but 6 of these pairs. In this note, byfocusing on K-stability of the pairs $(X, (1-\beta)D)$, we show that some of these remaining cases provide counterexamples to Conjecture 1.2 (see Sect. 2). In addition we provide other counterexamples in higher dimensions and investigate the subtlety involved (see Sect. 3). Here is one of the main results in the paper:

Theorem 1.3 (see Remark *2.10*) *Let* $\bar{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ *be a smooth curve of bi-degree* $(1, 2)$*, let* $0 \in \bar{C}$ *be a ramification point of the double cover* $\bar{C} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^1$*. Let* $S \to \mathbb{P}^1 \times \mathbb{P}^1$ *be the blowup at* $0 \in \mathbb{P}^1 \times \mathbb{P}^1$ *and let* $C \subset S$ *be the proper transform of* $\bar{C}$*. Then the pair* (S, C) *is an asymptotically log del Pezzo pair with* $(S + C)^2 = 0$ *such that* $(S, (1-\beta)C)$ *does not admit KEE metric with sufficiently small cone angle* β *along* C*.*

2 Examples and Counterexamples in Dimension 2

2.1 Preliminaries

In this section, we let (S, C) be an asymptotically log del Pezzo pair with both S and C are smooth. Assume that $(K_S + C)^2 = 0$. For the sufficient part, in dimension 2, it is useful to divide into two cases: when $K_S + C \sim 0$ and when $K_S + C \nsim 0$. In the first case existence (and hence the Conjecture 1.2) follows from [14, Corollary 1] which resolved a conjecture of Donaldson [9]. In the second case, Cheltsov–Rubinstein's classification of asymptotically log del Pezzo surfaces [4, Theorem 2.1] reduces the task to $(\mathbb{F}_1, C)$ with $C \in |2Z_1 + 2F|$ or (S_r, C_r), where $\mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1))$, $Z_1 \subset \mathbb{F}_1$ is the (-1)-curve, $F \subset \mathbb{F}_1$ is a fiber of $\mathbb{F}_1/\mathbb{P}^1$, and S_r the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at r-points on a bi-degree $(2, 1)$ curve with no two on the same $(0, 1)$ curve and C_r the proper transform of the bi-degree $(2, 1)$ curve on $\mathbb{P}^1 \times \mathbb{P}^1$. The surface (S_0, C_0) and $(\mathbb{F}_1, C)$ were successfully treated using α-invariant techniques in [4, Propositions 4.4 and 4.5]. And in a recent article [6] all but 6 of the pairs (S_r, C_r) were handled using δ-invariant techniques (see [6, Theorem 1.3]). Thus the conjecture seemed plausible, at least in dimension 2. Somewhat surprisingly, we show that nevertheless some of (S_1, C_1) and (S_2, C_2) provide subtle counterexamples to Conjecture 1.2.

More precisely, we will show in Sects. 2.3 and 2.4 that, for $r = 1$ and $r = 2$, some special configurations of (S_r, C_r) are not uniformly K-stable. Here by 'special' we mean that the blown up points on the $(1, 2)$ curve are chosen in a specific way.

To do this, we make use of the delta invariant defined by Fujita–Odaka [10] and we will show that, for some special (S, C) from above, one has

$$\delta\big(S, (1-\beta)C\big) \leq 1, \ \text{for sufficiently small}\ \beta.$$

This means that $(S, (1-\beta)C)$ is not uniformly K-stable by [1]. Then some further argument will imply the non-existence of small cone angle KEE metrics (see Remarks 2.10 and 2.11).

To bound δ-invariants from above, we use the following characterization of δ-invariant (see [1, 10]):

$$\delta\big(S, (1-\beta)C\big) = \inf_Z \frac{A_{S,(1-\beta)C}(Z)}{S_{S,(1-\beta)C}(Z)}. \tag{2.1}$$

Here Z runs through all the prime divisors over the surface S, i.e., there is a birational morphism $\sigma\colon \tilde{S} \to S$ with $\tilde{S}$ smooth such that Z is a prime divisor on $\tilde{S}$, and $A_{S,(1-\beta)C}(Z)$ denotes the log discrepancy of Z, that is,

$$A_{S,(1-\beta)C}(Z) := 1 + \mathrm{ord}_Z(K_{\tilde{S}} - \sigma^*(K_S + (1-\beta)C)).$$

For simplicity, we will write $A(Z) := A_{S,(1-\beta)C}(Z)$ in the following. Moreover, the quantity $S(Z) := S_{S,(1-\beta)C}(Z)$ is called the *expected vanishing order* of $-K_S - (1-\beta)C$ along Z, which is defined by

$$S(Z) := \frac{1}{(-K_S - (1-\beta)C)^2} \int_0^{\tau(Z)} \mathrm{Vol}(-K_S - (1-\beta)C - xZ)dx,$$

where $\tau(Z)$ denotes the pseudo-effective threshold of $-K_S - (1-\beta)C$ with respect to Z, i.e.,

$$\tau(Z) := \sup\{\tau \in \mathbb{R}_{>0} \mid \mathrm{Vol}(-K_S - \tau Z) > 0\}.$$

And as we will see, in some cases, the infimum in (2.1) is obtained by some specific Z over S.

2.2 Basic Setup and Notation

In this subsection, we fix some notation, which will be used throughout this section. Set

$$\overline{S} := \mathbb{P}^1 \times \mathbb{P}^1, \qquad \overline{C} := \text{a smooth curve of bi-degree } (1, 2) \subset \overline{S}.$$

Denote by $\overline{F}$ a general vertical line of bi-degree $(1, 0)$ and by $\overline{H}$ a general horizontal (with respects to the first projection $pr_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$) line of bi-degree $(0, 1)$.

Let $([s : t], [u : v])$ be the bi-homogeneous coordinate system on $\overline{S}$. Then, up to a linear change of coordinates, we may assume that $\overline{C}$ is cut out by the equation $sv^2 = tu^2$.

The linear system $|\overline{F}|$ contains exactly two curves that are tangent to $\overline{C}$. Denote them by $\overline{F}_0$, $\overline{F}_\infty$, and let

$$\overline{p_0} := (\overline{F}_0 \cap \overline{C})_{\mathrm{red}}, \qquad \overline{p_\infty} := (\overline{F}_\infty \cap \overline{C})_{\mathrm{red}}.$$

In $([s : t], [u : v])$ coordinates, one simply has

$$\overline{F}_0 = \{t = 0\},\ \overline{F}_\infty = \{s = 0\},\ \overline{p}_0 = ([1 : 0], [1; 0]) \text{ and } \overline{p}_\infty = ([0 : 1], [0 : 1]).$$

We also put $\overline{H}_0$ and $\overline{H}_\infty$ to be the horizontal $(0, 1)$ curves that intersect $\overline{C}$ transversely at $\overline{p}_0$ and $\overline{p}_\infty$ respectively. So $\overline{H}_0 = \{v = 0\}$ and $\overline{H}_\infty = \{u = 0\}$ (Fig. 1).

Choose some $r \in \mathbb{N}$ and let $\overline{F}_1, \ldots, \overline{F}_r$ be distinct bi-degree $(1, 0)$ curves in $\overline{S}$ that are all different from the curves $\overline{F}_0$ and $\overline{F}_\infty$. Then each intersection $\overline{F}_i \cap \overline{C}$ consists of two points. For each $i = 1, \ldots r$, let

$$\overline{p}_i \in \overline{F}_i \cap \overline{C}$$

be one of these two points.

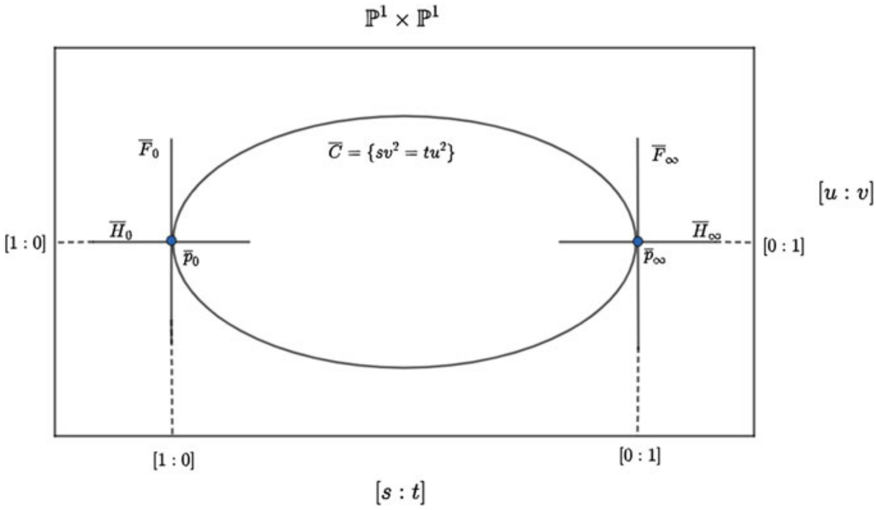

Fig. 1

Set

$$I := \{i_1, \ldots, i_r\} \subset \{0, 1, ..., r, \infty\},$$

and let $S := S_I$ denote the blow-up of $\overline{S}$ at the r points $\{\overline{p}_i\}_{i \in I} \subset \{\overline{p}_0, \overline{p}_1, ..., \overline{p}_r, \overline{p}_\infty\}$, with $\pi \colon S \to \overline{S}$ being the blowup morphism. Let us denote by

$$E_j := \pi^{-1}(\overline{p}_j), \qquad j \in I,$$

the exceptional curves of π. To be precise, we note that we are blowing-up r of the $r + 2$ points $\{\overline{p}_0, \overline{p}_1, ..., \overline{p}_r, \overline{p}_\infty\}$. Denote by

$$F_0, F_1, \ldots, F_r, F_\infty$$

the proper transform on the surface S of the curves $\overline{F}_0, \overline{F}_1, \ldots, \overline{F}_r, \overline{F}_\infty$ (note that exactly r of these are -1-curves and the remaining two are 0-curves). We also set $\overline{H}_i$ to be the horizontal (0, 1) curve passing through $\overline{p}_i$ and let H_i be its proper transform on S.

Let C be the proper transform of the curve $\overline{C}$, so

$$C = \pi^* \overline{C} - \sum_{j \in I} E_j \sim \pi^*(\overline{F} + 2\overline{H}) - \sum_{j \in I} E_j.$$

For any sufficiently small rational number $\beta > 0$, we put

$$\overline{L}_\beta := -K_{\overline{S}} - (1 - \beta)\overline{C}.$$

Then we have

$$\overline{L}_\beta \sim_{\mathbb{Q}} (1+\beta)\overline{F} + 2\beta\overline{H} \sim_{\mathbb{Q}} \overline{F} + \beta\overline{C}.$$

Let

$$L_\beta := -K_S - (1-\beta)C.$$

Then we have

$$L_\beta \sim_{\mathbb{Q}} \pi^*\overline{L}_\beta - \beta\sum_{j\in I} E_j \sim_{\mathbb{Q}} \pi^*\big((1+\beta)\overline{F} + 2\beta\overline{H}\big) - \beta\sum_{j\in I} E_j \sim_{\mathbb{Q}} \pi^*\overline{F} + \beta C.$$

Note that L_β is an ample $\mathbb{Q}$-line bundle for sufficiently small β, so that the pair (S, C) is asymptotically log del Pezzo.

2.3 Blowing up Two Special Points

In this part we set $r = 2$ and $I = \{0, \infty\}$. So (S, C) is obtained by blowing up $\overline{p}_0$ and $\overline{p}_\infty$. In this case, L_β is ample for any $\beta \in (0, 1]$. The main result is the following.

Proposition 2.2 $(S, (1-\beta)C)$ *admits a KEE metric with cone angle* β *along* C *for* $\beta \in (0, 1]$

To show this, we use Tian's α_G-invariant, where we take

$$G := \mathbb{C}^* \rtimes \mathbb{Z}_2.$$

Note that $G \subset \mathrm{Aut}(\mathbb{P}^1)$. The action is simply given by multiplication and involution. If we embed $\mathbb{P}^1$ into $\overline{S} = \mathbb{P}^1 \times \mathbb{P}^1$ as the $(1, 2)$ curve $\overline{C}$ (the map is given by $[x : y] \mapsto ([x^2 : y^2], [x : y])$), then the G-action extends to $(\overline{S}, \overline{C})$. Namely, $G \subset \mathrm{Aut}(\overline{S}, \overline{C})$. More specifically, for any $(\lambda, \iota) \in G$ and $([s : t], [u : v]) \in \overline{S}$, the induced action is given by

$$\lambda \cdot ([s : t], [u : v]) = ([s : \lambda^2 t], [u : \lambda v]),\ \ \iota \cdot ([s : t], [u : v]) = ([t : s], [v : u]).$$

This G-action lifts to (S, C) since we are blowing up $\overline{p}_0$ and $\overline{p}_\infty$. (Note that $G \cdot \{\bar{p}_0, \bar{p}_\infty\} = \{\bar{p}_0, \bar{p}_\infty\}$.) In particular, the curves $F_0, E_0, H_0, F_\infty, E_\infty, H_\infty$ and C are all $\mathbb{C}^*$-invariant and (Fig. 2)

$$\iota(F_0) = F_\infty,\ \ \iota(E_0) = E_\infty,\ \ \iota(H_0) = H_\infty,\ \ \iota(C) = C.$$

Remark 2.3 As G is positive dimensional, the pair $(S, (1-\beta)C)$ is not K-stable. Proposition 2.2 implies that the log Fano pair $\big(S, (1-\beta)C\big)$ is log K-polystable. See [24] for example. Moreover, when $\beta = 1$, this recovers the well-known existence of KE metrics on $Bl_3\mathbb{P}^2$.

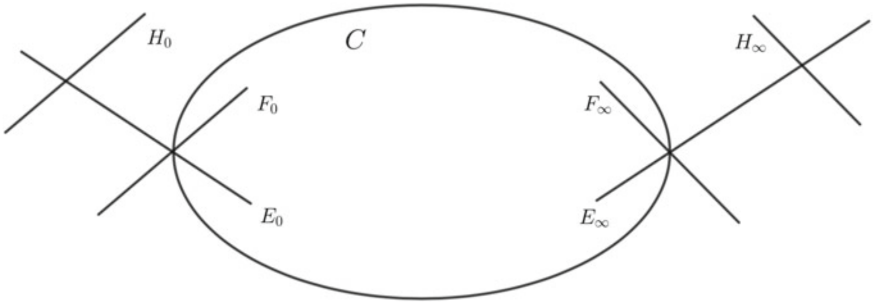

Fig. 2

To prove Proposition 2.2, it is enough (see [14, Theorem 2, Lemma 6.11]) to show the following:

Proposition 2.4 *One has* $\alpha_G\big(S, (1-\beta)C\big) = 1$ *for* $\beta \in (0, 1]$.

Here $\alpha_G\big(S, (1-\beta)C\big)$ is defined as

$$\alpha_G\big(S, (1-\beta)C\big) := \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (S, (1-\beta)C + \lambda\mathcal{D}) \text{ is log canonical}\\ \text{for every G-invariant } \mathbb{Q} - \text{linear system } \mathcal{D} \sim_{\mathbb{Q}} L_\beta\end{array}\right\}.$$

(See [7, (1.2)] for example.)

Remark 2.5 In fact, to compute our α_G, it suffices to consider G-invariant divisors. Indeed, as $\mathbb{C}^*$ is abelian, so any $\mathbb{C}^*$-invariant linear system must contain a $\mathbb{C}^*$-invariant divisor D by Borel's fixed point theorem. Then it is enough to look at $\frac{1}{2}(D + \iota(D))$, as in [7, Sect. 7]. Indeed, $\{D + \iota(D)\} \subset 2\mathcal{D}$ is a G-invariant $\mathbb{Q}$-sublinear system.

Proof of Proposition 2.4 We will show that for any effective G-invariant divisor $D \sim_{\mathbb{Q}} L_\beta$ the pair $(S, (1-\beta)C + D)$ is log canonical, but $(S, (1-\beta)C + \lambda D)$ is not for $\lambda > 1$. The Picard group of S has basis $[H] := [\pi^*\overline{H}_0]$, $[F] := [\pi^*\overline{F}_0]$, $[E_0]$ and $[E_\infty]$. In this basis we have $[F_0] = [F] - [E_0]$, $[F_\infty] = [F] - [E_\infty]$, $[H_0] = [H] - [E_0]$ and $[H_\infty] = [H] - [E_\infty]$. An anti-canonical divisor is given by

$$-K_S = 2H + 2F - E_0 - E_\infty. \tag{2.6}$$

We claim that the $\mathbb{C}^*$-invariant curves on S are given by $E_0, E_\infty, F_0, F_\infty, H_0, H_\infty$ and the strict transforms $C_{[\alpha:\beta]}$ of the curves $\overline{C}_{[\alpha:\beta]} = [\alpha s v^2 = \beta t u^2] \subset \overline{S}$ for $[\alpha : \beta] \in \mathbb{P}^1 \setminus \{0, \infty\}$. Indeed, all $\mathbb{C}^*$-invariant curves on S are given by the closures of 1-dimensional $\mathbb{C}^*$-orbits. The curves $C_{[\alpha:\beta]}$ are all linearly equivalent to $2H + F - E_0 - E_\infty$ and each of them passes through the intersection points $p_0 = E_0 \cap F_0$ and $p_\infty = E_\infty \cap F_\infty$ intersecting all four curves transversely. The curves $C_{[\alpha:\beta]}$ also intersect each other pairwise transversely in these two points.

Being G-invariant the divisor $(1-\beta)C + D \sim -K_S$ has to have the form

$$e(E_0 + E_\infty) + f(F_0 + F_\infty) + h(H_0 + H_\infty) + \sum \gamma_i C_{[\alpha_i:\beta_i]}.$$

Since $(1-\beta)C + D$ is effective, we have $e, f, h, \gamma_i \geq 0$. We set $\gamma := \sum_i \gamma_i$. Passing to the classes in the Picard group of S we get

$$[(1-\beta)C + D] = (e - f - h - \gamma)([E_0] + [E_\infty]) + (2f + \gamma) \cdot [F] + (2h + 2\gamma) \cdot [H].$$

Comparing coefficient with $-K_S$ in (2.6) gives $h = 1 - \gamma$, $f = 1 - \gamma/2$ and $e = f + h + \gamma - 1 = 1 - \gamma/2$. Therefore all coefficients of $(1-\beta)C + D$ are less or equal to 1. A log resolution of $\big(S, (1-\beta)C + D\big)$ is given by further blowing up S in p_0 and p_∞. The multiplicity of $(1-\beta)C + D$ in these points is $\gamma + e + f = 2$. By applying inversion of adjunction for the exceptional curves of the blowups of S in p_0 and p_∞, the pair $(S, (1-\beta)C + D)$ is log canonical, since $\gamma, e, f \leq 1$. □

2.4 Counterexamples

In this part we carry out some explicit computation and give upper bounds for $\delta\big(S, (1-\beta)C\big)$. This will give us some counterexamples to Conjecture 1.2 (see Remarks 2.10 and 2.11).

Recall that, for any prime divisor Z over the surface S, we have the expected vanishing order

$$S(Z) := S_\beta(Z) := \frac{1}{(L_\beta)^2} \int_0^{\tau(Z)} \mathrm{Vol}(L_\beta - xZ)dx.$$

Proposition 2.7 *For any $r \geq 1$, one has*

$$S(Z) = \begin{cases} \frac{1}{2} + \frac{6-r}{8}\beta + O(\beta^2), \;\; Z = E_i, \;\; i \in I; \\ \frac{1}{2} + \frac{6-r}{8}\beta + O(\beta^2), \;\; Z = F_i, \;\; i \in I; \\ \frac{\beta}{2} + \frac{r-4}{24}\beta^2 + O(\beta^3), \;\; Z = C; \\ \frac{1}{2} + \frac{4-r}{8}\beta + O(\beta^2), \;\; Z \in |\pi^*\overline{F}|. \end{cases}$$

This follows from explicit computation; see [6, Lemma 4.5] for more details. Note that, in the above proposition, the prime divisor Z is on the surface S. We can also consider Z *over* S.

Proposition 2.8 *We have the following*

(1) Suppose that $p \in C$ is away from any F_i or E_i, where $i \in I$. Let $\widetilde{S} \xrightarrow{\sigma} S$ be the blow-up of p and let Z be the exceptional curve of σ. Then

$$S(Z) = \frac{1}{2} + \frac{12-r}{8}\beta + O(\beta^2),\ r \geq 0.$$

(2) Suppose that $0 \in I$, i.e. $\overline{p}_0$ is blown up. Put $p_0 = E_0 \cap C$. Let $\widetilde{S} \xrightarrow{\sigma} S$ be the blow-up of p_0 and let Z be the exceptional curve of σ. Then

$$S(Z) = 1 + \frac{6-r}{4}\beta + O(\beta^2),\ r \geq 1.$$

Moreover, for $r = 2$, we have exactly

$$S(Z) = 1 + \beta;$$

(3) Suppose that $p = E_i \cap C$ or $p = F_i \cap C$ for some $i \in I$ and $i \neq 0$ or ∞. Let $\widetilde{S} \xrightarrow{\sigma} S$ be the blow-up of p and let Z be the exceptional curve of σ. Then

$$S(Z) = \frac{1}{2} + \frac{14-r}{8}\beta + O(\beta^2),\ r \geq 1;$$

(4) Suppose $0 \notin I$. Let $p_0 = F_0 \cap C$. Let $\widetilde{S} \xrightarrow{\sigma} S$ be the blow-up of p_0 and let G be the exceptional curve of σ. Let $\widetilde{C}$ be the proper transform of C on $\widetilde{S}$. Put $q_0 = G \cap \widetilde{C}$. Let $\widehat{S} \xrightarrow{\tau} \widetilde{S}$ be the blow-up of q_0 and let Z be the exceptional curve of τ. Then we have

$$S(Z) = \begin{cases} 1 + 2\beta,\ r = 0, \\ 1 + \frac{8-r}{4}\beta + O(\beta^2),\ r \geq 1. \end{cases}$$

Again, this follows from elementary computation. For the reader's convenience, we include the proof of case (2) with $r = 2$. The computation for other cases is similar.

Proof of Proposition 2.8(2) with $r = 2$ In this case, S is obtained by blowing up $\overline{p}_0$ and another point $\overline{p}_i$ (possibly $\overline{p}_\infty$) on $\overline{C}$. Then we have

$$L_\beta \sim_{\mathbb{Q}} \pi^*\big((1+\beta)\overline{F_0} + \beta\overline{H}_0 + \beta\overline{H}_i\big) - \beta E_0 - \beta E_i = (1+\beta)(F_0 + E_0) + \beta(H_0 + H_i).$$

Now let $\widetilde{S} \xrightarrow{\sigma} S$ be the blow-up of $p_0 = E_0 \cap C$ with Z being the exceptional curve of σ. Let $\widetilde{F}_0$, $\widetilde{E}_0$, $\widetilde{H}_0$ and $\widetilde{H}_i$ be the proper transforms of F_0, E_0, H_0 and H_i on $\widetilde{S}$ respectively. Then we have

$$\sigma^* L_\beta - xZ \sim_{\mathbb{Q}} (1+\beta)(\widetilde{F}_0 + \widetilde{E}_0) + \beta(\widetilde{H}_0 + \widetilde{H}_i) + (2 + 2\beta - x)Z.$$

Note that

$$\begin{cases}(\sigma^* L_\beta - xZ)\cdot \widetilde{F}_0 = (\sigma^* L_\beta - xZ)\cdot \widetilde{E}_0 = \beta - x,\\ (\sigma^* L_\beta - xZ)\cdot \widetilde{H}_0 = (\sigma^* L_\beta - xZ)\cdot \widetilde{H}_i = 1,\\ (\sigma^* L_\beta - xZ)\cdot Z = x.\end{cases}$$

So $\sigma^* L_\beta - xZ$ is nef for $x \in [0, \beta]$. Thus we have

$$\mathrm{Vol}\big(\sigma^* L_\beta - xZ\big) = \big(\sigma^* L_\beta - xZ\big)^2 = 4\beta + 2\beta^2 - x^2,\ x \in [0, \beta].$$

And for $x \geq \beta$, [8, Corollary 2.8] implies

$$\mathrm{Vol}\big(\sigma^* L_\beta - xZ\big) = \mathrm{Vol}\left(\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)\right).$$

Note that

$$\begin{cases}\big(\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)\big)\cdot \widetilde{F}_0 = \big(\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)\big)\cdot \widetilde{E}_0 = 0,\\ \big(\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)\big)\cdot \widetilde{H}_0 = \big(\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)\big)\cdot \widetilde{H}_i = 1 - \frac{x-\beta}{2},\\ \big(\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)\big)\cdot Z = \beta.\end{cases}$$

So $\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)$ is nef for $x \in [\beta, 2+\beta]$. Thus for $x \in [\beta, 2+\beta]$ we have

$$\mathrm{Vol}\big(\sigma^* L_\beta - xZ\big) = \left(\sigma^* L_\beta - xZ - \frac{x-\beta}{2}(\widetilde{F}_0 + \widetilde{E}_0)\right)^2 = 4\beta + 3\beta^2 - 2\beta x.$$

Now for $x \geq 2+\beta$, we use Zariski decomposition [8, Corollary 2.7]. Solve the following equations:

$$\begin{cases}\big(a\widetilde{F}_0 + b\widetilde{E}_0 + c\widetilde{H}_0 + d\widetilde{H}_i + (2+2\beta-x)Z\big)\cdot \widetilde{F}_0 = 0\\ \big(a\widetilde{F}_0 + b\widetilde{E}_0 + c\widetilde{H}_0 + d\widetilde{H}_i + (2+2\beta-x)Z\big)\cdot \widetilde{E}_0 = 0\\ \big(a\widetilde{F}_0 + b\widetilde{E}_0 + c\widetilde{H}_0 + d\widetilde{H}_i + (2+2\beta-x)Z\big)\cdot \widetilde{H}_0 = 0\\ \big(a\widetilde{F}_0 + b\widetilde{E}_0 + c\widetilde{H}_0 + d\widetilde{H}_i + (2+2\beta-x)Z\big)\cdot \widetilde{H}_i = 0\end{cases}$$

We get

$$a = b = c = d = 2 + 2\beta - x.$$

This implies, for $x \in [2+\beta, 2+2\beta]$, we have

$$\begin{aligned}\operatorname{Vol}\big(\sigma^* L_\beta - xZ\big) &= \operatorname{Vol}\big((2+2\beta-x)(\widetilde{F}_0+\widetilde{E}_0+\widetilde{H}_0+\widetilde{H}_i+Z)\big)\\ &= (2+2\beta-x)^2\big(\widetilde{F}_0+\widetilde{E}_0+\widetilde{H}_0+\widetilde{H}_i+Z\big)^2\\ &= (2+2\beta-x)^2.\end{aligned}$$

So we can compute

$$\begin{aligned}\int_0^{\tau(Z)} \operatorname{Vol}(\sigma^* L_\beta - xZ\big)dx &= \int_0^{2+2\beta} \operatorname{Vol}(\sigma^* L_\beta - xZ\big)dx\\ &= \int_0^{\beta}(4\beta+2\beta^2-x^2)dx + \int_\beta^{2+\beta}(4\beta+3\beta^2-2\beta x)dx +\\ &\quad + \int_{2+\beta}^{2+2\beta}(2+2\beta-x)^2 dx\\ &= 4\beta+6\beta^2+2\beta^3.\end{aligned}$$

Thus we get

$$S(Z) = \frac{1}{L_\beta^2}\int_0^{\tau(Z)} \operatorname{Vol}(\sigma^* L_\beta - xZ\big)dx = \frac{4\beta+6\beta^2+2\beta^3}{4\beta+2\beta^2} = 1+\beta.$$

□

Note that Proposition 2.8 has the following consequence.

Corollary 2.9 *We have the following upper bound for δ-invariant.*

(1) If $0 \in I$, then

$$\delta\big(S,(1-\beta)C\big) \le 1 + \frac{r-2}{4}\beta + O(\beta^2),\ r \ge 1.$$

Moreover, when $r = 2$, we have exactly

$$\delta\big(S,(1-\beta)C\big) = 1,$$

and the infimum of (2.1) is obtained by the Z in Proposition 2.7(2).

(2) If $0 \notin I$, then

$$\delta\big(S,(1-\beta)C\big) \le 1 + r\beta + O(\beta^2),\ r \ge 0.$$

Moreover, when $r = 0$, we have exactly

$$\delta\big(S,(1-\beta)C\big) = 1,$$

and the infimum of (2.1) is obtained by the Z in Proposition 2.7(4).

Proof (1) Let Z be the divisor in Proposition 2.8(2). Then we have

$$A(Z) = 2 - (1-\beta) = 1 + \beta.$$

Using (2.1), for $r \geq 1$ we get

$$\delta\big(S, (1-\beta)C\big) \leq \frac{A(Z)}{S(Z)} = \frac{1+\beta}{1 + \frac{6-r}{4}\beta + O(\beta^2)} = 1 + \frac{r-2}{4}\beta + O(\beta^2).$$

When $r = 2$, we have

$$\delta\big(S, (1-\beta)C\big) \leq \frac{A(Z)}{S(Z)} = \frac{1+\beta}{1+\beta} = 1.$$

To see this is actually an equality, we need to use some deeper results. Suppose that $\infty \in I$, so (S, C) is obtained by blowing up $\overline{p}_0$ and $\overline{p}_\infty$. Then Proposition 2.2 implies that $(S, (1-\beta)C)$ is K-polystable, so [1, 10] imply that we have the other direction:

$$\delta\big(S, (1-\beta)C\big) \geq 1.$$

If $\infty \notin I$, then the $\mathbb{C}^*$-action on $(\overline{S}, \overline{C})$ (see Sect. 2.3 for details about this action) induces a K-polystable degeneration of the log Fano pair $(S, (1-\beta)C)$. To be more precise, we are blowing up $\overline{p}_0$ and another point $\overline{p}_i$ on the $(1, 2)$ curve $\overline{C}$. Then $\mathbb{C}^*$-action on $(\overline{S}, \overline{C})$ fixes $\overline{p}_0$ but moves $\overline{p}_i$ towards $\overline{p}_\infty$. So this action induces a degeneration from (S, C) towards the above K-polystable pair obtained by blowing up $\overline{p}_0$ and $\overline{p}_\infty$. By the lower semi-continuity of δ-invariant (cf. [2]), we again obtain

$$\delta\big(S, (1-\beta)C\big) \geq 1.$$

(2) Let Z be the divisor in Proposition 2.8(4). Then we clearly have

$$A(Z) = A_S(Z) - (1-\beta)\,\mathrm{ord}(\tau^*\sigma^*C) = 3 - 2(1-\beta) = 1 + 2\beta.$$

Using (2.1), for $r \geq 0$ we get

$$\delta\big(S, (1-\beta)C\big) \leq \frac{A(Z)}{S(Z)} = \frac{1+2\beta}{1 + \frac{8-r}{4}\beta + O(\beta^2)} = 1 + r\beta + O(\beta^2).$$

When $r = 0$, we have

$$\delta\big(S, (1-\beta)C\big) \leq \frac{A(Z)}{S(Z)} = \frac{1+2\beta}{1+2\beta} = 1.$$

To see this is actually an equality, we use [4, Proposition 7.4], which implies that $(S, (1-\beta)C)$ is K-polystable, so we have the other direction:

$$\delta\big(S, (1-\beta)C\big) \geq 1.$$

□

Remark 2.10 Suppose that $I = \{0\}$, then Corollary 2.9(1) implies that

$$\delta\big(S, (1-\beta)C\big) < 1$$

for sufficiently small β. This means that $(S, (1-\beta)C)$ does not admit a KEE metric with sufficiently small cone angle β. So Conjecture 1.2 fails in this case. Note that there is another way to obtain $\delta\big(S, (1-\beta)C\big) < 1$, which relies on the toric calculation in [1, Sect. 7]. Indeed, if $I = \{0\}$, then $S \cong Bl_2\mathbb{P}^2$ is a toric surface. One can determine the polytope P_β of L_β and its barycenter bc_β. Let Z be the divisor in Proposition 2.8(2). Then by [1, Corollary 7.7], $S(Z)$ can be explicitly computed as Z gives rise to a toric valuation v_Z. More specifically, following the notation therein, we have

$$S(Z) = \langle bc_\beta, v_Z\rangle - \psi(v_Z) = \frac{4(1+\beta)^2}{4+3\beta},$$

where $bc_\beta = \left(\frac{-(4\beta^2+9\beta+6)}{3(4+3\beta)}, \frac{-(7\beta^2+12\beta)}{3(4+3\beta)}\right)$, $v_Z = (2, 1)$ and $\psi(v_Z) = -(2+3\beta)$. So that

$$\delta\big(S, (1-\beta)C\big) \leq \frac{A(Z)}{S(Z)} = \frac{4+3\beta}{4+4\beta} < 1,$$

where $A(Z) := A_{S,(1-\beta)C}(Z)$ (Fig. 3).

Remark 2.11 Let us take a closer look at the case when $I = \{0, i\}$ with $i \neq \infty$. Then $\mathrm{Aut}(S, C)$ is discrete. We claim that $(S, (1-\beta)C)$ does not admit a KEE metric with sufficiently small cone angle. If this is not the case, then the existence of a KEE metric implies the properness of K-energy, and hence the pair $(S, (1-\beta)C)$

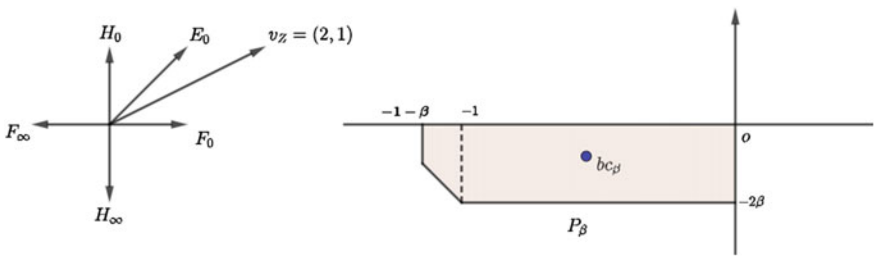

Fig. 3 The fan of S and the polytope P_β of L_β

Table 1 K-stability for (S_I, C)

I	K-stability
$\{i_1\} \subset \{0, \infty\}$	K-unstable (Corollary 2.9)
$\{i_1\} \not\subset \{0, \infty\}$	?
$\{i_1, i_2\} = \{0, \infty\}$	Strictly K-polystable (Proposition 2.2)
$\emptyset \neq \{i_1, i_2\} \cap \{0, \infty\} \neq \{0, \infty\}$	Strictly K-semistable (Remark 2.11)
$\{i_1, i_2\} \cap \{0, \infty\} = \emptyset$	?
$3 \leq \#I \leq 6$	?
$\#I \geq 7$	K-stable ([6, Theorem 1.3])

is uniformly K-stable (see [19, Corollary 1.2] for example). So we should have $\delta\big(S, (1-\beta)C\big) > 1$, contradicting Corollary 2.9(1). So Conjecture 1.2 fails in this case as well. There is another way to see this. Indeed, as we have seen in the proof of Corollary 2.9(1), $(S, (1-\beta)C)$ admits a K-polystable degeneration, which implies that $(S, (1-\beta)C)$ cannot be K-stable. So $(S, (1-\beta)C)$ is strictly K-semistable and it cannot admit a KEE metric. In other words, we get a family of strictly K-semistable log Fano surfaces degenerating to a K-polystable log Fano surface. This can be thought of as a 2-dimensional log version of Tian's Mukai-Umemura example (see [22]).

In the following table we summarize what is known about the K-stability of the asymptotically log del Pezzo surfaces $(S, C) = (S_I, C)$, where "strictly K-polystable" (resp., "strictly K-semistable") stands for "K-polystable but not K-semistable" (resp., "K-semistable but not K-polystable").

Remark 2.12 In the cases where we know the answer according to Table 1, it turns out that K-(semi/poly-)stability coincides with the GIT-(semi/poly-)stability of the point configuration on $\overline{C} \cong \mathbb{P}^1$ consisting of the blowup centers and the two special points $\overline{p}_0$ and $\overline{p}_\infty$. More precisely, we consider $(\overline{p}_0, \overline{p}_\infty, (\overline{p}_i)_{i\in I}) \in (\mathbb{P}^1)^{\#I+2}$ and ask for the stability in the GIT sense of this point with respect to the diagonal $\mathrm{SL}(2)$-action on $(\mathbb{P}^1)^{\#I+2}$ and the unique $\mathrm{SL}(2)$-linearization of $\mathcal{O}(1, \ldots, 1)$. By [20, Chapt. 4, Sect. 1], a point configuration $(\overline{p}_0, \overline{p}_\infty, (\overline{p}_i)_{i\in I})$ of $\mathbb{P}^1$ is GIT stable (resp. GIT semistable) if and only if at most $\lfloor \frac{\#I+1}{2} \rfloor$ (resp. at most $\lfloor \frac{\#I+2}{2} \rfloor$) points from the configuration coincide. In the light of this observation it is natural to expect that the remaining cases should all be K-stable, as the corresponding point configurations are indeed GIT-stable.

Postscript Remark. After the appearance of this work on arXiv, the Cheltsov–Rubinstein conjecture in dimension 2 has been completely settled by K. Fujita [12] and the previous expectation on the relation between K-stability and GIT stability has been verified affirmatively.

3 Higher Dimensional Counterexamples and Further Discussion

In this section we investigate the Cheltsov–Rubinstein program in higher dimensions.

3.1 Product Spaces

There are also simple counterexamples to Conjecture 1.2 by taking products of log Fano pairs. Let X_1 and X_2 be two smooth Fano varieties. Suppose that $F \in |-K_X|$ is a smooth divisor. Put

$$X := X_1 \times X_2,\ \ D := F \times X_2.$$

Then in particular, $-K_X = p_1^*(-K_{X_1}) + p_1^*(-K_{X_2})$ and $D \in |-p_1^* K_{X_1}|$, where p_1 and p_2 are the natural projections from X to X_1 and X_2 respectively. It is clear that (X, D) is an asymptotically log Fano pair, as for any $\beta \in (0, 1]$,

$$-K_X - (1-\beta)D \sim_{\mathbb{Q}} \beta \cdot p_1^*(-K_{X_1}) + p_2^*(-K_{X_2})$$

is ample. Moreover, $-(K_X + D) = p_2^*(-K_{X_2})$ is a nef divisor with $(K_X + D)^n = 0$.

On the other hand, from the definition of δ-invariant, we clearly have

$$\delta\big(X, (1-\beta)D\big) \le \min\{\delta\big(X_1, (1-\beta)F\big), \delta(X_2)\} \tag{3.1}$$

(see Remark 3.3). So in particular, if X_2 is a K-unstable Fano manifold with $\delta(X_2) < 1$, then $\delta\big(X, (1-\beta)D\big) < 1$ as well. So in this case the pair $\big(X, (1-\beta)D\big)$ cannot admit any KEE metric.

Example 3.2 Take $X_1 = \mathbb{P}^2$ and $X_2 = \mathrm{Bl}_p\mathbb{P}^2$. Let F be a smooth cubic curve on X_1. Then the pair (X, D) we constructed above is asymptotically log Fano with $(K_X + D)^4 = 0$. And the log pair $\big(X, (1-\beta)D\big)$ does not admit any KEE metric for $\beta \in (0, 1]$. So Conjecture 1.2 fails in this case.

Remark 3.3 (3.1) is actually an equality by the recent work [23].

3.2 K-stability of the Base

By Shokurov's base-point-free theorem [15, Theorem 3.3], it is easy to see that if (X, D) is asymptotically log Fano, then the divisor $-(K_X + D)$ is semi-ample and we let $\phi : X \to Y$ be its corresponding ample model (i.e. ϕ has connected fibers and $-(K_X + D) = \phi^* L$ for some ample divisor L on Y). Since $-(K_X + D)$ is not big by assumption, we have $\dim X > \dim Y$ and in particular, ϕ is not birational. As $K_X + D \sim_{\phi,\mathbb{Q}} 0$ and (X, D) is lc, we can write $K_X + D \sim_{\mathbb{Q}} \phi^*(K_Y + B + M)$ for some effective divisor B (the boundary part) and some pseudo-effective divisor M (the moduli part) by the canonical bundle formula [16, Theorem 8.5.1]. Note that $(Y, B + M)$ is a generalized pair, i.e., $M = \pi_* M'$ for some nef divisor M' on some log resolution $\pi\colon Y' \to Y$ and B is an effective $\mathbb{Q}$-divisor on Y such that $K_Y + B + M$ is $\mathbb{Q}$-Cartier (see [3, Sect. 4]). The example of product varieties above suggests that in order for Conjecture 1.2 to be true, we may need to impose some conditions on the K-stability of the generalized pair $(Y, B + M)$. Here we give a definition of uniform K-stability and K-semistability of a generalized klt log Fano pair similar to the valuative criterion of Fujita [13] and Li [17].

Definition 3.4 Let $(Y, B + M)$ be a projective generalized klt pair such that $-(K_Y + B + M)$ is ample.

(1) For any prime divisor E over Y, We define

$$S_{Y,B+M}(E) := \frac{1}{\mathrm{Vol}(-K_Y - B - M)} \int_0^\infty \mathrm{Vol}(-K_Y - B - M - tE)dt.$$

(2) We say that $(Y, B + M)$ is *K-semistable* if for any prime divisor E over Y, we have $A_{Y,B+M}(E) \geq S_{Y,B+M}(E)$.
(3) We say that $(Y, B + M)$ is *uniformly K-stable* if there exists $\epsilon > 0$ such that for any prime divisor E over Y, we have $A_{Y,B+M}(E) \geq (1 + \epsilon) S_{Y,B+M}(E)$.

In the following proposition we show that K-semistability of the base $(Y, B + M)$ is necessary for Conjecture 1.2 to hold for (X, D).

Proposition 3.5 *Notation as above. Assume that $(X, (1 - \beta)D)$ admits KEE metric for all sufficiently small cone angle $\beta > 0$. Then $(Y, B + M)$ is K-semistable.*

Proof Let E be a prime divisor over Y. Let $\pi_Y : Y' \to Y$ be a proper birational morphism that extracts E as a Cartier divisor. Let X' be the normalization of the main component of $X \times_Y Y'$ with projections $\pi_X : X' \to X$ and $\phi' : X' \to Y'$. Let $L' = \pi_Y^* L$ and $D' = \pi_X^* D$. Then or any ample line bundle L_X on X, we set

$$\mathrm{Vol}(L_X - tE) := \mathrm{Vol}(\pi_X^* L_X - t\phi'^*(E))$$

where $n = \dim X$. We define the expected vanishing order $S_{X,(1-\beta)D}(E)$ of the log Fano pair $(X, (1 - \beta)D)$ along E by

$$S_{X,(1-\beta)D}(E) = \frac{1}{\mathrm{Vol}(-K_X - (1 - \beta)D)} \int_0^\infty \mathrm{Vol}(-K_X - (1 - \beta)D) - tE)\mathrm{d}t.$$

We claim that

$$\liminf_{\beta \to 0+} S_{X,(1-\beta)D}(E) \geq S_{Y,B+M}(E). \tag{3.6}$$

Let $r = \dim X - \dim Y \geq 1$, let $C = \binom{n}{r}$ and let F be a general fiber of ϕ. As (X, D) is asymptotically log Fano, $\beta D \sim_{\phi.\mathbb{Q}} -(K_X + (1-\beta)D) = \phi^* L + \beta D$ is ϕ-ample for some $0 < \beta \ll 1$ and we have

$$\operatorname{Vol}(-K_X - (1-\beta)D) = \left((-K_X - (1-\beta)D)^n\right) = C\beta^r((\phi^* L)^{n-r} \cdot D^r) + O(\beta^{r+1})$$

since ϕ has relative dimension r. As L is ample and D is ϕ-ample, it is easy to see that $((\phi^* L)^{n-r} \cdot D^r) = (L^{n-r})(D^r \cdot F) > 0$, hence

$$\operatorname{Vol}(-K_X - (1-\beta)D) = C\beta^r(L^{n-r})(D^r \cdot F) + O(\beta^{r+1}). \tag{3.7}$$

For any $t \geq 0$ such that $\operatorname{Vol}(L - tE) > 0$ and any $\epsilon > 0$, by Fujita's approximation theorem (see e.g. [18, Theorem D]) we may assume that (after possibly replacing π_Y by another birational morphism) there exists $\mathbb{Q}$-divisors A and N on Y' such that A is ample, N is effective, $L' - tE = A + N$ and $\operatorname{Vol}(A) = (A^{n-r}) > \operatorname{Vol}(L - tE) - \epsilon$. As D is ϕ-ample, D' is ϕ'-ample, thus $\phi'^* A + \beta D'$ is ample for sufficiently small $\beta > 0$. It follows that

$$\begin{aligned}
\operatorname{Vol}(-K_X - (1-\beta)D - tE) &\geq \operatorname{Vol}(\phi'^*(L' - tE) + \beta D') \\
&\geq \operatorname{Vol}(\phi'^* A + \beta D') \\
&= \left((\phi'^* A + \beta D')^n\right) = C\beta^r(A^{n-r})(D^r \cdot F) + O(\beta^2)
\end{aligned}$$

where the last equality follows from the projection formula and the ampleness (resp. ϕ-ampleness) of A (resp. D') as before. In particular, we have

$$\begin{aligned}
\liminf_{\beta \to 0+} \beta^{-r} \operatorname{Vol}(-K_X - (1-\beta)D - tE) &\geq C \cdot (A^{n-r})(D^r \cdot F) \\
&> C \cdot (\operatorname{Vol}(L - tE) - \epsilon)\,(D^r \cdot F).
\end{aligned}$$

As this holds for all $\epsilon > 0$, we obtain

$$\liminf_{\beta \to 0+} \beta^{-1} \operatorname{Vol}(-K_X - (1-\beta)D - tE) \geq C \cdot \operatorname{Vol}(L - tE) \cdot (D^r \cdot F).$$

Therefore by Fatou's lemma we see that

$$\liminf_{\beta \to 0+} \frac{1}{\beta^r} \int_0^\infty \operatorname{Vol}(-K_X - (1-\beta)D - tE)\mathrm{d}t \geq C \cdot (D^r \cdot F) \int_0^\infty \operatorname{Vol}(L - tE)\mathrm{d}t. \tag{3.8}$$

The claimed inequality (3.6) then follows by combining (3.7) and (3.8) as $L \sim_{\mathbb{Q}} -(K_Y + B + M)$.

We now proceed to show that $(Y, B + M)$ is K-semistable, i.e. $A_{Y,B+M}(E) \geq S_{Y,B+M}(E)$ for all prime divisors E over Y. We keep the notation as above. By construction (see [16]), after possibly replacing the birational morphism $\pi_Y : Y' \to Y$, we may assume that if we write $K_{Y'} + B' + M' \sim_{\mathbb{Q}} \pi_Y^*(K_Y + B + M)$, then M'

is nef and the coefficient of E in B' is $1 - \mathrm{lct}_E(X', G; \phi'^* E)$ where (X', G) is the crepant pullback of (X, D) and the lct is taken only over the generic point of E. In particular, $A_{Y,B+M}(E) = \mathrm{lct}_E(X', G; \phi'^* E)$. Since $(X, (1-\beta)D)$ admits KEE metric for all sufficiently small cone angle $\beta > 0$, we have $\delta(X, (1-\beta)D) \geq 1$. In particular, if (X', G_β) is the crepant pullback of $(X, (1-\beta)D)$, then $(X', G_\beta + S_{X,(1-\beta)D}(E) \cdot \phi'^* E)$ is lc. Letting $\beta \to 0$ and using (3.6), we deduce that $(X', G + S_{Y,B+M}(E) \cdot \phi'^* E)$ is lc, hence $S_{Y,B+M}(E) \leq \mathrm{lct}_E(X', G; \phi'^* E) = A_{Y,B+M}(E)$ and we obtain $\beta_{Y,B+M}(E) \geq 0$ as desired. □

Unfortunately, the example from Remark 2.10 shows that only assuming K-semistability of the base is still not enough for Conjecture 1.2 to be true: in that example $Y = \mathbb{P}^1$, $B = \frac{1}{2}([0] + [\infty])$ and $M = 0$ by a direct calculation using the formula from [16]. So it seems to to the authors that the existence of KEE metrics on a asymptotically log Fano pair is a subtle problem and the condition $(K_X + D)^{\dim X} = 0$ is only necessary. More complicated structures, such as the fibration to the ample model of $-(K_X + D)$, should be taken into consideration.

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References

1. Blum, H., Jonsson, M: Thresholds, valuations, and K-stability. Adv. Math. **365**, 107062, 57 (2020)
2. Blum, H., Liu, Y.: Openness of uniform K-stability in families of $\mathbb{Q}$-Fano varieties. To appear in Ann. Sci. Éc. Norm. Supér. (2018). arXiv:1808.09070
3. Birkar, C., Zhang, D.-Q.: Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs. Publ. Math. Inst. Hautes Études Sci. **123**, 283–331 (2016)
4. Cheltsov, I.A., Rubinstein, Y.A.: Asymptotically log Fano varieties. Adv. Math. **285**, 1241–1300 (2015)
5. Cheltsov, I.A., Rubinstein, Y.A.: On flops and canonical metrics. Ann. Sc. Norm. Super. Pisa Cl. Sci. **18**(1), (5) 283–311 (2018)
6. Cheltsov, I.A., Rubinstein, Y.A., Zhang, K.: Basis log canonical thresholds, local intersection estimates, and asymptotically log del Pezzo surfaces. Selecta Math. (N.S.) **25**(2), 25:34 (2019)
7. Cheltsov, I., Shramov, C.: Kähler-Einstein Fano Threefolds of Degree, vol. 22 (2018). arXiv:1803.02774
8. Cheltsov, I., Zhang, K.: Delta invariants of smooth cubic surfaces. Eur. J. Math. **5**(3), 729–762 (2019)
9. Donaldson, S.K.: Discussion of the Kähler-Einstein Problem. (2009)
10. Fujita, K., Odaka, Y.: On the K-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. **70**(4), (2) 511–521 (2018)
11. Fujita, K.: On log K-stability for asymptotically log Fano varieties. Ann. Fac. Sci. Toulouse Math. **25**(5), (6), 1013–1024 (2016)
12. Fujita, K.: On the Uniform k-stability for some Asymptotically Log Del Pezzo Surfaces (2019)
13. Fujita, K.: A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. J. Reine Angew. Math. **751**, 309–338 (2019)

14. Jeffres, T., Mazzeo, R., Rubinstein, Y.A.: Kähler-Einstein metrics with edge singularities. Ann. Math. **183**(1), (2), 95–176 (2016)
15. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1998). With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original
16. Kollár, J.: Kodaira's canonical bundle formula and adjunction. In: Flips for 3-folds and 4-Folds, volume 35 of Oxford Lecture Series in Mathematics and its Applications, pp. 134–162. Oxford University Press, Oxford (2007)
17. Li, C.: K-semistability is equivariant volume minimization. Duke Math. J. **166**(16), 3147–3218 (2017)
18. Lazarsfeld, R., Mustaţă, M.: Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Supér. **42**(5), (4), 783–835 (2009)
19. Li, C., Tian, G., Wang, F.: The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties (2021). To appear in Peking Math. J. arXiv:1903.01215
20. Mumford, D., Fogarty, J., Kirwan, F.: Geometric invariant theory, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], 3rd ed. Springer, Berlin (1994)
21. Rubinstein, Y.A.: Smooth and singular Kähler-Einstein metrics. In: Geometric and Spectral Analysis, volume 630 of Contemporary Mathematics, pp. 45–138. American Mathematical Society, Providence, RI (2014)
22. Tian, G.: Kähler-Einstein metrics with positive scalar curvature. Invent. Math. **130**(1), 1–37 (1997)
23. Zhuang, Z.: Product theorem for K-stability. Adv. Math. **371**, 107250, 18 (2020)
24. Zhuang, Z.: Optimal destabilizing centers and equivariant K-stability. Invent. Math. **226**(1), 195–223 (2021)

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