Check for updates

Kernelization for Feedback Vertex Set via Elimination Distance to a Forest

David Dekker and Bart M. P. Jansen(✉)

Eindhoven University of Technology, P.O. Box 513,
5600 MB Eindhoven, The Netherlands
b.m.p.jansen@tue.nl

Abstract. We study efficient preprocessing for the undirected Feedback Vertex Set problem, a fundamental problem in graph theory which asks for a minimum-sized vertex set whose removal yields an acyclic graph. More precisely, we aim to determine for which parameterizations this problem admits a polynomial kernel. While a characterization is known for the related Vertex Cover problem based on the recently introduced notion of bridge-depth, it remained an open problem whether this could be generalized to Feedback Vertex Set. The answer turns out to be negative; the existence of polynomial kernels for structural parameterizations for Feedback Vertex Set is governed by the elimination distance to a forest. Under the standard assumption NP $\not\subseteq$ coNP/poly, we prove that for any minor-closed graph class $\mathcal{G}$, Feedback Vertex Set parameterized by the size of a modulator to $\mathcal{G}$ has a polynomial kernel if and only if $\mathcal{G}$ has bounded elimination distance to a forest. This captures and generalizes all existing kernels for structural parameterizations of the Feedback Vertex Set problem.

Keywords: Feedback Vertex Set · Kernelization · Elimination distance

1 Introduction

For NP-complete problems, a polynomial time algorithm solving any problem instance exactly is unlikely to exist. However, as one is often interested in solving specific instances, one can try to exploit characteristics of problem instances and develop algorithms that are fast when the input has certain properties. We therefore associate a parameter with each problem instance. In our context, a problem instance is a graph for which we ask for the existence of a vertex set of size at most ℓ having certain properties. Such a parameterized instance can be denoted with a triple (G, ℓ, k), where we are asking for the existence of a solution of size at most ℓ for a graph G with parameter k. We say that an algorithm is *fixed parameter tractable* (FPT) for such a parameterization if it

B. M. P. Jansen—This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 803421, ReduceSearch).

© The Author(s) 2022

M. A. Bekos and M. Kaufmann (Eds.): WG 2022, LNCS 13453, pp. 158–172, 2022.
https://doi.org/10.1007/978-3-031-15914-5_12

solves any instance (G, ℓ, k) of size n, as described above, in time bounded by $f(k)n^{\mathcal{O}(1)}$ for some computable function $f\colon \mathbb{N} \to \mathbb{N}$.

A strongly related field is that of *kernelization.* This field focuses on reducing a parameterized instance (G, ℓ, k) in polynomial time to an equivalent instance (G', ℓ', k') whose size is bounded by a computable function of the parameter. We speak of a polynomial kernel when this function is a polynomial. It is known that a decidable parameterized problem is fixed parameter tractable if and only if it admits a kernelization (cf. [12, Proposition 4.7.1]). In our quest for determining which parameterizations enable efficient algorithms, it is therefore interesting to determine those that allow a polynomial kernel.

This paper focuses on polynomial kernels for the undirected Feedback Vertex Set problem, which is an NP-complete problem in graph theory as originally identified by Karp [21]. For an undirected graph G, a vertex set $X \subseteq V(G)$ is a *feedback vertex set* if the graph is acyclic after removal of X. We call a vertex set whose removal yields a graph in some graph class $\mathcal{G}$ a $\mathcal{G}$-modulator and define the deletion distance to $\mathcal{G}$ as its minimum size. The Feedback Vertex Set problem then asks for the minimum size of such a feedback vertex set, or equivalently, the deletion distance to a forest. For a graph G, we let $\textsc{fvs}(G)$ (the *feedback vertex number* of G) denote that minimum size. Our main question is for which parameterizations the Feedback Vertex Set problem admits a polynomial kernel.

Before exploring the Feedback Vertex Set problem further, we should mention the related Vertex Cover problem. It asks for a minimum set of vertices hitting all edges in a graph. While a kernel in the solution size with a linear number of vertices can be obtained using various techniques [1,7–9,24], a polynomial kernel in a structurally smaller parameter was only discovered in 2011, when Jansen and Bodlaender developed a polynomial kernel in the feedback vertex number of a graph [18]. From there, many polynomial kernels for Vertex Cover were discovered in modulators to even larger graph classes [4,13,15,23]. In 2020, Bougeret, Jansen and Sau proved the following characterization under common hardness assumptions: Vertex Cover admits a polynomial kernel in the size of a modulator to a minor-closed graph family $\mathcal{G}$ if and only if $\mathcal{G}$ has bounded *bridge-depth* [3]. With this result, they generalized all existing work on kernels in the size of modulators to minor-closed graph families, and they proved that their results cannot be improved further under common hardness assumptions.

For Feedback Vertex Set, the first polynomial kernel with size bound $\mathcal{O}(k^{11})$ was obtained in 2006 and it was subsequently improved to a quadratic kernel [2,6,26]. After the improvements for Vertex Cover, researchers also tried to develop polynomial kernels in smaller parameters for Feedback Vertex Set [17,20,22]. It remained an open problem whether these results could be generalized further or whether there exists some parameter that characterizes Feedback Vertex Set similarly to how bridge-depth characterizes Vertex Cover. In particular, Bougeret, Jansen and Sau suggested in their paper on Vertex Cover that the deletion distance to constant bridge-depth might also be an interesting parameter to consider for problems such as Feedback Vertex Set. We therefore aim to answer the question for which graph families $\mathcal{G}$ the Feedback Vertex Set problem admits a polynomial kernel when parameterized by the size of a $\mathcal{G}$-modulator.

Our Results. To our initial surprise, the results for Vertex Cover cannot be generalized to Feedback Vertex Set. It turns out that a minor-closed graph family $\mathcal{G}$ must have bounded *elimination distance to a forest* (Definition 1), in order to allow a polynomial kernel in a $\mathcal{G}$-modulator. This concept was introduced by Bulian and Dawar [5] and is another generalization of the more common parameter treedepth [25]. The elimination distance to a forest is the minimum number of rounds needed to transform the graph into a forest when removing one vertex from each connected component in each round. Our result is described in Theorem 1. Proofs of statements marked ($\star$) are deferred to the full version [10].

Theorem 1 ($\star$). *Assume* NP $\not\subseteq$ coNP/poly *and let $\mathcal{G}$ be a minor-closed graph family. Then* Feedback Vertex Set *admits a polynomial kernel in the size of a $\mathcal{G}$-modulator if and only if $\mathcal{G}$ has bounded elimination distance to a forest.*

The minor-closed and hardness assumptions are only needed for the lower bound. To the best of our knowledge, our kernel generalizes all known polynomial kernels for the Feedback Vertex Set problem. Both the kernel and its correctness proof follow the structure of the kernel for $\mathcal{F}$-Minor Free Deletion in the deletion distance to a graph of constant treedepth by Jansen and Pieterse [20]. The correctness proof of their kernel crucially relies on their Lemma 3 whose technical proof spans thirty pages. We require a variation of this lemma. On the one hand, our variation is more involved since it deals with elimination distance to a forest rather than treedepth; on the other hand, it is simpler since it concerns only Feedback Vertex Set rather than $\mathcal{F}$-Minor Free Deletion. As a result of this simplification, we can formulate the lemma without the use of minors. Roughly speaking, the lemma says that in a graph G of bounded elimination distance to a forest, if no minimum feedback vertex set exists which simultaneously hits a prescribed set of *partial* cycles (single vertices in a set S or paths between two terminals in a set T), then the same holds for some sets $S^* \subseteq S$ and $T^* \subseteq T$ of constant size. As shown in previous work, this limited sensitivity with respect to whether optimal solutions can break all partial forbidden structures is crucial for the kernelization complexity. As one of our main contributions, we prove this lemma using a strategy that differs significantly from the one followed in earlier work [20].

Lemma 1 ($\star$). *Let G be a connected graph with disjoint vertex sets $S, T \subseteq V(G)$. Suppose that any minimum feedback vertex set X of G misses some vertex from S or leaves two vertices from T connected in $G - X$. Then there exist sets $S^* \subseteq S$ and $T^* \subseteq T$ whose sizes only depend on the elimination distance to a forest of G, such that any minimum feedback vertex set X of G misses some vertex from S^* or leaves two vertices from T^* connected in $G - X$.*

Once Lemma 1 is proven, the kernelization upper bounds follow similarly to earlier work [20]. As for the lower bound in Theorem 1, we are also able to generalize our proof for other $\mathcal{F}$-Minor Free Deletion problems as described in Theorem 2.

Theorem 2 (⋆). *Let $\mathcal{G}$ be a minor-closed family of graphs and let $\mathcal{F}$ be a finite set of biconnected planar graphs on at least three vertices. If $\mathcal{G}$ has unbounded elimination distance to an $\mathcal{F}$-minor free graph, then $\mathcal{F}$-*Minor Free Deletion *does not admit a polynomial kernel in the size of a $\mathcal{G}$-modulator, unless* NP $\subseteq$ coNP/poly.

Organization. Section 2 introduces all relevant terminology. Section 3 presents our kernel and thereby proves the 'if' direction of Theorem 1. Then Sect. 4 contains the proof of Theorem 2, thereby also proving the 'only if' direction of Theorem 1. Lastly, Sect. 5 contains our conclusions and discusses future work.

2 Preliminaries

For a positive integer n, we use the shorthand $[n]$ for the set of all natural numbers i with $1 \leq i \leq n$. All graphs we consider are finite, undirected and simple. When G is a graph, we let $V(G)$ denote the vertex set of G and $E(G)$ the edge set. For $S \subseteq V(G)$, the graph $G - S$ is the graph where all vertices in S and all incident edges are removed, and the graph $G[S]$ is the subgraph of G induced by the vertices in S. When an edge exists between two vertices in G, we say that these vertices are *adjacent*. The *neighbors* of v in G, denoted with $N_G(v)$, are the vertices adjacent to a vertex $v \in V(G)$ in G. For $S \subseteq V(G)$, we say that $v \in V(G - S)$ is adjacent to S if there exists some edge between v and a vertex in S. The set $N_G(S)$ contains all vertices $v \in V(G - S)$ for which this holds. We will sometimes slightly abuse notation and speak of a vertex being adjacent to some subgraph, rather than to the vertices in that subgraph. We say that two vertices are *connected* in G when they are in the same connected component. The set $\text{cc}(G)$ denotes the set of connected components (or shortly components) of G. For sets $S, T \subseteq V(G)$, we say that S *separates* T if each component of $G - S$ contains at most one vertex from T. Notice that we do not require S and T to be disjoint. Such a set S is a *vertex multiway cut* of T in G. We use the notation $\mathcal{O}_\eta(f(n))$ to describe the functions in η and n which can be bounded by $g(\eta) \cdot f(n)$ for some computable function g.

A concept that will be used extensively is the concept of graph minors. This uses the notion of *edge contraction*. When uv is an edge in a graph G, contracting this edge replaces vertices u and v by a new vertex whose set of neighbors is $N_G(\{u, v\})$. Now H is a *minor* of G if H can be obtained from G by removing vertices, removing edges and contracting edges. Alternatively, one can define H to be a minor of G if there exists a *minor model* $\phi \colon V(H) \to 2^{V(G)}$, such that for any $v \in V(H)$ the graph $G[\phi(u)]$ is connected, for any distinct $u, v \in V(H)$ we have $\phi(u) \cap \phi(v) = \emptyset$, and for any edge uv in H there exists an edge between a vertex in $\phi(u)$ and a vertex in $\phi(v)$ in G.

A graph G has an *$\mathcal{H}$-minor* for a set of graphs $\mathcal{H}$ if G contains some graph $H \in \mathcal{H}$ as a minor. For a minor model ϕ of H in G and a set $S \subseteq V(H)$, we use the shorthand notation $\phi(S) := \bigcup_{v \in S} \phi(v)$. We say that a minor model ϕ of H in G is *minimal*, if there does not exist a minor model ϕ' of H in G with $\phi'(V(H)) \subsetneq \phi(V(H))$. A minor model ϕ of H in G and a minor model ϕ' of H' in G *intersect* if $\phi(V(H)) \cap \phi'(V(H')) \neq \emptyset$.

2.1 Elimination Distance

Our work relies crucially on the concept of elimination distance as introduced by Bulian and Dawar [5].

Definition 1 (Elimination distance). *Let G be a graph and $\mathcal{G}$ a graph family. Then the elimination distance of G to $\mathcal{G}$ is*

$$\textsc{ed}_{\mathcal{G}}(G) = \begin{cases} 0 & \text{if } G \in \mathcal{G}, \\ \max_{G' \in \textsc{cc}(G)} \textsc{ed}_{\mathcal{G}}(G') & \text{if } |\textsc{cc}(G)| > 1, \\ \min_{v \in V(G)} \textsc{ed}_{\mathcal{G}}(G - \{v\}) + 1 & \text{otherwise.} \end{cases}$$

We only consider graph families $\mathcal{G}$ that are minor-closed. We use the shorthand $\mathcal{G}_F$ for the graph family containing precisely all forests. We say that a graph family $\mathcal{G}$ has bounded elimination distance to some graph class $\mathcal{H}$ if there is a constant $c \in \mathbb{N}$ such that all graphs $G \in \mathcal{G}$ satisfy $\textsc{ed}_{\mathcal{H}}(G) \leq c$. The elimination distance of a graph G to the empty graph is called the *treedepth* of G and is denoted with $\textsc{td}(G)$. More intuitively, the elimination distance to a graph class $\mathcal{G}$ can be interpreted as the minimum number of 'elimination iterations' that are necessary to obtain a graph where every connected component is in $\mathcal{G}$. In such an iteration, one is allowed to remove one vertex from each connected component. This interpretation leads to the notion of an elimination forest.

Definition 2 ($\mathcal{G}$-elimination forest). *Let G be a graph and $\mathcal{G}$ a graph family. A $\mathcal{G}$-elimination forest of G is a tuple $\big(F, (B_u)_{u \in V(F)}\big)$ where F is a rooted forest and where each vertex $v \in V(F)$ has a bag $B_v \subseteq V(G)$ such that:*

- *The bags define a partition of $V(G)$, i.e. for any vertex $v \in V(G)$ there is a unique vertex $u \in V(F)$ with $v \in B_u$.*
- *For any non-leaf u of F, its bag B_u contains precisely one vertex.*
- *For any leaf u of F, the graph $G[B_u]$ is connected and $G[B_u] \in \mathcal{G}$.*
- *For any edge uv in G, let $s, t \in V(F)$ be the vertices such that $u \in B_s$ and $v \in B_t$. Then s is an ancestor of t, or t is an ancestor of s in F.*

We define the height of an elimination forest F to be the maximum number of edges on a path from the root to a leaf in F. One can prove with induction that this height is equal to the elimination distance we defined earlier.

We will use these elimination forests extensively for our kernel and therefore introduce some shorthand notation. Let $(F, (B_u)_{u \in V(F)})$ be a $\mathcal{G}$-elimination forest. Let v be a vertex in F. The *tail* of v, denoted with $\textsc{tail}(v)$, is defined as the union of B_u over all proper ancestors u of v. The closed tail $\textsc{tail}[v]$ also includes B_v. Similarly, $\textsc{tree}(v)$ denotes the union of B_u over all proper descendants u of v and $\textsc{tree}[v]$ also includes B_v. The subgraph of G induced by all vertices in $\textsc{tree}[v]$ is denoted with G_v. We will sometimes slightly abuse notation and use G_v as a vertex set. We use the shorthand G_v^+ to describe the induced subgraph on the vertices in $\textsc{tree}[v] \cup \textsc{tail}[v]$.

We will also introduce the notion of *bridge-depth* as introduced by Bougeret, Jansen and Sau [3]. A *bridge* in a graph G is an edge whose removal increases the number of connected components of G. The concept of bridge-depth now allows us to delete a set of vertices S as long as $G[S]$ is connected and each edge in $G[S]$ is a bridge in G. Such a structure $G[S]$ is called a *tree of bridges*. Observe that a single vertex is always a tree of bridges.

Definition 3 (Bridge-depth). *Let G be a graph. The bridge-depth of G is defined as*

$$\textsc{bd}(G) = \begin{cases} 0 & \text{if } G \text{ is the empty graph,} \\ \max\limits_{G' \in \textsc{cc}(G)} \textsc{bd}(G') & \text{if } |\textsc{cc}(G)| > 1, \\ \min\limits_{\substack{S \subseteq V(G): \\ G[S] \text{ is a tree of bridges}}} \textsc{bd}(G - S) + 1 & \text{otherwise.} \end{cases}$$

Cf. [3] for equivalent definitions. Lastly, we sometimes use the more common concept of *treewidth*. The treewidth of a graph G is denoted with $\textsc{tw}(G)$. We mention some useful properties of these concepts in Proposition 1.

Proposition 1 ($\star$). *Let G be a graph with $\mathcal{G}_F$-elimination forest $(F, (B_u)_{u \in V(F)})$ and let η be an integer such that $\textsc{ed}_{\mathcal{G}_F}(G) \le \eta$. Let X be a minimum feedback vertex set in G and let v be a leaf in F. Then the following claims hold.*

1. $\textsc{tw}(G) \le \textsc{bd}(G) \le \textsc{ed}_{\mathcal{G}_F}(G) + 1$.
2. *X contains at most η vertices from B_v.*
3. *If there exists a path in G from a vertex in B_v to a vertex outside B_v, then this path contains a vertex in $\textsc{tail}(v)$.*
4. *Let $u \in V(F)$, then u has at most η children c where $X \cap G_c$ is not a minimum feedback vertex set in G_c.*

3 Kernelization Upper Bounds

Our kernel follows the structure of the polynomial kernel for $\mathcal{F}$-Minor Free Deletion when parameterized by a treedepth-η modulator for some integer η [20]. Our kernel relies crucially on the reduction rule specified in Lemma 2.

Lemma 2 ($\star$ Cf. [20], Lemma 6). *There is a polynomial-time algorithm that, given a graph G with modulator $X \subseteq V(G)$ such that $\textsc{ed}_{\mathcal{G}_F}(G - X) \le \eta$ for a constant η, outputs an induced subgraph G' of G together with an integer Δ such that $\textsc{fvs}(G) = \textsc{fvs}(G') + \Delta$ and $G' - X$ has at most $|X|^{\mathcal{O}_\eta(1)}$ components.*

We can use this reduction rule to obtain a graph G' where $G' - X$ has a bounded number of connected components. We can then identify a set of vertices $Y \subseteq V(G' - X)$ with $|Y| \leq |X|^{\mathcal{O}_\eta(1)}$ such that $\text{ED}_{\mathcal{G}_F}(G' - X - Y) < \eta$. By definition of elimination distance, every connected component C of $G' - X$ contains a vertex whose removal decreases $\text{ED}_{\mathcal{G}_F}(C)$. As we limited the number of connected components by applying Lemma 2, these vertices constitute a suitable set Y. Now observe that $X \cup Y$ is a modulator to a graph with elimination distance to a forest $\eta - 1$ and that $|X \cup Y|$ is bounded by a polynomial in $|X|$. One can therefore provide an inductive argument which repeatedly applies Lemma 2 and increases the modulator such that the elimination distance to a forest of the remaining graph decreases every iteration. Once we obtain a modulator to a graph with elimination distance to a forest 1 (which is a forest), we can apply a known polynomial kernel in the size of a feedback vertex set [16].

The reduction rule of Lemma 2 follows from the following key lemma using known techniques. We therefore focus our attention on the proof of Lemma 1.

Lemma 1 (⋆). *Let G be a connected graph with disjoint vertex sets $S, T \subseteq V(G)$. Suppose that any minimum feedback vertex set X of G misses some vertex from S or leaves two vertices from T connected in $G - X$. Then there exist sets $S^* \subseteq S$ and $T^* \subseteq T$ whose sizes only depend on the elimination distance to a forest of G, such that any minimum feedback vertex set X of G misses some vertex from S^* or leaves two vertices from T^* connected in $G - X$.*

We can split up Lemma 1 into two parts. Lemma 3 will bound the number of vertices in the $\mathcal{G}_F$-elimination tree that contain a vertex in S or T. This part corresponds to the original treedepth formulation in [20, Lemma 3], but is significantly simplified for our restricted setting. Lemma 4 bounds the number of vertices in S and T in a bag of the elimination tree. This covers the generalization to elimination distance to a forest and concludes the proof of Lemma 1. In the full version [10], we show how these lemmas imply Lemma 1.

Lemma 3. *Let G be a connected graph with disjoint vertex sets $S, T \subseteq V(G)$. Let $(R, (B_u)_{u \in V(R)})$ be a $\mathcal{G}_F$-elimination tree of G of height η. Suppose that any minimum feedback vertex set X of G misses a vertex from S or leaves two vertices from T connected. Then this also holds for some subsets $S^* \subseteq S$ and $T^* \subseteq T$, such that any vertex in the elimination tree has at most $3\eta \cdot 2^\eta$ children u for which G_u contains a vertex from S^* or T^*.*

Proof. In analogy to the original formulation in [20], a *labeled vertex* is a vertex in S or T. When we remove a label from a vertex, we remove the vertex from S and T while the vertex remains in the graph. Suppose that we pick S^* and T^* such that no minimum feedback vertex set contains S^* and separates T^*, while the latter property does not hold for any other pair of sets S', T' with $S' \subseteq S^*$ and $T' \subseteq T^*$. We claim that for such sets S^* and T^*, any vertex in the elimination tree R has at most $3\eta \cdot 2^\eta$ children u for which G_u contains a labeled vertex. We will also refer to the set T^* as the set of terminals.

Assume towards a contradiction that vertex v has more child subtrees with labels. Let these children be $c_1, \ldots, c_\ell$. For each of these children c_i, there exists a minimum feedback vertex set X_i in G that witnesses the fact that the labels cannot be removed from G_{c_i}. This set X_i will therefore miss a vertex in $S^* \cap G_{c_i}$ or leave a vertex in $T^* \cap G_{c_i}$ connected to some other vertex in T^*, while X_i contains all vertices in $S^* \setminus G_{c_i}$ and separates all vertices in $T^* \setminus G_{c_i}$. Define $Z_i := \textsc{tail}[v] \setminus X_i$.

Now fix a set $Z \subseteq \textsc{tail}[v]$. We will bound the number of children c_i for which $Z_i = Z$ by 3η. Suppose towards a contradiction that there are $3\eta + 1$ of these children. Let C be the set containing these vertices. Pick some child $c_j \in C$ and observe the following.

- By Proposition 1.4, there are at most η children $c_i \in C$ where $X_j \cap G_{c_i}$ is not a minimum feedback vertex set in G_{c_i}.
- There are at most η children $c_i \in C$ with $i \neq j$ such that a terminal in G_{c_i} is connected to a vertex in Z in $G^+_{c_i} - X_j$, i.e. (recall the notation in Sect. 2.1) in the induced subgraph on the remaining vertices in the subtree and tail of c_i. Otherwise, two children other than c_j have a terminal connected to the same vertex in Z, while X_j separates all terminals outside G_{c_j}.
- There are at most $\eta - 1$ children $c_i \in C$ such that in $G^+_{c_i} - X_j$, there exists a path between distinct vertices in Z that uses some vertex in G_{c_i}. Otherwise, we claim that we can directly construct a cycle in $G - X_j$. Consider for example the auxiliary (multi)graph on vertex set Z which contains, for each child $c_i \in C$ for which $G^+_{c_i} - X_j$ contains such a path, say between $z_1, z_2 \in Z$, one edge $z_1 z_2$. This auxiliary graph contains a cycle since it has too many edges to be acyclic, which implies that there exists a cycle in $G - X_j$.

Pick a child $c_k \in C$ which is neither c_j nor in the list of $3\eta - 1$ children above. As $|C| > 3\eta$, such a vertex exists. By the first item above, we can deduce that $X_j \cap G_{c_k}$ is a minimum feedback vertex set in G_{c_k}. Besides, this set contains $S^* \cap G_{c_k}$, it separates all terminals in $T^* \cap G_{c_k}$, and it separates $T^* \cap G_{c_k}$ from Z. Furthermore, no path exists in $G^+_{c_k} - X_j$ that connects two vertices in Z and also contains some vertex in G_{c_k}.

Claim 1 (⋆). The set $X' := (X_k \setminus G_{c_k}) \cup (X_j \cap G_{c_k})$ is a minimum feedback vertex set in G which contains S^* and separates T^*.

Claim 1 contradicts that any minimum feedback vertex set in G misses a vertex in S^* or leaves two vertices in T^* connected. We conclude that there are at most 3η children c_i of v for which a witnessing minimum feedback vertex set has $Z_i = Z$. As there are at most 2^η subsets of $\textsc{tail}[v]$ for any non-leaf v, this leads to the bound of at most $3\eta \cdot 2^\eta$ children for which the labels cannot be removed. □

Lemma 4. *Let G be a connected graph with disjoint vertex sets $S, T \subseteq V(G)$. Let $\big(R, (B_u)_{u \in V(R)}\big)$ be a $\mathcal{G}_F$-elimination tree of G of height η. Suppose that any minimum feedback vertex set X of G misses a vertex from S or leaves two* *vertices from T connected. Then this also holds for some subsets $S^* \subseteq S$ and $T^* \subseteq T$, such that for any leaf u in the elimination tree, the set B_u contains at most η vertices from S^* and at most $\mathcal{O}(\eta^2)$ from T^*.*

Proof. Pick some leaf v of elimination tree R, for which we want to ensure that there are $\mathcal{O}(\eta^2)$ vertices with labels among vertices in $Y := B_v$. Define $S_Y := S \cap Y$ and $T_Y := T \cap Y$. Our goal is to obtain subsets $S_Y^* \subseteq S_Y$ and $T_Y^* \subseteq T_Y$ whose sizes are $\mathcal{O}(\eta^2)$, such that every minimum feedback vertex set misses a vertex from $S_Y^* \cup (S \backslash Y)$ or leaves a pair of terminals in $T_Y^* \cup (T \backslash Y)$ connected. By applying this operation to all leaves of the elimination tree, we obtain the sets promised by Lemma 4.

The construction of S_Y^* is straightforward. If $|S_Y| > \eta + 1$, we let S_Y^* be an arbitrary subset of S_Y of size $\eta + 1$. Otherwise, $S_Y^* = S_Y$.

Claim 2. Let X be a minimum feedback vertex set in G. If X misses a vertex in S, then it also misses a vertex in $S_Y^* \cup (S \backslash Y)$.

Proof. If X misses a vertex in $S \backslash Y$, then the implication is trivial. Therefore assume X misses a vertex in S_Y. If this vertex is not in S_Y^*, then $|S_Y^*| = \eta + 1$ by construction. By Proposition 1.2, we know that $|X \cap S_Y^*| \leq \eta$ so X misses a vertex in S_Y^*. ■

For the construction of T_Y^* we distinguish two cases. First, we assume that T_Y cannot be separated with η vertices in $G[Y]$ and make the following observation.

Proposition 2 (⋆). *Let G be a tree and $T \subseteq V(G)$. If T cannot be separated with η vertices, then there exist $\eta + 1$ vertex-disjoint paths whose endpoints are distinct vertices in T.*

We define T_Y^* by taking the $2\eta + 2$ endpoints of the paths guaranteed by Proposition 2. Observe that these vertices are all in T_Y.

Claim 3. Suppose that T_Y cannot be separated with η vertices in $G[Y]$. Let X be a minimum feedback vertex set in G. Then X leaves two vertices in T_Y^* connected.

Proof. By Proposition 1.2, X can only intersect η of the $\eta + 1$ vertex-disjoint paths that were obtained through Proposition 2. Therefore, at least one path is disjoint from X, so its endpoints in T_Y^* are connected in $G - X$. ■

It remains to consider the case where T_Y can be separated with η vertices. Let Z be a vertex multiway cut of T_Y in $G[Y]$ with $|Z| \leq \eta$ and let $\mathcal{C} := \textsc{cc}(G[Y] - Z)$. Observe that each of these connected components is a tree with at most one vertex in T_Y. Let $\mathcal{C}_T \subseteq \mathcal{C}$ be the set of components that contain a vertex in T_Y. We are now going to mark components. For each $z \in Z$, mark $\eta + 2$ components in $\mathcal{C}_T$ that are adjacent to z in $G[Y]$, or all if there are fewer. Similarly, for each $u \in \textsc{tail}(v)$ we mark up to $\eta + 2$ components in $\mathcal{C}_T$ that are adjacent to v in G_v^+. Then we define T_Y^* to be the union of all vertices in T_Y in the marked components, together with $Z \cap T_Y$. These are at most $\eta(\eta + 2) + \eta(\eta + 2) + \eta = \mathcal{O}(\eta^2)$ vertices.

Claim 4. Suppose that T_Y can be separated with η vertices in $G[Y]$. Let X be a minimum feedback vertex set in G and suppose that X leaves two vertices in T connected. Then X also leaves two vertices in $T_Y^* \cup (T \setminus Y)$ connected.

Proof. Let Z be the vertex multiway cut used in the construction of T_Y^* and let $t_1, t_2 \in T$ be two terminals that are connected in $G - X$. If they are both in $T_Y^* \cup (T \setminus Y)$, then the implication is trivial, so assume that $t_1 \in T_Y$ but not in T_Y^*. Observe that therefore $t_1 \notin Z$. Let P be a path from t_1 to t_2 in $G - X$ and let z be the first vertex on this path that is not in $G[Y] - Z$. We now distinguish two cases. If $z \in Z$, then observe that t_1 was in a component in $\mathcal{C}_T$ that was not marked. Then there are $\eta + 2$ marked components in $\mathcal{C}_T$ adjacent to z in $G[Y]$ of which the terminals are in T_Y^*. Only η of these components can be intersected by X by Proposition 1.2, so there exists a path between two terminals in T_Y^* in $G[Y]$. If $z \notin Z$, then we obtain that $z \in \textsc{tail}(v)$ by Proposition 1.3 and the case follows analogously. ■

This concludes the construction of the sets S_Y^* and T_Y^*. If any minimum feedback vertex set in G misses a vertex in S or leaves a pair of terminals in T connected, then it also misses a vertex in $S_Y^* \cup (S \setminus Y)$ or leaves a pair of terminals in $T_Y^* \cup (T \setminus Y)$ connected. By applying this operation to all leaves of the elimination tree, we obtain the promised sets S^* and T^* which concludes the proof of Lemma 4. □

With Lemma 3 and Lemma 4 proven, we conclude the proof of Lemma 1: if any minimum feedback vertex set in a graph G misses a vertex from a set $S \subseteq V(G)$ or leaves two terminals in a set $T \subseteq V(G)$ connected, then this property also holds for sets $S^* \subseteq S$ and $T^* \subseteq T$ whose sizes only depend on $\textsc{ed}_{\mathcal{G}_F}(G)$. This is the key ingredient for the proof of Lemma 2, which leads to the kernel upper bound.

4 Kernelization Lower Bounds

In this section we summarize the main ideas behind the lower bound. We first introduce the notion of a necklace, which turns out to be a crucial structure.

Definition 4. *Let G be a graph and let $\mathcal{F}$ be a collection of connected graphs. G is an $\mathcal{F}$-*necklace *of length t if there exists a partition of $V(G)$ into $S_1, \ldots, S_t$ such that*

- *$G[S_i] \in \mathcal{F}$ for each $i \in [t]$ (these subgraphs are the* beads *of the necklace),*
- *G has precisely one edge between S_i and S_{i+1} for each $i \in [t-1]$,*
- *G has no edges between any other pair of sets S_i and S_j.*

When the length of the necklace is not relevant, we simply speak of an $\mathcal{F}$-necklace. The following definition specifies a special type of necklace.

Definition 5. *Let $\mathcal{F}$ be a collection of connected graphs. Let G be an $\mathcal{F}$-necklace of length t. We say that G is a* uniform necklace *if it satisfies two additional conditions.*

– *There exists a graph $H \in \mathcal{F}$ such that each bead $G[S_i]$ is isomorphic to H.*
– *There exist $x, y \in V(H)$ and graph isomorphisms $f_i \colon V(H) \to V(G[S_i])$ for each bead $G[S_i]$, such that for each $i \in [t-1]$, the edge between $G[S_i]$ and $G[S_{i+1}]$ has precisely the endpoints $f_i(x)$ and $f_{i+1}(y)$.*

These concepts are used to derive the following characterization. We say that a set contains arbitrarily long necklaces if there does not exist a constant c such that each necklace in the set has length at most c.

Lemma 5 ($\star$). *Let $\mathcal{F}$ be a finite collection of connected planar graphs. Any minor-closed graph family $\mathcal{G}$ with unbounded elimination distance to an $\mathcal{F}$-minor free graph contains arbitrarily long uniform $\mathcal{F}$-necklaces.*

Then we will prove the following lemma by giving a reduction from CNF Satisfiability parameterized by the number of variables [11].

Lemma 6 ($\star$). *Let $\mathcal{F}$ be a finite set of biconnected planar graphs on at least three vertices and let $\mathcal{G}$ be a minor-closed graph family. If $\mathcal{G}$ contains arbitrarily long uniform $\mathcal{F}$-necklaces, then $\mathcal{F}$-Minor Free Deletion does not admit a polynomial kernel in the size of a $\mathcal{G}$-modulator, unless* $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.

Lemma 5 and Lemma 6 together directly imply Theorem 2. We will explain the main ideas of the proof of Lemma 5 here. Our proof follows the proof by Bougeret et al. when they characterize graph families with unbounded bridge-depth [3]. Similar to their work, we define $\textsc{nm}_{\mathcal{F}}(G)$ to be the length of the longest $\mathcal{F}$-necklace that a graph G contains as a minor for a family of connected graphs $\mathcal{F}$. Our goal is now to prove the existence of a small set X such that $\textsc{nm}_{\mathcal{F}}(G - X) < \textsc{nm}_{\mathcal{F}}(G)$ as described in Lemma 7.

Lemma 7 ($\star$). *Let $\mathcal{F}$ be a collection of connected planar graphs. Then there exists a polynomial function $f_{\mathcal{F}} \colon \mathbb{N} \to \mathbb{N}$ such that for any connected graph G with $\textsc{nm}_{\mathcal{F}}(G) = t$, there exists a set $X \subseteq V(G)$ with $|X| \leq f_{\mathcal{F}}(t)$ such that $\textsc{nm}_{\mathcal{F}}(G - X) < t$.*

Bougeret et al. showed that one can derive a bounding function when the considered structures satisfy the Erdős-Pósa property [3]. This also is the case for $\mathcal{F}$-necklaces when the graphs in $\mathcal{F}$ are connected and planar, so this approach would be suitable for our purposes as well. To derive a polynomial bound on the size of X, we use a different argument that uses treewidth and grid minors. We start with the following property of planar graphs.

Proposition 3 ($\star$). *Any planar graph G on n vertices is a minor of the $4n \times 4n$ grid.*

Together with the Excluded Grid Theorem, this leads to the following treewidth bound.

Lemma 8 ($\star$). *Let $\mathcal{F}$ be a collection of connected planar graphs of at most n vertices each. There exists a polynomial $f \colon \mathbb{N} \to \mathbb{N}$ with $f(g) = \mathcal{O}(g^{19} \mathrm{poly} \log g)$ such that for any graph G with $\textsc{nm}_{\mathcal{F}}(G) = t$, it holds that $\textsc{tw}(G) < f(4n(t+1))$.*

To use this treewidth bound, we need a property similar to [3, Lemma 4.6].

Proposition 4 (⋆). *For any family of connected graphs $\mathcal{F}$ and connected graph G with $\text{NM}_{\mathcal{F}}(G) > 0$, any pair of minor models of $\mathcal{F}$-necklaces of length $\text{NM}_{\mathcal{F}}(G)$ in G must intersect.*

Proposition 4 is a generalization of the idea that in any connected graph, two paths of maximum length must intersect at a vertex. Given a graph G with a tree decomposition, we can use this property to identify a vertex in the tree decomposition such that the removal of all vertices in its bag decreases $\text{NM}_{\mathcal{F}}(G)$. This result is described in Lemma 9.

Lemma 9 (⋆). *Let $\mathcal{F}$ be a collection of connected graphs. Let G be a connected graph with $\text{TW}(G) = w$ and $\text{NM}_{\mathcal{F}}(G) = t$. Then there exists a set $Z \subseteq V(G)$ with $|Z| \leq w + 1$ such that $\text{NM}_{\mathcal{F}}(G - Z) < t$.*

The proof of Lemma 7 follows directly by combining Lemma 8 and Lemma 9. An inductive argument, analogous to [3, Theorem 4.8], remains to use this result to prove Lemma 5.

5 Conclusion and Discussion

We conclude that the elimination distance to a forest characterizes the Feedback Vertex Set problem in terms of polynomial kernelization. For a minor-closed graph family $\mathcal{G}$, the problem admits a polynomial kernel in the size of a $\mathcal{G}$-modulator if and only if $\mathcal{G}$ has bounded elimination distance to a forest, assuming NP $\not\subseteq$ coNP/poly. In particular, this implies that Feedback Vertex Set does not admit a polynomial kernel in the deletion distance to a graph of constant bridge-depth under the mentioned hardness assumption. We also generalize the lower bound to other $\mathcal{F}$-Minor Free Deletion problems where $\mathcal{F}$ contains only biconnected planar graphs on at least three vertices. It remains unknown whether such a lower bound also generalizes to collections of graphs $\mathcal{F}$ that contain non-planar graphs.

An interesting open problem is whether similar polynomial kernels can be obtained for other $\mathcal{F}$-Minor Free Deletion problems. Regarding the field of fixed parameter tractable algorithms, it was recently shown [19] that for any set $\mathcal{F}$ of connected graphs, $\mathcal{F}$-Minor Free Deletion admits an FPT algorithm when parameterized by the elimination distance to an $\mathcal{F}$-minor free graph (or even $\mathcal{H}$-treewidth when $\mathcal{H}$ is the class of $\mathcal{F}$-minor free graphs). This generalizes known FPT algorithms for the natural parameterization by solution size. Regarding polynomial kernels, $\mathcal{F}$-Minor Free Deletion problems admit a polynomial kernel in the solution size when $\mathcal{F}$ contains a planar graph [14]. Do polynomial kernels exist when the problem is parameterized by a modulator to a graph of constant elimination distance to being $\mathcal{F}$-minor free?

References

1. Abu-Khzam, F.N., Fellows, M.R., Langston, M.A., Suters, W.H.: Crown structures for vertex cover kernelization. Theor. Comput. Syst. **41**(3), 411–430 (2007). https://doi.org/10.1007/s00224-007-1328-0
2. Bodlaender, H.L., van Dijk, T.C.: A cubic kernel for feedback vertex set and loop cutset. Theor. Comput. Syst. **46**(3), 566–597 (2010). https://doi.org/10.1007/s00224-009-9234-2
3. Bougeret, M., Jansen, B.M.P., Sau, I.: Bridge-depth characterizes which structural parameterizations of vertex cover admit a polynomial kernel. In: Czumaj, A., Dawar, A., Merelli, E. (eds.) 47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, 8–11 July 2020, Saarbrücken, Germany (Virtual Conference). LIPIcs, vol. 168, pp. 1–19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020). https://doi.org/10.4230/LIPIcs.ICALP.2020.16
4. Bougeret, M., Sau, I.: How much does a treedepth modulator help to obtain polynomial kernels beyond sparse graphs? Algorithmica **81**(10), 4043–4068 (2018). https://doi.org/10.1007/s00453-018-0468-8
5. Bulian, J., Dawar, A.: Fixed-parameter tractable distances to sparse graph classes. Algorithmica **79**(1), 139–158 (2016). https://doi.org/10.1007/s00453-016-0235-7
6. Burrage, K., Estivill-Castro, V., Fellows, M., Langston, M., Mac, S., Rosamond, F.: The undirected feedback vertex set problem has a poly(k) kernel. In: Bodlaender, H.L., Langston, M.A. (eds.) IWPEC 2006. LNCS, vol. 4169, pp. 192–202. Springer, Heidelberg (2006). https://doi.org/10.1007/11847250_18
7. Chen, J., Kanj, I.A., Jia, W.: Vertex cover: further observations and further improvements. J. Algorithms **41**(2), 280–301 (2001). https://doi.org/10.1006/jagm.2001.1186
8. Chlebík, M., Chlebíková, J.: Crown reductions for the minimum weighted vertex cover problem. Discret. Appl. Math. **156**(3), 292–312 (2008). https://doi.org/10.1016/j.dam.2007.03.026
9. Chor, B., Fellows, M., Juedes, D.: Linear kernels in linear time, or how to save k colors in $O(n^2)$ steps. In: Hromkovič, J., Nagl, M., Westfechtel, B. (eds.) WG 2004. LNCS, vol. 3353, pp. 257–269. Springer, Heidelberg (2004). https://doi.org/10.1007/978-3-540-30559-0_22
10. Dekker, D., Jansen, B.M.P.: Kernelization for feedback vertex set via elimination distance to a forest. CoRR abs/2206.04387 (2022). https://doi.org/10.48550/arXiv. 2206.04387
11. Dell, H., van Melkebeek, D.: Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses. J. ACM **61**(4), 1–27 (2014). https://doi.org/10.1145/2629620
12. Downey, R.G., Fellows, M.R.: Fundamentals of Parameterized Complexity. TCS, Springer, London (2013). https://doi.org/10.1007/978-1-4471-5559-1
13. Fellows, M.R., Lokshtanov, D., Misra, N., Mnich, M., Rosamond, F.A., Saurabh, S.: The complexity ecology of parameters: an illustration using bounded max leaf number. Theor. Comput. Syst. **45**(4), 822–848 (2009). https://doi.org/10.1007/s00224-009-9167-9
14. Fomin, F.V., Lokshtanov, D., Misra, N., Saurabh, S.: Planar $\mathcal{F}$-Deletion: approximation, kernelization and optimal FPT algorithms. In: 53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, 20–23 October 2012, pp. 470–479. IEEE Computer Society (2012). https://doi.org/10.1109/FOCS.2012.62

15. Hols, E.C., Kratsch, S.: Smaller parameters for vertex cover kernelization. In: Lokshtanov, D., Nishimura, N. (eds.) 12th International Symposium on Parameterized and Exact Computation, IPEC 2017, 6–8 September 2017, Vienna, Austria. LIPIcs, vol. 89, pp. 1–12. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2017). https://doi.org/10.4230/LIPIcs.IPEC.2017.20
16. Iwata, Y.: Linear-time kernelization for feedback vertex set. In: Chatzigiannakis, I., Indyk, P., Kuhn, F., Muscholl, A. (eds.) 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, 10–14 July 2017, Warsaw, Poland. LIPIcs, vol. 80, pp. 1–14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2017). https://doi.org/10.4230/LIPIcs.ICALP.2017.68
17. Jansen, B., Raman, V., Vatshelle, M.: Parameter ecology for feedback vertex set. Tsinghua Sci.Technol. **19**(4), 387–409 (2014). https://doi.org/10.1109/TST.2014.6867520
18. Jansen, B.M.P., Bodlaender, H.L.: Vertex cover kernelization revisited. Theor. Comput. Syst. **53**(2), 263–299 (2012). https://doi.org/10.1007/s00224-012-9393-4
19. Jansen, B.M.P., de Kroon, J.J.H., Włodarczyk, M.: Vertex deletion parameterized by elimination distance and even less. In: Khuller, S., Williams, V.V. (eds.) STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, 21–25 June 2021, pp. 1757–1769. ACM (2021). https://doi.org/10.1145/3406325.3451068
20. Jansen, B.M.P., Pieterse, A.: Polynomial kernels for hitting forbidden minors under structural parameterizations. Theor. Comput. Sci. **841**, 124–166 (2020). https://doi.org/10.1016/j.tcs.2020.07.009
21. Karp, R.M.: Reducibility among combinatorial problems. In: Miller, R.E., Thatcher, J.W. (eds.) Proceedings of a symposium on the Complexity of Computer Computations, Held 20–22 March 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, USA. The IBM Research Symposia Series, pp. 85–103. Plenum Press, New York (1972). https://doi.org/10.1007/978-1-4684-2001-2_9
22. Majumdar, D., Raman, V.: Structural parameterizations of undirected feedback vertex set: FPT algorithms and kernelization. Algorithmica **80**(9), 2683–2724 (2018). https://doi.org/10.1007/s00453-018-0419-4
23. Majumdar, D., Raman, V., Saurabh, S.: Polynomial kernels for vertex cover parameterized by small degree modulators. Theor. Comput. Syst. **62**(8), 1910–1951 (2018). https://doi.org/10.1007/s00224-018-9858-1
24. Nemhauser, G.L., Jr., Trotter, L.E.: Vertex packings: structural properties and algorithms. Math. Program. **8**(1), 232–248 (1975). https://doi.org/10.1007/BF01580444
25. Nešetřil, J., Ossona de Mendez, P.: Sparsity - Graphs, Structures, and Algorithms, Algorithms and Combinatorics, vol. 28. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-27875-4
26. Thomassé, S.: A quadratic kernel for feedback vertex set. In: Mathieu, C. (ed.) Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2009, New York, NY, USA, 4–6 January 2009, pp. 115–119. SIAM (2009). http://dl.acm.org/citation.cfm?id=1496770.1496783

Open Access This chapter is licensed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license and indicate if changes were made.

The images or other third party material in this chapter are included in the chapter's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the chapter's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.