Chapter 4
Representations of Polynomial Covariance Type Commutation Relations by Linear Integral Operators on L_p Over Measure Spaces

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Abstract Representations of polynomial covariance type commutation relations by linear integral operators on L_p over measures spaces are constructed. Conditions for such representations are described in terms of kernels of the corresponding integral operators. Representation by integral operators are studied both for general polynomial covariance commutation relations and for important classes of polynomial covariance commutation relations associated to arbitrary monomials and to affine functions. Examples of integral operators on L_p spaces representing the covariance commutation relations are constructed. Representations of commutation relations by integral operators with special classes of kernels such as separable kernels and convolution kernels are investigated.

Keywords Integral operators · Covariance commutation relations · Convolution

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4.1 Introduction

Commutation relations of the form

$$AB = BF(A) \tag{4.1}$$

where A, B are elements of an associative algebra and F is a function of the elements of the algebra, are important in many areas of Mathematics and applications. Such commutation relations are usually called covariance relations, crossed product relations or semi-direct product relations. Elements of an algebra that satisfy (4.1) are called a representation of this relation in that algebra. Representations of covariance commutation relations (4.1) by linear operators are important for the study of actions and induced representations of groups and semigroups, crossed product operator algebras, dynamical systems, harmonic analysis, wavelets and fractals analysis and applications in physics and engineering [4, 5, 16–18, 26–28, 34, 35, 42].

A description of the structure of representations for the relation (4.1) and more general families of self-adjoint operators satisfying such relations by bounded and unbounded self-adjoint linear operators on a Hilbert space use reordering formulas for functions of the algebra elements and operators satisfying covariance commutation relation, functional calculus and spectral representation of operators and interplay with dynamical systems generated by iteration of maps involved in the commutation relations [3, 7–13, 19–21, 29–34, 36–40, 42–55].

In this paper, we construct representations of the covariance commutation relations (4.1) by linear integral operators on Banach spaces L_p over measure spaces. When $B = 0$, the relation (4.1) is trivially satisfied for any A. Thus, we focus on construction and properties of nontrivial representations of (4.1). We consider representations by the linear integral operators defined by kernels satisfying different conditions. We derive conditions on such kernel functions so that the corresponding operators satisfy (4.1) for polynomial F when both operators are of linear integral type. Representations of polynomial covariance type commutation relations by linear integral operators on L_p over measure spaces are constructed. Conditions for such representations are described in terms of kernels of the corresponding integral operators. Representation by integral operators are studied both for general polynomial covariance commutation relations and for important classes of polynomial covariance commutation relations associated to arbitrary monomials and to affine functions. Examples of integral operators on L_p spaces representing the covariance commutation relations are constructed. Representations of commutation relations by integral operators with special classes of kernels such as separable kernels and convolution kernels are investigated. In particular, we prove that there are no nonzero one sided convolution linear integral operators representing covariance type commutation relation for monomial t^m, where m a nonnegative integer except 1. This paper is organized in four sections. After the introduction, we present in Sect. 4.2 some preliminaries, notations, basic definitions and two useful lemmas. In Sect. 4.3, we present some representations when both operators A and B are linear integral operators acting on the Banach spaces L_p. In particular, we consider cases when operators are convolution type and operators with separable kernels.

4.2 Preliminaries and Notations

In this section we present preliminaries, basic definitions and notations for this article [1, 2, 6, 14, 22–24, 41].

Let $\mathbb{R}$ be the set of all real numbers, X be a non-empty space, and $S \subseteq X$. Let (S, Σ, μ) be a σ-finite measure space, where Σ is a $\sigma-$algebra with measurable subsets of S, and S can be covered with at most countably many disjoint sets $E_1, E_2, E_3, \ldots$ such that $E_i \in \Sigma$, $\mu(E_i) < \infty$, $i = 1, 2, \ldots$ and μ is a measure. For $1 \le p < \infty$, we denote by $L_p(S, \mu)$, the set of all classes of equivalent (different on a set of zero measure) measurable functions $f : S \to \mathbb{R}$ such that $\int\limits_S |f(t)|^p d\mu < \infty$. This is a Banach space (Hilbert space when $p = 2$) with norm $\|f\|_p = \left(\int\limits_S |f(t)|^p dt\right)^{\frac{1}{p}}$. We denote by $L_\infty(S, \mu)$ the set of all classes of equivalent measurable functions $f : S \to \mathbb{R}$ such that exists $C > 0$, $|f(t)| \le C$ almost everywhere. This is a Banach space with norm $\|f\|_\infty = \text{ess sup}_{t\in S} |f(t)|$. The support of a function $f : X \to \mathbb{R}$ is $\text{supp} f = \{t \in X : f(t) \neq 0\}$. We will use notation

$$Q_G(u, v) = \int\limits_G u(t)v(t)d\mu \tag{4.2}$$

for $G \in \Sigma$ and such functions $u, v : G \to \mathbb{R}$ that integral exists and is finite. The convolution of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is defined by $(f \star g)(t) = \int\limits_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$.

Now we will consider two useful lemmas for integral operators which will be used throughout the article. Lemma 1 is used in the proof of Theorem 1 and Lemma 2 is used in the proof of Theorem 2.

Lemma 1 *Let (X, Σ, μ) be a σ-finite measure space. Let $f, g \in L_q(X, \mu)$ for $1 \le q \le \infty$ and let $G_1, G_2 \in \Sigma$ such that $\mu(G_i) < \infty$, $i = 1, 2$. Let $G = G_1 \cap G_2$. Then the following statements are equivalent:*

1. *For all $x \in L_p(X, \mu)$, $1 \le p \le \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$,*

$$Q_{G_1}(f, x) = \int\limits_{G_1} f(t)x(t)d\mu = \int\limits_{G_2} g(t)x(t)d\mu = Q_{G_2}(g, x).$$

2. *The following conditions hold:*

(a) *for almost every* $t \in G$, $f(t) = g(t)$,
(b) *for almost every* $t \in G_1 \setminus G$, $f(t) = 0$,
(c) *for almost every* $t \in G_2 \setminus G$, $g(t) = 0$.

Proof $2 \Rightarrow 1$ By additivity of the measure of integration μ on Σ,

$$\int_{G_1} f(t)x(t)d\mu = \int_{G_1\setminus G} f(t)x(t)d\mu + \int_G f(t)x(t)d\mu = \int_G f(t)x(t)d\mu$$
$$= \int_G g(t)x(t)d\mu = \int_{G_2\setminus G} g(t)x(t)d\mu + \int_G g(t)x(t)d\mu = \int_{G_2} g(t)x(t)d\mu.$$

$1 \Rightarrow 2$ For the indicator function $x(t) = I_{H_1}(t)$ of the set $H_1 = G_1 \cup G_2$,

$$\int_{G_1} f(t)x(t)d\mu = \int_{G_2} g(t)x(t)d\mu = \int_{G_1} f(t)d\mu = \int_{G_2} g(t)d\mu = \eta,$$

where η is a constant. Now by taking $x(t) = I_{G_1\setminus G}$ we get

$$\int_{G_1} f(t)x(t)d\mu = \int_{G_2} g(t)x(t)d\mu = \int_{G_1\setminus G} f(t)d\mu = \int_{G_2} g(t)\cdot 0 d\mu = 0.$$

Then $\int_{G_1\setminus G} f(t)d\mu = 0$. Analogously by taking $x(t) = I_{G_2\setminus G}(t)$ we get $\int_{G_2\setminus G} g(t)d\mu = 0$ We claim that $f(t) = 0$ for almost every $t \in G_1 \setminus G$ and $g(t) = 0$ for almost every $t \in G_2 \setminus G$. We take a partition $S_1, S_2, \ldots, S_n, \ldots$ of the set $G_1 \setminus G$ such that each set S_i, $i = 1, 2, 3, \ldots$ has positive measure. For each $x_i(t) = I_{S_i}(t)$, $i = 1, 2, 3, \ldots$ we have $\int_{G_1} f(t)x(t)d\mu = \int_{G_2} g(t)x(t)d\mu = \int_{S_i} f(t)d\mu = \int_{G_2} g(t)\cdot 0d\mu = 0$. Thus, $\int_{S_i} f(t)d\mu = 0,\ i = 1, 2, 3, \ldots$ Since we can choose arbitrary partition with positive measure on each of its elements, $f(t) = 0$ for almost every $t \in G_1 \setminus G$. Analogously, $g(t) = 0$ for almost every $t \in G_2 \setminus G$. Therefore $\eta = \int_{G_1} f(t)d\mu = \int_{G_2} g(t)d\mu = \int_G f(t)d\mu = \int_G g(t)d\mu$. Then, for all function $x \in L_p(X, \mu)$ we have $\int_G f(t)x(t)d\mu = \int_G g(t)x(t)d\mu \Leftrightarrow \int_G [f(t) - g(t)]x(t)d\mu = 0$. This implies that $f(t) = g(t)$ for almost every $t \in G$. □

Let n be a positive integer, $(\mathbb{R}^n, \Sigma, \mu)$ be the standard Lebesgue measure space and $\Omega \in \Sigma$. We denote by $C(\Omega)$ the set of all continuous functions $f : \Omega \to \mathbb{R}$. This is a Banach space with norm $\|f\| = \max_{t\in\Omega} |f(t)|$. We denote by $C_c(\mathbb{R}^n)$ the set of all continuous functions with compact support.

The following statement is similar to Lemma 1 under conditions: $X = \mathbb{R}^n$ and sets G_1, G_2 can have infinite measure.

Lemma 2 *Let* $(\mathbb{R}^n, \Sigma, \mu)$ *be the standard Lebesgue measure space and* $f, g \in L_q(\mathbb{R}^n, \mu)$ *for* $1 < q < \infty$, $G_1 \in \Sigma$ *and* $G_2 \in \Sigma$. *Let* $G = G_1 \cap G_2$. *Then the following statements are equivalent:*

1. *For all* $x \in L_p(\mathbb{R}^n, \mu)$, *where* $1 \le p < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$,

$$Q_{G_1}(f, x) = \int\limits_{G_1} f(t)x(t)d\mu = \int\limits_{G_2} g(t)x(t)d\mu = Q_{G_2}(g, x).$$

2. *The following conditions hold:*
   (a) *for almost every* $t \in G$, $f(t) = g(t)$;
   (b) *for almost every* $t \in G_1 \setminus G$ $f(t) = 0$,
   (c) *for almost every* $t \in G_2 \setminus G$ $g(t) = 0$.

Proof $2 \Rightarrow 1$ This follows by direct computation as in the proof of Lemma 1.
$1 \Rightarrow 2$ Suppose that 2 is true. If $G_1 \in \Sigma$ and $G_2 \in \Sigma$ have finite measure then it follows from Lemma 1. Suppose that either G_1 has infinite measure or G_2 has infinite measure. For any $\alpha > 0$ and $\Omega_\alpha = [-\alpha, \alpha]^n \subset \mathbb{R}^n$, the set $V_\alpha = \{x \in C_c(\mathbb{R}^n) : x(t) = 0, \forall t \in \mathbb{R}^n \setminus \Omega_\alpha\}$ is a subspace of $C_c(\mathbb{R}^n)$. Since condition 1 is satisfied for any $x \in V_\alpha$, and any $x \in V_\alpha$ vanishes outside the set Ω_α, with finite measure, we have from Lemma 1:

(a) for almost every $t \in G \cap \Omega_\alpha$, $f(t) = g(t)$;
(b) for almost every $t \in (G_1 \cap \Omega_\alpha) \setminus G$, $f(t) = 0$;
(c) for almost every $t \in (G_2 \cap \Omega_\alpha) \setminus G$, $g(t) = 0$.

These conclusions are true for any fixed $\alpha > 0$, and so for the corresponding Ω_α, V_α. Since $f, g \in L_p(\mathbb{R}^n, \mu)$ $1 < p < \infty$ then there exist compact sets K_m such that

$$\lim_{m\to+\infty} \mu(\{t \in \mathbb{R}^n \setminus K_m : f(t) > 0\}) = \lim_{m\to+\infty} \mu(\{t \in \mathbb{R}^n \setminus K_m : g(t) > 0\}) = 0.$$

Hence condition 1 holds for all $x \in C_c(\mathbb{R}^n)$ if and only if condition 2 holds. The conclusion follows from [6, Theorem 4.3 and Theorem 4.12], that is, the set $C_c(\mathbb{R}^n)$ is dense in $L_p(\mathbb{R}^n, \mu)$, for $1 < p < \infty$. □

4.3 Representations by Linear Integral Operators

Let (X, Σ, μ) be a σ-finite measure space. In this section we consider representations of the covariance type commutation relation (4.1) when both A and B are linear integral operators acting from the Banach space $L_p(X, \mu)$ to itself for a fixed p such that $1 \le p \le \infty$ defined as follows:

$$(Ax)(t) = \int\limits_{S_A} k_A(t,s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{S_B} k_B(t,s)x(s)d\mu_s,$$

almost everywhere, where the index in μ_s indicates the variable of integration, $S_A, S_B \in \Sigma$, $\mu(S_A) < \infty$, $\mu(S_B) < \infty$, $k_A(t,s) : X \times S_A \to \mathbb{R}$, $k_B(t,s) : X \times S_B \to \mathbb{R}$ are measurable functions satisfying conditions bellow. For $1 < p < \infty$ we have from [15] that the operators $A : L_p(X,\mu) \to L_p(X,\mu)$ and $B : L_p(X,\mu) \to L_p(X,\mu)$ are well-defined if kernels satisfy the following conditions

$$\int\limits_X \left(\int\limits_{S_A} |k_A(t,s)|^q d\mu_s \right)^{p/q} d\mu_t < \infty, \ \int\limits_X \left(\int\limits_{S_B} |k_B(t,s)|^q d\mu_s \right)^{p/q} d\mu_t < \infty, \tag{4.3}$$

where $1 < q < \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. For $p = 1$, operators $A : L_1(X,\mu) \to L_1(X,\mu)$ and $B : L_1(X,\mu) \to L_1(X,\mu)$ are well-defined if kernels satisfy the following conditions

$$\int\limits_X \operatorname*{ess\,sup}_{s \in S_A} |k_A(t,s)| d\mu_t < \infty, \qquad \int\limits_X \operatorname*{ess\,sup}_{s \in S_B} |k_B(t,s)| d\mu_t < \infty. \tag{4.4}$$

For $p = \infty$, operators $A : L_\infty(X,\mu) \to L_\infty(X,\mu)$ and $B : L_\infty(X,\mu) \to L_\infty(X,\mu)$ are well-defined if kernels satisfy the following conditions

$$\operatorname*{ess\,sup}_{t \in X} \left(\int\limits_{S_A} |k_A(t,s)| d\mu_s \right) < \infty, \qquad \operatorname*{ess\,sup}_{t \in X} \left(\int\limits_{S_B} |k_B(t,s)| d\mu_s \right) < \infty. \tag{4.5}$$

Theorem 1 *Let (X, Σ, μ) be a σ-finite measure space. Let $A : L_p(X,\mu) \to L_p(X,\mu)$, $B : L_p(X,\mu) \to L_p(X,\mu)$, $1 \le p \le \infty$ be nonzero operators defined as follows*

$$(Ax)(t) = \int\limits_{G_A} k_A(t,s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{G_B} k_B(t,s)x(s)d\mu_s,$$

almost everywhere, where the index in μ_s indicates the variable of integration, $G_A, G_B \in \Sigma$, $\mu(G_A) < \infty$, $\mu(G_B) < \infty$, $k_A(t,s) : \mathbb{R} \times S_A \to \mathbb{R}$, $k_B(t,s) : \mathbb{R} \times S_B \to \mathbb{R}$ are measurable functions satisfying either relation (4.3) or (4.4) or (4.5), respectively. Consider a polynomial $F : \mathbb{R} \to \mathbb{R}$ defined by $F(z) = \sum\limits_{j=0}^{n} \delta_j z^j$, where $\delta_j \in \mathbb{R}$, $j = 0, 1, 2, \ldots, n$. Set $G = G_A \cap G_B$, and

$$k_{0,A}(t,s)=k_A(t,s),\ k_{m,A}(t,s)=\int\limits_{G_A} k_A(t,\tau)k_{m-1,A}(\tau,s)d\mu_\tau,\ m=1,2,3,\dots,n$$

$$F_n(k_A(t,s))=\sum_{j=1}^{n}\delta_j k_{j-1}(t,s),\quad n=1,2,3,\dots$$

Then $AB=BF(A)$ if and only if the following conditions are fulfilled :

1. *for almost every $(t,\tau)\in X\times G$,*

$$\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s-\delta_0\tilde{k}(t,\tau)=\int\limits_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s,$$

   where μ_s indicates that integration is taken with respect to the variable s;

2. *for almost every $(t,\tau)\in X\times(G_B\setminus G)$, $\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s=\delta_0 k_B(t,\tau)$;*
3. *for almost every $(t,\tau)\in X\times(G_A\setminus G)$, $\int\limits_{G_B} k_B(t,s)F_n(k(s,\tau))d\mu_s=0$.*

Proof By applying Fubini theorem from [1] and iterative kernels from [25] we have

$$(A^2x)(t)=\int\limits_{G_A} k_A(t,s)(Ax)(s)d\mu_s=\int\limits_{G_A} k_A(t,s)\left(\int\limits_{G_A} k_A(s,\tau)x(\tau)d\mu_\tau\right)d\mu_s$$

$$=\int\limits_{G_A}\big(\int\limits_{G_A} k_A(t,s)k(s,\tau)d\mu_s\big)x(\tau)d\mu_\tau=\int\limits_{G_A} k_{1,A}(t,\tau)x(\tau)d\mu_\tau,$$

$$k_{1,A}(t,s)=\int\limits_{G_A} k_A(t,\tau)k_A(\tau,s)d\mu_\tau;$$

$$(A^3x)(t)=\int\limits_{G_A} k_A(t,s)(A^2x)(s)d\mu_s=\int\limits_{G_A} k_A(t,s)\big(\int\limits_{G_A} k_{1,A}(s,\tau)x(\tau)d\mu_\tau\big)d\mu_s$$

$$=\int\limits_{G_A}\big(\int\limits_{G_A} k_A(t,s)k_1(s,\tau)d\mu_s\big)x(\tau)d\mu_\tau=\int\limits_{G_A} k_{2,A}(t,\tau)x(\tau)d\mu_\tau,$$

$$k_{2,A}(t,s)=\int\limits_{G_A} k_A(t,\tau)k_{1,A}(\tau,s)d\mu_\tau;$$

$$(A^nx)(t)=\int\limits_{G_A} k_{n-1,A}(t,s)x(s)d\mu_s,\quad n\geq 1,$$

$$k_{m,A}(t,s)=\int\limits_{G_A} k_A(t,\tau)k_{m-1,A}(\tau,s)d\mu_\tau,\ m=1,2,3,\dots,n,$$

$$k_{0,A}(t,s)=k_A(t,s);$$

$$(F(A)x)(t)=\delta_0x(t)+\sum_{j=1}^{n}\delta_j(A^jx)(t)$$

$$=\delta_0x(t)+\sum_{j=1}^{n}\delta_j\int\limits_{G_A} k_{j-1,A}(t,s)x(s)d\mu_s$$

$$=\delta_0x(t)+\int\limits_{G_A} F_n(k_A(t,s))x(s)d\mu_s,$$

$$
\begin{aligned}
&F_n(k_A(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1,A}(t,s), \quad n = 1,2,3,\ldots;\\
&(BF(A)x)(t) = \int_{G_B} k_B(t,s)(F(A)x)(s)d\mu_s\\
&= \int_{G_B} k_B(t,s)\left(\delta_0 x(s) + \int_{G_A} F_n(k_A(s,\tau)x(\tau)d\mu_\tau)\right)d\mu_s\\
&= \delta_0 \int_{G_B} k_B(t,s)x(s)d\mu_s + \int_{G_A}\left(\int_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s\right)x(\tau)d\mu_\tau\\
&= \delta_0 \int_{G_B} k_B(t,s)x(s)d\mu_s + \int_{G_A} k_{BFA}(t,\tau)x(\tau)d\mu_\tau,\\
&k_{BFA}(t,\tau) = \int_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s;\\
&(ABx)(t) = \int_{G_B} k_A(t,s)(Bx)(s)d\mu_s = \int_{G_A} k_A(t,s)\left(\int_{G_B} k_B(s,\tau)x(\tau)d\mu_\tau\right)d\mu_s\\
&= \int_{G_B}\left(\int_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s\right)x(\tau)d\mu_\tau = \int_{G_B} k_{AB}(t,\tau)x(\tau)d\mu_\tau,\\
&k_{AB}(t,\tau) = \int_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s.
\end{aligned}
$$

Therefore, $(ABx)(t) = (BF(A)x)(t)$ for all $x \in L_p(X,\mu)$ if and only if

$$\int_{G_B} [k_{AB}(t,\tau) - \delta_0 k_B(t,\tau)]x(\tau)d\mu_\tau = \int_{G_A} k_{BFA}(t,\tau)x(\tau)d\mu_\tau.$$

By applying Lemma 1 we have $AB = BF(A)$ if and only if

1. for almost every $(t,\tau) \in X \times G$,

$$\int_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s - \delta_0 k_B(t,\tau) = \int_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s;$$

2. for almost every $(t,\tau) \in X \times (G_B \setminus G)$, $\int_{G_A} k_A(t,s)k_A(s,\tau)d\mu_s = \delta_0 \tilde{k}(t,\tau)$;
3. for almost every $(t,\tau) \in X \times (G_A \setminus G)$, $\int_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s = 0$. □

Remark 1 In Theorem 1 when $G_A = G_B = G$ conditions 2 and 3 are taken on set of measure zero so we can ignore them. Thus, we only remain with condition 1. When $G_A \neq G_B$, then we need to check conditions 2 and 3 outside the intersection $G = G_A \cap G_B$. Moreover, condition 3 that for almost every $(t,\tau) \in X \times (G_A \setminus G)$,

$$\int_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s = 0, \tag{4.6}$$

does not imply $B\left(\sum_{k=1}^{n}\delta_k A^k\right) = 0$ because its kernel has to satisfy (4.6) only on the set $X \times (G_A \setminus G)$ and not on the whole set of definition. On the other hand, the same kernel has to satisfy condition 1, which is, for almost every $(t,\tau) \in X \times G$,

$$\int_{G_A} k(t,s)\tilde{k}(s,\tau)d\mu_s - \delta_0\tilde{k}(t,\tau) = \int_{G_B} \tilde{k}(t,s)F_n(k(s,\tau))d\mu_s.$$

Note that Theorem 1 does not imply $\sum_{k=1}^{n}\delta_k A^k = 0$. In fact, $\sum_{k=1}^{n}\delta_k A^k = 0$ implies $B\left(\sum_{k=1}^{n}\delta_k A^k\right) = 0$ but as mentioned above it can be non zero in general.

Example 1 Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space. Consider integral operators acting on $L_p(\mathbb{R},\mu)$ for $1 < p < \infty$. Let $A : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $B : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $1 < p < \infty$ defined as follows

$$(Ax)(t) = \int_0^{\pi} k_A(t,s)x(s)d\mu_s, \quad (Bx)(t) = \int_0^{\pi} k_B(t,s)x(s)d\mu_s,$$

almost everywhere, where the index in μ indicates the variable of integration,

$$\begin{aligned} k_A(t,s) &= I_{[\alpha,\beta]}(t)\tfrac{2}{\pi}(\cos t\cos s + \sin t\sin s + \cos t\sin s), \\ k_B(t,s) &= I_{[\alpha,\beta]}(t)\tfrac{2}{\pi}(\cos t\cos s + 2\sin t\sin s), \end{aligned}$$

almost everywhere $(t,s) \in \mathbb{R} \times [0,\pi]$, α, β are real constants such that $\alpha \le 0$, $\beta \ge \pi$ and $I_E(t)$ is the indicator function of the set E. These operators are well defined, since the kernels satisfy (4.3). In fact,

$$\begin{aligned} &\int_{\mathbb{R}} \Big(\int_0^{\pi} |k_A(t,s)|^q d\mu_s\Big)^{\frac{p}{q}} d\mu_t \\ &= \int_{\alpha}^{\beta} \Big(\int_0^{\pi} \Big|\frac{2}{\pi}(\cos t\cos s + \sin t\sin s + \cos t\sin s)\Big|^q d\mu_s\Big)^{\frac{p}{q}} d\mu_t \\ &\le \int_{\alpha}^{\beta} \tfrac{6^p}{\pi^{p-1}}dt = \tfrac{6^p(\beta-\alpha)}{\pi^{p-1}} < \infty, \end{aligned}$$

$$\int_{\mathbb{R}} \Big(\int_0^{\pi} |k_B(t,s)|^q d\mu_s\Big)^{p/q} d\mu_t$$

$$= \int_{\alpha}^{\beta} \Big(\int_{0}^{\pi} \left|\tfrac{2}{\pi}(\cos t \cos s + 2 \sin t \sin s)\right|^q d\mu_s\Big)^{p/q} d\mu_t \le \int_{\alpha}^{\beta} \tfrac{6^p}{\pi^{p-1}} dt = \tfrac{6^p(\beta-\alpha)}{\pi^{p-1}} < \infty,$$

where $q \ge 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. In the estimations above we used the inequalities:

$$|2(\cos t \cos s + \sin t \sin s + \cos t \sin s)|^q \le 2^q \cdot 3^q = 6^q,$$
$$|2(\cos t \cos s + 2 \sin t \sin s)|^q \le 2^q \cdot 3^q = 6^q,\ 1 < q < \infty.$$

Note that in this case conditions 1, 2 and 3 of Theorem 1 reduce just to condition 1 because the sets $G_A = G_B = [0, \pi]$, and so $G = [0, \pi]$, $G_A \setminus G = G_B \setminus G = \emptyset$. Therefore, according to Remark 1 conditions 2 and 3 are taken on a set of measure zero.

Consider the polynomial $F(t) = t^2, t \in \mathbb{R}$. These operators satisfy $AB = BF(A)$. In fact, by applying Theorem 1 we have $\delta_0 = \delta_1 = 0$, $\delta_2 = 1$, $n = 2$,

$$\begin{aligned} k_{AB}(t,\tau) &= \int_0^{\pi} k_A(t,s) k_B(s,\tau) d\mu_s \\ &= \tfrac{4}{\pi^2} \int_0^{\pi} I_{[\alpha,\beta]}(t)(\cos(t)\cos(s) + \sin(t)\sin(s) + \cos(t)\sin(s)) \cdot \\ &\quad I_{[\alpha,\beta]}(s)(\cos(s)\cos(\tau) + 2 \sin s \sin \tau) d\mu_s \\ &= \tfrac{4}{\pi} I_{[\alpha,\beta]}(t) \left(\tfrac{\cos t \cos \tau}{2} + \cos t \sin \tau + \sin t \sin \tau\right) \\ &= \tfrac{2}{\pi} I_{[\alpha,\beta]}(t)(\cos t \cos \tau + 2 \cos t \sin \tau + 2 \sin t \sin \tau), \end{aligned}$$

for almost every $(t, \tau) \in \mathbb{R} \times [0, \pi]$. Moreover,

$$\begin{aligned} F_2(k_A(t,s)) = k_{1,A}(t,s) &= \int_0^{\pi} k_A(t,\tau) k_A(\tau,s) d\mu_\tau = \\ &= \tfrac{4}{\pi^2} \int_0^{\pi} I_{[\alpha,\beta]}(t)(\cos t \cos \tau + \sin t \sin \tau + \cos t \sin \tau) \cdot \\ &\quad I_{[\alpha,\beta]}(\tau)(\cos \tau \cos s + \sin \tau \sin s + \cos \tau \sin s) d\mu_\tau \\ &= \tfrac{4}{\pi} I_{[\alpha,\beta]}(t) \left(\tfrac{\cos t \cos s}{2} + \cos t \sin s + \tfrac{\sin t \sin s}{2}\right) \\ &= \tfrac{2}{\pi} I_{[\alpha,\beta]}(t)(\cos t \cos s + 2 \cos t \sin s + \sin t \sin s), \end{aligned}$$

for almost every $(t, s) \in \mathbb{R} \times [0, \pi]$. Therefore,

$$\begin{aligned}
k_{BFA}(t,\tau) &= \int\limits_0^{\pi} k_B(t,s)F_2(k_A(s,\tau))d\mu_s \\
&= \tfrac{4}{\pi^2}\int\limits_0^{\pi} I_{[\alpha,\beta]}(t)(\cos(t)\cos(s)+2\sin(t)\sin(s)) \\
&\qquad\qquad\qquad \cdot I_{[\alpha,\beta]}(s)(\cos s\cos\tau + 2\cos s\sin\tau + \sin s\sin\tau)d\mu_s \\
&= \tfrac{4}{\pi} I_{[\alpha,\beta]}(t)\left(\tfrac{\cos t\cos\tau}{2} + \cos t\sin\tau + \sin t\sin\tau\right) \\
&= \tfrac{2}{\pi} I_{[\alpha,\beta]}(t)(\cos t\cos\tau + 2\cos t\sin\tau + 2\sin t\sin\tau),
\end{aligned}$$

for almost every $(t,\tau)\in\mathbb{R}\times[0,\pi]$, which coincides with the kernel k_{AB}. Thus, conditions of Theorem 1 are fulfilled and so $AB = BA^2$. Moreover, $BA^2 \neq 0$ as mentioned in Remark 1, in fact

$$(BA^2x)(t) = \tfrac{2}{\pi} I_{[\alpha,\beta]}(t)\int\limits_0^{\pi/2}(\cos t\cos\tau + 2\cos t\sin\tau + 2\sin t\sin\tau)x(\tau)d\mu_\tau$$

almost everywhere.

The following corollary is a special case of Theorem 1 for the important class of covariance commutation relations, associated to affine (degree 1) polynomials F.

Corollary 1 *Let (X,Σ,μ) be a σ-finite measure space. Let $A: L_p(X,\mu)\to L_p(X,\mu)$, $B: L_p(X,\mu)\to L_p(X,\mu)$, $1\le p\le\infty$ be nonzero operators defined as follows*

$$(Ax)(t) = \int\limits_{G_A} k_A(t,s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{G_B} k_B(t,s)x(s)d\mu_s,$$

where the index in μ_s indicates variable of integration, $G_A,\ G_B\in\Sigma$, $\mu(G_A)<\infty$, $\mu(G_B)<\infty$, $k_A(t,s): X\times G_A\to\mathbb{R}$, $k_B(t,s): X\times G_B\to\mathbb{R}$ are measurable functions satisfying either relation (4.3) or (4.4) or (4.5). Let $F:\mathbb{R}\to\mathbb{R}$ be a polynomial of degree at most 1 given by $F(z)=\delta_0+\delta_1 z$, where $\delta_0,\ \delta_1\in\mathbb{R}$. We set $G = G_A\cap G_B$.

Then $AB-\delta_1 BA = \delta_0 B$ if and only if the following conditions are fulfilled

1. *for almost every $(t,\tau)\in X\times G$,*

$$\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s - \delta_0 k_B(t,\tau) = \delta_1\int\limits_{G_B} k_B(t,s)k_A(s,\tau)d\mu_s.$$

2. *for almost every $(t,\tau)\in X\times(G_B\setminus G)$, $\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s = \delta_0 k_B(t,\tau)$.*
3. *for almost every $(t,\tau)\in X\times(G_A\setminus G)$, $\delta_1\int\limits_{G_B} k_B(t,s)k_A(s,\tau)d\mu_s = 0$.*

The following corollary of Theorem 1 is concerned with representations by integral operators of another important family of covariance commutation relations associated to monomials F.

Corollary 2 *Let (X, Σ, μ) be a σ-finite measure space. Let $A : L_p(X, \mu) \to L_p(X, \mu)$, $B : L_p(X, \mu) \to L_p(X, \mu)$, $1 \le p \le \infty$ be nonzero operators defined as follows*

$$(Ax)(t) = \int\limits_{G_A} k_A(t, s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{G_B} k_B(t, s)x(s)d\mu_s,$$

where the index in μ_s indicates variable of integration, G_A, $G_B \in \Sigma$, $\mu(G_A) < \infty$, $\mu(G_B) < \infty$, $k_A(t, s) : X \times G_A \to \mathbb{R}$, $k_B(t, s) : X \times G_B \to \mathbb{R}$ are measurable functions satisfying either relation (4.3) or (4.4) or (4.5). Let $F : \mathbb{R} \to \mathbb{R}$ be a monomial defined by $F(z) = \delta z^d$, where $\delta \neq 0$ is a real number and d is a positive integer. Let $G = G_A \cap G_B$ and

$$k_{0,A}(t, s) = k_A(t, s), \ k_{m,A}(t, s) = \int\limits_{G_A} k_A(t, \tau)k_{m-1,A}(\tau, s)d\mu_\tau, \ m = 1, 2, 3, \ldots, d.$$

Then $AB = \delta BA^d$ if and only if the following conditions are fulfilled

1. *for almost every $(t, \tau) \in X \times G$,*

$$\int\limits_{G_A} k_A(t, s)k_B(s, \tau)d\mu_s = \delta \int\limits_{G_B} k_B(t, s)k_{d-1,A}(s, \tau)d\mu_s.$$

2. *for almost every $(t, \tau) \in X \times (G_B \setminus G)$, $\int\limits_{G_A} k_A(t, s)k_B(s, \tau)d\mu_s = 0$.*
3. *for almost every $(t, \tau) \in X \times (G_A \setminus G)$, $\int\limits_{G_B} k_B(t, s)k_{d-1,A}(s, \tau)d\mu_s = 0$.*

Remark 2 Example 1 describes a specific case for Corollary 2 when $G_A = G_B = [0, \pi]$, $\delta = 1$, $d = 2$.

Consider now the case when $X = \mathbb{R}^l$ and μ is the Lebesgue measure. In the following theorem we allow the sets G_A and G_B to have infinite measure.

Theorem 2 *Let $(\mathbb{R}^l, \Sigma, \mu)$ be the standard Lebesgue measure space. Let*

$$A : L_p(\mathbb{R}^l, \mu) \to L_p(\mathbb{R}^l, \mu), \ B : L_p(\mathbb{R}^l, \mu) \to L_p(\mathbb{R}^l, \mu), \ 1 < p < \infty$$

be nonzero operators defined by

$$(Ax)(t) = \int\limits_{G_A} k_A(t, s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{G_B} k_B(t, s)x(s)d\mu_s,$$

where the index in μ indicates the variable of integration, $G_A \in \Sigma$ and $G_B \in \Sigma$, and kernels $k_A(t,s) : \mathbb{R}^l \times G_A \to \mathbb{R}$, $k_B(t,s) : \mathbb{R}^l \times G_B \to \mathbb{R}$ are measurable functions satisfying either relation (4.3) or (4.4). Consider a polynomial $F : \mathbb{R} \to \mathbb{R}$ defined by $F(z) = \sum\limits_{j=0}^{n} \delta_j z^j$, where $\delta_j \in \mathbb{R}$, $j = 0, 1, 2, \ldots, n$. Let $G = G_A \cap G_B$ and

$$k_{A,0}(t,s) = k_A(t,s),\ k_{A,m}(t,s) = \int\limits_{G_A} k_A(t,\tau) k_{A,m-1}(\tau,s) d\mu_\tau,\ m = 1, 2, 3, \ldots, n$$

$$F_m(k_A(t,s)) = \sum_{j=1}^{m} \delta_j k_{A,j-1}(t,s), \quad m = 1, 2, 3, \ldots, n.$$

Then $AB = BF(A)$ if and only if the following conditions are fulfilled:

1. *for almost every $(t,\tau) \in \mathbb{R}^n \times G$,*

$$\int\limits_{G_A} k_A(t,s) k_B(s,\tau) d\mu_s - \delta_0 k_B(t,\tau) = \int\limits_{G_B} k_B(t,s) F_n(k_A(s,\tau)) d\mu_s.$$

2. *for almost every $(t,\tau) \in \mathbb{R}^n \times (G_B \setminus G)$, $\int\limits_{G_A} k_A(t,s) k_B(s,\tau) d\mu_s = \delta_0 k_B(t,\tau)$.*
3. *for almost every $(t,\tau) \in \mathbb{R}^n \times (G_A \setminus G)$, $\int\limits_{G_B} k_B(t,s) F_n(k_A(s,\tau)) d\mu_s = 0$.*

Proof By applying Fubini theorem from [1] and iterative kernels from [25] we have

$$\begin{aligned}
(A^2 x)(t) &= \int\limits_{G_A} k_A(t,s)(Ax)(s) d\mu_s = \int\limits_{G_A} k_A(t,s) \Big(\int\limits_{G_A} k_A(s,\tau) x(\tau) d\mu_\tau \Big) d\mu_s \\
&= \int\limits_{G_A} \Big(\int\limits_{G_A} k_A(t,s) k_A(s,\tau) ds \Big) x(\tau) d\tau = \int\limits_{G_A} k_{1,A}(t,\tau) x(\tau) d\mu_\tau, \\
k_{1,A}(t,s) &= \int\limits_{G_A} k_A(t,\tau) k_A(\tau,s) d\mu_\tau; \\
(A^3 x)(t) &= \int\limits_{G_A} k_A(t,s)(A^2 x)(s) d\mu_s = \int\limits_{G_A} k_A(t,s) \Big(\int\limits_{G_A} k_{1,A}(s,\tau) x(\tau) d\mu_\tau \Big) d\mu_s \\
&= \int\limits_{G_A} \Big(\int\limits_{G_A} k_A(t,s) k_{1,A}(s,\tau) d\mu_s \Big) x(\tau) d\mu_\tau = \int\limits_{G_A} k_{2,A}(t,\tau) x(\tau) d\mu_\tau, \\
k_{2,A}(t,s) &= \int\limits_{G_A} k_A(t,\tau) k_{1,A}(\tau,s) d\mu_\tau; \\
(A^n x)(t) &= \int\limits_{G_A} k_{n-1,A}(t,s) x(s) d\mu_s, \quad n \geq 1 \\
k_{m,A}(t,s) &= \int\limits_{G_A} k_A(t,\tau) k_{m-1,A}(\tau,s) d\mu_\tau,\ m = 1, 2, 3, \ldots, n,\ k_{0,A}(t,s) = k_A(t,s); \\
(F(A)x)(t) &= \delta_0 x(t) + \sum_{j=1}^{n} \delta_j (A^j x)(t) = \delta_0 x(t) + \sum_{j=1}^{n} \delta_j \int\limits_{G_A} k_{j-1,A}(t,s) x(s) d\mu_s
\end{aligned}$$

$$
\begin{aligned}
&= \delta_0 x(t) + \int\limits_{G_A} F_n(k_A(t,s))x(s)d\mu_s,\\
F_n(k_A(t,s)) &= \sum_{j=1}^{n} \delta_j k_{j-1,A}(t,s),\ n=1,2,3,\ldots;\\
(BF(A)x)(t) &= \int\limits_{G_B} k_B(t,s)(F(A)x)(s)d\mu_s =\\
&= \int\limits_{G_B} k_B(t,s)\big(\delta_0 x(s) + \int\limits_{G_A} F_n(k_A(s,\tau)x(\tau)d\mu_\tau)\big)d\mu_s\\
&= \delta_0 \int\limits_{G_2} k_B(t,s)x(s)d\mu_s + \int\limits_{G_A}\big(\int\limits_{G_B} k_B(t,s)F_n(k_B(s,\tau))d\mu_s\big)x(\tau)d\mu_\tau =\\
&= \delta_0 \int\limits_{G_B} k_B(t,s)x(s)d\mu_s + \int\limits_{G_A} k_{BF}(t,\tau)x(\tau)d\mu_\tau\\
k_{BF}(t,\tau) &= \int\limits_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s;\\
(ABx)(t) &= \int\limits_{G_B} k_A(t,s)(Bx)(s)d\mu_s = \int\limits_{G_A} k_A(t,s)\big(\int\limits_{G_B} k_B(s,\tau)x(\tau)d\mu_\tau\big)d\mu_s\\
&= \int\limits_{G_B}\big(\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s\big)x(\tau)d\mu_\tau = \int\limits_{G_B} k_{AB}(t,\tau)x(\tau)d\mu_\tau,\\
k_{AB}(t,\tau) &= \int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s.
\end{aligned}
$$

Thus for all $x \in L_p(\mathbb{R}^l,\mu)$, $1 < p < \infty$ we have $(ABx)(t) = (BF(A)x)(t)$ almost everywhere if and only if $\int\limits_{G_B} [k_{AB}(t,\tau) - \delta_0\tilde{k}(t,\tau)]x(\tau)d\mu_\tau = \int\limits_{G_A} k_{BF}(t,\tau)x(\tau)d\mu_\tau$ almost everywhere. By Lemma 2 we have $AB = BF(A)$ if and only if

1. for almost every $(t,\tau) \in \mathbb{R}\times G$,
$$\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s - \delta_0 k_B(t,\tau) = \int\limits_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s;$$
2. for almost every $(t,\tau) \in \mathbb{R}\times(G_B\setminus G)$, $\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s = \delta_0 k_B(t,\tau)$.
3. for almost every $(t,\tau) \in \mathbb{R}\times(G_A\setminus G)$, $\int\limits_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s = 0$. □

Remark 3 Similar to Remark 1, in Theorem 2 when $G_A = G_B = G$ conditions 2 and 3 are taken on set of measure zero so we can ignore them. Thus, we only remain with condition 1. When $G_A \neq G_B$ we need to check also conditions 2 and 3 outside the intersection $G = G_A \cap G_B$. Moreover condition 3, which is, for almost every $(t,\tau) \in \mathbb{R}^n \times (G_A \setminus G)$,

$$\int\limits_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s = 0. \tag{4.7}$$

does not imply $B\left(\sum_{k=1}^{n}\delta_k A^k\right) = 0$ because its kernel has to satisfy (4.7) only on the set $\mathbb{R}^n \times (G_A \setminus G)$ and not on the whole set of definition. On the other hand, the same kernel has to satisfy condition 2, that for almost every $(t,\tau) \in \mathbb{R}^n \times G$,

$$\int_{G_A} k(t,s)\tilde{k}(s,\tau)d\mu_s - \delta_0\tilde{k}(t,\tau) = \int_{G_B} \tilde{k}(t,s)F_n(k(s,\tau))d\mu_s.$$

Note that Theorem 2 does not imply $\sum_{k=1}^{n}\delta_k A^k = 0$. In fact, $\sum_{k=1}^{n}\delta_k A^k = 0$ implies $B\left(\sum_{k=1}^{n}\delta_k A^k\right) = 0$ but as mentioned above it can be non zero in general.

Proposition 1 *Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space. Let $A : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $B : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $1 < p < \infty$ be nonzero operators defined as follows*

$$(Ax)(t) = \int_{\mathbb{R}} \tilde{k}_A(t-s)x(s)d\mu_s, \quad (Bx)(t) = \int_{\mathbb{R}} \tilde{k}_B(t-s)x(s)d\mu_s,$$

where the index in μ indicates the variable of integration, kernels $\tilde{k}_A(\cdot) \in L_1(\mathbb{R},\mu)$, $\tilde{k}_B(\cdot) \in L_1(\mathbb{R},\mu)$, that is,

$$\int_{\mathbb{R}} |\tilde{k}_A(t)|d\mu_t < \infty, \qquad \int_{\mathbb{R}} |\tilde{k}_B(t)|d\mu_t < \infty.$$

Consider a polynomial $F : \mathbb{R} \to \mathbb{R}$, $F(z) = \sum_{j=0}^{n}\delta_j z^j$, where $\delta_j \in \mathbb{R}$, $j = 0, 1, 2, \ldots, n$. Then $AB = BF(A)$ if and only if for almost every $t \in \mathbb{R}$,

$$\tilde{k}_B \star \left(\tilde{k}_A - \delta_0 - \sum_{j=1}^{n}\delta_j(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{j\ times})\right)(t) = 0. \tag{4.8}$$

In particular, if $\delta_0 = 0$, that is, $F(z) = \delta_1 t + \delta_2 z^2 + \ldots + \delta_n z^n$, then $AB = BF(A)$ if and only if the set $\operatorname{supp} K_B \cap \operatorname{supp}\left(K_A - \sum_{j=1}^{n}\delta_j K_A^j\right)$ *has measure zero in $\mathbb{R}$, where*

$$K_B(s) = \int_{-\infty}^{\infty} \exp(-st)\tilde{k}_B(t)d\mu_t, \quad K_A(s) = \int_{-\infty}^{\infty} \exp(-st)\tilde{k}_A(t)d\mu_t.$$

Proof Operators A and B are well defined by Young theorem ([6], Theorem 4.15). By Fubbini theorem for composition of operators A, B and A^n, similarly to the proof of Theorem 2 when $k_A(t,s)=\tilde{k}_A(t-s)$, $k_B(t,s)=\tilde{k}_B(t-s)$ and $G_A=G_B=\mathbb{R}$ we get from Lemma 2 that $AB=BF(A)$ if and only if for almost every $(t,s)\in\mathbb{R}^2$,

$$\int_{\mathbb{R}} \tilde{k}_A(t-\tau)\tilde{k}_B(\tau-s)d\mu_\tau - \delta_0\tilde{k}_B(t-s) = \int_{\mathbb{R}} \tilde{k}_B(t-\tau)F_n(k_A(\tau,s))d\mu_\tau, \quad (4.9)$$

where

$$\tilde{k}_{0,A}(t,s)=\tilde{k}_A(t-s),\ \tilde{k}_{m,A}(t,s)=\int_{\mathbb{R}} \tilde{k}_A(t-\tau)k_{m-1,A}(\tau,s)d\mu_\tau, \quad m=1,2,3,\ldots,n$$

$$F_m(\tilde{k}_A(t,s))=\sum_{j=1}^{m}\delta_j\tilde{k}_{j-1,A}(t,s), \quad m=1,2,3,\ldots,n.$$

Computing $\tilde{k}_{m,A}(t,s)$ we have for $m=1$,

$$\tilde{k}_{1,A}(t,s)=\int_{\mathbb{R}}\tilde{k}_A(t-\tau)\tilde{k}_A(\tau-s)d\mu_\tau=\int_{\mathbb{R}}\tilde{k}_A(t-s-\nu)\tilde{k}_A(\nu)d\mu_\nu=(\tilde{k}_A\star\tilde{k}_A)(t-s),$$

for $m=2$,

$$\begin{aligned}\tilde{k}_{2,A}(t,s) &= \int_{\mathbb{R}}\tilde{k}_A(t-\tau)(\tilde{k}_A\star\tilde{k}_A(\tau-s)d\mu_\tau = \\ &= \int_{\mathbb{R}}\tilde{k}_A(t-s-\nu)(\tilde{k}_A\star\tilde{k}_A(\nu)d\mu_\nu=(\tilde{k}_A\star\tilde{k}_A\star\tilde{k}_A)(t-s).\end{aligned}$$

and for all $2\le m\le n$, $\tilde{k}_{m-1,A}(t,s)=(\underbrace{\tilde{k}_A\star\tilde{k}_A\star\ldots\star\tilde{k}_A}_{m\text{ times}})(t-s)$. Thus, for all $1\le m\le n$

$$F_m(\tilde{k}_A(t,s))=\sum_{j=1}^{m}\delta_j(\underbrace{\tilde{k}_A\star\tilde{k}_A\star\ldots\star\tilde{k}_A}_{j\text{ times}})(t-s),$$

$$\int_{\mathbb{R}}\tilde{k}_B(t-s)F_n(\tilde{k}_A(s,\tau))d\mu_s=\int_{\mathbb{R}}\tilde{k}_B(t-\tau)\cdot\sum_{j=1}^{n}\delta_j(\underbrace{\tilde{k}_A\star\tilde{k}_A\star\ldots\star\tilde{k}_A}_{j\text{ times}})(\tau-s)d\mu_\tau$$

$$=\int_{\mathbb{R}}\sum_{j=1}^{n}\delta_j\tilde{k}_B(t-s-\nu)\cdot(\underbrace{\tilde{k}_A\star\tilde{k}_A\star\ldots\star\tilde{k}_A}_{j\text{ times}})(\nu)d\mu_\nu$$

$$= \sum_{j=1}^{n} \delta_j \tilde{k}_B \star (\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{j \text{ times}})(t-s).$$

Therefore, for almost every pairs $(t, s) \in \mathbb{R}^2$, the equality (4.9) is equivalent to

$$(\tilde{k}_A \star \tilde{k}_B)(t-s) = \delta_0 \tilde{k}_B(t-s) + \sum_{j=1}^{n} \delta_j \tilde{k}_B \star \left(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{j \text{ times}} \right)(t-s)$$

which is equivalent to (4.8). If $\delta_0 = 0$, then by applying the two-sided Laplace transform we get that (4.8) is equivalent to

$$\int_{-\infty}^{\infty} \exp(-st)\tilde{k}_B \star \left(\tilde{k}_A - \sum_{j=1}^{n} \delta_j \tilde{k}_A^{\star j} \right)(t) d\mu_t = 0,$$

which is equivalent to

$$K_B(s) \cdot (K_A(s) - \textstyle\sum_{j=1}^{n} \delta K_A^j(s)) = 0, \tag{4.10}$$

where $K_B(s) = \int\limits_{-\infty}^{\infty} \exp(-st)\tilde{k}_B(t) d\mu_t,\ K_A(s) = \int\limits_{-\infty}^{\infty} \exp(-st)\tilde{k}_A(t) d\mu_t.$

Equation (4.10) is equivalent to the set $\operatorname{supp} K_B \cap \operatorname{supp} \left(K_A - \sum_{j=1}^{n} \delta_j K_A^j \right)$ to have measure zero in $\mathbb{R}$. □

Proposition 2 *Let* $(\mathbb{R}, \Sigma, \mu)$ *be the standard Lebesgue measure space. Let*

$$A : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu),\ B : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu),\ 1 < p < \infty$$

be non-zero operators defined as follows

$$(Ax)(t) = \int\limits_{\mathbb{R}} \tilde{k}_A(t-s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{\mathbb{R}} \tilde{k}_B(t-s)x(s)d\mu_s, \tag{4.11}$$

where $\tilde{k}_A(\cdot) \in L_1(\mathbb{R}, \mu)$, $\tilde{k}_B(\cdot) \in L_1(\mathbb{R}, \mu)$, *that is,*

$$\int\limits_{\mathbb{R}} |\tilde{k}_A(t)| d\mu_t < \infty, \quad \int\limits_{\mathbb{R}} |\tilde{k}_B(t)| d\mu_t < \infty. \tag{4.12}$$

and the index in μ *indicates the variable of integration. Suppose that*

$$\int_{-\infty}^{\infty} \exp(-st)\tilde{k}_A(t)d\mu_t = K_A(s), \quad \int_{-\infty}^{\infty} \exp(-st)\tilde{k}_B(t)d\mu_t = K_B(s)$$

exist and the domain of $K_A(\cdot)$ is equal to the domain of $K_B(\cdot)$ with exception of a set of measure zero. Then, $AB = \delta BA^n$, for a fixed $n \in \mathbb{Z}$, $n \geq 2$ and $\delta \in \mathbb{R} \setminus \{0\}$ if and only if $(\tilde{k}_A \star \tilde{k}_B)(t) = 0$ almost everywhere.

Proof Operators A and B are well defined by Young theorem ([6], Theorem 4.15). Let $n \geq 1$. By Fubbini Theorem for composition of operators A, B and A^n, similarly to the proof of Theorem 2 when $k_A(t,s) = \tilde{k}_A(t-s)$, $k_B(t,s) = \tilde{k}_B(t-s)$, $G_A = G_B = \mathbb{R}$, we get from Lemma 2 that $AB = \delta BA^n$ if and only if, for almost every $(t,s) \in \mathbb{R}^2$,

$$\int_{\mathbb{R}} \tilde{k}_A(t-\tau)\tilde{k}_B(\tau-s)d\mu_\tau = \int_{\mathbb{R}} \delta\tilde{k}_B(t-\tau)\tilde{k}_{n-1,A}(\tau,s)d\mu_\tau, \tag{4.13}$$

$$\tilde{k}_{0,A}(t,s) = \tilde{k}_A(t-s),\ \tilde{k}_{n,A}(t,s) = \int_{\mathbb{R}} \tilde{k}_A(t-\tau)\tilde{k}_{n-1,A}(\tau,s)d\mu_\tau,\ n \geq 1.$$

Computing $\tilde{k}_{n,A}(t,s)$, we get for $n = 1$,

$$\tilde{k}_{1,A}(t,s) = \int_{\mathbb{R}} \tilde{k}_A(t-\tau)\tilde{k}_A(\tau-s)d\mu_\tau = \int_{\mathbb{R}} \tilde{k}_A(t-s-\nu)\tilde{k}_A(\nu)d\mu_\nu = (\tilde{k}_A \star \tilde{k}_A)(t-s),$$

for $n = 2$,

$$\begin{aligned}\tilde{k}_{2,A}(t,s) &= \int_{\mathbb{R}} \tilde{k}_A(t-\tau)(\tilde{k}_A \star \tilde{k}_A(\tau-s)d\mu_\tau = \\ &= \int_{\mathbb{R}} \tilde{k}_A(t-s-\nu)(\tilde{k}_A \star \tilde{k}_A(\nu)d\mu_\nu = (\tilde{k}_A \star \tilde{k}_A \star \tilde{k}_A)(t-s).\end{aligned}$$

and for all $n \geq 2$,

$$\tilde{k}_{n-1,A}(t,s) = (\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t-s), \tag{4.14}$$

$$\begin{aligned}\int_{\mathbb{R}} \tilde{k}_B(t-s)(\tilde{k}_{n-1,A}(s,\tau))d\mu_s &= \int_{\mathbb{R}} \tilde{k}_B(t-\tau)\cdot(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(\tau-s)d\mu_\tau \\ &= \int_{\mathbb{R}} \tilde{k}_B(t-s-\nu)\cdot(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(\nu)d\mu_\nu.\end{aligned}$$

Therefore, for almost all pairs $(t,s) \in \mathbb{R}^2$, the equality (4.13) is equivalent to

$$(\tilde{k}_A \star \tilde{k}_B)(t-s) = \delta \tilde{k}_B \star (\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t-s),$$

which is equivalent to

$$(\tilde{k}_A \star \tilde{k}_B)(t) = \delta \tilde{k}_B \star (\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t) \tag{4.15}$$

almost everywhere. By applying the two-sided Laplace transform in both cases $n \geq 2$ we get that (4.15) is equivalent to $\int\limits_{-\infty}^{\infty} \exp(-st)\tilde{k}_B \star \left(\tilde{k}_A - \delta \tilde{k}_A^{\star n}\right)(t) d\mu_t = 0$ almost everywhere, which can be written as follows

$$K_B(s) \cdot \big(K_A(s) - \delta K_A^n(s)\big) = 0, \ n \geq 2, \tag{4.16}$$

almost everywhere, $K_B(s) = \int\limits_{-\infty}^{\infty} \exp(-st)\tilde{k}_B(t) d\mu_t$, $K_A(s) = \int\limits_{-\infty}^{\infty} \exp(-st)\tilde{k}_A(t) d\mu_t$. Equation (4.16) is equivalent to the set $\operatorname{supp} K_B \cap \operatorname{supp} \left(K_A - \delta K_A^n\right)$, $n \geq 2$, to have measure zero in $\mathbb{R}$, that is, $K_B(\cdot) \cdot I_{(\operatorname{supp}(K_A - \delta K_A^n))}(\cdot) = 0$ almost everywhere and $(K_A(\cdot) - \delta K_A^n(\cdot)) \cdot I_{(\operatorname{supp} K_B)}(\cdot) = 0$ almost everywhere, where $I_E(\cdot)$ is the indicator function of the set E. If $\operatorname{supp}(K_A - \delta K_A^n) = \mathbb{R}$ then $\operatorname{supp} K_B$ has measure zero, that is, $B = 0$. Similarly, if $\operatorname{supp} K_B = \mathbb{R}$ then $A = 0$. Suppose that $\operatorname{supp} K_B \neq \mathbb{R}$ and has positive measure. If $(K_A(\cdot) - \delta K_A^n(\cdot)) \cdot I_{(\operatorname{supp} K_B)}(\cdot) = 0$ almost everywhere, then $K_A(s) - \delta K_A^n(s) = 0$ for almost every $s \in \operatorname{supp} K_B$. Let $p(z) = z - \delta z^n$. Suppose that $p(z)$ has $m > 0$ roots z_i, $i = 1, 2, \ldots, m$, $m \leq n$, $n \geq 2$. We consider the following cases:

- If $n > 1$ and $p(z)$ has $m \geq 2$ roots z_i, $i = 1, 2, \ldots, m$, $m \leq n$, then $\tilde{k}_A(t) = \sum\limits_{i=1}^{m} z_i \Delta(t - z_i)$, where $\Delta(t - t_0)$, $t, t_0 \in \mathbb{R}$, is the Dirac function defined as follows $\Delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$. In this case $K_A(s) - \delta K_A^n(s) = 0$ for almost every s in $\operatorname{supp} K_B$. But this implies $K_A(s) = 0$ for almost every s in $\operatorname{supp} K_B$ since the Dirac function $\Delta(\cdot)$ is equivalent to zero function.
- If $n > 1$ and $p(z)$ has only one real root, which is $z = 0$, then $\operatorname{supp} \left(K_A - \delta K_A^n\right) = \operatorname{supp} K_A$ for all s in $\operatorname{supp} K_B$. This implies that Equality (4.16) is satisfied if and only if $K_A(\cdot) = 0$ almost everywhere in $\operatorname{supp} K_B$.

In both cases we conclude that $K_A(\cdot) = 0$ almost everywhere in $\operatorname{supp} K_B$. Outside of $\operatorname{supp} K_B$, the function $K_A(\cdot)$ can be nonzero. This implies that Equality (4.16) is equivalent to $K_A(s)K_B(s) = 0$ almost everywhere. This is equivalent to $(\tilde{k}_A \star \tilde{k}_B)(t) = 0$ almost everywhere.

Remark 4 Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space. The operators A and B defined in (4.11) as $A : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu)$ and $B : L_p(\mathbb{R}, \mu) \to$

$L_p(\mathbb{R},\mu)$, $1<p<\infty$, $(Ax)(t)=\int\limits_{\mathbb{R}}\tilde{k}_A(t-s)x(s)d\mu_s$, $(Bx)(t)=\int\limits_{\mathbb{R}}\tilde{k}_B(t-s)$ $x(s)d\mu_s$, almost everywhere, with $\tilde{k}_A(\cdot)\in L_1(\mathbb{R},\mu)$, $\tilde{k}_B(\cdot)\in L_1(\mathbb{R},\mu)$, commute, that is $AB=BA$. In fact, by applying Fubbini theorem for composition of A, B and Lemma 2,

$$AB=BA \Leftrightarrow \int\limits_{\mathbb{R}}\tilde{k}_A(t-s)\tilde{k}_B(s-\tau)ds=\int\limits_{\mathbb{R}}\tilde{k}_B(t-s)\tilde{k}_A(s-\tau)d\mu_s$$
$$\Leftrightarrow (\tilde{k}_A\star\tilde{k}_B)(t-\tau)=(\tilde{k}_B\star\tilde{k}_A)(t-\tau) \text{ for almost every } (t,\tau)\in\mathbb{R}^2,$$

which holds true by the commutativity property of convolution.

Remark 5 If operators A and B commute then they satisfy, simultaneously, the following relations $AB=BF(A)$, $BA=F(A)B$ and $B(A-F(A))=0$. In fact, if A and B commute, then $AB=BA$, $BF(A)=F(A)B$, and thus $AB=BF(A)$ is equivalent to $BA=F(A)B$, which can be then written also as $B(A-F(A))=0$.

Proposition 3 *Let* $([0,\infty),\Sigma,\mu)$ *be the standard Lebesgue measure space. Let*

$$A:L_p([0,\infty),\mu)\to L_p([0,\infty),\mu),\ B:L_p([0,\infty),\mu)\to L_p([0,\infty),\mu), 1<p<\infty$$

be non-zero operators defined by

$$\begin{aligned}
&(Ax)(t)=\int\limits_0^\infty \tilde{k}_A(t-s)I_{[0,\infty)}(t-s)x(s)d\mu_s,\\
&(Bx)(t)=\int\limits_0^\infty \tilde{k}_B(t-s)I_{[0,\infty)}(t-s)x(s)d\mu_s,\\
&\tilde{k}_A(\cdot)\in L_1([0,\infty),\mu),\ \tilde{k}_B(\cdot)\in L_1([0,\infty),\mu)\\
&\text{that is, } \int\limits_0^\infty|\tilde{k}_A(t)|d\mu_t<\infty,\ \int\limits_0^\infty|\tilde{k}_B(t)|d\mu_t<\infty,
\end{aligned} \tag{4.17}$$

where $I_E(\cdot)$ *is the indicator function of the set* E *and the index in* μ *is the variable of integration. Then, there are no non-zero operators* A *and* B *satisfying* $AB=\delta BA^n$ *for a fixed* $n\in\mathscr{Z}$, $n\ge 2$, $\delta\in\mathbb{R}\setminus\{0\}$.

Proof Operators A and B are well defined by Young's theorem ([6], Theorem 4.15). Let $n\ge 1$. By applying Fubbini theorem for composition of operators A, B and A^n, similarly to the proof of Theorem 2 when $k_A(t,s)=\tilde{k}_A(t-s)I_{[0,\infty)}$, $k_B(t,s)=\tilde{k}_B(t-s)I_{[0,\infty)}(t-s)$ and $G_1=G_2=[0,\infty)$, we get from Lemma 2 that $AB=\delta BA^n$ if and only if for almost every $(t,s)\in\mathbb{R}^2$,

$$\begin{aligned}
&\int\limits_0^\infty \tilde{k}_A(t-\tau)I_{[0,\infty)}(t-\tau)\tilde{k}_B(\tau-s)I_{[0,\infty)}(\tau-s)d\mu_\tau \\
&\qquad = \int\limits_0^\infty \tilde{k}_B(t-\tau)(\tilde{k}_{n-1,A}(\tau,s))d\mu_\tau,
\end{aligned} \tag{4.18}$$

$$\tilde{k}_{0,A}(t,s) = \tilde{k}_A(t-s)I_{[0,\infty)}(t-s),$$

$$\tilde{k}_{n,A}(t,s) = \int\limits_0^\infty \tilde{k}_A(t-\tau)I_{[0,\infty)}(t-\tau)k_{n-1,A}(\tau,s)d\mu_\tau, \quad n \geq 1.$$

Computing $\tilde{k}_{n-1,A}(t,s)$ for $n \geq 1$, using (4.14), yields

$$\begin{aligned}
\tilde{k}_{n-1,A}(t,s) &= (\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t-s)I_{[0,\infty)}(t-s) = \\
&= \int\limits_s^t \tilde{k}_A(t-\tau)\cdot(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n-1 \text{ times}})(\tau-s)d\mu_\tau = \\
&= \int\limits_0^{t-s} \tilde{k}_A(t-s-\nu)\cdot(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n-1 \text{ times}})(\nu)d\mu_\nu \\
&= (\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t-s).
\end{aligned}$$

Therefore, from (4.18) we have for $n \geq 2$,

$$\int\limits_0^{t-s} \tilde{k}_A(t-s-\tau)\tilde{k}_B(\tau)d\mu_\tau = \int\limits_0^{t-s} \tilde{k}_B(t-s-\tau)\delta(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(\tau)d\mu_\tau$$

which we can write as follows

$$(\tilde{k}_A \star \tilde{k}_B)(t-s) = \delta(\tilde{k}_B \star (\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}}))(t-s). \tag{4.19}$$

By commutativity, linearity of convolution and the Titchmarsh convolution theorem, (4.19) is equivalent to either

$$\tilde{k}_B(t-s) = 0 \text{ or } \delta(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t-s) = \tilde{k}_A(t-s)$$

for almost every $(t,s) \in \mathbb{R}^2$ such that $t \geq 0$, $0 \leq s \leq t$. This is equivalent to either

$$\tilde{k}_B(t) = 0 \text{ or } \delta(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t) = \tilde{k}_A(t)$$

almost everywhere, $n \geq 2$. Suppose that $\tilde{k}_B(t) \neq 0$ for almost every t in a set of positive measure. Then $\delta(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t) = \tilde{k}_A(t)$ for almost every $t \in [0, \infty)$. By applying the one sided Laplace transform $K_A(s) = \int\limits_0^\infty \tilde{k}_A(t) \exp(-ts) dt$, which exists for certain $s > 0$ since $\exp(-st) \in L_p([0, \infty), \mu)$, $1 < p < \infty$, we have for $n \geq 2$ $\delta(\underbrace{\tilde{k}_A \star \tilde{k}_A \star \ldots \star \tilde{k}_A}_{n \text{ times}})(t) = \tilde{k}_A(t) \Longleftrightarrow \delta K_A^n(s) = K_A(s)$. Let $p(z) = z - \delta z^n$ and suppose that $p(z)$ has $m > 0$ roots z_i, $i = 1, 2, \ldots, m$, $m \leq n$, $n > 1$. We consider the following cases:

- If $n > 1$ and $p(z)$ has $m \geq 2$ roots, then $\tilde{k}_A(t) = \sum\limits_{i=1}^{m} z_i \Delta(t - z_i)$, In this case $K_A(s) - \delta K_A^n(s) = 0$ for all s in the domain of $K_A(\cdot)$. But this implies $A = 0$ since the Dirac function $\Delta(\cdot)$ is equivalent to zero function.
- If $n > 1$ and $p(z)$ has only one real root, which is $z = 0$, then $K_A(s) - \delta K_A^n(s) = 0$ implies $A = 0$. □

Remark 6 Let $([0, \infty), \Sigma, \mu)$ be the standard Lebesgue measure space. The operators $A : L_p([0, \infty), \mu) \to L_p([0, \infty), \mu)$, $B : L_p([0, \infty), \mu) \to L_p([0, \infty), \mu)$, $1 \leq p < \infty$, defined in (4.17) as

$$(Ax)(t) = \int\limits_0^\infty \tilde{k}_A(t - s) \cdot I_{[0,\infty)}(t - s) x(s) d\mu_s, \quad (Bx)(t) = \int\limits_0^\infty \tilde{k}_B(t - s) \cdot I_{[0,\infty)} x(s) d\mu_s,$$

almost everywhere, with $\tilde{k}_A(\cdot) \in L_1([0, \infty), \mu)$, $\tilde{k}_B(\cdot) \in L_1([0, \infty), \mu)$ (where $I_E(\cdot)$ denotes the indicator function of the set E, and the index in μ indicates the variable of integration) commute, $AB = BA$. In fact, by applying Fubbini theorem for composition of operators A, B and Lemma 2 we have $AB = BA$ if and only if

$$\begin{aligned}
&\int\limits_0^\infty \tilde{k}_A(t - s) \cdot I_{[0,\infty)}(t - s) \tilde{k}_B(s - \tau) \cdot I_{[0,\infty)}(s - \tau) d\mu_s \\
&\quad = \int\limits_0^\infty \tilde{k}_B(t - s) \cdot I_{[0,\infty)}(t - s) \tilde{k}_A(s - \tau) \cdot I_{[0,\infty)}(s - \tau) d\mu_s \quad \Leftrightarrow \\
&\int\limits_\tau^t \tilde{k}_A(t - s) \tilde{k}_B(s - \tau) d\mu_s = \int\limits_\tau^t \tilde{k}_B(t - s) \cdot \tilde{k}_A(s - \tau) d\mu_s \Leftrightarrow \\
&\int\limits_0^{t-\tau} \tilde{k}_A(t - \tau - \nu) \tilde{k}_B(\nu) d\mu_\nu = \int\limits_0^{t-\tau} \tilde{k}_B(t - \tau - \nu) \cdot \tilde{k}_A(\nu) d\mu_\nu,
\end{aligned} \tag{4.20}$$

for almost every $(t, \tau) \in \mathbb{R}^2$. By changing variable $\xi = t - \tau - \nu$ on the right hand side of (4.20) we get

$$\int_{0}^{t-\tau} \tilde{k}_B(t-\tau-\nu)\cdot\tilde{k}_A(\nu)d\mu_\nu = -\int_{t-\tau}^{0} \tilde{k}_B(\xi)\cdot\tilde{k}_A(t-\tau-\xi)d\mu_\xi$$
$$= \int_{0}^{t-\tau} \tilde{k}_A(t-\tau-\xi)\cdot\tilde{k}_B(\xi)d\mu_\xi$$

which proves (4.20). This completes the proof.

In the following theorem we consider a special case of operators in Theorem 1 when the kernels have the separated variables.

Theorem 3 *Let (X,Σ,μ) be σ-finite measure space. Let $A: L_p(X,\mu)\to L_p(X,\mu)$, $B: L_p(X,\mu)\to L_p(X,\mu)$, $1\le p\le\infty$ be nonzero operators defined as follows*

$$(Ax)(t)=\int_{G_A} a(t)b(s)x(s)d\mu_s,\quad (Bx)(t)=\int_{G_B} c(t)e(s)x(s)d\mu_s, \tag{4.21}$$

almost everywhere, where the index in μ_s indicates the variable of integration, $G_A\in\Sigma$ and $G_B\in\Sigma$ with finite measure, $a,c\in L_p(X,\mu)$, $b\in L_q(G_A,\mu)$, $e\in L_q(G_B,\mu)$, $1\le q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Consider a polynomial $F:\mathbb{R}\to\mathbb{R}$ defined by $F(z)=\sum_{j=0}^{n}\delta_j z^j$, where $\delta_j\in\mathbb{R}$ $j=0,1,2,\ldots,n$. let $G=G_A\cap G_B$, and

$$k_1=\sum_{j=1}^{n}\delta_j Q_{G_A}(a,b)^{j-1}Q_{G_B}(a,e),\quad k_2=Q_{G_B}(b,c),$$

where $Q_\Lambda(u,v)$, $\Lambda\in\Sigma$, is defined by (4.2). Then $AB=BF(A)$ if and only if the following conditions are fulfilled:

1. *(a) for almost every $(t,s)\in \operatorname{supp} c\times[(\operatorname{supp} e)\cap G]$, we have;*
   - *(i) if $k_2\neq 0$ then $b(s)k_1=\lambda e(s)$ and $a(t)=\frac{(\delta_0+\lambda)c(t)}{k_2}$ for some real scalar λ,*
   - *(ii) if $k_2=0$ then $k_1 b(s)=-\delta_0 e(s)$.*

   (b) If $t\notin\operatorname{supp} c$ then either $k_2=0$ or $a(t)=0$ for almost all $t\notin\operatorname{supp} c$.

   (c) If $s\in G\setminus\operatorname{supp} e$ then either $k_1=0$ or $b(s)=0$ for almost all $s\in G\setminus\operatorname{supp} e$.
2. *$k_2a(t)-\delta_0c(t)=0$ for almost every $t\in X$ or $e(s)=0$ for almost every $s\in G_B\setminus G$.*
3. *$k_1=0$ or $b(s)=0$ for almost every $s\in G_A\setminus G$.*

Proof We observe that since $a,c\in L_p(X,\mu)$, $1\le p\le\infty$, $b\in L_q(G_A,\mu)$, $e\in L_q(G_B,\mu)$, where $1\le q\le\infty$, with $\frac{1}{p}+\frac{1}{q}=1$, then either condition (4.3) or (4.4) or (4.5) is satisfied and therefore the operators A and B are well-defined. By direct calculation, we have

$$(A^2x)(t) = \int\limits_{G_A} a(t)b(s)(Ax)(s)d\mu_s = \int\limits_{G_A} a(t)b(s)a(s)d\mu_s \int\limits_{G_A} b(\tau_1)x(\tau_1)d\mu_{\tau_1}$$
$$= Q_{G_A}(a,b)(Ax)(t),$$

$$(A^3x)(t) = A(A^2x)(t) = Q_{G_A}(a,b)(A^2x)(t) = Q_{G_A}(a,b)^2(Ax)(t)$$

almost everywhere. We suppose that

$$(A^m x)(t) = Q_{G_A}(a,b)^{m-1}(Ax)(t), \quad m = 1, 2, \ldots$$

almost everywhere. Then

$$(A^{m+1}x)(t) = A(A^m x)(t) = Q_{G_A}(a,b)^{m-1}(A^2x)(t) = Q_{G_A}(a,b)^m(Ax)(t)$$

almost everywhere. Then, we compute

$$\begin{aligned}(ABx)(t) &= \int\limits_{G_A} a(t)b(s)c(s)d\mu_s \int\limits_{G_B} e(\tau_1)x(\tau_1)d\mu_{\tau_1} \\ &= k_2 \int\limits_{G_B} a(t)e(\tau_1)x(\tau_1)d\mu_{\tau_1},\end{aligned} \tag{4.22}$$

$$(F(A)x)(t) = \delta_0 x(t) + a(t)\sum_{j=1}^{n} \delta_j \left(Q_{G_A}(a,b)\right)^{j-1} \int\limits_{G_A} b(\tau)x(\tau)d\mu_\tau,$$

$$(BF(A)x)(t) = \delta_0 c(t) \int\limits_{G_B} e(\tau_1)x(\tau_1)d\mu_{\tau_1} \tag{4.23}$$

$$+ c(t)\sum_{j=1}^{n} \delta_j \left(Q_{G_A}(a,b)\right)^{j-1} \int\limits_{G_B} e(\tau)a(\tau)d\mu_\tau \int\limits_{G_B} b(\tau_1)x(\tau_1)d\mu_{\tau_1}$$

$$= \delta_0 c(t) \int\limits_{G_B} e(\tau_1)x(\tau_1)d\mu_{\tau_1} + c(t)k_1 \int\limits_{G_A} b(\tau_1)x(\tau_1)d\mu_{\tau_1}. \tag{4.24}$$

Thus, $(ABx)(t) = (BF(A)x)(t)$ for all $x \in L_p(X,\mu)$ if and only if

$$\int\limits_{G_B} [k_2 a(t) - \delta_0 c(t)]e(s)x(s)d\mu_s = \int\limits_{G_A} k_1 c(t)b(s)x(s)d\mu_s.$$

Then by Lemma 1, $AB = BF(A)$ if and only if

1. for almost every $(t,s) \in X \times G$,

$$[k_2a(t)-\delta_0c(t)]e(s)=k_1c(t)b(s);$$

2. $k_2a(t)-\delta_0c(t)=0$ for almost every $t\in X$ or $e(s)=0$ for almost every $s\in G_B\setminus G$;
3. $k_1=0$ or $c(t)=0$ for almost every $t\in X$ or $b(s)=0$ for almost every $s\in G_A\setminus G$.

We can rewrite the first condition as follows:

(a) Suppose $(t,s)\in \operatorname{supp} c\times[(\operatorname{supp} e)\cap G]$.
   (i) If $k_2\neq 0$, then $k_1\frac{b(s)}{e(s)}=k_2\frac{a(t)}{c(t)}-\delta_0=\lambda$ for some real scalar λ. From this, it follows that $k_1b(s)=e(s)\lambda$ and $a(t)=\frac{\delta_0+\lambda}{k_2}c(t)$.
   (ii) If $k_2=0$ then $-\delta_0c(t)e(s)=k_1c(t)b(s)$ from which we get that $k_1b(s)=-\delta_0e(s)$.

(b) If $t\notin \operatorname{supp} c$ then $k_2a(t)e(s)=0$ from which we get that either $k_2=0$ or $a(t)=0$ for almost all $t\notin \operatorname{supp} c$ or $e(s)=0$ almost everywhere (this implies $B=0$).
(c) If $s\in G\setminus \operatorname{supp} e$, then $k_1c(t)b(s)=0$ which implies that either $k_1=0$ or $b(s)=0$ for almost all $s\in G\setminus \operatorname{supp} e$, or $c(t)=0$ almost everywhere (this implies that $B=0$). □

Remark 7 Observe that operators A and B as defined in (4.21) take the form $(Ax)(t)=a(t)\phi(x)$ and $(Bx)(t)=c(t)\psi(x)$ for some functions $a,c\in L_p(X,\mu)$, $1\le p\le\infty$ and linear functionals $\phi,\psi: X\to\mathbb{R}$. In this case $AB=BF(A)$ if and only in $\phi(\psi(x)c(t))a(t)=\psi(F(\phi(x)a(t)))c(t)$ in $L_p(X,\mu)$, $1\le p\le\infty$.

Corollary 3 *Let (X,Σ,μ) be a σ-finite measure space. Let $A: L_p(X,\mu)\to L_p(X,\mu)$, $B: L_p(X,\mu)\to L_p(X,\mu)$, $1\le p\le\infty$ be nonzero operators such that*

$$(Ax)(t)=\int\limits_G a(t)b(s)x(s)d\mu_s,\quad (Bx)(t)=\int\limits_G c(t)e(s)x(s)d\mu_s,$$

(the index in μ_s indicates the variable of integration)

almost everywhere, $G\in\Sigma$ is a set with finite measure, $a,c\in L_p(X,\mu)$, $b,e\in L_q(G,\mu)$, $1\le q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Consider a polynomial $F(z)=\delta_0+\delta_1z+\ldots+\delta_nz^n$, where $z\in\mathbb{R}$, $\delta_j\in\mathbb{R}$, $j=0,1,2,\ldots,n$. Set

$$k_1=\sum_{j=1}^{n}\delta_jQ_G(a,b)^{j-1}Q_G(a,e),\quad k_2=Q_G(b,c),$$

Then $AB=BF(A)$ if and only if the following is true

1. *for almost every $(t,s)\in \operatorname{supp} c\times \operatorname{supp} e$, we have*
   a) *If $k_2\neq 0$, then $k_1b(s)=e(s)\lambda$ and $a(t)=\frac{\delta_0+\lambda}{k_2}c(t)$ for some $\lambda\in\mathbb{R}$.*
   b) *If $k_2=0$ then $k_1b(s)=-\delta_0e(s)$;*

2. *If* $t \notin \operatorname{supp} c$ *then either* $k_2 = 0$ *or* $a(t) = 0$ *for almost all* $t \notin \operatorname{supp} c$.
3. *If* $s \in G \setminus \operatorname{supp} e$, *then either* $k_1 = 0$ *or* $b(s) = 0$ *for almost all* $s \in G \setminus \operatorname{supp} e$

Proof This follows immediately from Theorem 3 as $G_A = G_B = G$. □

Remark 8 From Theorem 3 and Corollary 3 we observe that if $k_1, k_2 \neq 0$, then given operator B as defined by (4.21), we can obtain the kernel of operator A using relations, $a(t) = \frac{\delta_0+\lambda}{k_2}c(t)$ and $b(s) = \frac{\lambda}{k_1}e(s)$ for some $\lambda \in \mathbb{R}$. In the next two propositions we state necessary and sufficient conditions for the choice of λ.

Proposition 4 *Let* (X, Σ, μ) *be a* σ*-finite measure space. Let*

$$A : L_p(X,\mu) \to L_p(X,\mu),\ \ B : L_p(X,\mu) \to L_p(X,\mu),\ \ 1 \le p \le \infty$$

be nonzero operators such that

$$(Ax)(t) = \int\limits_G a(t)b(s)x(s)d\mu_s,\ \ (Bx)(t) = \int\limits_G c(t)e(s)x(s)d\mu_s,$$

(the index in μ_s *indicates the variable of integration)*

almost everywhere, $G \in \Sigma$ *is a set with finite measure, and* $a, c \in L_p(X,\mu)$, $b, e \in L_q(G,\mu)$, $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Consider a polynomial* $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, *where* $z \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0, 1, 2, 3, \ldots, n$. *Set*

$$k_1 = \sum_{j=1}^{n} \delta_j Q_G(a,b)^{j-1} Q_G(a,e), \quad k_2 = Q_G(b,c).$$

Suppose that $AB = BF(A)$. *If* $k_2 \neq 0$ *and* $k_1 \neq 0$ *in condition 1(a) in Corollary 3, then the corresponding nonzero* λ *satisfy*

$$F(\lambda + \delta_0) = \lambda + \delta_0. \tag{4.25}$$

Proof By definition $k_1 = \sum_{j=1}^{n} \delta_j Q_G(a,b)^{j-1} Q_G(a,e)$, $\quad k_2 = Q_G(b,c)$. If $k_1 \neq 0$, $k_2 \neq 0$, by condition 1(a) in Corollary 3 we have $a(t) = \frac{\lambda+\delta_0}{k_2}c(t)$, $\quad b(s) = \frac{\lambda}{k_1}e(s)$ almost everywhere. If $\lambda \neq 0$ then we replace $k_2 = Q_G(b,c) = Q_G(\frac{\lambda}{k_1}e, c)$ in the following equality $k_1 = \sum_{j=1}^{n} \delta_j Q_G\left(\frac{\lambda+\delta_0}{k_2}c, \frac{\lambda}{k_1}e\right)^{j-1} Q_G\left(\frac{\lambda+\delta_0}{k_2}c, e\right)$. Then, by using the bilinearity of $Q_G(\cdot,\cdot)$ and after simplification, this is equivalent to $\lambda = \sum_{j=1}^{n} \delta_j (\lambda+\delta_0)^j$. By adding δ_0 on both sides we can write this as (4.25). □

Proposition 5 *Let* (X, Σ, μ) *be a* σ*-finite measure space. Let*

$$A : L_p(X,\mu) \to L_p(X,\mu),\ \ B : L_p(X,\mu) \to L_p(X,\mu),\ \ 1 \le p \le \infty$$

be nonzero operators such that

$$(Ax)(t) = \int_G a(t)b(s)x(s)d\mu_s, \quad (Bx)(t) = \int_G c(t)e(s)x(s)d\mu_s,$$

(the index in μ_s indicates the variable of integration)

almost everywhere, $G \in \Sigma$ is a set with finite measure, $a, c \in L_p(X, \mu)$, $b, e \in L_q(G, \mu)$, $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Consider a polynomial $F(z) = \delta_0 + \delta_1 z + \ldots + \delta_n z^n$, where $z \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0, 1, 2, 3, \ldots, n$. Suppose that for almost every $(t, s) \in \operatorname{supp} c \times \operatorname{supp} e$, we have

$$a(t) = \frac{\lambda + \delta_0}{k_2} c(t), \quad b(s) = \frac{\lambda}{k_1} e(s)$$

for nonzero constants λ, k_1 and k_2. If $F(\lambda + \delta_0) = \lambda + \delta_0$ and $k_2 = \frac{\lambda}{k_1} Q_G(e, c)$, then

1. $A = \frac{\lambda+\delta_0}{Q_G(e,c)} B$,
2. *for all $x \in L_p(X, \mu)$ and almost all $t \in \operatorname{supp} c$, $(ABx)(t) = (BF(A)x)(t)$.*

Proof We have, almost everywhere,

$$(Ax)(t) = \int_G a(t)b(s)x(s)d\mu_s = \frac{(\lambda + \delta_0)\lambda}{k_1 k_2} \int_G c(t)e(s)x(s)d\mu_s = \frac{(\lambda + \delta_0)\lambda}{k_1 k_2} (Bx)(t)$$

$$(ABx)(t) = \frac{(\lambda + \delta_0)\lambda}{k_1 k_2} (B^2 x)(t) = \frac{(\lambda + \delta_0)\lambda}{k_1 k_2} Q_G(c, e)(Bx)(t),$$

$$(A^2 x)(t) = \left(\frac{(\lambda + \delta_0)\lambda}{k_1 k_2}\right)^2 (B^2 x)(t) = \left(\frac{(\lambda + \delta_0)\lambda}{k_1 k_2}\right)^2 Q_G(e, c)(Bx)(t).$$

Similarly, for $m \ge 2$, almost everywhere

$$(A^m x)(t) = \left(\frac{(\lambda + \delta_0)\lambda}{k_1 k_2}\right)^m Q_G(c, e)^{m-1}(Bx)(t)$$

$$(F(A)x)(t) = \delta_0 (Bx)(t) + \sum_{j=1}^{n} \delta_j \left(\frac{(\lambda + \delta_0)\lambda}{k_1 k_2}\right)^j Q_G(c, e)^{j-1}(Bx)(t).$$

Therefore, almost everywhere,

$$\begin{aligned}(BF(A)x)(t) &= \delta_0 (B^2 x)(t) + \sum_{j=1}^{n} \delta_j \left(\frac{(\lambda + \delta_0)\lambda}{k_1 k_2}\right)^j Q_G(c, e)^{j-1}(B^2 x)(t) \\ &= \delta_0 Q_G(c, e)(Bx)(t) + \sum_{j=1}^{n} \delta_j \left(\frac{(\lambda + \delta_0)\lambda}{k_1 k_2}\right)^j Q_G(c, e)^{j}(Bx)(t) \\ &= F\left(\frac{(\lambda + \delta_0)\lambda}{k_1 k_2} Q_G(c, e)\right)(Bx)(t),\end{aligned}$$

Hence, $(ABx)(t) = (BF(A)x)(t)$, for all $x \in L_p(X,\mu)$ and almost all $t \in \operatorname{supp} c$ if and only if

$$\frac{(\lambda+\delta_0)\lambda}{k_1k_2} Q_G(c,e) = F\left(\frac{(\lambda+\delta_0)\lambda}{k_1k_2} Q_G(c,e)\right) \tag{4.26}$$

for almost every $t \in \operatorname{supp} c$. If $k_2 = \frac{\lambda}{k_1} Q_G(c,e)$ and λ satisfies (4.25), then (4.26) holds. □

Corollary 4 *Let (X,Σ,μ) be a σ-finite measure space. Let $A : L_p(X,\mu) \to L_p(X,\mu)$, $B : L_p(X,\mu) \to L_p(X,\mu)$, $1 < p < \infty$ be nonzero operators such that*

$$(Ax)(t) = \int_G a(t)b(s)x(s)d\mu_s, \quad (Bx)(t) = \int_G c(t)e(s)x(s)d\mu_s,$$

(the index in μ_s indicates the variable of integration)

almost everywhere, $G \in \Sigma$ is a set with finite measure, $a, c \in L_p(X,\mu)$, $b, e \in L_q(G,\mu)$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Consider a polynomial $F(z) = \delta_0 + \delta_1 z + \delta_2 z^2$, where $z \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0, 1, 2$. Suppose that for almost every $(t,s) \in \operatorname{supp} c \times \operatorname{supp} e$, $a(t) = \frac{\lambda+\delta_0}{k_2} c(t)$, $b(s) = \frac{\lambda}{k_1} e(s)$ for nonzero constants λ, k_1 and k_2. If $k_2 = \frac{\lambda}{k_1} Q_G(e,c)$, then $(ABx)(t) = (BF(A)x)(t)$, for all $x \in L_p(G,\mu)$ and almost all $t \in \operatorname{supp} c$ if either $\delta_0\delta_2 < 0$, or $\delta_0\delta_2 \geq 0$ and either $\delta_1 \geq 1 + 2\sqrt{\delta_0\delta_2}$ or $\delta_1 \leq 1 - 2\sqrt{\delta_0\delta_2}$.

Proof From Propositions 4 and 5 we have that $AB = BF(A)$ if $F(\lambda+\delta_0) = \lambda + \delta_0$. This is equivalent to

$$\delta_2\lambda^2 + (2\delta_0\delta_2 + \delta_1 - 1)\lambda + \delta_2\delta_0^2 + \delta_1\delta_0 = 0 \tag{4.27}$$

Equation (4.27) has real solutions if and only if $(\delta_1 - 1)^2 - 4\delta_0\delta_2 \geqslant 0$. This is equivalent to either $\delta_0\delta_2 < 0$, or $\delta_0\delta_2 \geq 0$ and either $\delta_1 \geq 1 + 2\sqrt{\delta_0\delta_2}$ or $\delta_1 \leq 1 - 2\sqrt{\delta_0\delta_2}$, which completes the proof. □

Example 2 Let $(\mathbb{R},\Sigma,\mu)$ be the standard Lebesgue measure space. Let

$$A : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu),\ B : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu),\ 1 < p < \infty$$

be nonzero operators defined as follows

$$(Ax)(t) = \int_0^1 a(t)b(s)x(s)ds, \quad (Bx)(t) = \int_0^1 c(t)e(s)x(s)ds,$$

where $a \in L_p(\mathbb{R},\mu)$, $b \in L_q([0,1],\mu)$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $c(t) = tI_{[0,1]}(t)$, $e(s) = s + 1$. Consider the polynomial $F(z) = z^2 + z - 1$ and suppose that for almost every $(t,s) \in \operatorname{supp} c \times \operatorname{supp} e$, $a(t) = \frac{\lambda+\delta_0}{k_2}c(t)$, $b(s) = \lambda e(s)$ for nonzero constants λ and $k_2 = \lambda Q_{[0,1]}(e,c) = \frac{5}{6}\lambda$. From Propositions 4 and 5 we have that $AB = BF(A)$ if $F(\lambda-1) = \lambda-1$, or $\lambda^2 - 2\lambda = 0$. Therefore, we take $\lambda = 2$. Then, $A = \frac{\lambda+\delta_0}{Q_{[0,1]}(e,c)}B = \frac{6}{5}B$. Hence, $A^2 = \left(\frac{6}{5}B\right)\left(\frac{6}{5}B\right) = \left(\frac{6}{5}\right)^2 B^2$. But

$$(B^2x)(t) = \int_0^1 tI_{[0,1]}(t)(s+1)\int_0^1 sI_{[0,1]}(s)(\tau+1)x(\tau)d\tau = \frac{5}{6}(Bx)(t).$$

Therefore, $A^2 = (\frac{6}{5})^2B^2 = \frac{6}{5}B = A$. Thus, $F(A) = A^2 + A - I = 2A - I = \frac{12}{5}B - I$ and $BF(A) = B\left(\frac{12}{5}B - I\right) = \frac{12}{5}B^2 - B = \frac{12}{5}\cdot\frac{5}{6}B - B = B$. Finally, $AB = \frac{6}{5}B^2 = \frac{6}{5}\cdot\frac{5}{6}B = B = BF(A)$.

Remark 9 Example 2 is a case when operator $B\left(\sum_{k=1}^{1}\delta_k A^k\right) \neq 0$ as mentioned in Remark 1 and Remark 3. In this case we have $G_A = G_B = G = [0,1]$, operators $A : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $B : L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $1 < p < \infty$ are defined as follows $(Ax)(t) = \frac{6}{5}(Bx)(t)$, $(Bx)(t) = \int_0^1 tI_{[0,1]}(t)(s+1)x(s)ds$ almost everywhere, the polynomial is $F(z) = -1 + z + z^2$ with coefficients $\delta_0 = -1$, $\delta_1 = \delta_2 = 1$. We have that Condition 2 and 3 are satisfied because they are taken on the set $\mathbb{R}\times\emptyset = \emptyset$ which has measure zero in $\mathbb{R}\times[0,1]$. Condition 1 is satisfied as showed in Example 2. Moreover, $B(A+A^2) = 2BA = 2\cdot\frac{6}{5}B^2 = 2B = 2\int_0^1 tI_{[0,1]}(t)(s+1)x(s)ds \neq 0$, and $A + A^2 = 2A = \dfrac{12}{5}B = \displaystyle\int_0^1 tI_{[0,1]}(t)(s+1)x(s)ds \neq 0$.

Remark 10 Let (X, Σ, μ) be a σ-finite measure space. From Proposition 5 we have that if $A : L_p(X,\mu) \to L_p(X,\mu)$, $B : L_p(X,\mu) \to L_p(X,\mu)$, $1 \le p \le \infty$ are nonzero operators defined as follows $(Ax)(t) = \int_G a(t)b(s)x(s)d\mu_s$, $(Bx)(t) = \int_G c(t)e(s)x(s)d\mu_s$, almost everywhere, $G \in \Sigma$ is a set with finite measure, $a, c \in L_p(X,\mu)$, $b, e \in L_q(G,\mu)$, $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $F(z) = \delta_0 + \delta_1 z + \ldots + \delta_n z^n$, where $z \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0,1,2,3,\ldots,n$. If we suppose that for almost every $(t,s) \in \operatorname{supp} c \times \operatorname{supp} e$, $a(t) = \frac{\lambda+\delta_0}{k_2}c(t)$ and $b(s) = \frac{\lambda}{k_1}e(s)$ for some nonzero constants λ, k_1 and k_2 and if $F(\lambda+\delta_0) = \lambda+\delta_0$ and $k_2 = \frac{\lambda}{k_1}Q_G(e,c)$, then $A = \frac{\lambda+\delta_0}{Q_G(e,c)}B$ and $AB = BF(A)$. Now suppose that $A = \omega B$ for some $\omega \in \mathbb{R}$, then $AB = BF(A)$ if and only if

$$F(\omega Q_G(c,e)) = \omega Q_G(c,e). \tag{4.28}$$

This relation is the same as Equation (4.26) with $\omega = \frac{(\lambda+\delta_0)\lambda}{k_1k_2}$.

Corollary 5 *Let (X, Σ, μ) be a σ-finite measure space. Let $A : L_p(X, \mu) \to L_p(X, \mu)$, $B : L_p(X, \mu) \to L_p(X, \mu)$, $1 \le p \le \infty$ be nonzero operators defined by $(Ax)(t) = \int\limits_{G_A} a(t)b(s)x(s)d\mu_s$, $(Bx)(t) = \int\limits_{G_B} c(t)e(s)x(s)d\mu_s$, almost everywhere, where $G_A \in \Sigma$, $G_B \in \Sigma$ are sets with finite measure, $a, c \in L_p(X, \mu)$, $b \in L_q(G_A, \mu)$, $e \in L_q(G_B, \mu)$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider a polynomial $F(z) = \delta_0 + \delta_1 z + \ldots + \delta_n z^n$, where $z \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0, 1, 2, 3, \ldots, n$. Let $G = G_A \cap G_B$, and*

$$k_1 = \sum_{j=1}^{n} \delta_j Q_{G_A}(a, b)^{j-1} Q_{G_B}(a, e),\ k_2 = Q_{G_B}(b, c).$$

Then,

1. *if $k_1 \neq 0$, $k_2 \neq 0$, then $AB = BF(A)$ if and only if $A = \omega B$, for some constant ω which satisfies (4.28);*
2. *if $k_2 = 0$ then $AB = 0$ and, $AB = BF(A)$ if and only if $BF(A) = 0$. Moreover,*
   - *(a) if $k_1 \neq 0$ then $BF(A) = 0$ if and only if $b(s) = -\frac{\delta_0}{k_1} e(s) I_G(s)$ almost everywhere;*
   - *(b) if $k_1 = 0$ then $AB = BF(A)$ if $\delta_0 = 0$, that is, $F(t) = \sum_{j=1}^{n} \delta_j t^j$;*
3. *if $k_2 \neq 0$ and $k_1 = 0$ then $AB = BF(A)$ if and only if $AB = \delta_0 B$, that is*

$$(Ax)(t) = \frac{\delta_0}{k_2} \int\limits_{G_A} c(t)b(s)x(s)d\mu_s.$$

Proof 1. By applying Theorem 3 if $k_1 \neq 0$ and $k_2 \neq 0$ we have $AB = BF(A)$ if and only if the following is true:

- for almost every $t \in \operatorname{supp} c$ $a(t) = \frac{\delta_0 + \lambda}{k_2} c(t)$ and $b(s) = \frac{\lambda}{k_2} e(s)$ for almost every $s \in G \cap \operatorname{supp} e$ and nonzero constant λ satisfying (4.26);
- $e(s) = 0$ for almost every $s \in G_B \setminus G$;
- $b(s) = 0$ for almost every $s \in G_A \setminus G$;

From which we have,

$$(Ax)(t) = \int\limits_{G} a(t)b(s)x(s)d\mu_s + \int\limits_{G_A \setminus G} a(t)b(s)x(s)d\mu_s =$$
$$= \frac{(\lambda + \delta_0)\lambda}{k_1 k_2} \int\limits_{G} c(t)e(s)x(s)d\mu_s = \frac{(\lambda + \delta_0)\lambda}{k_1 k_2}(Bx)(t)$$

almost everywhere. If $\lambda = 0$ then $A = 0$.

2. If $k_2 = 0$ then from (4.22) we have $AB = 0$ and, hence $AB = BF(A)$ if and only if $BF(A) = 0$. Moreover, by applying Theorem 3 we have

(a) if $k_1 \neq 0$ then $AB = BF(A)$ if and only if for almost every $s \in \operatorname{supp} e \cap G$, $b(s) = -\frac{\delta_0}{k_1} e(s)$, $b(s) = 0$ for almost every $s \in G \setminus \operatorname{supp} e$, $e(s) = 0$ for almost every $s \in G_B \setminus G$ and $b(s) = 0$ for almost every $s \in G_A \setminus G$. Therefore, almost everywhere, $b(s) = -\frac{\delta_0}{k_1} e(s) I_G(s)$.

(b) if $k_1 = 0$ and $\delta_0 = 0$, then $AB = BF(A)$.

3. By applying Theorem 3 and if $k_2 \neq 0$, $k_1 = 0$ we have $AB = BF(A)$ if and only if for almost every $t \in \operatorname{supp} c$, $a(t) = \frac{\delta_0 + \lambda}{k_2} c(t)$ and $\lambda e(s) = 0$ for almost every $s \in G \cap \operatorname{supp} e$, from which we get $\lambda = 0$. Therefore, $a(t) = \frac{\delta_0}{k_2} c(t)$ almost everywhere. So we can write $(Ax)(t) = \int\limits_{G_A} a(t) b(s) x(s) d\mu_s = \frac{\delta_0}{k_2} \int\limits_{G_A} c(t) b(s) x(s) d\mu_s$ almost everywhere. Hence, almost everywhere,

$$
\begin{aligned}
(ABx)(t) &= \int\limits_{G_A} c(t) b(s) \left(\int\limits_{G_B} e(\tau) x(\tau) d\mu_\tau \right) d\mu_s \\
&= \frac{\delta_0}{k_2} c(t) \int\limits_{G_A} c(s) b(s) d\mu_s \int\limits_{G_B} e(\tau) x(\tau) d\mu_\tau \\
&= \frac{\delta_0}{k_2} Q_G(b, c) \int\limits_{G_B} c(t) e(\tau) x(\tau) d\mu_\tau = \delta_0 (Bx)(t)
\end{aligned}
$$

On the other hand, from (4.24) follows that $BF(A) = \delta_0 B$ if $k_1 = 0$. □

Example 3 Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space. Let

$$
A : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu), \;\; B : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu), \;\; 1 < p < \infty
$$

be nonzero operators $(Ax)(t) = \int\limits_0^1 a(t) b(s) x(s) ds$, $(Bx)(t) = \int\limits_0^1 c(t) e(s) x(s) ds$, where $a(t) = t^2 I_{[0,1]}(t)$, $b(s) = s^3$, $c(t) = -6t^2 I_{[0,1]}(t)$ and $e(s) = s$. Consider a polynomial $F(t) = \delta_0 + \delta_1 t + \delta_2 t^2$, where $t \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0, 1, 2$. We have $k_2 = Q_{[0,1]}(b, c) = \int\limits_0^1 b(s) c(s) ds = \int\limits_0^1 -6s^3 s^2 ds = -1$. If

$$
k_1 = \delta_1 Q_{[0,1]}(a, e) + \delta_2 Q_{[0,1]}(a, b) Q_{[0,1]}(a, e) = 0,
$$

then choose δ_i, $i = 1, 2$ such that $0 = \delta_1 + \delta_2 Q_{[0,1]}(a, b) = \delta_1 - \frac{1}{6} \delta_2 Q_{[0,1]}(c, b) = \delta_1 + \frac{1}{6} \delta_2$. Thus $\delta_2 = -6\delta_1$ and $\frac{\delta_0}{k_2} = -\frac{1}{6}$ from which we get $\delta_0 = \frac{1}{6}$. Hence, $F(t) = -6\delta_1 t^2 + \delta_1 t + \frac{1}{6}$. We have almost everywhere

$$(Ax)(t) = \int_0^1 t^2 I_{[0,1]}(t) s^3 x(s) ds, \ (Bx)(t) = -6 \int_0^1 t^2 I_{[0,1]}(t) s x(s) ds,$$

and thus almost everywhere

$$(ABx)(t) = \int_0^1 t^2 I_{[0,1]}(t) s^3 \left(-6 \int_0^1 s^2 I_{[0,1]}(s) \tau x(\tau) d\tau \right) ds = \frac{1}{6}(Bx)(t),$$
$$(A^2x)(t) = \int_0^1 t^2 I_{[0,1]}(t) s^3 \left(\int_0^1 s^2 I_{[0,1]}(s) \tau^3 x(\tau) d\tau \right) ds = \frac{1}{6}(Ax)(t).$$

Finally, we have

$$BF(A) = B\left(-6\delta_1 A^2 + \delta_1 A + \frac{1}{6} I\right) = -\delta_1 BA + \delta_1 BA + \frac{1}{6}B = \frac{1}{6}B = AB.$$

Example 4 Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space. Let

$$A : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu), \ B : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu), \ 1 < p < \infty$$

be nonzero operators $(Ax)(t) = \int_\alpha^\beta a(t)b(s)x(s)ds$, $(Bx)(t) = \int_\alpha^\beta c(t)e(s)x(s)ds$, where $\alpha, \beta \in \mathbb{R}$, $-\infty < \alpha \le \beta < \infty$, $a, c \in L_p(\mathbb{R}, \mu)$, $b, e \in L_q([\alpha, \beta], \mu)$ where $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Consider a polynomial $F(t) = \delta_0 + \delta_1 t + \delta_2 t^2$, where $t \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0, 1, 2$. We set

$$k_2 = Q_{[\alpha,\beta]}(b, c) = \int_\alpha^\beta b(s)c(s)ds, \ k_1 = \delta_1 Q_{[\alpha,\beta]}(a, e) + \delta_2 Q_{[\alpha,\beta]}(a, b) Q_{[\alpha,\beta]}(a, e).$$

If $k_2 \neq 0$ and $k_1 = 0$ then we choose either $Q_{[\alpha,\beta]}(a, e) = 0$ or δ_i, $i = 1, 2$ such that $\delta_1 + \delta_2 Q_{[\alpha,\beta]}(a, b) = 0$. Thus from Corollary 5 we have $a(t) = \frac{\delta_0}{k_2} c(t)$ almost everywhere. Thus $k_1 = 0$ implies that either $Q_{[\alpha,\beta]}(a, e) = 0$ or $\delta_1 + \frac{\delta_0}{k_2}\delta_2 k_2 = 0$. We choose coefficients δ_j, $j = 0, 1, 2$ such that $\delta_1 = -\delta_0\delta_2$, and hence $F(t) = \delta_2 t^2 - \delta_0\delta_2 t + \delta_0$. Then, the operators

$$(Ax)(t) = \frac{\delta_0}{k_2} \int_\alpha^\beta c(t)b(s)x(s)ds, \ (Bx)(t) = \int_\alpha^\beta c(t)e(s)x(s)ds$$

almost everywhere, satisfy the relation

$$AB = \delta_2 BA^2 - \delta_0\delta_2 BA + \delta_0 B. \tag{4.29}$$

In fact,

$$(ABx)(t) = \frac{\delta_0}{k_2}\int_\alpha^\beta c(t)b(s)\left(\int_\alpha^\beta c(s)e(\tau)x(\tau)d\tau\right)ds = \delta_0(Bx)(t),$$

$$(A^2x)(t) = \frac{\delta_0}{k_2}\int_\alpha^\beta c(t)b(s)\left(\frac{\delta_0}{k_2}\int_\alpha^\beta c(s)b(\tau)x(\tau)d\tau\right)ds = \delta_0(Ax)(t),$$

almost everywhere. Finally, we have $BF(A) = B\left(\delta_2A^2 - \delta_2\delta_0A + \delta_0 I\right) = \delta_2\delta_0BA - \delta_2\delta_0BA + \delta_0B = \delta_0B = AB$. In particular, if $\alpha = 0$, $\beta = 1$, $b(s) = s$ and $c(t) = t^2 I_{[0,1]}(t)$, $e(s) = s^3$ we have $k_2 = Q_{[0,1]}(b, c) = \frac{1}{4}$. Hence the operators

$$(Ax)(t) = 4\delta_0\int_0^1 t^2 I_{[0,1]}(t)sx(s)ds,\ (Bx)(t) = \int_0^1 t^2 I_{[0,1]}(t)s^3x(s)ds \tag{4.30}$$

satisfy the Relation (4.29). In particular, if $\delta_2 = 1$ and $\delta_0 = -1$, that is, $F(t) = t^2 + t - 1$ then the corresponding operators in (4.30) satisfy $AB = BA^2 + BA - B$.

Corollary 6 *Let (X, Σ, μ) be a σ-finite measure space. Let $A : L_p(X, \mu) \to L_p(X, \mu)$, $B : L_p(X, \mu) \to L_p(X, \mu)$, $1 \le p \le \infty$ be nonzero operators defined by*

$$(Ax)(t) = \int_{G_A} a(t)b(s)x(s)d\mu_s,\ (Bx)(t) = \int_{G_B} c(t)e(s)x(s)d\mu_s,$$

almost everywhere, $G_A \in \Sigma$, $G_B \in \Sigma$ are sets with finite measure, $a, c \in L_p(X, \mu)$, $b \in L_q(G_A, \mu)$, $e \in L_q(G_B, \mu)$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider a polynomial $F(z) = \delta_0 + \delta_1 z + \ldots + \delta_n z^n$, where $z \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, $j = 0, 1, 2, 3, \ldots, n$. Let $G = G_A \cap G_B$ and $k_1 = \sum_{j=1}^n \delta_j Q_{G_A}(a, b)^{j-1} Q_{G_B}(a, e)$, $k_2 = Q_{G_B}(b, c)$. If $k_2 \neq 0$ and $Q_{G_B}(a, e) = 0$, then $AB = BF(A)$ if and only if $AB = \delta_0 B$, that is $a(t) = \frac{\delta_0}{k_2}c(t)$, almost everywhere.

Proof This follows by Corollary 5 since $k_2 \neq 0$ and $k_1 = 0$. □

Corollary 7 *Let (X, Σ, μ) be a σ-finite measure space. Let $A : L_p(X, \mu) \to L_p(X, \mu)$, $B : L_p(X, \mu) \to L_p(X, \mu)$, $1 \le p \le \infty$ be nonzero operators defined by*

$$(Ax)(t) = \int_{G_A} a(t)b(s)x(s)d\mu_s, \quad (Bx)(t) = \int_{G_B} c(t)e(s)x(s)d\mu_s,$$

almost everywhere, $G_A \in \Sigma$, $G_B \in \Sigma$ are sets with finite measure, $a, c \in L_p(X, \mu)$, $b \in L_q(G_A, \mu)$, $e \in L_q(G_B, \mu)$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider a monomial

$F(z) = \delta z^d$, where $z \in \mathbb{R}$, d is a positive integer and $\delta \neq 0$ is a real number. Let $G = G_A \cap G_B$ and $k_1 = \delta Q_{G_A}(a,b)^{d-1} Q_{G_B}(a,e),\ k_2 = Q_{G_B}(b,c)$. Then $AB = \delta BA^d$ if and only the following conditions are fulfilled:

1. (a) *for almost every $(t,s) \in \operatorname{supp} c \times [(\operatorname{supp} e) \cap G]$ we have the following:*
   - (i) *If $k_2 \neq 0$, then $k_1 b(s) = e(s)\lambda$ and $a(t) = \frac{\lambda}{k_2} c(t)$ for some $\lambda \in \mathbb{R}$.*
   - (ii) *If $k_2 = 0$ then either $k_1 = 0$ or $b(s) = 0$ for almost all $s \in \operatorname{supp} e \cap G$.*

   (b) *If $t \notin \operatorname{supp} c$ then either $k_2 = 0$ or $a(t) = 0$ for almost all $t \notin \operatorname{supp} c$.*

   (c) *If $s \in G \setminus \operatorname{supp} e$ then either $k_1 = 0$ or $b(s) = 0$ for almost all $s \in G \setminus \operatorname{supp} e$.*
2. *$k_2 = 0$, or $e(s) = 0$ for almost every $s \in G_B \setminus G$.*
3. *$k_1 = 0$ or $b(s) = 0$ for almost every $s \in G_A \setminus G$.*

Proof This follows from Theorem 3 and the fact that $\delta_0 = 0$ in this case. □

Example 5 Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space. Let

$$A : L_2([\alpha,\beta],\mu) \to L_2([\alpha,\beta],\mu),\ \ B : L_2([\alpha,\beta],\mu) \to L_2([\alpha,\beta],\mu)$$

be defined by $(Ax)(t) = \int\limits_\alpha^\beta a(t)b(s)x(s)ds,\ \ (Bx)(t) = \int\limits_\alpha^\beta c(t)e(s)x(s)ds,$ where α, β are real numbers with $\alpha < \beta$, $a, b, c, e \in L_2([\alpha,\beta],\mu)$, such that $a \perp b$ and $b \perp c$, that is, $\int\limits_\alpha^\beta a(t)b(t)dt = \int\limits_\alpha^\beta b(t)c(t)dt = 0$. Then the above operators satisfy $AB = \delta BA^d$, $d = 2, 3, \ldots$. In fact, by using Corollary 7 and putting

$$F(t) = \delta t^d, \quad d = 2, 3, \ldots$$
$$k_1 = Q_{[\alpha,\beta]}(a,b)^{d-1} Q_{[\alpha,\beta]}(a,e), \quad k_2 = Q_{[\alpha,\beta]}(b,c),$$

we get $k_1 = k_2 = 0$. So we have all conditions in Corollary 7 satisfied. In particular, if $a(t) = \left(\frac{5}{3}t^3 - \frac{3}{2}t\right) I_{[-1,1]}(t)$, $b(s) = \frac{3}{2}s^2 - \frac{1}{2}$ and $c(t) = tI_{[-1,1]}(t)$, then the operators

$$(Ax)(t) = \int\limits_{-1}^{1} \left(\frac{5}{3}t^3 - \frac{3}{2}t\right) I_{[-1,1]}(t) \left(\frac{3}{2}s^2 - \frac{1}{2}\right) x(s)ds,$$

$$(Bx)(t) = \int\limits_{-1}^{1} tI_{[-1,1]}(t)e(s)x(s)ds$$

satisfy the relation $AB = BA^d$, $d = 2, 3, \ldots$. In fact $a,\ b,\ c$ are pairwise orthogonal in $[-1, 1]$.

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