Chapter 6
Valuation and Optimal Strategies for American Options Under a Markovian Regime-Switching Model

Lu Jin, Marko Dimitrov, and Ying Ni

Abstract In this research, we consider the pricing of American options when the underlying asset is governed by the Markovian regime-switching process. We assume that the price dynamics depend on the economy, the state of which transits based on a discrete-time Markov chain. The underlying economy cannot be known directly but can be partially observed by receiving a signal stochastically related to the real state of the economy. The pricing procedure and optimal stopping problem are formulated using a partially observable Markov decision process. Some structural properties of the American option prices are derived under certain assumptions. These properties establish the existence of a monotonic optimal exercise policy with respect to the holding time, asset price, and economic conditions.

Keywords Decision policy · Hidden Markov chain · Optimal strategy · Partially observable Markov decision process · Totally positive of order 2

MSC 2020 62M05

6.1 Introduction

Pricing of American options is a much harder problem than its European counterpart due to its feature of early-exercising. Exact analytical formulas do not exist even under the classical Black–Scholes model. At each time epoch and for each underlying process state, we can compare the value of early exercising to the value of holding

L. Jin
Department of Informatics, University of Electro-Communications, Tokyo, Japan
e-mail: jin.lu@uec.ac.jp

M. Dimitrov · Y. Ni (✉)
Division of Mathematics and Physics, Mälardalen University, Västerås, Sweden
e-mail: ying.ni@mdu.se

M. Dimitrov
e-mail: marko.dimitrov@mdu.se

A. Malyarenko et al. (eds.), *Stochastic Processes, Statistical Methods, and Engineering Mathematics*, Springer Proceedings in Mathematics & Statistics 408,
https://doi.org/10.1007/978-3-031-17820-7_6

the option to decide on an optimal strategy. The option's values are then determined by optimal strategies. In this way, we can also split the range of the prices of the underlying asset into a region for exercising and a region for holding.

Consider, for example, the standard American put options under the Black-Scholes model, at each time epoch t before maturity, there exists a unique critical underlying asset price s^* such that it is optimal to exercise the option for every underlying asset price below this critical price. The function $s^* = s^*(t)$ is called the *optimal exercise boundary*. This boundary is in general unknown and needs to be determined simultaneously with the solution to the pricing problem. Knowing the structural properties of the optimal exercising strategies and the optimal exercising boundary is very useful for American option pricing. For example, numerical methods like Monte-Carlo simulation can then be applied to the valuation of the American option.

There have been numerous studies on the analytical properties of the optimal exercise/hold regions and on the optimal exercise boundary for models with a one-dimensional underlying process. Under the simplest binomial tree model, [6] studied the properties of the optimal exercise boundary in the case of constant volatility and later in the case of deterministic volatility in [7]. For a geometric Brownian motion underlying process, [2] proved that the optimal exercising boundary is non-decreasing in the underlying asset price for American put options and non-increasing for American call options with continuous dividend yields. It should also be mentioned that non-standard American options, for example, options with different types of monotone payoff functions, were analyzed in [9, 10]. These monotonicity properties were also presented in other works, see [5] for a complete survey on optimal exercising regions for American options. A more recent reference on American option pricing and the corresponding optimal exercising strategies, under more advanced models, can be found in the comprehensive bibliographic remarks given in books [16, 17].

Our model is a regime-switching model with different volatility regimes. Comparing to the stochastic volatility models where the stochastic volatility is modeled by a mean-reverting diffusion process with an unclear motivation behind, regime-switching models provide a more natural and convenient way to describe the impact of changes in economic conditions on price dynamics. The reference [3] pioneered applications of regime-switching models in economic analysis, in particular, in modeling and explaining business cycles. In addition to Hamilton's research in [3], there are many applications of regime-switching models in option pricing. The reference [8] developed a discrete-time model in which the volatility is stochastic and controlled by a discrete-time Markov chain (and also a continuous-time model). Their discrete-time model can be seen as an extension of the binomial tree model with Markov regime-switching volatility. They concluded that if the underlying Markov chain is stochastically monotone, a European option's price is an increasing function of the current volatility. For a similar extension of the binomial tree model, [1] provided an algorithm for constructing a recombining tree to facilitate the pricing of European and American options. The reference [12] developed a model for two-state European options using a regime-switching model and analyzed options-based trading strategies to hedge against the risk of jumps in the return volatility. Reference [15] considered the pricing of variance swaps modulated by a continuous-time finite-state observable Markov chain.

Some previous researchers have considered a non-standard callable American option with which both the issuer and the holder have the right to terminate/exercise the option. The valuation process as a coupled stochastic game for the optimal stopping problem is then formulated within a Markov decision process. The reference [13] considered this problem under an observable economic condition. Some analytical properties of the optimal stopping rules for the issuer and the holder were derived. In practice, the states of the underlying economy are not directly observable. Reference [14] extended their previous model in [13] for the valuation of callable American options under a partially observable economic condition. They showed that there exists a unique optimal value for the callable option and derived sufficient conditions for establishing structural properties of the optimal investing strategies.

In this research, we consider the pricing and the optimal exercising problems for a standard American option. We formulate the pricing process as a partially observable Markov decision process using the same procedure as in [14]. However, we don't use one of the key assumptions in [14]. More specifically, we don't assume that the random variable of the underlying price relative at time t can be ordered across different economy states with respect to stochastic increasing (see explanation in Sect. 6.2). Our model is novel and can be seen as an extension of the binomial tree model with volatility parameter governed by a discrete-time hidden Markov chain.

The object of interest here lies in the structural properties of the prices of American options and the corresponding optimal exercise boundaries. Our contribution is to provide a set of sufficient conditions on the transition probability matrix of economy evolution and conditional probabilities of observations, to prove analytical properties related to American option prices and to illustrate these properties with numerical examples. The forms of the corresponding conditions derived in [14] are given as implicit functions. Our conditions are more explicit and clarify the relationship between the properties of American options and economic conditions and observations.

The paper is organized as follows. Section 6.2 presents the model of this research. Section 6.3 formulates the pricing procedure of American options using a partially observable Markov decision process. Section 6.4 studies the properties of American option prices and optimal exercise boundaries. In Sect. 6.5, these properties are illustrated via numerical examples. Finally, concluding remarks and future research are addressed in Sect. 6.6.

6.2 A Markovian Regime-Switching Model

We consider the pricing of American options when the underlying asset is governed by a Markovian regime-switching process under a variable economic situation Z. We assume that the financial market under consideration is frictionless and arbitrage-free.

Let T be the maturity time of the option, the interval $[0, T]$ is divided into several equidistant time periods with a length h. We assume that h is chosen in such a way that T/h is an integer. A time epoch t takes a discrete value from $\{0, 1, ..., T/h\}$. Suppose that Z_t takes a value in a finite state space: $\mathbb{Z} = \{1, 2, ..., n\}$. The numbers are ordered to reflect the progress of the economic situation, with 1 referring to the worst situation and n the best. At the beginning of each time period, the economic situation Z changes based on a known transition law: $\mathbf{P} = [p_{ij}]_{i,j \in \mathbb{Z}}$, of which p_{ij} is the probability that the economic situation transits to level j from level i.

We present a model for the price dynamics of the underlying risky assets under the partially observable Markovian regime-switching process. Without loss of generality, we suppose that the underlying asset is a dividend-paying equity stock with continuously compounded dividend yield $\delta > 0$. Denote S_t as the price of this underlying asset at a discrete-time t, and define the *price relative* dynamics as $X_t = \frac{S_t}{S_{t-1}}$. Assume that X_t depends on the situation of the economy Z at time t, more specifically X^j, which is the price dynamics given $Z = j$, follows a probability distribution

$$P(X^j = x_j) = \begin{cases} q_j & x_j = u_j \\ 1 - q_j & x_j = d_j = 1/u_j \end{cases} \tag{6.1}$$

with $q_j = \frac{e^{(r-\delta)h} - d_j}{u_j - d_j}$, $u_j = e^{\sigma_j \sqrt{h}}$, where $\sigma_j > \sigma_{j'}$ for $j < j'$. Here $r > 0$ is the continuously compounded risk-free rate and to exclude arbitrage opportunities assume

$$d_j < e^{(r-\delta)h} < u_j, \quad j \in \mathbb{Z}. \tag{6.2}$$

Note that our economy is classified according to the volatility σ_i, with high volatility referring to the bad economy and low volatility to good economy. Indeed, low volatility indicates usually a stable market.

By the risk-neutral option valuation theory, the price of a European option is the expected payoff under a risk-neutral probability measure, discounted at the risk-free interest rate. We assume here that a risk-neutral probability measure $\mathbb{Q}$ is chosen by the market and is given to us. All expectations and distributions in this paper are under this probability measure $\mathbb{Q}$. Indeed, the probability distribution (6.1) given above is the well-known risk-neutral probability distribution for a binomial model with a continuous dividend yield.

Important

Our model is not a particular case of the generic model in [14] as it does not satisfy its assumption 3.1 (i). Indeed, Assumption 3.1 (i) in [14] states that X^j is stochastically less than or equal to $X^{j'}$ for two different states if $j \leq j'$. However, in our model, it is trivial to show that $E[X^j] = e^{(r-\delta)h}$ for any state j, and $\mathrm{Var}[X^j] > \mathrm{Var}[X^{j'}]$ for $j < j'$, implying that X^j and $X^{j'}$ cannot be ordered in terms of stochastic increasing.

! Attention

It is never optimal to early exercise an American call if the underlying is a non-dividend-paying stock and $r > 0$, which is a well-known model-free result on American call options. Hence to avoid triviality we assume the dividend yield $\delta > 0$ in all our discussions on American call options below.

At each time period, the economic situation in the spot market cannot be observed directly. However, we are able to receive observation Y, such as economic indicators that provide incomplete information related to the real economic situation Z. Observation Y comes from a finite set, $\mathbb{Y} = \{1, 2, ..., m\}$. Let $\mathbf{\Gamma} = [\gamma_{j\theta}]_{j\in\mathbb{Z},\theta\in\mathbb{Y}}$ be an observed conditional probability matrix that describes the relationships between the economic situation and the observations. Here, $\gamma_{j\theta} = P(Y = \theta|Z = j)$ is the element of $\mathbf{\Gamma}$ in j-th row and θ-th column.

Let $\boldsymbol{\pi} = (\pi_1, ..., \pi_n)$ be a probability vector that expresses the information about the economic situation. Here, $\pi_i = P(Z = i)$, $\sum_{i=1}^{n} \pi_i = 1$. In this research, $\boldsymbol{\pi}$ is called the *economy information vector*. At any time period, the pair $(s, \boldsymbol{\pi})$ is called a *process state*, meaning that the current asset price is s and the information vector which reflects the economic situation is $\boldsymbol{\pi}$.

At the beginning of every time period, the holder can select one of two actions: early exercise or hold. If the holder decides to early exercise, a payoff of $v^e(s_t) = \max\{K - s_t, 0\}$ (reps. $v^e(s_t) = \max\{s_t - K, 0\}$) is received for put (reps. call) option, where K is the strike price and s_t is the underlying asset price at time t.

6.3 Pricing of American Options

At the beginning of every time period, an early-exercising decision is made based on the current process state $(s, \boldsymbol{\pi})$. Under the decision to hold for one more time period, the information vector at the beginning of the next time period is updated to $\mathbf{T}(\boldsymbol{\pi}, \theta)$, given the observation θ with probability $\psi(\theta|\boldsymbol{\pi})$. Here,

$$\psi(\theta|\boldsymbol{\pi}) = \sum_{j=1}^{n} \sum_{i=1}^{n} \pi_i p_{ij} \gamma_{j\theta}, \tag{6.3}$$

and the j-th element of the updated information vector $\mathbf{T}(\boldsymbol{\pi}, \theta)$ is

$$T_j(\boldsymbol{\pi}, \theta) = \frac{\sum_{i=1}^{n} \pi_i p_{ij} \gamma_{j\theta}}{\psi(\theta|\boldsymbol{\pi})}. \tag{6.4}$$

We now formulate the optimal stopping problem using a partially observable Markov decision process. Let N be the number of the remaining time periods to maturity. i.e. $N = T/h$ at the beginning of option transaction, and $N = 0$ at maturity.

Consider an American put option with the current process state $(s, \boldsymbol{\pi})$, strike price K and remaining periods to maturity N. The option price $v_N(s, \boldsymbol{\pi})$, is given by

$$v_N(s, \boldsymbol{\pi}) = \max \begin{cases} \max\{K - s, 0\} = v_N^e(s) \\ \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v_{N-1}\left[s x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta)\right] P(x_j^k) = v_N^h(s, \boldsymbol{\pi}) \end{cases}$$

where $x_j^1 = u_j$, $x_j^2 = d_j$, and $\beta = e^{-rh}$ $(0 < \beta < 1)$ is the discount factor.

Let the quantity $v_N^e(s, \boldsymbol{\pi})$ be the value/payoff if the holder exercises the option at the beginning of the current time period, and $v_N^h(s, \boldsymbol{\pi})$ be the value if the holder decides to hold and follow the optimal strategy in the remaining periods. Note that $v_N^h(s, \boldsymbol{\pi})$ is valued as the discounted expected payoff for one time period, i.e. in a similar way to an European option. Since the payoff of early exercise does not depend on the remaining time periods N, we use $v^e(s)$ instead of $v_N^e(s)$ in the following.

When the time period expires, $v_0^h(s, \boldsymbol{\pi}) = 0$, hence

$$v_0(s, \boldsymbol{\pi}) = \max\{v^e(s), v_0^h(s, \boldsymbol{\pi})\} = v^e(s). \tag{6.5}$$

6.4 Some Properties for Optimal Strategy

In this section, the structural properties of the optimal total payoff function are derived.

6.4.1 Preliminaries

First define a *totally positive property of order 2* (see [4]), abbreviated as TP_2, which is used in this research.

Definition 1 If for two vectors $\mathbf{x} = (x_1, x_2, ..., x_n)$, and $\mathbf{y} = (y_1, y_2, ..., y_n)$

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \geq 0, \quad 1 \leq i < j \leq n,$$

holds, it is said that $\mathbf{y}$ dominates $\mathbf{x}$ in the sense of totally positive ordering of order 2, denoted by $\mathbf{x} \overset{TP_2}{\prec} \mathbf{y}$.

Definition 2 Let $\mathbf{X} = [x_{ij}]_{ij}$ be an $n \times m$ matrix for which $\det(\mathbf{B}) \geq 0$ for every submatrix $\mathbf{B} = [x_{i_k j_l}]_{kl}$ of dimensions 2×2 where $1 \leq i_1 < i_2 \leq n$, $1 \leq j_1 < j_2 \leq$

m. Matrix $\mathbf{X}$ is said to have a property of totally positive of order two, denoted by $\mathbf{X} \in \mathrm{TP}_2$.

In this research, the following important conditions are assumed. Note that we consider functions as increasing or decreasing in the weak sense throughout this paper.

(A-1) The transition probability matrix for economic situation $\mathbf{P}$ has a TP_2 property.
(A-2) The conditional probability matrix for observation $\boldsymbol{\Gamma}$ has a TP_2 property.

Assumption (A-1) asserts that, as the economy gets better, it tends to move to a more progressing situation in the next time period. Assumption (A-2) implies that a better economic situation gives rise to higher output levels for the observations probabilistically.

6.4.2 Lemmata

From the assumptions (A-1) and (A-2), we obtain the following lemmata and properties to establish the structural properties of American option prices. We begin our preparation by citing two results on TP_2 ordering from [4, 11]. These are fundamental properties of TP_2 vectors and matrices.

Lemma 1 ([4]) *If $f(i)$ is a decreasing/increasing function of i, then $\sum_{i=1}^{n} \pi_i^1 f(i)$ decreases/increases in $\boldsymbol{\pi}$ in the sense of TP_2 ordering.*

Lemma 2 ([11]) *If $\mathbf{P}$ is a $(k_P \times k)$ TP_2 matrix, and $\mathbf{Q}$ is a $(k \times k_Q)$ TP_2 matrix, then $\mathbf{PQ}$ is a $(k_P \times k_Q)$ TP_2 matrix.*

Lemma 3 *Under assumptions* (A-1) *and* (A-2), $\mathbf{T}(\boldsymbol{\pi}, \theta_1) \overset{TP_2}{\prec} \mathbf{T}(\boldsymbol{\pi}, \theta_2)$ *holds for any $\boldsymbol{\pi}$ and $1 \le \theta_1 < \theta_2 \le m$.*

Lemma 4 *Under assumptions* (A-1) *and* (A-2), $\mathbf{T}(\boldsymbol{\pi}^{\mathbf{1}}, \theta) \overset{TP_2}{\prec} \mathbf{T}(\boldsymbol{\pi}^{\mathbf{2}}, \theta)$ *holds for any θ and $\boldsymbol{\pi}^{\mathbf{1}} \overset{TP_2}{\prec} \boldsymbol{\pi}^{\mathbf{2}}$.*

Lemmas 3 and 4 establish the monotonicity on θ and $\boldsymbol{\pi}$ of $\mathbf{T}(\boldsymbol{\pi}, \theta)$ in the sense of TP_2, respectively. Here, $\mathbf{T}(\boldsymbol{\pi}, \theta)$ is the updated information vector of the next time period given the current information vector $\boldsymbol{\pi}$. We omit the proofs for Lemmas 3 and 4 since they can be obtained by developing Eq. (6.4) from assumptions (A-1) and (A-2).

Lemma 5 *The following inequality holds for $j < j'$:*

$$\sum_{k=1}^{2} \max(K - s x_j^k, 0) P(x_j^k) \ge \sum_{k=1}^{2} \max(K - s x_{j'}^k, 0) P(x_{j'}^k).$$

Proof Recall that

$$P(x_j^1) = P(sx_j = su_j) = q_j, \quad P(x_j^2) = P(sx_j = sd_j) = 1 - q_j,$$
$$P(x_{j'}^1) = P(sx_{j'} = su_{j'}) = q_{j'}, \quad P(x_{j'}^2) = P(sx_{j'} = sd_{j'}) = 1 - q_{j'}.$$

It is straightforward to show that $E[sx_j] = E[sx_{j'}] = e^{(r-\delta)h}s$. Hence, the following is true

$$q_j su_j + (1 - q_j)sd_j = q_{j'}su_{j'} + (1 - q_{j'})sd_{j'}, \tag{6.6}$$

and Eq. (6.6) is the starting point of our proof. Note that $s > 0$ and $sd_j < sd_{j'} < su_{j'} < su_j$. Depending on the value of strike price K, we have one of the following cases: **(i)** $K < sd_j$, **(ii)** $sd_j \le K < sd_{j'}$, **(iii)** $sd_{j'} \le K < su_{j'}$; **(iv)** $su_{j'} \le K < su_j$ and **(v)** $K \ge su_j$.

We prove the lemma for each of the above cases. For case **(i)**

$$\sum_{k=1}^{2} \max(K - sx_j^k, 0)P(x_j^k) = \sum_{k=1}^{2} \max(K - sx_{j'}^k, 0)P(x_{j'}^k) = 0.$$

The lemma holds with equality. For case **(ii)**, the lemma holds due to the facts that

$$\sum_{k=1}^{2} \max(K - sx_j^k, 0)P(x_j^k) > 0, \quad \sum_{k=1}^{2} \max(K - sx_{j'}^k, 0)P(x_{j'}^k) = 0.$$

For case **(iii)**, the lemma reduces to $(1 - q_j)(K - sd_j) \ge (1 - q_{j'})(K - sd_{j'})$. In Eq. (6.6), we replace su_j in the left hand side and $su_{j'}$ by the strike price K in the right hand side. Since $su_j > su_{j'}$ and $q_j, q_{j'} \in (0, 1)$, by Eq. (6.6) we obtain $q_j K + (1 - q_j)sd_j < q_{j'}K + (1 - q_{j'})sd_{j'}$, which leads to $(1 - q_j)(K - sd_j) > (1 - q_{j'})(K - sd_{j'})$. Then, the proof for case **(iii)** is complete. Consider case **(iv)**. The lemma now reduces to $(1 - q_j)(K - sd_j) \ge q_{j'}(K - su_{j'}) + (1 - q_{j'})(K - sd_{j'})$. Since $K < su_j$, we have $(1 - q_j)sd_j < q_{j'}su_{j'} + (1 - q_{j'})sd_{j'} - q_j K$ from Eq. (6.6), and the above can be written as $(1 - q_j)(K - sd_j) > q_{j'}(K - su_{j'}) + (1 - q_{j'})(K - sd_{j'})$. The proof for this case is complete. Finally for case **(v)**, the lemma in this case is $q_j(K - su_j) + (1 - q_j)(K - sd_j) \ge q_{j'}(K - su_{j'}) + (1 - q_{j'})(K - sd_{j'})$ and it holds with equality from Eq. (6.6). □

Lemma 6 *The following is true for for* $j < j'$:

$$\sum_{k=1}^{2} \max(sx_j^k - K, 0)P(x_j^k) \ge \sum_{k=1}^{2} \max(sx_{j'}^k - K, 0)P(x_{j'}^k).$$

Proof Let $f_1 = \sum_{k=1}^{2} \max(sx_j^k - K, 0)P(x_j^k), \quad f_2 = \sum_{k=1}^{2} \max(K - sx_j^k, 0)P(x_j^k)$
Note that

$$f_1 - f_2 = \sum_{k=1}^{2}[\max(sx_j^k - K, 0) - \max(K - sx_j^k, 0)]P(x_j^k)$$
$$= \sum_{k=1}^{2}(sx_j^k - K)P(x_j^k) = e^{(r-\delta)h}s - K$$

Hence $f_1 = f_2 + e^{(r-\delta)h}s - K$. Now it is obvious that f_1 and f_2 have the same behavior with respect to the economy state, which completes the proof. □

Lemma 7 *For both American put and call options, for $\boldsymbol{\pi}^{\mathbf{1}} \overset{TP_2}{\prec} \boldsymbol{\pi}^{\mathbf{2}}$, the inequality $v_1^h(s, \boldsymbol{\pi}^{\mathbf{1}}) \geq v_1^h(s, \boldsymbol{\pi}^{\mathbf{2}})$ holds.*

Proof We prove the lemma for American put options. From Eq. (6.3),

$$v_1^h(s, \boldsymbol{\pi}^1) = \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}^{\mathbf{1}}) \sum_{k=1}^{2} v_0\left[sx_j^k, \mathbf{T}(\boldsymbol{\pi}^{\mathbf{1}}, \theta)\right] P(x_j^k)$$
$$= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}^{\mathbf{1}}) \sum_{k=1}^{2} v^e(sx_j^k)P(x_j^k) = \beta \sum_{j=1}^{n}\sum_{i=1}^{n} \pi_i^1 p_{ij} \sum_{k=1}^{2} \max(sx_j^k - K, 0)P(x_j^k)$$

Let $\boldsymbol{\pi}\mathbf{P}$ be a vector with $\sum_{i=1}^{n} \pi_i p_{ij}$ as the j-th element, then $\boldsymbol{\pi}^{\mathbf{1}}\mathbf{P} \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}^{\mathbf{2}}\mathbf{P}$ for $\boldsymbol{\pi}^{\mathbf{1}} \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}^{\mathbf{2}}$ from Assumption (A-1) and Definition 1. Therefore, we obtain

$$v_1^h(s, \boldsymbol{\pi}^1) = \beta \sum_{j=1}^{n}\sum_{i=1}^{n} \pi_i^1 p_{ij} \sum_{k=1}^{2} \max(sx_j^k - K, 0)P(x_j^k)$$
$$\geq \beta \sum_{j=1}^{n}\sum_{i=1}^{n} \pi_i^2 p_{ij} \sum_{k=1}^{2} \max(sx_j^k - K, 0)P(x_j^k) = v_1^h(s, \boldsymbol{\pi}^2)$$

from Lemmata 1 and 5. The case for American call can be proved similarly from Lemmata 1 and 6. □

6.4.3 Properties

From the above assumptions and lemmata, we provide some properties which are important for investigating the optimal strategies for an American option. A decision of the option buyer is made at the beginning of every time period. This discretization scheme of decision making enables us to use induction on the steps of option price iteration as a proof technique.

At first, we obtain some properties on the value function for holding $v_N^h(s, \boldsymbol{\pi})$.

Proposition 1 *For a put (call) option,* $v_N^h(s, \boldsymbol{\pi})$ *is decreasing (increasing) in* s *for any* N *and* $\boldsymbol{\pi}$ *under the assumptions* (A-1) *and* (A-2).

Proof We focus on American put options first and prove the claim by mathematical induction. For $N = 1$ the following holds for $s < s'$ using Eq. (6.5):

$$
\begin{aligned}
v_1^h(s, \boldsymbol{\pi}) &= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v_0 \left[s x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta) \right] P(x_j^k) \\
&= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v^e(s x_j^k) P(x_j^k) \geq \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v^e(s' x_j^k) P(x_j^k) = v_1^h(s', \boldsymbol{\pi}).
\end{aligned}
$$

For $N = n - 1$, assume that $v_{n-1}^h(s, \boldsymbol{\pi})$ is decreasing in s. Next, prove the monotonicity of s for $v_n^h(s, \boldsymbol{\pi})$. Since for $N = n$ it follows that

$$
\begin{aligned}
v_n^h(s, \boldsymbol{\pi}) &= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v_{n-1} \left[s x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta) \right] P(x_j^k) \\
&= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} \max \left\{ v^e(s x_j^k), v_{n-1}^h \left[s x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta) \right] \right\} P(x_j^k) \\
&\geq \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} \max \left\{ v^e(s' x_j^k), v_{n-1}^h \left[s' x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta) \right] \right\} P(x_j^k) = v_n^h(s', \boldsymbol{\pi})
\end{aligned}
$$

holds for $s < s'$ using the inductive hypothesis of $N = n - 1$, which proves the claim.

The increasing of holding value function for American call can be derived similarly. □

Proposition 2 *For both put and call options,* $v_N^h(s, \boldsymbol{\pi})$ *is increasing in remaining time periods* N *for any* s *and* $\boldsymbol{\pi}$ *under the assumptions* (A-1) *and* (A-2).

Proof Prove the proposition using mathematical induction for the case of American put. For $N = 1$, it is obvious that $v_1^h(s, \boldsymbol{\pi}) \geq v_0^h(s, \boldsymbol{\pi})$. For $N = n - 1$, assume that $v_{n-1}^h(s, \boldsymbol{\pi}) \geq v_{n-2}^h(s, \boldsymbol{\pi})$, then for $N = n$,

$$\begin{aligned}
v_n^h(s,\boldsymbol{\pi}) &= \beta\sum_{\theta=1}^{m}\psi(\theta|\boldsymbol{\pi})\sum_{k=1}^{2}v_{n-1}\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi},\theta)\right]P(x_j^k)\\
&= \beta\sum_{\theta=1}^{m}\psi(\theta|\boldsymbol{\pi})\sum_{k=1}^{2}\max\left\{v^e(sx_j^k),v_{n-1}^h\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi},\theta)\right]\right\}P(x_j^k)\\
&\geq \beta\sum_{\theta=1}^{m}\psi(\theta|\boldsymbol{\pi})\sum_{k=1}^{2}\max\left\{v^e(sx_j^k),v_{n-2}^h\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi},\theta)\right]\right\}P(x_j^k)\\
&= \beta\sum_{\theta=1}^{m}\psi(\theta|\boldsymbol{\pi})\sum_{k=1}^{2}v_{n-2}\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi},\theta)\right]P(x_j^k)=v_{n-1}^h(s,\boldsymbol{\pi})
\end{aligned}$$

from inductive hypothesis of $N=n-1$. Therefore, Proposition 2 holds true, and we obtain the same result for American call. □

Proposition 3 *For both put and call options, $v_N^h(s,\boldsymbol{\pi})$ is decreasing in $\boldsymbol{\pi}$ in the sense of TP_2 for any N and s under the assumptions (A-1) and (A-2).*

Proof We consider American put first. Since $v_0^h(s,\boldsymbol{\pi})=0$ for every s and $\boldsymbol{\pi}$, we have the fact that

$$v_0^h(s,\boldsymbol{\pi}^1)=v_0^h(s,\boldsymbol{\pi}^2),$$

and

$$v_1^h(s,\boldsymbol{\pi}^1)\geq v_1^h(s,\boldsymbol{\pi}^2)$$

holds for $\boldsymbol{\pi}^1\overset{\mathrm{TP}_2}{\prec}\boldsymbol{\pi}^2$ from Lemma 7.

Next, assume that

$$v_{n-1}^h(s,\boldsymbol{\pi}^1)\geq v_{n-1}^h(s,\boldsymbol{\pi}^2)\quad\text{for}\quad\boldsymbol{\pi}^1\overset{\mathrm{TP}_2}{\prec}\boldsymbol{\pi}^2\tag{6.7}$$

and prove that

$$v_n^h(s,\boldsymbol{\pi}^1)\geq v_n^h(s,\boldsymbol{\pi}^2)\quad\text{for}\quad\boldsymbol{\pi}^1\overset{\mathrm{TP}_2}{\prec}\boldsymbol{\pi}^2\tag{6.8}$$

holds for $N=n$. We focus on $\sum_{k=1}^{2}v_{n-1}[sx_j^k,\mathbf{T}(\boldsymbol{\pi}^1,\theta)]P(x_j^k)$ first.

$$\begin{aligned}
&\sum_{k=1}^{2}v_{n-1}\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi}^1,\theta)\right]P(x_j^k)\\
&=\sum_{k=1}^{2}\max\left\{v^e(sx_j^k),v_{n-1}^h\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi}^1,\theta)\right]\right\}P(x_j^k)\\
&\geq\sum_{k=1}^{2}\max\left\{v^e(sx_j^k),v_{n-1}^h\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi}^1,\theta')\right]\right\}P(x_j^k)\\
&=\sum_{k=1}^{2}v_{n-1}\left[sx_j^k,\mathbf{T}(\boldsymbol{\pi}^1,\theta')\right]P(x_j^k)
\end{aligned}$$

for $\theta < \theta'$ from the induction hypothesis given by Eq. (6.7) and Lemma 3, and this means $\sum_{k=1}^{2} v_{n-1}\left[sx_j^k, \mathbf{T}(\boldsymbol{\pi}^{\mathbf{1}}, \theta)\right] P(x_j^k)$ is a decreasing function of θ. Similarly,

$$\sum_{k=1}^{2} v_{n-1}\left[sx_j^k, \mathbf{T}(\boldsymbol{\pi}^{\mathbf{1}}, \theta)\right] P(x_j^k) \geq \sum_{k=1}^{2} v_{n-1}\left[sx_j^k, \mathbf{T}(\boldsymbol{\pi}^{\mathbf{2}}, \theta)\right] P(x_j^k) \tag{6.9}$$

Next, look at $\boldsymbol{\psi}(\cdot|\boldsymbol{\pi}) = (\psi(1|\boldsymbol{\pi}), ..., \psi(m|\boldsymbol{\pi}))$. Since

$$\boldsymbol{\psi}(\cdot|\boldsymbol{\pi}^{\mathbf{1}}) \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\psi}(\cdot|\boldsymbol{\pi}^{\mathbf{2}}) \tag{6.10}$$

for $\boldsymbol{\pi}^{\mathbf{1}} \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}^{\mathbf{2}}$ under assumptions (A-1) and (A-2) from Lemma 2. From Eqs. (6.9) and (6.10), the following holds

$$\begin{aligned} v_n^h(s, \boldsymbol{\pi}^{\mathbf{1}}) &= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}^{\mathbf{1}}) \sum_{k=1}^{2} v_{n-1}(sx_j^k, \mathbf{T}(\boldsymbol{\pi}^{\mathbf{1}}, \theta)) P(x_j^k) \\ &\geq \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}^{\mathbf{2}}) \sum_{k=1}^{2} v_{n-1}(sx_j^k, \mathbf{T}(\boldsymbol{\pi}^{\mathbf{1}}, \theta)) P(x_j^k) \\ &\geq \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}^{\mathbf{2}}) \sum_{k=1}^{2} v_{n-1}(sx_j^k, \mathbf{T}(\boldsymbol{\pi}^{\mathbf{2}}, \theta)) P(x_j^k) = v_n^h(s, \boldsymbol{\pi}^{\mathbf{2}}) \end{aligned}$$

on the basis of Lemmata 1 and 4. This establishes Eq. (6.8).

This property for American call can be derived in the same way. □

Since the value of early exercise is given by $v^e(s) = \max\{K - s, 0\}$ ($v^e(s) = \max\{s - K, 0\}$) which is a decreasing (increasing) function of s, we can obtain the following properties of the functions for American put/call option price from the above properties.

Proposition 4 *Under the assumptions* (A-1) *and* (A-2)*,* $v_N(s, \boldsymbol{\pi})$ *is monotonically decreasing (increasing) in* s *for any* $\boldsymbol{\pi}$ *for American put (call) option.*

Proposition 5 $v_N(s, \boldsymbol{\pi})$ *is increasing in remaining time periods* N *for any* s *and* $\boldsymbol{\pi}$ *under the assumptions* (A-1) *and* (A-2)*.*

Proposition 6 *Under the assumptions* (A-1) *and* (A-2)*,* $v_N(s, \boldsymbol{\pi})$ *is monotonically decreasing in* $\boldsymbol{\pi}$ *(in the sense of* TP_2 *ordering) for any* s *for both American put and call options.*

Propositions 4, 5, and 6 provide a set of sufficient conditions under which the American option price is monotonic in N, s and $\boldsymbol{\pi}$. Note that in this research we investigate the strategy in the sense of TP_2 ordering of $\boldsymbol{\pi}$. This means that the price of the American option is monotonic in the remaining time, asset price and the progression of the economy.

To explore the optimal investment strategy for buyers, we also need to investigate the relationship between the value functions under holding and early exercising decisions.

Define the holding value premium $L_N(s, \boldsymbol{\pi}) = \max\left\{0, v_N^h(s, \boldsymbol{\pi}) - v^e(s)\right\}$.

We study the properties of $L_N(s, \boldsymbol{\pi})$ in N and $\boldsymbol{\pi}$ for American options.

Proposition 7 *For a put (call) option, (i) $v_N^h(s, \boldsymbol{\pi})$ is a convex function of s, (ii) the decreasing (increasing) rate of $v_N^h(s, \boldsymbol{\pi})$ in s is less than 1 for any $\boldsymbol{\pi}$ under the assumptions* (A-1) *and* (A-2).

Proof First, prove the convexity of $v_N^h(s, \boldsymbol{\pi})$ in s for any given $\boldsymbol{\pi}$ inductively. For $N = 0$, $v_0^h(s, \boldsymbol{\pi}) = 0$. For $N = 1$, the following is true

$$v_1^h(s, \boldsymbol{\pi}) = \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v_0\left[sx_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta)\right] P(x_j^k)$$

$$= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v^e(sx_j^k) P(x_j^k) = \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} \max\left\{K - sx_j^k, 0\right\} P(x_j^k)$$

is a convex function of s.

Next, assume that $v_{n-1}^h(s, \boldsymbol{\pi})$ is a convex function of s for $N = n - 1$, then

$$\lambda v_{n-1}^h(s_1, \boldsymbol{\pi}) + (1 - \lambda) v_{n-1}^h(s_2, \boldsymbol{\pi}) \geq v_{n-1}^h\left[\lambda s_1 + (1 - \lambda)s_2, \boldsymbol{\pi}\right]$$

for $0 < \lambda < 1$ and $s_1 < s_2$ from Jensen's inequality. Since $v_{n-1}(s, \boldsymbol{\pi})$, which is a convex linear combination of $v_{n-1}^h(s, \boldsymbol{\pi})$ and $v^e(s) = \max\{K - s, 0\}$, is a convex function of s, then

$$\lambda v_{n-1}(s_1, \boldsymbol{\pi}) + (1 - \lambda) v_{n-1}(s_2, \boldsymbol{\pi}) \geq v_{n-1}\left[\lambda s_1 + (1 - \lambda)s_2, \boldsymbol{\pi}\right] \tag{6.11}$$

for $0 < \lambda < 1$ and $s_1 < s_2$.

For $N = n$, it follows that

$$\lambda v_n^h(s_1, \boldsymbol{\pi}) + (1 - \lambda) v_n^h(s_2, \boldsymbol{\pi})$$

$$= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} \left\{\lambda v_{n-1}\left[s_1 x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta)\right] + (1 - \lambda) v_{n-1}\left[s_2 x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta)\right]\right\} P(x_j^k)$$

$$\geq \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} \left\{v_{n-1}\left[(\lambda s_1 + (1 - \lambda)s_2) x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta)\right]\right\} P(x_j^k)$$

$$= v_n^h\left[(\lambda s_1 + (1 - \lambda)s_2) x_j^k, \boldsymbol{\pi}\right]$$

from the inductive hypothesis of the convexity of $v_{n-1}(s, \boldsymbol{\pi})$ given by Eq. (6.11) for $N = n - 1$.

Next, investigate the decreasing rate of $v_{n-1}^h(s, \boldsymbol{\pi})$ in s. For $N = 0$,

$$v_0^h(s_1, \boldsymbol{\pi}) - v_0^h(s_2, \boldsymbol{\pi}) = 0 \le s_2 - s_1$$

for $s_1 < s_2$. Assume $v_{n-1}^h(s_1, \boldsymbol{\pi}) - v_{n-1}^h(s_2, \boldsymbol{\pi}) \le s_2 - s_1$ for $N = n - 1$. Since $v_{n-1}(s_1, \boldsymbol{\pi}) - v_{n-1}(s_2, \boldsymbol{\pi}) = \max\left\{v_{n-1}^h(s_1, \boldsymbol{\pi}), v^e(s_1)\right\} - \max\left\{v_{n-1}^h(s_2, \boldsymbol{\pi}), v^e(s_2)\right\} \le s_2 - s_1$, then,

$$\begin{aligned}
&v_n^h(s_1, \boldsymbol{\pi}) - v_n^h(s_2, \boldsymbol{\pi}) \\
&= \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} \left\{v_{n-1}\left[s_1 x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta)\right] - v_{n-1}\left[s_2 x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta)\right]\right\} P(x_j^k) \\
&\le \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} (s_2 - s_1)\, x_j^k P(x_j^k) \\
&= e^{-rh} e^{(r-\delta)h} (s_2 - s_1) = e^{-\delta h} (s_2 - s_1) \le s_2 - s_1,
\end{aligned}$$

since $0 < e^{-\delta h} < 1$. Hence the decreasing rate of $v_n^h(s, \boldsymbol{\pi})$ is thus less than 1.

This result can be proven for a call option. □

Proposition 8 *For an American put or call option, $L_N(s, \boldsymbol{\pi})$ is increasing in N for any s and $\boldsymbol{\pi}$ under the assumptions* (A-1) *and* (A-2).

Proposition 9 *For an American put or call option, $L_N(s, \boldsymbol{\pi})$ is decreasing in $\boldsymbol{\pi}$ in the sense of TP_2 ordering for any N and s under the assumptions* (A-1) *and* (A-2).

Propositions 8 and 9 follow directly from the fact that $v^e(s)$ is constant in both N and $\boldsymbol{\pi}$.

6.4.4 Optimal Strategy

Based on the properties obtained in Sect. 6.4.3, we study the structural properties of the optimal strategy for an American option. Define the stopping region and the holding region for any N as follows:

- Stopping region for early exercise

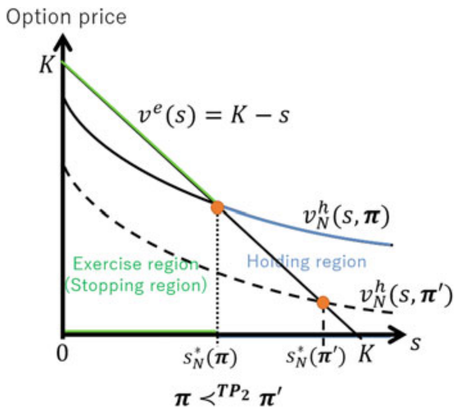
$$D_N^e = \left\{(s, \boldsymbol{\pi}) \mid v_N^h(s, \boldsymbol{\pi}) < v^e(s)\right\} = \left\{(s, \boldsymbol{\pi}) \mid v_N(s, \boldsymbol{\pi}) = v^e(s)\right\}$$

- Holding region

$$D_N^h = \left\{(s, \boldsymbol{\pi}) \mid v_N^h(s, \boldsymbol{\pi}) > v^e(s)\right\} = \left\{(s, \boldsymbol{\pi}) \mid v_N(s, \boldsymbol{\pi}) = v_N^h(s, \boldsymbol{\pi})\right\}.$$

We consider first American put options. Figure 6.1 plots the values for holding and for early exercising as functions of s. From Propositions 1 and 7, we know that $v_N^h(s, \boldsymbol{\pi})$ is a convex and decreasing function of s. Moreover the decreasing rate of

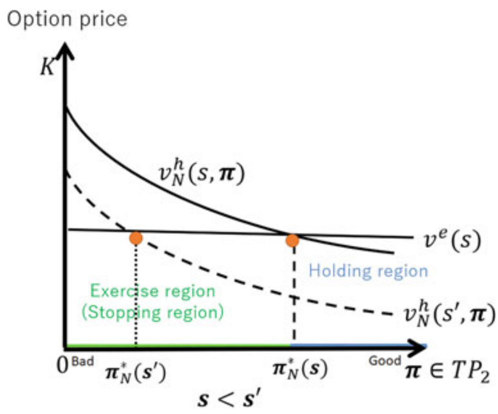
Fig. 6.1 Relationship between holding value, exercise value and asset price s for the case of American put option



$v_N^h(s, \boldsymbol{\pi})$ is thus less than 1. For a put option, the decreasing rate of $v_N^e(s)$ is -1 for $s \in [0, K)$ and 0 for $s \in [K, \infty)$. Consequently, there is at most one threshold $s_N^*(\boldsymbol{\pi})$ for $\boldsymbol{\pi}$ when the remaining number of periods is N. As shown in Fig. 6.1, the thresholds separate the space of s into two regions: stopping (early exercise) region and holding region. Furthermore, $s_N^*(\boldsymbol{\pi})$ increases with $\boldsymbol{\pi}$ since $v_N^h(s, \boldsymbol{\pi})$ decreases with $\boldsymbol{\pi}$ from Proposition 3. As shown in Fig. 6.1, $s_N^*(\boldsymbol{\pi}) < s_N^*(\boldsymbol{\pi}')$ for $\boldsymbol{\pi} \overset{\text{TP}_2}{\prec} \boldsymbol{\pi}'$. This means that it is preferable to hold the option under a worse economy situation i.e. in a more volatile market.

Figure 6.2 plots the values of holding and early exercise as functions of $\boldsymbol{\pi}$ ($\in \text{TP}_2$). From Proposition 3, we know that $v_N^h(s, \boldsymbol{\pi})$ is a decreasing function of $\boldsymbol{\pi}$ in the sense of TP_2, and $v^e(s)$ is constant for any $\boldsymbol{\pi}$. Therefore, there exists at most one threshold $\boldsymbol{\pi}^*(s)$ (as shown in Fig. 6.2). From Proposition 1, $\boldsymbol{\pi}^*(s)$ decreases with s which implies that it is better to exercise earlier if the asset price is lower.

Fig. 6.2 Relationship between holding value, exercise value and economy situation $\boldsymbol{\pi}^*(s)$ for the case of American put option



Similar properties for American call options can also be derived. This means that the information space of $(s, \boldsymbol{\pi})$ is divided into two regions for both American put and call options. We illustrate these regions using numerical examples in Sect. 6.5.

6.5 Numerical Examples

In this section, numerical examples are introduced to show the monotonicity of the mentioned functions. A tree model was used for computation of the American put option and American call option with dividend yield for three states.

6.5.1 Model Implementation

Consider a three-state economy with three pieces of information. Let $\mathbf{P}$ and $\boldsymbol{\Gamma}$ be a transition probability matrix and a conditional probability matrix, respectively, defined by $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$, $\boldsymbol{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$, and the economy information vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, 1 - \pi_1 - \pi_2)$. Assume that $\mathbf{P}, \boldsymbol{\Gamma} \in \mathrm{TP}_2$.

Let S_0 be the initial asset price, and K strike price. Consider a tree of M steps, with expiration time T, volatility $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, interest rate r, and economy information vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$. The time duration of a step is $h = T/M$.

The asset price after n steps on the tree depends on the state, and the number of times the price went up, and down. Denote with n_i the number of times the price increased in state i, $i = 1, 2, 3$, and with m_j the number of times the price decreased in state j, $j = 1, 2, 3$. Hence, $s_n(n_1, n_2, n_3, m_1, m_2, m_3) = S_0 u_1^{n_1} u_2^{n_2} u_3^{n_3} d_1^{m_1} d_2^{m_2} d_3^{m_3}$ where $n_1 + n_2 + n_3 + m_1 + m_2 + m_3 = n$. Note that $s_N = s_{M-n}$ $(n = M - N)$ where N is the number of the remaining time periods to the maturity.

The pricing of American options is explained in Sect. 6.3. Notice that the information vector $\boldsymbol{\pi}$ also yields a tree. In step n, each node represents a vector $\boldsymbol{\pi}_n^{i,j} = (\pi_{1,n}^{i,j}, \pi_{2,n}^{i,j}, \pi_{3,n}^{i,j})$, $j = 1, 2, ..., 3^{n-1}$, $i = 3j - 2, 3j - 1, 3j$, where $\boldsymbol{\pi}_0^{ij} = \boldsymbol{\pi}$. The index j represents a node in step $n - 1$ to which it is connected, e.g. nodes $\boldsymbol{\pi}_3^{i,3}$, $i = 7, 8, 9$ are connected to node $\boldsymbol{\pi}_2^{3,1}$. Therefore, in step n there is a total of $3^{n-1} \times 3 = 3^n$ nodes. To obtain vectors $\boldsymbol{\pi}_n^{i,j}$, first find 3^{n-1} probability mass functions $\mathbf{h}_n^j$, $j = 1, 2, ..., 3^{n-1}$, from $\mathbf{h}_n^j = (h_{1,n}^j, h_{2,n}^j, h_{3,n}^j) = \boldsymbol{\pi}_{n-1}^{j,k}\mathbf{P}$, where k is the only integer in set $\left\{\frac{j+2}{3}, \frac{j+1}{3}, \frac{j}{3}\right\}$ Then, for $\boldsymbol{\pi}_n^{i,j}$, and $j = 1, 2, ..., 3^{n-1}$, the following is true

$$\boldsymbol{\pi}_n^{3j-2,j} = (\pi_{1,n}^{3j-2,j}, \pi_{2,n}^{3j-2,j}, \pi_{3,n}^{3j-2,j}) = \left(\frac{h_{1,n}^j \gamma_{1,1}}{f_n^{1,j}}, \frac{h_{2,n}^j \gamma_{2,1}}{f_n^{1,j}}, \frac{h_{3,n}^j \gamma_{3,1}}{f_n^{1,j}}\right)$$

$$\boldsymbol{\pi}_n^{3j-1,j} = (\pi_{1,n}^{3j-1,j}, \pi_{2,n}^{3j-1,j}, \pi_{3,n}^{3j-1,j}) = \left(\frac{h_{1,n}^j \gamma_{1,2}}{f_n^{2,j}}, \frac{h_{2,n}^j \gamma_{2,2}}{f_n^{2,j}}, \frac{h_{3,n}^j \gamma_{3,2}}{f_n^{2,j}}\right)$$

$$\boldsymbol{\pi}_n^{3j,j} = (\pi_{1,n}^{3j,j}, \pi_{2,n}^{3j,j}, \pi_{3,n}^{3j,j}) = \left(\frac{h_{1,n}^j \gamma_{1,3}}{f_n^{3,j}}, \frac{h_{2,n}^j \gamma_{2,3}}{f_n^{3,j}}, \frac{h_{3,n}^j \gamma_{3,3}}{f_n^{3,j}}\right)$$

where $\mathbf{f}_n^j = (f_n^{1,j}, f_n^{2,j}, f_n^{3,j}) = \mathbf{h}_n^j \boldsymbol{\Gamma}$. The American put option pricing starts at each of the final nodes and ends at the tree's first node. In the M^{th} step, the value of the American put option is $v_M(\mathbf{x}_{nm}) = [K - s_M(\mathbf{x}_{nm})]^+ = \max\{K - s_M(\mathbf{x}_{nm}), 0\}$, where $\mathbf{x}_{nm} = (n_1, n_2, n_3, m_1, m_2, m_3)$, and $n_1 + n_2 + n_3 + m_1 + m_2 + m_3 = M$. For step k, each node in the original tree has 3^k option prices for $j = 1, 2, ..., 3^k$,

$$v_k^j(\mathbf{x}_{nm}; \mathbf{h}_k^j) = \max\{[K - s_k(\mathbf{x}_{nm})]^+, \tilde{\mathscr{B}} v_{k+1}\}$$

where $\tilde{\mathscr{B}} v_{k+1}(j, \mathbf{h}_{k+1}^j) = e^{-rh} \sum_{i=1}^3 \sum_{\theta=1}^3 h_{i,k}^j \gamma_{i,\theta} B_{k+1}(j, \mathbf{h}_{k+1}^l, i)$, and for option prices in the next step $B_{k+1}(j, \mathbf{h}_{k+1}^l, i)$, $l = 1, 2, ..., 3^{k+1}$ is given by

$$\begin{aligned}
B_{k+1}(j, \mathbf{h}_{k+1}^l, 1) &= q_1 v_{k+1}\Big(n_1 + 1, n_2, n_3, m_1, m_2, m_3; \mathbf{h}_{k+1}^l\Big) \\
&\quad + (1 - q_1) v_k\Big(n_1, n_2, n_3, m_1 + 1, m_2, m_3; \mathbf{h}_{k+1}^l\Big), \\
B_{k+1}(j, \mathbf{h}_{k+1}^l, 2) &= q_2 v_{k+1}\Big(n_1, n_2 + 1, n_3, m_1, m_2, m_3; \mathbf{h}_{k+1}^l\Big) \\
&\quad + (1 - q_2) v_k\Big(n_1, n_2, n_3, m_1, m_2 + 1, m_3; \mathbf{h}_{k+1}^l\Big), \\
B_{k+1}(j, \mathbf{h}_{k+1}^l, 3) &= q_3 v_{k+1}\Big(n_1, n_2, n_3 + 1, m_1, m_2, m_3; \mathbf{h}_{k+1}^l\Big) \\
&\quad + (1 - q_3) v_k\Big(n_1, n_2, n_3, m_1, m_2, m_3 + 1; \mathbf{h}_{k+1}^l\Big).
\end{aligned}$$

Let S be a set of initial asset prices S_0, and Π a family of sets Π_i, $i \in \mathfrak{I}$, where $\mathfrak{I}$ is an index set, for which if economy information vectors $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \Pi_i$, $i \in \mathfrak{I}$ then $\boldsymbol{\pi}, \boldsymbol{\pi}'$ are TP_2 comparable.

The thresholds in the numerical examples are obtained as follows. First, take a finite subset $S^* \subseteq S$, and a set $TP_2^* \in \Pi$. Then, for a fixed economy information vector $\boldsymbol{\pi} \in TP_2^*$ compute the option price for every $S_0 \in S^*$. Initial asset price $s^* \in S^*$ is the one-threshold that splits the set S^* into a (early) exercise region D_N^e and a hold region D_N^h.

The tree obtained from the information vector has exponential growth. It can be controlled by discretizing the $\boldsymbol{\pi}$ space as follows. First, choose evenly spread a finite number of information vectors from the $\boldsymbol{\pi}$ space. Then, find the closest information vector from the finite set of information vectors to the one acquired in the node and substitute it. In that way the number of different information vectors can be controlled.

6.5.2 Numerical Results

In this subsection, unless it is said otherwise, the parameters used for computation are given in Table 6.1.

The transition probability matrix (TPM) and the probabilistic relation between a signal and a state of the economy, the conditional probability matrix (CPM), are, respectively, given by $\mathbf{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.5 \\ 0.05 & 0.25 & 0.7 \end{bmatrix}$, $\boldsymbol{\Gamma} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.4 & 0.5 \\ 0.05 & 0.4 & 0.55 \end{bmatrix}$. This choice of parameters satisfy assumptions (A–1) and (A–2). It can be seen that both matrices have the property of TP_2.

(i) To show monotonicity of $v_N^h(s, \boldsymbol{\pi})$, $v_N(s, \boldsymbol{\pi})$, and $L_N(s, \pi)$ in $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ (in sense of TP_2 ordering) for American put options and every s and N, a set of all economy information vectors such that

$$\pi_2 = 0.05, \quad \pi_1 = \pi_2 + 0.03 \times i, \quad i = 0, 1, ..., 30, \quad \pi_3 = 1 - \pi_1 - \pi_2, \tag{6.12}$$

denoted by TP_2^*, was used. Note that sequence $\mathrm{TP}_2^* \in \Pi$, is in its reverse order. Indeed, let $\boldsymbol{\pi}_1 = (p_1, p_2, 1 - p_1 - p_2)$ and $\boldsymbol{\pi}_2 = (q_1, q_2, 1 - q_1 - q_2)$ such that $\boldsymbol{\pi}_1 \neq \boldsymbol{\pi}_2$. If $p_1 \neq 0$ and $p_1 = q_1$, then $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are not TP_2 comparable. If $p_1 \geq q_1$ and $p_2 = q_2$, then $\boldsymbol{\pi}_1 \overset{\mathrm{TP}_2}{\preceq} \boldsymbol{\pi}_2$. This claim can be easily derived from Definition 1 so we omit the proof here. By this result, we note also that other information vectors from the above partitioned space cannot be added into this

Table 6.1 General model test parameters

Name	Notation	Parameters
Maturity time	T	8/252
Number of steps	M	4
Time duration of a step	h	2/252
Volatility vector	$\boldsymbol{\sigma}$	(0.5, 0.3, 0.1)
Strike price	K	100
Interest rate	r	0.02
Dividend yield (American call)	δ	0.1
TPM	$\mathbf{P}$	$[p_{ij}]_{i,j=1,2,3}$
CPM	$\boldsymbol{\Gamma}$	$[\gamma_{ij}]_{i,j=1,2,3}$

sequence, as each of them is not comparable to at least one of the information vector in TP_2^*.

Note that in all relevant figures in this section, if the horizontal axis refers to π information vectors, we plot these information vectors in descending order with respect to the TP_2 ordering.

Figures 6.3, 6.4, and 6.5 show that, respectively, $\nu_N^h(s, \boldsymbol{\pi})$, $L_N(s, \boldsymbol{\pi})$, and $\nu_N(s, \boldsymbol{\pi})$ are decreasing in $\boldsymbol{\pi}$ for initial asset price $s = 93.3$ and $N = M = 4$. Figure 6.6 shows that $\nu_N(s, \boldsymbol{\pi})$ is decreasing in $\boldsymbol{\pi}$ for an American call option with dividend yield $\delta = 0.1$ and initial asset price $s = 105$. Note that the no-arbitrage condition (6.2) is satisfied with our choice of parameter values.

(ii) To show monotonicity of $\nu_N(s, \boldsymbol{\pi})$ in the remaining time periods N for every s and $\boldsymbol{\pi}$, a set $N^* = \{3, 6, ..., 90\}$ of remaining time periods N was used. Figure 6.7 shows that $\nu_N(s, \boldsymbol{\pi})$ is increasing in remaining time periods N for $s = 93.3$ and $\boldsymbol{\pi} = (0.92, 0.04, 0.04)$ for an American put option.

(iii) To show monotonicity of $\nu_N(s, \boldsymbol{\pi})$ in s for every $\boldsymbol{\pi}$ and N, a set S^*

$$S^* = \left\{ \left(0.7 + \frac{0.4}{30} \times i \right) \times K : i \in \{0, 1, ..., 30\} \right\}$$

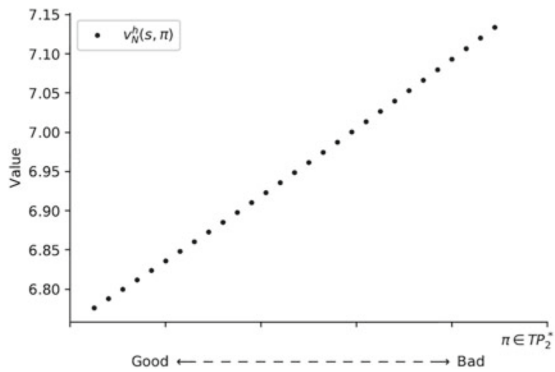


Fig. 6.3 An example of the monotonicity in $\boldsymbol{\pi}$ for the holding value of an American put $\nu_N^h(s, \boldsymbol{\pi})$ with parameters given in Table 6.1 and in (i)

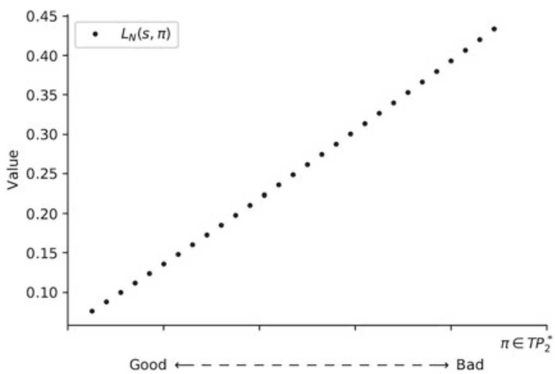


Fig. 6.4 An example of the monotonicity in $\boldsymbol{\pi}$ of holding value premium $L_N(s, \boldsymbol{\pi})$ with parameters given in Table 6.1 and in (i)

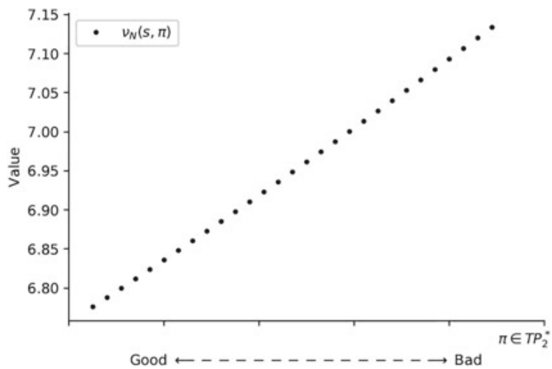


Fig. 6.5 An example of the monotonicity in $\boldsymbol{\pi}$ of the value of an American put option $\nu_N(s, \boldsymbol{\pi})$ with parameters given in Table 6.1 and in (i)

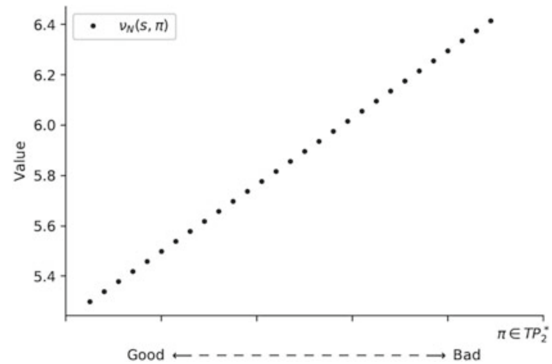


Fig. 6.6 An example of the monotonicity in $\boldsymbol{\pi}$ of the value of an American call option with dividend yield $\nu_N(s, \boldsymbol{\pi})$ with parameters given in Table 6.1 and in (i)

of initial asset prices s was used for an American put option.
Figure 6.8 shows that $\nu_N(s, \boldsymbol{\pi})$ is decreasing in s for $N = M = 4$ and $\boldsymbol{\pi} = (0.92, 0.04, 0.04)$.

(iv) In Sect. 6.4.4, we discussed the existence of one-threshold for the early exercising decisions. To show the exercise and hold regions, as well as the monotonicity of threshold in $\boldsymbol{\pi}$, sets TP_2^* and S^* defined by Eq. (6.12) and

$$S^* = \left\{\left(0.7 + \frac{0.3}{5000} \times i\right) \times K : i \in \{0, 1, ..., 5000\}\right\}$$

for an American put, and by

$$S^* = \left\{\left(1 + \frac{0.3}{5000} \times i\right) \times K : i \in \{0, 1, ..., 5000\}\right\}$$

for an American call option with dividend yield $\delta = 0.1$ were used.
Figures 6.9 and 6.10 show that the threshold is decreasing/increasing in $\boldsymbol{\pi}$ for $N = M = 4$, as well as the exercise and hold regions for the buyer and both American call option with dividend yield and American put, respectively.

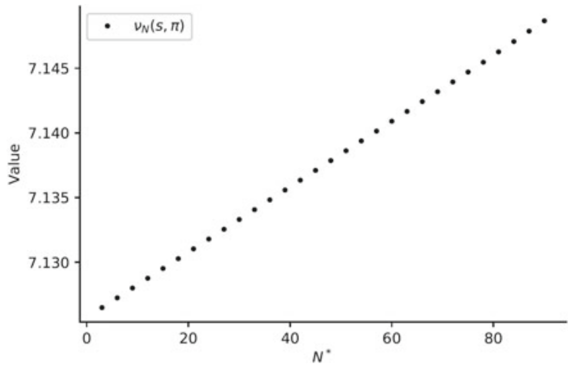


Fig. 6.7 An example of the monotonicity in N of the value of an American put option $v_N(s, \boldsymbol{\pi})$ with parameters given in Table 6.1 and in (ii)

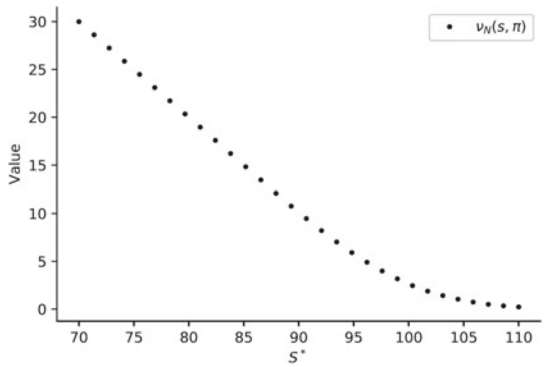


Fig. 6.8 An example of the monotonicity in s of the value of an American put option $v_N(s, \boldsymbol{\pi})$ with parameters given in Table 6.1 and in (iii)

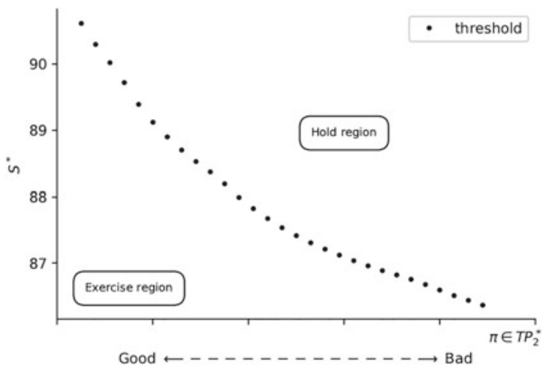


Fig. 6.9 An example of the optimal stopping regions for an American put option and the monotonicity of the threshold in $\boldsymbol{\pi}$ with parameters given in Table 6.1 and in (iv)

Figures 6.11 and 6.12 show the monotonicity of threshold in $\boldsymbol{\pi}$ for different choice of π_2 in Eq. (6.12), $\pi_2 = 0.02$ and $\pi_2 = 0.07$, respectively.

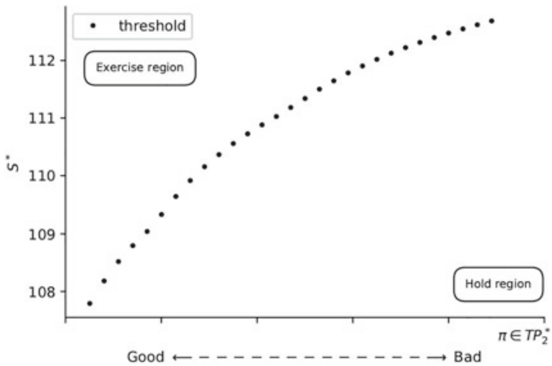


Fig. 6.10 An example of the optimal stopping regions for an American call option with dividend yield and the monotonicity of the threshold in $\boldsymbol{\pi}$ with parameters given in Table 6.1 and in (iv)

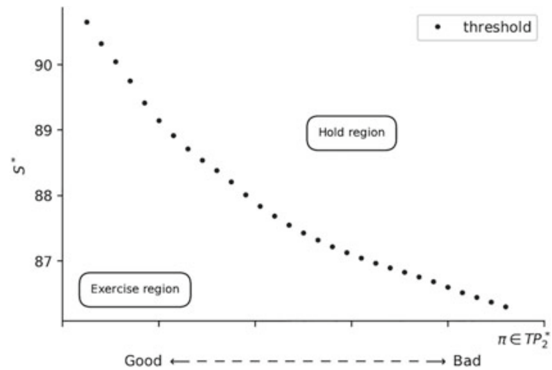


Fig. 6.11 An example of the optimal stopping regions for an American put option and the monotonicity of the threshold in $\boldsymbol{\pi}$ with parameters given in Table 6.1 and in (iv) for $\pi_2 = 0.02$

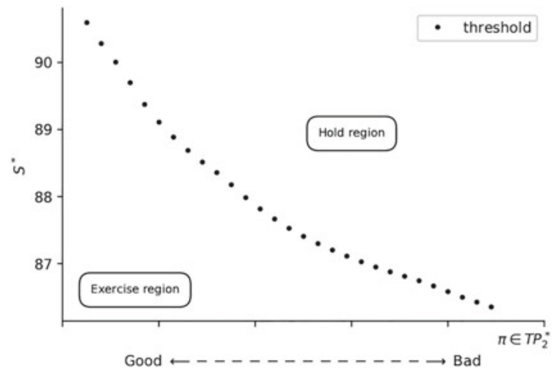


Fig. 6.12 An example of the optimal stopping regions for an American put option and the monotonicity of the threshold in $\boldsymbol{\pi}$ with parameters given in Table 6.1 and in (iv) for $\pi_2 = 0.07$

6.6 Conclusion and Future Research

We have studied the American option pricing and the corresponding optimal exercising strategies under a novel model. Under our model, the asset price follows an extended binomial tree with the volatility parameter governed by a discrete-time hidden Markov chain. We have formulated the problem using a partially observable Markov decision process and derived analytical structural properties for the American option prices and optimal exercising strategies, under a set of sufficient conditions on the transition probability matrix of the economy evolution and the conditional probabilities of observations. Our analytical results are fully illustrated with numerical examples.

For future research, we consider generalizing our model by permitting a more general probability distribution for the asset price. We plan also to conduct extensive numerical studies on the structural properties under less restrict conditions. Such information is useful when we use this model in practice.

The results of this research are limited to the pricing of short-maturity options because the changes in the economic situation are simple. For a long time period, decision-makers often face more complex situations in the economy. So as future work, we would also like to study extensions of our model for options with longer maturities.

References

1. Aingworth, D.D., Das, S.R., Motwani, R.: A simple approach for pricing equity options with Markov switching state variables. Quant. Finance **6**(2), 95–105 (2006)
2. Broadie, M., Detemple, J.: American options on dividend-paying assets. Fields Inst. Commun. **22**, 69–97 (1999)
3. Hamilton, J.D.: A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica **57**, 357–384 (1989)
4. Karlin, S.: Total Positivity, Volume I. Stanford University Press (1968)
5. Jösson, H.: Optimal stopping domains and Reward Funcions for Discrete Time American Type Options. Doctoral Dissertation, No. **22**, Mälardalen University (2005)
6. Kim, I.J., Byun, S.J.: Optimal exercise boundary in a binomial option pricing model. J. Financ. Eng. **3**(2), 137–158 (1994)
7. Kim, I.J., Byun, S.J., Lim, S.: Valuing and hedging American options under time varying volatility. J. Derivat. Account. **1**, 195–204 (2004)
8. Kijima, M., Yoshida, T.: A simple option pricing model with Markovian volatilities. J. Oper. Res So. Janpan **36**(3), 149–166 (1993)
9. Kukush, A.G., Silvestrov, D.S.: Strcuture of optimal stopping strategies for American type options. In: Uryasev, P. (ed.), Probabilistic Constrained Optimization: Methodology and Applications, Nonconvex Optim. Appl., vol. 49, pp. 173–185. Kluwer, Dordrecht (2000)
10. Kukush, A.G., Silvestrov, D.S.: Optimal pricing of American type options with discrete time. Theory Stoch. Process. **10(26)**(1-2), 72–96 (2004)
11. Marshall, A.W., Olkin, I., Arnold, B.C.: Inequalities: Theory of Majorization and Its Applications. Academic Press Inc, Orlando (1979)
12. Naik, V.: Option valuation and hedging strategies with jumps in the volatility of asset returns. J. Financ. **48**, 1969–1984 (1993)

13. Sato, K., Sawaki, K.: The dynamic pricing for callable securities with Markov-modulated prices. J. Oper. Res. Soc. Japan **57**, 87–103 (2014)
14. Sato, K., Sawaki, K.: The dynamic valuation of callable contingent claims with a partially observable regime switch. (2018). Available at SSRN: https://ssrn.com/abstract=3284489
15. Shen, Y., Siu T.-K.: Pricing variance swaps under a stochastic interest rate and volatility model with regime-switching. Oper. Res. Lett. **41**, 180–187 (2013)
16. Silvestrov, D.: American-Type Options. Stochastic Approximation Methods. Volume 1. De Gruyter Studies in Mathematics, vol. 56, p. x+509. Walter de Gruyter, Berlin (2014)
17. Silvestrov, D.: American-Type Options. Stochastic Approximation Methods. Volume 2. De Gruyter Studies in Mathematics, vol. 57, p. xi+558. Walter de Gruyter, Berlin (2015)