Total Coloring of Planar Graphs Without Some Adjacent Cycles

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Abstract. Let $G = (V, E)$ be a graph. If $x, y \in V \cup E$ are two adjacent or incident elements, then a k-total-coloring of graph G is a mapping φ from $V \cup E$ to $\{1, 2, \ldots, k\}$ on condition that $\varphi(x) \neq \varphi(y)$. In this paper, we define G to be a planar graph with maximum degree $\Delta \geq 8$. We prove that if for each vertex $v \in V(G)$, there exist two integers i_v, $j_v \in \{3, 4, 5, 6, 7, 8\}$ on condition that v is not incident with adjacent i_v-cycles and j_v-cycles, then G has a $(\Delta + 1)$-total-coloring.

Keywords: Total coloring · Planar graph · Short cycle

1 Introduction

In this paper, all graphs mentioned are finite, simple and undirected. Undefined notions and terminologies can be referred to [1]. Suppose G is a graph, then V and $d(v)$ are used to denote the vertex set and the degree of v. We use F, $d(f)$ and E to denote the face set, the degree of f and the edge set respectively. Then $\Delta = \max\{d(v)|v \in V\}$ is the maximum degree of a graph and $\delta = \min\{d(v)|v \in V\}$ is the minimum degree. We use n-vertex, n^+-vertex, or n^--vertex to denote the vertex v when $d(v) = n$, $d(v) \geq n$, or $d(v) \leq n$ respectively. A n-face, n^+-face, or n^--face are analogously defined. We use $(n_1, n_2, \ldots, n_k)$ to denote a k-face and its boundary vertices are n_i-vertex $(i = 1, 2 \ldots, k)$. Similarly, we can define a $({n_1}^+, {n_2}^-, \ldots, n_k)$-face. For instance, a (l, m^+, n^-)-face is a 3-face whose boundary vertices are l-vertex, m^+-vertex and n^--vertex respectively. If two cycles or faces have at least one common edge, then we call they are adjacent. We use $n_k(f)$ to denote the number of k-vertices that is incident with f. The number of k^+-face incident with f is denoted as $n_{k^+}(f)$ and the number of k^--face incident with f is denoted as $n_{k^-}(f)$. We use $n_k(v)$ to denote the number of k-vertices adjacent to v and use $f_k(v)$ to denote the number of k-faces incident with v. If G has a k-total-coloring, then we say that G can be totally colored by k colors. For the convenience of description, we say that G is total-k-colorable when G can be totally colored by k colors. If G can be totally colored by at least k colors, then k is the total chromatic number of G that is defined as $\chi^{''}$. It is easy to know that $\chi^{''}(G) \geq \Delta + 1$. For the upper bound of $\chi^{''}$, Behzad [2] and Vizing [3] put forth the Total Coloring Conjecture (for short, TCC):

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Conjecture 1. For any graph, $\Delta + 1 \leq \chi^{''}(G) \leq \Delta + 2$.

TCC has attracted lots of researchers' attention. However, this conjecture remains open even for planar graphs. In 1971, Rosenfeld [4] and Vijayaditya [5] confirmed TCC for all graphs with $\Delta \leq 3$ independently. Kostochka [6] proved that $\chi^{''}(G) \leq \Delta + 2$ when $\Delta \leq 5$. For a planar graph, TCC is unsolved only when $\Delta = 6$ (see [6,18]). With the advances in research, some researchers found that $\chi^{''}(G)$ of some specific graphs have an exact upper bound $\Delta + 1$. In 1989, Sánchez-Arroyo [7] demonstrated that it is a NP-complete problem to determine whether $\chi^{''}(G) = \Delta + 1$ for a specified graph G. Moreover, for every fixed $k \geq 3$, McDiarmid and Sánchez-Arroyo [8] demonstrated that to determine whether a specific k-regular bipartite graph is total-$(\Delta + 1)$-colorable or not is also a NP-complete problem. However, it is possible to prove that $\chi^{''}(G) = \Delta + 1$ when G is a planar graph having large maximum degree. It has been proved that $\chi^{''}(G) = \Delta + 1$ on condition that G is a planar graph when $\Delta(G) \geq 11$ [9], $\Delta(G) = 10$ [10] and $\Delta(G) = 9$ [11]. It is still open to determine whether a planar graph is total-$(\Delta+1)$-colorable when $\Delta = 6$, 7 and 8. If G is a planar graph and $\Delta(G) = 8$, then there are some relevant results obtained by adding some restrictions. For instance, for a planar graph with $\Delta(G) \geq 8$, it is proved that G is total-$(\Delta+1)$-colorable if G does not contain k-cycles $(k = 5, 6)$ [13], or adjacent 3-cycles [12], or adjacent 4-cycles [14]. Wang et al. [15] proved $\chi^{''}(G) = \Delta + 1$ if there exist two integers i, $j \in \{3, 4, 5\}$ such that G does not contain adjacent i-cycles and j-cycles. Recently, a result has been proved in [20] for a planar graph with $\Delta(G) = 8$, that is, if for each vertex $v \in V$, there exist two integers i_v, $j_v \in \{3, 4, 5, 6, 7\}$ on condition that v is not incident with adjacent i_v-cycles and j_v-cycles, then G is total-$(\Delta+1)$-colorable. Now we improve some former results and get the following theorem.

Theorem 1. *Suppose G is a planar graph with maximum degree $\Delta \geq 8$. If for each vertex $v \in V$, there exist two integers i_v, $j_v \in \{3, 4, 5, 6, 7, 8\}$ on condition that v is not incident with adjacent i_v-cycles and j_v-cycles. Then G is total-$(\Delta + 1)$-colorable.*

2 Reducible Configurations

Theorem 1 has been proved for $\Delta \geq 9$ in [11]. So we presume that $\Delta = 8$ in the rest of this paper. Suppose $G = (V, E)$ is a minimal counterexample to Theorem 1, that is, $|V| + |E|$ is as small as possible. In other words, G cannot be totally colored by $\Delta + 1$ colors, but every proper subgraph of G can be totally colored with $\Delta + 1$ colors. In this section, we give some information of configurations for our minimal counterexample G. A configuration is called to be reducible if it cannot occur in the minimal counterexample G. Firstly, we show some known properties of G.

Lemma 1. ([9]). *(a) G is 2-connected.*

(b) Suppose v_1v_2 is an edge of G. If $d(v_1) \leq 4$, then $d(v_1) + d(v_2) \geq \Delta + 2 = 10$.

(c) Suppose G_{28} is a subgraph of G that is induced by the edges joining 2-vertices to 8-vertices. Then G_{28} is a forest.

Lemma 2. ([16]). *G has no subgraph isomorphic to the configurations depicted in Fig. 1, where $7-v$ is used to denote the vertex of degree of seven. If a vertex is marked by •, then it has no more neighbors that are not depicted in G.*

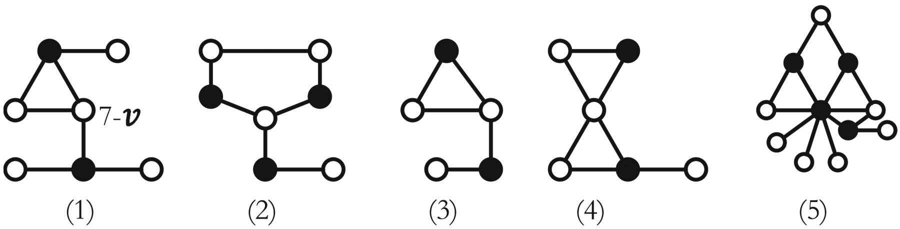

Fig. 1. Reducible configurations of Lemma 2

Lemma 3. ([19]). *Suppose $v \in V$, $d(v) = d$ and $d \geq 6$. Let v be clockwise adjacent to $v_1, \ldots, v_d$ and incident with $f_1, f_2, \ldots, f_d$ such that v_i is the common vertex of f_{i-1} and f_i ($i \in \{1, 2, \ldots, d\}$). Notice that f_0 and f_d denote a same face. Let $d(v_1) = 2$ and $N(v_1) = \{v, u_1\}$. Then G contains none of the following configurations.(see Fig. 2):*

(1) there exists an integer k ($2 \leq k \leq d-1$) such that $d(v_{k+1}) = 2$, $d(v_i) = 3$ ($2 \leq i \leq k$) and $d(f_j) = 4$ ($1 \leq j \leq k$).
(2) there exist two integers k and t ($2 \leq k < t \leq d-1$) such that $d(v_k) = 2$, $d(v_i) = 3$ ($k+1 \leq i \leq t$), $d(f_t) = 3$ and $d(f_j) = 4$ ($k \leq j \leq t-1$).
(3) there exist two integers k and t ($3 \leq k \leq t \leq d-1$) such that $d(v_i) = 3$ ($k \leq i \leq t$), $d(f_{k-1}) = d(f_t) = 3$ and $d(f_j) = 4$ ($k \leq j \leq t-1$).
(4) there exists an integer k ($2 \leq k \leq d-2$) such that $d(v_d) = d(v_i) = 3$ ($2 \leq i \leq k$), $d(f_k) = 3$ and $d(f_j) = 4$ ($0 \leq j \leq k-1$).

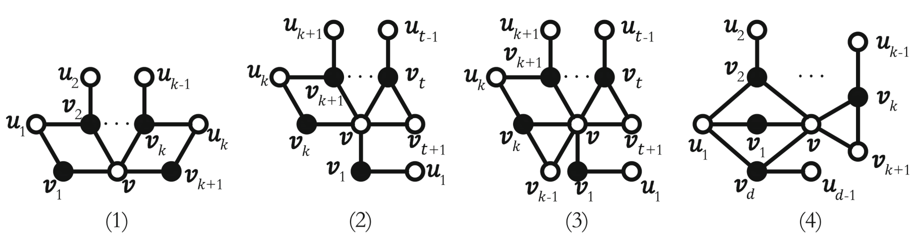

Fig. 2. Reducible configurations of Lemma 3

Lemma 4. ([20]). *Suppose u is a 6-vertex. If u is incident with one 3-cycle which is incident with a 4-vertex, then $n_{5^+}(u) = 5$.*

Lemma 5. ([17]). *G contains no $(6, 6, 4^+)$-cycles.*

Lemma 6. *Suppose $v \in G$. If $d(v) = 8$ and $n_2(v) \geq 1$, then $n_{5^+}(v) \geq 1$.*

Proof. Suppose $G^{'}$ is a subgraph of G. The mapping φ is said to be a nice coloring of G if $G^{'} = G - \{v|v \in V, d(v) \leq 4\}$ has a $(\Delta + 1)$-total-coloring. It is clear that a nice coloring can be easily extended to a $(\Delta + 1)$-total-coloring of G, because a 4^--vertex has at most 8 forbidden colors. Hence, in the following, we will always assume that every 4^--vertex is colored in the end.

Contrarily, we assume that G contains a configuration with $d(v) = 8$, $n_2(v) \geq 1$, and $n_{5^+}(v) = 0$. Suppose v is a 8-vertex. Let v be clockwise adjacent to $v_1, v_2, \ldots, v_8$ and incident with $e_1, e_2, \ldots, e_8$ such that v_i is incident with e_i $(i = 1, 2, \ldots, 8)$. Since $d(v_i) \leq 4$ $(i = 1, 2, \ldots, 8)$, we uncolor the adjacent vertices of v and color them in the end. We may assume that $d(v_1) = 2$. Then the one edge incident with v_1 is e_1, and the other edge incident with v_1 is denoted as e_9. Because of the minimality of G, $H = G - e_1$ has a nice coloring. Firstly, suppose $\varphi(e_9) = 9$. Otherwise, we color e_1 with 9 to get a nice coloring of G, which is a contradiction, so $\varphi(e_9) = 9$. We recolor v with 9, and color e_1 with 1 to get a nice coloring of G, which is a contradiction. □

3 Discharging

In this section, we will accomplish the proof of Theorem 1 by using discharging method. The discharging method is a familiar and important way to solve coloring problems for a planar graph. By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

We define $w(x)$ of $x \in V \cup F$ to be the original charge function. Let $w(v) = 2d(v) - 6$ for every $v \in V$ and $w(f) = d(f) - 6$ for every $f \in F$. So $\sum_{v \in V \cup F} w(x) < 0$. We use $\omega(x \to y)$ to denote the amount of total charge from x to y. We shall give proper discharging rules and transfer the original charge to get a new charge. We have two rounds of discharging rules. We use $w^*(x)$ to denote the charge of $x \in V \cup F$ after the first round of discharging and use $w^{'}(x)$ to denote the charge of $x \in V \cup F$ after the second round of discharging. If there is no discharging rule for $x \in V \cup F$, then the last charge of x is equal to the original charge of x. Notice that the total charge of G is unchangeable after redistributing the original charge, so $\sum_{x \in V \cup F} w^{'}(x) = \sum_{x \in V \cup F} w(x) = -6\chi(\Sigma) = -12 < 0$. We will get an obvious contradiction by proving that $\sum_{x \in V \cup F} w^{'}(x) \geq 0$.

These are the discharging rules:

R1. Suppose v is a 2-vertex. If u is adjacent to v, then $\omega(u \to v) = 1$.

R2. Let f be a face which is incident with v. Suppose $d(v) = 4$ or 5. If $d(f) = 4$, then $\omega(v \to f) = \frac{1}{2}$. If $d(f) = 5$, then $\omega(v \to f) = \frac{1}{3}$. Finally v sends the surplus charge to 3-faces incident with it evenly.
R3. If a 3-face is incident with 6-vertices and 7^+-vertices, then it receives $\frac{5}{4}$ from 7^+-vertices.
R4. Every 7^+-face sends $\frac{d(f)-6}{d(f)}$ to its adjacent 3-faces.

If $w^*(f) < 0$ of a 5^--face after the first round discharging, then we have the second round discharging:

R5. If $w^*(f) < 0$, then f receives $|\frac{w^*(f)}{n_{6^+}(v)}|$ from every 6^+-vertices incident it which do not give any charge to f.

Lemma 7. *Suppose f is a face which is incident with v.*

1. If $d(v) = 6$, then

$$\omega(v \to f) \leq \begin{cases} \frac{5}{4}, & \text{if } d(f) = 3 \text{ and } n_4(f) = 1, \\ \frac{11}{10}, & \text{if } d(f) = 3 \text{ and } n_5(f) \geq 1, \\ 1, & \text{if } d(f) = 3 \text{ and } n_{6^+}(f) = 3, \\ \frac{7}{8}, & \text{if } d(f) = 3, n_{5^-}(f) = 0 \text{ and } n_{7^+}(f) = 1, \\ \frac{1}{2}, & \text{if } d(f) = 3 \text{ and } n_{7^+}(f) = 2, \\ \frac{2}{3}, & \text{if } d(f) = 4 \text{ and } n_{3^-}(f) = 1, \\ \frac{1}{2}, & \text{if } d(f) = 4 \text{ and } n_{3^-}(f) = 0, \\ \frac{1}{3}, & \text{if } d(f) = 5. \end{cases}$$

2. If $d(v) \geq 7$, then

$$\omega(v \to f) \leq \begin{cases} \frac{3}{2}, & \text{if } d(f) = 3 \text{ and } n_{3^-}(f) = 1, \\ \frac{5}{4}, & \text{if } d(f) = 3 \text{ and } n_{3^-}(f) = 0, \\ 1, & \text{if } d(f) = 4 \text{ and } n_{3^-}(f) = 2, \\ \frac{3}{4}, & \text{if } d(f) = 4, n_{3^-}(f) = 1 \text{ and } n_4(f) = 1, \\ \frac{2}{3}, & \text{if } d(f) = 4, n_{3^-}(f) = 1 \text{ and } n_{5^+}(f) = 3, \\ \frac{1}{2}, & \text{if } d(f) = 4 \text{ and } n_{3^-}(f) = 0, \\ \frac{1}{3}, & \text{if } d(f) = 5. \end{cases}$$

Proof. Suppose v is incident with a 4^+-face f. Then it is clear that Lemma 7 is correct by R2 and R5. Now we think about that f is a 3-face that is incident with v. If $d(v) = 6$, then there exist no 3^--vertices adjacent to v by Lemma 1(b). If there exists a 4-vertex incident with f, then f is incident with a 7^+-vertex by Lemma 5. So $\omega(v \to f) \leq 3 - \frac{5}{4} - \frac{1}{4} = \frac{5}{4}$. If $n_5(f) = 1$ and $n_{6^+}(v) = 2$, then $\omega(v \to f) \leq \frac{3-\frac{4}{5}}{2} = \frac{11}{10}$. Suppose $n_5(f) = 2$. If there exists one 5-vertex incident with five 3-faces, then the other 5-vertex is incident with at least two 6^+-faces. So $\omega(v \to f) \leq 3 - \frac{4}{5} - \frac{4}{3} \leq \frac{11}{10}$. If there exists one 5-vertex incident with four

3-faces, then all of the two 5-vertices are incident with at least one 6^+-face. So $\omega(v \to f) \leq 3 - 1 \times 2 \leq \frac{11}{10}$. Suppose $n_{6^+}(f) = 3$. Then $\omega(v \to f) \leq \frac{3}{3} = 1$. If $n_{5^-}(f) = 0$ and $n_{7^+}(f) = 1$, then the 7^+-vertex sends $\frac{5}{4}$ to f by R4, so $\omega(v \to f) \leq \frac{3-\frac{5}{4}}{2} = \frac{7}{8}$. If $n_{7^+}(f) = 2$, then $\omega(v \to f) \leq 3 - \frac{5}{4} \times 2 = \frac{1}{2}$. If $d(v) \geq 7$, then there exists at most one 3^--vertex adjacent to v, so $\omega(v \to f) \leq \frac{3}{2}$. If $n_{3^-}(f) = 0$, then $\omega(v \to f) \leq \frac{3-\frac{1}{2}}{2} = \frac{5}{4}$. □

Lemma 8. *Suppose $d(v) = 8$. Let v be clockwise adjacent to $v_1, v_2, \ldots, v_n$ ($n \geq 3$) and incident with $f_1, f_2, \ldots, f_{n-1}$ such that f_j is incident with v_j and v_{j+1}. Clearly, f_0 and f_d denote a same face. If $d(v_1) = d(v_n) = 2$ and $d(v_i) \geq 3$ ($i = 2, 3, \ldots, n-1$), then $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq \frac{5}{4}n - \frac{9}{4}$.*

Proof. By Lemma 2, we know that $d(f_1) \geq 4$ and $d(f_{n-1}) \geq 4$. Firstly, suppose $d(f_1) = 4$ and $d(f_{n-1}) = 4$. If $\min\{d(f_2), d(f_3), \ldots, d(f_{n-2})\} \geq 5$, then $n \geq 4$, so $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq 1 \times 2 + \frac{1}{3}(n-3) \leq \frac{5}{4}n - \frac{9}{4}$. If $\min\{d(f_2), d(f_3), \ldots, d(f_{n-2})\} = 4$ and $\max\{d(f_2), d(f_3), \ldots, d(f_{n-2})\} = 5$, then $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq n - 2 + \frac{1}{3} \leq \frac{5}{4}n - \frac{9}{4}$. If $d(f_2) = d(f_3) = \ldots = d(f_{n-2}) = 4$, then $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq n - 3 + \frac{3}{4} \times 2 \leq \frac{5}{4}n - \frac{9}{4}$ by Lemma 3. Suppose $\min\{d(f_2), d(f_3), \ldots, d(f_{n-2})\} = 3$ and $\max\{d(f_2), d(f_3), \ldots, d(f_{n-2})\} = 4$. If $d(f_2) = 4$ or $d(f_{n-2}) = 4$, then $\omega(v \to f_1) + \omega(v \to f_2) \leq \max\{1 \times 2, \frac{3}{4} + \frac{5}{4}\} = 2$ and $\omega(v \to f_{n-2}) + \omega(v \to f_{n-1}) \leq \max\{1 \times 2, \frac{3}{4} + \frac{5}{4}\} = 2$. Moreover, v sends more charge to 3-faces than 4-faces, so we assume that v is incident with 3-faces as more as possible. Hence, $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq 2 \times 2 + \frac{5}{4} \times (n-5) \leq \frac{5}{4}n - \frac{9}{4}$. Suppose $d(f_2) = d(f_3) = \ldots = d(f_{n-2}) = 3$, then f_j ($2 \leq j \leq n-2$) receives at most $\frac{5}{4}$ from v by Lemma 3. Hence, $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq \frac{3}{4} \times 2 + \frac{5}{4} \times (n-3) \leq \frac{5}{4}n - \frac{9}{4}$. Secondly, suppose $\min\{d(f_1), d(f_{n-1})\} = 4$ and $\max\{d(f_1), d(f_{n-1})\} \geq 5$. If $d(f_2) = d(f_3) = \ldots = d(f_{n-2}) = 3$, then $\sum_{i}^{n-1} \omega(v \to f_i) \leq \frac{3}{4} + \frac{1}{3} + \frac{3}{2} + \frac{5}{4} \times (n-4) \leq \frac{5}{4}n - \frac{9}{4}$. If $\max\{d(f_2), d(f_3), \ldots, d(f_{n-2})\} = 4$, then $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq 1 \times 2 + \frac{1}{3} + \frac{3}{2} + \frac{5}{4} \times (n-5) \leq \frac{5}{4}n - \frac{9}{4}$. Finally, suppose $\min\{d(f_1), d(f_{n-1})\} \geq 5$. Then $\sum_{i=1}^{n-1} \omega(v \to f_i) \leq \frac{1}{3} \times 2 + \frac{3}{2} \times 2 + \frac{5}{4} \times (n-5) \leq \frac{5}{4}n - \frac{9}{4}$. □

In the rest of this paper, we can check that $w'(x) \geq 0$ for every $x \in V \cup F$ which is a contradiction to our assumption. Let $f \in F$. If $d(f) \geq 7$, then $w'(f) \geq w(f) - \frac{d(f)-6}{d(f)} \times d(f) = 0$ by R4. If f is a 6-face, then $w'(f) = w(f) = 0$. Suppose $d(f) \leq 5$. If $n_{6^+}(f) \geq 1$, then $w'(f) \geq 0$ by R5. If $n_{6^+}(f) = 0$, then $n_5(f) = d(f)$. Suppose $d(f) = 3$ and the boundary vertices of f are consecutively v_1, v_2 and v_3. Then $d(v_i) = 5$ ($i = 1, 2, 3$). By R2, 4^+-face receives at most $\frac{1}{2}$ from incident 4-vertices or 5-vertices. Suppose $f_3(v_i) \leq 3$ ($i = 1, 2, 3$). Then $\omega(v_i \to f) \geq 1$, so $w'(f) \geq 3 - 6 + 1 \times 3 = 0$. Suppose there exists $f_3(v_i) \geq 4$. Without loss of generality, assume that $f_3(v_3) \geq 4$. Then we have $f_3(v_1) \leq 4$ and $f_3(v_2) \leq 4$. Otherwise, $f_3(v_1) = 5$ or $f_3(v_2) = 5$, then for any integers $j, k \in \{3, 4, 5, 6, 7, 8\}$, there exists a vertex incident with adjacent j-cycles and k-cycles. So we get a contradiction to the condition of Theorem 1. If $f_3(v_1) = 4$, then v_1 is incident with a 9^+-face and v_2 is incident with at least two 6^+-faces, so $\omega(v_1 \to f) \geq 1$

and $\omega(v_2 \to f) \geq 1$. Consequently, $w^{'}(f) \geq 3 - 6 + \frac{4}{5} + 1 + \frac{4}{3} > 0$. Similarly, we know that if $f_3(v_2) = 4$, then $w^{'}(f) > 0$. Suppose $f_3(v_1) = f_3(v_2) = 3$. Then v_1 and v_2 is incident with at least one 6^+-face, so $\omega(v_i \to f) \geq \frac{4-\frac{1}{2}}{3} = \frac{7}{6}$, $(i = 1, 2)$. Consequently, $w^{'}(f) \geq 3 - 6 + \frac{4}{5} + \frac{7}{6} \times 2 > 0$. If $d(f) = 4$, then $w^{'}(f) \geq 4 - 6 + \frac{1}{2} \times 4 = 0$ by R2. If $d(f) = 5$, then $w^{'}(f) \geq 5 - 6 + \frac{1}{3} \times 5 > 0$ by R2. So for every $f \in F$, we prove that $w^{'}(f) \geq 0$. Next, we consider that $v \in V$. Suppose $d(v) = 2$. Then it is clear that $w(v) = -2$, so $w^{'}(v) = -2 + 1 \times 2 = 0$ by R1. If $d(v) = 3$, then $w^{'}(v) = w(v) = 0$. Suppose $d(v) = 4$ or $d(v) = 5$. Then $w^{'}(v) = 0$ by R2.

If v is a 6^+-vertex of G. Let v be clockwise adjacent to $v_1, \ldots, v_d$ and incident with $f_1, f_2, \ldots, f_d$ such that v_i is the common vertex of f_{i-1} and f_i $(i \in \{1, 2, \ldots, d\})$. Notice that f_0 and f_d denote the same face. Suppose $d(v) = 6$. Then there exist no 3^--vertices incident with v by Lemma 1 (b). Clearly, $w(v) = 2d(v) - 6 = 6$. By Lemma 4, there exist at most two 3-faces incident with a 4-vertex. Hence, if $f_3(v) \leq 3$, then $w^{'}(v) \geq 6 - \frac{5}{4} \times 2 - \frac{11}{10} \times 1 - \frac{2}{3} \times 3 > 0$ by R4. Suppose $f_3(v) = 4$. If $f_{5^+}(v) \geq 1$, then $w^{'}(v) \geq 6 - \frac{5}{4} \times 2 - \frac{11}{10} \times 2 - \frac{2}{3} - \frac{1}{3} > 0$. If $f_4(v) = 2$, then there exist three boundary vertices of the two 4-faces adjacent v, that is, all of the two 4-faces are incident with four 4^+-vertices. Hence, $w'(v) \geq 6 - \frac{5}{4} \times 2 - \frac{11}{10} \times 2 - \frac{1}{2} \times 2 > 0$. Suppose $f_3(v) \geq 5$. If v is adjacent to a 5-vertex v_0 and f is a 3-face incident with v and v_0, then $f_3(v_0) \leq 3$, so $\omega(v_0 \to f) \geq 1$ and $\omega(v \to f) \leq 1$. Suppose $f_3(v) = 5$. If $f_{5^+}(v) = 1$, then $w^{'}(v) \geq 6 - \frac{5}{4} \times 2 - 1 \times 3 - \frac{1}{3} > 0$. If $f_4(v) = 1$, then there exist three boundary vertices of the 4-faces adjacent to v, that is, the 4-face is incident with four 4^+-vertices. Hence, $w^{'}(v) \geq 6 - \frac{5}{4} \times 2 - 1 \times 3 - \frac{1}{2} = 0$.

Suppose $f_3(v) = 6$, that is, $d(f_i) = 3$ $(i = 1, 2, \ldots, 6)$. By Lemma 4, v is incident with at most one 4-vertex. So we may assume that $d(v_6) = 4$, then $d(v_1) \geq 7$ and $d(v_5) \geq 7$ by Lemma 5. Suppose $f_{6^+}(v_6) = 2$. Then $\omega(v_6 \to f_5) \geq 1$ and $\omega(v_6 \to f_6) \geq 1$, so $\omega(v \to f_5) \leq 1$ and $\omega(v \to f_6) \leq 1$. Therefore, $w'(v) \geq 6 - 1 \times 6 = 0$. Otherwise, $f_{5^-}(v) \geq 3$. Let f_x be the 5^--face incident with v_6 except f_5 and f_6. Suppose $d(f_x) = 5$. Then we get a contradiction to the condition of Theorem 1. Suppose $d(f_x) = 4$. Then v_6 is adjacent to v_4 and v_1 is adjacent to v_3. So we know that $f_{6^+}(v_6) = 1$ and $\omega(v_6 \to f_i) \geq \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$ $(i = 5, 6)$. Therefore, $\omega(v \to f_i) \leq 3 - \frac{5}{4} - \frac{3}{4} \leq 1$ $(i = 5, 6)$, and $w^{'}(v) \geq 6 - 1 \times 6 = 0$. Suppose $d(f_x) = 3$. Then each of the boundary vertices of f is adjacent to v. If v_6 is adjacent to v_4 and v_1 is adjacent to v_4, then $d(v_4) \geq 7$ by Lemma 5. So $\omega(v_4 \to f_4) = \frac{5}{4}$ and $\omega(v_5 \to f_4) = \frac{5}{4}$, then $\omega(v \to f_4) \leq \frac{1}{2}$ and $w^{'}(v) \geq 6 - \frac{5}{4} \times 2 - 1 \times 3 - \frac{1}{2} = 0$. If v_6 is adjacent to v_3 and v_1 is adjacent to v_3, then $d(v_3) \geq 7$ by Lemma 5. Suppose $d(v_2) \geq 6$ and $d(v_4) \geq 6$. Then $\omega(v \to f_i) \leq \frac{3-\frac{5}{4}}{2} = \frac{7}{8}$ $(i = 1, 2, 3, 4)$. Hence, $w^{'}(v) \geq 6 - \frac{5}{4} \times 2 - \frac{7}{8} \times 4 = 0$. Suppose $d(v_2) = 5$ or $d(v_4) = 5$. Without of generality, assume that $d(v_4) = 5$. Then $\omega(v_4 \to f_3) \geq 1$ and $\omega(v_4 \to f_4) \geq 1$. So $\omega(v \to f_3) \leq 3 - 1 - \frac{5}{4} = \frac{3}{4}$ and $\omega(v \to f_4) \leq 3 - 1 - \frac{5}{4} = \frac{3}{4}$. Therefore, $w^{'}(v) \geq 6 - \frac{5}{4} \times 2 - 1 \times 2 - \frac{3}{4} \times 2 = 0$.

Suppose $d(v) = 7$. Then it is easy to know that $f_3(v) \leq 6$ and v is not adjacent to a 2^--vertices by Lemma 1 (b). Clearly, $w(v) = 2d(v) - 6 = 8$. Suppose there exist no 3-faces incident with a 3-vertex. If $f_3(v) = 6$, then $f_{9^+}(v) = 1$, so $w^{'}(v) \geq 8 - \frac{5}{4} \times 6 > 0$ by Lemma 7. If $f_3(v) = 5$, then there exist no 4-faces incident with two 3^--vertex. So $w^{'}(v) \geq 8 - \frac{5}{4} \times 5 - \frac{3}{4} \times 2 > 0$ by Lemma 7. If $f_3(v) \leq 4$, then $w^{'}(v) \geq 8 - \frac{5}{4} \times 4 - 1 \times 3 = 0$. Now we presume that there exists at least one 3-face that is incident with a 3-vertex. Then all of the 4-faces are incident with at most one 3^--vertex. By Lemma 2, there exist at most two 3-faces incident with a 3-vertex. If $f_3(v) = 6$, then $f_{9^+}(v) = 1$, so $w^{'}(v) \geq 8 - \frac{3}{2} \times 2 - \frac{5}{4} \times 4 = 0$ by Lemma 7. Suppose $f_3(v) = 5$. If v is incident with at least one 5^+-face, then $w^{'}(v) \geq 8 - \frac{3}{2} \times 2 - \frac{5}{4} \times 3 - \frac{2}{3} - \frac{1}{3} > 0$ by Lemma 7. Otherwise, $f_4(v) = 2$, then there exist three boundary vertices of the 4-face adjacent to v, so $w^{'}(v) \geq 8 - \frac{3}{2} \times 2 - \frac{5}{4} \times 3 - \frac{3}{4} - \frac{1}{2} = 0$. Suppose $f_3(v) \leq 4$. Then $w^{'}(v) \geq 8 - \frac{3}{2} \times 2 - \frac{5}{4} \times 2 - \frac{3}{4} \times 3 > 0$ by Lemma 7. If $d(v) = 8$, then we know that $w(v) = 2 \times 8 - 6 = 10$, $f_3(v) \leq 6$ and $n_2(v) \leq 7$ by Lemma 6. By Lemma 7 and Lemma 8, we shall consider the following cases by discussing the number of $n_2(v)$.

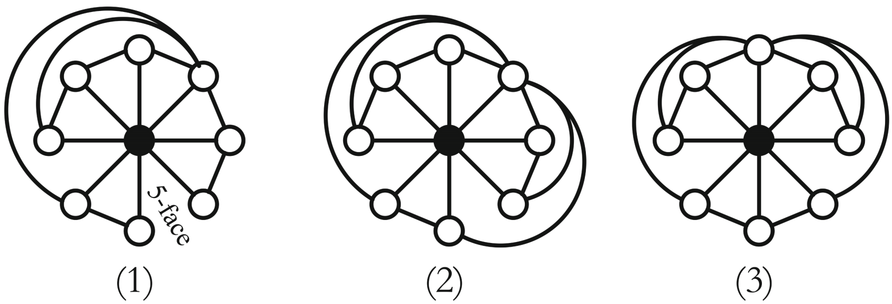

Fig. 3. $n_2(v) = 0$ and $f_3(v) = 6$

Case 1. $n_2(v) = 0$. Suppose $f_3(v) = 6$. If $f_{6^+}(v) \geq 1$ or $f_{5^+}(v) \geq 2$, then $w^{'}(v) \geq 10 - \frac{3}{2} \times 6 - 1 = 0$ by Lemma 7. Otherwise, $f_{6^+}(v) = 0$ and $f_{5^+}(v) \leq 1$. Suppose $f_4(v) = 1$ and $f_5(v) = 1$. According to the condition of Theorem 1, there is only one case in which the location of the faces satisfying the condition of 1. We depict this case in Fig. 3 (1). It is clear that there exist three boundary vertices of the 4-faces adjacent to v, and there is at least one 3-face which is not incident with a 3-vertex by Lemma 2. If the 4-face is incident with at most one 3-vertex, then $w^{'}(v) \geq 10 - \frac{3}{2} \times 5 - \frac{5}{4} - \frac{3}{4} - \frac{1}{3} > 0$. Otherwise, there exist two 3-vertex incident with the 4-face, then there exist at least two 3-faces that are not incident with a 3-vertex by Lemma 2. Hence, $w^{'}(v) \geq 10 - \frac{3}{2} \times 4 - \frac{5}{4} \times 2 - \frac{3}{4} - \frac{1}{3} > 0$. Suppose $f_4(v) = 2$. There are only two cases satisfying the condition of Theorem 1. We depict these cases in Fig. 3(2) and (3). In Fig. 3(2), there exist at least four 3-faces all of which are adjacent to a 8^+-face. By R4, if there exists a 8^+-face adjacent to a 3-face, then 8^+-face sends $\frac{1}{4}$ to the 3-face, so each of the 3-face adjacent to a 8^+-face receives at most $\frac{3-\frac{1}{4}}{2} = \frac{11}{8}$ from the boundary vertices. There exist at most one 4-face incident with two 3-vertices in Fig. 3(2). By Lemma 2, there exist at least one 3-face that is not incident with a 3-vertex, so $w^{'}(v) \geq 10 - \frac{3}{2} - \frac{11}{8} \times 4 - \frac{5}{4} - 1 - \frac{3}{4} = 0$. In Fig. 3(3), there exist at least four 3-faces all of which are adjacent to a 8^+-face. By Lemma 2, there is at most one 4-face incident with two 3-vertices. If all of the two 4-faces are incident with at most one 3-vertex, then $w^{'}(v) \geq 10 - \frac{3}{2} \times 2 - \frac{11}{8} \times 4 - \frac{3}{4} \times 2 = 0$. Otherwise, there exists one 4-face that is incident with two 3-vertices, then there exist at least three 3-faces that are not incident with a 3-vertex by Lemma 2. Hence, $w^{'}(v) \geq 10 - \frac{3}{2} \times 3 - \frac{5}{4} \times 3 - 1 - \frac{3}{4} = 0$. Suppose $f_3(v) = 5$. Then by the condition of Theorem 1, we know that $f_{5^+}(v) \geq 1$, so $w^{'}(v) \geq 10 - \frac{3}{2} \times 5 - 1 \times 2 - \frac{1}{3} \times 2 > 0$.

Case 2. $n_2(v) = 1$. Then $2 \times 8 - 6 - 1 = 9$.

Case 2.1. Let the 2-vertex be incident with a 3-cycle. It is clear that $f_3(v) \leq 6$ and there exist no 3-faces incident with a 3-vertex by Lemma 2. So v is incident with at most one 3-face that receives $\frac{3}{2}$ from v. If $f_3(v) = 6$, then by the condition of Theorem 1, we know that $f_{6^+}(v) \geq 1$ or $f_{5^+}(v) \geq 2$, so $w^{'}(v) \geq 9 - \frac{3}{2} - \frac{5}{4} \times 5 > 0$ by Lemma 7. Suppose $f_3(v) = 5$. If $f_4(v) = 3$, then there are at least two $(8, 4^+, 4^+, 2^+)$-faces between the three 4-faces by Lemma 2. Hence, $w^{'} \geq 9 - \frac{3}{2} - \frac{5}{4} \times 4 - 1 - \frac{3}{4} \times 2 = 0$. If $f_4(v) \leq 2$, then we have $w^{'}(v) \geq 9 - \frac{3}{2} - \frac{5}{4} \times 4 - 1 \times 2 - \frac{1}{3} > 0$. Suppose $f_3(v) = 4$. If $f_4(v) = 4$, then there exist at least two $(8, 4^+, 4^+, 2^+)$-faces between the four 4-faces by Lemma 2. Hence, $w^{'} \geq 9 - \frac{3}{2} - \frac{5}{4} \times 3 - 1 \times 2 - \frac{3}{4} \times 2 > 0$. If $f_4(v) \leq 3$, then $w^{'}(v) \geq 9 - \frac{3}{2} - \frac{5}{4} \times 3 - 1 \times 3 - \frac{1}{3} > 0$. If $f_3(v) \leq 3$, then $w^{'}(v) \geq 9 - \frac{3}{2} - \frac{5}{4} \times 2 - 1 \times 5 = 0$.

Case 2.2. Let the 2-vertex not be incident with a 3-cycle. Then $f_3(v) \leq 6$. Suppose $f_3(v) = 6$. Then the six 3-faces are adjacent and $f_{9^+}(v) = 1$, so there exist at least four $(8, 4^+, 4^+)$-faces between the six 3-faces by Lemma 3. Therefore, $w^{'}(v) \geq 9 - \frac{3}{2} \times 2 - \frac{5}{4} \times 4 - 1 \times 1 > 0$ by Lemma 7. Suppose $f_3(v) = 5$. It is easy to know that $f_{6^+}(v) \geq 1$ by the condition of Theorem 1. If $f_4(v) = 2$, then there exist three the boundary vertices of the two 4-faces adjacent to v. So v is incident with at least two $(8, 4^+, 4^+)$-faces and one $(8, 4^+, 4^+, 2^+)$-face by Lemma 3. Hence, $w^{'}(v) \geq 9 - \frac{3}{2} \times 3 - \frac{5}{4} \times 2 - 1 \times 1 - \frac{3}{4} \times 1 > 0$. If $f_4(v) = 1$, then there exists at least one $(8, 4^+, 4^+)$-face between the five 3-faces. by Lemma 3. Hence, $w^{'}(v) \geq 9 - \frac{3}{2} \times 4 - \frac{5}{4} \times 1 - 1 \times 1 - \frac{1}{3} \times 2 > 0$. If $f_4(v) = 0$, then $w^{'}(v) \geq 9 - \frac{3}{2} \times 5 - \frac{1}{3} \times 3 > 0$. Suppose $f_3(v) = 4$. Then we have $f_4(v) \leq 3$ by the condition of Theorem 1. If $f_4(v) = 3$, then there exist at least two $(8, 4^+, 4^+)$-faces between four the 3-faces. Hence, $w^{'}(v) \geq 9 - \frac{3}{2} \times 2 - \frac{5}{4} \times 2 - 1 \times 3 - \frac{1}{3} \times 1 > 0$. If $f_4(v) \leq 2$, then $w^{'}(v) \geq 9 - \frac{3}{2} \times 4 - 1 \times 2 - \frac{1}{3} \times 2 > 0$. Suppose $f_3(v) = 3$. If there exists a 5^+-face incident with v, then $w^{'}(v) \geq 9 - \frac{3}{2} \times 3 - 1 \times 4 - \frac{1}{3} > 0$. Otherwise, $f_4(v) = 5$, then there exist at least three $(8, 4^+, 4^+, 2^+)$-faces between the five 4-faces. Hence, $w^{'}(v) \geq 9 - \frac{3}{2} \times 3 - 1 \times 2 - \frac{3}{4} \times 3 > 0$. If $f_3(v) \leq 2$, then $w^{'}(v) \geq 9 - \frac{3}{2} \times 2 - 1 \times 6 = 0$.

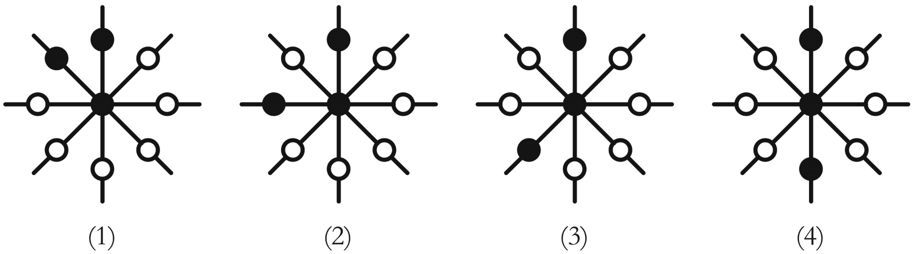

Fig. 4. $n_2(v) = 2$

Case 3. $n_2(v) = 2$. Then $2 \times 8 - 6 - 2 = 8$ and there are four cases in which 2-vertices are located. We depict these cases in Fig. 4. In Fig. 4 (1), $w^{'}(v) \geq 8 - (\frac{5}{4} \times 8 - \frac{9}{4}) > 0$ by Lemma 8. In Fig. 4 (2), $w^{'}(v) \geq 8 - (\frac{5}{4} \times 7 - \frac{9}{4}) - (\frac{5}{4} \times 3 - \frac{9}{4}) = 0$. In Fig. 4 (3), $w^{'}(v) \geq 8 - (\frac{5}{4} \times 6 - \frac{9}{4}) - (\frac{5}{4} \times 4 - \frac{9}{4}) = 0$. In Fig. 4 (4), $w^{'}(v) \geq 8 - (\frac{5}{4} \times 5 - \frac{9}{4}) \times 2 = 0$ by Lemma 8.

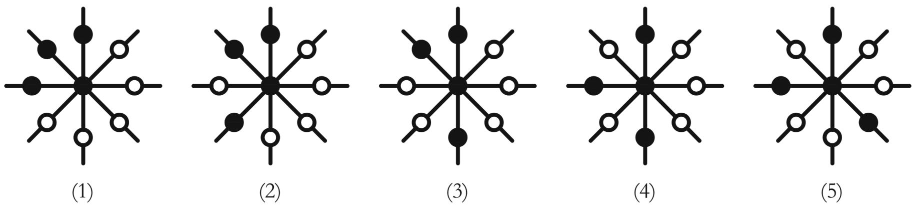

Fig. 5. $n_2(v) = 3$

Case 4. $n_2(v) = 3$. Then $2 \times 8 - 6 - 3 = 7$ and there are five cases in which 2-vertices are located. We depict these cases in Fig. 5. In Fig. 5(1), $w^{'}(v) \geq 7 - (\frac{5}{4} \times 7 - \frac{9}{4}) > 0$ by Lemma 8. In Fig. 5(2), $w^{'}(v) \geq 7 - (\frac{5}{4} \times 6 - \frac{9}{4}) - (\frac{5}{4} \times 3 - \frac{9}{4}) > 0$. In Fig. 5(3), $w^{'}(v) \geq 7 - (\frac{5}{4} \times 5 - \frac{9}{4}) - (\frac{5}{4} \times 4 - \frac{9}{4}) > 0$. In Fig. 5(4), $w^{'}(v) \geq 7 - (\frac{5}{4} \times 5 - \frac{9}{4}) - (\frac{5}{4} \times 3 - \frac{9}{4}) \times 2 = 0$. In Fig. 5(5), $w^{'}(v) \geq 7 - (\frac{5}{4} \times 3 - \frac{9}{4}) - (\frac{5}{4} \times 4 - \frac{9}{4}) \times 2 = 0$ by Lemma 8.

Case 5. $n_2(v) = 4$. Then $2 \times 8 - 6 - 4 = 6$ and there are eight cases in which 2-vertices are located. We depict these cases in Fig. 6. In Fig. 6(1), $w^{'}(v) \geq 6 - (\frac{5}{4} \times 6 - \frac{9}{4}) > 0$ by Lemma 8. In Fig. 5(2) and (4), $w^{'}(v) \geq 6 - (\frac{5}{4} \times 5 - \frac{9}{4}) - (\frac{5}{4} \times 3 - \frac{9}{4}) > 0$. In Fig. 6(3) and (7), $w^{'}(v) \geq 6 - (\frac{5}{4} \times 4 - \frac{9}{4}) \times 2 > 0$. In Fig. 6(5) and (6), $w^{'}(v) \geq 6 - (\frac{5}{4} \times 3 - \frac{9}{4}) \times 2 - (\frac{5}{4} \times 4 - \frac{9}{4}) > 0$. In Fig. 6(8), $w^{'}(v) \geq 6 - (\frac{5}{4} \times 3 - \frac{9}{4}) \times 4 = 0$ by Lemma 8.

Case 6. $n_2(v) \geq 5$. Suppose $n_2(v) = 5$. Then $2 \times 8 - 6 - 5 = 5$ and $f_3(v) \leq 2$. Suppose $f_3(v) = 2$. Then $f_{6^+}(v) \geq 4$ by Lemma 2. Consequently, $w^{'}(v) \geq 5 - \frac{3}{2} \times 2 - 1 \times 2 = 0$ by Lemma 7. If $f_3(v) = 1$, then $f_{6^+}(v) \geq 3$ and $f_4(v) \leq 4$. If $f_4(v) = 4$. then all of the four 4-faces are $(8, 4^+, 4^+, 2^+)$-faces. Hence, $w^{'}(v) \geq$

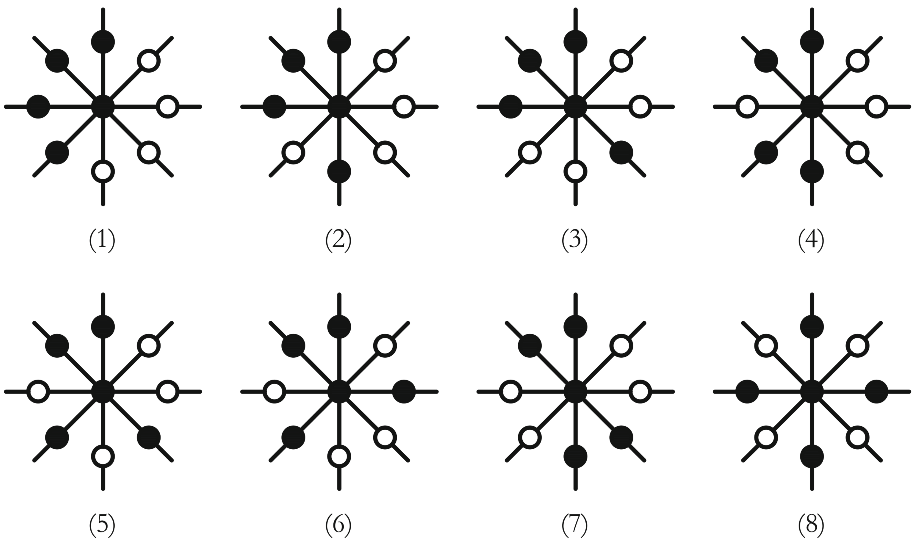

Fig. 6. $n_2(v) = 4$

$5 - \frac{3}{2} \times 1 - \frac{3}{4} \times 4 > 0$. If $f_4(v) \leq 3$, then $w^{'}(v) \geq 5 - \frac{3}{2} \times 1 - 1 \times 3 - \frac{1}{3} > 0$. Suppose $f_3(v) = 0$. Then $f_{6^+}(v) \geq 2$. If $f_4(v) = 6$, then all of the six 4-faces are $(8, 4^+, 4^+, 2^+)$-faces. Hence, $w^{'}(v) \geq 5 - \frac{3}{4} \times 6 > 0$. If $f_4(v) = 5$, then there exist at least four $(8, 4^+, 4^+, 2^+)$-faces between the five 4-faces. Hence, $w^{'}(v) \geq 5 - 1 \times 1 - \frac{3}{4} \times 4 - \frac{1}{3} \times 1 > 0$. If $f_4(v) \leq 4$, then $w^{'}(v) \geq 5 - 1 \times 4 - \frac{1}{3} \times 2 > 0$. Suppose $n_2(v) = 6$. Then $2 \times 8 - 6 - 6 = 4$ and $f_3(v) \leq 1$. If $f_3(v) = 1$, then $f_{6^+}(v) \geq 5$ and $f_4(v) \leq 2$. So $w^{'}(v) \geq 4 - \frac{3}{2} - 1 \times 2 > 0$. If $f_3(v) = 0$, then $f_{6^+}(v) \geq 4$. Hence, $w^{'}(v) \geq 4 - 1 \times 4 = 0$. Suppose $n_2(v) = 7$. Then by Lemma 2 we know that $f_{6^+}(v) \geq 6$ by and $f_3(v) = 0$, so $w^{'}(v) \geq 10 - 7 - 1 \times 2 > 0$.

In summary, we know that $w^{'}(x) \geq 0$ for every $x \in V \cup F$, so $\sum_{x \in V \cup F} w^{'}(x) \geq 0$. Hence, we get the desired contradiction and finish the proof of Theorem 1.

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