Chapter 14
Stochastic Differential Equations Driven by Additive Volterra–Lévy and Volterra–Gaussian Noises

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Abstract We study the existence and uniqueness of solutions to stochastic differential equations with Volterra processes driven by Lévy noise. For this purpose, we study in detail smoothness properties of these processes. Special attention is given to two kinds of Volterra–Gaussian processes that generalize the compact interval representation of fractional Brownian motion to stochastic equations with such process.

Keywords Volterra process · Lévy process · Gaussian process · Sonine pair · continuity · Hölder property · weak solution · strong solution

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14.1 Introduction

The main object that is studied in the present paper are stochastic differential equations with additive noise, admitting the form

$$dX_t = u(X_t)dt + dY_t,\ \ t \ge 0,\ \ X|_{t=0} = X_0 \in \mathbb{R}, \tag{14.1}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a measurable function, and $Y = \{Y_t, t \ge 0\}$ is a Volterra–Lévy process. Equations of the form (14.1), with different coefficients and different noises, were the subject of long and careful considerations. Namely, the most popular case

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is the Langevin equation, where $u(x) = ax$, $x \in \mathbb{R}$, with some coefficient $a \neq 0$, and a Wiener process as a noise. Such process is called the Ornstein–Uhlenbeck process, or the Vasicek process, and it serves as mathematical model in many areas of science. Initially the Eq. (14.1) was proposed as a model for velocity of particles in the theory of the Brownian motion in [9], then the corresponding mathematical theory was developed in [21, 24], see, e. g. the book [22] for applications of the Ornstein–Uhlenbeck process in physics. Since the seminal paper by Vasicek [23], the Ornstein–Uhlenbeck process has become a very popular model in mathematical finance, see e. g. [6, 7, 10, 11, 17–19, 25].

A Volterra–Lévy process has the form $Y_t = \int_0^t g(t,s)\,dZ_s$, where $g(t,s)$ is a given deterministic Volterra-type kernel, and Z is a Lévy process. The conditions on g and Z supplying the existence of Volterra–Lévy processes were studied in [2] together with a theory of pathwise stochastic integration with respect to such processes. Some approximations and first numerical results can be found in [1]. The goal of the present paper is to study stochastic differential equations with additive noise represented by a Volterra–Lévy process.

We start with investigation of continuity and Hölder properties of Volterra–Lévy processes. In order to apply the Kolmogorov–Chentsov theorem, we establish moment upper bounds for increments of these processes. In particular, we study in detail the case when the kernel g satisfies certain power restrictions. Two examples of such kernels are considered, namely, the Molchan–Golosov kernel, which arises in the compact interval representation of fractional Brownian motion, and a sub-fractional kernel, which corresponds to sub-fractional Brownian motion. For both kernels, it holds that sample paths of the corresponding Volterra–Lévy processes satisfy Hölder condition up to order $H - \frac{1}{2}$, where H denotes the Hurst index. However, in the particular case of Gaussian Z, one has Hölder continuity up to order H. This agrees with the theory of fractional Brownian motion and with the paper [20], where the authors study the case, when $g(t,s)$ is the Molchan–Golosov kernel and Z is a Lévy process without Gaussian component.

Special attention in the paper is given to Volterra–Gaussian processes that arise in the case when Lévy process Z is a Brownian motion. We investigate two types of kernels that generalize the Molchan–Golosov kernel of fractional Brownian motion. One of these kernels corresponds to fractional Brownian motion with Hurst index $H > \frac{1}{2}$. It was introduced in [12], where conditions for its existence and Hölder continuity were investigated. Also, in [12] the inverse representation of underlying Wiener process via Volterra–Gaussian process was studied. This study was based on the properties of Sonine pairs. In the present paper we introduce also another type of Volterra–Gaussian process that extends fractional Brownian motion with $H < \frac{1}{2}$. We study smoothness of this process. We also derive the inverse operators for both types of Volterra–Gaussian processes in terms of generalized fractional integrals and derivatives for Sonine pairs.

Then we apply the results mentioned above for investigation of stochastic differential equations with Volterra–Lévy processes. We start with a deterministic analog of the Eq. (14.1), where the stochastic term Y_t is replaced by a non-random function that is locally integrable or locally bounded. We study solvability of this equation under Lipschitz condition on the drift coefficient u. Then we prove that the stochastic Eq. (14.1) with locally Lipschitz coefficient of linear growth has a unique solution under certain conditions on the underlying Volterra process Z and power restrictions on the kernel $g(t,s)$.

We also study stochastic differential equations with two kinds of Volterra–Gaussian processes. In this case we can prove solvability of the equation under weaker assumptions on the drift coefficient. Namely, we assume sublinear growth of this coefficient and its Hölder continuity. We generalize the results of [14], where the noise was fractional Brownian motion, to the case of more general Volterra–Gaussian noise. We prove the existence and uniqueness of a weak solution, the pathwise uniqueness of two weak solutions and the existence and uniqueness of a strong solution.

The paper is organized as follows. In Sect. 14.2 we recall the definition of a Volterra–Lévy processes, necessary conditions for its existence, and a priory estimates for its moments. Section 14.3 is devoted to Hölder properties of Volterra–Lévy processes. As auxiliary results, we establish upper bounds for the incremental moments in general case (Subsect. 14.3.1) as well as in the case of power restrictions on the kernel (Subsect. 14.3.2). In Subsect. 14.3.3 we apply these bounds for investigation of continuity and Hölder properties of three types of Volterra–Lévy processes. Two examples of appropriate kernels are given in Subsect. 14.3.4. In Subsect. 14.3.5 two kinds of Volterra–Gaussian processes are studied. Section 14.4 is devoted to the existence and uniqueness of solution to the Eq. (14.1). The stochastic differential equations with Volterra–Gaussian processes are studied in Sect. 14.5. In Appendix A we prove some auxiliary results related to fractional calculus for Sonine pairs. In the Appendix B a deterministic analog of the Eq. (14.1) is investigated.

Throughout the paper, we shall use notation C for various constants whose value is not important and may change from line to line and even in the same line.

14.2 Brief Description of Volterra–Lévy Processes

We start with a Lévy process Z. In order to describe it, define $\tau(z) := \begin{cases} z, & |z| \le 1, \\ \frac{z}{|z|}, & |z| > 1. \end{cases}$ Then the characteristic function of Z_t can be represented in the following form (see, e. g., [16]): $\mathbf{E}\exp\{i\mu Z_t\} = \exp\{t\Psi(\mu)\}$, where

$$\Psi(\mu) = ib\mu - \frac{a\mu^2}{2} + \int_{\mathbb{R}} \left(e^{i\mu x} - 1 - i\mu\tau(x)\right)\pi(dx),$$

$b \in \mathbb{R}$, $a \ge 0$, π is a Lévy measure on $\mathbb{R}$, that is a σ-finite Borel measure satisfying $\int_{\mathbb{R}} \left(x^2 \wedge 1\right)\pi(dx) < \infty$, with $\pi(\{0\}) = 0$. The triplet (a, b, π) is shortly called the *characteristic triplet of* Z. Let us fix some $T > 0$ and introduce the following Volterra–Lévy process

$$Y_t = \int_0^t g(t,s)\,dZ_s, \quad t \in [0,T], \tag{14.2}$$

where $g(t,s)$ is a given deterministic Volterra-type kernel. The integral in (14.2) is understood in the sense of [15] as the limit in probability of elementary integrals. Its construction is described in [2, Thm. 2.2]. According to [2], in order to guarantee the existence of the process Y and of its moments, we need more strict assumptions on the here called *base-process* Z and the kernel $g(t,s)$. More precisely, in what follows we assume that the Volterra–Lévy process (14.2) has $b = 0$ (i. e., Z is a Lévy process without drift), the measure π is symmetric and one of the following conditions holds:

(A1) There exists $p \in [1,2)$ such that $g = g(t,\cdot) \in L_p([0,t])$ for any $t \in [0,T]$; $a = 0$ and $\int_{\mathbb{R}} |x|^p\, \pi(dx) < \infty$;
(A2) There exists $p \geq 2$ such that $g = g(t,\cdot) \in L_p([0,t])$ for any $t \in [0,T]$ and $\int_{\mathbb{R}} |x|^p\, \pi(dx) < \infty$.

Then, according to [2, Thm. 2.2], the integral $\int_0^t g(t,s)\,dZ_s$ exists for any $t \in [0,T]$. Moreover, in the case when condition (A1) holds, we have the following a priori estimate

$$\mathbf{E}\left|\int_0^t g(t,s)\,dZ_s\right|^p \leq C\|g(t,\cdot)\|^p_{L_p([0,t])} \int_{\mathbb{R}} |x|^p \pi(dx), \tag{14.3}$$

and in the case when condition (A2) holds, we have the following a priori estimate

$$\begin{aligned} &\mathbf{E}\left|\int_0^t g(t,s)\,dZ_s\right|^p \\ &\leq C\left(a^{p/2}\|g(t,\cdot)\|^p_{L_2([0,t])} + \|g(t,\cdot)\|^p_{L_p([0,t])}\int_{\mathbb{R}} |x|^p\, \pi(dx)\right). \end{aligned} \tag{14.4}$$

The constant C in (14.3) and (14.4) does not depend on the function g. However, it may depend on p and T.

14.3 Moment Upper Bounds and Hölder Properties of Volterra–Lévy Processes

In our approach, in order to consider a Volterra–Lévy process as a noise, we need in the smoothness properties of its trajectories. So, the present section is devoted to its Hölder properties. Obviously, these properties depend both on the properties of the kernel g and the Lévy baseprocess Z.

14.3.1 General Upper Bounds for the Incremental Moments

In this subsection we establish upper bounds for $\mathbf{E}\,|Y_t - Y_s|^p$ under assumptions (A1) and (A2).

Lemma 1 *Consider* $0 \le s \le t \le T$. *Let assumption (A1) hold. Then*

$$\mathbf{E}\,|Y_t - Y_s|^p \le C \int_{\mathbb{R}} |x|^p \pi(dx) \left(\int_s^t |g(t,u)|^p \, du + \int_0^s |g(t,u) - g(s,u)|^p \, du \right). \tag{14.5}$$

Let assumption (A2) hold. Then

$$\begin{aligned}\mathbf{E}\,|Y_t - Y_s|^p \le C \int_{\mathbb{R}} |x|^p \pi(dx) \left(\int_s^t |g(t,u)|^p \, du + \int_0^s |g(t,u) - g(s,u)|^p \, du \right) \\ + C a^{p/2} \left(\left(\int_s^t |g(t,u)|^2 \, du \right)^{p/2} + \left(\int_0^s |g(t,u) - g(s,u)|^2 \, du \right)^{p/2} \right).\end{aligned} \tag{14.6}$$

Proof Note that the increment of Y is given by

$$\begin{aligned} Y_t - Y_s &= \int_0^t g(t,u) dZ_u - \int_0^s g(s,u) dZ_u \\ &= \int_s^t g(t,u) dZ_u + \int_0^s (g(t,u) - g(s,u)) dZ_u. \end{aligned}$$

Therefore,

$$\mathbf{E}\,|Y_t - Y_s|^p \le C \left(\mathbf{E} \left| \int_s^t g(t,u) dZ_u \right|^p + \mathbf{E} \left| \int_0^s (g(t,u) - g(s,u)) dZ_u \right|^p \right). \tag{14.7}$$

In order to conclude the proof, it suffices to apply the bounds (14.3) and (14.4) to the integrals in the right-hand side of (14.7). □

We remark that the Hölder continuity of paths is a central property also e. g. in the rough-paths approach to the study of stochastic (partial) differential equations. Our results can then find application in that framework. We refer to e. g. [8] for a study of Volterra-driven stochastic differential equations with multiplicative noise via rough-paths. Note that, different from our work, the starting base-process is Hölder continuous.

14.3.2 Incremental Moments and Hölder Continuity Under Power Restrictions on the Kernel g

As one can see from the inequalities (14.5) and (14.6), the incremental moments of Y are bounded by some integrals containing g, its powers and its increments. Now let us consider more specific class of the kernels g. Assume that the function g satisfies the following power restrictions with some $p \geq 1$.

(B1) There exist constants $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{p}$ and $\gamma > -\frac{1}{p}$ such that

$$|g(t,u)| \leq C t^{\alpha} u^{\beta} (t-u)^{\gamma} \quad \text{for all } 0 < u < t \leq T.$$

(B2) There exist a constant $\delta > 0$ and a function $h(t,s,u)$ such that

$$|g(t,u) - g(s,u)| \leq |t-s|^{\delta}\, h(t,s,u) \quad \text{for all } 0 < u < s < t \leq T,$$

and $\displaystyle\sup_{0<s<t\leq T} \int_0^s |h(t,s,u)|^p \, du < \infty.$

As we shall see further on in the examples, these conditions on the kernel are well motivated by the fractional and sub-fractional Brownian motions. An extension of condition (B1) is provided in Remark 1 at the end of the next subsection.

Our goal in this and the next subsection is to obtain an inequality of the form $\mathbf{E}\,|Y_t - Y_s|^p \leq C\,|t-s|^c$ with some $c > 0$. In particular, if we get such an inequality with $c > 1$, we will be able to apply the Kolmogorov continuity theorem and to investigate Hölder properties of Y. Taking into account Lemma 1, we need to estimate the integrals of the form $\int_s^t |g(t,u)|^p \, du$ and $\int_0^s |g(t,u) - g(s,u)|^p \, du$. Obviously, the second integral under the assumption (B2) satisfies the inequality

$$\int_0^s |g(t,u) - g(s,u)|^p \, du \leq C\,|t-s|^{\delta p}\,. \tag{14.8}$$

The study of the first integral is more delicate. We start with the following auxiliary result.

Lemma 2 *Let $\mu > -1$ and $\nu > -1$. Then for all $0 \le s < t \le T$,*

$$\int_s^t u^\mu (t-u)^\nu \, du \le C t^\mu (t-s)^{\nu+1}. \tag{14.9}$$

The positive constant C in (14.9) may depend on μ, ν and T.

Proof Write

$$\int_s^t u^\mu (t-u)^\nu \, du = \int_s^{\frac{s+t}{2}} u^\mu (t-u)^\nu \, du + \int_{\frac{s+t}{2}}^t u^\mu (t-u)^\nu \, du =: I_1 + I_2. \tag{14.10}$$

For $s \le u \le t$, we have $(t-u)^\nu = (t-u)^{\nu+1}(t-u)^{-1} \le (t-s)^{\nu+1}(t-u)^{-1}$. Therefore,

$$\begin{aligned} I_1 &\le (t-s)^{\nu+1} \int_s^{\frac{s+t}{2}} \frac{u^\mu}{t-u} \, du = (t-s)^{\nu+1} t^{-1} \int_s^{\frac{s+t}{2}} \frac{u^\mu (t-u+u)}{t-u} \, du \\ &= (t-s)^{\nu+1} t^{-1} \int_s^{\frac{s+t}{2}} u^\mu \, du + (t-s)^{\nu+1} t^{-1} \int_s^{\frac{s+t}{2}} \frac{u^{\mu+1}}{t-u} \, du =: I_{11} + I_{12}. \end{aligned} \tag{14.11}$$

The term I_{11} can be bounded as follows:

$$I_{11} = C(t-s)^{\nu+1} t^{-1} \left(\left(\frac{s+t}{2} \right)^{\mu+1} - s^{\mu+1} \right) \le C t^\mu (t-s)^{\nu+1}, \tag{14.12}$$

since $\left(\frac{s+t}{2}\right)^{\mu+1} - s^{\mu+1} \le \left(\frac{s+t}{2}\right)^{\mu+1} \le t^{\mu+1}$.

In order to bound I_{12}, we use the inequality $u^{\mu+1} \le \left(\frac{s+t}{2}\right)^{\mu+1} \le t^{\mu+1}$. We get

$$\begin{aligned} I_{12} &\le (t-s)^{\nu+1} t^\mu \int_s^{\frac{s+t}{2}} \frac{du}{t-u} = (t-s)^{\nu+1} t^\mu \left(\log(t-s) - \log \frac{t-s}{2} \right) \\ &= t^\mu (t-s)^{\nu+1} \log 2 = C t^\mu (t-s)^{\nu+1}. \end{aligned} \tag{14.13}$$

Consider I_2. Note that for $\frac{s+t}{2} < u < t$,

$$\begin{aligned} u^\mu &\le \left(\frac{s+t}{2} \right)^\mu \le \left(\frac{t}{2} \right)^\mu && \text{if } \mu < 0, \\ u^\mu &\le t^\mu && \text{if } \mu \ge 0. \end{aligned}$$

Hence, in both cases we have the bound $u^\mu \le Ct^\mu$. Therefore,

$$I_2 \le Ct^\mu \int_{\frac{s+t}{2}}^{t} (t-u)^\nu \, du = Ct^\mu \left(t - \frac{s+t}{2}\right)^{\nu+1} = Ct^\mu (t-s)^{\nu+1}. \tag{14.14}$$

Combining (14.10)–(14.14), we get (14.9). □

Lemma 2 allows us to obtain an upper bound for the integral $\int_s^t |g(t,u)|^p \, du$.

Lemma 3 *Assume that condition (B1) holds with some $p \ge 1$. Then*

$$\int_s^t |g(t,u)|^p \, du \le C(t-s)^{\kappa p+1}, \quad \textit{for all } 0 \le s < t \le T,$$

where

$$\kappa = \kappa(\alpha,\beta,\gamma) = \begin{cases} \alpha+\beta+\gamma, & \textit{if } \alpha+\beta < 0, \\ \gamma, & \textit{if } \alpha+\beta \ge 0. \end{cases} \tag{14.15}$$

The constant C may depend on α, β, γ, p and T.

Proof According to condition (B1), $\int_s^t |g(t,u)|^p \, du \le Ct^{\alpha p} \int_s^t u^{\beta p}(t-u)^{\gamma p} \, du$. Applying the upper bound (14.9), one gets $\int_s^t |g(t,u)|^p \, du \le Ct^{(\alpha+\beta)p}(t-s)^{\gamma p+1}$. If $\alpha+\beta < 0$, then $t^{(\alpha+\beta)p} \le (t-s)^{(\alpha+\beta)p}$, and we obtain the inequality $\int_s^t |g(t,u)|^p \, du \le C(t-s)^{(\alpha+\beta+\gamma)p+1}$. If $\alpha+\beta \ge 0$, then $t^{(\alpha+\beta)p} \le T^{(\alpha+\beta)p}$, hence, $\int_s^t |g(t,u)|^p \, du \le C(t-s)^{\gamma p+1}$. □

14.3.3 Application of the Upper Bounds for the Incremental Moments to Volterra–Lévy Processes of Three Types

Now, basing on Lemma 3, we can better specify the upper bounds (14.5) and (14.6) for the moments of increments of the Volterra–Lévy process Y satisfying (B1)–(B2). Also, as a consequence, we shall state its Hölder properties. We consider three cases: (1) Z is a Lévy process without Brownian part; (2) Z is a Brownian motion; (3) Z is a Lévy process of a general form.

14.3.3.1 Lévy–Based Process Without Brownian Part

We start with the case of a Lévy process in (14.2) without Brownian part, that is, $a = 0$.

Lemma 4 *Assume that* $p \geq 1$, $a = 0$, $\int_{\mathbb{R}} |x|^p \, \pi(dx) < \infty$, *the conditions (B1) and (B2) hold with some* $\alpha \in \mathbb{R}$, $\delta > 0$, $\beta > -\frac{1}{p}$, $\gamma > -\frac{1}{p}$ *and such that* $\alpha + \beta + \gamma > -\frac{1}{p}$. *Then for all* $0 \leq s < t \leq T$, $\mathbf{E}\,|Y_t - Y_s|^p \leq C(t-s)^{\min\{\kappa p+1, \delta p\}}$, *where* κ *is defined by* (14.15). *If* $\kappa > 0$ *and* $\delta > \frac{1}{p}$, *then the trajectories of* Y *are a. s. Hölder continuous up to order* $\min\left\{\kappa, \delta - \frac{1}{p}\right\}$.

Proof According to Lemma 1, we have

$$\mathbf{E}\,|Y_t - Y_s|^p \leq C\left(\int_s^t |g(t,u)|^p \, du + \int_0^s |g(t,u) - g(s,u)|^p \, du\right).$$

Applying Lemma 3 and (14.8), we get

$$\begin{aligned}\mathbf{E}\,|Y_t - Y_s|^p &\leq C(t-s)^{\kappa p+1} + C(t-s)^{\delta p} \\ &\leq CT^{\kappa p+1}\left(\frac{t-s}{T}\right)^{\min\{\kappa p+1,\delta p\}} + CT^{\delta p}\left(\frac{t-s}{T}\right)^{\min\{\kappa p+1,\delta p\}} \\ &\leq C(t-s)^{\min\{\kappa p+1,\delta p\}}.\end{aligned}$$

Hölder continuity follows from the Kolmogorov continuity theorem. □

14.3.3.2 The Brownian Case

Lemma 5 *Assume that* Z *is a Brownian motion, the conditions (B1) and (B2) hold with* $p = 2$, $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{2}$, $\gamma > -\frac{1}{2}$ *such that* $\alpha + \beta + \gamma > -\frac{1}{2}$. *Then for all* $p \geq 2$ *and all* $0 \leq s < t \leq T$, $\mathbf{E}\,|Y_t - Y_s|^p \leq C(t-s)^{p\min\left\{\kappa+\frac{1}{2},\delta\right\}}$, *where* κ *is defined by* (14.15). *If* $\kappa > -\frac{1}{2}$, *then the trajectories of* Y *are a. s. Hölder continuous up to order* $\min\left\{\kappa + \frac{1}{2}, \delta\right\}$.

Proof In the Brownian case, (14.6) becomes

$$\mathbf{E}\,|Y_t - Y_s|^p \leq C\left(\left(\int_s^t |g(t,u)|^2 \, du\right)^{p/2} + \left(\int_0^s |g(t,u) - g(s,u)|^2 \, du\right)^{p/2}\right).$$

Then by Lemma 3 and (14.8), we get

$$\mathbf{E}\,|Y_t - Y_s|^p \leq C(t-s)^{\frac{p}{2}(2\kappa+1)} + C(t-s)^{\delta p} \leq C(t-s)^{p\min\left\{\kappa+\frac{1}{2},\delta\right\}}.$$

By the Kolmogorov continuity theorem, if $p\min\left\{\kappa + \frac{1}{2}, \delta\right\} > 1$, then the trajectories of Y are a. s. Hölder up to order $\min\left\{\kappa + \frac{1}{2}, \delta\right\} - \frac{1}{p}$. Since p can be chosen arbitrarily large, we get Hölder continuity up to order $\min\left\{\kappa + \frac{1}{2}, \delta\right\}$, if $\kappa > -1/2$. □

14.3.3.3 Lévy–Based Process of a General Form

Now let us consider a Lévy process Z of a general form. In this case we need to assume that $p \geq 2$ in order to guarantee the existence of Y and its moments, see [2, Thm. 2.2]. It turns out that under this assumption we have the same upper bound for the incremental moment as in the case $a = 0$.

Lemma 6 *Assume that for some $p \geq 2$ we have $\int_{\mathbb{R}} |x|^p \, \pi(dx) < \infty$ and the conditions (B1) and (B2) hold with some $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{p}$, $\gamma > -\frac{1}{p}$ such that $\alpha + \beta + \gamma > -\frac{1}{p}$. Then for all $0 \leq s < t \leq T$, $\mathbf{E}\,|Y_t - Y_s|^p \leq C(t-s)^{\min\{\kappa p+1, \delta p\}}$, where κ is defined by (14.15). If $\kappa > 0$ and $\delta > \frac{1}{p}$, then the trajectories of Y are a. s. Hölder continuous up to order $\min\left\{\kappa, \delta - \frac{1}{p}\right\}$.*

Proof Applying Lemma 1, Lemma 3 and (14.8), we obtain

$$\begin{aligned}
\mathbf{E}\,|Y_t - Y_s|^p &\leq C \left(\int_s^t |g(t,u)|^p \, du + \int_0^s |g(t,u) - g(s,u)|^p \, du \right.\\
&\quad \left. + \left(\int_s^t |g(t,u)|^2 \, du \right)^{p/2} + \left(\int_0^s |g(t,u) - g(s,u)|^2 \, du \right)^{p/2} \right)\\
&\leq C(t-s)^{\kappa p+1} + C(t-s)^{\delta p} + C(t-s)^{\frac{p}{2}(\kappa p+1)}\\
&\leq C(t-s)^{\min\left\{\kappa p+1, \delta p, \frac{p}{2}(\kappa p+1)\right\}} = C(t-s)^{\min\{\kappa p+1, \delta p\}}.
\end{aligned}$$

Hölder continuity follows from the Kolmogorov continuity theorem. □

Remark 1 The assumption (B1) can be replaced by the following more general condition:

(B1′) There exist constants $\alpha_i \in \mathbb{R}$, $\beta_i > -\frac{1}{p}$ and $\gamma_i > -\frac{1}{p}$, $i = 1, 2, \dots, m$, such that for all $0 < u < t \leq T$, $|g(t,u)| \leq C \sum_{i=1}^{m} t^{\alpha_i} u^{\beta_i} (t-u)^{\gamma_i}$.

In this case the statements of Lemmas 3–6 hold true with $\kappa = \min_{1 \leq i \leq m} \kappa_i$, where $\kappa_i = \kappa(\alpha_i, \beta_i, \gamma_i)$, $i = 1, \dots, m$, are defined by (14.15). Indeed, in order to proof Lemma 3 under the assumption (B1′), it suffices to apply the bound $(x_1 + \cdots + x_m)^p \leq C\left(x_1^p + \cdots + x_m^p\right)$ and follow the same reasoning as in the case of the condition (B1). Other lemmas are then easily deduced from Lemma 3.

14.3.4 Examples of Volterra–Lévy Processes with Power Restrictions on the Kernel

14.3.4.1 The Molchan–Golosov Kernel

Let us verify the assumptions (B1) and (B2) for the Molchan–Golosov kernel, which is defined as

$$K_H(t,s) = C_H s^{\frac{1}{2}-H} \left(t^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\tfrac{1}{2}) \int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\, du \right), \tag{14.16}$$

where $H \in (0,1)$, $C_H = \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}$. This kernel arises in the compact interval representation of the fractional Brownian motion as an integral with respect to a Wiener process W, see, e. g., [13, Sect. 2.8]. More precisely, the Volterra process

$$B_t^H = \int_0^t K_H(t,s)\, dW_s,\ t \geq 0 \tag{14.17}$$

is a fractional Brownian motion with the Hurst parameter H, that is a zero mean Gaussian process with covariance function $\mathbf{E} B_t^H B_s^H = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$. Note that the precise value of C_H is irrelevant in the context of our study, the following results concerning Hölder continuity of Volterra processes are valid for any $C > 0$ instead of C_H.

Hereafter we consider the Volterra process

$$Y_t^H = \int_0^t K_H(t,s)\, dZ_s, \quad t \in [0,T], \tag{14.18}$$

where Z is a Lévy base-process. We recall that if Z is without Gaussian component, then the process (14.18) is known as *fractional Lévy process by Molchan–Golosov transformation*. It was introduced and studied in [20].

Proposition 1 *Let* $H \in (0,1)$*,* $\varepsilon \in (0,H)$*.*

1. *Let* $0 < \int_{\mathbb{R}} x^2\, \pi(dx) < \infty$*. Then for all* $0 \leq s < t \leq T$*,*

$$\mathbf{E}\left|Y_t^H - Y_s^H\right|^2 \leq C(t-s)^{2(H-\varepsilon)}.$$

If $H \in (\frac{1}{2},1)$*, then the trajectories of* Y^H *are* $\varkappa$*-Hölder continuous for any* $\varkappa \in (0, H-\frac{1}{2})$*.*

2. *Let* Z *be a Brownian motion. Then for all* $p \geq 2$ *and all* $0 \leq s < t \leq T$*,*

$$\mathbf{E}\left|Y_t^H - Y_s^H\right|^p \le C(t-s)^{p(H-\varepsilon)},$$

and the trajectories of Y^H *are* $\varkappa$*-Hölder continuous for any* $\varkappa \in (0, H)$.

Proof We prove both statements simultaneously. Without loss of generality, assume that $0 < \varepsilon < \min\left\{1 - H, \frac{1}{2}\right\}$. Indeed, if the result of the proposition holds for some $\varepsilon = \varepsilon^* > 0$, then it holds also for all $\varepsilon > \varepsilon^*$. We consider the cases $H = \frac{1}{2}$, $H > \frac{1}{2}$ and $H < \frac{1}{2}$ separately.

Case $H = \frac{1}{2}$. Note that if $H = \frac{1}{2}$, then $K_H \equiv$ const. Hence, for any p, (B1) and (B2) are valid with $\alpha = \beta = \gamma = 0$ and with any $\delta > 0$. If $\int_{\mathbb{R}} x^2 \pi(dx) < \infty$, then, by Lemma 6, $\mathbf{E}\left|Y_t^H - Y_s^H\right|^2 \le C(t-s)$ for all $0 \le s < t \le T$. If Z is a Brownian motion, then by Lemma 5, $\mathbf{E}\left|Y_t^H - Y_s^H\right|^p \le C(t-s)^{\frac{p}{2}}$ for all $0 \le s < t \le T$ and $p \ge 2$. Hence, both statements of the proposition hold even for $\varepsilon = 0$ (consequently, they hold for any $\varepsilon > 0$).

Case $H \in (\frac{1}{2}, 1)$. In this case the kernel (14.16) can be rewritten using integration by parts in the following form:

$$K_H(t,s) = C s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}}\, du. \tag{14.19}$$

For $0 < s < t \le T$, we have

$$|K_H(t,s)| \le C s^{\frac{1}{2}-H} t^{H-\frac{1}{2}} \int_s^t (u-s)^{H-\frac{3}{2}}\, du = C t^{H-\frac{1}{2}} s^{\frac{1}{2}-H} (t-s)^{H-\frac{1}{2}}.$$

Therefore the condition (B1) holds with $\alpha = H - \frac{1}{2}$, $\beta = \frac{1}{2} - H$, $\gamma = H - \frac{1}{2}$.

In order to verify the condition (B2), we need to estimate the difference $|K_H(t,u) - K_H(s,u)|$. We have for $0 < u < s < t \le T$,

$$\begin{aligned} |K_H(t,u) - K_H(s,u)| &= C u^{\frac{1}{2}-H} \int_s^t z^{H-\frac{1}{2}} (z-u)^{H-\frac{3}{2}}\, dz \\ &\le C u^{\frac{1}{2}-H} \int_s^t (z-u)^{2H-2}\, dz + C \int_s^t (z-u)^{H-\frac{3}{2}}\, dz \end{aligned} \tag{14.20}$$

(here we have used the inequality $z^{H-\frac{1}{2}} \le (z-u)^{H-\frac{1}{2}} + u^{H-\frac{1}{2}}$). Let $\varepsilon \in (0, 1-H)$. Then the integrals in the right-hand side of (14.20) can be bounded as follows:

$$\int_s^t (z-u)^{2H-2}\,dz \le (s-u)^{H+\varepsilon-1}\int_s^t (z-s)^{H-\varepsilon-1}\,dz$$
$$= C(s-u)^{H+\varepsilon-1}(t-s)^{H-\varepsilon},$$

$$\int_s^t (z-u)^{H-\frac{3}{2}}\,dz \le (s-u)^{\varepsilon-\frac{1}{2}}\int_s^t (z-s)^{H-\varepsilon-1}\,dz = C(s-u)^{\varepsilon-\frac{1}{2}}(t-s)^{H-\varepsilon}.$$

Hence,

$$|K_H(t,u)-K_H(s,u)| \le (t-s)^{H-\varepsilon}h(s,u),$$

where

$$h(s,u) = C\left(u^{\frac{1}{2}-H}(s-u)^{H+\varepsilon-1} + (s-u)^{\varepsilon-\frac{1}{2}}\right).$$

If

$$p < \frac{1}{H-\frac{1}{2}}, \quad p < \frac{1}{1-H-\varepsilon}, \quad \text{and} \quad p \le \frac{1}{\frac{1}{2}-\varepsilon}, \tag{14.21}$$

then

$$\int_0^s |h(s,u)|^p\,du \le C\int_0^s \left(u^{(\frac{1}{2}-H)p}(s-u)^{(H+\varepsilon-1)p} + (s-u)^{(\varepsilon-\frac{1}{2})p}\right)du$$
$$= Cs^{(\varepsilon-\frac{1}{2})p+1} \le CT^{(\varepsilon-\frac{1}{2})p+1} < \infty.$$

Thus, the condition (B2) holds with $\delta = H-\varepsilon$ for all p satisfying (14.21) (in particular, for $p=2$).

According to Lemma 6, if $0 < \int_{\mathbb{R}} x^2\,\pi(dx) < \infty$, then for all $0 \le s < t \le T$,

$$\mathbf{E}\left|Y_t^H - Y_s^H\right|^2 \le C(t-s)^{2(H-\varepsilon)},$$

and the trajectories of Y^H are $\varkappa$-Hölder continuous for any $\varkappa \in (0, H-\frac{1}{2})$.

If Z is a Brownian motion, then, by Lemma 5, for all $p \ge 2$ and all $0 \le s < t \le T$,

$$\mathbf{E}\left|Y_t^H - Y_s^H\right|^p \le C(t-s)^{p(H-\varepsilon)},$$

and the trajectories of Y^H are $\varkappa$-Hölder continuous for any $\varkappa \in (0, H)$.

Case $H \in (0, \frac{1}{2})$. Denote

$$K_H^{(1)}(t,s) = t^{H-\frac{1}{2}}s^{\frac{1}{2}-H}(t-s)^{H-\frac{1}{2}}, \quad K_H^{(2)}(t,s) = s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\,du.$$

Then (14.16) implies that

$$|K_H(t,s)| \le C\left(K_H^{(1)}(t,s) + K_H^{(2)}(t,s)\right).$$

According to Remark 1, we can treat $K_H^{(1)}(t,s)$ and $K_H^{(2)}(t,s)$ separately. Evidently, the kernel $K_H^{(1)}(t,s)$ satisfies (B1) with $\alpha_1 = H - \frac{1}{2}$, $\beta_1 = \frac{1}{2} - H$, $\gamma_1 = H - \frac{1}{2}$. Then $\kappa_1 = \gamma_1 = H - \frac{1}{2}$, see (14.15).

In order to bound $K_H^{(2)}(t,s)$, we make a substitution $z = \frac{u-s}{s}$ in the integral. We get

$$\begin{aligned} K_H^{(2)}(t,s) &= s^{H-\frac{1}{2}} \int_0^{\frac{t-s}{s}} \frac{z^{H-\frac{1}{2}}}{(1+z)^{\frac{3}{2}-H}}\, dz \le s^{H-\frac{1}{2}} \int_0^{\infty} \frac{z^{H-\frac{1}{2}}}{(1+z)^{\frac{3}{2}-H}}\, dz \\ &= \mathrm{B}\left(H + \tfrac{1}{2}, 1 - 2H\right) s^{H-\frac{1}{2}}. \end{aligned}$$

Therefore, $K_H^{(2)}(t,s)$ satisfies (B1) with $\alpha_2 = 0$, $\beta_2 = H - \frac{1}{2}$, $\gamma_2 = 0$. Consequently, $\kappa_2 = \alpha_2 + \beta_2 + \gamma_2 = H - \frac{1}{2} = \kappa_1$. Thus, $K_H(t,s)$ satisfies (B1$'$), and the corresponding value of κ equals $H - \frac{1}{2}$.

Now let us verify the assumption (B2). Let $0 < u < s < t \le T$. We have

$$\begin{aligned} |K_H(t,u) - K_H(s,u)| \le C &\left|K_H^{(1)}(t,u) - K_H^{(1)}(s,u)\right| \\ + C &\left|K_H^{(2)}(t,u) - K_H^{(2)}(s,u)\right|. \end{aligned} \tag{14.22}$$

The first term in the right-hand side can be decomposed as follows:

$$\begin{aligned} \left|K_H^{(1)}(t,u) - K_H^{(1)}(s,u)\right| &= \left|t^{H-\frac{1}{2}} u^{\frac{1}{2}-H} (t-u)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} u^{\frac{1}{2}-H} (s-u)^{H-\frac{1}{2}}\right| \\ &\le u^{\frac{1}{2}-H} (t-u)^{H-\frac{1}{2}} \left|t^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right| + u^{\frac{1}{2}-H} s^{H-\frac{1}{2}} \left|(t-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}\right| \\ &=: K_H^{(1,1)}(t,s,u) + K_H^{(1,2)}(t,s,u). \end{aligned} \tag{14.23}$$

Let $\varepsilon \in (0, \frac{1}{2})$. For $K_H^{(1,1)}(t,s,u)$ we have

$$\begin{aligned} K_H^{(1,1)}(t,s,u) &= Cu^{\frac{1}{2}-H}(t-u)^{H-\frac{1}{2}}\int_s^t z^{H-\frac{3}{2}}\,dz \\ &\le Cu^{\frac{1}{2}-H}(t-s)^{H-\frac{1}{2}}s^{H-1+\varepsilon}\int_s^t z^{-\frac{1}{2}-\varepsilon}\,dz \\ &= Cu^{\frac{1}{2}-H}(t-s)^{H-\frac{1}{2}}s^{H-1+\varepsilon}\left(t^{\frac{1}{2}-\varepsilon}-s^{\frac{1}{2}-\varepsilon}\right) \\ &\le Cu^{\frac{1}{2}-H}s^{H-1+\varepsilon}(t-s)^{H-\varepsilon}. \end{aligned} \tag{14.24}$$

Similarly,

$$\begin{aligned} K_H^{(1,2)}(t,s,u) &= Cu^{\frac{1}{2}-H}s^{H-\frac{1}{2}}\int_s^t (z-u)^{H-\frac{3}{2}}\,dz \\ &\le Cu^{\frac{1}{2}-H}s^{H-\frac{1}{2}}(s-u)^{\varepsilon-\frac{1}{2}}\int_s^t (z-s)^{H-1-\varepsilon}\,dz, \end{aligned}$$

where we have used the inequality

$$(z-u)^{H-\frac{3}{2}} = (z-u)^{\varepsilon-\frac{1}{2}}(z-u)^{H-1-\varepsilon} \le (s-u)^{\varepsilon-\frac{1}{2}}(z-s)^{H-1-\varepsilon}.$$

Therefore,

$$K_H^{(1,2)}(t,s,u) \le Cu^{\frac{1}{2}-H}s^{H-\frac{1}{2}}(s-u)^{\varepsilon-\frac{1}{2}}(t-s)^{H-\varepsilon}. \tag{14.25}$$

Let us consider $\left|K_H^{(2)}(t,u) - K_H^{(2)}(s,u)\right|$. We have

$$\left|K_H^{(2)}(t,u) - K_H^{(2)}(s,u)\right| = u^{\frac{1}{2}-H}\int_s^t z^{H-\frac{3}{2}}(z-u)^{H-\frac{1}{2}}\,du.$$

Using the bounds $(z-u)^{H-\frac{1}{2}} \le (s-u)^{H-\frac{1}{2}}$ and $z^{H-\frac{3}{2}} = z^{\varepsilon-\frac{1}{2}}z^{H-1-\varepsilon} \le s^{\varepsilon-\frac{1}{2}}z^{H-1-\varepsilon}$, we obtain

$$\begin{aligned} \left|K_H^{(2)}(t,u) - K_H^{(2)}(s,u)\right| &\le u^{\frac{1}{2}-H}s^{\varepsilon-\frac{1}{2}}(s-u)^{H-\frac{1}{2}}\int_s^t z^{H-1-\varepsilon}\,du \\ &= Cu^{\frac{1}{2}-H}s^{\varepsilon-\frac{1}{2}}(s-u)^{H-\frac{1}{2}}\left(t^{H-\varepsilon}-s^{H-\varepsilon}\right) \\ &\le Cu^{\frac{1}{2}-H}s^{\varepsilon-\frac{1}{2}}(s-u)^{H-\frac{1}{2}}(t-s)^{H-\varepsilon}. \end{aligned} \tag{14.26}$$

Combining (14.22)–(14.26), we get

$$|K_H(t,u) - K_H(s,u)| \le (t-s)^{H-\varepsilon} h(s,u),$$

where

$$h(s,u) = C u^{\frac{1}{2}-H} \left(s^{H-1+\varepsilon} + s^{H-\frac{1}{2}} (s-u)^{\varepsilon-\frac{1}{2}} + s^{\varepsilon-\frac{1}{2}} (s-u)^{H-\frac{1}{2}} \right).$$

It is straightforward to check that if $p < \frac{1}{\frac{1}{2}-\varepsilon}$ and $p < \frac{1}{\frac{1}{2}-H}$, then

$$\int_0^s |h(s,u)|^p \, du \le C s^{1-(\frac{1}{2}-\varepsilon)p} \le C T^{(\varepsilon-\frac{1}{2})p+1} < \infty.$$

This means, in particular, that the condition (B2) is satisfied with $\delta = H - \varepsilon$ and $p = 2$.

According to Lemma 6, if $\int_{\mathbb{R}} x^2 \, \pi(dx) < \infty$, then for all $0 \le s < t \le T$,

$$\mathbf{E} \left| Y_t^H - Y_s^H \right|^2 \le C(t-s)^{2(H-\varepsilon)}.$$

If Z is a Brownian motion, then, by Lemma 5, for all $p \ge 2$ and all $0 \le s < t \le T$,

$$\mathbf{E} \left| Y_t^H - Y_s^H \right|^p \le C(t-s)^{p(H-\varepsilon)},$$

and the trajectories of Y^H are $\varkappa$-Hölder continuous for any $\varkappa \in (0, H)$. □

Remark 2 If $H < \frac{1}{2}$ and Z is a non-Gaussian Lévy process, then the Kolmogorov–Chentsov theorem does not guarantee continuity of Y^H, since $2(H-\varepsilon) < 1$. Moreover, if Z is a Lévy process without Gaussian component, then according to [20, Prop. 3.7], Y^H has discontinuous sample paths with positive probability.

14.3.4.2 The Sub-fractional Kernel

Let us consider another example for a kernel satisfying (**B1**)–(**B2**), namely

$$L_H(t,s) = C s^{\frac{3}{2}-H} \left(t^{-1} \left(t^2 - s^2\right)^{H-\frac{1}{2}} + \int_s^t z^{-2} \left(z^2 - s^2\right)^{H-\frac{1}{2}} dz \right), \quad (14.27)$$

where $H \in (0,1)$, $C > 0$. This kernel arises in the compact interval representation of the sub-fractional Brownian motion [13, Sect. 2.8] (see also [4]).

Let us consider the Volterra process $U_t^H = \int_0^t L_H(t,s) \, dZ_s$, $t \in [0,T]$. It turns out that its properties are similar to those of the process Y^H in (14.18).

Proposition 2 *Let* $H \in (0, 1)$*,* $\varepsilon \in (0, H)$*.*

1. *Let* $0 < \int_{\mathbb{R}} x^2 \pi(dx) < \infty$*. Then, for all* $0 \le s < t \le T$*,*

$$\mathbf{E}\left|U_t^H - U_s^H\right|^2 \le C(t-s)^{2(H-\varepsilon)}.$$

If $H \in (\frac{1}{2}, 1)$*, then the trajectories of* U^H *are* $\varkappa$*-Hölder continuous for any* $\varkappa \in (0, H - \frac{1}{2})$*.*

2. *Let* Z *be a Brownian motion. Then for all* $p \ge 2$ *and all* $0 \le s < t \le T$*,*

$$\mathbf{E}\left|U_t^H - U_s^H\right|^p \le C(t-s)^{p(H-\varepsilon)},$$

and the trajectories of U^H *are* $\varkappa$*-Hölder continuous for any* $\varkappa \in (0, H)$*.*

Proof *Case* $H = \frac{1}{2}$. Observe that $L_H \equiv$ const in this case, and the statement holds, see the proof of Proposition 1.

Case $H \in (\frac{1}{2}, 1)$. It is not hard to see that (14.27) can be written in the following form: $L_H(t,s) = Cs^{\frac{3}{2}-H} \int_s^t \left(z^2 - s^2\right)^{H-\frac{3}{2}} dz$. Then,

$$L_H(t,s) = Cs^{\frac{3}{2}-H} \int_s^t (z-s)^{H-\frac{3}{2}} (z+s)^{H-\frac{3}{2}} \, dz \le C \int_s^t (z-s)^{H-\frac{3}{2}} \, dz,$$

because $(z+s)^{H-\frac{3}{2}} \le s^{H-\frac{3}{2}}$. Therefore, $L_H(t,s) \le C(t-s)^{H-\frac{1}{2}}$, and (B1) holds with $\alpha = \beta = 0$ and $\gamma = H - \frac{1}{2}$.

Let us verify (B2). For $0 < u < s < t \le T$ we have

$$\begin{aligned}|L_H(t,u) - L_H(s,u)| &= Cu^{\frac{3}{2}-H} \int_s^t \left(z^2 - u^2\right)^{H-\frac{3}{2}} dz \\ &= Cu^{\frac{3}{2}-H} \int_s^t (z-u)^{H-\frac{3}{2}} (z+u)^{H-\frac{3}{2}} \, dz.\end{aligned}$$

Using the bound $(z+u)^{H-\frac{3}{2}} = (z+u)^{-1}(z+u)^{H-\frac{1}{2}} \le u^{-1}(2z)^{H-\frac{1}{2}}$, we get

$$|L_H(t,u) - L_H(s,u)| \le Cu^{\frac{1}{2}-H} \int_s^t (z-u)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} \, dz = C\left|K_H(t,u) - K_H(s,u)\right|,$$

see (14.20). Thus, the condition (B2) holds with $\delta = H - \varepsilon$ for all p satisfying (14.21) (in particular, for $p = 2$), see the proof of Proposition 1. Similarly to the case of the Molchan–Golosov kernel, we can conclude that the proposition holds in the case $H > \frac{1}{2}$.

Case $H \in (0, \frac{1}{2})$. It follows from (14.27) that $|L_H(t,s)| \le C\left(L_H^{(1)}(t,s) + L_H^{(2)}(t,s)\right)$, with $L_H^{(1)}(t,s) = s^{\frac{3}{2}-H} t^{-1}\left(t^2-s^2\right)^{H-\frac{1}{2}}$, $L_H^{(2)}(t,s) = s^{\frac{3}{2}-H}\int_s^t z^{-2}\left(z^2-s^2\right)^{H-\frac{1}{2}}\,dz$. Applying the estimate

$$\begin{aligned}\left(t^2-s^2\right)^{H-\frac{1}{2}} &= (t-s)^{H-\frac{1}{2}}(t+s)^{H-\frac{1}{2}} = (t-s)^{H-\frac{1}{2}}(t+s)^{H+\frac{1}{2}}(t+s)^{-1}\\ &\le (t-s)^{H-\frac{1}{2}}(2t)^{H+\frac{1}{2}}s^{-1} = Ct^{H+\frac{1}{2}}s^{-1}(t-s)^{H-\frac{1}{2}}, \end{aligned} \tag{14.28}$$

we obtain

$$L_H^{(1)}(t,s) \le Cs^{\frac{1}{2}-H}t^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} = CK_H^{(1)}(t,s). \tag{14.29}$$

For the term $L_H^{(2)}(t,s)$ we also use (14.28) and arrive at

$$L_H^{(2)}(t,s) \le Cs^{\frac{1}{2}-H}\int_s^t z^{H-\frac{3}{2}}(z-s)^{H-\frac{1}{2}}\,dz = CK_H^{(2)}(t,s). \tag{14.30}$$

From the bounds (14.29) and (14.30) we deduce that the condition (B1′) is satisfied with the same constants $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$, as in the case of the kernel $K_H(t,s)$, see the proof of Proposition 1.

Now we consider the difference

$$|L_H(t,u) - L_H(s,u)| \le C\left|L_H^{(1)}(t,u) - L_H^{(1)}(s,u)\right| + C\left|L_H^{(2)}(t,u) - L_H^{(2)}(s,u)\right|,$$

where $0 < u < s < t \le T$. For the first term in the right-hand side we have

$$\begin{aligned}\left|L_H^{(1)}(t,u) - L_H^{(1)}(s,u)\right| &= u^{\frac{3}{2}-H}\left|t^{-1}\left(t^2-u^2\right)^{H-\frac{1}{2}} - s^{-1}\left(s^2-u^2\right)^{H-\frac{1}{2}}\right|\\ &\le u^{\frac{3}{2}-H}\left(t^2-u^2\right)^{H-\frac{1}{2}}\left|t^{-1}-s^{-1}\right| + u^{\frac{3}{2}-H}s^{-1}\left|\left(t^2-u^2\right)^{H-\frac{1}{2}} - \left(s^2-u^2\right)^{H-\frac{1}{2}}\right|\\ &=: L_H^{(1,1)}(t,s,u) + L_H^{(1,2)}(t,s,u).\end{aligned}$$

Consider $L_H^{(1,1)}(t,s,u) = Cu^{\frac{3}{2}-H}(t-u)^{H-\frac{1}{2}}(t+u)^{H-\frac{1}{2}}\int_s^t z^{-2}\,dz$. Since

$$(t+u)^{H-\frac{1}{2}} \le u^{H-\frac{1}{2}},\ z^{-2} = z^{H-\frac{3}{2}}z^{-H-\frac{1}{2}} \le z^{H-\frac{3}{2}}u^{-H-\frac{1}{2}}$$

we see that

$$L_H^{(1,1)}(t,s,u) \le Cu^{\frac{1}{2}-H}(t-u)^{H-\frac{1}{2}}\int_s^t z^{H-\frac{3}{2}}\,dz \le CK_H^{(1,1)}(t,s,u).$$

by (14.24). The term $L_H^{(1,2)}(t,s,u)$ can be rewritten as follows:

$$L_H^{(1,2)}(t,s,u) = Cu^{\frac{3}{2}-H} s^{-1} \int_{s^2}^{t^2} \left(z-u^2\right)^{H-\frac{3}{2}} \, dz$$
$$= Cu^{\frac{3}{2}-H} s^{-1} \int_{s}^{t} \left(x^2-u^2\right)^{H-\frac{3}{2}} \, x \, dx$$
$$= Cu^{\frac{3}{2}-H} s^{-1} \int_{s}^{t} (x-u)^{H-\frac{1}{2}} (x+u)^{H-\frac{3}{2}} x \, dx.$$

Using the bound

$$(x+u)^{H-\frac{3}{2}} x = (x+u)^{H-\frac{3}{2}} x s^{H-\frac{1}{2}} s^{\frac{1}{2}-H}$$
$$\le (x+u)^{H-\frac{3}{2}} (x+u) s^{H-\frac{1}{2}} (x+u)^{\frac{1}{2}-H} = s^{H-\frac{1}{2}}$$

we obtain

$$L_H^{(1,2)}(t,s,u) \le Cu^{\frac{3}{2}-H} s^{H-\frac{3}{2}} \int_{s}^{t} (x-u)^{H-\frac{1}{2}} \, dx$$
$$= C\frac{u}{s} K_H^{(1,2)}(t,s,u) \le CK_H^{(1,2)}(t,s,u).$$

Finally, applying the inequality (14.28), we get

$$\left| L_H^{(2)}(t,u) - L_H^{(2)}(s,u) \right| = u^{\frac{3}{2}-H} \int_{s}^{t} z^{-2} \left(z^2-u^2\right)^{H-\frac{1}{2}} \, dz$$
$$\le Cu^{\frac{1}{2}-H} \int_{s}^{t} z^{H-\frac{3}{2}} (z-u)^{H-\frac{1}{2}} \, dz = C \left| K_H^{(2)}(t,u) - K_H^{(2)}(s,u) \right|.$$

Thus, we have established that

$$|L_H(t,u) - L_H(s,u)| \le CK_H^{(1,1)}(t,s,u) + CK_H^{(1,2)}(t,s,u)$$
$$+ C \left| K_H^{(2)}(t,u) - K_H^{(2)}(s,u) \right|.$$

The proof is concluded by applying the bounds and the arguments from the proof of Proposition 1. □

14.3.5 Sonine Pairs and Two Kinds of Volterra–Gaussian Processes

Hereafter we discuss some family of kernels providing in turn Volterra–Gaussian processes with good paths regularity. The characterization of the kernels is based on the so-called Sonine pairs. As a motivation, consider the compact interval representation (14.17) of the fractional Brownian motion, where the kernel is given by (14.16). We shall consider (14.16) in the two cases $H \in (\frac{1}{2}, 1)$ and $H \in (0, \frac{1}{2})$. This will lead to different kind of considerations on the family of kernels.
(a) Let us consider first $H \in (\frac{1}{2}, 1)$. In this case, the kernel K_H can be simplified to

$$K_H(t,s) = \left(H - \tfrac{1}{2}\right) C_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} \, du. \tag{14.31}$$

This leads us to consider the following Gaussian process

$$Y_t = \int_0^t K(t,s) \, dW_s, \quad t \in [0,T], \tag{14.32}$$

where $W = \{W_t, t \in [0,T]\}$ is a Wiener process, and the Volterra kernel $K(t,s)$ has the following form

$$K(t,s) = a(s) \int_s^t b(u) c(u-s) \, du. \tag{14.33}$$

The functions $a, b, c \colon [0,T] \to \mathbb{R}$ are measurable and satisfy the following assumptions

(C1) Functions $a \in L_p([0,T])$, $b \in L_q([0,T])$, and $c \in L_r([0,T])$ for $p \in [2,\infty]$, $q \in [1,\infty]$, $r \in [1,\infty]$ such that $1/p + 1/q + 1/r \le 3/2$.
(C2) Functions a, b are positive a. e. on $[0,T]$.
(C3) Function c creates a Sonine pair with some $h \in L_1([0,T])$.

Recall the definition of Sonine pairs as given in [12].

Definition 1 The function c creates a Sonine pair on the interval $[0,T]$ with some function $h \in L_1([0,T])$ if, for any $t \in [0,T]$,

$$\int_0^t c(t-s) h(s) \, ds = 1.$$

It was established in [12] that under the assumption (C1),

$$\sup_{t\in[0,T]} \|K(t,\cdot)\|_{L_2([0,t])} < \infty.$$

This means that for any Wiener process $W = \{W_t, t \in [0,T]\}$, the process

$$Y_t = \int_0^t K(t,s)\,dW_s, \quad t \in [0,T],$$

is well defined, see [12, Thm. 1].

Remark 3 If $H \in (\frac{1}{2}, 1)$, $a(s) = Cs^{\frac{1}{2}-H}$, $b(s) = s^{H-\frac{1}{2}}$, $c(s) = s^{H-\frac{3}{2}}$, then $K(t,s)$ is the Molchan–Golosov kernel (14.19), hence Y is a fractional Brownian motion with the Hurst index H. Moreover, in this case the assumptions (C1)–(C3) are satisfied, see [12]. Therefore, the kernel K is an analog of the kernel K_H with $H > \frac{1}{2}$. In this case $h(s) = s^{\frac{1}{2}-H}$. Other examples of Sonine pairs (c, h) are given in [12].

Let us consider the operator $\mathscr{K}$ associated with the kernel $K(t,s)$ in (14.33):

$$\mathscr{K} f(t) = \int_0^t K(t,s) f(s)\,ds = \int_0^t a(s) \int_s^t b(u)c(u-s)\,du\; f(s)\,ds. \tag{14.34}$$

In order to find an inverse operator to $\mathscr{K}$, let us apply the elements of "fractional" calculus related to Sonine pair (c, h). More precisely, we use the notions similar to notions of the fractional integral and the fractional derivative, as given in Definition 3 from Appendix A, see also [12].

In terms of the fractional integral I^c_{0+} from Definition 3, the operator $\mathscr{K}$ can be rewritten as follows:

$$\mathscr{K} f(t) = \int_0^t b(u) \int_0^s a(s)c(u-s) f(s)\,ds\,du = \int_0^t b(u) I^c_{0+}(af)(u)\,du. \tag{14.35}$$

Lemma 7 *Consider the equation*

$$\mathscr{K} f(t) = \int_0^t a(s) \int_s^t b(u)c(u-s)\,du f(s)\,ds = \int_0^t u(z)\,dz, \quad t \in [0,T].$$

Then its solution has a form

$$f(t) = a^{-1}(t) D^h_{0+}(ub^{-1})(t), \tag{14.36}$$

under the assumption that the right-hand side of (14.36) *is well-defined and* $D^h_{0+}(ub^{-1}) \in L_1([0,T])$. *Here* D^h_{0+} *stands for fractional derivative, see Definition 3.*

Proof According to (14.35), $b(t)I^c_{0+}(af)(t) = u(t)$ a.e. or

$$I^c_{0+}(af)(t) = b^{-1}(t)u(t). \tag{14.37}$$

Assume that $af \in L_1([0, T])$ and apply Lemma 10, item (i) to (14.37). As a result, we arrive to $af(t) = D^h_{0+}(b^{-1}u)(t)$, and the proof follows. □

As already mentioned, condition (C1) is sufficient for the existence of process Y. However, in order to guarantee its Hölder continuity, a stronger assumption is required. The following proposition summarizes the results in Lemma 1 and Theorem 3 of [12].

Proposition 3 *1. Let the coefficients a, b, c satisfy the assumption*

(C4) $a \in L^p([0, T])$, $b \in L^q([0, T])$, $c \in L^r([0, T])$, *where* $p \geq 2$, $q, r \geq 1$, $\frac{1}{p} + \frac{1}{r} \leq \frac{1}{2}$, *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1 + \varepsilon$ *for some* $\varepsilon \in (0, 1/2)$.

Then the stochastic process Y has a modification satisfying Hölder condition up to order $\nu = \frac{3}{2} - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} > 1/2 - \varepsilon$.

2. Let the coefficients a, b, c satisfy the assumption

(C5) for any $t_1 \geq 0$, $t_2 \geq 0$, $t_1 + t_2 < T$,

$$\begin{aligned} a &\in L^p([0, T]) \cap L^{p_1}([t_1, T]), \text{ where } 2 \leq p \leq p_1, \\ b &\in L^q([0, T]) \cap L^{q_1}([t_1 + t_2, T]), \text{ where } 1 < q \leq q_1, \\ c &\in L^r([0, T]) \cap L^{r_1}([t_2, T]), \text{ where } 1 \leq r \leq r_1, \end{aligned}$$

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{3}{2}, \text{ and } \frac{1}{q_1} + \max\left(\frac{1}{2}, \frac{1}{p} + \frac{1}{r_1}, \frac{1}{p_1} + \frac{1}{r}\right) < 1.$$

Then the process Y on any interval $[t_1 + t_2, T]$ *has a modification that satisfies Hölder condition up to order* $\mu = \frac{3}{2} - \frac{1}{q_1} - \max\left(\frac{1}{2}, \frac{1}{p} + \frac{1}{r_1}, \frac{1}{p_1} + \frac{1}{r}\right) > 1/2$.

In [12] details about the value of these conditions for fractional Brownian motion are given. Briefly, condition (C4) supplies its Hölder property up to order $1/2$, and condition (C5) supplies Hölder property up to order H on any interval separated from zero.

(b) In the present paper, we consider the kernel (14.16) with Hurst index $H \in (0, 1/2)$. Then we introduce its generalization in the form

$$\widehat{K}(t, s) = \hat{a}(s)\left[\hat{b}(t)\hat{c}(t - s) - \int_s^t \hat{b}'(u)\hat{c}(u - s)\,du\right], \tag{14.38}$$

where $\hat{a}, \hat{b}, \hat{c}\colon [0, T] \to \mathbb{R}$ are measurable functions. In what follows, we assume that the following conditions hold.

($\widehat{\mathbf{C}}1$) The function $\hat{a}$ is nondecreasing, $\hat{b}$ is absolutely continuous, $\hat{a}\hat{b}$ is bounded, $\hat{c} \in L_2([0,T])$, and

$$A(T) = \int_0^T\int_0^T \left|\hat{b}'(u)\right| \left|\hat{b}'(z)\right| \int_0^{u\wedge z} \hat{a}^2(s) \left|\hat{c}(u-s)\right| \left|\hat{c}(z-s)\right| \, ds\, du\, dz < \infty.$$

($\widehat{\mathbf{C}}2$) Functions $\hat{a}, \hat{b}$ are positive a. e. on $[0,T]$.
($\widehat{\mathbf{C}}3$) Function $\hat{c}$ creates a Sonine pair with some $\hat{h} \in L_1([0,T])$.

Remark 4 Sufficient condition for ($\widehat{\mathbf{C}}1$) is

$$(\widehat{\mathbf{C}}1') \quad \int_0^T \left|\hat{b}'(u)\right| \left(\int_0^u \hat{a}^2(z)\, \hat{c}^2(u-z)\, dz \right)^{\frac{1}{2}} du < \infty.$$

Indeed, under ($\widehat{\mathbf{C}}1'$)

$$\begin{aligned}
&\int_0^T\int_0^T \left|\hat{b}'(u)\right| \left|\hat{b}'(z)\right| \int_0^{u\wedge z} \hat{a}^2(s) \left|\hat{c}(u-s)\right| \left|\hat{c}(z-s)\right| \, ds\, du\, dz \\
&\quad \le \int_0^T\int_0^T \left|\hat{b}'(u)\right| \left|\hat{b}'(z)\right| \left(\int_0^{u\wedge z} \hat{a}^2(s) \left|\hat{c}(u-s)\right|^2 ds \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^{u\wedge z} \hat{a}^2(s) \left|\hat{c}(z-s)\right|^2 ds \right)^{\frac{1}{2}} du\, dz \\
&\quad \le \left(\int_0^T \left|\hat{b}'(u)\right| \left(\int_0^u \hat{a}^2(s) \left|\hat{c}(u-s)\right|^2 ds \right)^{\frac{1}{2}} du \right)^2 < \infty.
\end{aligned}$$

Remark 5 We observe that in the case of fractional Brownian motion with $H < 1/2$ it holds that $\hat{a}(s) = Cs^{\frac{1}{2}-H}$, $\hat{b}(s) = \hat{c}(s) = s^{H-\frac{1}{2}}$ and $\hat{h}(s) = s^{-\frac{1}{2}-H}$, so, these functions indeed satisfy conditions ($\widehat{\mathbf{C}}1$)–($\widehat{\mathbf{C}}3$). Indeed, from the remark above we can see that

$$\begin{aligned}
&\int_0^T \left|\hat{b}'(u)\right| \left(\int_0^u \hat{a}^2(s)\ \left|\hat{c}(u-s)\right|^2 ds \right)^{\frac{1}{2}} du \\
&\quad = C\left(\tfrac{1}{2} - H\right) \int_0^T u^{H-\frac{3}{2}} \left(\int_0^u s^{1-2H} (u-s)^{2H-1}\, ds \right)^{\frac{1}{2}} du
\end{aligned}$$

$$= C\left(\tfrac{1}{2} - H\right) \int_0^T u^{H-1}\, du < \infty.$$

Lemma 8 *Under assumption (*$\widehat{\mathbf{C}}1$*),* $\sup_{t\in[0,T]} \|\widehat{K}(t,\cdot)\|_{L_2([0,t])} < \infty$ *holds, and for any Wiener process* $W = \{W_t, t \in [0,T]\}$ *a process* $\widehat{Y}_t = \int_0^t \widehat{K}(t,s)\, dW_s$, $t \in [0,T]$, *is well defined.*

Proof Obviously,

$$\|\widehat{K}(t,\cdot)\|^2_{L_2([0,t])} \le C\hat{b}^2(t) \int_0^t \hat{a}^2(s)\hat{c}^2(t-s)\, ds + C \int_0^t \hat{a}^2(s) \left(\int_s^t \hat{b}(u)\hat{c}(u-s)\, du \right)^2 ds.$$

If $\hat{a}\hat{b}$ is bounded, $\hat{a}$ is nondecreasing, and $\hat{c} \in L_2([0,T])$, then

$$\hat{b}^2(t) \int_0^t \hat{a}^2(s)\hat{c}^2(t-s)\, ds \le \hat{b}^2(t)\hat{a}^2(t) \int_0^t \hat{c}^2(t-s)\, ds \le \left(\hat{a}\hat{b}\right)^2(t)\|\hat{c}\|^2_{L_2([0,T])} < \infty.$$

Furthermore,

$$\int_0^t \hat{a}^2(s) \left(\int_s^t \hat{b}(u)\hat{c}(u-s)\, du \right)^2 ds \le \int_0^t \hat{a}^2(s) \int_s^t \hat{b}(u) \left|\hat{c}(u-s)\right| du \int_s^t \hat{b}(v) \left|\hat{c}(v-s)\right| dv\, ds$$
$$= \int_0^t \int_0^t \hat{b}(u)\hat{b}(v) \int_0^{u\wedge v} \hat{a}^2(s) \left|\hat{c}(u-s)\right| \left|\hat{c}(v-s)\right| ds\, du\, dv \le A(T) < \infty,$$

and the proof follows. □

Let us now consider the operator $\widehat{\mathscr{K}}$ associated with the kernel $\widehat{K}(t,s)$ in (14.38) (similarly to the operator $\mathscr{K}$ from (14.34) associated with $K(t,s)$). In this case $\widehat{\mathscr{K}}$ has the form

$$\widehat{\mathscr{K}} f(t) = \int_0^t \widehat{K}(t,s) f(s)\,ds$$
$$= \hat{b}(t) I_{0+}^{c}(\hat{a} f)(t) - \int_0^t (\hat{a} f)(s) \int_s^t \hat{b}'(u)\hat{c}(u-s)\,du\,ds, \quad f \in L_2([0,T]),$$

and under the assumptions ($\widehat{\mathbf{C}}1$)–($\widehat{\mathbf{C}}3$) we can apply the Fubini theorem and get

$$\widehat{\mathscr{K}} f(t) = \hat{b}(t) I_{0+}^{c}(\hat{a} f)(t) - \int_0^t \hat{b}'(u) \int_0^u \hat{c}(u-s)(\hat{a} f)(s)\,ds\,du$$
$$= \int_0^t \hat{b}(u) \frac{d}{du} \int_0^u \hat{c}(u-z)(\hat{a} f)(z)\,dz\,du = \int_0^t \hat{b}(u) D_{0+}^{\hat{c}}(\hat{a} f)(u)\,du.$$

Consider the following Gaussian process

$$\widehat{Y}_t = \int_0^t \widehat{K}(t,s)\,dW_s, \quad t \in [0,T], \tag{14.39}$$

where $W = \{W_t, t \in [0,T]\}$ is a Wiener process. Under assumptions ($\widehat{\mathbf{C}}1$)–($\widehat{\mathbf{C}}3$) it is well defined on $[0,T]$. Taking Lemma 10 from Appendix A into account, it is easy to establish, similarly to Lemma 7, the following result.

Lemma 9 *Consider the equation* $\widehat{\mathscr{K}} f(t) = \int_0^t u(z)\,dz, \quad z \in [0,T]$. *Then its solution has a form* $f(t) = \hat{a}^{-1} I_{0+}^{h} \left(\hat{b}^{-1} u\right)(t)$.

Furthermore, we prove the following result on the Hölder continuity of paths.

Theorem 1 *Let the conditions (*$\widehat{\mathbf{C}}1$*)–(*$\widehat{\mathbf{C}}3$*) hold, together with the following assumptions:*

1. $\left|\hat{a}(t)\hat{b}'(t)\right| \le C t^{-1}$, $t \in [0,T]$;
2. *there exists* $\gamma \in (0,2)$ *such that*

$$\int_0^t \hat{c}^2(s)\,ds \le C t^{\gamma}, \quad t \in [0,T],$$
$$\int_0^{T-t} \left(\hat{c}(t+s) - \hat{c}(s)\right)^2 ds \le C t^{\gamma}, \quad t \in [0,T].$$

Then the trajectories of the process $\widehat{Y}$ satisfy δ-Hölder condition a. s. for any $\delta \in (0, \gamma/2)$.

Remark 6 In the case when $\hat{a}(s) = Cs^{\frac{1}{2}-H}$, $\hat{b}(s) = \hat{c}(s) = s^{H-\frac{1}{2}}$ we have that $\hat{a}(s)\hat{b}'(s) = Cs^{-1}$, $\int_0^t \hat{c}^2(s)\,ds = Ct^{2H}$,

$$\int_0^{T-t} \left(\hat{c}(t+s) - \hat{c}(s)\right)^2 ds = \int_0^{T-t} \left((t+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds$$
$$= t^{2H} \int_0^{\frac{T}{t}-1} \left((1+z)^{H-\frac{1}{2}} - z^{H-\frac{1}{2}}\right)^2 dz < t^{2H} \int_0^{\infty} \left((1+z)^{H-\frac{1}{2}} - z^{H-\frac{1}{2}}\right)^2 dz$$
$$\leq Ct^{2H},$$

since $(1+z)^{H-\frac{1}{2}} - z^{H-\frac{1}{2}} \sim z^{H-\frac{3}{2}}, z \to \infty$, and so $\int_0^\infty \left((1+z)^{H-\frac{1}{2}} - z^{H-\frac{1}{2}}\right)^2 dz \leq C$. Therefore, in this case we can put $\gamma = 2H$, and ($\widehat{\mathbf{C}}$4)–($\widehat{\mathbf{C}}$5) hold.

Proof For $t_1 < t_2$,

$$\mathbf{E}\left(\widehat{Y}_{t_2} - \widehat{Y}_{t_1}\right)^2 = \mathbf{E}\left(\int_0^{t_1} \left(\widehat{K}(t_2, s) - \widehat{K}(t_1, s)\right) dW_s + \int_{t_1}^{t_2} \widehat{K}(t_2, s)\,dW_s\right)^2$$
$$= \int_0^{t_1} \left(\widehat{K}(t_2, s) - \widehat{K}(t_1, s)\right)^2 ds + \int_{t_1}^{t_2} \widehat{K}^2(t_2, s)\,ds$$
$$\leq 2\left(\int_0^{t_1} \hat{a}^2(s)\left(\hat{b}(t_2)\hat{c}(t_2 - s) - \hat{b}(t_1)\hat{c}(t_1 - s)\right)^2 ds\right.$$
$$+\hat{b}^2(t_2)\int_{t_1}^{t_2} \hat{a}^2(s)\hat{c}^2(t_2 - s)\,ds + \int_0^{t_1} \hat{a}^2(s)\left(\int_{t_1}^{t_2} \hat{b}'(u)\hat{c}(u - s)\,du\right)^2 ds$$
$$\left.+\int_{t_1}^{t_2} \hat{a}^2(s)\left(\int_s^{t_2} \hat{b}'(u)\hat{c}(u - s)\,du\right)^2 ds\right) =: 2(I_1 + I_2 + I_3 + I_4).$$

Let us show that each term in the right-hand side is bounded by $C(t_2 - t_1)^\gamma$. We make the analysis term by term.

1. The first term can be rewritten as follows:

$$\begin{aligned} I_1 &= \int_0^{t_1} \hat{a}^2(s)\left(\hat{b}(t_2)\hat{c}(t_2-s) - \hat{b}(t_1)\hat{c}(t_1-s)\right)^2 ds \\ &\leq 2\hat{b}^2(t_2)\int_0^{t_1} \hat{a}^2(s)\left(\hat{c}(t_2-s)-\hat{c}(t_1-s)\right)^2 ds \\ &\quad + 2\left(\hat{b}(t_2)-\hat{b}(t_1)\right)^2 \int_0^{t_1} \hat{a}^2(s)\hat{c}^2(t_1-s)\,ds =: J_1 + J_2. \end{aligned} \tag{14.40}$$

The first term in the right-hand side of (14.40) is bounded as follows:

$$\begin{aligned} &\hat{b}^2(t_2)\int_0^{t_1} \hat{a}^2(s)\left(\hat{c}(t_2-s)-\hat{c}(t_1-s)\right)^2 ds \\ &\quad\leq \hat{b}^2(t_2)\hat{a}^2(t_2)\int_0^{t_1}\left(\hat{c}(t_2-s)-\hat{c}(t_1-s)\right)^2 ds \\ &\quad\leq C\int_0^{t_1}\left(\hat{c}(t_2-t_1+z)-\hat{c}(z)\right)^2 dz \leq C(t_2-t_1)^{\gamma}, \end{aligned}$$

and the second one can be bounded as follows:

$$\begin{aligned} &\left(\hat{b}(t_2)-\hat{b}(t_1)\right)^2\int_0^{t_1}\hat{a}^2(s)\hat{c}^2(t_1-s)\,ds = \int_0^{t_1}\hat{a}^2(s)\hat{c}^2(t_1-s)\left(\int_{t_1}^{t_2}\hat{b}'(v)\,dv\right)^2 ds \\ &\leq \int_0^{t_1}\hat{c}^2(t_1-s)\left(\int_{t_1}^{t_2}\left|\hat{a}(v)\hat{b}'(v)\right|dv\right)^2 ds \leq Ct_1^{\gamma}\left(\int_{t_1}^{t_2} v^{-1}\,dv\right)^2 \\ &= C\left(\int_{t_1}^{t_2} t_1^{\frac{\gamma}{2}}v^{-1}\,dv\right)^2 \leq C\left(\int_{t_1}^{t_2} v^{\frac{\gamma}{2}-1}\,dv\right)^2 \leq C\left(t_2^{\frac{\gamma}{2}}-t_1^{\frac{\gamma}{2}}\right)^2 \leq C\,(t_2-t_1)^{\gamma}. \end{aligned}$$

Here we have used the monotonicity of $\hat{a}$ and then the conditions ($\widehat{\mathbf{C}}4$) and ($\widehat{\mathbf{C}}5$).

2. The second term can be bounded with the help of conditions ($\widehat{\mathbf{C}}1$) and ($\widehat{\mathbf{C}}5$):

$$I_2 = \hat{b}^2(t_2) \int_{t_1}^{t_2} \hat{a}^2(s)\hat{c}^2(t_2 - s)\, ds \le \hat{a}^2(t_2)\hat{b}^2(t_2) \int_{t_1}^{t_2} \hat{c}^2(t_2 - s)\, ds$$
$$\le C \int_0^{t_2 - t_1} \hat{c}^2(z)\, dz \le C\,(t_2 - t_1)^\gamma\,.$$

3. By Fubini's theorem and monotonicity of $\hat{a}$, the third term can be estimated as follows:

$$I_3 = \int_0^{t_1} \hat{a}^2(s) \left(\int_{t_1}^{t_2} \hat{b}'(u)\hat{c}(u - s)\, du \right)^2 ds$$
$$= \int_0^{t_1} \hat{a}^2(s) \int_{t_1}^{t_2} \hat{b}'(u)\hat{c}(u - s)\, du \int_{t_1}^{t_2} \hat{b}'(v)\hat{c}(v - s)\, dv\, ds$$
$$= \int_{t_1}^{t_2}\int_{t_1}^{t_2} \hat{b}'(u)\hat{b}'(v) \int_0^{t_1} \hat{a}^2(s)\hat{c}(u - s)\hat{c}(v - s)\, ds\, du\, dv$$
$$\le \int_{t_1}^{t_2}\int_{t_1}^{t_2} \left|\hat{a}(u)\hat{b}'(u)\right| \left|\hat{a}(v)\hat{b}'(v)\right| \int_0^{t_1} \left|\hat{c}(u - s)\hat{c}(v - s)\right|\, ds\, du\, dv.$$

Then applying successively the condition ($\widehat{\mathbf{C}}4$), the Cauchy–Schwarz inequality and the condition ($\widehat{\mathbf{C}}5$) we obtain:

$$I_3 \le C \int_{t_1}^{t_2}\int_{t_1}^{t_2} u^{-1}v^{-1} \int_0^{t_1} \left|\hat{c}(u - s)\hat{c}(v - s)\right|\, ds\, du\, dv$$
$$\le C \int_{t_1}^{t_2}\int_{t_1}^{t_2} u^{-1}v^{-1} \left(\int_0^{t_1} \hat{c}^2(u - s)\, ds \right)^{\frac{1}{2}} \left(\int_0^{t_1} \hat{c}^2(v - s)\, ds \right)^{\frac{1}{2}} du\, dv$$
$$\le C \int_{t_1}^{t_2}\int_{t_1}^{t_2} u^{\frac{\gamma}{2}-1} v^{\frac{\gamma}{2}-1}\, du\, dv \le C \left(t_2^{\frac{\gamma}{2}} - t_1^{\frac{\gamma}{2}} \right)^2 \le C\,(t_2 - t_1)^\gamma\,.$$

4. The fourth term can be bounded similarly to the third one:

$$
\begin{aligned}
I_4 &= \int_{t_1}^{t_2} \hat{a}^2(s) \left(\int_s^{t_2} \hat{b}'(u)\hat{c}(u-s)\,du \right)^2 ds \\
&= \int_{t_1}^{t_2}\int_s^{t_2}\int_s^{t_2} \hat{a}^2(s)\hat{b}'(u)\hat{c}(u-s)\hat{b}'(v)\hat{c}(v-s)\,du\,dv\,ds \\
&\le \int_{t_1}^{t_2}\int_s^{t_2}\int_s^{t_2} \left|\hat{a}(u)\hat{b}'(u)\right| \left|\hat{a}(v)\hat{b}'(v)\right| \left|\hat{c}(u-s)\hat{c}(v-s)\right| du\,dv\,ds \\
&\le C \int_{t_1}^{t_2}\int_s^{t_2}\int_s^{t_2} u^{-1}v^{-1} \left|\hat{c}(u-s)\hat{c}(v-s)\right| du\,dv\,ds \\
&= C \int_{t_1}^{t_2}\int_{t_1}^{t_2} u^{-1}v^{-1} \int_{t_1}^{u\wedge v} \left|\hat{c}(u-s)\hat{c}(v-s)\right| ds\,du\,dv \\
&\le C \int_{t_1}^{t_2}\int_{t_1}^{t_2} u^{-1}v^{-1} \left(\int_{t_1}^{u\wedge v} \hat{c}^2(u-s)\,ds \right)^{\frac{1}{2}} \left(\int_{t_1}^{u\wedge v} \hat{c}^2(v-s)\,ds \right)^{\frac{1}{2}} du\,dv \\
&\le C \int_{t_1}^{t_2}\int_{t_1}^{t_2} u^{-1}v^{-1} \left(\int_{t_1}^{u} \hat{c}^2(u-s)\,ds \right)^{\frac{1}{2}} \left(\int_{t_1}^{v} \hat{c}^2(v-s)\,ds \right)^{\frac{1}{2}} du\,dv \\
&\le C \int_{t_1}^{t_2}\int_{t_1}^{t_2} u^{-1}v^{-1} (u-t_1)^{\frac{\gamma}{2}} (v-t_1)^{\frac{\gamma}{2}}\,du\,dv \\
&\le C \int_{t_1}^{t_2}\int_{t_1}^{t_2} (u-t_1)^{\frac{\gamma}{2}-1} (v-t_1)^{\frac{\gamma}{2}-1}\,du\,dv \le C\,(t_2-t_1)^{\gamma}\,.
\end{aligned}
$$

Combining the bounds we get $\mathbf{E}\left(\widehat{Y}_{t_2} - \widehat{Y}_{t_1}\right)^2 \le C\,(t_2-t_1)^{\gamma}$, whence the result follows. □

14.4 Equations with Locally Lipschitz Drift of Linear Growth

In this section we study stochastic differential equations with additive Volterra–Lévy noise. The noise considered has Hölder regularity of the paths as discussed in the first part of this work. We shall adopt pathwise considerations and, for this reason, we start the study taking deterministic equations into account, then we move to discuss the stochastic cases.

Let $T > 0$ be fixed, $f = f(t), t \in [0, T]$, and coefficient $u = u(x), x \in \mathbb{R}$, be the measurable functions. Introduce the equation

$$X_t = \int_0^t u(X_s)\, ds + f(t),\ \ t \in [0, T],\ \ X|_{t=0} = X_0 \in \mathbb{R}. \tag{14.41}$$

This equation is studied in Appendix B. Now let us return to the Eq. (14.1), that is, let us consider the Volterra–Lévy process $Y_t = \int_0^t g(t, s)\, dZ_s$ instead of the deterministic function f. According to Lemma 11 in Appendix B, in order to obtain the existence and uniqueness of a solution, it suffices to establish either local integrability or local boundedness of Y.

First, we study the sufficient conditions for integrability. Namely, we present the conditions supplying $\mathbf{E} \int_0^T |Y_t|\ dt < \infty$. If the assumption (A1) holds, then by (14.3),

$$\mathbf{E} \int_0^T |Y_t|\ dt \le \int_0^T \left(\mathbf{E}\, |Y_t|^p\right)^{\frac{1}{p}}\ dt \le C \left(\int_{\mathbb{R}} |x|^p \pi(dx)\right)^{\frac{1}{p}} \int_0^T \|g(t, \cdot)\|_{L_p([0,t])}\, dt,$$

therefore, the sufficient condition for integrability is $\int_0^T \|g(t, \cdot)\|_{L_p([0,t])}\, dt < \infty$.

Similarly, if the assumption (A2) holds, then using (14.4) we get

$$\begin{aligned}\mathbf{E} \int_0^T |Y_t|\ dt &\le C a^{\frac{1}{2}} \int_0^T \|g(t, \cdot)\|_{L_2([0,t])}\, dt \\ &\quad + C \left(\int_{\mathbb{R}} |x|^p \pi(dx)\right)^{\frac{1}{p}} \int_0^T \|g(t, \cdot)\|_{L_p([0,t])}\, dt.\end{aligned}$$

Since $p \ge 2$, we see that again the sufficient condition for integrability has the form $\int_0^T \|g(t, \cdot)\|_{L_p([0,t])}\, dt < \infty$. In the Gaussian case the second term vanishes, hence a weaker condition is required, namely $\int_0^T \|g(t, \cdot)\|_{L_2([0,t])}\, dt < \infty$.

Now let the kernel g satisfy the assumption (B1). Then

$$\int_0^T \|g(t,\cdot)\|_{L_p([0,t])}\,dt \le C\int_0^T t^\alpha \left(\int_0^t s^{\beta p}(t-s)^{\gamma p}\,ds\right)^{\frac{1}{p}} dt$$
$$\le C\int_0^T t^{\alpha+\beta+\gamma+\frac{1}{p}}\,dt,$$

where we have used the equality $\int_0^t s^{\beta p}(t-s)^{\gamma p}\,ds = \mathrm{B}(\beta p+1,\gamma p+1)t^{\beta p+\gamma p+1}$ (assuming that $\beta > -\frac{1}{p}$, $\gamma > -\frac{1}{p}$). Consequently, under the assumption (B1) the condition $\int_0^T \|g(t,\cdot)\|_{L_p([0,t])}\,dt < \infty$ holds, if $\alpha+\beta+\gamma+\frac{1}{p} > -1$.

Similarly to Lemmas 4–6, we can consider three cases. Thus, we arrive at the following result.

Theorem 2 *Assume that one of the following assumptions holds:*

1. $p \ge 1$, $a = 0$, $\int_{\mathbb{R}} |x|^p\,\pi(dx) < \infty$, *the condition (B1) holds with some* $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{p}$, $\gamma > -\frac{1}{p}$ *such that* $\alpha+\beta+\gamma > -\frac{1}{p} - 1$;
2. $p \ge 2$, $\int_{\mathbb{R}} |x|^p\,\pi(dx) < \infty$, *the condition (B1) holds with some* $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{p}$, $\gamma > -\frac{1}{p}$ *such that* $\alpha+\beta+\gamma > -\frac{1}{p} - 1$;
3. *Z is a Brownian motion, the condition (B1) holds with* $p = 2$, $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{2}$, $\gamma > -\frac{1}{2}$ *such that* $\alpha+\beta+\gamma > -\frac{3}{2}$.

Then $\mathbf{E}\int_0^T |Y_t|\,dt < \infty$. *Consequently, if the coefficient u satisfies the assumption (D1) 1) of Lemma 11, then the Eq.* (14.1) *has a unique solution.*

Now we adapt the condition (D2) 3) of Lemma 11 to the stochastic case. Since continuity is a sufficient condition for local boundedness, we obtain the following corollary from Lemmas 4–6.

Theorem 3 *Assume that one of the following assumptions holds:*

1. $p \ge 1$, $a = 0$, $\int_{\mathbb{R}} |x|^p\,\pi(dx) < \infty$, *the conditions (B1) and (B2) hold with some* $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{p}$, $\gamma > -\frac{1}{p}$, $\delta > \frac{1}{p}$ *such that* $\alpha+\beta+\gamma > -\frac{1}{p}$, $\kappa > 0$;
2. $p \ge 2$ *we have* $\int_{\mathbb{R}} |x|^p\,\pi(dx) < \infty$ *and the conditions (B1) and (B2) hold with some* $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{p}$, $\gamma > -\frac{1}{p}$, $\delta > \frac{1}{p}$ *such that* $\alpha+\beta+\gamma > -\frac{1}{p}$, $\kappa > 0$;
3. *Z is a Brownian motion, the conditions (B1) and (B2) hold with* $p = 2$, $\alpha \in \mathbb{R}$, $\beta > -\frac{1}{2}$, $\gamma > -\frac{1}{2}$, $\delta > 0$ *such that* $\alpha+\beta+\gamma > -\frac{1}{2}$, $\kappa > -\frac{1}{2}$.

Then Y has a.s. continuous (hence, locally bounded) sample paths. Consequently, if the coefficient u satisfies the assumptions (D2) 1), 2) of Lemma 11, then the Eq. (14.1) *has a unique solution.*

We remark that it seems that there no general results about solutions of stochastic differential equations (14.1) with Volterra–Lévy noise without some form of Lipschitz continuity assumptions. There are instead some papers dealing with some classes of such equations also with exploding drift. We refer e. g. to [3] for a short survey and the study of a class of such equations.

In the next section we address another class of equation without Lipschitz drift. We focus on Volterra–Gaussian processes. The particular case of fractional Brownian motion was considered in [14].

14.5 Equations with Volterra–Gaussian Processes

Now our goal is to consider equations with additive noise represented by various Volterra–Gaussian processes, some of which were introduced in [12]. Our aim is to relax the conditions on the drift coefficient, in a similar fashion to what was done in the paper [14]. Remark that, in [14], the noise was fractional Brownian motion, but here we deal with more general noise.

14.5.1 Girsanov Theorem. Definition of Weak and Strong Solutions

Let $\left\{\mathscr{F}_t^V, t \in [0,T]\right\}$ denote the natural filtration of V, where V can be either Y defined by (14.32) and (14.33), or it can be $\widehat{Y}$ is defined by (14.39) and (14.38). For some process $u = \{u_t, t \in [0,T]\}$ with integrable trajectories, denote

$$z(s) = \left(a^{-1} D^h_{0+}\left(ub^{-1}\right)\right)(s), \quad \hat{z}(s) = \left(\hat{a}^{-1} I^{\hat{h}}_{0+}\left(\hat{b}^{-1}u\right)\right)(s).$$

an let, respectively,

$$\xi_T = \mathbf{E}p\left\{-\int_0^T z(s)\,dW_s - \frac{1}{2}\int_0^T z^2(s)\,ds\right\},$$

$$\widehat{\xi}_T = \mathbf{E}p\left\{-\int_0^T \hat{z}(s)\,dW_s - \frac{1}{2}\int_0^T z^2(s)\,ds\right\}.$$

Theorem 4 *(1) Let the assumptions (C1)–(C3) hold, and let $u = \{u_t, t \in [0,T]\}$ be a F^Y-adapted process with integrable trajectories. Consider the transformation*

$$V_0(t) = Y_t + \int_0^t u_s\,ds. \tag{14.42}$$

Assume that

1. $z \in L_2([0,T])$ *a. s.,*
2. $\mathbf{E}\xi_T = 1$.

Then V_0 *can be represented as* $V_0(t) = \int_0^t K(t,s)\,dB_s,\ t \in [0,T]$, *where* B *is a* $\mathscr{F}^Y$*-Wiener process under the new probability* P_B *defined by* $d\mathsf{P}_B/d\mathsf{P} = \xi_T$.

(2) Let the assumptions $(\widehat{\mathbf{C}}1)$*–*$(\widehat{\mathbf{C}}3)$ *hold, and let* $u = \{u_t, t \in [0,T]\}$ *be a* $F^{\widehat{Y}}$*-adapted process with integrable trajectories. Consider the transformation*

$$\widehat{V}_0(t) = \widehat{Y}_t + \int_0^t u_s\,ds.$$

Assume that

1. $\hat{z} \in L_2([0,T])$ *a. s., and*
2. $\mathbf{E}\widehat{\xi}_T = 1$

Then $\widehat{V}_0$ *can be represented as* $\widehat{V}_0(t) = \int_0^t \widehat{K}(t,s)\,d\widehat{B}_s,\ t \in [0,T]$, *where* $\widehat{B}$ *is a* $\mathscr{F}^{\widehat{Y}}$*-Wiener process under the new probability* $\mathsf{P}_{\widehat{B}}$ *defined by* $d\mathsf{P}_{\widehat{B}}/d\mathsf{P} = \widehat{\xi}_T$.

Proof Let us prove only (1) since both statements are proved similarly. Inserting (14.32) into (14.42) yields $V_0(t) = \int_0^t K(t,s)\,dW_s + \int_0^t u_s\,ds = \int_0^t K(t,s)\,dB_s$, where $B_t = W_t + \int_0^t \mathscr{K}^{-1}\left(\int_0^\cdot u_s\,ds\right)(r)\,dr$. Using (14.36), we get

$$B_t = W_t + \int_0^t a^{-1}(r) D^h_{0+}\left(ub^{-1}\right)(r)\,dr.$$

Finally, by the standard Girsanov theorem, B is a $\mathscr{F}^Y$-Wiener process under the probability P_B. □

In the sequel, we study two stochastic differential equations

$$X_t = x + V_t + \int_0^t u(s, X_s)\,ds, \quad t \in [0,T], \tag{14.43}$$

where $x \in \mathbb{R}$, $u\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a measurable function, $V = Y, \widehat{Y}$, where Y is defined by (14.32) and (14.33), while $\widehat{Y}$ is defined by (14.39) and (14.38). We shall consider both strong and weak solutions according to the definition below.

Definition 2 (i) By a weak solution of Eq. (14.43) we mean a couple of processes (V, X) on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}^V, \mathsf{P})$, such that

$$V_t = \int_0^t K(t,s)\, dW_s \quad \text{or} \quad V_t = \int_0^t \widehat{K}(t,s)\, dW_s, \tag{14.44}$$

respectively, with some Wiener process W, and (V, X) satisfy (14.43).

(ii) By a strong solution of equation (14.43) we understand a process X on $(\Omega, \mathscr{F}, \mathbb{F}^V, \mathsf{P})$, and V is of the form (14.44) with the fixed Wiener process W.

14.5.2 Weak Existence and Weak Uniqueness

Let the coefficients a, b, c satisfy the assumptions (C1)–(C4). Then, according to Proposition 3, the stochastic process Y has a modification satisfying Hölder condition up to order $\nu \in (0, 1/2)$.

Theorem 5 *(i) Assume that $u(s, x)$ satisfies the sublinear growth condition: there exist such $0 < \alpha < 1$ and $C > 0$ that*

$$|u(t,x)| \le C(1 + |x|^\alpha), \tag{14.45}$$

and Hölder condition in space and time: there exist $0 < \beta \le 1$, $0 < \gamma < 1$ and $C > 0$ such that for any $s, t \in [0, T]$ and any $x, y \in \mathbb{R}$

$$|u(t,x) - u(s,y)| \le C\left(|t-s|^\beta + |y-x|^\gamma\right).$$

Additionally to (C1)–(C4), let also functions a, b and h satisfy the following assumption: there exist $C > 0$ and $\nu' \in (0, \nu)$ such that

$$\int_0^T a^{-2}(s) h^2(s) b^{-2}(s)\, ds \le C,$$

$$\int_0^T a^{-2}(t) \left(\int_0^t \left|h'(t-r)\right| \left|b^{-1}(t) - b^{-1}(r)\right| dr \right)^2 dt \le C, \tag{14.46}$$

$$\int_0^T a^{-2}(t) b^{-2}(t) \left(\int_0^t \left|h'(t-r)\right| (t-r)^\beta\, dr \right)^2 dt \le C,$$

$$\int_0^T a^{-2}(t) b^{-2}(t) \left(\int_0^t \left|h'(t-r)\right| (t-r)^{\gamma\nu'}\, dr \right)^2 dt \le C.$$

Then the equation (14.43) with $V = Y$ has a unique weak solution.

(ii) Assume that $u(s,x)$ satisfies the sublinear growth condition (14.45)*, and, additionally to (*$\widehat{\mathbf{C}}1$*)–(*$\widehat{\mathbf{C}}3$*), functions $\hat{a}$, $\hat{b}$ and $\hat{h}$ satisfy following assumption: there exists $C > 0$ such that*

$$\hat{a}^{-1}(s) \int_0^s \left|\hat{h}(s-r)\right| \left|\hat{b}^{-1}(r)\right| dr \le C. \tag{14.47}$$

Then Eq. (14.43) *with $V = \widehat{Y}$ has a unique weak solution.*

Remark 7 Let us check the conditions (14.46) and (14.47) in the case when V is a fractional Brownian motion.
(i) Let $H > \frac{1}{2}$, $a(s) = s^{\frac{1}{2}-H}$, $b(s) = s^{H-\frac{1}{2}}$, $c(s) = s^{H-\frac{3}{2}}$, $h(s) = s^{\frac{1}{2}-H}$. Then,

$$\int_0^T a^{-2}(s)h^2(s)b^{-2}(s)\,ds = \int_0^T s^{1-2H}\,ds = (2-2H)^{-1}T^{2-2H};$$

$$\int_0^T a^{-2}(t)\left(\int_0^t \left|h'(t-r)\right| \left|b^{-1}(t) - b^{-1}(r)\right| dr\right)^2 dt$$

$$= C\int_0^T t^{2H-1}\left(\int_0^t (t-r)^{-\frac{1}{2}-H}\left(r^{\frac{1}{2}-H} - t^{\frac{1}{2}-H}\right) dr\right)^2 dt$$

$$= C\int_0^T t^{2H-1} \cdot t^{-1-2H} \cdot t^{1-2H} t^2\,dt \left(\int_0^1 (1-r)^{-\frac{1}{2}-H}\left(r^{\frac{1}{2}-H} - 1\right) dr\right)^2$$

$$= CT^{2-2H}\left(\int_0^1 (1-r)^{-\frac{1}{2}-H}\left(r^{\frac{1}{2}-H} - 1\right) dr\right)^2.$$

Integral $\int_0^1 (1-r)^{-\frac{1}{2}-H}\left(r^{\frac{1}{2}-H} - 1\right) dr$ is finite, since around zero,

$$(1-r)^{-\frac{1}{2}-H}\left(r^{\frac{1}{2}-H} - 1\right) \sim r^{\frac{1}{2}-H} - 1$$

and around 1, $(1-r)^{-\frac{1}{2}-H}\left(r^{\frac{1}{2}-H} - 1\right) \sim (1-r)^{\frac{1}{2}-H}$. Further,

$$\int_0^T \left(a^{-2}b^{-2}\right)(t) \left(\int_0^t \left|h'(t-r)\right| (t-r)^{\beta}\, dr\right)^2 dt$$
$$= C\int_0^T \left(\int_0^t (t-r)^{-\frac{1}{2}-H+\beta}\, dr\right)^2 dt \le C$$

if $-\frac{1}{2} - H + \beta > -1$, or $\beta > H - \frac{1}{2}$. Finally,

$$\int_0^T \left(a^{-2}b^{-2}\right)(t) \left(\int_0^t \left|h'(t-r)\right| (t-r)^{\gamma\nu'}\, dr\right)^2 dt$$
$$= C\int_0^T \left(\int_0^t (t-r)^{-\frac{1}{2}-H+\gamma\nu'}\, dr\right)^2 dt \le C$$

if $-\frac{1}{2} - H + \gamma\nu'$ or $\gamma\nu' > H - \frac{1}{2}$. But in this case ν' can be any number from 0 to H, therefore, condition $\gamma\nu' > H - \frac{1}{2}$ holds if $\gamma H > H - \frac{1}{2}$, or $\gamma > 1 - \frac{1}{2H}$. Therefore assumptions (14.46) hold for $\beta > H - \frac{1}{2}$, $\gamma > 1 - \frac{1}{2H}$.
(ii) Let $H < \frac{1}{2}$. Then $\hat{a}(s) = Cs^{\frac{1}{2}-H}$, $\hat{b}(s) = \hat{c}(s) = s^{H-\frac{1}{2}}$, $\hat{h}(s) = s^{-\frac{1}{2}-H}$, therefore

$$\hat{a}^{-1}(s)\int_0^s \left|\hat{h}(s-r)\right| \left|\hat{b}^{-1}(r)\right| dr = Cs^{H-\frac{1}{2}}\int_0^s (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H}\, dr = Cs^{\frac{1}{2}-H} \le C,$$

so (14.47) holds.

Proof First, we give some upper bounds for $z(s)$ and $\hat{z}(s)$ in order to confirm that the theorem assumptions supply Novikov conditions for ξ_T and $\widehat{\xi}_T$, and therefore ξ_T and $\widehat{\xi}_T$ satisfy Theorem 4. Then the proofs of (i) and (ii) are similar, therefore we continue only with the second statement, dividing the proof into several steps and refer to the paper [14] for additional detail.

Concerning $z(s)$, by Lemma 10 (iii), we have that

$$\begin{aligned} z(s) &= \left(a^{-1}hb^{-1}\right)(s)u\left(s, Y_s + x\right) \\ &\quad + a^{-1}(s)\int_0^s \left(u\left(z, Y_z + x\right) b^{-1}(z) - u\left(s, Y_s + x\right) b^{-1}(s)\right) h'(s-z)\, dz \\ &= J_1(s) + J_2(s). \end{aligned}$$

Let us construct upper bounds for J_1 and J_2. Namely, we are interested in two integrals. First,

$$\int_0^T J_1^2(s)\,ds \le C\left(1+\sup_{0\le s\le T}|Y_s+x|^{2\alpha}\right)\int_0^T\left(a^{-2}h^2b^{-2}\right)(s)\,ds \le C\left(1+\sup_{0\le s\le T}|Y_s|^{2\alpha}\right), \tag{14.48}$$

according to the first assumption in (14.46).

Second,

$$\begin{aligned}\int_0^T J_2^2(s)\,ds &\le C\left(1+\sup_{0\le s\le T}|Y_s+x|^{2\alpha}\right)\\ &\quad\times\int_0^T a^{-2}(s)\left(\int_0^s\left|b^{-1}(z)-b^{-1}(s)\right|\left|h'(s-z)\right|dz\right)^2 ds\\ &\quad+\int_0^T\left(a^{-2}b^{-2}\right)(s)\left(\int_0^s|u(s,Y_s+x)-u(z,Y_z+x)|\left|h'(s-z)\right|dz\right)^2 ds\\ &= M_1+M_2.\end{aligned}$$

Obviously,

$$M_1 \le C\left(1+\sup_{0\le s\le T}|Y_s|^{2\alpha}\right), \tag{14.49}$$

according to the second assumption in (14.46). Concerning M_2, it admits the following upper bound:

$$\begin{aligned}M_2 &\le C\int_0^T(ab)^{-2}(s)\left(\int_0^s(s-z)^\beta\left|h'(s-z)\right|dz\right)^2 ds\\ &\quad+C\int_0^T(ab)^{-2}(s)\left(\int_0^s|Y_s-Y_z|^\gamma\left|h'(s-z)\right|dz\right)^2 ds = N_1+N_2.\end{aligned}$$

According to 3rd assumption in (14.46), $N_1 \le C$. Further, due to the 4th assumption from (14.46),

$$N_2 \le C \left(\sup_{0 \le s < t \le T} \frac{|Y_s - Y_t|^\gamma}{(t-s)^{\gamma\nu'}} \right)^2 \int_0^T (ab)^{-2}(s) \left(\int_0^s (s-z)^{\gamma\nu'} \left| h'(s-z) \right| dz \right)^2 ds$$
$$\le C \left(\sup_{0 \le s < t \le T} \frac{|Y_s - Y_t|}{(t-s)^{\nu'}} \right)^{2\gamma} =: CG,$$

and due to the fact that $2\gamma < 1$ and to [5], $\mathbf{E} \exp\{CG\} < \infty$ for any $G > 0$. Combining this with (14.48) and (14.49), we conclude that $\mathbf{E} \exp\left\{ \frac{1}{2} \int_0^T z_s^2 \, ds \right\} < \infty$, and Novikov condition holds, consequently, $\widehat{\xi}_T$ is indeed a density function.

Concerning $\hat{z}(s)$, let us provide the following calculations:

$$\hat{z}(s) = \hat{a}^{-1}(s) \int_0^s \hat{h}(s-r) \, \hat{b}^{-1}(r) \, u\left(r, \widehat{Y}_r + x\right) dr$$

and, according to (14.45),

$$\left|\hat{z}(s)\right|^2 \le C\hat{a}^{-2}(s) \left(\int_0^s \left|\hat{h}(s-r)\right| \left|\hat{b}^{-1}(r)\right| \left(1 + \left|\widehat{Y}_r + x\right|^\alpha\right) dr \right)^2$$
$$\le C \left(1 + \sup_{r \in [0,T]} \left|\widehat{Y}_r\right|^{2\alpha} \right) \hat{a}^{-2}(s) \left(\int_0^s \left|\hat{h}(s-r)\right| \left|\hat{b}^{-1}(r)\right| dr \right)^2. \tag{14.50}$$

Under assumption (14.47), $\left|\hat{z}(s)\right|^2 \le C \left(1 + \sup_{r \in [0,T]} \left|\widehat{Y}_r\right|^{2\alpha} \right)$. Then it follows from the fact that $2\alpha < 2$ and integrability of supremum of Gaussian process [5] that $\sup_{0 \le s \le T} \mathbf{E} \exp\left\{ \rho \sup_{0 \le s \le T} \left|\hat{z}(s)\right|^2 \right\} < \infty$ for any $\rho > 0$, and this inequality supplies Novikov condition for $\widehat{\xi}_T$.

Now we continue with the proof of (ii). We consider the two cases of V.

(a) Together with Theorem 4, we can conclude that $\widetilde{Y}$ is a Volterra–Gaussian process of the form $\widetilde{Y}_t = \int_0^t \widehat{K}(t,s) \, d\widetilde{B}_s$, where $\widetilde{B}$ is a Wiener process with respect to the probability measure $\mathsf{P}_{\widetilde{B}}$ defined by $d\mathsf{P}_{\widetilde{B}}/d\mathsf{P} = \widehat{\xi}_T$, where

$$\widehat{\xi}_T = \exp\left\{ \int_0^T \hat{z}(s) \, dW_s - \frac{1}{2} \int_0^T \hat{z}^2(s) \, ds \right\}.$$

It means that the couple $(\widetilde{Y}, \widehat{Y} + x)$ creates a weak solution of (14.43) with $V = \widehat{Y}$.

(b) Now let us apply and modify the approach from [14] concerning the proof of uniqueness in law and pathwise uniqueness of the equations with additive fractional noise. Namely, consider any solution of the equation

$$X_t = x + \int_0^t u(s, X_s)\, ds + \widehat{Y}_t,$$

where $\widehat{Y}_t = \int_0^t \widehat{K}(t, s)\, dB_s$, B is some Wiener process, and define

$$\hat{z}(s) = \hat{a}^{-1}(s) \int_0^s h(s-r)\hat{b}^{-1}(r) u(r, X_r)\, dr.$$

Note that $X \in \mathbb{C}([0, T])$, therefore, due to sublinear growth condition,

$$\sup_{0\le v\le r} |u(v, X_v)| \le C\left(1 + \sup_{0\le v\le r} |X_v|^{2\alpha}\right) < \infty \quad \text{a.s.}$$

Also, $\sup_{0\le t\le T} \left|\widehat{Y}_t\right| < \infty$ a. s. Therefore, from Gronwall inequality, we get $\sup_{0\le t\le T} |X_t| \le \left(|x| + \sup_{0\le t\le T} \left|\widehat{Y}_t\right| + CT\right) e^{CT}$, and in turn it implies that, similarly to (14.50), under assumption (14.47), $\left|\hat{z}(s)\right|^2 \le C\left(1 + \sup_{0\le t\le T} |X_t|^{2\alpha}\right) \le C_1\left(1 + \sup_{0\le t\le T} \left|\widehat{Y}_t\right|^{2\alpha}\right)$ for any $s \in [0, T]$. It means that w. r. t. the measure $\widehat{\mathsf{P}}$ such that

$$\frac{d\widehat{\mathsf{P}}_T}{d\mathsf{P}_T} = \exp\left\{-\int_0^T \hat{z}(s) dB_s - \frac{1}{2}\int_0^T \hat{z}^2(s)\, ds\right\}, \tag{14.51}$$

$X_t - x$ has the same distribution as the process $\int_0^t \hat{K}(t, s)\, dV_s$, where V is a Wiener process, $V_s = B_s + \int_0^s \hat{z}(u) du$, and the right-hand side of (14.51) indeed defines a probability measure. Further, for any bounded measurable functional Φ on $\mathbb{C}([0, T])$,

$$\begin{aligned}
\mathbf{E}_{\mathsf{P}} \Phi(X - x) &= \int_\Omega \Phi(\xi - x) \frac{d\mathsf{P}_T}{d\widehat{\mathsf{P}}_T}(\xi)\, d\widehat{\mathsf{P}}_T \\
&= \mathbf{E}_{\widehat{\mathsf{P}}}\left(\Phi(X - x) \exp\left\{\int_0^T \hat{z}(s)\, dB_s + \frac{1}{2}\int_0^T \hat{z}^2(s)\, ds\right\}\right) \\
&= \mathbf{E}_{\widehat{\mathsf{P}}}\left(\Phi(X - x) \exp\left\{\int_0^T \hat{a}^{-1}(s) \int_0^s h(s-r)\hat{b}^{-1}(r) u(r, X_r)\, dr\, dB_s\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}\int_0^T \left(\hat{a}^{-1}(s)\int_0^s h(s-r)\hat{b}^{-1}(r)u(r,X_r)\,dr\right)^2 ds\Bigg\}\Bigg) \\
&= \mathbf{E}_{\widehat{\mathsf{P}}}\Bigg(\Phi(X-x)\exp\Bigg\{\int_0^T \hat{a}^{-1}(s)\int_0^s h(s-r)\hat{b}^{-1}(r)u(r,X_r)\,dr\,dV_s \\
&- \frac{1}{2}\int_0^T \left(\hat{a}^{-1}(s)\int_0^s h(s-r)\hat{b}^{-1}(r)u(r,X_r)\,dr\right)^2 ds\Bigg\}\Bigg) \\
&= \mathbf{E}_{\mathsf{P}}\Phi\left(\int_0^{\cdot} \widehat{K}(\cdot,s)\,dB_s\right) \\
&\times \exp\Bigg\{\int_0^T \hat{a}^{-1}(s)\int_0^s h(s-r)\hat{b}^{-1}(r)\; u\left(r, x+\int_0^T \widehat{K}(r,z)\,dB_z\right)dr\,dB_s \\
&- \frac{1}{2}\int_0^T \left(\hat{a}^{-1}(s)\int_0^s h(s-r)\hat{b}^{-1}(r)u\left(r,x+\int_0^T \widehat{K}(r,z)\,dB_z\right)dr\right)^2 ds\Bigg\} \\
&= \mathbf{E}_{\mathsf{P}}\Phi\left(\int_0^{\cdot} \widehat{K}(\cdot,s)\,dV_s\right).
\end{aligned}
\tag{14.52}
$$

Taking (14.52) into account, we conclude that any two weak solutions have the same distribution, so we established weak uniqueness. □

14.5.3 Pathwise Uniqueness of Weak Solution. Existence and Uniqueness of Strong Solution

Now we consider only equation

$$
X_t = x + Y_t + \int_0^t u(s, X_s)\,ds, \quad t \in [0, T], \tag{14.53}
$$

where $x \in \mathbb{R}$, $u\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is a measurable function, Y is defined by (14.32) and (14.33).

Theorem 6 *Let coefficients a, b, c satisfy assumptions (C1)–(C3) and (C5). Let also coefficient $u(s,x)$ satisfy conditions of item (i), Theorem 5. Then any two weak solu-*

tions of Eq. (14.53) *with the same Wiener process W occurring in the representation of Y, coincide a. s.*

Proof According to Proposition 3, the condition (C5) supplies that the process Y on any interval $[t_1 + t_2, T]$ has a modification that satisfies Hölder condition up to order $\mu = \frac{3}{2} - \frac{1}{q_1} - \max\left(\frac{1}{2}, \frac{1}{p} + \frac{1}{r_1}, \frac{1}{p_1} + \frac{1}{r}\right) > \frac{1}{2}$. So, consider any $0 < \varepsilon < T$, and on the interval $[\varepsilon, T]$ apply Itô formula to the process $\max\left(X_t^1, X_t^2\right)$, where X^1 and X^2 are two weak solutions with the same Wiener process W. Observing that X^1 and X^2 are Hölder up to order $\mu > \frac{1}{2}$ on $[\varepsilon, T]$, which implies that the quadratic variation of $X^1 - X^2$ is zero, we get that for any $t \in [\varepsilon, T]$

$$\begin{aligned}\max\left(X_t^1, X_t^2\right) - \max\left(X_\varepsilon^1, X_\varepsilon^2\right) &= X_t^1 - X_\varepsilon^1 + \left(X_t^2 - X_t^1\right)_+ - \left(X_\varepsilon^2 - X_\varepsilon^1\right)_+ \\ &= Y_t - Y_\varepsilon + \int_\varepsilon^t u\left(s, X_s^1\right) ds + \int_\varepsilon^t \left(u\left(s, X_s^2\right) - u\left(s, X_s^1\right)\right) \mathbb{1}_{\{X_s^2 > X_s^1\}} ds \\ &= Y_t - Y_\varepsilon + \int_\varepsilon^t u\left(s, \max\left(X_s^1, X_s^2\right)\right) ds.\end{aligned}$$

Let $\varepsilon \to 0$. Then it follows from continuity of Y and u that $Y_\varepsilon \to 0$ a. s., and

$$\int_\varepsilon^t u\left(s, \max\left(X_s^1, X_s^2\right)\right) ds \to \int_0^t u\left(s, \max\left(X_s^1, X_s^2\right)\right) ds \quad \text{a. s.}$$

Moreover, $\max\left(X_\varepsilon^1, X_\varepsilon^2\right) \to x$ a. s.

Finally, $\max\left(X_t^1, X_t^2\right) = x + Y_t + \int_0^t u\left(s, \max\left(X_s^1, X_s^2\right)\right) ds$. It means that $\max\left(X_t^1, X_t^2\right)$ (and similarly $\min\left(X_s^1, X_s^2\right)$) satisfies Eq. (14.53). Due to the weak uniqueness proved in Theorem 5, $\max\left(X_t^1, X_t^2\right)$ and $\min\left(X_s^1, X_s^2\right)$ have the same distribution, whence $X_t^1 = X_t^2$ a. s., and from continuity of X^1 and X^2, $X_t^1 = X_t^2$, $t \in [0, T]$, a. s. □

Remark 8 1. Condition (C5) is fulfilled in the case when $Y = B^H$ with $H > \frac{1}{2}$. In this case we can put $p_1 = q_1 = r_1 = \frac{3}{\varepsilon}$, where $0 < \varepsilon < \min\left\{(H - \frac{1}{2}), 3(1 - H), \frac{1}{2}\right\}$, $\frac{1}{p} = H = \frac{1}{2} + \frac{\varepsilon}{3}$, $\frac{1}{q} = \frac{\varepsilon}{3}$, $\frac{1}{r} = \frac{3}{2} - H + \frac{\varepsilon}{3}$. Then

$$\mu = \frac{3}{2} - \frac{\varepsilon}{3} - \max\left\{\frac{1}{2}, \frac{3}{2} - H + \frac{2\varepsilon}{3}, H - \frac{1}{2} + \frac{2\varepsilon}{3}\right\} = H - \varepsilon > \frac{1}{2}.$$

2. In the case when we cannot guarantee that Y is Hölder up to some order $\mu > \frac{1}{2}$ (for example, in the case when $Y = B^H$ with $H < \frac{1}{2}$) the Itô formula for $\max\left(X_t^1, X_t^2\right)$ has a different form, and the statement like Theorem 5 is an open problem.

We conclude with a straightforward consequence of Theorems 5 and 6.

Theorem 7 *Under the assumptions of Theorem 6, Eq.* (14.53) *has a unique strong solution.*

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Appendix A: Elements of Fractional Calculus for Sonine Pairs

Here we consider some notions similar to the notions of the fractional integral and of the fractional derivative proper to classical fractional calculus.

Definition 3 Let functions c and h from $L_1([0,T])$ create a Sonine pair. Introduce the operators, similar to operators of fractional integral and fractional derivative:

$$\left(I^c_{0+}f\right)(t) = \int_0^t c(t-s)f(s)\,ds, \quad f \in L_1([0,T]),$$

$$\left(D^h_{0+}f\right)(t) = \frac{d}{ds}\left(\int_0^t h(t-s)f(s)\,ds\right),$$

where $f\colon [0,T] \to \mathbb{R}$ is such that $\int_0^t h(t-s)f(s)\,ds \in AC([0,T])$.

Now we can here establish some properties of the operators I^c_{0+} and D^h_{0+}. Denote

$$I^c_{0+}\left(L_1([0,T])\right) = \left\{\psi\colon [0,T] \to \mathbb{R} : \psi(t) = \left(I^c_{0+}\varphi\right)(t),\ \varphi \in L_1([0,T])\right\}.$$

Lemma 10 *(i) Let $f \in L_1([0,T])$. Then $\left(D^h_{0+}I^c_{0+}f\right)(t) = f(t)$ a.e.*

(ii) Let $f \in I^c_{0+}\left(L_1([0,T])\right)$. Then $\left(I^c_{0+}D^h_{0+}f\right)(t) = f(t),\ t \in [0,T]$.

(iii) Let $h \in C^1(0,T)$, there exist $\beta > 0$ such that $\lim_{s\to 0} s^{\beta+1}h'(s) < \infty$. Also, let f be a Hölder function of order γ, and $\gamma > \beta$. Then for any $t \in [0,T]$, $\left(D^h_{0+}f\right)(t) = h(t)f(t) + \int_0^t [f(z) - f(t)]h'(t-z)\,dz$.

Proof (i) Obviously,

$$\begin{aligned}\left(D^h_{0+}I^c_{0+}f\right)(t) &= \frac{d}{dt}\left(\int_0^t h(t-s)\left(\int_0^s c(s-u)f(u)\,du\right)ds\right)\\ &= \frac{d}{dt}\left(\int_0^t f(u)\left(\int_u^t h(t-s)c(s-u)\,ds\right)du\right)\\ &= \frac{d}{dt}\left(\int_0^t f(u)\,du\right) = f(t)\ \text{a.e.}\end{aligned}$$

(ii) Let $f(t) = \left(I^c_{0+}\varphi\right)(t)$, $\varphi \in L_1([0,T])$. Then, according to (i),

$$\left(I^c_{0+}D^h_{0+}f\right)(t) = \left(I^c_{0+}D^h_{0+}I^c_{0+}\varphi\right)(t) = \left(I^c_{0+}\varphi\right)(t) = f(t), \quad t \in [0,T].$$

(iii) For any $t \in (0,T)$ and $\Delta t > 0$ (other values can be considered similarly),

$$\begin{aligned}\Delta_f &:= \int_0^{t+\Delta t} h(t+\Delta t-s)f(s)\,ds - \int_0^t h(t-s)f(s)\,ds\\ &= \int_0^t \big(h(t+\Delta t-s)-h(t-s)\big)f(s)\,ds + \int_t^{t+\Delta t} h(t+\Delta t-s)f(s)\,ds\\ &= \int_0^t \big(h(t+\Delta t-s)-h(t-s)\big)\big(f(s)-f(t)\big)ds\\ &\quad + \int_t^{t+\Delta t} h(t+\Delta t-s)\big(f(s)-f(t)\big)\,ds + f(t)\int_t^{t+\Delta t} h(s)\,ds.\end{aligned}$$

Evidently, $\frac{1}{\Delta t}\left(f(t)\int_t^{t+\Delta t} h(s)\,ds\right) \to f(t)h(t)$, a. e., as $\Delta t \to 0$. Furthermore,

$$\frac{1}{\Delta t}\left|\int_t^{t+\Delta t} h(t+\Delta t-s)[f(s)-f(t)]ds\right| = |h(t+\Delta t-\theta_t)|\,|f(\theta_t)-f(t)|\,,$$

where $\theta_t \in [t, t+\Delta t]$. According to condition (iii) and L'Hôpital's rule, for some constant $C > 0$, $\lim_{\Delta t\to 0} |h(t+\Delta t-\theta_t)|\,|f(\theta_t)-f(t)| \le C \lim\limits_{\Delta t\to 0} \Delta t^{\gamma-\beta} = 0$. Finally, for $0 < \varepsilon < t$,

$$\left|\int_0^t \left(\frac{h(t+\Delta t-s)-h(t-s)}{\Delta t}-h'(t-s)\right)\big(f(s)-f(t)\big)ds\right|$$
$$=\left|\int_0^t \big(h'(\theta_t-s)-h'(t-s)\big)\big(f(s)-f(t)\big)ds\right|$$
$$\leq \left|\int_0^{t-\varepsilon} \big(h'(\theta_t-s)-h'(t-s)\big)\big(f(s)-f(t)\big)ds\right|$$
$$+\left|\int_{t-\varepsilon}^t \big(h'(\theta_t-s)-h'(t-s)\big)\big(f(s)-f(t)\big)ds\right|$$
$$\leq \left|\int_0^{t-\varepsilon} \big(h'(\theta_t-s)-h'(t-s)\big)\big(f(s)-f(t)\big)ds\right|$$
$$+\int_{t-\varepsilon}^t |h'(\theta_t-s)||f(s)-f(t)|ds+\int_{t-\varepsilon}^t \left|h'(t-s)\right||f(s)-f(t)|\,ds.$$

The first term, $\left|\int_0^{t-\varepsilon}\big(h'(\theta_t-s)-h'(t-s)\big)\big(f(s)-f(t)\big)ds\right|$, tends to 0 as $\Delta t \to 0$ for any $\varepsilon > 0$. Concerning the second term, it can be bounded as follows. For sufficiently small ε, it follows from condition (iii) that

$$\int_{t-\varepsilon}^t \left|h'(\theta_t-s)\right||f(s)-f(t)|\,ds \leq C\int_{t-\varepsilon}^t (t-s)^{-1-\beta}(t-s)^{\alpha}\,ds = C\varepsilon^{\alpha-\beta},$$

and, the same is true for $\int_{t-\varepsilon}^t \left|h'(t-s)\right||f(s)-f(t)|\,ds$, and the proof follows.

Appendix B: Deterministic Equations with Locally Lipschitz Drift of Linear Growth

In this appendix we investigate the deterministic Eq. (14.41).

Lemma 11 *Let any of two following groups of conditions hold.*

(D1) (1) The coefficient u is Lipschitz: there exists $C > 0$ such that for any $x, y \in \mathbb{R}$,

$$|u(x)-u(y)| \leq C\,|x-y|\,.$$

(2) The function f is locally integrable.

(D2) (1) The coefficient u is of linear growth: there exists $C > 0$ such that for any $x \in \mathbb{R}$,

$$|u(x)| \le C(1 + |x|).$$

(2) the coefficient u is locally Lipschitz: for any $R > 0$ there exists $C_R > 0$ such that for any $x, y \in \mathbb{R}$, $|x|, |y| < R$,

$$|u(x) - u(y)| \le C_R |x - y|.$$

(3) The function f is locally bounded.

Then the Eq. (14.41) has a unique solution X on $[0, T]$. If condition (D1) holds, then X is locally integrable. If condition (D2) holds, then X is locally bounded.

Proof First, we assume that (D1) holds. Let $t_0 > 0$ be some number. We apply successive approximations with $X_t^{(0)} = 0$, $X_t^{(1)} = f(t) \in L_1([0, t_0])$,

$$X_t^{(n)} = \int_0^t u\left(X_s^{(n-1)}\right) ds + f(t) \in L_1([0, t_0]). \tag{14.54}$$

Then for any $0 < t \le t_0$,

$$\begin{aligned}\int_0^t \left|X_s^{(n)} - X_s^{(n-1)}\right| ds &\le \int_0^t \int_0^s \left|u\left(X_v^{(n-1)}\right) - u\left(X_v^{(n-2)}\right)\right| dv\, ds \\ &\le C \int_0^t \left|X_v^{(n-1)} - X_v^{(n-2)}\right| (t - v)\, dv \le \cdots \le C^{n-1} \int_0^t |f(s)| \frac{(t-s)^{n-1}}{(n-1)!} ds \\ &\le \frac{(Ct)^{n-1}}{(n-1)!} \int_0^t |f(s)|\, ds.\end{aligned}$$

This means that $X^{(n)}$ is a Cauchy sequence in $L_1([0, t_0])$, therefore there exists a limit $X_t = \lim_{n\to\infty} X_t^{(n)}$ in $L_1([0, t_0])$. It is clear that X is a solution of (14.41). Uniqueness follows from the Gronwall inequality.

Now let us consider the case when holds. As before, let $t_0 > 0$ be fixed, and $f(t) \le C = C(t_0)$. With $X_t^{(0)} = 0$, $X_t^{(1)} = f(t)$ is locally bounded, and every $X^{(n)}$ that is defined by (14.54) is locally bounded as well. Moreover,

$$\left|X_t^{(n)}\right| \leq |f(t)| + Ct + C\int_0^t \left|X_s^{(n-1)}\right| ds$$
$$\leq |f(t)| + Ct + C\int_0^t (|f(s)| + Cs)ds + C^2\int_0^t \left|X_s^{(n-2)}\right| (t-s)\, ds \leq \cdots \leq$$
$$\leq |f(t)| + Ct + C\int_0^t (|f(s)| + Cs)e^{C(t-s)}ds,$$

therefore, $X^{(n)}$ are totally locally bounded. Existence of the limit that is a unique solution of (14.41) is evident.

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